SCATTERING EQUATIONS
&
S-MATRICES

by

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AUTHOR'S DECLARATION

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

ABSTRACT

We introduce a new formulation for the tree-level S-matrix in theories of massless particles. Sitting at the core of this formulation are the scattering equations, which yield a map from the kinematics of a scattering process to the moduli space of punctured Riemann spheres. The formula for an amplitude is constructed by an integration of a certain rational function over this moduli space, which is localized by the scattering equations. We provide a detailed analysis of the solutions to these equations and introduce this new formulation. After presenting some illustrative examples we show how to apply this formulation to the construction of closed formulas for actual amplitudes in various theories, using only a limited set of building blocks. Examples are amplitudes in Einstein gravity, Yang–Mills, Dirac–Born–Infeld, the $U(N)$ non-linear sigma model, and a special Galileon theory. The consistency of these formulas is checked by systematically studying locality and unitarity. In the end we discuss the implication of this formulation to the Kawai–Lewellen–Tye relations among amplitudes.

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to my parents
Xinhai Yuan & Yuqin Yao

with Love and Gratitude

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INTRODUCTION

1.1 THE S-MATRIX IN QUANTUM FIELD THEORIES

One of the main purposes of quantum field theories is to provide a quantitative explanation of the nature of elementary particles. A crucial physical observable in this study is the *S-matrix*, which underlies the computation of both cross-sections and decay rates that particle experiments measure. Intuitively, a scattering process in particle experiments is idealized to the interaction of a set of particles prepared in the infinite past, whose product is a certain set of particles to be detected in the infinite future. An element in the S-matrix, called a *scattering amplitude*, is a complex-valued function of the kinematics of the scattering process, whose norm squares to the probability for this process to happen.

1.1.1 *Fields and Particles*

In modern language what we call *particles* are excitations of quantum fields, obtained by quantizing classical fields. Classically a field is described by a function of space-time, valued in a certain target space, and its dynamics is dictated by a given Lagrangian density $\mathcal{L}$ (or Hamiltonian density $\mathcal{H}$). This target space is normally a certain representation space of the product of the Lorentz group with possibly some additional compact group encoding internal symmetries. For instance, a scalar field $\phi(x)$ is valued in $\mathbb{R}$ or $\mathbb{C}$, while a vector field $A_\mu(x)$ is a real field with a Lorentz index. When non-trivial internal symmetries are present, the field may carry additional indices associated to it, which are then called the *flavor indices*. Typical examples of internal symmetries are $U(N)$, $SU(N)$, $SO(N)$, etc.

The notion of a particle is most clear in the context of free fields, where the Lagrangian density is quadratic in the field variables. In the simplest case, for a free real massless scalar field we have (we choose the signature for the Minkowski metric $\eta_{\mu\nu}$ such that the component $\eta_{00} = -1$)

$$\mathcal{L}^{\text{scalar}} := -\frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi\,. \tag{1.1}$$

The typical example of a gauge field, the photon field, is given by

$$\mathcal{L}^{\text{photon}} := -\frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu}\,, \quad F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{1.2}$$

which is characterized by its invariance under the gauge transformation

$$A_\mu(x) \longrightarrow A_\mu(x) - \partial_\mu\,\Lambda(x)\,, \tag{1.3}$$

for any function $\Lambda(x)$.

The equations of motion for free fields are linear in the field variables, and after Fourier transformation they turn into one equation for each Fourier mode, resembling that of a single harmonic oscillator (SHO). So it is convenient to express the classical solutions of the fields in terms of a Fourier expansion. In D dimensions, for example, the general solution of the real scalar in (1.1) is (denote $x^\mu = (t, \vec{x})$)

$$\phi(x) = \int \frac{d^{D-1}\vec{k}}{(2\,\pi)^{D-1}}\,\frac{1}{\sqrt{2\,E_{\vec{k}}}}\,\left(a_{\vec{k}}\,e^{-i\,E_{\vec{k}}\,t + i\,\vec{k}\cdot\vec{x}} + a^\dagger_{\vec{k}}\,e^{i\,E_{\vec{k}}\,t - i\,\vec{k}\cdot\vec{x}}\right), \tag{1.4}$$

with $k_\mu\,k^\mu = -E^2_{\vec{k}} + \vec{k}^2 = 0$, which is called the (massless) *on-shell condition*. To solve the photon field (1.2) we need to fix the gauge redundancy (1.3) first. A convenient choice is the Lorentz gauge $\partial_\mu A^\mu = 0$, with which the solution is

$$A^\mu(x) = \int \frac{d^{D-1}\vec{k}}{(2\,\pi)^{D-1}}\,\frac{1}{\sqrt{2\,E_{\vec{k}}}}\,\sum_{I=1}^{D-2}\left(a_{I,\vec{k}}\,\epsilon^\mu_{I,\vec{k}}\,e^{-i\,E_{\vec{k}}\,t + i\,\vec{k}\cdot\vec{x}} + a^\dagger_{I,\vec{k}}\,\epsilon^{\mu\,\dagger}_{I,\vec{k}}\,e^{i\,E_{\vec{k}}\,t - i\,\vec{k}\cdot\vec{x}}\right), \tag{1.5}$$

where the polarization vector $\epsilon^\mu_{I,\vec{k}}$ satisfies $k_\mu\,\epsilon^\mu_{I,\vec{k}} = 0$ for any $I, \vec{k}$. Note that in addition the gauge redundancy (1.3) identifies $\epsilon^\mu_{I,\vec{k}} \sim \epsilon^\mu_{I,\vec{k}} + \ell\,k^\mu$ (for arbitrary constant ℓ). These two conditions indicate that the polarization vector only possesses $D-2$ degrees of freedom. A convenient choice is to pick $D-2$ ϵ's that are transverse to k^μ and are orthonormal, and we label them by an index I. Hence in the classical solutions (1.4) and (1.5), the a's are numbers that we can freely choose.

From classical fields to quantum fields, one promotes the a's and $a^\dagger$'s to annihilation and creation operators and imposes the canonical quantization condition (in analogy with that of the SHO), e.g., for the scalar field we have

$$[a_{\vec{k}}, a^\dagger_{\vec{l}}] = (2\,\pi)^{D-1}\,\delta^{D-1}(\vec{k} - \vec{l})\,. \tag{1.6}$$

A similar procedure can be done for the gauge fields by properly implementing the gauge fixing condition; see, e.g., Section 8.3 of [1]. Then starting with the free theory vacuum $|0\rangle$ one applies the creation operators and builds up eigenstates of the Hamiltonian $H = \int d^{D-1}\vec{x}\,\mathcal{H}$, which form a basis for the Hilbert space of the free theory. A generic eigenstate thus has the form

$$|\psi_{\vec{k}_1,\vec{k}_2,\ldots,\vec{k}_m}\rangle := a^\dagger_{\vec{k}_1}\, a^\dagger_{\vec{k}_2} \cdots a^\dagger_{\vec{k}}\, |0\rangle \tag{1.7}$$

in the scalar theory (note the operators above commute with each other, which is consistent with the statistics of bosons) and

$$|\psi^{\mu_1,\mu_2,\ldots,\mu_m}_{I_1,\vec{k}_1\, I_2,\vec{k}_2,\ldots,I_m,\vec{k}_m}\rangle := \epsilon^{\mu_1}_{I_1}\,\epsilon^{\mu_2}_{I_2} \cdots \epsilon^{\mu_m}_{I_m}\, a^\dagger_{I_1,\vec{k}_1}\, a^\dagger_{I_2,\vec{k}_2} \cdots a^\dagger_{I_m,\vec{K}}\, |0\rangle \tag{1.8}$$

in the gauge theory. Such an eigenstate is interpreted as the state of m on-shell particles, each of which is specified by its corresponding momentum, together with possibly additional indices for its polarization (if not a scalar) and possibly even more indices for the internal symmetries.

1.1.2 *Interactions and the S-Matrix*

Several theories that are relevant for the description of nature, such as Quantum Electrodynamics (QED), admit perturbative expansion. The Hamiltonian H for such theories can be decomposed into two parts

$$H = H_0 + H_{\text{int}}\,, \tag{1.9}$$

where H_0 is identical to the Hamiltonian of a free theory and H_{int} collects the remaining terms. These terms are in general of order higher than two in the number of field variables, and depend on a set of small parameters (the *couplings*) such that they vanish as the parameters are taken to zero.

We define the scattering states, the in-state $|\psi^{\text{in}}\rangle$ and the out-state $|\psi^{\text{out}}\rangle$, by

$$|\psi^{\text{in}}\rangle := \lim_{t\to-\infty} e^{i\,H\,t}\, e^{-i\,H_0\,t}\, |\psi^{\text{in}}_0\rangle\,, \quad |\psi^{\text{out}}\rangle := \lim_{t\to+\infty} e^{i\,H\,t}\, e^{-i\,H_0\,t}\, |\psi^{\text{out}}_0\rangle\,, \tag{1.10}$$

for some eigenstates $|\psi^{\text{in}}_0\rangle$ and $|\psi^{\text{out}}_0\rangle$ of H_0. An element of the S-matrix measures the transition amplitude from the in-state to the out-state

$$\langle\psi^{\text{out}}|\psi^{\text{in}}\rangle = \lim_{\substack{t_1\to-\infty\\ t_2\to+\infty}} \langle\psi^{\text{out}}_0|U(t_2,t_1)|\psi^{\text{in}}_0\rangle\,, \tag{1.11}$$

where

$$U(t_2,t_1) := e^{i\,H_0\,t_2}\, e^{-i\,H\,(t_2-t_1)}\, e^{-i\,H_0\,t_1} = \mathcal{T}\left\{e^{-i\int_{t_2}^{t_1} d\tau\, H_{\text{int}}(\tau)}\right\}. \tag{1.12}$$

In the above $\mathcal{T}$ denotes the time-ordering operator. Hence the precise definition for the *S-matrix*, which we denote by **S**, is

$$\mathbf{S} := \lim_{\substack{t_1 \to -\infty \\ t_2 \to +\infty}} U(t_2, t_1)\,. \tag{1.13}$$

An element of the S-matrix is specified by an in-state and an out-state, both of which acquire the interpretation of a collection of free on-shell particles.

Obviously the S-matrix contains a trivial identity **1** (corresponding to the case when no interaction occurs at all), and we decompose it as

$$\mathbf{S} = \mathbf{1} + i\,\mathbf{T}. \tag{1.14}$$

It is the matrix **T** that we are interested in, which encodes all the information on the interactions. When the in-state and the out-state is specified, we denote its corresponding element as $\mathcal{M}_n(\psi_0^{\text{in}} \to \psi_0^{\text{out}}) := \langle \psi_0^{\text{out}} | \mathbf{T} | \psi_0^{\text{in}} \rangle$, where n counts the total number of particles in the scattering, i.e., the number of particles in $|\psi_0^{\text{in}}\rangle$ plus the number of those in $|\psi_0^{\text{out}}\rangle$. Although not manifest in the definition (1.13), the S-matrix enjoys the so-called *crossing symmetry*: any particle in the in-state with momentum k^μ can be equally regarded as its anti-particle in the out-state with momentum $-k^\mu$ and vice versa, without changing the corresponding element in the S-matrix. Hence without loss of generality one can assume that all the particles in a scattering process to be in either the in-state or the out-state (consequently these particles together are called *external particles* or *external states*). Note that the S-matrix is translation invariant, which implies momentum conservation after a Fourier transformation, and so a generic scattering amplitude comes in the form of a distribution

$$\mathcal{M}_n(\{k, \epsilon\}) = M_n(\{k, \epsilon\})\,(2\,\pi\,i)^D\,\delta^D\Big(\sum_{a=1}^{D} k_a^\mu\Big)\,, \tag{1.15}$$

where k_a^μ denotes the momentum of the a^{th} particle and the delta functions encode the constraint from momentum conservation. The quantity $M_n(\{k, \epsilon\})$ is purely a function of the *kinematics data* (i.e., momenta and possibly polarizations of the particles). We call $\mathcal{M}_n$ the *scattering amplitude* (or simply *amplitude*), and M_n the (momentum-conservation) *stripped amplitude*. It is for the latter that we are going to introduce a novel formulation in this thesis.

With H_{int} small, in perturbation theory we expand (1.11) into Dyson series

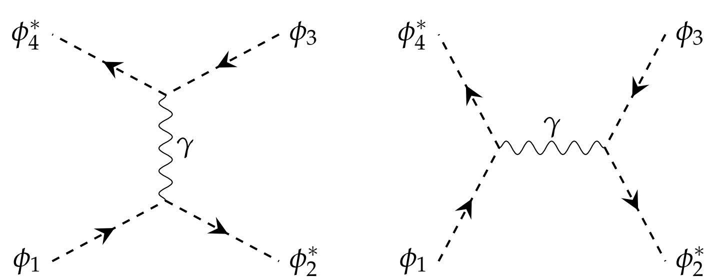
$$\langle \psi_0^{\text{out}} | \mathbf{S} | \psi_0^{\text{in}} \rangle = \langle \psi_0^{\text{out}} | \sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \int_{-\infty}^{+\infty} d\tau_1 \cdots d\tau_m \, \mathcal{T}\{H_{\text{int}}(\tau_1) \cdots H_{\text{int}}(\tau_m)\} | \psi_0^{\text{in}} \rangle \,. \quad (1.16)$$

Using the field expansion as in (1.1) and (1.2), H_{int} is nothing but a combination of creation and annihilation operators in the free theory (dressed with wave functions), and so the standard computing procedure is to apply Wick contraction to the creation and annihilation operators and the external states in all possible ways and then sum them up. This gives rise to an elegant diagrammatic interpretation: each term in H_{int} is treated as a vertex, and each Wick contraction between an annihilation operator and a creation operator (or between an annihilation/creation operator and an in/out-state) is treated as an edge. These are the famous *Feynman diagrams*. With these, each vertex comes with an expression that can be easily read off from H_{int} and each internal edge is attached by the propagator resulting from the Wick contraction. Momenta flow along the edges and respect momentum conservation at each vertex. For every given diagram one multiplies these together and integrate out the momenta flowing inside the diagram, and then the scattering amplitude at each order in the series (1.16) is obtained by summing over all connected amputated diagrams[1].

As an illustrative example, in the massless scalar–QED theory with the following Lagrangian density (here the scalar has to be complex)

$$\mathcal{L}^{\text{scalar–QED}} := -\mathcal{D}_\mu \phi^* \, \mathcal{D}^\mu \phi - \frac{1}{4} F_{\mu\nu} \, F^{\mu\nu} \,, \quad \mathcal{D}_\mu := \partial_\mu + i\, e\, A_\mu \,, \quad (1.17)$$

let us consider the amplitude for the scattering of a pair of scalar and anti-scalar (1,2) into another pair of scalar and anti-scalar (3,4). At the lowest non-trivial order there are two contributing diagrams



Again we choose the Lorentz gauge $\partial_\mu A^\mu = 0$ in reading off the Feynman rules from the Lagrangian (1.17). In this gauge, the interaction term needed in this computation

1 In principle Wick contraction may produce other types of diagrams, but they are either not interesting for the given scattering process or are responsible for renormalization, and can be systematically eliminated.

is $ie(\partial_\mu\phi^*\,\phi - \phi^*\,\partial_\mu\phi)A^\mu$, which gives rise to the vertex $ie(p-q)^\mu$, where p^μ and q^μ are the momenta flowing in each of the two scalar lines (assuming both out-going). In addition, the only propagator involved is the photon propagator $\frac{i\,\eta_{\mu\nu}}{p^2}$. So their total contribution is

$$-e^2\,\frac{(k_1-k_2)\cdot(k_4-k_3)}{s_{1,2}} - e^2\,\frac{(k_3-k_2)\cdot(k_4-k_1)}{s_{2,3}}\,, \tag{1.18}$$

where $s_{a,b} := -(k_a+k_b)^2$ denotes the Mandelstam variables, and from now on we use the center dot to denote contraction of Lorentz indices.

In the above example both diagrams have the topology of a tree. We call the summation of Feynman diagrams of this topology the *tree-level amplitude*, which contributes to the entire amplitude at the lowest order in terms of the couplings, which is associated to the classical process in the scattering. In addition to these there are also diagrams with the topology of loops, which are responsible for quantum corrections. The summation of diagrams with g loops is called a *g-loop amplitude*. It is easy to observe that for fixed external states, the more loops the higher the order of the couplings is. Hence the perturbative calculation of the amplitude for a given scattering process is equally organized by the number of loops g

$$M_n = \sum_{g=0}^{\infty} M_n^{(g)}\,. \tag{1.19}$$

In actual computation, one normally organizes elements of the S-matrix by both the number of particles n (called *multiplicity*) and the number of loops g.

In this thesis we focus on tree-level amplitudes, and so we will suppress the superscripts and simply use M_n to denote a tree-level amplitude.

1.2 THE S-MATRIX PROGRAM

It appears the computation of amplitudes by Feynman diagrams is not a hard task: the method is conceptually simple, and the Feynman rules are easily read from H_{int}. However, this computation becomes impractical immediately when going beyond the simplest cases (that is why standard QFT textbooks rarely discuss scattering among more than four particles). The main sources of complexity are as follows [2]: (i) In general the number of diagrams grows rapidly with the multiplicity. (ii) A single interaction vertex may have a complicated appearance, e.g., the quartic gluon self-interaction. (iii) The number of kinematic variables grows fast with the multiplicity as well, allowing for arbitrarily complicated expressions.

Given the above facts, it might sound surprising that there exist theories in which, despite of the complexity of detailed computations, the final results for certain amplitudes turn out to be extremely simple. Perhaps the most remarkable example is the Parke–Taylor formula for the so-called MHV amplitudes in the Yang–Mills theory in four dimensions (amplitudes with two negative helicity gluons and arbitrary number of positive ones) [3]

$$M_n^{\mathrm{MHV}} = \mathrm{tr}(T^{I_1}\, T^{I_2} \cdots T^{I_n})\, \frac{\langle ij \rangle^4}{\langle 12 \rangle \, \langle 23 \rangle \cdots \langle n1 \rangle} + (\text{permutations})\,, \tag{1.20}$$

by the use of the spin-helicity formalism in parametrizing the 4d kinematics[2]. This is an indication of the existence of formulations alternative to Feynman diagrams, which lead to the results as in (1.20) directly.

The Lagrangian formulation also has a typical disadvantage that for a single theory the Lagrangian is not unique: two Lagrangians may appear very different (and thus different Feynman rules), but they are related by a non-trivial field re-definition and thus lead to the same S-matrix.

After all, a tree-level scattering amplitude is no more than a rational function of the kinematic variables. When regarding the kinematic variables to be complex, it is a meromorphic function with only simple poles, due to the locality of the S-matrix (which we will explain in Section 1.2.3) [5]. In addition, the residue at each pole acquires a physical interpretation. Hence it is possible that we can determine tree-level amplitudes straightforwardly.

In the rest of this section we review the general properties of scattering amplitudes that are relevant to this program. Along the way we set up some of the notations for later chapters.

1.2.1 *Particle Contents and Symmetries*

When talking about a scattering process, the very first thing to keep in mind is the type of the external particles. As discussed before the notion of particles is independent of the detailed contents of interactions.

In fact, a systematic classification of one-particle states is available even before the introduction of quantum fields, by studying irreducible representations of the inho-

2 In this thesis we are interested in a formulation in generic spacetime dimensions. We suggest interested readers to [2, 4] for a thorough review of techniques in four dimensions and of the Parke–Taylor formula (1.20).

mogeneous Lorentz group [1]. The idea is roughly as follows. Firstly, any one-particle state is labeled by its on-shell momentum k^μ. In the case of massless particles discussed in this thesis it means $k_\mu k^\mu = 0$. Upon this, one classify representations of the *little group*, which is the subgroup of the Lorentz group that leaves k^μ invariant.

Scalars are in the trivial representation of the little group. Photon/Gluons or gravitons transform non-trivially. In order to describe them in a local formulation, we need to start from Lorentz vectors ϵ^μ and tensors $\epsilon^{\mu\nu}$, even though they are not the proper representations. The problem is that they have extra degrees of freedom, and the way to solve this is to identify

$$\epsilon^\mu \sim \epsilon^\mu + \ell\, k^\mu\,, \quad \text{and} \quad \epsilon^{\mu\nu} \sim \epsilon^{\mu\nu} + \xi^\mu\, k^\nu + k^\mu\, \xi^\nu\,, \tag{1.21}$$

with arbitrary constant ℓ and ξ^μ.

The amplitude has to transform according to its external particles under the Lorentz group. So when a particle a is a photon/gluon or graviton the amplitude has to be linear in its polarization

$$M_n = \epsilon_a^\mu\, G_\mu \quad (\text{photon/gluon})\,, \quad \text{or} \quad M_n = \epsilon_a^{\mu\nu}\, G_{\mu\nu} \quad (\text{graviton})\,, \tag{1.22}$$

with G whatever remaining part of the amplitude that is independent of ϵ_a^μ. Moreover, the identification (1.21) indicates

$$M_n\big|_{\epsilon_a^\mu \mapsto k_a^\mu} = k_a^\mu\, G_\mu = 0\,. \tag{1.23}$$

for photons/gluons (gauge invariance), and

$$M_n\big|_{\epsilon_a^{\mu\nu} \mapsto \xi^\mu\, k_a^\nu + k_a^\mu\, \xi^\nu} = (\xi^\mu\, k_a^\nu + k_a^\mu\, \xi^\nu)\, G_{\mu\nu} = 0\,, \tag{1.24}$$

for gravitons (diffeomorphism invariance).

In addition to the above constraints, very often an amplitude has to be invariant under certain discrete symmetries. In the case of bosons, the amplitude has to be invariant under the exchange of labels for any two particles of the same type, and in the case of fermions it has to pick up a minus sign. This is a direct consequence of the statistics of the external particles, which is obvious in (1.7) and (1.8). Especially, any amplitude whose external states are of the same type of bosons must be fully permutation invariant.

1.2.2 *Color Decomposition and Partial Amplitudes*

Scattering amplitudes can be decomposed into smaller pieces in the presence of a non-trivial internal symmetry or gauge redundancy. A particle carries a flavor index for the former and a color index for the latter. In this thesis we focus on the case when the corresponding flavor/color group is $U(N)$ and the particles transform in the adjoint representation. In this situation interaction vertices are normally dressed with a flavor/color factor that consists of the structure constants

$$f^{I_a,I_b,I_c} := \frac{1}{\sqrt{2}\,i}\,\mathrm{tr}(T^{I_a}\,T^{I_b}\,T^{I_c}) - \frac{1}{\sqrt{2}\,i}\,\mathrm{tr}(T^{I_a}\,T^{I_c}\,T^{I_b})\,, \tag{1.25}$$

where T are the generators of the corresponding algebra. Hence the scattering amplitude also contains a flavor/color factor.

However, it is usually more practical to apply (1.25) and use the generators T^I. The advantage is that for $U(N)$ there is an additional identity

$$\sum_I (T^I)^{i}_{\bar{j}}\,(T^I)^{k}_{\bar{l}} = \delta^{i}_{\bar{l}}\,\delta^{k}_{\bar{j}}\,, \tag{1.26}$$

which provides a way to glue flavor/color traces

$$\sum_{I_a} \mathrm{tr}(\cdots T^{I_b}\,T^{I_a})\,\mathrm{tr}(T^{I_a}\,T^{I_c}\cdots) = \mathrm{tr}(\cdots T^{I_b}\,T^{I_c}\cdots)\,. \tag{1.27}$$

In a theory of a single type of particle (e.g., pure Yang–Mills) this leads to the consequence that a tree-level amplitude can be decomposed into a summation where in each term the flavor/color factor forms a single trace

$$M_n = \sum_{\alpha\in S_n/\mathbb{Z}_n} \mathrm{tr}(T^{I_{\alpha(1)}}\,T^{I_{\alpha(2)}}\cdots T^{I_{\alpha(n)}})\,M_n[\alpha]\,. \tag{1.28}$$

Hence for each ordering α we have a corresponding quantity $M_n[\alpha]$, which is called the *(color-ordered) partial amplitude*.

To illustrate this decomposition, let us look at the scalar–QCD

$$\mathcal{L}^{\text{scalar–QCD}} := -(\mathcal{D}_\mu\Phi)^{*I}\,(\mathcal{D}^\mu\Phi)^I - \frac{1}{4}\,F^I_{\mu\nu}\,F^{\mu\nu,I}\,, \quad F_{\mu\nu} := [\mathcal{D}_\mu, \mathcal{D}_\nu]\,, \tag{1.29}$$

which is the non-Abelian version of the scalar–QED (1.17). To calculate the amplitude for the same four-scalar scattering, the only change is that the cubic vertex is now dressed by a structure constant, which induces the following color factors for each of the two diagrams

$$\sum_{I_a} f^{I_1,I_2,I_a}\,f^{I_a,I_3,I_4}\,, \quad \sum_{I_a} f^{I_2,I_3,I_a}\,f^{I_a,I_4,I_1}\,. \tag{1.30}$$

Hence the full amplitude is decomposed to

$$e^2 \left(\mathrm{tr}(1234)\, \frac{s^2+t^2+u^2}{st} - \mathrm{tr}(1243)\, \frac{t-u}{s} - \mathrm{tr}(1324)\, \frac{s-u}{t} \right), \tag{1.31}$$

where we abbreviate $\mathrm{tr}(abcd) := \mathrm{tr}(T^{I_a} T^{I_b} T^{I_c} T^{I_d})$, and $s = s_{1,2}$, $t = s_{1,4}$, $u = s_{1,3}$ are the usual Mandelstam variables.

More generally, if a theory involves several types of particles and certain particles do not carry the flavor/color indices (such as Yang–Mills coupled to gravity), then in the above decomposition the flavor/color factor may contain several traces.

1.2.3 *Locality and Unitarity*

By definition (1.13) the S-matrix $\mathbf{S}$ is a unitary operator, satisfying

$$\mathbf{S}^\dagger\, \mathbf{S} = \mathbf{1}\,. \tag{1.32}$$

Using the decomposition (1.14) we obtain the relation

$$-i\,(\mathbf{T} - \mathbf{T}^\dagger) = \mathbf{T}^\dagger\, \mathbf{T}\,. \tag{1.33}$$

To extract a specific entry, we specify two states $|\Psi_0^{\mathrm{in}}\rangle$ and $|\Psi_0^{\mathrm{out}}\rangle$. Since the RHS of (1.33) is a product, we can insert a complete set of intermediate states (which are eigenstates of H_0), which yields

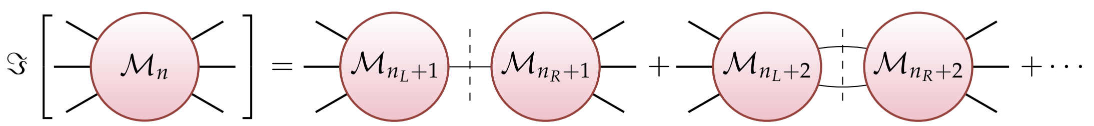
$$\Im(\langle \Psi_0^{\mathrm{out}} | \mathbf{T} | \Psi_0^{\mathrm{in}} \rangle) = \sum_m \left(\prod_{i=1}^m \int \frac{d^D k_i}{(2\,\pi)^D}\, \delta^{(+)}(k_i^2) \right) \langle \Psi_0^{\mathrm{out}} | \mathbf{T}^\dagger | m \rangle\, \langle m | \mathbf{T} | \Psi_0^{\mathrm{in}} \rangle\,, \tag{1.34}$$

where $|m\rangle$ denotes an m-particle state, and the delta constraint $\delta^+(k_i^2) = \delta^+(k_{i,\mu} k_i^\mu)$ restricts k_i^μ on the future light cone. This is called the *optical theorem*. The physical reason why we can insert a complete set of intermediate states is as follows. Imagine that the interaction in a scattering process occurs in two steps, each of which happens in a localized area in the Minkowski space far from the other, then the virtual particles propagated in between approximate to real particles, as long as wave packets are properly defined to separate them sufficiently apart that there is no other intermediate interaction. Then interpreted in the momentum space they form an eigenstate in the Hilbert space of H_0. Hence the ability in doing this is due to the fact that the theory we are considering involves *local* interactions only.

Recall the definition $\mathcal{M}_n = \langle \Psi_0^{\text{out}} | \mathbf{T} | \Psi_0^{\text{in}} \rangle$. Due to the crossing symmetry as discussed in Section 1.1.2, let us move all the external particles to the out-state and define the momenta to be all out-going, then (1.34) takes the form

$$\Im(\mathcal{M}_n) = \sum_m \left(\prod_{i=1}^{m} \int \frac{d^D p_i}{(2\,\pi)^D} \, \delta^{(+)}(p_i^2) \right) \mathcal{M}_{n_L+m} \, \mathcal{M}_{n_R+m} \,, \tag{1.35}$$

where n_L counts the number of particles in $|\Psi_0^{\text{out}}\rangle$ and n_R counts the number of particles in $|\Psi_0^{\text{in}}\rangle$, and $n = n_L + n_R$. Let us denote $P^2 := (\sum_{a \in \psi_0^{\text{in}}} k_a)^2$ as the Mandelstam variable associated to all the particles in $|\Psi_0^{\text{in}}\rangle$. Then the LHS of (1.35) just computes the discontinuity of $\mathcal{M}_n(P^2)$ when regarding P^2 as a complex variable. Pictorially we can express (1.35) as



(1.36)

The first term corresponds to propagating one intermediate on-shell particle (or *internal particle*). It is normally absent from the standard textbooks, because a generic kinematics configuration for the LHS excludes the contribution from this term. However, it does exist if the kinematics becomes singular, i.e., $P^2 \to 0$. When we restrict to the tree level, this is the only contribution from the RHS since as shown in (1.36) all the other terms necessarily form loops. Note that each amplitude $\mathcal{M}$ is defined to be dressed with the delta constraint of momentum conservation as in (1.15), and so the integration over the phase space is localized in this term. Explicitly, if we assume ψ_0^{in} consists of particles labeled by $\{1, 2, \ldots, n_R\}$, then this term is

$$\begin{aligned} &\int \frac{d^D p}{2\,\pi} \, \delta^{(+)}(p^2) \, M_{n_L+1} \, (2\pi)^D \delta^D(k_{n_R+1} + \cdots + k_{n-1} + k_n + p) \\ &\qquad\qquad \times M_{n_R+1} \, (2\pi)^D \delta^D(k_1 + k_2 + \cdots + k_{n_R} - p) \\ =&\delta^{(+)}(P^2) \, \big(M_{n_L+1}\big|_{p^\mu \to P^\mu}\big) \, \big(M_{n_R+1}\big|_{p^\mu \to P^\mu}\big) \, (2\pi)^D \delta^D(k_1 + k_2 + \cdots + k_n). \end{aligned} \tag{1.37}$$

Note that

$$\delta^{(+)}(P^2) = \Im \frac{1}{P^2 - i\epsilon} \,, \tag{1.38}$$

with ϵ an infinitesimal number. Comparing the LHS of (1.36), this means that M_n must contain a pole in P^2, as long as neither M_{n_L+1} nor M_{n_R+1} vanishes (which can happen due to symmetries of the theory, as we will see in later chapters). And furthermore, the residue at this pole factorizes into two parts, each of which looks like an amplitude. Explicitly we have

$$M_n = M_{n_L+1} \, \frac{1}{P^2} \, M_{n_R+1} + \mathcal{O}(1) \tag{1.39}$$

as $P^2 \to 0$. And so we call P^2 a *factorization channel*. This relation was written by assuming that the internal particle is a scalar. When it is in a non-trivial representation of the little group (e.g., a photon/gluon or graviton), then we also need to sum over its polarizations ϵ_I

$$M_n = \sum_{\epsilon_I} M_{n_L+1}(\epsilon_I)\, \frac{1}{P^2}\, M_{n_R+1}(\epsilon_I) + \mathcal{O}(1)\,. \tag{1.40}$$

From (1.39) and (1.40) we see that a tree-level amplitude contains only simple poles, and the residue upon each simple pole factorizes into two small amplitudes. In the study of the S-matrix, the former is usually referred to as *locality*, and the latter as *unitarity*.

1.3 The scattering equations and the CHY representation

In this thesis we introduce a general formulation (*CHY representation*) for tree-level amplitudes in theories of massless particles, which I developed together with Freddy Cachazo and Song He over the past two years [6–9]. This new formulation was largely motivated by encoding locality and unitarity in an auxiliary punctured Riemann sphere instead of the traditional Feynman diagrams.

To be precise, the CHY representation for a general n-point amplitude of massless particles is realized by an integral over the moduli space of n-punctured Riemann spheres $\mathfrak{M}_{0,n}$

$$M_n = \int {\prod_{a=1}^{n}}' d\sigma_a {\prod_{a=1}^{n}}' \delta\Big(\sum_{\substack{b=1\\ b\neq a}}^{n} \frac{k_a\cdot k_b}{\sigma_a-\sigma_b}\Big)\; I_n(\{k,\epsilon,\sigma\}) =: \int d\mu_n\, I_n\,, \tag{1.41}$$

where the σ's are holomorphic coordinates specifying the locations of the punctures, and the integrand I_n is some rational function of the kinematics data and the σ's [6]. The delta constraints impose the *scattering equations* [10, 11]

$$\sum_{\substack{b=1\\ b\neq a}}^{n} \frac{k_a\cdot k_b}{\sigma_a-\sigma_b} = 0\,, \quad \forall a\,. \tag{1.42}$$

The primes in (1.41) indicate that there are redundancies in both the variables and the constraints. To make the integral well-defined, for each product one has to delete three labels and compensate by a factor, i.e., $\prod'_a := \sigma_{a',b'}\sigma_{b',c'}\sigma_{c',a'}\prod_{a\neq a',b',c'}$ (we denote $\sigma_{a,b} := \sigma_a - \sigma_b$), which is independent of the choice of $\{a',b',c'\}$.

Since the number of variables and that of the contraints are the same, in actual computation the integral (1.41) is fully localized by the scattering equations (1.42), reducing to a summation over points in $\mathfrak{M}_{0,n}$. These discrete solutions to the scattering equations possess a nice feature that they capture the information of factorization channels by exploring boundaries of $\mathfrak{M}_{0,n}$ where the n-punctured sphere degenerates accordingly, which is a crucial fact underlying the self-consistency of a general CHY representation. The origin of the scattering equations and properties of their solutions are going to be discussed in full detail in Chapter 2.

Note that in this formulation (1.41) the only thing that depends on the amplitude under study is the integrand $I_n(\{k,\epsilon,\sigma\})$. Since the entire formula has to be invariant under any $SL(2,\mathbb{C})$ transformation acting on the σ's, the integrand I_n has to tranform in an appropriate way. As long as this criterion is satisfied, one can make any proposal for I_n and test whether the resulting formula yields expressions that are physically sensible (especially, whether they are indeed local and unitary). As will be discussed in Chapter 3, it turns out there exist a special class of integrands that possess certain correspondence to scalar Feynman diagrams, i.e., there is a systematic way to translate one such integrand to a diagram and vice versa.

