Semistable rank 2 co-Higgs bundles over Hirzebruch surfaces

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

It has been observed by S. Rayan that the complex projective surfaces that potentially admit non-trivial examples of semistable co-Higgs bundles must be found at the lower end of the Enriques-Kodaira classification. Motivated by this remark, we study the geometry of these objects (in the rank 2 case) over Hirzebruch surfaces, giving special emphasis to $\mathbb{P}^1 \times \mathbb{P}^1$. Two main topics can be identified throughout the dissertation: non-emptiness of the moduli spaces of rank 2 semistable co-Higgs bundles over Hirzebruch surfaces, and the description of these moduli spaces over $\mathbb{P}^1 \times \mathbb{P}^1$.

The existence problem consists in determining for which pairs of Chern classes (c_1, c_2) there exists a non-trivial semistable rank 2 co-Higgs bundle with Chern classes c_1 and c_2. We approach this problem from two different perspectives. On one hand, we restrict ourselves to certain natural choices of c_1 and give necessary and sufficient conditions on c_2 that guarantee the existence of non-trivial semistable co-Higgs bundles with these Chern classes; we do this for arbitrary polarizations when $c_2 \leq 2$. On the other hand, for arbitrary c_1, we also provide necessary and sufficient conditions on c_2 that ensure the existence of non-trivial semistable co-Higgs bundles; however, we only do this for the standard polarization.

As for the description of the moduli spaces $\mathcal{M}^{\mathrm{co}}(c_1, c_2)$ of rank 2 semistable co-Higgs bundles over $\mathbb{P}^1 \times \mathbb{P}^1$, we restrict ourselves to the standard polarization. We then discuss how to use the spectral construction and the Hitchin correspondence to understand generic rank 2 semistable co-Higgs bundles. Furthermore, we give an explicit description of the moduli spaces when $c_2 = 0, 1$ for certain choices of c_1. Finally, we explore the first order deformations of points in the moduli space $\mathcal{M}^{\mathrm{co}}(c_1, c_2)$.

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Chapter 1

Introduction

A Higgs bundle on a complex projective manifold X is a pair (E, Φ) consisting of a holomorphic vector bundle E over X together with a Higgs field $\Phi\colon E \to E \otimes T^\vee$ taking values in the holomorphic cotangent bundle $T^\vee$ of X such that $\Phi \wedge \Phi \in \mathrm{H}^0(\operatorname{End} E \otimes \wedge^2 T^\vee)$ is identically zero. Higgs bundles were introduced almost 30 years ago by Hitchin in [19] and by Simpson in his PhD dissertation [30]. These objects have several interesting applications to both physics and mathematics, and have been extensively studied by many other authors, including Bradlow, García-Prada, Gothen, Wentworth; see for instance [7, 8, 14, 15, 16, 20, 31, 32, 35]. Co-Higgs bundles, on the other hand, are holomorphic vector bundles E paired with Higgs fields $\Phi\colon E \to E \otimes T$ taking values in the holomorphic tangent bundle T of X, instead of its holomorphic cotangent bundle $T^\vee$, and satisfying the same integrability condition $\Phi \wedge \Phi = 0 \in \mathrm{H}^0(\operatorname{End} E \otimes \wedge^2 T)$. Their study is fairly recent. They first appeared in the work of Gualtieri [17], and were further studied by Hitchin in [22, 23] and Rayan in [27, 28, 29]. As Rayan pertinently points out in his PhD dissertation [27], the study of co-Higgs bundles goes beyond idle curiosity, as these objects appear naturally in geometry; for example, in generalized complex geometry and in the theory of twisted quiver bundles.

In the realm of generalized complex geometry, as introduced by Hitchin in [21] and developed by Gualtieri in [17], co-Higgs bundles emerge as generalized holomorphic vector bundles over complex manifolds (regarded as generalized complex manifolds). Indeed, as defined by Gualtieri in [17], a generalized holomorphic bundle on a complex manifold (regarded as a generalized complex manifold) is a smooth vector bundle E together with a differential operator $\bar{D} : C^\infty \to C^\infty(E \otimes \bar{T}^\vee \oplus T)$ such that $\bar{D}(fs) = \bar{\partial} f s + f \bar{D} s$ for any smooth function f and smooth section s, and $\bar{D}^2 = 0 \in C^\infty(\operatorname{End} E \otimes \wedge^2(\bar{T}^\vee \oplus T))$. A consecuence of the definition is that $\bar{D}$ can be written as $\bar{D} = \bar{\partial} + \Phi$, for an operator $\bar{\partial} : C^\infty(E) \to C^\infty(E \otimes \bar{T}^\vee)$ and a linear operator $\Phi : C^\infty(E) \to C^\infty(E \otimes T)$ satisfying:

1. $\bar{\partial}^2 = 0 \in C^\infty(\operatorname{End} E \otimes \wedge^2 \bar{T}^\vee)$,
2. $\bar{\partial}\Phi = 0 \in C^\infty(\operatorname{End} E \otimes T \otimes \bar{T}^\vee)$ and
3. $\Phi \wedge \Phi = 0 \in C^\infty(\operatorname{End} E \otimes \wedge^2 T)$.

Condition (1) means that E is a holomorphic vector bundle, condition (2) says that Φ is a holomorphic section of $\operatorname{End} E \otimes T$ and condition (3) implies that Φ is integrable. Hence, the generalized holomorphic structure of E yields naturally the structure of a co-Higgs bundle.

Co-Higgs bundles, just as Higgs bundles, also fit in the realm of twisted quiver bundles as developed by Álvarez-Cónsul and García-Prada in [1]. A quiver Q consists of a set Q_0 of vertices $v, v', \ldots$ and a set Q_1 of arrows $a : v \to v$ connecting the vertices. Given a quiver and a compact Kähler manifold, a quiver bundle is defined by assigning a holomorphic vector bundle E_v to a finite number of vertices, and a homomorphism $\Phi_a : E_v \to E'_v$ to a finite number of arrows. If a collection of holomorphic vector bundles M_a parametrized by the set of arrows is fixed, and the morphisms Φ_a are twisted by the corresponding bundles M_a, $\Phi_a : E_v \otimes M_a \to E'_v$, a twisted quiver bundle is obtained. Thus, a co-Higgs bundle can be thought of as a quiver bundle formed by one vertex and one arrow (with the homomorphism satisfying the integrability condition) whose head and tail coincide, and the twisting bundle is the holomorphic cotangent bundle $T^\vee$.

Co-Higgs bundles come with a natural stability condition, analogous to the one discovered by Hitchin in [19] for Higgs bundles, which allows the study of their moduli spaces. Rayan has already given a complete characterization of (rank 2) semistable co-Higgs bundles over Riemann surfaces, but very little is known about these objects in higher dimensions. In his PhD dissertation [27] and in [28, 29], Rayan makes a thorough investigation of semistable rank 2 co-Higgs bundles over the Riemann sphere and constructs some examples over the projective plane. He also proves a non-existence result for non-trivial (i.e., non-zero Higgs field) stable co-Higgs bundles over K3 and general type surfaces, suggesting that some of the interesting examples must be found at the lower end of the Enriques–Kodaira classification of (compact) complex surfaces. Motivated by this fact, in this dissertation, we investigate rank 2 semistable co-Higgs bundles over Hirzebruch surfaces.

We organize this thesis as follows. Chapter 2 can be thought of as a foundational chapter in the sense that we introduce most of the concepts that will play a central role in the rest of this document. We begin by recalling the notion of slope stability for bundles (in the sense of Mumford–Takemoto) and review some useful properties of stable bundles. As this dissertation is concerned with Hirzebruch surfaces, we also introduce them here and recall some of their properties. We then conclude the chapter with the introduction of co-Higgs bundles and their appropriate Hitchin-type stability condition. Some basic properties of co-Higgs bundles, that are immediate generalizations of the analogous results for stable bundles, are also included. Finally, we offer a brief review of some known facts of semistable co-Higgs bundles over curves and surfaces.

In Chapter 3, we explore the existence of rank 2 semistable co-Higgs bundles over Hirzebruch surfaces in two directions. On the one hand, while considering arbitrary polarizations, we reduce the first Chern class c_1 of the bundles by tensoring them with an appropriate line bundle (Lemma 3.1), and give necessary conditions on their second Chern class c_2 in order to ensure the existence of non-trivial semistable co-Higgs pairs of rank 2 (Theorem 3.3). More precisely, let $\mathbb{F}_n$ denote the n-th Hirzebruch surface, and let F and C_0 denote the two classes of divisors that freely generate $\text{Pic}(\mathbb{F}_n)$ (here F denotes a general fibre of the ruling and C_0 the negative section). Any rank 2 vector bundle E over $\mathbb{F}_n$ has, aside from the Chern classes that determine its topological type, two numerical invariants, d_E and r_E, which are used to express E as an extension in a canonical way (see Section 2.2.2 for the definition of these invariants). By making use of stability and these numerical invariants, we prove:

Theorem A. *Let H be an ample divisor and let E be a rank 2 vector bundle over $\mathbb{F}_n$. Suppose (E, Φ) is H-semistable.*

1. *If $c_1(E) = 0$, then $c_2(E) \geq 0$. Furthermore, when equality holds, E is an extension of line bundles.*
2. *If $c_1(E) = -F$, then $c_2(E) \geq 0$. Furthermore, when equality holds, E is an extension of line bundles.*
3. *If $c_1(E) = -C_0$, then $c_2(E) \geq -\frac{n}{2}$. Furthermore, when equality holds, E is an extension of line bundles, and if $d_E \neq 0$, then $c_2(E) > 0$.*
4. *If $c_1(E) = -C_0 - F$, then $c_2(E) \geq -\frac{n-1}{2}$. Furthermore, when equality holds, E is an extension of line bundles, and if $d_E \neq 0$, then $c_2(E) > 0$.*

Then, we constructively show that the necessary conditions described above are also sufficient. While for $c_2 \leq 1$ we do so by carefully analyzing the ample cone of $\mathbb{F}_n$ (Theorems 3.8 and 3.9), for $c_2 \geq 2$ we only work with the standard polarization $H = C_0 + (n+1)F$ (Theorem 3.13).

On the other hand, by again fixing the standard polarization H (which naturally extends the notion of degree from $\mathbb{P}^1$ to Hirzebruch surfaces), we give necessary and sufficient conditions on the second Chern class of the bundle in order to guarantee the existence of non-trivial (i.e., non-zero Higgs field) semistable co-Higgs bundles of rank 2 (Theorem 3.15). Indeed, if we let $\mathcal{M}^{\mathrm{co}}_{\mathrm{H}}(c_1, c_2)$ denote the moduli space of rank 2 H-semistable co-Higgs bundles over $\mathbb{F}_n$ with fixed Chern classes c_1 and c_2, we prove:

Theorem B. *Let $c_1 = \alpha C_0 + \beta F$ and $c_2 = \gamma$. Fix the standard polarization $H = C_0 + (n+1)F$. Then, the moduli space $\mathcal{M}^{co}_H(c_1, c_2)$ is non-empty (and moreover it contains a non-trivial co-Higgs pair) if and only if one of the following holds:*

1. *α and β are both even and $4\gamma \geq \alpha(2\beta - n\alpha)$;*
2. *α is even, β is odd and $4\gamma \geq \alpha(2\beta - n\alpha)$;*
3. *α is odd, β is even and $4\gamma \geq 2\alpha\beta - n(1+\alpha^2)$;*
4. *α and β are both odd and $4\gamma \geq 2(\alpha\beta - 2) - n(1+\alpha^2)$.*

In Chapter 4, we specialize to the 0-th Hirzebruch surface, $\mathbb{P}^1 \times \mathbb{P}^1$, and fix the polarization $H = C_0 + F$. In this setting, we describe some of the moduli spaces of semistable rank 2 co-Higgs bundles for fixed values of c_1 and c_2 over it. Two important tools in constructing examples of semistable co-Higgs pairs or in understanding their moduli spaces are spectral covers and deformation theory. In this chapter we cover both. In fact, we divide the chapter into three main sections.

The first section is devoted to spectral covers and the Hitchin correspondence. Given a rank 2 co-Higgs bundle over the complex projective manifold X, we can associate to it a spectral manifold, which is a double cover of X naturally living in the total space of its tangent bundle. By the work of Hitchin and Simpson, it is well known that, under certain genericity conditions, one can construct rank 2 stable co-Higgs pairs over X in the following fashion: Take any rank 1 torsion-free coherent sheaf $\mathcal{F}$ over the spectral cover of X and push it down to obtain the underlying bundle of the co-Higgs pair, take also the push down of the multiplication map associated to $\mathcal{F}$ to obtain the Higgs field (one would of course need to check that the integrability condition is satisfied). In the $\mathbb{P}^1 \times \mathbb{P}^1$ setting, we show that the generic elements in the moduli space are such that the underlying bundles are not decomposable (Proposition 4.4). Finally, in certain non-generic cases, we are able to describe the spectral covers as trivial elliptic fibrations over $\mathbb{P}^1$; in these cases, the fibres of the Hitchin map do contain co-Higgs pairs with decomposable underlying bundles (Proposition 4.5).

In the second section, we explicitly construct some moduli spaces $\mathcal{M}^{\mathrm{co}}(c_1, c_2)$. For $c_2 = 0$, there are only three possibilities for the reduced first Chern classes: $c_1 = 0, -F$ and $-C_0$. For the case where $c_1 = -F$ (or $-C_0$), we have a complete description of the moduli space (Theorem 4.8):

Theorem C. *The moduli space $\mathcal{M}^{co}(-F, 0)$ of rank 2 stable co-Higgs bundles over $\mathbb{P}^1 \times \mathbb{P}^1$ with first Chern class $-F$ and second Chern class 0 is a 6-dimensional smooth variety isomorphic to the moduli space $\mathcal{M}^{co}_{\mathbb{P}^1}(-1)$ of rank 2 stable co-Higgs bundles of degree -1 over $\mathbb{P}^1$ (the latter is described in [28, Section 7]).*

For $c_1 = 0$, we were not able to give such an explicit description. Nonetheless, we show that there are only three underlying bundles that admit semistable Higgs fields: $\mathcal{O} \oplus \mathcal{O}$, $\mathcal{O}(1,0) \oplus \mathcal{O}(-1,0)$ and $\mathcal{O}(0,1) \oplus \mathcal{O}(0,-1)$. Also, we fully describe the Higgs fields that make $\mathcal{O} \oplus \mathcal{O}$ strictly semistable, and in Proposition 4.11 we prove that the Higgs fields of points in $\mathcal{M}^{\mathrm{co}}(0,0)$ with underlying bundle $\mathcal{O}(1,0) \oplus \mathcal{O}(-1,0)$ ($\mathcal{O}(0,1) \oplus \mathcal{O}(0,-1)$) are naturally parametrized by $\mathrm{H}^0(\mathcal{O}(4,0))$ ($\mathrm{H}^0(\mathcal{O}(0,4))$, respectively).

For $c_2 = 1$, we consider the case $c_1 = -F$. We first show that any underlying bundle in the moduli space $\mathcal{M}^{\mathrm{co}}_{\mathrm{H}}(-F, 1)$ is an extension of $\mathcal{O}(-1,1)$ by $\mathcal{O}(0,-1)$ (Proposition 4.15). Then, for the sake of being explicit, we describe all the Higgs fields that these bundles admit. Finally, we give an explicit description of the moduli space in Theorem 4.23 (see also Propositon 4.21):

Theorem D. *The moduli space $\mathcal{M}^{co}_{H}(-F, 1)$ is a 7-dimensional algebraic variety whose singular locus are the points $(E, 0)$ for any non-trivial extension E of $\mathcal{O}(-1,1)$ by $\mathcal{O}(0,-1)$.*

In the third section, we focus on the deformation theory of rank 2 semistable co-Higgs bundles over $\mathbb{P}^1 \times \mathbb{P}^1$. After reviewing the general theory, adapted by Rayan from the existing deformation theory for Higgs bundles, we explore the deformation theory for the co-Higgs bundles in the moduli spaces analyzed in the second section of this chapter.

In Chapter 5, we briefly outline some of the possible directions in which one can take this work.

Finally, we include two appendices in which we discuss further some of the underlying bundles of semistable co-Higgs pairs. In Appendix A, we focus on decomposable bundles. First, we describe all the decomposable stable underlying bundles (and their Higgs fields) with $c_1 = 0$ or $-F$, and $c_2 = 1$ over any Hirzebruch surface. Then, we describe all the decomposable underlying bundles of stable co-Higgs pairs over $\mathbb{P}^1 \times \mathbb{P}^1$ for reduced first Chern classes and $c_2 \geq 2$. In Appendix B, we consider semistable co-Higgs pairs with $c_1 = 0, -C_0, -C_0 - F$ and $c_2 = 1$ over $\mathbb{P}^1 \times \mathbb{P}^1$. Since in Chapter 4, when $c_2 = 1$, we only considered $c_1 = -F$ when describing the moduli spaces, here we determine the underlying bundles for the other cases.

Chapter 2

Stability, Hirzebruch surfaces and co-Higgs bundles

We begin this chapter by recalling some basic facts about stability of holomorphic vector bundles over complex projective manifolds. However, we quickly specialize to the case of rank 2 vector bundles over surfaces, which is what we are interested in for this dissertation. Then, we briefly introduce Hirzebruch surfaces, and give some important facts about vector bundles over them. We conclude the chapter with a section on co-Higgs bundles, where we formally introduce the concept and lay the foundation for the remainder of this thesis.

2.1 The Notion of Stability

Stability of bundles has been an interesting topic of study for many years; one apparent reason being that, under many circumstances, these objects form sufficiently "nice" moduli spaces. In this section, we recall the definition of stability in the Mumford–Takemoto sense (or slope stability), and review, without proofs, some basic properties of stable bundles. Although a lot can be said about stable bundles and their properties, we try only to introduce those ideas that later in the chapter will be generalized to co-Higgs bundles or that will be used in subsequent chapters. Many of these facts and their proofs can be found in [9, 13, 24], to name a few. We rely heavily on Friedman's book [13, Chapter 4], and follow his presentation quite closely.

Throughout this section we let X denote a complex projective manifold of dimension d. Whenever we say "vector bundle", we mean *holomorphic* vector bundle, and we make no distinction between locally free sheaves and vector bundles.

Let H be an ample divisor on X. Then, for any torsion-free coherent sheaf V over X, we define the *degree* of V with respect to H as:

$$\deg_H(V) = c_1(V) \cdot H^{d-1}.$$

Note that, whereas the Chern classes are topological invariants of the sheaves, the degree is defined only up to the choice of H.

Definition 2.1. Let V be a torsion-free coherent sheaf over X. The *H-slope* of V is given by

$$\mu_H(V) = \frac{\deg_H(V)}{\operatorname{rk}(V)}.$$

When H is understood, we omit the subscript, and simply say the *slope* of V, which we denote by $\mu(V)$.

The following property, which follows almost immediately from the definition of slope and the Whitney product formula (see [13, Chapter 4, Lemma 2]), is an important one to keep in mind.

Lemma 2.2. *Suppose that*

$$0 \to V_1 \to V \to V_2 \to 0$$

is an exact sequence of nonzero torsion-free coherent sheaves on X and H is an ample divisor on X. Let $\mu = \mu_H$. Then

$$\min(\mu(V_1), \mu(V_2)) \leq \mu(V) \leq \max(\mu(V_1), \mu(V_2)),$$

and equality holds at either end if and only if $\mu(V_1) = \mu(V_2) = \mu(V)$.

As we have mentioned before, in this thesis, we are mainly concerned with rank 2 vector bundles, so the following lemma is an important one to keep in mind.

Lemma 2.3. *Suppose $E = L_1 \oplus L_2$ is a decomposable rank 2 vector bundle over X, and H is an ample divisor on X. Then*

$$\mu_H(E) = \frac{\mu_H(L_1) + \mu_H(L_2)}{2}.$$

Proof. We have that

$$\mu_H(E) = \frac{c_1(E) \cdot H}{2} = \frac{(c_1(L_1) + c_1(L_2)) \cdot H}{2} = \frac{\mu_H(L_1) + \mu_H(L_2)}{2},$$

and so the slope of E is the average of the slopes of L_1 and L_2. □

The next lemma relates the slope of a torsion free sheaf with the slopes of its subsheaves. We will mainly focus on the moreover part of the lemma.

Lemma 2.4. *Suppose that W is a subsheaf of the torsion-free coherent sheaf V, with* $\operatorname{rk} W = \operatorname{rk} V$. *Then $\mu(W) \leq \mu(V)$. Moreover, if W and V are vector bundles, then either $\mu(W) < \mu(V)$ or $W = V$.*

Recall that by a *sub-bundle* of a vector bundle V over X we mean a subsheaf which is a vector bundle. When V is of rank 2, the non-zero proper sub-bundles are called *sub-line* bundles.

We are now ready to define the notion of stability.

Definition 2.5. Let V be a torsion-free coherent sheaf over X. We say that V is *H-stable* (respectively, *H-semistable*) if, for all nonzero proper coherent subsheaves W of V, we have that

$$\mu_H(W) < \mu_H(V) \tag{2.1}$$

(respectively $\leq$). We call V *unstable* if it is not semistable, and *strictly semistable* if it is semistable but not stable. Finally, a nonzero proper subsheaf W of V is *destabilizing* if $\mu_H(W) \geq \mu_H(V)$.

Remark 2.6. There are several things to point out. Some of them follow immediately from the definition, and some are a bit more elaborate (or follow from well known facts about sheaves). For details see either [13, Chapter 4] or [27, Chapter 1].

1. Line bundles are always stable.

2. Recall that if X is a surface, two divisors D_1 and D_2 are numerically equivalent if $(D_1 - D_2) \cdot E = 0$ for all divisors E. If H is numerically equivalent to H', then the notion of H-stability is equivalent to H'-stability. Indeed, if H and H' are numerically equivalent, $\mu_H(V) = \mu_{H'}(V)$ for any V. Thus, we consider stability up to a choice of numerical equivalence class. In addition, a choice of a numerical equivalence class of an ample divisor H is often referred to as a choice of *polarization.*

3. When checking inequality (2.1) it is enough to consider subsheaves W such that the quotient V/W is torsion free.

4. If X is a curve, the slope of V is independent of the choice of H (as $\mu_H(V) = c_1(V)$), and so stability does not depend on the choice of polarization. Furthermore, there is no need to check inequality (2.1) for all nonzero proper subsheaves of V; checking it for proper sub-bundles is enough.

5. If X is a surface and E is a vector bundle of rank 2, we can simplify the criterion for the stability of E. Indeed, it is enough to check inequality (2.1) for proper sub-line bundles of E. The main idea is that, given any proper subsheaf W of E, its double dual $(W^\vee)^\vee$, which is locally free, only differs from it at finitely many points, and thus they have the same slope. Thus, for any proper subsheaf of E, it is possible to construct a sub-line bundle of E with the same slope.

We have the following useful property.

Lemma 2.7. *Let V be a torsion-free coherent sheaf over X. The following are equivalent:*

(i) V is stable (semistable).

(ii) There exists a line bundle L such that $V \otimes L$ is stable (semistable).

(iii) For all line bundles L, $V \otimes L$ is stable (semistable).

Thus, stability of a rank r vector bundle V does not change when we tensor it by a line bundle L, but recall that its Chern classes do. In fact, we have that:

$$c_k(V \otimes L) = \sum_{i=0}^{k} \binom{r-i}{k-i} c_i(V) \cdot c_1(L)^{k-i}. \tag{2.2}$$

See, for example, [24, Section 1.2].

Lemma 2.8. *Let*

$$0 \to V_1 \to V \to V_2 \to 0$$

be an exact sequence of torsion-free coherent sheaves, with $\mu(V_1) = \mu(V) = \mu(V_2)$. Then V is not stable. Moreover, V is semistable if and only if V_1 and V_2 are semistable. In particular, if V_1 and V_2 both have rank 1, then V is strictly semistable.

Before we specialize to rank 2 vector bundles, let us give one more interesting fact about stable torsion-free coherent sheaves; for a proof, see [13, Chapter 4, Propostion 7 and Corollary 8].

Proposition 2.9. *If V is a stable torsion-free coherent sheaf, then V is simple, i.e., the only endomorphisms of V are scalar multiples of the identity:*

$$\mathrm{H}^0(X, \operatorname{End} V) = \{\lambda \cdot \mathrm{Id}_V \;:\; \lambda \in \mathbb{C}\}.$$

2.1.1 Unstable and strictly semistable rank 2 bundles

We now turn our attention to vector bundles of rank 2. The following two propositions and their corollaries help us to better understand the structure of unstable and strictly semistable bundles, and will be used repeatedly in subsequent chapters. Again, the propositions and their proofs can be found in [13, Chapter 4].

Proposition 2.10. *Suppose that E is an unstable rank 2 bundle over X. Then there exists a unique sub-line bundle G of E with torsion-free quotient such that $\mu(G) > \mu(E)$. Indeed, if L is a sub-line bundle of E such that $\mu(L) \geq \mu(E)$, then L is a subsheaf of G and $\mu(L) \leq \mu(G)$, with equality if and only if $L = G$.*

Corollary 2.11. *Let $E = L_1 \oplus L_2$ be a decomposable unstable rank 2 bundle over X, and let $\mu = \mu_H$. Then either $\mu(L_1) > \mu(E)$ or $\mu(L_2) > \mu(E)$. Hence, it is always one of the summands of E that makes it into an unstable bundle.*

Proof. Since we know that the slope of E is the average of the slopes of L_1 and L_2 (see Lemma 2.3), we have that either both L_1 and L_2 have the same slope as E, or one of them has slope strictly larger than the slope of E (and consequently, the other one has slope strictly smaller than the slope of E). In the latter case, the result is obvious. However, when $\mu(E) = \mu(L_1) = \mu(L_2)$, by Lemma 2.8, it follows that E is semistable, which contradicts the assumption, and thus it cannot happen. □

Proposition 2.12. *Let E be a strictly semistable rank 2 bundle. Then exactly one of the following holds:*

1. *There is a unique sub-line bundle L of E with $\mu(L) = \mu(E)$. The quotient E/L is necessarily torsion-free, and E is given canonically as an extension*

$$0 \to L \to E \to L' \otimes I_Z \to 0,$$

where Z is a codimension 2 locally complete intersection in X or it is empty.

2. *There are exactly two distinct sub-line bundles* L_1 *and* L_2 *with* $\mu(L_1) = \mu(L_2) = \mu(E)$. *In this case* $E = L_1 \oplus L_2$.

3. $E = L \oplus L$ *and there are infinitely many sub-line bundles with slope* $\mu(E)$*, exactly corresponding to the choice of a line in* $\mathrm{H}^0(E \otimes L^\vee)$.

More precisely, the following holds: Suppose that E *is an arbitrary rank 2 vector bundle which is given as an extension*

$$0 \to L_1 \to E \to L_2 \otimes I_Z \to 0$$

with $\mu(L_1) = \mu(E)$*. Then* E *is semistable and either* L_1 *is the unique destabilizing sub-line bundle of* E *with torsion-free quotient or* $Z = \emptyset$ *and* $E = L_1 \oplus L_2$*, i.e., the extension splits.*

Lemma 2.13. *Let* $E = L_1 \oplus L_2$ *be a decomposable rank 2 bundle over* X*. Then,* E *is strictly semistable if and only if* $\mu(E) = \mu(L_1)$*. Moreover, if* E *is strictly semistable, then for every sub-line bundle* G *of* E *such that* $\mu(G) = \mu(E)$*, we have that either* $G \cong L_1$ *or* $G \cong L_2$.

Proof. If E is strictly semistable, Lemma 2.3 clearly implies that $\mu(E) = \mu(L_1) = \mu(L_2)$. For the converse direction, note that if $\mu(E) = \mu(L_1)$, then $\mu(L_2) = \mu(E)$ as well. Now, the strict semistability of E follows by applying the "in particular" clause of Lemma 2.8.

For the "moreover" part of the statement, assume that E is strictly semistable and let G be a sub-line bundle of E. Then, $0 \neq \mathrm{H}^0(G^\vee \otimes E) = \mathrm{H}^0(G^\vee \otimes L_1) \oplus \mathrm{H}^0(G^\vee \otimes L_2)$. So either $\mathrm{H}^0(G^\vee \otimes L_1) \neq 0$ or $\mathrm{H}^0(G^\vee \otimes L_2) \neq 0$. In the former case, $G^\vee \otimes L_1$ must be equal to $\mathcal{O}_X(D)$ for some effective divisor D, but then $0 = \mu(L_1) - \mu(G) = \mu(\mathcal{O}_X(D))$, and so $D = 0$, implying $G \cong L_1$. In the second case, a similar argument shows that $G \cong L_2$.

□

2.1.2 Different Polarizations, Walls and Chambers

The notions of walls and chambers provide the right framework to answer the question: Given two ample divisors H_1 and H_2, when does there exist a vector bundle which is H_1-stable but is not H_2-stable? Walls and chambers have been studied by Friedmann and Qin, among others, and [4, 25, 26] together with [13, Chapter 4] are useful references. In [26], Qin presents the theory for complex projective manifolds of arbitrary dimension; however, we let X be a complex projective surface. Also, E continues to represent a rank 2 vector bundle over X.

Let $\mathbf{C}_X$ denote the ample cone of X and let c_1 and c_2 be fixed Chern classes. Recall that $\mathrm{Num}(X)$ is the subgroup of divisors of X that are numerically equivalent to zero, and recall further that $\mathbf{C}_X$ is open and convex in $\mathrm{Num}(X) \otimes \mathbb{R}$. An element $\zeta \in \mathrm{Num}(X) \otimes \mathbb{R}$ is called a *class of type* (c_1, c_2) if it is the numerical equivalence class of $(2D - c_1)$ for some divisor D, and satisfies the condition $-(4c_2 - c_1^2) \leq \zeta^2 < 0$. For such ζ, we define the *wall*

$$W^\zeta = \mathbf{C}_X \cap \{x \in \mathrm{Num}(X) \otimes \mathbb{R} : x \cdot \zeta = 0\}.$$

Walls W^ζ corresponding to classes ζ of type (c_1, c_2) are called *walls of type* (c_1, c_2). Note that distinct numerical equivalence classes may yield the same wall.

Let $\mathcal{W}(c_1, c_2)$ be the union of W^ζ, where ζ runs over all classes of type (c_1, c_2). A *chamber of type* (c_1, c_2) is a connected component of the set $\mathbf{C}_X \backslash \mathcal{W}(c_1, c_2)$. Note that every chamber has an upper and a lower wall. A chamber $\mathcal{C}$ is said to be *below* or *above* a wall W^ζ if $\zeta \cdot H < 0$, respectively $\zeta \cdot H > 0$, for any $H \in \mathcal{C}$.

Now that we have introduced the appropriate definitions, we can give the following two propositions (see [13, Chapter 4]), which are key to answering the question posed at the beginning of this subsection. Whereas here we simply present the statements for completeness purposes, we provide further insight on their content when we introduce their analogous versions for co-Higgs bundles in Section 2.3.

Proposition 2.14. *Let H_1 and H_2 be two polarizations. Let E be an H_1-stable rank 2 bundle. Then, E is not H_2-stable if and only if there exists a sub-line bundle $\mathcal{O}_X(D)$ of E such that*

$$H_1 \cdot (2D - c_1(E)) < 0 \leq H_2 \cdot (2D - c_1(E))$$

and

$$-(4c_2(E) - c_1(E)^2) \leq (2D - c_1(E))^2 < 0.$$

Moreover, $\mathcal{O}_X(D)$ is the unique sub-line bundle of E with torsion free quotient with the above properties. Finally, E is strictly semistable with respect to an ample divisor which is a convex combination of H_1 and H_2.

Proposition 2.15. *Suppose that H_1 and H_2 are two ample divisors, and that W^ζ is the unique wall of type (c_1, c_2) separating H_1 and H_2, and assume further that $\zeta \cdot H_1 < 0 < \zeta \cdot H_2$. Suppose that E is such that $c_1(E) = c_1$, $c_2(E) = c_2$, and that it is given by a non-split exact sequence*

$$0 \to \mathcal{O}_X(D) \to E \to \mathcal{O}_X(c_1 - D) \otimes I_Z \to 0,$$

where $\zeta = 2D - c_1$. Then E is H_1-stable and H_2-unstable.

2.2 Hirzebruch Surfaces

As the title of this dissertation suggests and as we mentioned in the introduction, we will focus our study of co-Higgs bundles on Hirzebruch surfaces. In this section we give a very short introduction to this type of surfaces and present the properties of its (rank 1 and 2) vector bundles that will be relevant to our study. Some good references for the subject are [9, Chapter 5, Section 5], [13, Chapter 5] and [18, Chapter 5, Section 2].

Hirzebruch surfaces are ruled surfaces where the base curve is $\mathbb{P}^1$. In particular, Hirzebruch surfaces are rational and have Kodaira dimension $-\infty$. Even though Hirzebruch surfaces can be obtained by a sequence of blow-ups and blow-downs from the complex projective plane, we do not take this approach. Instead, we define them as the projectivization of a rank 2 vector bundle over $\mathbb{P}^1$. More precisely, we define the n-th Hirzebruch surface, $\mathbb{F}_n$, as:

$$\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)) \xrightarrow{\pi} \mathbb{P}^1,$$

where $n \geq 0$.

Recall that, by the Birkhoff–Grothendieck theorem, all vector bundles over $\mathbb{P}^1$ are decomposable. Thus, if we choose any rank 2 vector bundle on $\mathbb{P}^1$, it will be of the form

$\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$; nonetheless, the choice of the rank 2 bundle for the definition of $\mathbb{F}_n$ is justified by the following proposition. For a proof of this fact, see for example [18, Chapter 5, Section 2].

Proposition 2.16. *If E_1 and E_2 are two vector bundles of rank 2 on $\mathbb{P}^1$, then $\mathbb{P}(E_1)$ and $\mathbb{P}(E_2)$ are isomorphic as ruled surfaces over $\mathbb{P}^1$ if and only if there is a line bundle L on $\mathbb{P}^1$ such that $E_1 = E_2 \otimes L$.*

We denote by C_0 the negative section of $\pi : \mathbb{F}_n \to \mathbb{P}^1$, i.e., $C_0^2 = -n$, and by F a general fibre of π. These two divisors freely generate $\mathrm{Pic}(\mathbb{F}_n)$ (which, in this case, is isomorphic to $\mathrm{Num}(\mathbb{F}_n)$). A divisor $aC_0 + bF$ on $\mathbb{F}_n$ is ample (equivalently, very ample) if and only if $a > 0$ and $b > an$ (see [18, Chapter 5]).

From now on, we let $\mathcal{O}(aC_0 + bF)$ denote the line bundle over $\mathbb{F}_n$ corresponding to the divisor $aC_0 + bF$. The tangent bundle T of a Hirzebruch surface $\mathbb{F}_n$ is the rank 2 decomposable bundle

$$T = \mathcal{O}(2F) \oplus \mathcal{O}(2C_0 + nF).$$

Lemma 2.17. *Let D be a divisor on $\mathbb{F}_n$, and suppose that $D \cdot F = m \geq 0$. Then $\pi_*\mathcal{O}(D)$ is a vector bundle of rank $m+1$ on $\mathbb{P}^1$. Moreover, $\pi_*\mathcal{O} = \mathcal{O}_{\mathbb{P}^1}$.*

Lemma 2.18. *Let D be a divisor on $\mathbb{F}_n$, and assume that $D \cdot F \geq 0$. Then $R^i\pi_*\mathcal{O}(D) = 0$ for $i > 0$; and for all i,*

$$\mathrm{H}^i(\mathbb{F}_n; \mathcal{O}(D)) \cong \mathrm{H}^i(\mathbb{P}^1, \pi_*\mathcal{O}(D)).$$

2.2.1 Cohomology

In this section, we compute the cohomology of line bundles over $\mathbb{F}_n$, which will be useful for our study. We will compute the cohomology by taking the push-forward or higher direct image (as appropriate) of line bundles on $\mathbb{F}_n$ to $\mathbb{P}^1$, and then using the Leray spectral sequence. In order to do so, first note that at a point $p \in \mathbb{P}^1$,

$$[R^i\pi_*(\mathcal{O}(aC_0))]_p \cong \mathrm{H}^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a)).$$

Then, it is clear that:

1. $\pi_*\mathcal{O}(aC_0)$ is a vector bundle of rank $a+1$ over $\mathbb{P}^1$ when $a \geq 0$, and it is zero otherwise.
2. $R^1\pi_*(\mathcal{O}(aC_0))$ is a vector bundle or rank $-(a+1)$ over $\mathbb{P}^1$ when $a \leq -2$, and it is zero otherwise.
3. $R^i\pi_*\mathcal{O}(aC_0) = 0$ for all a when $i \geq 2$.

Now, when $a \geq 0$, in order to see which rank $a+1$ bundle we obtain by pushing forward $\mathcal{O}(aC_0)$, recall that $\pi_*\mathcal{O}(aC_0) = \mathrm{S}^a(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$, where $S^a(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$ is the a-th symmetric product of the bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$ (see [18, Chapter 3, Section 8]). Then, since

$$\mathrm{S}^a(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)) = \oplus_{j=0}^a \mathcal{O}_{\mathbb{P}^1}(-jn),$$

we obtain that

$$\pi_*\mathcal{O}(aC_0) = \begin{cases} \oplus_{j=0}^{a}\mathcal{O}_{\mathbb{P}^1}(-jn), & \text{if } a \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand, recall that for $a \leq -2$ (see [18, Chapter 3, Section 8]),

$$R^1\pi_*\mathcal{O}(aC_0) = [\pi_*\mathcal{O}(-(a+2)C_0)] \otimes \mathcal{O}_{\mathbb{P}^1}(n).$$

Then, since $-(a+2) \geq 0$, by the previous result, we get that $\pi_*\mathcal{O}(-(a+2)C_0) = \oplus_{j=0}^{-(a+2)}\mathcal{O}_{\mathbb{P}^1}(-jn)$, and thus

$$R^1\pi_*\mathcal{O}(aC_0) = \begin{cases} \oplus_{j=0}^{-(a+2)}\mathcal{O}_{\mathbb{P}^1}((1+j)n), & \text{if } a \leq -2, \\ 0, & \text{otherwise.} \end{cases}$$

Finally, to obtain the cohomology we use the Leray spectral sequence and the projection formula. Hence,

$$\mathrm{H}^0(\mathbb{F}_n, \mathcal{O}(aC_0 + bF)) = \begin{cases} \oplus_{j=0}^{a} \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b - jn)), & \text{if } a \geq 0, \\ \\ 0, & \text{otherwise,} \end{cases} \tag{2.3}$$

$$\mathrm{H}^1(\mathbb{F}_n, \mathcal{O}(aC_0 + bF)) = \begin{cases} \oplus_{j=0}^{a} \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(jn - b - 2)), & \text{if } a \geq 0, \\ \\ 0, & \text{if } a = -1, \\ \\ \oplus_{j=0}^{-(a+2)} \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b + (1+j)n), & \text{otherwise,} \end{cases} \tag{2.4}$$

and

$$\mathrm{H}^2(\mathbb{F}_n, \mathcal{O}(aC_0 + bF)) = \begin{cases} \oplus_{j=0}^{-(a+2)} \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-(b + (1+j)n + 2)), & \text{if } a \leq -2, \\ \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

We will often be interested in knowing when there are no non-zero global sections on a line bundle $\mathcal{O}(aC_0 + bF)$. From (2.3), it is obvious that

$$\mathrm{H}^0(\mathbb{F}_n, \mathcal{O}(aC_0 + bF)) = 0 \text{ if and only if } a < 0 \text{ or } b < 0.$$

Even though everything described above is obviously valid for the 0-th Hirzebruch surface, $\mathbb{P}^1 \times \mathbb{P}^1$, in this case computing the cohomology follows from more standard results (the Künneth formula and Serre duality). For convenience of the reader we do this below. However, before doing so, let us introduce the standard notation for line bundles over $\mathbb{P}^1 \times \mathbb{P}^1$. The line bundle $\mathcal{O}(aC_0 + bF)$ corresponds to

$$\mathcal{O}(b, a) := \mathrm{pr}_1^* \mathcal{O}_{\mathbb{P}^1}(b) \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^1}(a),$$

where pr_i denotes the projection from $\mathbb{P}^1 \times \mathbb{P}^1$ onto the i-th copy of $\mathbb{P}^1$.

