# Adaptive policies and drawdown problems in insurance risk models

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

## Abstract

Ruin theory studies an insurer's solvency risk, and to quantify such a risk, a stochastic process is used to model the insurer's surplus process. In fact, research on ruin theory dates back to the pioneer works of Lundberg (1903) and Cramér (1930), where the classical compound Poisson risk model (also known as the Cramér-Lundberg model) was first introduced. The research was later extended to the Sparre Andersen risk model, the Markov arrival risk model, the Lévy insurance risk model, and so on. However, in most analysis of the risk models, it is assumed that the premium rate per unit time is constant, which does not always reflect accurately the insurance environment. To better reflect the surplus cash flows of an insurance portfolio, there have been some studies (such as those related to dividend strategies and tax models) which allow the premium rate to take different values over time. Recently, Landriault et al. (2012) proposed the idea of an adaptive premium policy where the premium rate charged is based on the behaviour of the surplus process itself. Motivated by their model, the first part of the thesis focuses on risk models including certain adjustments to the premium rate to reflect the recent claim experience. In Chapter 2, we generalize the Gerber-Shiu analysis of the adaptive premium policy model of Landriault et al. (2012). Chapter 3 proposes an experience-based premium policy under the compound Poisson dynamic, where the premium rate changes are based on the increment between successive random review times. In Chapter 4, we examine a drawdown-based regime-switching Lévy insurance model, where the drawdown process is used to model an insurer's level of financial distress over time, and to trigger regime-switching (or premium changes).

Similarly to ruin problems which examine the first passage time of the risk process below a threshold level, drawdown problems relate to the first time that a drop in value from a historical peak exceeds a certain level (or equivalently the first passage time of the reflected process above a certain level). As such, drawdowns are fundamentally relevant from the viewpoint of risk management as they are known to be useful to detect, measure and manage extreme risks. They have various applications in many research areas, for instance,

mathematical finance, applied probability and statistics. Among the common insurance surplus processes in ruin theory, drawdown episodes have been extensively studied in the class of spectrally negative Lévy processes, or more recently, its Markov additive generalization. However, far less attention has been paid to the Sparre Andersen risk model, where the claim arrival process is modelled by a renewal process. The difficulty lies in the fact that such a process does not possess the strong Markov property. Therefore, in the second part of the thesis (Chapter 5), we extend the two-sided exit and drawdown analyses to a renewal risk process.

In conclusion, the general focus of this thesis is to derive and analyze ruin-related and drawdown-related quantities in insurance risk models with adaptive policies, and assess their risk management impacts. Chapter 6 ends the thesis by some concluding remarks and directions for future research.

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## Dedication

To my dear parents

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# Chapter 1

# Introduction

Risk theory has been widely applied by decision-makers in the areas of insurance, finance, and security investment to manage the risk in quantitative analysis and forecasting. Ruin theory, playing an important role in risk theory, utilizes analytical tools developed in applied probability to study an insurer's surplus process. The importance of studying the insurance surplus processes lies in the fact that it helps to measure and manage an insurer's solvency risk. For example, if one insurance portfolio has a ruin probability that is significantly large, the insurer should take appropriate measures to lower the risk, such as increasing the initial capital, transferring the risk by reinsurance, or using other risk management arrangements. It is therefore imperative that models of an increased complexity be analyzed in this context to better reflect the cash flow dynamics of an insurance portfolio and further incorporates additional features/recent trends in the insurance industry. Therefore, in this thesis, the primary goals are:

- Extend the Gerber-Shiu analysis in some existing models;
- Develop practical models with adaptive policies and assess their risk management impacts;
- Analyze two-sided exit problems and drawdown-related quantities in more general

models of interest in insurance.

The rest of this chapter intends to provide a brief literature review, introduce the insurance risk models of interest and some common ruin-related and drawdown-related quantities, as well as summarize some of the main mathematical tools in the ensuing ruin and drawdown analyses. This chapter is concluded by presenting an outline of the thesis.

## 1.1 Background

### 1.1.1 Ruin theory

The research on ruin theory dates back to 1903 when Lundberg first introduced the classical compound Poisson risk model (also known as the Cramér-Lundberg model). For a century, the research on ruin theory has remained a fascinating subject; see, e.g., Gerber (1979), Grandell (1991), Rolski et al. (1999) and Asmussen and Albrecher (2010). The central focus is to investigate quantities related to the time to ruin (which is a particular first passage time for insurance risk processes), such as the ruin probability, which provide insights into the insurer's ability to meet its obligations as well as its vulnerability to solvency.

One typical methodology to analyze these ruin-related quantities is through the Gerber-Shiu expected discounted penalty function or Gerber-Shiu function in its short form (see Section 1.3.1 for more details) introduced in Gerber and Shiu (1998). They showed that the expected discounted penalty function of the surplus prior to ruin and the deficit at ruin satisfies a defective renewal equation in the classical compound Poisson risk model. To describe a more general insurance surplus process, the Gerber-Shiu analysis was later extended to various generalizations of the classical compound Poisson risk model. For instance, an ordinary Sparre Andersen (renewal) risk model assumes a general renewal process rather than a Poisson process for the claim arrival process; see, e.g., Dickson and Hipp (2001), Li and Garrido (2004a), Li and Garrido (2005a). By assuming a dependence structure between the interclaim time and its subsequent claim size, the dependent Sparre Andersen risk model is obtained (see Section 1.2.1 for more references). Furthermore, to relax the independence assumption of interclaim times, the Markov arrival risk model is proposed; see, e.g., Ahn and Badescu (2007), Badescu et al. (2005) and Cheung and Landriault (2009). In addition, a trend of adding a diffusion process to traditional risk models is arising (see, e.g., Dufresne and Gerber (1991), Tsai and Willmot (2002), and Li and Garrido (2005b)), and more generally, the Lévy insurance model is considered; see, e.g., Klüppelberg et al. (2004), Garrido and Morales (2006) and Kyprianou (2013).

In most of the literature mentioned above, the premium rate is assumed to be constant, which does not always reflect accurately the insurance environment. Thus, to better reflect the surplus cash flows of an insurance portfolio, there has been some studies in which the premium rate is allowed to take different values over time. One typical research direction is to work with risk models with dividend strategies; see Avanzi (2009) for a comprehensive review on the topic. Three surplus-dependent dividend strategies are of particular interest. The first one is the horizontal dividend barrier strategy where all original premium income is paid out as dividend whenever the surplus level reaches a certain level; see for example, Gerber (1979), Lin et al. (2003) and Li and Garrido (2004b). The second one is the threshold strategy where dividends are paid at a rate that is less than the premium rate when the surplus exceeds a constant level (threshold) and no dividends are paid otherwise; see for example, Albrecher and Hartinger (2007) and Lin and Pavlova (2006). Lin and Sendova (2008) further considered a multi-threshold strategy in the classical compound Poisson risk model. The last one is a time-dependent barrier strategy, for which the barrier itself is an increasing function of time and if the risk process touches the barrier, it stays at the barrier until the next claim occurs and the additional premium income is paid out as dividends. See Asmussen and Albrecher (2010) and references therein for more information. Another research direction for changing premium rates is to involve credibility theory, such as the Bühlmann (1967) or Bühlmann and Straub (1970) credibility models; see, e.g., Tsai and

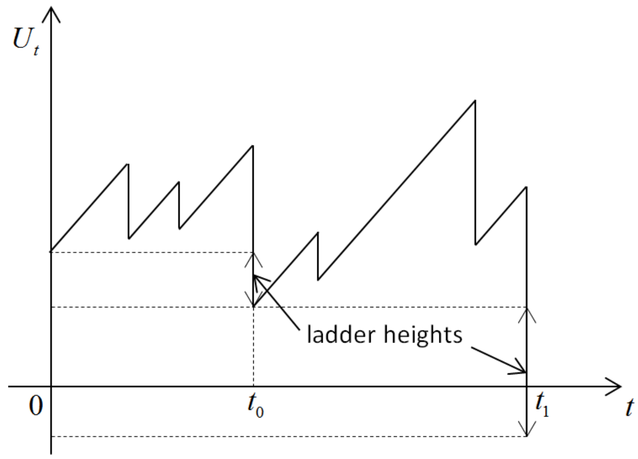


Figure 1.1: The ladder heights of the surplus process $U$

Parker (2004), Afonso et al. (2010), and Loisel and Trufin (2013). Also, Landriault et al. (2012) considered a risk model with an adaptive premium policy, where the choice of the premium rate depends on the time elapsed between successive ladder heights (see Figure 1.1).

Motivated by Landriault et al. (2012), we propose some models with different adaptive adjustments (to the premium rate) to reflect recent claim experience. In other words, we assume that the surplus regime (or the premium rate) will no longer be deterministic but rather responsive to the recent claim experience as is done in practice. For example, if an insurer incurs many claims in a short period of time or reaches a significant low surplus level, it may consider to charge a higher premium to prevent ruin from happening (subject to some competitive constraints). The opposite is also true: if the insurer incurs few claims in a long period of time or has a high surplus level, it may consider charging a smaller premium to be more competitive and attract new clients.

### 1.1.2 Drawdown analysis

The concept of drawdown is being used increasingly in risk analysis, as it provides surplus-related information similar to ruin-related quantities. Drawdown is a performance risk measure of the decline in value from a historical peak (see Figure 1.2), which can be the drop of a

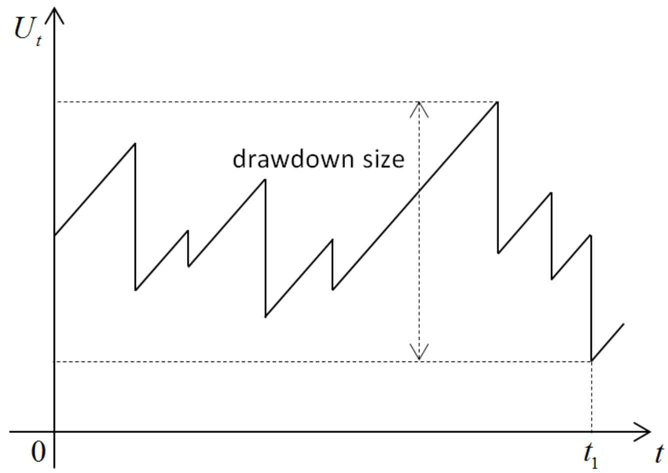

Figure 1.2: Drawdown size at time $t_1$

stock price, index or value of a portfolio relative to its historical running maximum. As such, drawdowns can be used to characterize extreme risks from a risk management standpoint. Mathematically speaking, similarly to the time to ruin (first passage time of the risk process below level 0), the drawdown time is the first passage time of the reflected process above a certain level. The research on drawdowns is of both practical and mathematical interest. A few examples are given next.

In the mutual fund industry, drawdown is frequently quoted by mutual fund managers and commodity trading advisors through performance ratios, such as the Calmar ratio, Sterling ratio, Burke ratio, Martin ratio and Pain ratio, where drawdown becomes an alternative measurement for volatility. Volatility measures the uncertainty of both positive and negative performance of assets returns, while drawdown measures are more desirable when the downward risks are of primary interest. Schuhmacher and Eling (2011) showed that from a decision-theoretic perspective, drawdown-based performance measures are as good as the Sharpe ratio for returns satisfying the location and scale conditions (see Meyer (1987)). In practice, drawdown-based performance measures are preferred because they are highly related with fund redemptions.

In an insurance context, drawdown problems have close ties with the constant dividend barrier strategy in insurance surplus analysis. By reflection, the time to ruin and the deficit at ruin of a risk process with a constant dividend barrier are distributed as the drawdown time and the overshoot of drawdown associated with a similar risk process without dividend barrier. Also, Avram et al. (2007) and Loeffen (2008) showed that the famous De Finetti's optimal dividend problem (De Finetti (1957)) can be connected to drawdowns when a dividend barrier strategy is optimal. As for the drawdown insurance design, Carr et al. (2011) introduced some vanilla digital drawdown insurance contracts and proposed semi-static hedging strategies using barrier and vanilla options. Zhang et al. (2013) studied the valuation of a vanilla drawdown insurance, the cancellable drawdown insurance, drawdown insurance with drawup contingency and drawdown insurance on a defaultable stock under the geometric Brownian motion dynamics.

In finance, drawdowns are popular in portfolio optimization problems. Grossman and Zhou (1993) solved a portfolio optimization problem subject to a linear drawdown constraint in the Black-Scholes framework. Cherny and Obloj (2013) further studied the same problem under non-linear drawdown constraints in a semimartingale framework. Chekhlov et al. (2005) proposed the Conditional Drawdown (CDD) risk measures and studied the portfolio optimization with drawdown measure. Pospisil and Vecer (2010) studied the sensitivities to the running maximum and the maximum drawdown of an underlying asset. The pricing of Russian options constitutes another application of drawdowns in mathematical finance; see, e.g., Shepp and Shiryaev (1993), Asmussen et al. (2004) and Avram et al. (2004).

In applied probability, most of the research has focused on the distributional studies of the size of the maximum drawdown and other drawdown-related quantities. The reader is referred to Section 1.3.2 for a detailed literature review of drawdown-related quantities analyzed in the context of spectrally negative Lévy processes or their Markov additive generalizations. Aside from the magnitude of drawdown, some attention has been paid to the duration and frequency of drawdowns. Landriault et al. (2014) examined the Laplace transform of the first time the duration of drawdowns exceeds a pre-specified time threshold in a spectrally negative Lévy process with positive phase-type jumps. Landriault et al. (2015c) studied the frequency rate of drawdowns and drawdown-related quantities at the $n$-th drawdown time in the Brownian motion processes, and they proposed some insurance contracts against the risk of frequent drawdowns.

In addition, drawdowns have many applications in statistics, for instance, drawdown and its dual drawup are used as stopping rules for the sequential analysis technique CUSUM (e.g., Khan (2008), Poor and Hadjiliadis (2009) and Zhang et al. (2014)).

## 1.2 Risk models

### 1.2.1 Dependent Sparre Andersen risk model

The insurer's surplus process $\{U_t; t \geq 0\}$ is defined as

$$U_t = u + ct - \sum_{i=1}^{N_t} Y_i,$$

where $u = U_0 \geq 0$ is the initial surplus level, $c > 0$ is the premium rate per unit time, and $\{Y_i; i \geq 1\}$ are the claim size random variables. Also, let $\{N_t; t \geq 0\}$ be the number of claims process defined through the sequence of interclaim times $\{V_i; i \geq 1\}$ with $V_1$ being the time of the first claim and $V_i$ for the time between the $(i-1)$th claim and $i$th claim. By specifying the distribution of the interclaim times and/or claim sizes and their dependencies, various risk models will be obtained. A *dependent Sparre Andersen risk model* has the following assumptions: the claim sizes $\{Y_i; i \geq 1\}$ are independent and identically distributed (iid) with probability density function (pdf) $p(\cdot)$, cumulative distribution function (cdf) $P(x) = 1 - \overline{P}(x)$ and mean $\mu$; the interclaim times $\{V_i; i \geq 1\}$ are a sequence of iid random variables with pdf $k_V(\cdot)$, cdf $K_V(t) = 1 - \overline{K}_V(t)$ and mean $1/\lambda$; and the pairs $\{(V_i, Y_i); i \geq 1\}$ are iid with joint density $f(t, y)$, so that $V_i$ and $Y_i$ may be dependent. The requirement of a positive security loading is $c\mathbb{E}[V] > \mathbb{E}[Y]$, where $V$ and $Y$ denote a representative of $\{V_i; i \geq 1\}$ and $\{Y_i; i \geq 1\}$, respectively. See e.g., Albrecher and Boxma

(2004), Badescu et al. (2009), Boudreault et al. (2006), and Cossette et al. (2008) for more references.

There are some well-known special cases of the dependent Sparre Andersen risk model. For instance, if $V_i$ and $Y_i$ are independent for all $i$, i.e., $f(t,y) = k(t)p(y)$, the model becomes the (ordinary) *Sparre Andersen risk model*; see, e.g., Andersen (1957), Li and Garrido (2005a) and Gerber and Shiu (2005). Also, if the interclaim times $\{V_i; i \geq 1\}$ are further assumed to be exponentially distributed, i.e., $f(t,y) = \frac{1}{\lambda}e^{-t/\lambda}p(y)$, the model reduces to the *classical compound Poisson risk model*; see, e.g., Gerber (1979), Grandell (1991) and Rolski et al. (1999). For an ordinary Sparre Andersen risk model, if we assume that the distribution of the time of the first claim $V_1$ differs from the distribution of the interclaim times $\{V_i; i \geq 2\}$, the model is referred to as the *delayed Sparre Andersen risk model*; see, e.g., Willmot and Lin (2001) and Willmot (2004). Moreover, if the distribution of $V_1$ is the equilibrium distribution of $\{V_i; i \geq 2\}$, i.e., $k_{V,1}(t) = k_{V,e}(t) = \overline{K}_V(t)/E(V)$, we refer to this risk model as the *stationary Sparre Andersen risk model*; see, e.g., Willmot and Dickson (2003) and Willmot et al. (2004). In addition, a delayed dependent Sparre Andersen risk model is studied in Woo (2010).

### 1.2.2 Spectrally negative Lévy process

Spectrally negative Lévy processes, also known as Lévy insurance risk models, have become popular in modelling the surplus process of an insurance portfolio, because they allow for a diffusion component and have only downward jumps, which is consistent with the insurance practice.

A spectrally negative Lévy process is a special type of Lévy process with only downward jumps. We shall start from the definition of Lévy process (see, e.g., Bertoin (1996) and Kyprianou (2006)). A strong Markov process $X = \{X_t; t \geq 0\}$ with càdlàg paths defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a *Lévy process* if it has the properties that

$\mathbb{P}(X_0 = 0) = 1$ and for each $0 \leq s \leq t$, the increment $X_t - X_s$ is independent of $\mathcal{F}_s$ and has the same distribution as $X_{t-s}$.

From their definition, Lévy processes have stationary and independent increments. Using the characteristic exponent of the infinitely divisible distribution, there exists a function $\Psi$ such that

$$\mathbb{E}(e^{isX_t}) = e^{-t\Psi(s)},$$

for $t \geq 0$ and $s \in \mathbb{R}$. The general form of $\Psi$ is given by the Lévy-Khintchine formula (see, e.g., Kyprianou (2006)). That is, for $s \in \mathbb{R}$,

$$\Psi(s) = ias + \frac{1}{2}\sigma^2 s^2 + \int_{\mathbb{R}} (1 - e^{isx} + isx1_{\{|x|<1\}})\Pi(dx),$$

where $a \in \mathbb{R}$, $\sigma \geq 0$ and $\Pi$ is a measure concentrated on $\mathbb{R}\backslash\{0\}$ such that $\int_{\mathbb{R}}(x^2 \wedge 1)\Pi(dx) < \infty$. In such an expression, the triplet $(a, \sigma, \Pi)$ fully characterizes a Lévy process. Also, from the Lévy-Itô decomposition (see, e.g., Kyprianou (2006)), the Lévy process $X$ can be viewed as the independent sum $X = X^{(1)} + X^{(2)} + X^{(3)}$, where $X^{(1)}$ is a linear Brownian motion with drift $-a$ and volatility $\sigma$, $X^{(2)}$ is a compound Poisson process with Poisson intensity rate $\Pi(\mathbb{R}/(-1,1))$ and iid jumps distributed as $\Pi(dx)/\Pi(\mathbb{R}/(-1,1))$, and $X^{(3)}$ determined by the Lévy measure $\Pi$ is a square integrable martingale with an almost surely countable number of jumps on each finite time interval which are of magnitude less than 1.

For example, a Poisson process and a compound Poisson process have the characteristic exponent $\Psi(s) = \lambda(1 - e^{is})$ and $\Psi(s) = \lambda \int_{\mathbb{R}}(1 - e^{isx})P(dx)$, respectively, where $\lambda$ is the Poisson intensity rate and $P$ is the distribution function for the iid jumps.

If the Lévy measure $\Pi$ is restricted on $(-\infty, 0)$, i.e., $\Pi(0, \infty) = 0$, such a Lévy process is called a spectrally negative Lévy process. Since there is no positive jumps, the Laplace exponent can be used to characterize the spectrally negative Lévy process, which is defined as

$$\psi(\lambda) = \frac{1}{t}\log \mathbb{E}(e^{\lambda X_t}) = -\Psi(-i\lambda), \tag{1.1}$$

for $\lambda \geq 0$. Given the triplet $(a, \sigma, \Pi)$ where $\Pi \subseteq (-\infty, 0)$, we have

$$\psi(\lambda) = -a\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{(-\infty,0)} (e^{\lambda x} - 1 - \lambda x 1_{\{x>-1\}})\Pi(dx).$$

It is easy to see $\psi(\lambda)$ is infinitely differentiable and strictly convex. Also, $\lim_{\lambda\to\infty} \psi(\lambda) = \infty$. Further results related to the spectrally negative Lévy process are given in Section 1.4.4.

### 1.2.3 Spectrally negative Markov additive process

Another risk model of interest is the spectrally negative Markov additive process (MAP), which is a generalization of the spectrally negative Lévy process. Consider a process $X = \{X_t; t \geq 0\}$ and an irreducible continuous time Markov process $J = \{J_t; t \geq 0\}$ with a finite state space $\{1, \ldots, n\}$ and infinitesimal generator $\mathbf{Q}$. We say the bivariate process $(X, J)$ is a MAP if given $\{J_t = i\}$, the pair $(X_{t+s} - X_t, J_{t+s})$ is independent of $(X_s, J_s)$ for all $0 \leq s \leq t$ and has the same law as $(X_s - X_0, J_s)$ given $\{J_0 = i\}$ for all $s, t \geq 0$ and $i \in \{1, \ldots, n\}$. The additive component $X$ evolves as some spectrally negative process $X^i$ when $J_t = i$. The processes $X^1, X^2, \ldots, X^n$ are assumed to be independent. In addition, a transition of $J$ from $i$ to $j \neq i$ triggers a downward jump of $X$ whose (absolute) size has the distribution function $P_{i,j} \geq 0$ for $i, j \in S$. Such a model is studied in, e.g., Kyprianou and Palmowski (2008) and Ivanovs and Palmowski (2012). Define the Laplace exponent of $X^i$ through $\mathbb{E}[e^{zX_t^i}] = e^{\psi_i(z)t}$. Define $\mathbf{F}^{(q)}(z)$ to be the matrix analogue of the Laplace exponent of $X$, namely

$$\mathbb{E}\left[e^{-qt+zX_t}; J_t = j | J_0 = i\right] = \left(e^{\mathbf{F}^{(q)}(z)t}\right)_{ij}.$$

Thus,

$$\mathbf{F}^{(q)}(z) = \text{diag}\{\psi_i(z)\}_{i=1}^n + \mathbf{Q} \circ \mathbf{G}(z) - q\mathbf{I}, \tag{1.2}$$

where $\mathbf{I}$ is the identity matrix, $\mathbf{G}(z)_{ij} = \mathbb{E}[e^{-zP_{ij}}]$ and the notation $A \circ B = (a_{ij}b_{ij})$ stands for the entry-wise matrix product.

## 1.3 Quantities of interest

### 1.3.1 Ruin-related quantities and Gerber-Shiu functions

In the previous section, various risk models were introduced. Now we are interested in the issue that at some time point the surplus level of an insurance portfolio will not be sufficient to cover claim amounts, i.e., the surplus level drops below 0 and triggers the so-called "ruin" event. Define the time to ruin

$$T = \inf\{t \geq 0 | U_t < 0\},$$

with $T = \infty$ if the surplus never drops below 0. The (ultimate) ruin probability is defined as

$$m(u) = \mathbb{P}(T < \infty | U_0 = u). \tag{1.3}$$

The quantity $m(u)$ is important for risk management purposes, while other ruin-related quantities are also of much interest. The most popular quantities to be considered in a typical ruin analysis are the surplus prior to ruin $U_{T^-}$ and the deficit at ruin $|U_T|$. Then it is obvious that the claim causing ruin $Y_{N_T}$ has the representation $Y_{N_T} = U_{T^-} + |U_T|$.

To analyze these quantities, Gerber and Shiu (1998) introduced a comprehensive analytic tool known as the *Gerber-Shiu expected discounted penalty function*, or simply the *Gerber-Shiu function*, defined as

$$m_\delta(u) = \mathbb{E}\left[e^{-\delta T} w(U_{T^-}, |U_T|) 1_{\{T<\infty\}} | U_0 = u\right], u \geq 0,$$

where $1_A$ is the indicator function of the event $A$, and $w(x, y)$ is a function of the surplus prior to ruin ($x$) and the deficit at ruin ($y$). The so-called *penalty function* $w(x, y)$ is assumed to satisfy mild integrability conditions. Also, $\delta \geq 0$ is a real number, which can be viewed as a discount factor (i.e., force of interest or spot rate) or a Laplace transform argument.

Using a variety of techniques related to Laplace transforms, renewal arguments and integro-differential equations, Gerber and Shiu (1998) showed that, under the classical compound Poisson risk model, the Gerber-Shiu function satisfies a defective renewal equation whose solution can be expressed in terms of a compound geometric tail.

The Gerber-Shiu function allows its user to extract information about certain quantities related to the time to ruin $T$, such as the surplus prior to ruin $U_{T^-}$ and the deficit at ruin $|U_T|$. For example, if we choose $w(x, y) = 1$, the Gerber-Shiu function reduces to the Laplace transform of the time to ruin, and furthermore if we assume $\delta = 0$, it reduces to the ruin probability defined in (1.3). The choice of $w(x, y) = e^{-sx-zy}$ leads to the trivariate Laplace transform of the triplet $(T, U_{T^-}, |U_T|)$. The choice of $\delta = 0$ and $w(x, y) = 1_{\{x \leq x_1\}} 1_{\{y \leq y_1\}}$ yields the analysis of the joint and marginal defective distribution function of $U_{T^-}$ and $|U_T|$.

Now if we take one step further to invert the Laplace transform of the time to ruin, we obtain the density of the time to ruin, which will be quite useful to the study of finite time ruin probability. Landriault et al. (2011b) and Shi and Landriault (2013) study the finite-time ruin problem by incorporating the number of claims until ruin into the Gerber-Shiu analysis, and the joint density of the time to ruin and the number of claims until ruin provides further probabilistic interpretations of the series expansion of the density of the time to ruin. This can be viewed as one generalization of the Gerber-Shiu function. Other generalizations of the Gerber-Shiu function studied in the literature on ruin theory are as follows. Cai et al. (2009) studied the expected present value of the total operating costs up to the time of default. Also, Cheung et al. (2010) incorporated the surplus level immediately after the second last claim before ruin and Biffis and Morales (2010) incorporated the minimum surplus level before ruin into the Gerber-Shiu function. Cheung and Feng (2013) extended the results in Cai et al. (2009) by examining all the moments of the discounted claim costs until ruin, and investigated a more general function which allows the cost function to depend on the surplus level immediately after the second last claim before ruin.

### 1.3.2 Drawdown-related quantities

For any insurance model, besides the ruin-related quantities, the drawdown-related quantities are also of interest, since they will give the insurer timely warnings before a capital shortfall occurs.

The drawdown process (or reflected process) $Y = \{Y_t; t \geq 0\}$ of a stochastic process $X$ is defined as

$$Y_t = M_t - X_t,$$

where $M_t = \sup_{0 \leq s \leq t} X_s$ is the running maximum of $X$ at time $t$. For a given $a > 0$, the drawdown time $\tau_a$ is defined as

$$\tau_a = \inf \{t \geq 0 : Y_t \geq a\}.$$

We also define the running minimum of $X$ at time $t$ as $m_t = \inf_{0 \leq s \leq t} X_s$. Path-dependent properties of the first drawdown episode include the running maximum (minimum) at the drawdown time $M_{\tau_a}$ ($m_{\tau_a}$), the drawdown size $Y_{\tau_a}$, as well as the last time a running maximum is reached prior to $\tau_a$ denoted by $G_{\tau_a}$, where

$$G_t = \sup\{0 \leq s \leq t : M_s = X_s\}.$$

Note that we follow the convention $\inf \emptyset = \infty$ and $\sup \emptyset = 0$ throughout the thesis.

In the literature, drawdown-related quantities are usually analyzed in the context of spectrally negative Lévy processes, or more recently, in their Markov additive generalizations. For instance, Taylor (1975) first derived the joint Laplace transform of $(\tau_a, M_{\tau_a})$ for Brownian motion processes. Later, Lehoczky (1977) generalized Taylor's work to a time homogeneous diffusion process. Avram et al. (2004) analyzed the joint Laplace transform of $(\tau_a, Y_{\tau_a})$ in the spectrally negative Lévy process. Zhang and Hadjiliadis (2012) studied the joint Laplace transform of $(G_{\tau_a}, M_{\tau_a}, \tau_a - G_{\tau_a})$ for a general time-homogeneous diffusion process. Douady et al. (2000) and Magdon-Ismail et al. (2004) derived the density and expectation of the maximum drawdown before time $t$ defined as $MDD_t = \sup_{0\le s\le t} Y_s$ for a Brownian motion and a Brownian motion with drift, respectively. For a spectrally negative Lévy process, Mijatovic and Pistorius (2012) derived the joint Laplace transform of $(\tau_a, G_{\tau_a}, M_{\tau_a}, m_{\tau_a}, a - Y_{\tau_a^-}, Y_{\tau_a} - a)$. Breuer (2012) studied the joint distribution of $(\tau_a, M_{\tau_a}, G_{\tau_a})$ in a Markov-modulated Brownian motion. Ivanovs and Palmowski (2012) analyzed the joint Laplace transform of $(\tau_a, M_{\tau_a}, Y_{\tau_a} - a)$ in the framework of MAPs.

## 1.4 Mathematical preliminaries

In this section, we present some of the mathematical tools and their properties that will be useful in the following chapters.

### 1.4.1 Erlangization

The method of *Erlangization* was first proposed in finance in the exercise of pricing American options, where the technique is often referred to as *randomization* (see, e.g., Carr (1998)). In general, randomization describes a three-step procedure: the first step is to randomize a parameter by assuming a plausible distribution for it; the second step is to somehow calculate the expected value of the dependent variable in this random parameter setting; the final step is to let the variance of the distribution governing the parameter approach zero, holding the mean of the distribution constant at the fixed parameter value. Erlangization, as a special case of randomization, uses an Erlang random variable to approximate a fixed parameter value.

The idea of Erlangization has also been used in ruin theory (Asmussen and Albrecher (2010), Chapter IX, Section 8 and Asmussen et al. (2002)), where a finite-time ruin problem of interest was approximated by the corresponding probability of a ruin prior to an Erlang distributed horizon. More precisely, if we are interested in the finite-time ruin probability denoted as $\psi(u,t)$ (which is a function of the initial surplus level $u$ and the finite time $t$), the first step is to replace the deterministic time horizon $t$ by a random variable $H$ that has an Erlang distribution with $k$ stages and mean $t$ (i.e., with variance $t^2/k$). For the second step, we compute the ruin probability $\psi(u) = \mathbb{E}\left[\psi(u,H)\right]$. Since the variance of the Erlang distribution goes to 0 as $k$ goes to $\infty$, we can prove that $\psi(u)$ converges to $\psi(u,t)$ as $k$ goes to $\infty$.

In the adaptive premium policy model, Landriault et al. (2012) used Erlangization as an approximation method that replaces fixed parameter values by Erlang distributed random variables (see Chapter 2).

### 1.4.2 Dickson-Hipp operator and Laplace transform

Another analytic tool which has been shown to be relevant in Gerber-Shiu type analysis is the Dickson-Hipp operator. A special case of the Dickson-Hipp operator is the Laplace transform, and it turns out that the Laplace transform argument is one of the common methods employed to derive the defective renewal equation of Gerber-Shiu functions.

Let $s$ and $r$ be any complex number with non-negative real part and $f(x)$ be any integrable real-valued function. Define

$$\mathcal{T}_r f(x) = \int_x^\infty e^{-r(y-x)} f(y) dy,$$

and

$$\widetilde{f}(s) = \int_0^\infty e^{-sy} f(y) dy,$$

where the former is referred to as the Dickson-Hipp operator (or transform), while the latter is the Laplace transform (LT) of the function $f$. Note that the LT is a special case of the Dickson-Hipp operator, since $\widetilde{f}(s) = \mathcal{T}_s f(0)$.

It is of interest to present the properties of repeated applications of the Dickson-Hipp operator. Thus, for any complex numbers $r_1, r_2, \ldots, r_n$ and $n = 1, 2, \ldots$, define

$$\mathcal{T}_{r_1,r_2,\ldots,r_n} f(x) = \mathcal{T}_{r_1}\mathcal{T}_{r_2}\ldots\mathcal{T}_{r_n} f(x).$$

For $n = 2$,

$$\mathcal{T}_{r_1,r_2} f(x) = \mathcal{T}_{r_2,r_1} f(x) = \frac{\mathcal{T}_{r_2} f(x) - \mathcal{T}_{r_1} f(x)}{r_1 - r_2}, \text{ if } r_1 \neq r_2, \tag{1.4}$$

and

$$\mathcal{T}_{r_1,r_1} f(x) = \mathcal{T}_{r_1}^2 f(x) = \int_x^\infty (y-x)e^{-r_1(y-x)} f(y)dy, \text{ if } r_1 = r_2.$$

In particular,

$$\mathcal{T}_{r_1}^2 f(0) = \int_0^\infty y e^{-r_1 y} f(y)dy = -\left(\frac{d}{ds}\mathcal{T}_s f(0)\right)\Bigg|_{s=r_1}.$$

A comprehensive list of properties of the Dickson-Hipp operator can be found in e.g., Dickson and Hipp (2001), Li and Garrido (2004a), and Gerber and Shiu (2005).

### 1.4.3 Rouche's theorem

In the Gerber-Shiu analysis, Rouche's theorem is used to show that there are a certain number of solutions to Lundberg's fundamental equation in the classical compound Poisson risk model and their generalizations when other risk models are considered. These solutions will help solve the unknown constants in the (matrix form) defective renewal equation and enable an explicit expression for the Gerber-Shiu function of interest. As such, we present the statement of Rouche's theorem in Theorem 1.4.1 (see, e.g., Titchmarsh (1939)) as well as its generalization for matrices in Theorem 1.4.2 (see, e.g., Dshalalow (1995)).

**Theorem 1.4.1.** *If $f(z)$ and $g(z)$ are analytic inside and on a closed contour $D$ and $|g(z)| < |f(z)|$ on $D$, then $f(z)$ and $g(z) + f(z)$ have the same number of zeros inside $D$.*

**Theorem 1.4.2.** *Let $A(z) = (a_{ij}(z))$ and $B(z) = (b_{ij}(z))$ be complex $n \times n$ matrices, where $B(z)$ is diagonal. The elements $a_{ij}$ and $b_{ij}$, $1 \leq i \leq n$, $1 \leq j \leq n$, are meromorphic functions* *in a simply connected region $S$ in which $T$ is the set of all poles of these functions. $C$ is a rectifiable closed Jordan curve in $S-T$. $\mathcal{N}_B$ (or $\mathcal{N}_{A+B}$) is the number of zeros inside $C$ of* $\det B(z)$ *(or* $\det(A(z)+B(z))$ *and $\mathcal{P}_B$ (or $\mathcal{P}_{A+B}$) the number of poles inside $C$. If*

$$|b_{ii}(z)| > \sum_{j=1}^{n} |a_{ij}(z)| \ \text{ on } C \ \text{ for all } i = 1, \ldots, n$$

*then on $C$*

$$\det(A(z)+B(z)) \neq 0, \ \det B(z) \neq 0,$$

*and*

$$\mathcal{N}_{A+B} - \mathcal{P}_{A+B} = \mathcal{N}_B - \mathcal{P}_B.$$

### 1.4.4 Scale functions and exit problems

In this subsection, some well known results for spectrally negative Lévy processes and MAPs are presented, which will be quite useful in Chapters 4 and 5.

For a spectrally negative Lévy process $X$, its Laplace exponent is given in (1.1). For any given $q \geq 0$, let $\Phi(q)$ to be the largest (real) solution to $\psi(\lambda) = q$. For $q \geq 0$, the $q$-scale function $W^{(q)}(\cdot) : \mathbb{R} \mapsto [0, \infty)$ is the unique function supported on $[0, \infty)$ with Laplace transform

$$\int_0^\infty e^{-sx} W^{(q)}(x) \mathrm{d}x = \frac{1}{\psi(s) - q}, \quad s > \Phi(q).$$

It is known that $W^{(q)}(\cdot)$ is continuous and increasing on $[0, \infty)$. The existence of scale functions is shown in Kuznetsov et al. (2012). In the sequel, we write $W(\cdot)$ for $W^{(0)}(\cdot)$. The scale function is closely related to exit problems for spectrally negative Lévy processes. For $x \in \mathbb{R}$, we define the first passage times of $X$ as

$$T_x^{+(-)} = \inf \{ t \geq 0 : X_t > (<) x \}.$$

We recall the following well-known fluctuation identities; e.g., Section 8.2 of Kyprianou (2006), which are fundamental quantities in the study of occupation times, Parisian ruin problems, some tax models and so on; see, e.g., Landriault et al. (2011a), Czarna and Palmowski (2011), Albrecher et al. (2008) and Kyprianou and Zhou (2009).

**Theorem 1.4.3.** *For $q \geq 0$, the one-sided exit results are*

$$\mathbb{E}_u \left[ e^{-qT_0^-} 1_{\{T_0^- < \infty\}} \right] = Z^{(q)}(u) - \frac{q}{\Phi(q)} W^{(q)}(u),$$

*for any $u \geq 0$, and*

$$\mathbb{E}_u \left[ e^{-qT_x^+} 1_{\{T_x^+ < \infty\}} \right] = e^{-\Phi(q)(x-u)},$$

*for $0 \leq u \leq x$. The two-sided exit results are*

$$\mathbb{E}_u \left[ e^{-qT_x^+} 1_{\left\{T_x^+ < T_0^-\right\}} \right] = \frac{W^{(q)}(u)}{W^{(q)}(x)},$$

*and*

$$\mathbb{E}_u \left[ e^{-qT_0^-} 1_{\left\{T_0^- < T_x^+\right\}} \right] = Z^{(q)}(u) - Z^{(q)}(x) \frac{W^{(q)}(u)}{W^{(q)}(x)},$$

*for $0 \leq u \leq x$, with the second scale function $Z^{(q)}$ defined as $Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(y) dy$, for $x \in \mathbb{R}$.*

Note that without confusion, we write $\mathbb{E}_u[\cdot]$ for the conditional expectation $\mathbb{E}[\cdot | X_0 = u]$. For brevity, $\mathbb{E}[\cdot] = \mathbb{E}_0[\cdot]$.

Similarly, for a MAP $(U, J)$ with Laplace exponent $\mathbf{F}^{(q)}$ given in (1.2), the scale functions are generalized to the scale matrices, and the two-sided exit results can be found in, e.g., Ivanovs and Palmowski (2012, Theorem 1 and Corollary 3). Define $T_x^{U,+(-)} = \inf\{t \geq 0 : U_t > (<) x\}$.

**Theorem 1.4.4.** *For $0 \leq x \leq a$,*

$$\mathbb{E}\left[e^{-qT_a^{U,+}} 1_{\{T_0^{U,-} > T_a^{U,+}, J_{T_a^{U,+}}\}} | U_0 = x, J_0\right] = \mathbf{W}^{(q)}(x)\mathbf{W}^{(q)}(a)^{-1},$$

*and*

$$\mathbb{E}\left[e^{-qT_0^{U,-}} e^{-s|U_{T_0^{U,-}}|} 1_{\{T_0^{U,-} < T_a^{U,+}, J_{T_0^{U,-}}\}} | U_0 = x, J_0\right] = \mathbf{Z}^{(q)}(s,x) - \mathbf{W}^{(q)}(x)\mathbf{W}^{(q)}(a)^{-1}\mathbf{Z}^{(q)}(s,a)$$

*where $\mathbf{W}^{(q)}(x)$ is the q-scale matrix defined through its LT*

$$\int_0^\infty e^{-sx}\mathbf{W}^{(q)}(x)dx = \mathbf{F}^{(q)}(s)^{-1},$$

*and $\mathbf{Z}^{(q)}(s,x)$ is the second scale matrix defined as*

$$\mathbf{Z}^{(q)}(s,x) = e^{sx}\left(\mathbf{I} - \int_0^x e^{-sy}\mathbf{W}^{(q)}(y)dy\mathbf{F}^{(q)}(s)\right).$$

More results will be cited regarding the exit and drawdown problems when needed in the following chapters.

## 1.5 Outline and contributions of the thesis

In this thesis, my primary contributions can be categorized into one of the four aspects (which correspond to Chapters 2-5), which will be detailed in the following paragraphs.

- Generalize the results in the adaptive premium policy model by considering the surplus prior to ruin, deficit at ruin, as well as the total discounted premium paid until ruin;
- Propose an experience-based premium policy model and show its merit from a risk management viewpoint;

- Propose a drawdown-based regime-switching Lévy insurance model and connect it to existing risk models;
- Analyze drawdown-related quantities in the renewal insurance risk process where the strong Markov property does not hold.

The thesis is organized as follows. In Chapter 2, we generalize the results of Landriault et al. (2012) to the general Gerber-Shiu analysis in the risk model with an adaptive premium policy. With the assumption of mixed Erlang distributed claim sizes, the explicit expression for a more general Gerber-Shiu function than the ruin probability is derived through the defective renewal equation and an asymptotic formula is obtained. Also, we introduce a cost function, and get the expression for the total discounted premium paid until ruin. Finally, a related premium rate changing strategy is briefly presented.

Chapter 3 considers another adaptive premium policy, which is called the experience-based premium policy. The premium rate changes based on the increments of the surplus process at the review times. Two main directions for generalizing the classical risk model are considered: premium changes reflect recent claim experience, and a random period between premium rate reviews. Under the framework of the classical compound Poisson risk model and with combination of exponentials distributed review times, we examine the distribution of the increments between successive review times by characterizing their two one-sided LTs. The matrix-form defective renewal equation of the LT of the time to ruin is derived. Numerical examples are studied to show the merit of the proposed model. In addition, as variants of the proposed model, we incorporate a random performance level and a premium policy review conducted at claim occurrence.

Chapter 4 examines an adaptive policy based on the drawdown size of the insurance risk process. In this drawdown-based regime-switching Lévy insurance model (DBRS), the underlying drawdown process is used to model an insurer's level of financial distress over time, and to trigger regime-switching transitions. Using analytical arguments, we derive explicit formulas for a generalized two-sided exit problem. We specifically state conditions under which the survival probability is not trivially zero (which corresponds to the positive security loading conditions of the proposed model). The regime-dependent occupation time until ruin is later studied. As a special case of the general DBRS model, a regime-switching premium model is given further consideration. Connections with other existing risk models are established.

In Chapter 5, we extend the analysis of drawdown-related quantities to the context of the renewal insurance risk process with general interarrival times and phase-type distributed jump sizes. We make use of some recent results on the two-sided exit problem of the MAP and a fluid flow analogy. The two-sided exit quantities are shown to be central to the analysis of drawdown quantities including the drawdown time, the drawdown size, the running maximum (minimum) at the drawdown time, the last running maximum time prior to drawdown, the number of jumps before drawdown and the number of excursions from running maximum before drawdown. Finally, as another application of the fluid flow methodology, the expected discounted dividend payments until ruin is considered in the presence of a constant dividend barrier model.

Chapter 6 concludes the thesis and discusses some directions for future research. Note that the results in Chapters 3 and 4 are published in Insurance: Mathematics and Economics (see Li et al. (2015) and Landriault et al. (2015a) in the bibliography), and the results in Chapter 5 have been submitted for publication (see Landriault et al. (2015b) in the bibliography).

# Chapter 2

# Some generalizations to the adaptive premium policy

## 2.1 Introduction

In this chapter, we generalize the results of the adaptive premium policy model introduced in Landriault et al. (2012) in several directions. As such, a more detailed description of such a risk model is as follows. The surplus process of this risk model is defined as

$$U(t) = u + \int_0^t c(s)ds - \sum_{i=1}^{N(t)} Y_i, \tag{2.1}$$

where $u$, $\{N(t); t \geq 0\}$, $\{Y_i; i \geq 1\}$ are all defined the same way as in the classical compound Poisson risk model, and $\{c(t); t \geq 0\}$ represents the premium changing process (if it is fixed to a constant $c$, then the model reduces to the classical compound Poisson risk model). Actually, Landriault et al. (2012) fix $m$ premium rates $c_1, \ldots, c_m$, where $c_1 > \cdots > c_m > 0$ and $m-1$ thresholds $x_1, \ldots, x_{m-1}$, where $0 = x_0 < x_1 < \cdots < x_{m-1} < x_m = \infty$. Let $\tau_i$ be the time of the $i$th descending ladder height of the surplus process, i.e.,

$$\tau_i = \inf_{t > \tau_{i-1}} \{U(t) < U(\tau_{i-1})\},$$

with $\tau_0 = 0$. The premium rate is changed only at time $\tau_i$, and if $T_i = \tau_i - \tau_{i-1} \in (x_{j-1}, x_j]$, then the premium rate is fixed to $c_j$ for $j = 1, \ldots, m$. Hence the $2m - 1$ parameters $(c_1, \ldots, c_m, x_1, \ldots, x_{m-1})$ characterize this adaptive premium policy.

