Mccarthy, P. J., & Nielsen, C. (2021). Global Sampled-Data Regulation of a Class of Fully Actuated Invariant Systems on Simply Connected Nilpotent Matrix Lie Groups. IEEE Transactions on Automatic Control, 1–1. https://doi.org/10.1109/TAC.2021.3058053

Global Sampled-Data Regulation of a Class of Fully Actuated Invariant Systems on Simply Connected Nilpotent Matrix Lie Groups

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***Abstract*—We examine a regulator problem for a class of fully actuated continuous-time invariant systems on Lie groups, using a discrete-time controller with constant sampling period. We present a smooth discrete-time control law that achieves global regulation on simply connected nilpotent Lie groups. We first solve the problem when both the plant state and exosystem state are available for feedback, then in the case where the plant state and plant output are available for feedback. The class of plant outputs considered includes, for example, the quantity to be regulated. This class of outputs allows us to use the classical Luenberger observer to estimate the exosystem states. In the full-information case, the regulation quantity on the Lie algebra is shown to decay exponentially to zero, which implies that it tends asymptotically to the identity on the Lie group.**

I. Introduction

The regulator problem is central to control theory; it combines reference tracking, disturbance rejection, and stabilization. The problem was completely solved in the continuous-time linear case in the seminal papers [1], [2], [3]. These linear results were extended to nonlinear systems in [4], wherein the continuous-time nonlinear regulator equations—the celebrated Francis-Byrnes-Isidori equations—were developed; this work inspired analogous results for the discrete-time regulator problem in [5]. More recently, researchers have tried to specialize continuous-time regulator problems to classes of systems evolving on Lie groups. The output regulation problem was solved for a class of systems evolving on the special Euclidean group $\mathsf{SE}(n)$ in [6] by identifying a separation principle. In [7], an almost-global solution to the output regulation problem for a class of systems on $\mathsf{SE}(3)$ was proposed; these results were extended to local results on arbitrary Lie groups in [8].

Many engineering systems can be modelled on Lie groups, which eliminates dependence on local coordinates, thereby avoiding representational singularities in the dynamical model. The motion of robots in a plane is modelled on the solvable Lie group $\mathsf{SE}(2)$ [9], and their motion in space, such as that of UAVs', is modelled on $\mathsf{SE}(3)$ [10]. Quantum systems evolve on the unitary groups $\mathsf{U}(n)$ [11] and $\mathsf{SU}(n)$ [12], [13]. Even the noise responses of some circuits evolve on Lie groups [14], specifically the solvable Lie group of invertible upper-triangular matrices. Control on the nilpotent Heisenberg group has also been the object of much study [15], [16]. In continuous-time, certain classes of vector fields can be approximated as being on nilpotent Lie algebras [17], [18]. This is of interest because of the relatively simple Lie algebraic structure of nilpotent Lie algebras. Additionally, systems in chained form [19] are expressions of an invariant system on a particular nilpotent Lie group of unipotent matrices [20] in exponential coordinates of the second kind. In this paper, we study the regulator problem for *sampled-data systems* on nilpotent Lie groups.

To the authors' knowledge, the literature on the sampled-data regulator problem for systems on Lie groups is sparse, currently comprising only [21], which developed a control law for invariant systems that achieves global tracking on certain Lie groups for sufficiently fast sampling, our preliminary work [22] on commutative Lie groups, and step tracking using passivity for general Lie groups [23].

The sampled-data setup is ubiquitous in applied control [24]. The design of a discrete-time controller for a discretized plant is called *direct design*. Since nonlinear ODEs generally do not have closed-form solutions, the plant dynamics must usually be approximately discretized. Approximation-based direct design has two main weaknesses [25]: 1) closed-loop stability may be impossible for a given discretization method; 2) when closed-loop stability is possible, it relies on fast sampling, which may be infeasible. For example, when using machine vision, the sampling rate may be limited by the framerate of the camera [26]. The latter issue is also the main weakness of *emulation*: solving the control problem in continuous-time, but implementing a discrete-time controller that approximates the continuous-time controller at the sampling instants [27].

Systems belonging to the class of left- and right-invariant systems on Lie groups have trajectories that can be expressed in closed-form using the matrix exponential [28]. Thus, direct design may be performed without resorting to approximations. In this paper, we examine a regulation problem for invariant systems on simply connected nilpotent Lie groups. The study of this class of systems is partially motivated by the nilpotent approximation techniques developed in [17], [18]. We show that, when the group is nilpotent and the plant is fully actuated, that the origin of the Lie algebra can be made semiglobally exponentially stable; as a corollary, the identity of the Lie group is globally asymptotically stable. We show that when the trajectories of the exosystem are bounded, the intersample behaviour of the closed-loop system is also bounded.

One of the main contributions of this work is to elucidate how the stability results from [29] can be used to design feedback controllers that solve a regulation problem. This allows for a significant generalization of the results in [22], which applied only to commutative Lie groups. We show how the restrictive assumptions of [29] (Property 1 in the current paper) can be enforced by a judicious choice of state feedback.

*Partially supported by the Ontario Graduate Scholarship (OGS).
†Supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

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A. Notation and Terminology

If $N \in \mathbb{N}$, then $\mathbb{N}_N := \{1, \ldots, N\}$. Given a matrix $M \in \mathbb{C}^{n \times n}$, $\sigma(M)$ is its spectrum, and $\rho(M)$ its spectral radius. If $x \in \mathbb{C}^n$, then $\|x\|$ is its norm (the choice of norm is arbitrary but fixed); if $M \in \mathbb{C}^{n \times n}$, then $\|M\|$ is its induced norm. A *word* ω in a Lie algebra $\mathfrak{g}$ with *length* $|\omega| \in \mathbb{N}$ over the $n \in \mathbb{N}$ *letters* $X_1, \ldots, X_n \in \mathfrak{g}$ is a (nested) Lie bracket $[X_{\omega_1}, [X_{\omega_2}, [\ldots X_{\omega_{|\omega|}}] \cdots]$, where $X_{\omega_i} \in \{X_1, \ldots, X_n\}$.

II. Sampled-Data Regulator Problem

We study the sampled-data control problem for the system illustrated in Figure 1.