Remark 2.19. Let $\mathcal{O}(a,b)$ (note the change on the roles of a and b) be a line bundle over $\mathbb{P}^1\times\mathbb{P}^1$. The Künneth formula gives us a relation between the cohomology groups of two topological spaces and their product space. In particular,

$$\mathrm{H}^0(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O}(a,b))=\mathrm{H}^0(\mathbb{P}^1,\mathcal{O}(a))\otimes\mathrm{H}^0(\mathbb{P}^1,\mathcal{O}(b)).$$

It then follows that

$$\mathrm{H}^0(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O}(a,b))=0 \text{ if and only if } a<0 \text{ or } b<0.$$

We also get that

$$\mathrm{H}^1(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O}(a,b))=(\mathrm{H}^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(a))\otimes\mathrm{H}^1(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(b)))\oplus(\mathrm{H}^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(b))\otimes\mathrm{H}^1(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(a))).$$

Now, we can see that $\mathrm{H}^1(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O}(a,b))$ vanishes if and only if $a<0$ and $b<0$, or $a\geq -1$ and $b\geq -1$. Finally, we can use Serre duality to obtain

$$\mathrm{H}^2(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O}(a,b))=[\mathrm{H}^0(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O}(-(a+2),-(b+2)))]^\vee,$$

and so this vanishes if and only if $a\geq -1$ or $b\geq -1$.

From now on, unless otherwise specified, the notation $\mathrm{H}^i(\mathcal{F})$, where $\mathcal{F}$ is a coherent torsion-free sheaf over $\mathbb{F}_n$, will stand for $\mathrm{H}^i(\mathbb{F}_n;\mathcal{F})$.

2.2.2 On the existence of stable rank 2 bundles over $\mathbb{F}_n$

In this section we present some of the results that Aprodu, Brînzănescu and Marchitan conveniently gathered and summarized in [4], but for more details we refer the reader to [2, 3, 5, 9].

A rank 2 bundle E over $\mathbb{F}_n$ is always an extension of the form

$$0\to L_1\to E\to L_2\otimes I_Z\to 0,$$

where Z is a finite set of points in $\mathbb{F}_n$. Recall that the Chern classes of E are given by

$$\begin{aligned}c_1(E)&=c_1(L_1)+c_1(L_2),\\ c_2(E)&=c_1(L_1)\cdot c_1(L_2)+\ell(Z),\end{aligned}$$

where $\ell(Z)=|Z|$. Besides the Chern classes, which determine the topological type of E, there are two numerical invariants describing it as an extension in a canonical manner. Let us recall these invariants and some of their properties.

The first invariant d_E is defined by the splitting type on the general fibre F: if $E_{|F}\cong\mathcal{O}_{\mathbb{P}^1}(d)\oplus\mathcal{O}_{\mathbb{P}^1}(d')$ with $d\geq d'$, then $d_E=d$. The second invariant r_E is obtained from a push-forward as follows. Note that the bundle $\pi_*(E(-dC_0))$ is either of rank one or two, according to whether $d>d'$ or $d=d'$, respectively. If $d>d'$, we put $r_E=r=\deg(\pi_*(E(-dC_0)))$. If $d=d'$, then $\pi_*(E(-dC_0))=\mathcal{O}_{\mathbb{P}^1}(r)\oplus\mathcal{O}_{\mathbb{P}^1}(s)$ with $r\geq s$ and we put $r_E=r$.

Then, a rank 2 vector bundle E with numerical invariants d and r can be expressed as an extension

$$0\to\mathcal{O}(dC_0+rF)\to E\to\mathcal{O}(d'C_0+r'F)\otimes I_Z\to 0, \tag{2.6}$$

where Z is a finite set of points in $\mathbb{F}_n$. This extension is unique if either $d>d'$ or $d=d'$ and $s<r$, where s is the extra invariant described above.

Remark 2.20. Note that the length of Z depends only on the Chern classes c_1, c_2 and on the invariants d and r, hence it could be denoted by $\ell(c_1, c_2, d, r)$. If $c_1 = \alpha C_0 + \beta F$, then

$$\ell(c_1, c_2, d, r) = c_2 + \alpha(dn - r) - \beta d + 2dr - d^2 n.$$

These numerical invariants help us to better understand rank 2 bundles over Hirzebruch surfaces. Indeed, let

$$M(c_1, c_2, d, r) = \{E \to \mathbb{F}_n : c_1(E) = c_1, c_2(E) = c_2, d_E = d, r_E = r\}/\sim,$$

where $\sim$ denotes the equivalence relation of vector bundle isomorphism, be the set of rank 2 bundles with fixed Chern classes c_1 and c_2, and fixed numerical invariants d and r. The following theorem tells us when this moduli space is non-empty.

Theorem 2.21. *Put $c_1 = \alpha C_0 + \beta F$. The set $M(c_1, c_2, d, r)$ is non-empty if and only if $\ell := \ell(c_1, c_2, d, r) \geq 0$ and one of the following conditions is satisfied:*

1. *$2d > \alpha$, or*
2. *$2d = \alpha$, $\beta - 2r \leq \ell$.*

Taking a little detour, but still talking about rank 2 bundles over $\mathbb{F}_n$, we present the following lemma which will be useful in subsequent chapters.

Lemma 2.22. *Let E be a decomposable rank 2 bundle over $\mathbb{F}_n$. If L is a sub-line bundle of $E = G_1 \oplus G_2$, then*

$$\deg(L) \leq \max\{\deg(G_1), \deg(G_2)\}.$$

Proof. Let $H = h_1 C_0 + h_2 F$ be any ample divisor on $\mathbb{F}_n$ and, without loss of generality, assume that $\deg_H(G_1) \leq \deg_H(G_2)$. Now, let $G_1 = \mathcal{O}(a_1 C_0 + b_1 F)$ and $G_2 = \mathcal{O}(a_2 C_0 + b_2 F)$. For $L = \mathcal{O}(aC_0 + bF)$ to be a possible sub-line bundle of E, it must be the case that $\mathrm{H}^0(\mathrm{Hom}(L, E)) \neq 0$. As such, either $a \leq a_1$ and $b \leq b_1$, or $a \leq a_2$ and $b \leq b_2$. Then,

$$\begin{aligned} \deg_H(L) &= a(h_2 - nh_1) + bh_1 \\ &\leq a_2(h_2 - nh_1) + b_2 h_1 \\ &= \deg_H(G_2). \end{aligned}$$

Thus, $\deg_H(L) \leq \max\{\deg_H(G_1), \deg_H(G_2)\}$. □

Now, since in this thesis we are concerned with stability of bundles, given a rank 2 bundle E, it seems natural to ask when does there exist an ample divisor H on $\mathbb{F}_n$ such that E is H-stable. The following theorem answers that.

Theorem 2.23. *For $E \in M(c_1, c_2, d, r)$, let $c_1 = \alpha C_0 + \beta F$. Then, there exists an ample divisor H on $\mathbb{F}_n$ such that E is H-stable if and only if $2r < \beta$ and the extension (2.6) is non-trivial. In this case, we have that if $2d > \alpha$, then E is stable with respect to the chamber below the wall W^ζ, where $\zeta = (2d - \alpha)C_0 + (2r - \beta)F$.*

Let us now discuss the existence of stable bundles on $\mathbb{F}_n$ for fixed $n \geq 0$. All of these results appear in [4]. Fix Chern classes c_1 and c_2 with $4c_2 - c_1^2 \geq 0$. It can be shown that when equality holds, there is no stable rank 2 bundle over $\mathbb{F}_n$; hence we may assume $4c_2 - c_1^2 > 0$. For any chamber $\mathcal{C}$ of type (c_1, c_2), let $\mathcal{M}_{\mathcal{C}}(c_1, c_2)$ denote the moduli space of stable bundles over $\mathbb{F}_n$ with respect to a polarization $H \in \mathcal{C}$ and with Chern classes c_1 and c_2. Note that, by Proposition 2.15, this is well-defined; i.e., it does not depend on the choice of polarization $H \in \mathcal{C}$. The first result is the following:

Proposition 2.24. *Assume $n \neq 0$. Let $\mathcal{C}$ be any chamber of type (c_1, c_2) different from $\mathcal{C}_F$, the chamber containing the class of F on the boundary. Then, the moduli space $\mathcal{M}_{\mathcal{C}}(c_1, c_2)$ is non-empty.*

For a polarization $H \in \mathcal{C}_F$, the definition of r and the necessary and sufficient conditions for the non-emptiness of $M(c_1, c_2, d, r)$, appearing in Theorem 2.21, yield

Proposition 2.25. *The moduli space $\mathcal{M}_{\mathcal{C}_F}(c_1, c_2)$ is non-empty if and only if α is even and the intersection $[\beta/2 - (c_2 - c_1^2/4)/2, \beta/2) \cap \mathbb{Z}$ is non-empty, where $c_1 = \alpha C_0 + \beta F$.*

When $n = 0$, C_0 defines the other axis $[C_0]$ of the boundary of the ample cone. Denote by $\mathcal{C}_{C_0}$ the chamber that has the $[C_0]$-axis on its boundary.

Proposition 2.26. *If $n = 0$, then $\mathcal{M}_{\mathcal{C}_{C_0}}(c_1, c_2)$ is non-empty if and only if β is even and the intersection $[\alpha/2 - (c_2 - c_1^2/4)/2, \alpha/2) \cap \mathbb{Z}$ is non-empty, where $c_1 = \alpha C_0 + \beta F$.*

For polarizations H lying on walls, we have

Proposition 2.27. *Suppose $n \neq 0$. Let $H = aC_0 + bF$ be an ample divisor lying on some non-empty wall W of type (c_1, c_2). Assume that either $\zeta \cdot F \geq 2$ for all numerical equivalence classes ζ which represent the wall W and are such that $\zeta \cdot F \geq 0$, or that $4c_2 - c_1^2 > 2b/a - n$. Then $\mathcal{M}_H(c_1, c_2)$ is non-empty.*

2.3 Co-Higgs Bundles

We are now ready to formally introduce co-Higgs bundles. In this section we give the basic definitions and properties of (semistable) co-Higgs bundles (some of which are analogous to those presented in the first section of this chapter) over complex projective manifolds. We then talk about known results about semistable co-Higgs bundles over curves and surfaces.

Let us begin with the definition.

Definition 2.28. If X is a complex projective manifold with tangent bundle T, then a *co-Higgs bundle* or a *co-Higgs pair* on X is a vector bundle $V \to X$ together with a map $\Phi \in \mathrm{H}^0(X; \operatorname{End} V \otimes T)$ for which $\Phi \wedge \Phi \in \mathrm{H}^0(X; \operatorname{End} V \otimes \wedge^2 T)$ is identically zero. We refer to such a Φ as a *Higgs field* of V.

Note that the integrability condition, $\Phi \wedge \Phi = 0$, is trivial when X is a curve. However, it plays a central role in higher dimensions.

Remark 2.29. It is worth mentioning how the wedge, $-\wedge-$, is computed. It acts as the commutator in elements of $\operatorname{End} V$ and as the usual wedge in elements of T. For instance, when X is a surface, and $\Psi, \Phi \in \mathrm{H}^0(\operatorname{End} V \otimes T)$, if we work locally, $\Psi = \Psi_1\partial_1 + \Psi_2\partial_2$ and $\Phi = \Phi_1\partial_1 + \Phi_2\partial_2$, so that

$$\begin{aligned}\Psi \wedge \Phi &= [\Psi_1, \Phi_2]\partial_1 \wedge \partial_2 + [\Psi_2, \Phi_1]\partial_2 \wedge \partial_1\\ &= ([\Psi_1, \Phi_2] - [\Psi_2, \Phi_1])\partial_1 \wedge \partial_2.\end{aligned}$$

A morphism of co-Higgs bundles (V, Φ) and (V', Φ') is a commutative diagram

$$\begin{tikzcd} V \arrow[r, "\psi"] \arrow[d, "\Phi"] & V' \arrow[d, "\Phi'"'] \\ V \otimes T \arrow[r, "\psi\otimes\mathrm{Id}"] & V' \otimes T \end{tikzcd}$$

in which $\psi : V \to V'$ is a homomorphism of vector bundles. The pairs (V, Φ) and (V', Φ') are said to be isomorphic if ψ is an isomorphism of vector bundles. In particular, (V, Φ) and (V, Φ') are isomorphic if and only if there exists an automorphism ψ of V such that $\psi \circ \Phi \circ \psi^{-1} = \Phi'$.

Co-Higgs bundles come with a natural stability condition analogous to the one discovered by Hitchin in the setting of Higgs bundles (see [20]), generalizing Mumford–Takemoto stability for vector bundles.

Definition 2.30. Fix a polarization $H \in \mathbf{C}_X$. A co-Higgs bundle (V, Φ) on a complex projective manifold X is stable (respectively, semistable) if

$$\mu_H(W) < \mu_H(V) \tag{2.7}$$

(respectively, $\leq$) for each non-zero proper subsheaf W of V that is Φ-invariant; i.e., $\Phi(W) \subseteq W \otimes T$.

Note that the usual notion of (semi)stability can be recovered by setting $\Phi = 0$. More precisely, V is stable if and only if $(V, 0)$ is a stable co-Higgs pair. By a *non-trivial* co-Higgs pair, we mean that Φ is non-zero. Also, when V is fixed, we refer to Φ as (semi)stable whenever the pair (V, Φ) is (semi)stable.

Given any $\Phi \in \mathrm{H}^0(X, \operatorname{End} V \otimes T)$, the trace-free part of Φ is defined as

$$\Phi_0 := \Phi - \left(\frac{\mathrm{Tr}\Phi}{\mathrm{rk}(V)}\right)\mathrm{Id} \in \mathrm{H}^0(\operatorname{End}_0 V \otimes T).$$

It is immediate to check that (V, Φ) is (semi)stable if and only if (V, Φ_0) is (semi)stable. Hence, from now on, whenever we have a co-Higgs pair (V, Φ), Φ is assumed to be trace-free. Furthermore, when we say “Higgs field” we really mean “trace-free Higgs field”, but, for economy, we omit the word ”trace-free”.

As before, we have that tensoring a (semi)stable co-Higgs bundle by a line bundle does not affect (semi)stability.

Lemma 2.31. *The co-Higgs bundle (V, Φ) is (semi)stable if and only if $(V \otimes L, \Phi \otimes \mathrm{Id}_L)$ is (semi)stable for any line bundle L over X.*

Proof. It suffices to show the forward implication. Let (V, Φ) be a stable co-Higgs bundle, and let L be any line bundle over X. The case where (V, Φ) is strictly semistable can be done in a similar fashion. Towards a contradiction, assume that there exists W, a Φ-invariant sub-sheaf of $V \otimes L$, such that $\mu(W) \geq \mu(V \otimes L)$. If this is the case, then it is clear that $W \otimes L^\vee$ is a Φ-invariant subsheaf of V:

$$\begin{aligned} \Phi(W \otimes L^\vee) &= (\Phi \otimes \mathrm{Id}_L)(W) \otimes L^\vee \\ &\subseteq W \otimes L^\vee \otimes T. \end{aligned}$$

Furthermore,

$$\begin{aligned} \mu(W \otimes L^\vee) &= \mu(W) - \mu(L) \\ &\geq \mu(V \otimes L) - \mu(L) \\ &= \mu(V) + \mu(L) - \mu(L) \\ &= \mu(V), \end{aligned}$$

contradicting the stability of (V, Φ). Hence $(V \otimes L, \Phi)$ is a stable co-Higgs bundle. ☐

We also have the following lemma:

Lemma 2.32. *Let W be a sub-bundle of V and*

$$S := \{\varphi \in \mathrm{H}^0(X; \mathrm{Hom}(W, V \otimes T)) \mid \varphi = \Phi|_W \text{ for some } \Phi \in \mathrm{H}^0(X; \mathrm{End}\, V \otimes T)\}.$$

Moreover, let $\boldsymbol{\iota} : \mathrm{H}^0(X; \mathrm{Hom}(W, W \otimes T)) \to \mathrm{H}^0(X; \mathrm{Hom}(W, V \otimes T))$ be the map induced by the inclusion $\iota : W \to V$. If $S \subseteq \mathrm{Im}(\boldsymbol{\iota})$, then W is Φ-invariant for any $\Phi \in \mathrm{H}^0(X; \mathrm{End}\, V \otimes T)$.

Proof. Take $\Phi \in \mathrm{H}^0(X; \mathrm{End}\, V \otimes T)$ and consider $\Phi|_W$, which is an element of S. Since $S \subseteq \mathrm{Im}(\boldsymbol{\iota})$, we must have that $\Phi|_W = \boldsymbol{\iota}(\psi)$ for some $\psi \in \mathrm{H}^0(X; \mathrm{Hom}(W, W \otimes T))$. Hence W is Φ-invariant. ☐

For the most part, in what follows, the above lemma will be used in situations where we actually have that $\boldsymbol{\iota} : \mathrm{H}^0(X; \mathrm{Hom}(W, W \otimes T)) \to \mathrm{H}^0(X; \mathrm{Hom}(W, V \otimes T))$ is an isomorphism.

We now discuss Proposition 2.14 in the setting of co-Higgs bundles. Even though Proposition 2.14 is about stable bundles, a very similar result holds for semistable ones. Here we present both statements, though we only prove the semistable case (as the proofs are identical), and we say what these statements mean in terms of preserving the notion of (semi)stability within chambers.

Proposition 2.33. *Let X be a complex projective surface and let E be a rank 2 bundle over X. Let H_1 and H_2 be two polarizations. Let (E, Φ) be an H_1-semistable co-Higgs bundle. Then, (E, Φ) is not H_2-semistable if and only if there exists a Φ-invariant sub-line bundle $\mathcal{O}_X(D)$ of E such that*

$$H_1 \cdot (2D - c_1(E)) \leq 0 < H_2 \cdot (2D - c_1(E))$$

and

$$-(4c_2(E) - c_1(E)^2) \leq (2D - c_1(E))^2 < 0.$$

Proof. Suppose there exists such a Φ-invariant sub-line bundle $\mathcal{O}_X(D)$. We have that

$$\mu_{H_2}(\mathcal{O}_X(D)) - \mu_{H_2}(E) = \frac{(2D - c_1(E)) \cdot H_2}{2} > 0.$$

Thus (E, Φ) is not H_2-semistable.

For the converse, assume that (E, Φ) is not H_2-semistable. Then, there exists a Φ-invariant sub-line bundle $\mathcal{O}_X(D)$ of E such that

$$H_2 \cdot (2D - c_1(E)) > 0.$$

Since $\mathcal{O}_X(D)$ is Φ-invariant and (E, Φ) is H_1-semistable, we have that

$$H_1 \cdot (2D - c_1(E)) \leq 0.$$

We can find a convex combination $H := \alpha H_1 + (1 - \alpha) H_2$ such that $H \cdot (2D - c_1(E)) = 0$; i.e., take

$$\alpha = \frac{(2D - c_1(E)) \cdot H_2}{(2D - c_1(E)) \cdot (H_2 - H_1)}.$$

By the Hodge index theorem (see [13, Chapter 1, Theorem 11]) $(2D - c_1(E))^2 \leq 0$ with equality if and only if $2D - c_1(E)$ is numerically equivalent to zero. However,

$$(2D - c_1(E)) \cdot H_2 > 0,$$

and thus, $(2D - c_1(E))^2 < 0$.

For the inequality $-(4c_2(E) - c_1(E)^2) \leq (2D - c_1(E))^2$, note that there is an exact sequence

$$0 \to \mathcal{O}_X(D) \to E \to \mathcal{O}_X(c_1(E) - D) \otimes I_Z \to 0,$$

where Z is a finite set of points of X. Thus,

$$c_2(E) = -D^2 + D \cdot c_1(E) + \ell(Z) \geq -D^2 + D \cdot c_1(E),$$

and so

$$(2D - c_1(E))^2 = 4D^2 - 4D \cdot c_1(E) + c_1(E)^2 \geq -(4c_2(E) - c_1(E)^2).$$

$\square$

Note that, taking the wall of type (c_1, c_2), W^ζ, corresponding to $\zeta = 2D - c_1(E)$, we have that $\zeta \cdot H_1 \leq 0$ and $\zeta \cdot H_2 > 0$, and so either H_1 lies on the wall or H_1 and H_2 are not in the same chamber. Therefore, we have the following:

Corollary 2.34. *In any chamber of type (c_1, c_2) and its upper wall, $\mathcal{C} \cup W^+$, the notion of H-semistability for co-Higgs bundles is independent of the choice of polarization H.*

One can analogously prove:

Proposition 2.35. *Let X be a complex projective surface and let E be a rank 2 bundle over X. Let H_1 and H_2 be two polarizations. Let (E, Φ) be an H_1-stable co-Higgs bundle. Then, (E, Φ) is not H_2-stable if and only if there exists a Φ-invariant sub-line bundle $\mathcal{O}(D)$ of E such that*

$$H_1 \cdot (2D - c_1(E)) < 0 \leq H_2 \cdot (2D - c_1(E))$$

and

$$-(4c_2(E) - c_1(E)^2) \leq (2D - c_1(E))^2 < 0.$$

Again, note that, taking the wall of type (c_1, c_2), W^ζ, corresponding to $\zeta = 2D - c_1(E)$, we have that $\zeta \cdot H_1 < 0$ and $\zeta \cdot H_2 \geq 0$, and so H_1 and H_2 are not in the same chamber. Therefore, we have the following:

Corollary 2.36. *In any chamber of type (c_1, c_2), $\mathcal{C}$, the notion of H-stability for co-Higgs bundles is independent of the choice of polarization H.*

Since we will be working with moduli spaces of semistable co-Higgs bundles, we need to determine when two pairs in the moduli space represent the same object. Semistable co-Higgs pairs, as semistable vector bundles, are subject to S-equivalence. We introduce this notion next.

If (V, Φ) is strictly semistable, we can find a Φ-invariant proper sub-bundle U of V for which $\mu(U) = \mu(V)$ and $\mu(G) < \mu(V)$ for all Φ-invariant subsheaves $U \subsetneq G \subsetneq V$. Clearly, (U, Φ) is semistable (by abuse of notation, we use the symbol Φ to denote the restriction of Φ to U). Moreover, $(V/U, \Phi)$ is stable (again, by abuse of notation, we use the symbol Φ to denote the quotient Higgs field). Indeed, any proper Φ-invariant subsheaf of V/U has the form G/U for some Φ-invariant sheaf $U \subsetneq G \subsetneq V$, and so

$$\mu(G/U) = \frac{\deg G - \deg U}{\operatorname{rk} G - \operatorname{rk} U} = \frac{\mu(G)\operatorname{rk}(G) - \mu(U)\operatorname{rk}(U)}{\operatorname{rk} G - \operatorname{rk} U} < \mu(U) = \mu(V/U),$$

where in the inequality we use $\mu(G) < \mu(U)$. If we let $V_m = V$, $V_{m-1} = U$ and continue this process, which terminates eventually, we obtain what is known as a *Jordan-Hölder filtration* of (V, Φ):

$$0 = V_0 \subset \cdots \subset V_m = V$$

for some m. Here (V_j, Φ) is semistable for $1 \leq i \leq m-1$, $(V_j/V_{j-1}, \Phi)$ is stable, and $\mu(V_j) = \mu(V_j/V_{j-1}) = \mu(V)$ for $1 \leq j \leq m$. In these pairs, Φ always denotes the appropriate quotient Higgs field. While this filtration is not unique, the isomorphism class of the following object is:

$$\operatorname{gr}(V, \Phi) := \bigoplus_{j=1}^{m} (V_j/V_{j-1}, \Phi).$$

This object is called the *associated graded object* of (V, Φ). Then, two semistable pairs (V, Φ) and (V', Φ') are said to be *S-equivalent* whenever their graded objects are isomorphic as co-Higgs bundles, i.e., $\mathrm{gr}(V,\Phi) \cong \mathrm{gr}(V',\Phi')$. If a pair is stable, then the underlying bundle has the trivial Jordan-Hölder filtration consisting of itself and the zero bundle, and so the isomorphism class of the graded object is nothing more than the isomorphism class of the original pair.

When working with decomposable rank 2 vector bundles, the following lemma is an important one to keep in mind.

Lemma 2.37. *Let $E = G_1 \oplus G_2$ be a decomposable rank 2 bundle over X, and let (E,Φ) be strictly semistable with $\Phi = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \in \mathrm{H}^0(\mathrm{End}_0 E \otimes T)$. If G_1 is Φ-invariant, then*

$$\mathrm{gr}\,(E,\Phi) = \left(E, \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}\right).$$

Proof. Since (E,Φ) is strictly semistable and G_1 is Φ-invariant, $\mu(G_1) = \mu(E)$ (see Lemma 2.13). Then, it is clear that a Jordan-Hölder filtration of E is:

$$0 \subset G_1 \subset E.$$

We simply note that, by Lemma 2.4, any Φ-invariant G such that $G_1 \subsetneq G \subsetneq E$ must satisfy $\mu(G) < \mu(E)$. Hence

$$\mathrm{gr}(E,\Phi) = (G_1,\Phi_1) \oplus (G_2,\Phi_2),$$

with $\Phi_1 = A$ and $\Phi_2 = -A$. ∎

We conclude this section with the following useful lemma.

Lemma 2.38. *Let $E = G_1 \oplus G_2$ be a decomposable rank 2 bundle over X. If $\mu(G_1) > \mu(E)$ and $\mathrm{H}^0(G_1^\vee \otimes G_2 \otimes T) \neq 0$, then there exists a Higgs field Φ such that (E,Φ) is semistable. Moreover, any Higgs field with non-zero $(2,1)$-entry makes (E,Φ) into a semistable pair.*

Proof. Note that having $\mu(G_1) > \mu(E)$ implies that E is unstable, with G_1 being the unique sub-line bundle that destabilizes E. Any Higgs field is an element of $\mathrm{H}^0(\mathrm{End}_0 E \otimes T)$ which is integrable. In particular,

$$\Phi = \begin{pmatrix} A & B \\ C & -A \end{pmatrix},$$

with $A \in \mathrm{H}^0(T)$, $B \in \mathrm{H}^0(G_1 \otimes G_2^\vee \otimes T)$ and $C \in \mathrm{H}^0(G_1^\vee \otimes G_2 \otimes T)$. Since $\mathrm{H}^0(G_1^\vee \otimes G_2 \otimes T) \neq 0$, there exists Φ with non-zero C. In that case, we have that G_1 is not Φ-invariant, and so (E,Φ) is semistable. ∎

2.3.1 Semistable co-Higgs bundles over curves

Everything presented here is due to Rayan and can be found in [27, 28]. The first thing to note is that the study of semistable co-Higgs bundles over curves reduces to the study of semistable co-Higgs bundles on $\mathbb{P}^1$. This is since for $g = 1$, semistable co-Higgs bundles are simply semistable Higgs bundles; and this have been studied extensively by Franco, García-Prada and Newstead in [11, 12]. Furthermore, for $g > 1$, semistable co-Higgs bundles are simply semistable bundles (as in this case there are no non-zero Higgs fields). Hence, we only focus on $\mathbb{P}^1$. Since any rank 2 vector bundle over $\mathbb{P}^1$ can be tensored by an appropriate line bundle to make it into a bundle of degree either 0 or -1, when studying semistable co-Higgs bundles, one needs only to consider the even (degree 0) and the odd (degree -1) cases.

First of all, Rayan proves that in the even case there are only two underlying bundles that yield semistable co-Higgs pairs: $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$; while in the odd case, there is only one bundle to consider: $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

In the odd degree case, where the notion of semistability and stability coincide, Rayan gives an explicit description of the moduli space as a six-dimensional subvariety of

$$\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(2)) \times \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(4)),$$

which is a universal elliptic curve (see [28, Section 7]).

Even though an explicit description is not as obvious in the even degree case, Rayan still describes the fibres of the Hitchin map (we will define this in Chapter 4). For details see [28, Section 8].

2.3.2 Semistable co-Higgs bundles over surfaces

As we mentioned in the introduction, not much is known about semistable co-Higgs bundles over surfaces. In this subsection, we present a very brief summary of Rayan's work over surfaces and, at the end, we also lay the foundation for our subsequent work on Hirzebruch surfaces. Details about Rayan's work can be found in [27, 29].

Non-Existence Results

The existence of non-trivial semistable co-Higgs bundles of rank 2 seems to be skewed to the non-positive end of the Kodaira spectrum, as the following theorem suggests.

Theorem 2.39. *[29, Section 4] Let X be a surface of general type or birational to a $K3$ surface. Then, if (E, Φ) is a stable, trace-free rank 2 co-Higgs bundle on X with $\mathrm{c}_1(E) = 0$, we must have that $\Phi = 0$.*

A key ingredient in the proof of this theorem is the fact that for X as above, $\mathrm{H}^0(X, \mathrm{S}^2(T))$ vanishes. Thus, the existence of stable rank 2 co-Higgs bundles is tied to the availability of holomorphic sections of $\mathrm{S}^2(T)$. Both on the projective plane and on Hirzebruch surfaces, there are plenty.

Co-Higgs bundles over $\mathbb{P}^2$

It is well known that $\mathbb{P}^1 \times \mathbb{P}^1$ is a double cover of $\mathbb{P}^2$, and so one can construct rank 2 vector bundles over $\mathbb{P}^2$ by pushing down line bundles over $\mathbb{P}^1 \times \mathbb{P}^1$. Rank 2 bundles over $\mathbb{P}^2$ obtained in this fashion are known as Schwarzenberger bundles. In [29], Rayan investigates co-Higgs bundles over $\mathbb{P}^2$, for which the underlying bundles are Schwarzenberger. Indeed, these turn out to naturally be co-Higgs bundles with Higgs fields also descending from the double cover. The locus of an element $\rho \in \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ defines a non-singular conic as well as a double cover of $\mathbb{P}^2$ by $\mathbb{P}^1 \times \mathbb{P}^1$ branched over the conic. If one allows the branched conic to vary, Rayan shows that the resulting moduli spaces of Schwarzenberger semistable co-Higgs bundles over $\mathbb{P}^2$ are 8-dimensional. Then, as an application of the deformation theory of co-Higgs bundles (which we will discuss in Chapter 4), Rayan also shows that starting with a non-trivial Schwarzenberger co-Higgs pair, nearby deformations are again Schwarzenberger. Hence, semistable Schwarzenberger co-Higgs bundles are rigid.

Co-Higgs bundles over Hirzebruch surfaces

From now on, we will be studying semistable (trace-free) rank 2 co-Higgs bundles over Hirzebruch surfaces and their moduli spaces. In the next chapter, we will talk about the existence of these objects, but for now let us investigate what the integrability condition implies in this case.

Let E be a rank 2 bundle over $\mathbb{F}_n$. Since the tangent bundle of $\mathbb{F}_n$ is decomposable ($T = \mathcal{O}(2F) \oplus \mathcal{O}(2C_0 + nF)$), it is clear that any $\Phi \in \mathrm{H}^0(\mathrm{End}_0\, E \otimes T)$ is of the form $\Phi = \Phi_1 + \Phi_2$ with $\Phi_1 \in \mathrm{H}^0(\mathrm{End}_0\, E(2F))$ and $\Phi_2 \in \mathrm{H}^0(\mathrm{End}_0\, E(2C_0 + nF))$. Working locally on an open set, where $\mathrm{End}_0\, E$ and T are trivial, we can write

$$\Phi = \Phi_1 + \Phi_2 = \begin{pmatrix} A_1 & B_1 \\ C_1 & -A_1 \end{pmatrix} + \begin{pmatrix} A_2 & B_2 \\ C_2 & -A_2 \end{pmatrix},$$

where A_i, B_i, C_i are complex valued functions for $i = 1, 2$. Also, on this open set,

$$\Phi \wedge \Phi = 2[\Phi_1, \Phi_2],$$

so we can locally write

$$\Phi \wedge \Phi = 2 \begin{pmatrix} B_1C_2 - C_1B_2 & 2(A_1B_2 - B_1A_2) \\ 2(C_1A_2 - A_1C_2) & -(B_1C_2 - C_1B_2) \end{pmatrix}.$$

Thus, we see that Φ is an (integrable) Higgs field if and only if, in each local trivialization, we have that

$$\begin{aligned} B_1C_2 &= C_1B_2 \\ A_1B_2 &= B_1A_2 \\ C_1A_2 &= A_1C_2. \end{aligned} \tag{2.8}$$

Let us finish this chapter with a remark on what the integrability of Φ entails geometrically (in terms of eigenspaces). To do this, we will need the following basic lemma from linear algebra.

Lemma 2.40. *Suppose M_1 is an $n \times n$ complex matrix with distinct eigenvalues. If M_2 is such that $[M_1, M_2] = 0$, then M_1 and M_2 have the same eigenvectors. Moreover, if M_2 has distinct eigenvalues, then M_1 and M_2 have the same eigenspaces.*

Proof. Let λ be an eigenvalue of M_1 and $v \neq 0$ be in the eigenspace corresponding to λ. Then, we have that

$$\begin{aligned}
\mathbf{0} &= \mathbf{0}v \\
&= [M_1, M_2]v \\
&= M_1 M_2 v - M_2 M_1 v \\
&= M_1(M_2 v) - \lambda(M_2 v),
\end{aligned}$$

and so $M_1(M_2v) = \lambda(M_2v)$. This implies that M_2v is an element of the eigenspace corresponding to λ, and so it can be written as a complex multiple of v. Hence $M_2v = \lambda' v$, and the result follows. □

Remark 2.41. We have seen that the integrability of Φ is equivalent to $[\Phi_1, \Phi_2] = 0$. Thus, from the above lemma, for those points of $\mathbb{F}_n$ where Φ_1 and Φ_2 have distinct eigenvalues, we must have that Φ_1 and Φ_2 share the same eigenspaces.

Chapter 3

Existence of semistable rank 2 co-Higgs bundles over Hirzebruch surfaces

This chapter is devoted to the existence of semistable rank 2 co-Higgs bundles over Hirzebruch surfaces. Given a Hirzebruch surface, $\mathbb{F}_n$, if one fixes the standard polarization $H = C_0 + (n+1)F$, it is possible to give a complete answer to the question: For which values of c_1 and c_2 are the moduli spaces of semistable rank 2 co-Higgs bundles over $\mathbb{F}_n$ non-empty? (See Theorem 3.15). On the other hand, if a polarization is not fixed, but we instead reduce the first Chern class by tensoring by a line bundle, it is also possible to give necessary and sufficient conditions for the existence of semistable rank 2 co-Higgs bundles over $\mathbb{F}_n$ for $c_2 \leq 1$.

3.1 Normalizing the first Chern class and bounding the second Chern class

Recall that, by Lemma 2.31, semistability of co-Higgs bundles is preserved after tensoring by a line bundle. Thus, the following lemma will be useful to, in many circumstances, simplify our study.

Lemma 3.1. *Let E be a rank 2 vector bundle over $\mathbb{F}_n$. Then there is a line bundle L such that $c_1(E \otimes L) = 0$ or $c_1(E \otimes L) = -F$ or $c_1(E \otimes L) = -C_0$ or $c_1(E) = -C_0 - F$.*

Proof. Let $c_1(E) = \alpha C_0 + \beta F$. There are four cases to consider:

(i) If both α and β are even, consider the line bundle

$$L = \mathcal{O}\left(-\left(\frac{\alpha}{2}\right) C_0 - \left(\frac{\beta}{2}\right) F\right),$$

so that $c_1(E \otimes L) = 0$ (see equation (2.2)).

(ii) If α is even and β is odd, consider the line bundle

$$L = \mathcal{O}\left(-\left(\frac{\alpha}{2}\right)C_0 - \left(\frac{1+\beta}{2}\right)F\right),$$

so that $c_1(E \otimes L) = -F$.

(iii) If α is odd and β is even, consider the line bundle

$$L = \mathcal{O}\left(-\left(\frac{1+\alpha}{2}\right)C_0 - \left(\frac{\beta}{2}\right)F\right),$$

so that $c_1(E \otimes L) = -C_0$.

(iv) If both α and β are odd, consider the line bundle

$$L = \mathcal{O}\left(-\left(\frac{1+\alpha}{2}\right)C_0 - \left(\frac{1+\beta}{2}\right)F\right),$$

so that $c_1(E \otimes L) = -C_0 - F$.

□

When we work with a rank 2 vector bundle E over $\mathbb{F}_n$, and we tensor it by a line bundle to obtain one of the first Chern classes $0, -C_0, -F$ or $-C_0 - F$, which from now on will be referred to as *reduced classes*, we also modify its second Chern class. That is,

$$c_2(E \otimes L) = c_2(E) + c_1(E) \cdot c_1(L) + c_1(L)^2, \tag{3.1}$$

(see equation (2.2)), and so we have:

Corollary 3.2. *Let E be a rank 2 vector bundle over $\mathbb{F}_n$ with $c_1(E) = \alpha C_0 + \beta F$ and $c_2(E) = \gamma$. Then, for any line bundle L such that $c_1(E \otimes L)$ is a reduced class, we have*

1. *If $c_1(E \otimes L) = 0$, then $c_2(E \otimes L) = \gamma + \frac{\alpha(n\alpha - 2\beta)}{4}$.*
2. *If $c_1(E \otimes L) = -F$, then $c_2(E \otimes L) = \gamma + \frac{\alpha(n\alpha - 2\beta)}{4}$.*
3. *If $c_1(E \otimes L) = -C_0$, then $c_2(E \otimes L) = \gamma + \frac{\alpha(n\alpha - 2\beta) - n}{4}$.*
4. *If $c_1(E \otimes L) = -C_0 - F$, then $c_2(E \otimes L) = \gamma + \frac{\alpha(n\alpha - 2\beta) + 2 - n}{4}$.*

Proof. This follows immediately from a direct computation using Lemma 3.1 and equation (3.1). □

Let us now work with the reduced classes, and give necessary conditions on c_2 in order to have a semistable co-Higgs pair. Recall that for any rank 2 vector bundle E, the numerical invariant d_E was introduced in Section 2.2.2.

