The Master Equality Polyhedron: Two-Slope Facets and Separation Algorithm

by

Xiaojing Wang

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Mathematics
in
Combinatorics and Optimization

Waterloo, Ontario, Canada, 2015

Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

This thesis presents our findings about the Master Equality Polyhedron (MEP), an extension of Gomory's Master Group Polyhedron. We prove a theorem analogous to Gomory and Johnson's two-slope theorem for the case of the MEP. We then show how such theorem can lead to facet defining inequalities for MEPs or extreme inequalities for an extension of the infinite group model. We finally study certain coefficient-restricted inequalities for the MEP and how to separate them.

Acknowledgements

First and the most important, I would like to thank my supervisor, Ricardo Fukasawa, for his great guidance and support throughout my pursuit of a Master's Degree. He has been an exceptional mentor of mine, and I learned a tremendous amount of not only knowledge, but also how to conduct research, from him. Without his patience and dedication, I would not have accomplished this work.

I also thank the readers for taking their time to read my thesis and giving precious suggestions for improvement.

Last but not least, I thank my family and friends for their constant support throughout my studies towards my Master's.

To Infinity and Beyond

Table of Contents

List of Figures viii

1 Introduction 1

1.1 Basic Polyhedral Theory . . . 2

1.2 Cutting Planes and GMI Cuts . . . 5

1.3 The Corner Polyhedron . . . 6

1.4 The Master Cyclic Group Polyhedron . . . 7

1.5 The Master Equality Polyhedron . . . 9

1.6 The Infinite Relaxation Model . . . 12

1.7 Outline . . . 14

2 Two-Slope Theorem for The MEP 16

2.1 Basic Properties . . . 17

2.2 The Two-Slope Theorem . . . 22

3 A Modified Infinite Relaxation Model 30

3.1 Derivation . . . 30

3.2 Facet Defining Inequalities of $MEPm(n, r)$ . . . 31

3.3 Basic Properties of the Modified Infinite Relaxation Model . . . 33

3.4 Using Inequalities for $MEP(n, r)$ to Build Inequalities for $IRm(q, r)$ . . . 35

4 Two-Slope Theorem for Extreme Functions **40**

4.1 Extremality of Extended Two-Slope Inequalities 40

4.2 Extremality of Regular Two-Slope Functions 46

4.3 MIP Extension of $IRm(q, r)$ 52

5 The Separation Problem Over A Class of Inequalities **55**

5.1 An Non-Decreasing Normalization 56

5.2 A Partial Separation Algorithm for Subproblems of $MEP(n, r)$ 60

6 Conclusion **66**

References **68**

List of Figures

4.1 4-subdivision of an inequality π for MEP(3, 1) 41

4.2 A Regular Two-Slope Function 47

5.1 An inequality π that is subadditive over I for MEP(8, 5) with $\pi(-n) = 0$. 57

5.2 A renormalization of π, namely π', with $\pi'(-1) = 0$ 58

5.3 Positioning of b_m's relative to c_l's 61

Chapter 1

Introduction

Integer programming (IP) is a branch of modern optimization that utilizes mathematical tools such as polyhedral theory to formulate the theoretical framework where optimization problems are described. Some well-known problems that can be formulated as an IP include the maximum weight matching problem, the set covering problem, and the famous traveling salesman problem (TSP). Specifically, all the variables in these problems can only have integer values. When some but not all of the variables are required to be integral, the problem is called a mixed integer programming (MIP) problem.

On the other hand, IPs and MIPs with linear constraints, i.e. integer linear programs (ILP) and mixed integer linear programs (MILP), can be viewed as a linear program (LP) with additional integrality constraints. A number of algorithms have been developed to solve LPs, such as the simplex method and the ellipsoid method [17]. The ellipsoid method was the first polynomial-time algorithm for LPs, even though it is not necessarily computationally practical. Meanwhile, the simplex method is widely used in practice and efficient in most cases, but many of its variants are an exponential time algorithms for certain LPs, as shown by Klee and Minty [16] and others. To date, no polynomial time worst case bound has been prove for any variant of the simplex method. When it comes to IPs and MIPs, the addition of the integrality constraint significantly increases the difficulty of the problem. Some specific classes of IPs have polynomial time algorithms, meaning that solutions to them can be computed efficiently. Nevertheless, as of now, there are no known algorithms to solve general IPs in polynomial time, and they are NP-hard.

Numerous techniques are developed to tackle IPs and MIPs. Some result in advancements toward solving a specific problem such as the TSP, and some improve general models, potentially making them easier to solve. Cutting planes are a common tool applicable to general IPs and MIPs. Ideal cutting planes describe the convex hull of the feasible solutions of the original IP or MIP, so that the optimal solution for a LP over the convex hull is feasible for the original problem. However, such cutting planes are hard to find. Our research considers strong cutting planes related to a framework named master equality polyhedron (MEP), which is generic for IPs. To enable further discussion about the MEP, we introduce some polyhedral theory first.

1.1 Basic Polyhedral Theory

Polyhedral theory is one of the common frameworks to use when it comes to studying general IPs. To begin, we formally define what a polyhedron is.

Definition 1.1.1 *A* ***polyhedron*** $P \subseteq \mathbb{R}^n$ *is a set of points that satisfy a finite number of linear inequalities; that is,* $P := \{x \in \mathbb{R}^n : \mathcal{A}x \leq \beta\}$*, where* $\mathcal{A}, \beta$ *have real entries only [17, p.85].*

In mixed integer programming, usually we are interested in feasible solutions of some general system

$$\begin{array}{rrcl} max & c^T x & & \\ s.t. & Ax & = & b \\ & x & \geq & 0 \\ & x_I & \in & \mathbb{Z}^{|I|} \end{array} \tag{1.1}$$

where A is an $m \times n$ rational matrix, b is a rational vector, and I is the index set for the integer variables. If $I = \{1, \ldots, n\}$, (1.1) is an *pure integer program*, otherwise it is a *mixed integer program*. Notice we are not losing any generality by having the equality in $Ax = b$, or assuming the constraint $x \geq 0$, because any MIP can be re-written into this form. We may also assume A has full row rank, for if not, we may delete certain rows for A to make it so, or the problem is trivially infeasible.

The dimension is an important parameter of a polyhedron. Before giving the formal definition of dimension, we introduce two important concepts, linear independence and affine independence.

Definition 1.1.2 *A set of points* $x^1, \ldots, x^k \in \mathbb{R}^n$ *is* ***linearly independent*** *if* $\lambda_i = 0$ *for* $i \in \{1, \ldots, k\}$ *is the unique solution to* $\sum_{i=1}^k \lambda_i x^i = 0$ *[17, p.83].*

Definition 1.1.3 *A set of points $x^1, \dots, x^k \in \mathbb{R}^n$ is **affinely independent** if $\lambda_i = 0$ for $i \in \{1, \dots, k\}$ is the unique solution to $\sum_{i=1}^k \lambda_i x^i = 0$ and $\sum_{i=1}^k \lambda_i = 0$ [17, p.84].*

Note that linear independence implies affine independence, but affine independence does not necessarily imply linear independence. Moreover, a set of points $x^1, \dots, x^k \in \mathbb{R}^n$ are affinely independent if and only if $(x^2 - x^1, x^3 - x^1, \dots, x^k - x^1)$ are linearly independent. The dimension of a polyhedron is defined based on affine independence.

Definition 1.1.4 *A polyhedron P has **dimension** $dim(P) = k$ if the maximum number of affinely independent points in P is $k+1$ [17, p.86].*

Notice the feasible region, S, of (1.1) is generally not convex, which is computationally hard to handle. However, it turns out solving IPs is equivalent to solving LPs over the *convex hull* of S, denoted as $conv(S)$, which is the smallest convex set that contains S. Recall we mentioned cutting planes are frequently used for IPs and MIPs. They are essentially inequalities satisfied by all feasible points of a given problem, which are exactly the valid inequalities. The formal definition is the following.

Definition 1.1.5 *The inequality $\pi^T x \geq \pi_0$ [or (π, π_0)] is called a **valid inequality** for a polyhedron P if for any $x_0 \in P$, $\pi^T x_0 \geq \pi_0$ [17, p.88].*

When there are no ambiguities, we may say (π, π_0) is valid for P, if $\pi^T x \geq \pi_0$ is a valid inequality for P. Moreover, if the value of π_0 is known, we may say π valid for P.

Moreover, up to normalization, π_0 could only have three possible values, $1, 0$, and -1. Accordingly, by considering the normalized version of valid inequalities, we divide them into three types.

Definition 1.1.6 *Suppose (π, π_0) is valid for a polyhedron P. If $\pi_0 = 1$, π is 1**-valid**; if $\pi_0 = 0$ and π is not 1-valid for P, then π is 0**-valid**; and if $\pi_0 = -1$ and π is not 0-valid for P, then π is -1**-valid**.*

Since we are interested in cutting planes, it is necessary to consider strong cuts for general polyhedra. Intuitively speaking, the most desirable cutting planes for a polyhedron is its facets. Here we define them in mathematical terms.

Definition 1.1.7 *If $\pi^T x \geq \pi_0$ is a valid inequality for P, and $F = \{x \in P : \pi^T x = \pi_0\}$, then F is called a* ***face*** *of P [17, p. 88].*

We say F is the face defined by $\pi^T x \geq \pi_0$, and that $\pi^T x \geq \pi_0$ defines face F.

Definition 1.1.8 *A face F of P is a* ***facet*** *of P if $dim(F) = dim(P) - 1$ [17, p. 89]. An inequality $\pi x \geq \pi_0$ is* ***facet defining*** *for P if the face it defines is a facet of P.*

Now, we introduce another important class of faces known as extreme points.

Definition 1.1.9 *A non-empty face F of P is an* ***extreme point*** *of P if $dim(F) = 0$.*

In some sense, facets and extreme points are the "largest" and "smallest" proper faces of a polyhedron. Notice that inequalities that define facets of $conv(S)$ give very strong cuts because any polyhedron P can be described by its necessary facets. Therefore, facet defining inequalities are considered the most important and desired cutting planes.

Observe that for a polyhedron in $\mathbb{R}^n$ with extreme points, there always exists n linearly independent and *tight* facet defining inequalities meeting at any extreme point, i.e. these facet defining inequalities are at equality at this point. By "linearly independent", we mean none of these facet defining inequalities can be written as a linear combination of the rest. It is also notable that an extreme point p cannot be written as a convex combination of any $x^1, x^2 \in P$, unless $x^1 = x^2 = p$.

For MIPs, it is usually hard to directly obtain all facet defining inequalities for $conv(S)$. However, the LP relaxation of P, S', can be easily obtained, and $conv(S) \subseteq S'$. So it is useful to add additional constraints, ideally facet defining inequalities of $conv(S)$, to S'.The facets of P can be artificially partitioned into trivial facets and non-trivial facets. Generally, the facets directly implied by the LP relaxation are called *trivial* facets. As the name implies, they are not often considered in details because they are already part of the LP relaxation. Consequently, the remaining facets of P, which are likely to be more interesting, are considered to be *non-trivial*. Note that whether a facet is trivial or not solely depends on the researcher's choice. By convention, the facets defined by non-negativity constraints are generally considered trivial, but this may vary in different polyhedra.

In situations where the non-trivial facet defining inequalities are difficult to obtain, it is necessary to consider a class of inequalities that are slightly weaker, but still retain some desired properties. They are called minimal inequalities.

Definition 1.1.10 *A valid inequality $\pi^T x \geq \pi_0$ for a polyhedron $P \subseteq \{x : x \geq 0\}$ is* ***minimal*** *if there does not exist $\pi' \leq \pi$ and $\pi' \neq \pi$ such that $\pi' x \geq \pi_0$ is valid for P, and there does not exist $\pi'_0 > \pi_0$ such that $\pi^T x \geq \pi'_0$ is valid. When there is no ambiguity, we may equivalently say that π is minimal.*

Subadditivity is a property that often comes with minimal inequalities. We will discuss the connection between these two properties in later chapters. For now we give the definition of a subadditive function.

Definition 1.1.11 *A function $f : S \to \mathbb{R}$ is* ***subadditive*** *over S if for any $x_1, x_2, x_1+x_2 \in S$, $f(x_1) + f(x_2) \geq f(x_1 + x_2)$.*

In the next section, we briefly discuss some history of cutting planes and introduce Gomory mixed integer cuts.

1.2 Cutting Planes and GMI Cuts

One of the first applications of cutting planes dates back to 1954, when a breakthrough on the TSP was made by Dantzig, Fulkerson, and Johnson [4]. In particular, they solved a specific case of this problem over 49 cities by first relaxing it to an LP and then adding additional constraints. While this may sound like a standard approach for IPs, it was surely an innovative idea at its time.

Inspired by Dantzig et al.'s work and several others, in 1958, Ralph Gomory gave a brief description of a cutting-plane method for general IPs and gave proof of its convergence, i.e. this algorithm gives an integer solution in finite steps [9]. Furthermore, based on the simplex tableau, he constructed a systematic algorithm to solve IPs by finding a class of cutting planes that is later referred as the Gomory's cuts. They are constructed as follows.

Consider the LP-relaxation of system (1.1) again. Since A is assumed to have full row rank, we can always write the constraint matrix of (1.1) as

$$A = [B \quad N],$$

where B is an $m \times m$ non-singular submatrix of A, and N consists of the remaining columns of A. We say B is a *basis* of A. Use x_B to denote the *basic variables* and x_N for the *non-basic variables*, which are variables whose index corresponds to a column in (or not in) the basis. Notice the constraint matrix becomes

$$Bx_B + Nx_N = b.$$

Multiplying by B^{-1} on both sides, we obtain an equivalent equation.

$$x_B = B^{-1}b - B^{-1}Nx_N.$$

The solution given by this basis, or the *basic optimal solution*, is simply $x_B = B^{-1}b$ and $x_N = 0$. However, recall we are considering an LP relaxation of an MIP. If all integer variables in the basic optimal solution have integer values, we have found an optimal solution to the MIP. If that is not the case, there exists a basic variable x_i such that the i-th row in the simplex tableau $x_i = \bar{a}_{i0} - \sum\limits_{j \in N} \bar{a}_{ij}x_j$ has a non-integer $\bar{a}_{i0}$ value and x_i is an integer variable in the MIP, where $\bar{a}_{ij}$ is the appropriate entry in the corresponding matrix and/or vector. This configuration is used repeatedly to introduce several polyhedra.

A universal cut for system (1.1) is the *Gomory mixed integer* (GMI) cut. Use I and J to denote the index sets for integer and continuous variables, respectively. Let a_{ij} be the ij-th entry of A and $\hat{t} = t - \lfloor t \rfloor$ for all $t \in \mathbb{R}$. Then the GMI cut is the following:

$$\sum_{j \in J, a_j > 0} \frac{a_j}{\hat{b}} x_j + \sum_{j \in J, a_j < 0} \frac{a_j}{1 - \hat{b}} x_j + \sum_{i \in I, \hat{a}_i \leq \hat{b}} \frac{\hat{a}_i}{\hat{b}} x_i + \sum_{i \in I, \hat{a}_i \leq \hat{b}} \frac{1 - \hat{a}_i}{1 - \hat{b}} x_i \geq 1. \tag{1.2}$$

The detailed derivation of this cut can be found in [14]. The GMI cut is one of the most useful ones in practice because it can be used repeatedly to solve IPs [10]. Therefore, a lot of research on cutting planes focuses on finding cuts that improve the efficiency of the cutting-plane method. Specifically, one approach is to develop well-structured and generic models in order to get strong cuts. In the next few sections, we introduce a number of such models that motivated our research.

1.3 The Corner Polyhedron

In addition to providing an algorithm to solve general IPs, Gomory introduced the corner polyhedron, which he considered his best work in the field of integer programming [12]. Using the notation from the last section, the corner polyhedron is the convex hull of the following.

$$\begin{aligned} x_B &= B^{-1}b - B^{-1}Nx_N \\ x_N &\geq 0, \\ x_B, x_N &\in \mathbb{Z}. \end{aligned}$$

In other words, the corner polyhedron is obtained by picking a specific basis B of the constraint matrix A, re-writing an equivalent version of the constraints based on it, and dropping the non-negativity constraints for the basic variables.

The corner polyhedron is an unbounded relaxation of the original IP [11]. The facets of the corner polyhedron are clearly valid inequalities for the convex hull of feasible points of the IP, and therefore can serve as cutting planes. Another thing to note is that the shape of the corner polyhedron is a translated cone, whose extreme points and extreme rays are likely to be a lot simpler to describe compared to the corresponding LP relaxation. This is useful to generate valid inequalities for the convex hull of the feasible solutions of (1.1). Therefore, the geometric structure makes the corner polyhedron desirable.

As one might expect, the corner polyhedron has some downsides as well. One major disadvantage is that the corner polyhedron is not a generic model. Each corner polyhedron is only applicable to one specific problem, so every time a new problem is posed, new corner polyhedra need to be derived to study the new problem. Also, there could be a large number of corner polyhedra for one IP, and it might not be very efficient to study them one-by-one. To avoid these problems, we introduce a more generic model that serves as a relaxation for all MIPs in the next section.

1.4 The Master Cyclic Group Polyhedron

After the introduction of the corner polyhedron, the idea of master polyhedra soon emerged. One of the first master polyhedra, named the master cyclic group polyhedron (MCGP), was developed by Gomory and Johnson [13]. To derive it, assume we are working with an IP. Using a similar notation as in the last section, we can start by considering the defining equation of the corner polyhedron:

$$x_B = B^{-1}b - B^{-1}Nx_N.$$

As discussed earlier, we focus on the case where the corresponding basic solution is not integral for some integer variables. Then there exists an integer i such that the i-th row the the simplex tableau is $x_i = \bar{a}_{i0} - \sum_{j \in N} \bar{a}_{ij}x_j$, where a_{i0} is not an integer. Recall x_i is an integer variable and observe the following equation holds:

$$\sum_{j \in N} \hat{\bar{a}}_{ij} x_j \equiv \hat{\bar{a}}_{i0} \mod 1. \tag{1.3}$$

Since all variables and coefficients we are treating are rational numbers, they have a least common multiple $n \in \mathbb{Z}_+$. By scaling 1.3 by n, we get $\sum_{j \in N} n\hat{\hat{a}}_{ij} x_j \equiv n\hat{\hat{a}}_{i0} \mod n$. Based on this, the master cyclic group polyhedron is defined as the convex hull of [5]

$$\begin{array}{rcl} \sum_{i=1}^{n-1} iw_i & \equiv & r \mod n \\ w_i & \in & \mathbb{Z}+, \end{array} \tag{1.4}$$

where $r, n \in \mathbb{Z}_+$ and $r < n$. We denote the polyhedron above as $MCGP(n, r)$. It is a generic model that is applicable to all IPs, with n and r being its parameters. Notice the defining equation presents a lot of structural symmetry, which is a reason why MCGPs are well studied.

Here we give an example of how to re-write a single-row constraint into the form of the MCGP. Consider the constraint $2.3x_1 + 5.8x_2 - 6.7x_3 = 11.9$, where $x_i \in \mathbb{Z}_+$ for $i \in \{1, 2, 3\}$. Take the fractional part of all coefficients and the right hand side, so that we get $0.3x_1 + 0.8x_2 + 0.3x_3 \equiv 0.9 \mod 1$. Rescaling this equation gives $3(x_1 + x_3) + 8x_2 = 9$. For $i \in \{1, ..., 10\}$, $w_i \in \mathbb{Z}$, let $w_3 = x_1 + x_3$, $w_8 = x_2$, and $w_i = 0$ for all remaining w_i's. Then this equation becomes $\sum\limits_{i=1}^{10} iw_i = 9$, which is in the form of a MCGP. Through this example, we can see the MCGP is general enough for practical purposes, since IP problems in practice would only have rational coefficients and a finite number of variables.

Comparing to the corner polyhedra, the MCGP is highly regular and not problem-specific, making it a lot easier and feasible to study. The coefficients of $MCGP(n, r)$'s non-trivial facets are characterized as extreme points of a well-structured polyhedron, which contains coefficients for exactly all the minimal valid inequalities for $MCGP(n, r)$. Therefore, they can be solved using LP methods, and serve as cutting planes for the original IP problem. The characterization is the following.

Theorem 1.4.1 *An inequality $\sum_{i=1}^{n-1} \pi(i) w_i \geq 1$ defines a non-trivial facet of $MCGP(n, r)$ if and only if π is an extreme point of the solution set of the following inequalities [11].*

$$\begin{array}{rcll} \pi(i) + \pi(j) & \geq & \pi((i+j) \mod n) & i, j \in \{1, ..., n-1\} \\ \pi(i) + \pi((r-i) \mod n) & = & \pi(r) & i \in \{1, ..., n-1\} \\ \pi(i) & \geq & 0 & i \in \{1, ..., n-1\} \\ \pi(r) & = & 1 & . \end{array}$$

Being able to describe and compute the facets of $MCGP(n, r)$ makes them advantageous to work with. Although certain information might get lost during the relaxation process, this model still has the potential to produce strong cuts for IPs. Moreover, there exists a class of inequalities named the two-slope inequalities that are easily identifiable and facet defining for the MCGP.

Definition 1.4.2 *An inequality* $\sum_{i=1}^{i=n-1} \pi(i) w_i \geq 1$ *for* $MCGP(n, r)$ *is* ***two-slope*** *if* $\pi(i) - \pi(i-1) = \pi(1)$ *or* $-\pi(n-1)$*, for all* $i \in \{2, ..., n-1\}$*.*

Theorem 1.4.3 *Every two-slope inequality for* $MCGP(n, r)$ *that is a feasible point of the polyhedron in Theorem 1.4.1 is facet defining for* $MCGP(n, r)$ *[13].*

Although it is not very extensive, this theorem provides a simplified alternative to find certain facet defining inequalities for the MCGP. For example, consider the GMI cut described by (1.2) for the MCGP. The continuous variables can be dropped, so that only the latter two summations are remaining. Notice that the GMI cut is indeed a two-slope inequality, since the numerators of in the GMI cut correspond to the coefficients of the variables (or one minus the coefficients of the variables), which are consecutive numbers. Therefore, the theorem implies the GMI cut is facet defining for the MCGP.

In the next section, we introduce another master polyhedron that is an generalization of the MCGP.

1.5 The Master Equality Polyhedron

A few other master polyhedra were built after the formulation of the MCGP. One of them is the master equality polyhedron, which was developed by Dash, Fukasawa, and Günlük in 2007 [6]. It is defined as the convex hull of

$$\begin{array}{rcl} \sum_{i=-n}^{n} i w_i & = & r \\ w_i & \in & \mathbb{Z}_+, \end{array} \tag{1.5}$$

where $r, n \in \mathbb{Z}_+$ and $r \leq n$. We denote this polyhedron as $MEP(n, r)$. The derivation of the MEP is similar to that of the MCGP. For an IP, we can obtain the i-th row of the simplex tableau $x_i = \bar{a}_{i0} - \sum_{j \in \mathcal{N}} \bar{a}_{ij} x_j$, where $\bar{a}_{i0}$ is not an integer and x_i is an integer variable. Consequently, we can manipulate the equation and re-index the variables to get

$$\sum_{j=1}^{m} a'_j x_j = \bar{a}_{i0},$$

for some appropriate coefficients $a'_1, \ldots, a'_m \in \mathbb{Q}$.

Once again, we can find the common denominator d for all a'_j's and $\bar{a}_{i0}$, such that $\bar{a}_{i0} = \frac{r}{d}$ for some $r \in \mathbb{Z}_+$. Multiply the equation above by d and define a new set of variables in the same fashion as in the MCGP case, and the defining equation for the MEP can be obtained.

Similar to other master polyhedra, the MEP can be used as a relaxation of any IP. However, note that MEP does not require working with modular arithmetic. It is a generalization of the MCGP, as $MCGP(n, r)$ is actually a face of $MEP(n, r)$ [6]. But this comes with a price. As more information from the original IP is retained, the complexity of the problem grows. As a result, the MEP is a lot less structured, making it much more difficult to analyze.