The technique of Erlangization is used here to replace the fixed parameter $x_j$ by Erlang distributed random variables. More precisely, the threshold differences $g_j := x_j - x_{j-1}$ are replaced by independent Erlang random variables $G_j$ with shape parameter $n_j$ and rate parameter $n_j/g_j$, $j = 1, \ldots, m-1$, where $n_j \in \mathbb{N}$, and then by increasing $n_j$, we can approximate the constant $g_j$ with arbitrary precision, since we have that $\mathbb{E}(G_j) = g_j$ and $\mathrm{Var}(G_j) = g_j^2/n_j$ and thus $\lim_{n_j \to \infty} \mathrm{Var}(G_j) = 0$. Without loss of generality, we can choose $g_j = \gamma_j x$ for some $x$ and $n_j = \gamma_j n$ so that the scale parameter of $G_j$ is $1/v$, where $v = n_j/g_j = n/x$. Thus, the threshold $x_j = (x_j - x_{j-1}) + \cdots + (x_1 - x_0)$ can be approximated by the random variable $D_j = G_1 + \cdots + G_j$ which has Erlang distribution with shape parameter $\widetilde{n}_j = n\Gamma_j$ and rate parameter $v$, for $j = 1, \ldots, m-1$, where $\Gamma_j = \sum_{k=1}^{j} \gamma_j$ and define $D_0 = 0$, $\widetilde{n}_0 = 0$, $\Gamma_0 = 0$, $D_m = \infty$, $\widetilde{n}_m = \infty$, and $\Gamma_m = \infty$.

Also, we assume that the claim sizes are iid mixed Erlang random variables with pdf

$$p(x) = \sum_{k=1}^{\infty} q_k e_{\beta,k}(x), \;\; x \geq 0,$$

where $\{q_k;\; k \geq 1\}$ are the mixing weights satisfying $\sum_{k=1}^{\infty} q_k = 1$ and

$$e_{\beta,k}(x) = \beta^k x^{k-1} e^{-\beta x}/(k-1)!,$$

is the Erlang pdf with shape parameter $k$ and rate parameter $\beta$ (with corresponding cdf denoted by $E_{\beta,k}(x)$). Thus, the cdf and mean are given by $P(x) = \sum_{k=1}^{\infty} q_k E_{\beta,k}(x)$ and

$\mu = \sum_{k=1}^{\infty} q_k k/\beta$.

Also, for later use, we have to introduce $k_i(t, y)$, which is the joint defective density of the time $t$ and size $y$ of the first drop given a premium rate $c_i$. Landriault and Willmot (2009) shows that under the assumption of mixed Erlang claim sizes,

$$k_i(t, y) = k_{i,1}(t, y) + k_{i,2}(t, y), \tag{2.2}$$

where

$$k_{i,1}(t, y) = \lambda e^{-\lambda t} \sum_{k=1}^{\infty} q_k e_{\beta,k}(c_i t + y),$$

and

$$k_{i,2}(t, y) = \lambda e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} \int_0^{c_i t} \frac{x}{c_i t} \sum_{k=1}^{\infty} q_k^{*n} e_{\beta,k}(c_i t - y) \sum_{r=1}^{\infty} q_r e_{\beta,k}(x + y) dx.$$

Here $\{q_k^{*n};\ k \geq 1\}$ are the mixing weights associated with the $n$-fold ($n > 1$) convolution of the pdf $p(x)$, i.e., $p^{*n}(x) = \sum_{k=1}^{\infty} q_k^{*m} e_{\beta,k}(x)$, which yields for $k \geq n$, $q_k^{*n} = \sum_{j=1}^{k-1} q_j^{*(n-1)} q_{k-j}$ (with $q_k^{*1} = q_k$) and for $k < n$, $q_k^{*n} = 0$.

## 2.2 The general Gerber-Shiu function

In Landriault et al. (2012), the goal is to compute the probability of ruin, which is defined as

$$\psi_i(u) = \Pr(T < \infty | U_0 = u,\ c(0) = c_i),\ \text{for } i = 1, \ldots, m, \tag{2.3}$$

where $T$ is the time to ruin.

Here we are interested in the Gerber-Shiu function defined as

$$m_{\delta,i}(u) = \mathbb{E}\left[e^{-\delta T} w_1\left(U_{T^-}\right) w_2\left(|U_T|\right) 1_{\{T<\infty\}} | U_0 = u,\ c(0) = c_i\right], \tag{2.4}$$

which allow us to generalize the results in Landriault et al. (2012) to the Gerber-Shiu function with the penalty function $w(x, y)$ of the form $w_1(x)w_2(y)$, since with the choice of $\delta = 0$, $w_1(x) = 1$ and $w_2(y) = 1$, (2.4) reduces to (2.3). Note that even though this type of Gerber-Shiu function seems not to be the most general one, it contains many of the most popular forms used in the literature, such as $w(x, y) = e^{-s_1 x - s_2 y}$ and $w(x, y) = 1_{\{x<x_1\}}1_{\{y<y_1\}}$. Furthermore, theoretically we can derive the Gerber-Shiu function with any penalty function $w(x, y)$ by integrating over the joint density of the surplus prior to ruin and the deficit at ruin which can be obtained by the LT inversion of the Gerber-Shiu function with penalty function $w(x, y) = e^{-s_1 x - s_2 y}$.

In the following, our focus is on deriving an explicit expression of $m_{\delta,i}(u)$ in (2.4) and the main result is shown in Theorem 2.2.1. To achieve this, we first show that $\{m_{\delta,i}(u);\ i = 1, \dots, m\}$ satisfies a matrix-form defective renewal equation. By conditioning on the first drop in surplus, we have

$$m_{\delta,i}(u) = \sum_{j=1}^{m} \int_0^u m_{\delta,j}(u-y) h_{\delta,i,j}(y) dy + v_{\delta,i}(u), \tag{2.5}$$

where $v_{\delta,i}(u)$ is given by the equation (2.33) in Gerber and Shiu (1998),

$$v_{\delta,i}(u) = \frac{\lambda}{c_i} \int_u^\infty \int_0^\infty e^{-\rho_i(x-u)} w_1(x) w_2(y) p(x+y) dy dx,\ \ u \geq 0, \tag{2.6}$$

and using the density of $k_i(t, y)$ and the Erlangization technique, we have

$$\begin{aligned} h_{\delta,i,j}(y) &= \int_0^\infty e^{-\delta t} k_i(t, y) \Pr(D_{j-1} < t \leq D_j) dt \\ &= \frac{1}{v} \int_0^\infty e^{-\delta t} k_i(t, y) \sum_{k=\widetilde{n}_{j-1}+1}^{\widetilde{n}_j} e_{v,k}(t) dt. \end{aligned} \tag{2.7}$$

We can rewrite (2.5) in a matrix-form

$$\mathbf{m}_\delta(u) = \int_0^u \mathbf{H}_\delta(y) \mathbf{m}_\delta(u-y) dy + \mathbf{v}_\delta(u), \tag{2.8}$$

where $\mathbf{m}_\delta(u) = (m_{\delta,1}(u), \ldots, m_{\delta,m}(u))^T$, $\mathbf{v}_\delta(u) = (v_{\delta,1}(u), \ldots, v_{\delta,m}(u))^T$ and $\mathbf{H}_\delta(u) = (h_{\delta,i,j}(u))_{i,j=1}^m$.

By taking the LT on both sides of (2.8), we have

$$\widetilde{\mathbf{m}}_\delta(z) = \widetilde{\mathbf{H}}_\delta(z)\widetilde{\mathbf{m}}_\delta(z) + \widetilde{\mathbf{v}}_\delta(z), \tag{2.9}$$

or

$$\widetilde{\mathbf{m}}_\delta(z) = \left(\mathbf{I} - \widetilde{\mathbf{H}}_\delta(z)\right)^{-1} \widetilde{\mathbf{v}}_\delta(z), \tag{2.10}$$

where $\widetilde{\mathbf{m}}_\delta(z) = (\widetilde{m}_{\delta,1}(z), \ldots, \widetilde{m}_{\delta,m}(z))^T$, $\widetilde{\mathbf{v}}_\delta(z) = (\widetilde{v}_{\delta,1}(z), \ldots, \widetilde{v}_{\delta,m}(z))^T$, $\widetilde{\mathbf{H}}_\delta(z) = \left(\widetilde{h}_{\delta,i,j}(z)\right)_{i,j=1}^m$ and $\mathbf{I}$ is the $m \times m$ identity matrix. Now we have to identify $v_{\delta,i}(u)$ and $h_{\delta,i,j}(y)$ and especially their LTs.

**Lemma 2.2.1.** *The LT of $v_{\delta,i}(u)$ is given by*

$$\widetilde{v}_{\delta,i}(z) = \frac{\lambda}{c_i \beta} \sum_{k=1}^{\infty} q_k \sum_{s=1}^{k} l_{s,k} \frac{\widetilde{w}^*_{1,s}(\rho_i) - \widetilde{w}^*_{1,s}(z)}{z - \rho_i}, \tag{2.11}$$

*for $i = 1, \ldots, m$, where $\rho_i$ is the unique non-negative solution to the Lundberg's fundamental equation*

$$\delta + \lambda - c_i z = \lambda \widetilde{p}(z),$$

$l_{s,k} = \int_0^\infty w_2(y) e_{\beta,k-s+1}(y) dy$ *and* $w^*_{1,s}(x) = w_1(x) e_{\beta,s}(x)$.

*Proof.* By plugging in the mixed Erlang claim size density in (2.6), we have

$$\begin{aligned}
v_{\delta,i}(u) &= \frac{\lambda}{c_i} \int_u^\infty \int_0^\infty e^{-\rho_i(x-u)} w_1(x) w_2(y) p(x+y) dy dx \\
&= \frac{\lambda}{c_i} \int_u^\infty \int_0^\infty e^{-\rho_i(x-u)} w_1(x) w_2(y) \{\sum_{k=1}^{\infty} q_k \beta^{-1} \sum_{s=1}^{k} e_{\beta,s}(x) e_{\beta,k-s+1}(y)\} dy dx \\
&= \frac{\lambda}{c_i \beta} \sum_{k=1}^{\infty} q_k \sum_{s=1}^{k} [\int_u^\infty e^{-\rho_i(x-u)} w_1(x) e_{\beta,s}(x) dx] \times [\int_0^\infty w_2(y) e_{\beta,k-s+1}(y) dy],
\end{aligned}$$

where we use the fact that $e_{\beta,k}(x+y) = \beta^{-1} \sum_{s=1}^{k} e_{\beta,s}(x) e_{\beta,k-s+1}(y)$.

Therefore,

$$p(x+y)=\sum_{k=1}^{\infty}q_k e_{\beta,k}(x+y)=\sum_{k=1}^{\infty}q_k\beta^{-1}\sum_{s=1}^{k}e_{\beta,s}(x)e_{\beta,k-s+1}(y).$$

Taking the LT on both sides of $v_{\delta,i}(u)$, it follows that

$$\begin{aligned}
\widetilde{v}_{\delta,i}(z) &= \int_0^\infty e^{-zu}v_{\delta,i}(u)du \\
&= \frac{\lambda}{c_i\beta}\sum_{k=1}^{\infty}q_k\sum_{s=1}^{k}\int_0^\infty e^{-zu}\int_u^\infty e^{-\rho_i(x-u)}w_1(x)e_{\beta,s}(x)dxdu\int_0^\infty w_2(y)e_{\beta,k-s+1}(y)dy \\
&= \frac{\lambda}{c_i\beta}\sum_{k=1}^{\infty}q_k\sum_{s=1}^{k}\int_0^\infty w_2(y)e_{\beta,k-s+1}(y)dyT_zT_{\rho_i}\{w_1(0)e_{\beta,s}(0)\} \\
&= \frac{\lambda}{c_i\beta}\sum_{k=1}^{\infty}q_k\sum_{s=1}^{k}l_{s,k}\frac{\widetilde{w}^*_{1,s}(\rho_i)-\widetilde{w}^*_{1,s}(z)}{z-\rho_i},
\end{aligned}$$

where $l_{s,k}$ and $w^*_{1,s}(x)$ are defined as above. The property of the Dickson-Hipp operator in (1.4) is used from the third line to the forth line. □

**Example 2.2.1.** *If $w_2(y)=1$,*

$$l_{s,k}=\int_0^\infty w_2(y)e_{\beta,k-s+1}(y)dy=1.$$

*Furthermore, if $w_1(x)=1$, by (2.11),*

$$\begin{aligned}
\widetilde{v}_{\delta,i}(z) &= \frac{\lambda}{c_i\beta}\sum_{k=1}^{\infty}q_k\sum_{s=1}^{k}l_{s,k}\frac{\widetilde{e}_{\beta,s}(\rho_i)-\widetilde{e}_{\beta,s}(z)}{z-\rho_i} \\
&= \frac{\lambda}{c_i\beta}\sum_{k=1}^{\infty}q_k\sum_{s=1}^{k}l_{s,k}\frac{1}{\beta}\sum_{t=1}^{s}\left(\frac{\beta}{\beta+z}\right)^t\left(\frac{\beta}{\beta+\rho_i}\right)^{s+1-t} \\
&= \frac{\lambda}{c_i\beta^2}\sum_{t=1}^{\infty}\sum_{s=t}^{\infty}\sum_{k=s}^{\infty}q_k\left(\frac{\beta}{\beta+\rho_i}\right)^{s+1-t}\left(\frac{\beta}{\beta+z}\right)^t,
\end{aligned} \tag{2.12}$$

*which is consistent with the result in Landriault et al. (2012), and from line 1 to line 2 we* *use the fact*

$$
\begin{aligned}
\frac{\widetilde{e}_{\beta,s}(\rho_i)-\widetilde{e}_{\beta,s}(z)}{z-\rho_i} &= \mathcal{T}_z\mathcal{T}_{\rho_i}\{e_{\beta,s}(0)\} \\
&= \int_0^\infty e^{-zx}\int_0^\infty e^{-\rho_i y}e_{\beta,s}(x+y)dydx \\
&= \int_0^\infty e^{-zx}\int_0^\infty e^{-\rho_i y}\beta^{-1}\sum_{t=1}^{s}e_{\beta,t}(x)e_{\beta,s-t+1}(y)dydx \\
&= \frac{1}{\beta}\sum_{t=1}^{s}\left(\frac{\beta}{\beta+z}\right)^t\left(\frac{\beta}{\beta+\rho_i}\right)^{s+1-t}.
\end{aligned}
$$

**Lemma 2.2.2.** *The LT of $h_{\delta,i,j}(u)$ is given by*

$$
\widetilde{h}_{\delta,i,j}(z)=\sum_{n=1}^{\infty}\zeta_{\delta,i,j,n}\left(\frac{\beta}{\beta+z}\right)^n, \tag{2.13}
$$

*for $i,j=1,\ldots,m$, where*

$$
\zeta_{\delta,i,j,n}=\frac{\lambda}{\beta c_i}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=1}^{\infty}q_k^{*m}q_{n+s-1}\frac{s}{k+s}\left(\frac{\lambda}{\lambda+\delta}\right)^m f_{NB}(m;k+s,\frac{\beta c_i}{\lambda+\delta+\beta c_i})\eta_{\delta,i,j}(m+k+s-1), \tag{2.14}
$$

*with $q_k^{*0}=1_{\{k=0\}}$ and*

$$
\eta_{\delta,i,j}(s)=F_{NB}\left(s;\widetilde{n}_{j-1},\frac{v}{\lambda+\delta+\beta c_i+v}\right)-F_{NB}\left(s;\widetilde{n}_j,\frac{v}{\lambda+\delta+\beta c_i+v}\right).
$$

*Note that the negative binomial (NB) distribution has pf $f_{NB}(n;p,\theta)=\binom{n+p-1}{n}\theta^p(1-\theta)^n$ and cdf $F_{NB}(s;p,\theta)=\sum_{n=0}^{s}f_{NB}(n;p,\theta)$.*

*Proof.* By (2.7) and (2.2), $\widetilde{h}_{\delta,i,j}(z)=\widetilde{h}_{\delta,i,j,1}(z)+\widetilde{h}_{\delta,i,j,2}(z)$, where

$$
\widetilde{h}_{\delta,i,j,r}(z)=\int_0^\infty e^{-zy}\frac{1}{v}\int_0^\infty e^{-\delta t}k_{i,r}(t,y)\sum_{p=\widetilde{n}_{j-1}+1}^{\widetilde{n}_j}e_{v,p}(t)dtdy,
$$

for $r=1,2$.

When $r = 1$,

$$\begin{aligned}
\widetilde{h}_{\delta,i,j,1}(z) &= \frac{1}{v}\int_0^{\infty} e^{-zy}\int_0^{\infty} e^{-\delta t}\lambda e^{-\lambda t}\sum_{k=1}^{\infty} q_k e_{\beta,k}(c_i t+y)\sum_{p=\widetilde{n}_{j-1}+1}^{\widetilde{n}_j} e_{v,p}(t)dtdy \\
&= \frac{\lambda}{\beta v}\sum_{k=1}^{\infty}\sum_{l=1}^{k} q_k\int_0^{\infty} e^{-zy}e_{\beta,k+1-l}(y)dy\int_0^{\infty} e^{-(\lambda+\delta)t}e_{\beta,l}(c_i t)\sum_{p=\widetilde{n}_{j-1}+1}^{\widetilde{n}_j} e_{v,p}(t)dt \\
&= \frac{\lambda}{\beta v}\sum_{k=1}^{\infty}\sum_{l=1}^{k} q_k\left(\frac{\beta}{\beta+z}\right)^{k+1-l}\sum_{p=\widetilde{n}_{j-1}+1}^{\widetilde{n}_j}\frac{\beta(\beta c_i)^{l-1}v^p}{(\lambda+\delta+\beta c_i+v)^{l+p-1}}\binom{l+p-2}{l-1} \\
&= \frac{\lambda}{\lambda+\delta+\beta c_i}\sum_{k=1}^{\infty}\sum_{l=1}^{k} q_k\left(\frac{\beta}{\beta+z}\right)^{k+1-l}\left(\frac{\beta c_i}{\lambda+\delta+\beta c_i}\right)^{l-1}\eta_{\delta,i,j}(l-1) \\
&= \sum_{l=1}^{\infty}\sum_{n=1}^{\infty}\frac{\lambda}{\beta c_i}q_{n+l-1}\left(\frac{\beta}{\beta+z}\right)^{n}\left(\frac{\beta c_i}{\lambda+\delta+\beta c_i}\right)^{l}\eta_{\delta,i,j}(l-1) \\
&= \sum_{n=1}^{\infty}\zeta_{\delta,i,j,n,1}\left(\frac{\beta}{\beta+z}\right)^{n}, \qquad (2.15)
\end{aligned}$$

where $\zeta_{\delta,i,j,n,1} = \sum_{l=1}^{\infty}\frac{\lambda}{\beta c_i}q_{n+l-1}\left(\frac{\beta c_i}{\lambda+\delta+\beta c_i}\right)^{l}\eta_{\delta,i,j}(l-1)$, and

$$\begin{aligned}
\eta_{\delta,i,j}(s) &= \frac{\lambda+\delta+\beta c_i}{v}\sum_{p=\widetilde{n}_{j-1}+1}^{\widetilde{n}_j}\frac{(\lambda+\delta+\beta c_i)^s v^p}{(\lambda+\delta+\beta c_i+v)^{s+p}}\binom{s+p-1}{s} \\
&= \frac{\lambda+\delta+\beta c_i}{v}\sum_{p=\widetilde{n}_{j-1}+1}^{\widetilde{n}_j} f_{NB}(s;p,\frac{v}{\lambda+\delta+\beta c_i+v}) \\
&= \frac{\lambda+\delta+\beta c_i}{v}\{\sum_{p=1}^{\widetilde{n}_j} f_{NB}(s;p,\frac{v}{\lambda+\delta+\beta c_i+v}) - \sum_{p=1}^{\widetilde{n}_{j-1}} f_{NB}(s;p,\frac{v}{\lambda+\delta+\beta c_i+v})\} \\
&= F_{NB}(s;\widetilde{n}_{j-1},\frac{v}{\lambda+\delta+\beta c_i+v}) - F_{NB}(s;\widetilde{n}_j,\frac{v}{\lambda+\delta+\beta c_i+v}).
\end{aligned}$$

Here for the last line, we used the fact that $F_{NB}(s;n,\theta) = 1-\frac{1-\theta}{\theta}\sum_{p=1}^{n} f_{NB}(s;p,\theta)$.

For $r = 2$, Landriault et al. (2012) provides the following result,

$$\sum_{r=1}^{\infty} q_r\int_0^{c_i t}\frac{x}{c_i t}e_{\beta,k}(c_i t-x)e_{\beta,r}(x+y)dx = \frac{1}{\beta}\sum_{s=1}^{\infty}\sum_{n=1}^{\infty} q_{n+s-1}\frac{s}{k+s}e_{\beta,n}(y)e_{\beta,k+s}(c_i t).$$

Hence,

$$
\begin{aligned}
\widetilde{h}_{\delta,i,j,2}(z) &= \int_0^\infty e^{-zy}\frac{1}{v}\int_0^\infty e^{-\delta t}k_{i,2}(t,y)\sum_{p=\widetilde{n}_{j-1}+1}^{\widetilde{n}_j} e_{v,p}(t)dtdy \\
&= \frac{1}{v}\int_0^\infty e^{-zy}\int_0^\infty e^{-\delta t}\lambda e^{-\lambda t}\sum_{m=1}^\infty \frac{(\lambda t)^m}{m!}\sum_{k=1}^\infty q_k^{*m} \\
&\quad \times\frac{1}{\beta}\sum_{s=1}^\infty\sum_{n=1}^\infty q_{n+s-1}\frac{s}{k+s}e_{\beta,n}(y)e_{\beta,k+s}(c_i t)\sum_{p=\widetilde{n}_{j-1}+1}^{\widetilde{n}_j} e_{v,p}(t)dtdy \\
&= \frac{\lambda}{\beta c_i}\sum_{m=1}^\infty\sum_{k=1}^\infty\sum_{s=1}^\infty\sum_{n=1}^\infty q_k^{*m}q_{n+s-1}\frac{s}{k+s}\left(\frac{\beta}{\beta+z}\right)^n\frac{\lambda^m\beta^{k+s}c_i^{k+s}}{(\lambda+\delta+\beta c_i)^{m+k+s}}\binom{m+k+s-1}{m} \\
&\quad \times\frac{\lambda+\delta+\beta c_i}{v}\sum_{p=\widetilde{n}_{j-1}+1}^{\widetilde{n}_j}\left(\frac{v}{\lambda+\delta+\beta c_i+v}\right)^p\left(\frac{\lambda+\delta+\beta c_i}{\lambda+\delta+\beta c_i+v}\right)^{m+k+s-1}\binom{m+k+s+p-2}{p-1} \\
&= \frac{\lambda}{\beta c_i}\sum_{m=1}^\infty\sum_{k=1}^\infty\sum_{s=1}^\infty\sum_{n=1}^\infty q_k^{*m}q_{n+s-1}\frac{s}{k+s}\left(\frac{\beta}{\beta+z}\right)^n\left(\frac{\lambda}{\lambda+\delta}\right)^m \\
&\quad \times f_{NB}(m;k+s,\frac{\beta c_i}{\lambda+\delta+\beta c_i})\eta_{\delta,i,j}(m+k+s-1) \\
&= \sum_{n=1}^\infty \zeta_{\delta,i,j,n,2}\left(\frac{\beta}{\beta+z}\right)^n, \qquad (2.16)
\end{aligned}
$$

where

$$
\zeta_{\delta,i,j,n,2} = \frac{\lambda}{\beta c_i}\sum_{m=1}^\infty\sum_{k=1}^\infty\sum_{s=1}^\infty q_k^{*m}q_{n+s-1}\frac{s}{k+s}\left(\frac{\lambda}{\lambda+\delta}\right)^m f_{NB}(m;k+s,\frac{\beta c_i}{\lambda+\delta+\beta c_i})\eta_{\delta,i,j}(m+k+s-1).
$$

Therefore, by combining (2.15) and (2.16), we have

$$
\widetilde{h}_{\delta,i,j}(z) = \sum_{n=1}^\infty \zeta_{\delta,i,j,n}\left(\frac{\beta}{\beta+z}\right)^n,
$$

where $\zeta_{\delta,i,j,n} = \zeta_{\delta,i,j,n,1} + \zeta_{\delta,i,j,n,2}$ is given in (2.14).

Now we have computed $\widetilde{h}_{\delta,i,j}(z)$ using the Erlang approximation, which finishes the second step in the Erlangization technique. In what follows, we need to replace the Erlang distributed thresholds by their real values, which can be achieved by letting the parameters $n_j$ go to infinity as mentioned above. From Lemmas 2.2.1 and 2.2.2, we see that $n_j$

only appears in $\widetilde{h}_{\delta,i,j}(z)$, or more precisely, in the term $\eta_{\delta,i,j}(s)$. Also, the NB distribution $F_{NB}(s;n,\frac{1}{1+p})$ has the probability generating function (pgf) $(1+p(1-z))^{-n}$, so if $np=\lambda$, then

$$\lim_{n\to\infty}\left(1+\frac{\lambda(1-z)}{n}\right)^{-n}=e^{\lambda(z-1)},$$

which is the pgf of the Poisson distribution with mean $\lambda$ denoted as $F_P(s;\lambda)$. Using the fact that the Poisson distribution can be viewed as a limiting NB distribution, the limit of $\eta_{\delta,i,j}(s)$ becomes

$$\begin{aligned}\lim_{\{n_j\}_{j=1}^{m-1}\to\infty}\eta_{\delta,i,j}(s) &= \lim_{\widetilde{n}_{j-1}\to\infty}F_{NB}\left(s;\widetilde{n}_{j-1},\frac{v}{\lambda+\delta+\beta c_i+v}\right)\\ &\quad -\lim_{\widetilde{n}_j\to\infty}F_{NB}\left(s;\widetilde{n}_j,\frac{v}{\lambda+\delta+\beta c_i+v}\right)\\ &= F_P(s;\lambda_{i,j-1})-F(s;\lambda_{i,j})\triangleq\eta^*_{\delta,i,j}(s),\end{aligned}$$

where $\lambda_{i,j}=\widetilde{n}_j\times\frac{\lambda+\delta+\beta c_i}{v}=n\Gamma_j\times\frac{\lambda+\delta+\beta c_i}{n/x}=x(\lambda+\delta+\beta c_i)\Gamma_j$ with $F_P(s;0)=1$ and $F_P(s;\infty)=0$. $\square$

Now we are ready to provide the main theorem in this chapter.

**Theorem 2.2.1.** *The Gerber-Shiu function defined in (2.4) has the explicit expression*

$$m_{\delta,i}(u)=\sum_{k=0}^{\infty}\sum_{s=1}^{\infty}\sum_{j=1}^{m}\kappa_{\delta,i,k,s,j}\int_0^u e_{\beta,k}(u-x)T_{\rho_j}\{w_1(x)e_{\beta,s}(x)\}dx,$$

*where* $\kappa_{\delta,i,k,s,j}=\sum_{l=0}^{\infty}\zeta^{*l}_{\delta,i,j,k}\frac{\lambda}{c_i\beta}\sum_{r=s}^{\infty}q_r l_{s,r}$.

*Proof.* In order to invert (2.10) w.r.t. $z$, we need to use the identity

$$\left(\mathbf{I}-\widetilde{\mathbf{H}}_\delta(z)\right)^{-1}=\sum_{l=0}^{\infty}\widetilde{\mathbf{H}}^l_\delta(z),$$

which holds when the spectral radius of $\widetilde{\mathbf{H}}_\delta(z)$ is less than one. In fact, $h_{\delta,i}(y)=\sum_{j=1}^{m}h_{\delta,i,j}(y)$

has a defective probability generating function (see Willmot and Woo (2012))

$$\widetilde{h}_{\delta,i}(z) = \sum_{l=1}^{\infty} \widetilde{q}_{\delta,i,l} \left( \frac{\beta}{\beta+z} \right)^l,$$

and

$$\sum_{j=1}^{m} \left| \widetilde{h}_{\delta,i,j}(z) \right| = \widetilde{h}_{\delta,i}(z) < \sum_{l=1}^{\infty} \widetilde{q}_{\delta,i,l} < 1,$$

so the spectral radius of $\widetilde{\mathbf{H}}_\delta(z)$ denoted as $r\left(\widetilde{\mathbf{H}}_\delta(z)\right) \leq \max_i \sum_{j=1}^{m} \left| \widetilde{h}_{\delta,i,j}(z) \right| < 1$.

Hence now, (2.10) can be written as

$$\widetilde{\mathbf{m}}_\delta(z) = \sum_{l=0}^{\infty} \widetilde{\mathbf{H}}_\delta^l(z) \widetilde{\mathbf{v}}_\delta(z), \tag{2.17}$$

i.e.,

$$\begin{aligned}
\widetilde{m}_{\delta,i}(z) &= \sum_{l=0}^{\infty} \sum_{j=1}^{m} \sum_{k=0}^{\infty} \zeta_{\delta,i,j,k}^{*l} \left( \frac{\beta}{\beta+z} \right)^k \times \frac{\lambda}{c_j \beta} \sum_{r=1}^{\infty} q_r \sum_{s=1}^{r} l_{s,r} \frac{\widetilde{w}_{1,s}^*(\rho_j) - \widetilde{w}_{1,s}^*(z)}{z-\rho_j} \\
&= \sum_{k=0}^{\infty} \sum_{s=1}^{\infty} \sum_{j=1}^{m} \sum_{l=0}^{\infty} \zeta_{\delta,i,j,k}^{*l} \frac{\lambda}{c_j \beta} \sum_{r=s}^{\infty} q_r l_{s,r} \left( \frac{\beta}{\beta+z} \right)^k \frac{\widetilde{w}_{1,s}^*(\rho_j) - \widetilde{w}_{1,s}^*(z)}{z-\rho_k},
\end{aligned}$$

where $\zeta_{\delta,i,j,k}^{*0} = 1_{\{i=j,k=0\}}$,

$$\zeta_{\delta,i,j,k}^{*1} = \frac{\lambda}{\beta c_i} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{s=1}^{\infty} q_k^{*m} q_{n+s-1} \frac{s}{k+s} \left( \frac{\lambda}{\lambda+\delta} \right)^m f_{NB}(m; k+s, \frac{\beta c_i}{\lambda+\delta+\beta c_i}) \eta_{\delta,i,j}^*(m+k+s-1)$$

and $\zeta_{\delta,i,j,k}^{*l} = \sum_{r=1}^{m} \sum_{v=1}^{k-1} \zeta_{\delta,i,r,v}^{*(l-1)} \zeta_{\delta,i,j,k-v}$.

After taking the LT inversion w.r.t. $z$, we obtain

$$m_{\delta,i}(u) = \sum_{k=0}^{\infty} \sum_{s=1}^{\infty} \sum_{j=1}^{m} \sum_{l=0}^{\infty} \zeta_{\delta,i,j,k}^{*l} \frac{\lambda}{c_j \beta} \sum_{r=s}^{\infty} q_r l_{s,r} \int_0^u e_{\beta,k}(u-x) T_{\rho_j} \{ w_1(x) e_{\beta,s}(x) \} dx.$$

$\square$

**Example 2.2.2.** *If $w_1(x) = 1$, then by (2.17),*

$$\begin{aligned}\widetilde{m}_{\delta,i}(z) &= \sum_{l=0}^{\infty}\sum_{j=1}^{m}\sum_{k=0}^{\infty}\zeta^{*l}_{\delta,i,j,k}\left(\frac{\beta}{\beta+z}\right)^{k}\times\frac{\lambda}{c_j\beta^2}\sum_{t=0}^{\infty}\sum_{s=t+1}^{\infty}\sum_{r=s}^{\infty}q_r l_{s,r}\left(\frac{\beta}{\beta+\rho_j}\right)^{s-t}\left(\frac{\beta}{\beta+z}\right)^{t+1}\\ &= \sum_{w=0}^{\infty}\left(\frac{\beta}{\beta+z}\right)^{w+1}\sum_{l=0}^{\infty}\sum_{j=1}^{m}\sum_{t=0}^{w}\sum_{s=t+1}^{\infty}\sum_{r=s}^{\infty}\zeta^{*l}_{\delta,i,j,w-t}\frac{\lambda}{c_j\beta^2}q_r l_{s,r}\left(\frac{\beta}{\beta+\rho_j}\right)^{s-t},\end{aligned}$$

*thus we have*

$$m_{\delta,i}(u) = \sum_{w=0}^{\infty}\kappa_{\delta,i,w}e_{\beta,w+1}(u),$$

*where* $\kappa_{\delta,i,w} = \sum_{l=0}^{\infty}\sum_{j=1}^{m}\sum_{t=0}^{w}\sum_{s=t+1}^{\infty}\sum_{r=s}^{\infty}\zeta^{*l}_{\delta,i,j,w-t}\frac{\lambda}{c_j\beta^2}q_r l_{s,r}\left(\frac{\beta}{\beta+\rho_j}\right)^{s-t}$.

*Furthermore, if $w_2(y) = 1$ and $\delta = 0$, the Gerber-Shiu function reduces to the probability of ruin. We have $l_{s,k} = \int_0^{\infty} w_2(y)e_{\beta,k-s+1}(y)dy = 1$, so*

$$m_{0,i}(u) = \sum_{w=0}^{\infty}\kappa_{0,i,w}e_{\beta,w+1}(u),$$

*where* $\kappa_{0,i,w} = \sum_{l=0}^{\infty}\sum_{j=1}^{m}\sum_{t=0}^{w}\sum_{s=t+1}^{\infty}\sum_{r=s}^{\infty}\zeta^{*l}_{i,j,w-t}\frac{\lambda}{c_j\beta^2}q_r\left(\frac{\beta}{\beta+\rho_j}\right)^{s-t}$. *This result is consistent with Theorem 1 in Landriault et al. (2012).*

## 2.3 Discounted total premium paid

In this section, we are interested in the discounted total premium paid until ruin, which is defined as

$$\phi_{\delta,i}(u) = \mathbb{E}[\int_0^T e^{-\delta t}c(t)dt1_{\{T<\infty\}}|U_0 = u,\ c(0) = c_i].$$

By conditioning on the first drop in surplus, we have

$$\phi_{\delta,i}(u) = \sum_{j=1}^{m}\int_0^u \phi_{\delta,j}(u-y)h_{\delta,i,j}(y)dy + v^*_{\delta,i}(u),$$

where $h_{\delta,i,j}(y)$ is as defined in (2.7) and

$$\begin{aligned} v^*_{\delta,i}(u) &= c_i \int_u^\infty \int_0^\infty \frac{1-e^{-\delta t}}{\delta} k_i(t,y)dtdy \\ &= \frac{c_i}{\delta}\left[\int_u^\infty h_i(y)dy - \int_u^\infty \int_0^\infty e^{-\delta t} k_i(t,y)dtdy\right] \\ &\triangleq \frac{c_i}{\delta}\left[v_{0,i,1}(u) - v_{\delta,i,2}(u)\right], \end{aligned} \tag{2.18}$$

where $v_{0,i,1}(u) = \int_u^\infty h_i(y)dy$ and $v_{\delta,i,2}(u) = \int_u^\infty \int_0^\infty e^{-\delta t} k_i(t,y)dtdy$.

Actually, $v_{0,i,1}(u)$ is given in Landriault et al. (2012) (also the same as the inversion of (2.12) with $\delta = 0$), which is

$$v_{0,i,1}(u) = \frac{\lambda}{c_i \beta^2} \sum_{t=1}^\infty \sum_{s=t}^\infty \sum_{k=s}^\infty q_k \left(\frac{\beta}{\beta+\rho_{0,i}}\right)^{s+1-t} e_{\beta,t}(u). \tag{2.19}$$

As for $v_{\delta,i,2}(u)$, we first consider the LT of the time to ruin given by

$$\overline{G}_{\delta,i}(u) = \mathbb{E}\left[e^{-\delta T} 1_{\{T<\infty\}} | U_0 = u,\ c(0) = c_i\right],$$

which is a special case of the Gerber-Shiu function by letting the penalty function $w_1(x) = 1$ and $w_2(x) = 1$. Then by (2.5), we find that $\overline{G}_{\delta,i}(u)$ satisfies

$$\overline{G}_{\delta,i}(u) = \sum_{j=1}^m \int_0^u \overline{G}_{\delta,j}(u-y) h_{\delta,i,j}(y)dy + v_{\delta,i,2}(u),$$

where $v_{\delta,i,2}(u) = \int_u^\infty \int_0^\infty e^{-\delta t} k_i(t,y)dtdy$ is the quantity that we are looking for.

Given by the inversion of (2.12) with $\delta \neq 0$,

$$v_{\delta,i,2}(u) = \frac{\lambda}{c_i \beta^2} \sum_{t=1}^\infty \sum_{s=t}^\infty \sum_{k=s}^\infty q_k \left(\frac{\beta}{\beta+\rho_{\delta,i}}\right)^{s+1-t} e_{\beta,t}(u). \tag{2.20}$$

Substitution of (2.19) and (2.20) into (2.18) yields

$$
\begin{aligned}
v^*_{\delta,i}(u) &= \frac{c_i}{\delta}\left[v_{0,i,1}(u) - v_{\delta,i,2}(u)\right] \\
&= \frac{\lambda}{\delta\beta^2}\sum_{t=1}^{\infty}\sum_{s=t}^{\infty}\sum_{k=s}^{\infty} q_k \left[\left(\frac{\beta}{\beta+\rho_{\delta,i}}\right)^{s+1-t} - \left(\frac{\beta}{\beta+\rho_{0,i}}\right)^{s+1-t}\right] e_{\beta,t}(u).
\end{aligned}
$$

**Proposition 2.3.1.** *In the adaptive premium policy model, the discounted total premium paid before ruin is given by*

$$\phi_{\delta,i}(u) = \sum_{w=0}^{\infty} e_{\beta,w+1}(u)\kappa^*_{\delta,i,w},$$

*where* $\kappa^*_{\delta,i,w} = \sum_{l=0}^{\infty}\sum_{j=1}^{m}\sum_{t=0}^{w}\sum_{s=t+1}^{\infty}\sum_{r=s}^{\infty}\zeta^{*l}_{\delta,i,j,w-t}\frac{\lambda}{\delta\beta^2}q_r l_{s,r}\left[\left(\frac{\beta}{\beta+\rho_{\delta,j}}\right)^{s-t} - \left(\frac{\beta}{\beta+\rho_{0,j}}\right)^{s-t}\right]$.

The proof follows the same procedure as that used in Section 2.2.

## 2.4 An asymptotic result for the Gerber-Shiu function

Here we are trying to get the defective renewal equation and asymptotic result of the Gerber-Shiu function $m_{\delta,i}(u)$ defined in (2.4) in a two-premium case.

**Proposition 2.4.1.** *The Gerber-Shiu function* $m_{\delta,i}(u)$ *for* $i = 1, 2$ *satisfies the defective renewal equation*

$$m_{\delta,i}(u) = \int_0^u m_{\delta,i}(u-y)g_\delta(y)dy + r_{\delta,i}(u), \tag{2.21}$$

*where* $g_\delta(\cdot)$ *and* $r_{\delta,i}(\cdot)$ *are defined through their LTs*

$$\widetilde{g}_\delta(z) = \widetilde{h}_{\delta,1,1}(z) + \widetilde{h}_{\delta,2,2}(z) - \widetilde{h}_{\delta,1,1}(z)\widetilde{h}_{\delta,2,2}(z) + \widetilde{h}_{\delta,2,1}(z)\widetilde{h}_{\delta,1,2}(z),$$

$$\widetilde{r}_{\delta,1}(z) = (1 - \widetilde{h}_{\delta,2,2}(z))\widetilde{v}_{\delta,1}(z) + \widetilde{h}_{\delta,1,2}(z)\widetilde{v}_{\delta,2}(z),$$

*and*

$$\widetilde{r}_{\delta,2}(z) = (1 - \widetilde{h}_{\delta,1,1}(z))\widetilde{v}_{\delta,2}(z) + \widetilde{h}_{\delta,2,1}(z)\widetilde{v}_{\delta,1}(z).$$

*Furthermore, suppose that $e^{Rx}r_{\delta,i}(x)$ is directly Riemann integrable on $(0,\infty)$, then the asymptotic result is as follows*

$$\lim_{u\to\infty} e^{Ru} m_{\delta,i}(u) = \frac{\int_0^\infty e^{Ry} r_{\delta,i}(y)dy}{\int_0^\infty y e^{Ry} g_\delta(y)dy}, \tag{2.22}$$

*where $R$ is the unique positive solution (if it exists) to*

$$\int_0^\infty e^{Rx} g_\delta(x)dx = 1.$$

*Proof.* Here we just need to prove that equation (2.21) is a defective renewal equation, and then the asymptotic result (2.22) follows by Theorem 9.1.3 in Willmot and Lin (2001).

By (2.9), the LT of the Gerber-Shiu function is given by

$$\widetilde{m}_{\delta,i}(z) = \widetilde{h}_{\delta,i,1}(z)\widetilde{m}_{\delta,1}(z) + \widetilde{h}_{\delta,i,2}(z)\widetilde{m}_{\delta,2}(z) + \widetilde{v}_{\delta,i}(z),$$

for $i = 1, 2$. The solutions $\widetilde{m}_{\delta,1}(z)$ and $\widetilde{m}_{\delta,2}(z)$ to the above system of linear equations are

$$\widetilde{m}_{\delta,1}(z) = \frac{(1-\widetilde{h}_{\delta,2,2}(z))\widetilde{v}_{\delta,1}(z) + \widetilde{h}_{\delta,1,2}(z)\widetilde{v}_{\delta,2}(z)}{1-\widetilde{h}_{\delta,1,1}(z)-\widetilde{h}_{\delta,2,2}(z)+\widetilde{h}_{\delta,1,1}(z)\widetilde{h}_{\delta,2,2}(z)-\widetilde{h}_{\delta,1,2}(z)\widetilde{h}_{\delta,2,1}(z)} = \frac{\widetilde{r}_{\delta,1}(z)}{1-\widetilde{g}_\delta(z)}, \tag{2.23}$$

and

$$\widetilde{m}_{\delta,2}(z) = \frac{(1-\widetilde{h}_{\delta,1,1}(z))\widetilde{v}_{\delta,2}(z) + \widetilde{h}_{\delta,2,1}(z)\widetilde{v}_{\delta,1}(z)}{1-\widetilde{h}_{\delta,1,1}(z)-\widetilde{h}_{\delta,2,2}(z)+\widetilde{h}_{\delta,1,1}(z)\widetilde{h}_{\delta,2,2}(z)-\widetilde{h}_{\delta,1,2}(z)\widetilde{h}_{\delta,2,1}(z)} = \frac{\widetilde{r}_{\delta,2}(z)}{1-\widetilde{g}_\delta(z)}. \tag{2.24}$$

Thus, we can rewrite (2.23) and (2.24) as

$$\widetilde{m}_{\delta,i}(z) = \widetilde{m}_{\delta,i}(z)\widetilde{g}_\delta(z) + \widetilde{r}_{\delta,i}(z), \tag{2.25}$$

i.e.,

$$m_{\delta,i}(u) = \int_0^u m_{\delta,i}(u-y)g_\delta(y)dy + r_{\delta,i}(u). \tag{2.26}$$

Therefore, now we need to show that $g_\delta(\cdot)$ is a defective density. We point out that $\widetilde{h}_{\delta,i}(z) = \widetilde{h}_{\delta,i,1}(z) + \widetilde{h}_{\delta,i,2}(z) \leq \widetilde{h}_{\delta,i}(0) < 1$, which implies

$$1-\widetilde{g}_\delta(z) = (1-\widetilde{h}_{\delta,1,1}(z))(1-\widetilde{h}_{\delta,2,2}(z))-\widetilde{h}_{\delta,1,2}(z)\widetilde{h}_{\delta,2,1}(z) > \widetilde{h}_{\delta,1,2}(z)\widetilde{h}_{\delta,2,1}(z)-\widetilde{h}_{\delta,1,2}(z)\widetilde{h}_{\delta,2,1}(z) = 0,$$

and

$$1-\widetilde{g}_\delta(z) = (1-\widetilde{h}_{\delta,1,1}(z))(1-\widetilde{h}_{\delta,2,2}(z))-\widetilde{h}_{\delta,1,2}(z)\widetilde{h}_{\delta,2,1}(z) < 1-\widetilde{h}_{\delta,1,2}(z)\widetilde{h}_{\delta,2,1}(z) < 1,$$

i.e., $\widetilde{g}_\delta(z) \in (0,1)$ for any $z$, therefore, $g_\delta(\cdot)$ is a defective density. In fact, if we recall the form of $\widetilde{h}_{\delta,i,j}(z)$ in equation (2.13), we have

$$\widetilde{g}_\delta(z) \;=\; \widetilde{h}_{\delta,1,1}(z)+\widetilde{h}_{\delta,2,2}(z)-\widetilde{h}_{\delta,1,1}(z)\widetilde{h}_{\delta,2,2}(z)+\widetilde{h}_{\delta,2,1}(z)\widetilde{h}_{\delta,1,2}(z) = \sum_{n=1}^{\infty} g_{\delta,n}\left(\frac{\beta}{\beta+z}\right)^n,$$

where $g_{\delta,1} = \zeta_{\delta,1,1,1}+\zeta_{\delta,2,2,1}$ and $g_{\delta,n} = \zeta_{\delta,1,1,n}+\zeta_{\delta,2,2,n}-\sum_{k=1}^{n-1}\zeta_{\delta,1,1,k}\zeta_{\delta,2,2,n-k}+\sum_{k=1}^{n-1}\zeta_{\delta,2,1,k}\zeta_{\delta,1,2,n-k}$ for $n \geq 2$ and can be identified. Therefore, $g_\delta(\cdot)$ has a combination of Erlangs which is non-arithmetic. Note that if we recall the form of $\widetilde{h}_{\delta,i,j}(z)$ and $\widetilde{v}_{\delta,i}(z)$ in equations (2.13) and (2.11), we may identify $\widetilde{r}_{\delta,i}(z)$. □

## 2.5 Another variant of the adaptive premium policy

Here we assume that at each review time, the current premium rate can only increase or decrease to the adjacent levels of the premium rates (with still $m$ levels of the premium rates). More concretely, suppose that the premium rate at the beginning of a given period is $c_i$, if the time between ladder heights $T \in (x_i, \infty]$, then the premium rate changes to $c_{i+1}$, if $T \in (0, x_{i-1}]$, then the premium rate changes to $c_{i-1}$, and if $T \in (x_{i-1}, x_i]$, then the premium rate stays the same $c_i$. This variant seems to be more practical, because even if the insurer experiences a good (or bad) period in which it has a long (or short) time to reach the next ladder height, he may not want to change the premium rate to a quite low (or high) level suddenly.