Theorem 3.3. *Let H be an ample divisor and let E be a rank 2 vector bundle over $\mathbb{F}_n$. Suppose (E, Φ) is H-semistable.*

1. *If $c_1(E)=0$, then $c_2(E)\geq 0$. Furthermore, when equality holds, E is an extension of line bundles.*

2. *If $c_1(E)=-F$, then $c_2(E)\geq 0$. Furthermore, when equality holds, E is an extension of line bundles.*

3. *If $c_1(E)=-C_0$, then $c_2(E)\geq -\frac{n}{2}$. Furthermore, when equality holds, E is an extension of line bundles, and if $d_E\neq 0$, then $c_2(E)>0$.*

4. *If $c_1(E)=-C_0-F$, then $c_2(E)\geq -\frac{n-1}{2}$. Furthermore, when equality holds, E is an extension of line bundles, and if $d_E\neq 0$, then $c_2(E)>0$.*

Proof. Let $c_1(E)=\alpha C_0+\beta F$, where $(\alpha,\beta)\in\{(0,0),(-1,0),(0,-1),(-1,-1)\}$. Hence, by (2.6), E fits into an exact sequence of the form

$$0\to\mathcal{O}(dC_0+rF)\to E\to\mathcal{O}((\alpha-d)C_0+(\beta-r)F)\otimes I_Z\to 0, \tag{3.2}$$

where Z is a finite set of points in $\mathbb{F}_n$. Then, $c_2(E)=d(nd+\beta-2r-n\alpha)+r\alpha+\ell(Z)$.

Let us now work by cases:

(i) $(\alpha,\beta)=(0,0)$: Since $E_{|F}\cong\mathcal{O}_{\mathbb{P}^1}(d)\oplus\mathcal{O}_{\mathbb{P}^1}(-d)$, we have that $d\geq 0$. In this case, $c_2(E)=d(nd-2r)+\ell(Z)$. Towards a contradiction, assume that $c_2(E)<0$, or $c_2(E)=0$ and $\ell(Z)>0$. We then have that $d>0$ and $d(nd-2r)<0$.

(ii) $(\alpha,\beta)=(0,-1)$: As in case (i), $d\geq 0$, but now $c_2(E)=d(nd-1-2r)+\ell(Z)$. Towards a contradiction, assume that $c_2(E)<0$, or $c_2(E)=0$ and $\ell(Z)>0$. We then have that $d>0$ and $d(nd-1-2r)<0$.

(iii) $(\alpha,\beta)=(-1,0)$: Since $E_{|F}\cong\mathcal{O}_{\mathbb{P}^1}(d)\oplus\mathcal{O}_{\mathbb{P}^1}(-1-d)$, we have that $d\geq -1-d$, and so $d\geq 0$. In this case, $c_2(E)=d(nd+n-2r)-r+\ell(Z)$. Towards a contradiction, assume that either $c_2(E)<-\frac{n}{2}$, or $c_2(E)=-\frac{n}{2}$ and $\ell(Z)>0$. We then have that: If $d=0$, then $2r-n>0$, and if $d>0$, then $r>0$.

(iv) $(\alpha,\beta)=(-1,-1)$: As in case (iii), $d\geq 0$, but now $c_2(E)=d(nd-1-2r+n)-r+\ell(Z)$. Towards a contradiction, assume that either $c_2(E)<-\frac{n-1}{2}$, or $c_2(E)=-\frac{n-1}{2}$ and $\ell(Z)>0$. We then have that: If $d=0$, then $2r+1-n>0$, and if $d>0$, then $r\geq 0$.

Now, since $T=\mathcal{O}(2F)\oplus\mathcal{O}(2C_0+nF)$, by plugging in the corresponding values of (α,β), using (2.3) and the corresponding bounds on d and r described in (i) to (iv) above, one can easily check that, in all four cases,

$$\begin{aligned}\mathrm{H}^0(\mathcal{O}((\alpha-2d)C_0+(\beta-2r)F)\otimes T\otimes I_Z) &= \mathrm{H}^0(\mathcal{O}((\alpha-2d)C_0+(\beta-2r+2)F)\otimes I_Z)\\ &\oplus \mathrm{H}^0(\mathcal{O}((\alpha-2d+2)C_0+(\beta-2r+n)F)\otimes I_Z)\\ &=0.\end{aligned}$$

By tensoring (3.2) with $\mathcal{O}(dC_0+rF)^\vee\otimes T$ and passing to the long exact sequence in cohomology, we get

$$0\to\mathrm{H}^0(T)\to\mathrm{H}^0(\mathcal{O}(dC_0+rF)^\vee\otimes E\otimes T)\to\mathrm{H}^0(\mathcal{O}((\alpha-2d)C_0+(\beta-2r)F)\otimes T\otimes I_Z)\to\dots.$$

However, since $\mathrm{H}^0(\mathcal{O}((\alpha-2d)C_0+(\beta-2r)F)\otimes T\otimes I_Z)=0$ we get that $\mathrm{H}^0(T)=\mathrm{H}^0(\mathcal{O}(dC_0+rF)^\vee\otimes E\otimes T)$, which, by Lemma 2.32, implies that $\mathcal{O}(dC_0+rF)$ is Φ-invariant for any $\Phi\in\mathrm{H}^0(\mathrm{End}_0\,E\otimes T)$. Furthermore, let $H=h_1C_0+h_2F$, and note that $\mu_H(\mathcal{O}(dC_0+rF))=h_1(r-nd)+h_2d$, while

$$\mu_H(E)=h_1\left(\frac{\beta-\alpha n}{2}\right)+h_2\left(\frac{\alpha}{2}\right).$$

Thus, we have that

$$\begin{aligned}\mu_H(\mathcal{O}(dC_0+rF))-\mu_H(E)&=h_1\left(r-nd+\frac{\alpha n-\beta}{2}\right)+h_2\left(d-\frac{\alpha}{2}\right)\\&\geq h_1\left(r-nd+\frac{\alpha n-\beta}{2}\right)+(nh_1+1)\left(d-\frac{\alpha}{2}\right)\\&=h_1\left(r-\frac{\beta}{2}\right)+\left(d-\frac{\alpha}{2}\right),\end{aligned}$$

which in all four cases, is a strictly positive number. This contradicts H-semistability of (E,Φ), and the result follows. □

3.2 Walls and Chambers for $c_2\leq 1$

Given c_1 and c_2, we have seen that the concept of semistability for co-Higgs bundles is independent of the choice of polarization within chambers and their upper walls of type (c_1,c_2) (see Proposition 2.33 and Corollary 2.34). Hence, in order to have a more pragmatic approach to dealing with arbitrary polarizations, it would be useful to better understand the ample cone $\mathbf{C}_{\mathbb{F}_n}$. In this section we do so for the reduced first Chern classes and for $c_2\leq 1$.

Keeping the same notation as in subsection 2.1.2, a class ζ of type (c_1,c_2) is said to be *normalized* if $\zeta\cdot F\geq 0$. From now on we work with normalized ζ. Let $c_1=\alpha C_0+\beta F$ and $c_2=\gamma$. Moreover, let $\zeta=\zeta_1C_0+\zeta_2F$ be a normalized class of type (c_1,c_2). We make the following remarks:

Remark 3.4.

1. Since ζ is normalized, we must have $\zeta_1\geq 0$.
2. Since ζ has to be in the same numerical equivalence class as $2D-c_1(E)$ for some divisor $D=d_1C_0+d_2F$, we have, in particular, that

$$(\zeta-(2D-c_1(E)))\cdot C_0=(\zeta-(2D-c_1(E)))\cdot F=0.$$

Since

$$\zeta-(2D-c_1(E))=((\zeta_1+\alpha-2d_1)C_0+(\zeta_2+\beta-2d_2)F),$$

this implies that $\zeta_1=2d_1-\alpha$ and $\zeta_2=n(\zeta_1+\alpha-2d_1)-\beta+2d_2$. Thus, we get the following table

c_1	ζ_1	ζ_2
0	even	even
$-C_0$	odd	even
$-F$	even	odd
$-C_0-F$	odd	odd

3. Recall that $x = x_1C_0 + x_2F \in \mathbf{C}_{\mathbb{F}_n}$ if and only if $x_1 > 0$ and $x_2 > nx_1$. Hence, if the wall
$$W^\zeta = \{x \in \mathbf{C}_{\mathbb{F}_n} \ : \ x \cdot \zeta = 0\}$$
is non-empty, we must have $\zeta_2 \leq 0$.

4. Since ζ satisfies $-(4c_2 - c_1^2) \leq \zeta^2 < 0$, we have
$$-(4\gamma + \alpha^2 n - 2\alpha\beta) \leq -\zeta_1^2 n + 2\zeta_1\zeta_2 < 0.$$
Thus, we obtain the following tables.

   For $c_2(E) \leq 0$, we get

c_1	
0	$0 \leq -\zeta_1^2 n + 2\zeta_1\zeta_2 < 0$
$-C_0$	$-n \leq -(4\gamma + n) \leq -\zeta_1^2 n + 2\zeta_1\zeta_2 < 0$
$-F$	$0 \leq -\zeta_1^2 n + 2\zeta_1\zeta_2 < 0$
$-C_0-F$	$-n+2 \leq -(4\gamma + n - 2) \leq -\zeta_1^2 n + 2\zeta_1\zeta_2 < 0$

   For $c_2(E) = 1$, we get

c_1	
0	$-4 \leq -\zeta_1^2 n + 2\zeta_1\zeta_2 < 0$
$-C_0$	$-4-n \leq -\zeta_1^2 n + 2\zeta_1\zeta_2 < 0$
$-F$	$-4 \leq -\zeta_1^2 n + 2\zeta_1\zeta_2 < 0$
$-C_0-F$	$-n-2 \leq -\zeta_1^2 n + 2\zeta_1\zeta_2 < 0$

With this is mind, we can now prove the following two propositions.

Proposition 3.5. *Let c_1 be one of the reduced classes, and let $c_2 \leq 0$. Then the ample cone $\mathbf{C}_{\mathbb{F}_n}$ has only one chamber of type (c_1, c_2). In particular, to study semistability of co-Higgs bundles in this case, it suffices to consider the standard polarization $H = C_0 + (n+1)F$.*

Proof. We will prove that, in this case, there are no walls and, by Proposition 2.33 and its corollary, it will follow that we may consider any polarization. It immediately follows from Remark 3.4 that there are no values of ζ_1, ζ_2 satisfying the conditions, except when $c_1 = -C_0$. In this case, the only possible values for ζ_1 and ζ_2 are $\zeta_1 = 1$ and $\zeta_2 = 0$. However, $\zeta = C_0$ describes the wall W^ζ given by the boundary of the ample cone containing $C_0 + nF$. Hence, there are no walls and the result follows. □

Proposition 3.6. *Let c_1 be one of the reduced classes, and let $c_2 = 1$. Then, in the cases where $c_1 = 0$, or $c_1 = -F$ and $n \neq 0$, there is only one chamber of type $(c_1, 1)$ in $\mathbf{C}_{\mathbb{F}_n}$. Otherwise, there are exactly two chambers of type $(c_1, 1)$.*

Proof. Again, from Remark 3.4, we see that when $c_1 = 0$, there is only one solution to the inequality when $n = 1$, given by $\zeta_1 = 2$ and $\zeta_2 = 0$, but this solution describes the wall W^ζ that is the boundary containing $C_0 + F$. Hence, there are no walls. When $c_1(E) = -F$ and $n \neq 0$, we see there are no values of ζ_1, ζ_2 that satisfy the conditions, and so there are again no walls. Now, when $c_1 = -F$ and $n = 0$, $\zeta_1 = 2$ and $\zeta_2 = -1$ satisfy the conditions, and so we get the wall

$$W^\zeta = \{x_1 C_0 + x_2 F \in \mathbf{C}_{\mathbb{F}_n} \ : \ x_1 = 2x_2\},$$

yielding two chambers. When $c_1(E) = -C_0$, the only possible values for ζ_1, ζ_2 satisfying the conditions are $\zeta_1 = 1$ and $\zeta_2 = 0, -2$. However, note that $\zeta = C_0$ describes the boundary of the ample cone containing $C_0 + nF$, so we only need to take $\zeta = C_0 - 2F$ into account. This ζ describes the wall

$$W^\zeta = \{x_1 C_0 + x_2 F \in \mathbf{C}_{\mathbb{F}_n} \ : \ x_2 = (n+2)x_1\},$$

and so there are two chambers. Finally, when $c_1 = -C_0 - F$, the only possible values for ζ_1, ζ_2 satisfying the conditions are $\zeta_1 = 1$ and $\zeta_2 = -1$. This ζ describes the wall

$$W^\zeta = \{x_1 C_0 + x_2 F \in \mathbf{C}_{\mathbb{F}_n} \ : \ x_2 = (n+1)x_1\},$$

and so there are again exactly two chambers. □

Note that we could continue to play the same game for larger values of c_2; however, even for $n = 0$ (which is the simplest case to consider), this becomes an arduous task. Fix $n = 0$ and $c_1 = -F$ for the moment; it is clear that, every time c_2 increases, the number of walls increases as well (as it will always include the previous ones). Hence, for example, for $c_2 = 3$ we obtain three walls of type $(-F, 3)$, for $c_2 = 4$ we obtain four walls of type $(-F, 4)$, for $c_2 = 5$ we obtain six walls of type $(-F, 5)$, for $c_2 = 6$, we obtain eight walls of type $(-F, 6)$, and so on. However, there is no obvious pattern to predict how many walls we will get for an arbitrary value of c_2, as this depends on the divisibility of the number $4c_2$. For this reason, this approach does not provide a strategic way of dealing with the problem. Nonetheless, using the results of Propositions 3.5 and 3.6, as well as Theorem 3.3, the following section deals with non-emptiness results for the moduli spaces, for arbitrary polarizations, where c_1 is one of the reduced classes and $c_2 \leq 1$. After that, we fix the standard polarization and work with arbitrary c_1 and c_2.

3.3 Non-emptiness of Moduli Spaces

This section is divided into two parts. While in the first one we discuss the existence of semistable co-Higgs pairs for reduced first Chern classes, in the second one we work with arbitrary c_1. For most of part one, we will work with arbitrary polarizations; however, we fix the standard polarization for the second part. From now on, we will denote the moduli space of rank 2 H-semistable co-Higgs bundles with fixed Chern classes c_1 and c_2 by $\mathcal{M}^{\mathrm{co}}_{\mathrm{H}}(c_1, c_2)$.

For a decomposable bundle E, let us give the general shape of an element in $\mathrm{H}^0(\mathrm{End}_0\, E \otimes T)$. We do so in the following remark, so that we can later refer to it.

Remark 3.7. Let $E = \mathcal{O}(a_1C_0 + b_1F) \oplus \mathcal{O}(a_2C_0 + b_2F)$ be a decomposable rank 2 bundle over $\mathbb{F}_n$, and recall that $T = \mathcal{O}(2F) \oplus \mathcal{O}(2C_0 + nF)$. Then, any element of $\mathrm{H}^0(\mathrm{End}_0 E \otimes T)$ is of the form $\Phi = \Phi_1 + \Phi_2$, with $\Phi_1 \in \mathrm{H}^0(\mathrm{End}_0 E \otimes \mathcal{O}(2F))$ and $\Phi_2 \in \mathrm{H}^0(\mathrm{End}_0 E \otimes \mathcal{O}(2C_0 + nF))$. More explicitly,

$$\Phi = \Phi_1 + \Phi_2 = \begin{pmatrix} A_1 & B_1 \\ C_1 & -A_1 \end{pmatrix} + \begin{pmatrix} A_2 & B_2 \\ C_2 & -A_2 \end{pmatrix},$$

with $A_1 \in \mathrm{H}^0(\mathcal{O}(2F))$, $B_1 \in \mathrm{H}^0(\mathcal{O}((a_1 - a_2)C_0 + (2 + b_1 - b_2)F))$, $C_1 \in \mathrm{H}^0(\mathcal{O}((a_2 - a_1)C_0 + (2 + b_2 - b_1)F))$ and $A_2 \in \mathrm{H}^0(\mathcal{O}(2C_0 + nF))$, $B_2 \in \mathrm{H}^0(\mathcal{O}((2 + a_1 - a_2)C_0 + (n + b_1 - b_2)F))$, $C_2 \in \mathrm{H}^0(\mathcal{O}((2 + a_2 - a_1)C_0 + (n + b_2 - b_1)F))$.

3.3.1 Non-emptiness for reduced first Chern class

In this section, we work only with reduced first Chern classes. We will show that the necessary conditions imposed on c_2 to guarantee the existence of semistable co-Higgs pairs in $\mathcal{M}^{\mathrm{co}}_{\mathrm{H}}(c_1, c_2)$ presented in Theorem 3.3 are indeed sufficient.

Second Chern class $c_2 \leq 0$.

Theorem 3.8. *Let H be any polarization. Suppose that c_1 is a reduced first Chern class and that $c_2 \leq 0$. Then, $\mathcal{M}^{co}_{H}(c_1, c_2)$ is non-empty if and only if c_2 satisfies the following conditions:*

1. *$c_2 = 0$ when $c_1 = 0, -F$.*
2. *$-\frac{n}{2} \leq c_2 \leq 0$ when $c_1 = -C_0$.*
3. *$-\frac{(n-1)}{2} \leq c_2 \leq 0$ when $c_1 = -C_0 - F$.*

In fact, $\mathcal{M}^{co}_{H}(c_1, c_2)$ contains non-trivial stable co-Higgs pairs (with decomposable underlying bundle) in all three cases.

Proof. The forward direction follows from Theorem 3.3. For the converse, recall that, by Propostion 3.5, there is only one chamber, and so it suffices to work with the standard polarization $H = C_0 + (n + 1)F$. We consider each case separately.

Case 1. (a) Suppose $c_1 = 0$ and $c_2 = 0$. Take, for example, $E = \mathcal{O}(F) \oplus \mathcal{O}(-F)$ together with a Higgs field of the form

$$\Phi = \Phi_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & -A_1 \end{pmatrix},$$

with $A_1 \in \mathrm{H}^0(\mathcal{O}(2F))$, $B_1 \in \mathrm{H}^0(\mathcal{O}(4F))$ and non-zero $C_1 \in \mathrm{H}^0(\mathcal{O})$. The only destabilizing sub-line bundle of E is $\mathcal{O}(F)$ (see Proposition 2.10). It follows that (E, Φ) is H-stable, as $C_1 \neq 0$, and so $\mathcal{O}(F)$ is not Φ-invariant. Thus, $(E, \Phi) \in \mathcal{M}^{\mathrm{co}}_{\mathrm{H}}(0, 0)$.

Case 1. (b) Suppose $c_1 = -F$ and $c_2 = 0$. Take $E = \mathcal{O} \oplus \mathcal{O}(-F)$ and a Higgs field of the form

$$\Phi = \Phi_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & -A_1 \end{pmatrix},$$

with $A_1 \in \mathrm{H}^0(\mathcal{O}(2F))$, $B_1 \in \mathrm{H}^0(\mathcal{O}(3F))$ and non-zero $C_1 \in \mathrm{H}^0(\mathcal{O}(F))$. The only destabilizing sub-line bundle of E is $\mathcal{O}$ (see Proposition 2.10). It follows that (E, Φ) is H-stable, as $C_1 \neq 0$, and so $\mathcal{O}$ is not Φ-invariant. Thus, $(E, \Phi) \in \mathcal{M}^{\mathrm{co}}_{\mathrm{H}}(-F, 0)$.

Case 2. Suppose $c_1 = -C_0$ and $-\frac{n}{2} \leq c_2 \leq 0$. Let $\gamma = c_2$ and consider $E = \mathcal{O}(-\gamma F) \oplus \mathcal{O}(-C_0 + \gamma F)$. Following the notation of Remark 3.7, any element of $\mathrm{H}^0(\mathrm{End}_0\, E \otimes T)$ has the form

$$\Phi = \Phi_1 + \Phi_2 = \begin{pmatrix} A_1 & B_1 \\ 0 & -A_1 \end{pmatrix} + \begin{pmatrix} A_2 & B_2 \\ C_2 & -A_2 \end{pmatrix},$$

with $A_1 \in \mathrm{H}^0(\mathcal{O}(2F))$, $B_1 \in \mathrm{H}^0(\mathcal{O}(C_0 + (2 - 2\gamma)F))$ and $A_2 \in \mathrm{H}^0(\mathcal{O}(2C_0 + nF))$, $B_2 \in \mathrm{H}^0(\mathcal{O}(3C_0 + (n - 2\gamma)F))$ and $C_2 \in \mathrm{H}^0(\mathcal{O}(C_0 + (n + 2\gamma)F))$. Any non-zero C_2 (which exists since $n + 2\gamma \geq 0$) will not leave $\mathcal{O}(-\gamma F)$ Φ-invariant (by Proposition 2.10, this is the only destabilizing sub-line bundle of E). Taking the integrability condition into account, equations (2.8) imply that $A_1 = B_1 = 0$, and so E together with any Higgs field $\Phi = \Phi_2$ with non-zero C_2 yields a stable pair.

Case 3. Suppose $c_1 = -C_0 - F$ and $-\frac{(n-1)}{2} \leq c_2 \leq 0$. Let $\gamma = c_2$ and consider $E = \mathcal{O}(-\gamma F) \oplus \mathcal{O}(-C_0 + (\gamma - 1)F)$. Again, following the notation of Remark 3.7, any element of $\mathrm{H}^0(\mathrm{End}_0\, E \otimes T)$ is of the form

$$\Phi = \Phi_1 + \Phi_2 = \begin{pmatrix} A_1 & B_1 \\ 0 & -A_1 \end{pmatrix} + \begin{pmatrix} A_2 & B_2 \\ C_2 & -A_2 \end{pmatrix},$$

with $A_1 \in \mathrm{H}^0(\mathcal{O}(2F))$, $B_1 \in \mathrm{H}^0(\mathcal{O}(C_0 + (3 - 2\gamma)F))$ and $A_2 \in \mathrm{H}^0(\mathcal{O}(2C_0 + nF))$, $B_2 \in \mathrm{H}^0(\mathcal{O}(3C_0 + (n + 1 - 2\gamma)F))$ and $C_2 \in \mathrm{H}^0(\mathcal{O}(C_0 + (n - 1 + 2\gamma)F))$. Any non-zero C_2 (which exists since $2\gamma + n - 1 \geq 0$) would not leave $\mathcal{O}(-\gamma F)$ Φ-invariant (by Proposition 2.10, this is the only destabilizing sub-line bundle of E). Taking the integrability condition into account, equations (2.8) imply that $A_1 = B_1 = 0$, and so E together with any Higgs field $\Phi = \Phi_2$ with non-zero C_2 yields a stable pair.

□

Second Chern class $c_2 = 1$.

Theorem 3.9. *Let H be any polarization. Suppose that c_1 is a reduced first Chern class and that $c_2 = 1$. Then, the moduli space $\mathcal{M}^{co}_H(c_1, 1)$ is non-empty and contains non-trivial semistable pairs. Moreover, in each of the following cases, there is a non-trivial stable co-Higgs pair.*

1. *For $c_1 = 0$, whenever n is odd.*
2. *For $c_1 = -F$, whenever H does not lie on a wall.*
3. *For $c_1 = -C_0$, whenever H does not lie on a wall, or $n \geq 2$.*
4. *For $c_1 = -C_0 - F$, whenever H does not lie on a wall, or $n \geq 1$.*

Proof. By Proposition 3.6, in each case, there are at most two chambers of type $(c_1, 1)$ to consider. Whenever there is a wall, we use the notation $\mathcal{C}_+, \mathcal{C}_-$ to denote the chambers above and below the wall, respectively. $\mathcal{C}_+$ corresponds to the chamber $\mathcal{C}_F$ containing F on the boundary, and $\mathcal{C}_-$ corresponds to the chamber containing $C_0 + nF$ on the boundary. We treat each reduced first Chern class separately.

Case 1. Suppose $c_1 = 0$. We know, by Proposition 3.6, that we may pick any polarization, and so we fix $H = C_0 + (n+1)F$. Consider a bundle E with numerical invariants $d = 0$, $r = 0$ (and with $c_1 = 0$ and $c_2 = 1$). Note that such a bundle exists by Theorem 2.21, and fits into an exact sequence of the form:

$$0 \to \mathcal{O} \to E \to I_x \to 0, \tag{3.3}$$

with x a point in $\mathbb{F}_n$. We show that E is semistable and that it admits non-trivial Higgs fields. The fact that E is semistable follows immediately since any possible sub-line bundle of E has slope at most 0 (and $\mathcal{O}$ is indeed a sub-line bundle of E). To see that any such E admits non-trivial Higgs fields, we prove that $\mathrm{H}^0(\mathrm{End}_0 E(2F))$ is non-zero. To see this, note that we can take the dual sequence of (3.3), tensor it by $E(2F)$ and pass to the long exact sequence in cohomology to get

$$0 \to \mathrm{H}^0(E(2F)) \to \mathrm{H}^0(\mathrm{End}\, E(2F)) \to \mathrm{H}^0(E(2F) \otimes I_x) \to \dots$$

Hence, it is enough to show that $\mathrm{h}^0(E(2F)) \geq 4$, as then $\mathrm{h}^0(\mathrm{End}\, E(2F)) \geq 4$ as well, and so $\mathrm{h}^0(\mathrm{End}_0\, E(2F)) = \mathrm{h}^0(\mathrm{End}\, E(2F)) - \mathrm{h}^0(\mathcal{O}(2F)) \geq 1$. To see that $\mathrm{h}^0(E(2F)) \geq 4$, start by tensoring (3.3) by $\mathcal{O}(2F)$ to get

$$0 \to \mathrm{H}^0(\mathcal{O}(2F)) \to \mathrm{H}^0(E(2F)) \to \mathrm{H}^0(I_x(2F)) \to \mathrm{H}^1(\mathcal{O}(2F)) \to \dots$$

One can easily see that $\mathrm{h}^0(\mathcal{O}(2F)) = 3$, $\mathrm{h}^1(\mathcal{O}(2F)) = 0$ and that $\mathrm{h}^0(I_x(2F)) = 2$, and so $\mathrm{h}^0(E(2F)) = 5$.

For the moreover part, assume n is odd and consider the bundle $\mathcal{O}\left(C_0 + \left(\frac{n-1}{2}\right) F\right) \oplus \mathcal{O}\left(-C_0 - \left(\frac{n-1}{2}\right) F\right)$ together with a Higgs field of the form

$$\Phi = \Phi_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & -A_2 \end{pmatrix},$$

with $A_2 \in \mathrm{H}^0(\mathcal{O}(2C_0 + nF))$, $B_2 \in \mathrm{H}^0(\mathcal{O}(4C_0 + (2n-1)F))$ and non-zero $C_2 \in \mathrm{H}^0(\mathcal{O}(C_0 + F))$. In Proposition A.1 we show that such a co-Higgs pair is indeed stable.

Case 2. (a) Suppose $c_1 = -F$ and $n \neq 0$. We know, by Proposition 3.6, that there is only one chamber, and so we work with the standard polarization $H = C_0 + (n+1)F$. Consider the bundle $E = \mathcal{O}(-C_0) \oplus \mathcal{O}(C_0 - F)$ together with a Higgs field of the form

$$\Phi = \Phi_2 = \begin{pmatrix} A_2 & B_2 \\ 0 & -A_2 \end{pmatrix},$$

with $A_2 \in \mathrm{H}^0(\mathcal{O}(2C_0))$ and non-zero $B_2 \in \mathrm{H}^0(\mathcal{O}(F))$. It is easy to verify that (E, Φ) is H-stable, as $\mathcal{O}(C_0 - F)$ is not Φ-invariant.

Case 2. (b) Suppose $c_1 = -F$ and $n = 0$. We know, by Proposition 3.6, that there are exactly two chambers. The bundle $E = \mathcal{O}(-C_0) \oplus \mathcal{O}(C_0 - F)$ admits Higgs fields Φ_+ and

Φ_- such that $(E, \Phi_+) \in \mathcal{M}^{\mathrm{co}}_{\mathcal{C}_+}(-F, 1)$ is stable, and $(E, \Phi_-) \in \mathcal{M}^{\mathrm{co}}_{\mathcal{C}_-}(-F, 1)$ is also stable. Indeed, take

$$\Phi_+ = \begin{pmatrix} A_1 & 0 \\ C_1 & -A_1 \end{pmatrix},$$

with $A_1 \in \mathrm{H}^0(\mathcal{O}(2F))$ and non-zero $C_1 \in \mathrm{H}^0(\mathcal{O}(2C_0 + 2F))$. One can easily check that (E, Φ_+) is $(3C_0 + F)$-stable, where $3C_0 + F \in \mathcal{C}_+$, as $\mathcal{O}(-C_0)$ is not Φ_+-invariant (by picking non-zero C_1). Now, take

$$\Phi_- = \begin{pmatrix} A_2 & B_2 \\ 0 & -A_2 \end{pmatrix},$$

with $A_2 \in \mathrm{H}^0(\mathcal{O}(2C_0))$ and non-zero $B_2 \in \mathrm{H}^0(\mathcal{O}(F))$. Again, it is easy to verify that (E, Φ_-) is $(C_0 + F)$-stable, where $C_0 + F \in \mathcal{C}_-$, as $\mathcal{O}(C_0 - F)$ is not Φ_--invariant.

Now, if we pick a polarization lying on a wall, $2C_0 + F$ say, any co-Higgs pair with underlying bundle E, which is of the form (E, Φ_+) or (E, Φ_-), is strictly $(2C_0 + F)$-semistable. This follows since both $\mathcal{O}(-C_0)$ and $\mathcal{O}(C_0 - F)$ have the same $(2C_0 + F)$-slope as E (namely -1), and one of them will necessarily be invariant under the Higgs field.

Case 3. Suppose $c_1 = -C_0$. We know, by Proposition 3.6, that there are exactly two chambers. The bundle $E = \mathcal{O}(-F) \oplus \mathcal{O}(-C_0 + F)$ admits Higgs fields Φ_+ and Φ_- such that $(E, \Phi_+) \in \mathcal{M}^{\mathrm{co}}_{\mathcal{C}_+}(-C_0, 1)$ is stable, and $(E, \Phi_-) \in \mathcal{M}^{\mathrm{co}}_{\mathcal{C}_-}(-C_0, 1)$ is stable as well. Indeed, take

$$\Phi_+ = \begin{pmatrix} A_2 & B_2 \\ C_2 & -A_2 \end{pmatrix},$$

with $A_2 \in \mathrm{H}^0(\mathcal{O}(2C_0 + nF))$, $B_2 \in \mathrm{H}^0(\mathcal{O}(3C_0 + (n-2)F))$ and non-zero $C_2 \in \mathrm{H}^0(\mathcal{O}(C_0 + (n+2)F))$. One can easily check that (E, Φ_+) is $(C_0 + (n+3)F)$-stable, where $C_0 + (n+3)F \in \mathcal{C}_+$, as $\mathcal{O}(-F)$ is not Φ_+-invariant (by picking non-zero C_2). Now, take

$$\Phi_- = \begin{pmatrix} A_1 & B_1 \\ 0 & -A_1 \end{pmatrix},$$

with $A_1 \in \mathrm{H}^0(\mathcal{O}(2F))$ and non-zero $B_1 \in \mathrm{H}^0(\mathcal{O}(C_0))$. Again, it is easy to verify that (E, Φ_-) is $(C_0 + (n+1)F)$-stable, where $C_0 + (n+1)F \in \mathcal{C}_-$, as $\mathcal{O}(-C_0 + F)$ is not Φ_--invariant.

Now, if we pick a polarization lying on a wall, $C_0 + (n+2)F$ say, any possible co-Higgs pair with underlying bundle E, which is of the form (E, Φ_+) or (E, Φ_-), is strictly $(C_0 + (n+2)F)$-semistable, unless $n \geq 2$. As both $\mathcal{O}(-F)$ and $\mathcal{O}(-C_0 + F)$ have the same $(C_0 + (n+2)F)$-slope as E (namely, -1). When $n \geq 2$, the pairs (E, Φ_+) will not leave these two sub-line bundles invariant, as long as we pick non-zero B_2 and C_2.

Case 4. Suppose $c_1 = -C_0 - F$. We know, by Proposition 3.6, that there are exactly two chambers. The bundle $E = \mathcal{O}(-C_0) \oplus \mathcal{O}(-F)$ admits Higgs fields Φ_+ and Φ_- such that $(E, \Phi_+) \in \mathcal{M}^{\mathrm{co}}_{\mathcal{C}_+}(-C_0 - F, 1)$ is stable, and $(E, \Phi_-) \in \mathcal{M}^{\mathrm{co}}_{\mathcal{C}_-}(-C_0 - F, 1)$ is stable as well. Indeed, take

$$\Phi_+ = \begin{pmatrix} A_2 & B_2 \\ C_2 & -A_2 \end{pmatrix},$$

with $A_2 \in \mathrm{H}^0(\mathcal{O}(2C_0{+}nF))$, non-zero $B_2 \in \mathrm{H}^0(\mathcal{O}(C_0{+}(n{+}1)F))$ and $C_2 \in \mathrm{H}^0(\mathcal{O}(3C_0{+}(n{-}1)F))$. One can easily check that (E, Φ_+) is $(C_0{+}(n{+}2)F)$-stable, where $C_0{+}(n{+}2)F \in \mathcal{C}_+$, as $\mathcal{O}(-F)$ is not Φ_+-invariant (by picking non-zero B_2). Now, take

$$\Phi_- = \begin{pmatrix} A_1 & 0 \\ C_1 & -A_1 \end{pmatrix},$$

with $A_1 \in \mathrm{H}^0(\mathcal{O}(2F))$ and non-zero $C_1 \in \mathrm{H}^0(\mathcal{O}(C_0 + F))$. Again, it is easy to verify that (E, Φ_-) is $(2C_0 + (2n+1)F)$-stable, where $2C_0 + (2n+1)F \in \mathcal{C}_-$, as $\mathcal{O}(-C_0)$ is not Φ_--invariant.

Now, if we pick a polarization lying on a wall, $C_0 + (n+1)F$ say, any possible co-Higgs pair with underlying bundle E, which is of the form (E, Φ_+) or (E, Φ_-), is strictly $(C_0 + (n+1)F)$-semistable, unless $n \geq 1$. This is as both $\mathcal{O}(-C_0)$ and $\mathcal{O}(-F)$ have the same $(C_0 + (n+1)F)$-slope as E (namely -1). When $n \geq 1$, the pairs (E, Φ_-) will not leave these two sub-line bundles invariant, as long as we pick non-zero B_2 and C_2. □

Second Chern class $c_2 \geq 2$.

As we had mentioned before, as c_2 becomes larger, it becomes increasingly difficult to work with walls and chambers, and thus we do not attempt to approach this problem in general. However, we can characterize the non-emptiness of the moduli spaces $\mathcal{M}^{\mathrm{co}}_{\mathrm{H}}(c_1, c_2)$, when H is the standard polarization.

We show that whenever the moduli space $\mathcal{M}^{\mathrm{co}}_{\mathrm{H}}(c_1, c_2)$ is non-empty, it actually contains a stable bundle E, which we can equip with the zero Higgs field in order to yield a stable co-Higgs pair. However, we want to be able to exhibit non-trivial co-Higgs bundles, showing that these objects do constitute an enlargement of the class of semistable bundles. Note that we have already dealt with this issue in the proof of Proposition 3.9, and in that case, it was fairly straightforward to show that the bundles in question had non-trivial Higgs fields. In general, if E is in an exact sequence of the form

$$0 \to L_1 \to E \to L_2 \otimes I_Z \to 0, \tag{3.4}$$

and we want to prove that it admits non-zero Higgs fields $\Phi \in \mathrm{H}^0(\mathrm{End}_0\, E \otimes T)$, it suffices to show that $\mathrm{H}^0(\mathrm{End}_0\, E(2F)) \neq 0$ or $\mathrm{H}^0(\mathrm{End}_0\, E(2C_0 + nF)) \neq 0$ (since $T = \mathcal{O}(2F) \oplus \mathcal{O}(2C_0 + nF)$). We prove that $\mathrm{H}^0(\mathrm{End}_0\, E(2F)) \neq 0$. Taking the dual of (3.4), tensoring it by $E(2F)$ and passing to the long exact sequence in cohomology, it is clear that, if one can prove that $\mathrm{H}^0(L_2^\vee \otimes E(2F)) \geq 4$ as in Proposition 3.9, the result follows. However, in general, this is too strong of a condition and we can prove that a much simpler condition suffices to guarantee that $\mathrm{H}^0(\mathrm{End}_0\, E(2F)) \neq 0$. We first recall a basic fact about modules that will be used in Proposition 3.11.

Lemma 3.10. *Let*

$$0 \to A \xrightarrow{\iota} B_1 \oplus B_2 \xrightarrow{p} C$$

be an exact sequence of R-modules. If $A \neq 0$ and $p_{|B_2}$ is injective, then $B_1 \neq 0$.

Proposition 3.11. *Let E be a rank 2 vector bundle over $\mathbb{F}_n$ that fits into the exact sequence*

$$0 \to L_1 \xrightarrow{\iota} E \xrightarrow{p} L_2 \otimes I_Z \to 0. \tag{3.5}$$

If $\mathrm{H}^0(L_2^\vee \otimes E(2F)) \neq 0$, then $\mathrm{H}^0(\mathrm{End}_0\, E(2F)) \neq 0$.