Dash et al. [6] were able to give a similar but slightly more complex characterization of non-trivial facet defining inequalities for the MEP. Let $I := \{-n, \ldots, n\}$, $I_+ = \{1, \ldots, n\}$, and $I_- = \{-1, \ldots, -n\}$ for some $n \in \mathbb{Z}_+$. We assume the trivial facets of (1.5) are the ones defined by the non-negativity constraints. Then the following theorem holds.

Theorem 1.5.1 *An inequality $\sum_{i=-n}^{n} \pi(i) w_i \geq 1$ is equivalent to $w_{-n} \geq 0$ or defines a non-trivial facet of $MEP(n, r)$ if and only if π is an extreme point of the following polyhedron [6].*

$$\begin{array}{rcll}
\pi(i) + \pi(j) & \geq & \pi(i+j) & i, j \in I, i+j \in I_+ \\
\pi(i) + \pi(j) + \pi(k) & \geq & \pi(i+j+k) & i \in I, j, k, i+j+k \in I_+ \\
\pi(i) + \pi(r-i) & = & \pi(r) & i \in \{r-n, \ldots, \lfloor \frac{r}{2} \rfloor\} \\
\pi(r) & = & 1 & \\
\pi(0) & = & 0 & \\
\pi(-n) & = & 0 &
\end{array} \tag{T}$$

In our discussion, we refer to the constraints in the first, second, and third row of (T) as *double subadditivity* constraints, *triple subadditivity* constraints, and *complementarity* constraints, respectively. In particular, note complementarity refers to the property where $\pi(i) + \pi(r - i) = \pi(r)$, for all $r - n \leq i \leq n$. Furthermore, Dash et al. provided a number of useful properties of the polyhedron (T) and facet defining inequalities of $MEP(n, r)$. We start with a non-trivial fact about facet defining inequalities.

Lemma 1.5.2 *Let $\sum_{i=-n}^{n} \pi(i) w_i \geq 1$ be a non-trivial facet defining inequality of $MEP(n, r)$, then π is subadditive over I [6].*

In addition, for any non-trivial facet defining inequality $\sum_{i=-n}^{n} \pi(i)w_i \geq 1$ such that $\pi(-n) = 0$, the ranges of each $\pi(i)$ is determined, based on the constraints of (T). Later in this thesis, we will provide some improved bounds on some of these $\pi(i)$'s. The following lemma is an extension of a result by Dash et al., and the proof is identical to the one in their paper [6].

Lemma 1.5.3 *Let* $\sum_{i=-n}^{n} \pi(i)w_i \geq 1$ *such that* $\pi(j) = 0$ *for some* $j \in I_-$, *then* $0 \leq \pi(i) \leq \lceil \frac{i}{r} \rceil$ *for* $i > 0$ *and* $-\lceil \frac{i}{r} \rceil \leq \pi(i) \leq \lceil \frac{n}{r} \rceil$ *for* $i < 0$.

It is also important to note that (T) contains only 1-valid inequalities.

Lemma 1.5.4 *Let* $\pi \in T$, *then* $\sum_{i=-n}^{n} \pi(i)w_i \geq 1$ *is a* 1*-valid inequality for* $MEP(n, r)$ *[6].*

Moreover, Dash et al. proved a sufficient condition for an inequality to be 1-valid. It helps us to recognize certain valid inequalities in our research.

Lemma 1.5.5 $\sum_{i=-n}^{n} \pi(i)w_i \geq 1$ *is a* 1*-valid inequality for* $MEP(n, r)$ *if* π *satisfies the following conditions [6].*

$$\begin{array}{rcll} \pi(i) + \pi(j) & \geq & \pi(i+j) & \qquad i, j \in I, i+j \in I_+ \\ \pi(i) + \pi(j) + \pi(k) & \geq & \pi(i+j+k) & \qquad i \in I, j, k, i+j+k \in I_+ \\ \pi(r) & \geq & 1 & \end{array}$$

Note the similarities between the characterization of non-trivial facets for $MCGP(n, r)$ and (1.5). Since there exists a two-slope theorem for $MCGP(n, r)$, it is natural to consider whether an analogous theorem for two-slope inequalities in the MEP case would still hold. Our research eventually proves such a theorem.

Next, we introduce another more relaxed system for general IPs that has a similar structure to (1.5). Along with (1.5), they are the two relaxations that are well studied in our research.

1.6 The Infinite Relaxation Model

Just as how the previously introduced polyhedra are derived, the infinite relaxation model is developed by considering a basis of the constraint matrix of any general MIP and picking a row of the simplex tableau: $x_i = \bar{a}_{i0} - \sum_{j \in N} \bar{a}_{ij} x_j$, which is equivalent to

$$1 - x_i = \sum_{j \in N} \bar{a}_{ij} x_j - \bar{a}_{i0} + 1.$$

Let $\bar{x} := 1 - x_i$ and $f := 1 - \bar{a}_{i0}$. Since x_i is non-negative, we must have $\bar{x} \in \{-\infty, ..., 0, 1\}$. Recall this is a MIP, so we can partition the non-basic variables into integral variables and continuous variables. Use y'_i to denote the integral non-basic variables with coefficient i and s'_i to denote the continuous non-basic variables with coefficient i, we can relax the previous equation and obtain the following system.

$$\begin{array}{rcl} \bar{x} & = & f + \sum_{t \in \mathbb{R}} t s'_t + \sum_{t \in \mathbb{R}} t y'_t \\ \bar{x} & \in & \{-\infty, \ldots, 0, 1\} \\ s'_t & \geq & 0 \\ y'_t & \in & \mathbb{Z}_+ \\ (s', y') & \text{has} & \text{finite support.} \end{array} \tag{1.6}$$

This is the infinite relaxation model. The model was originally derived for 0, 1-knapsack problems with $\bar{x} \in \{0, 1\}$, so the bound on $\bar{x}$ is indeed $\{-\infty, \ldots, 0, 1\}$ after relaxing the non-negativity constraints [1]. Moreover, this model assumes that $f \in (0, 1)$. Just like the MEP, the infinite relaxation model does not require any modular arithmetic, and therefore conserves more information about the original system comparing to $MCGP(n, r)$. We later show system (1.6) has a strong relationship with $MEP(n, r)$. Observe we may assume (s', y') has finite support because we are only interested in problems with a finite number of variables in practice. For simplicity, from this point on, we will use the variable x instead of $\bar{x}$.

Comparing to the MEP, the infinite model could be advantageous some times, because it admits coefficient values from $\mathbb{R}$. Since rounding is often used in practice, it might be difficult to distinguish whether a decimal number has an infinite number of digits, but this problem is avoided in (1.6). Although not very practical, this also means irrational coefficients are allowed. Another advantage of (1.6) appears to be handling coefficients with a large number of decimal places well. A variable with such a coefficient would result in a large n value in (1.5), potentially resulting in a problem with large size and a great number variables that we are not interested in. The infinite relaxation model, on the other hand, would not have this issue because it does not have any parameters, and all the coefficients are accepted.

However, the infinite relaxation model has some drawbacks as well, with the major problem being its complexity. It has infinitely many variables, so the continuity of the coefficient functions might become an issue when exploring valid inequalities. Since the infinite model is not a polyhedron, the previous definition does not apply anymore, and we need give a definition for valid functions for (1.6) that are akin to the previous corresponding definitions for polyhedra.

Definition 1.6.1 *The inequality* $\sum_{t\in\mathbb{R}} \psi(t)s_t + \sum_{t\in\mathbb{R}} \phi(t)y_t \geq \phi_0$ *[or* (ψ, ϕ, ϕ_0)*] is* ***valid*** *for* (1.6) *if for any* $(s', y') \in$(1.6)*,* $\sum_{t\in\mathbb{R}} \psi(t)s'_t + \sum_{t\in\mathbb{R}} \phi(t)y'_t \geq \phi_0$.

Moreover, up to normalization, if $\phi_0 = 1$*,* (ψ, ϕ) *is* 1***-valid****; if* $\phi_0 = 0$*, and* (ψ, ϕ) *is not* 1*-valid, then* (ψ, ϕ) *is* 0***-valid****; and if* $\phi_0 = -1$ *and* (ψ, ϕ) *is not* 0*-valid, then* (ψ, ϕ) *is* -1***-valid****.*

In particular, We may say (ψ, ϕ) is a pair of *valid functions* when ϕ_0 is specified. Similarly, the definition of a minimal function for (1.6) is given. It is very intuitive because the idea is the same as in the polyhedron case.

Definition 1.6.2 *A pair of* ϕ_0*-valid functions* (ψ, ϕ) *is* ***minimal*** *for* (1.6) *if there does not exist* $(\psi', \phi') \leq (\psi, \phi)$ *and* $(\psi', \phi') \neq (\psi, \phi)$ *such that* (ψ', ϕ') *is* ϕ_0*-valid for* (1.6)*, and there does not exist* $\phi'_0 > \phi_0$ *such that* (ψ, ϕ, ϕ'_0) *is valid for* (1.6)*.*

When no ambiguity arises, we may say (ψ, ϕ) is a pair of *minimal functions*.

In 2013, Cornuéjols et al. studied (1.6) [3]. They proved the following lemma about minimal functions for (1.6), which is a fair implication of the strong correlation between minimality and subadditivity.

Lemma 1.6.3 *If* (ψ, ϕ) *is a pair of minimal* 1*-valid functions for* (1.6)*, then* ϕ *is subadditive [3].*

Moreover, to be able to discuss cutting planes, we need some concept equivalent to facets in the finite case to represent a strong cut for (1.6). So, we introduce extreme functions.

Definition 1.6.4 *A pair of* 1*-valid functions* (ψ, ϕ) *is* ***extreme*** *for* (1.6) *if there do not exist two pairs of distinct functions* (ψ_1, ϕ_1) *and* (ψ_2, ϕ_2)*, such that* $\sum_{t\in\mathbb{R}} \psi_i(t)s_t + \sum_{t\in\mathbb{R}} \phi_i(t)y_t \geq 1$ *for* $i \in \{1, 2\}$*,* $\psi = \frac{1}{2}(\psi_1 + \psi_2)$*,* $\phi = \frac{1}{2}(\phi_1 + \phi_2)$*.*

In other words, we may say (ψ, ϕ) is a pair of *extreme functions* for (1.6).

In the research of Cornuéjols et al., a class of inequalities named the optimized wedge inequalities were established. Correspondingly, we call their coefficent functions the optimized wedge functions. They have proven the extremality of the optimized wedge functions.

Definition 1.6.5 *The* ***optimized wedge function*** *is a pair of functions* (ψ, ϕ)*, where* $\psi(t) = \begin{cases} -\frac{t}{f} & t < 0 \\ \frac{t}{1-f} & t \geq 0 \end{cases}$ *and* $\phi_\alpha(t) = \min\{\frac{-t+\lceil \alpha t\rceil}{f}, \frac{t}{1-f} - \frac{\lfloor \alpha t\rfloor(1-\alpha(1-f))}{\alpha f(1-f)}\}$*, where* $\alpha \in (0, 1]$*.*

Theorem 1.6.6 *For all* $\alpha \in (0, 1]$*, the optimized wedge function is extreme for* (1.6) *[3].*

Considering the strength of this result and the certain structural similarities between the MEP and the infinite relaxation model, we decided to explore the potential relationship between them. It is essential to note that the wedge inequalities has exactly two slopes, which inspired us to consider if a similar two-slope result holds for the MEP and/or the infinite model. Such a result would provide a simpler characterization for facet defining inequalities and extreme functions in their respective model. Another interest to us was the connection between the extreme functions in the infinite model and facet defining inequalities in the MEP. This might give a new method to find facet defining inequalities for the MEP, and vice versa.

1.7 Outline

The main goal of Chapter 2 is to prove an analogous two-slope theorem for (1.5). We start by discussing some basic properties such as minimality and subadditivity for valid functions of the MEP. Ideas such as normalization are introduced as well. We show when these properties might hold, and present the close relationship between them. To show these properties are essential, we prove the two-slope theorem using them.

Chapter 3 introduces a modified version of (1.6), IRm(q, r). Its important basic properties are discussed. Additionally, we demonstrate how it is related to (1.5) by introducing a modified version of (1.5) as well. By showing a method to extend coefficient functions for (1.5) to coefficient functions for IRm(q, r), a solid link between them is established. In particular, we prove the extended function for IRm(q, r) is 1-valid and minimal if the original function for (1.5) is 1-valid and minimal. Such extensions are used repeatedly to prove many results in the next Chapter.

The two-slope theorem is then extended to the modified infinite relaxation model in Chapter 4. A number of important results are proven in this chapter, including that the continuous extensions of a class of two-slope inequalities are extreme for IRm(q, r), and certain regular two-slope functions are extreme for IRm(q, r). The extremality of the optimized wedge cut is implied directly from these results. An MIP version of IRm(q, r) is also introduced, which shows how this model can be applied to MIPs as well.

At last, we give a separation algorithm over an important class of valid inequalities for subproblems of the MEP. It starts by describing a normalization of $MEP(n,r)$ that guarantees the coefficient functions of all minimal inequalities are non-decreasing. Such a normalization enables a straight forward enumeration of classes of inequalities called $\frac{1}{k}$-inequalities. We then study certain properties of $\frac{1}{k}$-inequalities under the specific normalization. Based on these properties, we provide an algorithm to solve separation problems over a class of $\frac{1}{2}$-inequalities of a corresponding subproblem of $MEP(n,r)$. The time-complexity of the algorithm is then discussed.

Chapter 2

Two-Slope Theorem for The MEP

As discussed in Chapter 1, the two-slope theorem for the MCGP is a highly applicable characterization for a class of its facets. Considering the structural similarities of the MCGP and the MEP, it is natural to consider whether some version of a similar result exists in the MEP. In the context of the MEP, a two-slope equation has the following definition.

Definition 2.0.1 *A function $\pi : I \to \mathbb{R}$ is **two-slope** if $\pi(i+1) - \pi(i) = \pi(1)$ or $\pi(i+1) - \pi(i) = -\pi(-1)$ for $i \in I \backslash \{n\}$.*

*Moreover, it is k-**partially two-slope** if $\pi(i+1) - \pi(i) = \pi(1)$ or $\pi(i+1) - \pi(i) = -\pi(-1)$ for $i \in \{k, ..., n-1\}$.*

The definitions above apply analogously to the inequality $\pi^T w \geq \pi_0$, with the value of π_0 specified. Based on these definitions, we will prove the following two-slope theorem for the MEP. It turns out we only need the function to be partially two-slope for the theorem to hold. However, as a trade-off, the 1-validity and minimality of such an inequality is not guaranteed, so we need to explicitly state them as conditions of the theorem.

Theorem 2.0.2 *Suppose π is a minimal, 1-valid and $(r-n)$-partially two-slope inequality for $MEP(n,r)$ such that $\pi(-n) = 0$, then it is two-slope and facet defining for $MEP(n,r)$.*

This Chapter is devoted to the proof of Theorem 2.0.2. We start by noting some fundamental concepts and properties of the MEP and its cutting planes, such as the subadditivity of minimal inequalities, and the 1-validity of subadditive inequalities. Based on these facts, we build our way to the theorem eventually.

2.1 Basic Properties

To enable a deeper discussion of the facet defining inequalities of $MEP(n, r)$, we need to take a closer look at certain interesting properties it holds. One property we are highly interested in is minimality, since all facet defining inequalities are indeed minimal. To begin, recall (T) is the polyhedron whose extreme points correspond to non-trivial facet defining inequalities or $\pi(-n) \geq 0$. Notice for any $\pi \in$ (T), the definition of (T) naturally poses a lower bound for every $\pi(i)$, $\forall i \in I_- := \{-n, ..., -1\}$ (although dependent of the choice of π).

Definition 2.1.1 *For all $i \in I_-$, let the **minimum valid value** of $\pi(i)$ be* $m_i = \max\{\max_{-i \leq j \leq n} \{\pi(i+j) - \pi(j)\}, \max_{j,k,i+j+k \in I_+} \{\pi(i+j+k) - \pi(j) - \pi(k)\}\}$. *Moreover, we define $m_0 = 0$.*

Notice the minimum valid value for a coefficient $\pi(i)$, $i \in I_-$, is the smallest value that $\pi(i)$ can take with π still being in (T), assuming all other coefficients are fixed. In other words, m_i is the minimum value that $\pi(i)$ could take that still satisfies the double and triple subadditivity constraints.

One useful result for MCGP(n, r) is that its minimal and 1-valid inequalities are subadditive over their domains. As a relaxation of MCGP(n ,r), $MEP(n, r)$ has a lot of similar structural advantages. So it is no surprise that an analogous result holds true in the $MEP(n, r)$ case.

Lemma 2.1.2 *If π is a minimal 1-valid inequality for $MEP(n, r)$, then it is subadditive over I.*

Proof. Suppose π is not subadditive over I. Then there exists $j, k \in I$ such that $j + k \in I$ and $\pi(j) + \pi(k) < \pi(j + k)$. We define an inequality π' for $MEP(n, r)$ as follows:

$$\pi'(i) = \begin{cases} \pi(j) + \pi(k) & i = j + k, \\ \pi(i) & \text{otherwise.} \end{cases}$$

Given $w \in MEP(n, r)$, we define w' with

$$w'_i = \begin{cases} w_j + w_{j+k} & i = j, \\ w_k + w_{j+k} & i = k, \\ 0 & i = j + k, \\ w_i & \text{otherwise.} \end{cases}$$

By these definitions, it can be verified that $\sum_{i=-n}^{n} iw_i = \sum_{i=-n}^{n} iw'_i$ and $\sum_{i=-n}^{n} \pi'(i)w_i = \sum_{i=-n}^{n} \pi(i)w'_i$. Since $w' \in$ (1.5), and π is 1-valid for (1.5), we get $\sum_{i=-n}^{n} \pi'(i)w_i = \sum_{i=-n}^{n} \pi(i)w'_i \geq 1$.

Then the minimality of π guarantees $\pi(j) + \pi(k) \geq \pi(j+k)$. $\square$

As mentioned before, Lemma 2.1.2 builds a connection between minimality and subadditivity. They are two key properties in our research, because facet defining inequalities for $MEP(n,r)$ possess them. In the rest of this section, we strive to prove a stronger result, that all subadditive functions π with domain I and $\pi(r) \geq 1$ are 1-valid for $MEP(n,r)$. The following lemma shows if we have a sequence of k numbers with their sum in I, then they can be permuted such that the sum of the first j terms in the sequence is in I, for any $2 \leq j \leq k$. As unrelated as it may seem, notice any coefficient function π for $MEP(n,r)$ has domain I. If π is subadditive, the following lemma allows for a permutation where subadditivity can be applied repeatedly such that every intermediate term is defined. We later use this technique to prove subadditive coefficient functions for $MEP(n,r)$ are 1-valid.

Lemma 2.1.3 *Given $a_1, a_2, ..., a_k$ with $\sum_{i=1}^{k} a_i \in I$, there exists a permutation $\sigma : [k] \to [k]$ such that $\sum_{i=1}^{m} a_{\sigma(i)} \in I$, for any integer $1 \leq m \leq k$.*

Proof. We prove this lemma by induction.

Base case: If $k = 2$, the trivial permutation would suffice.

Induction hypothesis: For any $a_1, a_2, ..., a_{k-1}$, $\sum_{i=1}^{k-1} a_i \in I$, such a permutation exists.

Inductive step: Suppose $k > 2$. We may assume that for all $j \in \mathbb{Z}$ with $1 \leq j \leq k$, either $\sum_{i=1}^{k} a_i - a_j > n$ or $\sum_{i=1}^{k} a_i - a_j < -n$. In other words, either $a_j \leq \sum_{i=1}^{k} a_i - n - 1$ or $a_j \geq \sum_{i=1}^{k} a_i + n + 1$.

Since $\sum_{i=1}^{k} a_i \in I$, it is clear that not both of $\sum_{i=1}^{k} a_i - n - 1$ and $\sum_{i=1}^{k} a_i + n + 1$ are in I. On the other hand, if neither of them is in I, then a_j is not in I for all $1 \leq j \leq k$, which is a contradiction. Without loss of generality, assume $\sum_{i=1}^{k} a_i - n - 1 \in I$ and $\sum_{i=1}^{k} a_i + n + 1 \notin I$.

Therefore, for all $1 \leq j \leq k$, $a_j \leq \sum\limits_{i=1}^{k} a_i - n - 1$. Summing over all possible values of j, we get $\sum\limits_{j=1}^{k} a_j \leq k \sum\limits_{i=1}^{k} a_i - k(n+1)$, or $k(n+1) \leq (k-1)\sum_{i=1}^{k} a_i$. However, this contradicts $\sum_{i=1}^{k} a_i \leq n$. Thus, there exists an a_j for some integer $1 \leq j \leq k$ such that $\sum\limits_{i=1}^{k} a_i - a_j \in I$.
$\square$

As discussed, we can use Lemma 2.1.3 and the definition of subadditive functions to easily prove that subadditive functions for $MEP(n, r)$ are 1-valid.

Lemma 2.1.4 *Let $\pi : I \to \mathbb{R}$ be subadditive over I such that $\pi(r) \geq 1$. Then π is 1-valid for $MEP(n, r)$.*

PROOF. Observe it suffices to show $\pi^T w \geq 1$ for all $w \in$ MEP(n ,r)$\cap\mathbb{Z}^{2n}$, because all extreme points of MEP(n ,r) are integral. For any feasible point w for $MEP(n, r)$, recall that each component of w is non-negative. Moreover, by Lemma 2.1.3, we may assume that the $\pi(i)$'s in $\sum\limits_{i=-n}^{n} \pi(i)w_i$ can be permuted such that the partial sums of the first k terms are in I, for all $1 \leq k \leq \sum\limits_{i=-n}^{n} w_i$ (note w_i simply counts the number of times each $\pi(i)$ is added). Then subadditivity can be applied repeatedly to the permuted series to obtain the following.

$$\begin{aligned} \sum_{i=-n}^{n} \pi(i)w_i &= \pi(\sum_{i=-n}^{n} w_i) \\ &= \pi(r) \\ &\geq 1. \end{aligned}$$

Observe all inequalities in the computations result from subadditivity of π. Hence π is 1-valid for $MEP(n, r)$. $\square$

The lemma above is quite significant because it provides an easy-to-check condition for inequalities that are sufficiently 1-valid. In addition, Dash et al. proved that all non-trivial facets of $MEP(n, r)$ are 1-valid under the normalization $\pi(-n) = 0$ [6]. Moreover, their result is easily extendable to any normalization $\pi(i) = 0$, for any $i \in I_-$. So we have the following lemma.

Lemma 2.1.5 *Let $\pi^T w \geq \pi_0$ be a non-trivial facet defining inequality for $MEP(n, r)$ such that $\pi(i) = 0$ for any $i \in I_-$, then $\pi_0 > 0$.*

The proof for Lemma 2.1.5 is identical to lemma 2.11 in the paper of Dash et al [6]. In particular, note that this implies every non-trivial facet defining inequality for $MEP(n, r)$ has a 1-valid representation under the normalization $\pi(i) = 0$ for any $i \in I_-$. Based on this observation, we may derive the following lemma.

Lemma 2.1.6 *Suppose $n > r$. Let $\pi^T w \geq \pi_0$ be a non-trivial facet defining inequality for $MEP(n, r)$ such that $\pi(j) = 0$ for some $r < j \leq n$, then $\pi_0 > 0$.*

Proof. By Lemma 2.1.5, we may assume π is 1-valid under normalization $\pi(i) = 0$ for any $i \in I_-$. For this to happen, under the normalization $\pi(-n) = 0$, π must satisfy that $\pi(i) > \frac{i}{r}$, for some $i \in I_- \backslash \{-n\}$.

Moreover, by Theorem 1.5.1, $\pi(j) + \pi(r - j) = 1$ for any $r < j \leq n$. Observe $r - j < 0$, so under the normalization $\pi(-n) = 0$, $\pi(j) = 1 - \pi(r - j) < 1 - \frac{r-j}{r} = \frac{j}{r}$. Then, when changing the normalization to $\pi(j) = 0$ from $\pi(-n) = 0$, we will subtract less than $\frac{1}{r}$ times of the defining equation from π, which implies π still has positive right hand side, i.e. $\pi_0 > 0$ under the normalization $\pi(j) = 0$. $\square$

Similarly, observe Lemma 2.1.6 implies that every non-trivial facet defining inequality for $MEP(n, r)$ has a 1-valid representation under the normalization $\pi(j) = 0$ for any $r < j \leq n$. The two lemmas above allow for switching between certain normalizations without losing the 1-validity of facet defining inequalities of $MEP(n, r)$. Such a property is crucial for our results in Chapter 4.