In a large variety of theories of massless bosons, we can obtain a single compact formula for every element of the corresponding S-matrix at the tree level (in this case we say the formula is *closed*). In addition, the integrands in all these formulas are built out of several simple building blocks that are constructed naturally by the consideration of the general constraints on an amplitude as discussed in the previous section.

Here let us summarize what these building blocks are. Firstly there is the so-called Parke–Taylor factor, depending on a planar ordering α

$$C_n[\alpha] := \frac{1}{\sigma_{\alpha(1),\alpha(2)}\,\sigma_{\alpha(2),\alpha(3)}\cdots\sigma_{\alpha(n),\alpha(1)}}\,, \tag{1.43}$$

which is natural for the partial amplitudes after color decomposition. When a formula for the full amplitude is needed, we can simply substitute it by

$$\mathcal{C}_n := \sum_{\alpha\in S_n/\mathbb{Z}_n} \mathrm{tr}\big(T^{I_{\alpha(1)}}\,T^{I_{\alpha(2)}}\cdots T^{I_{\alpha(n)}}\big)\,C_n[\alpha]\,. \tag{1.44}$$

The other building blocks are all permutation invariant quantities, which come in the form of a Pfaffian of some anti-symmetric matrix. These are

$$\mathrm{Pf}\mathcal{X}_n\,,\quad \mathrm{Pf}X_n\,,\quad \mathrm{Pf}'A_n\,,\quad \mathrm{Pf}'\Psi_n\,,\quad \mathrm{Pf}'\Pi\,. \tag{1.45}$$

Here both $\mathcal{X}_n$ and A_n have size $n \times n$, defined as

$$(\mathcal{X}_n)_{a,b} := \begin{cases} \frac{\delta^{I_a,I_b}}{\sigma_a - \sigma_b}, & a \neq b \\ 0, & a = b, \end{cases} \qquad (A_n)_{a,b} := \begin{cases} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b}, & a \neq b \\ 0, & a = b, \end{cases} \tag{1.46}$$

where I_a, I_b are the flavor/color indices. The matrix X_n is obtained from $\mathcal{X}_n$ just by identifying all the flavor/color indices $\delta^{I,J} \equiv 1$. The matrix Ψ_n is of size $2n \times 2n$ with the block structure

$$\Psi_n := \left(\begin{array}{c|c} A_n & -C_n^{\mathrm{T}} \\ \hline C_n & B_n \end{array} \right), \tag{1.47}$$

where the block A_n is the same as that in (1.46), while B_n and C_n depend on the polarization vectors ϵ^μ, defined by

$$(B_n)_{a,b} := \begin{cases} \frac{\epsilon_b \cdot \epsilon_b}{\sigma_a - \sigma_b}, & a \neq b \\ 0, & a = b, \end{cases} \qquad (C_n)_{a,b} := \begin{cases} \frac{\epsilon_a \cdot k_b}{\sigma_a - \sigma_b}, & a \neq b \\ -\sum_{c \neq a} (C_n)_{a,c}, & a = b. \end{cases} \tag{1.48}$$

And finally the matrix Π is obtained by applying a "squeezing" operation on the matrix Ψ_n. Since a generic Π matrix has a more complicated structure, we leave its definition to the detailed discussion in Chapter 5. Note that the matrices A_n, Ψ_n and Π all have corank two on the support of the scattering equations, their Pfaffian vanish, and so the appropriate quantity associated with them is a reduced Pfaffian (denoted by a prime), which is to be defined in Chapter 4.

With these building blocks, the theories for which we have discovered closed formulas together with the corresponding integrands are listed in Table 1. In this table, we use

Table 1: List of Theories with Their Corresponding Integrands

Theory	Integrand	Section
Einstein gravity	$\mathrm{Pf}'\Psi_n \, \mathrm{Pf}'\Psi_n$	4.5
Yang–Mills	$\mathcal{C}_n \, \mathrm{Pf}'\Psi_n$	4.4.1
Φ^3 flavored in $U(N) \times U(\tilde{N})$	$\mathcal{C}_n \, \mathcal{C}_n$	4.2.1
Einstein–Maxwell	$\mathrm{Pf}[\mathcal{X}_n]_\gamma \, \mathrm{Pf}'[\Psi_n]_{:\hat{\gamma}} \, \mathrm{Pf}'\Psi_n$	5.1.3
Einstein–Yang–Mills	$\mathcal{C}_{\mathrm{tr}_1} \cdots \mathcal{C}_{\mathrm{tr}_t} \, \mathrm{Pf}'\Pi(\mathrm{h}; \mathrm{tr}_1 \ldots, \mathrm{tr}_t) \, \mathrm{Pf}'\Psi_n$	5.2
Yang–Mills–Scalar	$\mathcal{C}_n \, \mathrm{Pf}[\mathcal{X}_n]_{\mathrm{s}} \, \mathrm{Pf}'[\Psi_n]_{:\hat{\mathrm{s}}}$	5.1.1
generalized Yang–Mills–Scalar	$\mathcal{C}_n \, \mathcal{C}_{\mathrm{tr}_1} \cdots \mathcal{C}_{\mathrm{tr}_t} \, \mathrm{Pf}'\Pi(\mathrm{g}; \mathrm{tr}_1 \ldots, \mathrm{tr}_t)$	5.2.4
Born–Infeld	$\mathrm{Pf}'\Psi_n \, (\mathrm{Pf}' A_n)^2$	4.4.3
Dirac–Born–Infeld	$\mathrm{Pf}[\mathcal{X}_n]_{\mathrm{s}} \, \mathrm{Pf}'[\Psi_n]_{:\hat{\mathrm{s}}} \, (\mathrm{Pf}' A_n)^2$	5.1.2
extended Dirac–Born–Infeld	$\mathcal{C}_{\mathrm{tr}_1} \cdots \mathcal{C}_{\mathrm{tr}_t} \, \mathrm{Pf}'\Pi(\gamma; \mathrm{tr}_1 \ldots, \mathrm{tr}_t) \, (\mathrm{Pf}' A_n)^2$	5.2.5
$U(N)$ non-linear sigma model	$\mathcal{C}_n \, (\mathrm{Pf}' A_n)^2$	4.2.3
special Galileon	$(\mathrm{Pf}' A_n)^4$	4.2.6

$[\cdots]$ to denote a minor of the matrix, and details about the notation is explained in Chapter 5. We include a third column showing the sections where the corresponding theory is explained.

Using the previously defined building blocks, we are going to first construct all possible integrands for scalar amplitudes in Chapter 4. Most of them already give rise to closed formulas for some of the theories listed in Table 1. For those that do not form a closed formula, additional types of particles are involved in the same theory. The way we extend them to a closed formula for the entire theory is to introduce in Chapter 5 three operations that act on the matrices: the standard compactification, the "squeezing", and a "generalized compactification" [9]. By exhausting the application of these operations, we find out closed formulas for several other theories as well. In particular, with these operations, all the formulas in Table 1 can be regarded as descending from that for the amplitudes in pure gravity. The explicit connections among them are summarized at the end of Chapter 5. It would be interesting to understand better whether the latter two operations acquire some physical interpretation.

A fact that we need to point out here is that some of these formulas still remain as conjectures. Nevertheless, all of them have passed abundant non-trivial checks, both numerically and analytically. On the one hand, most of the checks are performed by comparing the results of these formulas with the corresponding results from the usual Feynman diagram computations, with random kinematics data and in arbitrary spacetime dimensions. These are summarized in Table 4. On the other hand, the general CHY formulation provides a very convenient way to study behavior of the amplitudes both in soft limits and in a generic factorization channel. Detailed analysis in these two limits provides a strong all-multiplicity check of consistency of these formulas with locality and unitarity. In Chapter 6, we first study the behavior of these formulas in the single soft limits. There we only focus on the leading order, aimed at explaining the universality of Weinberg's soft theorems [12] from the view of the CHY representaion. Then we describe a systematic procedure for the study of factorization limits and explain how an on-shell internal particle emerges in this formulation. By applying similar analysis to the two soft particle emissions, for several classes of amplitudes we also discover several new soft theorems up to subleading orders for the emission of two soft particles [13].

Another elegant feature that is universal to the CHY representation is related to the Kawai–Lewellen–Tye (KLT) relations [14, 15]. In their original form in the field theory context, these relations connect a gravity amplitude to the "product" of two Yang–Mills amplitudes

$$M_n^{\mathrm{GR}} = \sum_{\alpha,\beta \in S_{n-3}} M_n^{\mathrm{YM}}[1,\alpha,n-1,n]\,\mathcal{K}_n[\alpha|\beta]\,M_n^{\mathrm{YM}}[1,\beta,n,n-1]\,, \tag{1.49}$$

where the so-called KLT momentum kernel $\mathcal{K}_n[\alpha|\beta]$ is an $(n-3)! \times (n-3)!$ matrix purely depending on the Mandelstam variables [16]. When expressed in terms of the CHY representation, we are going to show in Chapter 7 that (1.49) is equivalent to a theorem called KLT orthogonality [11, 17], which states that

$$\sum_{\alpha,\beta \in S_{n-3}} C_n^{(i)}[1,\alpha,n-1,n]\,\mathcal{K}_n[\alpha|\beta]\,C_n^{(j)}[1,\beta,n,n-1] = \delta^{i,j}\,(\det{}'\Phi_n)^{(i)}\,, \tag{1.50}$$

where the superscript "(i)" means to evaluate on the i^{th} solution to the scattering equations (1.42), and the factor $\det{}'\Phi_n$ is the Jacobian arising from the localized integration (1.41), which will be discussed in full detail later in Chapter 7. An important consequence of the KLT orthogonality is that the KLT relations are not something special to gravity and YM: one can consider the KLT construction from partial amplitudes in any theories what acquire similar color decomposition [7].

In Chapter 8 we end this thesis by commenting on several interesting directions for future explorations on the CHY representation.

Before going on to the detailed discussions on the scattering equations and the CHY representation, we would like to point out that the scattering equations have made an appearance in various contexts in previous literature [18–23]. While this thesis only focus on the tree-level S-matrix for massless bosons, there have also been explorations in connecting this to ambi-twistor strings [24–34] and ordinary string theory [35, 36], and extending this formulation to loop levels [24, 27, 31, 37] and to massive particles [38–40]. We are going to comment on these further in Chapter 8.

This thesis is a summary of the work collaborated with Freddy Cachazo and Song He, appearing in a series of papers:

1. Freddy Cachazo, Song He, **Ellis Ye Yuan**,
   New Double Soft Emission Theorems,
   arXiv: 1503.04816.
2. Freddy Cachazo, Song He, **Ellis Ye Yuan**,
   Scattering Equations and Matrices: From Einstein To Yang-Mills, DBI and NLSM,
   arXiv: 1412.3479.
3. Freddy Cachazo, Song He, **Ellis Ye Yuan**,
   Einstein–Yang–Mills Scattering Amplitudes from Scattering Equations,
   JHEP 01 (2015) 121, arXiv: 1409.8256.
4. Freddy Cachazo, Song He, **Ellis Ye Yuan**,
   Scattering of Massless Particles: Scalars, Gluons and Gravitons,
   JHEP 07 (2014) 033, arXiv: 1309.0885.
5. Freddy Cachazo, Song He, **Ellis Ye Yuan**,
   Scattering of Massless Particles in Arbitrary Dimensions,
   Phys. Rev. Lett. 113 171601 (2014), arXiv: 1307.2199.
6. Freddy Cachazo, Song He, **Ellis Ye Yuan**,
   Scattering Equations and Kawai–Lewellen–Tye Orthogonality,
   Phys. Rev. D 90 065001 (2014), arXiv: 1306.6575.

as well as the following two supplementary notes posted online:

1. Freddy Cachazo, Song He, **Ellis Ye Yuan**,
   Detailed Proof of Double Soft Theorems Up to the Sub-Leading Order,
   https://ellisyeyuan.wordpress.com/2015/03/14/double-soft-theorems-subleading/.
2. Freddy Cachazo, Song He, **Ellis Ye Yuan**,
   Soft Limits and Factorizations of the Pfaffian Formula,
   https://ellisyeyuan.wordpress.com/2013/07/07/soft-limits-and-factorizations/.

2

SCATTERING EQUATIONS

In this chapter we introduce one of the main subjects of this thesis, the *scattering equations* [10, 11]

$$\sum_{\substack{b=1\\b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0\,, \quad \forall a \in \{1,2,\ldots,n\}\,, \tag{2.1}$$

and discuss in detail properties of their solutions.

2.1 THE AUXILIARY RIEMANN SPHERE

All possible configurations of the kinematics associated to a scattering among n massless particles in D dimensions form a space that we call the kinematic space, defined as

$$\mathfrak{K}_{D,n} := \{(k_1^\mu, k_2^\mu, \ldots, k_n^\mu)| \sum_{a=1}^{n} k_a^\mu = 0,\ k_1^2 = k_2^2 = \cdots = k_n^2 = 0\}/SO(1,D-1)\,. \tag{2.2}$$

Since we need Lorentz invariant quantities, it is convenient to introduce the Mandelstam variables $s_{a_1,a_2,\ldots,a_r} := (k_{a_1} + k_{a_2} + \cdots + k_{a_r})^2$, and use an independent subset of them as the coordinates of $\mathfrak{K}_{n,D}$.

The equation $s_{a_1,a_2,\ldots,a_r} = 0$ carves out a subspace of $\mathfrak{K}_{n,D}$. Since $s_{a_1,a_2,\ldots,a_r} = 0$ corresponds to a kinematic configuration that is singular (in the sense that the amplitude diverges), we call this subspace a codimension-1 singularity of the amplitude.

In general we can set a sequence of Mandelstam variables to zero, which leads to singularities of higher codimensions. Let us denote a generic Mandelstam variable as $s_{a_1,a_2,\ldots,a_r} := (k_{a_1} + k_{a_2} + \cdots + k_{a_r})^2$. Then for example, setting $s_{1,2} = s_{1,2,3} = 0$ yields a codimension-2 singularity, and setting $s_{1,2} = s_{3,4} = s_{1,2,3,4} = 0$ yields a codimension-3 singularity. In comparison, e.g., $s_{1,2} = s_{2,3} = 0$ is not physically interesting because the amplitude behaves ambiguously when approaching this kinematics configuration (in this case we say these two factorization channels are inconsistent), so we exclude such points from the singularities. Obviously the highest codimension of a singularity is $n-3$. This is a qualitative characterization of the unitarity of a generic scattering amplitude. Hence we see the structure of the kinematic singularities is complicated, which is one of the main reason why it is in general hard to find out a closed expression or formula for amplitudes to all multiplicities in a given theory.

To avoid this difficulty, a practical strategy is to introduce an auxiliary object that characterizes the singularities of $\mathfrak{K}_{D,n}$ in a cleaner way. To be precise, we look for a certain space whose singularity structure is well understood and easy to analyze, together with a map from $\mathfrak{K}_{D,n}$ to this new space such that (at best) singularities (of the amplitude) are mapped to singularities (in the auxiliary space).

For this purpose, let us consider a Riemann sphere $\mathbb{CP}^1$ and mark out n distinct points (or punctures) on it. The moduli space of all such n-punctured Riemann spheres is known as $\mathfrak{M}_{0,n}$. It is an $n-3$ dimensional complex space, and can be nicely parametrized by a set of n holomorphic variables $\{\sigma_1, \sigma_2, \dots, \sigma_n\}$ quotienting out the $SL(2,\mathbb{C})$ transformations, i.e.,

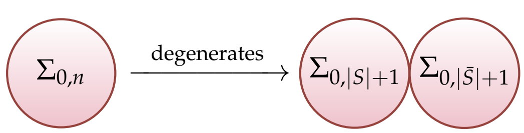
$$\{\sigma_1, \sigma_2, \dots, \sigma_n\} \sim \{\psi(\sigma_1), \psi(\sigma_2), \dots, \psi(\sigma_n)\}\,, \tag{2.3}$$

where

$$\psi(\sigma) := \frac{\alpha\,\sigma + \beta}{\gamma\,\sigma + \delta}\,, \quad \alpha, \beta, \gamma, \delta \in \mathbb{C},\ \alpha\,\delta - \beta\,\gamma = 1\,. \tag{2.4}$$

Here σ_a specifies the location of the a^{th} point on the Riemann sphere. These coordinates controls the shape of the punctured Riemann sphere. Due to the presence of the $SL(2,\mathbb{C})$ redundancy, one can fix any three variables at distinct points on the complex plane $\mathbb{C}$ and leave the rest free, which forms an open patch of $\mathfrak{M}_{0,n}$; and $\mathfrak{M}_{0,n}$ is covered by the union of all such patches.

$\mathfrak{M}_{0,n}$ has boundaries. Let us consider a set of labels $S \subset \{1, 2, \dots, n\}$ of size at least two and denote its complement as $\bar{S}$. A codimension-1 boundary corresponds to the region where the original punctured Riemann sphere $\Sigma_{0,n}$ degenerates into two generic Riemann spheres $\Sigma_{0,|S|+1}$ and $\Sigma_{0,|\bar{S}|+1}$, the former containing points in S and the latter those in $\bar{S}$, and the two spheres are glued at a point that is distinct from the original n points on each sphere

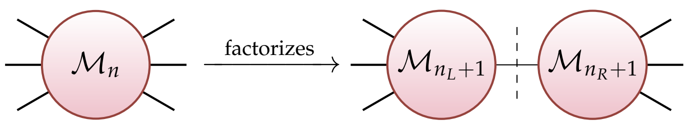


. (2.5)

Here $|S|$ denotes the cardinality of S. With the original coordinates, we can make different choices of three variables to fix in order to see either of the two new Riemann spheres. In one choice, if we fix variables for two points in S and one point in $\bar{S}$, then $\sigma_c - \sigma_d \to 0\ \forall c, d \in \bar{S}$, i.e., they all pinch at the gluing point as seen on $\Sigma_{|S|+1}$. In another choice, if we fix variables for one point in S and two points in $\bar{S}$, then similarly $\sigma_a - \sigma_b \to 0 \; \forall a, b \in S$.

Hence the total complex degrees of freedom left over is $(|S| - 1) + (|\bar{S}| - 1) = n - 4$, justifying that this boundary has codimension one. Upon this, one can further explore codimension-1 boundaries of either of the two new Riemann spheres, which leads to a codimension-2 boundary of $\mathfrak{M}_{0,n}$, and the picture thus attached is a set of three Riemann spheres glued together. This procedure can be iterated until one reaches a codimension-$(n-3)$ boundary, where each of the $n-2$ new Riemann spheres contains exactly three punctures, counting the gluing points.

Now assume that whenever we see a punctured Riemann sphere we literally associate it to a generic scattering amplitude, by identifying the a^{th} external particle to the a^{th} marked point. Recall the term in (1.36) that is implemented as a tree-level factorization (1.39) or (1.40)



. (2.6)

Comparing (2.6) with (2.5), we observe a strong correspondence between the two sides: factorization of amplitudes can be unambiguously mapped to degeneration of Riemann spheres. Moreover, when two factorization channels are inconsistent with each other, there is no way to let the Riemann sphere to degenerate twice, each corresponding to one of the channels. This means the moduli space $\mathfrak{M}_{0,n}$ of n-punctured Riemann spheres already captures all the qualitative requirement of locality and unitarity, i.e., of how an amplitude should factorize.

Therefore, we propose to choose the auxiliary space to be $\mathfrak{M}_{0,n}$, the moduli space of n-punctured Riemann spheres. What is still missing is a map

$$\varphi : \mathfrak{K}_{D,n} \longrightarrow \mathfrak{M}_{0,n} \,, \tag{2.7}$$

that fulfills the above correspondence between n-point amplitudes and n-punctured Riemann spheres in a quantitative way. This is achieved by the scattering equations that we are going to introduce in the next section. Note that so far our discussion depends neither on specific theories nor on the types of external particles, hence we expect that this map is going to be universal.

As discussed in the introduction, every scattering process is specified by a set of *kinematics data*, which consist of the momentum k_a^μ and possibly the polarization ϵ_a (depending on the representation) for each external particle a. When we draw the correspondence between an n-point amplitude and an n-punctured sphere, we should associate each k_a^μ and ϵ_a^μ locally to the marked point a.

Let us make this precise. Since we have a Riemann sphere $\mathbb{CP}^1$, natural objects that we can construct upon it are functions and more generally differential forms. Recall the constraints from momentum conservation

$$\sum_{a=1}^{n} k_a^\mu = 0\,, \quad \forall \mu\,. \tag{2.8}$$

We define a Lorentz-vector-valued meromorphic form ω^μ on $\mathbb{CP}^1$, such that it only possess a simple pole on each marked point a, with the corresponding residue specified by k_a^μ, i.e.,

$$k_a^\mu = \frac{1}{2\,\pi\,i} \oint_{|z-\sigma_a|=\epsilon} \omega^\mu\,, \quad \forall a \in \{1,2,\ldots,n\}\,. \tag{2.9}$$

Then the momentum conservation (2.8) implies that ω^μ has no pole at $z=\infty$. Once we specify a point in $\mathfrak{M}_{0,n}$ by fixing the σ's, we find a unique solution for ω^μ satisfying (2.9), which is

$$\omega^\mu = dz \sum_{a=1}^{n} \frac{k_a^\mu}{z-\sigma_a}\,. \tag{2.10}$$

One can explicitly check that there is no pole at $z=\infty$.

Due to our purpose in finding the map (2.7), $\{\sigma_a\}$ cannot be independent of $\{k_a\}$, and so ω^μ has to be constrained. In order to find it out, we first contract ω^μ with itself to produce a quadratic differential

$$Q = \omega_\mu\, \omega^\mu = dz^2 \sum_{1\leq a<b\leq n} \frac{2\,k_a \cdot k_b}{(z-\sigma_a)\,(z-\sigma_b)}\,. \tag{2.11}$$

In the second equality above we applied massless on-shell conditions $k_a \cdot k_a = 0$ ($\forall a$), so that explicitly there are still only simple poles.

Let us study this quantity in the simplest examples to acquire some intuition. When $n=3$, we have $k_1 \cdot k_2 = k_1 \cdot k_3 = k_2 \cdot k_3 = 0$, which means $Q \equiv 0$. Next we move to $n=4$ which is the first non-trivial case. Here we can choose to fix three punctures, say $\{\sigma_1,\sigma_2,\sigma_3\} = \{0,1,\infty\}$. Recalling our desired correspondence between the amplitude and the Riemann sphere discussed in the previous section, σ_4 has to satisfy

$$\begin{cases} \sigma_4 \to 0 & \text{when} \quad k_1 \cdot k_4 \to 0\,, \\ \sigma_4 \to 1 & \text{when} \quad k_1 \cdot k_3 \to 0\,, \\ \sigma_4 \to \infty & \text{when} \quad k_1 \cdot k_2 \to 0\,. \end{cases} \tag{2.12}$$

Since we are studying tree-level amplitudes, the simplest thing to try is to impose σ_4 as a rational function of the kinematic variables, which leads to a unique solution (up to momentum conservation)

$$\sigma_4 = -\frac{k_1 \cdot k_4}{k_1 \cdot k_2}\,. \tag{2.13}$$

Remarkably, when we plug the solution (2.13) back into (2.11), again we find

$$Q = 2\,dz^2 \left(\frac{k_1 \cdot k_2}{z\,(z-1)} + \frac{k_1 \cdot k_4}{z\,(z-\sigma_4)} + \frac{k_2 \cdot k_4}{(z-1)\,(z-\sigma_4)} \right) \equiv 0\,. \tag{2.14}$$

This leads to the proposal that for any multiplicity n

$$Q(z) \equiv 0\,. \tag{2.15}$$

Note that Q/dz is again a meromorphic form with only a simple pole at each σ_a, then the constraint (2.15) is equivalent to

$$\frac{1}{4\,\pi\,i} \oint_{|z-\sigma_a|} \frac{Q}{dz} = \sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0\,, \quad a \in \{1, 2, \ldots, n\}\,, \tag{2.16}$$

which we name as the *scattering equations*[1].

Since these equations are obtained from the form ω^μ on $\mathbb{CP}^1$, it is natural to expect that they provide a genuine map from $\mathfrak{K}_{D,n}$ to $\mathfrak{M}_{0,n}$. To further confirm this, we explicitly show that (2.16) are covariant under any $SL(2,\mathbb{C})$ transformation

$$\begin{aligned} \sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{k_a \cdot k_b}{\psi(\sigma_a) - \psi(\sigma_b)} &= \sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b}\,(\gamma\,\sigma_a + \delta)\,(\gamma\,\sigma_b + \delta) \\ &= (\gamma\,\sigma_a + \delta)^2 \sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} - (\gamma\,\sigma_a + \delta)\,\gamma \sum_{\substack{b=1 \\ b\neq a}}^{n} k_a \cdot k_b\,, \end{aligned} \tag{2.17}$$

where the second term vanishes due to momentum conservation and the fact that particle a is massless on-shell $k_a^2 = 0$. Hence any solution to the scattering equations indeed specifies a set of n points on $\mathbb{CP}^1$.

1 Here we choose to normalize in order to remove the trivial overall constant.

Although there are in total n equations, since before imposing the constraint (2.15) Q has $n-3$ complex degrees of freedom, we would expect only $n-3$ independent equations. Indeed, the following identities

$$\sum_{a=1}^{n} \sigma_a^m \sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0\,, \quad m = 0,1,2\,, \tag{2.18}$$

hold for generic values of σ's. Among these three identities, the one for $m=0$ is valid merely by anti-symmetry. In addition, we have

$$\sum_{a=1}^{n} \sigma_a \sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = \frac{1}{2} \sum_{\substack{a,b=1 \\ a\neq b}}^{n} k_a \cdot k_b = 0 \tag{2.19}$$

by momentum conservation and on-shell conditions, and

$$\sum_{a=1}^{n} \sigma_a^2 \sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = \sum_{a=1}^{n} \sigma_a \sum_{\substack{b=1 \\ b\neq a}}^{n} k_a \cdot k_b + \sum_{\substack{a,b=1 \\ a\neq b}}^{n} \frac{\sigma_a\, \sigma_b\, k_a \cdot k_b}{\sigma_a - \sigma_b} = 0 \tag{2.20}$$

due to the same reasons as the previous two.

2.3 COUNTING THE NUMBER OF SOLUTIONS

Since the number of independent scattering equations is $n-3$, which is identical to the dimension of $\mathfrak{M}_{0,n}$, there are only a finite number of solutions for any given kinematics data $\{k_a\}$. It turns out that the total number of solutions is always $(n-3)!$. As the first example, we have already seen exactly one solution (2.13) at $n=4$ in the previous section (when $n=3$ there are no equations).

A physical way to understand this counting, as described in [11], is by exploring a single soft limit, i.e., by taking one of the momentum to zero while preserving momentum conservation. Suppose we already know the counting for a generic $(n-1)$-particle scattering is $(n-4)!$. Starting from the kinematics for an n-particle scattering, we can continuously change k_n^μ so that it approaches zero. In the $k_n^\mu \to 0$ limit, the n^{th} scattering equation decouples, while the remaining $n-1$ equations become the scattering equations for an $(n-1)$-particle scattering and so there are $(n-4)!$ solutions for $\{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$. When we slightly turn on k_n^μ, since an integer cannot jump during a continuous change this number counting remains the same. Now take the n^{th} equation and evaluate $\{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$ on one of the $(n-4)!$ solutions, this equation becomes a polynomial equation of degree $n-3$ for σ_n. So in total we obtain $(n-3)!$ solutions for $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$. Due to the same reason, as we deform the kinematics data back to their original value, this number $(n-3)!$ stays the same.

The above proof also provides an algorithm in solving the scattering equations for large n, when it becomes hard to solve them directly.

In fact, after fixing the $SL(2,\mathbb{C})$ redundancy by setting, say $\{\sigma_1,\sigma_2,\sigma_n\} = \{\infty,,\sigma^*,0\}$ (where σ^* is some fixed non-zero number), one can show that the scattering equations (2.16) are equivalent to the following $n-3$ polynomial equations [38]

$$\sum_{S_i\subset\{2,3,\ldots,n-1\}} s_{\{1\}\cup S}\,\sigma_S = 0\,, \quad i\in\{1,2,\ldots,n-3\}\,, \tag{2.21}$$

where S_i denote a subset of size i and the summation is over all such subsets, and $s_S := \sum_{a<b\in S} k_a\cdot k_b$ and $\sigma_S := \prod_{a\in S}\sigma_a$. Obviously, each of these equations are multi-linear in each σ_a, and the total degrees go from 1 to $n-3$. Since these equations are independent, the counting $(n-3)!$ is a direct consequence of Bézout's theorem. The polynomial equations (2.21) are also easier to solve as compared to the original form (2.16). But as we will see in the next section, the original form (2.1)

$$\sum_{\substack{b=1\\b\neq a}}^{n} \frac{k_a\cdot k_b}{\sigma_a-\sigma_b} = 0\,, \quad \forall a\in\{1,2,\ldots,n\}\,, \tag{2.22}$$

manifestly has a physical meaning, which makes it easier to analyse properties of amplitudes.

2.4 VARIOUS LIMITS

In this section we analyze the behavior of the solutions in different limits of the kinematics data, with the purpose of providing a detailed description of how scattering equations encodes locality and unitarity. This serves as the first step in the consistency checks of various formulas we are going to construct in the next two chapters, which will be discussed in Chapter 6.

2.4.1 *Soft Limits*

In the previous section we used a single soft limit to argue about the total number of solutions to the scattering equations. Here we investigate how the solutions behave in such limit. We are going to consider two types of soft limits: the single soft limit, and the double soft limit.

In the single soft limit, we fix the convention of always taking the momentum for the last particle to be soft. More precisely, we introduce a small parameter τ and let $k_n^\mu = \tau\, p^\mu$ with a certain fixed null vector p^μ, and we choose values for the other null momenta which satisfy the momentum conservation

$$k_1^\mu + k_2^\mu + \cdots + k_{n-1}^\mu + \tau\, p^\mu = 0\,, \tag{2.23}$$

and stay finite in the $\tau \to 0$ limit.

As discussed before, when $\tau = 0$ the system reduces to that for a generic $(n-1)$-particle scattering, and so each of the $(n-4)!$ solutions for $\{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$ is non-degenerate. Staying in the neighborhood of $\tau = 0$ this continues to be true. Let us pick up the n^{th} scattering equation

$$\tau \sum_{b=1}^{n-1} \frac{p \cdot k_b}{\sigma_n - \sigma_b} = 0\,. \tag{2.24}$$

Since $p \cdot k_b$ is generically finite for any b, if we assume that $|\sigma_n - \sigma_{b^*}| \sim \tau$ for some b^*, it contradicts the above equation as we send $\tau \to 0$. This means that in the single soft limit, all the $(n-3)!$ solutions are non-degenerate, i.e., $|\sigma_a - \sigma_b| \sim \tau^0$ $\forall a, b$.

For the double soft limit, we consider the situation when two soft particles are simultaneously emitted (say, the last two particles). In other words, we assume $k_{n-1}^\mu = \tau\, p^\mu$ and $k_n^\mu = \tau\, q^\mu$, with the null vectors p^μ and q^μ fixed, hence they are controled by the same scale. Again we choose values for the remaining null momenta to satisfy the momentum conservation

$$k_1^\mu + k_2^\mu + \cdots + k_{n-2}^\mu + \tau\,(p^\mu + q^\mu) = 0\,, \tag{2.25}$$

which stay finite in the $\tau \to 0$ limit.

The situation is more interesting in this case. When $\tau = 0$ the system reduces to that for a generic $(n-2)$-particle scattering and the solutions for $\{\sigma_1\, \sigma_2, \ldots, \sigma_{n-2}\}$ are therefore non-degenerate. Now let us focus on the neighborhood of $\tau = 0$. To make the structure explicit, let us re-define

$$\sigma_{n-1} = \rho - \frac{\xi}{2}\,, \quad \sigma_n = \rho + \frac{\xi}{2}\,, \tag{2.26}$$

i.e., ρ is the center position and ξ the difference of σ_n and σ_{n-1}. With these new variables, we take the summation and the difference of the $(n-1)^{\text{th}}$ and n^{th} scattering equations, which yields

$$\sum_{b=1}^{n-2}\left(\frac{p\cdot k_b}{\rho-\frac{\xi}{2}-\sigma_b}+\frac{q\cdot k_b}{\rho+\frac{\xi}{2}-\sigma_b}\right)=0\,, \tag{2.27}$$

$$\sum_{b=1}^{n-2}\left(\frac{p\cdot k_b}{\rho-\frac{\xi}{2}-\sigma_b}-\frac{q\cdot k_b}{\rho+\frac{\xi}{2}-\sigma_b}\right)-\frac{2\,\tau\,p\cdot q}{\xi}=0\,. \tag{2.28}$$

We consider the second equation above as constraining the variables ξ. Obviously, while there exist solutions for ξ that stay finite as $\tau\to 0$, there also exist solutions where $\xi\sim\tau$ (degenerate). For the latter case, we can perturbatively expand ξ and by (2.28) find a unique solution, which at the leading order is

$$\xi=\tau\left(\frac{1}{2\,p\cdot q}\sum_{b=1}^{n-2}\frac{k_b\cdot(p-q)}{\rho-\sigma_b}\right)^{-1}+\mathcal{O}(\tau^2)\,. \tag{2.29}$$

With this solution for ξ, we can also perturbatively expand $\rho=\rho_0+\mathcal{O}(\tau)$, and the leading term is solved by the leading part of (2.27), i.e.,

$$\sum_{b=1}^{n-2}\frac{k_b\cdot(p+q)}{\rho_0-\sigma_b}=0\,, \tag{2.30}$$

which is equivalent to a degree-$(n-4)$ polynomial for ρ_0 (naively it is of degree $n-3$, but the leading coefficient is of order τ and thus should be neglected).

As a consequence, we see that in the double soft limit, $(n-4)!$ of the solutions become degenerate, where $|\sigma_{n-1}-\sigma_n|\sim\tau$. The other $(n-4)!\,(n-4)$ solutions remain non-degenerate.

In analogy, one can define triple soft limit and so on as well. There in general the solutions will fall into several types according to the pattern how the punctured Riemann sphere degenerates. However, the double soft limit singles out among all these higher order soft limits, because there exist soft theorems for the double soft limit in certain classes of theories, and there we are going to see that the degenerate solutions play a crucial role in deriving these theorems. Details are presented in Chapter 6.

2.4.2 *Factorization Channels*

Now let us have a look at a generic factorization channel. Without lost of generality, we explore the channel defined by

$$k_I^2 := (k_1 + k_2 + \cdots + k_{n_L})^2 \longrightarrow 0\,, \tag{2.31}$$

with $2 \leq n_L \leq n-2$, and we denote $L = \{1, 2, \ldots, n_L\}$ and R its complement, with $n_R = n - n_L$.