Proof. Start by taking the dual of the exact sequence (3.5), tensor it by $E(2F)$ and pass to the exact sequence in cohomology to get

$$0 \to \mathrm{H}^0(L_2^\vee \otimes E(2F)) \to \mathrm{H}^0(\operatorname{End} E(2F)) \to \mathrm{H}^0(L_1^\vee \otimes E(2F) \otimes I_Z) \to \dots$$

The map $\mathrm{H}^0(\operatorname{End} E \otimes \mathcal{O}(2F)) \to \mathrm{H}^0(L_1^\vee \otimes E \otimes I_Z \otimes \mathcal{O}(2F))$ is the induced one from

$$\epsilon \otimes \mathrm{Id}_{\mathcal{O}(2F)} : \operatorname{End} E \otimes \mathcal{O}(2F) \to (L_1^\vee \otimes E \otimes I_Z) \otimes \mathcal{O}(2F),$$

where ϵ takes h to $h \circ \iota$. Writing $\operatorname{End} E = \operatorname{End}_0 E \oplus \mathcal{O}$, and noting that Id_E generates $\mathcal{O}$ in $\operatorname{End} E$, we get that the map induced by $\epsilon \otimes \mathrm{Id}_{\mathcal{O}(2F)}$, restricted to $\mathrm{H}^0(\mathcal{O} \otimes \mathcal{O}(2F))$, is injective. By Lemma 3.10, $\mathrm{H}^0(\operatorname{End}_0 E(2F)) \neq 0$, as desired. □

We now prove that the bundles that will serve as examples of underlying bundles for the non-trivial co-Higgs pairs in Theorem 3.13 are indeed stable.

Lemma 3.12. *Let E be such that it either fits into the exact sequence*

$$0 \to \mathcal{O}(-F) \to E \to I_Z \to 0,$$

with $\ell(Z) \geq 1$, or

$$0 \to \mathcal{O}(-F) \to E \to \mathcal{O}(-C_0 + F) \otimes I_Z \to 0,$$

with $\ell(Z) \geq 1$. Then E is stable.

Proof. In the first case, we prove that E is stable by showing that, if there is a non-zero map $\mathcal{O}(aC_0 + bF) \to E$, then

$$\mu(\mathcal{O}(aC_0 + bF)) = a + b < -\frac{1}{2} = \mu(E).$$

We have

$$0 \to \mathcal{O}(-aC_0 - (b+1)F) \to E(-aC_0 - bF) \to I_Z(-aC_0 - bF) \to 0,$$

and

$$0 \to I_Z(-aC_0 - bF) \to \mathcal{O}(-aC_0 - bF) \to \mathcal{O}/I_Z \to 0.$$

First note that if $a > 0$ or $b > 0$, then $\mathrm{H}^0(\mathcal{O}(-aC_0 - (b+1)F)) = 0$ and $\mathrm{H}^0(I_Z(-aC_0 - bF)) = 0$ since $\mathrm{H}^0(\mathcal{O}(-aC_0 - bF)) = 0$, in which case $\mathrm{H}^0(E(-aC_0 - bF)) = 0$. We thus assume $a, b \leq 0$. But then

$$\mu(\mathcal{O}(aC_0 + bF)) \leq -1 < \mu(E),$$

unless $a = b = 0$. Moreover, if $a = b = 0$, then $\mathrm{H}^0(\mathcal{O}(-F)) = \mathrm{H}^0(I_Z) = 0$, so that $\mathrm{H}^0(E) = 0$, implying that there are no non-zero maps $\mathcal{O} \to E$. Consequently, if $\mathrm{H}^0(E(-aC_0 - bF)) \neq 0$, then $\mu(\mathcal{O}(aC_0 + bF)) < \mu(E)$.

Similarly, in the second case, one can show that if $\mu(\mathcal{O}(aC_0 + bF)) = a + b \geq 0$, then $\mathrm{H}^0(E(-aC_0 - bF)) = 0$; otherwise we have

$$\mu(\mathcal{O}(aC_0 + bF)) \leq -1 < \mu(E),$$

so E is in fact stable. □

We can now prove:

Theorem 3.13. *Let H be the standard polarization. Suppose that c_1 is a reduced first Chern class and that $c_2 \geq 2$. Then, the moduli space $\mathcal{M}_H^{co}(c_1, c_2)$ is non-empty and it contains non-trivial stable co-Higgs pairs.*

Proof. To see that $\mathcal{M}_{\mathrm{H}}^{\mathrm{co}}(0, c_2) \neq \emptyset$, consider a rank-2 vector bundle E with $c_1(E) = 0$, $c_2(E) \geq 2$, and numerical invariants $d = r = 0$ (such a bundle exists by Theorem 2.21). We can write E as

$$0 \to \mathcal{O} \to E \to I_Z \to 0$$

with $\ell(Z) = c_2$. Since $\mu(E) = \mu(\mathcal{O}) = 0$, E is semistable (see the "more precisely" clause of Proposition 2.12). Thus, it suffices to show that $\mathrm{H}^0(\mathrm{End}_0 E(2F)) \neq 0$. This follows from Proposition 3.11, as $\mathrm{H}^0(E(2F)) \neq 0$.

To see that $\mathcal{M}_{\mathrm{H}}^{\mathrm{co}}(-F, c_2) \neq \emptyset$, consider a rank-2 vector bundle E with $c_1(E) = -F$, $c_2(E) \geq 2$, and numerical invariants $d = 0$, $r = -1$ (such a bundle exists by Theorem 2.21). We can write E as

$$0 \to \mathcal{O}(-F) \to E \to I_Z \to 0$$

with $\ell(Z) = c_2$. We proved in Lemma 3.12 that such an E is stable. Thus, it suffices to show that $\mathrm{H}^0(\mathrm{End}_0 E(2F)) \neq 0$. This follows from Proposition 3.11, as $\mathrm{H}^0(E(2F)) \neq 0$.

To see that $\mathcal{M}_{\mathrm{H}}^{\mathrm{co}}(-C_0, c_2) \neq \emptyset$, consider a rank-2 vector bundle E with $c_1(E) = -C_0$, $c_2(E) \geq 2$, and numerical invariants $d = 0$, $r = -1$ (such a bundle exists by Theorem 2.21). We can write E as

$$0 \to \mathcal{O}(-F) \to E \to \mathcal{O}(-C_0 + F) \otimes I_Z \to 0$$

with $\ell(Z) = c_2 - 1$. Again, we proved in Lemma 3.12 that such an E is stable. Thus, it suffices to show that $\mathrm{H}^0(\mathrm{End}_0 E(2F)) \neq 0$. This follows from Proposition 3.11, as $\mathrm{H}^0(E(2F)) \neq 0$.

Finally, to see that $\mathcal{M}_{\mathrm{H}}^{\mathrm{co}}(-C_0 - F, c_2) \neq \emptyset$, consider a rank-2 vector bundle E with $c_1(E) = -C_0 - F$, $c_2(E) \geq 2$, and numerical invariants $d = 0$, $r = -1$ (such a bundle exists by Theorem 2.21). We can write E as

$$0 \to \mathcal{O}(-F) \to E \to \mathcal{O}(-C_0) \otimes I_Z \to 0$$

with $\ell(Z) = c_2(E) - 1$. Since $\mu(E) = \mu(\mathcal{O}(-F)) = -1$, E is semistable (see the "more precisely" clause of Proposition 2.12). Thus, it suffices to show that $\mathrm{H}^0(\mathrm{End}_0 E(2F)) \neq 0$. This follows from Proposition 3.11, as $\mathrm{H}^0(E(2F)) \neq 0$. □

3.3.2 Non-emptiness for arbitrary first Chern class

In this section, we work with arbitrary first Chern class c_1. For the standard polarization, we give a complete characterization of when $\mathcal{M}_{\mathrm{H}}^{\mathrm{co}}(c_1, c_2)$ is non-empty. Before doing so, let us fix the second Chern class to be 0. Then, one can say something about the non-emptiness of the moduli space $\mathcal{M}_{\mathrm{H}}^{\mathrm{co}}(c_1, 0)$ for arbitrary H. The next proposition addresses this, by giving necessary conditions for the existence of stable co-Higgs bundles.

Proposition 3.14. *Let H be any polarization and $c_2 = 0$. Then $\mathcal{M}_H^{co}(c_1, 0)$ is non-empty if c_1 satisfies one of the following three conditions:*

1. *$c_1 \cdot H > 0$, and $2F - c_1 \geq 0$ or $2C_0 + nF - c_1 \geq 0$.*
2. *$c_1 \cdot H < 0$, and $2F + c_1 \geq 0$ or $2C_0 + nF + c_1 \geq 0$.*
3. *$c_1 \cdot H = 0$, and $-2C_0 - nF \leq c_1 \leq 2C_0 + nF$.*

Moreover, if any of the above holds, we can, in fact, find a non-trivial stable co-Higgs pair.

Proof. Let $c_1 = \alpha C_0 + \beta F$ and consider $E = \mathcal{O} \oplus \mathcal{O}(\alpha C_0 + \beta F)$. Any element of $\mathrm{H}^0(\mathrm{End}_0 E \otimes T)$ is of the form

$$\Phi = \Phi_1 + \Phi_2 = \begin{pmatrix} A_1 & B_1 \\ C_1 & -A_1 \end{pmatrix} + \begin{pmatrix} A_2 & B_2 \\ C_2 & -A_2 \end{pmatrix},$$

with $A_1 \in \mathrm{H}^0(\mathcal{O}(2F))$, $B_1 \in \mathrm{H}^0(\mathcal{O}(2F - c_1))$, $C_1 \in \mathrm{H}^0(\mathcal{O}(2F + c_1))$ and $A_2 \in \mathrm{H}^0(\mathcal{O}(2C_0 + nF))$, $B_2 \in \mathrm{H}^0(\mathcal{O}(2C_0 + nF - c_1))$, $C_2 \in \mathrm{H}^0(\mathcal{O}(2C_0 + nF + c_1))$. In the first case, the condition $c_1 \cdot H > 0$ yields that $\mathcal{O}(\alpha C_0 + \beta F)$ is the destabilizing sub-line bundle of E. Moreover, since either $2F - c_1$ or $2C_0 + nF - c_1$ is effective, we get that either B_1 or B_2 is non-zero (possibly both), and so it is possible to choose Φ such that it is integrable and $\mathcal{O}(\alpha C_0 + \beta F)$ is not Φ-invariant. Hence, at least one of the pairs (E, Φ_1) or (E, Φ_2) is stable.

In the second case, the condition $c_1 \cdot H < 0$ yields that $\mathcal{O}$ is the destabilizing sub-line bundle of E. This time, since either $2F + c_1$ or $2C_0 + nF + c_1$ is effective, we get that either C_1 or C_2 is non-zero (possibly both), and so it is possible to choose Φ such that it is integrable and $\mathcal{O}$ is not Φ-invariant. Hence, at least one of the pairs (E, Φ_1) or (E, Φ_2) is stable.

Finally, in the third case, the condition $c_1 \cdot H = 0$ tells us that $\mu(E) = \mu(\mathcal{O}) = \mu(\mathcal{O}(\alpha C_0 + \beta F)) = 0$. Consequently, E is semistable by Lemma 2.22. Moreover, $\mathcal{O}$ and $\mathcal{O}(\alpha C_0 + \beta F)$ are the only sub-line bundles of E of slope 0 (see Proposition 2.12). We need a Higgs field Φ such that these two are not Φ-invariant. The condition $-2C_0 - nF \leq c_1 \leq 2C_0 + nF$ assures that we can pick $\Phi = \Phi_2$ such that B_2 and C_2 are non-zero, so that $\mathcal{O}$ and $\mathcal{O}(\alpha C_0 + \beta F)$ are not Φ-invariant. Hence (E, Φ_2) is stable. □

We now fix H to be the standard polarization $C_0 + (n+1)F$, and give necessary and sufficient conditions on c_2 in order to guarantee that $\mathcal{M}_{\mathrm{H}}^{\mathrm{co}}(c_1, c_2)$ is non-empty.

Theorem 3.15. *Let $c_1 = \alpha C_0 + \beta F$ and $c_2 = \gamma$. Fix the standard polarization $H = C_0 + (n+1)F$. Then, the moduli space $\mathcal{M}_H^{co}(c_1, c_2)$ is non-empty (and moreover it contains a non-trivial co-Higgs pair) if and only if one of the following holds:*

1. *α and β are both even and $4\gamma \geq \alpha(2\beta - n\alpha)$;*
2. *α is even, β is odd and $4\gamma \geq \alpha(2\beta - n\alpha)$;*
3. *α is odd, β is even and $4\gamma \geq 2\alpha\beta - n(1 + \alpha^2)$;*

4. α and β are both odd and $4\gamma \geq 2(\alpha\beta - 2) - n(1+\alpha^2)$.

Proof. For the forward direction, take $E \in \mathcal{M}_{\mathrm{H}}^{\mathrm{co}}(c_1, c_2)$ and tensor it by the appropriate line bundle L, so that its first Chern class is one of the reduced ones (see the proof of Lemma 3.1). Then Corollary 3.2 tells us how the second Chern class changes after tensoring E by L. We can then apply Theorem 3.3 to obtain the desired result. The converse follows from the results in this section on the existence of semistable co-Higgs bundles with one of the reduced classes, see Theorems 3.8, 3.9 and 3.13. $\square$

We finish the chapter by showing that not every stable bundle E admits a non-zero Higgs field. We exhibit this phenomenon for vector bundles over $\mathbb{P}^1 \times \mathbb{P}^1$ with first Chern class $-F$. We work with the standard polarization $H = C_0 + F$.

First, note that if $d > 0$, $r \leq -1-d$, and $c_2 + d(1+2r) > 0$, then $M(-F, c_2, d, r) \neq \emptyset$ by Theorem 2.21. Moreover, by Theorem 2.23, every $E \in M(-F, c_2, d, r)$ is stable, as $(2dC_0 + (2r+1)F) \cdot (C_0 + F) < 0$. We now have:

Proposition 3.16. *Suppose that $d > 1$, $r \leq -1-d$ and $c_2 \geq 3 - d(1+2r)$, or that $d = 1$, $r \leq -2$ and $c_2 \geq -4r-1$. If $E \in M(-F, c_2, d, r)$, then E has no non-trivial Higgs fields.*

In order to prove the above proposition, we first show the following technical lemmas.

Lemma 3.17. *Let Z be a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Then*

$$\mathrm{H}^0(I_Z(2,0)) = \mathrm{H}^0(I_Z(0,2)) = \mathbb{C}^{3-\ell(Z)},$$

with the convention that $\mathbb{C}^q = 0$ whenever $q \leq 0$. Also,

$$\mathrm{H}^1(I_Z(2,0)) = \mathrm{H}^1(I_Z(0,2)) = \mathbb{C}^{\ell(Z)-3}.$$

Proof. Start by tensoring the exact sequence

$$0 \to I_Z \to \mathcal{O} \to \mathcal{O}_Z \to 0$$

by $\mathcal{O}(2,0)$, and pass to the long exact sequence in cohomology

$$\begin{array}{ccccccc} 0 & \to & \mathrm{H}^0(I_Z(2,0)) & \to & \mathrm{H}^0(\mathcal{O}(2,0)) & \to & \mathrm{H}^0(\mathcal{O}_Z) \\ & \to & \mathrm{H}^1(I_Z(2,0)) & \to & \mathrm{H}^1(\mathcal{O}(2,0)) & \to & \mathrm{H}^1(\mathcal{O}_Z) \\ & \to & \mathrm{H}^2(I_Z(2,0)) & \to & \mathrm{H}^2(\mathcal{O}(2,0)) & \to & \mathrm{H}^2(\mathcal{O}_Z) \quad \to 0. \end{array}$$

Now note that $\mathcal{O}_Z$ is a skyscraper sheaf supported at Z, and so $\mathrm{H}^0(\mathcal{O}_Z) \cong \mathbb{C}^{\ell(Z)}$ and $\mathrm{H}^1(\mathcal{O}_Z) = \mathrm{H}^2(\mathcal{O}_Z) = 0$. Also $\mathrm{H}^0(\mathcal{O}(2,0)) \cong \mathbb{C}^3$ and $\mathrm{H}^1(\mathcal{O}(2,0)) = \mathrm{H}^2(\mathcal{O}(2,0)) = 0$, so we have

$$0 \to \mathrm{H}^0(I_Z(2,0)) \to \mathbb{C}^3 \to \mathbb{C}^{\ell(Z)} \to \mathrm{H}^1(I_Z(2,0)) \to 0.$$

Since we can interpret elements of $\mathrm{H}^0(I_Z(2,0))$ as homogeneous polynomials of degree 2 that vanish at $\ell(Z)$ points, it is clear that $\mathrm{H}^0(I_Z(2,0)) \cong \mathbb{C}^{3-\ell(Z)}$, and as a consequence of exactness we get that $\mathrm{H}^1(I_Z(2,0)) \cong \mathbb{C}^{\ell(Z)-3}$.

The proof that $\mathrm{H}^0(I_Z(0,2)) \cong \mathbb{C}^{3-\ell(Z)}$ and $\mathrm{H}^1(I_Z(0,2)) \cong \mathbb{C}^{\ell(Z)-3}$ is identical; just replace $\mathcal{O}(2,0)$ by $\mathcal{O}(0,2)$ above. $\square$

Lemma 3.18. *Let $d > 0$, $r \leq -1-d$, and $c_2 \geq 3-d(1+2r)$. For every $E \in M(-F, c_2, d, r)$, we have that $\mathrm{H}^0(\mathrm{End}_0\, E(2,0)) = 0$. If furthermore $d = 1$ and $c_2 \geq -4r-1$, or if $d > 1$, then $\mathrm{H}^0(\mathrm{End}_0\, E(0,2)) = 0$.*

Proof. Recall that E fits into an exact sequence of the form

$$0 \to \mathcal{O}(r,d) \to E \to I_Z(-r-1,-d) \to 0, \tag{3.6}$$

with $\ell(Z) = c_2 + d(1+2r) \geq 3$. By taking the dual of (3.6), tensoring it with $E(2,0)$, and passing to the long exact sequence in cohomology, we get:

$$0 \to \mathrm{H}^0(E(r+3,d)) \to \mathrm{H}^0(\mathrm{End}\, E(2,0)) \to \mathrm{H}^0(E(-r+2,-d) \otimes I_Z) \to \dots$$

Let us show that $\mathrm{h}^0(E(r+3,d)) = 0$ and $\mathrm{h}^0(E(-r+2,-d) \otimes I_Z) \leq 3$, as such $\mathrm{h}^0(\mathrm{End}\, E(2,0)) \leq 3$, and so $\mathrm{H}^0(\mathrm{End}_0\, E(2,0)) = 0$.

To show that $\mathrm{H}^0(E(r+3,d)) = 0$, tensor (3.6) with $\mathcal{O}(r+3,d)$ to get

$$0 \to \mathrm{H}^0(\mathcal{O}(2r+3,2d)) \to \mathrm{H}^0(E(r+3,d)) \to \mathrm{H}^0(I_Z(2,0)) \to \dots$$

Now, $\mathrm{h}^0(\mathcal{O}(2r+3,2d)) = 0$ as $2r+3 \leq -1$, and $\mathrm{H}^0(\mathcal{O}(2,0) \otimes I_Z) = 0$ by Lemma 3.17. Hence $\mathrm{H}^0(E(r+3,d)) = 0$.

To see that $\mathrm{h}^0(E(-r+2,-d) \otimes I_Z) \leq 3$, start by tensoring (3.6) by $\mathcal{O}(-r+2,-d)$ to get:

$$0 \to \mathrm{H}^0(\mathcal{O}(2,0)) \to \mathrm{H}^0(E(-r+2,-d)) \to \mathrm{H}^0(I_Z(-2r-1,-2d)) \to \dots$$

Since $-2d < 0$, we have that $\mathrm{h}^0(I_Z(-2r-1,-2d)) = 0$, and so $\mathrm{h}^0(E(-r+2,-d)) = \mathrm{h}^0(\mathcal{O}(2,0)) = 3$. Now tensor

$$0 \to I_Z \to \mathcal{O} \to \mathcal{O}_Z \to 0 \tag{3.7}$$

by $E(-r+2,-d)$, and pass to the exact sequence in cohomology to get

$$0 \to \mathrm{H}^0(E(-r+2,-d) \otimes I_Z) \to \mathrm{H}^0(E(-r+2,-d)) \to \mathrm{H}^0(E(-r+2,-d) \otimes \mathcal{O}_Z) \to \dots$$

From here it is clear that $\mathrm{h}^0(E(-r+2,-d) \otimes I_Z) \leq 3$, and so $\mathrm{H}^0(\mathrm{End}_0\, E(2,0)) = 0$.

We now show that if $d = 1$ and $c_2 \geq -4r-1$, or $d > 1$, then $\mathrm{H}^0(\mathrm{End}_0\, E(0,2)) = 0$. Taking the dual of (3.6), and tensoring it by $E(0,2)$ we get:

$$0 \to \mathrm{H}^0(E(r+1,d+2)) \to \mathrm{H}^0(\mathrm{End}\, E(0,2)) \to \mathrm{H}^0(E(-r,-d+2) \otimes I_Z) \to \dots \tag{3.8}$$

This time $\mathrm{h}^0(E(r+1,d+2)) = 0$ and $\mathrm{h}^0(E(-r,-d+2) \otimes I_Z) \leq 3$, giving us $\mathrm{h}^0(\mathrm{End}\, E(0,2)) \leq 3$ and $\mathrm{H}^0(\mathrm{End}_0\, E(0,2)) = 0$. Let us prove it.

To show that $\mathrm{h}^0(E(r+1,d+2)) = 0$, tensor (3.6) with $\mathcal{O}(r+1,d+2)$ to get

$$0 \to \mathrm{H}^0(\mathcal{O}(2r+1,2d+2)) \to \mathrm{H}^0(E(r+1,d+2)) \to \mathrm{H}^0(I_Z(0,2)) \to \dots$$

Now, $\mathrm{h}^0(\mathcal{O}(2r+1,2d+2)) = 0$ as $2r+1 \leq -3$, and $\mathrm{H}^0(I_Z(0,2)) = 0$ by Lemma 3.17. Hence, $\mathrm{h}^0(E(r+1,d+2)) = 0$.

To see that $\mathrm{h}^0(E(-r,-d+2)\otimes I_Z)\leq 3$, start by tensoring (3.6) with $\mathcal{O}(-r,-d+2)$ to get:

$$0\to \mathrm{H}^0(\mathcal{O}(0,2))\to \mathrm{H}^0(E(-r,-d+2))\to \mathrm{H}^0(\mathcal{O}(-2r-1,-2d+2)\otimes I_Z)\to \dots$$

If $d>1$, then $-2d+2<0$ and $\mathrm{h}^0(I_Z(-2r-1,-2d+2))=0$. Thus $\mathrm{h}^0(E(-r,-d+2))=\mathrm{h}^0(\mathcal{O}(0,2))=3$. If $d=1$ and $c_2\geq -4r-1$, then $\ell(Z)+2r\geq 0$. Moreover, using a similar argument to the one of the proof of Lemma 3.17, we have $\mathrm{h}^0(I_Z(-2r-1,0))=-2r-\ell(Z)$. Thus

$$\mathrm{h}^0(E(-r,-d+2))=\mathrm{h}^0(E(-r,1))=\mathrm{h}^0(\mathcal{O}(0,2))+\mathrm{h}^0(\mathcal{O}(-2r-1,0)\otimes I_Z)=3-2r-\ell(Z)\leq 3,$$

since one can show, again using a similar argument to that of Lemma 3.17, that $\mathrm{h}^0(\mathcal{O}(-2r-1,0)\otimes I_Z)=-2r-\ell(Z)$. Finally, tensor the exact sequence (3.7) by $E(-r,-d+2)$ to get

$$0\to \mathrm{H}^0(E(-r,-d+2)\otimes I_Z)\to \mathrm{H}^0(E(-r,-d+2))\to \mathrm{H}^0(E(-r,-d+2)\otimes I_Z)\to \dots$$

From here it is clear that $\mathrm{h}^0(E(-r,-d+2)\otimes I_Z)\leq 3$ in both cases. $\square$

Proposition 3.16 now follows immediately from Lemma 3.18, since

$$\mathrm{H}^0(\operatorname{End}_0 E\otimes T)=\mathrm{H}^0(\operatorname{End}_0 E(2,0))\oplus \mathrm{H}^0(\operatorname{End}_0 E(0,2)).$$

Chapter 4

Moduli Spaces of rank 2 semistable co-Higgs bundles over $\mathbb{P}^1 \times \mathbb{P}^1$

In this chapter, we work only over the 0-th Hirzebruch surface, $\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\mathrm{pr}_1} \mathbb{P}^1$. We also fix the standard polarization $H = C_0 + F$. After giving a brief review of what spectral curves are and how the Hitchin map is defined for co-Higgs bundles over $\mathbb{P}^1$, we discuss the analogous notions in the $\mathbb{P}^1 \times \mathbb{P}^1$ setting. We then move on to the description of the moduli spaces of rank 2 semistable co-Higgs bundles for $c_2 = 0$ (and any of the reduced classes for c_1). In the case of $c_2 = 1$, we also give an example (when $c_1 = -F$) of how the moduli space $\mathcal{M}_{\mathrm{H}}^{\mathrm{co}}(-F, 1)$ looks like. In this case, a technical obstacle in giving an explicit description of the moduli space is obtaining the Higgs fields for non-trivial extensions of a line bundle by another line bundle (which are not decomposable). Though the idea of how to achieve this is straightforward, the execution is computationally heavy. We finish the chapter with a section on deformation theory, where we apply this tool to points in the moduli spaces described in the present chapter.

4.1 Spectral correspondence and Hitchin map

4.1.1 A brief review of spectral curves over $\mathbb{P}^1$

In this section, we briefly recall the notions of spectral curve and the Hitchin correspondence over $\mathbb{P}^1$. We let $\mathcal{M}_{\mathbb{P}^1}^{\mathrm{co}}(2)$ be the moduli space of rank 2 semistable co-Higgs bundles over $\mathbb{P}^1$.

The Hitchin map is given by

$$\begin{array}{rccc} H: & \mathcal{M}_{\mathbb{P}^1}^{\mathrm{co}}(2) & \to & \mathrm{H}^0(\mathbb{P}^1, T^2) \\ & (E, \Phi) & \mapsto & \operatorname{char}\Phi, \end{array}$$

where $\operatorname{char}\Phi$ is the characteristic polynomial of Φ. Recall that, in this case, $T = \mathcal{O}_{\mathbb{P}^1}(2)$, and so $T^2 = \mathcal{O}_{\mathbb{P}^1}(4)$. Since we are assuming that Φ is trace-free,

$$\operatorname{char}\Phi = \eta^2(y) + \det\Phi,$$

where η is the tautological section of the pullback of $\mathcal{O}_{\mathbb{P}^1}(2)$ to its own total space, and y ranges in $\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(2))$. Thus, we can identify $\mathrm{char}\,\Phi$ with $\det\Phi \in \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4))$. This map is surjective and proper.

Now let $\rho \in \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4))$ be a generic section and fix $(E, \Phi) \in H^{-1}(\rho)$. The spectral curve X_ρ associated to $\mathrm{char}\,\Phi = \rho$ is a smooth subvariety of $\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(2))$ given by the equation

$$\eta^2(y) + \det\Phi(\pi(y)) = 0.$$

Since ρ is irreducible, by genericity, X_ρ is an irreducible curve. Furthermore, it is a double cover of $\mathbb{P}^1$. To see this, first equip X_ρ with the projection

$$\pi \colon X_\rho \to \mathbb{P}^1$$

induced from the projection $\pi : \mathcal{O}_{\mathbb{P}^1}(2) \to \mathbb{P}^1$. Note that if $p \in \mathbb{P}^1$, then

$$\pi^{-1}(p) = \{b \in \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(2)) \ : \ \mathrm{char}\,\Phi(p)(b) = 0\},$$

and so $\pi^{-1}(p)$ are the eigenvalues of $\Phi(p)$. Since ρ was choosen to be generic, this implies that $\pi : X_\rho \to \mathbb{P}^1$ is indeed a 2-sheeted branched covering. The ramification of the spectral curve occurs at finitely many points, where Φ has repeated eigenvalues. These are precisely the 4 points in $\mathbb{P}^1$ where ρ vanishes. By the Riemann-Hurwitz formula, the genus of X_ρ is one, hence an elliptic curve. It follows from more general arguments presented in both [6, 10] that, for generic $\rho \in \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4))$, the fibre $H^{-1}(\rho)$ is isomorphic to $\mathrm{Pic}(X_\rho)$ (which is again an elliptic curve). This is the Hitchin correspondence and, more precisely, it works as follows:

Given any line bundle $L \to X_\rho$, the push-forward $\pi_* L$ yields a rank 2 bundle E over $\mathbb{P}^1$. Moreover, the push-forward of the multiplication map

$$- \otimes \eta : L \to L \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(2)$$

is a Higgs field for E with $\mathrm{char}\,\Phi = \rho$. Because X_ρ is irreducible, this Higgs field leaves no sub-line bundle of E invariant, and thus (E, Φ) is trivially stable. In other words, $(E, \Phi) \in H^{-1}(\rho)$.

Conversely, one can associate to any $(E, \phi) \in H^{-1}(\rho)$ a line bundle over X_ρ as follows: First consider the pull-back of both $\pi^* E$, which is a rank 2 bundle over X_ρ, and $\pi^*\Phi : \pi^* E \to \pi^* E \otimes \mathcal{O}_{\mathbb{P}^1}(2)$. The eigenline bundle L_e is defined as the kernel of the map

$$\pi^*\Phi - \eta\,\mathrm{Id} : \pi^* E \to \pi^*(E \otimes \mathcal{O}_{\mathbb{P}^1}(2))$$

tensored by R, where R is the ramification divisor of X_ρ. This is the desired element in $\mathrm{Pic}(X_\rho)$ that we wish to push-forward. Indeed, the restriction of $\pi^*\Phi$ to L_e is precisely the multiplication map, and so applying the above construction to $(L_e, \pi^*\Phi)$ yields the pair (E, Φ) we started with.

To conclude this section we review the results of Rayan in [27, 29] that address this spectral correspondence in the case of rank 2 vector bundles of degree 0 and -1 over $\mathbb{P}^1$.

Let ρ be a generic section of $\mathcal{O}_{\mathbb{P}^1}(4)$ and $(E, \Phi) \in H^{-1}(\rho)$. If $\deg E = -1$, as we mentioned in Section 2.3.1, $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. By Grothendieck–Riemann–Roch, the line bundles L over the spectral curve X_ρ that correspond to $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ under the

Hitchin correspondence have degree 1. To determine the corresponding Higgs field, one has to look at the point $p \in X_\rho$ where all the elements of $\mathrm{H}^0(X_\rho, L)$ vanish (because $\mathrm{H}^0(X_\rho, L)$ is one-dimensional in this case). If p is a ramification point, the Higgs field can be expressed as

$$\begin{pmatrix} 0 & a_1 + a_2 z + a_3 z^2 + a_4 z^3 \\ z & 0 \end{pmatrix},$$

where z is a coordinate on $\mathbb{P}^1$ centered at $z_0 := \pi(p)$ and $\rho = -z(a_1 + a_2 z + a_3 z^2 + a_4 z^3)$. If p is unramified, then there are two possibilities, either $p = (z_0, \sqrt{a_0})$, in which case Φ has the form

$$\begin{pmatrix} \sqrt{a_0} & a_1 + a_2 z + a_3 z^2 + a_4 z^3 \\ z & -\sqrt{a_0} \end{pmatrix},$$

or $p = (z_0, -\sqrt{a_0})$, in which case Φ has the form

$$\begin{pmatrix} \sqrt{a_0} & a_1 + a_2 z + a_3 z^2 + a_4 z^3 \\ z & -\sqrt{a_0} \end{pmatrix}.$$

Finally, if E has degree 0, we have seen in Section 2.3.1 that E is either $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ or $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. In this case, pushing down the degree 2 line bundle $\pi^*\mathcal{O}_{\mathbb{P}^1}(1)$ over X_ρ yields $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ with Higgs field

$$\begin{pmatrix} 0 & -\rho \\ 1 & 0 \end{pmatrix},$$

which is stable. Moreover, pushing down any other line bundle of degree 2 on X_ρ yields a stable co-Higgs pair with underlying bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$.

4.1.2 Spectral surfaces over $\mathbb{P}^1 \times \mathbb{P}^1$

In this section, we present spectral surfaces over $\mathbb{P}^1 \times \mathbb{P}^1$ and discuss the Hitchin correspondence in this setting.

The Hitchin map H that goes from the moduli space of semistable rank 2 co-Higgs bundles over $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathcal{M}^{\mathrm{co}}(2)$, to the global sections of $\mathrm{S}^2(T) = \mathcal{O}(4,0) \oplus \mathcal{O}(2,2) \oplus \mathcal{O}(0,4)$ is defined as follows:

$$\begin{array}{rccc} H: & \mathcal{M}^{\mathrm{co}}(2) & \to & \mathrm{H}^0(\mathrm{S}^2(T)) \\ & (E, \Phi) & \mapsto & \operatorname{char} \Phi. \end{array}$$

Here we are again identifying $\operatorname{char} \Phi$ with $\det \Phi \in \mathrm{H}^0(\mathrm{S}^2(T))$, as $\operatorname{char} \Phi = \eta^2(y) + \det \Phi$ (since Φ is trace-free). Recall that η denotes the tautological section of the pullback of T to its own total space. Explicitly, the Hitchin map is given as follows: Let $(E, \Phi) \in \mathcal{M}^{\mathrm{co}}(2)$, then working on an open set $\mathcal{U}$, we can write

$$\Phi = \Phi_1 + \Phi_2 = \begin{pmatrix} A_1 & B_1 \\ C_1 & -A_1 \end{pmatrix} + \begin{pmatrix} A_2 & B_2 \\ C_2 & -A_2 \end{pmatrix},$$

where $A_1, B_1, C_1 \in \mathrm{H}^0(\mathcal{U}, \mathcal{O}(2,0))$ and $A_2, B_2, C_2 \in \mathrm{H}^0(\mathcal{U}, \mathcal{O}(0,2))$. Then

$$H(E, \Phi) = (\det \Phi_1, -2A_1 A_2 - 2B_1 C_2, \det \Phi_2) \in \mathrm{H}^0(\mathcal{O}(4,0) \oplus \mathcal{O}(2,2) \oplus \mathcal{O}(0,4)).$$

We first note that the Hitchin map is not surjective. Indeed, if $H(E, \Phi) = (\rho_1, \rho_{1,2}, \rho_2)$, then by the above equation and the integrability of Φ, we see that $\rho_{1,2}^2 = 4\rho_1\rho_2$, and so H is clearly not onto.

Definition 4.1. Let $(E, \Phi) \in \mathcal{M}^{\mathrm{co}}(2)$. The spectral surface S_ρ associated to $\rho = \operatorname{char} \Phi$, is given by those points $y \in \operatorname{Tot}(T)$ such that

$$\operatorname{char} \Phi(y) = \eta^2(y) + \det \Phi(\theta(y)) = 0,$$

where $\theta : \operatorname{Tot}(T) \to \mathbb{P}^1 \times \mathbb{P}^1$. We equip S_ρ with the restriction $\theta|_{S_\rho}$.

The equations of S_ρ can be written as follows: if $y = (y_1, y_2) \in \operatorname{Tot}(T)$ and $\rho = (\rho_1, \rho_{1,2}, \rho_2)$, then the spectral surface is given by

$$\begin{cases} \eta_1^2(y_1) + \rho_1 = 0 \\ \eta_2^2(y_2) + \rho_2 = 0 \\ 2\eta_1(y_1)\eta_2(y_2) + \rho_{1,2} = 0, \end{cases} \tag{4.1}$$

where η_1 and η_2 are the tautological sections of the pullback of $\mathcal{O}(2,0)$ and $\mathcal{O}(0,2)$, respectively, to their own total spaces.

Note that multiplying the first equation by $\eta_2^2(y_2)$, and using the second equation and the fact that $\rho_{1,2}^2 = 4\rho_1\rho_2$, yields

$$\begin{aligned} 0 &= (\eta_1^2(y_1) + \rho_1)\eta_2^2(y_2) \\ &= \eta_1^2(y_1)\eta_2^2(y_2) - \rho_1\rho_2 \\ &= (\eta_1(y_1)\eta_2(y_2) + \rho_{1,2}/2)(\eta_1(y_1)\eta_2(y_2) - \rho_{1,2}/2). \end{aligned}$$

Thus the first two equations yield a reducible surface in $\operatorname{Tot}(T)$. Clearly, as the third equation appears as one of the factors above, it cuts out a 2-dimensional subvariety of this surface.

Analogous to the case of curves, the elements of S_ρ lying above a point in $\mathbb{P}^1 \times \mathbb{P}^1$ are pairs where the first entry is an eigenvalue of Φ_1, and the second entry is an eigenvalue of Φ_2. Moreover, we claim that for generic ρ, S_ρ is a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$. To see this, let λ_i^1 and λ_i^2 be the eigenvalues of Φ_i at an unramified point $p \in \mathbb{P}^1 \times \mathbb{P}^1$. Recall that, since Φ is integrable, by Remark 2.41, Φ_1 and Φ_2 have the same eigenspaces, and so we assume that the eigenspace of λ_1^j is equal to the eigenspace of λ_2^j for $j = 1, 2$. We now check that the third equation of S_ρ is equivalent to $(\lambda_1^i, \lambda_2^j) \in S_\rho$ if and only if $i = j$. In other words, the points of S_ρ, at unramified points of $\mathbb{P}^1 \times \mathbb{P}^1$, are pairs of eigenvalues of Φ_1 and Φ_2 sharing the same eigenspace. First note that since Φ_1 and Φ_2 commute, λ_1^j and λ_2^j sharing the same eigenspaces is equivalent to $\lambda_1^j\lambda_2^j$ being an eigenvalue of $\Phi_1\Phi_2$, and so it must satisfy the characteristic polynomial $\operatorname{char} \Phi_1\Phi_2$:

$$\eta_1^2(\lambda_1^j)\eta_2^2(\lambda_2^j) - \operatorname{tr}(\Phi_1\Phi_2)\eta_1(\lambda_1^j)\eta_2(\lambda_2^j) + \det(\Phi_1\Phi_2) = 0.$$

After some algebraic manipulation, the above equation reduces to

$$2\eta_1(\lambda_1^j)\eta_2(\lambda_2^j) + \rho_{1,2} = 0,$$

which is precisely saying that $(\lambda_1^j, \lambda_2^j)$ satisfies the third equation. We can thus conclude that the points in S_ρ lying above $p \in \mathbb{P}^1 \times \mathbb{P}^1$ are $(\lambda_1^1, \lambda_2^1)$ and $(\lambda_1^2, \lambda_2^2)$, showing that S_ρ is indeed a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$.

Remark 4.2. 1. Let ρ be generic. Unlike the curves case, in order to get a Hitchin correspondence, one needs to push-forward rank 1 torsion free sheaves over S_ρ instead of only elements of $\mathrm{Pic}(S_\rho)$ (see [33, 34]).

2. All of the above discussion also holds for an arbitrary Hirzebruch surface $\mathbb{F}_n$ after replacing $\mathcal{O}(2,0) \oplus \mathcal{O}(0,2)$ by $\mathcal{O}(2F) \oplus \mathcal{O}(2C_0 + nF)$.