Now, we switch gears a little to discuss some useful results based on minimality of a function. We would like to prove that all minimal and 1-valid functions for $MEP(n, r)$ satisfying complementarity, i.e. $\pi(i) + \pi(r - i) = \pi(r)$, for all $i \in \{r - n, \ldots, n\}$.

Lemma 2.1.7 *If π is minimal and 1-valid for $MEP(n, r)$, then $\pi(i) + \pi(r - i) = \pi(r) = 1$ for all $i \in \{r - n, \ldots, n\}$.*

Proof. Since $MEP(n, r) \subseteq \{x : x \geq 0\}$, we can write $\pi(i) \geq \sum_{k \in K} \lambda_k \pi^k + \alpha i$ for all $i \in \{r - n, \ldots, n\}$ and $1 \leq \sum_{k \in K} \lambda_k \pi_0^k + \alpha r$, where (π^k, π_0^k) are non-trivial facets of $MEP(n, r)$, K is a index set, and some $\alpha \in \mathbb{R}$. Since $\pi_0^k = 1$, and π is minimal and 1-valid, we get $\pi(i) = \sum_{k \in K} \lambda_k \pi^k + \alpha i$ and $1 = \sum_{k \in K} \lambda_k + \alpha r$.

By minimality of π, we must have $\pi = \sum_{k\in K} \lambda_k \pi^k$ and $\pi_0 = \sum_{k\in K} \lambda_k \pi_0^k = 1$. Moreover, Theorem 1.5.1 gives $\pi^k(i) + \pi^k(r-i) = \pi(r)$ for all $i \in \{r-n, \ldots, n\}$. Therefore,

$$\begin{array}{rcl} \pi(i) + \pi(r-i) & = & \sum_{k\in K} \pi^k(i) + \alpha i + \sum_{k\in K} \pi^k(r-i) + \alpha(r-i) \\ & = & \sum_{k\in K}(\pi^k(i) + \pi^k(r-i)) + \alpha r \\ & = & \sum_{k\in K} \pi^k(r) + \alpha r \\ & = & \pi(r) \\ & = & 1. \end{array}$$

□

Notice that the following observation follows directly from Lemma 2.1.7, since $\pi(0) + \pi(r) = \pi(r) = 1$.

Observation 2.1.8 *If π is minimal and 1-valid for $MEP(n,r)$, then $\pi(0) = 0$.*

Using Lemma 2.1.7, we give a characterization of minimal inequalities for $MEP(n,r)$.

Lemma 2.1.9 *Let π be a 1-valid inequality for $MEP(n,r)$. Then π is minimal if and only if $\pi(r) = 1$, $\pi(i) + \pi(r-i) = 1$ for all $i \in \{r-n, \ldots, n\}$ and $\pi(i) = m_i$, $\forall i \in \{-n, ..., r-n-1\}$.*

Proof. ($\Rightarrow$) Assume π is minimal, then Lemma 2.1.7 guarantees that $\pi(i)+\pi(r-i) = 1$ for all $i \in \{r-n, \ldots, n\}$. Notice that if there exists $i \in I_-\setminus$ such that $\pi(i) < m_i$, then a double or triple subadditivity constraint is violated, which implies a pairwise subadditivity constraint is violated. By Lemma 2.1.2, π is not minimal, contradicting our assumption. Therefore, we must have $\pi(i) \geq m_i$ for all $i \in I_-$. If $\pi(i) > m_i$, then by Lemma 2.1.4, $\pi'(i) := \min\{\pi(i), m_i\}$ is 1-valid. Since $\pi' \leq \pi$ by definition, this contradicts that π is minimal as well. Therefore, we must have $\pi(i) = m_i$ for all $i \in I_-$.

($\Leftarrow$) Now let π be a 1-valid inequality such that $\pi(i) = m_i$ for all $i \in I_-$ and $\pi(i)+\pi(r-i) = 1$ for all $i \in \{r-n, \ldots, n\}$. Assume π is not minimal, then there exists a minimal and 1-valid inequality π' for $MEP(n,r)$ such that $\pi' \leq \pi$ and $\pi' \neq \pi$.

Recall $\pi(i) + \pi(r-i) = 1$ for all $i \in \{r-n, ..., n\}$. By Lemma 2.1.7, π' satisfies $\pi'(i) + \pi'(r-i) = 1$ as well. So if π' satisfies $\pi'(i) < \pi(i)$ for some $i \in \{r-n, ..., n\}$, then $\pi'(r-i) > \pi(r-i)$, since π' also satisfies complementarity by Lemma 2.1.7. Hence $\pi'(i) = \pi(i)$ for all $i \in \{r-n, ..., n\}$. For all $i \in \{-n, ..., r-n-1\}$, we have $\pi(i) = m_i$. In other words, there either exists $j_1 \in \{1, .., n\}$ such that $\pi(i) = \pi(i+j_1) - \pi(j_1)$ and

$i + j_1 \in I_+$, or there exist $j_2, k_2 \in I_+$ such that $\pi(i) = \pi(i + j_2 + k_2) - \pi(j_2) - \pi(k_2)$. If $\pi'(i) < \pi(i)$, since $\pi' \leq \pi$, we must have $\pi'(i+j_1) < \pi(i+j_1)$ or $\pi'(i+j_2+k_2) < \pi(i+j_2+k_2)$.

Note $i+j_1, i+j_2+k_2 \in I_+$, so they must satisfy the complementarity constraints. Then $\pi'(r-i-j_1) > \pi(r-i-j_1)$ or $\pi'(r-i-j_2-k_2) > \pi(r-i-j_2-k_2)$, which contradicts $\pi' \leq \pi$. Thus, if π is 1-valid, then π is minimal if and only if $\pi(i) + \pi(r-i) = 1$ for all $i \in \{r-n, \ldots, n\}$ and $\pi(i) = m_i$, $\forall i \in \{-n, ..., r-n-1\}$. □

Lemma 2.1.9 provides a straight forward characterization for necessary and sufficient condition for minimal 1-valid inequalities for $MEP(n,r)$. Moreover, note it is a necessary condition for non-trivial facet defining inequalities of $MEP(n,r)$. With this tool, we are ready to prove the two-slope theorem for $MEP(n,r)$.

2.2 The Two-Slope Theorem

The proof of Theorem 2.0.2 breaks down into two parts: to prove a minimal and $(r-n)$-partially two-slope inequality $\sum_{i=-n}^{n} \pi(i)w_i \geq 1$ is facet defining for $MEP(n,r)$, and to prove such a π is two-slope. We proceed to prove the facet-defining-ness of π. This proof directly counts the number of linearly independent constraints of (T) that are satisfied at equality by π, and therefore showing π is an extreme point of (T).

Lemma 2.2.1 *Suppose π is a minimal, 1-valid, and $(r-n)$-partially two-slope inequality for $MEP(n,r)$ such that $\pi(-n) = 0$, then π defines a non-trivial facet of $MEP(n,r)$.*

Proof. Because π is $(r-n)$-partially two-slope, for $i = 2, ..., n$, we have either $\pi(1) + \pi(i-1) = \pi(i)$ or $\pi(i) + \pi(-1) = \pi(i-1)$. This gives a total of $n-1$ linearly independent subadditivity constraints that are tight. Meanwhile, there are $n-r$ complementarity constraints that involves a $\pi(i)$ with $i < 0$. This gives $n-r-1$ tight constraints that are linearly independent from the previous $n-1$ constraints, since $\pi(r+1) + \pi(-1) = \pi(r)$ is counted previously.

By Lemma 2.1.9, there exists a tight constraint at each i for $i \in \{-n, ..., -n+r-1\}$. Moreover, these indices are all negative, and each constraint of (T) contains at most one $\pi(i)$ with a negative i value, so these constraints are linearly independent from each other. Therefore, there exists r tight constraints that are linearly independent from the ones described above, each containing exactly one index in $\{-n, ..., -n+r-1\}$. This gives $2n-2$ linearly independent tight constraints in total.

Now consider the equality $\pi(-n) = 0$. Out of all the tight constraints we have found so far, exactly one of them has the term $\pi(-n)$ in it. We denote it as $\pi(-n) + \pi(j) + \pi(k) = \pi(j+k-n)$, for some $j, k, j+k-n \in I_+$. If $\pi(-n) = 0$ is linearly dependent to the $2n-2$ linearly independent tight constraints, then $\pi(j) + \pi(k) = \pi(j+k-n)$ must be a linear combination of the $2n-3$ linearly independent tight constraints that do not involve $\pi(-n)$. However, this is not possible, since the sum of the indices of the left equals the sum of the indices on the right in the $2n-3$ linearly independent tight constraints written in a form where each term has a positive sign, but this is not the case in $\pi(j) + \pi(k) = \pi(j+k-n)$. Then, with the addition of $\pi(-n) = 0$, we have obtained $2n-1$ linearly independent tight constraints.

Finally, to see that $\pi(r) = 1$ is linearly independent from the $2n-1$ linearly independent tight constraints, notice that it has a non-zero constant term. Since all previous tight constraints have 0 as their constant term, we can conclude that $\pi(r) = 1$ is not a linear combination of them. Thus, we can find a total of $2n$ linearly independent tight constraints for any minimal, 1-valid, and $(r-n)$-partially two-slope inequality for $MEP(n, r)$. Consequently, such an inequality is facet defining for $MEP(n, r)$. Since π is 1-valid, the facet it defines is non-trivial. $\square$

Observe the condition $\pi(-n) = 0$ is simply a result of normalization. Any 1-valid inequality π would be 1-valid under this normalization if $\pi(-n) > -\frac{n}{r}$. In addition, if we refer to the non-negativity constraints and any inequalities directly drawn from the defining equation as *trivial inequalities*, then the following observation can be made.

Observation 2.2.2 *Suppose π is a 1-valid, minimal, and non-trivial inequality for $MEP(n, r)$, then $\pi(-n) > -\frac{n}{r}$.*

Proof. Suppose $\pi(-n) \leq -\frac{n}{r}$. Since π is 1-valid, for all $i \in I$, we have $\pi(i) \geq \sum_{k \in K} \lambda_k \pi^k(i) + \alpha i$, and $1 \leq \sum_{k \in K} \lambda_k + \alpha r$, where $\lambda^k \geq 0$, $\alpha \in \mathbb{R}$, and $\{\pi^k : k \in K\}$ is the set of all the facet defining inequalities of $MEP(n, r)$, with the normalization $\pi^k(-n) = 0$. Moreover, since π is minimal, the previous two inequalities hold at equality.

Based on our assumption, we get that $-\frac{n}{r} \geq \pi(-n) = \sum_{k \in K} \lambda_k \pi^k(-n) + \alpha(-\frac{n}{r}) = \alpha(-\frac{n}{r})$, so $\alpha \geq 1$. On the other hand, we have $1 = \sum_{k \in K} \lambda_k + \alpha r \geq \sum_{k \in K} \lambda_k + 1 \geq 1$, so $\lambda_k = 0$ for all $k \in K$, and $\alpha = r = 1$. This implies that π corresponds to the inequality $\sum_{i=-n}^{n} i w_i \geq 1$. This is a trivial inequality by definition, so our assumption is contradicted. Therefore, our assumption is false, and we must have $\pi(-n) > -\frac{n}{r}$. $\square$

Note a non-trivial inequality does not necessarily define a non-trivial facet of $MEP(n, r)$. With Observation 2.2.2, we prove that any 1-valid, minimal, non-trivial, and $(r-n)$-partially two-slope inequality π can be re-written with $\pi(-n) = 0$ while remaining 1-valid.

Lemma 2.2.3 *Suppose π is a minimal, 1-valid, non-trivial, and $(r-n)$-partially two-slope inequality for $MEP(n,r)$, then it's normalization $\bar{\pi}$ with $\bar{\pi}(-n)=0$ is also minimal, 1-valid, and $(r-n)$-partially two-slope for $MEP(n,r)$.*

PROOF. Note that $\bar{\pi}$ is defined by the following: $\bar{\pi}(i) := \frac{\pi(i)+\frac{i}{n}\pi(-n)}{1+\frac{r}{n}\pi(-n)}$, $\forall i \in I$. Then for any $w \in MEP(n,r)$,

$$\begin{aligned}
\textstyle\sum_{i=-n}^{n} \bar{\pi}(i) w_i &= \textstyle\sum_{i=-n}^{n} \frac{\pi(i)+\frac{i}{n}\pi(-n)}{1+\frac{r}{n}\pi(-n)} w_i \\
&= \frac{\sum_{i=-n}^{n} \pi(i) w_i + \frac{\pi(-n)}{n} \sum_{i=-n}^{n} i w_i}{1+\frac{r}{n}\pi(-n)} \\
&\geq \frac{1+\frac{\pi(-n)}{n} r}{1+\frac{r}{n}\pi(-n)} \qquad \text{since } \pi(-n) > -\tfrac{n}{r} \text{ by observation 2.2.2} \\
&= 1.
\end{aligned}$$

Hence $\bar{\pi}$ is 1-valid for $MEP(n,r)$.

If $\bar{\pi}$ is not minimal, then there exists a 1-valid function $\bar{\pi}'$ for $MEP(n,r)$ such that $\bar{\pi}' \leq \bar{\pi}$ and $\bar{\pi}' \neq \bar{\pi}$. However, by multiplying $1+\frac{r}{n}\pi(-n)$ and subtracting $\frac{i}{n}\pi(-n)$ from each $\bar{\pi}'(i)$ for all $i \in I$, we can obtain a 1-valid inequality π' for $MEP(n,r)$ such that $\pi' \leq \pi$ and $\pi' \neq \pi$, which contradicts the minimality of π. So, $\bar{\pi}$ is minimal.

Moreover, observe for $i \in \{r-n, \ldots, n-1\}$, $\bar{\pi}(i+1) - \bar{\pi}(i) = \frac{\pi(i+1)-\pi(i)+\frac{1}{n}\pi(-n)}{1+\frac{r}{n}\pi(-n)} = \frac{\pi(1)+\frac{1}{n}\pi(-n)}{1+\frac{r}{n}\pi(-n)}$ or $\frac{-\pi(-1)+\frac{1}{n}\pi(-n)}{1+\frac{r}{n}\pi(-n)} = \bar{\pi}(1)$ or $-\bar{\pi}(-1)$. Therefore $\bar{\pi}$ is an $(r-n)$-partially two-slope inequality as well.

By Lemma 2.2.1, $\bar{\pi}$ defines a facet of $MEP(n,r)$, since $\bar{\pi}$ and π are equivalent, we conclude π defines the same facet of $MEP(n,r)$. $\square$

One quick but powerful corollary follows directly from Lemma 2.2.1 and Lemma 2.2.3 is the following.

Corollary 2.2.4 *Suppose π is a 1-valid, minimal, non-trivial, and $(r-n)$-partially two-slope inequality for $MEP(n,r)$ such that $\pi(-n) > -\frac{n}{r}$, then π defines a non-trivial facet of $MEP(n,r)$.*

PROOF. We continue with the notation used in Lemma 2.2.3. In the proof of Lemma 2.2.3, we have shown if some $w \in \mathbb{R}^{2n+1}$ satisfies $\sum_{i=-n}^{n} \pi(i) w_i \geq 1$, then $\sum_{i=-n}^{n} \bar{\pi}(i) w_i \geq 1$ as well.

Now we consider some $w' \in \mathbb{R}^{2n+1}$ such that $\sum_{i=-n}^{n} \bar{\pi}(i) w'_i \geq 1$. Observe that $\pi(i) = \bar{\pi}(i)(1 + \frac{r}{n}\pi(-n)) - \frac{i}{n}\pi(-n)$, then

$$\begin{array}{rcl} \sum_{i=-n}^{n} \pi(i) w'_i & = & (1 + \frac{r}{n}\pi(-n)) \sum_{i=-n}^{n} \bar{\pi}(i) w'_i - \frac{\pi(-n)}{n} \sum_{i=-n}^{n} i w'_i \\ & \geq & (1 + \frac{r}{n}\pi(-n)) - \frac{\pi(-n)}{n} \cdot r \qquad \text{since } \pi(-n) > -\frac{n}{r} \text{ by observation 2.2.2} \\ & = & 1. \end{array}$$

This implies if one of π and $\bar{\pi}$ defines a face of $MEP(n,r)$, the other one must define the same face. Then by Lemma 2.2.1, $\bar{\pi}$ and π define a non-trivial facet of $MEP(n,r)$. □

Now only the second part of the theorem is unproven. Before we prove that part, it is useful to consider Lemma 2.2.5. As irrelevant and uninteresting as it may seem, it clarifies the relationship of the minimum valid values and lower bounds provided by the subadditivity constraints.

Lemma 2.2.5 *Let $i_0 \in \{-n, \ldots, r-n-1\}$, $l, m \in \mathbb{Z}_+$, $j_1, \ldots, j_l, k_1, \ldots, k_m \in I_+$, and $\sum_{a=1}^{l} j_a, \sum_{b=1}^{m} k_b \in I_+$. If $\sum_{i=-n}^{n} \pi(i) w_i \geq 1$ is an $(r-n)$-partially two-slope inequality for $MEP(n,r)$, then $\left(\pi(i_0 + \sum_{a=1}^{l} j_a) - \sum_{a=1}^{l} \pi(j_a)\right) - \left(\pi(i_0 + \sum_{b=1}^{m} k_b) - \sum_{b=1}^{m} \pi(k_b)\right)$ $= c\,(\pi(1) + \pi(-1))$, for some $c \in \mathbb{Z}$.*

Proof. Without loss of generality, assume $l \geq m$. Observe that for any $j_0, k_0 \in \{1, \ldots, n\}$, the difference between $\pi(j_0)$ and $\pi(k_0)$ can be expressed as $x\pi(1) - (j_0 - k_0 - x)\pi(-1)$ for some $x \in \mathbb{Z}$. This is because π is $(r-n)$-partially two-slope, and $\pi(j_0)$ and $\pi(k_0)$ differ by exactly $j_0 - k_0$ indices. Each index in the difference represents either $\pi(1)$ or $-\pi(-1)$.

By applying this observation to the desired difference, we obtain

$$
\begin{aligned}
& \left(\pi(i_0 + \sum_{a=1}^{l} j_a) - \sum_{a=1}^{l} \pi(j_a)\right) - \left(\pi(i_0 + \sum_{b=1}^{m} k_b) - \sum_{b=1}^{m} \pi(k_b)\right) \\
= & \left(\pi(i_0 + \sum_{a=1}^{l} j_a) - \pi(i_0 + \sum_{b=1}^{m} k_b)\right) - \sum_{b=1}^{m} (\pi(j_b) - \pi(k_b)) \\
& - \sum_{a=m+1}^{l} (\pi(j_a) - \pi(0)) \\
= & \left(x\pi(1) - \left(\sum_{a=1}^{l} j_a - \sum_{b=1}^{m} k_b - x\right) \pi(-1)\right) \\
& - \sum_{b=1}^{m} (x_b \pi(1) - (j_b - k_b - x_b)\pi(-1)) \\
& - \sum_{a=m+1}^{l} (x_a \pi(1) - (j_b - x_a)\pi(-1)) \\
= & \left(x - \sum_{a=1}^{l} x_a\right) (\pi(1) + \pi(-1)),
\end{aligned}
$$

for some $x, x_1, ..., x_l \in \mathbb{Z}$. To see the lemma is true, define $c := x - \sum_{a=1}^{l} x_a$. $\square$

In particular, we can see that by manipulating the variables, each subadditivity constraint with $\pi(i)$ on the left for some i can serve as a lower bound for m_i. Then the difference between any two such lower bounds is an integer multiple of $\pi(1) + \pi(-1)$. This observation is formally proven through a series of lemmas and it completes the proof of the second part of Theorem 2.0.2. The first two lemmas give a measure of the bound provided by a double or a triple subadditivity inequalities for a minimum valid value, assuming this bound is not tight.

Before we go into the proofs of the following lemmas, notice π is referred to as a "function" instead of an "inequality" in the lemma statements. This is because π is a coefficient function for the variables of $MEP(n, r)$ that implies the inequality $\pi^T w \geq \pi_0$ when the value of π_0 is specified. However, in the following lemmas, we only focus on the properties it has as a function, while its corresponding inequality is not considered. Naturally, we view π as a function in this case.

Lemma 2.2.6 *Suppose π is an $(r-n)$-partially two-slope function such that $\pi(1)+\pi(-1) \geq 0$. Then for any $i \in \{-n, \ldots, r-n-1\}$ and $j_0, i+j_0 \in I_+$, if $m_i > \pi(i+j_0) - \pi(j_0)$, then $m_i \geq \pi(i+j_0) - \pi(j_0) + \pi(1) + \pi(-1)$.*

PROOF. By Definition 2.1.1, either there exist $j_1, i+j_1 \in I_+$ such that $m_i = \pi(i+j_1) - \pi(j_1)$ or there exist $j_2, k_2 \in I_+$ such that $\pi(i+j_2+k_2) - \pi(j_2) - \pi(k_2)$. Since $m_i > \pi(i+j_0) - \pi(j_0)$, Lemma 2.2.5 gives either $m_i - (\pi(i+j_0) - \pi(j_0)) = (\pi(i+j_1) - \pi(j_1)) - (\pi(i+j_0) - \pi(j_0)) > 0$ or $m_i - (\pi(i+j_0) - \pi(j_0)) = (\pi(i+j_2+k_2) - \pi(j_2) - \pi(k_2)) - (\pi(i+j_0) - \pi(j_0)) > 0$ is an integer multiple of $\pi(1) + \pi(-1)$. Recall $\pi(1) + \pi(-1) \geq 0$, so $m_i - (\pi(i+j_0) - \pi(j_0)) \geq \pi(1) + \pi(-1)$. □

Lemma 2.2.7 *Suppose π is an $(r-n)$-partially two-slope function such that $\pi(1)+\pi(-1) \geq 0$. Then for any $i \in \{-n, \ldots, r-n-1\}$ and $j_0, k_0, i+j_0+k_0 \in I_+$, if $m_i > \pi(i+j_0+k_0) - \pi(j_0) - \pi(k_0)$, then $m_i \geq \pi(i+j_0+k_0) - \pi(j_0) - \pi(k_0) + \pi(1) + \pi(-1)$.*

PROOF. By Definition 2.1.1, either there exist $j_1, i+j_1 \in I_+$ such that $m_i = \pi(i+j_1) - \pi(j_1)$ or there exist $j_2, k_2 \in I_+$ such that $\pi(i+j_2+k_2) - \pi(j_2) - \pi(k_2)$. Since $m_i > \pi(i+j_0+k_0) - \pi(j_0) - \pi(k_0)$, Lemma 2.2.5 gives either $m_i - (\pi(i+j_0+k_0) - \pi(j_0) - \pi(k_0)) = (\pi(i+j_1) - \pi(j_1)) - (\pi(i+j_0+k_0) - \pi(j_0) - \pi(k_0)) > 0$ or $m_i - (\pi(i+j_0+k_0) - \pi(j_0) - \pi(k_0)) = (\pi(i+j_2+k_2) - \pi(j_2) - \pi(k_2)) - (\pi(i+j_0+k_0) - \pi(j_0) - \pi(k_0)) > 0$ is an integer multiple of $\pi(1)+\pi(-1)$. Recall $\pi(1)+\pi(-1) \geq 0$, so $m_i - (\pi(i+j_0+k_0) - \pi(j_0) - \pi(k_0)) \geq \pi(1)+\pi(-1)$.
□

With the aid of Lemma 2.2.6 and Lemma 2.2.7, we are ready to prove the second part of Theorem 2.0.2.