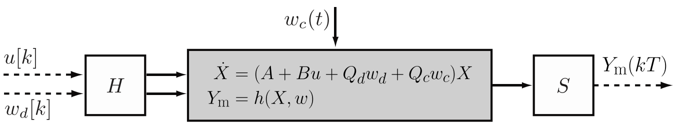

Fig. 1. Sampled-data plant on a Lie group G. The blocks H and S in Figure 1 are the ideal hold and sample operators, respectively, operating synchronously at a common, fixed sampling period $T > 0$.

The plant is modelled by the differential equation

$$\dot{X}(t) = (A + Bu(t) + Q_d w_d(t) + Q_c w_c(t))X(t). \tag{1}$$

The system has a **measured output** Y_{m}, which models the information from the plant and exosystem that is available to the controller. It is convenient to model a **plant output**

$$Y(t) = \exp(C + D_d w_d(t) + D_c w_c(t))X(t), \tag{2}$$

which models a signal that is available for feedback, which is a function of the plant state and exostates. This signal could be, for example, the quantity to be regulated.

We assume, as is typical, that the exogenous signals w_d, w_c, depicted in Figure 1, evolve according to known dynamics, modelled as

$$\begin{aligned} w_d[k+1] &= S_d w_d[k] \\ \dot{w}_c(t) &= S_c w_c(t). \end{aligned} \tag{3}$$

In (1) and (2), $X \in \mathsf{G}$ where $\mathsf{G} \subseteq \mathsf{GL}(N, \mathbb{C})$ is an n-dimensional simply connected nilpotent Lie group—which is formalized in Assumption 1 below—over the complex field $\mathbb{C}$ which includes, as a special case, real Lie groups. The associated Lie algebra is $\mathfrak{g}$, which is an n-dimensional vector space over the field $\mathbb{F}$, which is equal to either $\mathbb{C}$ or $\mathbb{R}$. The control input is $u(t) \in \mathbb{F}^n$, the discrete- and continuous-time exostates are $w_d[k] \in \mathbb{F}^{r_d}$ and $w_c(t) \in \mathbb{F}^{r_c}$, respectively, and $S_d \in \mathbb{F}^{r_d \times r_d}$, $S_c \in \mathbb{F}^{r_c \times r_c}$. The quantities A and C are elements of $\mathfrak{g}$, and $B : \mathbb{F}^n \to \mathfrak{g}$, $Q_d : \mathbb{F}^{r_d} \to \mathfrak{g}$, $Q_c : \mathbb{F}^{r_c} \to \mathfrak{g}$, $D_d : \mathbb{F}^{r_d} \to \mathfrak{g}$, and $D_c : \mathbb{F}^{r_c} \to \mathfrak{g}$ are linear maps.

Equation (1) is an invariant system evolving on a Lie group G, where the plant output (2) is some quantity that could be available for feedback. The exosystem (3) comprises both discrete- and continuous-time subsystems. This enables modelling of, for example, physical plants that are subject to continuous-time disturbances, but are sent reference signals from a computer. We impose four standing assumptions.

Assumption 1. *The Lie group G is simply connected, and nilpotent with nilindex p.*

Under Assumption 1, the Lie Group G is diffeomorphic to its Lie algebra $\mathfrak{g}$, which is isomorphic to $\mathbb{F}^n$; in particular, the exponential map $\exp : \mathfrak{g} \to \mathsf{G}$ is a (global) diffeomorphism. This is easily verified by nilpotency of $\mathfrak{g}$ and the Baker-Campbell Hausdorff formula [30, Theorem 1.2.1].

Assumption 2. *The spectra of S_d and S_c lie outside the open unit disc and in the closed right half plane, respectively.*

Assumption 2 incurs no loss of generality; if necessary, S_d and S_c can be redefined as their restrictions to their respective unstable modal subspaces [31, §2.3].

Assumption 3. *The plant is fully actuated:* $\operatorname{Im} B = \mathfrak{g}$.

The foregoing assumption is necessary for the linearization of the tracking error dynamics to be stabilizable. The following technical assumption greatly simplifies our analysis, and guarantees well-posedness of the closed-loop dynamics, as it obviates use of the Magnus expansion in deriving the exact discretization of (1). See the proof of Proposition II.1.

Assumption 4. *The image of Q_c is in the centre of $\mathfrak{g}$.*

Assumption 4 can be interpreted as the continuous exostate acting as a purely linear disturbance in the plant dynamics on the Lie algebra. Under Assumption 4, letting $X[k] := X(kT)$, $u[k] := u(kT)$, and $w_c[k] := w_c(kT)$, the plant (1) and exosystem (3) have exact discretizations, as we will prove at the end of this section:

$$X^+ = \exp\left(TA + TBu + TQ_d w_d + Q_c \int_0^T \mathrm{e}^{\tau S_c} \mathrm{d}\tau \, w_c \right) X, \tag{4}$$

and

$$\begin{bmatrix} w_d^+ \\ w_c^+ \end{bmatrix} = \underbrace{\begin{bmatrix} S_d & 0 \\ 0 & \mathrm{e}^{T S_c} \end{bmatrix}}_{=:S} \underbrace{\begin{bmatrix} w_d \\ w_c \end{bmatrix}}_{=:w}. \tag{5}$$

To simplify the notation, let $r := r_d + r_c$ and identify $\mathbb{F}^{r_d} \times \mathbb{F}^{r_c}$ with $\mathbb{F}^r$, define $Q : \mathbb{F}^r \to \mathfrak{g}$, $(w_d, w_c) \mapsto TQ_d w_d + Q_c \int_0^T \mathrm{e}^{\tau S_c} \mathrm{d}\tau w_c$, and $D : \mathbb{F}^r \to \mathfrak{g}$, $(w_d, w_c) \mapsto D_d w_d + D_c w_c$. Equipped with this notation, we rewrite the discretized plant dynamics (4) and the plant output in the compact form

$$\begin{aligned} X^+ &= \exp(TA + TBu + Qw)X \\ Y &= \exp(C + Dw)X. \end{aligned} \tag{6}$$

Proposition II.1. *The dynamics (6) are the exact discretization of (1), in the sense that for all $k \geq 0$, $X[k] = X(kT)$.*

Proof. Consider an ODE $\dot{X} = U(t)X(t)$, where $X \in \mathsf{G}$ and $U : \mathbb{R} \to \mathfrak{g}$ is a piecewise continuous time-varying vector field. It is well-known, and can be verified from the Magnus expansion [32], that if for all $t_1, t_2 \geq 0$, $[U(t_1), U(t_2)] = 0$, then the unique solution to this ODE is $X(t) = \exp\left(\int_0^t U(\tau)\mathrm{d}\tau\right) X(0)$.