In this setting, the Gerber-Shiu function satisfies

$$m_{\delta,i}(u) = \sum_{j=\max(1,i-1)}^{\min(m,i+1)} \int_0^u m_{\delta,j}(u-y)h^*_{\delta,i,j}(y)dy + v_{\delta,i}(u),$$

for $1 \le i \le m$, where

$$h^*_{\delta,i,i-1}(y) = \int_0^\infty e^{-\delta t}k_i(t,y)\Pr(t \le D_{i-1})dt,$$

$$h^*_{\delta,i,i}(y) = \int_0^\infty e^{-\delta t}k_i(t,y)\Pr(D_{i-1} < t \le D_i)dt,$$

$$h^*_{\delta,i,i-1}(y) = \int_0^\infty e^{-\delta t}k_i(t,y)\Pr(t > D_i)dt,$$

and $v_{\delta,i}(u)$ is the same as defined in (2.6). This model can be solved by following the same procedure as in Section 2.2.

# Chapter 3

# Experience-based premium policy

## 3.1 Introduction

Recall that the classical compound Poisson risk model is defined as

$$U_t = u + Z_t,\ t \geq 0,$$

where $u \geq 0$ is the initial surplus level, $c > 0$ is the constant premium rate, and $Z_t \equiv Z_{t,c} = ct - S_t$. In what follows, the dependence of $Z_t$ on $c$ is for the most part silently assumed (except in e.g., Proposition 3.3.1).The aggregate claim amount process $\{S_t; t \geq 0\}$ is defined as

$$S_t = \begin{cases} \sum_{i=1}^{N_t} P_i, & N_t > 0, \\ 0, & N_t = 0, \end{cases}$$

where $\{N_t;\ t \geq 0\}$ is an homogeneous Poisson process with arrival rate $\lambda > 0$ and the claim sizes $\{P_i;\ i \geq 1\}$ are a sequence of iid random variables with density $p$ and mean $1/\mu$, independent of $\{N_t;\ t \geq 0\}$.

Following the idea of an adaptive premium policy discussed in Chapter 2, in this chapter, we propose another strategy where the incoming premium rate is allowed to vary based on the recent claim experience of a particular insurance portfolio. We use the name "experience-based premium policy" for the proposed premium strategy. This can be viewed as a mechanism to have a premium setting policy which is somehow responsive to the recent claim experience, a practice supported by *credibility theory* in insurance mathematics (see, e.g., Klugman et al. (2012)). The premium review policy described in this chapter can also be regarded as a different allocation of the insurer's revenues over time, which we will show has great merit from a risk management standpoint. Indeed, the proposed premium strategy is expected to provide a better matching of the cash inflows and outflows of an insurer over time, reducing its solvency risk.

In the same spirit as credibility theory, the main idea of the insurer's premium policy is to generate supplementary premium income following a period of bad claim experience, while reducing the incoming premium rate when a period of good claim experience is observed. Such a strategy is often consistent with the insurer's new perception of the risk insured (even though the risk itself may have remained unchanged). The experience-based premium policy we intend to examine will operate under the following mechanism. We consider an insurer with $m$ different premium rate options: $\{c_i\}_{i=1}^m$ with $c_i < c_j$ for $i < j$. For instance, these premium rates can result from the application of a set of security loadings to the underlying risk. We assume that the insurer has the ability to modify the incoming premium rate at some review time points based on the increment value of the surplus process since the last review time. Suppose the premium rate at the beginning of a given period is $c_i$. If the increment of the surplus process until the next review time is negative, then the premium rate increases to $c_{\min(i+1,m)}$. If the increment of the surplus process until the next review time is positive, then the premium rate decreases to $c_{\max(i-1,1)}$. Also, as in Albrecher et al. (2011, 2013), ruin will be monitored at these discrete random review time points only. There are two main reasons for choosing the discrete random review time. First, in practice, it is more reasonable to assume that the company checks their surplus level on a periodic basis instead of continuously. However, the problem with discrete-type risk model is that it usually does not lead to explicit solutions and then is difficult to gain structural insight on the influence of parameters and compare to other strategies. Therefore, in order to get the explicit analytical results, we choose the random review time structure. To conclude, we generalize the classical risk model in two main directions: premium changes based on recent claim experience and a random period between premium rate reviews.

Mathematically, we assume that the risk process can only be reviewed at random times $\{X_k; k \geq 1\}$, where $X_k$ is the $k$-th review time with $X_0 = 0$ and $X_k > X_{k-1}$ a.s. Thus, to analyze the ruin event, it suffices to consider the surplus process at the review times $\{X_k; k \geq 1\}$ only. Let $\mathcal{U}_k$ be the surplus process value at time $X_k$, and define $\eta_k$ ($\eta_k \in \{c_i\}_{i=1}^m$) to be the effective premium rate between the successive review times $X_{k-1}$ and $X_k$. We define $\mathcal{U}_k$ as

$$\mathcal{U}_k = u + \sum_{j=1}^{k} Y_j, \tag{3.1}$$

where $Y_j = \eta_j T_j - \left(S_{X_j} - S_{X_{j-1}}\right)$ and $T_j = X_j - X_{j-1}$ is the $j$-th inter-review time. Conditional on $\{\eta_k; k \geq 1\}$, the inter-review times $\{T_k; k \geq 1\}$ are mutually independent, as well as independent of the aggregate claim process $\{S_t; t \geq 0\}$.

We further assume that when $\eta_k = c_i$:

- the inter-review time $T_k$ is distributed as a rv $K_i$ with density $k_i$ and mean $\kappa_i$;
- the next effective premium rate $\eta_{k+1}$ is

$$\eta_{k+1} = \begin{cases} c_{\min(i+1,m)}, & \text{if } c_i T_k - \left(S_{X_k} - S_{X_{k-1}}\right) \leq 0, \\ c_{\max(i-1,1)}, & \text{if } c_i T_k - \left(S_{X_k} - S_{X_{k-1}}\right) > 0. \end{cases} \tag{3.2}$$

**Remark 3.1.1.** *Note that marginally, the premium rate process $\{\eta_k; k \geq 1\}$ is a (time homogeneous) discrete-time Markov chain with transition probability matrix $\mathbf{Q} = [q_{i,j}]_{i,j=1}^m$ where $q_{i,j} = \mathbb{P}(\eta_k = c_j | \eta_{k-1} = c_i)$. We have that $q_{i,\min(i+1,1)}$ and $q_{i,\max(i-1,1)}$ are the respective probabilities associated to events on the first and second row of (3.2), while all other transition* *probabilities are 0.*

For risk model (3.1), we define the time to ruin $T^*$ as $T^* = X_{k^*}$ where $k^* = \inf\{k \geq 1 : \mathcal{U}_k < 0\}$ (with $T^* = \infty$ if $\mathcal{U}_k \geq 0$ for $k = 1, 2, \ldots$). Also, let $\mathcal{U}_{k^*-1}$ and $|\mathcal{U}_{k^*}|$ be the surplus prior to ruin and the deficit at ruin, respectively. See Figure 3.1 for a sample path illustration (and the traditional ruin-related quantities, such as the time to ruin $T$, the surplus prior to ruin $\mathcal{U}_{T^-}$ and the deficit at ruin $|\mathcal{U}_T|$, are given in grey colour).

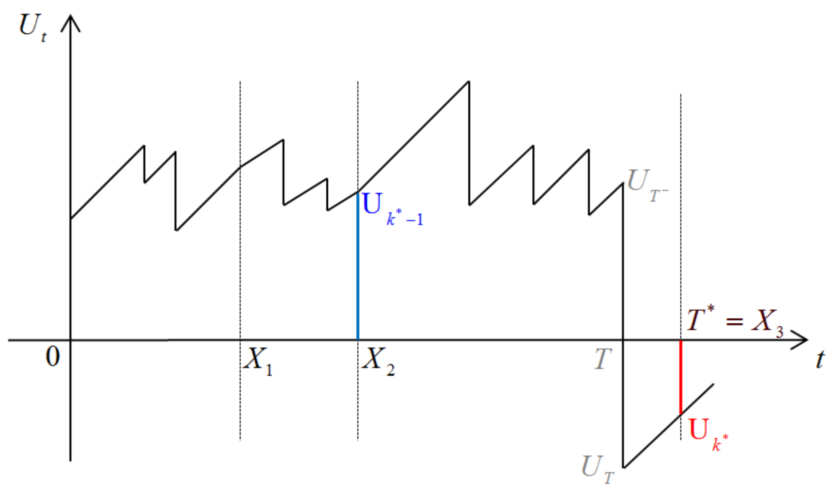


Figure 3.1: A sample path of risk process $\mathcal{U}$

In this chapter, we will focus our attention on the analysis of these ruin-related quantities through the Gerber-Shiu function. The Gerber-Shiu function of interest in this context is

$$m_{i,j,\delta}(u) = \mathbb{E}\left[e^{-\delta T^*} w\left(\mathcal{U}_{k^*-1}, |\mathcal{U}_{k^*}|\right) 1_{\{\eta_{k^*}=c_j\}} 1_{\{T^*<\infty\}} |\mathcal{U}_0 = u, \eta_1 = c_i\right], \tag{3.3}$$

for $i, j = 1, \ldots, m$, where $\delta \geq 0$, $w(x, y)$ is a penalty function which satisfies mild integrability conditions, and $1_A$ is the indicator function of the event $A$.

To analyze the Gerber-Shiu function (3.3), it will be particularly helpful to examine the distribution of the increments of the surplus process $\{U_t; t \geq 0\}$ over an exponentially distributed time horizon (which is studied in the next section), since in our main result in Section 3.3, we will assume that the inter-review times are distributed as a combination of exponentials.

The rest of this chapter is structured as follows. In Section 3.2, we characterize the two one-sided densities of surplus increments over an exponentially distributed time horizon, quantities central to the later analysis. In Section 3.3, when review times are distributed as a combination of exponentials and claim arrivals follow a compound Poisson process, a matrix-form defective renewal equation for the Gerber-Shiu function is derived. By employing Rouche's theorem and the initial value theorem, we derive explicit expressions for the density of some ruin-related quantities. Section 3.4 illustrates the benefit of the proposed premium policy from a risk management standpoint via some numerical examples. Section 3.5 generalizes the experience-based premium policy under a random barrier framework and proposed a similar premium policy conducted at claim occurrence.

Most results in this chapter have already been published in Li, Landriault and Lemieux (2015), except for Section 3.4.3 and Section 3.5.

## 3.2 The two one-sided densities of $Z_{e_\alpha}$

Let $e_\alpha$ be a generic inter-review time, which is assumed to be exponentially distributed with mean $1/\alpha$ in this section and will be generalized to the combination of exponentials in the next section. Also, define $Z_{e_\alpha}$ to be the increment of the surplus process over this exponentially distributed time horizon.

The two one-sided densities of $Z_{e_\alpha}$, namely $g_+$ and $g_-$, defined respectively through their one-sided LTs as

$$\mathbb{E}\left[e^{-sZ_{e_\alpha}}1_{\{Z_{e_\alpha}>0\}}\right] = \int_0^\infty e^{-sx}g_+(x)dx,$$

and

$$\mathbb{E}\left[e^{-s(-Z_{e_\alpha})}1_{\{Z_{e_\alpha}<0\}}\right] = \int_0^\infty e^{-sx}g_-(x)dx,$$

will be examined.

Our objective is to identify $g_+$ and $g_-$ in the classical risk model with exponential random review time $e_\alpha$. The main results can be found in Propositions 3.2.1 and 3.2.2. We point out that Kyprianou (2006, Corollary 8.9) also presents these results in the more general class of spectrally negative Lévy processes. However, we suggest a simpler proof to this result in the context of the classical risk model, which relies on LT arguments only.

### 3.2.1 Density of $Z_{e_\alpha} 1_{\{Z_{e_\alpha}>0\}}$

To examine $\mathbb{E}\left[e^{-sZ_{e_\alpha}} 1_{\{Z_{e_\alpha}>0\}}\right]$, we first define the first passage time

$$\tau_0^- = \inf\{t \geq 0 | U_t < 0\},$$

(with $\tau_0^- = \infty$ if the surplus never drops below 0). Thus by conditioning on whichever of the review time $e_\alpha$ or the first passage time $\tau_0^-$ occurs first, we have

$$\begin{aligned}
\mathbb{E}\left[e^{-sZ_{e_\alpha}} 1_{\{Z_{e_\alpha}>0\}}\right] &= \mathbb{E}\left[e^{-sZ_{e_\alpha}} 1_{\{Z_{e_\alpha}>0\}} 1_{\{e_\alpha<\tau_0^-\}}\right] + \mathbb{E}\left[e^{-sZ_{e_\alpha}} 1_{\{Z_{e_\alpha}>0\}} 1_{\{e_\alpha\geq\tau_0^-\}}\right] \\
&= \mathbb{E}\left[e^{-sZ_{e_\alpha}} 1_{\{e_\alpha<\tau_0^-\}}\right] + \mathbb{E}\left[e^{-sZ_{e_\alpha}} 1_{\{Z_{e_\alpha}>0\}} 1_{\{e_\alpha\geq\tau_0^-\}}\right]. \qquad (3.4)
\end{aligned}$$

To obtain an expression for the first term on the right-hand side of (3.4), we define

$$\psi_\alpha(u) = \mathbb{E}\left[e^{-sU_{e_\alpha}} 1_{\{e_\alpha<\tau_0^-\}} | U_0 = u\right],$$

for which $\mathbb{E}\left[e^{-sZ_{e_\alpha}} 1_{\{e_\alpha<\tau_0^-\}}\right] = \psi_\alpha(0)$. Note that $\psi_\alpha(u)$ is the LT of $U_{e_\alpha}$ killed if the surplus process reaches negative values before the generic time $e_\alpha$. The term "killed" is used here to specify that all sample paths such that $e_\alpha > \tau_0^-$ are discarded.

**Lemma 3.2.1.** *The LT of $U_{e_\alpha}$ for all sample paths of $\{U_t; t \geq 0\}$ with $\{e_\alpha < \tau_0^-\}$ is given by*

$$\psi_\alpha(u) = \alpha \left\{ \frac{1}{s+\rho} v_{\alpha,c}(u) - \int_0^u e^{-sx} v_{\alpha,c}(u-x)dx \right\},$$

*where $v_{\alpha,c}(u)$ is defined on $[0,\infty)$ through its LT*

$$\widetilde{v}_{\alpha,c}(z) = \frac{1}{cz - \lambda(1-\widetilde{p}(z)) - \alpha}, \tag{3.5}$$

*and $\rho = \rho_{\alpha,c}(\alpha)$ is the unique non-negative solution of Lundberg's fundamental equation*

$$cz - \lambda(1-\widetilde{p}(z)) - \alpha = 0. \tag{3.6}$$

*Proof.* By conditioning on the first occurrence between a claim instant and a review time, we have

$$\begin{aligned}
\psi_\alpha(u) &= \int_0^\infty \lambda e^{-(\lambda+\alpha)t}\left\{\int_0^{u+ct}\psi_\alpha(u+ct-y)p(y)dy\right\}dt + \int_0^\infty \alpha e^{-(\lambda+\alpha)t}e^{-s(u+ct)}dt \\
&= \frac{\lambda}{c}\int_u^\infty e^{-\frac{\lambda+\alpha}{c}(x-u)}\int_0^x \psi_\alpha(x-y)p(y)dydx + \frac{\alpha}{\lambda+\alpha+cs}e^{-su} \\
&= \frac{\lambda}{c}\mathcal{T}_{\frac{\lambda+\alpha}{c}}r_\alpha(u) + \frac{\alpha}{\lambda+\alpha+cs}e^{-su},
\end{aligned} \tag{3.7}$$

where

$$r_\alpha(x) = \int_0^x \psi_\alpha(x-y)p(y)dy$$

and $\mathcal{T}_r f(x) = \int_x^\infty e^{-r(y-x)}f(y)dy$ is the Dickson-Hipp operator defined in Section 1.4.2. Taking the LT on both sides of (3.7), one obtains

$$\begin{aligned}
\widetilde{\psi}_\alpha(z) &= \frac{\lambda}{c}\frac{\widetilde{r}_\alpha(\frac{\lambda+\alpha}{c}) - \widetilde{r}_\alpha(z)}{z-\frac{\lambda+\alpha}{c}} + \frac{\alpha}{\lambda+\alpha+cs}\frac{1}{z+s} \\
&= \frac{\lambda}{c}\frac{\widetilde{\psi}_\alpha(\frac{\lambda+\alpha}{c})\widetilde{p}(\frac{\lambda+\alpha}{c}) - \widetilde{\psi}_\alpha(z)\widetilde{p}(z)}{z-\frac{\lambda+\alpha}{c}} + \frac{\alpha}{\lambda+\alpha+cs}\frac{1}{z+s}.
\end{aligned} \tag{3.8}$$

A simple rearrangement of (3.8) yields

$$\{cz - \lambda(1-\widetilde{p}(z)) - \alpha\}\,\widetilde{\psi}_\alpha(z) = \left\{\lambda\widetilde{\psi}_\alpha\left(\frac{\lambda+\alpha}{c}\right)\widetilde{p}\left(\frac{\lambda+\alpha}{c}\right) + \frac{\alpha c}{\lambda+\alpha+cs}\right\} - \frac{\alpha}{z+s}. \tag{3.9}$$

The first term on the right-hand side of (3.9) does not depend on $z$, and by taking $z = \rho$, we can express it as

$$\lambda \widetilde{\psi}_\alpha \left( \frac{\lambda + \alpha}{c} \right) \widetilde{p} \left( \frac{\lambda + \alpha}{c} \right) + \frac{\alpha c}{\lambda + \alpha + cs} = \frac{\alpha}{\rho + s}. \tag{3.10}$$

Substituting (3.10) into (3.9), we get

$$\{cz - \lambda(1 - \widetilde{p}(z)) - \alpha\} \widetilde{\psi}_\alpha(z) = \alpha \left( \frac{1}{\rho + s} - \frac{1}{s + z} \right),$$

i.e.,

$$\widetilde{\psi}_\alpha(z) = \alpha \left( \frac{1}{s + \rho} - \frac{1}{s + z} \right) \widetilde{v}_{\alpha,c}(z).$$

Taking the LT inversion wrt $z$, one obtains

$$\psi_\alpha(u) = \alpha \left\{ \frac{1}{s + \rho} v_{\alpha,c}(u) - \int_0^u e^{-sx} v_{\alpha,c}(u - x) dx \right\}.$$

□

Note that $v_{\alpha,c}(u)$ is known as the *$\alpha$-scale function* in the literature on Lévy processes (see, e.g., Kyprianou, 2006). Also, we remark that the inversion of $\psi_\alpha(u)$ wrt $s$ yields

$$\mathbb{E}\left\{\mathbb{P}(U_{e_\alpha} \in (x, x + dx), e_\alpha < \tau_0^- | U_0 = u)\right\} \cong \alpha \left\{ e^{-\rho x} v_{\alpha,c}(u) - v_{\alpha,c}(u - x) 1_{\{u > x\}} \right\} dx,$$

which is consistent with Kyprianou (2006, Corollary 8.8).

From Lemma 3.2.1, it is immediate that

$$\mathbb{E}\left[ e^{-sZ_{e_\alpha}} 1_{\{e_\alpha < \tau_0^-\}} \right] = \psi_\alpha(0) = \alpha \frac{1}{s + \rho} v_{\alpha,c}(0) = \frac{\alpha}{c} \frac{1}{s + \rho}, \tag{3.11}$$

given that the initial value theorem for the LTs implies that

$$v_{\alpha,c}(0) = \lim_{z\to\infty} z\widetilde{v}_{\alpha,c}(z) = \lim_{z\to\infty} \frac{z}{cz - \lambda(1-\widetilde{p}(z)) - \alpha} = \frac{1}{c}.$$

We now consider the second term on the right-hand side of (3.4).

**Lemma 3.2.2.** *The LT of the one-sided density $Z_{e_\alpha} 1_{\{Z_{e_\alpha}>0\}}$ together with $\{e_\alpha \geq \tau_0^-\}$ is given by*

$$\mathbb{E}\left[e^{-sZ_{e_\alpha}} 1_{\{Z_{e_\alpha}>0\}} 1_{\{e_\alpha \geq \tau_0^-\}}\right] = \int_0^\infty \left\{\frac{\lambda}{c}\mathcal{T}_\rho p(y)\right\} e^{-\rho y} \mathbb{E}\left[e^{-sZ_{e_\alpha}} 1_{\{Z_{e_\alpha}>0\}}\right] dy. \tag{3.12}$$

*Proof.* For $\tau_0^- \leq e_\alpha$, we shall first condition on the distribution of the deficit at ruin $|U_{\tau_0^-}|$ together with the event $\{\tau_0^- \leq e_\alpha\}$, i.e., the distribution of $|U_{\tau_0^-}| 1_{\{\tau_0^- \leq e_\alpha\}}$. This corresponds to the discounted density of the deficit at ruin, which was obtained by Gerber and Shiu (1998, Equation 3.4) with $\delta = \alpha$. For an initial surplus of 0, the discounted density of deficit at ruin is

$$\mathbb{E}\left[e^{-\alpha\tau_0^-} 1_{\{|U_{\tau_0^-}| \in (y, y+dy)\}} | U_0 = 0\right] \cong \frac{\lambda}{c}\mathcal{T}_\rho p(y) dy.$$

From a deficit of $y$, the skip-free upward surplus process must then return to level 0 before the exponential time $e_\alpha$, which is of probability $e^{-\rho y}$ (see Asmussen and Albrecher (2010, Chapter V, Lemma 3.1)). The process restarts at this return time to 0 by the strong Markov property. Thus,

$$\begin{aligned}
&\mathbb{E}\left[e^{-sZ_{e_\alpha}} 1_{\{Z_{e_\alpha}>0\}} 1_{\{e_\alpha \geq \tau_0^-\}}\right] \\
=& \int_0^\infty \mathbb{P}\left(|U_{\tau_0^-}| \in (y, y+dy), e_\alpha \geq \tau_0^- | U_0 = 0\right) e^{-\rho y} \mathbb{E}\left[e^{-sZ_{e_\alpha}} 1_{\{Z_{e_\alpha}>0\}}\right] \\
=& \int_0^\infty \mathbb{E}\left[e^{-\alpha\tau_0^-} 1_{\{|U_{\tau_0^-}| \in (y, y+dy)\}} | U_0 = 0\right] e^{-\rho y} \mathbb{E}\left[e^{-sZ_{e_\alpha}} 1_{\{Z_{e_\alpha}>0\}}\right] \\
=& \int_0^\infty \left\{\frac{\lambda}{c}\mathcal{T}_\rho p(y)\right\} e^{-\rho y} \mathbb{E}\left[e^{-sZ_{e_\alpha}} 1_{\{Z_{e_\alpha}>0\}}\right] dy.
\end{aligned}$$

This completes the proof of Lemma 3.2.2. $\square$

We now make use of Lemma 3.2.2 together with Equation (3.11) to identify the one-sided density $g_+$.

**Proposition 3.2.1.** *The defective density of* $Z_{e_\alpha} 1_{\{Z_{e_\alpha}>0\}}$ *is*

$$g_+(x) = \alpha \Phi_{\alpha,c} e^{-\rho x}, \;\; x > 0, \tag{3.13}$$

*where*

$$\Phi_{\alpha,c} = \frac{1}{c - \lambda \mathcal{T}_\rho^2 p(0)} > 0, \tag{3.14}$$

*and* $\mathcal{T}_\rho^2 p(0) = \int_0^\infty y e^{-\rho y} p(y) dy$.

*Proof.* Substituting (3.11) and (3.12) into (3.4) yields

$$\mathbb{E}\left[e^{-sZ_{e_\alpha}} 1_{\{Z_{e_\alpha}>0\}}\right] = \frac{\alpha}{c}\frac{1}{s+\rho} + \left\{\int_0^\infty e^{-\rho y}\frac{\lambda}{c}\mathcal{T}_\rho p(y) dy\right\} \mathbb{E}\left[e^{-sZ_{e_\alpha}} 1_{\{Z_{e_\alpha}>0\}}\right],$$

which gives

$$\mathbb{E}\left[e^{-sZ_{e_\alpha}} 1_{\{Z_{e_\alpha}>0\}}\right] = \frac{\frac{\alpha}{c}\frac{1}{s+\rho}}{1 - \int_0^\infty e^{-\rho y}\frac{\lambda}{c}\mathcal{T}_\rho p(y) dy} = \frac{\alpha}{c}\frac{1}{1-\frac{\lambda}{c}\mathcal{T}_\rho^2 p(0)}\frac{1}{s+\rho}. \tag{3.15}$$

The LT inversion of (3.15) wrt $s$ yields the defective density of $Z_{e_\alpha} 1_{\{Z_{e_\alpha}>0\}}$, which is given by

$$g_+(x) = \frac{\alpha}{c}\frac{1}{1-\frac{\lambda}{c}\mathcal{T}_\rho^2 p(0)} e^{-\rho x} = \alpha \Phi_{\alpha,c} e^{-\rho x}, \;\; x > 0.$$

In the following, we will show that $\Phi_{\alpha,c} > 0$, a fact that will be used later. If we substitute

$$\begin{aligned}
\int_0^\infty e^{-\rho y} y p(y) dy &= -e^{-\rho y} y \bar{P}(y)\Big|_{y=0}^\infty + \int_0^\infty \left(e^{-\rho y} - \rho y e^{-\rho y}\right) \bar{P}(y) dy \\
&= \widetilde{\bar{P}}(\rho) - \int_0^\infty \rho y e^{-\rho y} \bar{P}(y) dy \\
&= \frac{1-\widetilde{p}(\rho)}{\rho} - \int_0^\infty \rho y e^{-\rho y} \bar{P}(y) dy
\end{aligned}$$

back into (3.14), we have

$$\begin{aligned}
\Phi_{\alpha,c} &= \frac{1}{c-\lambda\int_0^\infty e^{-\rho y} y p(y)dy} \\
&= \frac{1}{c-\lambda\frac{1-\widetilde{p}(\rho)}{\rho}+\lambda\int_0^\infty \rho y e^{-\rho y}\bar{P}(y)dy} \\
&= \frac{1}{\frac{\alpha}{\rho}+\lambda\int_0^\infty \rho y e^{-\rho y}\bar{P}(y)dy} > 0,
\end{aligned}$$

since using the definition of $\rho$, it follows

$$c-\lambda\frac{1-\widetilde{p}(\rho)}{\rho}=\frac{\alpha}{\rho}.$$

$\square$

We point out that this result is consistent with Kyprianou (2006, Corollary 8.9).

The following example shows that in some special cases, we can explicitly express the density $g_+(x)$ using Equation (3.13).

**Example 3.2.1.** *We assume that claim sizes are exponentially distributed with mean $1/\mu$. Let $\rho > 0$ and $-R < 0$ to be the two solutions of the characteristic equation*

$$s^2+(\mu-\frac{\lambda+\alpha}{c})s-\frac{\alpha\mu}{c}=0.$$

*It follows that*

$$\begin{aligned}
g_+(x) &= \frac{\alpha}{c}\frac{1}{1-\frac{\lambda}{c}\mathcal{T}_\rho^2 p(0)}e^{-\rho x} \\
&= \frac{\alpha}{c}\frac{1}{1-\frac{\lambda}{c}\frac{\mu}{(\mu+\rho)^2}}e^{-\rho x} \\
&= \frac{\alpha}{c}\frac{\mu+\rho}{R+\rho}e^{-\rho x},
\end{aligned}$$

*for $x > 0$, since $(\mu+\rho)^2-\frac{\lambda}{c}\mu=(\mu+\rho)(\rho+R)$. Note that this result is consistent with*

*Albrecher et al. (2013, Example 4.1).*

### 3.2.2 Density of $-Z_{e_\alpha} 1_{\{Z_{e_\alpha}<0\}}$

We are now interested in the other one-sided density of $Z_{e_\alpha}$, namely $-Z_{e_\alpha} 1_{\{Z_{e_\alpha}<0\}}$. Define

$$\tau_b^+ = \inf\{t \geq 0 | U_t \geq b\},$$

which is the first passage time of $\{U_t;\ t \geq 0\}$ at level $b$. Conditioning on the first excursion below 0 (before $e_\alpha$), and then on whichever of the review time or the recovery time $\tau_0^+$ occurs first, we have

$$\mathbb{E}\left[e^{-s(-Z_{e_\alpha})} 1_{\{Z_{e_\alpha}<0\}}\right] = \int_0^\infty \left\{\frac{\lambda}{c}\mathcal{T}_\rho p(y)\right\} \left\{e^{-\rho y}\mathbb{E}\left[e^{-s(-Z_{e_\alpha})} 1_{\{Z_{e_\alpha}<0\}}\right] + \phi_\alpha(y)\right\} dy, \quad (3.16)$$

where

$$\phi_\alpha(y) = \mathbb{E}\left[e^{-s(-U_{e_\alpha})} 1_{\{e_\alpha<\tau_0^+\}} | U_0 = -y\right].$$

An explicit expression for $\phi_\alpha(y)$ is given in Lemma 3.2.3.

**Lemma 3.2.3.** *The ruin quantity $\phi_\alpha(y)$ can be expressed as*

$$\phi_\alpha(y) = \alpha\left(e^{-\rho y} - e^{-sy}\right)\widetilde{v}_{\alpha,c}(s), \quad (3.17)$$

*where $\widetilde{v}_{\alpha,c}(s)$ and $\rho$ are as defined in (3.5) and (3.6), respectively.*

*Proof.* By reflection, we obtain

$$\phi_\alpha(y) = \mathbb{E}\left[e^{-sR_{e_\alpha}} 1_{\{e_\alpha<\tau_0^{*-}\}} | R_0 = y\right],$$

where $R_t = u - Z_t,\ t \geq 0$ is the dual risk model, and $\tau_0^{*-} = \inf\{t \geq 0 | R_t \leq 0\}$ is the first passage time of $\{R_t;\ t \geq 0\}$ at level 0. Thus, $\phi_\alpha(y)$ is the LT of $R_{e_\alpha}$ given that the review time $e_\alpha$ occurs before $\tau_0^{*-}$. Intuitively, it is clear that

$$\phi_\alpha(y) = e^{-s\epsilon}\phi_\alpha(y-\epsilon) + e^{-\rho(y-\epsilon)}\phi_\alpha(\epsilon),$$

for all $\epsilon \in [0, y]$. Integrating over $\epsilon$ from 0 to $y$, it follows that

$$y\phi_\alpha(y) = \int_0^y e^{-s\epsilon}\phi_\alpha(y-\epsilon)d\epsilon + \int_0^y e^{-\rho(y-\epsilon)}\phi_\alpha(\epsilon)d\epsilon. \tag{3.18}$$

Taking the LT on both sides of (3.18), we obtain

$$\int_0^\infty e^{-zy} y\phi_\alpha(y)dy = \left(\frac{1}{s+z} + \frac{1}{\rho+z}\right)\widetilde{\phi}_\alpha(z).$$

Note that

$$\int_0^\infty e^{-zy} y\phi_\alpha(y)dy = \frac{d}{dz}\widetilde{\phi}_\alpha(z).$$

Thus, solving this ordinary differential equation followed by a LT inversion yields

$$\phi_\alpha(y) = C(s)\left(e^{-sy} - e^{-\rho y}\right), \tag{3.19}$$

where $C(s)$ is a constant involving $s$.

To identify $C(s)$, we condition on the time and amount of the first jump, i.e.,

$$\begin{aligned}
\phi_\alpha(y) &= \int_0^{y/c} \lambda e^{-(\lambda+\alpha)t}\left\{\int_0^\infty \phi_\alpha(y-ct+x)p(x)dx\right\}dt + \int_0^{y/c} \alpha e^{-(\lambda+\alpha)t}e^{-s(y-ct)}dt \\
&= C(s)\left\{\frac{\lambda}{\lambda+\alpha-cs}\widetilde{p}(s)(e^{-sy} - e^{-(\lambda+\alpha)y/c}) - \frac{\lambda}{\lambda+\alpha-c\rho}\widetilde{p}(\rho)(e^{-\rho y} - e^{-(\lambda+\alpha)y/c})\right\} \\
&\quad + \frac{\alpha}{\lambda+\alpha-cs}(e^{-sy} - e^{-(\lambda+\alpha)y/c}) \\
&= \left\{C(s)\frac{\lambda}{\lambda+\alpha-cs}\widetilde{p}(s) + \frac{\alpha}{\lambda+\alpha-cs}\right\}e^{-sy} - C(s)e^{-\rho y} \\
&\quad - \left\{C(s)\frac{\lambda}{\lambda+\alpha-cs}\widetilde{p}(s) - C(s) + \frac{\alpha}{\lambda+\alpha-cs}\right\}e^{-(\lambda+\alpha)y/c}.
\end{aligned}$$

Matching the coefficients of $e^{-sy}$, we get

$$C(s) = C(s)\frac{\lambda}{\lambda+\alpha-cs}\widetilde{p}(s) + \frac{\alpha}{\lambda+\alpha-cs},$$

or alternatively

$$C(s) = \frac{\alpha}{\alpha+\lambda-cs-\lambda\widetilde{p}(s)} = -\alpha\widetilde{v}_{\alpha,c}(s). \tag{3.20}$$

Thus, substituting (3.20) into (3.19) yields

$$\phi_\alpha(y) = \alpha\left(e^{-\rho y} - e^{-sy}\right)\widetilde{v}_{\alpha,c}(s).$$

□

We are now in a position to provide a closed form expression for the one-sided density $g_-$.

**Proposition 3.2.2.** *The defective density of $-Z_{e_\alpha}1_{\{Z_{e_\alpha}<0\}}$ is given by*

$$g_-(x) = \alpha\left\{\Phi_{\alpha,c}e^{\rho x} - v_{\alpha,c}(x)\right\}, \ x>0, \tag{3.21}$$

*where $\Phi_{\alpha,c}$ is as defined in (3.14) and $v_{\alpha,c}(u)$ is defined on $[0,\infty)$ through its LT (3.5).*

*Proof.* Substituting (3.17) back into the second term of (3.16), we get

$$\begin{aligned}
\int_0^\infty \left\{\frac{\lambda}{c}\mathcal{T}_\rho p(y)\right\}\phi_\alpha(y)dy &= \int_0^\infty \left\{\frac{\lambda}{c}\mathcal{T}_\rho p(y)\right\}\left\{\alpha\left(e^{-\rho y}-e^{-sy}\right)\widetilde{v}_{\alpha,c}(s)\right\}dy \\
&= \alpha\widetilde{v}_{\alpha,c}(s)\left\{\frac{\lambda}{c}\mathcal{T}_\rho^2 p(0) - \frac{\lambda}{c}\frac{\widetilde{p}(\rho)-\widetilde{p}(s)}{s-\rho}\right\} \\
&= \alpha\widetilde{v}_{\alpha,c}(s)\left\{\frac{\lambda}{c}\mathcal{T}_\rho^2 p(0) - 1\right\} + \frac{\alpha}{c}\frac{1}{s-\rho},
\end{aligned} \tag{3.22}$$

where we use the following identity to move from the third to the fourth lines

$$\widetilde{v}_{\alpha,c}(s) = \frac{1}{cs-\lambda(1-\widetilde{p}(s))-\alpha} = \frac{1}{c(s-\rho)-\lambda(\widetilde{p}(\rho)-\widetilde{p}(s))} = \frac{1}{c(s-\rho)}\frac{1}{1-\frac{\lambda}{c}\frac{\widetilde{p}(\rho)-\widetilde{p}(s)}{s-\rho}}.$$

Substituting (3.22) into (3.16) yields

$$\mathbb{E}\left[e^{-s(-Z_{e_\alpha})}1_{\{Z_{e_\alpha}<0\}}\right] = \frac{\lambda}{c}\mathcal{T}_\rho^2 p(0)\mathbb{E}\left[e^{-s(-Z_{e_\alpha})}1_{\{Z_{e_\alpha}<0\}}\right] + \alpha\widetilde{v}_{\alpha,c}(s)\left\{\frac{\lambda}{c}\mathcal{T}_\rho^2 p(0) - 1\right\} + \frac{\alpha}{c}\frac{1}{s-\rho},$$

which implies that

$$\begin{aligned} \mathbb{E}\left[e^{-s(-Z_{e_\alpha})}1_{\{Z_{e_\alpha}<0\}}\right] &= \frac{\alpha\widetilde{v}_{\alpha,c}(s)\left\{\frac{\lambda}{c}\mathcal{T}_\rho^2 p(0) - 1\right\} + \frac{\alpha}{c}\frac{1}{s-\rho}}{1-\frac{\lambda}{c}\mathcal{T}_\rho^2 p(0)} \\ &= \alpha\Phi_{\alpha,c}\frac{1}{s-\rho} - \alpha\widetilde{v}_{\alpha,c}(s). \end{aligned} \tag{3.23}$$

The inversion of (3.23) wrt $s$ yields the density of $-Z_{e_\alpha}1_{\{Z_{e_\alpha}<0\}}$,

$$g_-(x) = \alpha\left\{\Phi_{\alpha,c}e^{\rho x} - v_{\alpha,c}(x)\right\},\ x>0,$$

which is consistent with Kyprianou (2006, Corollary 8.9). □

The following example shows that in some special cases, we can explicitly express the density $g_-(x)$ using Equation (3.21).

**Example 3.2.2.** *Following the same assumptions as in Example 3.2.1, we have*

$$\begin{aligned} g_-(x) &= \alpha\{\Phi_{\alpha,c}e^{\rho x} - v_{\alpha,c}(x)\} \\ &= \frac{\alpha}{c}\frac{\mu+\rho}{R+\rho}e^{\rho x} + \frac{\alpha}{c}\frac{\mu-R}{\rho+R}e^{-Rx} - \frac{\alpha}{c}\frac{\mu+\rho}{\rho+R}e^{\rho x} \\ &= \frac{\alpha}{c}\frac{\mu-R}{\rho+R}e^{-Rx},\ x>0, \end{aligned}$$

*where we use the inversion of the $\widetilde{v}_{\alpha,c}(s)$ in*

$$\alpha\widetilde{v}_{\alpha,c}(s) = -\frac{\alpha}{\alpha-cs+\lambda\left(1-\widetilde{p}(s)\right)} = \frac{\alpha}{c}\frac{\mu+\rho}{\rho+R}\frac{1}{s-\rho} - \frac{\alpha}{c}\frac{\mu-R}{\rho+R}\frac{1}{s+R}.$$

*This result is in agreement with Albrecher et al. (2013, Example 4.1).*

Before we move to the next section, we need to examine the two one-sided "discounted" densities of $Z_{e_\alpha}$, namely $g^\delta_{+(-)}$, defined respectively through their one-sided LTs as

$$\mathbb{E}\left[e^{-\delta e_\alpha} e^{-sZ_{e_\alpha}} 1_{\{Z_{e_\alpha}>(<)0\}}\right] = \int_0^\infty e^{-sx} g^\delta_{+(-)}(x)dx, \text{ for } \delta \geq 0. \tag{3.24}$$

It follows that

$$\mathbb{E}\left[e^{-\delta e_\alpha} e^{-sZ_{e_\alpha}} 1_{\{Z_{e_\alpha}>(<)0\}}\right] = \mathbb{E}\left[e^{-sZ_{e_\alpha}} 1_{\{Z_{e_\alpha}>(<)0\}} 1_{\{e^*_\delta > e_\alpha\}}\right] = \frac{\alpha}{\alpha+\delta}\mathbb{E}\left[e^{-sZ_{e_{\alpha+\delta}}} 1_{\{Z_{e_{\alpha+\delta}}>(<)0\}}\right], \tag{3.25}$$

where $e^*_\delta$ is an exponential random variable with mean $1/\delta$, independent of $e_\alpha$. The second equation uses the fact that $\min(e^*_\delta, e_\alpha)$ is distributed as $e_{\alpha+\delta}$, $1_{\{e^*_\delta > e_\alpha\}}$ is Bernoulli distributed with mean $\frac{\alpha}{\alpha+\delta}$, and $\min(e^*_\delta, e_\alpha)$ and $1_{\{e^*_\delta > e_\alpha\}}$ are independent (see Ross (2010)). Thus,

$$g^\delta_{+(-)}(x) = \frac{\alpha}{\alpha+\delta} g_{+(-)}(x), \text{ for } x > 0,$$

where $g_+(x)$ and $g_-(x)$ are given in (3.13) and (3.21) respectively, but with the parameter $\alpha$ substituted for $\alpha+\delta$. Therefore, the two one-sided discounted densities of $Z_{e_\alpha}$ are given by,

$$g^\delta_+(x) = \alpha\Phi_{\alpha+\delta,c} e^{-\rho_{\alpha+\delta,c}x}, \; x > 0, \tag{3.26}$$

$$g^\delta_-(x) = \alpha\left\{\Phi_{\alpha+\delta,c} e^{\rho_{\alpha+\delta,c}x} - v_{\alpha+\delta,c}(x)\right\}, \; x > 0. \tag{3.27}$$

## 3.3 Defective renewal equation and discounted joint density

In this section, we assume that the generic inter-review time $K_i$ has density

$$k_i(t) = \sum_{k=1}^n \xi_{ik}\alpha_{ik} e^{-\alpha_{ik}t}, \; t > 0, \tag{3.28}$$

where $\alpha_{ik} > 0$ for $\forall\ i, k$ and $\sum_{k=1}^{n} \xi_{ik} = 1$. Thus, the mean of $K_i$ is $\kappa_i = \sum_{k=1}^{n} \xi_{ik} \frac{1}{\alpha_{ik}}$.

**Remark 3.3.1.** *Note that the class (3.28) of combinations of exponentials is dense in the set of all continuous probability distributions defined on the positive axis (see, e.g., Dufresne (2007)). Also, one may follow the idea of randomization (or Erlangization) and use combinations of exponentials to approximate the fixed time value. Thus, the results in this section can be used as an approximation for the corresponding results in the discrete-time risk model. In fact, by letting* $\alpha_{ik} = \frac{n(n+1)}{2k\kappa_i}$ *and* $\xi_{ik} = \prod_{j=1, j\neq k}^{n} \frac{\alpha_{ij}}{\alpha_{ij}-\alpha_{ik}}$*, it is not hard to show that* $\mathbb{E}(K_i) = \kappa_i$ *and* $Var(K_i) = \frac{2(2n+1)}{3n(n+1)}\kappa_i^2$ *(see Klugman et al. (2012)). Therefore, as n goes to infinity, the mean of the inter-review time* $K_i$ *stays the same while its variance goes to 0.*

**Proposition 3.3.1.** *Let* $g_{i,+(-)}^{\delta}$ *be defined through*

$$\mathbb{E}\left[e^{-\delta K_i} e^{-s|Z_{K_i,c_i}|} 1_{\{Z_{K_i,c_i} > (<) 0\}}\right] = \int_0^\infty e^{-sx} g_{i,+(-)}^{\delta}(x) dx.$$

*For* $x > 0$*, we have*

$$g_{i,+}^{\delta}(x) = \sum_{k=1}^{n} \xi_{ik} \alpha_{ik} \Phi_{ik} e^{-\rho_{ik} x}, \tag{3.29}$$

*and*

$$g_{i,-}^{\delta}(x) = \sum_{k=1}^{n} \xi_{ik} \alpha_{ik} \left\{\Phi_{ik} e^{\rho_{ik} x} - v_{ik}(x)\right\}, \tag{3.30}$$

*where* $\rho_{ik} = \rho_{\alpha_{ik}+\delta, c_i}$*,* $\Phi_{ik} = \Phi_{\alpha_{ik}+\delta, c_i}$ *and* $v_{ik}(x) = v_{\alpha_{ik}+\delta, c_i}(x)$*.*

*Proof.* By the definition of the LT of $g_{i,+}^{\delta}$, along with (3.24) and (3.26), it follows

$$\begin{aligned}
\int_0^\infty e^{-sx} g_{i,+}^{\delta}(x) dx &= \mathbb{E}\left[e^{-\delta K_i} e^{-s Z_{K_i,c_i}} 1_{\{Z_{K_i,c_i} > 0\}}\right] \\
&= \sum_{k=1}^{n} \xi_{ik} \mathbb{E}\left[e^{-\delta e_{\alpha_{ik}}} e^{-s Z_{e_{\alpha_{ik}},c_i}} 1_{\{Z_{e_{\alpha_{ik}},c_i} > 0\}}\right] \\
&= \int_0^\infty e^{-sx} \left\{\sum_{k=1}^{n} \xi_{ik} \alpha_{ik} \Phi_{ik} e^{-\rho_{ik} x}\right\} dx.
\end{aligned}$$

By the uniqueness of LT, we obtain (3.29). Similar arguments apply to (3.30). $\square$

### 3.3.1 Laplace transform of the Gerber-Shiu function

With the one-sided discounted densities (3.29) and (3.30), we now consider ruin-related quantities in the risk model (3.1) with the experience-based premium policy.