Lemma 2.2.8 *Let π be an $(r-n)$-partially two-slope function such that for any $i \in \{-n, \ldots, r-n\}$ the following conditions are satisfied.*

- *$\pi(i) + \pi(j_1) \geq \pi(i+j_1)$ for all $j_1, i+j_1 \in I_+$.*
- *$\pi(i) + \pi(j_2) + \pi(k_2) \geq \pi(i+j_2+k_2)$ for all $j_2, k_2, i+j_2+k_2 \in I_+$, $j_2 \leq k_2$.*
- *$\pi(1) + \pi(-1) \geq 0$.*

Then for $i \in \{-n, \ldots, r-n-1\}$, $m_{i+1} - m_i = \pi(1)$ or $-\pi(-1)$.

PROOF. For $i \in \{-n, \ldots, r-n\}$, let $S_i^d := \{j_1 \in I_+ : -i+1 \leq j_1 \in I_+, m_i = \pi(i+j_1) - \pi(j_1)\}$, $S_i^t := \{(j_2, k_2) \in I_+^2 : j_2, k_2, i+j_2+k_2 \in I_+, m_i = \pi(i+j_2+k_2) - \pi(j_2) - \pi(k_2)\}$, and observe at least one of S_i^d and S_i^t is non-empty. By definition of m_i, the double and triple subadditivity inequalities can provide lower bounds for m_i. We denote the bounds by l_{ij}^d and l_{ijk}^t, respectively.

Consider any $i \in \{-n, \ldots, r-n-1\}$. For j_1 such that $j_1, i+j_1 \in I_+$, since $-i+1 \leq j_i \leq n$, we have $-i \leq j_1 - 1 \leq n-1$. Then there are the following two cases.

- Case 1: $j_1 - 1 \in S^d_{i+1}$. Then

$$\begin{aligned} m_i &\geq \pi(i+j_1) - \pi(j_1) \\ &= \pi(i+j_1) - \pi(j_1-1) + \pi(j_1-1) - \pi(j_1) \\ &= m_{i+1} + \pi(j_1-1) - \pi(j_1) \\ &= m_{i+1} - \pi(1) \text{ or } m_{i+1} + \pi(-1) \\ &= l^d_{ij_1} \\ &\geq m_{i+1} - \pi(1) \qquad \text{since } \pi(1) + \pi(-1) \geq 0 \end{aligned}$$

- Case 2: $j_1 - 1 \notin S^d_{i+1}$. Recall that $-i \leq j_1 - 1 \leq n-1$. In this case, we have

$$\begin{aligned} m_i &\geq \pi(i+j_1) - \pi(j_1) \\ &= \pi(i+j_1) - \pi(j_2-1) + \pi(j_1-1) - \pi(j_1) \\ &= (\pi(i+j_1) - \pi(j_1-1)) - \pi(1) \text{ or } (\pi(i+j_1) - \pi(j_1-1)) + \pi(-1) \\ &= l^d_{ij_1} \end{aligned}$$

  If there exists some $j'_1 \in S^d_{i+1}$, then by Lemma 2.2.6 and the facet that $\pi(1)+\pi(-1) \geq 0$, we get $l^d_{ij_1} = m_{i+1} - 2\pi(1) - \pi(-1) \leq m_{i+1} - \pi(1) \leq l^d_{ij'_1}$ or $l^d_{ij_1} = (\pi(i+j_1) - \pi(j_1-1)) + \pi(-1) \leq m_{i+1} - \pi(1) \leq l^d_{ij'_1}$. So the bound for m_i provided by these j_1's are at most as much as the one provided by j'_1.

  If such a j'_1 does not exist, then there exists (j'_2, k'_2) with $j'_2, k'_2 \in I_+$, $j'_2 \leq k'_2$ such that either $(j'_2 - 1, k'_2) \in S^t_{i+1}$ or $(j'_2, k'_2 - 1)$ is in S^t_{i+1}. By the argument below about the bounds provided by the triple subadditivity inequalities, we can see that $l^t_{ij_1} \leq m_{i+1} - \pi(1) \leq l^t_{ij'_2k'_2}$. Therefore, $l^d_{ij_1}$ can be neglected when looking for the possible values of m_i, since it does not provide a tighter bounds than the tight double and triple subadditivity inequalities at $i+1$.

We consider the bounds provided by the triple subadditivity inequalities for m_i in a similar fashion. For any (j_2, k_2) such that $j_2, k_2, i + j_2 + k_2 \in I_+$ and $j_2 < k_2$, we have the following cases to consider.

- Case 1: $(j_2 - 1, k_2) \in S^t_{i+1}$ and $j_2 > 1$. Then

$$\begin{aligned} m_i &\geq \pi(i+j_2+k_2) - \pi(j_2) - \pi(k_2) \\ &= \pi(i+j_2+k_2) - \pi(j_2-1) - \pi(k_2) + \pi(j_2-1) - \pi(j_2) \\ &= m_{i+1} + \pi(j_2-1) - \pi(j_2) \\ &= m_{i+1} - \pi(1) \text{ or } m_{i+1} + \pi(-1) \\ &= l^t_{ij_2k_2} \\ &\geq m_{i+1} - \pi(1) \qquad \text{since } \pi(1) + \pi(-1) \geq 0 \end{aligned}$$

- Case 2: $(j_2-1,k_2)\notin S^t_{i+1}$ and $j_2>1$. Then

$$\begin{aligned}
m_i &\geq \pi(i+j_2+k_2)-\pi(j_2)-\pi(k_2)\\
&= \pi(i+j_2+k_2)-\pi(j_2-1)-\pi(k_2)+\pi(j_2-1)-\pi(j_2)\\
&= (\pi(i+j_2+k_2)-\pi(j_2-1)-\pi(k_2))-\pi(1)\\
&\quad \text{ or } (\pi(i+j_2+k_2)-\pi(j_2-1)-\pi(k_2))+\pi(-1)\\
&= l^t_{ij_2k_2}
\end{aligned}$$

  By Lemma 2.2.7 and $\pi(1)+\pi(-1)\geq 0$, we have either $l^t_{ij_2k_2}=(\pi(i+j_2+k_2)-\pi(j_2-1)-\pi(k_2))-\pi(1)\leq m_{i+1}-2\pi(1)-\pi(-1)\leq m_{i+1}-\pi(1)$ or $l^t_{ij_2k_2}=(\pi(i+j_2+k_2)-\pi(j_2-1)-\pi(k_2))+\pi(-1)\leq m_{i+1}-\pi(1)$. Similar to the case 2 in the double subadditivity inequalities case, we conclude $l^t_{ij_2k_2}$ can be discarded when looking for the possible values of m_i.

- Case 3: $j_2=1$. In this case, if $k_2\neq 1$, the argument follows analogously from case 1 and case 2, depending on whether $(j_2,k_2-1)\in S^t_{i+1}$, such that the best bound given by these inequalities are $m_{i+1}-\pi(1)$ or $m_{i+1}+\pi(-1)$. Moreover, if $k_2=1$, we simply have $m_i\geq\pi(i+1+1)-\pi(1)-\pi(1)$. Since $i<0$ and $i+2>0$, we must have $i=-1$. Then $m_{i+1}=m_0=0$, and the bound given by the inequality is exactly $m_i\geq-\pi(1)=m_{i+1}-\pi(1)$.

Thus, considering all the possible bounds, we get $m_i\geq m_{i+1}-\pi(1)$ or $m_i\geq m_{i+1}+\pi(-1)$, with one of these two inequalities holding at equality. □

It is clear that Theorem 2.0.2 follows directly from Lemma 2.1.2, Lemma 2.2.1 and Lemma 2.2.8. Note minimality and partially two-slope-ness of an inequality are easy to check, and this theorem proved the strength of inequalities with these properties, which is a fairly powerful result. That is to say, Theorem 2.0.2 provides a characterization for some facets of $MEP(n,r)$, where only a few simple conditions need to be satisfied.

Chapter 3

A Modified Infinite Relaxation Model

As briefly mentioned in Chapter 1, the clearly noticeable structural similarities between the MEP and the infinite relaxation model is a major motivation of our research. In this chapter, we study the link between these two models by considering a modified version of the infinite relaxation model. The relation between the modified model and the MEP is then carefully examined. In Chapter 4, a two-slope theorem for the modified model will be proven, so some elementary results related to that are introduced in this chapter as well.

3.1 Derivation

Consider system (1.6), the infinite relaxation model. Since the MEP is a model for pure IPs, to study its relationship with the infinite model, we need to consider the pure integer version of the infinite relaxation model. In other words, we may assume $s'_t = 0$, for all $t \in \mathbb{R}$. Then the defining equation becomes $x = f + \sum_{t \in \mathbb{R}} t y'_t$. Note that for both the MEP and the pure integer version of the infinite relaxation model, we can easily re-introduce the continuous variables back if necessary.

In practice, the pure integer version of the infinite relaxation model is used as a relaxation of some finite IP so that reasonably good cuts may be obtained. Consequently, there are only a finite number of relevant y'_t variables in this problem. Then there exist $l_1, l_2 \in \mathbb{R}_+$ such that there are no relevant y'_t's, $\forall t > l_1$ and $\forall t < l_2$.

In addition, recall all coefficients and constants are assumed to be rational, so there exists q, a least common denominator of f and all relevant y'_t's. Define $n := \max\{|l_1|, |l_2|\}q$,

$z = 1 - x$, and $r := (1-f)q$. We can rewrite the defining equation of system (1.6) as $\frac{r}{q} = z + \sum_{t \in \mathbb{R}} t y'_t$, where $r \in \mathbb{Z}_+$. Moreover, we introduce a new set of variables $y_t = y'_{t/q}$, so that the modified infinite relaxation model is obtained:

$$\begin{array}{rcl} r & = & qz + \sum_{t \in \mathbb{R}} t y_t \\ z, y_t & \in & \mathbb{Z}_+ \\ (y, z) & \text{has} & \text{finite support} \end{array} \tag{3.1}$$

Note q and r are the only parameters of this model, so we denote it as $IRm(q, r)$.

To see how the modified infinite relaxation and $MEP(n, r)$ are related, note that y_t is irrelevant for any t such that $|t| > n$, so any given IP in the form of $IRm(q, r)$ can be written as a problem in the following modified version of $MEP(n, r)$ instead:

$$\begin{array}{rcl} r & = & qz + \sum_{i=-n}^{n} i w_i \\ z, w_i & \in & \mathbb{Z}_+ \end{array} \tag{3.2}$$

where $r < q \leq n$ and $r, q, n \in \mathbb{Z}_+$.

Moreover, this system, denoted as $MEPm(n, r)$, the modified MEP, can be relaxed by a change of variable: $w_i := \begin{cases} y_i & i \neq q \\ y_i + z & i = q. \end{cases}$ This relaxation gives exactly $MEP(n, r)$. With this demonstrated, we are ready to present some properties of $MEPm(n, r)$.

3.2 Facet Defining Inequalities of $MEPm(n, r)$

Recall $MEP(n, r)$ is obtained by a change of variable that combines z and y_q, the variable in the summation with the same coefficient as z. So it is logical to expect the coefficient of z in a facet defining inequality of $MEPm(n, r)$ to be related to the coefficient of y_q. In particular, we prove the following lemma.

Lemma 3.2.1 *If $\rho z + \sum_{i=-n}^{n} \pi(i) w_i \geq 1$ is a non-trivial facet defining inequality for $MEPm(n, r)$, then $\rho = \pi(q)$.*

Proof. Suppose there exists a non-trivial facet defining inequality (ρ, π) for $MEPm(n, r)$ such that $\rho \neq \pi(q)$. Since (ρ, π) does not define the trivial facet $z \geq 0$, there exists a feasible solution (z, w) for $MEPm(n, r)$ such that $z \geq 1$ and $\rho z + \sum_{i=-n}^{n} \pi(i) w_i = 1$.

If $\rho > \pi(q)$, consider $z' := z - 1$, $w'_i := \begin{cases} w_i & i \neq q \\ w_q + 1 & i = q. \end{cases}$. Then clearly (z', w') is a feasible solution for system $MEPm(n, r)$. However, this implies

$$\rho z' + \sum_{i=-n}^{n} \pi(i) w'_i < \rho z + \sum_{i=-n}^{n} \pi(i) w_i = 1,$$

which contradicts the validity of (ρ, π).

Similarly, there exists a feasible solution $(z, w) \in MEPm(n, r)$ such that $w_q \geq 1$ and $\rho z + \sum_{i=-n}^{n} \pi(i) w_i = 1$. Then if $\rho < \pi(q)$, we may consider the feasible point (z'', w'') defined by $z'' := z + 1$, $w''_i = \begin{cases} w_i & i \neq q \\ w_q - 1 & i = q. \end{cases}$. This gives

$$\rho z'' + \sum_{i=-n}^{n} \pi(i) w''_i < \rho z + \sum_{i=-n}^{n} \pi(i) w_i = 1,$$

which contradicts the validity of (ρ, π) as well. Thus, we must have $\rho = \pi(q)$. $\square$

Moreover, it is important to note all non-trivial facet defining inequalities of $MEPm(n, r)$ can be obtained from facet defining inequalities of system $MEP(n, r)$.

Lemma 3.2.2 *An 1-valid inequality π defines a non-trivial facet of system $MEP(n, r)$ if and only if $(\pi(q), \pi)$ defines a non-trivial facet of system $MEPm(n, r)$.*

Proof. $(\Rightarrow)$ Suppose π is a 1-valid inequality that defines a non-trivial facet Q for $MEP(n, r)$. Then there exists $2n - 1$ affinely independent points in Q. Since Q is not defined by $w_q \geq 0$, there exists some $w \in Q$ such that $\sum_{i=-n}^{n} \pi(i) w_i = 1$, $w_q \geq 1$.

Observe that $(z', w') = (1, w - e_q) \in MEPm(n, r)$, where e_q represents the unit vector in the direction of w_q. Also, $\pi(q) z' + \sum_{i=-n}^{n} \pi(i) w'_i = 1$. The point (z', w') along with the $2n - 1$ affinely independent points in Q gives a total of $2n$ affinely independent points. Therefore, $(\pi(q), \pi)$ defines a non-trivial facet of $MEPm(n, r)$.

($\Leftarrow$) Suppose a 1-valid inequality π does not define a facet for $MEP(n,r)$, then there exist facet defining inequalities $\pi^1, \pi^2, ..., \pi^k$ for $MEP(n,r)$ such that for all $j \in I$, $\sum_{i=1}^{k} \lambda_i \pi^i(j) + cj \leq \pi(j)$, where $c \in \mathbb{R}$, $\sum_{i=1}^{k} \lambda_i = 1$ and $k \geq 2$.

As shown above, we know that $(\pi^i(q), \pi^i)$ is a facet defining inequality for $MEPm(n,r)$, $\forall 1 \leq i \leq k$. Then π can be written as a convex combination of at least two distinct facet defining inequalities for $MEPm(n,r)$. Hence, we can conclude that π is not facet defining for $MEPm(n,r)$. $\square$

Intuitively, $MEP(n,r)$ is a face of $MEPm(n,r)$, because $MEP(n,r)$ can be obtain by setting $z = 0$ in $MEPm(n,r)$. The lemma above supports this fact.

After discussing certain properties of $MEPm(n,r)$, we continue to introduce the main interest of study of this chapter, $IRm(q,r)$.

3.3 Basic Properties of the Modified Infinite Relaxation Model

Similar to the infinite relaxation model, $IRm(q,r)$ is not a polyhedron. To be able to discuss its properties, certain concepts such as validity, minimality, and extremality must be re-defined.

Definition 3.3.1 *Let $\gamma, \phi_0 \in \mathbb{R}$ and $\phi : \mathbb{R} \to \mathbb{R}$. Then we say (ϕ, γ, ϕ_0) is valid for $IRm(q,r)$ if the inequality $\gamma z + \sum_{t \in \mathbb{R}} \phi(t) y_t \geq \phi_0$ holds for all feasible points of $IRm(q,r)$. When ϕ_0 is specified, we may say (ϕ, γ) is valid for $IRm(q,r)$.*

*Moreover, up to normalization, if $\phi_0 = 1$, (ϕ, γ) is 1-**valid**; if $\phi_0 = 0$, and (ϕ, γ) is not 1-valid, then (ϕ, γ) is 0-**valid**; and if $\phi_0 = -1$ and (ϕ, γ) is not 0-valid, then (ϕ, γ) is -1-**valid**.*

Definition 3.3.2 *Suppose (ϕ, γ) is ϕ_0-valid. Then (ϕ, γ) is **minimal** for $IRm(q,r)$ if there does not exist $(\phi', \gamma') \leq (\phi, \gamma)$ and $(\phi', \gamma') \neq (\phi, \gamma)$ such that ϕ' is ϕ_0-valid for $IRm(q,r)$, and there does not exist $\phi'_0 > \phi_0$ such that (ϕ, γ, ϕ'_0) is valid for $IRm(q,r)$.*

The following two observations follow from the definitions above.

Observation 3.3.3 *If (ϕ, γ) is 1-valid and minimal for $IRm(q,r)$, then $\phi(0) = 0$.*

Observation 3.3.4 *If (ϕ, γ) is 1-valid and minimal for $IRm(q, r)$, then ϕ is subadditive over $\mathbb{R}$.*

The first observation can be obtained analogously as in the MEP case. For the second observation, notice that $IRm(q, r)$ contains a subset of system (1.6), with certain variables set to zero. So by Lemma 1.6.3, the second observation must be true.

Moreover, we can prove that any 1-valid and minimal pair (ϕ, γ) satisfies $\phi(q) = \gamma$. This is an analogous result to Lemma 3.2.1, and it builds a strong connection between γ and $\phi(q)$. Such connection is of high importance, because it implies the normalization $\pi(q) = 0$ would give $\gamma = 0$, which leaves only the ϕ function for us to study. This greatly reduces the complexity of the problem at hand.

Lemma 3.3.5 *If (ϕ, γ) is 1-valid and minimal for $IRm(q, r)$, then $\phi(q) = \gamma$.*

Proof. Suppose $\gamma \neq \phi(q)$, then we have two cases: $\gamma > \phi(q)$ or $\phi(q) > \gamma$.

Assume for some $\beta > 0$, $\gamma - \phi(q) = \beta$. Consider some feasible point $(y, z) \in IRm(q, r)$. Let $y'_t := y_t$ for all $t \neq q$, $y'_q := y_q + z$, and $z' = 0$, then (y', z') is also a feasible point of $IRm(q, r)$, which implies $\gamma z' + \sum_{t\in\mathbb{R}} \phi(t)y'_t \geq 1$. So, for any $(y, z) \in IRm(q, r)$, $\gamma z + \sum_{t\in\mathbb{R}} \phi(t)y_t = \beta z + \gamma z' + \sum_{t\in\mathbb{R}} \phi(t)y'_t \geq 1 + \beta z$.

Now let $\gamma' = \gamma - \beta = \phi(q)$, then $\gamma' z + \sum_{t\in\mathbb{R}} \phi(t)y_t = \gamma z + \sum_{t\in\mathbb{R}} \phi(t)y_t - \beta z \geq 1 + \beta z - \beta z = 1$. In other words, (ϕ, γ') is 1-valid for $IRm(q, r)$. However, this contradicts the minimality of (ϕ, γ).

Similarly, if $\phi(q) > \gamma$, then there exists some $\beta > 0$ such that $\phi(q) - \gamma = \beta$. For a feasible point $(y, z) \in IRm(q, r)$, let $y'_t := y_t$ for all $t \neq q$, $y'_q := 0$, and $z' = y_q + z$, then (y', z') is also a feasible point of $IRm(q, r)$. So, $\gamma z' + \sum_{t\in\mathbb{R}} \phi(t)y'_t \geq 1$, and $\gamma z + \sum_{t\in\mathbb{R}} \phi(t)y_t = \beta y_q + \gamma z' + \sum_{t\in\mathbb{R}} \phi(t)y'_t \geq 1 + \beta y_q$.

Consider ϕ' such that $\phi'(t) = \phi(t)$ for all $t \neq q$, and $\phi'(q) = \phi(q) - \beta = \gamma$. Observe that $\gamma z + \sum_{t\in\mathbb{R}} \phi'(t)y_t = \gamma z + \sum_{t\in\mathbb{R}} \phi(t)y_t - \beta y_q \geq 1 + \beta y_q - \beta y_q = 1$. So (ϕ', γ) is 1-valid for $IRm(q, r)$, contradicting the minimality of (ϕ, γ). Thus, we may conclude $\phi(q) = \gamma$. □

Similar to validity and minimality, the extremality of a function must be redefined for $IRm(q, r)$ as well.

Definition 3.3.6 *Let (ϕ, γ) be a 1-valid for $IRm(q, r)$ such that either $\gamma = 0$ or $\phi(t) = 0$ for some $t \in \mathbb{R}$. Suppose there exist two 1-valid pairs (ϕ_1, γ_1), (ϕ_2, γ_2) such that $\phi =$*

$\frac{1}{2}(\phi_1 + \phi_2)$ *and* $\gamma = \frac{1}{2}(\gamma_1 + \gamma_2)$ *with* $\phi_1(t) = \phi_2(t) = 0$ *if* $\gamma \neq 0$ *and* $\gamma_1 = \gamma_2 = 0$ *if* $\gamma = 0$. *Then* (ϕ, γ) *is* ***extreme*** *if the only possible choices for* (ϕ, γ) *and* (ϕ, γ) *are* $\gamma_1 = \gamma_2 = \gamma$ *and* $\phi_1 = \phi_2 = \phi$.

It is intuitive to think extreme functions are minimal. The following lemma formally proves it.

Lemma 3.3.7 *If* (ϕ, γ) *is* 1*-valid and extreme for* $IRm(q, r)$*, then* (ϕ, γ) *is minimal.*

PROOF. Suppose (ϕ, γ) is 1-valid, extreme, but not minimal for $IRm(q, r)$ with some normalization (either $\gamma = 0$ or $\phi(t) = 0$ for some $t \in \mathbb{R}$), then there exists (ϕ_1, γ_1) with the same normalization such that it is 1-valid for $IRm(q, r)$, $\phi_1 \leq \phi$, $\gamma_1 \leq \gamma$, and $(\phi_1, \gamma_1) \neq (\phi, \gamma)$. Notice that $(\phi_2, \gamma_2) = (\phi + (\phi - \phi_1), \gamma + (\gamma - \gamma_1))$ is also 1-valid, since $\phi - \phi_1, \gamma - \gamma_1 \geq 0$. Then $\phi = \frac{1}{2}(\phi_1 + \phi_2)$, $\gamma = \frac{1}{2}(\gamma_1 + \gamma_2)$, and ϕ_2 has the same normalization as ϕ and ϕ_1 contradicting the extremality of ϕ. $\square$

Lemma 3.3.5 and Lemma 3.3.7 imply that any extreme pair (ϕ, γ) for $IRm(q, r)$ satisfies $\phi(q) = \gamma$. Then notice there is a natural choice of normalization for (ϕ, γ): $\phi(q) = \gamma = 0$. This normalization is assumed for the rest of the discussions in Chapter 3 and Chapter 4. Observe the coefficient of z is then assumed to be 0, so that we only need to study properties of ϕ.

The next step is to study the connection between $IRm(q, r)$ and $MEP(n, r)$. In particular, we give a method to extend 1-valid inequalities for $MEP(n, r)$ to a 1-valid inequality for an MEP with higher dimensions or a 1-valid coefficient function for $IRm(q, r)$.

3.4 Using Inequalities for $MEP(n, r)$ to Build Inequalities for $IRm(q, r)$

Suppose we have a 1-valid inequality π for $MEP(n, r)$. One interesting question to consider is, could we "build upon" this inequality to get a 1-valid inequality for an MEP with higher dimensions? We start experimenting this idea by defining the discrete extension for a function $\pi : I \to \mathbb{R}$.

Definition 3.4.1 *Suppose* $\pi : \{-n, ..., n\} \to \mathbb{R}$ *is a function. For any* $m \in \mathbb{Z}_+$*, the* ***discrete extension*** *of* π *for MEP(n+m, r), denoted by* π^{n+m}*, is defined recursively as*

$$\pi^{n+m}(i) := \begin{cases} \pi^{n+m-1}(i) & i \in \{-n-m+1, ..., n+m-1\}; \\ \pi^{n+m-1}(r) - \pi^{n+m-1}(r-n-m) & i = n+m; \\ \max\limits_{j,k,i+j+k \in I_+} \{\pi(i+j+k) - \pi(j) - \pi(k)\} & i = -n-m, \end{cases}$$

where $\pi^n := \pi$.