For (1), for $t \in [kT, (k+1)T)$, we have $U(t) = (A + Bu(kT) + Q_d w_d(kT)) + Q_c w_c(t)$, where, under Assumption 4, $[U(t_1), U(t_2)] = 0$ for all t_1, t_2. Thus, over this sampling period, the ODE (1) is satisfied by

$$X(t) = \exp\Bigg((t-kT)A + (t-kT)Bu(kT) \\ + (t-kT)Q_d w_d(kT) + Q_c \int_{kT}^{t} w_c(\tau)\mathrm{d}\tau\Bigg) X(kT).$$

The rest of the proof follows from routine calculation. □

The goal of the regulator problem is to drive a regulation quantity to identity. We take the regulation quantity to be

$$Z(t) = \exp(F + Gw(t))X(t), \tag{7}$$

where $F \in \mathfrak{g}$ and $G : \mathbb{F}^r \to \mathfrak{g}$ is a linear map.

Given the plant with state $X \in \mathsf{G}$ evolving according to (6), measured output $Y_{\mathrm{m}} : \mathsf{G} \times \mathbb{F}^n \times \mathbb{F}^r \to \mathcal{Y}$, where $\mathcal{Y}$ is some Cartesian product of multiples of G and $\mathbb{F}^r$, and regulation quantity $Z \in \mathsf{G}$ given by (7), the objective of the sampled-data regulator problem is to find, if possible, a control law of the form

$$x_c^+ = f_c(x_c, Y_{\mathrm{m}}), \qquad u = h_c(x_c, Y_{\mathrm{m}}),$$

where x_c belongs to an *a priori* unspecified Lie group[1] such that

1) the closed-loop system is well-posed;
2) for all initial conditions, $Z[k] \to I$ as $k \to \infty$.

We impose no requirements on internal stability or the intersample behaviour. Concerning the latter, see Remark IV.2. Regarding the former, it follows from (7) that, when $w \equiv 0$, there is a unique constant $X^\star$ such that if $X = X^\star$, then $Z = I$; when $F = 0$, this constant is $X^\star = I$. Thus internal stability is trivially satisfied by any regulating control law. We consider two cases: 1) $Y_{\mathrm{m}} = (X[k], w[k])$; 2) $Y_{\mathrm{m}} = (X[k], Y[k])$. The former is called the **regulator problem with full information**, the latter the **regulator problem with plant state information**. In both cases, since there is no direct feedthrough from u to Y_{m}, the closed-loop system is well-posed.

III. Preliminaries

A. Nilpotent Lie Groups and Lie Algebras

We now define what it means for a Lie group or algebra to be nilpotent. We also state several algebraic properties of such Lie algebras used in our analysis.

Definition III.1 (Lower Central Series)**.** *The **lower central series** of a Lie algebra $\mathfrak{g}$ is defined recursively by $\mathfrak{g}^{(1)} := \mathfrak{g}$, $\mathfrak{g}^{(i+1)} := [\mathfrak{g}^{(i)}, \mathfrak{g}]$, for $i \geq 1$.*

There are two important consequences of Definition III.1: the algebras of the lower central series $\mathfrak{g}^{(i)}$ are ideals, and for all $i \geq 1$, $\mathfrak{g}^{(i)} \supseteq \mathfrak{g}^{(i+1)}$.

Definition III.2 (Nilpotent)**.** *A Lie algebra $\mathfrak{g}$ is nilpotent if there exists a finite p such that $\mathfrak{g}^{(p+1)} = 0$. The smallest such p is called the **nilindex** of $\mathfrak{g}$. A Lie group is **nilpotent** if its Lie algebra is nilpotent.*

[1]Possibly commutative, e.g., $\mathbb{R}^{n_c}$ as an additive group.

Theorem III.3 ([30, Theorem 1.2.1])**.** *If G is a simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$, then $\exp : \mathfrak{g} \to \mathsf{G}$ is an analytic diffeomorphism.*

By the foregoing theorem, the matrix exponential has a globally defined inverse, which we denote by $\mathrm{Log} : \mathsf{G} \to \mathfrak{g}$, thus the group G is globally diffeomorphic to a finite dimensional vector space.

B. The Class of Systems

To prove the validity of our proposed solution, we leverage (specializations of) the results in [29]. We first introduce notation that allows us to succinctly describe the class of systems studied therein.

Define the tensor product $(\mathbb{F}^n \otimes \mathfrak{g}, \otimes)$. If $f_1, \dots, f_m$ are series of words with letters $x_1, \dots, x_N \in \mathfrak{g}$, where in f_i, the scalar coefficients of the word ω is $c^i_\omega \in \mathbb{F}$, then

$$f(x) := \begin{bmatrix} f_1(x_1,\dots,x_N) \\ \vdots \\ f_m(x_1,\dots,x_N)\end{bmatrix} = \begin{bmatrix} \sum_\omega c^1_\omega \omega \\ \vdots \\ \sum_\omega c^m_\omega \omega\end{bmatrix} = \sum_\omega \begin{bmatrix} c^1_\omega \\ \vdots \\ c^m_\omega\end{bmatrix} \otimes \omega,$$

which we write compactly as

$$f(x) = \sum_\omega c_\omega \otimes \omega.$$

Relabelling r of the letters as $w_1, \dots, w_r \in \mathfrak{g}$, and redefining x as the remaining letters, we have

$$x^+ = f(x, w) = Ax + Pw + \sum_{|\omega| \geq 2} c_\omega \otimes \omega. \tag{8}$$

We are interested in systems of the form (9), where $f : \mathfrak{g}^N \times \mathfrak{g}^r \to \mathfrak{g}^N$ enjoys Property 1.