By conditioning on the surplus increment $Y_1$ over the first review period, the Gerber-Shiu function (3.3) can be expressed as

$$m_{i,j,\delta}(u) = \int_0^u m_{\min(i+1,m),j,\delta}(u-y)g_{i,-}^{\delta}(y)dy + b_{ij}(u) + \int_0^{\infty} m_{\max(i-1,1),j,\delta}(u+y)g_{i,+}^{\delta}(y)dy, \tag{3.31}$$

where $g_{i,+}^{\delta}$ and $g_{i,-}^{\delta}$ are as defined in (3.29) and (3.30) respectively, and

$$b_{ij}(u) = \left\{\int_u^{\infty} w(u, y-u)g_{i,-}^{\delta}(y)dy\right\} 1_{\{i=j\}}. \tag{3.32}$$

Taking the LT on both sides of (3.31), it follows that

$$\widetilde{m}_{i,j,\delta}(z) = \widetilde{m}_{\min(i+1,m),j,\delta}(z)\widetilde{g}_{i,-}^{\delta}(z)+\widetilde{b}_{ij}(z)+\sum_{k=1}^{n}\xi_{ik}\alpha_{ik}\Phi_{ik}\frac{\widetilde{m}_{\max(i-1,1),j,\delta}(\rho_{ik}) - \widetilde{m}_{\max(i-1,1),j,\delta}(z)}{z-\rho_{ik}}. \tag{3.33}$$

In matrix form, Equation (3.33) becomes

$$(\mathbf{I} - \mathbf{A}_{\delta}(z))\,\widetilde{\mathbf{m}}_{\delta}(z) = \widetilde{\mathbf{B}}(z) + \sum_{k=1}^{n}\mathbf{D}_k(z)\mathbf{C}_{k,\delta}, \tag{3.34}$$

where $\widetilde{\mathbf{m}}_{\delta}(z) = [\widetilde{m}_{i,j,\delta}(z)]_{i,j=1}^m$, $\widetilde{\mathbf{B}}(z) = \mathrm{diag}\left\{\widetilde{b}_{ii}(z)\right\}_{i=1}^m$, $\mathbf{D}_k(z) = \mathrm{diag}\left\{\frac{\xi_{ik}\alpha_{ik}\Phi_{ik}}{z-\rho_{ik}}\right\}_{i=1}^m$, $\mathbf{C}_{k,\delta} = \left[\widetilde{m}_{\max(i-1,1),j,\delta}(\rho_{ik})\right]_{i,j=1}^m$, and $\mathbf{I}$ is the identity matrix. Also, the matrix $\mathbf{A}_{\delta}(z) = [a_{i,j,\delta}(z)]_{i,j=1}^m$ has entries $a_{i,\max(i-1,1),\delta}(z) = \sum_{k=1}^n \frac{\xi_{ik}\alpha_{ik}\Phi_{ik}}{\rho_{ik}-z}$ and $a_{i,\min(i+1,m),\delta}(z) = \widetilde{g}_{i,-}^{\delta}(z)$, for $i = 1, \ldots, m$, while all other entries are 0.

**Remark 3.3.2.** *It is not difficult to check that the transition matrix $\mathbf{Q}$ for the discrete-time Markov chain $\{\eta_k; k \geq 1\}$ also corresponds to $\mathbf{A}_0(0)$. The stationary probabilities*

$\boldsymbol{\vartheta}=[\vartheta_1,\ldots,\vartheta_m]$ *of this Markov process satisfy*

$$\begin{cases} \boldsymbol{\vartheta}\mathbf{A}_0(0) = \boldsymbol{\vartheta}, \\ \sum_{i=1}^{m}\vartheta_i = 1. \end{cases} \tag{3.35}$$

*The semi-Markov process generated from the discrete-time Markov chain* $\{\eta_k; k \geq 1\}$, *where a random time with density* $k_i$ *is spent in the state* $\eta_k = c_i$ *has stationary probabilities* $\boldsymbol{\pi} = [\pi_1,\ldots,\pi_m]$, *where*

$$\pi_i = \frac{\vartheta_i \kappa_i}{\sum_{j=1}^{m}\vartheta_j\kappa_j}, \tag{3.36}$$

*for* $i = 1,\ldots,m$ *(see, e.g., Ross, 1996, Section 8.6.1). One concludes that the positive security loading condition for model (3.1) is*

$$\sum_{i=1}^{m}\pi_i c_i > \lambda/\mu, \tag{3.37}$$

*which can equivalently be represented as* $\sum_{i=1}^{m}\vartheta_i\mathbb{E}\left[c_iK_i - S_{K_i}\right] > 0$.

Assuming $\mathbf{I} - \mathbf{A}_\delta(z)$ is invertible, it follows that

$$\widetilde{\mathbf{m}}_\delta(z) = \frac{\operatorname{adj}\left(\mathbf{I} - \mathbf{A}_\delta(z)\right)\left(\widetilde{\mathbf{B}}(z) + \sum_{k=1}^{n}\mathbf{D}_k(z)\mathbf{C}_{k,\delta}\right)}{\det\left(\mathbf{I} - \mathbf{A}_\delta(z)\right)}, \tag{3.38}$$

where $\operatorname{adj}\left(\mathbf{I} - \mathbf{A}_\delta(z)\right)$ is the adjoint matrix of $\mathbf{I} - \mathbf{A}_\delta(z)$. Note that the matrices $\{\mathbf{C}_{k,\delta}\}_{k=1}^{n}$ in (3.38) contain $m \times m \times n$ unknown constants, namely $\widetilde{m}_{1,j,\delta}(\rho_{1k})$ and $\widetilde{m}_{(i-1),j,\delta}(\rho_{ik})$ for $i = 2,\ldots,m$, $j = 1,\ldots,m$ and $k = 1,\ldots,n$. Thus, our objective is to identify these constants in order to fully characterize the closed-form expression for $\widetilde{\mathbf{m}}_\delta(z)$ given in (3.38).

**Lemma 3.3.1.** *For* $\delta > 0$, *there are* $m \times n$ *non-negative solutions, namely* $\gamma_1,\ldots,\gamma_{mn}$, *to*

$\det\left(\mathbf{I} - \mathbf{A}_\delta(z)\right) = 0.$

*Proof.* Define the contour $D = \lim_{r\to\infty}\left(D_r \cup D_0\right)$, where $D_r = \{z : |z| = r, \operatorname{Re}(z) \geq 0\}$ and $D_0 = \{z : |z| < r, \operatorname{Re}(z) = 0\}$. We will show that $\sum_{j=1}^{m}|a_{i,j,\delta}(z)| < 1$ on $D$.

Let us now assume that

$$\left|\sum_{k=1}^{n} \frac{\xi_{ik}\alpha_{ik}\Phi_{ik}}{z-\rho_{ik}}\right| \leq \widetilde{g}_{i,+}^{\delta}(0), \tag{3.39}$$

holds for all $z$ in $D_r \cup D_0$. It follows that

$$\sum_{j=1}^{m} |a_{i,j,\delta}(z)| = \left|\sum_{k=1}^{n} \frac{\xi_{ik}\alpha_{ik}\Phi_{ik}}{z-\rho_{ik}}\right| + \left|\widetilde{g}_{i,-}^{\delta}(z)\right| \leq \widetilde{g}_{i,+}^{\delta}(0) + \widetilde{g}_{i,-}^{\delta}(0) < 1.$$

To show that (3.39) holds on the contour $D_r \cup D_0$ for $r$ sufficiently large, let us first consider the imaginary part of the contour. It is clear that for any $z$ such that $\mathrm{Re}(z) = 0$,

$$\left|\sum_{k=1}^{n} \frac{\xi_{ik}\alpha_{ik}\Phi_{ik}}{z-\rho_{ik}}\right| = \left|\widetilde{g}_{i,+}^{\delta}(-z)\right| \leq \widetilde{g}_{i,+}^{\delta}(0).$$

Also, for all $z \in D_r$ such that $r > r_0 = \max_{i,k} \rho_{ik} + \frac{\sum_{k=1}^{n}|\xi_{ik}\alpha_{ik}\Phi_{ik}|}{\sum_{k=1}^{n}\frac{\xi_{ik}\alpha_{ik}\Phi_{ik}}{\rho_{ik}}}$, we have

$$\left|\sum_{k=1}^{n} \frac{\xi_{ik}\alpha_{ik}\Phi_{ik}}{z-\rho_{ik}}\right| \leq \sum_{k=1}^{n} \frac{|\xi_{ik}\alpha_{ik}\Phi_{ik}|}{|z| - \max_{i,k}\rho_{ik}} \leq \sum_{k=1}^{n} \frac{\xi_{ik}\alpha_{ik}\Phi_{ik}}{\rho_{ik}} = \widetilde{g}_{i,+}^{\delta}(0).$$

Therefore, $\sum_{j=1}^{m} |a_{i,j,\delta}(z)| < 1$ holds on $D_r$ for any $r > r_0$ and the imaginary axis $D_0$.

Now we can apply the matrix form of Rouche's theorem (see Theorem 1.4.2). Since $\det \mathbf{I} = 1 \neq 0$, $\det(\mathbf{I} - \mathbf{A}_\delta(z))$ satisfies $\mathcal{N}_{\mathbf{I}-\mathbf{A}_\delta} - \mathcal{P}_{\mathbf{I}-\mathbf{A}_\delta} = 0$, where $\mathcal{N}_{\mathbf{I}-\mathbf{A}_\delta}$ and $\mathcal{P}_{\mathbf{I}-\mathbf{A}_\delta}$ are the number of zeros and poles inside $D$ of $\det(\mathbf{I} - \mathbf{A}_\delta(z))$, respectively. It is clear from the definition of $\mathbf{A}_\delta(z)$ that $\det(\mathbf{I} - \mathbf{A}_\delta(z))$ has $m \times n$ poles, namely $z = \rho_{ik}$ for $i = 1, \ldots, m$ and $k = 1, \ldots, n$. Therefore, $\det(\mathbf{I} - \mathbf{A}_\delta(z))$ must have $m \times n$ zeros inside $D$. □

Note that when $\delta = 0$, it is easy to see $\sum_{j=1}^{m} |a_{i,j,0}(0)| = 1$, therefore the matrix form of Rouche's theorem does not work in this case. It has not been proved, but numerically it is verified that when $\delta = 0$, if the positive security loading condition (3.37) is satisfied, there are still $m \times n$ non-negative solutions to $\det(\mathbf{I} - \mathbf{A}_0(z)) = 0$ among which one solution is 0.

Henceforth, we assume that $\gamma_1, \ldots, \gamma_{mn}$ are distinct. For $i = 1, \ldots, mn$, let the non-zero row vector $\underline{\mathbf{h}}_i = [\underline{h}_{i1}, \ldots, \underline{h}_{im}]$ be the left eigenvector of $(\mathbf{I} - \mathbf{A}_\delta(\gamma_i))$ associated with the eigenvalue 0. Now we are ready to provide an explicit expression for $\{\mathbf{C}_{k,\delta}\}_{k=1}^n$.

**Proposition 3.3.2.** *If the matrix* $\mathbf{V} = [\underline{\mathbf{h}}_i \mathbf{D}_k(\gamma_i)]_{i=1,k=1}^{mn,n}$ *is invertible, we have*

$$\begin{pmatrix} \mathbf{C}_{1,\delta} \\ \vdots \\ \mathbf{C}_{n,\delta} \end{pmatrix} = \mathbf{\Theta} \begin{pmatrix} \underline{\mathbf{h}}_1 \widetilde{\mathbf{B}}(\gamma_1) \\ \vdots \\ \underline{\mathbf{h}}_{mn} \widetilde{\mathbf{B}}(\gamma_{mn}) \end{pmatrix}, \tag{3.40}$$

*where* $\mathbf{\Theta} = [\theta_{i,j}]_{i,j=1}^{mn} = -\mathbf{V}^{-1}$.

*Proof.* By definition, for $i = 1, \ldots, mn$,

$$\underline{\mathbf{h}}_i \left(\mathbf{I} - \mathbf{A}_\delta(\gamma_i)\right) = \mathbf{0},$$

where $\mathbf{0}$ is a row vector of all 0s, which implies

$$\underline{\mathbf{h}}_i \left(\mathbf{I} - \mathbf{A}_\delta(\gamma_i)\right) \widetilde{\mathbf{m}}_\delta(\gamma_i) = \mathbf{0}\widetilde{\mathbf{m}}_\delta(\gamma_i) = \mathbf{0}. \tag{3.41}$$

Multiplying (3.34) at $z = \gamma_i$ by the left eigenvector $\underline{\mathbf{h}}_i$ and using (3.41), one finds

$$\underline{\mathbf{h}}_i \left(\mathbf{I} - \mathbf{A}_\delta(\gamma_i)\right) \widetilde{\mathbf{m}}_\delta(\gamma_i) = \underline{\mathbf{h}}_i \left( \widetilde{\mathbf{B}}(\gamma_i) + \sum_{k=1}^{n} \mathbf{D}_k(\gamma_i) \mathbf{C}_{k,\delta} \right) = \mathbf{0},$$

which results in the following system of linear equations:

$$\mathbf{V} \begin{pmatrix} \mathbf{C}_{1,\delta} \\ \vdots \\ \mathbf{C}_{n,\delta} \end{pmatrix} = - \begin{pmatrix} \underline{\mathbf{h}}_1 \widetilde{\mathbf{B}}(\gamma_1) \\ \vdots \\ \underline{\mathbf{h}}_{mn} \widetilde{\mathbf{B}}(\gamma_{mn}) \end{pmatrix}.$$

For $\mathbf{V}$ an invertible matrix, the result easily follows. $\square$

We point out that the matrix $\mathbf{V}$ is a generalized Cauchy matrix (see, e.g., Heinig, 1995) of the form

$$\left[\frac{\mathbf{z}_i^T\mathbf{y}_j}{c_i-d_j}\right]_{i,j=1}^{mn},$$

where $c_i = \gamma_i$, $d_j = \rho_{st}$, $\mathbf{z}_i^T = (\underline{\mathbf{h}}_i)^T$, $\mathbf{y}_j = \xi_{st}\alpha_{st}\boldsymbol{\Phi}'_{st}\mathbf{e}_s$ with $\mathbf{e}_s$ the canonical vectors, $s = j - \lfloor \frac{j}{m} \rfloor \times m$, and $t = \lceil \frac{j}{m} \rceil$. Sufficient conditions under which such a matrix is invertible have been widely analyzed in the literature (see, e.g., Heinig (1995, 1998)). When $\mathbf{V}$ is invertible, an application of Theorem 2.2 in Heinig (1995) leads to an expression for $\mathbf{W} = -\mathbf{V}^{-1}$. Let $\mathbf{Z} = \mathrm{col}(\mathbf{z}_i^T)_{i=1}^{mn}$ and $\mathbf{Y} = \mathrm{col}(\mathbf{y}_j^T)_{j=1}^{mn}$, then

$$\mathbf{W} = [w_{i,j}]_{i,j=1}^{mn} = \left[\frac{\mathbf{x}_i^T\mathbf{p}_j}{d_i-c_j}\right]_{i,j=1}^{mn},$$

where $\mathbf{X} = \mathrm{col}(\mathbf{x}_i^T)_{i=1}^{mn}$ and $\mathbf{P} = \mathrm{col}(\mathbf{p}_j^T)_{j=1}^{mn}$ are the solutions to $\mathbf{VX} = \mathbf{Z}$ and $\mathbf{P}^T\mathbf{V} = \mathbf{Y}^T$.

For the choice of parameters considered in the examples of Section 3.4, these conditions are satisfied and thus $\mathbf{V}$ is invertible in those cases.

Therefore, using (3.38) and (3.40), we have an explicit expression for $\widetilde{\mathbf{m}}_\delta(z)$ whose inversion results in an expression for $m_{ij,\delta}(u)$ in terms of the solutions $\gamma_1, \dots, \gamma_{mn}$.

### 3.3.2 Matrix-form defective renewal equation and discounted joint densities

Intuitively, we expect the Gerber-Shiu function to satisfy a matrix-form defective renewal equation, also known as Markov renewal equation in the ruin theory literature (see, e.g., Cheung and Feng (2013)). Interest in such a representation comes from the fact that its solution is known to possess some particular properties, such as uniqueness. Also, the asymptotics (or Cramér-Lundberg approximations) and the two-sided bounds of the solution are discussed in Miyazawa (2002) and Li and Luo (2005), respectively. Especially, by comparing the matrix-form defective renewal equation with the Gerber-Shiu function derived in last section, we can obtain the joint density of the surplus prior to ruin and deficit at ruin with an initial surplus level at 0.

Let $h^{*\delta}_{1,ij}(y|u)$ and $h^{*\delta}_{2,ij}(x,y|u)$ be the discounted density of the deficit at ruin $|\mathcal{U}_{k^*}|$ for ruin occurring at time $X_1$ and the discounted joint density of $(\mathcal{U}_{k^*-1}, |\mathcal{U}_{k^*}|)$ for ruin occurring after $X_1$, respectively. The arguments $i$ and $j$ in the above two discounted densities stand for the event $1_{\{\eta_{k^*}=c_j\}}|\eta_1 = c_i$. By conditioning on the first drop in surplus at a review time, the Gerber-Shiu function $m_{i,j,\delta}(u)$ can be represented as

$$m_{i,j,\delta}(u) = \sum_{l=1}^{m} \int_0^u m_{l,j,\delta}(u-y) h^{*\delta}_{il}(y|0)dy + f_{ij}(u), \tag{3.42}$$

where

$$h^{*\delta}_{ij}(y|u) = h^{*\delta}_{1,ij}(y|u) + \int_0^\infty h^{*\delta}_{2,ij}(x,y|u)dx,$$

and

$$f_{ij}(u) = \int_u^\infty w(u, y-u) h^{*\delta}_{1,ij}(y|0)dy + \int_u^\infty \int_0^\infty w(x+u, y-u) h^{*\delta}_{2,ij}(x,y|0)dxdy.$$

In a matrix form, we have

$$\mathbf{m}_\delta(u) = \mathbf{H} * \mathbf{m}_\delta(u) + \mathbf{F}(u), \tag{3.43}$$

where $\mathbf{m}_\delta(u) = [m_{i,j,\delta}(u)]_{i,j=1}^m$, $\mathbf{H}(y) = \left[h^{*\delta}_{ij}(y|0)\right]_{i,j=1}^m$, $\mathbf{F}(u) = [f_{ij}(u)]_{i,j=1}^m$ and the convolution of two matrices is defined as

$$[\mathbf{H} * \mathbf{m}_\delta(u)]_{ij} = \sum_{l=1}^{m} \int_0^u h^{*\delta}_{il}(y|0) m_{l,j,\delta}(u-y)dy.$$

Given that $\sum_{l=1}^m \int_0^\infty h^{*\delta}_{il}(y|0)dy = \mathbb{E}\left[e^{-\delta T^*} 1_{\{T^*<\infty\}} | \mathcal{U}_0 = 0, \eta_1 = c_i\right] < 1$ when $\delta > 0$ (or when (3.37) is satisfied with $\delta = 0$), (3.43) is a matrix-form defective renewal equation.

In the following, we complete the characterization of (3.43) by identifying the discounted densities $h^{*\delta}_{1,ij}(y|0)$ and $h^{*\delta}_{2,ij}(x,y|0)$.

**Proposition 3.3.3.** *The discounted densities $h^{*\delta}_{1,ij}(y|0)$ and $h^{*\delta}_{2,ij}(x,y|0)$ are given by*

$$h^{*\delta}_{1,ij}(y|0) = g^{\delta}_{i,-}(y)1_{\{i=j\}}, \tag{3.44}$$

*and*

$$h^{*\delta}_{2,ij}(x,y|0) = \sum_{k=1}^{n} \xi_{ik}\alpha_{ik}\Phi_{ik} \sum_{l=1}^{mn} \theta_{m(k-1)+i,l}\ \underline{h}_{lj} e^{-\gamma_l x} g^{\delta}_{j,-}(y+x). \tag{3.45}$$

*Proof.* Letting $u=0$ and $w(x,y)=e^{-s_1x-s_2y}$ in (3.42), we have

$$m_{i,j,\delta}(0) = \int_0^\infty e^{-s_2y} h^{*\delta}_{1,ij}(y|0)dy + \int_0^\infty \int_0^\infty e^{-s_1x-s_2y} h^{*\delta}_{2,ij}(x,y|0)dxdy. \tag{3.46}$$

Alternatively, an application of the initial value theorem (e.g., Spiegel (1965)) to $\widetilde{\mathbf{m}}_\delta(z)$ in (3.38) leads to

$$\mathbf{m}_\delta(0) = \mathbf{B}(0) + \sum_{k=1}^{n} \mathbf{D}_k\mathbf{C}_{k,\delta},$$

where $\mathbf{D}_k = \mathrm{diag}\,\{\xi_{ik}\alpha_{ik}\Phi_{ik}\}_{i=1}^m$. Thus, for $i,j=1,\ldots,m$,

$$m_{i,j,\delta}(0) = b_{ij}(0) + \sum_{k=1}^{n} \xi_{ik}\alpha_{ik}\Phi_{ik}\widetilde{m}_{\max(i-1,1),j,\delta}(\rho_{ik}). \tag{3.47}$$

Now, making use of (3.32) and (3.40), it is immediate that

$$b_{ij}(0) = \int_0^\infty e^{-s_2y} g^{\delta}_{i,-}(y)1_{\{i=j\}}dy, \tag{3.48}$$

and

$$\begin{aligned}
\widetilde{m}_{\max(i-1,1),j,\delta}(\rho_{ik}) &= \sum_{l=1}^{mn} \theta_{m(k-1)+i,l} \left( \sum_{s=1}^{m} \underline{h}_{ls} \widetilde{b}_{sj}(\gamma_l) \right) \\
&= \sum_{l=1}^{mn} \sum_{s=1}^{m} \theta_{m(k-1)+i,l}\ \underline{h}_{ls} \int_0^\infty e^{-\gamma_l u} \int_u^\infty e^{-s_1u-s_2(y-u)} g^{\delta}_{s,-}(y)dydu1_{\{s=j\}} \\
&= \int_0^\infty \int_0^\infty \sum_{l=1}^{mn} \theta_{m(k-1)+i,l}\ \underline{h}_{lj} e^{-\gamma_l x} e^{-s_1x-s_2y} g^{\delta}_{j,-}(y+x)dydx. \qquad (3.49)
\end{aligned}$$

Substituting (3.48) and (3.49) into (3.47), one obtains

$$m_{i,j,\delta}(0) = \int_0^\infty e^{-s_2 y} g_{i,-}^{\delta}(y) 1_{\{i=j\}} dy + \int_0^\infty \int_0^\infty \sum_{k=1}^{n} \xi_{ik}\alpha_{ik}\Phi_{ik} \sum_{l=1}^{mn} \theta_{m(k-1)+i,l}\ \underline{h}_{lj} e^{-\gamma_l x} e^{-s_1 x - s_2 y} g_{j,-}^{\delta}(y+x) dy dx. \quad (3.50)$$

A comparison of (3.46) and (3.50) immediately leads to (3.61) and (3.45). □

## 3.4 Numerical examples

In this section, we implement the theoretical results of Section 3.3 to show that the risk model (3.1) mitigates the risk of an insurer's insolvency.

In the following, we propose to compare ruin quantities in the risk model (3.1) to their counterparts in a *constant premium model with randomized reviews* (CPRR model). For our experience-based premium policy, we remove the dependence on $\eta_1$ by mixing the ruin quantities over the stationary probabilities $\boldsymbol{\vartheta}=[\vartheta_1, \dots, \vartheta_m]$ which satisfy (3.35), i.e., we consider

$$m_{st,\delta}(u) = \sum_{i=1}^{m} \vartheta_i m_{i,\delta}(u), \quad (3.51)$$

where $m_{i,\delta}(u) = \sum_{j=1}^{m} m_{i,j,\delta}(u)$. In addition, we assume that the review times' density does not depend on the premium rate in effect, i.e., $k_i(t) \equiv k(t)$ for $i = 1, \dots, m$.

On the other hand, the CPRR model used for comparative purposes is a special case of the model (3.1) where $c_i \equiv \bar{c}$ and $k_i(t) \equiv k(t)$ for $i = 1, \dots, m$. We assume that $\bar{c} = \sum_{j=1}^{m} \pi_j c_j$, where $\boldsymbol{\pi} = [\pi_1, \dots, \pi_m]$ are the stationary probabilities defined in (3.36). Note that the CPRR model is similar to the risk model studied in Albrecher et al. (2011, 2013).

### 3.4.1 Ruin probability

We begin our analysis with the ruin probability. Let $\psi_{st}(u)$ be the stationary ruin probability resulting from Equation (3.51) with $\delta = 0$ and $w(x, y) = 1$. Also, define $\psi_{\bar{c}}(u)$ to be the ruin probability of the CPRR model.

**Example 3.4.1.** *We consider an example with two premium rates* $c_1 = 11$ *and* $c_2 = 14$. *Claim sizes are assumed to be exponentially distributed with mean* 10*, while the inter-review times are also exponentially distributed with mean* $1/\alpha$. *Finally, the claim arrival rate is* $\lambda = 1$. *Results for* $\psi_{st}(u)$ *and* $\psi_{\bar{c}}(u)$ *are provided in Table 3.1 for different values of* $u$ *and* $\alpha$.

| | $\psi_{st}(u)$ | $\psi_{\bar{c}}(u)$ | $\psi_{st}(u)$ | $\psi_{\bar{c}}(u)$ | $\psi_{st}(u)$ | $\psi_{\bar{c}}(u)$ | $\psi_{st}(u)$ | $\psi_{\bar{c}}(u)$ |
|---|---|---|---|---|---|---|---|---|
| $\alpha$ | $u$=0 | | $u$=25 | | $u$=50 | | $u$=100 | |
| 0.1 | 0.5410 | 0.5158 | 0.3418 | 0.3458 | 0.2143 | 0.2318 | 0.0831 | 0.1042 |
| 0.5 | 0.7104 | 0.7053 | 0.4583 | 0.4772 | 0.2946 | 0.3229 | 0.1213 | 0.1478 |
| 1 | 0.7688 | 0.7666 | 0.5122 | 0.5289 | 0.3407 | 0.3650 | 0.1505 | 0.1737 |
| 10 | 0.8808 | 0.8807 | 0.6660 | 0.6700 | 0.5036 | 0.5098 | 0.2880 | 0.2950 |
| $\infty$ | 0.9091 | 0.9091 | 0.7243 | 0.7243 | 0.5770 | 0.5770 | 0.3663 | 0.3663 |

Table 3.1: Ruin probability with different values of $u$ and $\alpha$

From Table 3.1, we observe that:

1. As expected, the ruin probability is a decreasing function of the initial surplus $u$.

2. The ruin probability is an increasing function of $\alpha$. As the rate $\alpha$ increases, the frequency of solvency checks increases, making it more likely to identify a ruin event. Also, given that $c_1$ and $c_2$ have positive security loadings, a larger $\alpha$ implies that the premium review will be conducted more often to reduce the premium rate to $c_1$, and thus making the surplus process riskier.

   As expected, when $\alpha$ goes to $\infty$ (i.e., $\alpha$ is large enough), all ruin probabilities converge to the ruin probability in the (continuous time) classical risk model with a constant premium rate of $c_1$.

3. For relatively large surplus values, the ruin probabilities $\psi_{st}$ are smaller than $\psi_{\bar{c}}$, which implies that our experience-based premium policy reduces the risk of insolvency in the long run. However, the opposite conclusion is reached for small initial surplus values, an observation also made by Tsai and Parker (2004) and Loisel and Trufin (2013) in a similar context.

In the following example, our goal is to investigate the effect of the distribution of the inter-review times on the ruin probability.

**Example 3.4.2.** *We reconsider Example 3.4.1 under two alternative distributional assumptions:*

*M1:* $k(t) = (\alpha_1 e^{-\alpha_1 t} + \alpha_2 e^{-\alpha_2 t})/2$ *with* $(\alpha_1, \alpha_2)$ *such that the mean is* $1/\alpha$ *and the variance is* $1.5/\alpha^2 > 1/\alpha^2$.

*M2:* $k(t) = (3\alpha_1 e^{-\alpha_1 t} - \alpha_2 e^{-\alpha_2 t})/2$ *with* $(\alpha_1, \alpha_2)$ *such that the mean is* $1/\alpha$ *and the variance is* $0.5/\alpha^2 < 1/\alpha^2$.

| | $\psi_{st}(u)$ | $\psi_{\bar{c}}(u)$ | $\psi_{st}(u)$ | $\psi_{\bar{c}}(u)$ | $\psi_{st}(u)$ | $\psi_{\bar{c}}(u)$ | $\psi_{st}(u)$ | $\psi_{\bar{c}}(u)$ |
|---|---|---|---|---|---|---|---|---|
| $\alpha$ | $u$=0 | | $u$=25 | | $u$=50 | | $u$=100 | |
| 0.1 | 0.5486 | 0.5198 | 0.3444 | 0.3436 | 0.2164 | 0.2285 | 0.0858 | 0.1020 |
| 0.5 | 0.7126 | 0.7057 | 0.4612 | 0.4768 | 0.2992 | 0.3237 | 0.1261 | 0.1498 |
| 1 | 0.7696 | 0.7664 | 0.5160 | 0.5304 | 0.3465 | 0.3682 | 0.1564 | 0.1778 |
| 10 | 0.8799 | 0.8798 | 0.6668 | 0.6706 | 0.5054 | 0.5112 | 0.2903 | 0.2971 |
| $\infty$ | 0.9091 | 0.9091 | 0.7243 | 0.7243 | 0.5770 | 0.5770 | 0.3663 | 0.3663 |

Table 3.2: Ruin probability under M1

| | $\psi_{st}(u)$ | $\psi_{\bar{c}}(u)$ | $\psi_{st}(u)$ | $\psi_{\bar{c}}(u)$ | $\psi_{st}(u)$ | $\psi_{\bar{c}}(u)$ | $\psi_{st}(u)$ | $\psi_{\bar{c}}(u)$ |
|---|---|---|---|---|---|---|---|---|
| $\alpha$ | $u$=0 | | $u$=25 | | $u$=50 | | $u$=100 | |
| 0.1 | 0.5359 | 0.5123 | 0.3397 | 0.3458 | 0.2130 | 0.2330 | 0.0820 | 0.1055 |
| 0.5 | 0.7086 | 0.7047 | 0.4570 | 0.4781 | 0.2928 | 0.3236 | 0.1191 | 0.1478 |
| 1 | 0.7680 | 0.7665 | 0.5108 | 0.5291 | 0.3384 | 0.3644 | 0.1479 | 0.1725 |
| 10 | 0.8814 | 0.8814 | 0.6656 | 0.6698 | 0.5026 | 0.5088 | 0.2863 | 0.2936 |
| $\infty$ | 0.9091 | 0.9091 | 0.7243 | 0.7243 | 0.5770 | 0.5770 | 0.3663 | 0.3663 |

Table 3.3: Ruin probability under M2

Tables 3.2 and 3.3 contain the values of the resulting ruin probabilities. Similar conclusions as those provided for Example 3.4.1 are also valid here. As far as the distributional assumptions of inter-review times are concerned, we remark a tendency for the ruin probability to increase as the variance of the inter-review time distribution increases. However, this conclusion is not general, as when $u = 0$ and $\alpha = 10$, the opposite ordering is observed.

### 3.4.2 Deficit at ruin

We now shift our attention to the deficit at ruin, more precisely to its tail properties. By letting $\delta = 0$ and $w(x, y) = e^{-sy}$ in Equation (3.51), one finds that

$$\sum_{i=1}^{m} \vartheta_i \mathbb{E}\left[e^{-s|\mathcal{U}_{k^*}|} 1_{\{T^*<\infty\}} | \mathcal{U}_0 = u, \eta_1 = c_i\right] = \int_0^\infty e^{-sy} \mathbb{P}\left(L_{st} \in dy\right),$$

where $L_{st}$ corresponds to the deficit at ruin in the stationary risk model (3.1). Clearly, $L_{st}$ is a defective rv and we also consider the proper rv $L^*_{st} = L_{st}|T^* < \infty$. In what follows, we focus on the Value-at-Risk (VaR) of the mixing deficit at ruin $L_{st}$ (and $L^*_{st}$), which is defined as

$$\text{VaR}^{(*)}_{st,q} = \inf\left\{y \geq 0 : \mathbb{P}\left(L^{(*)}_{st} > y\right) \leq 1 - q\right\}. \tag{3.52}$$

In the CPRR model, the counterparts to (3.52) are denoted by $\text{VaR}_{\bar{c},q}$ and $\text{VaR}^*_{\bar{c},q}$, respectively.

**Example 3.4.3.** *We reconsider Example 3.4.2 under assumption M1 with $\alpha = 0.5$. Tables 3.4 and 3.5 contain the corresponding VaR values of both the defective and proper deficit at ruin.*

Tables 3.4 and 3.5 lead to similar conclusions as those for the ruin probabilities of Table 3.1, even though the impact is far less noticeable. Indeed, with the exception of small surplus levels, the values of VaR of the deficit at ruin (both defective and proper) in the proposed premium policy risk model are smaller than their counterparts in the CPRR model. This is

| | $\mathrm{VaR}_{st,q}$ | $\mathrm{VaR}_{\bar{c},q}$ | $\mathrm{VaR}_{st,q}$ | $\mathrm{VaR}_{\bar{c},q}$ | $\mathrm{VaR}_{st,q}$ | $\mathrm{VaR}_{\bar{c},q}$ | $\mathrm{VaR}_{st,q}$ | $\mathrm{VaR}_{\bar{c},q}$ | $\mathrm{VaR}_{st,q}$ | $\mathrm{VaR}_{\bar{c},q}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $q$ | $u$=0 | | $u$=25 | | $u$=50 | | $u$=100 | | $u$=200 | |
| 0.95 | 51.37 | 50.59 | 43.02 | 43.71 | 34.54 | 36.29 | 17.70 | 21.22 | 0 | 0 |
| 0.98 | 69.82 | 68.74 | 61.37 | 62.00 | 52.75 | 54.56 | 35.59 | 39.30 | 2.15 | 9.13 |
| 0.99 | 83.99 | 82.65 | 75.46 | 76.00 | 66.73 | 68.55 | 49.34 | 53.16 | 15.41 | 22.61 |
| 0.995 | 98.32 | 96.70 | 89.70 | 90.13 | 80.88 | 82.67 | 63.28 | 67.17 | 28.87 | 36.28 |
| 0.9995 | 146.86 | 144.14 | 137.89 | 137.76 | 128.77 | 130.27 | 110.60 | 114.50 | 74.92 | 82.77 |

Table 3.4: VaR of the defective deficit at ruin

| | $\mathrm{VaR}^*_{st,q}$ | $\mathrm{VaR}^*_{\bar{c},q}$ | $\mathrm{VaR}^*_{st,q}$ | $\mathrm{VaR}^*_{\bar{c},q}$ | $\mathrm{VaR}^*_{st,q}$ | $\mathrm{VaR}^*_{\bar{c},q}$ | $\mathrm{VaR}^*_{st,q}$ | $\mathrm{VaR}^*_{\bar{c},q}$ | $\mathrm{VaR}^*_{st,q}$ | $\mathrm{VaR}^*_{\bar{c},q}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $q$ | $u$=0 | | $u$=25 | | $u$=50 | | $u$=100 | | $u$=200 | |
| 0.95 | 58.15 | 57.46 | 58.50 | 58.47 | 58.59 | 58.81 | 58.60 | 58.98 | 58.58 | 59.00 |
| 0.98 | 76.72 | 75.72 | 77.11 | 76.97 | 77.20 | 77.40 | 77.19 | 77.60 | 77.17 | 77.62 |
| 0.99 | 90.97 | 89.70 | 91.36 | 91.11 | 91.44 | 91.59 | 91.42 | 91.81 | 91.39 | 91.84 |
| 0.995 | 105.38 | 103.82 | 105.76 | 105.35 | 105.82 | 105.88 | 105.78 | 106.12 | 105.74 | 106.15 |
| 0.9995 | 154.09 | 151.40 | 154.31 | 153.25 | 154.28 | 153.86 | 154.17 | 154.15 | 154.11 | 154.19 |

Table 3.5: VaR of the proper deficit at ruin

another numerical evidence of the merit of the experience-based premium policy proposed in this chapter from a risk management standpoint.

### 3.4.3 Comparison with discrete time ruin problem

In Subsections 3.4.1 and 3.4.2, some numerical examples are implemented to compare the experience-based premium model with the constant premium model with the same random review system. In this section, we want to compare the experience-based premium model with the discrete time ruin problem.

In the discrete model, we assume that we can only review the surplus process at some discrete times $\{T^d, 2T^d, 3T^d, \ldots\}$, where $T^d$ is a fixed constant. We also assume that there are two premium rates ($c_1 < c_2$), and the premium rate changes based on the increment of the surplus process at these discrete time as before. Note that the premium rates in effective denoted by $\{\eta_k^d; k \geq 1\}$ also form a discrete time Markov Chain with transition probability matrix $\mathbf{Q}^d = \left[q_{i,j}^d\right]_{i,j=1}^2$. One can calculate numerically through

$$\begin{aligned} q_{i,1}^d =& \mathbb{P}\left(\eta_k^d = c_1 | \eta_{k-1}^d = c_i\right) \\ =& \mathbb{P}\left(c_i T^d - \sum_{l=1}^{N(T^d)} P_l > 0 | \eta_{k-1}^d = c_i\right) \\ =& \mathbb{P}\left(\sum_{l=1}^{N(T^d)} P_l < c_i T^d\right) \\ =& \sum_{j=0}^{\infty} \mathbb{P}\left(\sum_{l=1}^{j} P_l < c_i T^d\right) \mathbb{P}(N(T^d) = j) \\ =& \sum_{j=0}^{\infty} \mathbb{P}\left(\sum_{l=1}^{j} P_l < c_i T^d\right) \frac{(\lambda T^d)^j e^{-\lambda T^d}}{j!}, \end{aligned}$$

and $q_{i,2}^d = \mathbb{P}\left(\eta_k^d = c_2 | \eta_{k-1}^d = c_i\right) = 1 - q_{i,1}^d$, for $i = 1, 2$. The stationary probabilities $\boldsymbol{\vartheta}^d = [\vartheta_1^d, \vartheta_2^d]$ of this Markov process satisfy

$$\begin{cases} \boldsymbol{\vartheta}^d \mathbf{Q}^d = \boldsymbol{\vartheta}^d, \\ \sum_{i=1}^2 \vartheta_i^d = 1. \end{cases} \tag{3.53}$$

In the discrete time ruin calculation, we still remove the dependence on $\eta_1^d$ by mixing the ruin quantities over the stationary probabilities $\boldsymbol{\vartheta}^d$. We are also interested in the counterparts of such a model in a constant premium model with discrete checks. The ruin probabilities in the discrete setting are denoted by $\psi_{st}^d$ and $\psi_{\bar{c}^d}$, where $\bar{c}^d = \sum_{i=1}^2 \vartheta_i^d c_i$.

Here is a numerical example from Monte Carlo simulation (using 10000 simulation paths for each estimate).

**Example 3.4.4.** *We reconsider Example 3.4.1 with $T^d = 1/\alpha$.*

From Table 3.6, we observe similar results as in Example 3.4.1. Also, by comparing with Example 3.4.2, we see that the fixed discrete time review model gives the smallest ruin probability, which reinforces the tendency for the ruin probability to increase as the variance

|  | $\psi^d_{st}(u)$ | $\psi_{\bar{c}^d}(u)$ | $\psi^d_{st}(u)$ | $\psi_{\bar{c}^d}(u)$ | $\psi^d_{st}(u)$ | $\psi_{\bar{c}^d}(u)$ | $\psi^d_{st}(u)$ | $\psi_{\bar{c}^d}(u)$ |
|---|---|---|---|---|---|---|---|---|
| $T^d$ | $u$=0 |  | $u$=25 |  | $u$=50 |  | $u$=100 |  |
| 10 | 0.5103 | 0.5028 | 0.3341 | 0.3562 | 0.2087 | 0.2474 | 0.0704 | 0.1114 |
| 2 | 0.6993 | 0.7015 | 0.4450 | 0.4736 | 0.2731 | 0.3134 | 0.1018 | 0.1373 |
| 1 | 0.7658 | 0.7679 | 0.4943 | 0.5196 | 0.3179 | 0.3499 | 0.1255 | 0.1538 |
| 0.1 | 0.8747 | 0.8745 | 0.6507 | 0.6563 | 0.4926 | 0.4997 | 0.2756 | 0.2835 |
| 0 | 0.9091 | 0.9091 | 0.7243 | 0.7243 | 0.5770 | 0.5770 | 0.3663 | 0.3663 |

Table 3.6: Ruin probability with different values of $u$ and $T^d$

of the inter-review time distribution increases.

## 3.5 Other related models

### 3.5.1 A random performance framework

In this subsection, we generalize the experience-based premium policy considered so far in this chapter by introducing a random performance level. In previous sections, we compare the increments between successive review times with a "natural" performance level 0. However, it is of interest to have a performance level other than 0, which enables the insurer to better manage the risk. Our first idea was to incorporate a fixed performance level, but as for the finite-time ruin problem, an incomplete integral does not lead to an explicit expression for the Gerber-Shiu function. Therefore, based on the idea of randomization, we introduce the following random performance framework.

We assume that at the exponential review time, if the premium rate in force is $c_i$, then another random variable distributed as $L_i$ (assumed to be independent of $\{N_t;\ t \geq 0\}$, $\{Y_i;\ i \geq 1\}$ and $e_\alpha$) is generated. Assume $L_i$ has combination of exponentials density $f_{L_i}(y) = \sum_{l=1}^{n^*} \zeta_{il}\beta_{il}e^{-\beta_{il}y}$ and LT $\widetilde{f}_{L_i}(s) = \sum_{l=1}^{n^*} \zeta_{il}\frac{\beta_{il}}{s+\beta_{il}}$. (Note that in this section, we only consider the $m = 2$ premium rates case and assume that the inter-review time $e_\alpha$ is exponential.) Suppose the premium rate at the beginning of a given period is $c_i$ (for $i = 1, 2$). If the increment of the surplus process until the next review time is negative or if the increment of the surplus process is positive but less than $L_i$, then the premium rate increases to $c_2$. If the increment of the surplus process until the next review time is positive and larger than $L_i$, then the premium rate becomes $c_1$. By conditioning on the increment of the surplus process between successive review time as well as on the random performance level $L_i$, we can express the Gerber-Shiu function associated to this model as

$$\begin{aligned} m_{i,j,\delta}(u) &= \int_0^u m_{2,j,\delta}(u-y) g_{i,-}^{\delta}(y)dy + b_{ij}(u) + \\ & \int_0^{\infty} m_{2,j,\delta}(u+y)\bar{F}_{L_i}(y) g_{i,+}^{\delta}(y)dy + \int_0^{\infty} m_{1,j,\delta}(u+y) F_{L_i}(y) g_{i,+}^{\delta}(y)dy, \end{aligned} \tag{3.54}$$

where $g_{i,+}^{\delta}$ and $g_{i,-}^{\delta}$ are as defined in (3.26) and (3.27) with $c = c_i$ respectively. Taking the LT on both sides of (3.54), one finds

$$\begin{aligned} \widetilde{m}_{i,j,\delta}(z) &= \widetilde{m}_{2,j,\delta}(z)\widetilde{g}_{i,-}^{\delta}(z) + \widetilde{b}_{ij}(z) \\ & +\alpha\Phi_i \left\{ \sum_{l=1}^{m} \zeta_{il} \frac{n_{i,j,l} - \widetilde{m}_{2,j,\delta}(z) + \widetilde{m}_{1,j,\delta}(z)}{z-(\rho_i+\beta_{il})} + \frac{\widetilde{m}_{1,j,\delta}(\rho_i) - \widetilde{m}_{1,j,\delta}(z)}{z-\rho_i} \right\} \end{aligned} \tag{3.55}$$

for $i = 1, 2$, where $n_{i,j,l} = \widetilde{m}_{2,j,\delta}(\rho_i+\beta_{il}) - \widetilde{m}_{1,j,\delta}(\rho_i+\beta_{il})$. We re-express (3.55) as

$$\widetilde{\mathbf{m}}_{\delta}(z) = \mathbf{A}_{\delta}(z)\widetilde{\mathbf{m}}_{\delta}(z) + \widetilde{\mathbf{B}}(z) + \sum_{l=1}^{n^*} \mathbf{D}_l(z)\mathbf{N}_l + \mathbf{D}_{m+1}(z)\mathbf{C}, \tag{3.56}$$

where $\widetilde{\mathbf{m}}_{\delta}(z) = [\widetilde{m}_{i,j,\delta}(z)]_{i,j=1}^2$, $\widetilde{\mathbf{B}}(z) = \left[\widetilde{b}_{ij}(z)\right]_{i,j=1}^2 = \text{diag}\{\widetilde{b}_{ii}(z)\}_{i=1}^2$, $\mathbf{D}_l(z) = \text{diag}\{\frac{\zeta_{il}\alpha\Phi_i}{z-(\rho_i+\beta_{il})}\}_{i=1}^2$ for $l = 1, \ldots, n^*$, $\mathbf{D}_{m+1}(z) = \text{diag}\{\frac{\alpha\Phi_i}{z-\rho_i}\}_{i=1}^2$, $\mathbf{N}_l = [n_{i,j,l}]_{i,j=1}^2$, $\mathbf{C} = [\widetilde{m}_{1,j,\delta}(\rho_i)]_{i,j=1}^2$ and the elements for $\mathbf{A}_{\delta}(z)$ are

$$a_{i,1,\delta}(z) = \alpha\Phi_i \left\{ \sum_{l=1}^{n^*} \zeta_{il} \frac{1}{z-(\rho_i+\beta_{il})} - \frac{1}{z-\rho_i} \right\},$$

and

$$a_{i,2,\delta}(z) = \widetilde{g}_{i,-}^{\delta}(z) - \alpha\Phi_i \sum_{l=1}^{n^*} \zeta_{il} \frac{1}{z-(\rho_i+\beta_{il})},$$

for $i=1,2$. Note that

$$a_{i,1,\delta}(0)=\alpha\Phi_i\left\{\sum_{l=1}^{n^*}\zeta_{il}\frac{\beta_{il}}{\rho_i(\rho_i+\beta_{il})}\right\},$$

and

$$\begin{aligned}
a_{i,2,\delta}(0) &= \widetilde{g}_{i,-}^{\delta}(0)+\alpha\Phi_i\sum_{l=1}^{n^*}\zeta_{il}\frac{1}{\rho_i+\beta_{il}} \\
&= \frac{\alpha}{\alpha+\delta}-\alpha\Phi_i\sum_{l=1}^{n^*}\zeta_{il}\frac{\beta_{il}}{\rho_i(\rho_i+\beta_{il})} \\
&= \frac{\alpha}{\alpha+\delta}-\alpha\Phi_i\widetilde{\overline{F}}_{L_i}(\rho_i)<1. \qquad (3.57)
\end{aligned}$$

Here we see that there are $4n^*+4$ unknown constants, namely $\widetilde{m}_{2,j,\delta}(\rho_i)$ and $n_{i,j,l}$ for $i,j=1,2$ and $l=1,\ldots,n^*$.