Observe the idea of Definition 3.4.1 is fairly simple: To extend π for $MEP(n,r)$ to π^{n+1} for $MEP(n+1,r)$, we begin by setting $\pi^{n+1}(i) = \pi(i)$ for all $i \in I$. Then, since $\pi(r-n-1)$ is known, $\pi^{n+1}(n+1)$ is defined based on complementarity. On the other hand, $\pi^{n+1}(-n-1)$ is defined as the smallest value such that the triple subadditivity constraints are not violated at $\pi^{n+1}(-n-1)$. Note no double subadditivity constraints applies at $pi^{n+1}(-n-1)$, so π^{n+1} always satisfies $\pi^{n+1}(-n-1) = m_{-n-1}$.

As the discrete extensions of π for $MEP(n,r)$ are introduced, certain notation must be clarified to avoid ambiguities. In particular, we define $I^n := \{-n, \dots, n\}$, and $T^{n,r}$ denotes the polyhedron (T) from Theorem 1.5.1 for $MEP(n,r)$.

If a function π for $MEP(n,r)$ possesses certain properties, it is interesting to consider whether its discrete extensions could "inherit" these properties from π. In this regard, we show minimality is indeed an inheritable property.

Lemma 3.4.2 *If π is a 1-valid and minimal inequality for $MEP(n,r)$, its discrete extension π^{n+1} is 1-valid and minimal for MEP(n+1, r).*

Proof. By the definition of discrete extension, π^{n+1} still satisfies complementarity, and $\pi^{n+1}(-n-1) = m_{-n-1}$. Hence Lemma 2.1.9 gives that π^{n+1} is minimal. □

Furthermore, we show the most important inequalities - the non-trivial facet defining ones - can be extended to obtain non-trivial facet defining inequalities for a higher dimensional MEP.

Lemma 3.4.3 *If a 1-valid inequality π defines a non-trivial facet of $MEP(n,r)$, then its discrete extension π^{n+1} defines a non-trivial facet of $MEP(n+1,r)$.*

Proof. Notice π defines a non-trivial facet of $MEP(n,r)$, so it is minimal. Additionally, by Theorem 1.5.1, we may assume $\pi(r) \geq 1$. Also, note π^{n+1} is 1-valid because it is in $T^{n,r}$.

Since π^n is facet defining for $MEP(n,r)$, there exist $2n$ linearly independent constraints at equality. These $2n$ linearly independent constraints at equality exist in π^{n+1} as well. Along with $\pi^{n+1}(n+1)+\pi^{n+1}(r-n-1) = \pi^{n+1}(r)$ and $\pi^{n+1}(-n-1)+\pi^{n+1}(k)+\pi^{n+1}(j) = \pi^{n+1}(j+k-n-1)$, for some $j,k \in I_+$ and $j+k > n+1$, we have $2n+2$ linearly independent constraints at equality in π^{n+1} for $MEP(n+1,r)$. Therefore, the 1-validity of π^{n+1} guarantees that it is facet defining for $MEP(n+1,r)$. $\square$

In addition, if we have a 1-valid, minimal and $(r-n)$-partially two-slope inequality, then all of its extensions are two-slope inequalities.

Lemma 3.4.4 *Suppose π is a 1-valid, minimal, and $(r-n)$-partially two-slope inequality for $MEP(n,r)$. Then π^{n+1} is 1-valid, minimal, and two-slope for MEP(n+1, r).*

Proof. By Lemma 3.4.2, π^{n+1} is 1-valid and minimal. So we only need to prove it is a two-slope inequality. Recall Lemma 2.1.2 implies π is subadditive, then Lemma 2.2.8 gives that π is a two-slope inequality. Observe that by Definition 3.4.1 and complementarity, we have

$$\begin{array}{rcl}\pi^{n+1}(n+1)-\pi^{n+1}(n) & = & \pi^{n+1}(r-n)-\pi^{n+1}(r-n-1) \\ & = & \pi(1) \text{ or } -\pi(-1).\end{array}$$

On the other hand, by Lemma 2.1.9 and Lemma 2.2.8, we immediately get $\pi^{n+1}(-n) - \pi^{n+1}(-n-1) = m_{-n} - m_{-n-1} = \pi^{n+1}(1)$ or $-\pi^{n+1}(-1)$. Therefore, π^{n+1} is a two-slope inequality. $\square$

Recall our goal is to relate $MEP(n,r)$ to $IRm(q,r)$. Therefore, it is essential to have a method that enables extensions of some π for $MEP(n,r)$ to a function for the entire $\mathbb{R}$. As a result, we define the continuous extension of π.

Definition 3.4.5 *Given a function $\pi : \{-n,...,n\} \to \mathbb{R}$, its* ***continuous extension*** *for (3.1) is* $\phi(t) := \begin{cases} \hat{t}\pi(\lceil t\rceil)+(1-\hat{t})\pi(\lfloor t\rfloor) & t\in[-n,n] \\ \hat{t}\pi^{\lceil t\rceil}(\lceil t\rceil)+(1-\hat{t})\pi^{\lceil t\rceil}(\lfloor t\rfloor) & \textit{otherwise,}\end{cases}$
where $\pi^{\lceil t\rceil}$ is the discrete extension of π in $MEP(\lceil t\rceil, r)$.

The definition of the continuous extension of π is also quite intuitive. Given some inequality π for $MEP(n,r)$, we can interpolate it to obtain a function defined on $[-n,n]$. By interpolating all discrete extensions of π, an continuous function on $\mathbb{R}$, which is the continuous extension in Definition 3.4.5, is obtained. Observe it is a piecewise linear function with integer breakpoints.

Similar to the discrete case, we prove the continuous extension for a 1-valid and minimal function π for $IRm(q,r)$, along with the coefficient of z being 0, is also 1-valid and minimal for $IRm(q,r)$.

Lemma 3.4.6 *If π is a 1-valid and minimal inequality for $MEP(n,r)$ with normalization $\pi(q) = 0$, then $(\phi, 0)$ is 1-valid and minimal for $IRm(q,r)$, where ϕ is the continuous extension of π, such that ϕ satisfies complementarity, i.e. $\phi(t)+\phi(r-t) = 1$, for all $t \in \mathbb{R}$.*

Proof. Recall that Lemma 3.4.2 tells us any discrete extension of π is still 1-valid and minimal, and therefore subadditive over it respective domain. Then by definition, the continuous extension ϕ of π is subadditive over $\mathbb{R}$ as well. Given a feasible solution (y, z) for $IRm(q,r)$, by subadditivity, we get

$$\begin{aligned}\sum_{t\in\mathbb{R}} \phi(t)y_t &\geq \sum_{t\in\mathbb{R}} \phi(ty_t) \\ &\geq \phi(\sum_{t\in\mathbb{R}} ty_t) \\ &= \phi(r-qz).\end{aligned}$$

Claim: $\phi(qz) \leq 0$, $\forall z \geq 0$.

We prove the claim by induction. When $z = 0$, $\phi(qz) = \phi(0) = 0$. For $z \geq 1$, notice that $\phi(qz) \leq \phi(q(z-1)) + \phi(q) = \phi(q(z-1)) \leq 0$, because $\phi(q) = 0$ by definition.

Therefore,

$$\begin{aligned}\sum_{t\in\mathbb{R}} \phi(t)y_t &\geq \phi(r-qz) \\ &\geq \phi(r) - \phi(qz) \\ &\geq \phi(r) = 1,\end{aligned}$$

since $\phi(r) = \pi(r) = 1$ by minimality of π. Then $(\phi, 0)$ is indeed 1-valid, and the computation above show any subadditivity ϕ is 1-valid for $IRm(q,r)$. In addition, complementarity holds for any extension of π, which means it must holds for any $t \in \mathbb{R}$ for ϕ.

Now assume $(\phi, 0)$ is not minimal, then there exists a 1-valid and minimal pair $(\phi', 0)$ such that $\phi' \leq \phi$ and $\phi' \neq \phi$. By $(\phi', 0)$'s 1-validity, we must have $\phi'(t)+\phi'(r-t) \geq 1$ for any $t \in \mathbb{R}$. However, since $\phi' \leq \phi$ and ϕ satisfies complementarity, we get $\phi'(t) + \phi'(r-t) \leq 1$ for any $t \in \mathbb{R}$. Then $\phi'(t) + \phi'(r-t) = 1$. Recall $\phi' \neq \phi$, so there exists $t_0 \in \mathbb{R}$ such that $\phi'(t_0) < \phi(t_0)$. But this implies $\phi'(r-t_0) > \phi(r-t_0)$, which contradicts $\phi' \leq \phi$. Therefore, such a ϕ' does not exist, and $(\phi, 0)$ is minimal. $\square$

It turns out the properties of continuous extensions play an essential role in proving a version of the two-slope theorem for $IRm(q, r)$. The full proof will be presented in the next Chapter.

Chapter 4

Two-Slope Theorem for Extreme Functions

This chapter focuses on the relationship between two-slope functions and extreme functions for $IRm(q,r)$. We start by showing the continuous extension of 1-valid and minimal two-slope inequalities are extreme for $IRm(q,r)$. Then we prove the theorem for the extremality of certain regular two-slope functions, from which the extremality of the optimized wedge function can be shown. This serves as a good example of how the theorem can be applied. At the end we give a brief introduction to an MIP version of $IRm(q,r)$, which has some interesting properties.

4.1 Extremality of Extended Two-Slope Inequalities

Recall the two-slope inequalities we have discussed are all under the framework of $MEP(n,r)$. To enable our discussion of two-slope functions for $IRm(q,r)$, we need to define it first.

Definition 4.1.1 *A continuous function ϕ is a **two-slope function** if the limits $\lim\limits_{a \to x^+} \frac{\phi(x)-\phi(a)}{x-a}$ and $\lim\limits_{a \to x^-} \frac{\phi(x)-\phi(a)}{x-a}$ exist for all $x \in \mathbb{R}$, and the value of these two limits are either s_1 or s_2 for some $s_1, s_2 \in \mathbb{R}$, given any $x \in \mathbb{R}$.*

Intuitively, given a 1-valid inequality π for $MEP(n,r)$, if $(\phi, 0)$ is be extreme such that ϕ is the continuous extension of π, π should be a strong cut. Moreover, the subdivisions of π, which are introduced below, must be strong cuts in their respective MEPs as well.

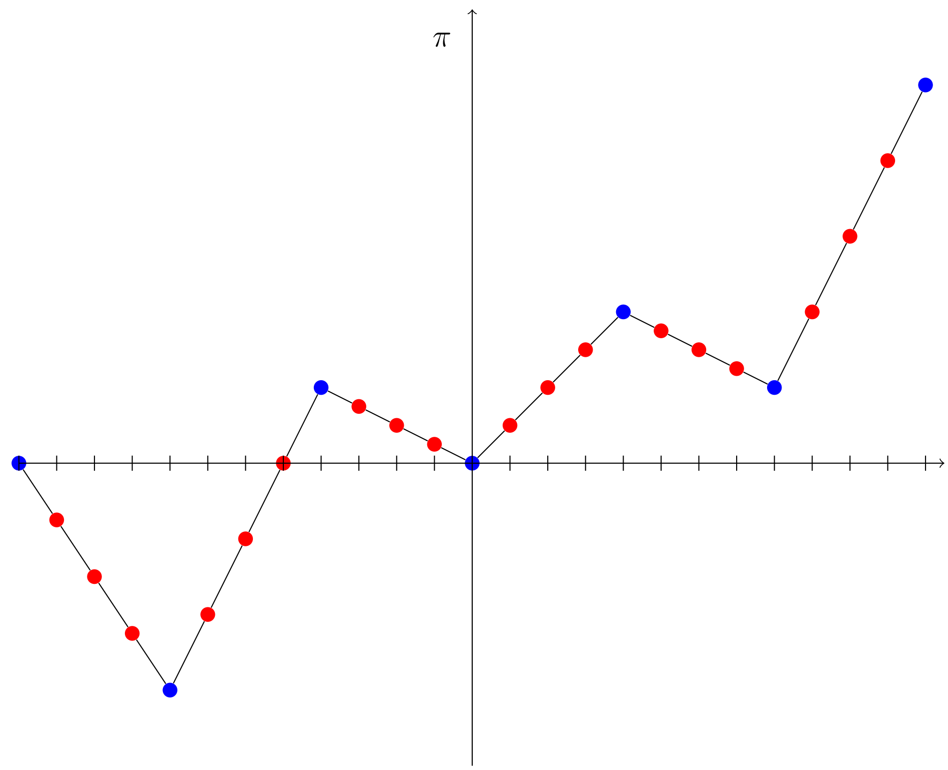

Figure 4.1: 4-subdivision of an inequality π for MEP(3, 1)

Definition 4.1.2 *Suppose $\pi^n : \{-n, ..., n\} \to \mathbb{R}$ is a function. Then the **k-subdivision** of π^n for $MEP(kn, kr)$ is $\pi^{n,k}(i) := \frac{p}{k}\pi^n(m+1) + \frac{k-p}{k}\pi^n(m)$, for all $i = km + p$, where $0 \le p < k$, $-n \le m \le n$, $p, m \in \mathbb{Z}$.*

As demonstrated in figure 4.1, π^3 consists of the blue vertices. Its 4-subdivision, $\pi^{3,4}$, interpolates between each two adjacent vertices, and divides the interpolated line segment into four line segments using the red vertices, such that the red vertices along with the blue vertices form $\pi^{3,4}$. In general, a k-subdivision of any inequality π^n uniformly partitions the line segment connecting any two adjacent values of π of consecutive indices into k segments. The function values of $\pi^{n,k}$ are then taken as the height of the breakpoints of each segment.

The main theorem of this section proves the extremality of the continuous extension of a facet defining inequality π for $MEP(n, r)$, if all k-subdivisions of any discrete extension of π is facet defining for the MEP with appropriate parameters, for any $k \in \mathbb{Z}_+$. This is somewhat intuitive, because the continuous extension of π is an interpolation of the discrete extensions of π. The strength of the discrete extensions of π and their subdivisions seems to imply the continuous extension is a strong cut for the modified infinite relaxation as well. We begin by proving a similar result for an infinite model that only admits rational coefficients, and then extend the theorem to $IRm(q, r)$. The rational coefficient infinite model is the following.

$$\begin{array}{rcl} r & = & qz + \sum_{t \in \mathbb{Q}} t y_t \\ z, y_t & \in & \mathbb{Z}_+ \\ (y, z) & \text{has} & \text{finite support} \end{array} \tag{4.1}$$

where $r < q \leq n$ and $r, q, n \in \mathbb{Z}_+$.

It is clear that system (4.1) is almost identical to system 3.1, except for the summation is conducted over the rationals. Definition of valid, α-valid, minimal, and extreme for system (4.1) are analogous to those of $IRm(q, r)$. Consequently, all results we proved for $IRm(q, r)$ still holds for system (4.1).

Lemma 4.1.3 *Let $n > r$. Suppose π is 1-valid and defines a non-trivial facet of $MEP(n, r)$ such that $\pi(q) = 0$, and for any $k, l \in \mathbb{Z}_+$ with $l > n$, the k-subdivision of the discrete extension of π for $MEP(ln, r)$, denoted as $\pi^{ln,k}$, is facet defining for MEP(kln, kr). Let $\phi'(t) := \phi(t), \forall t \in \mathbb{Q}$, where ϕ is the continuous extension of π, then $(\phi', 0)$ is extreme for system (4.1).*

Proof. Observe that Lemma 3.4.6 indicates $(\phi', 0)$ is 1-valid and minimal for system (4.1). ϕ' is continuous and piecewise linear with rational breakpoints by construction, and it has a natural normalization $\phi'(q) = 0$. For any (ϕ', γ) to be extreme, Lemma 3.3.7 tells us it must be minimal. By Lemma 3.3.5, it is impossible for (ϕ', γ) to be minimal unless $\gamma = 0$ as well. So we consider the pair $(\phi', 0)$.

Suppose $(\phi', 0)$ is not extreme for system (4.1), then by Definition 3.3.6, there exist $(\phi_1, 0)$, $(\phi_2, 0)$ 1-valid and minimal for system (4.1) such that $\phi' = \frac{1}{2}(\phi_1 + \phi_2)$, $\gamma_1 + \gamma_2 = 0$, $\phi_1(q) = \phi_2(q) = 0$, and $(\phi_1, 0) \neq (\phi_2, 0)$. Note there exists a point $t_0 \in \mathbb{Q}$ such that $\phi_1(t_0) < \phi'(t_0) < \phi_2(t_0)$. Moreover, since t_0 is rational, it can be uniquely expressed as $t_0 = \frac{a}{b}$, where $b > 0$, and $\gcd(|a|, b) = 1$. Let $l' := \lceil \frac{|t_0|}{n} \rceil$ and $k' = \frac{\text{lcm}(b,n)}{n}$.

Consider $MEP(k'l'n, k'r)$ and the corresponding functions $\pi^{l'n,k'}(i) := \phi'(\frac{i}{k'})$ and $\pi_j(i) := \phi_j(\frac{i}{k'})$, for $i \in \{-k'l'n, ..., k'l'n\}$, $j = 1, 2$. Given a feasible point $w \in$ MEP$(k'l'n, k'r)$, by letting $y_{i/k'} = w_i$, and $z = 0$, we can obtain a feasible point for system (4.1). Since ϕ', ϕ_1, and ϕ_2 are 1-valid for system (4.1), it is clear that π, π_1 and π_2 are 1-valid for $MEP(k'l'n, k'r)$. Meanwhile, $\pi^{l'n,k'}$ is the k'-subdivision of the extension of π in $MEP(l'n, r)$ by definition, so it is facet defining for $MEP(k'l'n, k'r)$. However, this contradicts $\pi_1(\frac{ak'}{b}) < \pi^{l'n,k'}(\frac{ak'}{b}) < \pi_2(\frac{ak'}{b})$. Therefore, such a point t_0 does not exist, and $\phi' = \phi_1 = \phi_2$, $\forall t \in \mathbb{Q}$. Thus ϕ' is extreme for system (4.1). $\square$

Moreover, the following lemma allows us to extend Lemma 4.1.3 to $IRm(q, r)$. It is inspired by a theorem from [7].

Lemma 4.1.4 *Let $\phi : \mathbb{R} \to \mathbb{R}$ be continuous, piecewise linear and subadditive over $\mathbb{R}$ such that $(\phi, 0)$ is 1-valid for $IRm(q, r)$, and $\phi(t) = ct$ for all $t \in [0, t_0]$, for some $t_0 \in \mathbb{R}$, $c > 0$. Suppose $\phi = \lambda\phi_1 + (1 - \lambda)\phi_2$ such that $0 < \lambda < 1$, and ϕ_1 and ϕ_2 are subadditive over $\mathbb{R}$ and 1-valid functions for $IRm(q, r)$, then ϕ_1 and ϕ_2 are continuous.*

Proof. For any $a, b \in [0, t_0]$ such that $a+b \in [0, t_0]$, by linearity we have $\phi(a)+\phi(b) = \phi(a+b)$. Recall ϕ_1 and ϕ_2 are both subadditive over $\mathbb{R}$, and ϕ is their convex combination. Then $\phi_i(a) + \phi_i(b) = \phi_i(a + b)$, for $i \in \{1, 2\}$. Therefore, ϕ_1 and ϕ_2 are linear on the interval $[0, t_0]$ as well, i.e. $\phi_i(t) = c_i t$, for $i \in \{1, 2\}$. Without loss of generality, we may assume $c_1 \geq c_2$. Moreover, suppose $c_i \leq 0$ for $i = 1$ or 2. Since there exists $a \in \mathbb{Z}_+$ such that $\frac{r}{a} \in [0, t_0]$, we obtain $\phi_i(\frac{r}{a}) \leq 0$. However, by subadditivity, we have $0 \geq a\phi_i(\frac{r}{a}) \geq \phi_i(r) \geq 1$, which is a contradiction, so $c_i > 0$ for $i \in \{1, 2\}$.

Note ϕ is a convex combination of ϕ_1 and ϕ_2, so if ϕ is left (or right) continuous at some point, then either both ϕ_1 and ϕ_2 are left (or right) continuous at that point, or they both are not left (or right) continuous at that point. For any $u \in \mathbb{R}$, suppose ϕ is right continuous at u, then ϕ is right differentiable. Use c' to denote the right derivative of ϕ at u. Since ϕ is right continuous at u and ϕ is piecewise linear, there exists $l > 0$ such that the slope of ϕ is c' in the interval $[u, u + l]$.

Assume ϕ_1 is not right continuous at u, then there exists $\epsilon > 0$ and $v \in [u, u + l]$ that satisfies $\delta = d(u, v) < \min\{\frac{\epsilon}{\alpha|c'|+\beta c_1}, \frac{t_0}{2}, l\}$, $\alpha = \frac{1}{1-\lambda} + 1$, $\beta = \max\{2, \frac{\lambda}{1-\lambda} + 1\}$ such that $|\phi_1(v)-\phi_1(u)| \geq \epsilon$. So, for some $k > 0$, $|\phi_1(v)-\phi_1(u)| = \alpha|c'|\delta+\beta c_1\delta+k$. By subadditivity and $\phi_1 = c_1 t$ for all $t \in [0, t_0]$, we have $\phi_1(u) - \phi_1(v) \geq -\phi_1(\delta) = -c_1\delta$. Since $\beta > 1$, we obtain $\phi_1(u) - \phi_1(v) = \alpha|c'|\delta + \beta c_1\delta + k$.

Notice that $\phi_2 = \frac{1}{\lambda}\phi - \frac{1-\lambda}{\lambda}\phi_1$, then $\phi_2(v)-\phi_2(u) = \frac{1}{\lambda}(\phi(v)-\phi(u)) - \frac{1-\lambda}{\lambda}(\phi_1(v)-\phi_1(u)) = \frac{c'\delta}{\lambda} + \frac{1-\lambda}{\lambda}(\alpha|c'|\delta + \beta c_1\delta + k) > \frac{c'\delta}{\lambda} + \frac{|c'|\delta}{\lambda} + c_1\delta \geq c_1\delta \geq c_2\delta$, i.e. $\phi_2(v) > \phi_2(u) + \phi_2(\delta)$, which contradicts the subadditivity of ϕ_2. Therefore, ϕ_1 and ϕ_2 are both right continuous at u.

Similarly, assume ϕ is left continuous at some $u \in \mathbb{R}$, and ϕ_1 is not left continuous at u. Use c' to denote the left derivative of ϕ at u. Since ϕ is left continuous at u and ϕ is piecewise linear, there exists $l > 0$ such that the slope of ϕ is c' in the interval $[u-l, u]$. Likewise, since ϕ_1 is not left continuous at u, there exists $\epsilon > 0$ such that we can find $v < u$ such tat $|\phi_1(v) - \phi_1(u)| \geq \epsilon$ and $\delta = d(v,u) = \min\{\frac{\epsilon}{\alpha|c'|+\beta c_1}, \frac{t_0}{2}, l\}$, where $\alpha = \frac{1}{1-\lambda} + 1$ and $\beta = \max\{2, \frac{\lambda}{1-\lambda} + 1\}$. By subadditivity and $\phi_1 = c_1 t$ for all $t \in [0, t_0]$, we have $\phi_1(v) - \phi_1(u) \geq -\phi_1(\delta) = -c_1\delta$, so $\phi_1(v) - \phi_1(u) = \alpha|c'|\delta + \beta c_1 \delta + k$, since $\beta > 1$. Observe $\phi_2(v) - \phi_2(u) = \frac{1}{\lambda}(\phi(v) - \phi(u)) - \frac{1-\lambda}{\lambda}(\phi_1(v) - \phi_1(u)) = -\frac{c'\delta}{\lambda} - \frac{1-\lambda}{\lambda}(\alpha|c'|\delta + \beta c_1\delta + k) < -\frac{c'\delta}{\lambda} - \frac{|c'|\delta}{\lambda} - c_1\delta \leq -c_1\delta \leq -c_2\delta$. This implies $\phi_2(v) + \phi_2(\delta) < \phi_2(u)$, which contradicts the subadditivity of ϕ_2. Since ϕ is continuous on $\mathbb{R}$, we can conclude ϕ_1 and ϕ_2 are continuous on $\mathbb{R}$ as well. $\square$

Note that $IRm(q,r)$ is a relaxation of system (4.1). Suppose a function $\phi : \mathbb{R} \to \mathbb{R}$ satisfies that $(\phi, 0)$ is extreme for $IRm(q,r)$. Define $\phi' : \mathbb{Q} \to \mathbb{R}$ as $\phi'(t) := \phi(t)$ for all $t \in \mathbb{Q}$, then $(\phi', 0)$ is extreme for system (4.1). Using this fact and Lemma 4.1.4, the following theorem is proven.