Property 1. *The function $f : \mathfrak{g}^N \times \mathfrak{g}^r \to \mathfrak{g}^N$ in (9) enjoys the following properties:*

(a) the origin of $\mathfrak{g}^N$ is a unique fixed point: for all $w \in \mathbb{F}^r$,

$$f(x, w) = 0 \iff x = 0;$$

(b) there exists an ideal $\mathfrak{h} \subseteq \mathfrak{g}$ with nilindex p, such that $\mathfrak{h} \supseteq [\mathfrak{g}, \mathfrak{g}]$, whereof each ideal in the lower central series of $\left(\mathfrak{h}^{(i)}\right)^N \subseteq \mathfrak{g}^N$ is invariant under f, i.e.,

$$f\left(\left(\mathfrak{h}^{(i)}\right)^N, \mathfrak{g}^r\right) \subseteq \left(\mathfrak{h}^{(i)}\right)^N.$$

If (8) enjoys Property 1(a), then $P = 0$ and every word ω has at least one letter in the set $\{x_1, \dots, x_N\}$ [29, Proposition 4]. Therefore, we will focus on systems of the form

$$x^+ = f(x, w) = Ax + \sum_{|\omega| \geq 2} c_\omega \otimes \omega. \tag{9}$$

There are two key results in [29] applicable to the regulator problem under consideration.

Theorem III.4 ([29, Theorem 4])**.** *Consider the dynamics (9). Let $\mathfrak{g}$ be a nilpotent Lie algebra and suppose $f : \mathfrak{g}^N \times \mathfrak{g}^r \to \mathfrak{g}^N$ enjoys Property 1, where Property 1(b) is satisfied with*

$\mathfrak{h} = \mathfrak{g}$. *If there exist* $\beta \geq 0$, $s \geq 1$ *such that* $\|w[k]\| \leq \beta s^k$, *and* $\rho(A) < s^{-\frac{p(p-1)}{2}}$, *then the origin of* $\mathfrak{g}^N$ *is semiglobally exponentially stable.*

Theorem III.5 ([29, Theorem 6])**.** *Consider the dynamics* (9). *Let* $\mathfrak{g}$ *be a nilpotent Lie algebra and suppose* $f : \mathfrak{g}^N \times \mathfrak{g}^r \to \mathfrak{g}^N$ *enjoys Property 1. If* $\rho(A) = 0$ *and for all* $k \geq 0$, $w[k] \in \mathfrak{h}^r$, *then* $x[k]$ *converges to zero in finite time.*

IV. The Solution

In this section, we show that the regulator problem has a solution under the standing assumptions of Section II. We first solve the regulator problem with full information, i.e, $Y_{\mathrm{m}} = (X[k], w[k])$, which is equivalent to $Y_{\mathrm{m}} = (Y[k], w[k])$, since the plant state can be computed algebraically as $X[k] = Y[k] \exp(C + Dw[k])^{-1}$. We then solve the regulator problem when the exostate is not measured, i.e., $Y_{\mathrm{m}} = (X[k], Y[k])$.

To prove our main results, we will invoke Theorems III.4 and III.5, which require the system to enjoy Property 1, which is quite restrictive. Much of our effort in the current paper is devoted to proving that by a judicious choice of state feedback and observer design, the closed-loop system does in fact enjoy the unlikely Property 1.

A. Regulator Problem with Full Information

We solve the regulator problem with full information in two steps: 1) make the tracking manifold $\mathcal{T} := \{(X, w) \in \mathsf{G} \times \mathbb{F}^r : Z = I\}$ positively invariant in discrete-time; 2) make the tracking manifold globally attractive. The tracking manifold $\mathcal{T}$ is positively invariant if there exist $\Pi : \mathbb{F}^r \to \mathsf{G}$ and $\Psi : \mathbb{F}^r \to \mathfrak{g}$ satisfying the regulator equations:

$$\Pi(Sw) = \exp(TA + TB\Psi(w) + Qw)\Pi(w)$$
$$I = \exp(F + Gw)\Pi(w).$$

Straightforward arithmetic yields the **state reference**

$$\Pi(w) = \exp(F + Gw)^{-1} \tag{10}$$

and the friend

$$\Psi(w) = \frac{1}{T} B^{-1} \left(\mathrm{Log}\left(\Pi(Sw)\Pi(w)^{-1}\right) - TA - Qw \right). \tag{11}$$

By construction, if the restriction of u to the tracking manifold equals the friend Ψ, then the tracking manifold is controlled-invariant under the dynamics (6).

Remark IV.1. *These regulator equations are always solvable, because* Z *is a product of group elements on* G, *and* B *is invertible under Assumption 3; this decouples the two equations, which makes it possible for* $X[k]$ *to track any* $\Pi(w[k])$ *when properly initialized.*

In particular, that B *is invertible allows us to choose* u *such that the discrete-time plant dynamics are of the form* $X^+ = UX$, *where* $U \in \mathsf{G}$ *can be set arbitrarily. Choosing* $U[k] = \Pi(Sw[k])X[k]^{-1}$ *results in one-step-ahead deadbeat reference tracking for any sampling period. Technically, such a control law would solve the regulator problem, but, in practice, such a control law would generally be impractical, as it would (attempt to) effect potentially very large actuations in very small time scales, e.g., rotating a* 0.5*-meter-long robotic arm by* 60 *degrees in* 1 *millisecond.*

Remark IV.2. *Regulation at the sampling instants does not imply intersample regulation. If* $u[k] = \Psi(w[k])$, *then the continuous-time plant dynamics* (1) *over the sampling period* $t \in [kT, (k+1)T)$ *are*

$$\dot{X} = \left(\frac{1}{T} \mathrm{Log}\left(\Pi(Sw[k])\Pi(w[k])^{-1}\right) + Q_c w_c - \frac{1}{T} Q_c \int_0^T \mathrm{e}^{\tau S_c} \mathrm{d}\tau w_c[k] \right) X[k].$$

Solving for $X(kT + \delta)$, *where* $\delta \in [0, T)$, *and setting* $X[k] = \Pi(w[k])$, *we have*

$$X(kT+\delta) = \exp\left(\frac{\delta}{T} \mathrm{Log}\left(\Pi(Sw[k])\Pi(w[k])^{-1}\right) + Q_c \left(\int_0^\delta \mathrm{e}^{\tau S_c} \mathrm{d}\tau - \frac{\delta}{T} \int_0^T \mathrm{e}^{\tau S_c} \mathrm{d}\tau \right) w_c[k] \right) \Pi(w[k]), \tag{12}$$

which shows $X(t) \neq \Pi(w[k])$ *for all* $t \in [kT, (k+1)T)$.