For convenience in studying the behavior of the solutions in this limit, we introduce a new variable ζ and re-define the σ's as

$$\sigma_a = \begin{cases} \frac{\zeta}{u_a}\,, & a \in L\,, \\ \frac{v_a}{\zeta}\,, & a \in R\,. \end{cases} \tag{2.32}$$

Recall that we always fix three variables to get rid of the $SL(2, \mathbb{C})$ redundancy. Now we have altogether $n+1$ variables so that we need to fix one more. A good choice is to fix two u's and two v's, say $\{u_1, u_2, v_{n-1}, v_n\}$. Hence in this case we regard ζ as a variable to be solved together with the rest by the scattering equations.

With this set-up, our expectation is that in the limit (2.31) there will be solutions in which $\zeta \to 0$ so that all the points in L pinch together and the Riemann sphere degenerates in the desired manner. For this purpose we can first assume that ζ is small and check that there indeed exist such solutions at the end.

We first study the scattering equations labeled by $a \in R$, and expand it with respect to ζ up to the sub-leading order

$$\zeta \sum_{b=1}^{n_L} \frac{k_a \cdot k_b}{v_a} + \zeta \sum_{\substack{b=n_L+1 \\ b \neq a}}^{n} \frac{k_a \cdot k_b}{v_a - v_b} + \zeta^3 \sum_{b=1}^{n_L} \frac{k_a \cdot k_b}{v_a^2\, u_b} + \mathcal{O}(\zeta^5) = 0\,. \tag{2.33}$$

Next we multiply this a^{th} equation by $\frac{v_a}{\zeta}$ and sum over all $a \in R$. Applying momentum conservation we obtain

$$-\frac{1}{2}\, k_I^2 + \zeta^2 \sum_{\substack{a \in R \\ b \in L}} \frac{k_a \cdot k_b}{v_a\, u_b} + \mathcal{O}(\zeta^4) = 0\,. \tag{2.34}$$

From the above equation we explicitly observe that the solutions fall into two types. The first type, $\zeta^2 \sim k_I^2$, are exactly the degenerate solutions that we were looking for.

There are also a second type of solutions where ζ remains finite and the Riemann sphere remains non-degenerate in the limit. At first sight this seems to cause problems, because, as we discussed at the beginning of this chapter, we want a map that maps kinematic singularities to the corresponding degenerate Riemann spheres. However, recall that the factorization of the original amplitude is a phenomenon only at the leading order. Hence it does not cause any issue if the contributions from these non-degenerate solutions are at most sub-leading in the factorization limit.

If fact, the extra non-degenerate solutions have to exist, by the counting of solutions to the scattering equations. At the leading order the n-point amplitude factorize into a left part and a right part, each of which is again an amplitude. Correspondingly we should expect that at the leading order the scattering equations reduce to two sets, one for the left amplitude and the other for the right one. Then obviously the total number of degenerate solutions has to be $(n_L-2)!(n_R-2)!$. However, this number is always smaller than $(n-3)!$, which means that there are some remaining solutions from the original system and that they have to be non-degenerate.

Now the remaining task is to verify that indeed the original equations reduce to those for the left and the right amplitudes. First let us look back at the equations (2.33) but this time only keep the leading order, which becomes

$$\zeta\left(\frac{k_a\cdot k_{I_R}}{v_a-0}+\sum_{b\in R\setminus\{a\}}^{n}\frac{k_a\cdot k_b}{v_a-v_b}\right)+\mathcal{O}(\zeta^3)=0\,, \tag{2.35}$$

where $k^\mu_{I_R}:=\sum_{b\in L}k^\mu_b$. In the above, we intentionally write out the "0" in the denominator of the first term, so that the leading part has exactly the appearance of the scattering equations for the right amplitude, with external particles labeled by $R\cup\{I_R\}$, where I_R denotes the gluing point on the right Riemann sphere. Note that the location of point I_R is already fixed at $v_{I_R}=0$. This is allowed because previously we only fixed v_{n-1} and v_n. One may also worry that we have not yet obtained the scattering equation labeled by I_R, but this is fine since only n_R-2 of the equations are independent.

Next we inspect the original scattering equations labeled by $a\in L$. To the leading order they become

$$-\sum_{b\in L\setminus\{a\}}\frac{u_a\,u_b}{\zeta}\,\frac{k_a\cdot k_b}{u_a-u_b}+\mathcal{O}(\zeta)=-\frac{u_a^2}{\zeta}\left(\sum_{b\in L\setminus\{a\}}\frac{k_a\cdot k_b}{u_a-u_b}+\frac{k_a\cdot k_{I_L}}{u_a-0}\right)+\mathcal{O}(\zeta)=0\,, \tag{2.36}$$

where $k^\mu_{I_L}:=-k^\mu_{I_R}$, and so we use I_L to denote the gluing point on the left Riemann sphere. Again, the leading terms become the scattering equations for the left amplitude, with the fixing $u_{I_L}=0$.

From the discussions by now, we see that the scattering equations (2.16) provide a map from $\mathfrak{K}_{D,n}$ to $\mathfrak{M}_{0,n}$ for any D and n that quantitatively fulfills the correspondence (2.7) between the two spaces. In all cases the map is 1 to $(n-3)!$, and so for whatever formula for the amplitudes that we construct with this map, we should expect the amplitudes only receive contribution from these $(n-3)!$ discrete points in $\mathfrak{M}_{0,n}$. The explicit construction is our task in the next chapter.

3 GENERAL FORMULATION

In this chapter we introduce the other main subject of this thesis, the Cachazo–He–Yuan (CHY) representaion of tree-level amplitudes for massless particles, using the scattering equations that we introduced in the previous chapter.

From the previous discussion, we associate $\mathfrak{M}_{0,n}$ to an n-particle scattering. Since this $\mathfrak{M}_{0,n}$ is merely an auxiliary space, in constructing a formula for scattering amplitudes it is natural to consider certain top differential form in $\mathfrak{M}_{0,n}$ and integrate it over the entire space. Moreover, since the $\mathfrak{M}_{0,n}$ was introduced to keep track of locality and unitarity, we want the integral to be further localized by the scattering equations. Hence a natural proposal is

$$\int \prod_{a=1}^{n} d\sigma_a \prod_{a=1}^{n} \delta\Big(\sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} \Big) \, I(\{k,\epsilon,\sigma\}) \,, \tag{3.1}$$

where the integrand $I(\{k,\epsilon,\sigma\})$ is some function of the kinematics data $\{k,\epsilon\}$ and the σ's.

However, (3.1) is problematic due to the redundancies in both the variables and the delta constraints. Firstly, it is standard to define an invariant measure by replacing

$$d\sigma_1 \wedge d\sigma_2 \wedge \cdots \wedge d\sigma_n \longmapsto \theta_n(a',b',c') \begin{vmatrix} 1 & 1 & 1 \\ \sigma_{a'} & \sigma_{b'} & \sigma_{c'} \\ \sigma_{a'}^2 & \sigma_{b'}^2 & \sigma_{c'}^2 \end{vmatrix} \underbrace{d\sigma_1 \wedge d\sigma_2 \wedge \cdots \wedge d\sigma_n}_{\text{deleting } d\sigma_{a'}, d\sigma_{b'}, d\sigma_{c'}} \,, \tag{3.2}$$

where $\theta_n(a',b',c')$ is a sign from re-arranging the wedge product in bringing $d\sigma_{a'} \wedge d\sigma_{b'} \wedge d\sigma_{c'}$ to the front, i.e., it is the signature of the permutation $(a',b',c',1,2,\ldots,n)$ where the "$\ldots$" excludes $\{a',b',c'\}$. Secondly, recalling the form of the three linear relations (2.18), it is obvious that we need to delete three delta constraints and pay by a Jacobian in the similar form as that in (3.2). So the problem in (3.1) is fixed by the replacement

$$\prod_{a=1}^{n} \longrightarrow \prod_{a=1}^{n}{}' := \theta_n(a',b',c')(\sigma_{a'} - \sigma_{b'})\,(\sigma_{b'} - \sigma_{c'})\,(\sigma_{c'} - \sigma_{a'}) \prod_{a\in\{1,2,\ldots,n\}\backslash\{a',b',c'\}} \,, \tag{3.3}$$

for arbitrary $\{a', b', c'\}$ (the choices for the two parts can be different). And we make the following proposal for amplitudes stripped off momentum conservation delta functions [6, 11]

$$M_n = \int \prod_{a=1}^{n}{}' d\sigma_a \prod_{a=1}^{n}{}' \delta\Big(\sum_{\substack{b=1\\ b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b}\Big)\, I_n(\{k,\epsilon,\sigma\}) =: \int d\mu_n\, I_n\,. \tag{3.4}$$

In the above, we introduced a notation for the part other than I_n for later convenience, and with a slight abuse of terminology we name $d\mu_n$ as the *measure*.

As a result of the previous section, the integration in (3.4) is actually localized to $(n-3)!$ points in $\mathfrak{M}_{0,n}$. So equivalently we have

$$M_n = \sum_{i=1}^{(n-3)!} \frac{I_n(\{k,\epsilon,\sigma\})}{J_n(\{k,\sigma\})}\Big|_{i^{\text{th}}\text{ solution}}, \tag{3.5}$$

where $J_n(\{k,\sigma\})$ is the Jacobian from both the redundancy fixing and solving the delta constraints. In calculating this Jacobian, we encounter a matrix Φ_n of size $n \times n$, defined as

$$(\Phi_n)_{a,b} := \begin{cases} \frac{k_a \cdot k_b}{(\sigma_a - \sigma_b)^2}\,, & a \neq b\,, \\ -\sum_{\substack{c=1\\ c\neq a}}^{n} (\Phi_n)_{a,c}\,, & a = b\,. \end{cases} \tag{3.6}$$

Then the Jacobian in (3.5) is

$$J_n(\{k,\sigma\}) = \det{}'\Phi_n := \frac{\det[\Phi_n]^{\hat{a}',\hat{b}',\hat{c}'}_{\hat{a}'',\hat{b}'',\hat{c}''}}{\sigma_{a',b'}\,\sigma_{b',c'}\,\sigma_{c',a'}\;\sigma_{a'',b''}\,\sigma_{b'',c''}\,\sigma_{c'',a''}}\,, \tag{3.7}$$

where we abbreviate $\sigma_{a,b} := \sigma_a - \sigma_b$, and $[\Phi_n]^{\hat{a}',\hat{b}',\hat{c}'}_{\hat{a}'',\hat{b}'',\hat{c}''}$ denotes the minor of Φ_n obtained by deleting the rows labeled by $\{a', b', c'\}$ and the columns labeled by $\{a'', b'', c''\}$.

Note that on the support of the scattering equations $\det \Phi_n = 0$ as the matrix Φ_n has corank 3, with its kernel spanned by the vectors $(\sigma_1^m, \sigma_2^m, \ldots, \sigma_n^m)^{\mathrm{T}}$ for $m = 0, 1, 2$. Thus the Jacobian J_n as defined in (3.7) is a natural invariant quantity associated to Φ_n, and we call it the *reduced determinant* of Φ_n, denoted as $\det'\Phi_n$.

Since we are considering tree-level amplitudes only, it is natural to expect that the function $I_n(\{k,\epsilon,\sigma\})$ is rational. But we cannot use an arbitrary rational function for I_n: the formula (3.4) has to be invariant under any $SL(2,\mathbb{C})$ transformation acting on σ's, since the Riemann sphere is auxiliary. To see what this implies on I, note that the measure $d\mu_n$ under the transformation (2.4) behaves as

$$d\mu_n \xrightarrow{\psi \in SL(2,\mathbb{C})} d\mu_n \prod_{a=1}^{n} (\gamma\,\sigma_a + \delta)^{-4}\,. \tag{3.8}$$

So for whatever integrand I_n we choose, it has to satisfy

$$I_n \xrightarrow{\psi\in SL(2,\mathbb{C})} I_n \prod_{a=1}^{n} (\gamma\,\sigma_a+\delta)^4\,. \tag{3.9}$$

In the rest of this chapter we present simple examples followed by a neat correspondence between a special class of integrands with the trivalent diagams of massless scalars. The aim is to provide some intuition about how the formulation introduced above works. Readers who are eager to see the construction of closed formulas for amplitudes in various theories can go directly to the next chapter.

3.1 Illustrative examples

Let us start by considering the case that $I_n=I_n(\{\sigma\})$ depends on the σ's only. Recall (3.9), the easiest way to construct a candidate for I_n is to use simple expressions which transform covariantly by themselves. The simplest expression of this kind is $(\sigma_a-\sigma_b)$ for some label a and b

$$(\sigma_a-\sigma_b) \xrightarrow{\psi\in SL(2,\mathbb{C})} \frac{(\sigma_a-\sigma_b)}{(\gamma\,\sigma_a+\delta)\,(\gamma\,\sigma_b+\delta)}\,. \tag{3.10}$$

So it is sufficient to construct an I_n purely by factors of this form, as long as for any label a the total number of times that it appears in the denominator minus that in the numerator is 4. In this chapter we only consider examples which have a trivial numerator 1.

3.1.1 $n=3$

When $n=3$, with the restrictions imposed before the integrand is uniquely fixed to be $I_3=\sigma_{1,2}^{-2}\,\sigma_{2,3}^{-2}\,\sigma_{3,1}^{-2}$. Since there is no constraints from the scattering equations in this case, we have

$$\int d\mu_3\, I_3 = \frac{(\sigma_{1,2}\,\sigma_{2,3}\,\sigma_{3,1})^2}{\sigma_{1,2}^2\,\sigma_{2,3}^2\,\sigma_{3,1}^2} = 1\,. \tag{3.11}$$

3.1.2 $n=4$

When $n=4$, we already obtained the unique solution in (2.13), and so let us stick to the same convention. Explicitly, we fix $\{\sigma_1,\sigma_2,\sigma_3\}=\{0,1,\infty\}$, and we have the solution $\sigma_4 = -\frac{k_1 \cdot k_4}{k_1 \cdot k_2}$. Suppose in the delta constraints we delete those labeled by $\{2, 3, 4\}$, then the Jacobian is

$$J_4 = \frac{k_1 \cdot k_4}{\sigma_{1,2}\, \sigma_{2,3}^2\, \sigma_{3,1}\, \sigma_{3,4}\, \sigma_{4,2}\, \sigma_{1,4}^2} \,. \tag{3.12}$$

In the first example, we choose the integrand to be

$$I_4 = \frac{1}{(\sigma_{1,2}\, \sigma_{2,3}\, \sigma_{3,4}\, \sigma_{4,1})\, (\sigma_{1,2}\, \sigma_{2,4}\, \sigma_{4,3}\, \sigma_{3,1})} \,. \tag{3.13}$$

One can check this produces the result

$$\int d\mu_4\, I_4 = -\frac{\sigma_4}{k_1 \cdot k_4} = \frac{2}{s_{1,2}} \,, \tag{3.14}$$

where $s_{1,2} = (k_1 + k_2)^2 = 2\, k_1 \cdot k_2$ denotes the usual Mandelstam variable. We see up to a constant factor we obtain a single (massless) scalar propagator.

In the second example, we choose an integrand with a cyclic symmetry

$$I_4 = \frac{1}{(\sigma_{1,2}\, \sigma_{2,3}\, \sigma_{3,4}\, \sigma_{4,1})^2} \,. \tag{3.15}$$

Following similar calculations we obtain

$$\int d\mu_4\, I_4 = \frac{\sigma_4 - 1}{k_1 \cdot k_4} = -2 \left(\frac{1}{s_{1,2}} + \frac{1}{s_{1,4}} \right) \,, \tag{3.16}$$

which looks like a summation of two four-point scalar diagrams.

3.1.3 $n = 5$

When $n = 5$, although in solving the scattering equations here we will encounter a quadratic equation, the calculation can still be done analytically. But due to its lengthy appearance we will not write out the detailed intermediate steps. Instead, we summarize the result for several types of integrands as follows:

Table 2: Five-Point Examples

Integrand	Result
$\frac{1}{(\sigma_{1,2}\,\sigma_{2,3}\,\sigma_{3,4}\,\sigma_{4,5}\,\sigma_{5,1})(\sigma_{1,2}\,\sigma_{2,4}\,\sigma_{4,5}\,\sigma_{5,3}\,\sigma_{3,1})}$	$-\frac{4}{s_{1,2}\,s_{4,5}}$
$\frac{1}{(\sigma_{1,2}\,\sigma_{2,3}\,\sigma_{3,4}\,\sigma_{4,5}\,\sigma_{5,1})(\sigma_{1,2}\,\sigma_{2,5}\,\sigma_{5,4}\,\sigma_{4,3}\,\sigma_{3,1})}$	$\frac{4}{s_{1,2}}\left(\frac{1}{s_{3,4}} + \frac{1}{s_{4,5}}\right)$
$\frac{1}{(\sigma_{1,2}\,\sigma_{2,3}\,\sigma_{3,4}\,\sigma_{4,5}\,\sigma_{5,1})^2}$	$4\left(\frac{1}{s_{1,2}\,s_{4,5}} + \frac{1}{s_{2,3}\,s_{5,1}} + \frac{1}{s_{3,4}\,s_{1,2}} + \frac{1}{s_{4,5}\,s_{2,3}} + \frac{1}{s_{5,1}\,s_{3,4}}\right)$
$\frac{1}{(\sigma_{1,2}\,\sigma_{2,3}\,\sigma_{3,4}\,\sigma_{4,5}\,\sigma_{5,1})(\sigma_{1,3}\,\sigma_{3,5}\,\sigma_{5,2}\,\sigma_{2,4}\,\sigma_{4,1})}$	0

In the previous section we intentionally listed out examples that share several common features. Firstly, the integrands were all written as a product of two parts. In particular, one easily see that each of the two transforms covariantly under $SL(2,\mathbb{C})$ but picks up a factor $\prod_{a=1}^{n}(\gamma\,\sigma_a+\delta)^2$ (instead of a power 4), and so in a sense they are "half-integrands". Secondly, all the results obtained from the integration, if not vanish, can be interpreted as a sum of trivalent diagrams of massless scalars.

In fact this continues to be true for arbitrary higher n, which gives rise to a correspondence between a special class of integrands and the trivalent massless scalar diagrams. Let us first introduce an object which generalizes the above "half-integrands". For a given ordering of α the n labels, we define

$$C_n[\alpha] := \frac{1}{\sigma_{\alpha(1),\alpha(2)}\,\sigma_{\alpha(2),\alpha(3)}\cdots\sigma_{\alpha(n-1),\alpha(n)}}\,. \tag{3.17}$$

This is named as the *Parke–Taylor factor*, due to the reason that it appeared for the first time in [41] in a twistor string formulation that leads to the Parke–Taylor formula (1.20). The special class of integrands is then constructed by

$$I_n[\alpha|\beta] := C_n[\alpha]\,C_n[\beta] \tag{3.18}$$

for two given orderings α and β. We call the formula $m_n[\alpha|\beta] := \int d\mu_n\, I_n[\alpha|\beta]$ the *double partial amplitude* [7], whose meaning will become clear in Section 4.2.1.

In addition, note that any tree diagram can always be embedded on a plane, by which its external points acquire an induced planar ordering α. In this case we call this tree diagram α-ordered. Obviously a single tree diagram can assume several different planar orderings.

Given these definitions, we have the following theorem that translates any integrand of the special class of the trivalent diagrams:

Proposition 3.2.1. *The function $m_n[\alpha|\beta]$ computes the sum of all trivalent massless scalar diagrams that can be regarded both as α-ordered and as β-ordered, where each diagram's contribution is given by the product of its propagators.*

More explicitly, let $\mathcal{T}[\alpha]$ denote the set of all α-ordered trivalent diagrams and $\mathcal{T}[\beta]$ that of the β-ordered ones. Then (up to an overall sign)

$$m_n[\alpha|\beta] = 2^{n-3}\sum_{g\in\mathcal{T}[\alpha]\cap\mathcal{T}[\beta]}\ \prod_{e\in E_g}\frac{1}{s_e}\,, \tag{3.19}$$

where $s_e := P_e^2$ with P_e^μ being the momentum flowing along the edge e in the set E_g of all edges in the trivalent diagram g. In particular, whenever $\mathcal{T}[\alpha] \cap \mathcal{T}[\beta] = \varnothing$ then $m_n[\alpha|\beta] = 0$. Moreover, for any single diagram, there exist two orderings α and β such that it is identical to $m_n[\alpha|\beta]$ (although the choice of such α and β is usually highly non-unique). More detailed explanations about this proposition, including the determination of the overall sign as well as an efficient algorithm to compute $m_n[\alpha|\beta]$, can be found in [7].

This leads to the fact that in principle any amplitude of massless particles acquires a formula of the form (3.4). The explicit procedure is to write out the expression in terms of Feynman diagrams, and translate the denominator in each diagram to certain integrand of the form (3.18), i.e., ("F.D.": Feynamn diagrams)

$$M_n = \sum_{g \in \text{F.D.}} \frac{N_g(\{k, \epsilon\})}{\prod_{e \in E_g} s_e} = \sum_{g \in \text{F.D.}} \frac{N_g(\{k, \epsilon\})}{2^{|E_g|}} \int d\mu_n \, C_n[\alpha_g] \, C_n[\beta_g] \,, \tag{3.20}$$

where E_g denotes the set of propagators in the diagram g, with $|E_g|$ its cardinality, and N_g denotes the numerator. Since N_g only depends on the kinematics data, we can pull the integration out and obtain

$$M_n = \int d\mu_n \left(\sum_{g \in \text{F.D.}} \frac{N_g(\{k, \epsilon\})}{2^{|E_g|}} \, C_n[\alpha_g] \, C_n[\beta_g] \right) . \tag{3.21}$$

The expression (3.21) looks merely as an alternative way of talking about Feynman diagrams and one may wonder what do we really gain from doing this translation. It turns out that for many interesting theories (3.21) unexpectedly simplifies to a simple and compact expression, which reveals interesting structures that can be directly generalized to any multiplicities. This is nowhere to be expected from the Feynman diagrams.

However, this is not the strategy that we are going to take in finding out the suitable integrand for amplitudes in a given theory. Instead, we are going to propose a set of compact building blocks (as we did in defining $C_n[\alpha]$) and use them to build up integrands, which we will further identify with certain class of amplitudes to all multiplicities and find what theory they describe.

In fact, as we are going to see in later chapters, a formula obtained in this way is often *closed* by itself, as it produces all possible tree-level amplitudes in a given theory and no new particle contents are observed in the study of a generic factorization channel. In this case we call the resulting formula the *CHY representation* of the corresponding theory.

4

AMPLITUDES WITH ONE TYPE OF BOSON

In this chapter we construct formulas for amplitudes in theories of bosons. As usual, by "boson" we refer to particles of spin zero (scalars), one (photons/gluons), or two (gravitons). Some of the formulas are so far conjectures, but all of them have undergone explicit comparison either with known formulas or with Feynman diagram computations, and have passed analytic checks regarding locality and unitarity. These formulas are valid in arbitrary dimensions as long as the corresponding theory is allowed.

In the following we first focus on scalar amplitudes only, and then introduce new building blocks needed for building up amplitudes of photons/gluons and gravitons. We identify the corresponding theories along the discussion.

4.1 BUILDING BLOCKS FOR SCALARS

We first introduce the necessary building blocks for scalar amplitudes. We already have one at hand, the Parke–Taylor factor, as defined in (3.17)

$$C_n[\alpha] := \frac{1}{\sigma_{\alpha(1)\,\alpha(2)}\,\sigma_{\alpha(2),\alpha(3)}\cdots\sigma_{\alpha(n),\alpha(1)}}\,. \tag{4.1}$$

As commented before, this factor is natural for representing partial amplitudes in theories with non-trivial flavor/color groups. To obtain the building block for the full amplitude, we dress it with the trace of the group generators accordingly, and sum over all inequivalent orderings, i.e.,

$$\mathcal{C}_n := \sum_{\alpha\in S_n/\mathbb{Z}_n} \mathrm{tr}(T^{\alpha(1)}T^{\alpha(2)}\cdots T^{\alpha(n)})\,C_n(\alpha)\,. \tag{4.2}$$

Apart from this, we have two other building blocks which are fully permutation invariant. Let us first define two $n\times n$ anti-symmetric matrices in terms of their entries

$$(X_n)_{a,b} := \begin{cases} \frac{1}{\sigma_a-\sigma_b}\,, & a\neq b\,,\\ 0\,, & a=b\,,\end{cases} \qquad (A_n)_{a,b} := \begin{cases} \frac{k_a\cdot k_b}{\sigma_a-\sigma_b}\,, & a\neq b\,,\\ 0\,, & a=b\,.\end{cases} \tag{4.3}$$

A natural invariant quantity associated to a matrix is its determinant (we could also study traces but here it is trivially zero). Due to anti-symmetry, the determinant becomes a perfect square and so we define a second building block

$$\mathrm{Pf}X_n = \sqrt{\det X_n}\,. \tag{4.4}$$

Here we only consider even n, because for odd n the determinant of an anti-symmetric matrix is trivially zero. An operationally more convenient way to define a Pfaffian is by summing over all *perfect matchings* (P.M.), i.e., all possible decompositions of the n labels into pairs. Explicitly,

$$\mathrm{Pf}X_n := \sum_{\alpha\in\text{P.M.}} \mathrm{sgn}(\alpha)\,(X_n)_{\alpha(1),\alpha(2)}\,(X_n)_{\alpha(3),\alpha(4)}\cdots(X_n)_{\alpha(n-1),\alpha(n)}\,, \tag{4.5}$$

where

$$\mathrm{sgn}(\alpha) := \begin{cases} +1\,, & \alpha\in\text{even permutations}\,,\\ -1\,, & \alpha\in\text{odd permutations}\,. \end{cases} \tag{4.6}$$

When scalars are flavored in $U(1)^m$, each will carry a flavor index I_a, but the only flavor structure that can appear in an amplitude is the flavor contraction δ^{I_a,I_b}. In this case, we modify the matrix X_n to

$$(\mathcal{X}_n)_{a,b} := \begin{cases} \frac{\delta^{I_a,I_b}}{\sigma_a-\sigma_b}\,, & a\neq b\,,\\ 0\,, & a=b\,, \end{cases} \tag{4.7}$$

and use $\mathrm{Pf}\mathcal{X}_n$ instead.

We have not defined a similar quantity $\mathrm{Pf}A_n$ for the matrix A_n. The reason is that when using A_n in any formulas the σ's therein are to be evaluated on the solutions to the scattering equations. On these equations the matrix A_n possesses a kernel of dimension 2, which is spanned by the vectors

$$(\sigma_1^m, \sigma_2^m, \ldots, \sigma_n^m)^{\mathrm{T}} \quad m = 0, 1\,. \tag{4.8}$$

While the case with $m = 0$ is a direct consequence of the scattering equations, the case with $m = 1$ is explicitly

$$\sum_{\substack{b=1\\ b\neq a}}^{n} \frac{k_a\cdot k_b\,\sigma_b}{\sigma_a-\sigma_b} = \sigma_a \sum_{\substack{b=1\\ b\neq a}}^{n} \frac{k_a\cdot k_b}{\sigma_a-\sigma_b} - \sum_{\substack{b=1\\ b\neq a}}^{n} k_a\cdot k_b = 0\,. \tag{4.9}$$

Due to this non-trivial kernel we have $\mathrm{Pf}A_n = 0$. In this case, we define the third building block to be

$$\mathrm{Pf}'A_n := -\frac{(-1)^{a+b}}{\sigma_a-\sigma_b}\,\mathrm{Pf}[A]_{\hat{a},\hat{b}}\,, \tag{4.10}$$

which we call the *reduced Pfaffian* of A_n. In the above, $[A]_{\hat{a},\hat{b}}$ denotes the minor of A_n obtained by deleting the two rows and two columns labeled by a and b. This quantity is independent of the choice of $\{a, b\}$. To understand why it is so, let us compare two matrices $[A]_{\hat{a},\hat{b}}$ and $[A]_{\hat{a},\hat{c}}$. In the first matrix we add all other rows to the c^{th} row (and do the same to the column), and in the second matrix we do the same but to the b^{th} row and column instead. Then by scattering equations we observe that the c^{th} row/column of $[A]_{\hat{a},\hat{b}}$ is $\frac{\sigma_a-\sigma_c}{\sigma_a-\sigma_b}$ times the b^{th} row/column of $[A]_{\hat{a},\hat{c}}$.

Recalling the requirement on the integrand as discussed in (3.9), the reason that we can use these building blocks is that they transform covariantly under $SL(2, \mathbb{C})$. Explicitly, we have

$$C_n[\alpha] \xrightarrow{\psi \in SL(2,\mathbb{C})} C_n[\alpha] \prod_{a=1}^{n} (\gamma\, \sigma_a + \delta)^2 \,, \tag{4.11}$$

and

$$(\text{Pf}) \xrightarrow{\psi \in SL(2,\mathbb{C})} (\text{Pf}) \prod_{a=1}^{n} (\gamma\, \sigma_a + \delta) \,, \quad \text{where } (\text{Pf}) \in \{\text{Pf}X_n, \text{Pf}\mathcal{X}_n, \text{Pf}' A_n\} \,, \tag{4.12}$$

which is due to the multi-linearity of the Pfaffian in the entries of the matrix.

4.2 Scalar Amplitudes

By comparing the behaviors under $SL(2, \mathbb{C})$ transformations of the building blocks $C_n[\alpha]$, $\text{Pf}X_n$, $\text{Pf}\mathcal{X}_n$ and $\text{Pf}' A_n$ (4.11) and (4.12) with that required on the entire integrand (3.9), we can make different products of them to construct proper integrands. Explicitly, these combinations are

$$\begin{aligned} &C_n[\alpha]\, C_n[\beta] \,, \quad C_n[\alpha]\, \text{Pf}X_n\, \text{Pf}' A_n \,, \quad C_n[\alpha]\, (\text{Pf}' A_n)^2 \,, \\ &(\text{Pf}X_n)^2\, (\text{Pf}' A_n)^2 \,, \quad \text{Pf}X_n\, (\text{Pf}' A_n)^3 \,, \quad (\text{Pf}' A_n)^4 \,. \end{aligned} \tag{4.13}$$

In the following we discuss case by case.

4.2.1 *The ϕ^3 Theory*

In the first case, we consider the integrands obtained by two copies of C_n's

$$I_n^{\Phi^3}[\alpha|\beta] := C_n[\alpha]\, C_n[\beta] \,, \tag{4.14}$$

which have already been defined in the previous chapter. Recall the result of an $I_n^{\Phi^3}[\alpha|\beta]$ is the summation of trivalent massless scalar diagrams which are both α-

ordered and β-ordered. Due to the presence of two copies of orderings, we may expect that this integrand computes the expression obtained from the full amplitude by decomposing certain flavor structure twice. This is why we call the integral $m_n[\alpha|\beta] = \int d\mu_n I_n^{\Psi^3}[\alpha|\beta]$ the double partial amplitude. Indeed, the integrand for the full amplitude can be written as

$$I_n^{\Phi^3} := \mathcal{C}_n\, \mathcal{C}_n\,, \tag{4.15}$$

and the corresponding theory is a scalar theory with a cubic self-interaction vertex, and the scalars therein has a flavor group of the form $U(N) \times U(\tilde{N})$ [7]. Explicitly, the Lagrangian is

$$\mathcal{L}^{\Phi^3} := -\frac{1}{2}\, \partial_\mu \Phi_{I,\tilde{I}}\, \partial^\mu \Phi^{I,\tilde{I}} - \frac{\lambda}{3!}\, f_{I,J,K}\, \tilde{f}_{\tilde{I},\tilde{J},\tilde{K}}\, \Phi^{I,\tilde{I}}\, \Phi^{J,\tilde{J}}\, \Phi^{K,\tilde{K}}\,. \tag{4.16}$$

In fact, we can use the same building block to construct an integrand for the ϕ^3 theory without any flavor structure [42]. It is well-known that any amplitude in this theory is merely a summation of all trivalent diagrams. To achieve this, note that the integrand $C_n^2[\alpha]$ leads to the summation of all diagrams that are α-ordered, then we just need to sum up all orderings to obtain the summation of all diagrams, only at the cost of an overall constant due to over counting. So we have

$$I^{\phi^3} := \frac{1}{2^{n-2}} \sum_{\alpha \in S_n} C_n^2[\alpha]\,, \tag{4.17}$$

corresponding to the Lagrangian

$$\mathcal{L}^{\phi^3} := -\frac{1}{2}\, \partial_\mu \phi\, \partial^\mu \phi - \frac{\lambda}{3!}\, \phi^3\,. \tag{4.18}$$

4.2.2 *Scalar Amplitudes in a Yang–Mills–Scalar Theory*

Next we fix one and only one factor to be $C_n[\alpha]$, so as to consider normal partial amplitudes with the planar ordering α. In the remaining part of the integrand we are allowed to insert two copies of Pfaffians previously introduced. First of all, for the combination $I_n := C_n[\alpha]\, (\mathrm{Pf}X_n)^2$ (or with $\mathcal{X}_n$), explicit calculation at four points shows that the result contains double poles, i.e.,

$$\int d\mu_4\, C_n[1234]\, (\mathrm{Pf}X_n)^2 = \frac{(s^2 + s\,t + t^2)^2}{s^2\, t^2\, u}\,, \tag{4.19}$$

where $s = s_{1,2}$, $t = s_{1,4}$ and $u = s_{1,3}$. This situation is excluded as it violates locality. Then with the building blocks at hand we are left with two choices.

The first choice is

$$I_n^{\text{gauged scalar:scalar}}[\alpha] := C_n[\alpha]\, \text{Pf}X_n\, \text{Pf}'A_n\,. \tag{4.20}$$

This integrand corresponds to scalar amplitudes in a gauged massless scalar theory. The full Lagrangian for this theory is

$$\mathcal{L}^{\text{gauged scalar}} := -\frac{1}{2}\,\text{tr}_\text{c}(D_\mu\Phi\, D^\mu\Phi) - \frac{1}{4}\,\text{tr}_\text{c}(F_{\mu\nu}\, F^{\mu\nu})\,. \tag{4.21}$$

This is a special case (only a single flavor) of the more general case

$$I_n^{\text{YMS:scalar}}[\alpha] := C_n[\alpha]\, \text{Pf}\mathcal{X}_n\, \text{Pf}'A_n\,, \tag{4.22}$$

which describes the scalar amplitudes in a gauged massless scalar theory with a quartic scalar self-interaction vertex that descends from the compactification of a pure Yang–Mills theory [9]. To emphasize its Yang–Mills origin we name it as Yang–Mills–Scalar (YMS). Explicitly, the corresponding Lagrangian is

$$\mathcal{L}^{\text{YMS}} := -\frac{1}{2}\,\text{tr}_\text{c}(D_\mu\Phi^I\, D^\mu\Phi^I) - \frac{1}{4}\,\text{tr}_\text{c}(F_{\mu\nu}\, F^{\mu\nu}) - \frac{g^2}{4}\,\text{tr}_\text{c}\left([\Phi^I, \Phi^J]^2\right)\,, \tag{4.23}$$

where we use the subscript "c" to indicate that the trace is associated to the color group.