Theorem 4.1.5 *Let $n > r$. Suppose π is 1-valid and defines a non-trivial facet of $MEP(n,r)$ with $\pi(q) = 0$, and for any $k, l \in \mathbb{Z}_+$ with $l > n$, the k-subdivision of the discrete extension of π for $MEP(ln,r)$, denoted as $\pi^{ln,k}$, is facet defining for MEP(kln, kr), then $(\phi, 0)$ is extreme for system $IRm(q,r)$, where ϕ is the continuous extension of π.*

PROOF. By Lemma 3.4.6, $(\phi, 0)$ is 1-valid and minimal for $IRm(q,r)$. Suppose $(\phi, 0)$ is not extreme for $IRm(q,r)$, then there exist $(\phi_1, 0)$, $(\phi_2, 0)$ 1-valid for system (4.1) such that $\phi = \frac{1}{2}(\phi_1 + \phi_2)$ and $\phi_1 \neq \phi_2$.

Now consider $\phi', \phi'_1, \phi'_2 : \mathbb{Q} \to \mathbb{R}$ such that $\phi'(t) := \phi(t)$ and $\phi'_i(t) = \phi_i(t)$ for all $t \in \mathbb{Q}$, $i \in \{1,2\}$. By Lemma 4.1.3, $(\phi', 0)$ is extreme for system (4.1), so $\phi' = \phi'_1 = \phi'_2$. In other words, the value of ϕ, ϕ_1, and ϕ_2 equal at all rational points.

Moreover, Lemma 4.1.4 implies ϕ_1 and ϕ_2 are both continuous. Since ϕ is continuous by construction, we obtain $\phi = \phi_1 = \phi_2$, which contradicts that assumption that $(\phi, 0)$ is not extreme for $IRm(q,r)$. $\square$

One thing to note is that certain 1-valid inequalities under the normalization $\pi(-n) = 0$ are not necessarily 1-valid under the normalization $\pi(q) = 0$. However, recall that all non-trivial facet defining inequalities for $MEP(n,r)$ are still 1-valid with $\pi(q) = 0$, so this normalization is sufficient for our needs.

Recall one condition of Theorem 4.1.5 is that any k-subdivision of a certain class of discrete extension of π defines a facet of an MEP with appropriate parameters. At the first glance, it may seem that this condition is too strong that makes this theorem impractical. Below, we show that any 1-valid, minimal, non-trivial, and $(r-n)$-partially two-slope inequality for $MEP(n,r)$ satisfies this condition. This result relates Theorem 4.1.5 to Theorem 2.0.2.

Lemma 4.1.6 *Let π be a 1-valid, minimal, non-trivial, and $(r-n)$-partially two-slope inequality for $MEP(n,r)$ with $\pi(-n)=0$. For any $k,l \in \mathbb{Z}_+$ with $l>n$, the k-subdivision of the extension of π in $MEP(ln,r)$, denoted as $\pi^{ln,k}$, is facet defining for MEP(kln, kr).*

Proof. Lemma 3.4.3 implies π^{ln} is facet defining for MEP(ln, r), so we may assume $\pi^{ln}(-ln) = \pi^{ln,k}(-kln) > -\frac{-ln}{r} = -\frac{kln}{kr}$. In addition, Lemma 2.1.2 then gives π^{ln} is subadditive over I^{ln}. As an interpolation of π^{ln}, we know that $\pi^{ln,k}$ must be subadditive over I^{kln}, and $\pi^{ln,k}(kr)=1$, so $\pi^{ln,k}$ is 1-valid for MEP(kln, kr), by Lemma 2.1.4. Meanwhile, Lemma 3.4.4 implies π^{ln} is a two-slope inequality, so $\pi^{ln,k}$ is also two-slope. Therefore, to prove $\pi^{ln,k}$ defines a facet of MEP(kln, kr), we only need to prove it is minimal.

Notice $\pi^{ln,k}$ satisfies complementarity, because π^{ln} defines a facet of MEP(ln, r). Let $m'_i = \max\{\max_{-i\le j\le kln}\{\pi^{ln,k}(i+j)-\pi^{ln,k}(j)\}, \max_{j_1,j_2,i+j_1+j_2\in I_+^{kln}}\{\pi^{ln,k}(i+j_1+j_2)\}-\pi^{ln,k}(j_1)-\pi^{ln,k}(j_2)\}\}$. Given $i\in\{-n,\ldots,r-n-1\}$, since π is minimal for $MEP(n,r)$ and $\pi^{ln,k}$ is 1-valid for MEP(kln, kr), we get $\pi^{ln,k}(ki)=\pi(i)=m_i\le m'_{ki}$. On the other hand, since $\pi^{ln,k}$ is subadditive, $m'_{ki}\le\pi^{ln,k}(ki)$. So, for $i\in\{-n,\ldots,r-n-1\}$, $m'_{ki}=\pi^{ln,k}(ki)$. Moreover, $\pi^{ln,k}(ki)=m'_{ki}$ $\forall i\in\{-ln,\ldots,-n-1\}$ by definition. Combining these, we get $\pi^{ln,k}(ki)=m'_{ki}$ $\forall i\in\{-ln,\ldots,r-n-1\}$.

Besides, applying lemma 2.2.8 gives $m'_{i+1}-m'_i=\pi^{ln,k}(1)$ or $-\pi^{ln,k}(-1)$, for all $i\in\{-lkn,...,kr-kn-1\}$. Since $\pi^{ln,k}(k(i+1))-\pi^{ln,k}(ki)=m'_{k(i+1)}-m'_{ki}=k\pi^{ln,k}(1)$ or $-k\pi^{ln,k}(-1)$, $\forall i\in\{-kn,\ldots,k(n-1)\}$, we can see that $\pi^{ln,k}(i)=m'_i$, $\forall i\in\{-kn,\ldots,kr-kn-1\}$, which indicates $\pi^{ln,k}$ is minimal by lemma 2.1.9. Thus, theorem 2.0.2 and Lemma 2.2.3 give that $\pi^{ln,k}$ is facet defining for MEP(kln, kr). $\square$

Combining Theorem 4.1.5 and Lemma 4.1.6, we obtain the following corollary, which directly implies the continuous extension of a 1-valid, minimal, non-trivial, and $(r-n)$-partially two-slope inequality π with $\pi(q)=0$ is extreme for $IRm(q,r)$ when the coefficient of z is zero.

Corollary 4.1.7 *Suppose π is a 1-valid, minimal, non-trivial, and $(r-n)$-partially two-slope inequality for $MEP(n,r)$ with $\pi(q)=0$ and $n>r$, then $(\phi,0)$ is extreme for $IRm(q,r)$, where ϕ is the continuous extension of π.*

The results in this section are fairly powerful because they enable us to construct extreme pairs for $IRm(q, r)$ based on a non-trivial facet defining inequality of $MEP(n, r)$ with certain properties. Moreover, an interesting fact is that the ϕ's are all two-slope functions. This motivates us to further our study in two-slope functions. In the next section, we provide yet another important result that gives a sufficient condition of extreme functions for $IRm(q, r)$ based on two-slope-ness.

4.2 Extremality of Regular Two-Slope Functions

Up to this point, we have proven the continuous extensions of certain two-slope inequalities for $MEP(n, r)$ is extreme for $IRm(q, r)$ with the coefficient of z, γ, being zero. It is natural to wonder if two-slope functions with $\gamma = 0$ are extreme for $IRm(q, r)$ in general. In this section, we prove that a class of two-slope functions with $\gamma = 0$, defined as follows, are extreme for $IRm(q, r)$.

Definition 4.2.1 *A continuous two-slope function $\phi : \mathbb{R} \to \mathbb{R}$ with slopes s_1, s_2 is **regular** if it satisfies the following: There exists $p \in \mathbb{R}_+$ such that $\phi(t) + \phi(p) = \phi(t + p)$ for all $t \geq 0$ and $\phi(t) + \phi(-p) = \phi(t - p)$ for all $t \leq 0$. p is called the **period** of ϕ.*

As shown in figure 4.2, a regular two-slope function is a two-slope function with a periodic pattern, such that its behavior is identical within each period on the positive side, and with in each period on the negative side. Notice the patterns on the positive side and the negative side do not have to be the same.

To prove the extremality of some specified regular two-slope functions with $\gamma = 0$, we prove that either they are the continuous extensions of a special class of non-trivial facet defining two-slope inequalities for $MEP(n, r)$, or they are horizontal compressions of functions of this kind. Either way, we begin by obtaining an inequality for $MEP(n, r)$ from ϕ.

For the rest of this section, we consider a function ϕ_{rts}, which is a regular two-slope function for $IRm(q, r)$ with rational breakpoints such that $p \in \mathbb{Q}$, $p \geq r$, ϕ_{rts} is subadditive over $\mathbb{R}$, ϕ_{rts} satisfies $\phi_{rts}(t) + \phi_{rts}(r - t) = 1$ for all $t \in \mathbb{R}$, and there exists some rational number $n \geq p$ with $\phi(-n) > -\frac{n}{r}$, such that there are only a finite number of breakpoints in $[-n, n]$. Let $d \in \mathbb{Z}_+$ be the least common denominator of n, p, and all breakpoints in $[-n, n]$. We define the function π_{rts} for MEP(dn, dr) such that $\pi_{rts}(i) := \phi_{rts}(\frac{i}{d})$, for all $i \in I^{dn}$. Observe π_{rts} captures all breakpoints of ϕ_{rts} in the range $[-n, n]$. To prove ϕ_{rts} is extreme for $IRm(q, r)$, we first show some properties of π_{rts}.

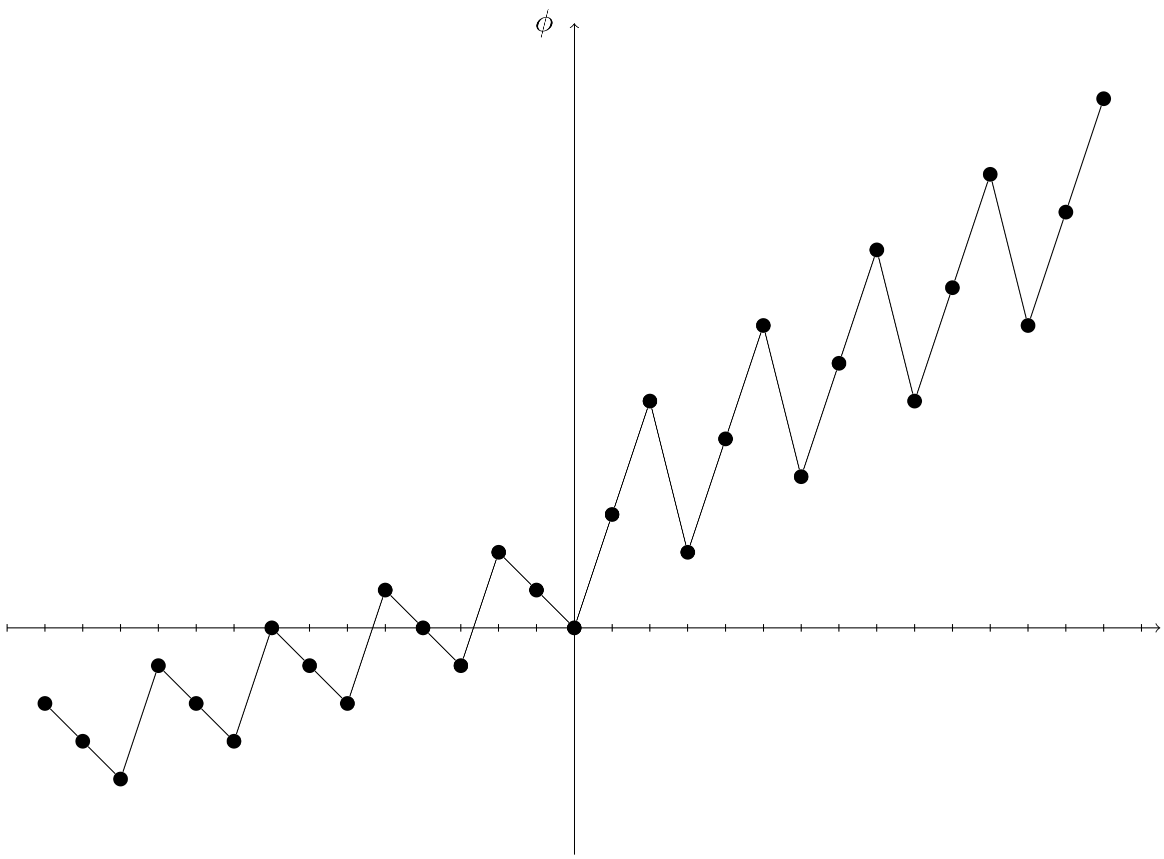

Figure 4.2: A Regular Two-Slope Function

Lemma 4.2.2 *The inequality π_{rts} is 1-valid, minimal, and two-slope for MEP(dn, dr) with $\pi_{rts}(-dn) > -\frac{dn}{dr}$.*

PROOF. By definition, π_{rts} is clearly a two-slope inequality that satisfies complementarity, subadditive over I^{dn}, and $\pi_{rts}(-dn) > -\frac{n}{r} = -\frac{dn}{dr}$. The subadditivity guarantees π_{rts} is 1-valid for MEP(dn, dr), by Lemma 2.1.4.

Moreover, for any $i \in \{-dn, \ldots, dr - dn - 1\}$, observe $\pi_{rts}(i) \geq m_i$ since π_{rts} is subadditive over I^{dn}. Moreover, since $n \geq p$, there exists a minimal $c \in \mathbb{Z}_+$ such that $dr - i - cdp \in I_+^{dn}$. By complementarity of and regularity of ϕ, we obtain $\pi_{rts}(i) = \pi_{rts}(dr) - \pi_{rts}(dr - i - cdp) - \phi_{rts}(cdp)$.

It is clear that $cdp > 0$. Since $r \leq p$, we get $dr - i - dp \leq -i \leq dn$, so $c = 1$, and $cdp = dp \leq n$. Then $\pi_{rts}(i) = \pi_{rts}(dr) - \pi_{rts}(dr - i - dp) - \pi_{rts}(dp) \leq m_i$. Hence we get $\pi_{rts}(i) = m_i$. By Lemma 2.1.9, π_{rts} is minimal for MEP(dn, dr). $\square$

Lemma 4.2.2 and Lemma 2.2.1 directly imply that π_{rts} defines a non-trivial facet for MEP(dn, dr).

Corollary 4.2.3 *The inequality π_{rts} defines a non-trivial facet of MEP(dn, dr).*

Use ϕ' to denote the continuous extension of π_{rts}, then we would like to prove $\phi'(t) = \phi_{rts}(\frac{t}{d})$ for all $t \in \mathbb{R}$. In other words, we would like to show π_{rts} is a horizontal compression of ϕ' by a factor of d.

Lemma 4.2.4 *$\phi'(t) = \phi_{rts}(\frac{t}{d})$ for all $t \in \mathbb{R}$.*

PROOF. Observe it is sufficient to show $\phi'(-dn-1) = \phi_{rts}(\frac{-dn-1}{d})$ as long as we do not use the condition $\phi_{rts}(t) > -\frac{n}{r}$. By subadditivity of ϕ_{rts} and Definition 3.4.5, we get $\phi'(-dn-1) = \max_{j \in I_+^{dn+1}}\{\phi'(j - dn - 1) - \phi'(j)\} = \max_{j \in I_+^{dn+1}}\{\phi_{rts}(\frac{j-dn-1}{d}) - \phi_{rts}(\frac{j}{d})\} \leq \phi_{rts}(\frac{-dn-1}{d})$.

By complementarity, $\phi_{rts}(-p) + \phi_{rts}(r+p) = \phi_{rts}(r)$. Meanwhile, by regularity, $\phi_{rts}(r) + \phi_{rts}(p) = \phi_{rts}(r+p)$. This implies $\phi_{rts}(p) = -\phi_{rts}(-p)$.

Note we may write $-n = -kp - l$, for some $k \in \mathbb{Z}_+$ and $0 \leq l < p$. By regularity of ϕ_{rts}, we have $\phi_{rts}(-n - \frac{1}{d}) = k\phi_{rts}(-p) + \phi_{rts}(-l - \frac{1}{d}) = -\phi_{rts}(kp) + \phi_{rts}(-l - \frac{1}{d}) = -\phi_{rts}(\frac{kdp}{d}) + \phi_{rts}(\frac{-d(n-kp)-1}{d})$. On the other hand, note $0 < kdp \leq np$, and observe that $\phi'(-dn-1) =$

$\max_{j\in I_{+}^{dn+1}}\{\phi_{rts}(\frac{j-dn-1}{d}) - \phi_{rts}(\frac{j}{d})\} \geq -\phi_{rts}(\frac{kdp}{d}) + \phi_{rts}(\frac{-d(n-kp)-1}{d}) = \phi_{rts}(-n-\frac{1}{d})$. Since $\phi'(-dn-1) \leq \phi_{rts}(\frac{-dn-1}{d})$, we get $\phi'(-dn-1) = \phi_{rts}(\frac{-dn-1}{d})$. □

And now, we are ready to prove our main theorem of this section, stating that ϕ_{rts} is extreme for $IRm(q,r)$.

Theorem 4.2.5 *Suppose ϕ_{rts} is a 1-valid and regular two-slope function with period $p \geq r$ and $\phi_{rts}(q) = 0$ for $IRm(q,r)$. Then $(\phi_{rts}, 0)$ is extreme for $IRm(q,r)$ if it satisfies the following conditions.*

- *ϕ_{rts} is subadditive over $\mathbb{R}$.*
- *All breakpoints of ϕ_{rts} are rational.*
- *There exists a rational number $n > p$ such that $\phi_{rts}(-n) > -\frac{n}{r}$.*
- *For all $t \in \mathbb{R}$, $\phi_{rts}(t) + \phi_{rts}(r-t) = 1$.*

Proof. Let $\pi_{rts}(i) := \phi_{rts}(\frac{i}{d})$, for all $i \in I^{dn}$, and let ϕ' be the continuous extension of π_{rts}. Then Lemma 4.2.2 and Corollary 4.1.7 implies ϕ' is extreme for IRm(dq, dr). Moreover, by Lemma 3.4.6, ϕ' is 1-valid for IRm(dq, dr). Recall that Lemma 4.2.4 gives us $\phi'(t) = \phi_{rts}(\frac{t}{d})$ for all $t \in \mathbb{R}$.

For any feasible point $(y,z) \in IRm(q,r)$, observe $(y',z) \in$ IRm(dq, dr), where $y'_t = y_{t/d}$ for all $t \in \mathbb{R}$. Then

$$1 \leq \sum_{t\in\mathbb{R}} \phi'(t)y'_t = \sum_{t\in\mathbb{R}} \phi_{rts}(\frac{t}{d})y_{t/d},$$

which means ϕ_{rts} is also 1-valid for $IRm(q,r)$.

Suppose ϕ_{rts} is not extreme for $IRm(q,r)$. Then, by Definition 3.3.6, there exists two 1-valid functions ϕ_1, ϕ_2 for $IRm(q,r)$, such that $\phi_{rts} = \frac{1}{2}(\phi_1 + \phi_2)$ and $\phi_1 \neq \phi_2$. Now we consider ϕ'_i, such that $\phi'_i(t) = \phi_i(\frac{t}{d})$ for all $t \in \mathbb{R}$ and $i \in \{1,2\}$. By a similar argument as above, we can see that ϕ'_1 and ϕ'_2 are 1-valid for IRm(dq, dr). However, this implies $\phi'_{rts} = \frac{1}{2}(\phi'_1 + \phi'_2)$ and $\phi'_1 \neq \phi'_2$, which contradicts the extremality of ϕ'_{rts}. Hence such ϕ_1 and ϕ_2 do not exist, which implies ϕ_{rts} is extreme for $IRm(q,r)$. □

Therefore, we have proven that regular two-slope functions that satisfy certain criteria are extreme for $IRm(q,r)$. This is a significant result because it provides a straightforward sufficient condition for extreme functions of $IRm(q,r)$. Now we use this theorem to show the optimized wedge cut is indeed extreme for $IRm(q,r)$.

As introduced in Chapter 1, the optimized wedge cut is defined as

$$\sum_{t\in\mathbb{R}} \bar{\psi}(t)s'_t + \sum_{t\in\mathbb{R}} \bar{\phi}_\alpha(t)y'_t \geq 1,$$

where

$$\bar{\psi}(t) =: \begin{cases} -\frac{t}{f} & t < 0 \\ \frac{t}{1-f} & t \geq 0, \end{cases}$$

$$\bar{\phi}_\alpha(t) := \min\left\{\frac{-t + \lceil \alpha t \rceil}{f}, \frac{t}{1-f} - \frac{\lfloor \alpha t \rfloor (1 - \alpha(1-f))}{\alpha f(1-f)}\right\},$$

where $f = \frac{r}{q}$ and α is a rational number such that $\alpha \in (0,1]$. Note there exists unique $a, b \in \mathbb{Z}$ such that $b > 0$, $gcd(a,b) = 1$ and $\alpha = \frac{a}{b}$. This cut is optimized in the sense that it is the cut with the smallest coefficients out of this family of cuts. It is proven to be extreme for the infinite relaxation model.

Since we are interested in studying $IRm(q,r)$, we would like to convert the optimized wedge cut to an appropriate function for this system. Let $s_t = 0$, for all $t \in \mathbb{R}$, and $y_t = y'_{t/q}$, then the wedge cut becomes $\sum_{t\in\mathbb{R}} \bar{\phi}_\alpha(\frac{t}{q})y_t \geq 1$. We define $\phi_\alpha(t) := \bar{\phi}_\alpha(\frac{t}{q})$ to get

$$\sum_{t\in\mathbb{Q}} \phi_\alpha(t)y_t \geq 1$$

with

$$\phi_\alpha(t) = \min\left\{\frac{-t + q\lceil \frac{at}{bq} \rceil}{q-r}, \frac{t}{r} - \frac{q\lfloor \frac{at}{bq} \rfloor (bq - ar)}{ar(q-r)}\right\}.$$

Observe that $\phi_\alpha(t)$ is a piece-wise linear function with only two slopes: $-\frac{1}{q-r}$ and $\frac{1}{r}$. Moreover, for any $k \in \mathbb{Z}$, if $t = \frac{kbq}{a}$, then

$$\begin{aligned} \frac{t}{r} - \frac{q\lfloor \frac{at}{bq} \rfloor (bq - ar)}{ar(q-r)} &= \frac{kbq^2 - kbqr}{ar(q-r)} - \frac{kbq^2 - kaqr}{ar(q-r)} \\ &= \frac{-kq(b-a)}{a(q-r)} \\ &= \frac{-t + q\lceil \frac{at}{bq} \rceil}{q-r}. \end{aligned}$$

Also, if $t = \frac{kbq}{a} + r$, then

$$\begin{aligned}
\frac{t}{r} - \frac{q\lfloor \frac{at}{bq} \rfloor (bq - ar)}{ar(q-r)} &= \frac{\frac{kbq}{a} + r}{r} - \frac{kq(bq - ar)}{ar(q-r)} \\
&= \frac{kbq^2 - kbqr + arq - ar^2 - kbq^2 + kqar}{ar(q-r)} \\
&= \frac{-kbq - ar + kaq + aq}{a(q-r)} \\
&= \frac{-\frac{kbq}{a} - r + (k+1)q}{q-r} \\
&= \frac{-t + q\lceil \frac{at}{bq} \rceil}{q-r}.
\end{aligned}$$

Since both $\frac{-t+q\lceil \frac{at}{bq} \rceil}{q-r}$ and $\frac{t}{r} - \frac{q\lfloor \frac{at}{bq} \rfloor (bq-ar)}{ar(q-r)}$ are both monotonic in the interval $(\frac{kbq}{a}, \frac{(k+1)bq}{a})$, all break points of $\phi_\alpha(t)$ are of the form $t = \frac{kbq}{a}$ or $t = \frac{kbq}{a} + r$, $\forall k \in \mathbb{Z}$. In addition, we may assume all these breakpoints are integers, for if not, we can define $\phi_\alpha = \bar{\phi}(\frac{\gcd(a,q)t}{aq})$ instead.