From (3.56), we have

$$(\mathbf{I}-\mathbf{A}_\delta(z))\,\widetilde{\mathbf{m}}_\delta(z)=\widetilde{\mathbf{B}}(z)+\sum_{l=1}^{n^*}\mathbf{D}_l(z)\mathbf{N}_l+\mathbf{D}_{m+1}(z)\mathbf{C},$$

so we now make use of Rouche's theorem to show that $\det(\mathbf{I}-\mathbf{A}_\delta(z))=0$ has $2n^*+2$ non-negative solutions. As long as we have the following lemma, similar results to those in Section 3.3 can easily be obtained.

**Lemma 3.5.1.** *For $\delta>0$, there are $2n^*+2$ non-negative solutions to* $\det(\mathbf{I}-\mathbf{A}_\delta(z))=0$.

*Proof.* Define $D=\lim_{r\to\infty}(D_r\cup D_0)$, where $D_r=\{z:\ |z|=r \text{ and } \mathrm{Re}(z)\geq 0\}$ and $D_0=\{z:\ |z|\leq r \text{ and } \mathrm{Re}(z)=0\}$. It can be shown that $|a_{i,1,\delta}(z)|+|a_{i,2,\delta}(z)|<1$ on $D$.

For the time being, let us assume that

$$\left|\sum_{l=1}^{n^*}\frac{\zeta_{il}}{z-(\rho_i+\beta_{il})}\right|\leq\sum_{l=1}^{n^*}\zeta_{il}\frac{1}{\rho_i+\beta_{il}} \qquad (3.58)$$

holds for all $z$ in $D_r \cup D_0$. It follows that

$$
\begin{aligned}
|a_{i,2,\delta}(z)| &= \left| \widetilde{g}^{\delta}_{i,-}(z) - \alpha\Phi_i \sum_{l=1}^{n^*} \zeta_{il} \frac{1}{z-(\rho_i+\beta_{il})} \right| \\
&\le \widetilde{g}^{\delta}_{i,-}(0) + \alpha\Phi_i \left| \sum_{l=1}^{n^*} \frac{\zeta_{il}}{z-(\rho_i+\beta_{il})} \right| \\
&\le \widetilde{g}^{\delta}_{i,-}(0) + \alpha\Phi_i \sum_{l=1}^{n^*} \zeta_{il} \frac{1}{\rho_i+\beta_{il}} = a_{i,2,\delta}(0) < 1, \qquad (3.59)
\end{aligned}
$$

where we use (3.57) to go from the third line to the fourth line.

To show (3.58) holds on the contour $D_r \cup D_0$ for $r$ sufficiently large, let us first consider the imaginary part of the contour. It is clear that for $\mathrm{Re}(z) = 0$,

$$
\begin{aligned}
\left| \sum_{l=1}^{n^*} \frac{\zeta_{il}}{z-(\rho_i+\beta_{il})} \right| &= \left| \int_0^\infty e^{zy} e^{-\rho_i y} \sum_{l=1}^{n^*} \zeta_{il} e^{-\beta_{il} y} dy \right| \\
&\le \int_0^\infty e^{-\rho_i y} \sum_{l=1}^{n^*} \zeta_{il} e^{-\beta_{il} y} dy = \sum_{l=1}^{n^*} \zeta_{il} \frac{1}{\rho_i+\beta_{il}}.
\end{aligned}
$$

Also, for all $z \in D_r$ such that $r > r_0 = \max_{i,l}(\rho_i+\beta_{il}) + \sum_{l=1}^{n^*} |\zeta_{il}| / \{\sum_{l=1}^{n^*} \zeta_{il} \frac{1}{\rho_i+\beta_{il}}\}$, we have

$$
\left| \sum_{l=1}^{n^*} \frac{\zeta_{il}}{z-(\rho_i+\beta_{il})} \right| \le \sum_{l=1}^{n^*} \frac{|\zeta_{il}|}{|z| - \max_{i,l}(\rho_i+\beta_{il})} \le \sum_{l=1}^{n^*} \zeta_{il} \frac{1}{\rho_i+\beta_{il}}.
$$

On the other hand, we can show that

$$
|a_{i,1,\delta}(z)| \le a_{i,1,\delta}(0) = \frac{\alpha}{\alpha+\delta} - a_{i,2,\delta}(0) \qquad (3.60)
$$

holds on the contour $D_r \cup D_0$ for $r$ sufficiently large. Let us first consider the imaginary part of the contour. It is clear that for $\mathrm{Re}(z) = 0$,

$$
|a_{i,1,\delta}(z)| = \left| \alpha\Phi_i \{ \int_0^\infty e^{zy} e^{-\rho_i y} (1 - \sum_{l=1}^{n^*} \zeta_{il} e^{-\beta_{il} y}) dy \} \right| = \alpha\Phi_i \sum_{l=1}^{n^*} \zeta_{il} \frac{\beta_{il}}{\rho_i(\rho_i+\beta_{il})} = a_{i,1,\delta}(0), \qquad (3.61)
$$

for any $\mathrm{Re}(z) = 0$. Also, for all $z \in D_r$ such that $r > r_0 = \max_{i,j}(\rho_i + \beta_{ij}) + \left( \frac{\sum_{l=1}^{n^*} |\zeta_{il}| \beta_{il}}{\sum_{l=1}^{n^*} \zeta_{il} \frac{\beta_{il}}{\rho_i(\rho_i + \beta_{il})}} \right)^{1/2}$, we have

$$
\begin{aligned}
|a_{i,1,\delta}(z)| &= \alpha \Phi_i \left| \sum_{l=1}^{n^*} \frac{\zeta_{il}}{z - (\rho_i + \beta_{il})} - \frac{1}{z - \rho_i} \right| \\
&\leq \alpha \Phi_i \sum_{l=1}^{n^*} \frac{|q_j| \, \beta_{il}}{[|z| - \max_{i,l}(\rho_i + \beta_{il})]^2} \\
&\leq \alpha \Phi_i \sum_{l=1}^{n^*} \zeta_{il} \frac{\beta_{il}}{\rho_i(\rho_i + \beta_{il})} \\
&= a_{i,1,\delta}(0)
\end{aligned}
$$

Thus, by (3.59) and (3.60), we have

$$
|a_{i,1,\delta}(z)| + |a_{i,2,\delta}(z)| \leq a_{i,2,\delta}(0) + \frac{\alpha}{\alpha + \delta} - a_{i,2,\delta}(0) < 1.
$$

Now we can apply the matrix form of Rouche's theorem (see Theorem 1.2). Since $\det \mathbf{I} = 1 \neq 0$, $\det(\mathbf{I} - \mathbf{A}_\delta(z))$ satisfies $\mathcal{N}_{\mathbf{I}-\mathbf{A}_\delta} - \mathcal{P}_{\mathbf{I}-\mathbf{A}_\delta} = 0$ inside $D$. On the other hand, we see that $\det(\mathbf{I} - \mathbf{A}_\delta(z))$ has $2n^* + 2$ poles, namely, $z = \rho_i$ and $z = \rho_i + \beta_{il}$ for $i = 1, 2$ and $l = 1, \ldots, n^*$, therefore, $\det(\mathbf{I} - \mathbf{A}_\delta(z))$ has $2n^* + 2$ zeros, i.e., there are $2n^* + 2$ non-negative solutions to $\det(\mathbf{I} - \mathbf{A}_\delta(z)) = 0$, which are distinct from $\rho_i$ and $\rho_i + \beta_{il}$ for $i = 1, 2$ and $l = 1, \ldots, n^*$. $\square$

A matrix-form defective renewal equation for the Gerber-Shiu function and the discounted joint densities of the surplus prior to ruin and deficit at ruin can be obtained using the same procedure as in Section 3.3.

### 3.5.2 Premium policy review conducted at claim occurrence

In this subsection, we consider a risk model where the premium rates are changed at some random claim occurrence, whose idea is similar to the random review time premium policy analyzed in previous sections. Let us consider the case of two premium rates ($c_1 < c_2$), where the premium rates are changed based on the increment value of the surplus process since the last review time. If the increment of the surplus process until the next review time is negative, then the premium rate increases to $c_2$. Otherwise, if the increment of the surplus process until the next review time is positive, then the premium rate decreases to $c_1$.

More specifically, we use a series of geometric rv's to describe the reviews conducted at claim occurrence. Let $\{M_i; i \geq 1\}$ to be a sequence of iid rv's with the generic geometric rv $M$ with probability mass function $P(M = i) = q(1-q)^{i-1}$ for $i = 1, 2, \ldots$. Thus, the review times $\{X_k; k \geq 1\}$ are defined as

$$X_k = \inf\{t : N_t = \sum_{i=1}^{k} M_i\},$$

i.e., the $k$-th review occurs at the $(\sum_{i=1}^{k} M_i)$-th claim occurrence. Define $T_M$ to be the generic inter-review time rv.

We assume that ruin can be detected only at the review times $\{X_k; k \geq 1\}$, i.e., same as the premium review times. With the newly defined $\{X_k; k \geq 1\}$, we can still use $\mathcal{U}_k$ defined in (3.1) as the new surplus process.

Following similar procedures as to those in Sections 3.2 and 3.3, we first need to examine the distribution of increments of the surplus process $\{U_t; t \geq 0\}$ over a geometric number of claims.

Define $Z_M = cT_M - \sum_{i=1}^{M} P_i$ to be the increment of the surplus process over a geometric number of claims. The two one-sided discounted densities of $Z_M$, namely, $g_+^{\delta,M}$ and $g_-^{\delta,M}$, defined respectively through their one-sided LTs as

$$\mathbb{E}\left[e^{-\delta T_M - sZ_M} 1_{\{Z_M > 0\}}\right] = \int_0^\infty e^{-sx} g_+^{\delta,M}(x)dx,$$

and

$$\mathbb{E}\left[e^{-\delta T_M - s(-Z_M)} 1_{\{Z_M < 0\}}\right] = \int_0^\infty e^{-sx} g_-^{\delta,M}(x)dx,$$

will be examined.

**Proposition 3.5.1.** *The discounted defective density of* $Z_M 1_{\{Z_M>0\}}$ *is*

$$g_+^{\delta,M}(x) = q\lambda\widetilde{p}(\rho_M)\Phi_M e^{-\rho_M x}, \tag{3.62}$$

*where* $\Phi_M = \frac{1}{c-(1-q)\lambda\mathcal{T}^2_{\rho_M}p(0)}$ *and* $\rho_M$ *is the positive solution to*

$$cz - \lambda\left(1-(1-q)\widetilde{p}(z)\right) - \delta = 0.$$

*Proof.* By conditioning on whichever of the $M$ or $N_{\tau_0^-}$ occurs first, we have

$$\begin{aligned}
\mathbb{E}\left[e^{-\delta T_M - sZ_M}1_{\{Z_M>0\}}\right] &= \mathbb{E}\left[e^{-\delta T_M - sZ_M}1_{\{Z_M>0\}}1_{\{M<N_{\tau_0^-}\}}\right] + \mathbb{E}\left[e^{-\delta T_M - sZ_M}1_{\{Z_M>0\}}1_{\{M>N_{\tau_0^-}\}}\right] \\
&= \psi^\delta_M(0) + \mathbb{E}\left[e^{-\delta T_M - sZ_M}1_{\{Z_M>0\}}1_{\{M>N_{\tau_0^-}\}}\right], \qquad (3.63)
\end{aligned}$$

where $N_{\tau_0^-}$ is the number of claims until ruin (including the claim causing ruin) and

$$\psi^\delta_M(u) = \mathbb{E}[e^{-\delta T_M - sU_{T_M}}1_{\{M<N_{\tau_0^-}\}}|U_0 = u].$$

Note that the event $\{M < N_{\tau_0^-}\} = \{U_1 > 0, \ldots, U_M > 0\}$.

Also, note that the geometric check is equivalent to having, for each claim occurred, a probability $q$ to review the surplus process. Thus, by conditioning on the time and amount of the first claim, one obtains

$$\begin{aligned}
\psi^\delta_M(u) &= \int_0^\infty \lambda e^{-(\lambda+\delta)t}\int_0^{u+ct} qe^{-s(u+ct-x)}p(x)dxdt \\
&\quad + \int_0^\infty \lambda e^{-(\lambda+\delta)t}\int_0^{u+ct}(1-q)\psi^\delta_M(u+ct-x)p(x)dxdt \\
&= \frac{q\lambda}{c}\int_u^\infty e^{-\frac{\lambda+\delta}{c}(y-u)}\int_0^y e^{-s(y-x)}p(x)dxdy + \frac{(1-q)\lambda}{c}\int_u^\infty e^{-\frac{\lambda+\delta}{c}(y-u)}\int_0^y \psi^\delta_M(y-x)p(x)dxdy.
\end{aligned} \tag{3.64}$$

Taking the LT on both sides of (3.64), we have

$$\widetilde{\psi}_M^\delta(z) \;=\; \frac{q\lambda}{c}\frac{\frac{1}{s+\xi}\widetilde{p}(\xi)-\frac{1}{s+z}\widetilde{p}(z)}{z-\xi}+\frac{(1-q)\lambda}{c}\frac{\widetilde{\psi}_M^\delta(\xi)\widetilde{p}(\xi)-\widetilde{\psi}_M^\delta(z)\widetilde{p}(z)}{z-\xi},$$

where $\xi=(\lambda+\delta)/c$, i.e.,

$$\{cz-\lambda\left(1-(1-q)\widetilde{p}(z)\right)-\delta\}\,\widetilde{\psi}_M^\delta(z)=\lambda\left\{q\frac{1}{s+\xi}\widetilde{p}(\xi)+(1-q)\widetilde{\psi}_M^\delta(\xi)\widetilde{p}(\xi)\right\}-q\lambda\frac{1}{s+z}\widetilde{p}(z). \tag{3.65}$$

The first term on the right-hand side of (3.65) does not depend on $z$, and by taking $z=\rho_M$, we can express it as

$$\lambda\left\{q\frac{1}{s+\xi}\widetilde{p}(\xi)+(1-q)\widetilde{\psi}_M^\delta(\xi)\widetilde{p}(\xi)\right\}=q\lambda\frac{1}{s+\rho_M}\widetilde{p}(\rho_M).$$

Let $v_M^\delta$ be a function defined through it LT

$$\widetilde{v}_M^\delta(z)=\frac{1}{cz-\lambda\left(1-(1-q)\widetilde{p}(z)\right)-\delta}.$$

Therefore, (3.65) can be represented as

$$\widetilde{\psi}_M^\delta(z) \;=\; q\lambda\widetilde{v}_M^\delta(z)\left\{\frac{1}{s+\rho_M}\widetilde{p}(\rho_M)-\frac{1}{s+z}\widetilde{p}(z)\right\}.$$

A LT inversion wrt $z$ yields

$$\psi_M^\delta(u)=q\lambda\left\{\frac{1}{s+\rho_M}\widetilde{p}(\rho_M)v_M^\delta(u)-\int_0^u e^{-sy}\int_0^{u-y}p(u-y-x)v_M^\delta(x)dxdy\right\},$$

and therefore

$$\psi_M^\delta(0)=q\lambda\frac{1}{s+\rho_M}\widetilde{p}(\rho_M)v_M^\delta(0)=\frac{q\lambda\widetilde{p}(\rho_M)}{c}\frac{1}{s+\rho_M}, \tag{3.66}$$

where $v_M^\delta(0)=1/c$ is easily seen from the initial value theorem of the LT.

For the second term on the right-hand side of (3.63), we shall first condition on the distribution of the deficit at ruin together with the event $\{N_{\tau_0^-} < M\}$. This corresponds to the discounted density of the deficit at ruin together with the number of claims before ruin, which is given by

$$\mathbb{E}\left[(1-q)^{N_{\tau_0^-}} e^{-\delta\tau_0^-} 1_{\{|U_{\tau_0^-}|\in(y,y+dy)\}}|U_0=0\right] = \frac{(1-q)\lambda}{c}\mathcal{T}_{\rho_M}p(y)dy. \tag{3.67}$$

From a deficit of $y$, the skip-free upward surplus process must then return to level 0 before the geometric number of claims, which is given in Landriault and Shi (2014, Equation 5),

$$\mathbb{E}\left[(1-q)^{N_{\tau_0^+}} e^{-\delta\tau_0^+} 1_{\{\tau_0^+<\infty\}}|U_0=-y\right] = e^{-\rho_M y}.$$

The process restarts at this return time to 0 by the strong Markov property. Thus

$$\begin{aligned}&\mathbb{E}\left[e^{-\delta T_M - sZ_M}1_{\{Z_M>0\}}1_{\{M>N_{\tau_0^-}\}}\right]\\ &= \int_0^\infty \frac{(1-q)\lambda}{c}\mathcal{T}_{\rho_M}p(y)e^{-\rho_M y}\mathbb{E}\left[e^{-\delta T_M - sZ_M}1_{\{Z_M>0\}}\right]dy.\end{aligned} \tag{3.68}$$

Substituting (3.66) and (3.68) back into (3.63) yields

$$\mathbb{E}\left[e^{-\delta T_M - sZ_M}1_{\{Z_M>0\}}\right] = \frac{q\lambda\widetilde{p}(\rho_M)}{c}\frac{1}{s+\rho_M}+\int_0^\infty \frac{(1-q)\lambda}{c}\mathcal{T}_{\rho_M}p(y)e^{-\rho_M y}dy\mathbb{E}\left[e^{-\delta T_M - sZ_M}1_{\{Z_M>0\}}\right],$$

which implies that

$$\mathbb{E}\left[e^{-\delta T_M - sZ_M}1_{\{Z_M>0\}}\right] = \frac{q\lambda\widetilde{p}(\rho_M)}{c-(1-q)\lambda\mathcal{T}^2_{\rho_M}p(0)}\frac{1}{s+\rho_M}. \tag{3.69}$$

The LT inversion of (3.69) wrt $s$ yields (3.62). □

**Proposition 3.5.2.** *The discounted defective density of $-Z_M 1_{\{Z_M<0\}}$ is*

$$g_-^{\delta,M}(x) = q\lambda\widetilde{p}(\rho_M)\Phi_M e^{\rho_M x} - q\lambda\int_0^x v_M^\delta(x-y)p(y)dy. \tag{3.70}$$

*Proof.* By conditioning on the first time the surplus process drops below 0 (before $M$ or at $M$), and on whichever of $M$ or $\tau_0^+$ occurs first, we have

$$\begin{aligned}\mathbb{E}\left[e^{-\delta T_M - s(-Z_M)} 1_{\{Z_M<0\}}\right] = \int_0^\infty \frac{(1-q)\lambda}{c} \mathcal{T}_{\rho_M} p(y) \left\{e^{-\rho_M y} \mathbb{E}\left[e^{-\delta T_M - s(-Z_M)} 1_{\{Z_M<0\}}\right] + \phi_M^\delta(y)\right\} dy \\ + \mathbb{E}\left[e^{-\delta T_M - s(-Z_M)} 1_{\{M=N_{\tau_0^-}\}}\right], \qquad (3.71)\end{aligned}$$

where

$$\phi_M^\delta(y) = \mathbb{E}[e^{-\delta T_M - s(-U_{T_M})} 1_{\{M \le N_{\tau_0^+}\}} | U_0 = -y].$$

To obtain an explicit expression for $\phi_M^\delta$, by reflection, we obtain

$$\phi_M^\delta(y) = \mathbb{E}\left[e^{-\delta T_M - s R_{T_M}} 1_{\{M \le \ N_{\tau_0^{*-}}\}} | R_0 = y\right],$$

where $R_t$ is the dual risk model. By conditioning on whether $M \le N_{\tau_\epsilon^{*-}}$ or not, it is easy to see

$$\phi_M^\delta(y) = e^{-s\epsilon} \phi_M^\delta(y-\epsilon) + e^{-\rho_M(y-\epsilon)} \phi_M^\delta(\epsilon),$$

for all $\epsilon \in [0, y]$. Integrating over $\epsilon$ from 0 to $y$, it follows that

$$y\phi_M^\delta(y) = \int_0^\infty e^{-s\epsilon} \phi_M^\delta(y-\epsilon) d\epsilon + \int_0^\infty e^{-\rho_M(y-\epsilon)} \phi_M^\delta(\epsilon) d\epsilon,$$

whose LT is

$$\frac{d}{dz} \widetilde{\phi}_M^\delta(z) = \int_0^\infty e^{-sy} y \phi_M^\delta(y) dy = \left(\frac{1}{s+z} + \frac{1}{\rho_M + z}\right) \widetilde{\phi}_M^\delta(z).$$

Thus, solving this ordinary differential equation followed by a LT inversion yields

$$\phi_M^\delta(y) = C_M(s)(e^{-sy} - e^{-\rho_M y}), \qquad (3.72)$$

where $C_M(s)$ is a constant involving $s$.

To identify $C_M(s)$, we condition on the time and amount of the the first jump, i.e.,

$$\begin{aligned}
\phi_M^\delta(y) =&(1-q)\int_0^{y/c}\lambda e^{-(\lambda+\delta)t}\left\{\int_0^\infty \phi_M^\delta(y-ct+x)p(x)\right\}dt + q\int_0^{y/c}\lambda e^{-(\lambda+\delta)t}\int_0^\infty e^{-s(y-ct+x)}p(x)dxdt\\
=&(1-q)C_M(s)\left\{\frac{\lambda}{\lambda+\delta-cs}\widetilde{p}(s)(e^{-sy}-e^{-(\lambda+\delta)y/c})-\frac{\lambda}{\lambda+\delta-c\rho_M}\widetilde{p}(\rho_M)(e^{-\rho_M y}-e^{-(\lambda+\delta)y/c})\right\}\\
&+q\frac{\lambda\widetilde{p}(s)}{\lambda+\delta-cs}(e^{-sy}-e^{-(\lambda+\delta)y/c})\\
=&\left\{(1-q)C_M(s)\frac{\lambda}{\lambda+\delta-cs}\widetilde{p}(s)+q\frac{\lambda\widetilde{p}(s)}{\lambda+\delta-cs}\right\}e^{-sy}-C_M(s)e^{-\rho_M y}\\
&-\left\{(1-q)C_M(s)\frac{\lambda}{\lambda+\delta-cs}\widetilde{p}(s)-C_M(s)+q\frac{\lambda\widetilde{p}(s)}{\lambda+\delta-cs}\right\}e^{-(\lambda+\delta)y/c}.
\end{aligned}$$

Matching the coefficients of $e^{-sy}$, we get

$$C_M(s)=\frac{q\lambda\widetilde{p}(s)}{\lambda+\delta-cs-(1-q)\lambda\widetilde{p}(s)}=-q\lambda\widetilde{p}(s)\widetilde{v}_M^\delta(s). \tag{3.73}$$

Thus, substituting (3.73) back into (3.72) yields

$$\phi_M^\delta(y)=q\lambda\widetilde{p}(s)\widetilde{v}_M^\delta(s)(e^{-\rho_M y}-e^{-sy}).$$

Therefore, one can calculate

$$\begin{aligned}
\int_0^\infty \frac{(1-q)\lambda}{c}\mathcal{T}_{\rho_M}p(y)\phi_M^\delta(y)dy =& \int_0^\infty \frac{(1-q)\lambda}{c}\mathcal{T}_{\rho_M}p(y)\left\{q\lambda\widetilde{p}(s)\widetilde{v}_M^\delta(s)(e^{-\rho_M y}-e^{-sy})\right\}dy\\
=&q\lambda\widetilde{p}(s)\widetilde{v}_M^\delta(s)\left\{\frac{(1-q)\lambda}{c}\mathcal{T}^2_{\rho_M}p(0)-1\right\}+q\lambda\widetilde{p}(s)\frac{1}{c(s-\rho_M)}.
\end{aligned} \tag{3.74}$$

Also, using (3.67), we have

$$\mathbb{E}\left[e^{-\delta T_M-s(-Z_M)}1_{\{M=N_{\tau_0^-}\}}\right]=q\mathbb{E}\left[(1-q)^{N_{\tau_0^-}-1}e^{-\delta\tau_0^- -s|U_{\tau_0^-}|}\right]=\frac{q\lambda}{c}\frac{\widetilde{p}(s)-\widetilde{p}(\rho_M)}{\rho_M-s}. \tag{3.75}$$

Finally, substituting (3.74) and (3.75) back into Equation (3.71), it becomes

$$
\begin{aligned}
&\mathbb{E}\left[e^{-\delta T_M - s(-Z_M)} 1_{\{Z_M<0\}}\right] \\
=&\int_0^\infty \frac{(1-q)\lambda}{c} \mathcal{T}_{\rho_M} p(y) \left\{ e^{-\rho_M y} \mathbb{E}\left[e^{-\delta T_M - s(-Z_M)} 1_{\{Z_M<0\}}\right] + \phi_M^\delta(y) \right\} dy + \frac{q\lambda}{c} \frac{\widetilde{p}(s) - \widetilde{p}(\rho_M)}{\rho_M - s} \\
=&\frac{(1-q)\lambda}{c} \mathcal{T}_{\rho_M}^2 p(0) \mathbb{E}\left[e^{-\delta T_M - s(-Z_M)} 1_{\{Z_M<0\}}\right] + q\lambda \widetilde{p}(s) \widetilde{v}_M^\delta(s) \left\{ \frac{(1-q)\lambda}{c} \mathcal{T}_{\rho_M}^2 p(0) - 1 \right\} + \frac{q\lambda \widetilde{p}(\rho_M)}{c(s-\rho_M)},
\end{aligned}
$$

which implies

$$
\mathbb{E}\left[e^{-\delta T_M - s(-Z_M)} 1_{\{Z_M<0\}}\right] = \frac{q\lambda \Phi_M \widetilde{p}(\rho_M)}{s - \rho_M} - q\lambda \widetilde{p}(s) \widetilde{v}_M^\delta(s). \tag{3.76}
$$

The inversion of (3.76) wrt s yields the discounted density (3.70).

□

With the discounted defective densities $g_+^{\delta,M}(x)$ and $g_-^{\delta,M}(x)$ given in (3.62) and (3.70), we can follow the same procedures in Section 3.3 to derive the matrix-form defective renewal equation for the Gerber-Shiu function and the discounted joint densities of the surplus prior to ruin and deficit at ruin.

# Chapter 4

# Drawdown-based regime-switching Lévy insurance model

## 4.1 Introduction

In this chapter, we propose and analyze a new drawdown-based regime-switching (DBRS) Lévy insurance model. For completeness, we recall the drawdown-related quantities defined in Section 1.3.2. For an insurance surplus process $X = \{X_t; t \geq 0\}$ defined on a filtered probability space $(\Omega, \mathcal{F}, \boldsymbol{F} = \{\mathcal{F}_t; t \geq 0\}, \mathbb{P})$ satisfying the usual conditions, its drawdown process $Y = \{Y_t; t \geq 0\}$ is defined as

$$Y_t = M_t - X_t,$$

where $M_t = \sup_{0 \leq s \leq t} X_s$ is the running maximum of $X$ at time $t$. For a fixed level $a$, the drawdown time is

$$\tau_a = \inf \{t \geq 0 : Y_t > a\}.$$

In this chapter, the level of an insurer's financial distress is measured by the drawdown size of the surplus process $X$. We use an auxiliary stochastic process $Q = \{Q_t; t \geq 0\}$ to describe the dynamic of the DBRS model between two regimes: the "non-distressed" regime ($Q_t = 1$) and the "distressed" regime ($Q_t = 2$). Here, the drawdown size triggering a distressed period is modeled by a constant $a > 0$. The end of the distressed period corresponds to the time the running maximum of $X$ is recovered. Hence, for $t \geq 0$, we define

$$Q_t = \begin{cases} 1, & \text{if } \sup_{l_t \leq s \leq t} Y_s < a, \\ 2, & \text{if } \sup_{l_t \leq s \leq t} Y_s \geq a, \end{cases}$$

where $l_t = \sup\{s \leq t : Y_s = 0\}$ is the last time the process $X$ is at its running maximum prior to or at time $t$. Note that $Q$ is not a Markov process since its transition rates are path-dependent.

We consider the DBRS Lévy insurance model with dynamics

$$\mathrm{d}X_t = \begin{cases} \mathrm{d}X_t^1, & \text{if } Q_t = 1, \\ \mathrm{d}X_t^2, & \text{if } Q_t = 2, \end{cases} \tag{4.1}$$

and initial surplus $X_0 = u > 0$. Here, $X^1$ and $X^2$ are two spectrally negative Lévy processes defined on $(\Omega, \mathcal{F}, \boldsymbol{F}, \mathbb{P})$. We exclude cases where $X^1$ or $X^2$ have monotone sample paths. Also, we assume the Lévy measures of $X^1$ and $X^2$ have no atoms.

The dynamic of the proposed DBRS model (4.1) can be interpreted as follows: with $Y_0 = 0$, the process stays in the non-distressed regime ($Q_t = 1$) until the drawdown time $\tau_a$. At that moment, the process enters the distressed regime ($Q_t = 2$) and eventually exits when the running maximum of $X$ is recovered. Figure 4.1 displays a sample path of $X$ to illustrate the dynamics of the DBRS risk process.

Here a natural question arises: *how does the DBRS risk model help characterize the real insurance business cycle?* We answer this question as follows. A significant drawdown in insurance surplus can be caused by various exogenous and endogenous risk factors: the occurrence of a natural catastrophe, a period of high claim frequency, a financial market

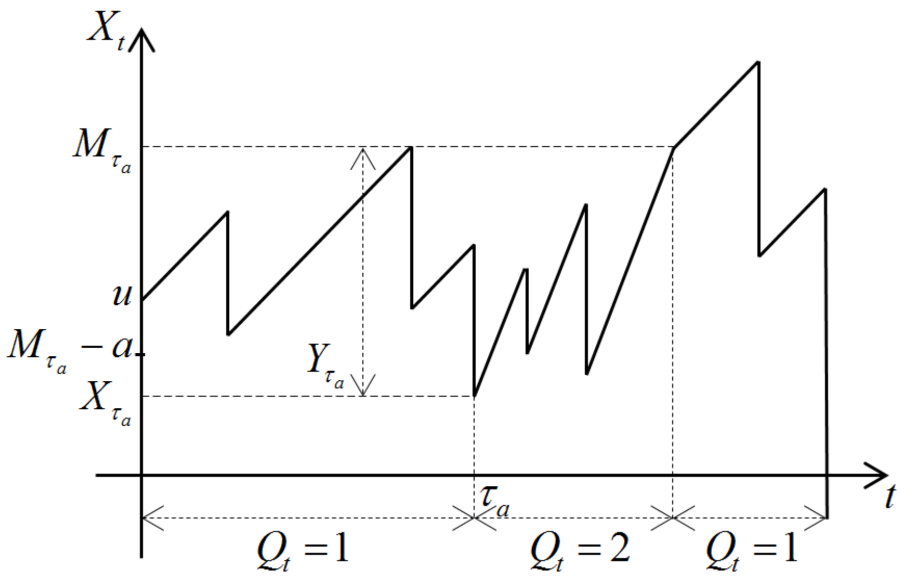


Figure 4.1: A sample path of the DBRS risk process $X$

crash, or suboptimal investment strategies, among others.

An advantage of using drawdown as a risk indicator is that *timely warnings* may be given to the insurer before a capital shortfall occurs. To resolve the financial distress, an insurer will likely go through a revision of its overall activities including its capital structure, insurance policies, investment strategies, as well as others. However, such adjustments may not be optimal in the long run (from a business standpoint) and shall preferably be adjusted back once the financial distress is resolved. For instance, a disaster will cause a large surplus drawdown due to the high volume of claims handled by the insurer. The insurer may have to charge higher premium or undersell financial investment products to honor its financial obligations. These reactions are usually not optimal in the long run. A higher premium will motivate policyholders to surrender and a lack of financial investment may result in the insurer missing the usual stock market rebound after a disaster. Therefore, in the DBRS model (4.1), we assume that all changes will be reverted back once the previous running maximum is recovered which can be viewed as the end of a business cycle.

The rest of the chapter is organized as follows. In Section 4.2, we review some preliminary results for the spectrally negative Lévy process and its drawdown related quantities. In

Section 4.3, we derive a generalized version of the two-sided exit problem for the DBRS model (4.1), and utilize this result in Section 4.4 to further examine conditions under which the survival probability of model (4.1) is not trivially zero. In Section 4.5, we study the regime-dependent occupation time in the DBRS risk model. In Section 4.6, we investigate a special jump diffusion model with regime-switching premium and build connections with other existing risk models in the literature. Most results in this chapter have already been published in Landriault, Li and Li (2015a).

## 4.2 Preliminaries

For ease of notation, we will adopt the following conventions throughout the chapter:

1. We use superscript 1 or 2 to distinguish quantities related to $X^1$ and $X^2$, respectively. Also, the superscript will be dropped for quantities related to $X$.
2. We write $\int_x^y \cdot \mathrm{d}z$ for an integral on the open interval $z \in (x, y)$ with $-\infty \le x < y \le \infty$.

For $k = 1, 2$, the Laplace exponent of $X^k$ is given by

$$\psi_k(s) = d_k s + \frac{1}{2}\sigma_k^2 s^2 + \int_{-\infty}^{0} (e^{sx} - 1 - sx1_{\{x>-1\}})\Pi_k(\mathrm{d}x),$$

where the triple $(d_k, \sigma_k, \Pi_k)$ fully characterizes the spectrally negative Lévy process. In the following, assuming $\int_{(-1,0)} |x|\Pi_k(\mathrm{d}x) < \infty$, we will use $c_k = d_k + \int_{(-1,0)} |x|\Pi_k(\mathrm{d}x)$ to denote the premium rates. For any given $q \ge 0$, the scale functions for $X^k$ are denoted as $W_k^{(q)}$ and $Z_k^{(q)}$. In the sequel, we write $W_k(\cdot)$ for $W_k^{(0)}(\cdot)$. For $x \in \mathbb{R}$, we define the first passage times of $X$ as

$$T_x^{+(-)} = \inf\{t \ge 0 : X_t > (<) x\}.$$

The first passage times of $X^k$ are similarly defined. For completeness, we recall the following well-known fluctuation identities given in Theorem 1.4.3.

**Proposition 4.2.1.** *For $k = 1, 2$, $q \geq 0$ and $0 < u < x$,*

$$\mathbb{E}_u[e^{-qT_x^{k,+}} 1_{\{T_x^{k,+} < T_0^{k,-}\}}] = \frac{W_k^{(q)}(u)}{W_k^{(q)}(x)}, \tag{4.2}$$

*and*

$$\mathbb{E}_u[e^{-qT_0^{k,-}} 1_{\{T_0^{k,-} < T_x^{k,+}\}}] = Z_k^{(q)}(u) - Z_k^{(q)}(x)\frac{W_k^{(q)}(u)}{W_k^{(q)}(x)}. \tag{4.3}$$

For drawdown estimates, we also recall Theorem 1 of Mijatovic and Pistorius (2012) and Theorem 2.1 of Landriault et al. (2014) where the joint law of $(\tau_a, M_{\tau_a}, Y_{\tau_a})$ is given.

**Theorem 4.2.1.** *For $q, x \geq 0$ and $y \geq a$,*

$$\mathbb{E}[e^{-q\tau_a} 1_{\{Y_{\tau_a} \in \mathrm{d}y, M_{\tau_a} \in \mathrm{d}x\}}] = \frac{W_1^{(q)\prime}(a)}{W_1^{(q)}(a)} e^{-\frac{W_1^{(q)\prime}(a)}{W_1^{(q)}(a)}x} \mathrm{d}x F_{Y_{\tau_a}}^{(q)}(\mathrm{d}y), \tag{4.4}$$

*where*

$$\begin{aligned} F_{Y_{\tau_a}}^{(q)}(\mathrm{d}y) &:= \mathbb{E}[e^{-q\tau_a} 1_{\{Y_{\tau_a} \in \mathrm{d}y\}}] \\ &= \int_0^a \left(\frac{W_1^{(q)}(a)}{W_1^{(q)\prime}(a)} W_1^{(q)\prime}(z) - W_1^{(q)}(z)\right)\Pi_1(z - \mathrm{d}y)\mathrm{d}z \\ &+ \frac{W_1^{(q)}(a)}{W_1^{(q)\prime}(a)} W_1^{(q)}(0+)\Pi_1(-\mathrm{d}y) + \frac{\sigma_1^2}{2}\left(W_1^{(q)\prime}(a) - \frac{W_1^{(q)}(a)}{W_1^{(q)\prime}(a)} W_1^{(q)\prime\prime}(a)\right)\delta_a(\mathrm{d}y), \end{aligned} \tag{4.5}$$

*and $\delta_a(\cdot)$ is the Dirac delta function with mass point at $a$.*

For ease of notation, we write $\overline{F}_{Y_{\tau_a}}^{(q)}(y) := \mathbb{E}[e^{-q\tau_a} 1_{\{Y_{\tau_a} > y\}}]$ and $F_{Y_{\tau_a}}(\cdot) := F_{Y_{\tau_a}}^{(0)}(\cdot)$. In particular, we have

$$\mathbb{E}[e^{-q\tau_a}] = Z_1^{(q)}(a) - q\frac{W_1^{(q)}(a)^2}{W_1^{(q)\prime}(a)}, \tag{4.6}$$

which can be obtained from Theorem 1 of Avram et al. (2004). By Theorem 4.2.1, it is easy to conclude that $M_{\tau_a}$ is exponentially distributed with mean $W_1(a)/W_1'(a)$. Moreover, Proposition 2.1 of Landriault et al. (2014) gives the following result.

**Proposition 4.2.2.** *For $q, x \geq 0$,*

$$\mathbb{E}[e^{-qT_x^+} 1_{\{M_{\tau_a} > x\}}] = e^{-\frac{W_1^{(q)\prime}(a)}{W_1^{(q)}(a)} x}. \tag{4.7}$$

## 4.3 Generalized two-sided exit problem

In this section, we study a generalization of the two-sided exit problem of Proposition 4.2.1. More specifically, for $s, q \geq 0$ and $0 < u < b$, we consider the following quantities:

$$\mathbb{E}_u[e^{-s\mathcal{T}_{b,1}^+ - q\mathcal{T}_{b,2}^+} 1_{\{T_b^+ < T_0^-\}}] \quad \text{and} \quad \mathbb{E}_u[e^{-s\mathcal{T}_{0,1}^- - q\mathcal{T}_{0,2}^-} 1_{\{T_0^- < T_b^+\}}],$$

where $\mathcal{T}_{x,k}^{+(-)} := \int_0^{T_x^{+(-)}} 1_{\{Q_t = k\}} \mathrm{d}t$ is the occupation time in regime $k$ ($k = 1, 2$) until the first passage time $T_x^{+(-)}$. Naturally, $T_x^{+(-)} = \mathcal{T}_{x,1}^{+(-)} + \mathcal{T}_{x,2}^{+(-)}$.

We define generalizations of the first and second scale functions, namely

$$\boldsymbol{W}_a^{(s,q)}(x) := \frac{W_1^{(s)}(x) W_1^{(s)}(a)}{W_1^{(s)}(x \vee a)} e^{\int_a^{x \vee a} C_{s,q}(z) \mathrm{d}z}, \tag{4.8}$$

and

$$\begin{aligned} \boldsymbol{Z}_a^{(s,q)}(x) := {} & Z_1^{(s)}(x) - \frac{W_1^{(s)}(x)}{W_1^{(s)}(x \vee a)} (Z_1^{(s)}(x \vee a) - Z_1^{(s)}(a) e^{\int_a^{x \vee a} C_{s,q}(z) \mathrm{d}z}) \\ & - \frac{W_1^{(s)}(x)}{W_1^{(s)}(x \vee a)} \int_a^{x \vee a} e^{\int_y^{x \vee a} C_{s,q}(z) \mathrm{d}z} D_{s,q}(y) \mathrm{d}y, \end{aligned} \tag{4.9}$$

respectively, for $s, q \geq 0$ and $x \in \mathbb{R}$, where $x \vee a = \max\{x, a\}$. Here, for $z > a$,

$$C_{s,q}(z) := \frac{W_1^{(s)\prime}(a)}{W_1^{(s)}(a)} (1 - \int_{[a,z)} \frac{W_2^{(q)}(z - y)}{W_2^{(q)}(z)} F_{Y_{\tau_a}}^{(s)}(\mathrm{d}y)), \tag{4.10}$$

and

$$D_{s,q}(z) := \frac{W_1^{(s)\prime}(a)}{W_1^{(s)}(a)} \int_{[a,\infty)} (Z_2^{(q)}(z - y) - Z_2^{(q)}(z) \frac{W_2^{(q)}(z - y)}{W_2^{(q)}(z)}) F_{Y_{\tau_a}}^{(s)}(\mathrm{d}y).$$

Note that $\boldsymbol{W}_a^{(s,q)}(\cdot)$ and $\boldsymbol{Z}_a^{(s,q)}(\cdot)$ reduce to the classical scale functions $W_1^{(q)}(\cdot)$ and $Z_1^{(q)}(\cdot)$, respectively, in certain cases (see Remark 4.3.1 for more details). Briefly, we write $\boldsymbol{W}_a^{(q)}(\cdot) = \boldsymbol{W}_a^{(q,q)}(\cdot)$, $\boldsymbol{Z}_a^{(q)}(\cdot) = \boldsymbol{Z}_a^{(q,q)}(\cdot)$, $\boldsymbol{W}_a(\cdot) = \boldsymbol{W}_a^{(0)}(\cdot)$, $\boldsymbol{Z}_a(\cdot) = \boldsymbol{Z}_a^{(0)}(\cdot)$, $C_q(\cdot) = C_{q,q}(\cdot)$ and $D_q(\cdot) = D_{q,q}(\cdot)$.