Remark IV.2 may seem ominous, but under the standard assumption that the exosystem (5) is neutrally stable, we can partially characterize the intersample behaviour.

Proposition IV.3. *If the trajectories of* (3) *are bounded and* $X[k] = \Pi(w[k])$, *then for all* $t \geq 0$, $X(t)$ *is bounded.*

Proof. Given a square matrix A, $\|\exp(A)\| \leq \exp(\|A\|)$. Applying this property to (12), we obtain

$$\|X(kT+\delta)\| \leq \exp\left(\frac{\delta}{T} \left\|\mathrm{Log}\left(\Pi(Sw[k])\Pi(w[k])^{-1}\right)\right\| + \left\| Q_c \left(\int_0^\delta \mathrm{e}^{\tau S_c} \mathrm{d}\tau - \frac{\delta}{T} \int_0^T \mathrm{e}^{\tau S_c} \mathrm{d}\tau \right) \right\| \|w_c[k]\| \right) \|\Pi(w[k])\|.$$

Since w is bounded, $\Pi(w)$, its inverse, and $\Pi(Sw)$ are bounded. Since $\mathrm{Log} : \mathsf{G} \to \mathfrak{g}$ is a diffeomorphism, the boundedness of $\Pi(Sw)\Pi(w)^{-1}$ implies that $\mathrm{Log}(\Pi(Sw)\Pi(w)^{-1})$ is bounded. Noting that w_c is bounded completes the proof. ☐

The next result addresses the important special case of step reference tracking and disturbance rejection.

Proposition IV.4. *If* w_d *and* w_c *are constant and* $X[k] = \Pi(w[k])$, *then* Z *is identity for all* $t \geq kT$.

Proof. Without loss of generality, we take $S_d = I$ and $S_c = 0$. Then (12) simplifies to $X(kT + \delta) = \Pi(w)$. Thus, $X(t)$ is constant. The result is immediate from (7) and (10). ☐

The two foregoing Propositions furnish analogous corollaries for the intersample behaviour of Z, which follow immediately from (7).

We now use state feedback to make the tracking manifold $\mathcal{T}$ globally attractive. Define the **state-tracking error** $E := X\Pi(w)^{-1}$. By definition, if $E = I$, then $(X, w) \in \mathcal{T}$. We will design a control law that stabilizes the Jacobian linearization of the tracking error dynamics on $\mathfrak{g}$; this of course implies local exponential stability of the tracking manifold $\mathcal{T}$ on any Lie group. We then use the results of [29] to show that such a controller achieves global regulation on nilpotent Lie groups.

We propose a controller of the form

$$u[k] = \Gamma(X[k], w[k]) + \Psi(w[k]),$$

where Ψ is given by (11), and $\Gamma(X, w) := K\,\mathrm{Log}(E)$, where $K : \mathfrak{g} \to \mathbb{F}^n$ is a yet-to-be-specified gain. The tracking manifold $\mathcal{T}$ is rendered invariant by the friend Ψ, and attractive by the state feedback Γ. Define $e := \mathrm{Log}(E)$ and $\pi := \mathrm{Log}\circ\Pi$; under Assumption 1, e is well-defined for all $E \in \mathsf{G}$. Using the proposed controller, the error dynamics on G are

$$\begin{aligned} E^+ &= X^+\Pi(w^+)^{-1} \\ &= \exp\left(TBKe + \mathrm{Log}\left(\Pi(Sw)\Pi(w)^{-1}\right)\right) X\Pi(Sw)^{-1} \\ &= \exp\left(TBKe + \mathrm{Log}\left(\Pi(Sw)\Pi(w)^{-1}\right)\right) E\left(\Pi(Sw)\Pi(w)^{-1}\right)^{-1}. \end{aligned} \tag{13}$$

We examine the tracking error dynamics (13) on the Lie algebra $\mathfrak{g}$. To facilitate this, we use a generalization of the Baker-Campbell-Hausdorff (BCH) formula [33], developed in [34, §5]. Given $A_1, \ldots, A_n \in \mathfrak{g}$,

$$\mathrm{Log}(\exp(A_1)\cdots\exp(A_n)) = \sum_{i=1}^{n} A_i + \frac{1}{2}\sum_{i<j}[A_i, A_j] + \cdots, \tag{14}$$

where the remaining terms comprise scalar multiples of all words of length at least three.

Applying the BCH to $\mathrm{Log}\left(\Pi(Sw)\Pi(w)^{-1}\right)$ in (11), and the generalized BCH to (13), performing some simplifications and rearranging, we obtain

$$\begin{aligned} e^+ &= (I + TBK)e + \frac{1}{2}[TBKe, e] + \frac{1}{2}[\pi(Sw), e] \\ &+ \frac{1}{2}[-\pi(w), e] + \frac{1}{2}[TBKe, \pi(w)] + \frac{1}{2}[TBKe, -\pi(Sw)] + \cdots, \end{aligned}$$

which can be written in the form of (9) thus

$$e^+ = (I + TBK)e + \sum_{\omega} c_\omega \otimes \omega, \tag{15}$$

where the words ω are over the letters $e, BKe, \pi(Sw), \pi(w)$; the disturbance signals are $\pi(Sw)$ and $\pi(w)$. Note that the pair (I, TB) is stabilizable if and only if Assumption 3 is satisfied. We now state the main result of this section.

Theorem IV.5. *If $\Pi : \mathbb{F}^r \to \mathsf{G}$ and $\Psi : \mathbb{F}^r \to \mathbb{F}^n$ are given by (10) and (11), respectively, then there exists $K : \mathfrak{g} \to \mathbb{F}^n$ such that if*

$$u = K\,\mathrm{Log}(X\Pi(w)^{-1}) + \Psi(w), \tag{16}$$

then (16) solves the regulator problem with full information.