Moreover, observe ϕ_α has slope $\frac{1}{r}$ in the interval $(\frac{kbq}{a}, \frac{kbq}{a} + r)$ and has slope $-\frac{1}{q-r}$ in the interval $(\frac{kbq}{a} + r, \frac{(k+1)bq}{a})$. Since $\phi_\alpha(0) = 0$, and $\phi_\alpha(\frac{(k+1)bq}{a}) - \phi_\alpha(\frac{kbq}{a}) = \frac{-q(b-a)}{a(q-r)}$, we can obtain an alternative expression of $\phi_\alpha(t)$: Given any $t = \frac{kbq}{a} + p$, where $0 \leq p < \frac{bq}{a}$,

$$\phi_\alpha(t) = \begin{cases} \frac{-kq(b-a)}{a(q-r)} + \frac{p}{r} & 0 \leq p \leq r, \\ \frac{-kq(b-a)}{a(q-r)} + \frac{q-p}{q-r} & r \leq p < \frac{bq}{a}. \end{cases}$$

It is important to note $\phi(q) = 0$ and $\phi_\alpha(\frac{kbq}{a}) > 0$ for all $k \in \mathbb{Z}_-$. Moreover, Cornuéjols et al. proved that ϕ_α is subadditive over $\mathbb{R}$.

Lemma 4.2.6 *ϕ_α is subadditive over $\mathbb{R}$ [3].*

To apply Theorem 4.2.5, we only need to show ϕ_α satisfies complementarity.

Lemma 4.2.7 *$\phi_\alpha(t) + \phi_\alpha(r - t) = 1$, for all $t \in \mathbb{R}$.*

Proof. We prove this lemma by direct computation. Suppose $t = \frac{kbq}{a} + p$, then there are two cases, either $0 \le p \le r$ or $r \le p < \frac{bq}{a}$.

If $0 \le p \le r$, then $r - t = -\frac{kbq}{a} + (r-p)$, where $0 \le r - p \le r$. Then

$$\phi_\alpha(t) + \phi_\alpha(r-t) = \frac{-kq(b-a)}{a(q-r)} + \frac{p}{r} + \frac{kq(b-a)}{a(q-r)} + \frac{r-p}{r} = 1.$$

Similarly, if $r \le p < \frac{bq}{a}$, then $r - t = -\frac{kbq}{a} + (r-p) = -\frac{(k+1)bq}{a} + (r - p + \frac{bq}{a})$, where $r \le r - p + \frac{bq}{a} < \frac{bq}{a}$. We get

$$\begin{aligned} \phi_\alpha(t) + \phi_\alpha(r-t) &= \tfrac{-kq(b-a)}{a(q-r)} + \tfrac{q-p}{q-r} + \tfrac{(k+1)q(b-a)}{a(q-r)} + \tfrac{q-r+p-\frac{bq}{a}}{q-r} \\ &= \tfrac{q(b-a)}{a(q-r)} + \tfrac{2q-r-\frac{bq}{a}}{q-r} \\ &= 1. \end{aligned}$$

Thus, $\phi_\alpha(t) + \phi_\alpha(r-t) = 1$, for all $t \in \mathbb{R}$. $\square$

Therefore, by Theorem 4.2.5, ϕ_α is extreme for $IRm(q,r)$. This is an alternative proof of the extremality of ϕ_α. In addition, this discussion serves as an example of applications of Theorem 4.2.5, demonstrating the value of this result.

In the next section, we introduce the MIP extension of $IRm(q,r)$, which gives some interesting results.

4.3 MIP Extension of $IRm(q,r)$

As its name suggests, the MIP extension of $IRm(q,r)$ adds continuous variables to the original system. In particular, two non-negative continuous variables are added: one has a positive coefficient and the other has a negative coefficient. The system is defined as

$$\begin{array}{rcl} r & = & s_+ - s_- + qz + \sum_{t \in \mathbb{R}} t y_t \\ z, y_t & \in & \mathbb{Z}_+ \\ s_+, s_- & \in & \mathbb{R}_+ \\ (s_+, s_-, y, z) & \text{has} & \text{finite support.} \end{array} \tag{4.2}$$

By construction, there seems to be a link between 1-valid functions for system (4.2) and certain 1-valid functions for $IRm(q,r)$. The following lemma demonstrates this link.

Lemma 4.3.1 *Suppose $\phi : \mathbb{R} \to \mathbb{R}$ is a function where $\phi_+ := \lim_{t \to 0^+} \frac{\phi(t)}{|t|}$ and $\phi_- := \lim_{t \to 0^-} \frac{\phi(t)}{|t|}$ exist. If ϕ is 1-valid and minimal for $IRm(q, r)$, then the following inequality*

$$\phi_+ s_+ + \phi_- s_- + \sum_{t \in \mathbb{R}} \phi(t) y_t \geq 1 \tag{4.3}$$

is 1-valid for system (4.2).

Proof. Let ϕ be a minimal valid function for $IRm(q, r)$. Suppose inequality (4.3) is not valid for system (4.2). Then there exists a feasible point (s_+, s_-, y, z) for system (4.2) such that $\phi_+ s_+ + \phi_- s_- + \sum_{t \in \mathbb{R}} \phi(t) y_t = 1 - \epsilon$, for some $\epsilon > 0$. By Lemma 1.6.3, ϕ is subadditive over $\mathbb{R}$, so $\phi_+ + \phi_- = \lim_{t \to 0} \frac{\phi(t) + \phi(-t)}{|t|} \geq 0$. If $\min\{s_+, s_-\} > 0$, then the feasible point $(s_+ - \min\{s_+, s_-\}, s_- - \min\{s_+, s_-\}, y, z)$ of system (4.2) also violates inequality (4.3). So we may assume $\min\{s_+, s_-\} = 0$. Without loss of generality, suppose $s_- = 0$.

By definition, there exists $M \in \mathbb{Z}$ such that $|\phi_+ - \frac{\phi(s_+/M)}{s_+/M}| < \frac{\epsilon}{s_+}$. Then $s_+ \phi_+ - \epsilon < M\phi(s_+/M) < s_+ \phi_+ + \epsilon$. Define $\bar{y}$ as

$$\bar{y}(t) = \begin{cases} y_t + M & t = \frac{s_+}{M}, \\ y_t & t \in \mathbb{R}, t \neq \frac{s_+}{M}. \end{cases}$$

Observe $\bar{y}$ is feasible for $IRm(q, r)$, and

$$\begin{array}{rcl} \sum_{t \in \mathbb{R}} \phi(t) \bar{y}_t & = & \sum_{t \in \mathbb{R}} \phi(t) y_t + M\phi(\frac{s_+}{M}) \\ & < & \sum_{t \in \mathbb{R}} \phi(t) y_t + \phi_+ s_+ + \epsilon \\ & = & 1. \end{array}$$

However, this contradicts that ϕ is 1-valid for $IRm(q, r)$. $\square$

Moreover, for any 1-valid inequality for $IRm(q, r)$, as long as such ϕ_+ and ϕ_- exist, each of them must be lower bounded.

Lemma 4.3.2 *If ϕ is 1-valid for $IRm(q, r)$ such that $\phi_+ := \lim_{t \to 0^+} \frac{\phi(t)}{|t|}$ and $\phi_- := \lim_{t \to 0^-} \frac{\phi(t)}{|t|}$ exist, then $\phi_+ \geq \frac{1}{r}$ and $\phi_- \geq \frac{1}{q-r}$.*

PROOF. Suppose in contrary that $\phi_+ < \frac{1}{r}$. Then there exists $k_1 \in \mathbb{Z}_+$ such that for all integers $M \geq k_1$, $\frac{\phi(r/M)}{|r/M|} < \frac{1}{r}$. Since $y_{r/M} = M$ is a feasible point of $IRm(q,r)$, we get $\phi(r/M)y_{r/M} < 1$, contradicting that ϕ is valid.

Similarly, suppose $\phi_- < \frac{1}{q-r}$, then there exists $k_2 \in \mathbb{Z}_+$ such that for all integers $M \geq k_2$, $\frac{\phi((r-q)/M)}{|(r-q)/M|} < \frac{1}{q-r}$. However, $y_{(r-q)/M} = M, z = 1$ is a feasible point of $IRm(q,r)$, and $\phi((r-q)/M)y_{(r-q)/M} < 1$, which is a contradiction. □

In this Chapter, we have considered which inequalities for $MEP(n,r)$ can be extended to $IRm(q,r)$ to obtain extreme functions. As a result, a sufficient condition for extreme functions for $IRm(q,r)$ was given. This condition serves as an alternative proof of that ϕ_α is indeed extreme for $IRm(q,r)$. We also discussed the MIP extension of $IRm(q,r)$, which is already an interesting topic by itself, since it makes this system applicable to more general problems.

In the next Chapter, we consider something that is almost disjoint from what we have discussed so far: we introduce a new normalization for $MEP(n,r)$ and discuss the separation problem based on it.

Chapter 5

The Separation Problem Over A Class of Inequalities

The separation problem is a classical problem in the field of optimization. When only considering linear inequalities, it can be described as follows: Given a family of linear inequalities $\{\pi_k^T w \geq \pi_0^k\}_{k\in\mathcal{K}}$ where $\mathcal{K}$ is an index set and $w \in \mathbb{R}^n$, to separate some given point $w^* \in \mathbb{R}^n$, we either find some $k \in \mathcal{K}$ such that $\pi_k^T w^* < \pi_0^k$, or show that $\pi_k^T w \geq \pi_0^k$ for all $k \in \mathcal{K}$.

When it comes to the MEP, a characterization of non-trivial facets is already given by Theorem 1.5.1, which means we can separate over $MEP(n, r)$ in time polynomial in n. However, for a generic IP, after converting it into the form of MEP, the corresponding n could get significantly large, making the computation time undesirably long. Also, it is possible that a large number of variables in the converted MEP are not represented in the original IP. Therefore, the only cases that are necessary to consider are when these variables are set to be zero. Consequently, attempting to solve the separation problem over the entire $MEP(n, r)$ might waste a considerable amount of computational power.

In this Chapter, we exploit these facts to give a separation algorithm over an important class of valid inequalities for subproblems of the MEP. We start by considering a specific normalization, $\pi(-1) = 0$, for $MEP(n, r)$, under which all 1-valid and minimal inequalities have non-decreasing coefficients. Then $\frac{1}{k}$-inequalities are introduced in a manner analogous to the ones used by Shim et al. [19] for the master knapsack polyhedron. Some additional properties related to subadditive and facet defining inequalities under this normalization will be discussed.

Another important concept used in our separation algorithm, namely a subproblem of

MEPs, is defined as follows.

Definition 5.0.3 *A* ***subproblem*** *of $MEP(n,r)$, denoted as MEP$_J$(n, r), is an MEP with an additional constraint: for any $j \notin J$, $w_j = 0$, where J is an index set such that $J \subseteq I$.*

Observe the convex hull of the feasible solutions of MEP$_J$(n, r) is $MEP(n,r) \cap \{w : w_i = 0, \forall i \notin J\}$. Given any $w \in \mathbb{R}^{2n+1}$, let its corresponding subproblem MEP$_J$(n, r) have index set $J = \{i : w_i \neq 0, i \in I\}$. Moreover, we may denote J as $\{c_1, ..., c_\kappa\}$ such that $-n \leq c_1 < ... < c_\kappa \leq n$. Under the normalization $\pi(-1) = 0$, we provide a separation algorithm for w over a class of interesting inequalities that are valid for both MEP$_J$(n, r)' and $MEP(n,r)$. Furthermore, we give a proof of the correctness and computational complexity of the algorithm, and discuss when it is favorable accordingly.

5.1 An Non-Decreasing Normalization

The new normalization discussed in this section, $\pi(-1) = 0$, is essential to our separation algorithm, because it enables a simple enumeration of the class of functions we are interested in. Here we formally introduce it, and study some unique properties certain coefficients display under this normalization.

Consider a function $\pi : \{-n, ..., n\} \to \mathbb{R}$ for $MEP(n,r)$ with normalization $\pi(-1) = 0$. Notice that if π is subadditive over I, then it is a non-decreasing function, because $\pi(-1) + \pi(i) \geq \pi(i-1)$ for all $i \in \{-n+1, ..., n\}$. In this setting, we would like to give a definition for $\frac{1}{k}$-functions. Moreover, note that all non-trivial facet defining inequalities are still 1-valid under this normalization, based on Lemma 2.1.5.

Definition 5.1.1 *For any $k \in \mathbb{Z}_+$, let $\mathcal{C}_k$ denote the union of all integers and integer multiples of $\frac{k}{2}$.*

The function $\pi : \{-n, ..., n\} \to \mathbb{Q}$ is a ***$\frac{1}{k}$-function*** *if k is a positive integer such that $\pi(i) = \frac{a}{k}$ for all $i \in \{-n, ..., n\}$, where $a \in \mathcal{C}_k$.*

Moreover, if π is a facet defining inequality for $MEP(n,r)$, and π is a $\frac{1}{k}$-function (or $\frac{1}{k}$-inequality), then π defines is a ***$\frac{1}{k}$-facet*** *of $MEP(n,r)$.*

In the definition above, one might wonder why a could be taken as an integer multiple of $\frac{k}{2}$. Observe that any minimal π with $\pi(r) = 1$ must satisfy $\pi(i) + \pi(r-i) = 1$ for all

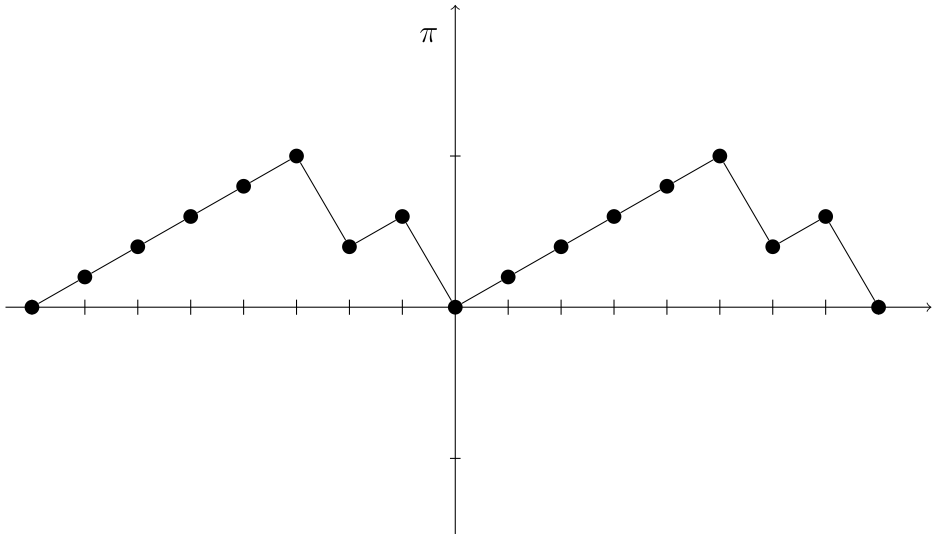

Figure 5.1: An inequality π that is subadditive over I for MEP(8, 5) with $\pi(-n) = 0$

$i \in \{r-n, \ldots, n\}$, by Lemma 2.1.9. In particular, if r is even, then $2\pi(\frac{r}{2}) = \pi(r) = 1$, which implies $\pi(\frac{r}{2}) = \frac{1}{2}$. To ensure this case is considered when all other coefficients are integer multiples of $\frac{1}{k}$, it is sufficient to assume $a \in \mathcal{C}_k$.

We also define a similar concept for MEP$_J$(n, r).

Definition 5.1.2 *For any index set $J \subseteq I$, an 1-valid inequality $\sum_{j \in J} \pi'(j)w \geq 1$ is a* ***$\frac{1}{k}$-inequality for MEP$_J$(n, r)*** *if there exists a corresponding $\frac{1}{k}$-inequality π for $MEP(n, r)$ such that π is 1-valid for $MEP(n, r)$ and $\pi(j) = \pi'(j)$, for all $j \in J$.*

One fundamental fact to note is, if $J \neq I$, then a $\frac{1}{k}$-inequality for some MEP$_J$(n, r) may be obtained from several distinct corresponding $\frac{1}{k}$-inequalities for $MEP(n, r)$.

A $\frac{1}{k}$-inequality π for $MEP(n, r)$ is especially nice if π is non-decreasing, because π would have the shape of a step function. For example, figure 5.1 plots an inequality π for MEP(8, 5) that is subadditive over I with normalization $\pi(-n) = 0$. Its renormalized version with $\pi'(-1) = 0$, π', is shown in figure 5.2. Observe π' resembles a non-decreasing step function, where there are specific "levels" that the function values would take.

In general, under the normalization $\pi(-1) = 0$, functions that are subadditive over I resemble step functions. In particular, they can be fully described by their breakpoints, i.e. the largest indices out of all indices with the same function value. Such a property will be heavily relied upon in our study of the separation algorithm.

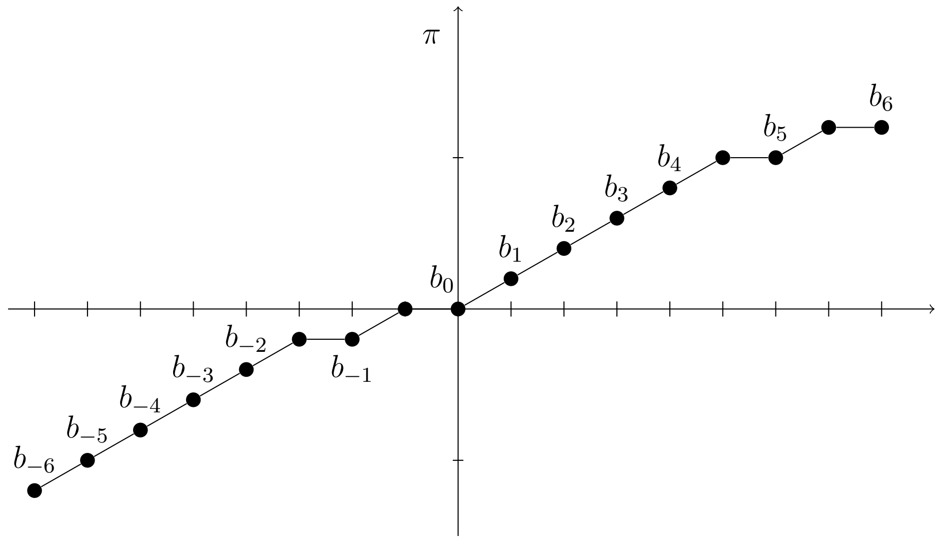

Figure 5.2: A renormalization of π, namely π', with $\pi'(-1) = 0$

Moreover, there is evidence to support further research of $\frac{1}{k}$-inequalities for $MEP(n, r)$. In [2], where the master knapsack polyhedron is studied, a major interest of study is the LP relaxation gap for a facet. This is obtained by first removing the respective knapsack facet π from the master knapsack polyhedron. Then the maximum gap over all possible objective function vectors $v \geq 0$ and all feasible points of the new polyhedron is computed. This maximum is called the LP relaxation gap for the facet π. The larger the LP relaxation gap of a facet is, the more weakened the LP relaxation is upon the removal of this facet. Therefore, facets with a larger LP relaxation gap are stronger.

Shim et al. [19] have both theoretically and experimentally shown that $\frac{1}{2}$-facets have the largest LP relaxation gap in the master knapsack polyhedron, with the next largest ones being $\frac{1}{3}$ and $\frac{1}{4}$-facets. They have also done shooting experiments, where the results imply that $\frac{1}{k}$-facets get weaker as k increases. Therefore, $\frac{1}{k}$-inequalities with small k values are the most important in the master knapsack polyhedron, which implies they might be important in the MEP case as well.

Now we define the index sequence b_m, which is the ending indices of each function value that π could take.

Definition 5.1.3 *Let b_m be the index such that $\pi(b_m) = \frac{m}{k}$ and $\pi(b_m + 1) > \frac{m}{k}$ where $m \in \mathcal{C}_k$. If $\pi(b_m + 1) > \frac{m+1}{k}$, we define $b_{m+1} := b_m$.*

As mentioned before, although the b_m's are not breakpoints in the traditional sense, we sometimes refer to them as the breakpoints of π, when there are no ambiguities, because they serve as indicators of where the function value would increase. Figure 5.2 gives an example of a sequence of b_m's corresponding to the plotted function, where $k = 5$. Based on the definition above, we can give a characterization of subadditive coefficient functions.

Lemma 5.1.4 *Let $\pi : \{-n, ..., n\} \to \mathbb{R}$ be a $\frac{1}{k}$-inequality. Then π is subadditive over I if and only if*

$$b_{m_1} + b_{m_2} \leq \begin{cases} b_{m_1+m_2} & \text{if } m_1 + m_2 \in \mathcal{C}_k \\ b_{\lfloor m_1+m_2 \rfloor} & \text{otherwise} \end{cases}$$

for all m_1, m_2 such that $\lfloor m_1 + m_2 \rfloor \leq k\pi(n)$ and $m_1 + m_2 \geq k\pi(-n)$.

Proof. ($\Rightarrow$) Suppose π is subadditive over I. For all m_1, m_2 such that $\lfloor m_1 + m_2 \rfloor \leq k\pi(n)$ and $m_1 + m_2 \geq k\pi(-n)$, note that $\pi(b_{m_1} + b_{m_2}) \leq \pi(b_{m_1}) + \pi(b_{m_2}) = \frac{m_1+m_2}{k}$. If $m_1 + m_2 \in \mathcal{C}_k$, then $\pi(b_{m_1} + b_{m_2}) \leq \frac{m_1+m_2}{k} = \pi(b_{m_1+m_2})$. Otherwise, $\lfloor m_1 + m_2 \rfloor$ must be the largest number in $\mathcal{C}_k$ that is less than $\frac{m_1+m_2}{k}$, so we obtain $\pi(b_{m_1} + b_{m_2}) \leq \frac{\lfloor m_1+m_2 \rfloor}{k} = \pi(b_{\lfloor m_1+m_2 \rfloor})$.

($\Leftarrow$) Assume $b_{m_1} + b_{m_2} \leq \begin{cases} b_{m_1+m_2} & m_1 + m_2 \in \mathcal{C}_k \\ b_{\lfloor m_1+m_2 \rfloor} & \text{otherwise} \end{cases}$, for all m_1, m_2 such that $\lfloor m_1 + m_2 \rfloor \leq k\pi(n)$ and $m_1 + m_2 \geq k\pi(-n)$. Given any i, j, $i + j \in I$, there exists b_{m_1}, b_{m_2} such that $\pi(i) = \pi(b_{m_1}) = \frac{m_1}{k}$ and $\pi(j) = \pi(b_{m_2}) = \frac{m_2}{k}$. This implies $i \leq b_{m_1}$ and $j \leq b_{m_2}$, so $\pi(i + j) \leq \pi(b_{m_1} + b_{m_2})$. Note either $\pi(b_{m_1} + b_{m_2}) \leq \pi(b_{\lfloor m_1+m_2 \rfloor})$ or $\pi(b_{m_1+m_2}) \leq \frac{m_1+m_2}{k} \leq \pi(b_{m_1}) + \pi(b_{m_2}) = \pi(i) + \pi(j)$, so π is buadditive over I. $\square$

Moreover, observe that certain coefficient functions π have a sequence of distinct b_m values, i.e. $b_m \neq b_{m'}$ for any $m, m' \in \mathcal{C}$.