3. One can show that, for generic ρ, S_ρ consists of two isomorphic copies of $\mathbb{P}^1 \times \mathbb{P}^1$ that intersect in four points.

Now, we aim to show that, the underlying bundle of the generic elements of $\mathcal{M}^{\mathrm{co}}(2)$ are indecomposable.

Lemma 4.3. *Let $E = L_1 \oplus L_2$ be a decomposable rank 2 vector bundle over $\mathbb{P}^1 \times \mathbb{P}^1$.*

1. *Suppse $\mu(L_1) > \mu(L_2)$. If $(E, \Phi = \Phi_1 + \Phi_2)$ is a semistable co-Higgs pair, then $\Phi_1 = 0$ or $\Phi_2 = 0$.*

2. *Suppose $\mu(L_1) = \mu(L_2)$. Then, either $\det \Phi_1$ is non-generic in $\mathrm{H}^0(\mathcal{O}(4,0))$ or $\det \Phi_2$ is non-generic in $\mathrm{H}^0(\mathcal{O}(0,4))$.*

Proof. Let $L_1 = \mathcal{O}(a_1, b_1)$ and $L_2 = \mathcal{O}(a_2, b_2)$. Then, any element of $\mathrm{H}^0(\mathrm{End}_0\, E \otimes T)$ is of the form

$$\Phi = \Phi_1 + \Phi_2 = \begin{pmatrix} A_1 & B_1 \\ C_1 & -A_1 \end{pmatrix} + \begin{pmatrix} A_2 & B_2 \\ C_2 & -A_2 \end{pmatrix},$$

with $A_1 \in \mathrm{H}^0(\mathcal{O}(2,0))$, $B_1 \in \mathrm{H}^0(\mathcal{O}(a_1 - a_2 + 2, b_1 - b_2)$, $C_1 \in \mathrm{H}^0(\mathcal{O}(a_2 - a_1 + 2, b_2 - b_1))$ and $A_2 \in \mathrm{H}^0(\mathcal{O}(0,2))$, $B_2 \in \mathrm{H}^0(\mathcal{O}(a_1 - a_2, b_1 - b_2 + 2))$, $C_2 \in \mathrm{H}^0(\mathcal{O}(a_2 - a_1, b_2 - b_1 + 2))$.

1. Suppose $\mu(L_1) > \mu(L_2)$. We have two cases to consider. If $a_1 > a_2$, then any element in $\mathrm{H}^0(\mathrm{End}_0\, E \otimes T)$ is such that $C_2 = 0$. If we were to have a Higgs field Φ for E such that (E, Φ) is semistable, then C_1 must not be identically zero, for otherwise it would leave L_1 invariant, contradicting semistability. The integrability condition, equations (2.8), implies that $A_2 = B_2 = 0$. Hence, $\Phi = \Phi_1$ with non-zero C_1. Similarly, if $b_1 > b_2$, we get that $\Phi = \Phi_2$ with non-zero C_2.

2. Suppose $\mu(L_1) = \mu(L_2)$. It is enough to consider the following three cases:

(i) If $a_1 > a_2$ and $b_2 > b_1$, then any element in $\mathrm{H}^0(\mathrm{End}_0\, E \otimes T)$ is such that $B_1 = 0$ and $C_2 = 0$. Then $\det \Phi_1 = -A_1^2$ and $\det \Phi_2 = -A_2^2$, so they are non-generic in $\mathrm{H}^0(\mathcal{O}(4,0))$ and $\mathrm{H}^0(\mathcal{O}(0,4))$, respectively.

(ii) If $a_2 > a_1$ and $b_1 > b_2$, then any element in $\mathrm{H}^0(\mathrm{End}_0\, E \otimes T)$ is such that $C_1 = 0$ and $B_2 = 0$. Then $\det \Phi_1 = -A_1^2$ and $\det \Phi_2 = -A_2^2$, so they are non-generic in $\mathrm{H}^0(\mathcal{O}(4,0))$ and $\mathrm{H}^0(\mathcal{O}(0,4))$, respectively.

(iii) If $a_1 = a_2$ and $b_1 = b_2$, then any element in $\mathrm{H}^0(\mathrm{End}_0\, E \otimes T)$ is such that A_1, B_1, C_1 are elements of $\mathrm{H}^0(\mathcal{O}(2,0))$ and A_2, B_2, C_2 are elements of $\mathrm{H}^0(\mathcal{O}(0,2))$. Note that we may assume that at least one entry in either Φ_1 or Φ_2 is non-zero, for otherwise the result follows. Without loss of generality let us assume that $A_2 \neq 0$. By the integrability condition, equations (2.8) imply that if $A_2 \neq 0$, we can pick a point $p_2 \in \mathbb{P}^1$, which is not a zero of A_2, evaluating both $A_1B_2 = B_1A_2$ and $C_1A_2 = A_1C_2$ on $p = (z_1, p_2)$, we get $B_1 = uA_1$ and $C_1 = vA_1$ for $u, v \in \mathbb{C}$. Hence, $\det \Phi_1 = -(1+uv)A_1^2$, which is non-generic in $\mathrm{H}^0(\mathcal{O}(4,0))$.

□

Proposition 4.4. *If ρ is generic, then $H^{-1}(\rho)$ does not contain co-Higgs pairs where the underlying bundle is decomposable. In particular, for $(E,\Phi) \in \mathcal{M}^{co}(2)$ generic, E is not decomposable.*

Proof. Let ρ be generic and assume that $H^{-1}(\rho)$ contains a pair with decomposable underlying bundle. Then, by Lemma 4.3 (1.), either $\rho = (\rho_1, 0, 0)$ or $\rho = (0, 0, \rho_2)$, or, by Lemma 4.3 (2.), $\rho = (\rho_1, \rho_{1,2}, \rho_2)$ with either ρ_1 or ρ_2 non-generic. Hence, ρ is not generic. □

Let us now discuss spectral surfaces in the case where either Φ_1 or Φ_2 is zero. We will be interested in the cases when $\rho = (\rho_1, 0, 0)$ or $(0, 0, \rho_2) \in \mathrm{H}^0(\mathrm{S}^2(T))$, where ρ_1 and ρ_2 are generic in $\mathrm{H}^0(\mathcal{O}(4,0))$ and $\mathrm{H}^0(\mathcal{O}(0,4))$, respectively.

When $\rho = (\rho_1, 0, 0)$, and ρ_1 is generic, any Higgs field Φ of a co-Higgs pair in the fibre of the Hitchin map above ρ must have the form $\Phi = \Phi_1$. To see this, let Φ be a Higgs field such that $\det \Phi = \rho$. Since $\det \Phi_2 = 0$, we have that $\lambda = 0$ is an eigenvalue of Φ_2 of algebraic multiplicity 2. Also, above all points where $\det \Phi_1 \neq 0$, we have a basis of eigenvectors for Φ_1. By the integrability of Φ and Remark 2.41, this is also a basis of eigenvectors of Φ_2. Hence, Φ_2 is diagonalizable and thus the zero matrix at all such points. Hence, $\Phi_2 = 0$. Moreover, in this case, the equations of the spectral surface reduce to

$$\begin{cases} \eta_1^2(y_1) + \det \Phi_1 = 0 \\ \eta_2^2(y_2) = 0. \end{cases} \tag{4.2}$$

Hence,

$$S_\rho = X_{\rho_1} \times \mathbb{P}^1,$$

where X_{ρ_1} is the spectral curve associated to ρ_1 (we view ρ_1 as an element of $\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4))$), which we have seen is an elliptic curve (see Section 4.1.1). Also, the projection $\theta : S_\rho \to \mathbb{P}^1 \times \mathbb{P}^1$ is given by $(\pi, \mathrm{Id}_{\mathbb{P}^1})$, where $\pi : X_\rho \to \mathbb{P}^1$.

Similar observations can be made when $\rho = (0, 0, \rho_2)$ is generic (in particular $S_\rho = \mathbb{P}^1 \times X_{\rho_2}$). Consequently, in both of these cases, we have a Hitchin correspondence on the spectral surface coming from the correspondence on the spectral curve. More precisely,

Proposition 4.5. *Suppose $\rho = (\rho_1, 0, 0) \in \mathrm{H}^0(\mathrm{S}^2(T))$ with ρ_1 generic. Then, there is a Hitchin correspondence between the line bundles of S_ρ and the elements (E, Φ) of $\mathcal{M}^{co}(2)$ with underlying bundle of the form $E = \mathcal{O}(a,m) \oplus \mathcal{O}(b,m)$ and $\Phi = \Phi_1 \in \mathrm{H}^0(\mathrm{End}_0\, E \otimes \mathcal{O}(2,0))$.*

Proof. Let M be a line bundle over S_ρ, then M is of the form $\mathrm{Pr}_1^* L \otimes \mathrm{Pr}_2^* \mathcal{O}_{\mathbb{P}^1}(m)$ (see [18, Chapter 3, Section 12]), where L is a line bundle over X_ρ and $m \in \mathbb{Z}$, and $\mathrm{Pr}_1, \mathrm{Pr}_2$ are the projections of S_ρ to X_ρ and $\mathbb{P}^1$, respectively. From the commutative diagram

$$\begin{tikzcd} S_\rho = X_\rho \times \mathbb{P}^1 \arrow[r, "(\pi, \mathrm{Id}_{\mathbb{P}^1})"] \arrow[d, "\mathrm{Pr}_1"] & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[d, "\mathrm{pr}_1"] \\ X_\rho \arrow[r, "\pi"] & \mathbb{P}^1 \end{tikzcd}$$

we see that $(\pi, \mathrm{Id}_{\mathbb{P}^1})_*(\mathrm{Pr}_1^*(L)) = \mathrm{pr}_1^*(\pi_*(L)) = \mathcal{O}(a,0) \oplus \mathcal{O}(b,0)$ for some $a, b \in \mathbb{Z}$ such that $\pi_*(L) = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$. Similarly, from the commutative diagram

$$\begin{tikzcd} S_\rho = X_\rho \times \mathbb{P}^1 \arrow[r, "(\pi, \mathrm{Id}_{\mathbb{P}^1})"] \arrow[d, "\mathrm{Pr}_2"] & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[d, "\mathrm{pr}_2"'] \\ \mathbb{P}^1 \arrow[r, "\mathrm{Id}_{\mathbb{P}^1}"] & \mathbb{P}^1 \end{tikzcd}$$

we see that $(\pi, \mathrm{Id}_{\mathbb{P}^1})_*(\mathrm{Pr}_2^*(\mathcal{O}_{\mathbb{P}^1}(m))) = \mathrm{pr}_2^*(\mathrm{Id}_{\mathbb{P}^1} *(\mathcal{O}_{\mathbb{P}^1}(m))) = \mathcal{O}(0,m)$. Therefore, $\theta_* M = \mathcal{O}(a,m) \oplus \mathcal{O}(b,m)$. Moreover, since the multiplication of elements in M by elements in S_ρ maps to $M \otimes \mathcal{O}(2,0)$, the push-forward of $- \otimes \eta$ yields a Higgs field Φ with $\Phi = \Phi_1$. The Higgs field Φ_1 is the pullback of the Higgs field obtained by pushing-down the multiplication map of L.

On the other hand, if we start with something of the form $\mathcal{O}(a,m) \oplus \mathcal{O}(b,m)$, to find the corresponding line bundle over S_ρ we first find the line bundle L over X_ρ corresponding to $(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b), \Phi_1)$. Then we tensor the pullback of the latter with $\mathrm{Pr}_2^* \mathcal{O}_{\mathbb{P}^1}(m)$. $\square$

Remark 4.6. A similar result holds when ρ is of the form $(0, 0, \rho_2)$ with ρ_2 generic.

We will discuss some examples of this nature in the following sections. See Remarks 4.9, 4.13 and 4.24.

4.2 The Moduli Spaces $\mathcal{M}^{\mathrm{co}}(c_1, c_2)$

We fix the standard polarization $H = C_0 + F$, and let $\mathcal{M}^{\mathrm{co}}(c_1, c_2)$ denote $\mathcal{M}^{\mathrm{co}}_{\mathrm{H}}(c_1, c_2)$. In this section, we consider only the reduced first Chern classes.

4.2.1 Second Chern Class $c_2 = 0$

Throughout this subsection, we let E denote a rank 2 vector bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ and assume $c_2 = 0$. By Lemma 3.1, we may assume that c_1 is 0, $-F$ or $-C_0$, and throughout this section we do so. Indeed, we assume that c_1 is either 0 or $-F$, since the case where $c_1 = -C_0$ is symmetric to the case $c_1 = -F$, in the sense that one simply interchanges the roles of the first and second copies of $\mathbb{P}^1$. By Proposition 3.5, we know that in this case there is only one chamber, and so having fixed $H = C_0 + F$ does not actually impact the results of this subsection, as any other ample divisor would yield the exact same results. Recall that, in this case, by Theorem 3.3, any semistable co-Higgs pair (E, Φ) is such that E is an extension of line bundles. Moreover, we can prove that, in this case, E is decomposable.

First Chern class $c_1 = -F$

We now analyze further the case $c_1 = -F$. Recall that in this case the notions of semistability and stability coincide. We begin by describing the possible co-Higgs pairs appearing in the moduli space.

Proposition 4.7. *Suppose that $c_1(E) = -F$ and $c_2(E) = 0$. If (E, Φ) is a stable co-Higgs pair, then $E = \mathcal{O} \oplus \mathcal{O}(-1, 0)$. Moreover, Φ is of the form*

$$\Phi = \Phi_1 + \Phi_2 = \begin{pmatrix} A_1 & B_1 \\ C_1 & -A_1 \end{pmatrix} + \begin{pmatrix} A_2 & B_2 \\ 0 & -A_2 \end{pmatrix},$$

with $A_1 \in H^0(\mathcal{O}(2, 0))$, $B_1 \in H^0(\mathcal{O}(3, 0))$, $C_1 \in H^0(\mathcal{O}(1, 0))$ and $A_2 \in H^0(\mathcal{O}(0, 2))$, $B_2 \in H^0(\mathcal{O}(1, 2))$.

Proof. By Theorem 3.3, E is an extension of line bundles. Let us first show that E is in fact decomposable. We know that E fits into an exact sequence of the form

$$0 \to \mathcal{O}(a, b) \to E \to \mathcal{O}(-1 - a, -b) \to 0,$$

and $0 = c_2(E) = -b(1+2a)$. Thus $b = 0$. A non-trivial extension corresponds to an element of $\mathrm{H}^1(\mathcal{O}(2a+1, 0)) = \mathrm{H}^1(\mathbb{P}^1; \mathcal{O}_{\mathbb{P}^1}(2a+1))$, and so it is the pullback to $\mathbb{P}^1 \times \mathbb{P}^1$ of a non-trivial extension V of $\mathcal{O}_{\mathbb{P}^1}(-a - 1)$ by $\mathcal{O}_{\mathbb{P}^1}(a)$ over $\mathbb{P}^1$. Since bundles over $\mathbb{P}^1$ are decomposable, $V = \mathcal{O}_{\mathbb{P}^1}(c) \oplus \mathcal{O}_{\mathbb{P}^1}(c')$, for some integers c and c', and $E = \mathrm{pr}_1^* V = \mathcal{O}(c, 0) \oplus \mathcal{O}(c', 0)$.

Then, the underlying bundle of a stable co-Higgs pair (E, Φ) with $c_1(E) = -F$ and $c_2(E) = 0$ is of the form $E = \mathcal{O}(a, 0) \oplus \mathcal{O}(-a-1, 0)$. Let us show that a can only take the values -1 or 0. Any element $\Phi \in \mathrm{H}^0(\mathrm{End}_0 E \otimes T)$ has the form

$$\Phi = \begin{pmatrix} A_1 & B_1 \\ C_1 & -A_1 \end{pmatrix} + \begin{pmatrix} A_2 & B_2 \\ C_2 & -A_2 \end{pmatrix},$$

where $A_1 \in \mathrm{H}^0(\mathcal{O}(2, 0))$, $B_1 \in \mathrm{H}^0(\mathcal{O}(2a + 3, 0))$, $C_1 \in \mathrm{H}^0(\mathcal{O}(1 - 2a, 0))$, and $A_2 \in \mathrm{H}^0(\mathcal{O}(0, 2))$, $B_2 \in \mathrm{H}^0(\mathcal{O}(2a + 1, 2))$, $C_2 \in \mathrm{H}^0(\mathcal{O}(-1 - 2a, 2))$. If $a \geq 1$, then $C_1 = C_2 = 0$, and $\mathcal{O}(a, 0)$ is Φ-invariant. However, $\mu(\mathcal{O}(a, 0)) > \mu(E)$, which contradicts stability. A similar argument, but interchanging the roles of the C_i's for the B_i's, and of $\mathcal{O}(a, 0)$ for $\mathcal{O}(-1 - a, 0)$, shows that $a > -2$. Hence, $a = 0, -1$ and thus $E = \mathcal{O} \oplus \mathcal{O}(-1, 0)$.

Let us now determine which Φ's yield stable pairs (E, Φ). Any element in $\mathrm{H}^0(\mathrm{End}_0 E \otimes T)$ is of the form:

$$\Phi = \Phi_1 + \Phi_2 = \begin{pmatrix} A_1 & B_1 \\ C_1 & -A_1 \end{pmatrix} + \begin{pmatrix} A_2 & B_2 \\ 0 & -A_2 \end{pmatrix},$$

with $A_1 \in \mathrm{H}^0(\mathcal{O}(2, 0))$, $B_1 \in \mathrm{H}^0(\mathcal{O}(3, 0))$, $C_1 \in \mathrm{H}^0(\mathcal{O}(1, 0))$ and $A_2 \in \mathrm{H}^0(\mathcal{O}(0, 2))$, $B_2 \in \mathrm{H}^0(\mathcal{O}(1, 2))$. Note that, if Φ were a Higgs field of E, then C_1 must be non-zero, as otherwise it would leave $\mathcal{O}$ invariant, contradicting stability. Also, taking into account the integrability condition, equations (2.8) imply that $A_2 = B_2 = 0$. Therefore, any possible Higgs field of E is of the form $\Phi = \Phi_1 \in \mathrm{H}^0(\mathrm{End}_0 E \otimes \mathcal{O}(2, 0))$, with Φ_1 as above, and non-zero C_1. Now, the fact that (E, Φ) is indeed stable for any of these Higgs fields follows from Lemma 2.38. ☐

Given Proposition 4.7, we now discuss the isomorphism classes of pairs $(\mathcal{O} \oplus \mathcal{O}(-1, 0), \Phi)$ with Φ as above. Recall that (E, Φ) is isomorphic to (E, Φ') when there exists an automorphism Ψ of E such that $\Phi' = \Psi \circ \Phi \circ \Psi^{-1}$. Now, an automorphism Ψ of E can be chosen of the form

$$\Psi = \begin{pmatrix} 1 & P \\ 0 & Q \end{pmatrix} \in \mathrm{H}^0(\mathrm{End}\, E),$$

where P and Q are global sections of $\mathcal{O}(1,0)$ and $\mathcal{O}$, respectively; moreover, $Q \neq 0$. Hence,

$$\Psi \circ \Phi \circ \Psi^{-1} = \begin{pmatrix} A_1 + PC_1 & -Q^{-1}(2A_1P - B_1 - C_1P) \\ QC_1 & -(A_1 + PC_1) \end{pmatrix}.$$

Since $C_1 \in \mathrm{H}^0(\mathcal{O}(1,0))$, we can locally write $C_1 = \alpha(z_1 - p)$. Then, $A_1 = A_1(p) + (z_1 - p)[A_1'(p) + A_1''(p)(z_1 - p)]$. It is not hard to see that, by choosing $P = -\alpha^{-1}[A_1'(p) + A_1''(p)(z_1 - p)]$ and $Q = \alpha^{-1}$, we have a representative of the conjugacy class of Φ of the form

$$\begin{pmatrix} A_1(p) & B_1' \\ z_1 - p & -A_1(p) \end{pmatrix},$$

where $B_1' \in \mathrm{H}^0(\mathcal{O}(3,0))$.

It follows from the above discussion that every Higgs field of a stable co-Higgs pair is the pullback of a Higgs field of the bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ over $\mathbb{P}^1$. Furthermore, every stable co-Higgs pair of degree -1 over $\mathbb{P}^1$ (see [27]) gives rise, by taking pullbacks, to a stable co-Higgs pair over $\mathbb{P}^1 \times \mathbb{P}^1$ of this form. Let us state these facts as:

Theorem 4.8. *The moduli space $\mathcal{M}^{co}(-F, 0)$ of rank 2 stable co-Higgs bundles over $\mathbb{P}^1 \times \mathbb{P}^1$ with first Chern class $-F$ and second Chern class 0 is a 6-dimensional smooth variety isomorphic to the moduli space $\mathcal{M}^{co}_{\mathbb{P}^1}(-1)$ of rank 2 stable co-Higgs bundles of degree -1 over $\mathbb{P}^1$.*

Proof. First recall that in [27], Rayan proved that $\mathcal{M}^{\mathrm{co}}_{\mathbb{P}^1}(-1)$ is a 6-dimensional smooth variety given by

$$\mathcal{V} := \{(y, \rho) \in \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(2)) \times \mathrm{H}^0(\mathbb{P}^1; \mathcal{O}_{\mathbb{P}^1}(4)) : \eta^2 = \rho(\pi(y))\},$$

where $\pi : \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(2)) \to \mathbb{P}^1$ is the natural projection, and η is the tautological section of the pullback of $\mathcal{O}_{\mathbb{P}^1}(2)$ to its own total space.

Now, consider the map

$$\begin{array}{rccc} f: & \mathcal{M}^{\mathrm{co}}_{\mathbb{P}^1}(-1) & \to & \mathcal{M}^{\mathrm{co}}(-F, 0) \\ & (V, \varphi) & \mapsto & ((\mathrm{pr}_1)^* V, (\mathrm{pr}_1)^* \varphi). \end{array}$$

We only check that this map is well-defined; the fact that it is an isomorphism is immediate. Since the only underlying bundle of a stable co-Higgs pair living in $\mathcal{M}^{\mathrm{co}}_{\mathbb{P}^1}(-1)$ is $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$, and since $(\mathrm{pr}_1)^*(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}) = \mathcal{O}(-1, 0) \oplus \mathcal{O}$, this map is simply given by $f(\varphi) = (\mathrm{pr}_1)^*(\varphi)$. Suppose $\varphi' = \psi \circ \varphi \circ \psi^{-1}$, where $\psi \in \mathrm{Aut}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1})$. Then

$$f(\varphi') = f(\psi \circ \varphi \circ \psi^{-1}) = (\mathrm{pr}_1)^*(\psi \circ \varphi \circ \psi^{-1}) = (\mathrm{pr}_1)^*(\psi) \circ (\mathrm{pr}_1)^*(\varphi) \circ (\mathrm{pr}_1)^*(\psi)^{-1},$$

where $(\mathrm{pr}_1)^*(\psi) \in \mathrm{Aut}((\mathrm{pr}_1)^*(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}))$. Thus $f(\varphi')$ and $f(\varphi)$ are indeed in the same conjugacy class. □

Remark 4.9. Note that, in this case, the spectral correspondence is as follows. We have seen in Section 4.1.2 that for $\rho = (\rho_1, 0, 0)$ with $\rho_1 \in \mathrm{H}^0(\mathcal{O}(4,0))$ generic, the spectral surface S_ρ is an elliptic fibration $X_{\rho_1} \times \mathbb{P}^1$, with the spectral correspondence coming from that of the elliptic curve X_{ρ_1}. Thus, if M is a line bundle over S_ρ corresponding to $\mathcal{O} \oplus \mathcal{O}(-1, 0) \in H^{-1}(\rho)$, we must have that $M = \mathrm{Pr}_1^*(L) \otimes \mathrm{Pr}_2^*(\mathcal{O}(m))$, where L is a line bundle over X_{ρ_1} and $\mathcal{O}(m)$ is a line bundle over $\mathbb{P}^1$. Moreover, L corresponds to a co-Higgs bundle $(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1), \phi)$, and so L has degree 1 (see [28, Section 8]). Note that $m = 0$. Therefore, the line bundles over S_ρ that are in correspondence with co-Higgs pairs $(\mathcal{O} \oplus \mathcal{O}(-1,0), \Phi) \in H^{-1}(\rho)$ are those of the form $\mathrm{Pr}_1^*(L)$ with L being a degree 1 line bundle over X_ρ. The Higgs fields are obtained by pulling-back the Higgs fields obtained from L in the $\mathbb{P}^1$ case (just as we discussed in Section 4.1.2).

First Chern class $c_1 = 0$

We now focus on the case $c_1 = 0$. In this case, we no longer have a single underlying bundle; nonetheless, there are only three possibilities.

Proposition 4.10. *Suppose that $c_1(E) = 0$ and $c_2(E) = 0$. If (E, Φ) is a semistable co-Higgs pair, then $E = \mathcal{O} \oplus \mathcal{O}$ or $\mathcal{O}(0,1) \oplus \mathcal{O}(0,-1)$ or $\mathcal{O}(1,0) \oplus \mathcal{O}(-1,0)$.*

Proof. It follows from Theorem 3.3 that E is an extension of line bundles. As in the proof of Propostion 4.7, one can easily check that E is decomposable and of the form $E = \mathcal{O}(a,0) \oplus \mathcal{O}(-a,0)$ or $E = \mathcal{O}(0,b) \oplus \mathcal{O}(0,-b)$. Without loss of generality, we now assume that $E = \mathcal{O}(a,0) \oplus \mathcal{O}(-a,0)$, as the other case is analogous. Let us show that a can only take the values -1, 0 or 1. Any Higgs field $\Phi \in \mathrm{H}^0(\mathrm{End}_0 E \otimes T)$ has the form:

$$\Phi = \begin{pmatrix} A_1 & B_1 \\ C_1 & -A_1 \end{pmatrix} + \begin{pmatrix} A_2 & B_2 \\ C_2 & -A_2 \end{pmatrix},$$

where $A_1 \in \mathrm{H}^0(\mathcal{O}(2,0))$, $B_1 \in \mathrm{H}^0(\mathcal{O}(2a+2,0))$, $C_1 \in \mathrm{H}^0(\mathcal{O}(2-2a,0))$, and $A_2 \in \mathrm{H}^0(\mathcal{O}(0,2))$, $B_2 \in \mathrm{H}^0(\mathcal{O}(2a,2))$, $C_2 \in \mathrm{H}^0(\mathcal{O}(-2a,2))$. If $a \geq 2$, then $C_1 = C_2 = 0$, and so $\mathcal{O}(a,0)$ would be Φ-invariant. However, $\mu(\mathcal{O}(a,0)) > \mu(E)$, which contradicts stability. A similar argument, but interchanging the roles of the C_i's for the B_i's, and of $\mathcal{O}(a,0)$ for $\mathcal{O}(-a,0)$, shows that $a > -2$. The result follows. □

We first consider $E = \mathcal{O} \oplus \mathcal{O}$, which is a strictly semistable bundle. It would be desirable to describe all the possible Higgs fields that E admits. However, we do not yet know the shape of the Higgs fields Φ for which (E, Φ) is stable; i.e., those Φ's for which no copy of $\mathcal{O}$ inside E is Φ-invariant. We do, however, describe those Higgs fields which are strictly semistable. We remind the reader that, in the moduli spaces $\mathcal{M}^{\mathrm{co}}(c_1, c_2)$, two objects are identified if they are S-equivalent (see Section 2.3).

Any element of $\mathrm{H}^0(\mathrm{End}_0 E \otimes T)$ is of the form

$$\Phi = \Phi_1 + \Phi_2 = \begin{pmatrix} A_1 & B_1 \\ C_1 & -A_1 \end{pmatrix} + \begin{pmatrix} A_2 & B_2 \\ C_2 & -A_2 \end{pmatrix}$$

with $A_1, B_1, C_1 \in \mathrm{H}^0(\mathcal{O}(2,0))$ and $A_2, B_2, C_2 \in \mathrm{H}^0(\mathcal{O}(0,2))$.

We now need to consider the Φ's as above, which are integrable; i.e., the $\Phi's$ that satisfy equations (2.8). Furthermore, we focus our attention in those Higgs fields which make (E, Φ) into a strictly semistable co-Higgs bundle.

A pair (E,Φ) is strictly semistable if and only if there is a Φ-invariant copy of $\mathcal{O}$ in E. This is equivalent to the existence of a non-zero $v \in \mathrm{H}^0(E)$ such that $\Phi(v) = v\otimes\lambda_1 + v\otimes\lambda_2$, where $\lambda_1 \in \mathrm{H}^0(\mathcal{O}(2,0))$ and $\lambda_2 \in \mathrm{H}^0(\mathcal{O}(0,2))$. Writing Φ as

$$\Phi = (M_0 + M_1 z_1 + M_2 z_1^2) + (N_0 + N_1 z_2 + N_2 z_2^2),$$

where the M_i's and the N_i's are 2×2 complex valued matrices, we see that $\Phi(v) = v\otimes\lambda_1 + v\otimes\lambda_2$ if and only if v is a common eigenvector of the M_i's and N_i's (note that, in this case, the coefficients of λ_1, λ_2 are the eigenvalues of the M_i's and the N_i's, respectively). If this is the case, by a change of basis, we may assume that the M_i's and N_i's are upper triangular. Therefore, (E,Φ) is strictly semistable if and only if Φ is upper triangular and its matrix coefficients admit a common eigenvector. In this case, by Lemma 2.37, we have that

$$\mathrm{gr}(E,\Phi) = \left(E, \begin{pmatrix} A_1 & 0 \\ 0 & -A_1 \end{pmatrix} + \begin{pmatrix} A_2 & 0 \\ 0 & -A_2 \end{pmatrix}\right).$$

Note that two graded objects $\mathrm{gr}(E,\Phi)$ and $\mathrm{gr}(E,\Phi')$ are isomorphic if and only if $A_1 + A_2 = \pm(A_1' + A_2')$. Indeed, two matrices $\begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}$ and $\begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix}$ live in the same S-equivalence class, that is

$$\begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} = \Psi \begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix} \Psi^{-1},$$

if and only if $A = \pm B$. Therefore, a set of representatives for the S-equivalence classes of strictly semistable pairs, with underlying bundle E, is given by

$$\left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \ : \ A \in \mathrm{H}^0(T) \right\} / \sim$$

where $\sim$ is defined by $A \sim B$ if and only if $A = \pm B$.

On the other hand, note that for the bundle $E = \mathcal{O}(1,0) \oplus \mathcal{O}(-1,0)$, any element of $\mathrm{H}^0(\mathrm{End}_0\, E \otimes T)$ is of the form

$$\Phi = \Phi_1 + \Phi_2 = \begin{pmatrix} A_1 & B_1 \\ C_1 & -A_1 \end{pmatrix} + \begin{pmatrix} A_2 & B_2 \\ 0 & -A_2 \end{pmatrix}$$

with $A_1 \in \mathrm{H}^0(\mathcal{O}(2,0))$, $B_1 \in \mathrm{H}^0(\mathcal{O}(4,0))$, $C_1 \in \mathrm{H}^0(\mathcal{O})$ and $A_2 \in \mathrm{H}^0(\mathcal{O}(0,2))$ and $B_2 \in \mathrm{H}^0(\mathcal{O}(2,2))$. Note that, if Φ were a Higgs field of E, then C_1 must be non-zero, as otherwise it would leave $\mathcal{O}(1,0)$ invariant, contradicting stability. Also, taking into account the integrability condition, equations (2.8) imply that $A_2 = B_2 = 0$. Therefore, any possible Higgs field of E is of the form $\Phi = \Phi_1 \in \mathrm{H}^0(\mathrm{End}_0\, E \otimes \mathcal{O}(2,0))$, with Φ_1 as above, and non-zero C_1.

Now, we observe that (E,Φ) is in fact co-Higgs stable for any Φ as above. Note that the sub-line bundles of E are of the form $\mathcal{O}(r,s)$ with $r \leq 1$ and $s \leq 0$ or $r \leq -1$ and $s \leq 0$. As such, the only sub-line bundles that could potentially contradict stability are $\mathcal{O}$ and $\mathcal{O}(1,0)$. However, any degree zero sub-line bundle of E; that is, any copy of $\mathcal{O}$ in E, is contained in $\mathcal{O}(1,0)$; and so, since the latter is not Φ-invariant, the result follows.

We now claim that a set of representatives for the isomorphism classes (recall that in this case S-equivalence reduces to co-Higgs isomorphism) is given by

$$\left\{ \begin{pmatrix} 0 & B \\ 1 & 0 \end{pmatrix} \;:\; B \in \mathrm{H}^0(\mathcal{O}(4,0)) \right\}.$$

Indeed, given any Higgs field Φ as described above, letting

$$\Psi = \begin{pmatrix} 1 & -A_1 \\ 0 & 1 \end{pmatrix}$$

we get

$$\Psi\Phi\Psi^{-1} = \begin{pmatrix} 0 & B_1 - A_1^2 \\ 1 & 0 \end{pmatrix}.$$

Similarly, for the bundle $E = \mathcal{O}(0,1) \oplus \mathcal{O}(0,-1)$, a set of representatives for the isomorphism class is

$$\left\{ \begin{pmatrix} 0 & B \\ 1 & 0 \end{pmatrix} \;:\; B \in \mathrm{H}^0(\mathcal{O}(0,4)) \right\}.$$

Following the description of [28, Section 8] for the case of curves, we now show that we can view the set of stable co-Higgs bundles, in $\mathcal{M}^{\mathrm{co}}(0,0)$, with underlying bundle $\mathcal{O}(1,0) \oplus \mathcal{O}(-1,0)$ or $\mathcal{O}(0,1) \oplus \mathcal{O}(0,-1)$ as sections of certain maps, as follows:

Consider the maps

$$\begin{array}{rccc} h_1 : & \mathcal{M}^{\mathrm{co}}(0,0) & \to & \mathrm{H}^0(\mathcal{O}(4,0)) \\ & (E, \Phi = \Phi_1 + \Phi_2) & \mapsto & \det \Phi_1 \end{array}$$

and

$$\begin{array}{rccc} h_2 : & \mathcal{M}^{\mathrm{co}}(0,0) & \to & \mathrm{H}^0(\mathcal{O}(0,4)) \\ & (E, \Phi = \Phi_1 + \Phi_2) & \mapsto & \det \Phi_2. \end{array}$$

Define

$$\begin{array}{rccc} Q_1 : & \mathrm{H}^0(\mathcal{O}(4,0)) & \to & \mathcal{M}^{\mathrm{co}}(0,0) \\ & \rho & \mapsto & \left(\mathcal{O}(1,0) \oplus \mathcal{O}(-1,0), \Phi_1 = \begin{pmatrix} 0 & -\rho \\ 1 & 0 \end{pmatrix} \right) \end{array}$$

and

$$\begin{array}{rccc} Q_2 : & \mathrm{H}^0(\mathcal{O}(0,4)) & \to & \mathcal{M}^{\mathrm{co}}(0,0) \\ & \rho & \mapsto & \left(\mathcal{O}(0,1) \oplus \mathcal{O}(0,-1), \Phi_2 = \begin{pmatrix} 0 & -\rho \\ 1 & 0 \end{pmatrix} \right). \end{array}$$

Clearly, Q_i is a section of h_i for $i = 1, 2$. Moreover, by the above discussion, we have:

Proposition 4.11. *The images of the sections Q_1, Q_2 in $\mathcal{M}^{co}(0,0)$ are precisely the set of stable co-Higgs bundles with underlying bundle $\mathcal{O}(1,0) \oplus \mathcal{O}(-1,0)$ and $\mathcal{O}(0,1) \oplus \mathcal{O}(0,-1)$, respectively.*

Remark 4.12. Note that every point in $\mathcal{M}^{\mathrm{co}}(0,0)$ with underlying bundle $\mathcal{O}(1,0) \oplus \mathcal{O}(-1,0)$ or $\mathcal{O}(0,1) \oplus \mathcal{O}(0,-1)$ is the pullback from $\mathbb{P}^1$ to $\mathbb{P}^1 \times \mathbb{P}^1$ of a stable co-Higgs pair with underlying bundle $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ with respect to the first or second projections, respectively. Similarly, any point in $\mathcal{M}^{\mathrm{co}}(0,0)$ with underlying bundle $\mathcal{O} \oplus \mathcal{O}$ can be obtained from a stable co-Higgs bundle $(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}, \phi)$ over $\mathbb{P}^1$ by taking the pullback of ϕ with respect to the two projections; i.e., $(\mathcal{O} \oplus \mathcal{O}, \mathrm{pr}_1^* \phi + \mathrm{pr}_2^* \phi)$.

Remark 4.13. Let us now discuss the spectral correspondence. Again, for $\rho = (\rho_1, 0, 0)$, where $\rho_1 \in \mathrm{H}^0(\mathcal{O}(4,0))$ is generic, we have that $S_\rho = X_{\rho_1} \times \mathbb{P}^1$. In this case, pushing-down the line bundle $M = \mathrm{Pr}_1^*(\pi^*\mathcal{O}_{\mathbb{P}^1}(1))$ yields the bundle $\mathcal{O}(1,0) \oplus \mathcal{O}(-1,0)$. Since the degree 2 bundle $\pi^*\mathcal{O}_{\mathbb{P}^1}(1)$ is the unique line bundle over X_{ρ_1} which corresponds to $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, the line bundle M is the unique line bundle over S_ρ that corresponds to $\mathcal{O}(1,0) \oplus \mathcal{O}(-1,0)$ with Higgs field

$$\Phi = \Phi_1 = \begin{pmatrix} 0 & -\rho_1 \\ 1 & 0 \end{pmatrix}.$$

This follows again from the discussion in Section 4.1.2. Similar observations can be made when $\rho = (0,0,\rho_2)$, where $\rho_2 \in \mathrm{H}^0(\mathcal{O}(0,4))$ is generic. Clearly, in this case, the correspondence yields the underlying bundle $\mathcal{O}(0,1) \oplus \mathcal{O}(0,-1)$ and Higgs fields of the form

$$\Phi = \Phi_2 = \begin{pmatrix} 0 & -\rho_2 \\ 1 & 0 \end{pmatrix}.$$

Finally, pushing-down any line bundle M over S_{ρ_i} (for $i = 1,2$) of the form $\mathrm{Pr}_i^*(L)$, where L is any other degree 2 bundle over X_{ρ_i} corresponds to a Higgs field of the form $\Phi = \Phi_i$ which makes $\mathcal{O} \oplus \mathcal{O}$ into a stable pair.

4.2.2 Second Chern Class $c_2 = 1$

We now turn our attention to co-Higgs pairs (E,Φ) over $\mathbb{P}^1 \times \mathbb{P}^1$ with $c_1 = -F$ and $c_2 = 1$. Once again, recall that, in this case, stability and semistability are identical notions. In studying the possible underlying bundles for stable co-Higgs pairs (E,Φ), we will show that E is an extension of $\mathcal{O}(-1,1)$ by $\mathcal{O}(0,-1)$.