Here we provide some intuition for the above identification. First of all, due to the presence of $\text{Pf}'A_n$ in the integrand, any amplitude of odd number of particles has to vanish. In particular this means any cubic scalar self-coupling can be eliminated.

Next let us have a look at the four-point amplitudes with the canonical ordering in the general case (with $\mathcal{X}_4$). If we assume that the flavor indices satisfy $I_1 = I_3$ and $I_2 = I_4$, we have

$$\int d\mu_4\, \frac{1}{\sigma_{1,2}\,\sigma_{2,3}\,\sigma_{3,4}\,\sigma_{4,1}}\, \frac{-1}{\sigma_{1,3}\,\sigma_{2,4}}\, \text{Pf}'A_4 = -1\,. \tag{4.24}$$

This is the first indication that there has to be a quartic scalar vertex and that this vertex does not involve any derivatives. In addition, if we assume that the flavor indices satisfy $I_1 = I_2$ and $I_3 = I_4$, we have

$$\int d\mu_4\, \frac{1}{\sigma_{1,2}\,\sigma_{2,3}\,\sigma_{3,4}\,\sigma_{4,1}}\, \frac{1}{\sigma_{1,2}\,\sigma_{3,4}}\, \text{Pf}'A_4 = -\frac{s_{1,3}}{s_{1,2}}\,. \tag{4.25}$$

Obviously this amplitude has to receive contribution from a trivalent diagram. Since there cannot be a three-point scalar vertex and the mass dimension of the expression is $[M]^0$, the three-point vertices therein have to involve one derivative and the propagator has to be that for a vector particle. At higher points, in general the mass dimension of the result produced by this integrand at n points is $[M]^{-\frac{n-2}{2}}$, by which we see there cannot exist any higher-point scalar self-interaction vertices, nor any higher point vertices of scalars coupled to vectors.

Detailed structure of the Lagrangian (4.23) can be verified by comparing results of Feyman diagrams and the result of the formula up to sufficient high points, which we will summarize in Table 4. Besides, the possible appearance of the gluon in a general factorization channel can be explicitly confirmed, which will be postponed to Chapter 6. When (4.23) is confirmed, the validity of the Lagrangian (4.21) is guaranteed by identifying all the flavor indices.

4.2.3 *The $U(N)$ Non-Linear Sigma Model*

The second choice with a factor $C_n[\alpha]$ is

$$I_n^{\text{NLSM}}[\alpha] := C_n[\alpha]\,(\text{Pf}'A_n)^2\,. \tag{4.26}$$

This describes a pure scalar theory with a non-trivial flavor group, the $U(N)$ non-linear sigma model (NLSM) [9, 43–45]. The usual form of the Lagrangian for this theory is defined with an $N \times N$ unitary matrix $\mathbf{U}$ (i.e., $\mathbf{U}^\dagger\,\mathbf{U} = \mathbf{1}$)

$$\mathcal{L}^{\text{NLSM}} := -\frac{1}{2\,\lambda^2}\,\text{tr}_\text{f}(\partial_\mu \mathbf{U}^\dagger\,\partial^\mu \mathbf{U})\,, \tag{4.27}$$

where λ is a dimensionful real coupling. However, in order to draw direct connection with the result from the above integrand, we do a field re-definition

$$\mathbf{U} = \frac{\mathbf{1} + i\,\lambda\,\Phi}{\mathbf{1} - i\,\lambda\,\Phi}\,, \tag{4.28}$$

such that Φ is a Hermitian matrix. Using the relations

$$\partial_\mu \frac{\mathbf{1}}{\mathbf{1} \mp i\,\lambda\,\Phi} = \pm \frac{\mathbf{1}}{\mathbf{1} \mp i\,\lambda\,\Phi}\,(i\,\lambda\,\partial_\mu \Phi)\,\frac{\mathbf{1}}{\mathbf{1} \mp i\,\lambda\,\Phi}\,, \tag{4.29}$$

the Lagrangian (4.27) becomes

$$\mathcal{L}^{\text{NLSM}} := -\frac{1}{2}\,\text{tr}_\text{f}\left(\frac{\mathbf{1}}{\mathbf{1} + \lambda^2\,\Phi^2}\,\partial_\mu \Phi\,\frac{\mathbf{1}}{\mathbf{1} + \lambda^2\,\Phi^2}\,\partial^\mu \Phi\right), \tag{4.30}$$

where the rational factors are to be understood in terms of Taylor expansions.

In fact, even before knowing anything about the Lagrangian, by studying a generic factorization channel one can confirm that the integrand (4.26) by itself leads to a closed formula for all amplitudes in certain theory, because of the following:

1. The internal particle observed in the factorization is always a scalar.

2. Whenever the formula factorizes at the leading order, the integrand in the two factorized pieces are both of the same type as original integrand (4.26).

This will be discussed in more detail in Chapter 6.

For a theory with scalars only, it is relatively easy to simply work out the Lagrangian from the amplitudes. As a first step, from (4.26) we know that the mass dimension of an n-point amplitude should be $[\mathrm{M}]^2$. So at each order of multiplicity, one can always make an ansatz of at most a finite number of contact terms with the correct mass dimension (in this case the contact terms always have exactly two derivatives). The coefficient in front of each contact term is then determined by comparing Feynman diagrams and the result produced by (4.26). As is expected, in general there will be many redundancies among the coefficients and one is allowed to arbitrarily fix them without changing the physics. For (4.26), a clever choice of fixing leads to a closed expression for the form of the interaction vertices, which is

$$V_n := (-1)^{\frac{n}{2}-1} \frac{i\,\lambda^{n-2}}{2} \sum_{l=0}^{\frac{n}{2}-1} \sum_{a=1}^{n} k_a \cdot k_{a+2l+1} \,. \tag{4.31}$$

These are exactly the vertices produced by the Lagrangian (4.30) [46].

4.2.4 *Scalar Amplitudes in an Einstein–Maxwell–Scalar Theory*

Next we consider integrands constructed exactly by four Pfaffians, which are permutation invariant. Again we may worry that a sufficient high power of $\mathrm{Pf}X_n$ violates locality, as was exemplified in (4.19). A quick computation at four points excludes $(\mathrm{Pf}X_n)^4$ and $(\mathrm{Pf}X_n)^3\,\mathrm{Pf}'A_n$, and so we are left with three choices.

The first choice is

$$I_n^{\text{EMS:scalar}} := (\mathrm{Pf}X_n)^2\,(\mathrm{Pf}'A_n)^2 \,. \tag{4.32}$$

In a generic factorization channel we may find the internal particle to be a rank-2 tensor. This describes the scalar amplitudes in a theory of scalars coupled to gravity [13]. As will be clear in the discussions in Section 5.1.3 this theory comes naturally from the compactification of gravity. More generally the theory also involves photons.

4.2.5 *The Scalar Sector of the Dirac–Born–Infeld Theory*

The second choice is

$$I_n^{\text{DBI:scalar}} := \text{Pf}X_n\, (\text{Pf}' A_n)^3\,. \tag{4.33}$$

This gives rise to a closed formula for all amplitudes in the scalar sector of the Dirac–Born–Infeld theory (DBI) [9] (For a review, see [47]). Restricting to this sector, the Lagrangian takes the explicit form

$$\mathcal{L}^{\text{DBI:scalar}} := \ell^{-2}\sqrt{1-\ell^2\,\partial_\mu\phi\,\partial^\mu\phi} - \ell^{-2}\,, \tag{4.34}$$

which is to be understood in terms of Taylor expansion.

This Lagrangian can be worked out from the amplitudes in similar way as that discussed in the case of NLSM. Again the general study of factorizations does not reveal any other internal state. According to (4.33), the mass dimension of an n-point amplitude is $[\text{M}]^n$, hence at each order of multiplicity the contact terms have one derivative associated to one leg on average. The simplest ansatz is to assume that in the Lagrangian the scalar field ϕ always enter in the form $\partial\phi$. With this ansatz, one can work out the interaction terms order by order

$$-\frac{\ell^2}{2!}\Big(\frac{(\partial\phi)^2}{2}\Big)^2 - \frac{3\,\ell^4}{3!}\Big(\frac{(\partial\phi)^2}{2}\Big)^3 - \frac{15\,\ell^5}{4!}\Big(\frac{(\partial\phi)^2}{2}\Big)^4 - \cdots\,, \tag{4.35}$$

which together with the kinetic term $\frac{1}{2}(\partial\phi)^2$ turns out to re-sum into (4.34). While (4.33) by itself forms a closed formula, as we will see in the next chapter, we can extended to a closed formula for amplitudes in the full DBI theory.

4.2.6 *A Special Galileon Theory*

The third choice is to use $\text{Pf}' A_n$ only

$$I_n^{\text{sGal}} := (\text{Pf}' A_n)^4\,. \tag{4.36}$$

As with NLSM and the scalar sector of DBI, one can first confirm that this integrand again yields a closed formula for a pure scalar theory. However, an n-particle amplitude in this theory has an even higher mass dimension, which is $[\text{M}]^{2n-2}$. While in principal one can still make an ansatz and try to determine the coefficients therein, this soon becomes impractical at higher points due to the huge amount of redundancies.

Luckily, the recent effort in classifying scalar theories by behaviors of amplitudes in soft limits reveals a special case of Galileon theory (sGal) [48, 49] whose order of vanishing in a soft limit is enhanced as compared to that of the generic Galileon theory [50, 51]. This order of vanishing is the same as what is predicted by the integrand (4.36) in studying this limit. In addition, the mass dimension discussed above matches that of a Galileon amplitude. By further explicit checks between the Feynamn diagram computations and the results of (4.36), it turns out this special Galileon theory is exactly described by (4.36).

We provide a brief review of the general Galileon theory and comment on the uniqueness of this special one from that point of view in Section 4.A at the end of this chapter. And here we merely summarize the Lagrangian for this special Galileon theory [49]

$$\mathcal{L}^{\text{sGal}} := -(\partial\phi)^2 \sum_{m=0}^{\infty} \frac{(-1)^m\, \partial_{\mu_1}\partial^{\nu_1}\phi \cdots \partial_{\mu_{2m}}\partial^{\nu_{2m}}\phi\, \varepsilon^{\mu_1\cdots\mu_{2m}\rho_{2m+1}\cdots\rho_D}\, \varepsilon_{\nu_1\cdots\nu_{2m}\rho_{2m+1}\cdots\rho_D}}{2\,(D-2m)!\,(2\,m+1)!}\,. \tag{4.37}$$

4.3 A BUILDING BLOCK WITH POLARIZATION VECTORS

We now consider building blocks for amplitudes of bosons with spin higher than zero. The first case of this type is amplitudes of photons or gluons, where to each particle we associate a polarization vector ϵ^μ. From Feynman diagram computation of Yang-Mills amplitudes, it is known that the final answer definitely has to involve terms with factors of the form $\epsilon \cdot k$ as well as $\epsilon \cdot \epsilon$. Since these factors can nowhere be produced from the integration, we have to insert them explicitly somewhere in a new building block.

The correct building block for our purpose turns out to be a reduced Pfaffian. But now it is the reduced Pfaffian of a $2n \times 2n$ anti-symmetric matrix Ψ_n instead [6]. Explicitly, this matrix has a block structure

$$\Psi_n := \left(\begin{array}{c|c} A_n & -(C_n)^{\mathrm{T}} \\ \hline C_n & B_n \end{array}\right), \tag{4.38}$$

where each $n \times n$ block is defined in terms entries as

$$A_n := \begin{cases} \frac{k_a\cdot k_b}{\sigma_a-\sigma_b}\,, \\ 0\,, \end{cases} \qquad B_n := \begin{cases} \frac{\epsilon_a\cdot \epsilon_b}{\sigma_a-\sigma_b}\,, \\ 0\,, \end{cases} \qquad C_n := \begin{cases} \frac{\epsilon_a\cdot k_b}{\sigma_a-\sigma_b}\,, & a\neq b\,, \\ -\sum_{\substack{c=1\\ c\neq a}}^{n}(C_n)_{a,c}\,, & a=b\,. \end{cases} \tag{4.39}$$

So this A_n block here is exactly the same as the matrix A_n we defined in (4.3). Let us set up some notations here for later convenience. Due to the block structure (4.38) of Ψ_n, we divide the range of its indices into two blocks as well, running as $\{1,2,\ldots,n:1,2,\ldots,n\}$. When a label a belongs to the first block, in the corresponding row/column we see the vector k_a^μ (expect for the diagonal entries in C_n), and when it belongs to the second block, we see the vector ϵ_a^μ. When we refer to a minor of Ψ_n, we either denote it as, e.g., $[\Psi_n]_{a,b:c}$, meaning that it is obtained by extracting rows and columns labeled by $\{a,b\}$ in the first block, and $\{c\}$ in the second block, or denote it as, e.g., $[\Psi_n]_{\hat{a},\hat{b}:\hat{c}}$, meaning to delete rather than to extract.

Similar to the matrix A_n, on the support of the scattering equations the matrix Ψ_n also has a two-dimensional kernel, spanned by the vectors

$$(\sigma_1^m,\sigma_2^m,\ldots,\sigma_n^m:\underbrace{0,0,\ldots,0}_{n})^{\mathrm{T}}\,, \tag{4.40}$$

and so an invariant quantity associated to this matrix is

$$\mathrm{Pf}'\Psi_n := -\frac{(-1)^{a+b}}{\sigma_a-\sigma_b}\,\mathrm{Pf}[\Psi_n]_{\hat{a},\hat{b}:}\,, \tag{4.41}$$

which we take as a new building block.

Some explanations is in order about why this quantity is natural. (i) Under a generic $SL(2,\mathbb{C})$ transformation each entry of the matrix transforms as $(\Psi_n)_{a,b}\to(\Psi_n)_{a,b}(\gamma\sigma_a+\delta)(\gamma\sigma_b+\delta)$. While this is obvious for most entries, the entries $(C_n)_{a,a}$ are a bit non-trivial

$$\begin{aligned}(C_n)_{a,a}\xrightarrow{\psi\in SL(2,\mathbb{C})}&(\gamma\,\sigma_a+\delta)\sum_{\substack{b=1\\b\neq a}}^{n}\frac{\epsilon_a\cdot k_b}{\sigma_a-\sigma_b}\,(\gamma\,\sigma_b+\delta)\\&=(\gamma\,\sigma_a+\delta)^2\sum_{\substack{b=1\\b\neq a}}^{n}\frac{\epsilon_a\cdot k_b}{\sigma_a-\sigma_b}-(\gamma\,\sigma_a+\delta)\,\gamma\sum_{\substack{b=1\\b\neq a}}^{n}\epsilon_a\cdot k_b\,,\end{aligned} \tag{4.42}$$

where the second term vanishes due to momentum conservation and that $\epsilon_a\cdot k_a=0$. Then we conclude that $\mathrm{Pf}'\Psi_n$ transforms covariantly under $SL(2,\mathbb{C})$

$$\mathrm{Pf}'\Psi_n\xrightarrow{\psi\in SL(2,\mathbb{C})}\mathrm{Pf}'\Psi_n\prod_{a=1}^{n}(\gamma\,\sigma_a+\delta)^2\,, \tag{4.43}$$

which is the same as that of the Parke–Taylor factor. (ii) By the definition of Pfaffian in terms of perfect matchings (4.5), $\mathrm{Pf}'\Psi_n$ is obviously multi-linear in each ϵ_a, which is required by the external vector particles. (iii) Gauge invariance of amplitudes in theories involving photons/gluons requires that whenever we substitute an ϵ_a^μ by the corresponding momentum k_a^μ the expression has to vanish. We see that if we use this building block in such amplitudes this is straightforwardly achieved by

$\mathrm{Pf}'\Psi_n$ itself, because after the substitution the rows/columns labeled by a in both the first and the second block are exactly proportional to each other, which forces $\mathrm{Pf}'\Psi_n = 0$.

4.4 PHOTON/GLUON AMPLITUDES

Now we go on to the integrands for amplitudes of photons/gluons, in which case we need exactly one copy of $\mathrm{Pf}'\Psi_n$. There are three choices here.

4.4.1 *The pure Yang–Mills Theory*

Firstly, we have

$$I_n^{\mathrm{YM}}[\alpha] := C_n[\alpha]\,\mathrm{Pf}'\Psi_n\,. \tag{4.44}$$

This is the integrand for partial amplitudes in the pure Yang–Mills theory (YM).

The validity of this integrand was first checked both analytically and numerically in explicit cases in [6]. In the same paper the correct behavior in a general factorization channel was also confirmed. Later on a proof by Britto–Cachazo–Feng–Witten (BCFW) recursion relations [52] was given in [42].

4.4.2 *Photon Amplitudes in an Einstein–Maxwell Theory*

Secondly, we have

$$I_n^{\mathrm{EM}} := \mathrm{Pf}X_n\,\mathrm{Pf}'A_n\,\mathrm{Pf}'\Psi_n\,. \tag{4.45}$$

This is the integrand for photon amplitudes in the Einstein–Maxwell theory (EM), which is obtained by compactifying gravity on a circle $\mathbb{R}^D \to \mathbb{R}^{D-1}\times S^1$. More generally we can study

$$I_n^{\mathrm{EM}} := \mathrm{Pf}\mathcal{X}_n\,\mathrm{Pf}'A_n\,\mathrm{Pf}'\Psi_n \tag{4.46}$$

instead, where the photons acquire several flavors, obtained by the compactification $\mathbb{R}^D \to \mathbb{R}^{D-m}\times S^m$ for some integer $m > 1$.

4.4.3 *The Born–Infeld Theory*

Finally, we also have

$$I_n^{\mathrm{BI}} := (\mathrm{Pf}' A_n)^2\, \mathrm{Pf}'\Psi_n\,. \tag{4.47}$$

This yields one more closed formula, for amplitudes in the photon sector of DBI, or generally referred to as Born–Infeld theory (BI). Its explicit Lagrangian is

$$\mathcal{L}^{\mathrm{BI}} := \ell^{-2}\sqrt{-\det(\eta_{\mu\nu} + \ell\, F_{\mu\nu})} - \ell^{-2}\,, \tag{4.48}$$

where $\eta_{\mu\nu}$ denotes the Minkowski flat metric. The closedness is again confirmed by the study of factorization.

4.5 GRAVITY AMPLITUDES

Finally let us consider gravity amplitudes. Since in perturbative gravity the gravitons are identified to spin-2 particles, in the amplitudes we need to associate to each graviton a polarization tensor $\epsilon^{\mu\nu}$, which is traceless symmetric. It is convenient to start with a tensor which is identical to a direct product of two polarization vectors $\epsilon^{\mu\nu} = \epsilon^{\mu}\,\tilde{\epsilon}^{\nu}$. For scattering among these states, the simplest guess is

$$I_n := \mathrm{Pf}'\Psi_n(\{k,\epsilon,\sigma\})\, \mathrm{Pf}'\Psi_n(\{k,\tilde{\epsilon},\sigma\})\,. \tag{4.49}$$

Recall that each reduced Pfaffian is multilinear in the polarization vectors, and so we can directly bring them out, i.e.,

$$\mathrm{Pf}'\Psi_n(\{k,\epsilon,\sigma\}) = (\mathrm{Pf}'\Psi_n)^{\mu_1\mu_2\cdots\mu_n}(\{k,\sigma\})\,\epsilon_{1,\mu_1}\epsilon_{2,\mu_2}\cdots\epsilon_{n,\mu_n}. \tag{4.50}$$

Then for scattering among generic states we can simply generalize (4.49) to

$$I_n^{\mathrm{GR}} := (\mathrm{Pf}'\Psi_n)^{\mu_1\mu_2\cdots\mu_n}\,(\mathrm{Pf}'\Psi_n)^{\nu_1\nu_2\cdots\nu_n}\,\epsilon_{1,\mu_1\nu_1}\epsilon_{2,\mu_2\nu_2}\cdots\epsilon_{n,\mu_n\nu_n}\,. \tag{4.51}$$

This turns out to produce a closed formula for amplitudes in Einstein gravity (GR) [6, 42]. Note that in the form (4.51) we are not forced to restrict to traceless symmetric tensors, and so more generally we also allow other irreducible components of a rank-2 tensor, i.e., the B-fields and the dilatons. But to simplify notation, later on we will only work with the form (4.49).

In Chapter 7 we are going to argue about the validity of this formula by relations between amplitudes in pure Yang–Mills and those in gravity.

A general Galileon theory is a theory of a real scalar ϕ with higher derivative interactions [51], equipped with a non-linearly realized Galileon symmetry

$$\delta\phi = a + b \cdot x\,, \tag{4.52}$$

where a and b^μ are parameters in the transformation, and x^μ denotes the space-time coordinates. Arising from various infrared modifications of gravity (see, e.g., [50, 53]), this theory has been of primary interest in cosmology over the recent years. It can also be viewed as Wess-Zumino terms in a particular spontaneously broken spacetime symmetry [54].

In a form commonly used in the literature, the most general Lagrangian for the Galileon is written as

$$\mathcal{L}^{\text{Gal}} := -\frac{(\partial\phi)^2}{2}\sum_{m=0}^{\infty}\frac{g_m}{(D-m)!}\,\partial_{\mu_1}\partial^{\nu_1}\phi\cdots\partial_{\mu_m}\partial^{\nu_m}\phi\,\varepsilon^{\mu_1\mu_2\cdots\mu_m\rho_{m+1}\cdots\rho_D}\,\varepsilon_{\nu_1\nu_2\cdots\nu_m\rho_{m+1}\cdots\rho_D}\,, \tag{4.53}$$

where ε denotes the Levi-Civita tensor in D space-time dimensions. So for a fixed D the summation terminates at $m = D$. For any multiplicity in the field ϕ we have exactly one coupling to be freely tuned. Since this is a massless scalar with higher derivative interaction, one can verify that the three-point amplitude always vanishes no matter what value g_1 assumes. Hence changing g_1 does not really change the physics, but nonetheless we can leave it to be generic.

In the special Galileon theory that the formula (4.36) describes, only g_1 and g_2 are free parameters, but $g_m = g_m(g_1, g_2)$ are determined to be certain functions for all $m \geq 2$. In a single soft limit, while a generic Galileon amplitude vanishes at the order τ^2, amplitudes in this special theory is vanishes at τ^3, which indicates that it should possess certain extra global symmetry so as to allow for additional cancellation between Feynman diagrams [48]. As was realized in [49], this is the unique Galileon whose global symmetry is enhanced to the second order

$$\delta\phi = s_{\mu\nu}\,(x^\mu + \beta\,\partial^\mu\phi)\,(x^\nu + \beta\,\partial^\nu\phi) + s_{\mu\nu}\,\alpha\,\partial^\mu\phi\,\partial^\nu\phi\,, \tag{4.54}$$

with α and β some constants and $s_{\mu\nu}$ a constant traceless symmetry tensor. Especially, if we set $g_1 = 0$, then for a fixed α we have the solution

$$g_m = \begin{cases} \frac{(-\alpha)^{\frac{m}{2}}}{(m+1)!}\,, & \text{even } m\,, \\ 0\,, & \text{odd } m\,. \end{cases} \tag{4.55}$$

5

THEORIES WITH SEVERAL BOSONS

In the previous chapter we built integrands for amplitudes that involve a single type of particles (at most with non-trivial flavor or color structure). While most of the integrands there give rise to closed formulas in certain theories, there are two cases which are incomplete because the study of factorizations reveals new particles. These cases are identified as a Yang–Mills–Scalar theory and an Einstein–Maxwell–(Scalar) theory. Furthermore, we also found integrands for two theories which form different sectors of the same more general theory, i.e., the Dirac–Born–Infeld theory.

In generic amplitudes in all these theories we observe particles with different spins. The task of this chapter is to find out closed formulas for each of them. As a by-product, we also discover a closed formula for a theory which seems to be new. Differing from the previous discussions, here we do not construct new building blocks straightforwardly. Instead, we are going to introduce three operations that act on the existing building block $\mathrm{Pf}'\Psi_n$: the standard compactification, the "squeezing", and a generalized compactification, which lead to new integrands that produce physically sensible results [9].

Due to the presence of different types of particles, in any amplitude under study we denote the set of graviton labels by "h", the set of photons as "γ", the set of gluons as "g", and finally the set of scalars as "s".

5.1 COMPACTIFICATION

The first operation we introduce has a natural physical interpretation: compactification. In the context of scattering amplitudes, which are on-shell, this is fulfilled by restricting momenta and polarization vectors.

To be precise, let us consider compactification from D-dimensional spacetime to d-dimensional spacetime, and define $m = D - d$. We denote the momenta and polarizations in D dimensions by K^M and $\mathcal{E}^M$ respectively, and those in d dimensions by k^μ and ϵ^μ. Since the momenta are fully restricted to d dimensions, for every $a \in \{1, 2, \ldots, n\}$ we have

$$K_a^M = (k_a^\mu | \underbrace{0, 0, \ldots, 0}_{m}) =: (k_a^\mu | \vec{0})\,. \tag{5.1}$$

For each polarization vector, however, we can make two choices. If it is also restricted to the d-dimensional external spacetime, then

$$\mathcal{E}_a^M = (\epsilon_a^\mu | \underbrace{0, 0, \ldots, 0}_{m}) =: (\epsilon_a^\mu | \vec{0})\,. \tag{5.2}$$

If it is restricted to the internal space, we denote it as e^I and

$$\mathcal{E}_a^M = (\underbrace{0, 0, \ldots, 0}_{d} | e_a^I) =: (\vec{0} | e_a^I)\,. \tag{5.3}$$

In this case, it is convenient to further choose the polarization vector to be in an orthonormal basis of the m-dimensional internal space so as to assign a flavor index I_a to particle a, i.e., $e_a^I := \delta^{I_a, I}$.

As a result, we always have $K_a \cdot K_b = k_a \cdot k_b$, while the other Lorentz products reduce to

$$K_a \cdot \mathcal{E}_b = \begin{cases} k_a \cdot \epsilon_b\,, & \mathcal{E}_b \text{ external}\,, \\ 0\,, & \mathcal{E}_b \text{ internal}\,, \end{cases} \qquad \mathcal{E}_a \cdot \mathcal{E}_b = \begin{cases} \epsilon_a \cdot \epsilon_b\,, & \text{both external}\,, \\ \delta^{I_a, I_b}\,, & \text{both internal}\,, \\ 0\,, & \text{else}\,. \end{cases} \tag{5.4}$$

For a photon or gluon amplitude, depending on whether we compactify an external state or not, we choose the corresponding polarization vector to be either internal or external.

For a graviton amplitude, we usually start from $\epsilon^{\mu\nu} = \epsilon^\mu \tilde{\epsilon}^\nu$ with the two polarization vectors distinguished. Then we can either choose to compactify one of them so as to obtain a photon (with a flavor index), or to compactify both so as to obtain a scalar (with two flavor indices).

5.1.1 *A Yang–Mills–Scalar Theory*

In the simplest case, we consider compactifying the pure Yang–Mills theory, which yields a certain theory of massless scalars coupled to Yang–Mills. For a given scattering, the gluons set and the scalar set are explicitly defined as

$$\mathrm{g} := \{a \in \{1, 2, \ldots, n\} | \mathcal{E}_a^M = (\epsilon_a^\mu, \vec{0})\}\,, \quad \mathrm{s} := \{a \in \{1, 2, \ldots, n\} | \mathcal{E}_a^M = (\vec{0}, e_a^I)\}\,. \tag{5.5}$$

Recall the integrand for Yang–Mills amplitudes $I_n^{\mathrm{YM}}[\alpha] = C_n[\alpha]\,\mathrm{Pf}'\Psi_n$. The only change imposed by the compactification is in the matrix Ψ_n, which becomes

$$
\begin{array}{c} \\ a \in \mathrm{g} \\ a \in \mathrm{s} \\ a \in \mathrm{g} \\ a \in \mathrm{s} \end{array}
\begin{array}{c}
\begin{array}{cc|cc} b \in \mathrm{g} & b \in \mathrm{s} & b \in \mathrm{g} & b \in \mathrm{s} \end{array} \\
\left(\begin{array}{cc|cc}
(A_n)_{ab} & (A_n)_{ab} & (-C_n^{\mathrm{T}})_{ab} & 0 \\
(A_n)_{a,b} & (A_n)_{a,b} & (-C_n^{\mathrm{T}})_{a,b} & 0 \\ \hline
(C_n)_{a,b} & (C_n)_{a,b} & (B_n)_{a,b} & 0 \\
0 & 0 & 0 & (\mathcal{X}_n)_{a,b}
\end{array}\right)
\end{array},
\tag{5.6}
$$

where the horizontal and vertical solid lines are to distinguish the original four blocks as shown in the definition of Ψ_n (4.38), and the dashed lines separate the scalar labels from the gluon labels. Entries at the bottom-right corner are exactly identical to those in the matrix $\mathcal{X}_n$ defined in (4.7), i.e., $(\mathcal{X}_n)_{a,b} = \frac{\delta^{I_a,I_b}}{\sigma_{a,b}}(1-\delta_{a,b})$. Hence the building block $\mathrm{Pf}'\Psi_n$ factorizes into

$$
\mathrm{Pf}'\Psi_n = \mathrm{Pf}[\mathcal{X}_n]_{:\mathrm{s}}\,\mathrm{Pf}'[\Psi_n]_{:\hat{\mathrm{s}}}\,,
\tag{5.7}
$$

where as before $[\mathcal{X}_n]_{:\mathrm{s}}$ denotes the minor of $\mathcal{X}_n$ obtained by extracting rows and columns with scalar labels in the second block. Also, $[\Psi_n]_{:\hat{\mathrm{s}}} = [\Psi_n]_{\mathrm{g,s:g}}$ denotes the minor of Ψ_n obtained by deleting rows and columns with scalar labels in the second block, i.e.,

$$
\begin{array}{c} \\ a \in \mathrm{g} \\ a \in \mathrm{s} \\ a \in \mathrm{g} \end{array}
\begin{array}{c}
\begin{array}{cc|c} b \in \mathrm{g} & b \in \mathrm{s} & b \in \mathrm{g} \end{array} \\
\left(\begin{array}{cc|c}
(A_n)_{a,b} & (A_n)_{a,b} & (-C_n^{\mathrm{T}})_{a,b} \\
(A_n)_{a,b} & (A_n)_{a,b} & (-C_n^{\mathrm{T}})_{a,b} \\ \hline
(C_n)_{a,b} & (C_n)_{a,b} & (B_n)_{a,b}
\end{array}\right)
\end{array}.
\tag{5.8}
$$

As a consequence, compactification of the pure Yang–Mills theory yields the Yang–Mill-Scalar theory (YMS), in which the most general amplitude is computed from the integrand

$$
I_n^{\mathrm{YMS}}[\alpha] := C_n[\alpha]\,\mathrm{Pf}[\mathcal{X}_n]_{:\mathrm{s}}\,\mathrm{Pf}'[\Psi_n]_{:\hat{\mathrm{s}}}\,.
\tag{5.9}
$$

As shown in (4.23) the Lagrangian of this theory is

$$
\mathcal{L}^{\mathrm{YMS}} := -\frac{1}{2}\,\mathrm{tr_c}(D_\mu\Phi^I\,D^\mu\Phi^I) - \frac{1}{4}\,\mathrm{tr_c}(F_{\mu\nu}\,F^{\mu\nu}) - \frac{g^2}{4}\,\mathrm{tr_c}\left([\Phi^I,\Phi^J]^2\right)\,,
\tag{5.10}
$$

Note that we are free to choose the sets g and s as long as $\mathrm{g}\cap\mathrm{s} = \varnothing$ and $\mathrm{g}\cup\mathrm{s} = \{1,2,\ldots,n\}$. In particular either of them can be an empty set. When $\mathrm{s}=\varnothing$ this goes back to the integrand for YM amplitudes but in d dimensions. In the other limit when $\mathrm{g}=\varnothing$, since

$$
\mathrm{Pf}'[\Psi_n]_{\{1,2,\ldots,n\}:} = \mathrm{Pf}'A_n\,,
\tag{5.11}
$$

we observe that this reduces to the scalar integrand we found in (4.22)

$$I_n^{\text{YMS:scalar}}[\alpha] = C_n[\alpha]\,\text{Pf}\mathcal{X}_n\,\text{Pf}'A_n\,. \tag{5.12}$$

Furthermore, in a generic factorization channel we do not observe any amplitudes with integrands of a different type. Hence the integrand (5.9) is the closed completion of (4.22). The full Lagrangian is easily worked out from the standard compactification of the pure Yang–Mills theory, which is exactly that shown in (4.23).

In the above discussion we assumed a generic $m > 1$. When $m = 1$ the internal space is one-dimensional and so all flavor indices are identified, in which case the matrix $\mathcal{X}_n$ reduces to X_n, and in the Lagrangian the quartic scalar vertex vanishes.

5.1.2 *The Dirac–Born–Infeld Theory*

Next we have a look at the compactification of the Born–Infeld theory, which, when viewed from the integrand, is a close analogue of the pure Yang–Mills theory. Recall the integrands for these two theories

$$I_n^{\text{YM}}[\alpha] = C_n[\alpha]\,\text{Pf}'\Psi_n\,,\quad I_n^{\text{BI}} = (\text{Pf}'A_n)^2\,\text{Pf}'\Psi_n\,. \tag{5.13}$$

The only difference is in the part which does not depend on any polarization vectors. And so the discussion is exactly the same as that in the previous subsection. Hence the integrand for a general amplitude in DBI is given as

$$I_n^{\text{DBI}} := \text{Pf}[\mathcal{X}_n]_{:\text{s}}\,\text{Pf}'[\Psi_n]_{:\hat{\text{s}}}\,(\text{Pf}'A_n)^2\,, \tag{5.14}$$

and we denote the photon set γ as the complement of s in $\{1, 2, \ldots, n\}$. When $\text{s} = \varnothing$ this returns to the closed formula for the photon sector (4.47), i.e., the BI amplitudes in d dimensions, and when $\gamma = \varnothing$ this reduces to the closed formula for the scalar sector in (4.33)

$$I_n^{\text{DBI:scalar}} = \text{Pf}X_n\,(\text{Pf}'A_n)^3\,. \tag{5.15}$$

Again by the standard compactification, it is well-known that the Lagrangian for DBI is

$$\mathcal{L}^{\text{DBI}} := \ell^{-2}\sqrt{-\det\left(\eta_{\mu\nu} + \ell^2\,\partial_\mu\phi\,\partial_\nu\phi + \ell\,F_{\mu\nu}\right)} - \ell^{-2}\,. \tag{5.16}$$

5.1.3 *An Einstein–Maxwell–(Scalar) Theory*

Finally, we study the compactification of gravity. As commented in Section 4.5, we consider the perturbative gravity that also involves the B-field an the dilaton, and so we represent the external state of an amplitude by $\epsilon^{\mu\nu} = \epsilon^{\mu}\,\tilde{\epsilon}^{\nu}$. Recall the integrand $I_n^{\text{GR}} = \text{Pf}'\Psi_n(\{\mathcal{E}\})\,\text{Pf}'\Psi_n(\{\tilde{\mathcal{E}}\})$. Since we have two copies of $\text{Pf}'\Psi_n$, we can choose to compactify the polarization vectors in either of them independently.