**Theorem 4.3.1.** *For $s, q \geq 0$ and $0 < u < b$,*

$$\mathbb{E}_u[e^{-s\mathcal{T}_{b,1}^{+}-q\mathcal{T}_{b,2}^{+}}1_{\{T_b^+<T_0^-\}}] = \frac{\boldsymbol{W}_a^{(s,q)}(u)}{\boldsymbol{W}_a^{(s,q)}(b)}. \tag{4.11}$$

*Proof.* For ease of notation, let $g(u) := \mathbb{E}_u[e^{-s\mathcal{T}_{b,1}^{+}-q\mathcal{T}_{b,2}^{+}}1_{\{T_b^+<T_0^-\}}]$ for $0 < u < b$. We first consider the case $a \leq u < b$. By the strong Markov property, (4.2), (4.4), and (4.7), we have

$$\begin{aligned}
g(u) &= \mathbb{E}_u[e^{-s\mathcal{T}_{b,1}^{+}-q\mathcal{T}_{b,2}^{+}}1_{\{\tau_a<T_b^+<T_0^-\}}] + \mathbb{E}_u[e^{-s\mathcal{T}_{b,1}^{+}-q\mathcal{T}_{b,2}^{+}}1_{\{T_b^+<\tau_a\}}] \\
&= \int_u^b \int_{[a,x)} \mathbb{E}_u[e^{-s\tau_a}1_{\{Y_{\tau_a}\in \mathrm{d}y, M_{\tau_a}\in \mathrm{d}x\}}]\mathbb{E}_{x-y}[e^{-qT_x^{2,+}}1_{\{T_x^{2,+}<T_0^{2,-}\}}]g(x) + \mathbb{E}_u[e^{-sT_b^+}1_{\{M_{\tau_a}>b\}}] \\
&= \frac{W_1^{(s)\prime}(a)}{W_1^{(s)}(a)}\int_u^b g(x)e^{-\frac{W_1^{(s)\prime}(a)}{W_1^{(s)}(a)}(x-u)}\int_{[a,x)}\frac{W_2^{(q)}(x-y)}{W_2^{(q)}(x)}F_{Y_{\tau_a}}^{(s)}(\mathrm{d}y)\mathrm{d}x + e^{-\frac{W_1^{(s)\prime}(a)}{W_1^{(s)}(a)}(b-u)}.
\end{aligned} \tag{4.12}$$

Differentiating (4.12) wrt $u$ and then utilizing (4.10), we obtain

$$g'(u) = \frac{W_1^{(s)\prime}(a)}{W_1^{(s)}(a)}(1 - \int_{[a,u)}\frac{W_2^{(q)}(u-y)}{W_2^{(q)}(u)}F_{Y_{\tau_a}}^{(s)}(\mathrm{d}y))g(u) = C_{s,q}(u)g(u). \tag{4.13}$$

The solution to the ordinary differential equation (ODE) (4.13) with boundary condition $\lim_{u\uparrow b} g(u) = 1$ is

$$g(u) = e^{-\int_u^b C_{s,q}(z)\mathrm{d}z}, \ a \leq u < b. \tag{4.14}$$

For $u < a < b$, by the strong Markov property, (4.2) and (4.14), it follows that

$$g(u) = \mathbb{E}_u[e^{-sT_a^{1,+}}1_{\{T_a^{1,+}<T_0^{1,-}\}}]g(a) = \frac{W_1^{(s)}(u)}{W_1^{(s)}(a)}e^{-\int_a^b C_{s,q}(z)\mathrm{d}z}.$$

Finally, for $u < b \leq a$, by (4.2),

$$g(u) = \mathbb{E}_u[e^{-sT_b^{1,+}} 1_{\{T_b^{1,+} < T_0^{1,-}\}}] = \frac{W_1^{(s)}(u)}{W_1^{(s)}(b)}.$$

From the definition of $\boldsymbol{W}_a^{(s,q)}(\cdot)$ in (4.8), it is straightforward to check that the expressions of $g(u)$ in the above three cases can be unified to the representation (4.11). $\square$

Now we turn to the problem of exiting from below.

**Theorem 4.3.2.** *For $s, q \geq 0$ and $0 < u < b$,*

$$\mathbb{E}_u[e^{-s\mathcal{T}_{0,1}^- - q\mathcal{T}_{0,2}^-} 1_{\{T_0^- < T_b^+\}}] = \boldsymbol{Z}_a^{(s,q)}(u) - \boldsymbol{Z}_a^{(s,q)}(b) \frac{\boldsymbol{W}_a^{(s,q)}(u)}{\boldsymbol{W}_a^{(s,q)}(b)}. \tag{4.15}$$

*Proof.* This theorem is proved in a similar fashion as Theorem 4.3.1. Let

$$h(u) := \mathbb{E}_u[e^{-s\mathcal{T}_{0,1}^- - q\mathcal{T}_{0,2}^-} 1_{\{T_0^- < T_b^+\}}].$$

We first consider the case $a \leq u < b$. By conditioning on the distributional properties of the first drawdown and on whether the process $X$ recovers (or not) its running maximum before $T_0^-$, we have

$$\begin{aligned} h(u) = &\int_u^b \int_{[a,x)} \mathbb{E}_u[e^{-s\tau_a} 1_{\{Y_{\tau_a} \in \mathrm{d}y, M_{\tau_a} \in \mathrm{d}x\}}] \mathbb{E}_{x-y}[e^{-qT_x^{2,+}} 1_{\{T_x^{2,+} < T_0^{2,-}\}}] h(x) \\ &+ \int_u^b \int_{[a,\infty)} \mathbb{E}_u[e^{-s\tau_a} 1_{\{Y_{\tau_a} \in \mathrm{d}y, M_{\tau_a} \in \mathrm{d}x\}}] \mathbb{E}_{x-y}[e^{-qT_0^{2,-}} 1_{\{T_0^{2,-} < T_x^{2,+}\}}], \end{aligned}$$

where $\mathbb{E}_{x-y}[e^{-qT_0^{2,-}} 1_{\{T_0^{2,-} < T_b^{2,+}\}}] = 1$ when $y > x$. Furthermore, by substituting (4.2)-(4.4) into the above equation, we have

$$h(u) = \frac{W_1^{(s)\prime}(a)}{W_1^{(s)}(a)} \int_u^b h(x) e^{-\frac{W_1^{(s)\prime}(a)}{W_1^{(s)}(a)}(x-u)} \mathrm{d}x \int_{[a,x)} \frac{W_2^{(q)}(x-y)}{W_2^{(q)}(x)} F_{Y_{\tau_a}}^{(s)}(\mathrm{d}y) + \int_u^b e^{-\frac{W_1^{(s)\prime}(a)}{W_1^{(s)}(a)}(x-u)} D_{s,q}(x) \mathrm{d}x. \tag{4.16}$$

Taking the derivative of (4.16) wrt $u$ yields the ODE

$$h'(u) = C_{s,q}(u)h(u) - D_{s,q}(u),$$

whose solution is

$$h(u) = \int_u^b e^{-\int_u^y C_{s,q}(z)\mathrm{d}z} D_{s,q}(y)\mathrm{d}y, \quad a \leq u < b, \tag{4.17}$$

using the boundary condition $\lim_{u\uparrow b} h(u) = 0$.

For $u < a < b$, by conditioning on whether $T_a^+$ or $T_0^-$ occurs first and using (4.17), we obtain

$$\begin{aligned}
h(u) &= \mathbb{E}_u[e^{-s\mathcal{T}_{0,1}^- - q\mathcal{T}_{0,2}^-}1_{\{T_0^- < T_a^+\}}] + \mathbb{E}_u[e^{-s\mathcal{T}_{0,1}^- - q\mathcal{T}_{0,2}^-}1_{\{T_a^+ < T_0^- < T_b^+\}}] \\
&= \mathbb{E}_u[e^{-sT_0^{1,-}}1_{\{T_0^{1,-} < T_a^{1,+}\}}] + \mathbb{E}_u[e^{-sT_a^{1,+}}1_{\{T_a^{1,+} < T_0^{1,-}\}}]h(a) \\
&= Z_1^{(s)}(u) - Z_1^{(s)}(a)\frac{W_1^{(s)}(u)}{W_1^{(s)}(a)} + \frac{W_1^{(s)}(u)}{W_1^{(s)}(a)}\int_a^b e^{-\int_a^y C_{s,q}(z)\mathrm{d}z} D_{s,q}(y)\mathrm{d}y,
\end{aligned}$$

Finally, for $u \leq b < a$, we immediately obtain

$$h(u) = \mathbb{E}_u[e^{-sT_0^{1,-}}1_{\{T_0^{1,-} < T_b^{1,+}\}}] = Z_1^{(s)}(u) - Z_1^{(s)}(b)\frac{W_1^{(s)}(u)}{W_1^{(s)}(b)}.$$

The expression of $h(u)$ in the above three cases can be unified to the representation (4.15). $\square$

When $s = q$ in (4.11) and (4.15), we obtain the following formulas for the two-sided exit problem of $X$, that is

$$\begin{aligned}
\mathbb{E}_u[e^{-qT_b^+}1_{\{T_b^+ < T_0^-\}}] &= \frac{\boldsymbol{W}_a^{(q)}(u)}{\boldsymbol{W}_a^{(q)}(b)}, \\
\mathbb{E}_u[e^{-qT_0^-}1_{\{T_0^- < T_b^+\}}] &= \boldsymbol{Z}_a^{(q)}(u) - \boldsymbol{Z}_a^{(q)}(b)\frac{\boldsymbol{W}_a^{(q)}(u)}{\boldsymbol{W}_a^{(q)}(b)}.
\end{aligned} \tag{4.18}$$

**Corollary 4.3.1.** *The DBRS model reduces to the spectrally negative Lévy model $X^1$ as $a \uparrow \infty$ or $(d_1, \sigma_1, \Pi_1) = (d_2, \sigma_2, \Pi_2)$.*

*Proof.* It is clear that $\boldsymbol{W}_a^{(q)}(z) = W_1^{(q)}(z)$ and $\boldsymbol{Z}_a^{(q)}(z) = Z_1^{(q)}(z)$ when $a > z$. This is also true when $(d_1, \sigma_1, \Pi_1) = (d_2, \sigma_2, \Pi_2)$ by making use of the following two identities (4.19) and (4.20).

For $z > a$,

$$W_1^{(q)}(z) - \int_{[a,z)} W_1^{(q)}(z-y) F_{Y_{\tau_a}}^{(q)}(\mathrm{d}y) = \frac{W_1^{(q)}(a)}{W_1^{(q)\prime}(a)} W_1^{(q)\prime}(z), \tag{4.19}$$

and

$$\int_{[a,\infty)} Z_1^{(q)}(z-y) F_{Y_{\tau_a}}^{(q)}(\mathrm{d}y) = Z_1^{(q)}(z) - \frac{W_1^{(q)}(a)}{W_1^{(q)\prime}(a)} Z_1^{(q)\prime}(z). \tag{4.20}$$

We prove identity (4.19) by a Laplace transform argument. For $s \geq 0$ and $z > a$, the Laplace transform of the left-hand side of (4.19) is

$$\int_a^\infty e^{-sz}(W_1^{(q)}(z) - \int_{[a,z)} W_1^{(q)}(z-y) F_{Y_{\tau_a}}^{(q)}(\mathrm{d}y))\mathrm{d}z = \int_a^\infty e^{-sz} W_1^{(q)}(z)\mathrm{d}z + \frac{1}{q-\psi_1(s)} \mathbb{E}[e^{-q\tau_a - sY_{\tau_a}}]. \tag{4.21}$$

By Theorem 1 of Avram et al. (2004), we have

$$\mathbb{E}[e^{-q\tau_a - sY_{\tau_a}}] = (1 - s\frac{W_1^{(q)}(a)}{W_1^{(q)\prime}(a)})(1 + (q-\psi_1(s)) \int_0^a e^{-sz} W_1^{(q)}(z)\mathrm{d}z) - (q-\psi_1(s)) e^{-as} \frac{W_1^{(q)}(a)^2}{W_1^{(q)\prime}(a)}. \tag{4.22}$$

Substituting (4.22) into (4.21), some algebraic simplifications result in

$$\begin{aligned}
&\int_a^\infty e^{-sz}(W_1^{(q)}(z) - \int_{[a,z)} W_1^{(q)}(z-y) F_{Y_{\tau_a}}^{(q)}(\mathrm{d}y))\mathrm{d}z \\
=& s\frac{W_1^{(q)}(a)}{W_1^{(q)\prime}(a)} \int_a^\infty e^{-sz} W_1^{(q)}(z)\mathrm{d}z - e^{-as}\frac{W_1^{(q)}(a)^2}{W_1^{(q)\prime}(a)} \\
=& \int_a^\infty e^{-sz} \frac{W_1^{(q)}(a)}{W_1^{(q)\prime}(a)} W_1^{(q)\prime}(z)\mathrm{d}z.
\end{aligned}$$

By the uniqueness of Laplace transforms, identity (4.19) emerges!

We use a similar argument to show (4.20). For $s \geq 0$ and $z > a$, the Laplace transform of the left-hand side of (4.20) is

$$\begin{aligned}
&\int_a^\infty e^{-sz}\int_{[a,z)} Z_1^{(q)}(z-y)F_{Y_{\tau_a}}^{(q)}(\mathrm{d}y)\mathrm{d}z+\int_a^\infty e^{-sz}\overline{F}_{Y_{\tau_a}}^{(q)}(z)\mathrm{d}z\\
&=\mathbb{E}[e^{-q\tau_a-sY_{\tau_a}}]\int_0^\infty e^{-sz}Z_1^{(q)}(z)\mathrm{d}z+\frac{1}{s}e^{-sa}\mathbb{E}[e^{-q\tau_a}]-\frac{1}{s}\mathbb{E}[e^{-q\tau_a-sY_{\tau_a}}].\\
&=-\frac{q}{s(q-\psi_1(s))}\mathbb{E}[e^{-q\tau_a-sY_{\tau_a}}]+\frac{1}{s}e^{-sa}\mathbb{E}[e^{-q\tau_a}].
\end{aligned}$$

By (4.22) followed by some calculations, one obtains

$$\begin{aligned}
&\int_a^\infty e^{-sz}\int_{[a,z)} Z_1^{(q)}(z-y)F_{Y_{\tau_a}}^{(q)}(\mathrm{d}y)\mathrm{d}z+\int_a^\infty e^{-sz}\overline{F}_{Y_{\tau_a}}^{(q)}(z)\mathrm{d}z\\
&=q\Big(\frac{1}{s}-\frac{W_1^{(q)}(a)}{W_1^{(q)\prime}(a)}\Big)\int_a^\infty e^{-sz}W_1^{(q)}(z)\mathrm{d}z+\frac{1}{s}e^{-sa}Z_1^{(q)}(a)\\
&=\frac{1}{s}e^{-sa}Z_1^{(q)}(a)+\frac{q}{s}\int_a^\infty e^{-sz}W_1^{(q)}(z)\mathrm{d}z-q\int_a^\infty e^{-sz}\frac{W_1^{(q)}(a)}{W_1^{(q)\prime}(a)}W_1^{(q)}(z)\mathrm{d}z\\
&=\int_a^\infty e^{-sz}\Big(Z_1^{(q)}(z)-q\frac{W_1^{(q)}(a)}{W_1^{(q)\prime}(a)}W_1^{(q)}(z)\Big)\mathrm{d}z.
\end{aligned}$$

Inversion of the Laplace transform results in identity (4.20). □

## 4.4 Survival probability

The (infinite-time) survival probability can be obtained by letting $q=0$ and $b\uparrow\infty$ in (4.18), that is

$$\mathbb{P}_u\left\{T_0^-=\infty\right\}=\frac{\boldsymbol{W}_a(u)}{\boldsymbol{W}_a(\infty)},\quad u>0. \tag{4.23}$$

In the following theorem, we state the positive security loading condition for the DBRS model (4.1), i.e., conditions under which ruin does not occur almost surely.

**Theorem 4.4.1.** *The following two statements are equivalent:*

*(i)* $\mathbb{E}[Y_{\tau_a}]<\infty$ *and* $\psi_2'(0+)>0$.

*(ii)* $\mathbb{P}_u\left\{T_0^-=\infty\right\}>0$ *for any* $u>0$.

*Proof.* From (4.23), (4.8) and (4.10), it is easy to see that $\mathbb{P}_u\left\{T_0^- = \infty\right\} \equiv 0$ if and only if

$$I_a := \int_a^\infty (1 - \int_{[a,x)} \frac{W_2(x-y)}{W_2(x)} F_{Y_{\tau_a}}(\mathrm{d}y))\mathrm{d}x = \infty.$$

(i)$\Longrightarrow$(ii). Since $W_2(\cdot)$ is an increasing function, we have

$$I_a \leq \frac{1}{W_2(a)} \int_a^\infty (W_2(x) - \int_{[a,x)} W_2(x-y) F_{Y_{\tau_a}}(\mathrm{d}y))\mathrm{d}x. \tag{4.24}$$

For $s > 0$, Fubini's theorem yields

$$\begin{aligned}
&\int_a^\infty e^{-sx}(W_2(x) - \int_{[a,x)} W_2(x-y) F_{Y_{\tau_a}}(\mathrm{d}y))\mathrm{d}x \\
&\leq \int_0^\infty e^{-sx} W_2(x)\mathrm{d}x - \int_{[a,\infty)} \int_y^\infty e^{-sx} W_2(x-y)\mathrm{d}x F_{Y_{\tau_a}}(\mathrm{d}y) \\
&= \frac{1 - \int_{[a,\infty)} e^{-sy} F_{Y_{\tau_a}}(\mathrm{d}y)}{\psi_2(s)}.
\end{aligned} \tag{4.25}$$

By (4.24), (4.25) and the monotone convergence theorem, it follows that

$$I_a \leq \frac{1}{W_2(a)} \lim_{s \downarrow 0} \frac{1 - \int_{[a,\infty)} e^{-sy} F_{Y_{\tau_a}}(\mathrm{d}y)}{\psi_2(s)} = \frac{\mathbb{E}[Y_{\tau_a}]}{W_2(a)\psi_2'(0+)}.$$

Under (i), it is clear that $I_a < \infty$ which leads to (ii).

(ii)$\Longrightarrow$(i). We prove it by the law of contrapositive, i.e. if $\mathbb{E}[Y_{\tau_a}] = \infty$ or $\psi_2'(0+) \leq 0$, the survival probability is zero for any $u > 0$. By Tonelli's theorem, we rewrite $I_a$ as

$$\begin{aligned}
I_a &= \int_a^\infty (1 - F_{Y_{\tau_a}}(x))\mathrm{d}x + \int_a^\infty \int_{[a,x)} \frac{W_2(x) - W_2(x-y)}{W_2(x)} F_{Y_{\tau_a}}(\mathrm{d}y)\mathrm{d}x \\
&= \int_a^\infty \overline{F}_{Y_{\tau_a}}(x)\mathrm{d}x + \int_{[a,\infty)} \int_y^\infty \frac{W_2(x) - W_2(x-y)}{W_2(x)} \mathrm{d}x F_{Y_{\tau_a}}(\mathrm{d}y).
\end{aligned} \tag{4.26}$$

Firstly, if $\mathbb{E}[Y_{\tau_a}] = \infty$, from (4.26), $I_a \geq \int_a^\infty \overline{F}_{Y_{\tau_a}}(x)\mathrm{d}x = \infty$, which implies $\mathbb{P}_u\left\{T_0^- = \infty\right\} \equiv 0$.

Secondly, if $\psi_2'(0+) < 0$, we have $\Phi_2(0) > 0$. By Exercise 8.5 of Kyprianou (2006),

$$\lim_{x\uparrow\infty} \frac{W_2(x-y)}{W_2(x)} = e^{-\Phi_2(0)y},$$

for any fixed $y \in [0, x)$. It follows that the integral

$$\int_y^\infty \frac{W_2(x) - W_2(x-y)}{W_2(x)} \mathrm{d}x = \infty$$

for any $y \geq a$. Therefore, by (4.26), we have $I_a = \infty$ which yields $\mathbb{P}_u \left\{T_0^- = \infty\right\} \equiv 0$.

Finally, for the third case $\psi_2'(0+) = 0$, we have $\Phi_2(0) = 0$, $W_2(\infty) = \infty$ and $W_2'(x) > 0$ for any $x > 0$. By Exercise 8.5 of Kyprianou (2006), we have

$$\lim_{x\uparrow\infty} \frac{W_2(x-y)}{W_2(x)} = \lim_{x\uparrow\infty} \frac{W_2'(x-y)}{W_2'(x)} = 1,$$

for any fixed $y \in [0, x)$. Then, for $y \in [0, b]$ and $b > a$,

$$K_y := \inf\left\{z > b : \frac{W_2'(x-y)}{W_2'(x)} > \frac{1}{2} \text{ for all } x > z\right\}, \tag{4.27}$$

exists and is finite. In particular, we have $K_0 = b$. For now we assume the following relation holds,

$$\max_{y\in[0,b]} K_y \leq K^* < \infty. \tag{4.28}$$

Therefore, for any $x > K^*$ and $y \in [0, b]$, we have

$$W_2'(x-y)/W_2'(x) > 1/2. \tag{4.29}$$

By the mean value theorem, there exists some $\theta_x \in (0, b)$ such that

$$W_2(x) - W_2(x-b) = bW_2'(x - \theta_x).$$

It follows from (4.29) that, for any $b > a$, we have

$$\int_b^\infty \frac{W_2(x) - W_2(x-b)}{W_2(x)} \mathrm{d}x = \int_b^\infty \frac{bW_2'(x-\theta_x)}{W_2(x)} \mathrm{d}x > \frac{b}{2} \int_{K^*}^\infty \frac{W_2'(x)}{W_2(x)} \mathrm{d}x = \infty.$$

From (4.26), one concludes that $I_a = \infty$, which implies that $\mathbb{P}_u\left\{T_0^- = \infty\right\} \equiv 0$.

Now, the only thing left is to show (4.28) holds. For convenience, we define

$$f(x,y) := \frac{W_2'(x-y)}{W_2'(x)},\ 0 \le y < x.$$

Then we have

$$K_y = \inf\left\{z > b : f(x,y) > \frac{1}{2} \text{ for all } x > z\right\}.$$

We prove (4.28) by contradiction. Suppose (4.28) fails, then there exists a sequence $\{y_n\}_{n\in\mathbb{N}}$ in $[0,b]$ such that $K_{y_n} \to \infty$. By the Bolzano–Weierstrass theorem, there exists a convergent subsequence $\{\tilde{y}_n\}_{n\in\mathbb{N}}$ such that $\tilde{y}_n \to y_0$ for some $y_0 \in [0,b]$. By the definition of $K_{y_0}$, for any fixed $x_0 > K_{y_0}$, we have

$$f(x_0, y_0) > \frac{1}{2}.$$

On the other hand, since $K_{\tilde{y}_n} \to \infty$, we have

$$f(x_0, \tilde{y}_n) \le \frac{1}{2}, \quad \text{for } n \text{ large enough,}$$

which contradicts the continuity of $f$ at $(x_0, y_0)$ as $\tilde{y}_n \to y_0$. Therefore, relation (4.28) holds.

$\square$

Note that the condition $\mathbb{E}[Y_{\tau_a}] < \infty$ relates to the process $X^1$ only, while $\psi_2'(0+) > 0$ is the positive security loading condition of $X^2$ (e.g., Exercise 7.3 of Kyprianou (2006)).

The following proposition further establishes a connection between $\mathbb{E}[Y_{\tau_a}]$ and the Lévy measure $\Pi_1$. This result is intuitive as drawdowns of size larger than $a$ occurs from jumps in $X^1$ governed by the Lévy measure $\Pi_1$.

**Proposition 4.4.1.** *We have $\mathbb{E}[Y_{\tau_a}] < \infty$ if and only if $\int_{-\infty}^{-1} |y|\Pi_1(dy) < \infty$.*

*Proof.* We focus on $\mathbb{E}[Y_{\tau_a} 1_{\{Y_{\tau_a} > a+1\}}]$ to determine if $\mathbb{E}[Y_{\tau_a}]$ is finite or not. By (4.5) with $q = 0$, and a subsequent change of variable,

$$\begin{aligned}
&\mathbb{E}[Y_{\tau_a} 1_{\{Y_{\tau_a} > a+1\}}] \\
&= \frac{W_1(a)W_1(0+)}{W_1'(a)} \int_{a+1}^{\infty} y\Pi_1(-\mathrm{d}y) + \int_0^a (\frac{W_1(a)}{W_1'(a)} W_1'(z) - W_1(z))\mathrm{d}z \int_{a+1}^{\infty} y\Pi_1(z - \mathrm{d}y) \\
&= \frac{W_1(a)W_1(0+)}{W_1'(a)} \int_{-\infty}^{-a-1} |y|\Pi_1(\mathrm{d}y) + \int_0^a (\frac{W_1(a)}{W_1'(a)} W_1'(z) - W_1(z))\mathrm{d}z \int_{-\infty}^{z-a-1} (z - y)\Pi_1(\mathrm{d}y).
\end{aligned} \tag{4.30}$$

Note that

$$\frac{W_1(a)}{W_1'(a)} W_1'(z) - W_1(z) \geq 0, \quad 0 < z \leq a, \tag{4.31}$$

as $(\frac{W_1(a)}{W_1'(a)} W_1'(z) - W_1(z))1_{\{0<z\leq a\}}\mathrm{d}z + \frac{W_1(a)W_1(0+)}{W_1'(a)}\delta_0(\mathrm{d}z)$ is the density of a potential measure (see Theorem 1 of Pistorius (2004), or Theorem 8.11 of Kyprianou (2006)).

Further, it is not difficult to see that

$$\int_{-\infty}^{-a-1} |y|\Pi_1(\mathrm{d}y) \leq \int_{-\infty}^{z-a-1} (z - y)\Pi_1(\mathrm{d}y) \leq a\Pi_1(-\infty, -1) + \int_{-\infty}^{-1} |y|\Pi_1(\mathrm{d}y). \tag{4.32}$$

The substitution of (4.32) into (4.30) yields a two-sided bound for $\mathbb{E}[Y_{\tau_a} 1_{\{Y_{\tau_a} > a+1\}}]$. Finally, given that the Lévy measure $\Pi_1(\cdot)$ is finite on any compact subset of $(-\infty, 0)$, one concludes that $\mathbb{E}[Y_{\tau_a} 1_{\{Y_{\tau_a} > a+1\}}] < \infty$ if and only if $\int_{-\infty}^{-1} |y|\Pi_1(dy) < \infty$. □

In what follows, the survival probability for two special DBRS models is further examined.

**Example 4.4.1.** *(Drifted Brownian motion) We consider the DBRS model where the process $X^k$ $(k = 1, 2)$ is a drifted Brownian motion with Laplace exponent $\psi_k(s) = c_k s + \frac{1}{2}\sigma_k^2 s^2$ $(s \geq 0)$, and scale function*

$$W_k(x) = \frac{1 - e^{-2c_k x/\sigma_k^2}}{c_k}, \quad x \geq 0.$$

*By (4.23), the survival probability is then given by*

$$\mathbb{P}_u\left\{T_0^- = \infty\right\} = \frac{1-e^{-2c_1u/\sigma_1^2}}{1-e^{-2c_1(u\vee a)/\sigma_1^2}}(1-e^{-2c_2(u\vee a)/\sigma_2^2})^r,$$

*where* $r = \frac{c_1\sigma_2^2(e^{2c_2a/\sigma_2^2}-1)}{c_2\sigma_1^2(e^{2c_1a/\sigma_1^2}-1)}$.

**Example 4.4.2.** *(Compound Poisson risk model with exponential jumps) We consider the DBRS model where* $X^k$ $(k = 1, 2)$ *is a drifted compound Poisson process with exponential jumps. The Laplace exponent of* $X^k$ *is given by* $\psi_k(s) = c_k s - \lambda_k s/(s + \beta_k)$ *for* $s \geq 0$, *where* $c_k, \lambda_k, \beta_k$ *are positive constants. Its scale function is known to be*

$$W_k(x) = \frac{1}{c_k - \lambda_k/\beta_k}(1 - \frac{\lambda_k}{c_k\beta_k}e^{-(\beta_k-\lambda_k/c_k)x}), \quad x \geq 0,$$

*while the drawdown overshoot* $Y_{\tau_a} - a$ *is known to be exponentially distributed with mean* $1/\beta_1$. *With some calculations, one finds that*

$$\begin{aligned}\mathbb{P}_u\left\{T_0^- = \infty\right\} = &\frac{c_1\beta_1 - \lambda_1 e^{-(\beta_1-\lambda_1/c_1)u}}{c_1\beta_1 - \lambda_1 e^{-(\beta_1-\lambda_1/c_1)(u\vee a)}}(1 - \frac{\lambda_2}{c_2\beta_2}e^{-(\beta_2-\lambda_2/c_2)(u\vee a)})^{\rho_2} \\ &\times \exp\left\{-\rho_1 e^{-\beta_1(u\vee a)}\ {}_2F_1(1, \frac{\beta_1}{\beta_2-\lambda_2/c_2}, \frac{\beta_1}{\beta_2-\lambda_2/c_2}+1; \frac{\lambda_2}{c_2\beta_2}e^{-(\beta_2-\lambda_2/c_2)(u\vee a)})\right\},\end{aligned}$$

*where*

$$\rho_1 = \frac{\lambda_1(c_1\beta_1-\lambda_1)e^{-(\beta_1-\lambda_1/c_1)a}}{c_1\beta_1(c_1\beta_1-\lambda_1 e^{-(\beta_1-\lambda_1/c_1)a})}(1 - \frac{\lambda_2\beta_1}{c_2\beta_2}\frac{1}{\beta_1-\beta_2+\lambda_2/c_2})e^{\beta_1 a},$$

$$\rho_2 = \frac{\lambda_1(\beta_1-\lambda_1/c_1)e^{-(\beta_1-\lambda_1/c_1)a}}{(\beta_2-\lambda_2/c_2)(c_1\beta_1-\lambda_1 e^{-(\beta_1-\lambda_1/c_1)a})}(\frac{\beta_1}{\beta_1-\beta_2+\lambda_2/c_2}e^{(\beta_2-\lambda_2/c_2)a} - 1),$$

*and the Gauss hypergeometric series* ${}_2F_1(a, b, c; z)$ *(e.g., Abramowitz and Stegun, 1965) is defined as*

$${}_2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\mathrm{d}t.$$

We conduct a numerical study of Example 4.4.2 where a sensitivity analysis of the parameters of the drawdown-based policy is performed. To this end, we assume that the drawdown-based policy has no influence on the claim arrival dynamics and thus, $\lambda_1 = \lambda_2$ and $\beta_1 = \beta_2$. For the numerical analysis, we set $\lambda_1 = \lambda_2 = 0.5$ and $\beta_1 = \beta_2 = 0.1$. The premium rate $c_1$ is assumed to be 6 (i.e., the security loading is of a magnitude of 20%). The endogenous parameters of the DBRS model, namely $c_2$ and $a$, will be examined to measure their impact on the survival probability. For this analysis, we assume that $c_2 \geq c_1$ and thus, a drawdown of size $a$ or larger implies a larger net premium rate thereafter.

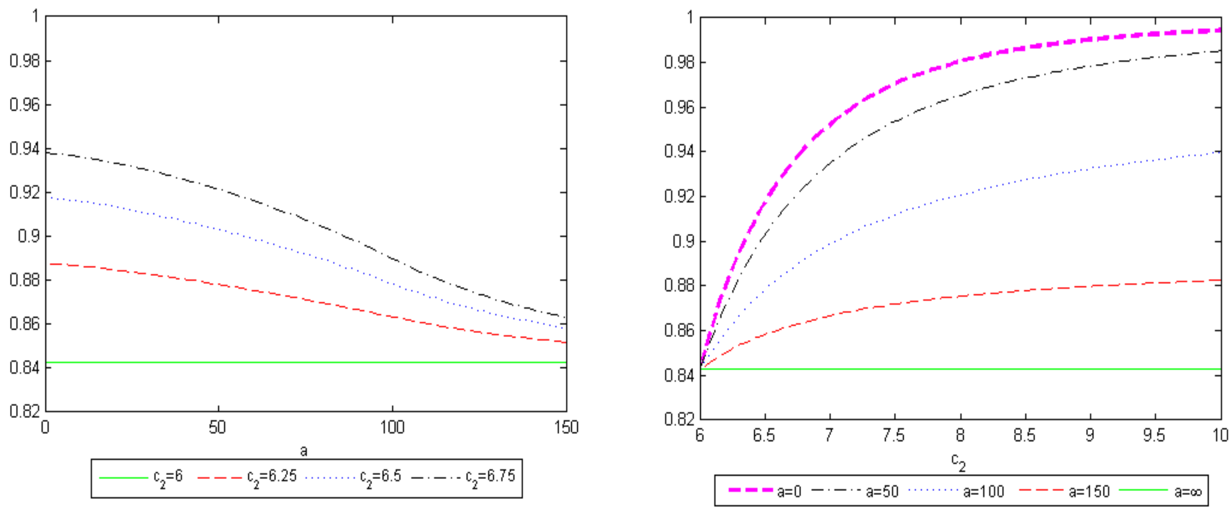


Figure 4.2: Survival Probability of Example 4.4.2

As expected, the survival probability decreases in the threshold size $a$ and increases in the premium rate $c_2$. In particular, when $c_2 = c_1$ or $a = \infty$, the model reduces to the classical risk model with (single) premium rate $c_1$. As for the sensitivity, we see that as $a$ becomes larger, the sensitivity of the survival probability wrt $c_2$ decreases. More interestingly, for fixed $c_2$, the sensitivity wrt $a$ first increases and then decreases. We point out that there are multiple combinations of ($a$,$c_2$) which can achieve a given survival probability. For instance, the combinations of $(25, 7.08)$ and $(50, 7.38)$ both lead to a survival probability of 95%.

## 4.5 Regime-dependent occupation time

We now examine the (discounted) regime-dependent occupation time until ruin. Due to their similarity, we only consider the occupation time in the non-distressed regime, that is

$$A_q := \int_0^{T_0^-} e^{-qt} 1_{\{Q_t=1\}} \mathrm{d}t, \quad q \geq 0.$$

In particular, we have $A_0 = \mathcal{T}_{0,1}^-$ whose Laplace transform can be obtained by letting $q = 0$ and $b \uparrow \infty$ in Theorem 4.3.2. Therefore, we only consider the case $q > 0$ in this section. The inclusion of the discounted component in $A_q$ may seem unnatural at first, but a connection with the discounted total tax payments of a loss-carry-forward tax model (e.g., Theorem 2 of Albrecher and Hipp (2007) Theorem 3.2 of Albrecher et al. (2008), and Theorem 1.2 of Kyprianou and Zhou (2009)) will be made in the next section.

In the following theorem, a recursive formula for the moments of $A_q$ is derived. The Laplace transform of $A_q$ can be obtained from its moments through the standard infinite sum representation.

**Theorem 4.5.1.** *For $q > 0$ and $k \in \mathbb{N}$, we have*

$$\mathbb{E}_u[(A_q)^k] = \begin{cases} \int_u^\infty e^{-\int_u^y C_{qk}(z)\mathrm{d}z}\left(\frac{W_1^{(qk)\prime}(a)}{W_1^{(qk)}(a)} m_{k-1}(y) - m_{k-1}'(y)\right)\mathrm{d}y, & u \geq a, \\ \sum\limits_{j=0}^{k} \binom{k}{j} \sum_{l=j}^{k} \binom{k-j}{l-j} \frac{(-1)^{l-j}}{q^{k-j}} \frac{W_1^{(lq)}(u)}{W_1^{(lq)}(a)} \mathbb{E}_a[(A_q)^j] & \\ \quad + \sum\limits_{j=0}^{k} \binom{k}{j} \frac{(-1)^j}{q^k}\left(Z_1^{(qj)}(u) - Z_1^{(qj)}(a)\frac{W_1^{(q)}(u)}{W_1^{(q)}(a)}\right), & 0 < u < a, \end{cases}$$

*where*

$$\begin{aligned} m_{k-1}(z) = {} & \sum_{j=1}^{k-1} \binom{k}{j} \sum_{l=j}^{k} \binom{k-j}{l-j} \frac{(-1)^{l-j} W_1^{(lq)\prime}(a)}{q^{k-j} W_1^{(lq)}(a)} \int_z^\infty \mathbb{E}_x[(A_q)^j] e^{-\frac{W_1^{(lq)\prime}(a)}{W_1^{(lq)}(a)}(x-z)} \mathrm{d}x \\ & \times \int_{[a,x)} \frac{W_2^{(jq)}(x-y)}{W_2^{(jq)}(x)} F_{Y_{\tau_a}}^{(lq)}(\mathrm{d}y) + \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j}{q^k}\left(Z_1^{(jq)}(a) - jq\frac{W_1^{(jq)}(a)^2}{W_1^{(jq)\prime}(a)}\right). \end{aligned} \tag{4.33}$$

*Proof.* Let $w_k(u) := \mathbb{E}_u[(A_q)^k]$. For $u \geq a$, by conditioning on $\mathcal{F}_{\tau_a}$, we have

$$\begin{aligned} w_k(u) &= \mathbb{E}_u[(\int_0^{\tau_a} e^{-qt}\mathrm{d}t + \int_{\tau_a}^{T_0^-} e^{-qt} 1_{\{Q_t=1\}}\mathrm{d}t)^k] \\ &= \sum_{j=0}^{k} \binom{k}{j} \mathbb{E}_u[(\int_0^{\tau_a} e^{-qt}\mathrm{d}t)^{k-j} \mathbb{E}[(\int_{\tau_a}^{T_0^-} e^{-qt} 1_{\{Q_t=1\}}\mathrm{d}t)^j \bigg| \mathcal{F}_{\tau_a}]]. \end{aligned} \tag{4.34}$$

Furthermore, for $j \geq 1$, by (4.2), one deduces that

$$\begin{aligned} \mathbb{E}[(\int_{\tau_a}^{T_0^-} e^{-qt} 1_{\{Q_t=1\}}\mathrm{d}t)^j \bigg| \mathcal{F}_{\tau_a}] &= e^{-jq\tau_a} \mathbb{E}_{M_{\tau_a}-Y_{\tau_a}}[e^{-jqT^{2,+}_{M_{\tau_a}}} 1_{\left\{T^{2,+}_{M_{\tau_a}} < T_0^{2,-}\right\}}] w_j(M_{\tau_a}) \\ &= e^{-jq\tau_a} \frac{W_2^{(jq)}(M_{\tau_a} - Y_{\tau_a})}{W_2^{(jq)}(M_{\tau_a})} w_j(M_{\tau_a}). \end{aligned} \tag{4.35}$$

The substitution of (4.35) into (4.34) results in

$$w_k(u) = \sum_{j=1}^{k} \binom{k}{j} \mathbb{E}_u[(\int_0^{\tau_a} e^{-qt}\mathrm{d}t)^{k-j} e^{-jq\tau_a} \frac{W_2^{(jq)}(M_{\tau_a} - Y_{\tau_a})}{W_2^{(jq)}(M_{\tau_a})} w_j(M_{\tau_a})] + \mathbb{E}[(\int_0^{\tau_a} e^{-qt}\mathrm{d}t)^k].$$

By separating the terms $j = k$ and $j < k$, it follows that

$$w_k(u) = \int_u^\infty \int_{[a,x)} \mathbb{E}_u[e^{-kq\tau_a} 1_{\{Y_{\tau_a}\in \mathrm{d}y, M_{\tau_a}\in \mathrm{d}x\}}] \frac{W_2^{(kq)}(x-y)}{W_2^{(kq)}(x)} w_k(x) + m_{k-1}(u), \tag{4.36}$$

where

$$\begin{aligned} m_{k-1}(u) := \sum_{j=1}^{k-1} \binom{k}{j} \mathbb{E}_u[\int_u^\infty \int_{[a,x)} (\int_0^{\tau_a} e^{-qt}\mathrm{d}t)^{k-j} e^{-jq\tau_a} 1_{\{Y_{\tau_a}\in \mathrm{d}y, M_{\tau_a}\in \mathrm{d}x\}}] \frac{W_2^{(jq)}(x-y)}{W_2^{(jq)}(x)} w_j(x) \\ + \mathbb{E}[(\int_0^{\tau_a} e^{-qt}\mathrm{d}t)^k]. \end{aligned} \tag{4.37}$$

By (4.4) and (4.6), it is straightforward to verify that (4.37) is consistent with (4.33).

Differentiating (4.36) wrt $u$ yields

$$w_k'(u) = C_{kq}(u) w_k(u) - \frac{W_1^{(kq)\prime}(a)}{W_1^{(kq)}(a)} m_{k-1}(u) + m_{k-1}'(u), \quad u > a. \tag{4.38}$$

The solution to the ODE (4.38) is

$$w_k(u) = e^{\int_a^u C_{kq}(z)\mathrm{d}z}(w_k(a) - \int_a^u e^{-\int_a^y C_{kq}(z)\mathrm{d}z}(\frac{W_1^{(kq)\prime}(a)}{W_1^{(kq)}(a)}m_{k-1}(y) - m'_{k-1}(y))\mathrm{d}y). \qquad (4.39)$$

Furthermore, since $C_{kq}(z) \geq \frac{W_1^{(s)\prime}(a)}{W_1^{(s)}(a)}(1-\mathbb{E}[e^{-kq\tau_a}])$ for $q > 0$ and $z > a$, we have $\int_a^\infty C_{kq}(z)dz = \infty$. On the other hand, since $w_k(\cdot) \leq \frac{1}{q^k}$, we conclude from (4.39) that

$$w_k(a) = \int_a^\infty e^{-\int_a^y C_{kq}(z)\mathrm{d}z}(\frac{W_1^{(kq)\prime}(a)}{W_1^{(kq)}(a)}m_{k-1}(y) - m'_{k-1}(y))\mathrm{d}y.$$

Substituting the above equation into (4.39) yields the desired result for $u \geq a$.

For $0 < u < a$, from the strong Markov property of $X$ at $T_a^+$, one has

$$\begin{aligned} w_k(u) &= \mathbb{E}_u[(\int_0^{T_a^+} e^{-qt}\mathrm{d}t 1_{\{T_a^+<T_0^-\}} + \int_{T_a^+}^{T_0^-} e^{-qt}1_{\{Q_t=1, T_a^+<T_0^-\}}\mathrm{d}t)^k] \\ &+ \mathbb{E}_u[(\int_0^{T_0^-} e^{-qt}\mathrm{d}t)^k 1_{\{T_0^-<T_a^+\}}] \\ &= \textstyle\sum_{j=0}^k \binom{k}{j}\frac{1}{q^{k-j}}\mathbb{E}_u[(1-e^{-qT_a^+})^{k-j}e^{-qjT_a^{1,+}}1_{\{T_a^{1,+}<T_0^{1,-}\}}]w_j(a) \\ &+ \frac{1}{q^k}\mathbb{E}_u[(1-e^{-qT_0^{1,-}})^k 1_{\{T_0^{1,-}<T_a^{1,+}\}}]. \end{aligned}$$

We complete the proof by using binomial expansion, (4.2) and (4.3).

□

In particular, when $k = 1$, we have the following corollary of Theorem 4.5.1, which will be used explicitly in the next section.

**Corollary 4.5.1.** *For $q > 0$ and $u \geq a$, the first moment of $A_q$ is*

$$\mathbb{E}_u[A_q] = (W_1^{(q)}(a) - \frac{W_1^{(q)\prime}(a)}{W_1^{(q)}(a)}\int_0^a W_1^{(q)}(x)\mathrm{d}x)\int_u^\infty e^{-\int_u^y C_q(z)\mathrm{d}z}\mathrm{d}y.$$

## 4.6 Regime-switching premium model and its relation with other risk models

In this section, we study a special case of the DBRS (4.1), namely the regime-switching premium model, and consider its relation with other risk models in the literature. As illustrated below, when the threshold level $a \downarrow 0$, the regime-switching premium model has interesting connections to both the loss-carry-forward tax model (e.g., Albrecher et al. (2009), Albrecher and Hipp (2007), and Li et al. (2013)) and a single premium model.

We assume $X^1$ and $X^2$ are jump-diffusion processes with $d_1 < d_2$, $\sigma_1 = \sigma_2 := \sigma$ and $\Pi_1(\cdot) = \Pi_2(\cdot) := \Pi(\cdot)$ with $\Pi(-\infty, 0) < \infty$. Equivalently, we can express the process $X$ as

$$X_t = u + \boldsymbol{c}_t t + \sigma B_t - \sum_{i=1}^{N_t} J_i, \qquad t \geq 0, \tag{4.40}$$

where

$$\boldsymbol{c}_t = \begin{cases} c_1 := d_1 + \int_{(-1,0)} |y| \Pi(\mathrm{d}y), & \text{if } Q_t = 1, \\ c_2 := d_2 + \int_{(-1,0)} |y| \Pi(\mathrm{d}y), & \text{if } Q_t = 2. \end{cases}$$

Clearly, $\boldsymbol{c}_t$ is $\mathcal{F}_t$-measurable. As usual, $N = \{N_t; t \geq 0\}$ is a Poisson process with rate $\lambda > 0$, $\{J_i, i \geq 1\}$ form an iid sequence of positive random variables with common distribution function $F(\cdot)$, and $B = \{B_t; t \geq 0\}$ is a standard Brownian motion. We assume that $N$, $\{J_i, i \geq 1\}$ and $B$ are mutually independent. In this case, we have $\lambda = \Pi(-\infty, 0)$ and $\Pi(-\mathrm{d}y) = \lambda F(\mathrm{d}y)$. This model essentially switches between premium rates $c_1$ and $c_2$ according to the DBRS dynamics. In the later analysis, the following asymptotic behavior of the scale function at 0+ (e.g., Kuznetsov et al. (2012)) will be called upon.

**Proposition 4.6.1.** *For $\Pi_k(-\infty, 0) < \infty$ $(k = 1, 2)$, we have*

$$W_k^{(q)}(0+) = \begin{cases} 0, & \sigma > 0, \\ 1/c_k, & \sigma = 0, \end{cases} \quad \text{and} \quad W_k^{(q)\prime}(0+) = \begin{cases} 2/\sigma^2, & \sigma > 0, \\ (q+\lambda)/(c_k)^2, & \sigma = 0. \end{cases}$$

### 4.6.1 Relation with the loss-carry-forward tax model: $\sigma = 0$

The following proposition indicates that, when $\sigma = 0$, the regime-switching premium model (4.40) reduces to the loss-carry-forward tax model (with tax rate $\gamma = (c_2 - c_1)/c_2$) as $a \downarrow 0$; e.g., Theorem 3.1 of Albrecher et al. (2008), where the result $\mathbb{E}_u[e^{-q\tau^+_{b,\gamma}} 1_{\{\tau^+_{b,\gamma} < \tau^-_{0,\gamma}\}}] = (\frac{W_2^{(q)}(u)}{W_2^{(q)}(b)})^{1/(1-\gamma)}$ is given with $\tau^{+(-)}_{b,\gamma}$ to be the exit time in the tax model.