We begin by proving a technical lemma.

Lemma 4.14. *Let x be a point in $\mathbb{P}^1 \times \mathbb{P}^1$ and suppose that $c_2(E) = 1$. If E fits into the exact sequence*

$$0 \to \mathcal{O} \xrightarrow{\iota} E \xrightarrow{p} \mathcal{O}(-1,0) \otimes I_x \to 0, \tag{4.3}$$

then $\mathcal{O}$ is Φ-invariant for any $\Phi \in \mathrm{H}^0(\mathrm{End}_0 E \otimes T)$.

Proof. First note that since $\mathrm{Ext}^1(\mathcal{O}(-1,0) \otimes I_x, \mathcal{O}) \cong \mathrm{H}^0(\mathcal{O}_x) \cong \mathbb{C}$, up to isomorphism, there is a unique E that fits into (4.3). Now, tensoring (4.3) with T and passing to the long exact sequence in cohomology, we get

$$0 \to \mathrm{H}^0(T) \xrightarrow{\iota} \mathrm{H}^0(E \otimes T) \xrightarrow{p} \mathrm{H}^0(\mathcal{O}(-1,0) \otimes I_x \otimes T) \to 0,$$

where we also denote by ι and p the induced map $\iota \otimes \mathrm{Id}_T$ and $p \otimes \mathrm{Id}_T$, respectively. Note that $\mathrm{Im}(\iota) = \mathrm{Ker}(p)$. Moreover, using Grothendieck–Riemann–Roch [18, Appendix A], one can check that $\pi_*(I_x) = \mathcal{O}_{\mathbb{P}^1}(-1)$. Therefore,

$$\begin{aligned} \mathrm{H}^0(\mathcal{O}(-1,0) \otimes I_x \otimes T) &= \mathrm{H}^0(\mathbb{P}^1 \times \mathbb{P}^1; I_x(1,0)) \\ &= \mathrm{H}^0(\mathbb{P}^1; \pi_*(I_x(1,0))) \\ &= \mathrm{H}^0(\mathbb{P}^1; \mathcal{O}_{\mathbb{P}^1}(1) \otimes \pi_*(I_x)) \\ &= \mathrm{H}^0(\mathbb{P}^1; \mathcal{O}_{\mathbb{P}^1}), \end{aligned}$$

so the non-zero elements of $\mathrm{H}^0(\mathcal{O}(-1,0)\otimes I_x\otimes T)$ are nowhere vanishing.

With this in mind, we now claim that if

$$S=\{\varphi\in \mathrm{H}^0(\mathrm{Hom}(\mathcal{O},E\otimes T)\mid \varphi=\Phi_{|\mathcal{O}},\Phi\in \mathrm{H}^0(\mathrm{End}_0\,E\otimes T)\},$$

$S\subseteq \mathrm{Im}(\iota)$, implying that $\mathcal{O}$ is Φ-invariant for any $\Phi\in \mathrm{H}^0(\mathrm{End}_0\,E\otimes T)$, by Lemma 2.32. Indeed, let $\varphi\in S$ so that $\varphi=\Phi\circ\iota$ for some $\Phi\in \mathrm{H}^0(\mathrm{End}_0\,E\otimes T)$. Then,

$$p(\varphi)=p\circ\Phi\circ\iota,$$

which vanishes at x since ι does (otherwise the quotient $E/\iota(\mathcal{O})$ would be locally free). Hence, $p(\varphi)=0$ because non-zero elements of $\mathrm{H}^0(\mathcal{O}(-1,0))\otimes I_x\otimes T$ are nowhere vanishing, proving that $\varphi\in \mathrm{Ker}(p)=\mathrm{Im}(\iota)$. □

We are now ready to prove that the only possible underlying bundles for stable co-Higgs pairs are extensions of $\mathcal{O}(-1,1)$ by $\mathcal{O}(0,-1)$.

Proposition 4.15. *Let (E,Φ) be a stable co-Higgs pair such that $c_1(E)=-F$ and $c_2(E)=1$. Then E is an extension of $\mathcal{O}(-1,1)$ by $\mathcal{O}(0,-1)$.*

Proof. Let E have invariants d and r. Then we know that E fits into an exact sequence of the form

$$0\to\mathcal{O}(r,d)\to E\to\mathcal{O}(-1-r,-d)\otimes I_Z\to 0, \tag{4.4}$$

with $\ell(Z)=1+d(2r+1)\geq 0$. Now, for such a rank-2 vector bundle to exist, we know by Theorem 2.21 that one of the following two conditions must be satisfied.

1. $d\geq 1$, or
2. $d=0$ and $r\geq -1$.

In case (1), we consider two subcases:

(i) $r\geq 0$. By tensoring (4.4) with $\mathcal{O}(r,d)^\vee\otimes T$, and passing to the long exact sequence in cohomology, we get

$$0\to \mathrm{H}^0(T)\to \mathrm{H}^0(E(-r,-d)\otimes T)\to \mathrm{H}^0(\mathcal{O}(-1-2r,-2d)\otimes T\otimes I_Z)\to 0,$$

where $\mathrm{H}^0(\mathcal{O}(-1-2r,-2d)\otimes T\otimes I_Z)=0$, and so, by Lemma 2.32, $\mathcal{O}(r,d)$ is Φ-invariant for any $\Phi\in \mathrm{H}^0(\mathrm{End}_0\,E\otimes T)$, and thus destabilizing. This contradicts stability, and thus this case cannot happen.

(ii) $r\leq -1$. If either $r<-1$, or $d>1$ and $r=-1$, then $\ell(Z)<0$, which is impossible. Hence $d=1$ and $r=-1$, so that $\ell(Z)=0$, and therefore E is an extension of $\mathcal{O}(0,-1)$ by $\mathcal{O}(-1,1)$. However, $\mathrm{H}^1(\mathcal{O}(-1,2))=0$, implying that $E=\mathcal{O}(-1,1)\oplus\mathcal{O}(0,-1)$.

In case (2) we consider three subcases:

(i) $r\geq 1$. By the exact same argument as above, one can check that $\mathcal{O}(r,d)$ is Φ-invariant for any $\Phi\in \mathrm{H}^0(\mathrm{End}_0\,E\otimes T)$, and thus destabilizing. Hence this case cannot happen.

(ii) $r = 0$. In this case, E fits into an exact sequence of the form

$$0 \to \mathcal{O} \to E \to \mathcal{O}(-1,0) \otimes I_x \to 0,$$

and so, by Lemma 4.14, $\mathcal{O}$ is Φ-invariant for any $\Phi \in \mathrm{H}^0(\operatorname{End} E \otimes T)$, and thus destabilizing. Again, this case cannot happen.

(iii) $r = -1$. In this case, E fits into an exact sequence of the form

$$0 \to \mathcal{O}(-1,0) \to E \to I_x \to 0.$$

Now, since $\operatorname{Ext}^1(I_x, \mathcal{O}(-1,0)) = \mathrm{H}^0(\mathcal{O}_x) = \mathbb{C}$, there is a unique bundle, up to isomorphism, that fits into this exact sequence. Hence, E is completely determined by the invariants $d = 0$ and $r = -1$, up to isomorphism. On the other hand, any non-trivial extension E' of $\mathcal{O}(-1,1)$ by $\mathcal{O}(0,-1)$ has invariants $d = 0$ and $r = -1$. Indeed, the restriction of E' to the generic fibre is a non-trivial extension of $\mathcal{O}_{\mathbb{P}^1}(1)$ by $\mathcal{O}_{\mathbb{P}^1}(-1)$ over $\mathbb{P}^1$, so $d = 0$. Moreover, pushing down the extension

$$0 \to \mathcal{O}(-1,1) \to E' \to \mathcal{O}(0,-1) \to 0$$

to $\mathbb{P}^1$, we obtain

$$(\mathrm{pr}_1)_*(E) = (\mathrm{pr}_1)_*(\mathcal{O}(-1,1)) = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

Thus, $r = -1$. Hence, $E \cong E'$ and E is a non-trivial extension of $\mathcal{O}(-1,1)$ by $\mathcal{O}(0,-1)$. $\square$

As before, now that we know the possible underlying bundles for stable co-Higgs pairs, we can check whether they admit (non-trivial) Higgs fields.

We start by working with the trivial extension $E = \mathcal{O}(0,-1) \oplus \mathcal{O}(-1,1)$. In this case, any element of $\mathrm{H}^0(\operatorname{End}_0 E \otimes T)$ is of the form

$$\Phi = \Phi_1 + \Phi_2 = \begin{pmatrix} A_1 & 0 \\ C_1 & -A_1 \end{pmatrix} + \begin{pmatrix} A_2 & B_2 \\ 0 & -A_2 \end{pmatrix},$$

with $A_1 \in \mathrm{H}^0(\mathcal{O}(2,0))$, $C_1 \in \mathrm{H}^0(\mathcal{O}(1,2))$, and $A_2 \in \mathrm{H}^0(\mathcal{O}(0,2))$, $B_2 \in \mathrm{H}^0(\mathcal{O}(1,0))$. Note that B_2 cannot be identically zero, for otherwise it would leave $\mathcal{O}(-1,1)$ invariant, contradicting stability. Again, taking into account the integrability condition, equations (2.8) imply that $A_1 = C_1 = 0$. Therefore, any possible Higgs field of E is of the form

$$\Phi = \Phi_2 \in \mathrm{H}^0(\operatorname{End}_0(0,2)),$$

with Φ_2 as above and B_2 not identically zero. The fact that (E, Φ) is indeed co-Higgs stable for these Higgs fields follows from Lemma 2.38.

Now, note that an automorphism ψ of $E = \mathcal{O}(0,-1) \oplus \mathcal{O}(-1,1)$ can be chosen of the form

$$\psi = \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix} \in \mathrm{H}^0(\operatorname{End} E),$$

where P is a non-zero global section of $\mathcal{O}$. We then have that

$$\psi \circ \Phi \circ \psi^{-1} = \begin{pmatrix} A_2 & P^{-1} B_2 \\ 0 & -A_2 \end{pmatrix}.$$

Since $B_2 \in \mathrm{H}^0(\mathcal{O}(1,0))$, we can locally write $B_2 = \alpha(z_1 - p)$, so by choosing $P = \alpha^{-1}$, we have a representative of the conjugacy class of Φ of the form

$$\Phi = \begin{pmatrix} A_2 & z_1 - p \\ 0 & -A_2 \end{pmatrix}. \tag{4.5}$$

We now turn our attention to the non-trivial extensions

$$0 \to \mathcal{O}(0,-1) \to E \to \mathcal{O}(-1,1) \to 0. \tag{4.6}$$

These, as they are stable bundles with respect to the standard polarization, admit the zero Higgs field. They nonetheless also admit non-zero Higgs fields. To prove this, we do the following:

(i) Check that $\mathrm{h}^0(\operatorname{End}_0 E \otimes T) \neq 0$.

(ii) Check which elements of $\mathrm{H}^0(\operatorname{End}_0 E \otimes T)$ satisfy the integrability condition.

A direct computation gives (i):

Lemma 4.16. *If E is a non-trivial extension of $\mathcal{O}(-1,1)$ by $\mathcal{O}(0,-1)$, then $\mathrm{h}^0(\operatorname{End}_0 E(2,0)) = 6$ and $\mathrm{h}^0(\operatorname{End}_0 E(0,2)) = 5$. In particular,*

$$\mathrm{h}^0(\operatorname{End}_0 E \otimes T) = 11.$$

Proof. The non-trivial extension E is given by a class in $\mathrm{H}^1(\mathcal{O}(1,-2)) = \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \oplus \mathrm{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2))$, which vanishes at a single point x_0 in the first factor of $\mathbb{P}^1 \times \mathbb{P}^1$. Therefore,

$$E_{|F_x} = \begin{cases} \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} & \text{if } x \neq x_0 \\ \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) & \text{if } x = x_0 \end{cases}$$

where $F_x = (\mathrm{pr}_1)^{-1}(x)$.

Now, in order to compute the dimension of $\mathrm{H}^0(\operatorname{End}_0 E(2,0))$, take the dual sequence of (4.6), tensor it by $E \otimes \mathcal{O}(2,0)$, and then push-forward it to the first copy of $\mathbb{P}^1$, we are left with:

$$0 \to (\mathrm{pr}_1)_* \operatorname{End} E(2,0) \to (\mathrm{pr}_1)_* E(2,1) \to R^1(\mathrm{pr}_1)_* E(3,-1) \to R^1(\mathrm{pr}_1)_* \operatorname{End} E(2,0) \to 0.$$

Note that both $R^1(\mathrm{pr}_1)_* E(3,-1)$ and $R^1(\mathrm{pr}_1)_* \operatorname{End} E(2,0)$ are skyscraper sheaves supported at x_0. Hence, $(\mathrm{pr}_1)_* \operatorname{End} E(2,0) \cong (\mathrm{pr}_1)_* E(2,1)$, and so we get that $\mathrm{H}^0(\operatorname{End} E(2,0)) = \mathrm{H}^0(E(2,1))$. Now, tensoring (4.6) by $\mathcal{O}(2,1)$ and passing to the long exact sequence in cohomology, we get

$$0 \to \mathrm{H}^0(\mathcal{O}(2,0)) \to \mathrm{H}^0(E(2,1)) \to \mathrm{H}^0(\mathcal{O}(1,2)) \to 0.$$

Hence, $\mathrm{h}^0(E(2,1)) = \mathrm{h}^0(\mathcal{O}(2,0)) + \mathrm{h}^0(\mathcal{O}(1,2)) = 9$, and so $\mathrm{h}^0(\operatorname{End} E(2,0)) = 9$. Finally, since $\operatorname{End} E(2,0) = \operatorname{End}_0 E(2,0) \oplus \mathcal{O}(2,0)$, we get that $\mathrm{h}^0(\operatorname{End}_0 E(2,0)) = 6$.

Now, in order to compute the dimension of $\mathrm{H}^0(\mathrm{End}_0 E(0,2))$, take the dual sequence of (4.6) and tensor it by $E(0,2)$ to get:

$$0 \to \mathrm{H}^0(E(1,1)) \to \mathrm{H}^0(\mathrm{End}\, E(0,2)) \to \mathrm{H}^0(E(0,3)) \to \mathrm{H}^1(E(1,1)) \to \cdots \tag{4.7}$$

In order to compute $\mathrm{h}^i(E(1,1))$, tensor (4.6) by $\mathcal{O}(1,1)$ and pass to the long exact sequence in cohomology:

$$0 \to \mathrm{H}^0(\mathcal{O}(1,0)) \to \mathrm{H}^0(E(1,1)) \to \mathrm{H}^0(\mathcal{O}(0,2)) \to 0.$$

Hence $\mathrm{h}^0(E(1,1)) = \mathrm{h}^0(\mathcal{O}(1,0)) + \mathrm{h}^0(\mathcal{O}(0,2)) = 5$ and $\mathrm{h}^1(E(1,1)) = 0$.

Now to compute $\mathrm{h}^0(E(0,3))$, tensor (4.6) by $\mathcal{O}(0,3)$ and pass to the long exact sequence in cohomology

$$0 \to \mathrm{H}^0(\mathcal{O}(0,2)) \to \mathrm{H}^0(E(0,3)) \to 0$$

Hence $\mathrm{h}^0(E(0,3)) = \mathrm{h}^0(\mathcal{O}(0,2)) = 3$. It now follows, from (4.7), that

$$\mathrm{h}^0(\mathrm{End}\, E(0,2)) = \mathrm{h}^0(E(1,1)) + \mathrm{h}^0(E(0,3)) = 8.$$

Finally, we have that $\mathrm{h}^0(\mathrm{End}_0 E(0,2)) = \mathrm{h}^0(\mathrm{End}\, E(0,2)) - \mathrm{h}^0(\mathcal{O}(0,2)) = 5$. Therefore,

$$\mathrm{h}^0(\mathrm{End}_0 E \otimes T) = \mathrm{h}^0(\mathrm{End}_0 E(2,0)) + \mathrm{h}^0(\mathrm{End}_0(0,2)) = 11$$

. $\square$

Let us now determine which elements of $\mathrm{H}^0(\mathrm{End}_0 E \otimes T)$ satisfy the integrability condition. We begin by giving a local description of $\mathrm{H}^0(\mathrm{End}_0 E \otimes T)$. In order to do so, we will need to explicitly know the transition functions of $\mathrm{End}_0 E \otimes T$, and so we will explain how to obtain the transition functions, $g_{ij}^{\mathrm{End}_0 E}$, of $\mathrm{End}_0 E$ from the transition functions, g_{ij}^E, of E. Obtaining the transition functions of $\mathrm{End}_0 E \otimes T$ from those of $\mathrm{End}_0 E$ is immediate. We fix the standard open cover of $\mathbb{P}^1 \times \mathbb{P}^1$:

$$\begin{aligned} V_1 &= \mathcal{U}_0^1 \times \mathcal{U}_0^2 \\ V_2 &= \mathcal{U}_0^1 \times \mathcal{U}_\infty^2 \\ V_3 &= \mathcal{U}_\infty^1 \times \mathcal{U}_0^2 \\ V_4 &= \mathcal{U}_\infty^1 \times \mathcal{U}_\infty^2, \end{aligned}$$

where $\mathcal{U}_0^i$ is the affine open subset of the i-th copy of $\mathbb{P}^1$ that does not contain the point at infinity, and $\mathcal{U}_\infty^i$ is the affine open subset of the i-th copy of $\mathbb{P}^1$ that does not contain zero. Let us work on the intersection $V_{ij} := V_i \cap V_j$. Given the trivializations φ_i, φ_j of E, we know we can obtain (local) frames $\{e_1, e_2\}, \{f_1, f_2\}$ by pulling back the standard basis of $\mathbb{C}^2$ under the maps φ_i, φ_j, respectively. Furthermore, we may obtain dual frames $\{e_1^*, e_2^*\}$ and $\{f_1^*, f_2^*\}$ for $E^\vee$, which also correspond to the pullbacks of the standard basis of $\mathbb{C}^2$ under $\varphi_i^* = (\varphi_i^{-1})^T$ and $\varphi_j^* = (\varphi_j^{-1})^T$, respectively. It is clear then that both $\mathcal{E} = \{e_1 \otimes e_1^*, e_1 \otimes e_2^*, e_2 \otimes e_1^*, e_2 \otimes e_2^*\}$ and $\mathcal{F} = \{f_1 \otimes f_1^*, f_1 \otimes f_2^*, f_2 \otimes f_1^*, f_2 \otimes f_2^*\}$ are frames for $\mathrm{End} E$ over V_i and V_j, respectively. Furthermore, $\mathrm{End}\, E = \mathrm{End}_0\, E \oplus \mathcal{O}$. Hence, we may obtain another frame for $\mathrm{End} E$ over V_i, $\mathcal{E}' = \{e_1 \otimes e_1^* - e_2 \otimes e_2^*, e_1 \otimes e_2^*, e_2 \otimes e_1^*, e_1 \otimes e_1^* + e_2 \otimes e_2^*\}$, where the first three elements form a frame for $\mathrm{End}_0 E$ over V_i, and the last one forms a frame for $\mathcal{O}$. Similarly, we get $\mathcal{F}' = \{f_1 \otimes f_1^* - f_2 \otimes f_2^*, f_1 \otimes f_2^*, f_2 \otimes f_1^*, f_1 \otimes f_1^* + f_2 \otimes f_2^*\}$, where the first three elements form a frame for $\mathrm{End}_0 E$ over V_j, and the last one forms a frame for $\mathcal{O}$. Hence, $g_{ij}^{End_0 E}$ will be the restriction to $\mathrm{End}_0 E$ of the change of basis matrix from $\mathcal{F}$ to $\mathcal{E}$. By linear algebra, we know that

$$g_{ij}^{\mathrm{End}_0 E \oplus \mathcal{O}} = P_{\mathcal{E}} g_{ij}^{\mathrm{End}\, E} P_{\mathcal{F}},$$

where $P_{\mathcal{E}}$ is the change of basis matrix from $\mathcal{E}$ to $\mathcal{E}'$, and $P_{\mathcal{F}}$ is the change of basis matrix from $\mathcal{F}'$ to $\mathcal{F}$. Hence, if we let $g_{ij}^E = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$ be the transition function of E on V_{ij}, then the transition function of $\mathrm{End} E$ on V_{ij} is given by

$$g_{ij}^{\mathrm{End}\, E} = \frac{1}{\det g_{ij}^E} \begin{pmatrix} G_{11}G_{22} & -G_{11}G_{21} & G_{12}G_{22} & -G_{12}G_{21} \\ -G_{11}G_{12} & G_{11}G_{11} & -G_{12}G_{12} & G_{12}G_{11} \\ G_{21}G_{22} & -G_{21}G_{21} & G_{22}G_{22} & -G_{22}G_{21} \\ -G_{21}G_{12} & G_{21}G_{11} & -G_{22}G_{12} & G_{22}G_{11} \end{pmatrix},$$

and so the transition function for $\mathrm{End}_0\, E$ on V_{ij} is the matrix

$$g_{ij}^{\mathrm{End}_0\, E} = \frac{1}{\det g_{ij}^E} \begin{pmatrix} G_{11}G_{22} + G_{21}G_{12} & -G_{11}G_{21} & G_{12}G_{22} \\ -2G_{11}G_{12} & G_{11}G_{11} & -G_{12}G_{12} \\ 2G_{21}G_{22} & -G_{21}G_{21} & G_{22}G_{22} \end{pmatrix}.$$

In particular, if we let $(uz_1 + v)z_2^{-1}$ be the (non-zero) element in $\mathrm{H}^1(\mathcal{O}(1, -2))$ that determines the (non-trivial) extension E, then we know that in V_{12} and V_{13} the transition functions of E are given by

$$g_{12}^E = \begin{pmatrix} z_2^{-1} & (uz_1 + v) \\ 0 & z_2 \end{pmatrix},$$

and

$$g_{13}^E = \begin{pmatrix} 1 & 0 \\ 0 & z_1^{-1} \end{pmatrix}.$$

Thus, letting $g_{ij}^{(2,0)}$ and $g_{ij}^{(0,2)}$ denote the transition functions of $\mathrm{End}_0\, E(2,0)$ and $\mathrm{End}_0\, E(0,2)$, respectively, we have that

$$g_{12}^{(2,0)} = \begin{pmatrix} 1 & 0 & uz_1z_2 + vz_2 \\ -2(uz_1z_2^{-1} + vz_2^{-1}) & z_2^{-2} & -(uz_1 + v)^2 \\ 0 & 0 & z_2^2 \end{pmatrix}$$

$$g_{13}^{(2,0)} = \begin{pmatrix} z_1^2 & 0 & 0 \\ 0 & z_1^3 & 0 \\ 0 & 0 & z_1 \end{pmatrix}$$

$$g_{12}^{(0,2)} = \begin{pmatrix} z_2^2 & 0 & uz_1z_2^3 + vz_2^3 \\ -2(uz_1z_2 + vz_2) & 1 & -(uz_1 + v)^2 z_2^2 \\ 0 & 0 & z_2^4 \end{pmatrix}$$

$$g_{13}^{(0,2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & z_1 & 0 \\ 0 & 0 & z_1^{-1} \end{pmatrix}.$$

Note that the above transition functions are 3×3 matrices; thus, for this purpose, we will treat the trace-free sections Φ_1 and Φ_2 as 3×1 vectors. Let Φ_j^i, $(j = 1, 2)$ be the trivialization of Φ_j on V_i.

We will work on the open set V_1. In order to describe $\Phi_1^1 \in \mathrm{H}^0(\mathrm{End}_0\, E(2,0))$, let

$$\Phi_1^1 = \begin{pmatrix} \sum\limits_{i,j\geq 0} a_{ij}^1 z_1^i z_2^j \\ \sum\limits_{i,j\geq 0} b_{ij}^1 z_1^i z_2^j \\ \sum\limits_{i,j\geq 0} c_{ij}^1 z_1^i z_2^j \end{pmatrix}.$$

Using the fact that $\Phi_1^1 = g_{13}^{(2,0)} \Phi_1^3$, a straightforward computation shows that $a_{ij}^1 = 0$ for $i > 2$, $b_{ij}^1 = 0$ for $i > 3$ and $c_{ij}^1 = 0$ for $i > 1$. Similarly, using the fact that $\Phi_1^1 = g_{12}^{(2,0)} \Phi_1^2$ we get that $a_{ij}^1 = 0$ for $j > 1$, $b_{ij}^1 = 0$ for $j > 0$ and $c_{ij}^1 = 0$ for $j > 2$. Furthermore, we get that

$$\begin{gathered} a_{00}^1 = \tfrac{1}{2} v c_{01}^1 \\ a_{01}^1 = v c_{02}^1 \\ a_{10}^1 = \tfrac{u}{2} c_{01}^1 \\ a_{11}^1 = u c_{02}^1 + v c_{12}^1 \\ a_{20}^1 = \tfrac{u}{2} c_{11}^1 \\ a_{21}^1 = u c_{12}^1 \\ b_{00}^1 = -v^2 c_{02}^1 \\ b_{10}^1 = -(v^2 c_{12}^1 + 2uv c_{02}^1) \\ b_{20}^1 = -(u^2 c_{02}^1 + 2uv c_{12}^1) \\ b_{30}^1 = -u^2 c_{12}^1, \end{gathered}$$

and so

$$\Phi_1^1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & -A_1 \end{pmatrix}, \tag{4.8}$$

where

$$\begin{aligned} A_1 &= \tfrac{1}{2} v c_{01}^1 + v c_{02}^1 z_2 + \tfrac{u}{2} c_{01}^1 z_1 + (u c_{02}^1 + v c_{12}^1) z_1 z_2 + \tfrac{u}{2} c_{11}^1 z_1^2 + u c_{12}^1 z_1^2 z_2, \\ B_1 &= -v^2 c_{02}^1 - (v^2 c_{12}^1 + 2uv c_{02}^1) z_1 - (u^2 c_{02}^1 + 2uv c_{12}^1) z_1^2 - u^2 c_{12}^1 z_1^3, \\ C_1 &= c_{00}^1 + c_{01}^1 z_2 + c_{02}^1 z_2^2 + c_{10}^1 z_1 + c_{11}^1 z_1 z_2 + c_{12}^1 z_1 z_2^2. \end{aligned} \tag{4.9}$$

Remark 4.17. Note that the above equations imply that, in Φ_1, A_1 and B_1 depend on C_1. In particular, if C_1 is zero, then $A_1 = B_1 = 0$ and $\Phi = 0$.

We will now describe $\Phi_2^1 \in \mathrm{H}^0(\mathrm{End}_0\, E(0,2))$. As before, let

$$\Phi_2^1 = \begin{pmatrix} \sum\limits_{i,j\geq 0} a_{ij}^2 z_1^i z_2^j \\ \sum\limits_{i,j\geq 0} b_{ij}^2 z_1^i z_2^j \\ \sum\limits_{i,j\geq 0} c_{ij}^2 z_1^i z_2^j \end{pmatrix}.$$

Using the fact that $\Phi_2^1 = g_{13}^{(0,2)}\Phi_2^3$, again, a straightforward computation shows that $a_{ij}^2 = 0$ for $i > 0$, $b_{ij}^2 = 0$ for $i > 1$ and $c_{ij}^2 = 0$ for all i, j. Similarly, using the fact that $\Phi_2^1 = g_{12}^{(0,2)}\Phi_2^2$, we get that $a_{ij}^2 = 0$ for $j > 2$ and $b_{ij}^2 = 0$ for $j > 1$. Furthermore, we get that

$$\begin{aligned} b_{01}^2 &= -2va_{02}^2 \\ b_{11}^2 &= -2ua_{02}^2 \end{aligned}$$

and so

$$\Phi_2^1 = \begin{pmatrix} A_2 & B_2 \\ 0 & -A_2 \end{pmatrix}, \tag{4.10}$$

where

$$\begin{aligned} A_2 &= a_{00}^2 + a_{01}^2 z_2 + a_{02}^2 z_2^2, \\ B_2 &= b_{00}^2 + b_{10}^2 z_1 - 2(uz_1 + v)a_{02}^2 z_2. \end{aligned} \tag{4.11}$$

For the following lemma we use the notation described above.

Lemma 4.18. *Let $\Phi \in \mathrm{H}^0(\mathrm{End}_0\, E \otimes T)$ be integrable. If $C_1 = 0$, then $\Phi = \Phi_2$. Otherwise, $\Phi = \Phi_1$.*

Proof. It suffices to prove the lemma on the open set V_1. Indeed, in any other standard open set, the Higgs field is a conjugation (by the transition functions of E) of its trivialization on V_1. Recall that, since Φ is integrable, it satisfies equations (2.8). Now, it is clear that if $C_1 = 0$, then $A_1 = B_1 = 0$ (this follows simply by the shape of A_1, B_1, C_1, see (4.9)), and so $\Phi = \Phi_2$. On the other hand, if $C_1 \neq 0$, then $A_2 = B_2 = 0$ (this follows from equations (2.8)), and so $\Phi = \Phi_2$. $\square$

We now aim to give a geometric description of the moduli space $\mathcal{M}^{\mathrm{co}}(-F, 1)$ of rank 2 stable co-Higgs bundles with first Chern class $-F$ and second Chern class 1. We have seen that if $(E, \Phi) \in \mathcal{M}^{\mathrm{co}}(-F, 1)$, then E is an extension of $\mathcal{O}(-1, 1)$ by $\mathcal{O}(0, -1)$. Such extensions are parametrized (up to strong isomorphism) by $\mathrm{H}^1(\mathcal{O}(1, -2)) = \mathbb{C}^2$, and have transition functions on V_{12} given by

$$\begin{pmatrix} g_{12} & uz_1 + v \\ 0 & g'_{12} \end{pmatrix},$$

for $(u, v) \in \mathbb{C}^2$. We will use the convenient notation $E = E_{u,v}$.

Lemma 4.19. *Let E and E' be extensions of $\mathcal{O}(-1, 1)$ by $\mathcal{O}(0, -1)$. If E and E' are isomorphic as vector bundles, then E and E' are weakly isomorphic extensions.*

Proof. Let $E = E_{u,v}$ and $E' = E_{u',v'}$. Also, let $p = uz_1 + v$ and $p' = u'z_1 + v'$. Now, suppose $\alpha \in \mathbb{P}^1$ is a zero of p. We have that for each $z \in \mathbb{P}^1$, $E_{|\{z\}\times\mathbb{P}^1} \cong E'_{|\{z\}\times\mathbb{P}^1}$. Since the only extensions of $\mathcal{O}(1)$ by $\mathcal{O}(-1)$ over $\mathbb{P}^1$ are the split one $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ and $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$, we have that

$$E_{|_{\{z\}\times\mathbb{P}^1}} = \begin{cases} \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} & \text{if } z \neq \alpha \\ \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) & \text{if } z = \alpha. \end{cases}$$

Since the same can be said of $E'_{|_{\{z\}\times\mathbb{P}^1}}$ and p', we have that p and p' have exactly the same zeroes. Hence, E and E' are weakly isomorphic. $\square$

Remark 4.20. Recall that, up to weak isomorphism, non-trivial extensions of $\mathcal{O}(-1,1)$ by $\mathcal{O}(0,-1)$ are parametrized by $\mathbb{P}(\mathrm{H}^1(\mathcal{O}(1,-2))) = \mathbb{P}^1$.

Let $X_0 := \{(E,\Phi) \in \mathcal{M}^{\mathrm{co}}(-F,1) :\ E \text{ is the trivial extension}\}$. By (4.5), we have that $X_0 = \mathbb{P}^1 \times \mathbb{C}^3$. Now, let us fix a non-trivial extension $E_{u,v}$, and let $X_{u,v}$ be the set of elements in $\mathcal{M}^{\mathrm{co}}(-F,1)$ with underlying bundle $E_{u,v}$. By (4.8), (4.9), (4.10), (4.11) and Lemma 4.18, we get that

$$X_{u,v} = \{(\bar{x},\bar{y}) \in \mathbb{C}^6 \times \mathbb{C}^5;\ x_i y_j = 0,\ 1 \le i \le 6 \text{ and } 1 \le j \le 5\}.$$

Since $X_{u,v}$ is the union of the two subspaces $\bar{x} = 0$ and $\bar{y} = 0$, $\dim X_{u,v} = 6$. It follows that:

Proposition 4.21. *The space $S = S_0 \cup S_1 \cup S_2$, where*

$$\begin{aligned} S_0 &= \{((u,v),(p,\bar{w}),(\bar{x},\bar{y})) \in \mathbb{C}^2 \times \mathbb{P}^1 \times \mathbb{C}^3 \times \mathbb{C}^6 \times \mathbb{C}^5 :\ u = v = \bar{x} = \bar{y} = 0\}, \\ S_1 &= \{((u,v),(p,\bar{w}),(\bar{x},\bar{y})) \in \mathbb{C}^2 \times \mathbb{P}^1 \times \mathbb{C}^3 \times \mathbb{C}^6 \times \mathbb{C}^5 :\ (u,v) \neq (0,0), p = \bar{w} = \bar{y} = 0\}, \\ S_2 &= \{((u,v),(p,\bar{w}),(\bar{x},\bar{y})) \in \mathbb{C}^2 \times \mathbb{P}^1 \times \mathbb{C}^3 \times \mathbb{C}^6 \times \mathbb{C}^5 :\ (u,v) \neq (0,0), p = \bar{w} = \bar{x} = 0\}, \end{aligned}$$

parametrizes rank 2 stable co-Higgs bundles with first Chern class $-F$ and second Chern class 1.

Remark 4.22. Note that $S_0 = X_0$ parametrizes the points of the form $(\mathcal{O}(0,-1) \oplus \mathcal{O}(-1,1), \Phi)$, S_1 the points (E,Φ_1) and S_2 the points (E,Φ_2), where E is a non-trivial extension.

By Lemma 4.19, it is clear that the moduli space $\mathcal{M}^{\mathrm{co}}(-F,1)$ is the quotient of S by a $\mathbb{C}^*$ action of weight 1 on $(u,v) \in \mathbb{C}^2$. Hence we have

Theorem 4.23. *$\mathcal{M}^{co}(-F,1)$ is a 7-dimensional algebraic variety whose singular locus are the points $(E,0)$ for any non-trivial extension E.*

Finally, before moving on to the next section, let us say a word about the spectral correspondence in this setting.

Remark 4.24. The first thing to note is that, in this case, we will indeed have to push-forward rank 1 torsion-free sheaves which are not locally free in order to get a correspondence. This follows immediately from Proposition 4.5 and Remark 4.6 by observing that any possible underlying bundle of a co-Higgs stable pair is an extension of $\mathcal{O}(-1,1)$ by $\mathcal{O}(0,-1)$.

For $\rho = (\rho_1, 0, 0)$ with $\rho_1 \in \mathrm{H}^0 = (\mathcal{O}(4,0))$ generic, we have seen that the spectral surface is of the form $S_\rho = X_{\rho_1} \times \mathbb{P}^1$, and recall that $\theta = (\pi, \mathrm{Id}_{\mathbb{P}^1}) : S_\rho \to \mathbb{P}^1 \times \mathbb{P}^1$, where π is the 2-sheeted covering map $X_{\rho_1} \to \mathbb{P}^1$. Let $\mathcal{L}$ be a rank 1 torsion-free coherent sheaf over S_ρ with $c_1(\mathcal{L}) = \alpha X_{\rho_1} + \beta \mathbb{P}^1$, where we are abusing notation and identifying X_{ρ_1} and $\mathbb{P}^1$ with the generators of $\mathrm{H}^2(S_\rho, \mathbb{Z})$, and $c_2 = \gamma X_{\rho_1} \times \mathbb{P}^1$ (again, by abuse of notation, $X_{\rho_1} \times \mathbb{P}^1$ is being identified with the generator of $\mathrm{H}^4(S_\rho, \mathbb{Z})$). We let

$$m = X_{\rho_1} \cdot X_{\rho_1} \text{ and } n = X_{\rho_1} \cdot \mathbb{P}^1.$$

We now apply Grothendieck–Riemann–Roch (see [18, Appendix A, Theorem 5.3]) in order to find the values of α, β and γ:

$$\mathrm{ch}(\theta_*(\mathcal{L})) \cdot \mathrm{Td}(\mathbb{P}^1 \times \mathbb{P}^1) = \theta_*(\mathrm{ch}(\mathcal{L}) \cdot \mathrm{Td}(X_{\rho_1} \times \mathbb{P}^1)), \tag{4.12}$$

where

$$\begin{array}{rcl} \mathrm{ch}(\theta_*(\mathcal{L})) & = & 2 - F - \mathbb{P}^1 \times \mathbb{P}^1 \\ \mathrm{Td}(\mathbb{P}^1 \times \mathbb{P}^1) & = & 1 + (C_0 + F) + \mathbb{P}^1 \times \mathbb{P}^1 \\ \mathrm{ch}(\mathcal{L}) & = & 1 + (\alpha X_{\rho_1} + \beta \mathbb{P}^1) + (\alpha\beta - \gamma) X_{\rho_1} \times \mathbb{P}^1 \\ \mathrm{Td}(X_{\rho_1} \times \mathbb{P}^1) & = & 1 + X_{\rho_1}. \end{array}$$

Then, the left hand side of 4.12 becomes $2 + 2C_0 + F$, and before pushing-forward the right hand side becomes $1 + (\alpha+1)X_{\rho_1} + \beta\mathbb{P}^1 + (\alpha\beta + \alpha m + \beta n - \gamma + \beta) X_{\rho_1} \times \mathbb{P}^1$. Finally, since X_{ρ_1} is a double cover of $\mathbb{P}^1$, pushing-down $\mathrm{ch}(\mathcal{L}) \cdot \mathrm{Td}(X_{\rho_1} \times \mathbb{P}^1)$ under θ, we get

$$2 + (\alpha+1)F + 2\beta C_0 + (\alpha\beta + \alpha m + \beta n - \gamma)\mathbb{P}^1 \times \mathbb{P}^1.$$

Thus, equating both sides, (4.12) yields $\alpha = 0, \beta = 1$ and $\gamma = n$. Hence, pushing-forward any rank 1 torsion-free coherent sheaf $\mathcal{L}$ over S_ρ with $c_1(\mathcal{L}) = \mathbb{P}^1$ and $c_2(\mathcal{L}) = X_{\rho_1} \cdot \mathbb{P}^1$ yields a stable co-Higgs pair with the underlying bundle a non-trivial extension of $\mathcal{O}(-1,1)$ by $\mathcal{O}(0,-1)$, and Higgs field of the form $\Phi = \Phi_1$, as described before.