Firstly, let us leave all $\mathcal{E}$'s to be external, while restricting a subset of the $\tilde{\mathcal{E}}$'s to the internal space, whose corresponding particles thus become photons. Similar to the case in YMS and DBI discussed before, the resulting integrand is

$$I_n^{\text{EM}} := \text{Pf}[\mathcal{X}_n]_{:\gamma}\,\text{Pf}'[\Psi_n]_{:\hat{\gamma}}(\{\tilde{\epsilon}\})\,\text{Pf}'\Psi_n(\{\epsilon\})\,, \tag{5.17}$$

which, as expected, describes a theory of photons coupled to gravity, obtained by compactifying gravity, or we call it Einstein–Maxwell (EM). The extreme case when $\mathsf{h} = \varnothing$ reduces to the photon amplitudes we obtained in Section 4.4.2

$$I_n^{\text{EM:photon}} = \text{Pf}\mathcal{X}_n\,\text{Pf}'A_n\,\text{Pf}'\Psi_n\,. \tag{5.18}$$

Staring from the integrand for Einstein–Maxwell (5.17), we can to a further compactification on the polarization vectors in the the other copy of $\text{Pf}'\Psi_n$. However, here we cannot restrict arbitrary $\mathcal{E}$'s to be internal, as it may lead to results that violate locality. Instead, we can only do it for particles which are already photons. This final result is thus

$$I_n^{\text{EMS}} := \text{Pf}[\mathcal{X}_n]_{:\gamma,\mathsf{s}}\,\text{Pf}'[\Psi_n]_{\mathsf{h},\gamma,\mathsf{s}:\mathsf{h}}(\{\tilde{\epsilon}\})\,\text{Pf}[\mathcal{X}_n]_{:\mathsf{s}}\,\text{Pf}'[\Psi_n]_{\mathsf{h},\gamma,\mathsf{s}:\mathsf{h},\gamma}(\{\epsilon\})\,, \tag{5.19}$$

in which $\mathsf{h}\cap\gamma = \mathsf{h}\cap\mathsf{s} = \gamma\cap\mathsf{s} = \varnothing$ and $\mathsf{h}\cup\gamma\cup\mathsf{s} = \{1,2,\ldots,n\}$. We call this theory Einstein–Maxwell–Scalar (EMS). In particular, when $\mathsf{h} = \gamma = \varnothing$, this reduces to the integrand for the scalar amplitudes considered in Section 4.2.4

$$I_n^{\text{EMS:scalar}} = (\text{Pf}X_n)^2\,(\text{Pf}'A_n)^2\,. \tag{5.20}$$

5.2 THE SQUEEZING PROCEDURE

In the previous section we use the standard compactification to complete the analysis in theories with several types of bosons that were found in the study of scalar amplitudes in Chapter 4, i.e., YMS, DBI, and EMS.

It is then natural to wonder that, since we have a closed formula for Einstein–Maxwell, do we have something similar for its natural non-Abelian version, i.e., Yang–Mills coupled to gravity? The answer is yes. In this chapter we are going to illustrate the procedure in detail in the case of Einstein–Yang–Mills (EYM). In short, one can take two points of views. In one we conjecture a natural generalization of the integrand for Einstein–Maxwell, while in the other we design a new operation on the matrix Ψ_n in the integrand for gravity that leads to an equivalent expression. The same analysis can then be applied to other integrands which contain a factor $\mathrm{Pf}'\Psi_n$, which leads to closed formulas for theories that we have not discussed previously.

5.2.1 *An Einstein–Yang–Mills Theory: First Approach*

In the first approach to the Einstein–Yang–Mills theory we start with the Einstein–Maxwell theory where the photons are flavored in $U(1)^m$. To make the structure explicit, let us consider an n-point amplitude in EM with $2t$ photons. We expand the factor $\mathrm{Pf}'[\mathcal{X}_n]_\gamma$ explicitly, and the integrand reads

$$I_n^{\mathrm{EM}} = \sum_{\{a,b\}\in \mathrm{P.M.}(\gamma)} \delta^{I_{a_1},I_{b_1}} \cdots \delta^{I_{a_t},I_{b_t}} \left(\frac{\mathrm{sgn}(\{a,b\})}{\sigma_{a_1,b_1}\cdots\sigma_{a_t,b_t}} \mathrm{Pf}'[\Psi_n]_{:\hat{\gamma}}(\{\tilde{\epsilon}\})\,\mathrm{Pf}'\Psi_n(\{\epsilon\}) \right), \quad (5.21)$$

where $\{a,b\} := (a_1,b_1,a_2,b_2,\cdots,a_t,b_t)$ denotes a permutation of the set of photon labels γ, and $[\Psi_n]_{:\hat{\gamma}} = [\Psi]_{\mathrm{h},\gamma:\mathrm{h}}$.

Note that since EM is Abelian, in the above integrand the flavor structure only enter in the form of δ^{I_a,I_b}. The key observation that leads to its corresponding non-Abelian version is that this Kronecker delta can be identified with the usual normalization condition of the generators in the non-Abelian group $U(N)$, i.e.,

$$\mathrm{tr}(T^{I_a}\,T^{I_b}) = \delta^{I_a,I_b}\,. \quad (5.22)$$

When we regard these generators as those in the color group in a Yang–Mills theory, the flavor indices are to be understood as the color indices therein. This means we can think of the photons in such amplitudes as gluons, such that the original flavor structure (for $2\,t$ photons)

$$\delta^{I_{a_1},I_{b_1}}\,\delta^{I_{a_2},I_{b_2}} \cdots \delta^{I_{a_t},I_{b_t}} \quad (5.23)$$

in (5.21) is the color structure in an EYM amplitude

$$\mathrm{tr}(T^{I_{a_1}}\,T^{I_{b_1}})\,\mathrm{tr}(T^{I_{a_2}}\,T^{I_{b_2}})\cdots\mathrm{tr}(T^{I_{a_t}}\,T^{I_{b_t}}) \quad (5.24)$$

where we have in total t traces and each trace involves exactly two gluons. Then then summand in (5.21) for a given perfect matching is actually identical to this special class of partial amplitudes in EYM. From the Feynman diagram point of view this is easy to understand: as the form of the color structure (5.24) excludes contributions from cubic or quartic gluon self-interactions, they interact only by exchanging virtual gravitons as if they were photons. Of course, we should modify the notation slightly by replacing γ by g.

It is pleasing that a simple formula such as (5.21) already computes a special class of EYM partial amplitudes, involving arbitrary number of gluon traces and arbitrary number of gravitons. To extend this special case to the most general amplitudes in EYM, the only thing we need to consider is how to increase the number of gluons in each individual trace.

To achieve this, we first re-arrange a "partial" amplitude of this special class, i.e., a single term in (5.21), to a slightly different form (ignoring the overall sign)

$$\frac{\text{tr}(T^{I_{a_1}}\,T^{I_{b_1}})}{\sigma_{a_1,b_1}\,\sigma_{b_1,a_1}}\,\frac{\text{tr}(T^{I_{a_2}}\,T^{I_{b_2}})}{\sigma_{a_2,b_2}\,\sigma_{b_2,a_2}}\cdots\frac{\text{tr}(T^{I_{a_t}}\,T^{I_{b_t}})}{\sigma_{a_t,b_t}\,\sigma_{b_t,a_t}}\,\mathcal{P}_{\{a,b\}}(\{\tilde{\epsilon}\})\,\text{Pf}'\Psi_n(\{\epsilon\})\,, \tag{5.25}$$

where we define a new quantity

$$\mathcal{P}_{\{a,b\}} := \text{sgn}(\{a,b\})\,\sigma_{a_1,b_1}\,\sigma_{a_2,b_2}\cdots\sigma_{a_t,b_t}\,\text{Pf}'[\Psi_n]_{\text{h},a_1,b_1,a_2,b_2,\cdots,a_t,b_t:\text{h}}\,. \tag{5.26}$$

In the above we explicitly write out $\{a_1, b_1, a_2, b_2, \ldots, a_t, b_t\}$ instead of just g, in order to emphasize that these labels also indicates the relative positions of the corresponding rows and columns in the minor, which may lead to a relative sign of the Pfaffian when specifying different ordering of the labels. This will be important later on.

The re-writing of (5.25) is useful because each pre-factors of the form $\frac{\text{tr}(T^{I_a}\,T^{I_b})}{\sigma_{a,b}\,\sigma_{b,a}}$ therein is exactly a special case of a building block, the Parke–Taylor factor after ordering summation, that we defined in Section 4.1

$$\mathcal{C}_n := \sum_{\alpha\in S_n/\mathbb{Z}_n} \text{tr}(T^{I_{\alpha(1)}}\,T^{I_{\alpha(2)}}\cdots T^{I_{\alpha(n)}})\,C_n(\alpha)\,, \tag{5.27}$$

where now $n = 2$ and the two labels are $\{a, b\}$. As commented before, this structure is natural for theories with a non-trivial flavor or color group. In the current context, imagine that we start from certain amplitude in EYM where the color structure of the gluons consists of several traces, and so in terms of Feynman diagrams the gluon lines separate into corresponding disconnected parts. Obviously the propagators linking these different parts have to be those of virtual gravitons. Then we can explore factorization channels corresponding to these virtual gravitons recursively, so as to extract smaller amplitudes each of which contains a single color trace. In any such small amplitude, we expect a building block similar to (5.27) to enter into the corresponding integrand, hence we generalize its definition to

$$\mathcal{C}_{\{a_1,a_2,\ldots,a_s\}} := \sum_{\rho\in S_s/\mathbb{Z}_s} \frac{\mathrm{tr}(T^{I_{\rho(a_1)}}\, T^{I_{\rho(a_2)}} \cdots T^{I_{\rho(a_s)}})}{\sigma_{\rho(a_1),\rho(a_2)}\, \sigma_{\rho(a_2),\rho(a_3)} \cdots \sigma_{\rho(a_s),\rho(a_1)}} \tag{5.28}$$

for a subset of labels $\{a_1, a_2, \ldots, a_s\} \subset \{1, 2, \ldots, n\}$. As a result of the above discussion, these $\mathcal{C}$ factors can account for the most general color structures of EYM amplitudes. It is convenient to introduce the notation tr_i for the set of labels for the gluons in the i^{th} trace, so that $\mathrm{g} = \mathrm{tr}_1 \cup \mathrm{tr}_2 \cup \cdots \cup \mathrm{tr}_t$, and now $|\mathrm{g}| \geq 2\,t$.

To generalize the formula to arbitrary EYM amplitudes, it is obvious that there should be a factor of $\mathrm{Pf}'\Psi_n$ providing gluon polarization vectors, and an additional copy of the polarization vectors that make up the graviton polarization tensor. Another clue is that given the trace structure $\mathrm{tr}_1, \mathrm{tr}_2, \ldots, \mathrm{tr}_t$, we need the corresponding $\mathcal{C}$ factors, i.e., the combination $\mathcal{C}_{\mathrm{tr}_1}\,\mathcal{C}_{\mathrm{tr}_2} \cdots \mathcal{C}_{\mathrm{tr}_t}$. The remaining problem is how to generalize $\mathcal{P}_{\{a,b\}}$.

The most natural generalization is as follows: we choose two labels $\{a_i, b_i\} \subset \mathrm{tr}_i$ for each i, compute the RHS of (5.26), and then sum over all choices, i.e.,

$$\sum_{\{a,b\}}{}'\mathcal{P}_{\{a,b\}} := \sum_{\substack{a_1<b_1\in\mathrm{tr}_1\\ \cdots \\ a_{t-1}<b_{t-1}\in\mathrm{tr}_{t-1}}} \mathrm{sgn}(\{a,b\})\, \sigma_{a_1b_1} \cdots \sigma_{a_{t-1}b_{t-1}}\, \mathrm{Pf}[\Psi]_{\mathrm{h},a_1,b_1,\ldots,a_{t-1},b_{t-1}:\mathrm{h}}\,. \tag{5.29}$$

Note that in (5.29) we did not take the t^{th} trace in to consideration, which seems to be a flaw. But recall that in the extreme case (5.26) we were using a reduced Pfaffian, and when writing it explicitly out we need to delete two labels and pay a factor of the form $\frac{1}{\sigma_{a,b}}$. Hence the generalization (5.29) is completely consistent with (5.26). That is why we put a prime in denoting this quantity $\sum_{\{a,b\}}{}'\mathcal{P}_{\{a,b\}}$. In fact, we can choose to delete any one of the t traces so that the RHS of (5.29) has no explicit dependence on it, and the result is independent of the choice we make. To see the reason for this independence, however, it is better to switch to an alternative expression equivalent to (5.29) that will be introduced in the next subsection, where this fact will be explicit. What is nice with this quantity is that it is manifestly $SL(2,\mathbb{C})$ covariant with respect to every graviton label and neutral with respect to all gluon labels

$$\sum_{\{a,b\}}{}'\mathcal{P}_{\{a,b\}} \xrightarrow{\psi\in SL(2,\mathbb{C})} \sum_{\{a,b\}}{}'\mathcal{P}_{\{a,b\}} \sum_{c\in\mathrm{h}} (\gamma\,\sigma_c + \delta)^2\,. \tag{5.30}$$

To summarize, the proposal for the integrand for the most general m-trace amplitudes in EYM, as obtained from our first approach, has the form

$$I_n^{\mathrm{EYM}}(\mathrm{tr}_1, \mathrm{tr}_2, \ldots, \mathrm{tr}_t; \mathrm{h}) := \left(\mathcal{C}_{\mathrm{tr}_1}\, \mathcal{C}_{\mathrm{tr}_2} \cdots \mathcal{C}_{\mathrm{tr}_t} \sum_{\{a,b\}}{}' \mathcal{P}_{\{a,b\}}(\{\tilde{\epsilon}\}) \right) \mathrm{Pf}'\Psi_n(\{\epsilon\})\,. \quad (5.31)$$

By (5.30) this integrand is well-defined.

5.2.2 *An Einstein–Yang–Mills Theory: Second Approach*

In the second approach to EYM, we start from a gravity amplitude, and turn a subset of the gravitons into gluons by "hiding half of the polarization tensor" for each of them. So the gravity theory considered here again involves the B-fields and the dilaton and we decompose $\epsilon^{\mu\nu} = \epsilon^{\mu}\,\tilde{\epsilon}^{\nu}$. Without lose of generality we choose to hide ϵ's to obtain gluons. This is achieved by "squeezing" the matrix $\Psi_n(\{\tilde{\epsilon}\})$ into a new matrix and restoring the $SL(2,\mathbb{C})$ covariance of its Pfaffian.

In the simplest case, let us consider how to produce the integrand for amplitudes with a single trace. As mentioned above, the operation leading to this acts on the matrix $\Psi_n(\{\tilde{\epsilon}\})$. We assume that particles $\{1,2,\ldots,n\}$ stay as gravitons while those labeled by $\{r+1, r+2, \ldots, n\}$ are converted into gluons. Recall the range of the indices $\{1,2,\ldots,n : 1,2,\ldots,n\}$ for the matrix Ψ_n that we introduced in Section 4.3, which separates into two blocks. The squeezing procedure consists of several steps:

1. Add all rows $\{r+1, r+2, \ldots, n-1\}$ from the first block of $\{1,2,\ldots,n : 1,2,\ldots,n\}$ to the n^{th} row in the first block. Do the same for the second block.

2. Repeat the same procedure on the columns.

3. Delete all rows and columns with labels in $\{r+1, r+2, \ldots, n-1\}$ from both blocks of $\{1,2,\ldots,n : 1,2,\ldots,n\}$, so as to obtain a $2\,(r+1) \times 2\,(r+1)$ matrix. So far we are only doing linear manipulations, and explicitly the resulting matrix has the form as if all the original rows and columns in $\mathrm{tr}_1 = \{r+1, r+2, \ldots, n\}$

are squeezed into the one labeled by n, in both blocks (it is in this sense that we call this operation "squeezing")

$$
\begin{array}{cc|cc|c}
b \in \mathrm{h} & n & b \in \mathrm{h} & n & \\
\hline
(A_n)_{a,b} & \sum\limits_{d\in \mathrm{tr}_1} (A_n)_{a,d} & -(C_n^{\mathrm{T}})_{a,b} & -\sum\limits_{d\in \mathrm{tr}_1} (C_n^{\mathrm{T}})_{a,d} & a \in \mathrm{h} \\
\sum\limits_{c\in \mathrm{tr}_1} (A_n)_{c,b} & 0 & -\sum\limits_{c\in \mathrm{tr}_1} (C_n^{\mathrm{T}})_{c,b} & -\sum\limits_{c,d\in \mathrm{tr}_1, c\neq d} (C_n^{\mathrm{T}})_{c,d} & n \\
\hline
(C_n)_{a,b} & \sum\limits_{d\in \mathrm{tr}_1} (C_n)_{a,d} & (B_n)_{a,b} & \sum\limits_{d\in \mathrm{tr}_1} (B_n)_{a,d} & a \in \mathrm{h} \\
\sum\limits_{c\in \mathrm{tr}_1} (C_n)_{c,b} & -\sum\limits_{c,d\in \mathrm{tr}_1, c\neq d} (C_n)_{c,d} & \sum\limits_{c\in \mathrm{tr}_1} (B_n)_{c,b} & 0 & n
\end{array}
\,, \tag{5.32}
$$

where $\mathrm{h} = \{1, 2, \ldots, r\}$. Note that the two original diagonal entries $(A_n)_{n,n}$ and $(B_n)_{n,n}$ remain zero after the manipulation, due to anti-symmetry. Obviously this resulting matrix is still anti-symmetric, and so we can define a (reduced) Pfaffian again. But there are two problems: (i) we have not yet hidden the polarization vectors $\tilde{\epsilon}$ for the gluons, and (ii) the Pfaffian of this matrix does not transform covariantly under $SL(2, \mathbb{C})$.

4. These two remaining problems are both fixed by a single step: replace $\tilde{\epsilon}_a^{\mu} \to \sigma_a\, k_a^{\mu}$ $\forall a \in \mathrm{tr}_1 = \{r+1, r+2, \ldots, n\}$. We denote the resulting new matrix as $\Pi(\mathrm{h}; \mathrm{tr}_1)$.

5. Replace $\mathrm{Pf}'\Psi_n$ in the integrand

$$
\mathrm{Pf}'\Psi_n \longrightarrow \mathcal{C}_{\mathrm{tr}_1}\, \mathrm{Pf}'\Pi(\mathrm{h}; \mathrm{tr}_1)\,. \tag{5.33}
$$

The definition of the reduced Pfaffian $\mathrm{Pf}'\Pi$ and the reason for the validity of this replacement will be explained shortly.

The explicit form of the $2\,(r+1) \times 2\,(r+1)$ matrix $\Pi(\mathrm{h}; \mathrm{tr}_1)$ is

$$
\begin{array}{cc|cc|c}
b \in \mathrm{h} & \underline{1} & b \in \mathrm{h} & \underline{1}' & \\
\hline
(A_n)_{a,b} & \sum\limits_{d\in \mathrm{tr}_1} \dfrac{k_a \cdot k_d}{\sigma_{ad}} & -(C_n^{\mathrm{T}})_{a,b} & \sum\limits_{d\in \mathrm{tr}_1} \dfrac{k_a \cdot k_d\, \sigma_d}{\sigma_{a,d}} & a \in \mathrm{h} \\
\sum\limits_{c\in \mathrm{tr}_1} \dfrac{k_c \cdot k_b}{\sigma_{c,b}} & 0 & \sum\limits_{c\in \mathrm{tr}_1} \dfrac{k_c \cdot \tilde{\epsilon}_b}{\sigma_{c,b}} & \sum\limits_{c,d\in \mathrm{tr}_1, c\neq d} k_c \cdot k_d & \underline{1} \\
\hline
(C_n)_{a,b} & \sum\limits_{d\in \mathrm{tr}_1} \dfrac{\tilde{\epsilon}_a \cdot k_d}{\sigma_{a,d}} & (B_n)_{a,b} & \sum\limits_{d\in \mathrm{tr}_1} \dfrac{\tilde{\epsilon}_a \cdot k_d\, \sigma_d}{\sigma_{a,d}} & a \in \mathrm{h} \\
\sum\limits_{c\in \mathrm{tr}_1} \dfrac{\sigma_c\, k_c \cdot k_b}{\sigma_{c,b}} & -\sum\limits_{c,d\in \mathrm{tr}_1, c\neq d} k_c \cdot k_d & \sum\limits_{c\in \mathrm{tr}_1} \dfrac{\sigma_c\, k_c \cdot \tilde{\epsilon}_b}{\sigma_{c,b}} & 0 & \underline{1}'
\end{array}
\,. \tag{5.34}
$$

Here we also introduced new notation for the rows and columns resulting from the squeezing procedure: $\underline{1}$ and $\underline{1}'$, instead of the original label n. The label $\underline{1}$ refers to the trace of gluons tr_1 in the first block, and we use a prime for the two rows/columns in the other block.

The above operation can be iterated to generate a Π matrix corresponding to multiple traces. For example, for the case of two traces $\mathrm{tr}_1 = \{r'+1, \ldots, n\}$, $\mathrm{tr}_2 = \{r+1, \ldots, r'\}$, we start from (5.34) (but with r replaced by r') and convert gravitons $\{r+1, \ldots, r'\}$ into gluons in the same way, obtaining a $2\,(r+2) \times 2\,(r+2)$ matrix, which we denote as $\Pi(\mathrm{h}; \mathrm{tr}_1, \mathrm{tr}_2)$.

In general for t traces, assuming r remaining gravitons, we obtain a $2\,(r+t) \times 2\,(r+t)$ matrix, $\Pi(\mathrm{h}; \mathrm{tr}_1, \mathrm{tr}_2, \ldots, \mathrm{tr}_t)$, by iterating the same operations t times. It is straightforward to implement this procedure, but notation-wise it is non-trivial to present the result explicitly. Nevertheless we present the most general Π matrix below, labeling its columns and rows by $a, b \in \mathrm{h}$, and $i, j, i', j' \in \{\mathrm{tr}\} \equiv \{\underline{1}, \ldots, \underline{t}\}$ for the traces:

$$\Pi = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{cc|cc|c} b \in \mathrm{h} & j \in \{\mathrm{tr}\} & b \in \mathrm{h} & j' \in \{\mathrm{tr}\} & \\ \hline (A_n)_{a,b} & \Pi_{a,j} & -(C_n^{\mathrm{T}})_{a,b} & \Pi_{a,j'} & a \in \mathrm{h} \\ \Pi_{i,b} & \Pi_{i,j} & \widetilde{\Pi}_{i,b} & \Pi_{i,j'} & i \in \{\mathrm{tr}\} \\ \hline (C_n)_{a,b} & \widetilde{\Pi}_{a,j} & (B_n)_{a,b} & \widetilde{\Pi}_{a,j'} & a \in \mathrm{h} \\ \Pi_{i',b} & \Pi_{i',j} & \widetilde{\Pi}_{i',b} & \Pi_{i',j'} & i' \in \{\mathrm{tr}\} \end{array}\,. \tag{5.35}$$

Note that here four blocks of the Π matrix are identical to those in Ψ_n, and we use a slight abuse of notation for the remaining twelve blocks: the blocks with different types of subscripts, such as i, b and i', b, or i, j, i', j and i', j' are distinct matrices, and in addition we denote $\widetilde{\Pi}$ those blocks where one subscript is a graviton label and the other a trace label. Explicitly, entries in eight of the remaining blocks are

$$\begin{gathered} \Pi_{i,b} = \sum_{c \in \mathrm{tr}_i} \frac{k_c \cdot k_b}{\sigma_{c,b}}, \quad \widetilde{\Pi}_{i,b} = \sum_{c \in \mathrm{tr}_i} \frac{k_c \cdot \tilde{\epsilon}_b}{\sigma_{c,b}}, \quad \Pi_{i',b} = \sum_{c \in \mathrm{tr}_{i'}} \frac{\sigma_c\, k_c \cdot k_b}{\sigma_{c,b}}, \quad \widetilde{\Pi}_{i',b} = \sum_{c \in \mathrm{tr}_{i'}} \frac{\sigma_c\, k_c \cdot \tilde{\epsilon}_b}{\sigma_{c,b}}, \\ \Pi_{i,j} = \sum_{c \in \mathrm{tr}_i,\, d \in \mathrm{tr}_j} \frac{k_c \cdot k_d}{\sigma_{c,d}}, \quad \Pi_{i',j} = \sum_{c \in \mathrm{tr}_{i'},\, d \in \mathrm{tr}_j} \frac{\sigma_c\, k_c \cdot k_d}{\sigma_{c,d}}, \quad \Pi_{i',j'} = \sum_{c \in \mathrm{tr}_{i'},\, d \in \mathrm{tr}_{j'}} \frac{\sigma_c\, k_c \cdot k_d\, \sigma_d}{\sigma_{c,d}}, \end{gathered} \tag{5.36}$$

while the other four blocks can be obtained from (5.36) by anti-symmetry. To save space, we suppressed the condition $c \neq d$ on the second line for diagonal entries $i = j$ and $i' = j'$.

Before writing down the final integrand for the amplitudes, note that Π has a two-dimensional kernel spanned by the vectors:

$$
\begin{aligned}
v_1 &= (\underbrace{1,\ldots,1}_{r},\underbrace{1,\ldots,1}_{t},\underbrace{0,\ldots,0}_{r},\underbrace{0,\ldots,0}_{t})^{\mathrm{T}}, \\
v_2 &= (\underbrace{\sigma_1,\ldots,\sigma_r}_{r},\underbrace{0,\ldots,0}_{t},\underbrace{0,\ldots,0}_{r},\underbrace{1,\ldots,1}_{t})^{\mathrm{T}},
\end{aligned}
\tag{5.37}
$$

which is reminiscent of the kernel of the original matrix Ψ_n. Recall the labels are arranged as $\{1,\ldots,r,\underline{1},\ldots,\underline{t} : 1,\ldots,r,\underline{1},\ldots,\underline{t}\}$. Given v_1, v_2, the reduced Pfaffian of Π can be defined as the Pfaffian of a reduced matrix obtained by deleting two rows and two columns in any of the following four equivalent ways, dressed by its corresponding Jacobian:

$$
\mathrm{Pf}'\Pi := \mathrm{Pf}[\Pi]_{\hat{i}:\hat{j}'} = \frac{(-1)^a}{\sigma_a}\mathrm{Pf}[\Pi]_{\hat{a},\hat{i}:} = -\frac{(-1)^a}{\sigma_a}\mathrm{Pf}[\Pi]_{\hat{a}:\hat{j}'} = \frac{(-1)^{a+b}}{\sigma_{a,b}}\mathrm{Pf}[\Pi]_{\hat{a},\hat{b}:}\,, \tag{5.38}
$$

with $i, j' \in \{\underline{1},\ldots,\underline{t}\}$, and (importantly) $a, b \in \{1,2,\ldots,r\}$ for the first r rows/columns. Here $[\Pi]$ with two hatted subscripts denotes Π with the two indicated rows and columns deleted, and the Jacobian factor in front is easily understood by the structure of v_1 and v_2. The reduced Pfaffian is independent of the labels being deleted, and in particular the first definition means we can eliminate any one of the t traces. This should sound familiar from the results in the previous subsection.

The final integrand for general multi-trace mixed amplitudes in EYM is then

$$
I_n^{\mathrm{EYM}}(\mathrm{h};\mathrm{tr}_1,\mathrm{tr}_2,\ldots,\mathrm{tr}_t) := \mathcal{C}_{\mathrm{tr}_1}\,\mathcal{C}_{\mathrm{tr}_2}\cdots\mathcal{C}_{\mathrm{tr}_t}\,\mathrm{Pf}'\Pi(\mathrm{h};\mathrm{tr}_1,\mathrm{tr}_2,\ldots,\mathrm{tr}_t)\,\mathrm{Pf}'\Psi_n\,. \tag{5.39}
$$

One of the advantages of having a formulation in terms of $\mathrm{Pf}'\Pi$ is that it makes the symmetries among the traces manifest, and is convenient for analyzing various properties of the amplitudes, such as soft limits. We did not show that the integrand (5.39) is well-defined. In fact, its $SL(2,\mathbb{C})$ covariance is guaranteed by its equivalence with the integrand (5.31), via the identity

$$
\mathrm{Pf}'\Pi(\mathrm{h};\mathrm{tr}_1,\mathrm{tr}_2,\ldots,\mathrm{tr}_t) = \sum_{\{a,b\}}{}'\,\mathcal{P}_{\{a,b\}}\,. \tag{5.40}
$$

The proof for this identity is summarized in Section 5.A.

5.2.3 *Explicit Examples*

In this subsection we provide several explicit examples of general EYM amplitudes, for the sake of a better understanding of its structure.

Single-Trace Amplitudes

In the first example, we discuss an arbitrary amplitudes involving gluons and (possibly) gravitons, where the color structure of the gluons form a single trace, which we name as single-trace amplitudes.

The matrix Π entering the integrand for such amplitudes has been worked out explicitly in (5.34). Note that in defining its reduced Pfaffian we are free to make any of the choices as listed in (5.38). A particularly nice choice for these amplitudes is to delete the only two rows and columns labeled by the unique trace. What is left is a familiar minor of the matrix Ψ_n, obtained by extracting graviton labels only. Hence the integrand can be written out solely with analogues of familiar building blocks in the pure gluon and pure gravity amplitudes. Explicitly, the integrand for the full amplitudes is

$$I_n^{\text{EYM:single-trace}} = \mathcal{C}_{\text{g}}\,\text{Pf}'[\Psi_n]_{\text{h:h}}\,\text{Pf}'\Psi_n\,. \tag{5.41}$$

Multi-Trace Gluon Amplitudes

In the second example, we study amplitudes of scattering among gluons only, where the color structure forms several traces (say t traces), which we call multi-trace gluon amplitudes. In this case the matrix Π only contain trace labels and it only depends on σ's and the Mandelstam variables. Explicitly, it reads

$$\Pi(\varnothing;\text{tr}_1,\text{tr}_2,\ldots,\text{tr}_t) = \begin{array}{c|c} j\in\{\text{tr}\} & j'\in\{\text{tr}\} \\ \end{array} \left(\begin{array}{c|c} \displaystyle\sum_{c\in\text{tr}_i,\,d\in\text{tr}_j}\frac{k_c\cdot k_d}{\sigma_{c,d}} & \displaystyle\sum_{c\in\text{tr}_i,\,d\in\text{tr}_j}\frac{\sigma_c\,k_c\cdot k_d}{\sigma_{c,d}} \\ \hline \displaystyle\sum_{c\in\text{tr}_i,\,d\in\text{tr}_j}\frac{k_c\cdot k_d\,\sigma_d}{\sigma_{c,d}} & \displaystyle\sum_{c\in\text{tr}_i,\,d\in\text{tr}_j}\frac{\sigma_c\,k_c\cdot k_d\,\sigma_d}{\sigma_{c,d}} \end{array}\right) \begin{array}{c} i\in\{\text{tr}\} \\ \hline i'\in\{\text{tr}\} \end{array}\,. \tag{5.42}$$

In the above, each block is labeled by $i,j\in\{\underline{1},\ldots,\underline{t}\}$, and in the diagonal entries we have $c\neq d$. According to (5.38), we define the reduced Pfaffian by deleting rows and columns for some i and j'.

In the simplest case the color structure forms a single trace $\mathcal{C}_{\text{tr}_1}\equiv\mathcal{C}_n$. Note that now the matrix Π is a 2×2 matrix of rank-0, and so by definition its reduced Pfaffian is trivially 1. In this situation the integrand is exactly that for pure Yang–Mills amplitudes, as is expected (if there is any internal graviton, the color structure has to break up into several traces).

The simplest non-trivial case is when the color structure forms two traces and the matrix Π becomes a 4×4 matrix. In defining its reduced Pfaffian we can further delete, say both rows and columns labeled by $\{\underline{2}:\underline{2}\}$. Then it reduces to a single Mandelstam variable, which is completely independent of the σ variables

$$\mathrm{Pf}'\Pi(\varnothing;\mathrm{tr}_1,\mathrm{tr}_2)=\sum_{\substack{c,d\in\mathrm{tr}_1\\ c\neq d}}\frac{\sigma_c\,k_c\cdot k_d}{\sigma_{c,d}}=\frac{1}{2}\sum_{\substack{c,d\in\mathrm{tr}_1\\ c\neq d}}k_c\cdot k_d=\frac{1}{2}\left(\sum_{c\in\mathrm{tr}_1}k_c\right)^2. \tag{5.43}$$

In this example we clearly see that the choice in defining $\mathrm{Pf}'\Pi$ preserves the symmetry among the traces, since $(\sum_{c\in\mathrm{tr}_1}k_c)^2=s_{\mathrm{tr}_1}=s_{\mathrm{tr}_2}$. Hence the integrand for the full amplitude is

$$\mathcal{I}_n^{\text{EYM:double-trace gluon}}=\frac{1}{2}\,\mathcal{C}_{\mathrm{tr}_1}\,\mathcal{C}_{\mathrm{tr}_2}\,s_{\mathrm{tr}_1}\,\mathrm{Pf}'\Psi_n\,. \tag{5.44}$$

5.2.4 *Another Yang–Mills–Scalar Theory*

When we obtained the formula for general EYM amplitudes from "squeezing" graviton amplitudes, the only change in the integrand is the replacement

$$\mathrm{Pf}'\Psi_n\longrightarrow\mathcal{C}_{\mathrm{tr}_1}\,\mathcal{C}_{\mathrm{tr}_2}\cdots\mathcal{C}_{\mathrm{tr}_t}\,\mathrm{Pf}'\Pi(\mathrm{h};\mathrm{tr}_1,\mathrm{tr}_2,\ldots,\mathrm{tr}_t)\,. \tag{5.45}$$

So in principle we are allowed to apply this operation to any formula where the integrand involves at least one copy of $\mathrm{Pf}'\Psi_n$ (we have already considered the case with two copies). Recalling the results in Section 4.4, such candidate can be amplitudes in the pure Yang–Mills theory, or the Born–Infeld theory. While the integrand for photon amplitudes in Eintein–Maxwell always contain a $\mathrm{Pf}'\Psi_n$ as well, the result of "squeezing" in general does not produce physically sensible results, so we do not consider this case.

We first study the formula from "squeezing" that for pure Yang–Mills

$$\mathcal{I}_n^{\text{gen.YMS}}:=\underbrace{\mathcal{C}_n}_{\text{color}}\,\underbrace{\mathcal{C}_{\mathrm{tr}_1}\,\mathcal{C}_{\mathrm{tr}_2}\cdots\mathcal{C}_{\mathrm{tr}_t}}_{\text{flavor}}\,\mathrm{Pf}'\Pi(\mathrm{g};\mathrm{tr}_1,\mathrm{tr}_2,\ldots,\mathrm{tr}_t)\,. \tag{5.46}$$

Note that the above integrand is for the full amplitude instead of a partial amplitude. Since the new particles arising from the "squeeze" are scalars, the factor $\mathcal{C}_{\mathrm{tr}_1}\,\mathcal{C}_{\mathrm{tr}_2}\cdots\mathcal{C}_{\mathrm{tr}_t}$ thus produced is associated to the flavor structure of the scalars, while the scalars are also gauged by the original gluons and the corresponding color structure is captured by the factor $\mathcal{C}_n$. The study of a general factorization channel shows no other internal propagating states. In particular this means there are no gravitons (nor B-fields nor dilatons), and so the color structure has to always form a single trace containing generators from every external state. Hence (5.46) again leads to a closed formula for a theory of massless scalars gauged by Yang–Mills.