To leverage the results of [29] in the proof of Theorem IV.5, we require the following Lemma.

Lemma IV.6. *There exists $K : \mathfrak{g} \to \mathbb{F}^n$ such that the tracking error dynamics on the Lie algebra (15) enjoy Property 1.*

Proof. We verify that (15) enjoys each of Properties 1(a) and 1(b), in order.

Claim 1. *There exists $K : \mathfrak{g} \to \mathbb{F}^n$, such that $(I + TBK)$ is Schur and every subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is BK-invariant.*

Proof. Fix $\alpha \in (0, 2)$ and $K = -\alpha(TB)^{-1}$. Then $BK = -\alpha T^{-1} I$ leaves any subspace invariant, and $(I + TBK) = (1 - \alpha)I$ is Schur. □

Fix $K : \mathfrak{g} \to \mathbb{F}^n$ such that $(I + TBK)$ is Schur and $BK\mathfrak{g}^{(i)} \subseteq \mathfrak{g}^{(i)}$ for all $i \in \mathbb{N}_p$. Such a K exists by Claim 1.

Claim 2. *The dynamics (15) enjoy Property 1(a).*

Proof. Note that $e \in \mathfrak{g}$ is a fixed point of (15) if and only if $E = \exp(e)$ is a fixed point of (13). Solving for the fixed points of (13),

$$\begin{aligned} &\Pi(Sw)\Pi(w)^{-1} = \exp\left(TBKe + \mathrm{Log}\left(\Pi(Sw)\Pi(w)^{-1}\right)\right) \\ &\iff \mathrm{Log}\left(\Pi(Sw)\Pi(w)^{-1}\right) = \\ &\qquad\qquad TBKe + \mathrm{Log}\left(\Pi(Sw)\Pi(w)^{-1}\right) \\ &\iff TBKe = 0. \end{aligned}$$

By the Spectral Mapping Theorem, $(I + TBK)$ is Schur only if 0 is not an eigenvalue of TBK, implying that TBK is an isomorphism. Thus, $TBKe = 0$ if and only if $e = 0$. □

Claim 3. *The dynamics (15) enjoy Property 1(b).*

Proof. By Claim 2, $e = 0$ is an equilibrium, so without loss of generality, we may assume that every word in (15) has at least one letter e [29, Proposition 3.1.10], so if $e \in \mathfrak{g}^{(i)}$, then since $\mathfrak{g}^{(i)}$ is an ideal, every word is in $\mathfrak{g}^{(i)}$. By our choice of K, $(I + TBK)\mathfrak{g}^{(i)} \subseteq \mathfrak{g}^{(i)}$. □

Claims 2, 3, and the initial argument verify the Lemma. □

Equipped with Lemma IV.6, we prove Theorem IV.5.

Proof of Theorem IV.5. Let $K = -(TB)^{-1}$. Then $(I + TBK) = 0$. By Lemma IV.6, Property 1 is satisfied. By Theorem III.5, the tracking error e converges to 0 in finite time. Consequently, E converges to identity in finite time. □

B. Rate of Convergence

In the proof of Theorem IV.5, we invoked Theorem III.5 to demonstrate the existence of $K : \mathfrak{g} \to \mathbb{F}^n$ such that the state-tracking error converges to zero in finite time; in the proof of Theorem III.5 in [29], this time is found to be bounded above by a linear combination of the dimensions of the ideals of the lower central series of $\mathfrak{g}$. In this section, we characterize the general rate of convergence. In anticipation of invoking Theorem III.4, we establish the following lemmas, which assert that the growth rates of the exogenous signals are independent of the choice of norm; this is important, because per Theorem III.4, this growth rate defines a sufficiently small spectral radius for stability.

Lemma IV.7. *For the discretized exosystem (5), there exists $s \geq 1$, such that given any norm $\|\cdot\| : \mathbb{F}^r \to \mathbb{R}$ and any initial condition $w[0]$, there exists $\beta \geq 0$ such that $\|w[k]\| \leq \beta s^k$.*

We emphasize that Lemma IV.7 establishes a bound on the rate of growth of w independent of the norm chosen.

Proof. Let $\|\cdot\| : \mathbb{F}^r \to \mathbb{R}$ be arbitrary. Fix $\varepsilon > 0$ and let $\|\cdot\|_\varepsilon : \mathbb{F}^r \to \mathbb{R}$ be a norm such that its induced norm satisfies $\|S\|_\varepsilon = \rho(S) + \varepsilon =: s$. Since all norms are equivalent on finite-dimensional vector spaces, there exists $\alpha > 0$ such that for all $w \in \mathbb{F}^r$, $\|w\| \leq \alpha \|w\|_\varepsilon$. Since the solution to (5) is $w[k] = S^k w[0]$, we have $\|w[k]\|_\varepsilon \leq s^k \|w[0]\|_\varepsilon$, so

$$\|w[k]\| \leq \alpha \|w[k]\|_\varepsilon = \underbrace{(\alpha \|w[0]\|_\varepsilon)}_{=:\beta} s^k.$$

Since ε was arbitrary, s can be made arbitrarily close to $\rho(S)$, which, under Assumption 2, is at least 1. □

Lemma IV.8. *There exists $s \geq 1$ such that given any norms on $\mathbb{F}^r$ and $\mathfrak{g}$, and any initial condition $w[0] \in \mathbb{F}^r$, there exists $\beta \geq 0$ such that*

$$\left\| \begin{bmatrix} \pi(Sw) \\ \pi(w) \\ C + Dw \end{bmatrix} \right\| \leq \beta s^k.$$

Proof. We first bound the norm of $\pi(Sw[k])$:

$$\|\pi(Sw[k])\| = \|F + GSw[k]\| \leq \|F\| + \|GS\| \|w[k]\|.$$

Applying Lemma IV.7,

$$\|\pi(Sw[k])\| \leq \|F\| + \|GS\| \beta' s^k \leq (\|F\| + \|GS\| \beta') s^k.$$

where we have used that $s \geq 1$. Similarly, we establish $\|\pi(w[k])\| \leq (\|F\| + \|G\|\beta') s^k$ and $\|C + Dw[k]\| \leq (\|C\| + \|D\|)\beta' s^k$. Let $\beta := \max\{\|F\| + \max\{\|GS\|, \|G\|\}, \|C\| + \|D\|\}\beta'$. □