Definition 5.1.5 *Consider a $\frac{1}{k}$-function π for $MEP(n, r)$ such that $\pi(-n) = \frac{-\alpha}{k}$ and $\pi(n) = \frac{\omega}{k}$, where α and ω are integer multiples of $\frac{1}{2}$. Then π* ***strict*** *if for any i that is an integer or an integer multiple of $\frac{k}{2}$ and $\alpha \leq i \leq \omega$, there exists $i_0 \in \{-n, ..., n\}$ such that $\pi(i_0) = \frac{i}{k}$.*

As a result, we can see that a $\frac{1}{k}$-inequality that satisfies the following conditions defines a facet of $MEP(n, r)$.

Corollary 5.1.6 *For any positive even integer k, a 1-valid, minimal, non-trivial, and strict $\frac{1}{k}$-inequality π with $\pi(-n) > -\frac{n}{r}$ defines a $\frac{1}{k}$-facet for $MEP(n,r)$.*

PROOF. Since π is minimal, $\pi(0) = 0$ and π is subadditive over I. So, $\pi(1) = 0$ or $\frac{1}{k}$. If $\pi(1) = 0$, then by subadditivity and the non-decreasing property, $\pi(i) = 0$ for all $i \in I^+$. Similarly, by applying subadditivity inductively to the coefficients with negative indices, we get $\pi(j) = 0$ for all $j \in I^-$. However, such a π is not 1-valid. So we only need to consider $\pi(1) = \frac{1}{k}$.

Since k is even, for any $i \in \{-n, ..., n-1\}$, $\pi(i+1) - \pi(i) = 0$ or $\frac{1}{k}$ by definition, because $\frac{1}{2}$ is an integer multiple of $\frac{1}{k}$. Since $\pi(-1) = 0$ and $\pi(1) = \frac{1}{k}$, we can see that π is a two-slope inequality. Then the corollary is true, based on Corollary 2.2.4. □

All what we have shown in this section gives a flavor of how interesting this new normalization could be. In the next section, we use some properties of this new normalization to give an algorithm to solve the separation problem over $MEP(n,r)$.

5.2 A Partial Separation Algorithm for Subproblems of $MEP(n,r)$

Given any $w \in \mathbb{R}^{2n+1}$, we consider the separation problem in the subproblem corresponding to w, over 1-valid $\frac{1}{2}$-inequalities that are subadditive over I with $\pi(-1) = 0$, $\pi(r) = 1$, $-\lceil \frac{n}{r} \rceil \leq \pi(-n) \leq 0$, and $0 \leq \pi(n) \leq \lceil \frac{n}{r} \rceil$.

One might wonder why we are only interested in $\frac{1}{2}$-inequalities. As mentioned before, in the study of Shim et al. [19], $\frac{1}{2}$-facets are found to be the inequalities with the largest LP relaxation gap in the master knapsack polyhedron. So in terms of the MEP, $\frac{1}{2}$-inequalities might also provide strong cuts. Moreover, this algorithm is only the first step in the consideration of the separation problem in the MEP setting. It can be potentially extended to more general cases where a given point can be separated over $\frac{1}{k}$-inequalities. Also note we are capable of separating over the $\frac{1}{2}$-inequalities that are subadditive over I because Lemma 5.1.4 provides a characterization of $\frac{1}{k}$-inequalities that are subadditive over I based on the breakpoints.

Our separation algorithm consists of two distinct algorithms: the main algorithm (algorithm 1) and the recursive sub-algorithm (algorithm 2). Algorithm 1 is the main separation algorithm that enumerates all possible values of $\pi(-n)$ and $\pi(n)$. It takes in n, r, the parameters of the MEP we are working with, as well as some $w \in \mathbb{R}^{2n+1}$ for which the

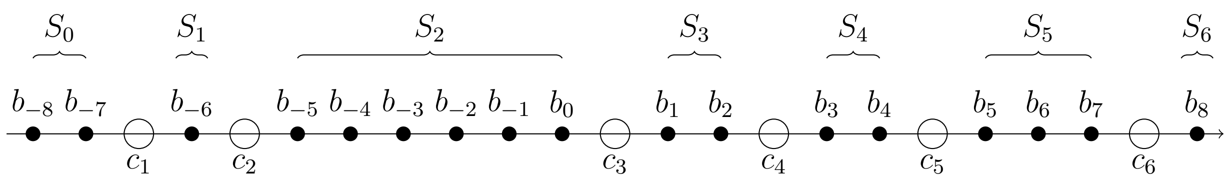

Figure 5.3: Positioning of b_m's relative to c_l's

separation problem is considered. A set $J = \{c_1, ..., c_\kappa\}$ for the corresponding subproblem is then generated based on w. In algorithm 1, the variables i and j are integers that record the values of $\pi(-n)$ and $\pi(n)$, i.e. $\pi(-n) = \frac{i}{2}$ and $\pi(n) = \frac{j}{2}$. (Since we are working with $\frac{1}{2}$-inequalities, $\mathcal{C}_2 = \mathbb{Z}$, so it is sufficient to assume i and j are integers.) These two integers are enumerated over their respective range to get all possible combinations of the $\pi(-n)$ and $\pi(n)$ values. Recall that by Lemma 1.5.3, $-\lceil \frac{n}{r} \rceil \leq \pi(-n) \leq \lceil \frac{n}{r} \rceil$ and $0 \leq \pi(n) \leq \lceil \frac{n}{r} \rceil$. Moreover, Since $\pi(-1) = 0$ and π is subadditive over I, $\pi(-n) \leq n\pi(-1) = 0$.

For each pair of i and j, we then enumerate the positions of the breakpoints b_m's, for $i \leq m \leq j$. Notice the c_l's partition I into $\kappa + 1$ intervals, considered as the 0-th, 1-st, ..., κ-th intervals. If b_m is put in the l-th interval, we denote this by putting b_m into a set S_l. For example, when $J = \{c_1, \ldots, c_6\}$, $i = -8$, and $j = 8$, figure 5.3 illustrates how the c_l's partition I, and how b_m's can be placed accordingly. In the figure, $S_0 = \{b_{-8}, b_{-7}\}$, $S_1 = \{b_{-6}\}$, etc. The resulting inequality has $\pi(c_1) = -\frac{6}{2}$, $\pi(c_2) = -\frac{5}{2}$, $\pi(c_3) = \frac{1}{2}$, $\pi(c_4) = \frac{3}{2}$, $\pi(c_5) = \frac{5}{2}$, and $\pi(c_6) = \frac{8}{2}$. This demonstrates that if b_m is in S_l, then b_m is between some c_l and c_{l+1} (inclusive on the left). Moreover, the values of $\pi(c_l)$'s can be determined once all the b_m's are placed. Note that it is possible that some S_l is empty. The sets $\mathcal{S}_l$ for all $1 \leq l \leq \kappa$ are generated in algorithm 1.

Algorithm 2 is a recursive sub-algorithm of algorithm 1, where each iteration takes inputs n, r, w, i, j, m, l_0, and $\mathcal{S}$, the collection of all S_l's. Similar to algorithm 1, n are r are still the parameters of the MEP we are working with, w is the point to be separated, and i and j record the values of $\pi(-n)$ and $\pi(n)$ enumerated in algorithm 1. The variables m and l_0 are two parameters implying that in the current pass, the algorithm will place the index b_m into some S_l, where $0 \leq l \leq l_0$. It continues recursively to place $b_{m'}$'s in the $S_{l'}$'s, for all $m' < m$ and $l' \leq l$. Once the positions for all b_m's are set, the values of $\pi(c_l)$'s are determined accordingly, for all $1 \leq l \leq \kappa$. Then whether w violates a π with these b_m values, i.e. whether $\sum_{p=-n}^{n} \pi(p) w_p < 1$, is checked.

If w violates the corresponding inequality, algorithm 2 checks the validity of such an inequality by finding a coefficient function π that is subadditive over I with $\pi(-1) = 0$ and $\pi(r) = 1$ that matches the inequality at the given breakpoints. Such a π is generated by solving an IP with b_m's being the variables, for $i \leq m \leq j$. The IP is designed by applying Lemma 5.1.4, such that it has a feasible solution if and only if π that is subadditive over I with $\pi(-1) = 0$ and $\pi(r) = 1$.

Since the enumeration checks all possible values of π at the non-zero variables for the subproblem, the output of the algorithm is a set of $\frac{1}{2}$-inequalities π that are 1-valid with $\pi(-1) = 0$, $\pi(r) = 1$, $-\lceil \frac{n}{r} \rceil \leq \pi(-n) \leq 0$, and $0 \leq \pi(n) \leq \lceil \frac{n}{r} \rceil$. When this set is empty, w is not separable by this class of $\frac{1}{2}$-inequalities. The complete separation algorithm is shown below.

Algorithm 1 Separation Algorithm over $\frac{1}{2}$-functions

Input: $n, r, w \in \mathbb{R}^{2n+1}$.
Let $J := \{c_1, \ldots, c_\kappa\}$ be indices in increasing order such that $w_{c_l} \neq 0$ for all $1 \leq l_0 \leq \kappa$.

```
for −2⌈n/r⌉ ≤ i ≤ 0 do
    π(−n) := i/2
  for 0 ≤ j ≤ 2⌈n/r⌉ do
    π(n) := j/2
    for 1 ≤ l ≤ κ − 1 do
        S_l := ∅
    end for
    S_κ := {b_j}
    𝒮 := {S_0, S_1, ..., S_κ}
    LC(n, r, w, i, j, j − 1, κ, 𝒮)
  end for
end for
```

To help understand the algorithm better, we give an example to illustrate how it works. Consider $w = [2 \quad 2 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1]^T$, we use the algorithm to find a 1-valid $\frac{1}{2}$-inequality for MEP(3, 2) that satisfies the respective criteria and is violated by w.

We start by obtaining the corresponding index set $J = \{-3, -2, 1, 3\}$. Set $\pi(-3) = -4$, $\pi(3) = 4$, $S_4 = \{3\}$. Then we consider $LC(3, 2, w, -4, 3, 2, 4, \mathcal{S})$. The first few inequalities generated by the recursion of algorithm 2 may violate $\sum_{p=-n}^{n} \pi(p)w_p \geq 1$, but they do not satisfy $\pi(-1) = 0$ and $\pi(r) = 1$, and are not necessarily subadditive. Through the iterations, the first inequality that gives $\sum_{p=-n}^{n} \pi(p)w_p < 1$ and has the conditions satisfied is $\pi = [-4 \quad -4 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1]^T$. It separates w from MEP(3, 2) because $\sum_{p=-n}^{n} \pi(p)w_p = -7$. To sum it up, the algorithm works by finding a violated inequality first, if any, and then decides if it satisfies our criteria for validity.

Algorithm 2 Locate b_m's and Check Subadditivity (LC)

Input: $n, r, w, i, j, m, l_0, \mathcal{S} = \{S_0, S_1, \ldots, S_\kappa\}$.

Let $J := \{c_1, \ldots, c_\kappa\}$ be indices in increasing order such that $w_{c_l} \neq 0$ for all $1 \leq l \leq \kappa$.

if $m = i - 1$ **then**
  $b_j := n$
  for $\kappa \geq l \geq 1$ **do**
    if $S_l \neq \emptyset$ **then**
      $\pi(c_l) = \frac{\nu}{2}$, where $\nu = \min_{b_m \in S_l}\{m\}$.
    else
      $\pi(c_l) = \pi(c_{l+1})$.
    end if
  end for
  if $\sum_{p=-n}^{n} \pi(p) w_p < 1$ **then**
    Determine if there exists a sequence of b_m's that satisfies the following constraints. If so, return the inequality π this sequence describes and continue; if not, continue.

$$
\begin{aligned}
b_{m_1} + b_{m_2} - b_{m_1+m_2} &\leq 0 \qquad \forall i \leq m_1, m_2, m_1 + m_2 \leq j && (5.1)\\
b_m - b_{m+1} &\leq 0 \qquad \forall i \leq m \leq j && (5.2)\\
b_{-1} &\leq -2 && (5.3)\\
-b_0 &\leq 1 && (5.4)\\
b_1 &\leq r - 1 && (5.5)\\
-b_2 &\leq -r && (5.6)\\
-b_m &\leq n \qquad b_m \in S_0 && (5.7)\\
-b_m &\leq -c_l \qquad b_m \in S_l, 1 \leq l \leq \kappa && (5.8)\\
b_m &\leq c_{l+1} - 1 \qquad b_m \in S_l, 1 \leq l \leq \kappa - 1 && (5.9)\\
b_m &\leq n - 1 \qquad b_m \in S_\kappa, m \neq j && (5.10)\\
b_j &= n && (5.11)\\
b_m &\in I \qquad \forall i \leq m \leq j && (5.12)
\end{aligned}
$$

  end if
else
  for $0 \leq l \leq l_0$ **do**
    $S_l = S_l \cup \{b_m\}$
    LC$(n, r, w, i, j, m - 1, l, \mathcal{S})$
  end for
end if

Now, we prove the separation algorithm given above returns all possible 1-valid and subadditive $\frac{1}{2}$-inequalities for MEP$_J$(n, r) that w violates. In other words, for each 1-valid and subadditive $\frac{1}{2}$-inequalities π' that w violates in MEP$_J$(n, r), there exists a 1-valid and subadditive $\frac{1}{2}$-inequality π for $MEP(n,r)$ such that π is in the output of the algorithm and π satisfies $\pi(j) = \pi'(j)$ for all $j \in J$. Moreover, we give the time-complexity of the algorithm in terms of n, r, and κ. It is necessary to note that the algorithm might also return some 1-valid inequalities that are subadditive over I but not necessarily facet defining for $MEP(n,r)$.

Theorem 5.2.1 *For any $w \in \mathbb{R}^{2n+1}$, our algorithm returns a set of 1-valid and subadditive $\frac{1}{2}$-inequalities π with $\pi(-1) = 0$ and $\pi(r) = 1$, such that each of which corresponds to exactly one distinct 1-valid $\frac{1}{2}$-inequalities π' for MEP$_J$(n, r) that are violated by w. Moreover, the algorithm's run time is $O((\frac{n}{r})^2 \cdot \kappa^{\frac{n}{r}}(\kappa + (\frac{n}{r})^{\frac{9}{2}\cdot\frac{n}{r}}))$.*

PROOF. In the algorithm, we enumerated the positions of the sequence of b_m's, each of which corresponds to one and only one distinct $\frac{1}{2}$-inequality π' for MEP$_J$(n, r). Then, if w violates π', we continue to the IP in the algorithm. Constraints (5.1), (5.2), and (5.12) in the IP imply that all produced π''s are subadditive over I, by Lemma 5.1.4; constraints (5.3) and (5.4) imply that $\pi'(-1) = 0$; and constraints (5.5) and (5.6) imply $\pi'(r) = 1$. So, if the IP has at least one feasible point, then the optimal solution of the IP is a sequence of b_m's that corresponds to a subadditive $\frac{1}{2}$-inequality π with $\pi(-1) = 0$ and $\pi(r) = 1$. Lemma 2.1.4 implies π is 1-valid. Further, constraints (5.7)-(5.11) guarantees that optimal solution of the IP corresponds to the violated π' for MEP$_J$(n, r). This proves the first statement in the theorem and the correctness of our algorithm.

In terms of the time-complexity of our algorithm, we first consider algorithm 1. It only has three **for** loops, where the first two having $O(\frac{n}{r})$ operations, and the last one having $O(\kappa)$ operations. Also, in the second **for** loop, there is a reference to algorithm 2. Suppose algorithm 2 has complexity $O(g)$. Then, the complexity of algorithm 1 is $O((\frac{n}{r})^2 \cdot (\kappa + g))$.

Now we aim to analyze the complexity of algorithm 2. Kannan [15] has proven that the time complexity of solving an n variable integer programming problem is $O(n^{\frac{9}{2}n}s)$, where s is the binary length of the input. The IP in algorithm 2 has $O(\frac{n}{r})$ variables. Moreover, observe it has $O((\frac{n}{r})^2)$ constraints. Therefore, the binary length of input is $O((\frac{n}{r})^3)$, and the IP in algorithm 2 can be solved with time complexity $O((\frac{n}{r})^{\frac{9}{2}\cdot\frac{n}{r}+3}) = O((\frac{n}{r})^{\frac{9}{2}\cdot\frac{n}{r}})$.

On the other hand, we also need to consider the **for** loop in algorithm 2. In particular, we prove its complexity is $O(\kappa^{a+1}(\kappa + h))$ with recursion, by induction on the difference between i and m, where the complexity of the IP is represented as $O(h)$, and $m - i = a$.

When $m - i = -1$, only the IP part of the algorithm 2 runs. Then the complexity of algorithm 2 is $O(\kappa + h)$, where the κ comes from determining the values of $\pi(c_l)$'s, for all $1 \leq l \leq \kappa$.

Now suppose the **for** loop has complexity $O(\kappa^a(\kappa+h))$ with recursion when $m-i = a-1$, and we consider the case where $m-i = a$. In this case, we have a total of $a+1$ breakpoints b_m's to place into the sets S_l's. When we had a such break points, i.e. when $m-i = a-1$, the complexity with recursion is $O(\kappa^a(\kappa+h))$. To place the one additional breakpoint into the sets S_l's, it would only take at most $O(\kappa)$ time, because there are only $O(\kappa)$ sets to add to. Therefore, the **for** loop has complexity $O(\kappa^{a+1}(\kappa+h))$ with recursion when $m-i = a$.

Since $0 \leq m - i \leq j - i \leq 4\lceil \frac{n}{r} \rceil$, the overall complexity of the **for** loop is $O(\kappa^{\frac{n}{r}}(\kappa + (\frac{n}{r})^{\frac{9}{2}\cdot\frac{n}{r}}))$. Therefore, the complexity of the entire algorithm is $O((\frac{n}{r})^2(\kappa+\kappa^{\frac{n}{r}}(\kappa+(\frac{n}{r})^{\frac{9}{2}\cdot\frac{n}{r}}))) = O((\frac{n}{r})^2 \cdot \kappa^{\frac{n}{r}}(\kappa + (\frac{n}{r})^{\frac{9}{2}\cdot\frac{n}{r}}))$. $\square$

Observe that when r has a comparable size to n, i.e. $\frac{n}{r}$ is some small constant c, the complexity of our algorithm is $O(c^2 \cdot \kappa^c(\kappa + c^{\frac{9}{2}c})) = O(\kappa^{c+1})$, which is polynomial in κ. Therefore, when $\kappa << n$ and $r = \Omega(n)$, such as $r = \frac{n}{2}$ or $\frac{n}{3}$, our algorithm is quite favorable. Moreover, since our complexity analysis is theoretical and considers the worst-case, its actual run time on a commercial solver might be significantly faster.

Our algorithm can be easily extended to guarantee the complementarity or strictness of the output, but it is unclear whether there exists a method to separate over $\frac{1}{k}$-inequalities that are minimal. Nevertheless, since 1-valid and minimal inequalities are subadditive over I, if a minimal inequality is violated, a separation over subadditive inequalities over I would return a violated inequality. Hence, this aspect of our algorithm is sufficient for practical purposes.

Chapter 6

Conclusion

We considered a number of problems that are related to non-trivial facet defining inequalities of $MEP(n, r)$. In particular, we proved a version of a two-slope theorem for $MEP(n, r)$, extended it for the system $IRm(q, r)$, and eventually proved a version of the two-slope theorem for IRm(n, r). Moreover, we considered properties of a non-trivial facet π of $MEP(n, r)$ under the normalization $\pi(-1) = 0$. A separation problem over a class of $\frac{1}{2}$-inequalities is then given under this normalization.

A recent paper by Yildiz and Cornuéjols [21] proved a result similar to Theorem 4.2.5. However, notice our result has a different flavor, because it gives an algorithmic construction to obtain extreme functions for $IRm(q, r)$.

Our research raises a number of non-trivial questions that could be an interest of future research. For example, the concept of k-subdivisions of an inequality for $MEP(n, r)$ was introduced in Chapter 4. We proved the k-subdivision of a 1-valid, minimal, and $(r-n)$-partially two-slope inequality defines a facet of MEP(kn, kr), but can this result be extended to more general cases? In other words, is the k-subdivision of any non-trivial facet defining inequality of $MEP(n, r)$ facet defining for MEP(kn, kr)? Even if this is not the case, it would be beneficial to characterize non-trivial facets of $MEP(n, r)$ that have this property.

Another possible research direction is to continue studying the relationship between facet defining inequalities for $MEP(n, r)$ and extreme function for $IRm(q, r)$. In particular, whether the continuous extension of a facet defining inequality of $MEP(n, r)$ is extreme can be explored. It would also be interesting to see if a two-slope theorem for extreme functions holds for the MIP version of $IRm(q, r)$.

There are also a number of potential extensions to the algorithm provided in Chapter 5. As we briefly mentioned, the algorithm can be easily extended to output inequalities that are strict or satisfy complementarity. But it might take a non-trivial effort to extend it to separate over $\frac{1}{k}$-inequalities.

In addition, the IP in our algorithm may be relaxed to accept any real solutions. In this setting, when a non-integer solution is obtained, it can be subdivided and interpreted as a valid inequality for a MEP with a larger n value. Although the details are unclear, it certainly remains a possible future direction to explore.

Last but not least, analogous to the study by Shim et al. [19], a worst-case analysis for $\frac{1}{k}$-facets of $MEP(n, r)$ can be performed, in terms of their LP relaxation gaps. This might provide additional evidence that $\frac{1}{k}$-inequalities are an interest of study when it comes to MEPs.

References

[1] E. Balas and A. Qualizza. Monoidal cut strengthening revisited. *Discrete Optimization*, 9(1):40–49, 2012.

[2] S. Chopra, S. Shim, and D. Steffy. A few strong knapsack facets. *MOPTA 2014*, 2014.

[3] G. Cornuéjols, T. Kis, and M. Molinaro. Lifting gomory cuts with bounded variables. *Operations Research Letters*, 41:142–146, 2013.

[4] G. Dantzig, R. Fulkerson, and S. Johnson. Solution of a large-scale traveling-salesman problem. *Journal of the Operations Research Society of America*, 2:393–410, 1954.

[5] S. Dash. Mixed integer rounding cuts and master group polyhedra. *Combinatorial Optimization: Methods and Applications*, pages 1–32, 2011.

[6] S. Dash, R. Fukasawa, and O. Günlük. On a generalization of the master cyclic group polyhedron. *Lecture Notes in Computer Science*, 4513:197–209, 2013.

[7] S. Dey, J. Richard, Y. Li, and L. Miller. On the extreme inequalities of infinite group problems. *Mathematical Programming*, 121(1):145–170, 2010.

[8] F. Eisenbrand. Fast integer programming in fixed dimension. *Lecture Notes in Computer Science*, 2832:196–207, 2003.

[9] R. Gomory. Outline of an algorithm for integer solution to linear programs. *Bulletin of the American Society*, 64:275–278, 1958.

[10] R. Gomory. Solving linear programming problems in integers. *Combinatorial Analysis*, pages 211–215, 1960.

[11] R. Gomory. Some polyhedra related to combinatorial problems. *Linear Algebra and Its Applications*, 2:451–558, 1969.

[12] R. Gomory. Early integer programming. *Operations Research*, 50:78–81, 2002.

[13] R. Gomory and E. Johnson. Some continuous functions related to corner polyhedra. *Mathematical Programming*, 3:23–85, 1972.

[14] M. Goycoolea. *Cutting Planes for Large Mixed Integer Programming Models*. PhD thesis, School of Industrial and Systems Engineering, Georgia Tech, 2006.

[15] R. Kannan. Minkowski's convex body theorem and integer programming. *Mathematics of Operations Research*, 12(3):415–440, 1986.

[16] V. Klee and G. Minty. How good is the simplex algorithm? *Inequalities*, III:159–175, 1972.

[17] G. Nemhauser and L. Wolsey. *Integer and Combinatorial Optimization*. Wiley, 1999.

[18] A. Schrijver. *Theory of Linear and Integer Programming*. Wiley, 1987.

[19] S. Shim, W. Cao, and S. Chopra. The worst case analysis of strong knapsack facets. 2014. Unpublished manuscript. http://www.researchgate.net/publication/263181439_The_worst_case_analysis_of_strong_knapsack_facets.

[20] L. Wolsey. *Integer Programming*. Wiley, 1998.

[21] S. Yildiz and G. Cornuéjols. Cut-generating functions for integer variables. 2014. Unpublished manuscript. http://integer.tepper.cmu.edu/webpub/draft.pdf.