At first sight this seems to be no different from the YMS that we discussed in Section 5.1.1, but there is a crucial difference. In (5.46) we can take the extreme case that the flavor structure forms a single trace only, i.e.,

$$\mathcal{C}_{\mathrm{tr}_1}\,\mathcal{C}_{\mathrm{tr}_2}\cdots\mathcal{C}_{\mathrm{tr}_t}\,\mathrm{Pf}'\Pi(\mathrm{g};\mathrm{tr}_1,\mathrm{tr}_2,\ldots,\mathrm{tr}_t)\longrightarrow\mathcal{C}_{\mathrm{tr}_1}\equiv\mathcal{C}_n\,. \tag{5.47}$$

We see that the integrand reduces to that for the Φ^3 theory where the scalars are flavored in $U(N)\times U(\tilde{N})$ (although one copy of $U(N)$ is now gauged). This indicates that the new theory at least contains a cubic scalar vertex with exactly the same form as that in the Lagrangian (4.16). It turns out this is all what we need apart from the terms already existing in the YMS Lagrangian (4.23). Hence we call this new theory the generalized Yang–Mill–Scalar (gen. YMS). Explicitly, the its Lagrangian is

$$\begin{aligned}\mathcal{L}^{\text{gen. YMS}} :=& -\frac{1}{2}\,\mathrm{tr_c}(D_\mu\Phi^I\,D^\mu\Phi^I)-\frac{1}{4}\,\mathrm{tr_c}(F_{\mu\nu}\,F^{\mu\nu})-\frac{g^2}{4}\,\mathrm{tr_c}\left([\Phi^I,\Phi^J]^2\right)\\&-\frac{\lambda}{3!}\,f^{\mathrm{f}}_{I,J,K}\,f^{\mathrm{c}}_{\tilde{I},\tilde{J},\tilde{K}}\,\Phi^{I,\tilde{I}}\,\Phi^{J,\tilde{J}}\,\Phi^{K,\tilde{K}}\,.\end{aligned} \tag{5.48}$$

5.2.5 *An Extended Dirac–Born–Infeld Theory*

Next we study what "squeezing" the formula for the Born–Infeld theory has to produce. The resulting integrand is easily obtained as

$$I_n^{\text{ext.DBI}} := \mathcal{C}_{\mathrm{tr}_1}\,\mathcal{C}_{\mathrm{tr}_2}\cdots\mathcal{C}_{\mathrm{tr}_t}\,\mathrm{Pf}'\Pi(\mathrm{h};\mathrm{tr}_1,\mathrm{tr}_2,\ldots,\mathrm{tr}_t)\,(\mathrm{Pf}'A_n)^2\,. \tag{5.49}$$

As will be discussed in detail in Chapter 6, the study of a general factorization channel confirms the consistency of this formula regarding locality and unitarity, and we do not observe new particles, and so this integrand leads to a closed formula for yet one more theory of photons and scalars, where the scalars are flavored in $U(N)$.

First of all, as before we can take two different extreme cases of this integrand. The first one is the scalar amplitudes in which the flavor factor forms a single trace, by which the integrand (5.49) reduces to that for NLSM (4.26)

$$I_n^{\text{NLSM}} = \mathcal{C}_n\,(\mathrm{Pf}'A_n)^2\,. \tag{5.50}$$

In the second case, we study the amplitudes with even number of scalars, where each flavor trace contains generators from exactly two scalars. Recall the normalization condition $\mathrm{tr}(T^{I_a}\,T^{I_b}) = \delta^{I_a,I_b}$, the integrand reduces to a single term in the integrand for DBI (5.14)

$$I_n^{\mathrm{DBI}} = \mathrm{Pf}[X_n]_{\mathrm{s}}\,\mathrm{Pf}'[\Psi_n]_{:\hat{\mathrm{s}}}\,(\mathrm{Pf}'A_n)^2 \tag{5.51}$$

after expanding $\mathrm{Pf}[X_n]_{\mathrm{s}}$. Hence we know that both NLSM and DBI are certain sectors of this new theory.

Apart from these, we expect to observe more interaction vertices in this theory. For example, the integrand (5.49) obviously produces a non-trivial amplitude for the scattering of a single photon with three scalars, which has to vanish if all vertices involving photons come from the Lagrangian for DBI. We name this theory as extended DBI, considering the fact that this is an extension of the ordinary DBI theory with the flavor structure of the scalars non-Abelian.

Given that any amplitude in this theory can be computed from (5.49), one can derive all the interaction vertices in the Lagrangian starting from lower orders. In fact, an educated ansatz can be made due to the observation of the two sectors pointed out before. When the vertices are worked out to all orders of multiplicity, interestingly they turn out to sum up into a square root again, as is in the DBI Lagrangian. The conjectured Lagrangian for this theory is

$$\mathcal{L}^{\text{ext. DBI}} = \ell^{-2}\sqrt{-\det\left(\eta_{\mu\nu} + \frac{\ell^2}{\lambda^2}\,\mathrm{tr}\left(\partial_\mu \mathbf{U}^\dagger\,\partial_\nu \mathbf{U}\right) + \ell^2\,W_{\mu\nu} + \ell\,F_{\mu\nu}\right)} - \ell^{-2}\,, \tag{5.52}$$

where $\mathbf{U} = \mathbf{U}(\Phi)$ is defined in (4.28) and expanding $\mathbf{U}$ in terms of Φ gives rise to the usual scalar kinetic term. The extra term $W_{\mu\nu}$ is

$$W_{\mu\nu} = \sum_{m=1}^{\infty}\sum_{k=0}^{m-1}\frac{2(m-k)}{2m+1}\,\lambda^{2m+1}\,\mathrm{tr}(\partial_{[\mu}\Phi\,\Phi^{2k}\,\partial_{\nu]}\Phi\,\Phi^{2(m-k)-1}). \tag{5.53}$$

We call this the extended Dirac–Born–Infeld theory (ext. DBI).

5.3 A UNIFIED POINT OF VIEW

In this section we introduce one more operation. While this does not leads to any formula for a new theory other than the ones we have discussed so far, it provides a convenient view point of how the integrands in different theories are connected together, as a result of which we can regard amplitudes in any of these theories as obtained by several steps of compactification or analogues of compactification from the gravity amplitudes.

To introduce the new operation, let us recall the standard compactification discussed in Section 5.1

$$\mathrm{Pf}'\Psi_n \longrightarrow \mathrm{Pf}[\mathcal{X}_n]_S\,\mathrm{Pf}'[\Psi_n]_{:\hat{S}}\,, \tag{5.54}$$

where S denotes certain subset of $\{1,2,\ldots,n\}$. When all the polarization in the $\mathrm{Pf}'\Psi_n$ are compactified (i.e., $S=\{1,2,\ldots,n\}$) we simply get

$$\mathrm{Pf}'\Psi_n \longrightarrow \mathrm{Pf}\mathcal{X}_n\,\mathrm{Pf}'A_n\,. \tag{5.55}$$

The important fact leading to this result is that for the compactified polarization vectors $\mathcal{E}_a\cdot\mathcal{E}_b=\delta^{I_a,I_b}$.

Now, instead of the usual compactification, we can imagine a compactification from $2\,d$ dimensions to d dimensions, and for the polarization vectors that are compactified we restrict them to be

$$\mathcal{E}_a^M=(\underbrace{0,0,\ldots,0}_{d},k_a^0,k_a^1,\ldots,k_a^{d-1})\,. \tag{5.56}$$

We call this "generalized compactification". Whenever we apply this compactification we always do it for all labels, which means $K_a\cdot\epsilon_b=0$ and $\epsilon_a\cdot\epsilon_b=k_a\cdot k_b$ for any a,b.

In this case the matrix Ψ_n again becomes block-diagonal (since $C_n=0$)

$$\Psi_n=\left(\begin{array}{c|c} A_n & -C_n^{\mathrm{T}} \\ \hline C_n & B_n \end{array}\right)=\left(\begin{array}{c|c} A_n(k,\sigma) & 0 \\ \hline 0 & A_n(k,\sigma) \end{array}\right). \tag{5.57}$$

If we naively compute $\mathrm{Pf}'\Psi$ we get zero, since Ψ has two additional null vectors due to the bottom-right A_n block

$$\frac{\mathrm{Pf}'_{\mathrm{old}}\Psi_n}{\mathrm{Pf}'A_n}=\mathrm{Pf}A_n=\sum_{b=1,b\neq a}^{n}(-1)^{a+b+1}\frac{s_{a,b}}{\sigma_{a,b}}\,\mathrm{Pf}[A_n]_{\hat{a},\hat{b}}=(\mathrm{Pf}'A_n)\sum_{b=1,b\neq a}^{n}s_{a,b}=0\,. \tag{5.58}$$

The correct way to implement this procedure is to extract the coefficient of the zero $\sum_{b=1,b\neq a}^{n}s_{a,b}=-s_{a,a}=0$, which naturally yields a non-trivial result. In other words, we define the reduced Pfaffian by deleting four rows and four columns, two for each A_n,

$$\mathrm{Pf}'_{\mathrm{new}}\Psi_n:=(\mathrm{Pf}'A_n)^2\,. \tag{5.59}$$

Hence this connects, e.g., gravity ampliutdes to BI amplitudes.

In this chapter we have introduced three operations on the integrand, whose effects are all certain replacement of $\text{Pf}'\Psi_n$, summarized as follows

$$\text{compactification} : \text{Pf}'\Psi_n \xrightarrow{\text{compactify}} \text{Pf}[\mathcal{X}_n]_S\, \text{Pf}'[\Psi_n]_{:\hat{S}}\,, \tag{5.60}$$

$$\text{squeezing} : \text{Pf}'\Psi_n \xrightarrow{\text{squeeze}} \mathcal{C}_{\text{tr}_1}\, \mathcal{C}_{\text{tr}_2} \cdots \mathcal{C}_{\text{tr}_t}\, \text{Pf}'\Pi(\bar{\text{tr}}; \text{tr}_1, \text{tr}_2, \ldots, \text{tr}_t), \tag{5.61}$$

$$\begin{matrix}\text{a generalized}\\ \text{compactification}\end{matrix} : \text{Pf}'\Psi_n \xrightarrow{\text{“compactify”}} (\text{Pf}'A_n)^2\,, \tag{5.62}$$

where $S, \text{tr}_1 \ldots, \text{tr}_t$ are some subsets of $\{1, 2, \ldots, n\}$ with $\text{tr}_i \cap \text{tr}_j = \varnothing\ \forall i, j$, which we can freely choose, and $\bar{\text{tr}}$ denotes the complement of $(\text{tr}_1 \cup \text{tr}_2 \cup \cdots \cup \text{tr}_t)$. There are several choices that are comparatively special. The first one is

$$\mathcal{C}_{\text{tr}_1}\, \mathcal{C}_{\text{tr}_2} \cdots \mathcal{C}_{\text{tr}_t}\, \text{Pf}'\Pi(\bar{\text{tr}}; \text{tr}_1, \text{tr}_2, \ldots, \text{tr}_t) \xrightarrow{|\text{tr}_i|=2\ \ \forall i} \begin{matrix} \frac{\delta^{I_{a_1,b_1}}}{\sigma_{a_1,b_1}} \frac{\delta^{I_{a_2,b_2}}}{\sigma_{a_2,b_2}} \cdots \frac{\delta^{I_{a_t,b_t}}}{\sigma_{a_t,b_t}}\, \text{Pf}'[\Psi_n]_{:\hat{S}} \\ \cap \\ \text{Pf}[\mathcal{X}_n]_S\, \text{Pf}'[\Psi_n]_{:\hat{S}} \end{matrix}\,, \tag{5.63}$$

where $S = \{a_1, b_1, a_2, b_2, \ldots, a_t, b_t\}$. In fact we can reverse the above arrow, which then becomes the approach that we discussed in Section 5.2.1 then produces the alternative expression $\sum_{\{a,b\}}{}'\mathcal{P}_{\{a,b\}}$ for $\text{Pf}'\Pi$. For example, this switches photons to gluons, so as to bring the integrand for Einstein–Maxwell to that for Einstein–Yang–Mills. The second one is

$$\mathcal{C}_{\text{tr}_1}\, \mathcal{C}_{\text{tr}_2} \cdots \mathcal{C}_{\text{tr}_t}\, \text{Pf}'\Pi(\bar{\text{tr}}; \text{tr}_1, \text{tr}_2, \ldots, \text{tr}_t) \xrightarrow{\text{tr}_1=\{1,2,\ldots,n\}} \mathcal{C}_n\,, \tag{5.64}$$

in other words all labels enter the same trace. While this is just a special case of the generic integrand on the LHS, it actually produces an integrand which is closed by itself, and thus singles out a certain sector of the original theory. For example, this produces the integrand for pure Yang–Mills from that for the Einstein–Yang–Mills, and we know very well that in a single-trace gluon amplitude there cannot be any graviton propagators from the point of view of Feynman diagrams.

As a consequence, we are able to start from the integrand for pure gravity $\text{Pf}'\Psi_n\, \text{Pf}'\Psi_n$, and interpret the closed integrands for all the theories we have found so far as obtained by applying a sequence of the operations discussed above. These are summarized in Table 3. Note that the integrand for gravity consists of two copies of $\text{Pf}'\Psi_n$ and the operation of the generlized compactfication commutes with the other operations, hence the integrands containing $(\text{Pf}'A_n)^2$ can be obtained by different procedures. However, in general we cannot freely compactify or squeeze both $\text{Pf}'\Psi_n$ in the integrand for pure gravity, as this may produce double poles. The only consistent

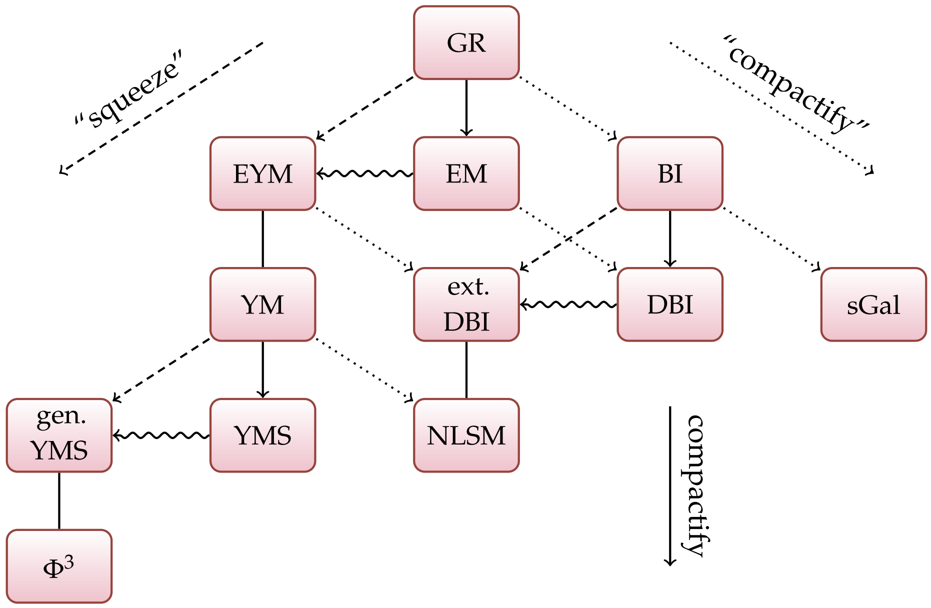


Table 3: Connections among integrands. Compactify: ⟶. Squeeze: - - - ->. "Compactify": ·······>. Non-Abelian: ⇝. Restrict to single trace: ——.

way is to compactify or squeeze in the second step those particles which are already photons/gluons from the first step. Due to this restriction, we did not list out EMS in Table 3.

As mentioned at the beginning of Chapter 4, most of the formulas that we identify with amplitudes in certain theories are conjectured. The ones that have been completely proven are the integrand for gravity, Yang–Mills, and Φ^3. Given these, we do not have worry about the formulas for EM(S) and YMS, since they are obtained just by the standard compactifications of amplitudes in gravity and YM. As will be discussed in detail in Chapter 6, we are able to study a generic soft limit or a generic factorization limit to confirm the consistency regarding locality and unitarity. However, in the absence of a complete proof explicit checks up to a sufficiently high order in the multiplicity n and to a sufficiently generic situation (spacetime dimensions, and specific configuration of external states when several types of particles are involved) are still necessary for the remaining formulas. We summarize the explicit checks that have been performed for each of them in Table 4.

Table 4: Existing Checks for the Formulas without a Proof

Theory	Ref.	Checks
EYM	[8, 9]	(numerically in four dimensions, against results in [55] and those from BCFW recursion) up to eight points, single-trace one-gluon amplitudes; all double- and triple-trace gluon amplitudes up to six points; double-trace four-gluon one-/two-graviton amplitudes; double-trace five-gluon one-graviton amplitude.
gen. YMS	[9]	(include the amplitudes in YMS) analytically up to eight points, all possible amplitudes.
ext. DBI	[9]	(include the amplitudes in DBI and in NLSM) analytically up to six points, all DBI amplitudes; numerically up to ten points, all possible amplitudes.
sGal	[9]	(in arbitrary dimensions) analytically up to six points; numerically at eight points.

5.A EXPANSION OF $\mathrm{pf}'\Pi$

In this appendix we provide a proof for equation (5.40) that $\mathrm{Pf}'\Pi$ can be expanded as a linear combination of Pfaffians of minors of matrix Ψ. Recall the convention there that we consider t traces of gluons and r gravitons[1]. In using the definition (5.38) for the reduced Pfaffian we choose to delete the two rows and columns corresponding to the t^{th} trace, so that the Jacobian is trivially 1, and the reduced matrix $[\Pi]_{\hat{t}:\hat{t}'}$ is of size $2(t+r-1)\times 2(t+r-1)$.

We express $\mathrm{Pf}'\Pi$ in terms of summing over perfect matchings

$$\mathrm{Pf}'\Pi = \sum_{\alpha\in\text{p.f.}} \mathrm{sgn}(\alpha(1),\ldots,\alpha(2(t+r-1)))\ \underbrace{\Pi_{\alpha(1),\alpha(2)}\cdots\Pi_{\alpha(2(t+r)-3),\alpha(2(t+r)-2)}}_{m+r-1}. \quad (5.65)$$

Here α denotes a permutation of the label set $\mathtt{h}\cup\{1,1',\ldots,(t-1),(t-1)'\}$, and restricted to inequivalent perfect matchings. For an entry $\Pi_{\alpha,\beta}$, the non-trivial situation

1 Here "gluon" and "graviton" are merely ways to name the entries, and they can equally be called "scalar" and "gluon"; (5.40) is a purely mathematical identity.

is when $\alpha \in \{1, \ldots, (t-1)\}$ or $\alpha \in \{1', \ldots, (t-1)'\}$ (the trace labels), in which it can be further expanded into

$$\Pi_{\alpha,\beta} = \sum_{a_\alpha \in \text{tr}_\alpha} \frac{k_{a_\alpha} \cdot \#_\beta}{\sigma_{a_\alpha,\beta}} \qquad \text{or} \qquad \Pi_{\alpha,\beta} = \sum_{a_\alpha \in \text{tr}_\alpha} \frac{\sigma_{a_\alpha} \left(k_{a_\alpha} \cdot \#_\beta\right)}{\sigma_{a_\alpha,\beta}}, \tag{5.66}$$

respectively, where $\#_b$ denotes some Lorentz vector depending on the label β. Similarly when β belongs to the trace labels we have instead

$$\Pi_{\alpha,\beta} = \sum_{b_\beta \in \text{tr}_\beta} \frac{\#_\alpha \cdot k_{b_\beta}}{\sigma_{\alpha,b_\beta}} \qquad \text{or} \qquad \Pi_{\alpha,\beta} = \sum_{b_\beta \in \text{tr}_\beta} \frac{\left(\#_\alpha \cdot k_{b_\beta}\right) \sigma_{b_\beta}}{\sigma_{\alpha,b_\beta}}. \tag{5.67}$$

After fully expanding the Π entries labeled by traces in (5.65), it is obvious that each term in the full expansion of $\text{Pf}'\Pi$ is again a product of $(t+r-1)$ factors of the form in (5.66) and (5.67) (since when $\alpha, \beta \in \mathtt{h}$ $\Pi_{\alpha,\beta}$ is also of this form), which are the same as those appearing in the entries of matrix Ψ, except for possible extra σ factors in the numerator.

Note that for every trace i, the summation over labels in tr_i always appears twice in the full expansion, due to the block structure of Π. Let us distinguish the particle labels for these two summations as a_i and b_i (though they both sum over tr_i), we see that in each term of the full expansion of (5.65), either σ_{a_i} or σ_{b_i} will appear, but they can neither both appear nor both be absent. So in each term, apart from the kinematic factors, the form of the σ factors is exactly

$$\sigma_{c_1} \sigma_{c_2} \cdots \sigma_{c_{m-1}}, \tag{5.68}$$

where c_i denotes either a_i or b_i. Now there are two cases which we discuss separately.

Case 1: If in a given term a_i,b_i appear in the same factor in the denominator, i.e.,

$$\text{term}^{\text{adj.}}_{a_i,b_i} = \text{sgn}(\ldots, i', i, \ldots) \cdots \frac{\sigma_{a_i} k_{a_i} \cdot k_{b_i}}{\sigma_{a_i} - \sigma_{b_i}} \cdots, \tag{5.69}$$

then in the full expansion we cannot find another term which is identical to

$$\text{sgn}(\ldots, i, i', \ldots) \cdots \frac{k_{a_i} \cdot k_{b_i} \sigma_{b_i}}{\sigma_{a_i,b_i}} \cdots, \tag{5.70}$$

since the summation in (5.65) is over perfect matchings rather than the full permutations. Hence fixing the other indices and summing over a_i, b_i results in

$$\begin{aligned} \sum_{a_i \in \text{tr}_i} \sum_{b_i \in \text{tr}_i} \text{term}^{\text{adj.}}_{a_i,b_i} &= \sum_{a_i < b_i \in \text{tr}_i} \sigma_{a_i,b_i} \, \text{sgn}(\ldots, i', i, \ldots) \cdots \frac{k_{a_i} \cdot k_{b_i}}{\sigma_{a_i,b_i}} \cdots, \\ &= \text{sgn}(i', i) \sum_{a_i < b_i \in \text{tr}_i} \text{sgn}(a_i, b_i) \, \sigma_{a_i,b_i} \, \text{sgn}(\ldots, a_i, b_i, \ldots) \cdots \frac{k_{a_i} \cdot k_{b_i}}{\sigma_{a_i,b_i}} \cdots. \end{aligned}$$

$$ \tag{5.71} $$

Case 2: If in a given term a_i, b_i appear in different factors in the denominator, i.e.,

$$\text{term}^{\text{non-adj.(1)}}_{a_i,b_i} = \text{sgn}(\ldots, i', \ldots, i, \ldots) \cdots \frac{\sigma_{a_i} k_{a_i} \cdot \#_c}{\sigma_{a_i,c}} \cdots \frac{k_{b_i} \cdot \#_d}{\sigma_{b_i,d}} \cdots , \tag{5.72}$$

the full expansion also contains the contribution from

$$\text{term}^{\text{non-adj.(2)}}_{a_i,b_i} = \text{sgn}(\ldots, i, \ldots, i', \ldots) \cdots \frac{k_{a_i} \cdot \#_c}{\sigma_{a_i,c}} \cdots \frac{\sigma_{b_i} k_{b_i} \cdot \#_d}{\sigma_{b_i,d}} \cdots . \tag{5.73}$$

The summation over a_i, b_i with the other indices fixed thus produces

$$\begin{aligned} \sum_{\substack{a_i \in \text{tr}_i \\ b_i \in \text{tr}_i}} \sum_{q=1,2} \text{term}^{\text{non-adj.(q)}}_{a_i,b_i} &= \sum_{a_i,b_i \in \text{tr}_i} \sigma_{a_i,b_i} \,\text{sgn}(\ldots, i', \ldots, i, \ldots) \cdots \frac{k_{a_i} \cdot \#_c}{\sigma_{a_i,c}} \cdots \frac{k_{b_i} \cdot \#_d}{\sigma_{b_i,d}} \cdots , \\ &= \text{sgn}(i', i) \sum_{a_i,b_i \in \text{tr}_i} \text{sgn}(a_i, b_i)\, \sigma_{a_i,b_i} \,\text{sgn}(\ldots, a_i, \ldots, b_i, \ldots) \\ &\qquad \cdots \frac{k_{a_i} \cdot \#_c}{\sigma_{a_i,c}} \cdots \frac{k_{b_i} \cdot \#_d}{\sigma_{b_i,d}} \cdots . \end{aligned} \tag{5.74}$$

By comparing (5.71) and (5.74), we see that they have the same form

$$\text{sgn}(i', i) \sum_{a_i < b_i \in \text{tr}_i} \text{sgn}(a_i, b_i)\, \sigma_{a_i,b_i} \cdots . \tag{5.75}$$

This applies to every trace label i, and the remaining factors depending on labels a_i, b_i are exactly the same as the entries of matrix A_n, and can be observed to re-sum back into $\text{Pf}[\Psi]_{\text{h},a_1,b_1,\ldots,a_{m-1},b_{m-1}:\text{h}}$ since during the above manipulations preserve the structure of the original Pfaffian expansion in (5.65), only switching the meaning of the labels and corresponding entries. Without loss of generality, we can choose to set $\text{sgn}(i', i) = 1$ $(\forall i)$. Therefore, the full expansion can be re-arranged into

$$\text{Pf}'\Pi(\text{h}; \text{tr}_1, \text{tr}_2, \ldots, \text{tr}_t) = \sum_{\{a,b\}}{}' \mathcal{P}_{\{a,b\}} , \tag{5.76}$$

with $\sum_{\{a,b\}}{}' \mathcal{P}_{\{a,b\}}$ defined as in (5.29).

6

LOCALITY AND UNITARITY

As promised in Chapter 4 and Chapter 5, in this chapter we explore two types of limits of amplitudes using the CHY representation: the soft limits and the factorization limits, for the purpose of a systematic consistency check regarding locality and unitarity for all the formulas we have discussed in the previous two chapters.

For later convenience, we abbreviate the scattering equations as

$$f_a^{(n)} := \sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0\,, \quad \forall a\,, \tag{6.1}$$

where the superscript in $f_a^{(n)}$ means that the kinematic variables $\{k\}$ therein are associated to an n-particle scattering. Also, in order to focus on the structure of the formulas in these limits, let us not worry about the overall signs.

6.1 SOFT THEOREMS I

We begin by studying the single soft limit. We follow the convention in Section 2.4.1 and assume k_n^μ to be soft: let $k_n^\mu = \tau\, p^\mu$ and take $\tau \to 0$ while keeping p^μ fixed.

In this limit, a graviton amplitudes satisfies a soft theorem upto the sub-sub-leading order [56, 57]

$$\mathcal{M}_n^{\text{GR}} = (S_{\text{h}}^{(0)} + S_{\text{h}}^{(1)} + S_{\text{h}}^{(2)})[\mathcal{M}_{n-1}^{\text{GR}}] + \mathcal{O}(\tau^2)\,, \tag{6.2}$$

where the soft operators are

$$S_{\text{h}}^{(0)} := \sum_{a=1}^{n-1} \frac{\epsilon_{n,\mu\nu}\, k_a^\mu\, k_a^\nu}{k_n \cdot k_a}\,, \tag{6.3}$$

$$S_{\text{h}}^{(1)} := -i \sum_{a=1}^{n-1} \frac{\epsilon_{n,\mu\nu}\, k_a^\mu\, (k_{n,\rho}\, J_a^{\rho\nu})}{k_n \cdot k_a}\,, \tag{6.4}$$

$$S_{\text{h}}^{(2)} := -\frac{1}{2} \sum_{a=1}^{n-1} \frac{\epsilon_{n,\mu\nu}\, (k_{n,\rho}\, J_a^{\rho\mu})\, (k_{n,\sigma}\, J_a^{\sigma\nu})}{k_n \cdot k_a}\,. \tag{6.5}$$

Here $J_a^{\mu\nu} := J_{a,\text{orbital}}^{\mu\nu} + J_{a,\text{splin}}^{\mu\nu}$ denotes the total angular momentum operator for the a^{th} particle, with

$$J_{a,\text{orbital}}^{\mu\nu} := i\big(k_a^\mu \frac{\partial}{\partial k_{a,\nu}} - k_a^\nu \frac{\partial}{\partial k_{a,\mu}}\big), \tag{6.6}$$

$$J_{a,\text{spin}}^{\mu\nu}[\epsilon_b^\rho] := i\,\delta_{ab}\,(\delta^{\nu\rho}\delta^\mu_\sigma - \delta^{\mu\rho}\delta^\nu_\sigma)\epsilon_a^\sigma\,, \tag{6.7}$$

acting on the orbital part and on the spin part respectively. Among these, the leading-order $S_\text{h}^{(0)}$ is just a multiplicative operator, corresponding to the Weinberg's soft theorem [12], which is actually universal in the sense that it is valid for the general situation that the soft graviton couples to any particles. This universality was first observed by the arguments from Feynman diagrams, and actually each term in $S_\text{h}^{(0)}$ for a specific a is largely the three-point vertex of the graviton n coupled to the external leg for the hard particle a in the limit $k_n^\mu \to 0$. Hence deriving this multiplicative soft operator bridges the CHY representation and Feynman diagrams, which at first sight appear very different from each other.

There exists a similar soft theorem for gluon amplitudes, but up to the sub-leading order

$$\mathcal{M}_n^{\text{YM}} = (S_\text{g}^{(0)} + S_\text{g}^{(1)})[\mathcal{M}_{n-1}^{\text{YM}}] + \mathcal{O}(\tau)\,, \tag{6.8}$$

where

$$S_\text{g}^{(0)} := \sum_{\alpha\in S_{n-1}} \text{tr}\big(T^{I_{\alpha(1)}}\cdots T^{I_{\alpha(n-2)}}\,[T^{I_{\alpha(n-1)}}, T^{I_n}]\big)\,\frac{\epsilon_n\cdot k_{\alpha(n-1)}}{k_n\cdot k_{\alpha(n-1)}}\,, \tag{6.9}$$

$$S_\text{g}^{(1)} := \sum_{\alpha\in S_{n-1}} \text{tr}\big(T^{I_{\alpha(1)}}\cdots T^{I_{\alpha(n-2)}}\,[T^{I_{\alpha(n-1)}}, T^{I_n}]\big)\,\frac{\epsilon_{n,\mu}\,(k_{\alpha(n-1),\nu}\,J^{\mu\nu}_{\alpha(n-1)})}{k_n\cdot k_{\alpha(n-1)}}\,, \tag{6.10}$$

where $S_\text{g}^{(0)}$ is again a multiplicative operator, and it also holds when the soft gluon couples to scalars.

Aimed at drawing connections between the CHY representation and the Feynman diagrams, in studying the single soft limit we only focus on the leading order. We first estimate the leading order scaling in τ of various amplitudes. After that we derive the leading order soft theorems for gravitons and gluons, emphasizing their universality as viewed from the CHY representation. Extension of this analysis to the subleading orders can be found in [58–60].

In a generic CHY formula, let us not exclude the variable σ_n, nor delete the delta constraint $\delta(f_n^{(n)})$. Then in the measure

$$d\mu_n = \prod_a{}' d\sigma_a \prod_a{}' \delta(f_A^{(n)})\,, \tag{6.11}$$

every $f_a^{(n)}$ approximates to $f_a^{(n-1)}$ except for $f_n^{(n)}$, which is

$$f_n^{(n)} = \tau \sum_{b=1}^{n-1} \frac{p\cdot k_b}{\sigma_n - \sigma_b}\,. \tag{6.12}$$

Hence the measure behaves as

$$d\mu_n = d\mu_{n-1}\,\frac{1}{\tau}\,d\sigma_n\,\delta(\sum_{b=1}^{n-1} \frac{p\cdot k_b}{\sigma_n - \sigma_b}) + \mathcal{O}(\tau^0)\,. \tag{6.13}$$

From Section 2.4.1 we know that in the single soft limit all the solutions to the scattering equations remain non-degenerate as $\tau \to 0$, i.e., $(\sigma_a - \sigma_b) \sim \tau^0$ for any a, b. So the leading order in τ of each building block only depends on whether it explicitly involves the soft momentum k_n^μ, and we quickly observe

$$\mathcal{C}_n[\alpha] \sim \tau^0\,, \quad \mathrm{Pf}X_n \sim \tau^0\,, \quad \mathrm{Pf}'A_n \sim \tau^1\,. \tag{6.14}$$

The situation for $\mathrm{Pf}'\Psi_n$ and $\mathrm{Pf}'\Pi(\mathrm{h};\mathrm{tr}_1,\mathrm{tr}_2,\ldots,\mathrm{tr}_t)$ is relatively involved. In both quantities if there are rows and columns labeled by n, at first sight there will be exactly one row and one column proportional to τ. But there is a unique entry that remains finite, i.e., the entries $(C_n)_{n,n}$. Thus we know

$$\mathrm{Pf}'\Psi_n \sim \tau^0\,, \quad \mathrm{Pf}'\Pi \sim \tau^0\,. \tag{6.15}$$

There is a subtlety for $\mathrm{Pf}'\Pi$ when $n \in \mathrm{tr}_i$ for some trace i. In most situations k_n^μ enters in a summation and thus sub-leading, which leads to the same conclusion (6.15). However, the only exception to this is when tr_i contains exactly two labels, say $\{a, n\}$ for some a. To understand the structure in this case, let us write out the matrix Π explicitly

$$\begin{array}{c} \\ \left(\begin{array}{c:c|c:c} b\in\mathrm{h} & j & b\in\mathrm{h} & j' \\ \displaystyle\sum_{c\in\{a,n\}}\frac{k_c\cdot k_b}{\sigma_{c,b}} & \displaystyle\sum_{c\in\{a,n\},d\in\mathrm{tr}_j}\frac{k_c\cdot k_d}{\sigma_{c,d}} & \displaystyle\sum_{c\in\{a,n\}}\frac{k_c\cdot \epsilon_b}{\sigma_{c,b}} & \displaystyle\sum_{c\in\{a,n\},d\in\mathrm{tr}_{j'}}\frac{k_c\cdot k_d\,\sigma_d}{\sigma_{c,d}} \\ \hline \displaystyle\sum_{c\in\{a,n\}}\frac{\sigma_c\,k_c\cdot k_b}{\sigma_{c,b}} & \displaystyle\sum_{c\in\{a,n\},d\in\mathrm{tr}_j}\frac{\sigma_c\,k_c\cdot k_d}{\sigma_{c,d}} & \displaystyle\sum_{c\in\{a,n\}}\frac{\sigma_c\,k_c\cdot \epsilon_b}{\sigma_{c,b}} & \displaystyle\sum_{c\in\{a,n\},d\in\mathrm{tr}_{j'}}\frac{\sigma_c\,k_c\cdot k_d\,\sigma_d}{\sigma_{c,d}} \end{array}\right)\begin{array}{c} i \\ i' \end{array}\end{array}. \tag{6.16}$$

To avoid unnecessary information, in the above we content to write out the two rows labeled by the i^{th} trace only, and one column for each of the four types. Since $\text{tr}_i = \{a, n\}$, from (6.16) it is easy to observe that if we take the first row therein, multiply by σ_a, and then subtract the second row, then in each entry in the resulting row the terms for $c = a$ cancel away, so that this linear combination is proportional to τ. Hence in this case we conclude

$$\text{Pf}'\Pi \sim \tau^1 \,. \tag{6.17}$$

One can observe that this case is a generalization for the product $\text{Pf}\mathcal{X}_n \, \text{Pf}' A_n \sim \tau^1$, which is an extreme case of $\text{Pf}'\Pi$ (when every trace contain exactly two labels). This happens in the amplitudes in EM and DBI, and in YMS and extended DBI when the flavor/color structure contains a factor of the form $\text{tr}(T^{I_a} T^{I_n})$. All these cases share the common feature that, from the Feynman diagram point of view the structure responsible for this is a three-point vertex involving different types of particles, e.g., two scalars coupled to one photon/gluon, or two photons coupled to a graviton.