Proposition IV.9. *There exists $K : \mathfrak{g} \to \mathbb{F}^n$ such that the origin of $\mathfrak{g}$ is semiglobally exponentially stable under (15).*

Proof. Fix $\alpha \in \left(1 - \rho(S)^{-\frac{p(p-1)}{2}}, 1 + \rho(S)^{-\frac{p(p-1)}{2}}\right)$ and $K = -\alpha(TB)^{-1}$. Then $I + TBK = (1-\alpha)I$, whose spectral radius is $|1 - \alpha| < \rho(S)^{-\frac{p(p-1)}{2}}$, and the letters ω in (15) reduce to $\{e, \pi(Sw), \pi(w)\}$, where the exogenous signals are $\pi(Sw)$ and $\pi(w)$. Stacking the exogenous signals into a single variable $W \in \mathfrak{g} \times \mathfrak{g}$, we apply Lemma IV.8. The result then follows by direct application of Theorem III.4. □

Since $\mathfrak{g}$ is diffeomorphic to G, we translate the Proposition IV.9 to the group.

Theorem IV.10. *There exists $K : \mathfrak{g} \to \mathbb{F}^n$ such that the identity of G is globally asymptotically stable under the group tracking-error dynamics (13).*

C. Regulator Problem with Plant State Information

Four natural choices of a measured output Y_m are 1) $Y_\mathrm{m} = (X[k], w[k])$; 2) $Y_\mathrm{m} = (Y[k], w[k])$; 3) $Y_\mathrm{m} = (Y[k], X[k])$; 4) $Y_\mathrm{m} = Y[k]$.

The first case is that of full information studied in the previous subsection. The second case is equivalent to the first, because it allows us to algebraically compute $X[k] = \exp(C + Dw[k])^{-1} Y[k]$. The third case includes, for example, the case where the plant state X and the regulation quantity Z are measured, i.e., $Y = Z$, $F = C$, and $D = G$. The fourth case characterizes the regulator problem with output information. In this section, we treat the third case, and leave the fourth case as a topic for future research.

When $Y_\mathrm{m} = (Y[k], X[k])$, at each sampling instant, we can compute

$$Dw[k] = \mathrm{Log}(Y[k]X[k]^{-1}) - C. \tag{17}$$

We therefore propose the linear observer

$$\hat{w}^+ = S\hat{w} + L(D\hat{w} - Dw), \tag{18}$$

which yields the estimation error dynamics

$$e_w^+ = (S + LD)e_w. \tag{19}$$

Under Assumption 2, $L : \mathfrak{g} \to \mathbb{F}^r$ can be chosen such that (19) is stable, if and only if the pair (D, S) is observable.

Theorem IV.11. *If the pair (D, S) is observable, then there exist $K : \mathfrak{g} \to \mathbb{F}^n$ and $L : \mathfrak{g} \to \mathbb{F}^r$ such that the control law defined by (10), (11), (18), and*

$$u = K\,\mathrm{Log}(X\Pi(\hat{w})^{-1}) + \Psi(\hat{w}) \tag{20}$$

solves the regulator problem with plant state information.

Proof. If (D, S) is observable, then there exists $L : \mathfrak{g} \to \mathbb{F}^r$ such that $\rho(S + LD) = 0$; fix such an L. Then for all $k \geq r$, $e_w[k] = 0$, or equivalently, $\hat{w}[k] = w[k]$. For all $k \geq r$, the control law (20) reduces to (16) from the full-information case.

Since all the dynamics under consideration are polynomial in the dynamical variables, none of the trajectories can exhibit finite escape time, so for $k \leq r$, the trajectories are well-defined, and for all $k \geq r$, the tracking error dynamics are (13) on the group, and (15) on the algebra. The proof follows from applying the arguments used in the proof of Theorem IV.5 to establish global attractivity of the origin under the tracking error dynamics. □

V. Simulations on the Heisenberg Group

To illustrate our results, we simulate Brockett's nonholonomic integrator [35] on the Heisenberg group $\mathsf{G} \subset \mathsf{GL}(3, \mathbb{R})$, which is a prototypical example for nonlinear control problems [36], further, it is a special case of a system in chained form, which can be expressed as an invariant system on the Heisenberg group [20, §4.1].

We choose the basis for the Heisenberg algebra $\mathfrak{g}$ to be $\{g_1, g_2, g_3\}$, where

$$g_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad g_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The lower central series is $\mathfrak{g} =: \mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \supset \mathfrak{g}^{(3)} = 0$, where $\mathfrak{g}^{(2)} = [\mathfrak{g}, \mathfrak{g}] = \mathrm{Lie}_\mathbb{R}\{g_3\} = \mathrm{span}_\mathbb{R}\{g_3\}$, thus the nilindex is $p = 2$.

We present several examples with different exosystems. In every case, $\rho(S) = 1$, and the observer gain L is chosen such that $\rho(S+LD) = 0$. We first consider a system with exosystem parameters $S_d = 1$ and $S_c = 0$, which both define steps in discrete- and continuous-time, respectively; plant parameters:

$$\begin{aligned} &A = g_1 + g_2 + g_3, \quad &&Q_{d1} = g_1 + g_2 + g_3, \quad &&Q_{c1} = g_3, \\ &B_1 = g_1, &&B_2 = g_2, &&B_3 = g_3, \end{aligned}$$

where $Bu = \sum_{i=1}^{3} B_i u_i$; plant output parameters:

$$C = g_1 + 2g_2 + 3g_3, \quad D_{d1} = g_1 - g_2, \quad D_{c1} = g_2 + g_3,$$

and regulation quantity parameters:

$$F = -3g_1 - 2g_2 - g_3,$$
$$G_{d1} = -g_1 + 2g_2 - 3g_3, \quad G_{c1} = 2g_1 + g_2 - 3g_3.$$

We use a sampling period of $T = 1$ and initialize with

$$X(0) = \exp(g_1 + 2g_2 - 3g_3),$$
$$w_d[0] = 1, \qquad \hat{w}_d[0] = 0,$$
$$w_c(0) = 1, \qquad \hat{w}_c[0] = 0.$$

We choose $K = -(1/2)I$, which yields $(I + TBK) = (1/2)I$, whose spectral radius is $1/2 \le \rho(S)^{-\frac{p(p-1)}{2}} = 1$. By Theorem IV.11, this choice of K and L furnishes a control law that solves the regulator problem. As predicted by Proposition IV.4, since w_d and w_c are constant, $Z(t) \to I$ as $t \to \infty$, as seen in Figure 2.