**Proposition 4.6.2.** *Consider the regime-switching premium model (4.40) with $\sigma = 0$. For $q \geq 0$ and $0 < u < b$,*

$$\lim_{a \downarrow 0} \mathbb{E}_u[e^{-qT_b^+} 1_{\{T_b^+ < T_0^-\}}] = \lim_{a \downarrow 0} \frac{\boldsymbol{W}_a^{(q)}(u)}{\boldsymbol{W}_a^{(q)}(b)} = \left( \frac{W_2^{(q)}(u)}{W_2^{(q)}(b)} \right)^{c_2/c_1}. \tag{4.41}$$

*Proof.* By letting $a \downarrow 0$ in (4.5) and using Proposition 4.6.1, it is immediate that

$$\lim_{a \downarrow 0} F^{(q)}_{Y_{\tau_a}}(\mathrm{d}y) = \frac{\lambda}{q+\lambda} F(\mathrm{d}y).$$

On the other hand, by letting $a \downarrow 0$ in (4.19) with $\sigma = 0$, we have

$$W_2^{(q)}(z) - \frac{\lambda}{q+\lambda} \int_0^z W_2^{(q)}(z-y) F(\mathrm{d}y) = \frac{c_2}{q+\lambda} W_2^{(q)\prime}(z), \quad z > a.$$

It follows that

$$\begin{aligned}
\lim_{a \downarrow 0} C_q(z) &= \lim_{a \downarrow 0} \frac{W_1^{(q)\prime}(a)}{W_1^{(q)}(a)} \Big(1 - \int_a^z \frac{W_2^{(q)}(z-y)}{W_2^{(q)}(z)} F^{(q)}_{Y_{\tau_a}}(\mathrm{d}y)\Big) \\
&= \frac{q+\lambda}{c_1} \Big(1 - \frac{\lambda}{q+\lambda} \int_0^z \frac{W_2^{(q)}(z-y)}{W_2^{(q)}(z)} F(\mathrm{d}y)\Big) \\
&= \frac{c_2}{c_1} \frac{W_2^{(q)\prime}(z)}{W_2^{(q)}(z)},
\end{aligned} \tag{4.42}$$

which easily leads to (4.41) using (4.18) and (4.8). ☐

Moreover, from Corollary 4.5.1, Proposition 4.6.1, and (4.42), for $q > 0$,

$$\lim_{a \downarrow 0}(c_2 - c_1)\mathbb{E}_u[A_q] = \frac{c_2 - c_1}{c_1} \int_u^\infty \left( \frac{W_2^{(q)}(u)}{W_2^{(q)}(y)} \right)^{c_2/c_1} \mathrm{d}y,$$

which implies that $(c_2 - c_1)\mathbb{E}_u[A_q]$ reduces to the expected discounted tax payment (with tax rate $\gamma = (c_2 - c_1)/c_2$) in the loss-carry-forward tax model; see Theorem 3.2 of Albrecher et al. (2008), where the result $\mathbb{E}_u[\gamma \int_0^{\tau_{0,\gamma}^-} e^{-\delta t} dD(t)] = \frac{\gamma}{1-\gamma} \int_u^\infty (\frac{W_2^{(q)}(u)}{W_2^{(q)}(y)})^{1/(1-\gamma)} \mathrm{d}y$ is given with $dD(t) = \frac{1}{1-\gamma} d(\max_{0 \le s \le t} X_t - X_0)$ .

### 4.6.2 Relation with the single premium model: $\sigma > 0$

However, when $\sigma > 0$, the following proposition shows that the regime-switching premium model (4.40) reduces to the jump diffusion model with single premium rate $c_2$ as $a \downarrow 0$.

**Proposition 4.6.3.** *Consider the regime-switching premium model (4.40) with $\sigma > 0$. For $q \geq 0$ and $0 < u < b$,*

$$\lim_{a \downarrow 0} \mathbb{E}_u[e^{-qT_b^+} 1_{\{T_b^+ < T_0^-\}}] = \lim_{a \downarrow 0} \frac{\boldsymbol{W}_a^{(q)}(u)}{\boldsymbol{W}_a^{(q)}(b)} = \frac{W_2^{(q)}(u)}{W_2^{(q)}(b)}.$$

*Proof.* By (4.5), (4.31) and Proposition 4.6.1, we have

$$\begin{aligned}
\frac{\overline{F}_{Y_{\tau_a}}^{(q)}(a)}{a} &= \frac{\lambda}{a} \int_0^a \left(\frac{W_1^{(q)}(a)}{W_1^{(q)\prime}(a)} W_1^{(q)\prime}(z) - W_1^{(q)}(z)\right) \int_a^\infty F(-z + \mathrm{d}y)\mathrm{d}z. \\
&\leq \frac{\lambda}{a} \int_0^a \left(\frac{W_1^{(q)}(a)}{W_1^{(q)\prime}(a)} W_1^{(q)\prime}(z) - W_1^{(q)}(z)\right)\mathrm{d}z \\
&\leq \frac{\lambda (W_1^{(q)}(a))^2}{a W_1^{(q)\prime}(a)}.
\end{aligned}$$

By Proposition 4.6.1 again, one deduces that $\lim_{a \downarrow 0} \frac{\lambda (W_1^{(q)}(a))^2}{a W_1^{(q)\prime}(a)} = 0$. Hence,

$$\overline{F}_{Y_{\tau_a}}^{(q)}(a) = o(a), \tag{4.43}$$

which stands for $\lim_{a\downarrow 0}\overline{F}^{(q)}_{Y_{\tau_a}}(a)/a=0$.

By (4.10) and (4.43), it follows that

$$\begin{aligned}C_q(z)&=\frac{W_1^{(q)\prime}(a)}{W_1^{(q)}(a)}(1-\frac{W_2^{(q)}(z-a)}{W_2^{(q)}(z)}F^{(q)}_{Y_{\tau_a}}(a)+o(a))\\&=\frac{W_1^{(q)\prime}(a)}{W_1^{(q)}(a)}(1-\frac{W_2^{(q)}(z-a)}{W_2^{(q)}(z)}+o(a)).\end{aligned}$$

Taking the limit as $a\downarrow 0$, one finds that

$$\begin{aligned}\lim_{a\downarrow 0}C_q(z)&=\lim_{a\downarrow 0}\frac{aW_1^{(q)\prime}(a)}{W_1^{(q)}(a)}(\frac{W_2^{(q)}(z)-W_2^{(q)}(z-a)}{aW_2^{(q)}(z)}+\frac{o(a)}{a})\\&=\frac{W_2^{(q)\prime}(z)}{W_2^{(q)}(z)},\end{aligned}$$

which completes the proof of Proposition 4.6.3 by (4.18) and (4.8). □

**Remark 4.6.1.** *In relation to the above analysis, we would also like to provide some intuitive explanations on the limiting cases of the regime-switching premium model (4.40) as* $a\downarrow 0$. *It is clear that, when* $a\downarrow 0$, *the premium rate of model (4.40) is* $c_1$ *when the process is at its running maximum and* $c_2$, *otherwise. Therefore, mathematically, model (4.40) reduces to the loss-carry-forward tax model (e.g., Albrecher and Hipp (2007) and Albrecher et al. (2008)) when* $\sigma=0$.

*However, when* $\sigma>0$, *the occupation time in a finite time period of the process (4.40) at its running maximum is almost surely zero. Roughly speaking, the drift term of (4.40) is dominated by the diffusion term in any infinitesimal time period. Therefore, as* $a\downarrow 0$, *the change of drift at running maxima has virtually no effect, and thus the model (4.40) reduces to the jump diffusion process with single premium rate* $c_2$.

# Chapter 5

# Drawdown risk analysis for the renewal insurance risk process

## 5.1 Introduction

In this chapter, we extend the drawdown analysis to the renewal insurance risk process with a fairly general and common modelling assumption for the claim arrival dynamics.

We first consider the two-sided exit problem of the renewal insurance risk process, a quantity which will be shown to play an important role in the subsequent study of drawdowns. Thanks to a fluid flow transformation of a truncated version of the renewal process of interest and some recent results on exit problems for spectrally negative MAPs by Ivanovs and Palmowski (2012), expressions for some two-sided exit quantities related to the renewal insurance risk process and its truncated version are derived. The analogous results for spectrally negative Lévy processes or MAPs are well known to be closely tied to the analysis of many other related problems such as those pertaining to occupation times (see, e.g., Landriault et al. (2011a), Li and Zhou (2013), and Loeffen et al. (2014)). Second, we consider the joint law of various drawdown-related quantities which include but are not limited to the drawdown time, the drawdown size, the running maximum and minimum at the drawdown time and the last running maximum time prior to drawdown. Finally, as an application of the aforementioned drawdown results, we derive the expected discounted dividend payments until ruin for a constant dividend barrier model under the renewal framework. Here again, an extensive literature can be found on the analysis of renewal-type risk models under a constant dividend barrier strategy when interarrival times are of a phase-type form (see, e.g., Li and Garrido (2004b) and Albrecher et al. (2005)). Note that the constant dividend barrier strategy is not optimal, i.e., does not maximize the expected dividend payments for the risk model under the Sparre Andersen framework (see, e.g., Albrecher and Hartinger (2006)). To the best of our knowledge, results are rather scarce on the renewal insurance risk process with an arbitrary interarrival distribution. As such, a few contributions and observations on this subject matter will be made.

The rest of the chapter is organized as follows. In Section 5.2, we formally define the renewal insurance risk process $X$, and introduce the two-sided exit and drawdown related quantities which will be the primary subject matter of Sections 5.3 and 5.4, respectively. Also, a few observations on the constant dividend barrier model superimposed to the process $X$ will be made at the end of Section 5.4.

## 5.2 Problem formulation

Consider the renewal insurance risk process $X = \{X_t; t \geq 0\}$ given by

$$X_t = u + ct - S_t, \tag{5.1}$$

where $u \in \mathbb{R}$, $c > 0$, and $\{S_t; t \geq 0\}$ is a compound renewal process. That is, $S_t$ is defined as

$$S_t = \begin{cases} \sum_{i=1}^{N_t} P_i, & N_t > 0, \\ 0, & N_t = 0, \end{cases} \tag{5.2}$$

where $\{N_t; t \geq 0\}$ is a renewal process defined through the sequence of independent and identically distributed (iid) positive interarrival times $\{T_i; i \in \mathbb{N}\}$ with distribution function (df) $K$ and LT $\widetilde{k}$, and $\{P_i; i \in \mathbb{N}\}$ is a sequence of iid positive random variables (rv's) with df $P$ and LT $\widetilde{p}$, independent of $\{N_t; t \geq 0\}$. To obtain explicit formulas in the later analysis, we assume that $\{P_i; i \in \mathbb{N}\}$ follow a phase-type distribution $\mathrm{PH}(\boldsymbol{\beta}, \mathbf{B})$ with df $P(x) = 1 - \boldsymbol{\beta} e^{\mathbf{B}x}\mathbf{e}$ for $x > 0$ and LT $\widetilde{p}(s) = \boldsymbol{\beta}(s\mathbf{I} - \mathbf{B})^{-1}\mathbf{b}$ for $s \geq 0$, where the $1 \times n$ row vector $\boldsymbol{\beta}$ contains the initial probabilities of an associated finite-state continuous-time Markov process, the $n \times n$ matrix $\mathbf{B}$ is a non-singular subintensity matrix, $\mathbf{b} = -\mathbf{Be}$ and $\mathbf{e}$ is a column vector of ones. Note that the class of phase-type distribution is dense in the sense of weak convergence for all distributions with positive support. The interested readers are referred to Chapter IX of Asmussen and Albrecher (2010) for more details about phase-type distributions.

For $x \in \mathbb{R}$, we define the first passage time of $X$ as

$$T_x^{X,+(-)} = \inf\left\{t \geq 0 : X_t > (<)x\right\}.$$

Recall from the drawdown-related quantities introduced in Section 1.3.2 that the drawdown process $Y = \{Y_t; t \geq 0\}$ is defined as

$$Y_t = M_t - X_t,$$

where $M_t = \sup_{0 \leq s \leq t} X_s$ is the running maximum of $X$ at time $t$. Note that in queueing theory, $Y$ is known as the workload of a GI/PH/1 queue. This observation is particularly relevant in connection with the fluid flow transformation between queues and risk processes discussed in Section 5.3.

For a fixed level $a$, the drawdown time is $\tau_a = \inf\left\{t \geq 0 : Y_t \geq a\right\}$, and $G_t := \sup\{0 \leq s \leq t : Y_s = 0\}$ is the last time the process $Y$ is at level 0 before or at time $t$. Heuristically, the first drawdown episode $[0, \tau_a]$ can be split into two parts: the rising part in $[0, G_{\tau_a}]$ and the subsequent crashing part in $[G_{\tau_a}, \tau_a]$. To study the number of excursions from the maximum of $X$, we further introduce a sequence of stopping times $\{\nu_m; m \in \{0\} \cup \mathbb{N}\}$ defined recursively as $\nu_m = \inf\left\{t > \nu_{m-1} : Y_t \neq 0, Y_{t_-} = 0\right\}$ with $\nu_0 = 0$. We define a process $\{\overline{N}_t; t \geq 0\}$ as

$$\overline{N}_t = \sup\{m \in \{0\} \cup \mathbb{N} : \nu_m \leq t\},$$

where $\overline{N}_t$ represents the number of excursions of $X$ from its running maximum until time $t$.

In this chapter, we are mainly interested in the following two types of problems pertaining to the renewal insurance risk process (5.1). Unless otherwise stated, we assume throughout that $a > 0$, $0 < r, \rho \leq 1$ and $\delta, q, v, z \geq 0$.

1. **Two-sided exit problems:** For $0 \leq u \leq a$,

$$\mathbb{E}[r^{N_{T_a^{X,+}}} e^{-\delta T_a^{X,+} - v(H^X_{T_a^{X,+}} - T_a^{X,+})} 1_{\{T_a^{X,+} < T_0^{X,-}\}} | X_0 = u], \tag{5.3}$$

   and

$$\mathbb{E}[r^{N_{T_0^{X,-}}} e^{-\delta T_0^{X,-}} 1_{\{T_0^{X,-} < T_a^{X,+}, |X_{T_0^{X,-}}| > z\}} | X_0 = u], \tag{5.4}$$

   where $H_t^X := \inf\{s > t \geq 0 : X_{s_-} > X_s\}$ corresponds to the first jump time of $X$ after time $t$.

2. **Drawdown problems:** The joint law of $(N_{\tau_a}, \overline{N}_{\tau_a}, \tau_a, G_{\tau_a}, M_{\tau_a}, Y_{\tau_a} - a)$ and the law of $m_{\tau_a}$, where $m_t = \inf_{0 \leq s \leq t} X_s$ is the running minimum of $X$ at time $t$, i.e.,

$$\mathbb{E}[r^{N_{\tau_a}} \rho^{\overline{N}_{\tau_a}} e^{-\delta \tau_a - q G_{\tau_a} - v(M_{\tau_a} - X_0)} 1_{\{Y_{\tau_a} - a > z\}}], \tag{5.5}$$

   and, for $u \geq 0$,

$$\mathbb{P}(m_{\tau_a} < 0 | X_0 = u). \tag{5.6}$$

It is easy to see that (5.5) is independent of $X_0$. Also, since $\mathbb{P}(m_{\tau_a} < x | X_0 = u) = \mathbb{P}(m_{\tau_a} < 0 | X_0 = u - x)$, we consider (5.6) without loss of generality.

**Remark 5.2.1.** *We find that $m_{\tau_a}$ is the most challenging drawdown-related quantity subject to analyze in this chapter. One can consider $m_{\tau_a}$ jointly with the sextuple $(N_{\tau_a}, \overline{N}_{\tau_a}, \tau_a, G_{\tau_a}, M_{\tau_a}, Y_{\tau_a} - a)$, but the analysis will be rather involved. Hence, for ease of presentation and sake of conciseness and transparency, we choose to study the law of $m_{\tau_a}$ separately from the other drawdown quantities in (5.5).*

## 5.3 Two-sided exit problems

In this section, we study the exit problems (5.3) and (5.4) for the renewal insurance risk process $X$ by making a connection between risk processes and their corresponding queues (via their fluid flow analogue process), as well as some recent results on the spectrally negative MAPs. We recall that such a fluid flow connection between risk processes and queues can be found in e.g. Asmussen (1995) and has been applied by many authors to study various Markov-type risk processes (see, e.g., Ahn et al. (2007), Badescu et al. (2007), Ramaswami (2006) and Frostig et al. (2012)).

As shown later, sample paths of $X$ will be (locally) mapped one-on-one to sample paths of a stochastic process $U = \{U_t; t \geq 0\}$ defined as

$$U_t = x + ct - \sum_{i=1}^{D_t} L_i. \tag{5.7}$$

Here, $\{D_t; t \geq 0\}$ is a renewal process with iid PH$(\boldsymbol{\beta}, c\mathbf{B})$ interarrival times $\{\mathfrak{T}_i; i \in \mathbb{N}\}$ with LT $\mathbb{E}[e^{-s\mathfrak{T}_1}] = \widetilde{p}(s/c)$, and the jump sizes $\{L_i; i \in \mathbb{N}\}$ form a sequence of iid rv's with LT $\mathbb{E}[e^{-sL_1}] = \widetilde{k}(cs)$, independent of $\{D_t; t \geq 0\}$. Furthermore, associated with the interarrival times $\{\mathfrak{T}_i; i \in \mathbb{N}\}$, there exists an underlying continuous-time Markov process $J = \{J_t; t \geq 0\}$ recording the phase of the interarrival time over time with state space $\{1, \ldots, n\}$ and infinitesimal generator $c\,(\mathbf{B} + \mathbf{b}\boldsymbol{\beta})$. Thus, the bivariate process $(U, J)$ is a special case of the spectrally negative MAP. For the MAP $(U, J)$, let $\mathbf{F}^{(\delta)}(s)$ be the matrix analogue of the

Laplace exponent of a Lévy process, namely

$$\mathbb{E}[e^{-\delta t+sU_t}1_{\{J_t=j\}}|J_0=i]=(e^{\mathbf{F}^{(\delta)}(s)t})_{ij},$$

which is well known to be of the form

$$\mathbf{F}^{(\delta)}(s)=(cs-\delta)\mathbf{I}+c\mathbf{B}+c\mathbf{b}\boldsymbol{\beta}\widetilde{k}(cs).$$

For $x\in\mathbb{R}$, let $T_x^{U,+(-)}=\inf\{t\geq 0:U_t>(<)x\}$ be the first passage time of $U$. The two-sided exit problem for a spectrally negative MAP is introduced in Theorem 1.4.4. For completeness, these results are recalled here. For $0\leq x\leq a$,

$$\mathbb{E}[e^{-\delta T_a^{U,+}}1_{\{T_a^{U,+}<T_0^{U,-},J_{T_a^{U,+}}\}}|U_0=x,J_0]=\mathbf{W}^{(\delta)}(x)\mathbf{W}^{(\delta)}(a)^{-1}, \tag{5.8}$$

$$\mathbb{E}[e^{-\delta T_0^{U,-}-s|U_{T_0^{U,-}}|}1_{\{T_0^{U,-}<T_a^{U,+},J_{T_0^{U,-}}\}}|U_0=x,J_0]=\mathbf{Z}^{(\delta)}(s,x)-\mathbf{W}^{(\delta)}(x)\mathbf{W}^{(\delta)}(a)^{-1}\mathbf{Z}^{(\delta)}(s,a), \tag{5.9}$$

where $\mathbf{W}^{(\delta)}(x)$ is the $\delta$-scale matrix defined through its LT

$$\int_0^\infty e^{-sx}\mathbf{W}^{(\delta)}(x)\mathrm{d}x=\mathbf{F}^{(\delta)}(s)^{-1}, \tag{5.10}$$

and $\mathbf{Z}^{(\delta)}(s,x)$ is the second scale matrix defined as

$$\mathbf{Z}^{(\delta)}(s,x)=e^{sx}\left(\mathbf{I}-\int_0^x e^{-sy}\mathbf{W}^{(\delta)}(y)\mathrm{d}y\mathbf{F}^{(\delta)}(s)\right). \tag{5.11}$$

Kyprianou and Palmowski (2008) showed the existence of the scale matrix $\mathbf{W}^{(\delta)}(x)$, and Ivanovs and Palmowski (2012) further provided a probabilistic construction of the scale matrix and identified its transform.

We first generalize the two-sided exit results (5.8) and (5.9) for the MAP process (5.7) by incorporating the number of jumps prior to the first exit time.

### 5.3.1 A generalized result

We consider a generalization of the two-sided exit problem for a subclass of the spectrally negative MAPs. For completeness, we recall the definition of this process which can be found in e.g., Ivanovs and Palmowski (2012) and references therein. Define a process $\mathcal{U} = \{\mathcal{U}_t; t \geq 0\}$ and an irreducible continuous time Markov process $\mathcal{J} = \{\mathcal{J}_t; t \geq 0\}$ with finite state space $\{1, \ldots, n\}$ and infinitesimal generator $\mathbf{D}$. We say the bivariate process $(\mathcal{U}, \mathcal{J})$ is a MAP if given $\{\mathcal{J}_t = i\}$, the pair $(\mathcal{U}_{t+h} - \mathcal{U}_t, \mathcal{J}_{t+h})$ is independent of $\{(\mathcal{U}_s, \mathcal{J}_s); 0 \leq s \leq t\}$ and has the same law as $(\mathcal{U}_h - \mathcal{U}_0, \mathcal{J}_h)$ given $\{\mathcal{J}_0 = i\}$ for all $t, h \geq 0$ and $i \in \{1, \ldots, n\}$.

In what follows, we assume that when $\{\mathcal{J}_t = i\}$, the process $\mathcal{U}$ evolves as $\mathcal{U}^i$, a compound Poisson process perturbed by an independent Brownian motion (rather than a spectrally negative Lévy process as in the general model). More specifically, we define $\mathcal{U}^i = \{\mathcal{U}^i_t; t \geq 0\}$ as

$$\mathcal{U}^i_t = c_i t + \sigma_i B^i_t - \sum_{l=1}^{N^i_t} P^i_l,$$

where $c_i > 0$, $\sigma_i \geq 0$, $\{B^i_t; t \geq 0\}$ is a standard Brownian motion, $N^i = \{N^i_t; t \geq 0\}$ is a Poisson process with intensity rate $\lambda_i > 0$, and the iid positive jumps $\{P^i_l; l \in \mathbb{N}\}$ with df $G_{ii}$ (where $G_{ii}(0) = 0$ without loss of generality). The processes $\mathcal{U}^1, \mathcal{U}^2, ..., \mathcal{U}^n$ are assumed to be independent. In addition, a transition of $\mathcal{J}$ from state $i$ to state $j \neq i$ triggers a downward jump of $\mathcal{U}$ whose (absolute) size has df $G_{ij}$ ($G_{ij}(0)$ is not necessarily 0). Such a model is also called the *Markov-modulated Poisson process with diffusion.*

**Remark 5.3.1.** *In what follows, for simplicity, we only consider the case where $\sigma_i > 0$ for any $i \in \{1, \ldots, n\}$. Cases where some or all $\sigma_i$'s are $0$ can be obtained similarly.*

Let $\psi_i^{(r,\delta)}(z)$ be the generalized Laplace exponent of $\mathcal{U}^i$ defined as

$$\mathbb{E}[r^{N^i_t} e^{-\delta t + z\mathcal{U}^i_t} | \mathcal{U}_0 = 0] = e^{\psi_i^{(r,\delta)}(z)t},$$

where

$$\psi_i^{(r,\delta)}(z) = c_i z + \frac{\sigma_i^2}{2} z^2 - (\lambda_i + \delta) + r\lambda_i \int_0^\infty e^{-zy} G_{ii}\,(\mathrm{d}y)\,.$$

Also, define $\mathbf{F}^{(r,\delta)}(z)$ to be the $\mathcal{U}$-matrix analogue of the generalized Laplace exponent, that is

$$\mathbb{E}[r^{N_t} e^{-\delta t + z\mathcal{U}_t} 1_{\{\mathcal{J}_t = j\}} | \mathcal{U}_0 = 0, \mathcal{J}_0 = i] = (e^{\mathbf{F}^{(r,\delta)}(z)t})_{ij},$$

where $N_t$ represents the number of positive jumps of $\mathcal{U}$ by time $t$. From e.g., Ivanovs and Palmowski (2012), it is known that

$$\mathbf{F}^{(r,\delta)}(z) = \mathrm{diag}\{\psi_i^{(r,\delta)}(z)\}_{i=1}^n + \mathbf{D} \circ \widetilde{\mathbf{H}}^{(r)}(z), \tag{5.12}$$

where $\widetilde{\mathbf{H}}^{(r)}(z)$ is the LT of $\mathbf{H}^{(r)}(y) = \mathbf{G}^{(r)}(y) - \mathbf{P}^{(r)}(y)$, $\mathbf{G}^{(r)}(y) = \{G_{ij}^{(r)}(y)\}_{i,j=1}^n$ and $\mathbf{P}^{(r)}(y) = \mathrm{diag}\{G_{ii}^{(r)}(y)\}_{i=1}^n$ with $G_{ij}^{(r)}(y) = G_{ij}\,(0) + r(G_{ij}(y) - G_{ij}(0))$. The notation $A \circ B = (a_{ij} b_{ij})$ stands for the entry-wise matrix product. Letting $\mathbf{C} = \mathrm{diag}\{c_i\}_{i=1}^n$, $\mathbf{\Sigma} = \mathrm{diag}\{\frac{\sigma_i^2}{2}\}_{i=1}^n$ and $\mathbf{\Lambda} = \mathrm{diag}\{\lambda_i\}_{i=1}^n$, (5.12) can also be rewritten as

$$\mathbf{F}^{(r,\delta)}(z) = \mathbf{C}z + \mathbf{\Sigma} z^2 - \mathbf{\Lambda} - \delta \mathbf{I} + \mathbf{\Lambda} \circ \widetilde{\mathbf{P}}^{(r)}(z) + \mathbf{D} \circ \widetilde{\mathbf{H}}^{(r)}(z). \tag{5.13}$$

**Remark 5.3.2.** *As a special case of the above MAP, the process (5.7) can be recovered by choosing* $\mathbf{C} = c\mathbf{I}$, $\mathbf{\Sigma} = \mathbf{0}$, $\mathbf{\Lambda} = \mathrm{diag}\{(c\mathbf{b}\boldsymbol{\beta})_{ii}\}_{i=1}^n$, $\mathbf{D} = c\mathbf{B} + c\mathbf{b}\boldsymbol{\beta}$, $G_{ii}(y) = K(\frac{y}{c})$ *($y \geq 0$), and* $G_{ij}(y) = \frac{c\mathbf{B}_{ij}}{\mathbf{D}_{ij}} + \frac{(c\mathbf{b}\boldsymbol{\beta})_{ij}}{\mathbf{D}_{ij}} K(\frac{y}{c})$ *($y \geq 0$) for $j \neq i$.*

In what follows, we show the generalized two-sided exit results for the *Markov-modulated Poisson process with diffusion* $(\mathcal{U}, \mathcal{J})$.

**Lemma 5.3.1.** *For the Markov-modulated Poisson process with diffusion* $(\mathcal{U}, \mathcal{J})$*,*

$$\mathbb{E}[r^{N_{T_a^{\mathcal{U},+}}} e^{-\delta T_a^{\mathcal{U},+}} 1_{\{T_a^{\mathcal{U},+} < T_0^{\mathcal{U},-}, \mathcal{J}_{T_a^{\mathcal{U},+}}\}} | \mathcal{U}_0 = u, \mathcal{J}_0] = \mathbf{W}^{(r,\delta)}(u) \mathbf{W}^{(r,\delta)}(a)^{-1}, \tag{5.14}$$

*and*

$$\mathbb{E}[r^{N_{T_0^{\mathcal{U},-}}} e^{-\delta T_0^{\mathcal{U},-} - s|\mathcal{U}_{T_0^{\mathcal{U},-}}|} 1_{\{T_0^{\mathcal{U},-} < T_a^{\mathcal{U},+}, \mathcal{J}_{T_0^{\mathcal{U},-}}\}} | \mathcal{U}_0 = u, \mathcal{J}_0]$$
$$= \mathbf{Z}^{(r,\delta)}(s,u) - \mathbf{W}^{(r,\delta)}(u)\mathbf{W}^{(r,\delta)}(a)^{-1}\mathbf{Z}^{(r,\delta)}(s,a), \tag{5.15}$$

*where* $\mathbf{W}^{(r,\delta)}(u)$ *and* $\mathbf{Z}^{(r,\delta)}(s,u)$ *are the generalizations of (5.10) and (5.11), respectively, with* $\mathbf{F}^{(\delta)}(z)$ *replaced by a generalization* $\mathbf{F}^{(r,\delta)}(z)$ *defined in (5.13).*

*Proof.* First, we prove (5.14). Let

$$\boldsymbol{\chi}(u) = \mathbb{E}[r^{N_{T_a^{\mathcal{U},+}}} e^{-\delta T_a^{\mathcal{U},+}} 1_{\{T_a^{\mathcal{U},+} < T_0^{\mathcal{U},-}, \mathcal{J}_{T_a^{\mathcal{U},+}} = j\}} | \mathcal{U}_0 = u, \mathcal{J}_0 = i], \quad 0 \le u \le a.$$

From the theory on piecewise deterministic Markov process (see, e.g., Davis 1984 and Rolski et al. 1999) and Equation (5.13), we know that $\boldsymbol{\chi}(u)$ satisfies the following system of integro-differential equations:

$$\mathbf{C}\boldsymbol{\chi}'(u)+\boldsymbol{\Sigma}\boldsymbol{\chi}''(u)-(\delta\mathbf{I}+\boldsymbol{\Lambda})\boldsymbol{\chi}(u)+\int_0^u(\boldsymbol{\Lambda}\circ\mathbf{P}^{(r)}(y))\boldsymbol{\chi}(u-y)\mathrm{d}y+\int_{[0,u)}(\mathbf{D}\circ\mathbf{H}^{(r)}(y))\boldsymbol{\chi}(u-y)\mathrm{d}y = \mathbf{0}, \tag{5.16}$$

for $0 < u < a$, with boundary conditions $\boldsymbol{\chi}(0) = \mathbf{0}$ and $\boldsymbol{\chi}(a) = \mathbf{I}$ by Remark 5.3.1.

We consider a general solution $\boldsymbol{\zeta}(u)$ of (5.16) when $u \ge 0$ such that $\boldsymbol{\zeta}(0) = \mathbf{0}$. Taking LTs on both sides of the resulting equation leads to

$$\left(\mathbf{C}z + \boldsymbol{\Sigma}z^2 - \delta\mathbf{I} - \boldsymbol{\Lambda} + \boldsymbol{\Lambda}\circ\widetilde{\mathbf{P}}^{(r)}(z) + \mathbf{D}\circ\widetilde{\mathbf{H}}^{(r)}(z)\right)\widetilde{\boldsymbol{\zeta}}(z) = \boldsymbol{\Sigma}\boldsymbol{\zeta}'(0),$$

for $z \ge 0$. It follows from (5.13) that

$$\mathbf{F}^{(r,\delta)}(z)\widetilde{\boldsymbol{\zeta}}(z) = \boldsymbol{\Sigma}\boldsymbol{\zeta}'(0)$$

or alternatively,

$$\widetilde{\boldsymbol{\zeta}}(z) = \mathbf{F}^{(r,\delta)}(z)^{-1}\boldsymbol{\Sigma}\boldsymbol{\zeta}'(0).$$

By a Laplace inversion and from (5.10) for the generalized scale matrix $\mathbf{W}^{(r,\delta)}$, we obtain

$$\boldsymbol{\zeta}(u) = \mathbf{W}^{(r,\delta)}(u)\boldsymbol{\Sigma}\boldsymbol{\zeta}'(0), \quad u \geq 0.$$

Finally, by letting $\boldsymbol{\zeta}'(0) = \boldsymbol{\Sigma}^{-1}\mathbf{W}^{(r,\delta)}(a)^{-1}$, one concludes that $\boldsymbol{\zeta}(u) = \mathbf{W}^{(r,\delta)}(u)\mathbf{W}^{(r,\delta)}(a)^{-1}$ solves the initial value problem (5.16). Hence, we have $\boldsymbol{\zeta}(u) = \boldsymbol{\chi}(u)$ for $0 \leq u \leq a$ which ends the proof of (5.14).

Now, we prove (5.15). Let

$$\boldsymbol{\Psi}(u) = \mathbb{E}[r^{N_{T_0^{\mathcal{U},-}}} e^{-\delta T_0^{\mathcal{U},-} - s|\mathcal{U}_{T_0^{\mathcal{U},-}}|} 1_{\{T_0^{\mathcal{U},-} < T_a^{\mathcal{U},+}, \mathcal{J}_{T_0^{\mathcal{U},-}}\}} | \mathcal{U}_0 = u, \mathcal{J}_0], \quad 0 \leq u \leq a,$$

and

$$\mathbf{m}(u) = \mathbb{E}[r^{N_{T_0^{\mathcal{U},-}}} e^{-\delta T_0^{\mathcal{U},-} - s|\mathcal{U}_{T_0^{\mathcal{U},-}}|} 1_{\{T_0^{\mathcal{U},-} < \infty, \mathcal{J}_{T_0^{\mathcal{U},-}}\}} | \mathcal{U}_0 = u, \mathcal{J}_0], \quad u \geq 0.$$

It follows from the skip-free upward property of $\mathcal{U}$, the strong Markov property of $(\mathcal{U}, \mathcal{J})$ and (5.14) that

$$\boldsymbol{\Psi}(u) = \mathbf{m}(u) - \boldsymbol{\chi}(u)\mathbf{m}(a) = \mathbf{m}(u) - \mathbf{W}^{(r,\delta)}(u)\mathbf{W}^{(r,\delta)}(a)^{-1}\mathbf{m}(a). \tag{5.17}$$

From the theory on piecewise deterministic Markov process, the matrix $\mathbf{m}(u)$ for $u > 0$ satisfies

$$\begin{aligned}
\mathbf{0} &= \mathbf{C}\mathbf{m}'(u) + \boldsymbol{\Sigma}\mathbf{m}''(u) - (\delta\mathbf{I} + \boldsymbol{\Lambda})\mathbf{m}(u) \\
&\quad + \int_0^u \left(\boldsymbol{\Lambda} \circ \mathbf{P}^{(r)}(y)\right)\mathbf{m}(u-y)\mathrm{d}y + \int_u^\infty \boldsymbol{\Lambda} \circ \mathbf{P}^{(r)}(y)e^{-s(y-u)}\mathrm{d}y \\
&\quad + \int_{[0,u)} \left(\mathbf{D} \circ \mathbf{H}^{(r)}(y)\right)\mathbf{m}(u-y)\mathrm{d}y + \int_u^\infty \left(\mathbf{D} \circ \mathbf{H}^{(r)}(y)\right)e^{-s(y-u)}\mathrm{d}y.
\end{aligned} \tag{5.18}$$

Taking the LT with respect to $u$ on both sides of (5.18) leads to

$$\mathbf{F}^{(r,\delta)}(z)\widetilde{\mathbf{m}}(z) = (\mathbf{C}+\mathbf{\Sigma}z)\mathbf{m}(0) + \mathbf{\Sigma}\mathbf{m}'(0) - \mathbf{\Lambda}\circ\frac{\widetilde{\mathbf{P}}^{(r)}(s)-\widetilde{\mathbf{P}}^{(r)}(z)}{z-s} - \mathbf{D}\circ\frac{\widetilde{\mathbf{H}}^{(r)}(s)-\widetilde{\mathbf{H}}^{(r)}(z)}{z-s}, \tag{5.19}$$

for $z \geq 0$, where $\mathbf{m}(0) = \mathbf{I}$ from Remark 5.3.1. On the other hand, it is not difficult to see from (5.13) that

$$\mathbf{\Lambda}\circ\frac{\widetilde{\mathbf{P}}^{(r)}(s)-\widetilde{\mathbf{P}}^{(r)}(z)}{z-s} + \mathbf{D}\circ\frac{\widetilde{\mathbf{H}}^{(r)}(s)-\widetilde{\mathbf{H}}^{(r)}(z)}{z-s} = \frac{\mathbf{F}^{(r,\delta)}(s)-\mathbf{F}^{(r,\delta)}(z)}{z-s} + \mathbf{C} + \mathbf{\Sigma}\,(z+s)\,. \tag{5.20}$$

Thus, substituting (5.20) into (5.19) leads to

$$\mathbf{F}^{(r,\delta)}(z)\widetilde{\mathbf{m}}(z) = \mathbf{\Sigma}\mathbf{m}'(0) - \mathbf{\Sigma}s - \frac{\mathbf{F}^{(r,\delta)}(s)-\mathbf{F}^{(r,\delta)}(z)}{z-s}.$$

It follows that

$$\begin{aligned}
\widetilde{\mathbf{m}}(z) &= \widetilde{\mathbf{W}}^{(r,\delta)}(z)\left(\mathbf{\Sigma}\mathbf{m}'(0) - \mathbf{\Sigma}s + \frac{\mathbf{F}^{(r,\delta)}(z)-\mathbf{F}^{(r,\delta)}(s)}{z-s}\right)\\
&= \frac{\mathbf{I}-\widetilde{\mathbf{W}}^{(r,\delta)}(z)\mathbf{F}^{(r,\delta)}(s)}{z-s} + \widetilde{\mathbf{W}}^{(r,\delta)}(z)(\mathbf{\Sigma}\mathbf{m}'(0)-\mathbf{\Sigma}s)\\
&= \widetilde{\mathbf{Z}}^{(r,\delta)}(s,z) + \widetilde{\mathbf{W}}^{(r,\delta)}(z)\,(\mathbf{\Sigma}\mathbf{m}'(0)-\mathbf{\Sigma}s)\,. 
\end{aligned} \tag{5.21}$$

A Laplace inversion of (5.21) with respect to $z$ yields

$$\mathbf{m}(u) = \mathbf{Z}^{(r,\delta)}(s,u) + \mathbf{W}^{(r,\delta)}(u)\,(\mathbf{\Sigma}\mathbf{m}'(0)-\mathbf{\Sigma}s)\,, \tag{5.22}$$

for $u \geq 0$. Finally, substituting (5.22) into (5.17) completes the proof of (5.15).

□

Now we turn to a limiting case of (5.14), and the result will be used later. By (5.14), we have

$$
\begin{aligned}
\lim_{a\to\infty} \mathbf{W}^{(r,\delta)}(a-y)\mathbf{W}^{(r,\delta)}(a)^{-1} &= \lim_{a\to\infty} \mathbb{E}[r^{N_{T_a^{\mathcal{U},+}}} e^{-\delta T_a^{\mathcal{U},+}} 1_{\{T_a^{\mathcal{U},+}<T_0^{\mathcal{U},-},\mathcal{J}_{T_a^{\mathcal{U},+}}\}} | \mathcal{U}_0 = a-y, \mathcal{J}_0] \\
&= \lim_{a\to\infty} \mathbb{E}[r^{N_{T_y^{\mathcal{U},+}}} e^{-\delta T_y^{\mathcal{U},+}} 1_{\{T_y^{\mathcal{U},+}<T_{-a}^{\mathcal{U},-},\mathcal{J}_{T_y^{\mathcal{U},+}}\}} | \mathcal{U}_0 = 0, \mathcal{J}_0] \\
&= \mathbb{E}[r^{N_{T_y^{\mathcal{U},+}}} e^{-\delta T_y^{\mathcal{U},+}} 1_{\{T_y^{\mathcal{U},+}<\infty,\mathcal{J}_{T_y^{\mathcal{U},+}}\}} | \mathcal{U}_0 = 0, \mathcal{J}_0] \\
&:= \mathbf{\Upsilon}\left(y\right).
\end{aligned}
$$

By the strong Markov property of $(\mathcal{U},\mathcal{J})$ and the fact that $\mathcal{U}$ is skip-free upward, it is clear that

$$
\mathbf{\Upsilon}\left(x+y\right) = \mathbf{\Upsilon}\left(x\right)\mathbf{\Upsilon}\left(y\right), \tag{5.23}
$$

with $\mathbf{\Upsilon}\left(0\right) = \mathbf{I}$. The solution to (5.23) is well known to be

$$
\mathbf{\Upsilon}\left(x\right) = e^{\mathbf{Q}x}, \quad x \geq 0, \tag{5.24}
$$

for some matrix $\mathbf{Q}$. From the theory on piecewise deterministic Markov process, $\mathbf{\Upsilon}\left(-u\right)$ (for $u \leq 0$) satisfies the following system of integro-differential equations:

$$
\begin{aligned}
&-\mathbf{C}\mathbf{\Upsilon}'\left(-u\right) + \mathbf{\Sigma}\mathbf{\Upsilon}''(-u) - \left(\delta\mathbf{I}+\mathbf{\Lambda}\right)\mathbf{\Upsilon}(-u) \\
&+ \int_0^\infty (\mathbf{\Lambda}\circ\mathbf{P}^{(r)}(y))\mathbf{\Upsilon}(y-u)\mathrm{d}y + \int_{[0,\infty)} (\mathbf{D}\circ\mathbf{H}^{(r)}(y))\mathbf{\Upsilon}(y-u)\mathrm{d}y = \mathbf{0}.
\end{aligned} \tag{5.25}
$$

Substituting (5.24) into (5.25), it follows that $\mathbf{Q}$ must be a solution to

$$
-\mathbf{C}\mathbf{Q}+\mathbf{\Sigma}\mathbf{Q}^2-(\delta\mathbf{I}+\mathbf{\Lambda})+\int_0^\infty \left(\mathbf{\Lambda}\circ\mathbf{P}^{(r)}(y)\right)e^{\mathbf{Q}y}\mathrm{d}y+\int_{[0,\infty)} \left(\mathbf{D}\circ\mathbf{H}^{(r)}(y)\right)e^{\mathbf{Q}y}\mathrm{d}y = \mathbf{0}. \tag{5.26}
$$

In particular, when $\det\mathbf{F}^{(r,\delta)}(\rho_i) = 0$ for $i = 1, ..., n$ with $\rho_i \neq \rho_j$ for $i \neq j$, we denote by $\boldsymbol{\theta}_i$ the right-eigenvector associated to the eigenvalue 0 of $\mathbf{F}^{(r,\delta)}(\rho_i)$, i.e. $\mathbf{F}^{(r,\delta)}(\rho_i)\boldsymbol{\theta}_i = \mathbf{0}$. For $\mathbf{\Theta} = (\boldsymbol{\theta}_1, ..., \boldsymbol{\theta}_n)$, it can be shown that $\mathbf{Q} = -\mathbf{\Theta}\,\mathrm{diag}\left\{\rho_i\right\}_{i=1}^n \mathbf{\Theta}^{-1}$.

**Remark 5.3.3.** *For the special MAP discussed in Remark 5.3.2 with* $r = 1$ *and* $\delta = 0$*, (5.26) can be simplified to*

$$-\mathbf{Q} + \mathbf{B} + \mathbf{b}\boldsymbol{\beta}\widetilde{k}(-c\mathbf{Q}) = \mathbf{0},$$

*which is an important result in connection with the ruin probability (5.46).*

### 5.3.2 Main results

In this subsection, we will make use of the connections between the risk process $X$ and its fluid flow analogue process to find explicit expressions for the exit problems (5.3) and (5.4).

To state the following result, we define $\mathbf{W}^{(r,\delta)}(x)$ and $\mathbf{Z}^{(r,\delta)}(s,x)$, respectively, in the same way as in (5.10) and (5.11) with $\mathbf{F}^{(\delta)}(z)$ replaced by a generalization $\mathbf{F}^{(r,\delta)}(z) := (cz-\delta)\mathbf{I} + c\mathbf{B} + rc\mathbf{b}\boldsymbol{\beta}\widetilde{k}(cz)$. When $r = 1$, we have $\mathbf{W}^{(1,\delta)}(\cdot) = \mathbf{W}^{(\delta)}(\cdot)$ and $\mathbf{Z}^{(1,\delta)}(\cdot) = \mathbf{Z}^{(\delta)}(\cdot)$ for $\delta \geq 0$. Furthermore, we write $\mathbf{W}^{(0)}(\cdot) = \mathbf{W}(\cdot)$ and $\mathbf{Z}^{(0)}(\cdot) = \mathbf{Z}(\cdot)$.