With the above estimation, we can easily read out the scaling in τ at the leading order for amplitudes in each theory, which are summarize in Table 5. In the third

Table 5: Leading Order Scaling of the Amplitudes

Theory	Integrand	Leading Scaling
gravity	$\text{Pf}'\Psi_n \, \text{Pf}'\Psi_n$	τ^{-1}
YM	$\mathcal{C}_n \, \text{Pf}'\Psi_n$	τ^{-1}
Φ^3	$\mathcal{C}_n \, \mathcal{C}_n$	τ^{-1}
EM	$\text{Pf}[\mathcal{X}_n]_\gamma \, \text{Pf}'[\Psi_n]_{:\hat{\gamma}} \, \text{Pf}'\Psi_n$	$\gamma : \tau^0$, $\text{h} : \tau^{-1}$
EYM	$\mathcal{C}_{\text{tr}_1} \cdots \mathcal{C}_{\text{tr}_t} \, \text{Pf}'\Pi(\text{h}; \text{tr}_1 \ldots, \text{tr}_t) \, \text{Pf}'\Psi_n$	$\text{g} : \tau^{-1}/\tau^0$, $\text{h} : \tau^{-1}$
YMS	$\mathcal{C}_n \, \text{Pf}[\mathcal{X}_n]_\text{s} \, \text{Pf}'[\Psi_n]_{:\hat{\text{s}}}$	$\text{s} : \tau^0$, $\text{g} : \tau^{-1}$
gen. YMS	$\mathcal{C}_n \, \mathcal{C}_{\text{tr}_1} \cdots \mathcal{C}_{\text{tr}_t} \, \text{Pf}'\Pi(\text{g}; \text{tr}_1 \ldots, \text{tr}_t)$	$\text{s} : \tau^{-1}/\tau^0$, $\text{g} : \tau^{-1}$
BI	$\text{Pf}'\Psi_n \, (\text{Pf}' A_n)^2$	τ^1
DBI	$\text{Pf}[\mathcal{X}_n]_\text{s} \, \text{Pf}'[\Psi_n]_{:\hat{\text{s}}} \, (\text{Pf}' A_n)^2$	$\text{s} : \tau^2$, $\gamma : \tau^1$
ext. DBI	$\mathcal{C}_{\text{tr}_1} \cdots \mathcal{C}_{\text{tr}_t} \, \text{Pf}'\Pi(\gamma; \text{tr}_1 \ldots, \text{tr}_t) \, (\text{Pf}' A_n)^2$	$\text{s} : \tau^1/\tau^2$, $\gamma : \tau^1$
NLSM	$\mathcal{C}_n \, (\text{Pf}' A_n)^2$	τ^1
sGal	$(\text{Pf}' A_n)^4$	τ^3

column of this table, for theories involving two types of particles the leading scaling depends on the type of the soft particle. Especially in EYM, gen. YMS and ext. DBI when the soft particle is the one with lower spin, the leading scaling has to further depend on the detailed flavor/color structure, due to the subtlety previously discussed.

In that same table, we also separate the theories into three groups, according to the power of τ. The theories that we are interested in regarding single soft theorems are those in the first group (top) and those in the second group (middle) when a particle of higher spin becomes soft. In all these cases the leading terms in the single soft limit diverges as τ^{-1}, which is a hint for the presence of a three-point vertex. Those in the third group (bottom) vanish in the soft limit, which indicates that the theory should possess non-trivial symmetries that force cancellations among Feynman diagrams [48].

6.1.2 *Single Soft Theorems at the Leading Order*

Now we derive the leading order soft theorems for the emission of a single soft graviton or gluon, which couples to any amplitude for which a CHY formula is known from previous chapters. The key observation is that the σ integrals can be equally viewed as a contour integral whose contour wraps the solutions to the scattering equations. Specifically, here we re-interpret $\int d\sigma_n$ as a contour integral, and so by the leading-order approximation (6.13) we obtain

$$M_n = \int d\mu_{n-1} \oint \frac{d\sigma_n}{2\,\pi\, i} \frac{\tau^{-1}}{\sum_{b=1}^{n-1} \frac{p\cdot k_b}{\sigma_n - \sigma_b}} I_n + (\text{sub-leading})\,. \tag{6.18}$$

The general strategy is to apply a contour deformation. But before doing this, we need to work out the approximation of I_n.

Soft Graviton

For single soft graviton emissions, we study amplitudes in gravity, EM(S) and EYM, which include the situations when the graviton couples to itself or photons/gluons or scalars. Since the graviton label never enters $\mathcal{C}_n$, we only need to approximate $\text{Pf}'\Psi_n$ and $\text{Pf}'\Pi(\text{h};\text{tr}_1,\ldots,\text{tr}_t)$ to the leading order. As discussed before, the leading terms in $\text{Pf}'\Psi_n$ are those containing $(C_n)_{n,n}$, and the explicit expansion is

$$\text{Pf}'\Psi_n = \text{Pf}'\Psi_{n-1} \sum_{b=1}^{n-1} \frac{\epsilon_n \cdot k_b}{\sigma_n - \sigma_b} + \mathcal{O}(\tau)\,. \tag{6.19}$$

The same applies to reduced pfaffians of minors of Ψ_n, as long as rows and columns labeled by n are involved. Since $n \in \mathrm{h}$, it is not affected when a "squeezing" procedure is applied, and so a general $\mathrm{Pf}'\Pi$ admits a similar expansion

$$\mathrm{Pf}'\Pi(\mathrm{h};\mathrm{tr}_1,\mathrm{tr}_2,\ldots,\mathrm{tr}_t) = \mathrm{Pf}'\Pi(\mathrm{h}\backslash\{n\};\mathrm{tr}_1,\mathrm{tr}_2,\ldots,\mathrm{tr}_t)\sum_{b=1}^{n-1}\frac{\epsilon_n\cdot k_b}{\sigma_n-\sigma_b} + \mathcal{O}(\tau)\,. \tag{6.20}$$

Hence in the soft limit any amplitudes in these theories behaves as

$$M_n = \int d\mu_{n-1}\, I_{n-1} \oint \frac{d\sigma_n}{2\,\pi\,i}\,\frac{\tau^{-1}\left(\sum_{b=1}^{n-1}\frac{\epsilon_n\cdot k_b}{\sigma_n-\sigma_b}\right)^2}{\sum_{b=1}^{n-1}\frac{p\cdot k_b}{\sigma_n-\sigma_b}} + \mathcal{O}(\tau^0)\,. \tag{6.21}$$

Obviously there is no pole at $\sigma_n = \infty$, and so we are free to deform the σ_n contour, which picks up a simple pole at $\sigma_n = \sigma_b$ for every $b \in \{1,2,\ldots,n-1\}$. The residue at each pole has the form of a factor times the corresponding lower-point amplitude, and in total we obtain

$$M_n = \left(\tau^{-1}\sum_{b=1}^{n-1}\frac{(\epsilon_n\cdot k_b)^2}{p\cdot k_b}\right) M_{n-1} + \mathcal{O}(\tau^0)\,. \tag{6.22}$$

Soft Gluon

For the single soft gluon emission, we consider amplitudes in YM, EYM, YMS and gen. YMS. Note that for ampliutdes in EYM we consider the case when the trace containing T^{I_n} at least has size three (here let us assume that $n \in \mathrm{tr}_1$ and that tr_1 contains t_1 labels). Since $\mathrm{Pf}'\Psi_n$ is fully permutation invariant, its behavior is the same as (6.19). In addition, one can work out

$$\mathrm{Pf}'\Pi(\mathrm{h};\mathrm{tr}_1,\ldots,\mathrm{tr}_t) = \mathrm{Pf}'\Pi(\mathrm{h};\mathrm{tr}_1\backslash\{n\},\mathrm{tr}_2,\ldots,\mathrm{tr}_t)\,, \tag{6.23}$$

$$\mathrm{Pf}'\Pi(\mathrm{g};\mathrm{tr}_1,\ldots,\mathrm{tr}_t) = \mathrm{Pf}'\Pi(\mathrm{g}\backslash\{n\};\mathrm{tr}_1,\ldots,\mathrm{tr}_t)\sum_{b=1}^{n-1}\frac{\epsilon_n\cdot k_b}{\sigma_n-\sigma_b}\,, \tag{6.24}$$

$$\mathcal{C}_n = \sum_{\alpha\in S_n/\mathbb{Z}_n}\mathrm{tr}(T^{I_{\alpha(1)}}\cdots T^{I_{\alpha(n-1)}}\,T^{I_n})\,\mathcal{C}_{n-1}\,\frac{\sigma_{\alpha(n-1),\alpha(1)}}{\sigma_{\alpha(n-1),n}\,\sigma_{n,\alpha(1)}}\,, \tag{6.25}$$

$$\mathcal{C}_{\mathrm{tr}_1} = \sum_{\alpha\in S_{t_1}/\mathbb{Z}_{t_1}}\mathrm{tr}(T^{I_{\alpha(1)}}\cdots T^{I_{\alpha(t_1-1)}}\,T^{I_n})\,\mathcal{C}_{\mathrm{tr}_1\backslash\{n\}}\,\frac{\sigma_{\alpha(t_1-1),\alpha(1)}}{\sigma_{\alpha(t_1-1),n}\,\sigma_{n,\alpha(1)}}. \tag{6.26}$$

With these, the formula for YM, YMS and gen. YMS approximates to

$$M_n = \int d\mu_{n-1}\sum_{\alpha\in S_{n-1}}\mathrm{tr}(T^{I_{\alpha(1)}}\cdots T^{I_{\alpha(n-1)}}\,T^{I_n})\,I_{n-1}[\alpha]\oint\frac{d\sigma_n}{2\,\pi\,i}\,\frac{\tau^{-1}\sum_{b=1}^{n-1}\frac{\epsilon_n\cdot k_b}{\sigma_n-\sigma_b}}{\sum_{b=1}^{n-1}\frac{p\cdot k_b}{\sigma_n-\sigma_b}}\,\frac{\sigma_{\alpha(n-1),\alpha(1)}}{\sigma_{\alpha(n-1),n}\,\sigma_{n,\alpha(1)}}\,. \tag{6.27}$$

Again, there is no pole at $\sigma_n = \infty$, but this time for each term in the summation the contour deformation only picks up two simple poles, at $\sigma_n = \sigma_{\alpha(n-1)}$ and $\sigma_n = \sigma_{\alpha(1)}$ respectively. The final result is

$$
\begin{aligned}
M_n &= \sum_{\alpha \in S_{n-1}} \mathrm{tr}(T^{I_{\alpha(1)}} \cdots T^{I_{\alpha(n-1)}}\, T^{I_n})\, \tau^{-1} \left(\frac{\epsilon_n \cdot k_{\alpha(n-1)}}{p \cdot k_{\alpha(n-1)}} - \frac{\epsilon_n \cdot k_{\alpha(1)}}{p \cdot k_{\alpha(1)}} \right) M_{n-1}[\alpha] \\
&= \sum_{\alpha \in S_{n-1}} \mathrm{tr}(T^{I_{\alpha(1)}} \cdots T^{I_{\alpha(n-2)}}\, [T^{I_{\alpha(n-1)}}, T^{I_n}])\, \tau^{-1} \frac{\epsilon_n \cdot k_{\alpha(n-1)}}{p \cdot k_{\alpha(n-1)}} M_{n-1}[\alpha]\,.
\end{aligned}
\tag{6.28}
$$

For the remaining theory EYM it is not hard to see that we obtain exactly the same result, but the summation is instead over the permutations of labels in tr_1.

6.2 Factorization

In the absence of a general proof for our formulas, one can nonetheless show that they behave correctly in a generic factorization channel. This analysis requires that one first do a careful re-parametrization to the σ moduli so as to see that the formula indeed possesses a *simple* pole when approaching any desired physical channel, and remains (at most) finite for any physical channels that are forbidden by the theory. In those desired channels, one further needs to verify that the given amplitude factorizes into two sub-amplitudes at leading order; most importantly, the internal particle thus produced has to be consistent with Feynman diagrams. When working with the CHY representation, we say that a given formula is *closed* if in any physical factorization channel it splits into two formulas at the leading order, both of which share the same type of integrand as the original one. We use this to verify that the formulas we obtained in Chapter 4 and Chapter 5 are self-consistent and closed.

6.2.1 *Setting Up the General Analysis*

We follow the analysis in Section 2.4.2 to specify a generic factorization channel as $k_I^2 = (k_1 + k_2 + \cdots + k_{n_L})^2 \to 0$, for which we re-define

$$
\sigma_a = \begin{cases} \frac{\zeta}{u_a}\,, & a \in L\,, \\ \frac{v_a}{\zeta}\,, & a \in R\,, \end{cases}
\tag{6.29}
$$

and fixing $\{u_1, u_2, v_{n-1}, v_n\}$ to get rid of the $SL(2,\mathbb{C})$ redundancy while leaving ζ as a variable. From those discussions we also know that the original scattering equations produce a constraint for ζ

$$-\frac{1}{2}\,k_I^2 + \zeta^2 \sum_{\substack{a\in R\\ b\in L}} \frac{k_a\cdot k_b}{v_a\,u_b} + \mathcal{O}(\zeta^4) = 0\,, \tag{6.30}$$

by which there exist a set of degenerate solutions such that $\zeta^2 \sim k_{I_R}^2$. For these solutions, we concluded that the new $\{u\}$ and $\{v\}$ variables satisfy scattering equations for the left amplitude and the right one respectively (at the leading order in ζ^2), where the variables for the internal particle u_{I_L} and v_{I_R} are fixed to 0.

When working with a specific CHY formula we need to do this more carefully, i.e., in the measure $d\mu_n$ we choose to exclude the delta constraint labeled by $\{1, 2, n-1\}$ and properly apply linear transformations to turn one of the remaining delta constraint (say the n^{th} one) to the one imposing the equation (6.30), and in addition one needs to carefully work out the Jacobian for the gauge fixing with the new variables. These are explained in detail in [61], which lead to the following expansion of a CHY formula (with a generic integrand) to the leading order

$$\begin{aligned} M_n = \int d\zeta^2\,\zeta^{2(n_L-n_R-3)} &\prod_{a\in L\backslash\{1,2\}} du_a \prod_{a\in R\backslash\{n-1,n\}} dv_a\, \frac{(u_{1,2}u_1u_2)^2(v_{n-1}v_nv_{n-1,n})^2}{(\prod u)^4} \\ &\times \delta(\zeta^2\,F - k_I^2) \prod_{a=3}^{n_L} \delta\Big(\sum_{\substack{b\in L\cup\{I_L\}\\ b\neq a}} \frac{k_a\cdot k_b}{u_{a,b}}\Big) \prod_{a=n_L+1}^{n-2} \delta\Big(\sum_{\substack{b\in R\cup\{I_R\}\\ b\neq a}} \frac{k_a\cdot k_b}{v_{a,b}}\Big)\, I_n(\{\frac{\zeta}{u}, \frac{v}{\zeta}\}) \\ &+ (\text{subleading})\,, \end{aligned} \tag{6.31}$$

where we denote the coefficient of ζ^2 in (6.30) as F since it is not important for the analysis. From (6.31), if the amplitude indeed factorizes we should expect (up to a sign)

$$I_n(\{\frac{\zeta}{u}, \frac{v}{\zeta}\}) \longrightarrow \zeta^{2(-n_L+n_R+2)}\,(\prod u)^4\, I_{n_L+1}(\{u\})\, I_{n_R+1}(\{v\}) \tag{6.32}$$

to the leading *non-vanishing* order in ζ, where the "+1" in the subscripts refers to the internal particle. If this is true, then the integration

$$\int \frac{d\zeta^2}{\zeta^2}\,\delta(\zeta^2\,F - k_I^2) \tag{6.33}$$

merely produces a $1/k_I^2$. We do not have to worry about the non-degenerate solutions to the original scattering equations, because their contribution can never diverge and so are always subleading. Furthermore, if I_{n_L+1} and I_{n_R+1} on the RHS of (6.32) belongs to the same kind of integrand as I_n on the LHS, then the unitarity of the amplitude described by this formula is confirmed, otherwise this indicates that the formula is not closed and we need to seek for its extension.

6.2.2 *Behavior of the Buildng Blocks for Scalars*

According (6.32) in the previous subsection, our task reduces to the study of the various building blocks in the factorization limit. We first study $C_n[\alpha]$, $\mathrm{Pf}'A_n$ and $\mathrm{Pf}X_n$.

For the Parke–Taylor factor, we first assume that it comes with the canonical ordering. Its expansion is easily worked out to be (up to a sign)

$$\begin{aligned} C_n[12\cdots n] =& \zeta^{-n_L+n_R+2}\,(\prod u)^2 \\ &\times \frac{1}{u_1\,u_{1,2}\cdots u_{n_L-1,n_L}\,u_{n_L}}\,\frac{1}{v_{n_L+1}\,v_{n_L+1,n_L+2}\cdots v_{n-1,n}\,v_n} \\ &+\mathcal{O}(\zeta^{-n_L+n_R+3})\,. \end{aligned} \tag{6.34}$$

Note that the two factors in the second line are exactly the Parke–Taylor factors for the two factorized parts. For a generic ordering α, if it can be split into two parts, such that one part contains all the labels $\{1,2,\ldots,n_L\}$ while the other its complement, then $C_n[\alpha]$ factorizes as (6.34) with exactly the same prefactor $(-1)^{n_L}\zeta^{-n_L+n_R+2}(\prod u)^2$ and differs only in the ordering of u's and v's. However, if this is not the case, then one can verify that the power of ζ will always be greater than $-n_L+n_R+2$, which is a hint that the entire formula stays finite in the factorization limit. Hence in terms of the fully color-dressed Parke–Taylor factor $\mathcal{C}_n$, we have

$$\mathcal{C}_n = \zeta^{-n_L+n_R+2}\,(\prod u)^2\,\mathcal{C}_{L\cup\{I_L\}}(\{u\})\,\mathcal{C}_{R\cup\{I_R\}}(\{v\})+\mathcal{O}(\zeta^{-n_L+n_R+3})\,. \tag{6.35}$$

More generally, if the flavor/color factor contains several traces and thus several smaller Parke–Taylor factors of the form (5.28) are present, say $\mathcal{C}_{\mathrm{tr}_1}\mathcal{C}_{\mathrm{tr}_2}\cdots\mathcal{C}_{\mathrm{tr}_t}$ (note that we only require $(\mathrm{tr}_1\cup\mathrm{tr}_2\cup\cdots\cup\mathrm{tr}_t)\subset\{1,2,\ldots,n\}$), there are several situations

1. For some $1\leq r\leq t$, we have $(\mathrm{tr}_1\cup\mathrm{tr}_2\cup\cdots\cup\mathrm{tr}_r)\subset L$ and $(\mathrm{tr}_{r+1}\cup\mathrm{tr}_{r+2}\cup\cdots\cup\mathrm{tr}_t)\subset R$. In this case it trivially splits and we do not see any internal labels arises

$$\begin{aligned} \mathcal{C}_{\mathrm{tr}_1}\cdots\mathcal{C}_{\mathrm{tr}_t} =& \zeta^{-|\mathrm{tr}_1|-\cdots-|\mathrm{tr}_r|+\cdots+|\mathrm{tr}_{r+1}|+|\mathrm{tr}_t|}\,(\prod_{a\in\mathrm{tr}_1\cup\cdots\cup\mathrm{tr}_r}u_a)^2 \\ &\times\mathcal{C}_{\mathrm{tr}_1}(\{u\})\cdots\mathcal{C}_{\mathrm{tr}_r}(\{u\})\,\mathcal{C}_{\mathrm{tr}_{r+1}}(\{v\})\cdots\mathcal{C}_{\mathrm{tr}_t}(\{v\}) \\ &+\mathcal{O}(\zeta^{1-|\mathrm{tr}_1|-\cdots-|\mathrm{tr}_r|+|\mathrm{tr}_{r+1}|+\cdots+|\mathrm{tr}_t|})\,, \end{aligned} \tag{6.36}$$

where || denotes the cardinality of the set.

2. For some $1 \leq r \leq t$, we have $\mathrm{tr}_r = \mathrm{tr}_r^{\mathrm{L}} \cup \mathrm{tr}_r^{\mathrm{R}}$ with $\mathrm{tr}_r^{\mathrm{L}} \cap \mathrm{tr}_r^{\mathrm{R}} = \varnothing$, and that $(\mathrm{tr}_1 \cup \cdots \cup \mathrm{tr}_{r-1} \cup \mathrm{tr}_r^{\mathrm{L}}) \subset L$ and $(\mathrm{tr}_r^{\mathrm{R}} \cup \mathrm{tr}_{r+1} \cup \cdots \cup \mathrm{tr}_t) \subset R$. In this case it factorizes into two parts, one containing r traces and the other $t-r+1$ traces, and two internal labels appear

$$\begin{aligned}\mathcal{C}_{\mathrm{tr}_1} \cdots \mathcal{C}_{\mathrm{tr}_t} =& \zeta^{2-|\mathrm{tr}_1|-\cdots-|\mathrm{tr}_{r-1}|-|\mathrm{tr}_r^{\mathrm{L}}|+|\mathrm{tr}_r^{\mathrm{R}}|+|\mathrm{tr}_{r+1}|+\cdots+|\mathrm{tr}_t|}\,(\prod_{a\in \mathrm{tr}_1\cup\cdots\cup\mathrm{tr}_{r-1}\cup\mathrm{tr}_r^{\mathrm{L}}} u_a)^2 \\ &\times \mathcal{C}_{\mathrm{tr}_1}(\{u\}) \cdots \mathcal{C}_{\mathrm{tr}_{r-1}}(\{u\})\, \mathcal{C}_{\mathrm{tr}_r^{\mathrm{L}}\cup\{I_L\}}(\{u\}) \\ &\times \mathcal{C}_{\{I_R\}\cup\mathrm{tr}_r^{\mathrm{R}}}(\{v\})\, \mathcal{C}_{\mathrm{tr}_{r+1}}(\{v\}) \cdots \mathcal{C}_{\mathrm{tr}_t}(\{v\}) \\ &+ \mathcal{O}(\zeta^{3-|\mathrm{tr}_1|-\cdots-|\mathrm{tr}_{r-1}|-|\mathrm{tr}_r^{\mathrm{L}}|+|\mathrm{tr}_r^{\mathrm{R}}|+|\mathrm{tr}_{r+1}|+\cdots+|\mathrm{tr}_t|})\,. \end{aligned} \tag{6.37}$$

3. There are two or more traces whose labels splits into two parts, one belonging to L and the other belonging to R. In this case at the leading order or $\mathcal{C}_{\mathrm{tr}_1}\mathcal{C}_{\mathrm{tr}_2}\cdots\mathcal{C}_{\mathrm{tr}_t}$ one will observe more than one pair of $\{I_L, I_R\}$, which is not sensible for tree-level factorizations. In fact, when combining this with the remaining part of the integrand (to be discussed later), one can verify that the total power of ζ in I_n is too high to produce a simple pole. Hence in this case the amplitudes at most stays finite, and we do not further discuss this.

Next we study the behavior of $\mathrm{Pf}'A_n$. In the factorization limit, the matrix A_n approximates to

$$\begin{array}{c} \begin{array}{cc} b\in L & b \in R \end{array} \\ \left(\begin{array}{c:c} -\frac{u_a u_b}{\zeta}\,\frac{k_a\cdot k_b}{u_a-u_b} & -\zeta\,\frac{k_a\cdot k_b}{v_b} \\ \hdashline \zeta\,\frac{k_a\cdot k_b}{v_a} & \zeta\,\frac{k_a\cdot k_b}{v_a-v_b} \end{array}\right) \begin{array}{c} a\in L \\ a \in R \end{array} \end{array} . \tag{6.38}$$

Without loss of generality we assume $1\in L$ and $n \in R$. In the definition of the reduced Pfaffian, it is convenient to exclude the rows and columns labeled by 1 and n. There are two situations

1. Both $|L|$ and $|R|$ are odd. In this case, dues to the fact that we use $\mathrm{Pf}'A_n = \mathrm{Pf}[A_n]_{\hat{1},\hat{n}}$, by the structure in (6.38) one immediately see that

$$\begin{aligned}\mathrm{Pf}'A_n =& \zeta^{\frac{-n_L+n_R+2}{2}}\,(\prod_{a\in L} u_a)\,\mathrm{Pf}'A_{L\cup\{I_L\}}(\{u\})\,\mathrm{Pf}'A_{R\cup\{I_R\}}(\{v\}) \\ &+ \mathcal{O}(\zeta^{\frac{-n_L+n_R+4}{2}})\,, \end{aligned} \tag{6.39}$$

where from direct computation $\mathrm{Pf}'A_{L\cup\{I_L\}} = \frac{1}{u_1}\mathrm{Pf}[A_{L\cup\{I_L\}}]_{\hat{1},\hat{I}_L}$ and $\mathrm{Pf}'A_{R\cup\{I_R\}} = \frac{1}{u_n}\mathrm{Pf}[A_{R\cup\{I_R\}}]_{\hat{n},\hat{I}_R}$.

2. Both $|L|$ and $|R|$ are even. In this case the leading piece in $\text{Pf}'A_n$ does not factorize cleanly, and its power in ζ is $\frac{-n_L+n_R+4}{2}$. Hence in most cases we would expect that the simple pole is not present. The only exception is when $\text{Pf}'A_n$ is accompanied by a $\text{Pf}X_n$, which will be discussed later.

We do not discuss in detail about $\text{Pf}X_n$ (or more generally $\text{Pf}\mathcal{X}_n$), because in all the formulas we found in Chapter 4 and Chapter 5 whenever it appears it always appears with a copy of $\text{Pf}'A_n$. Since as commented before $\text{Pf}\mathcal{X}_n\text{Pf}'A_n$ is a special case of $\mathcal{C}_{\text{tr}_1}\mathcal{C}_{\text{tr}_2}\cdot\mathcal{C}_{\text{tr}_t}\,\text{Pf}'\Pi$, we postpone this to the discussion of $\text{Pf}'\Pi$.

6.2.3 Φ^3, $U(N)$ *Non-Linear Sigma Model, and the Special Galileon*

Now we are well-equipped to study the amplitudes whose corresponding integrand can be constructed using $\mathcal{C}_n$ and $\text{Pf}'A_n$ only: which are Φ^3, NLSM, and sGal. From (6.35) and (6.39), there is never any polarization vectors produced for the internal particle, and so it has to be a scalar always. Explicitly, we see that

$$\Phi^3: \ \mathcal{C}_n\mathcal{C}_n \longrightarrow \sharp\,(\mathcal{C}_{L\cup\{I_L\}}\mathcal{C}_{L\cup\{I_L\}})\,(\mathcal{C}_{R\cup\{I_R\}}\mathcal{C}_{R\cup\{I_R\}})\,, \tag{6.40}$$

$$\text{NLSM}: \ \mathcal{C}_n(\text{Pf}'A_n)^2 \longrightarrow \sharp\,(\mathcal{C}_{L\cup\{I_L\}}(\text{Pf}'A_{L\cup\{I_L\}})^2)\,(\mathcal{C}_{R\cup\{I_R\}}(\text{Pf}'A_{R\cup\{I_R\}})^2)\,, \tag{6.41}$$

$$\text{sGal}: \ (\text{Pf}'A_n)^4 \longrightarrow \sharp\,(\text{Pf}'A_{L\cup\{I_L\}})^4\,(\text{Pf}'A_{R\cup\{I_R\}})^4\,, \tag{6.42}$$

where $\sharp = \zeta^{2(-n_L+n_R+2)}\,(\prod u)^4$ is exactly the correct factor in (6.32) that we expect in order that the formula factorizes, and the latter two are true only in odd particle channels (in even particle channels, one can explicitly check that the ζ power is higher). Hence we see these three formulas are closed by themselves.

In addition, if we focus on the double partial amplitudes $m_n[\alpha|\beta]$ in Φ^3 instead of the full amplitudes, we see that a given $m_n[\alpha|\beta]$ factorizes if and only if both α and β can be split into two parts, which consist of labels in L and R respectively.

6.2.4 *Behavior of Building Blocks with Polarization Vectors*

In this subsection we discuss the behavior of $\text{Pf}'\Psi_n$ and $\text{Pf}'\Pi$, where in particular we are going to see how an internal photon/gluon or graviton may arise from factorization. This is non-trivial because in the CHY representation there is nowhere any Feynman diagrams and for a generic amplitude it is not even sensible to talk about internal particles, not to say what particle it is.

We first have a look at the structure of the matrix Ψ_n in a generic factorization limit, which is

$$\begin{array}{cc|cc}
b\in L & b\in R & b\in L & b\in R
\end{array}$$
$$\left(\begin{array}{cc|cc}
-\frac{u_a\,u_b}{\zeta}\,\frac{k_a\cdot k_b}{u_a-u_b} & -\zeta\,\frac{k_a\cdot k_b}{v_b} & -\frac{u_a\,u_b}{\zeta}\,\frac{k_a\cdot \epsilon_b}{u_a-u_b} & -\zeta\,\frac{k_a\cdot \epsilon_b}{v_b} \\
\zeta\,\frac{k_a\cdot k_b}{v_a} & \zeta\,\frac{k_a\cdot k_b}{v_a-v_b} & \zeta\,\frac{k_a\cdot \epsilon_b}{v_a} & \zeta\,\frac{k_a\cdot \epsilon_b}{v_a-v_b} \\
\hline
-\frac{u_a\,u_b}{\zeta}\,\frac{\epsilon_a\cdot k_b}{u_a-u_b} & -\zeta\,\frac{\epsilon_a\cdot k_b}{v_b} & -\frac{u_a\,u_b}{\zeta}\,\frac{\epsilon_a\cdot \epsilon_b}{u_a-u_b} & -\zeta\,\frac{\epsilon_a\cdot \epsilon_b}{v_b} \\
\zeta\,\frac{\epsilon_a\cdot k_b}{v_a} & \zeta\,\frac{\epsilon_a\cdot k_b}{v_a-v_b} & \zeta\,\frac{\epsilon_a\cdot \epsilon_b}{v_a} & \zeta\,\frac{\epsilon_a\cdot \epsilon_b}{v_a-v_b}
\end{array}\right)
\begin{array}{c}
a\in L \\ a\in R \\ a\in L \\ a\in R
\end{array}\,. \tag{6.43}$$

The only exception to the above is the diagonal entries of the off-diagonal blocks, i.e., $(C_n)_{a,a}$. For these entries, when $a \in L$ we have

$$(C_n)_{a,a} = -\frac{u_a}{\zeta}\sum_{c\in L\setminus\{a\}}\frac{\epsilon_a\cdot k_c\,u_c}{u_a-u_c} + \mathcal{O}(\zeta) = -\frac{u_a^2}{\zeta}\sum_{c\in L\cup\{I_L\}\setminus\{a\}}\frac{\epsilon_a\cdot k_c}{u_a-u_c} + \mathcal{O}(\zeta)\,, \tag{6.44}$$

again with $u_{I_L} = 0$. And when $a \in R$ we have

$$(C_n)_{a,a} = \zeta(\sum_{c\in L}\frac{\epsilon_a\cdot k_c}{v_a} + \sum_{c\in R\setminus\{a\}}\frac{\epsilon_a\cdot k_c}{v_a-v_c}) + \mathcal{O}(\zeta^3) = \sum_{c\in R\cup\{I_R\}\setminus\{a\}}\frac{\epsilon_a\cdot k_c}{v_a-v_c} + \mathcal{O}(\zeta^3)\,, \tag{6.45}$$

with $v_{I_R} = 0$. Thus to the leading order all the entries with both labels in L (or R) are exactly the entries that should appear in $\Psi_{L\cup\{I_L\}}$ (or $\Psi_{R\cup\{I_R\}}$) for the factorized amplitude, up to a simple coefficient.

From (6.43), (6.44) and (6.45), we observe that

$$(\Psi_n)_{a,b} \sim \begin{cases} \zeta^{-1}\,, & a,b\in L\,, \\ \zeta\,, & \text{else}\,. \end{cases} \tag{6.46}$$

Recalling the definition of Pfaffian (4.5), this means that in the expansion of $\text{Pf}'\Psi_n$ in terms of perfect matchings, the more factors whose two labels are in L the more dominant the corresponding term is. In order to see the emergence of the internal particle, it is most convenient to exclude, say, $1 \in L$ and $n \in R$ in the first block of labels, i.e., $\text{Pf}'\Psi_n = -\frac{(-1)^{n+1}}{\sigma_1-\sigma_n}\,\text{Pf}[\Psi_n]_{\hat{1},\hat{n}:}$. Counting labels in both blocks together, we see there are altogether $2n_L - 1$ labels belonging to L, which is odd. This means at the leading order, all the terms are in the form

$$\underbrace{(\Psi_n)_{L,L}\cdots(\Psi_n)_{L,L}}_{n_L-1}\,(\Psi_n)_{L,R}\,\underbrace{(\Psi_n)_{R,R}\cdots(\Psi_n)_{R,R}}_{n_R-1}\,, \tag{6.47}$$

where for simplicity we use L and R to denote any label in the set. Note that the entry $(\Psi_n)_{L,R}$ always has the form of a Lorentz product $e_a \cdot e_b$. We insert a completeness relation into each of such factor

$$e_a \cdot e_b = e_a^\mu \, \eta_{\mu\nu} \, e_b^\nu = e_a^\mu \left(\sum_{\epsilon_I} \epsilon_{I_L,\mu} \, \epsilon_{I_R,\nu} + \frac{k_{I,\mu} \, k_{I,\nu}}{k_I^2} \right) e_b^\nu \Rightarrow \sum_{\epsilon_{I_L}} e_a \cdot \epsilon_{I_R} \, \epsilon_I \cdot e_b \, , \tag{6.48}$$

where $\epsilon_I = \epsilon_{I_L} = \epsilon_{I_R}$. The last step above is not an identity by itself, but as long as with this substitution we are able to confirm in the end that the original formula factorizes and each part from the factorization corresponds to an on-shell amplitude, then the $k_{I,\mu} k_{I,\nu}$ term merely projects out.