Similarly, one can show that for $\rho = (0, 0, \rho_2)$ with $\rho_2 \in \mathrm{H}^0(\mathcal{O}(0,4))$ generic, pushing-down a rank 1 torsion-free coherent sheaf $\mathcal{L}$ over $S_\rho = \mathbb{P}^1 \times X_{\rho_2}$ with $c_1(\mathcal{L}) = \mathbb{P}^1$ and $c_2(\mathcal{L}) = X_{\rho_2} \cdot \mathbb{P}^1$ yields a stable co-Higgs pair with underlying a non-trivial extension of $\mathcal{O}(-1,1)$ by $\mathcal{O}(0,-1)$ with Higgs field of the form $\Phi = \Phi_2$, as described before.

4.3 Hypercohomology and Deformation Theory

A useful tool in better understanding the moduli spaces $\mathcal{M}^{\mathrm{co}}(c_1, c_2)$ of co-Higgs semistable pairs (E, Φ) on $\mathbb{P}^1 \times \mathbb{P}^1$ is the study of their behaviour under infinitesimal deformations. The deformation theory of co-Higgs bundles is discussed, by Rayan, for curves and surfaces in [27, Chapter 2], and for arbitrary complex projective manifolds in [29]; more details can be found there. In what follows, we will only outline the methodology and recall some important facts for the case of surfaces.

The main idea is to obtain the hypercohomology $\mathbb{H}^\bullet$ for a given semistable pair $(E, \Phi) \in \mathcal{M}^{\mathrm{co}}(c_1, c_2)$, paying special attention to $\mathbb{H}^1$, as this can be interpreted as the Zariski tangent space to the local moduli space at (E, Φ). In order to obtain the hypercohomology $\mathbb{H}^\bullet$ for (E, Φ), note that a double complex arises in a natural way (from the fact that $\Phi \wedge \Phi = 0$) by taking as the horizontal map

$$0 \to \ \mathrm{End}_0 E \xrightarrow{-\wedge\Phi} \ \mathrm{End}_0 E \otimes T \xrightarrow{-\wedge\Phi} \mathrm{End}_0 E \otimes \wedge^2 T \to 0,$$

where $- \wedge \Phi$ acts as the commutator on $\mathrm{End}_0 E$ and as the usual wedge on T. We use this wedge as the horizontal map in the above complex. We take as the vertical map the Čech coboundary operator δ. More precisely, the integrability of Φ implies that the wedge map $- \wedge \Phi$ is a differential, and so we can define cohomology groups for $\mathrm{End}_0 E \otimes \wedge^i T$, relative to it. These two define the hypercohomology spaces for (E, Φ). The operation $- \wedge \Phi$ commutes with the Čech coboundary δ, making

$$(C^\bullet(\mathrm{End}_0 E \otimes \wedge^\bullet T); D = \delta + (- \wedge \Phi))$$

into a first quadrant double complex. A spectral sequence is defined by choosing the 0-th page to be

$$(\mathcal{E}_0^{p,q} = C^q(\mathrm{End}_0 E \otimes \wedge^p T); d_0 = \delta).$$

Note that $\mathcal{E}_0^{p,q} = 0$ for $p > 2$. Then proceed by setting:

(i) $\mathcal{E}_1^{p,q} = \mathrm{H}^q_{d_0}(\mathcal{E}_0^{p,\bullet}) = \mathrm{H}^q(\mathrm{End}_0 E \otimes \wedge^p T)$. Note that, for any surface, $\mathcal{E}_1^{p,q} = 0$ for p or $q > 2$, and for the cases we are considering here, we actually have $\mathcal{E}_1^{p,q} = 0$ for $p > 2$ or $q \geq 2$. Thus, we make this assumption throughout this section.

(ii) $d_1 = - \wedge \Phi : \mathcal{E}_1^{p,q} \to \mathcal{E}_1^{p+1,q}$ for $0 \leq p \leq 1$, $0 \leq q \leq 1$. For the other possible values of p, d_1 is clearly the zero map.

(iii) $\mathcal{E}_2^{p,q} = \mathrm{H}^p_{d_1}(\mathcal{E}_1^{\bullet,q})$ for $0 \leq p \leq 2$, $0 \leq q \leq 1$.

(iv) $d_2 : \mathcal{E}_2^{0,1} \to \mathcal{E}_2^{2,0}$ is given by $d_2(\Psi) = \theta \wedge \Phi$, where $\theta \in C^0(\mathrm{End}_0 E \otimes T)$ is a solution of the equation $\Psi \wedge \Phi - \delta\theta = 0 \in C^1(\mathrm{End}_0 E \otimes T)$.

It is the 2-nd page that encodes the hypercohomology of the double complex. For instance

$$\mathbb{H}^0 \cong \mathcal{E}_2^{0,0} \text{ and } \mathbb{H}^3 \cong \mathcal{E}_2^{2,1}.$$

Moreover, provided that $\mathrm{H}^2(\mathrm{End}_0 E \otimes \wedge^i T)) = 0$ for $i = 0, 1, 2$ (which we are assuming), we have that

$$0 \to \mathcal{E}_2^{1,0} \to \mathbb{H}^1 \to \mathcal{E}_2^{0,1} \xrightarrow{d_2} \mathcal{E}_2^{2,0} \to \mathbb{H}^2 \to \mathcal{E}_2^{1,1} \to 0.$$

Thus, if the d_2 map is zero, we obtain

$$\mathbb{H}^1 \cong \mathcal{E}_2^{1,0} \oplus \mathcal{E}_2^{0,1} \text{ and } \mathbb{H}^2 \cong \mathcal{E}_2^{2,0} \oplus \mathcal{E}_2^{1,1}.$$

Remark 4.25. (See [27, Section 2.2]). It is important to mention that $\mathcal{E}_2^{1,0}$ parametrizes the first order deformations of Φ, while $\mathcal{E}_2^{0,1}$ parametrizes the first order deformations of E compatible with Φ. As such, in the case where $d_2 = 0$, $\mathbb{H}^1 = \mathcal{E}_2^{1,0} \oplus \mathcal{E}_2^{0,1}$ provides valuable information regarding the deformations of (E, Φ).

Given that all the information we need in order to obtain the hypercohomology $\mathbb{H}^\bullet$ of a co-Higgs semistable pair (E, Φ) can be extracted from the second page of the spectral sequence, let us explicitly write $\mathcal{E}_2^{p,q}$ for $p = 0, 1, 2$, and $q = 0, 1$, so that in the future we can simply use the symbol $\mathcal{E}_2^{p,q}$ instead of the actual definition.

$$\begin{aligned}
\mathcal{E}_2^{0,0} &= \operatorname{Ker}\{d_1 : \mathrm{H}^0(\mathrm{End}_0 E) \to \mathrm{H}^0(\mathrm{End}_0 E \otimes T)\} \\
\mathcal{E}_2^{1,0} &= \frac{\operatorname{Ker}\{d_1 : \mathrm{H}^0(\mathrm{End}_0 E \otimes T) \to \mathrm{H}^0(\mathrm{End}_0 E \otimes \wedge^2 T)\}}{\operatorname{Im}\{d_1 : \mathrm{H}^0(\mathrm{End}_0 E) \to \mathrm{H}^0(\mathrm{End}_0 E \otimes T)\}} \\
\mathcal{E}_2^{2,0} &= \frac{\mathrm{H}^0(\mathrm{End}_0 E \otimes \wedge^2 T)}{\operatorname{Im}\{d_1 : \mathrm{H}^0(\mathrm{End}_0 E \otimes T) \to \mathrm{H}^0(\mathrm{End}_0 E \otimes \wedge^2 T)\}} \\
\mathcal{E}_2^{0,1} &= \operatorname{Ker}\{d_1 : \mathrm{H}^1(\mathrm{End}_0 E) \to \mathrm{H}^1(\mathrm{End}_0 E \otimes T)\} \\
\mathcal{E}_2^{1,1} &= \frac{\operatorname{Ker}\{d_1 : \mathrm{H}^1(\mathrm{End}_0 E \otimes T) \to \mathrm{H}^1(\mathrm{End}_0 E \otimes \wedge^2 T)\}}{\operatorname{Im}\{d_1 : \mathrm{H}^1(\mathrm{End}_0 E) \to \mathrm{H}^1(\mathrm{End}_0 E \otimes T)\}} \\
\mathcal{E}_2^{2,1} &= \frac{\mathrm{H}^1(\mathrm{End}_0 E \otimes \wedge^2 T)}{\operatorname{Im}\{d_1 : \mathrm{H}^1(\mathrm{End}_0 E \otimes T) \to \mathrm{H}^1(\mathrm{End}_0 E \otimes \wedge^2 T)\}}
\end{aligned}$$

In what follows, we consider deformations of semistable co-Higgs bundles in the moduli spaces $\mathcal{M}^{\mathrm{co}}(c_1, c_2)$ previously discussed in this chapter. Once we have fixed $(E, \Phi) \in \mathcal{M}^{\mathrm{co}}(c_1, c_2)$, we will proceed as follows:

1. Verify that the condition $\mathrm{H}^2(\mathrm{End}_0 E \otimes \wedge^i T)) = 0$ for $i = 0, 1, 2$ is satisfied.
2. Obtain $\mathcal{E}_2^{p,q}$ for $p = 0, 1, 2$, and $q = 0, 1$.
3. Verify that the d_2 map is indeed the zero map.
4. Interpret the results in terms of the deformation theory discussed above.

4.3.1 Deformations of points in $\mathcal{M}^{\mathrm{co}}(-F, 0)$

Proposition 4.26. *Let $(E, \Phi) \in \mathcal{M}^{co}(-F, 0)$. Then,*

$$\mathbb{H}^0 = 0 \qquad \mathbb{H}^1 = \mathbb{C}^6 \qquad \mathbb{H}^2 = \mathbb{C}^{18} \qquad \mathbb{H}^3 = 0.$$

Moreover, $\mathbb{H}^1 = \mathcal{E}_2^{1,0} \oplus \mathcal{E}_2^{0,1}$ with $\mathcal{E}_2^{1,0} = \mathbb{C}^6$ and $\mathcal{E}_2^{0,1} = 0$.

Interpretation. Since $\mathcal{E}_2^{0,1} = 0$, the bundle cannot be deformed. This is expected as $\mathcal{O} \oplus \mathcal{O}(-1, 0)$ is the only possible underlying bundle in this moduli space. On the other hand, since $\mathcal{E}_2^{1,0} = \mathbb{C}^6$ we see that it is possible to deform the Higgs field into any other stable Higgs field in the 6-parameter family described in Section 4.2.1.

Proof. Recall that, in this case, the only underlying bundle of co-Higgs pairs in $\mathcal{M}^{\mathrm{co}}(-F, 0)$ is $E = \mathcal{O} \oplus \mathcal{O}(-1, 0)$, and any Higgs field Φ yielding a stable pair is of the form

$$\Phi = \Phi_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & -A_1 \end{pmatrix},$$

with $A_1 \in \mathrm{H}^0(\mathcal{O}(2, 0))$, $B_1 \in \mathrm{H}^0(\mathcal{O}(3, 0))$ and non-zero $C_1 \in \mathrm{H}^0(\mathcal{O}(1, 0))$. Fix a Higgs field Φ. Using the facts that

1. $\mathrm{End}_0(E) = \mathcal{O} \oplus \mathcal{O}(1,0) \oplus \mathcal{O}(-1,0)$,
2. $\mathrm{End}_0(E) \otimes T = \mathcal{O}(2,0) \oplus \mathcal{O}(3,0) \oplus \mathcal{O}(1,0) \oplus \mathcal{O}(0,2) \oplus \mathcal{O}(1,2) \oplus \mathcal{O}(-1,2)$,
3. $\mathrm{End}_0(E) \otimes \wedge^2 T = \mathcal{O}(2,2) \oplus \mathcal{O}(3,2) \oplus \mathcal{O}(1,2)$,

and Remark 2.19, we get $\mathrm{H}^1(\mathrm{End}_0 E \otimes \wedge^i T) = \mathrm{H}^2(\mathrm{End}_0 E \otimes \wedge^i T) = 0$ for $i = 0, 1, 2$. Hence, $\mathcal{E}_2^{p,1} = 0$ for $p = 0, 1, 2$, and the d_2 map is zero. Now, for

$$\Psi = \begin{pmatrix} \alpha & \beta \\ 0 & -\alpha \end{pmatrix} \in \mathrm{H}^0(\mathrm{End}_0 E),$$

we have

$$d_1(\Psi) = \begin{pmatrix} \beta C_1 & 2(\alpha B_1 - \beta A_1) \\ -2\alpha C_1 & -\beta C_1 \end{pmatrix};$$

since C_1 is non-zero, we have that Ψ is in the kernel of d_1 if and only if Ψ is zero. Thus,

$$\mathcal{E}_2^{0,0} = \mathrm{Ker}\{d_1 : \mathrm{H}^0(\mathrm{End}_0 E) \to \mathrm{H}^0(\mathrm{End}_0 E \otimes T)\} = 0,$$

and

$$\mathrm{Im}\{d_1 : \mathrm{H}^0(\mathrm{End}_0 E) \to \mathrm{H}^0(\mathrm{End}_0 E \otimes T)\} = \mathrm{H}^0(\mathrm{End}_0 E) = \mathbb{C}^3.$$

On the other hand, if

$$\Psi = \Psi_1 + \Psi_2 = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & -\alpha_1 \end{pmatrix} + \begin{pmatrix} \alpha_2 & \beta_2 \\ 0 & -\alpha_2 \end{pmatrix} \in \mathrm{H}^0(\mathrm{End}_0 E \otimes T),$$

then by Remark 2.29

$$d_1(\Psi) = -[\Psi_2, \Phi_1] = \begin{pmatrix} \beta_2 C_1 & 2(\alpha_2 B_1 - \beta_2 A_1) \\ -2\alpha_2 C_1 & -\beta_2 C_1 \end{pmatrix}.$$

Again, since C_1 is not identically zero, Ψ is in the kernel of the d_1 map if and only if $\alpha_2 = \beta_2 = 0$ if and only if $\Psi_2 = 0$. Thus

$$\mathrm{Ker}\{d_1 : \mathrm{H}^0(\mathrm{End}_0 E \otimes T) \to \mathrm{H}^0(\mathrm{End}_0 E \otimes \wedge^2 T)\} = \mathrm{H}^0(\mathrm{End}_0 E(2,0)) = \mathbb{C}^9.$$

Hence $\mathcal{E}_2^{1,0} = \mathbb{C}^6$. Also,

$$\mathrm{Im}\{d_1 : \mathrm{H}^0(\mathrm{End}_0 E \otimes T) \to \mathrm{H}^0(\mathrm{End}_0 E \otimes \wedge^2 T)\} = \mathrm{H}^0(\mathrm{End}_0 E(0,2)) = \mathbb{C}^9,$$

and thus $\mathcal{E}_2^{2,0} = \mathbb{C}^{18}$. Therefore, the hypercohomology is

$$\mathbb{H}^0 = 0 \qquad \mathbb{H}^1 = \mathbb{C}^6 \qquad \mathbb{H}^2 = \mathbb{C}^{18} \qquad \mathbb{H}^3 = 0.$$

$\square$

4.3.2 Deformations of points in $\mathcal{M}^{\mathbf{co}}(0,0)$

Proposition 4.27. *Let $(E,\Phi) \in \mathcal{M}_H^{co}(0,0)$.*

1. *If $E = \mathcal{O} \oplus \mathcal{O}$, then $\mathcal{E}_2^{0,1} = 0$ and $\mathbb{H}^1 = \mathcal{E}_2^{1,0}$.*

   (a) *If $\Phi = 0$, then*

   $$\mathbb{H}^0 = \mathbb{C}^3 \qquad \mathbb{H}^1 = \mathbb{C}^{18} \qquad \mathbb{H}^2 = \mathbb{C}^{27} \qquad \mathbb{H}^3 = 0.$$

   (b) *If $\Phi \neq 0$, then*

   $$\mathbb{H}^0 = \mathbb{C} \qquad \mathbb{H}^1 = \mathbb{C}^{10} \qquad \mathbb{H}^2 = \mathbb{C}^{21} \qquad \mathbb{H}^3 = 0.$$

2. *If $E = \mathcal{O}(1,0) \oplus \mathcal{O}(-1,0)$, then*

   $$\mathbb{H}^0 = 0 \qquad \mathbb{H}^1 = \mathbb{C}^{6} \qquad \mathbb{H}^2 = \mathbb{C}^{21} \qquad \mathbb{H}^3 = 0.$$

   Moreover, $\mathbb{H}^1 = \mathcal{E}_2^{1,0} \oplus \mathcal{E}_2^{0,1}$ with $\mathcal{E}_2^{1,0} = \mathbb{C}^5$ and $\mathcal{E}_2^{0,1} = \mathrm{H}^1(\mathrm{End}_0\, E) = \mathbb{C}$. By symmetry, the same result holds for $E = \mathcal{O}(0,1) \oplus \mathcal{O}(0,-1)$.

Interpretation. In (1), $\mathcal{E}_2^{0,1} = 0$ and so, the bundle $\mathcal{O} \oplus \mathcal{O}$ cannot be deformed into any other bundle living in this moduli space. When $\Phi = 0$, we obtain that the point $(\mathcal{O} \oplus \mathcal{O}, 0) \in \mathcal{M}^{\mathrm{co}}(0,0)$ is a singular point (as the dimension of $\mathcal{M}^{\mathrm{co}}(0,0)$ is strictly smaller than the dimension of $\mathcal{E}_2^{1,0}$). When $\Phi \neq 0$, we see that a strictly semistable pair $(\mathcal{O} \oplus \mathcal{O}, \Phi)$ can be deformed into a stable pair $(\mathcal{O} \oplus \mathcal{O}, \Phi')$. Indeed, in Section 4.2.1, we saw that the possible Higgs fields Φ which make $(\mathcal{O} \oplus \mathcal{O}, \Phi)$ strictly semistable form a 6-parameter family, and here $\mathbb{H}^1 = \mathbb{C}^{10}$.

In (2), $\mathcal{E}_2^{0,1} = \mathrm{H}^1(\mathrm{End}_0\, E) = \mathbb{C}$ and so, we can deform the pair $(\mathcal{O}(1,0) \oplus \mathcal{O}(-1,0), \Phi)$ into a pair of the form $(\mathcal{O} \oplus \mathcal{O}, \Phi')$. On the other hand, since $\mathcal{E}_2^{1,0} = \mathbb{C}^5$, we can deform the Higgs field of $(\mathcal{O}(1,0) \oplus \mathcal{O}(-1,0), \Phi)$ into any other stable Higgs field in the 5-parameter family described in Section 4.2.1.

Proof. As we know from Corollary 4.10, there are three underlying bundles in this moduli space, namely $\mathcal{O} \oplus \mathcal{O}$, $\mathcal{O}(1,0) \oplus \mathcal{O}(-1,0)$ and $\mathcal{O}(0,1) \oplus \mathcal{O}(0,-1)$. Let us now work by cases.

Case 1. Consider $E = \mathcal{O} \oplus \mathcal{O}$. Since we do not have an explicit description of the stable co-Higgs bundles of the form $(\mathcal{O} \oplus \mathcal{O}, \Phi)$ (see section 4.2.1), we will only consider deformations of points $(\mathcal{O} \oplus \mathcal{O}, \Phi)$ that are strictly semistable. Using the fact that

(i) $\mathrm{End}_0(E) = \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}$,

(ii) $\mathrm{End}_0(E) \otimes T = \mathcal{O}(2,0) \oplus \mathcal{O}(2,0) \oplus \mathcal{O}(2,0) \oplus \mathcal{O}(0,2) \oplus \mathcal{O}(0,2) \oplus \mathcal{O}(0,2)$,

(iii) $\mathrm{End}_0(E) \otimes \wedge^2 T = \mathcal{O}(2,2) \oplus \mathcal{O}(2,2) \oplus \mathcal{O}(2,2)$, and Remark 2.19, we get $\mathrm{H}^1(\mathrm{End}_0 E \otimes \wedge^i T) = \mathrm{H}^2(\mathrm{End}_0 E \otimes \wedge^i T) = 0$ for $i = 0, 1, 2$. Hence, we have that $\mathcal{E}_2^{p,1} = 0$ for $p = 0, 1, 2$, and so the d_2 map is clearly zero.

For $\Phi = 0$, it is immediate that $\mathcal{E}_2^{0,0} = \mathrm{H}^0(\mathrm{End}_0 E) = \mathbb{C}^3$. Also, $\mathcal{E}_2^{1,0} = \mathrm{H}^0(\mathrm{End}_0 E \otimes T) = \mathbb{C}^{18}$, and $\mathcal{E}_2^{2,0} = \mathrm{H}^0(\mathrm{End}_0 E \otimes \wedge^2 T) = \mathbb{C}^{27}$. Hence, in this case, the hypercohomology is

$$\mathbb{H}^0 = \mathbb{C}^3 \qquad \mathbb{H}^1 = \mathbb{C}^{18} \qquad \mathbb{H}^2 = \mathbb{C}^{27} \qquad \mathbb{H}^3 = 0.$$

When $\Phi \neq 0$, we have

$$\mathcal{E}_2^{0,0} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} : \alpha \in \mathrm{H}^0(\mathcal{O}) \right\} = \mathbb{C}.$$

Now, by the rank-nullity Theorem, we have that

$$\mathrm{Im}\{d_1 : \mathrm{H}^0(\mathrm{End}_0 E) \to \mathrm{H}^0(\mathrm{End}_0 E \otimes T)\} = \mathbb{C}^2.$$

On the other hand, writing

$$\Phi = \Phi_1 + \Phi_2 = \begin{pmatrix} A_1 & 0 \\ 0 & -A_1 \end{pmatrix} + \begin{pmatrix} A_2 & 0 \\ 0 & -A_2 \end{pmatrix},$$

for any

$$\Psi = \Psi_1 + \Psi_2 = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & -\alpha_1 \end{pmatrix} + \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & -\alpha_2 \end{pmatrix} \in \mathrm{H}^0(\mathrm{End}_0 E \otimes T)$$

we have

$$d_1(\Psi) = [\Psi_1, \Phi_2] + [\Psi_2, \Phi_1] = \begin{pmatrix} 0 & -2\beta_1 A_2 + 2\beta_2 A_1 \\ 2\gamma_1 A_2 - 2\gamma_2 A_1 & 0 \end{pmatrix}.$$

Therefore, Ψ is in the kernel of d_1 if and only if $\gamma_1 A_2 = \gamma_2 A_1$ and $\beta_1 A_2 = \beta_2 A_1$. Since A_1 and A_2 are not both zero, $\mathrm{Ker}\{d_1 : \mathrm{H}^0(\mathrm{End}_0 E \otimes T) \to \mathrm{H}^0(\mathrm{End}_0 E \otimes \wedge T)\} = \mathbb{C}^{12}$ and $\mathcal{E}_2^{1,0} = \mathbb{C}^{10}$. Moreover, $\mathrm{Im}\{d_1 : \mathrm{H}^0(\mathrm{End}_0 E \otimes T) \to \mathrm{H}^0(\mathrm{End}_0 E \otimes \wedge^2 T)\} = \mathbb{C}^6$. Thus, $\mathcal{E}_2^{2,0} = \mathbb{C}^{21}$ and the hypercohomology is

$$\mathbb{H}^0 = \mathbb{C} \qquad \mathbb{H}^1 = \mathbb{C}^{10} \qquad \mathbb{H}^2 = \mathbb{C}^{21} \qquad \mathbb{H}^3 = 0.$$

<u>Case 2.</u> Consider $E = \mathcal{O}(1,0) \oplus \mathcal{O}(-1,0)$ with a fixed Higgs field Φ. We have that

(i) $\mathrm{End}_0(E) = \mathcal{O} \oplus \mathcal{O}(2,0) \oplus \mathcal{O}(-2,0)$,

(ii) $\mathrm{End}_0(E) \otimes T = \mathcal{O}(2,0) \oplus \mathcal{O}(2,2) \oplus \mathcal{O}(-2,2) \oplus \mathcal{O}(0,2) \oplus \mathcal{O}(0,4) \oplus \mathcal{O}(0,0)$,

(iii) $\mathrm{End}_0(E) \otimes \wedge^2 T = \mathcal{O}(2,2) \oplus \mathcal{O}(4,2) \oplus \mathcal{O}(0,2)$.

By Remark 2.19, $\mathrm{H}^2(\mathrm{End}_0 E \otimes \wedge^i T) = 0$ for $i = 0, 1, 2$. From Section 4.2.1, we know we can take a representative Higgs field of the form

$$\Phi = \Phi_1 = \begin{pmatrix} 0 & B \\ 1 & 0 \end{pmatrix},$$

where $B \in \mathrm{H}^0(\mathcal{O}(4,0))$. It then follows that, for

$$\Psi = \begin{pmatrix} \alpha & \beta \\ 0 & -\alpha_1 \end{pmatrix} \in \mathrm{H}^0(\mathrm{End}_0 E),$$

we have

$$d_1(\Psi) = [\Psi, \Phi] = \begin{pmatrix} \beta & 2\alpha B \\ -2\alpha & -\beta \end{pmatrix}.$$

Thus, we see that $\mathcal{E}_2^{0,0} = 0$ and $\mathrm{Im}\{d_1 : \mathrm{H}^0(\mathrm{End}_0 E) \to \mathrm{H}^0(\mathrm{End}_0 E \otimes T)\} = \mathrm{H}^0(\mathrm{End}_0 E) = \mathbb{C}^4$. For

$$\Psi = \Psi_1 + \Psi_2 = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & -\alpha_1 \end{pmatrix} + \begin{pmatrix} \alpha_2 & \beta_2 \\ 0 & -\alpha_2 \end{pmatrix} \in \mathrm{H}^0(\mathrm{End}_0 E \otimes T),$$

we have

$$d_1(\Psi) = -[\Psi_2, \Phi_1] = \begin{pmatrix} -\beta_2 & -2\alpha_2 B \\ 2\alpha_2 & \beta_2 \end{pmatrix}.$$

This time, Ψ is in the kernel of d_1 if and only if $\alpha_2 = \beta_2 = 0$ if and only if $\Psi_2 = 0$. Thus, $\mathrm{Ker}\{d_1 : \mathrm{H}^0(\mathrm{End}_0 E \otimes T) \to \mathrm{H}^0(\mathrm{End}_0 E \otimes \wedge^2 T)\} = \mathrm{H}^0(\mathrm{End}_0 E(2,0)) = \mathbb{C}^9$, and so $\mathcal{E}_2^{1,0} = \mathbb{C}^5$. We also have that $\mathrm{Im}\{d_1 : \mathrm{H}^0(\mathrm{End}_0 E \otimes T) \to \mathrm{H}^0(\mathrm{End}_0 E \otimes \wedge^2 T)\} = \mathrm{H}^0(\mathrm{End}_0 E(0,2)) = \mathbb{C}^9$. Since $\mathrm{H}^0(\mathrm{End}_0 E \otimes \wedge^2 T) = \mathbb{C}^{27}$, we get $\mathcal{E}_2^{2,0} = \mathbb{C}^{18}$.

Now, note that any $\Psi \in \mathrm{H}^1(\mathrm{End}_0 E)$ has the form

$$\Psi = \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}$$

with $\gamma \in \mathrm{H}^1(\mathcal{O}(-2,0))$. Applying the d_1 map, we get

$$d_1(\Psi) = [\Psi, \Phi_1] = \begin{pmatrix} -B\gamma & 0 \\ 0 & B\gamma \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

where $B\gamma \in \mathrm{H}^1(\mathcal{O}(2,0)) = 0$. The map $d_1 : \mathrm{H}^1(\mathrm{End}_0 E) \to \mathrm{H}^1(\mathrm{End}_0 E \otimes T)$ is thus zero, implying that $\mathcal{E}_2^{0,1} = \mathrm{H}^1(\mathrm{End}_0 E) = \mathbb{C}$ and $\mathrm{Im}\{d_1 : \mathrm{H}^1(\mathrm{End}_0 E) \to \mathrm{H}^1(\mathrm{End}_0 E \otimes T)\} = 0$. Moreover, since $\mathrm{H}^1(\mathrm{End}_0 E \otimes \wedge^2 T) = 0$, $\mathcal{E}_2^{1,1} = \mathrm{H}^1(\mathrm{End}_0 E \otimes T) = \mathbb{C}^3$ and $\mathcal{E}_2^{2,1} = 0$.

We now check that the d_2 map is zero. Let $\Psi \in \mathcal{E}_2^{0,1} = \mathrm{H}^1(\mathrm{End}_0 E)$. Since $\Phi = \Phi_1 \in \mathrm{H}^0(\mathrm{End}_0 E(2,0))$, the image of the map $d_1 : \mathrm{H}^1(\mathrm{End}_0 E) \to \mathrm{H}^1(\mathrm{End}_0 E \otimes T)$ lies in $\mathrm{H}^1(\mathrm{End}_0 E(2,0))$ and can pick $\theta = \theta_1 \in C^0(\mathrm{End}_0 E(2,0))$ such that $\Psi \wedge \Phi - \delta\theta = 0 \in C^1(\mathrm{End}_0 E \otimes T)$. We then see that

$$d_2(\Psi) = \theta \wedge \Phi = \theta_1 \wedge \Phi_1 = 0,$$

as desired. Hence, the hypercohomology is

$$\mathbb{H}^0 = 0 \qquad \mathbb{H}^1 = \mathbb{C}^6 \qquad \mathbb{H}^2 = \mathbb{C}^{21} \qquad \mathbb{H}^3 = 0.$$

□

4.3.3 Deformations of points in $\mathcal{M}^{\mathbf{co}}(-F, 1)$

Proposition 4.28. *Let $(E, \Phi) \in \mathcal{M}^{co}(-F, 1)$.*

1. *If $E = \mathcal{O}(0, -1) \oplus \mathcal{O}(-1, 1)$, then*

$$\mathbb{H}^0 = 0 \qquad \mathbb{H}^1 = \mathbb{C}^6 \qquad \mathbb{H}^2 = \mathbb{C}^{18} \qquad \mathbb{H}^3 = 0.$$

   Moreover, $\mathbb{H}^1 = \mathcal{E}_2^{1,0} \oplus \mathcal{E}_2^{0,1}$ with $\mathcal{E}_2^{1,0} = \mathbb{C}^4$ and $\mathcal{E}_2^{0,1} = \mathbb{C}^2$.

2. *If E is a non-trivial extension of $\mathcal{O}(-1, 1)$ by $\mathcal{O}(0, -1)$, then $\mathbb{H}^1 = \mathcal{E}_2^{1,0} \oplus \mathcal{E}_2^{0,1}$ and $\mathcal{E}_2^{0,1} = \mathrm{H}^1(\mathrm{End}_0 E) = \mathbb{C}$. Moreover,*

   (a) *if $\Phi = 0$, then $\mathcal{E}_2^{1,0} = \mathbb{C}^{11}$ and*

$$\mathbb{H}^0 = 0 \qquad \mathbb{H}^1 = \mathbb{C}^{12} \qquad \mathbb{H}^2 = \mathbb{C}^{24} \qquad \mathbb{H}^3 = 0.$$

   (b) *if $\Phi = \Phi_2 \neq 0$, then $\mathcal{E}_2^{1,0} = \mathbb{C}^5$ and*

$$\mathbb{H}^0 = 0 \qquad \mathbb{H}^1 = \mathbb{C}^6 \qquad \mathbb{H}^2 = \mathbb{C}^{18} \qquad \mathbb{H}^3 = 0.$$

   (c) *if $\Phi = \Phi_1 \neq 0$, then $\mathcal{E}_2^{1,0} = \mathbb{C}^6$ and*

$$\mathbb{H}^0 = 0 \qquad \mathbb{H}^1 = \mathbb{C}^7 \qquad \mathbb{H}^2 = \mathbb{C}^{19} \qquad \mathbb{H}^3 = 0.$$

Interpretation. In (1), $\mathcal{E}_2^{0,1} = \mathrm{H}^1(\mathrm{End}_0 E) = \mathbb{C}^2$ and so, the pair $(\mathcal{O}(0,-1) \oplus \mathcal{O}(-1,1), \Phi)$ can be deformed into a pair of the form $(\tilde{E}, \tilde{\Phi})$ for any non-trivial extension $\tilde{E}$ of $\mathcal{O}(-1,1)$ by $\mathcal{O}(0,-1)$. Since $\mathcal{E}_2^{1,0} = \mathbb{C}^4$, then we can deform the Higgs field into any other stable Higgs field in the 4-parameter family described in Section 4.2.2.

In (2), $\mathcal{E}_2^{0,1} = \mathbb{C}$ and so, we see that it is possible to deform the bundle E into any other extension of $\mathcal{O}(-1,1)$ by $\mathcal{O}(0,-1)$ (up to weak isomorphism). When $\Phi = 0$, the fact that $\mathcal{E}_2^{1,0} = \mathbb{C}^{11}$ confirms that this points are indeed singular points, as the moduli space has dimension 7. When $\Phi = \Phi_2 \neq 0$, $\mathcal{E}_2^{1,0} = \mathbb{C}^5$. Thus, we can deform the Higgs field into any other stable Higgs field in the 5-parameter family described in Section 4.2.2. Finally, when $\Phi = \Phi_1 \neq 0$, $\mathcal{E}_2^{1,0} = \mathbb{C}^6$. Thus, we can deform the Higgs field into any other stable Higgs field in the 6-parameter family described in Section 4.2.2.

Proof. As we saw in Section 4.2.2, in this case, every underlying bundle is an extension of $\mathcal{O}(-1,1)$ by $\mathcal{O}(0,-1)$. We work by cases.

<u>Case 1.</u> Consider the trivial extension, $E = \mathcal{O}(0,-1) \oplus \mathcal{O}(-1,1)$, with a fixed Higgs field Φ. We have that

(i) $\mathrm{End}_0(E) = \mathcal{O} \oplus \mathcal{O}(1,-2) \oplus \mathcal{O}(-1,2)$,

(ii) $\mathrm{End}_0(E) \otimes T = \mathcal{O}(2,0) \oplus \mathcal{O}(3,-2) \oplus \mathcal{O}(1,2) \oplus \mathcal{O}(0,2) \oplus \mathcal{O}(1,0) \oplus \mathcal{O}(-1,4)$,

(iii) $\mathrm{End}_0(E) \otimes \wedge^2 T = \mathcal{O}(2,2) \oplus \mathcal{O}(3,0) \oplus \mathcal{O}(1,4)$.

By Remark 2.19, $\mathrm{H}^2(\mathrm{End}_0 E \otimes \wedge^i T) = 0$ for $i = 0, 1, 2$. From Section 4.2.2, we know a Higgs field of E is of the form

$$\Phi = \Phi_2 = \begin{pmatrix} A_2 & B_2 \\ 0 & -A_2 \end{pmatrix},$$

with $A_2 \in \mathrm{H}^0(\mathcal{O}(0,2))$ and non-zero $B_2 \in \mathrm{H}^0(\mathcal{O}(1,0))$. It then follows that, for

$$\Psi = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \in \mathrm{H}^0(\mathrm{End}_0 E),$$

we have

$$d_1(\Psi) = [\Psi, \Phi] = \begin{pmatrix} 0 & 2\alpha B_2 \\ 0 & 0 \end{pmatrix}.$$

Since B_2 is non-zero, we see that $\mathcal{E}_2^{0,0} = 0$ and

$$\mathrm{Im}\{d_1 : \mathrm{H}^0(\mathrm{End}_0 E) \to \mathrm{H}^0(\mathrm{End}_0 E \otimes T)\} \cong \mathbb{C}.$$

For

$$\Psi = \Psi_1 + \Psi_2 = \begin{pmatrix} \alpha_1 & 0 \\ \gamma_1 & -\alpha_1 \end{pmatrix} + \begin{pmatrix} \alpha_2 & \beta_2 \\ 0 & -\alpha_2 \end{pmatrix} \in \mathrm{H}^0(\mathrm{End}_0 E \otimes T),$$

we have

$$d_1(\Psi) = [\Psi_1, \Phi_2] = \begin{pmatrix} -\gamma_1 B_2 & 2\alpha_1 B_2 \\ 2\gamma_1 A_2 & \gamma_1 B_2 \end{pmatrix}.$$

Again, since B_2 is non-zero, Ψ is in the kernel of d_1 if and only if $\Psi_1 = 0$. Thus, $\mathrm{Ker}\{d_1 : \mathrm{H}^0(\mathrm{End}_0 E \otimes T) \to \mathrm{H}^0(\mathrm{End}_0 E \otimes \wedge^2 T)\} = \mathrm{H}^0(\mathrm{End}_0 E(0,2)) \cong \mathbb{C}^5$, and so $\mathcal{E}_2^{1,0} \cong \mathbb{C}^4$. We also have that

$$\mathrm{Im}\{d_1 : \mathrm{H}^0(\mathrm{End}_0 E \otimes T) \to \mathrm{H}^0(\mathrm{End}_0 E \otimes \wedge^2 T)\} = \mathrm{H}^0(\mathrm{End}_0 E(0,2)) = \mathbb{C}^9.$$

Since $\mathrm{H}^0(\mathrm{End}\, E \otimes \wedge^2 T) \cong \mathbb{C}^{23}$, we get that $\mathcal{E}_2^{2,0} \cong \mathbb{C}^{14}$.

Now, note that any $\Psi \in \mathrm{H}^1(\mathrm{End}_0 E)$ has the form

$$\Psi = \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix},$$

with $\gamma \in \mathrm{H}^1(\mathcal{O}(1,-2))$. Applying the d_1 map, we get

$$d_1(\Psi) = [\Psi, \Phi_2] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -2\gamma A_2 \\ 0 & 0 \end{pmatrix},$$

with $2A_2\gamma \in \mathrm{H}^1(\mathcal{O}(1,0)) = 0$. The map $d_1 : \mathrm{H}^1(\mathrm{End}_0 E) \to \mathrm{H}^1(\mathrm{End}_0 E \otimes T)$ is thus the zero map, implying that $\mathcal{E}_2^{0,1} = \mathrm{H}^1(\mathrm{End}_0 E) = \mathbb{C}^2$ and

$$\mathrm{Im}\{d_1 : \mathrm{H}^1(\mathrm{End}_0 E) \to \mathrm{H}^1(\mathrm{End}_0 E \otimes T)\} = 0.$$

Moreover, since $\mathrm{H}^1(\mathrm{End}_0 E \otimes \wedge^2 T) = 0$, we get that $\mathcal{E}_2^{1,1} \cong \mathrm{H}^1(\mathrm{End}_0 E \otimes T) \cong \mathbb{C}^4$, and $\mathcal{E}_2^{2,1} = 0$.