**Lemma 5.3.2.** *For the MAP process (5.7) and* $0 \leq x \leq a$*,*

$$\mathbb{E}[r^{D_{T_a^{U,+}}} e^{-\delta T_a^{U,+}} 1_{\{T_a^{U,+} < T_0^{U,-}, J_{T_a^{U,+}}\}} | U_0 = x, J_0] = \mathbf{W}^{(r,\delta)}(x)\mathbf{W}^{(r,\delta)}(a)^{-1},$$

*and*

$$\mathbb{E}[r^{D_{T_0^{U,-}}} e^{-\delta T_0^{U,-} - s|U_{T_0^{U,-}}|} 1_{\{T_0^{U,-} < T_a^{U,+}, J_{T_0^{U,-}}\}} | U_0 = x, J_0] = \mathbf{V}^{(r,\delta,s)}(x,a),$$

*where*

$$\mathbf{V}^{(r,\delta,s)}(x,a) := \mathbf{Z}^{(r,\delta)}(s,x) - \mathbf{W}^{(r,\delta)}(x)\mathbf{W}^{(r,\delta)}(a)^{-1}\mathbf{Z}^{(r,\delta)}(s,a).$$

Furthermore, we naturally extend the definitions of $\mathbf{Z}^{(r,\delta)}(s,x)$ and $\mathbf{V}^{(r,\delta,s)}(x,a)$ to $x < 0$ by letting

$$\mathbf{V}^{(r,\delta,s)}(x,a) = \mathbf{Z}^{(r,\delta)}(s,x) = e^{sx}\mathbf{I}. \tag{5.27}$$

As for the fluid flow connection alluded above, it will be shown that the $X$-truncated process $A$ defined by

$$A_t = X_{t+T_1}, \quad t \geq 0_-, \tag{5.28}$$

plays a central role in the analysis of the renewal insurance risk process (5.1). We recall that $P_1$ is the size of the first downward jump of $X$ which occurs at time $T_1$. So given $T_1 = t$, we have $A_{0_-} = X_{T_{1-}} = x + ct$ and $A_0 = X_{T_1} = x + ct - P_1$.

The two-sided exit problem of $A$ is solved in the following lemma, where $T_x^{A,+(-)} := \inf\{t \geq 0_- : A_t > (<)x\}$. Also, we define $H_t^A := \inf\{s > t > 0 : A_{s_-} > A_s\}$ to be the time of the first jump of $A$ occurring after time $t$, and $N_t^A := N_{t+T_1}$ for $t \geq 0_-$ to be the number of jumps of $A$ by time $t$ with $N_{0_-}^A = 0$ and $N_0^A = 1$.

**Lemma 5.3.3.** *For $0 \leq y \leq a$, the process $A$ admits the following representation for its two-sided exit quantities:*

$$\begin{aligned}\alpha_1^{(r,\delta,v)}(y) &:= \mathbb{E}[r^{N^A_{T_a^{A,+}}} e^{-\delta T_a^{A,+} - v(H^A_{T_a^{A,+}} - T_a^{A,+})} 1_{\{T_a^{A,+} < T_0^{A,-}\}} | A_{0_-} = y] \\ &= e^{-\frac{\delta}{c}(a-y)} \boldsymbol{\beta} \mathbf{V}^{(r,\delta,\frac{v}{c})}(a-y, a)\mathbf{e},\end{aligned} \tag{5.29}$$

*and*

$$\begin{aligned}\alpha_2^{(r,\delta)}(y,z) &:= \mathbb{E}[r^{N^A_{T_0^{A,-}}} e^{-\delta T_0^{A,-}} 1_{\{T_0^{A,-} < T_a^{A,+}, |A_{T_0^{A,-}}| > z\}} | A_{0_-} = y] \\ &= r e^{\frac{\delta}{c} y} \boldsymbol{\beta} \mathbf{W}^{(r,\delta)}(a-y) \mathbf{W}^{(r,\delta)}(a)^{-1} e^{\mathbf{B}z} \mathbf{e}.\end{aligned} \tag{5.30}$$

*Proof.* By the so-called freezing/unfreezing transformation of sample paths (see, e.g., Ahn et al. (2007)), there exists a one-to-one (local) mapping between the sample paths of $A$ and $U$. More specifically, for $A_{0_-} = y$ and $U_0 = a - y$, it is not difficult to see that

$$(N^A_{T_a^{A,+}}, T_a^{A,+}, c(H^A_{T_a^{A,+}} - T_a^{A,+})) | T_a^{A,+} < T_0^{A,-} \overset{d}{=} (D_{T_0^{U,-}}, T_0^{U,-} + \frac{a-y}{c}, |U_{T_0^{U,-}}|) | T_0^{U,-} < T_a^{U,+}, \tag{5.31}$$

and

$$(N^A_{T_0^{A,-}}, T_0^{A,-}, |A_{T_0^{A,-}}|)|T_0^{A,-} < T_a^{A,+} \stackrel{d}{=} (D_{T_a^{U,+}} + 1, T_a^{U,+} - \frac{y}{c}, c(H^U_{T_a^{U,+}} - T_a^{U,+}))|T_a^{U,+} < T_0^{U,-}, \tag{5.32}$$

where $H_t^U := \inf\{s > t > 0 : U_{s_-} > U_s\}$. Figure 5.1 illustrates the sample-path mapping between $A$ and $U$, where the top and bottom pair of graphs relate to (5.31) and (5.32), respectively.

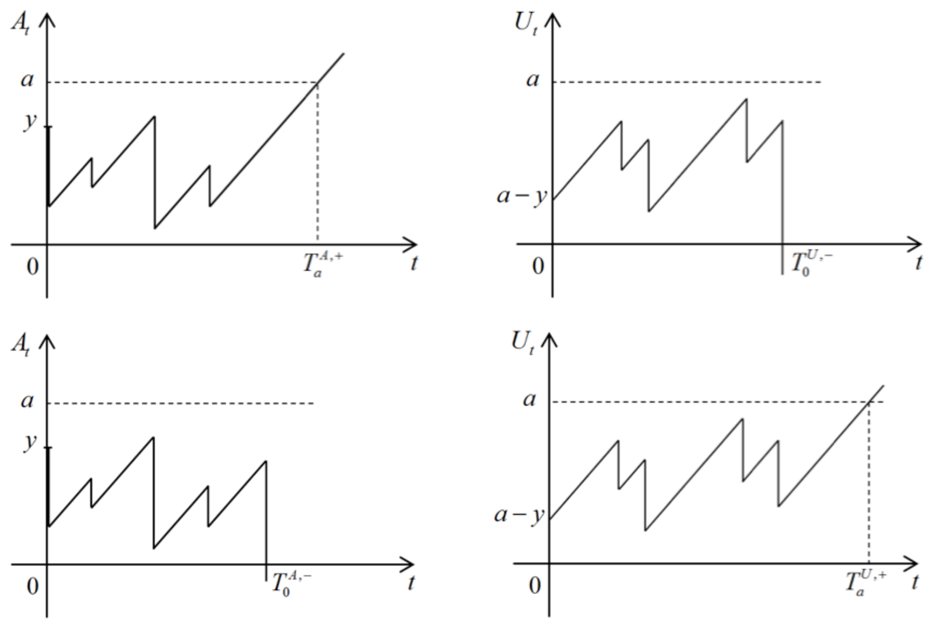


Figure 5.1: Sample-path mappings between $A$ and $U$

It follows from (5.31) and Lemma 5.3.2 that

$$\begin{aligned}\alpha_1^{(r,\delta,v)}(y) &= e^{-\frac{\delta}{c}(a-y)}\mathbb{E}[r^{D_{T_0^{U,-}}} e^{-\delta T_0^{U,-} - \frac{v}{c}|U_{T_0^{U,-}}|} 1_{\{T_0^{U,-} < T_a^{U,+}\}}|U_0 = a-y] \\ &= e^{-\frac{\delta}{c}(a-y)}\boldsymbol{\beta}\mathbf{V}^{(r,\delta,\frac{v}{c})}(a-y,a)\mathbf{e},\end{aligned}$$

which proves (5.29). Furthermore, for the MAP process (5.7), given $J_{T_a^{U,+}}$, it is well known that the overshoot $H^U_{T_a^{U,+}} - T_a^{U,+}$ is phase-type distributed with subintensity matrix $c\mathbf{B}$, independent of $\{(U_s, J_s)\,; s \le T_a^{U,+}\}$ (see, e.g., Chapter IX of Asmussen and Albrecher (2010)).

Hence, from (5.32) and Lemma 5.3.2, we have

$$\begin{aligned}\alpha_2^{(r,\delta,)}(y,z) &= re^{\frac{\delta}{c}y}\mathbb{E}[r^{D_{T_a^{U,+}}}e^{-\delta T_a^{U,+}}1_{\{T_a^{U,+}<T_0^{U,-},H^U_{T_a^{U,+}}-T_a^{U,+}>\frac{z}{c}\}}|U_0=a-y] \\ &= re^{\frac{\delta}{c}y}\boldsymbol{\beta}\mathbf{W}^{(r,\delta)}(a-y)\mathbf{W}^{(r,\delta)}(a)^{-1}e^{\mathbf{B}z}\mathbf{e},\end{aligned}$$

which completes the proof of (5.30). □

We are now ready to present results on the two-sided exit problems (5.3) and (5.4) of $X$.

**Theorem 5.3.1.** *For the renewal insurance risk process $X$, its two-sided exit quantities (5.3) and (5.4) can be expressed as, respectively,*

$$\begin{aligned}&\mathbb{E}[r^{N_{T_a^{X,+}}}e^{-\delta T_a^{X,+}-v(H^X_{T_a^{X,+}}-T_a^{X,+})}1_{\{T_a^{X,+}<T_0^{X,-}\}}|X_0=u] \\ &= e^{-\frac{\delta(a-u)}{c}}\int_0^\infty \boldsymbol{\beta}\mathbf{V}^{\left(r,\delta,\frac{v}{c}\right)}(a-u-ct,a)\mathbf{e}K(\mathrm{d}t),\end{aligned} \tag{5.33}$$

*and*

$$\begin{aligned}&\mathbb{E}[r^{N_{T_0^{X,-}}}e^{-\delta T_0^{X,-}}1_{\{T_0^{X,-}<T_a^{X,+},|X_{T_0^{X,-}}|>z\}}|X_0=u] \\ &= re^{\frac{\delta u}{c}}\int_0^{\frac{a-u}{c}}\boldsymbol{\beta}\mathbf{W}^{(r,\delta)}(a-u-ct)\mathbf{W}^{(r,\delta)}(a)^{-1}e^{\mathbf{B}z}\mathbf{e}K(\mathrm{d}t).\end{aligned} \tag{5.34}$$

*Proof.* By conditioning on $T_1$ (i.e., the first jump arrival time of $X$) and then considering the residual portion of the sample paths of $X$ after $T_1$ (namely, $A$), it follows from (5.29),

$$\begin{aligned}&\mathbb{E}[r^{N_{T_a^{X,+}}}e^{-\delta T_a^{X,+}-v(H^X_{T_a^{X,+}}-T_a^{X,+})}1_{\{T_a^{X,+}<T_0^{X,-}\}}|X_0=u] \\ &= \int_0^{\frac{a-u}{c}} e^{-\delta t}\alpha_1^{(r,\delta,v)}(u+ct)\,K(\mathrm{d}t) + e^{-\frac{\delta(a-u)}{c}}\int_{\frac{a-u}{c}}^\infty e^{-v\left(t-\frac{a-u}{c}\right)}K(\mathrm{d}t) \\ &= e^{-\frac{\delta(a-u)}{c}}\int_0^{\frac{a-u}{c}}\boldsymbol{\beta}\mathbf{V}^{\left(r,\delta,\frac{v}{c}\right)}(a-u-ct,a)\mathbf{e}K(\mathrm{d}t) + e^{-\frac{\delta(a-u)}{c}}\int_{\frac{a-u}{c}}^\infty e^{\frac{v}{c}(a-u-ct)}K(\mathrm{d}t) \\ &= e^{-\frac{\delta(a-u)}{c}}\int_0^\infty \boldsymbol{\beta}\mathbf{V}^{\left(r,\delta,\frac{v}{c}\right)}(a-u-ct,a)\mathbf{e}K(\mathrm{d}t),\end{aligned}$$

where the last step is due to the extended definition $\mathbf{V}^{\left(r,\delta,\frac{v}{c}\right)}(x,a) = e^{\frac{v}{c}x}\mathbf{I}$ for $x<0$ in (5.27).

Next we consider the exit of $X$ from below. Similarly, by conditioning on $T_1$ and using (5.30), we obtain

$$\begin{aligned}
&\mathbb{E}[r^{N_{T_0^{X,-}}} e^{-\delta T_0^{X,-}} 1_{\{T_0^{X,-}<T_a^{X,+}, |X_{T_0^{X,-}}|>z\}} | X_0 = u] \\
&= \int_0^{\frac{a-u}{c}} e^{-\delta t} \alpha_2^{(r,\delta)} (u+ct, z)\, K(\mathrm{d}t) \\
&= r e^{\frac{\delta u}{c}} \int_0^{\frac{a-u}{c}} \boldsymbol{\beta} \mathbf{W}^{(r,\delta)}(a-u-ct) \mathbf{W}^{(r,\delta)}(a)^{-1} e^{\mathbf{B}z} \mathbf{e} K(\mathrm{d}t).
\end{aligned}$$

This completes the proof of Theorem 5.3.1. □

## 5.4 Drawdown problems

The drawdown problems (5.5) and (5.6) are considered in Sections 5.4.1 and 5.4.2, respectively. As an application of our analysis, the constant dividend barrier problem for the renewal insurance risk process (5.1) will be investigated in Section 5.4.3.

### 5.4.1 Sextuple law of $(N_{\tau_a}, \overline{N}_{\tau_a}, \tau_a, G_{\tau_a}, M_{\tau_a}, Y_{\tau_a} - a)$

Thanks to the results on the two-sided exit problems in Section 5.3 , the joint law of $(N_{\tau_a}, \overline{N}_{\tau_a}, \tau_a, G_{\tau_a}, M_{\tau_a}, Y_{\tau_a} - a)$ is given in the following theorem. The process $A$ defined in (5.28) plays a central role in the proof of Theorem 5.4.1. Therefore, analogous to $X$, we define the following quantities related to the process $A$ by first recalling that $N_t^A := N_{t+T_1}$ for $t \geq 0_-$. Moreover, $\overline{N}_t^A := \overline{N}_{t+T_1}$ for $t \geq 0_-$ represents the number of excursions of $A$ from its running maximum. We also define $\tau_a^A := \inf\left\{t \geq 0 : Y_t^A \geq a\right\}$ where $Y_t^A := M_t^A - A_t$ and $M_t^A := \sup_{0_- \leq s \leq t} A_s$, and finally $G_t^A := \sup\{0_- \leq s \leq t : Y_s^A = 0\}$.

**Theorem 5.4.1.** *For the renewal insurance risk process (5.1), its sextuple law is given by*

$$\mathbb{E}[r^{N_{\tau_a}}\rho^{\overline{N}_{\tau_a}}e^{-\delta\tau_a-qG_{\tau_a}-v(M_{\tau_a}-X_0)}1_{\{Y_{\tau_a}-a>z\}}]=r\rho e^{\frac{\delta a}{c}}\widetilde{k}\left(\delta+q+cv\right)\frac{\boldsymbol{\beta}\mathbf{W}^{(r,\delta)}(0)\mathbf{W}^{(r,\delta)}(a)^{-1}e^{\mathbf{B}z}\mathbf{e}}{1-\rho\boldsymbol{\beta}\mathbf{V}^{(r,\delta+q,\frac{\delta+q}{c}+v)}(0,a)\mathbf{e}}.$$

*Proof.* By conditioning on $T_1$ and making use of the $X$-truncated process $A$, it is clear that

$$\mathbb{E}[r^{N_{\tau_a}}\rho^{\overline{N}_{\tau_a}}e^{-\delta\tau_a-qG_{\tau_a}-v(M_{\tau_a}-X_0)}1_{\{Y_{\tau_a}-a>z\}}]=\widetilde{k}\left(\delta+q+cv\right)\phi(z), \tag{5.35}$$

where

$$\phi(z):=\mathbb{E}[r^{N^A_{\tau_a^A}}\rho^{\overline{N}^A_{\tau_a^A}}e^{-\delta\tau_a^A-qG^A_{\tau_a^A}-v(M^A_{\tau_a^A}-A_{0\_})}1_{\{Y^A_{\tau_a^A}-a>z\}}]. \tag{5.36}$$

Note that $\phi(z)$ is independent of $A_{0\_}$. Therefore, without loss of generality, we assume that $A_{0\_}=a$ below. By conditioning on whether $T_0^{A,-}$ or $T_a^{A,+}$ occurs first and some exit-related quantities of $A$, it follows that

$$\begin{aligned}
\phi(z)&=\mathbb{E}[r^{N^A_{\tau_a^A}}\rho^{\overline{N}^A_{\tau_a^A}}e^{-\delta\tau_a^A-qG^A_{\tau_a^A}-v(M^A_{\tau_a^A}-a)}1_{\{Y^A_{\tau_a^A}-a>z,T_0^{A,-}<T_a^{A,+}\}}]\\
&\quad+\mathbb{E}[r^{N^A_{\tau_a^A}}\rho^{\overline{N}^A_{\tau_a^A}}e^{-\delta\tau_a^A-qG^A_{\tau_a^A}-v(M^A_{\tau_a^A}-a)}1_{\{Y^A_{\tau_a^A}-a>z,T_a^{A,+}<T_0^{A,-}\}}]\\
&=\rho\mathbb{E}[r^{N^A_{T_0^{A,-}}}e^{-\delta T_0^{A,-}}1_{\{|A_{T_0^{A,-}}|>z,T_0^{A,-}<T_a^{A,+}\}}]\\
&\quad+\rho\mathbb{E}[r^{N^A_{T_a^{A,+}}}e^{-(\delta+q)H^A_{T_a^{A,+}}-cv(H^A_{T_a^{A,+}}-T_a^{A,+})}1_{\{T_a^{A,+}<T_0^{A,-}\}}]\phi(z)\\
&=\rho\alpha_2^{(r,\delta)}\left(a,z\right)+\rho\alpha_1^{(r,\delta+q,\delta+q+cv)}\left(a\right)\phi(z).
\end{aligned}$$

Solving for $\phi(z)$ and then using Lemma 5.3.3, one obtains

$$\phi(z)=\frac{\rho\alpha_2^{(r,\delta)}\left(a,z\right)}{1-\rho\alpha_1^{(r,\delta+q,\delta+q+cv)}\left(a\right)}=\frac{r\rho e^{\frac{\delta a}{c}}\boldsymbol{\beta}\mathbf{W}^{(r,\delta)}(0)\mathbf{W}^{(r,\delta)}(a)^{-1}e^{\mathbf{B}z}\mathbf{e}}{1-\rho\boldsymbol{\beta}\mathbf{V}^{\left(r,\delta+q,\delta+q+\frac{v}{c}\right)}(0,a)\mathbf{e}}. \tag{5.37}$$

We complete the proof by substituting (5.37) into (5.35). $\square$

**Remark 5.4.1.** *Theorem 5.4.1 can be generalized by considering the so-called floored running* *maximum of $X$ defined as*

$$M_t = y \vee \sup_{0\le s\le t} X_s,$$

*where $y \ge X_0$ is a constant level and $y \vee x = \max\{y, x\}$. Without loss of generality, suppose that $X_0 = 0$, and hence $Y_0 = M_0 - X_0 = y$. In fact, we only need to consider the nontrivial case $0 \le y \le a$; otherwise $\tau_a = 0$ a.s. By (5.33), (5.34) and (5.36), one obtains*

$$\begin{aligned}
&\mathbb{E}[r^{N_{\tau_a}}\rho^{\overline{N}_{\tau_a}}e^{-\delta\tau_a-qG_{\tau_a}-v(M_{\tau_a}-X_0)}1_{\{Y_{\tau_a}-a>z\}}|Y_0=y]\\
&=\mathbb{E}[r^{N_{\tau_a}}\rho^{\overline{N}_{\tau_a}}e^{-\delta\tau_a-qG_{\tau_a}-vM_{\tau_a}}1_{\{Y_{\tau_a}-a>z,T_y^{X,+}<T_{y-a}^{X,-}\}}]\\
&+\mathbb{E}[r^{N_{\tau_a}}\rho^{\overline{N}_{\tau_a}}e^{-\delta\tau_a-qG_{\tau_a}-vM_{\tau_a}}1_{\{Y_{\tau_a}-a>z,T_{y-a}^{X,-}<T_y^{X,+}\}}]\\
&=e^{-vy}\mathbb{E}[r^{N_{T_y^{X,+}}}e^{-(\delta+q)T_y^{X,+}-(\delta+q+cv)(H^X_{T_y^{X,+}}-T_y^{X,+})}1_{\{T_y^{X,+}<T_{y-a}^{X,-}\}}]\phi(z)\\
&+e^{-vy}\mathbb{E}[r^{N_{T_{y-a}^{X,-}}}e^{-\delta T_{y-a}^{X,-}}1_{\{T_{y-a}^{X,-}<T_y^{X,+},|X_{T_{y-a}^{X,-}}|>z\}}]\\
&=e^{-\frac{\delta+q+cv}{c}y}\int_0^\infty \boldsymbol{\beta}\mathbf{V}^{(r,\delta+q,\frac{\delta+q}{c}+v)}(y-ct,a)\mathbf{e}K(\mathrm{d}t)\frac{r\rho e^{\frac{\delta a}{c}}\boldsymbol{\beta}\mathbf{W}^{(r,\delta)}(0)\mathbf{W}^{(r,\delta)}(a)^{-1}e^{\mathbf{B}z}\mathbf{e}}{1-\rho\boldsymbol{\beta}\mathbf{V}^{\left(r,\delta+q,\delta+q+\frac{v}{c}\right)}(0,a)\mathbf{e}}\\
&+re^{\frac{\delta(a-y)}{c}-vy}\int_0^{\frac{y}{c}}\boldsymbol{\beta}\mathbf{W}^{(r,\delta)}(y-ct)\mathbf{W}^{(r,\delta)}(a)^{-1}e^{\mathbf{B}z}\mathbf{e}K(\mathrm{d}t).
\end{aligned}\tag{5.38}$$

By letting $r = \rho = 1$ in Theorem 5.4.1 and using the fact that $\mathbf{Z}^{(\cdot)}(\cdot, 0) = \mathbf{I}$, we immediately obtain the following corollary.

**Corollary 5.4.1.** *For the renewal insurance risk process (5.1),*

$$\mathbb{E}[e^{-qG_{\tau_a}-v(M_{\tau_a}-X_0)}e^{-\delta(\tau_a-G_{\tau_a})}1_{\{Y_{\tau_a}-a>z\}}]=\frac{\widetilde{k}\,(q+cv)\,e^{\frac{\delta a}{c}}\boldsymbol{\beta}\mathbf{W}^{(\delta)}(0)\mathbf{W}^{(\delta)}(a)^{-1}e^{\mathbf{B}z}\mathbf{e}}{\boldsymbol{\beta}\mathbf{W}^{(q)}(0)\mathbf{W}^{(q)}(a)^{-1}\mathbf{Z}^{(q)}(\frac{q}{c}+v,a)\mathbf{e}}.$$

From Corollary 5.4.1, one concludes that the pairs $(G_{\tau_a}, M_{\tau_a})$ and $(\tau_a - G_{\tau_a}, Y_{\tau_a})$ are mutually independent. Intuitively, this independence means that the rising and crashing parts of a drawdown episode are independent in both time and level scales. It is known that such independence holds for Lévy processes. Not surprisingly, it also holds for the renewal process $X$ as it renews at jump instants.

By further letting $r = \rho = 1$ and $\delta = q = z = 0$ in (5.37), it follows that

$$\mathbb{E}[e^{-v(M^A_{\tau^A_a} - A_{0_-})}] = \frac{\boldsymbol{\beta}\mathbf{W}(0)\mathbf{W}(a)^{-1}\mathbf{e}.}{\boldsymbol{\beta}\mathbf{W}(0)\mathbf{W}(a)^{-1}\mathbf{Z}(v,a)\mathbf{e}}, \tag{5.39}$$

a result which will be called upon in Section 5.4.2. Next, we give a simple example of the result of Theorem 5.4.1.

**Example 5.4.1.** *Suppose that the jump sizes $\{P_i; i \in \mathbb{N}\}$ are exponentially distributed with mean $1/\mu$. By Theorem 5.4.1,*

$$\mathbb{E}[e^{-\delta\tau_a - qG_{\tau_a} - v(M_{\tau_a} - X_0)} 1_{\{Y_{\tau_a} - a > z\}}] = \frac{e^{\frac{\delta a}{c}} \widetilde{k}\,(\delta + q + cv)\, W^{(\delta+q)}(a)}{W^{(\delta)}(a) Z^{(\delta+q)}(\frac{\delta+q}{c} + v, a)} e^{-\mu z}.$$

*Furthermore, suppose that the interarrival times $\{T_i; i \in \mathbb{N}\}$ are also exponentially distributed with mean $1/\lambda$ for $\lambda > 0$. In this case, it is well known that*

$$W^{(\delta)}(x) = \frac{1}{c^2}\left(\frac{\lambda + cR_1}{R_1 - R_2} e^{R_1 x} + \frac{\lambda + cR_2}{R_2 - R_1} e^{R_2 x}\right),$$

*and*

$$Z^{(\delta)}(s,x) = \frac{\lambda + cR_1}{R_1 - R_2}\frac{s - R_2}{cs + \lambda} e^{R_1 x} + \frac{\lambda + cR_2}{R_2 - R_1}\frac{s - R_1}{cs + \lambda} e^{R_2 x},$$

*where $R_1$ and $R_2$ are the two solutions to $cs - \delta - c^2\mu s/(cs + \lambda) = 0$.*

### 5.4.2 Distribution of $m_{\tau_a}$

To provide a more comprehensive treatment of drawdowns, we now consider the problem (5.6), that is the law of $m_{\tau_a}$. A connection with the classical ruin probability for the renewal insurance risk model (5.1) will be made.

**Theorem 5.4.2.** *For the renewal process (5.1) with diagonalizable subintensity matrix* $\mathbf{B} =$

$\mathbf{\Omega}^{-1}\operatorname{diag}\{\gamma_i\}_{i=1}^n\mathbf{\Omega}$ *where the eigenvalues* $\{\gamma_i\}_{i=1}^n$ *are distinct, we have*

$$\mathbb{P}\left(m_{\tau_a}<0|X_0=u\right)=\boldsymbol{\beta}\mathbf{W}(0)\mathbf{W}(a)^{-1}\mathbf{\Omega}^{-1}\operatorname{diag}\left\{v_i\int_0^\infty\boldsymbol{\beta}\mathbf{V}^{(1,0,-\gamma_i)}(a-u-ct,a)\mathbf{e}K(\mathrm{d}t)\right\}_{i=1}^n\mathbf{\Omega}\mathbf{e}$$
$$+\int_0^{\frac{(a-u)\vee 0}{c}}\boldsymbol{\beta}\mathbf{W}(a-u-ct)\mathbf{W}(a)^{-1}\mathbf{e}K(\mathrm{d}t), \tag{5.40}$$

*where* $v_i:=\frac{1}{\boldsymbol{\beta}\mathbf{W}(0)\mathbf{W}(a)^{-1}\mathbf{Z}(-\gamma_i,a)\mathbf{e}}$.

*Proof.* We first consider the case $u\geq a$. By conditioning on $T_1$, we have

$$\mathbb{P}\left(m_{\tau_a}<0|X_0=u\right)=\int_0^\infty\sigma(u+ct)K(\mathrm{d}t),$$

where

$$\sigma(x):=\mathbb{P}\left(m^A_{\tau^A_a}<0|A_{0_-}=x\right),\quad x\geq 0,$$

with $m^A_t:=\inf_{0_-\leq s\leq t}A_s$. By conditioning on $M^A_{\tau^A_a}$ and using (5.30), one obtains

$$\begin{aligned}\sigma(x)&=\int_{[x,\infty)}\mathbb{P}\left(M^A_{\tau^A_a}\in\mathrm{d}y|A_{0_-}=x\right)\mathbb{P}\left(A_{T^{A,-}_{y-a}}<0|T^{A,-}_{y-a}<T^{A,+}_y,A_{0_-}=y\right)\\&=\int_{[x,\infty)}\mathbb{P}\left(M^A_{\tau^A_a}\in\mathrm{d}y|A_{0_-}=x\right)\frac{\boldsymbol{\beta}\mathbf{W}(0)\mathbf{W}(a)^{-1}e^{\mathbf{B}(y-a)}\mathbf{e}}{\boldsymbol{\beta}\mathbf{W}(0)\mathbf{W}(a)^{-1}\mathbf{e}}\\&=\frac{\boldsymbol{\beta}\mathbf{W}(0)\mathbf{W}(a)^{-1}\mathbb{E}[e^{\mathbf{B}(M^A_{\tau^A_a}-a)}|A_{0_-}=x]\mathbf{e}}{\boldsymbol{\beta}\mathbf{W}(0)\mathbf{W}(a)^{-1}\mathbf{e}}\\&=\frac{\boldsymbol{\beta}\mathbf{W}(0)\mathbf{W}(a)^{-1}e^{\mathbf{B}(x-a)}\mathbb{E}[e^{\mathbf{B}(M^A_{\tau^A_a}-A_{0_-})}]\mathbf{e}}{\boldsymbol{\beta}\mathbf{W}(0)\mathbf{W}(a)^{-1}\mathbf{e}}.\end{aligned}\tag{5.41}$$

It follows from the diagonalization $\mathbf{B}=\mathbf{\Omega}^{-1}\operatorname{diag}\{\gamma_i\}_{i=1}^n\mathbf{\Omega}$ and Equation (5.39) that

$$\sigma(x)=\boldsymbol{\beta}\mathbf{W}(0)\mathbf{W}(a)^{-1}\mathbf{\Omega}^{-1}\operatorname{diag}\{e^{(x-a)\gamma_i}v_i\}_{i=1}^n\mathbf{\Omega}\mathbf{e}, \tag{5.42}$$

where

$$v_i:=\frac{\mathbb{E}[e^{\gamma_i(M^A_{\tau^A_a}-A_{0_-})}]}{\boldsymbol{\beta}\mathbf{W}(0)\mathbf{W}(a)^{-1}\mathbf{e}}=\frac{1}{\boldsymbol{\beta}\mathbf{W}(0)\mathbf{W}(a)^{-1}\mathbf{Z}(-\gamma_i,a)\mathbf{e}}.$$

This further implies that

$$\mathbb{P}\left(m_{\tau_a} < 0 | X_0 = u\right) = \int_0^\infty \boldsymbol{\beta} \mathbf{W}(0) \mathbf{W}(a)^{-1} \boldsymbol{\Omega}^{-1} \operatorname{diag}\{e^{(u+ct-a)\gamma_i} v_i\}_{i=1}^n \boldsymbol{\Omega} \mathbf{e} K(\mathrm{d}t). \tag{5.43}$$

Next we consider the case $0 \le u < a$. By conditioning on whether $X$ crosses level 0 or level $a$ first (together with properties of this first passage time of $X$) and using Equation (5.42) and Theorem 5.3.1, we have

$$\begin{aligned}
&\mathbb{P}\left(m_{\tau_a} < 0 | X_0 = u\right) \\
=&\int_0^\infty \mathbb{P}(c(H^X_{T_a^{X,+}} - T_a^{X,+}) \in \mathrm{d}x, T_a^{X,+} < T_0^{X,-} | X_0 = u) \sigma(a+x) + \mathbb{P}\left(T_0^{X,-} < T_a^{X,+} | X_0 = u\right) \\
=&\boldsymbol{\beta} \mathbf{W}(0) \mathbf{W}(a)^{-1} \boldsymbol{\Omega}^{-1} \operatorname{diag}\{v_i \mathbb{E}[e^{c\gamma_i (H^X_{T_a^{X,+}} - T_a^{X,+})} 1_{\{T_a^{X,+} < T_0^{X,-}\}} | X_0 = u]\}_{i=1}^n \boldsymbol{\Omega} \mathbf{e} \\
&+ \int_0^{\frac{a-u}{c}} \boldsymbol{\beta} \mathbf{W}(a-u-ct) \mathbf{W}(a)^{-1} \mathbf{e} K(\mathrm{d}t) \\
=&\boldsymbol{\beta} \mathbf{W}(0) \mathbf{W}(a)^{-1} \boldsymbol{\Omega}^{-1} \operatorname{diag}\{v_i \int_0^\infty \boldsymbol{\beta} \mathbf{V}^{(1,0,-\gamma_i)}(a-u-ct, a) \mathbf{e} K(\mathrm{d}t)\}_{i=1}^n \boldsymbol{\Omega} \mathbf{e} \\
&+ \int_0^{\frac{a-u}{c}} \boldsymbol{\beta} \mathbf{W}(a-u-ct) \mathbf{W}(a)^{-1} \mathbf{e} K(\mathrm{d}t).
\end{aligned} \tag{5.44}$$

A unified representation of the cases $u \ge a$ and $0 \le u < a$ in (5.43) and (5.44) leads to (5.40). □

**Remark 5.4.2.** *In Theorem 5.4.2, it is assumed that the subintensity matrix* $\mathbf{B}$ *has distinct eigenvalues* $\{\gamma_i\}_{i=1}^n$*. If this assumption is not satisfied, i.e.* $\mathbf{B}$ *has eigenvalues with multiplicity greater than* 1*, one can make use of Theorem 8.2.2 in Rolski et al. (1999) to evaluate the matrix exponential function* $\mathbb{E}[e^{\mathbf{B}(M^A_{\tau_a^A} - A_{0\_})}]$ *in (5.41).*

The law of $m_{\tau_a}$ is particularly relevant in contexts where ruin and drawdown events are analyzed concurrently. Indeed, we have

$$\mathbb{P}\left(m_{\tau_a} < 0 | X_0 = u\right) = \mathbb{P}\left(T_0^{X,-} \le \tau_a | X_0 = u\right). \tag{5.45}$$

This implies that the ruin probability of $X$ can be recovered as a limiting case of (5.45) by letting $a \to \infty$. More specifically,

$$\begin{aligned}\mathbb{P}\left(T_0^{X,-} < \infty | X_0 = u\right) &= \lim_{a\to\infty} \mathbb{P}(T_0^{X,-} < T_a^{X,+} | X_0 = u) \\ &= \lim_{a\to\infty} \int_0^{\frac{a-u}{c}} \boldsymbol{\beta} \mathbf{W}(a-u-ct)\mathbf{W}(a)^{-1}\mathbf{e}K(\mathrm{d}t).\end{aligned}$$

It is shown in (5.24) that for any fixed $y \in (0, a)$,

$$\lim_{a\to\infty} \mathbf{W}(a-y)\mathbf{W}(a)^{-1} = e^{\mathbf{Q}y},$$

where $\mathbf{Q} = \mathbf{B} + \mathbf{b}\boldsymbol{x}$ and $\boldsymbol{x}$ solves the linear equation $\boldsymbol{x} = \boldsymbol{\beta}\widetilde{k}(-c(\mathbf{B} + \mathbf{b}\boldsymbol{x}))$. This immediately yields the ruin probability

$$\mathbb{P}\left(T_0^{X,-} < \infty | X_0 = u\right) = \boldsymbol{\beta}\widetilde{k}\left(-c\mathbf{Q}\right) e^{\mathbf{Q}u}\mathbf{e}, \tag{5.46}$$

a result stated in e.g., Theorem 4.4 of Asmussen and Albrecher (2010).

### 5.4.3 Constant dividend barrier risk model

Drawdown problems are closely related to the analysis of the so-called constant dividend barrier model in risk theory (see e.g., Chapter 6 of Kyprianou (2013) and references therein). For a fixed constant dividend barrier $a > 0$, we consider the process $X^a = \{X_t^a; t \geq 0\}$ defined as

$$\mathrm{d}X_t^a = \begin{cases} \mathrm{d}X_t, & X_t^a < a, \\ -\mathrm{d}S_t, & X_t^a = a, \end{cases} \tag{5.47}$$

with $X_0^a = u \in [0, a]$. We recall that $\{S_t; t \geq 0\}$ is the compound renewal process defined in (5.2). In this subsection, we are interested in the expected discounted dividend payments until ruin, that is,

$$\mathcal{V}(u) = \mathbb{E}[\int_0^{T_0^{X^a,-}} ce^{-\delta t} 1_{\{X_t^a = a\}} \mathrm{d}t | X_0^a = u],$$

where $\delta > 0$ is the discounted rate. A comment on the case $\delta = 0$ will follow the proof of Proposition 5.4.1.

**Proposition 5.4.1.** *For the constant dividend barrier model (5.47), the expected discounted dividend payments until ruin is given by*

$$\begin{aligned}\mathcal{V}(u) &= \frac{c}{\delta} e^{-\frac{\delta(a-u)}{c}} \int_0^\infty \boldsymbol{\beta} \mathbf{V}^{(1,\delta,0)}(a-u-ct, a)\mathbf{e} K(\mathrm{d}t) \\ &\quad - \frac{c}{\delta} e^{-\frac{\delta(a-u)}{c}} \frac{1 - \boldsymbol{\beta}\mathbf{V}^{(1,\delta,0)}(0,a)\mathbf{e}}{1 - \boldsymbol{\beta}\mathbf{V}^{\left(1,\delta,\frac{\delta}{c}\right)}(0,a)\mathbf{e}} \int_0^\infty \boldsymbol{\beta}\mathbf{V}^{\left(1,\delta,\frac{\delta}{c}\right)}(a-u-ct, a)\mathbf{e} K(\mathrm{d}t).\end{aligned}$$

*where $u \in [0, a]$ and $\delta > 0$.*

*Proof.* It is easy to see that the time to ruin $T_0^{X^a,-} = \inf\{t \geq 0 : X_t^a < 0\}$ has the same distribution as the drawdown time $\tau_a$ (associated to the process $X$) given that $Y_0 = a - u$. Moreover, since the cumulative dividend at time $t$ is given by $M_t = a \vee \sup_{0 \leq s \leq t} X_s$, we have

$$\mathcal{V}(u) = \mathbb{E}[\int_0^{\tau_a} e^{-\delta t} \mathrm{d}M_t | X_0 = u, Y_0 = a - u]. \tag{5.48}$$

To derive $\mathcal{V}(u)$, we first consider the $A$-analogue defined as $\mathcal{V}^A(a) := \mathbb{E}[\int_0^{\tau_a^A} e^{-\delta t} \mathrm{d}M_t^A | A_{0_-} = a]$. By conditioning on $(T_a^{A,+}, H^A_{T_a^{A,+}})$ whenever $\{T_a^{A,+} < T_0^{A,-}\}$, one finds

$$\begin{aligned}\mathcal{V}^A(a) &= \mathbb{E}[\int_{T_a^{A,+}}^{H^A_{T_a^{A,+}}} e^{-\delta t} 1_{\{T_a^{A,+} < T_0^{A,-}\}} \mathrm{d}M_t^A | A_{0_-} = a] + \mathbb{E}[\int_{H^A_{T_a^{A,+}}}^{\tau_a^A} e^{-\delta t} 1_{\{T_a^{A,+} < T_0^{A,-}\}} \mathrm{d}M_t^A | A_{0_-} = a] \\ &= \frac{c}{\delta}\left(\alpha_1^{(1,\delta,0)}(a) - \alpha_1^{(1,\delta,\delta)}(a)\right) + \alpha_1^{(1,\delta,\delta)}(a)\mathcal{V}^A(a).\end{aligned}$$

Solving for $\mathcal{V}^A(a)$ and then using Lemma 5.3.3, we have

$$\mathcal{V}^A(a) = \frac{c}{\delta} \frac{\alpha_1^{(1,\delta,0)}(a) - \alpha_1^{(1,\delta,\delta)}(a)}{1 - \alpha_1^{(1,\delta,\delta)}(a)} = \frac{c}{\delta} \frac{\boldsymbol{\beta}\left(\mathbf{V}^{(1,\delta,0)}(0,a) - \mathbf{V}^{\left(1,\delta,\frac{\delta}{c}\right)}(0,a)\right)\mathbf{e}}{1 - \boldsymbol{\beta}\mathbf{V}^{\left(1,\delta,\frac{\delta}{c}\right)}(0,a)\mathbf{e}}. \tag{5.49}$$

Similarly,

$$\begin{aligned}\mathcal{V}(u) &= \mathbb{E}[\int_{T_a^{X,+}}^{H^X_{T_a^{X,+}}} e^{-\delta t} 1_{\{T_a^{X,+}<T_0^{X,-}\}} \mathrm{d}M_t + \int_{H^X_{T_a^{X,+}}}^{\tau_a} e^{-\delta t} 1_{\{T_a^{X,+}<T_0^{X,-}\}} \mathrm{d}M_t | X_0 = u, Y_0 = a-u] \\ &= \mathbb{E}[\frac{c}{\delta}(e^{-\delta T_a^{X,+}} - e^{-\delta H^X_{T_a^{X,+}}}) 1_{\{T_a^{X,+}<T_0^{X,-}\}} | X_0 = u] + \mathbb{E}[e^{-\delta H^X_{T_a^{X,+}}} 1_{\{T_a^{X,+}<T_0^{X,-}\}} | X_0 = u] \mathcal{V}^A(a).\end{aligned}$$

The expression of $\mathcal{V}(u)$ follows immediately by substituting (5.33) and (5.49) into the above equation. □

Note that from (5.48), the case $\delta = 0$ is straightforward as $\mathcal{V}(u) = \mathbb{E}[M_{\tau_a} | X_0 = u, Y_0 = a-u]$, which can be obtained easily from (5.38) by a standard differentiating argument.

# Chapter 6

# Conclusion and Future Research

In this thesis, I have continued ongoing efforts in the broad field of quantitative risk management to more accurately measure, and better manage risks pertaining to an insurance portfolio. More precisely, I have developed a better understanding of the insurance risk process in more practical settings, and have shown that the overall risk is managed more effectively when some policy adjustments are adopted.

I first consider risk models in which adjustment policies are applied in response to the recent trend in claim experience. Several adjustment mechanisms have been considered in Chapters 2, 3 and 4. Chapter 2 considers the compound Poisson risk model with the adaptive premium policy, where the premium rate will change when there is a new drop in the surplus level. Chapter 3 involves the review time within the compound Poisson risk model, and the premium rate will change based on the increments between two successive review times. Chapter 4 proposes a regime-switching spectrally negative Lévy model, where not only the premium rate but also the claim arrival dynamic could change between two regimes based on the drawdown process. As one can see, these three adaptive policies have their own features to characterize the dynamic of the insurance risk processes, and it is shown that each policy helps to reduce the overall risk in comparison to a static strategy. The main approach to analyze the Gerber-Shiu function in the first two models is by conditioning on the first drop at a the renewal point to derive the defective renewal or integral equation, and the main approach to analyze the exit problem (including the ruin probability) for the third model is to make use of the strong Markov property to derive a differential equation satisfied by the quantities of interest. In the second part of this thesis (Chapter 5), the drawdown-related quantities and two-sided exit problems are analyzed in a Sparre Andersen (renewal) risk model, where the fluid flow technique is used to build connections with some existing Markov arrival processes.

Based on the research done in this thesis, there are several extensions that I intend to work on in the future.

The first direction is to study in depth the proposed risk models. For instance, future research can be done to analyze the LT or density of some occupation times (such as the total time spent below level 0) or Parisian ruin problem related to the Sparre Andersen risk model in Chapter 5 using the same fluid flow methodology; see, e.g., Landriault et al. (2011a) and Loeffen et al. (2013).

The second direction is to generalize some assumptions of the proposed risk models in Chapters 2, 3 and 4. For instance, in Chapter 4, although the drawdown-based regime-switching risk model captures some practical features of the changes in the insurance surplus process, there are several limitations of the current model which may inspire future extensions. First, the drawdown threshold $a$ is constant. According to other related studies (Kyprianou and Zhou (2009), Li et al. (2013), Zhang (2015)), it appears that this can be generalized to an increasing function of the running maximum. Second, the level of financial distress may ideally not only depend on the size of drawdown but also its duration (e.g., Landriault et al., (2015c)). Third, the condition to completely resolve the financial distress may be too restrictive as a historical running maximum may never be recovered. Therefore, we may choose another threshold level $b(< a)$, and suppose that the financial distress is recovered if the drawdown size is no greater than $b$.

The third direction is to build on these adaptive models to suggest new ones. For instance, in Chapter 3, the experience-based premium policy model could be considered in a more general claim arrival process, such as the Markov-type risk models (see, e.g., Albrecher and Boxma (2005), Ahn and Badescu (2007), Lu and Tsai (2007), and Asmussen and Albrecher (2010) for more details). An idea of an insurance risk model of interest is one where there are two states $\theta = 1, 2$, where $\theta$ might be an external environment parameter. When the surplus process is in state $i$, the inter-arrival time is exponentially distributed with mean $1/\lambda_i$ and the premium rate charged is $c_i$. However, given that the parameter $\theta$ is unknown and unobservable, one needs to predict the external environment parameter $\theta$ at review times to determine which premium rate should be effective over the next period. Since the premium rate can only be changed at random review times, so there may be a time delay to change the premium rates. Also, it is of interest to consider other variants for changing the premium rates. For example, a premium policy conducted at claim instants and a dynamic premium policy involving credibility theory (see, e.g., Tsai and Parker (2004) and Loisel and Trufin (2013)) is likely to yield analytically tractable results.

The last direction is to consider optimality questions in the context of these risk models. The models discussed in this thesis are parametric models, and hence it is of interest to examine how these parameters can be chosen to maximize (or minimize) some objective functions. In insurance, some common objective functions may be to minimize the ruin probability or maximize the cumulative discounted dividend before ruin (see, e.g., Browne (1995) and Hipp and Plum (2003)). In the context of drawdowns, some early warning criteria can be developed and thus classical optimization problems with exit times could be studied accordingly.

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