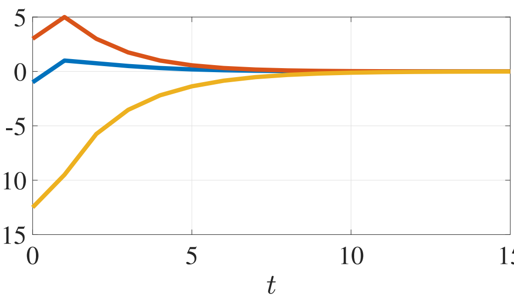

Fig. 2. Regulation quantity $\mathrm{Log}(Z)$ for constant w.

To illustrate non-step-tracking behaviour, we redefine the exosystem dynamics as

$$S_d = \begin{bmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{bmatrix}, \qquad S_c = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

which define discrete- and continuous-time sinusoids, respectively, both with unit frequency. We extend the plant, plant output, and regulation quantity definitions with the parameters

$$Q_{d2} = -g_1 - g_2, \qquad Q_{c2} = -g_3,$$
$$D_{d2} = g_1 + g_2 + g_3, \quad D_{c2} = g_1 + g_3$$
$$G_{d2} = g_3, \qquad G_{c2} = g_1 + 2g_2 + 3g_3,$$

where now $Q_d w_d = \sum_{i=1}^{2} Q_{di} w_{di}$, etc.

We use the same sampling period $T = 1$ and initial condition $X(0)$, and initialize the observer states at the origin, but now initialize the exostates at

$$w_d[0] = (0, 1), \quad w_c(0) = (1, -1).$$

We use the same tracking-error feedback gain K. Since the exostates are bounded, Proposition IV.3 predicts that $Z(t)$ will be bounded, as is implied in Figure 3.

Repeating the simulation again, but changing the discrete-time exosystem dynamics to

$$S_d = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

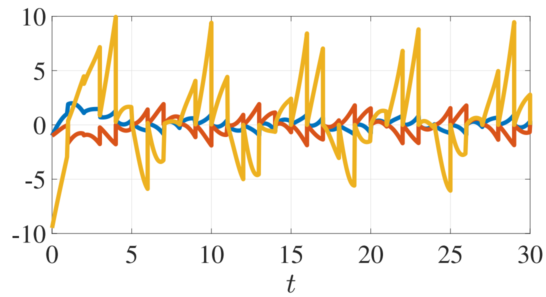

Fig. 3. Regulation quantity $\mathrm{Log}(Z)$ for sinusoidal w.

which defines a ramp, the regulation quantity exhibits the behaviour seen in Figure 4. At the sampling instants, $Z[k] \to I$, however, the intersample behaviour of $Z(t)$ is unbounded.

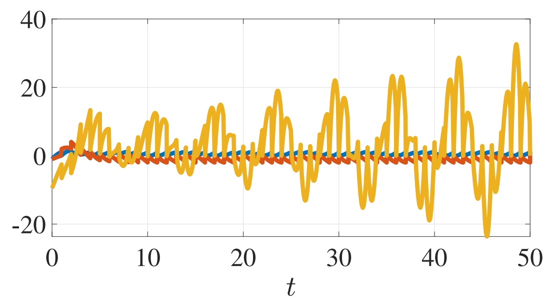

Fig. 4. Regulation quantity $\mathrm{Log}(Z)$ for ramp w_d and sinusoidal w_c.

However, if we remove the continuous-time exostate w_c, or equivalently set $w_c(0) = 0$, then we make the interesting observation that $Z(t)$ is bounded, as is implied in Figure 5. From (12), it is not surprising that eliminating the continuous-time disturbance improves intersample behaviour, and it seems plausible that when, in addition, the growth rate of w_d is bounded, that the intersample behaviour is bounded. However, due to the nonlinearity of (12), it is not obvious that this is always the case. We leave this as a topic for future research.

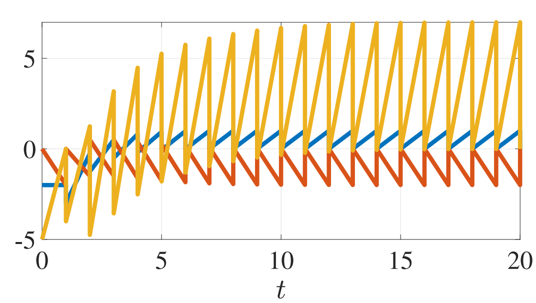

Fig. 5. Regulation quantity $\mathrm{Log}(Z)$ for ramp w_d and $w_c \equiv 0$.

VI. Summary and Future Research

We extended existing results for the sampled-data regulator problem for a class of invariant systems on commutative Lie groups [22], to simply connected nilpotent Lie groups in two cases: 1) when the plant state and exostate are available for feedback; 2) when the plant state and a so-called plant output are available for feedback. In the latter, we used a Luenberger observer to estimate the exostates, thereby furnishing a dynamical control law. In the full-information case, we showed that the origin of the Lie algebra is semiglobally exponentially stable under the tracking error dynamics.

Future work includes regulation when the plant is underactuated, but controllable, i.e., $\mathrm{Lie}_{\mathbb{F}} \operatorname{Im} B = \mathfrak{g}$, rather than $\operatorname{Im} B = \mathfrak{g}$. We conjecture that this could be done using multirate sampling. The case where only the plant output Y is available for feedback should also be addressed. The last simulation in Section V suggests that our conditions for bounded intersample behaviour can be refined. Another natural extension is to remove Assumption 4, and use the Magnus expansion to express the local trajectory of the plant's state and design control laws. It would also be of interest to identify conditions for robustness to noise and structural stability of the error dynamics. The utility of the methods described in this paper should be tested using the nilpotent approximation methods of [17], [18].

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