We now check that the d_2 map is zero. Let $\Psi \in \mathcal{E}_2^{0,1} = \mathrm{H}^1(\mathrm{End}_0 E)$. Since $\Phi = \Phi_2 \in \mathrm{H}^0(\mathrm{End}_0 E(0,2))$, the image of the map $d_1 : \mathrm{H}^1(\mathrm{End}_0 E) \to \mathrm{H}^1(\mathrm{End}_0 E \otimes T)$ lies in

$\mathrm{H}^1(\operatorname{End}_0 E(0,2))$ and we can pick $\theta = \theta_2 \in C^0(\operatorname{End}_0 E(0,2))$ such that $\Psi \wedge \Phi - \delta\theta = 0 \in C^1(\operatorname{End}_0 E \otimes T)$. We then see that

$$d_2(\Psi) = \theta \wedge \Phi = \theta_2 \wedge \Phi_2 = 0,$$

as desired. Hence, the hypercohomology is

$$\mathbb{H}^0 = 0 \qquad \mathbb{H}^1 = \mathbb{C}^6 \qquad \mathbb{H}^2 = \mathbb{C}^{18} \qquad \mathbb{H}^3 = 0.$$

Case 2. Consider E to be a non-trivial extension of $\mathcal{O}(-1,1)$ by $\mathcal{O}(0,-1)$. Using the fact that E is simple (since it is stable), and calculations similar to those of Lemma 4.16, we get

$\mathrm{h}^0(\operatorname{End}_0 E) = 0$	$\mathrm{h}^1(\operatorname{End}_0 E) = 1$	$\mathrm{h}^2(\operatorname{End}_0 E) = 0$
$\mathrm{h}^0(\operatorname{End}_0 E \otimes T) = 11$	$\mathrm{h}^1(\operatorname{End}_0 E \otimes T) = 1$	$\mathrm{h}^2(\operatorname{End}_0 E \otimes T) = 0$
$\mathrm{h}^0(\operatorname{End}_0 E \otimes \wedge^2 T) = 23$	$\mathrm{h}^1(\operatorname{End}_0 E \otimes \wedge^2 T) = 0$	$\mathrm{h}^2(\operatorname{End}_0 E \otimes \wedge^2 T) = 0$

Note that $\mathcal{E}_2^{0,0} = 0$ and $\mathcal{E}_2^{1,0} = \operatorname{Ker}\{d_1 : \mathrm{H}^0(\operatorname{End}_0 E \otimes T) \to \mathrm{H}^0(\operatorname{End}_0 E \otimes \wedge^2 T)\}$ in all three subcases because $\mathrm{h}^0(\operatorname{End}_0 E) = 0$. Moreover, $\mathcal{E}_2^{0,1} = \mathrm{H}^1(\operatorname{End}_0 E) = \mathbb{C}$ again in all three cases. Indeed, $\mathcal{E}_2^{0,1} = \operatorname{Ker}\{d_1 : \mathrm{H}^1(\operatorname{End}_0 E) \to \mathrm{H}^1(\operatorname{End}_0 E \otimes T)\}$ with $\mathrm{h}^1(\operatorname{End}_0 E) = \mathrm{h}^1(\operatorname{End}_0 E \otimes T) = 1$. However, $\mathcal{E}_2^{0,1} \neq 0$, since otherwise (E, Φ) would not admit deformations, contradicting the description of the Higgs fields of the non-trivial extensions of $\mathcal{O}(1,-1)$ by $\mathcal{O}(0,-1)$ in Section 4.2.2. Thus,

$$\mathcal{E}_2^{0,1} = \mathrm{H}^1(\operatorname{End}_0 E) = \mathbb{C}$$

and $d_1 : \mathrm{H}^1(\operatorname{End}_0 E) \to \mathrm{H}^1(\operatorname{End}_0 E \otimes T)$ is the zero map. This implies, in particular, that the d_2 is zero as well. To see this, let $\Psi \in \mathcal{E}_2^{1,0} \cong \mathrm{H}^1(\operatorname{End}_0 E)$. Since

$$\operatorname{Im}\{d_1 : \mathrm{H}^1(\operatorname{End}_0 E) \to \mathrm{H}^1(\operatorname{End}_0 E \otimes T)\} = 0,$$

$\Psi \wedge \Phi = 0 \in C^1(\operatorname{End}_0 E \otimes T)$, and so we can pick $\theta = 0$, implying that

$$d_2(\Psi) = \theta \wedge \Phi = 0.$$

To compute $\mathcal{E}_2^{1,0}$, $\mathcal{E}_2^{2,0}$, $\mathcal{E}_2^{1,1}$ and $\mathcal{E}_2^{2,1}$, we look at cases.

First we work with the case $\Phi = 0$. It is clear then that the d_1 map is zero. Therefore, $\mathcal{E}_2^{1,0} \cong \mathrm{H}^0(\operatorname{End}_0 E \otimes T) \cong \mathbb{C}^{11}$ and $\mathcal{E}_2^{2,0} = \mathrm{H}^0(\operatorname{End}_0 E \otimes \wedge^2 T) \cong \mathbb{C}^{23}$. We also have that $\mathcal{E}_2^{1,1} \cong \mathrm{H}^1(\operatorname{End}_0 E \otimes T) \cong \mathbb{C}$ and $\mathcal{E}_2^{0,1} = \mathrm{H}^1(\operatorname{End}_0 E \otimes \wedge^2 T) = 0$. Hence, the hypercohomology is

$$\mathbb{H}^0 = 0 \qquad \mathbb{H}^1 = \mathbb{C}^{12} \qquad \mathbb{H}^2 = \mathbb{C}^{24} \qquad \mathbb{H}^3 = 0.$$

We now work with the case where $\Phi = \Phi_2 \neq 0$. Referring to Lemma 4.18,

$$\Phi = \Phi_2 = \begin{pmatrix} A_2 & B_2 \\ 0 & -A_2 \end{pmatrix}.$$

For

$$\Psi = \Psi_1 + \Psi_2 = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & -\alpha_1 \end{pmatrix} + \begin{pmatrix} \alpha_2 & \beta_2 \\ 0 & -\alpha_2 \end{pmatrix} \in \mathrm{H}^0(\operatorname{End}_0 E \otimes T),$$

we have

$$d_1(\Psi) = [\Psi_1, \Phi_2] = \begin{pmatrix} -\gamma_1 B_2 & 2(\alpha_1 B_2 - \beta_1 A_2) \\ 2\gamma_1 A_2 & \gamma_1 B_2 \end{pmatrix}.$$

Since Φ is non-zero, we have that Ψ is in the kernel of d_1 if and only if $\gamma_1 = 0$ if and only if $\Psi_1 = 0$ by Remark 4.17. Thus,

$$\mathcal{E}_2^{1,0} = \operatorname{Ker}\{d_1 : \mathrm{H}^0(\operatorname{End}_0 E \otimes T) \to \mathrm{H}^0(\operatorname{End}_0 E \otimes \wedge^2 T)\} = \mathrm{H}^0(\operatorname{End}_0 E(0,2)) = \mathbb{C}^5$$

and

$$\operatorname{Im}\{d_1 : \mathrm{H}^0(\operatorname{End}_0 E \otimes T) \to \mathrm{H}^0(\operatorname{End}_0 E \otimes \wedge^2 T)\} = \mathrm{H}^0(\operatorname{End}_0 E(2,0)) = \mathbb{C}^6.$$

Hence, since $\mathrm{H}^0(\operatorname{End} E \otimes \wedge^2 T) \cong \mathbb{C}^{23}$, we get $\mathcal{E}_2^{2,0} \cong \mathbb{C}^{17}$. Note that $\mathrm{h}^0(\operatorname{End}_0 E(2,0))$ and $\mathrm{h}^0(\operatorname{End}_0 E(0,2))$ were computed in Lemma 4.16.

Now, since $\Phi \in \mathrm{H}^0(\operatorname{End}_0 E(0,2))$, for any $\Psi \in \mathrm{H}^1(\operatorname{End}_0 E)$, we have that $d_1(\Psi) \in \mathrm{H}^1(\operatorname{End}_0 E(0,2)) = 0$. Thus, $\mathcal{E}_2^{0,1} \cong \mathrm{H}^1(\operatorname{End}_0 E) \cong \mathbb{C}$. Furthermore, since $\mathrm{H}^1(\operatorname{End}_0 E \otimes \wedge^2 T) = 0$, we obtain $\mathcal{E}_2^{1,1} \cong \mathrm{H}^1(\operatorname{End}_0 E \otimes T) \cong \mathbb{C}$ and $\mathcal{E}_2^{2,1} = 0$.

Hence, the hypercohomology is

$$\mathbb{H}^0 = 0 \qquad \mathbb{H}^1 = \mathbb{C}^6 \qquad \mathbb{H}^2 = \mathbb{C}^{18} \qquad \mathbb{H}^3 = 0.$$

Finally, we work with the case where $\Phi = \Phi_1 \neq 0$. Referring to Lemma 4.18,

$$\Phi = \Phi_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & -A_1 \end{pmatrix}$$

with C_1 non-zero.

For

$$\Psi = \Psi_1 + \Psi_2 = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & -\alpha_1 \end{pmatrix} + \begin{pmatrix} \alpha_2 & \beta_2 \\ 0 & -\alpha_2 \end{pmatrix} \in \mathrm{H}^0(\operatorname{End}_0 E \otimes T),$$

we have

$$d_1(\Psi) = -[\Psi_2, \Phi_1] = \begin{pmatrix} -\beta_2 C_1 & 2(\beta_2 A_1 - \alpha_2 B_1) \\ 2\alpha_2 C_1 & \beta_2 C_1 \end{pmatrix}.$$

Since C_1 is non-zero, we have that Ψ is in the kernel of d_1 if and only if $\alpha_2 = \beta_2 = 0$ if and only if $\Psi_2 = 0$. Thus,

$$\mathcal{E}_2^{1,0} \cong \operatorname{Ker}\{d_1 : \mathrm{H}^0(\operatorname{End}_0 E \otimes T) \to \mathrm{H}^0(\operatorname{End}_0 E \otimes \wedge^2 T)\} = \mathrm{H}^0(\operatorname{End}_0 E(2,0)) \cong \mathbb{C}^6$$

and

$$\operatorname{Im}\{d_1 : \mathrm{H}^0(\operatorname{End}_0 E \otimes T) \to \mathrm{H}^0(\operatorname{End}_0 E \otimes \wedge^2 T)\} = \mathrm{H}^0(\operatorname{End}_0 E(0,2)) = \mathbb{C}^5.$$

Since $\mathrm{H}^0(\operatorname{End} E \otimes \wedge^2 T) \cong \mathbb{C}^{23}$, we get $\mathcal{E}_2^{2,0} \cong \mathbb{C}^{18}$. Hence, the hypercohomology is

$$\mathbb{H}^0 = 0 \qquad \mathbb{H}^1 = \mathbb{C}^7 \qquad \mathbb{H}^2 = \mathbb{C}^{19} \qquad \mathbb{H}^3 = 0.$$

□

Chapter 5

Outlook

In this brief chapter, we present some possible directions in which the ideas in this thesis can be taken further.

Even though in this thesis we addressed the existence problem of semistable rank 2 co-Higgs bundles over Hirzebruch surfaces quite thoroughly, we only offered descriptions of the moduli spaces $\mathcal{M}^{\mathrm{co}}(c_1, c_2)$ of rank 2 semistable co-Higgs bundles over $\mathbb{P}^1 \times \mathbb{P}^1$, for certain values of c_1 and c_2, and with respect to the standard polarization. The approach we took was the following: We picked certain choices of c_1 and c_2 for which $\mathcal{M}^{\mathrm{co}}(c_1, c_2)$ was non-empty. Then, in each case, we determined the underlying bundles of semistable co-Higgs pairs in $\mathcal{M}^{\mathrm{co}}(c_1, c_2)$. Finally, we gave an explicit description of the corresponding Higgs fields, making sure to properly identify isomorphic co-Higgs pairs in the moduli space. Given the exhaustive nature of our analysis, we were only able to fully carry it out for low values of c_2.

To study the moduli spaces $\mathcal{M}^{\mathrm{co}}(c_1, c_2)$ for other values of c_1 and c_2, we propose further exploiting the spectral correspondence. One can attempt to obtain new examples of semistable co-Higgs pairs over $\mathbb{P}^1 \times \mathbb{P}^1$ by pushing down rank 1 torsion-free sheaves over the spectral covers and their associated multiplication maps. One, of course, needs to check if the resulting Higgs fields satisfy the integrability condition, a non-trivial task. In order to do this, one needs to understand several features of these spectral surfaces: their tangent bundle, their Picard group and their intersection pairing, to name a few. A similar approach can be followed for abitrary Hirzebruch surfaces. In this setting, the first question to address is what type of surfaces arise as spectral covers. This seems to be a relevant problem to consider in a future project.

Another natural question to ask is whether some of the techniques we applied to Hirzebruch surfaces can be generalized to other types of manifolds. We expect that similar techniques to the ones presented in Chapters 3 and 4 can be used to study moduli spaces of semistable co-Higgs bundles over ruled surfaces. Indeed, many of the arguments presented in this dissertation rely more on the fact that the fibres of the Hirzebruch surfaces are copies of $\mathbb{P}^1$ rather than on the fact that the base of the fibration is $\mathbb{P}^1$. However, some technical difficulties may arise for general ruled surfaces. For instance, even though rank 2 vector bundles over a ruled surface can still be expressed canonically in terms of certain numerical invariants (see [9, Section 5.5]), these are not as easy to work with.

Finally, in this thesis, we only studied co-Higgs bundles of rank 2. The case of co-Higgs bundles of higher rank should be explored. Moreover, it would be interesting to determine what type of structures moduli spaces of co-Higgs bundles admit. For example, the moduli spaces we described over $\mathbb{P}^1 \times \mathbb{P}^1$ for $c_2 = 0$ and 1 are algebraic varieties. Is this always the case for co-Higgs bundles over an algebraic surface? What other structures does the moduli spaces inherit from the surface? These are just a few pertinent questions to be undertaken.

APPENDICES

Appendix A

Semistable decomposable co-Higgs bundles

A.1 The case of Hirzebruch surfaces

In the following proposition, working with an arbitrary polarization, we describe all the possible decomposable stable co-Higgs bundles with $c_2 = 1$, and $c_1 = 0$ or $-F$.

Proposition A.1. *Let H be an arbitrary polarization, and $c_2 = 1$.*

1. *Suppose $c_1 = 0$. Only when n is odd there are decomposable stable co-Higgs bundles in $\mathcal{M}_H^{co}(0,1)$. Furthermore, these are $\left(\mathcal{O}\left(C_0 + \left(\frac{n-1}{2}\right)F\right) \oplus \mathcal{O}\left(-C_0 - \left(\frac{n-1}{2}\right)F\right), \Phi\right)$ with*
$$\Phi = \Phi_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & -A_2 \end{pmatrix} \in \mathrm{H}^0(\mathrm{End}_0\, E(2C_0 + nF)),$$
and non-zero C_2.

2. *Suppose $c_1 = -F$. Only when n is even there are decomposable stable co-Higgs bundles in $\mathcal{M}_H^{co}(-F,1)$. Furthermore, these are $\left(\mathcal{O}\left(C_0 + \left(\frac{n}{2} - 1\right)F\right) \oplus \mathcal{O}\left(-C_0 - \left(\frac{n}{2}\right)F\right), \Phi\right)$ with*
$$\Phi = \Phi_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & -A_2 \end{pmatrix} \in \mathrm{H}^0(\mathrm{End}_0\, E(2C_0 + nF)),$$
and non-zero C_2.

Proof.

1. Since E is decomposable with $c_1 = 0$ and $c_2 = 1$, then $E = \mathcal{O}(aC_0 + bF) \oplus \mathcal{O}(-aC_0 - bF)$ and $1 = c_2 = a(an - 2b)$, so that $a = 1$ and $b = \frac{n-1}{2}$ or $a = -1$ and $b = -\frac{n-1}{2}$, hence n has to be odd. As there is only one chamber of type $(0,1)$, pick the standard polarization $H = C_0 + (n+1)F$, and note that $\mu_H(E) = 0$, while $\mu_H\left(\mathcal{O}\left(C_0 + \left(\frac{n-1}{2}\right)F\right)\right) = \frac{n+1}{2} > 0$. Hence $\mathcal{O}\left(C_0 + \left(\frac{n-1}{2}\right)F\right)$ is the destabilizing sub-line bundle of E. Finally, any element of $\mathrm{H}^0(\mathrm{End}_0\, E \otimes T)$ is of the form

$$\Phi = \Phi_1 + \Phi_2 = \begin{pmatrix} A_1 & B_1 \\ C_1 & -A_1 \end{pmatrix} + \begin{pmatrix} A_2 & B_2 \\ C_2 & -A_2 \end{pmatrix},$$

with $A_1 \in \mathrm{H}^0(\mathcal{O}(2F))$, $B_1 \in \mathrm{H}^0(\mathcal{O}(2C_0 + (n+1)F))$ and $A_2 \in \mathrm{H}^0(\mathcal{O}(2C_0 + nF))$, $B_2 \in \mathrm{H}^0(\mathcal{O}(4C_0+(2n-1)F))$, $C_2 \in \mathrm{H}^0(\mathcal{O}(C_0+F))$ (see Remark 3.7). Any non-zero C_2 would not leave $\mathcal{O}\left(C_0 + \left(\frac{n-1}{2}\right) F\right)$ Φ-invariant, and so taking the integrability condition into account, equations (2.8) imply that $A_1 = C_1 = 0$, and so the result follows.

2. The case $n = 0$ was considered in Theorem 3.9, so consider $n > 0$. Since E is decomposable with $c_1 = -F$ and $c_2 = 1$, then $E = \mathcal{O}(aC_0 + bF) \oplus \mathcal{O}(-aC_0 - (b+1)F)$ and $1 = c_2 = a(an - 2b - 1)$, so that $a = 1$ and $b = \frac{n}{2} - 1$ or $a = -1$ and $b = -\frac{n}{2}$, hence n has to be even. As there is only one chamber of type $(-F, 1)$, pick the standard polarization $H = C_0 + (n+1)F$, and note that $\mu_H(E) = -\frac{1}{2}$, while $\mu_H\left(\mathcal{O}\left(C_0 + \left(\frac{n}{2} - 1\right) F\right)\right) = \frac{n}{2} > 0$. Hence $\mathcal{O}\left(C_0 + \left(\frac{n}{2} - 1\right) F\right)$ is the destabilizing sub-line bundle of E. Finally, any element of $\mathrm{H}^0(\mathrm{End}_0 E \otimes T)$ is of the form

$$\Phi = \Phi_1 + \Phi_2 = \begin{pmatrix} A_1 & B_1 \\ C_1 & -A_1 \end{pmatrix} + \begin{pmatrix} A_2 & B_2 \\ C_2 & -A_2 \end{pmatrix},$$

with $A_1 \in \mathrm{H}^0(\mathcal{O}(2F))$, $B_1 \in \mathrm{H}^0(\mathcal{O}(2C_0 + (n+1)F))$ and $A_2 \in \mathrm{H}^0(\mathcal{O}(2C_0 + nF))$, $B_2 \in \mathrm{H}^0(\mathcal{O}(4C_0 + (2n-1)F))$, $C_2 \in \mathrm{H}^0(\mathcal{O}(F))$ (see Remark 3.7). Any non-zero C_2 would not leave $\mathcal{O}\left(C_0 + \left(\frac{n}{2} - 1\right) F\right)$ Φ-invariant, and so taking the integrability condition into account, equations (2.8) imply that $A_1 = C_1 = 0$, and so the result follows. $\square$

A.2 The case of $\mathbb{P}^1 \times \mathbb{P}^1$

We now turn our attention to the 0-th Hirzebruch surface, $\mathbb{P}^1 \times \mathbb{P}^1$. We fix the standard polarization $H = C_0 + F$ and only work with the reduced first Chern classes: $0, -F, -C_0$ and $-C_0 - F$. Note that, with respect to H, the reduced classes 0 and $-C_0 - F$ yield bundles of even degree, while the reduced classes $-F$ and $-C_0$ yield bundles of odd degree. Therefore, in the latter case, it is impossible to have strictly semistable bundles, and so the concepts of stability and semistability coincide.

In Chapter 4, we discussed the moduli spaces $\mathcal{M}^{\mathrm{co}}(c_1, c_2)$ when $c_2 = 0, 1$, so here we only focus on $c_2 \geq 2$.

Proposition A.2. *Let $c_1 = 0$ and $c_2 \geq 2$. Let E be a decomposable, rank 2 vector bundle over $\mathbb{P}^1 \times \mathbb{P}^1$. If (E, Φ) is a semistable co-Higgs bundle, then E is of the form $\mathcal{O}(a, -a) \oplus \mathcal{O}(-a, a)$ for $a \geq 1$.*

Proof. Let $E = \mathcal{O}(a, b) \oplus \mathcal{O}(-a, -b)$, then $c_2(E) = -2ab \geq 2$. Note that a and b must have opposite signs and that they both have absolute value strictly positive. Without loss of generality, assume that $a > 0$. Any Higgs field $\Phi \in \mathrm{H}^0(\mathrm{End}_0 E \otimes T)$ has the form:

$$\Phi = \begin{pmatrix} A_1 & B_1 \\ C_1 & -A_1 \end{pmatrix} + \begin{pmatrix} A_2 & B_2 \\ C_2 & -A_2 \end{pmatrix},$$

where $A_1 \in \mathrm{H}^0(\mathcal{O}(2, 0))$, $B_1 \in \mathrm{H}^0(\mathcal{O}(2a + 2, 2b))$, $C_1 \in \mathrm{H}^0(\mathcal{O}(2 - 2a, -2b))$, and $A_2 \in \mathrm{H}^0(\mathcal{O}(0, 2))$, $B_2 \in \mathrm{H}^0(\mathcal{O}(2a, 2b + 2))$, $C_2 \in \mathrm{H}^0(\mathcal{O}(-2a, 2 - 2b))$ satisfy equations 2.8.

Note that since we are assuming $a > 0$, $C_2 = 0$ and, unless $a = 1$, $C_1 = 0$ as well. We now show that $b = -a$. Towards a contradiction, assume that $b > -a$ or $b < -a$. In the former case, $C_1 = 0$ since if $a = 1$, then $b > -1$, but this is impossible. This leaves $\mathcal{O}(a, b)$ Φ-invariant, contradicting semistability. In the latter case, $B_1 = B_2 = 0$, as $b < -a \leq -1$, leaving $\mathcal{O}(-a, -b)$ Φ-invariant, again contradicting stability. $\square$

Proposition A.3. *Let $c_1 = -F$ or $-C_0$, and let $c_2 \geq 2$. Then, there are no semistable decomposable rank 2 co-Higgs bundles over $\mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. We will only deal with the case $c_1(E) = -F$. The proof for $c_1(E) = -C_0$ is analogous. Let $E = \mathcal{O}(a, b) \oplus \mathcal{O}(-1-a, -b)$, then $c_2(E) = -b(2a+1) \geq 2$. Any Higgs field $\Phi \in \mathrm{H}^0(\mathrm{End}_0\, E \otimes T)$ has the form:

$$\Phi = \begin{pmatrix} A_1 & B_1 \\ C_1 & -A_1 \end{pmatrix} + \begin{pmatrix} A_2 & B_2 \\ C_2 & -A_2 \end{pmatrix},$$

where $A_1 \in \mathrm{H}^0(\mathcal{O}(2, 0))$, $B_1 \in \mathrm{H}^0(\mathcal{O}(2a+3, 2b))$, $C_1 \in \mathrm{H}^0(\mathcal{O}(1-2a, -2b))$, and $A_2 \in \mathrm{H}^0(\mathcal{O}(0, 2))$, $B_2 \in \mathrm{H}^0(\mathcal{O}(2a+1, 2b+2))$, $C_2 \in \mathrm{H}^0(\mathcal{O}(-1-2a, 2-2b))$, satisfy equations (2.8).

We start by showing that a can only take the values -1 or 0. Indeed, if $a \geq 1$, then $C_1 = C_2 = 0$, and so $\mathcal{O}(a, b)$ would be Φ-invariant. By stability, this would yield $b \leq -2$. However, the latter implies that $B_1 = B_2 = 0$, and so $\mathcal{O}(a, b)$ and $\mathcal{O}(-1-a, -b)$ would both be Φ-invariant, contradicting stability (since at least one of the two would have non-negative slope). A similar argument, but now interchanging the roles of the C_i's for the B_i's, and of $\mathcal{O}(a, b)$ for $\mathcal{O}(-1-a, -b)$, shows that $a > -2$.

Note that $a = 0$ implies that $b \leq -2$, and $a = -1$ implies that $b \geq 2$, but this is impossible. Indeed, for the bundle $\mathcal{O}(0, b) \oplus \mathcal{O}(-1, -b)$, if $b \leq -2$, then $B_1 = B_2 = 0$, and so $\mathcal{O}(-1, -b)$ is Φ-invariant, contradicting stability. Similarly, for the bundle $\mathcal{O}(-1, b) \oplus \mathcal{O}(0, b)$, if $b \geq 2$, then $C_1 = C_2 = 0$, and so $\mathcal{O}(-1, b)$ is Φ-invariant, contradicting stability. Hence, there are no stable decomposable rank 2 co-Higgs bundles with $c_2 \geq 2$. $\square$

Hence, it follows that:

Corollary A.4. *Suppose that E is decomposable and (E, Φ) is a semistable co-Higgs pair. If $c_1(E) = -F$, then there are only two possibilities for E:*

1. *$\mathcal{O}(0, 0) \oplus \mathcal{O}(-1, 0)$ and*
2. *$\mathcal{O}(0, -1) \oplus \mathcal{O}(-1, 1)$.*

On the other hand, if $c_1(E) = -C_0$, then there are also only two possibilities for E:

1. *$\mathcal{O}(0, 0) \oplus \mathcal{O}(0, -1)$ and*
2. *$\mathcal{O}(-1, 0) \oplus \mathcal{O}(1, -1)$.*

Proposition A.5. *Let $c_1 = -C_0 - F$ and $c_2 \geq 2$. Let E be a decomposable rank 2 vector bundle over $\mathbb{P}^1 \times \mathbb{P}^1$. If (E, Φ) is a semistable co-Higgs bundle, then E is of the form $\mathcal{O}(a, -a-1) \oplus \mathcal{O}(-a-1, a)$ for $a \geq 1$.*

Proof. Let $E = \mathcal{O}(a, b) \oplus \mathcal{O}(-a-1, -b-1)$, then $c_2(E) = -2ab - a - b \geq 2$. Any Higgs field $\Phi \in \mathrm{H}^0(\mathrm{End}_0 E \otimes T)$ has the form:

$$\Phi = \begin{pmatrix} A_1 & B_1 \\ C_1 & -A_1 \end{pmatrix} + \begin{pmatrix} A_2 & B_2 \\ C_2 & -A_2 \end{pmatrix},$$

where $A_1 \in \mathrm{H}^0(\mathcal{O}(2, 0))$, $B_1 \in \mathrm{H}^0(\mathcal{O}(2a + 3, 2b + 1))$, $C_1 \in \mathrm{H}^0(\mathcal{O}(1 - 2a, -2b - 1))$, and $A_2 \in \mathrm{H}^0(\mathcal{O}(0, 2))$, $B_2 \in \mathrm{H}^0(\mathcal{O}(2a-1, 2b+3))$, $C_2 \in \mathrm{H}^0(\mathcal{O}(-2a-1, 1-2b))$, satisfy equations (2.8).

We now show that $b = -a - 1$. Towards a contradiction, assume that $b > -a - 1$ or $b < -a - 1$. In the former case, by considering the three possibilities $a > 0$, $a = 0$ and $a < 0$, one can show that $C_1 = C_2 = 0$, leaving $\mathcal{O}(a, b)$ Φ-invariant, contradicting stability. In the latter case, again, by considering the three possibilities $a > 0$, $a = 0$ and $a < 0$, one can show that $B_1 = B_2 = 0$, leaving $\mathcal{O}(-a - 1, -b - 1)$ Φ-invariant, again contradicting stability. □

Appendix B

Underlying Bundles of semistable co-Higgs pairs in the case of $c_2 = 1$

In this appendix, for the fixed standard polarization $H = C_0 + F$, we give a full description of the possible underlying bundles of a co-Higgs semistable pair over $\mathbb{P}^1 \times \mathbb{P}^1$ with reduced first Chern class and second Chern class 1. In Section 4.2.2 we saw that for the case $c_1 = -F$, any possible underlying bundle of a semistable co-Higgs pair is an extension of $\mathcal{O}(-1,1)$ by $\mathcal{O}(0,-1)$. Here, we discuss the cases where $c_1 = 0, -C_0$ and $-C_0 - F$. In the case of $c_1 = 0$, we will see that there are no underlying bundles which are extensions of line bundles, and in fact, up to weak isomorphism, any underlying bundle of a semistable co-Higgs pair is an extension of the form

$$0 \to \mathcal{O} \to E \to I_x \to 0,$$

where $x \in \mathbb{P}^1 \times \mathbb{P}^1$. In the case of $c_1 = -C_0$, we will show that any possible underlying bundle is an extension of $\mathcal{O}(1,-1)$ by $\mathcal{O}(-1,0)$. Thus, this case is completely analogous to the case of $c_1 = -F$; however, some of the proofs that rely on the use of the numerical invariants of the bundle (which, in turn, depend on the Chern classes) vary, and so we present those arguments here. Finally, in the case of $c_1 = -C_0 - F$, we will prove that the only possible underlying bundle of a semistable co-Higgs pair is $\mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1)$. In what follows, we offer no comment on the description of the Higgs fields in any of these cases. Nonetheless let us remark that in all these cases there exist non-trivial Higgs fields. Moreover, the Higgs fields in the case of $c_1 = -C_0$ are analogous to those of the case of $c_1 = -F$ (with the roles of the first and second copy of $\mathbb{P}^1$ interchanged, as usual), the Higgs fields for the case $c_1 = -C_0 - F$ are easy to describe, as the underlying bundle is decomposable, and the Higgs fields for the case $c_1 = 0$ can be obtained using a similar technique as the one used for $c_1 = -F$, but this is tedious and require some work.

B.1 First Chern class $c_1 = 0$

Let us start by showing that in this case there are no extensions of line bundles.

Lemma B.1. *Let E be rank 2 bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ with $c_1(E) = 0$ and $c_2(E) = 1$. Then E is not an extension of line bundles.*

Proof. Towards a contradiction, assume that E is an extension of line bundles; i.e., it fits into an exact sequence of the form

$$0 \to \mathcal{O}(a,b) \to E \to \mathcal{O}(-a,-b) \to 0,$$

with $1 = c_2(E) = -2ab$, but this is impossible as both a and b are integers. □

Now that we have seen that there are no possible underlying bundles which are extensions of line bundles, but we know (see Proposition 3.9) nonetheless that there are non-trivial semistable co-Higgs pairs with $c_1 = 0$ and $c_2 = 1$, we investigate which form the underlying bundles take.

Proposition B.2. *Suppose $c_1(E) = 0$ and $c_2(E) = 1$. If (E, Φ) is a semistable co-Higgs pair, then E is an extension of the form*

$$0 \to \mathcal{O} \to E \to I_x \to 0,$$

where $x \in \mathbb{P}^1 \times \mathbb{P}^1$, which is unique up to weak isomorphism.

Proof. Let E have invariants d and r, then we know that E fits into an exact sequence of the form

$$0 \to \mathcal{O}(r,d) \to E \to \mathcal{O}(-r,-d) \otimes I_Z \to 0, \tag{B.1}$$

with $\ell(Z) = 1 + 2rd > 0$ (as we have seen that E cannot be an extension of line bundles). Now, for such a rank 2 vector bundle to exist, we know by Theorem 2.21 that one of the following two conditions must be satisfied.

1. $d \geq 1$, or
2. $d = 0$ and $r \geq 0$.

In case (1) we consider two subcases:

(i) $r \geq 0$. By tensoring (B.1) with $\mathcal{O}(-r,-d) \otimes T$, and passing to the long exact sequence in cohomology, we get

$$0 \to \mathrm{H}^0(T) \to \mathrm{H}^0(E(-r,-d) \otimes T) \to \mathrm{H}^0(\mathcal{O}(-2r,-2d) \otimes T \otimes I_Z) \to 0,$$

where $\mathrm{H}^0(\mathcal{O}(-2r,-2d) \otimes T \otimes I_Z) = 0$, and so, by Lemma 2.32, $\mathcal{O}(r,d)$ is Φ-invariant for any $\Phi \in \mathrm{H}^0(\mathrm{End}_0 E \otimes T)$, and thus destabilizing. This contradicts semistability, and so it cannot happen.

(ii) $r \leq -1$. This yields $\ell(Z) < 0$, which is impossible.

In case (2) we again consider two subcases:

(i) $r \geq 1$. By the exact same argument as above, one can check that $\mathcal{O}(r,d)$ is Φ-invariant for any $\Phi \in \mathrm{H}^0(\mathrm{End}_0 E \otimes T)$, and thus destabilizing. Hence this case cannot happen.

(ii) $r = 0$. In this case E fits into an exact sequence of the form

$$0 \to \mathcal{O} \to E \to I_x \to 0,$$

where $x \in \mathbb{P}^1 \times \mathbb{P}^1$. Note that $\mathrm{Ext}^1(I_x, O) = \mathbb{C}$, and so, up to weak isomorphism, there is a unique bundle that fits into this exact sequence. Finally, note that this is a strictly semistable bundle (and we have seen in Proposition 3.9 that it admits non-trivial Higgs fields). □

B.2 First Chern class $c_1 = -C_0$

Using an identical argument to that presented in Section 4.2.2, one can prove an analogous result for Lemma 4.14. For convenience, let us write the statement below:

Lemma B.3. *Let x be a point in $\mathbb{P}^1 \times \mathbb{P}^1$ and suppose that $c_2(E) = 1$. If E fits into the exact sequence*

$$0 \to \mathcal{O} \to E \to \mathcal{O}(0,-1) \otimes I_x \to 0,$$

then $\mathcal{O}$ is Φ-invariant for any $\Phi \in \mathrm{H}^0(\mathrm{End}_0\, E \otimes T)$.

We can now prove the following proposition:

Proposition B.4. *Let (E, Φ) be a stable co-Higgs pair such that $c_1(E) = -C_0$ and $c_2(E) = 1$. Then E is an extension of $\mathcal{O}(1,-1)$ by $\mathcal{O}(-1,0)$.*

Proof. Let E have invariants d and r, then we know that E fits into an exact sequence of the form

$$0 \to \mathcal{O}(r,d) \to E \to \mathcal{O}(-r,-1-d) \otimes I_Z \to 0, \tag{B.2}$$

with $\ell(Z) = 1 + r(2d+1) \geq 0$. Now, for such a rank 2 vector bundle to exist, we know by Theorem 2.21 that $d \geq 0$. If $r \leq -2$, or $r = -1$ and $d > 0$, then $\ell(Z) < 0$, which is impossible, so $r \geq -1$. If $r > 0$, or $r = 0$ and $d > 0$, by tensoring (B.2) with $\mathcal{O}(-r,-d) \otimes T$, and passing to the long exact sequence in cohomology, we get

$$0 \to \mathrm{H}^0(T) \to \mathrm{H}^0(E(-r,-d) \otimes T) \to \mathrm{H}^0(\mathcal{O}(-2r,-1-2d) \otimes T \otimes I_Z) \to 0,$$

and moreover $\mathrm{H}^0(\mathcal{O}(-2r,-1-2d) \otimes T \otimes I_Z) = 0$, and so, by Lemma 2.32, $\mathcal{O}(r,d)$ is Φ-invariant for any $\Phi \in \mathrm{H}^0(\mathrm{End}_0\, E \otimes T)$, and thus destabilizing. Hence, none of these cases can happen. As such, we have that either (i) $r = 0$ and $d = 0$ or (ii) $r = -1$ and $d = 0$. In (i) we have that, by Lemma B.3, $\mathcal{O}$ is Φ-invariant for any $\Phi \in \mathrm{H}^0(\mathrm{End}_0\, E \otimes T)$, and thus destabilizing. Again, this case cannot happen. Finally, in (ii) we get that $\ell(Z) = 0$, and so E is an extension of line bundles.

An analogous argument to that of Proposition 4.15 shows that if E is an extension of line bundles, then it has to be an extension of $\mathcal{O}(1,-1)$ by $\mathcal{O}(-1,0)$. This concludes the proof. □

B.3 First Chern class $c_1 = -C_0 - F$

In the same spirit as Proposition B.4 we have:

Proposition B.5. *Let (E, Φ) be a semistable co-Higgs pair such that $c_1(E) = -C_0 - F$ and $c_2(E) = 1$. Then $E = \mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1)$.*

Proof. Let E have invariants d and r, then we know that E fits into an exact sequence of the form

$$0 \to \mathcal{O}(r,d) \to E \to \mathcal{O}(-1-r,-1-d) \otimes I_Z \to 0, \tag{B.3}$$

with $\ell(Z) = 1 + d + r + 2dr \geq 0$. Now, for such a rank 2 vector bundle to exist, we know by Theorem 2.21 that $d \geq 0$. If $r \leq -2$, or $r = -1$ and $d > 0$, then $\ell(Z) < 0$, which is impossible, so $r \geq -1$. If $r \geq 0$, by tensoring (B.3) with $\mathcal{O}(-r, -d) \otimes T$, and passing to the long exact sequence in cohomology, we get

$$0 \to \mathrm{H}^0(T) \to \mathrm{H}^0(E(-r, -d) \otimes T) \to \mathrm{H}^0(\mathcal{O}(-1 - 2r, -1 - 2d) \otimes T \otimes I_Z) \to 0,$$

and moreover $\mathrm{H}^0(\mathcal{O}(-1 - 2r, -1 - 2d) \otimes T \otimes I_Z) = 0$, and so, by Lemma 2.32, $\mathcal{O}(r, d)$ is Φ-invariant for any $\Phi \in \mathrm{H}^0(\operatorname{End}_0 E \otimes T)$, and thus destabilizing. Hence, none of these cases can happen. As such, we have that $r = -1$ and $d = 0$. Then, $\ell(Z) = 0$, and so E is an extension of line bundles.

Finally, E fits into an exact sequence of the form

$$0 \to \mathcal{O}(a, b) \to E \to \mathcal{O}(-1 - a, -1 - b) \to 0, \tag{B.4}$$

where $1 = c_2(E) = -a - b - 2ab$. Consequently, either $a = -1$ and $b = 0$ or $a = 0$ and $b = -1$. In the first case, we have that $E = \mathcal{O}(-1, 0) \oplus \mathcal{O}(0, -1)$ as $\mathrm{H}^1(\mathcal{O}(-1, 1)) = 0$. In the second case, we have that $\mathcal{O}(0, -1) \oplus \mathcal{O}(-1, 0)$ as $\mathrm{H}^1(\mathcal{O}(1, -1)) = 0$. $\square$

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