Clearly the substitution (6.48) fully disentangle the labels in L from the labels in R. After summing over all perfect matchings with the pattern (6.47), we observe that the leading terms factorize into two reduced Pfaffians again

$$\text{Pf}'\Psi_n = \zeta^{-n_L+n_R+2} \, (\prod_{a\in L} u_a)^2 \sum_{\epsilon_I} \text{Pf}'\Psi_{L\cup\{I_L\}}(\{u\}) \, \text{Pf}'\Psi_{R\cup\{I_R\}}(\{v\}) + \mathcal{O}(\zeta^{-n_L+n_R+3}) \, , \tag{6.49}$$

where $\text{Pf}'\Psi_{L\cup\{I_L\}} = \frac{1}{u_1}\text{Pf}[\Psi_{L\cup\{I_L\}}]_{\hat{1},\hat{I}_L:}$ and $\text{Pf}'\Psi_{R\cup\{I_R\}} = \frac{1}{v_n}\text{Pf}[\Psi_{R\cup\{I_R\}}]_{\hat{n},\hat{I}_R:}$, as one directly observe from the computation. Details about this re-summation is explained in [61].

It is helpful to comment on the structure $e_a \cdot e_b$ that enters (6.48) a bit more. In most cases e^μ can be proportional to either k^μ or ϵ^μ, but it can also a summation, e.g., $(C_n)_{a,a} \approx \zeta\epsilon_a \cdot (\sum_{b\in L\cup\{I_L\}\setminus\{a\}} k_b)$ when $a \in L$. Note that this fact continues to be true in a generic matrix Π (5.35). Recalling the squeezing procedure with which we obtain Π from Ψ_n, we should expect that $\text{Pf}'\Pi$ factorizes in the same way, as long as the factorization channel merely separates the labels that explicitly enter Π.

Behavior of $\text{Pf}'\Pi$

Since generically $\text{Pf}'\Pi$ corresponds to amplitudes with multi-trace, and is always accompanied by $\mathcal{C}_{\text{tr}_1}\mathcal{C}_{\text{tr}_2}\cdots\mathcal{C}_{\text{tr}_t}$, there are also three situations here.

1. For some $1 \leq r \leq t$, we have $(\text{tr}_1 \cup \text{tr}_2 \cup \cdots \cup \text{tr}_r) \subset L$ and $(\text{tr}_{r+1} \cup \text{tr}_{r+2} \cup \cdots \cup \text{tr}_t) \subset R$. In this case, the labels that enter the matrix Π, i.e., $\{1, 2, \ldots, r, \underline{1}, \underline{2}, \ldots, \underline{t}\}$, are simply separated into two groups, those belongs to L and those to R. With a slight abuse of notation we say $\underline{i} \in L$ iff $\text{tr}_i \subset L$, and similarly for R. By the observation at the end of the previous subsubsection, it is easy to see that the

leading piece of $\text{Pf}'\Pi$ factorizes in a way similar to that of $\text{Pf}'\Psi_n$. In fact, it is nice to combine the Parke–Taylor factors and $\text{Pf}'\Pi$ together, and a detailed derivation leads to

$$\begin{aligned}\mathcal{C}_{\text{tr}_1}\cdots\mathcal{C}_{\text{tr}_t}\,\text{Pf}'\Pi =&\zeta^{-n_L+n_R+2}\,(\prod_{a\in L}u_a)^2\sum_{\epsilon_I}\text{Pf}'\Psi_{L\cup\{I_L\}}(\{u\})\\&\times\mathcal{C}_{\text{tr}_1}\cdots\mathcal{C}_{\text{tr}_r}\,\text{Pf}'\Pi(\text{h}_L\cup\{I_L\};\text{tr}_1,\ldots,\text{tr}_r)\\&\times\mathcal{C}_{\text{tr}_{r+1}}\cdots\mathcal{C}_{\text{tr}_t}\,\text{Pf}'\Pi(\text{h}_R\cup\{I_R\};\text{tr}_{r+1},\ldots,\text{tr}_t)\\&+\mathcal{O}(\zeta^{-n_L+n_R+3})\,,\end{aligned}\tag{6.50}$$

where $\text{h}_L\subset L$, $\text{h}_R\subset R$ and $\text{h}=\text{h}_L\cup\text{h}_R$. So in this case the factorization produces a polarization vector for the internal particle.

2. For some $1\leq r\leq t$, we have $\text{tr}_r=\text{tr}_r^{\text{L}}\cup\text{tr}_r^{\text{R}}$ with $\text{tr}_r^{\text{L}}\cap\text{tr}_r^{\text{R}}=\varnothing$, and that $(\text{tr}_1\cup\cdots\cup\text{tr}_{r-1}\cup\text{tr}_r^{\text{L}})\subset L$ and $(\text{tr}_r^{\text{R}}\cup\text{tr}_{r+1}\cup\cdots\cup\text{tr}_t)\subset R$. In this case, it is easier to work with the representation of $\text{Pf}'\Pi$ in terms of the expansion $\sum_{\{a,b\}}{}'\mathcal{P}_{\{a,b\}}$ (5.29). Recall that in this expansion every term involves a Pfaffian instead of a reduced Pfaffian, and the Pfaffian is for a minor of the matrix Ψ with all the graviton labels and exactly two labels from each trace, except for one trace that we choose to exclude. Since we are able to exclude any of the traces, a particularly nice choice is to exclude $\underline{r}$, then with the experience in the behavior of Ψ_n we know that the Pfaffians entering the summation (5.29) cleanly factorize. Again let us combine $\text{Pf}'\Pi$ with the Parke–Taylor factors, and we obtain

$$\begin{aligned}\mathcal{C}_{\text{tr}_1}\cdots\mathcal{C}_{\text{tr}_t}\,\text{Pf}'\Pi =&\zeta^{-n_L+n_R+2}\,(\prod_{a\in L}u_a)^2\,\text{Pf}'\Psi_{L\cup\{I_L\}}(\{u\})\\&\times\mathcal{C}_{\text{tr}_1}\cdots\mathcal{C}_{\text{tr}_{r-1}}\,\mathcal{C}_{\text{tr}_r^{\text{L}}\cup\{I_L\}}\,\text{Pf}'\Pi(\text{h}_L;\text{tr}_1,\ldots,\text{tr}_{r-1},\text{tr}_r^{\text{L}}\cup\{I_R\})\\&\times\mathcal{C}_{\text{tr}_r^{\text{R}}\cup\{I_R\}}\mathcal{C}_{\text{tr}_{r+1}}\cdots\mathcal{C}_{\text{tr}_t}\,\text{Pf}'\Pi(\text{h}_R;\text{tr}_r^{\text{R}}\cup\{I_R\},\text{tr}_{r+1},\ldots,\text{tr}_t)\\&+\mathcal{O}(\zeta^{-n_L+n_R+3})\,.\end{aligned}\tag{6.51}$$

So the factorization of Π in this case does not produce any polarization vector for the internal particle.

3. There are two or more traces whose labels splits into two parts, one belonging to L and the other belonging to R. As mentioned before, by counting the leading power in ζ we know there cannot be any simple pole and so this case is excluded.

Some emphasis needs to be stressed on the combination $\text{Pf}[\mathcal{X}_n]_S\text{Pf}'[\Psi_n]_{:\hat{S}}$ for some $S\in\{1,2,\ldots,n\}$. As discussed before, if we expand $\text{Pf}[\mathcal{X}_n]_S$ in terms of perfect matchings, than each term in the expansion of $\text{Pf}[\mathcal{X}_n]_S\text{Pf}'[\Psi_n]_{:\hat{S}}$ can be regarded as a special case of $\mathcal{C}_{\text{tr}_1}\mathcal{C}_{\text{tr}_2}\cdots\mathcal{C}_{\text{tr}_t}\text{Pf}'\Pi(\text{h};\text{tr}_1,\text{tr}_2,\ldots,\text{tr}_t)$, where $|tr_i|=2\ \forall i$. Suppose a generic factorization channel splits the set S into $S^{\mathrm{L}} \in L$ and $S^{\mathrm{R}} \in R$, and also its complement into $\bar{S}^{\mathrm{L}}$ and $\bar{S}^{\mathrm{R}} \in R$. In analogy with the above discussion, this combination can factorize in two ways

1. If $|S^{\mathrm{L}}|$ is even (and so is $|S^{\mathrm{R}}|$), then we have

$$\begin{aligned}\mathrm{Pf}[\mathcal{X}_n]_S\mathrm{Pf}'[\Psi_n]_{:\hat{S}} =& \sum_{\epsilon_I} \zeta^{-n_L+n_R+2}\,(\prod_{a\in L} u_a)^2 \\ &\times \mathrm{Pf}[\mathcal{X}_{L\cup\{I_L\}}]_{S^{\mathrm{L}}}\mathrm{Pf}'[\Psi_{L\cup\{I_L\}}]_{:\hat{S}^{\mathrm{L}}}\,\mathrm{Pf}[\mathcal{X}_{R\cup\{I_R\}}]_{S^{\mathrm{R}}}\mathrm{Pf}'[\Psi_{R\cup\{I_R\}}]_{:\hat{S}^{\mathrm{R}}} \\ &+ \mathcal{O}(\zeta^{-n_L+n_R+3})\,,\end{aligned} \tag{6.52}$$

   which produces a polarization vector for the internal particle.

2. If $|S^{\mathrm{L}}|$ is edd (and so is $|S^{\mathrm{R}}|$), then we have

$$\begin{aligned}\mathrm{Pf}[\mathcal{X}_n]_S\mathrm{Pf}'[\Psi_n]_{:\hat{S}} =& \zeta^{-n_L+n_R+2}\,(\prod_{a\in L} u_a)^2 \\ &\times \mathrm{Pf}[\mathcal{X}_{L\cup\{I_L\}}]_{S^{\mathrm{L}}\cup\{I_L\}}\mathrm{Pf}'[\Psi_{L\cup\{I_L\}}]_{L\cup\{I_L\}:\bar{S}^{\mathrm{L}}} \\ &\times \mathrm{Pf}[\mathcal{X}_{R\cup\{I_R\}}]_{S^{\mathrm{R}}\cup\{I_R\}}\mathrm{Pf}'[\Psi_{R\cup\{I_R\}}]_{R\cup\{I_R\}:\bar{S}^{\mathrm{R}}} \\ &+ \mathcal{O}(\zeta^{-n_L+n_R+3})\,,\end{aligned} \tag{6.53}$$

   which does not yields any new polarization vector.

6.2.5 *Theories with Photons/Gluons and Gravitons*

In Section 4.2 we found several more formulas for scalar amplitudes whose factorizations have not been checked so far. These are (here we use the integrand for the full amplitude)

$$\begin{aligned} I_n^{\text{YMS:scalar}} &= \mathcal{C}_n\,\mathrm{Pf}\mathcal{X}_n\,\mathrm{Pf}'A_n\,, \quad I_n^{\text{EMS:scalar}} = (\mathrm{Pf}\mathcal{X}_n\,\mathrm{Pf}'A_n)^2\,, \\ I_n^{\text{DBI:scalar}} &= \mathrm{Pf}\mathcal{X}_n\,(\mathrm{Pf}'A_n)^3\,. \end{aligned} \tag{6.54}$$

We commented in Section 4.2 that the former two integrands are not closed. From the discussion at the end of the previous subsection this is obvious, since the combination $\mathrm{Pf}\mathcal{X}_n\mathrm{Pf}'A_n$ is a special case of $\mathrm{Pf}[\mathcal{X}]_S\mathrm{Pf}'[\Psi_n]_{:\hat{S}}$ (when $S = \{1,2,\ldots,n\}$), and in any even particle channel this part of scalars amplitudes in either YMS or EMS thus have to factorize according to (6.52), which produces a new polarization vector to be associated to the internal particles. Because $I_n^{\text{YMS:scalar}}$ contains one copy of $\mathrm{Pf}\mathcal{X}_n\mathrm{Pf}'A_n$, this internal particle has to be a gluon (photon is ruled out due to the presence of $\mathcal{C}_n$). And because $I_n^{\text{EMS:scalar}}$ is exactly two copies of this combination, the internal particle has to sum over the graviton states, the B-field states, and the dilaton state.

In comparison, the factorizations in even particle channels are excluded for the scalar amplitudes in DBI, due to the presence of extra two powers of $\mathrm{Pf}' A_n$. Recall that in such a channel $\mathrm{Pf}' A_n$ by itself scales as $\zeta^{\frac{-n_L+n_R+4}{2}}$, so that $I_n^{\text{DBI:scalar}}$ acquires too high a power to eliminate the simple pole. Hence these amplitudes can only factorize in an odd particle channel, and according to (6.39) and (6.53)

$$\begin{aligned}\mathrm{Pf}\mathcal{X}_n\,(\mathrm{Pf}' A_n)^3 =& \zeta^{2(-n_L+n_R+2)}\,(\textstyle\prod u)^4\\ &\times \mathrm{Pf}\mathcal{X}_{L\cup\{I_L\}}\,(\mathrm{Pf}' A_{L\cup\{I_L\}})^3\,\mathrm{Pf}\mathcal{X}_{L\cup\{I_R\}}\,(\mathrm{Pf}' A_{L\cup\{I_R\}})^3\\ &+\mathcal{O}(\zeta^{2(-n_L+n_R+3)})\,,\end{aligned} \tag{6.55}$$

verifying that $I_n^{\text{DBI:scalar}}$ is closed.

For amplitudes in GR, YM and BI, since the corresponding integrands are constructed from $\mathcal{C}_n$, $\mathrm{Pf}'\psi_n$ and $\mathrm{Pf}' A_n$, there is only a single type of facotrization channel, and by (6.34), (6.39) and (6.49), it is easy to verify that these factorizes correctly and are closed. The only thing special about BI is that the even particle channel is forbidden in this theory, due to the same reason as that for the scalar amplitudes in DBI as discussed above.

For amplitudes in EM, YMS and DBI, the corresponding integrands share the common factor $\mathrm{Pf}[\mathcal{X}_n]_S \mathrm{Pf}'[\Psi_n]_{:\hat{S}}$, and so by (6.52) and (6.53) there are two types of factorization channels according to whether the channel splits any trace structure, and the internal particles observed in these channels are different. As mentioned in the previous subsector these amplitudes can be regarded as special cases of amplitudes in EYM, gen. YMS and ext. DBI, respectively. By (6.50) and (6.51) the factorization pattern in these theories follows similarly. We do not repeat to write out the explicit expressions for their behaviors in a generic factorization channel here, but merely provide an example in YMS that is very neat. Recall the double-trace gluon amplitudes in EYM shown in (5.44)

$$I_n^{\text{EYM:double-trace g}} = \frac{1}{2}\,\mathcal{C}_{\mathrm{tr}_1}\,\mathcal{C}_{\mathrm{tr}_2}\,s_{\mathrm{tr}_1}\,\mathrm{Pf}'\Psi_n\,. \tag{6.56}$$

Let us explore the channel defined by $s_{\mathrm{tr}_1} = s_{\mathrm{tr}_2} \to 0$. In this channel it is very easy to see how $\mathcal{C}_{\mathrm{tr}_1}$, $\mathcal{C}_{\mathrm{tr}_2}$ and $\mathrm{Pf}'\Psi_n$ behave. For the remaining Mandeltam variable, while it vanishes, we need to know its leading behavior. Recalling the constraint (2.34), to the leading order we have

$$\frac{1}{2}\,s_{\mathrm{tr}_1}^2 \approx \zeta^2 \sum_{\substack{a\in\mathrm{tr}_1\\ b\in\mathrm{tr}_2}} \frac{k_a\cdot k_b}{u_a\,v_b} = \zeta^2 \sum_{\epsilon_I} (C_{\mathrm{tr}_1\cup\{I_L\}})_{I_L,I_L}\,(C_{\mathrm{tr}_2\cup\{I_R\}})_{I_R,I_R}\,, \tag{6.57}$$

and so altogether

$$I_n^{\text{EYM:double-trace g}} \approx \zeta^{2(-n_L+n_R+2)} \left(\prod u\right)^4 \mathcal{C}_{\text{tr}_1} \left(C_{\text{tr}_1 \cup \{I_L\}}\right)_{I_L, I_L} \text{Pf}'\Psi_{\text{tr}_1 \cup \{I_L\}} \\ \times \mathcal{C}_{\text{tr}_2} \left(C_{\text{tr}_2 \cup \{I_R\}}\right)_{I_R, I_R} \text{Pf}'\Psi_{\text{tr}_2 \cup \{I_R\}} \,. \tag{6.58}$$

Given the fact that

$$(C_{\text{tr}_1 \cup \{I_L\}})_{I_L, I_L} = \text{Pf}'\Pi(\{I_L\}; \text{tr}_1)\,, \quad (C_{\text{tr}_2 \cup \{I_R\}})_{I_R, I_R} = \text{Pf}'\Pi(\{I_R\}; \text{tr}_2)\,, \tag{6.59}$$

one immediately recognize that this formula correctly factorizes into two, each of which computes a single-trace amplitudes of gluons coupled to one graviton (the internal particle) in EYM, as is expected by Feynman diagrams.

6.3 SOFT THEOREMS II

In this section we present two types of double soft scalar theorems, each of which captures the behavior of amplitudes in a class of theories in this limit, respectively [13]. Double soft limits of scalars are of interests in the study of symmetries in the underlying theory, e.g., the moduli space of vacua and the $E_{7(7)}$ structure in $\mathcal{N} = 8$ supergravity [62]. As before, we stick to the convention in Section 2.4.1 and assume $k_{n-1}^{\mu} = \tau\, p^{\mu}$ and $k_n^{\mu} = \tau\, q^{\mu}$ to be soft ($\tau \to 0$).

The first class consists of theories with neither color nor flavor structure, which includes sGal, DBI, and EMS. For theories in the first class, any n-point amplitude behaves as

$$\mathcal{M}_n = (k_{n-1} \cdot k_n)^m \left(S^{(0)} + S^{(1)} + S^{(2)}\right) \mathcal{M}_{n-2} + \mathcal{O}(\tau^{2m+4})\,, \tag{6.60}$$

where $m = 1, 0, -1$ for sGal, DBI and EMS respectively, and

$$S^{(0)} = \frac{1}{4} \sum_{a=1}^{n} \left(\frac{(k_a \cdot (k_{n-1} - k_n))^2}{k_a \cdot (k_{n-1} + k_n) + k_{n-1} \cdot k_n} + k_a \cdot (k_{n-1} + k_n) + k_{n-1} \cdot k_n \right), \tag{6.61}$$

$$S^{(1)} = -i\, \frac{1}{2} \sum_{a=1}^{n} \frac{k_a \cdot (k_{n-1} - k_n)}{k_a \cdot (k_{n-1} + k_n) + k_{n-1} \cdot k_n} (k_{n-1,\mu} k_{n,\nu} J_a^{\mu\nu})\,, \tag{6.62}$$

$$S^{(2)} = -\frac{1}{2} \sum_{a=1}^{n} \frac{1}{k_a \cdot (k_{n-1} + k_n) + k_{n-1} \cdot k_n} \left((k_{n-1,\mu} k_{n,\nu} J_a^{\mu\nu})^2 + \frac{4m-3}{2} (k_{n-1} \cdot k_n)^2 \right). \tag{6.63}$$

Here $S^{(0)}$ is a multiplicative operator which has an expansion in τ starting at $\mathcal{O}(\tau^1)$, $S^{(1)}$ is a first-order differential operator starting at $\mathcal{O}(\tau^2)$, and $S^{(2)}$ is a second order differential operator starting at $\mathcal{O}(\tau^3)$.

Two comments are in order at this point. The first is that in (6.60) the kinematic invariant $(k_{n-1} \cdot k_n)$ plays the role of a natural "dimensionful parameter" needed to link amplitudes with different number of particles. This simple dimensional argument leads to universal formulas for $S^{(0)}$ and $S^{(1)}$, i.e., they are theory independent. The second is that the only dependence on the theory under consideration appears in the multiplicative piece of $S^{(2)}$.

The second contains theories with a $U(N)$ flavor/color group for the scalars, which include NLSM and YMS. The double soft scalar emission for a color-ordered partial amplitude is

$$\mathcal{M}_n[1,2,\ldots,n] = (k_{n-1} \cdot k_n)^m \left(S^{(0)} + S^{(1)}\right) \mathcal{M}_{n-2}[1,2,\ldots,n-2] + \mathcal{O}(\tau^{2m+2}), \quad (6.64)$$

with $m = 0, -1$ for NLSM and YMS respectively, and

$$S^{(0)} = \frac{1}{2}\left(\frac{k_{n-2} \cdot (k_{n-1} - k_n) + k_{n-1} \cdot k_n}{k_{n-2} \cdot (k_{n-1} + k_n) + k_{n-1} \cdot k_n} + \frac{k_1 \cdot (k_n - k_{n-1}) + k_n \cdot k_{n-1}}{k_1 \cdot (k_n + k_{n-1}) + k_n \cdot k_{n-1}}\right), \quad (6.65)$$

$$S^{(1)} = -i\left(\frac{k_{n-1,\mu} k_{n,\nu}}{k_{n-2} \cdot (k_{n-1} + k_n) + k_{n-1} \cdot k_n} J^{\mu\nu}_{n-2} + \frac{k_{n,\mu} k_{n-1,\nu}}{k_1 \cdot (k_n + k_{n-1}) + k_n \cdot k_{n-1}} J^{\mu\nu}_1\right). \quad (6.66)$$

In this formula $S^{(0)}$ starts at $\mathcal{O}(\tau^0)$ while $S^{(1)}$ starts at order $\mathcal{O}(\tau)$.

In the following we summarize the proof for these scalar soft theorems to the subleading order. Emphases are stressed on the origins of different parts in the soft operators.

6.3.1 *Scaling at the Leading Order*

Again we first investigate the leading scaling in τ of different formulas in the double soft limit. As a result of the discussions in Section 2.4.1, in this case there exist both degenerate solutions and non-degenerate ones, depending on whether the two punctures for the soft particles pinch together. It is helpful to count the leading scaling of the contributions from both types of solutions respectively, because they *may not* both contribute to the leading terms in the limit.

As mentioned before it is nice to re-define

$$\sigma_{n-1} = \rho - \frac{\xi}{2}, \quad \sigma_n = \rho + \frac{\xi}{2}. \quad (6.67)$$

Let us delete neither $d\sigma_{n-1}d\sigma_n$ nor $\delta(f^{(n)}_{n-1})\delta(f^{(n)}_n)$ in the measure $d\mu_n$, and do the transformation (we do not care a possible over sign)

$$d\sigma_{n-1}\, d\sigma_n\, \delta(f^{(n)}_{n-1})\, \delta(f^{(n)}_n) = -d\rho\, d\xi\, 2\, \delta(f^{(n)}_{n-1} + f^{(n)}_n)\, \delta(f^{(n)}_{n-1} - f^{(n)}_n). \tag{6.68}$$

Clearly, the remaining part of $d\mu_n$ merely reduce to the lower-point one $d\mu_{n-2}$ in the limit $\tau \to 0$. Recall that in the non-degenerate solutions $\xi \sim \tau^0$ while in the degenerate ones $\xi \sim \tau^1$. With the form on the RHS of (6.68), it is straightforward to see that for the non-degenerate solutions $d\mu_n \sim \tau^{-2}$ and for degenerate ones $d\mu_n \sim \tau^{-1}$.

We then go on to check the scaling behavior of the various building blocks:

1. With the new variables the Parke–Taylor factor (considering the two soft particles to be adjacent) is

$$\frac{1}{(\sigma_1 - \sigma_2)\cdots(\sigma_{n-2} - \rho + \frac{\xi}{2})(-\xi)(\rho + \frac{\xi}{2} - \sigma_1)}\,. \tag{6.69}$$

   Obviously it remains finite on non-degenerate solutions but diverges as τ^{-1} on degenerate ones. More generally, we may encounter multi-cycles of Parke-Taylor factors, where each is of size equal or bigger than two. Inspired by the above structure, the general case always stays finite on non-degenerate solutions, while on degenerate solutions there are several situations: (i) when a cycle of size greater than two contains $\sigma_{n-1,n}$, it scales as τ^{-1}; (ii) when still in this situation but the cycle is of size two, it scales as τ^{-2}; (iii) in all other situations it stays finite as well. Analogously, we always have $\text{Pf}\mathcal{X} \sim \tau^{-1}$ on degenerate solutions.

2. We then look at the building block $\text{Pf}'A_n$. In terms of the new variables matrix A_n has the form

$$A_n = \left(\begin{array}{c|cc} A_{n-2} & \vdots & \vdots \\ \hline \cdots & 0 & \frac{\tau^2\, p\cdot q}{-\xi} \\ \cdots & \frac{\tau^2\, p\cdot q}{\xi} & 0 \end{array}\right). \tag{6.70}$$

   All entries whose explicit expressions are suppressed always scales as τ, and so on non-degenerate solutions every term in the expansion of $\text{Pf}'A_n$ scales as τ^2. However, due to the lower right block, on degenerate solutions the leading scaling reduces to τ^1.

3. Finally we inspect $\text{Pf}'\Psi$. To see the structure clearly, let us rearrange the rows and columns of matrix Ψ so that the last four rows/columns are labeled by

$\{n-1, n : n-1, n\}$. Then it suffices to focus only on the entries explicitly shown in the following

$$\Psi_n = \left(\begin{array}{c:cc|cc} \Psi_{n-2} & \mathcal{O}(\tau) & \mathcal{O}(\tau) & \mathcal{O}(\tau^0) & \mathcal{O}(\tau^0) \\ \hdashline & 0 & \tau^2 \frac{p\cdot q}{-\xi} & -(C_n^{\mathrm{T}})_{n-1,n-1} & \tau \frac{p\cdot\epsilon_n}{-\xi} \\ & & 0 & \tau \frac{q\cdot\epsilon_{n-1}}{\xi} & -(C_n^{\mathrm{T}})_{n,n} \\ \hline & & & 0 & \frac{\epsilon_{n-1}\cdot\epsilon_n}{-\xi} \\ & & & & 0 \end{array}\right). \quad (6.71)$$

Note that both $(C_n)_{n-1,n-1}$ and $(C_n)_{n,n}$ remain finite on all solutions. Since $\mathrm{Pf}'\Psi_n$ contains a term $\mathrm{Pf}'\Psi_{n-2}(C_n)_{n-1,n-1}(C_n)_{n,n}$ and it is easy to verify that no term can ever diverge, we conclude that $\mathrm{Pf}'\Psi_n \sim \tau^0$ on all solutions.

We summarize the results so far in Table 6 (Here "adj." indicates that the two soft particles are adjacent in the trace, while "non-adj." means they are non-adjacent. When non-trivial flavor structure is present, we assume the two soft particles share the same flavor index). Combining these results, we can find the total behavior of formulas for different amplitudes, which we list out in Table 7.

Table 6: Scaling of Building Blocks in Double Soft Limits

building block	non-deg. soln.	deg. soln.
$d\mu_n$	τ^{-2}	τ^{-1}
$C_n[\alpha]$ (adj.)	τ^0	τ^{-1}
$C_n[\alpha]$ (non-adj.)	τ^0	τ^0
$\mathrm{Pf}\mathcal{X}_n$	τ^0	τ^{-1}
$\mathrm{Pf}'A_n$	τ^2	τ^1
$\mathrm{Pf}'\Psi_n$	τ^0	τ^0

6.3.2 *Proof of Scalar Double Soft Theorems $S^{(0)}$*

In this subsection we provide a proof for the double-soft theorem to the leading order. Clearly, we need to get rid of the ρ and ξ integrations in order to land on the lower-point amplitude. Starting from the formula after the transformation (6.68), the general idea is to localize the ξ integration using $\delta(f_{n-1}^{(n)} - f_n^{(n)})$, and regard the

Table 7: Scaling of Formulas in Double Soft Limits

Amplitude	Integrand	non-deg. soln.	deg. soln.
YM (adj.)	$C_n[\alpha]\,\mathrm{Pf}'\Psi_n$	τ^{-2}	τ^{-2}
YM (non-adj.)	$C_n[\alpha]\,\mathrm{Pf}'\Psi_n$	τ^{-2}	τ^{-1}
GR	$\det'\Psi_n$	τ^{-2}	τ^{-1}
YMS: ϕ (adj.)	$C_n[\alpha]\,\mathrm{Pf}\mathcal{X}_n\,\mathrm{Pf}'A_n$	τ^{0}	τ^{-2}
YMS: ϕ (non-adj.)	$C_n[\alpha]\,\mathrm{Pf}\mathcal{X}_n\,\mathrm{Pf}'A_n$	τ^{0}	τ^{-1}
BI	$\mathrm{Pf}'\Psi_n\,(\mathrm{Pf}'A_n)^2$	τ^{2}	τ^{1}
DBI: ϕ	$\mathrm{Pf}\mathcal{X}_n\,(\mathrm{Pf}'A_n)^3$	τ^{4}	τ^{1}
EMS: ϕ	$(\mathrm{Pf}\mathcal{X}_n)^2\,(\mathrm{Pf}'A_n)^2$	τ^{2}	τ^{-1}
NLSM (adj.)	$C_n[\alpha]\,(\mathrm{Pf}'A_n)^2$	τ^{2}	τ^{0}
NLSM (non-adj.)	$C_n[\alpha]\,(\mathrm{Pf}'A_n)^2$	τ^{2}	τ^{1}
sGal	$(\mathrm{Pf}'A_n)^4$	τ^{6}	τ^{3}

ρ integration as a contour integration whose contour wraps the zeros of $f^{(n)}_{n-1}+f^{(n)}_n$. This leads to

$$M_n=\oint\frac{d\rho}{2\pi i}\sum_{\xi\text{ solutions}}\int d\mu'_n\,\frac{1}{f^{(n)}_{n-1}+f^{(n)}_n}\,\frac{-2}{\frac{\partial}{\partial\xi}\left[f^{(n)}_{n-1}-f^{(n)}_n\right]}\,I_N(k,\sigma,\rho,\xi)\,,\tag{6.72}$$

where ξ is understood to be evaluated on its solutions to the equation $f^{(n)}_{n-1}-f^{(n)}_n=0$. We also use the notation $d\mu'_n$ to denote the remaining part of $d\mu_n$.

As a result from the previous subsection, we only need to consider the degenerate solutions, in which $\xi\sim\tau^1$. So we can perturbatively expand it as $\xi=\tau\,\xi_1+\mathcal{O}(\tau^2)$, and from (2.29) we know at the leading order it is uniquely fixed to be $\xi_1^{-1}=\frac{1}{2p\cdot q}\sum_{a=1}^{n-2}\frac{k_a\cdot(p-q)}{\rho-\sigma_a}$. Then (6.72) can be expanded to

$$M_n=-\int d\mu_{n-2}\oint\frac{d\rho}{2\pi i}\,\frac{1}{\sum_{a=1}^{n-2}\frac{k_a\cdot(p+q)}{\rho-\sigma_a}}\,\frac{\xi_1^2}{\tau\,p\cdot q}\,I_n+(\text{sub-leading})\,,\tag{6.73}$$

where the ρ-contour is now specified by $\sum_{a=1}^{n-2}\frac{k_a\cdot(p+q)}{\rho-\sigma_a}=0$.

To derive the leading-order soft theorems from (6.73), we need to expand I_n with respect to τ. From previous discussions it is easy to see the relevant building blocks behave as

$$C_n[1,2,\ldots,n] = C_{n-2}[1,2,\ldots,n-2]\,\frac{\sigma_{n-2}-\sigma_1}{(\sigma_{n-2}-\rho)\,(-\tau\,\xi_1)\,(\rho-\sigma_1)}+\mathcal{O}(\tau^0)\,, \quad (6.74)$$

$$\mathrm{Pf}X_n = -\frac{1}{\tau\,\xi_1}\,\mathrm{Pf}X_{n-2}+\mathcal{O}(\tau^0)\,, \quad (6.75)$$

$$\mathrm{Pf}'A_n = -\frac{\tau\,p\cdot q}{\xi_1}\,\mathrm{Pf}'A_{n-2}+\mathcal{O}(\tau^2)\,. \quad (6.76)$$

Combining these, for scalar amplitudes in sGal, DBI and EMS (with $m = 1, 0, -1$) we obtain

$$\begin{aligned} M_n &= -\int d\mu_{n-2}\,I_{n-2}\oint\frac{d\rho}{2\pi i}\,\frac{1}{\sum_{a=1}^{n-2}\frac{k_a\cdot(p+q)}{\rho-\sigma_a}}\,\frac{\xi_1^2}{\tau\,p\cdot q}\left(\frac{1}{\tau\,\xi_1}\right)^{1-m}\left(\frac{\tau\,p\cdot q}{\xi_1}\right)^{m+3}+\mathcal{O}(\tau^{2m+2}) \\ &= -\int d\mu_{n-2}\,I_{n-2}\,\frac{\tau\,(\tau^2\,p\cdot q)^m}{4}\oint\frac{d\rho}{2\pi i}\,\frac{\left(\sum_{a=1}^{n-2}\frac{k_a\cdot(p-q)}{\rho-\sigma_a}\right)^2}{\sum_{a=1}^{n-2}\frac{k_a\cdot(p+q)}{\rho-\sigma_a}}+\mathcal{O}(\tau^{2m+2})\,. \end{aligned} \quad (6.77)$$

Now we perform the ρ-integral by deforming the contour and use a residue theorem. Although there appears to be a simple pole at $\rho=\infty$, it is eliminated by an additional zero in the numerator due to momentum conservation. Thus we only encounter simple poles at $\rho=\sigma_a$ ($a=1,\ldots,n$), and the final result is

$$M_n = (\tau^2\,p\cdot q)^m\left(\frac{\tau}{4}\sum_{a=1}^{n-2}\frac{(k_a\cdot(p-q))^2}{k_a\cdot(p+q)}\right)M_{n-2}+\mathcal{O}(\tau^{2m+2})\,. \quad (6.78)$$

Note that the soft operator in the bracket agrees with $S^{(0)}$ in (6.61) at $\mathcal{O}(\tau)$, because the additional piece in $S^{(0)}$ is of higher order by momentum conservation, $\sum_{a=1}^{n-2} k_a\cdot(p+q) = -2\tau(p.q)^2$. This concludes our proof for the leading order soft theorem in these theories.

Similarly, for scalar partial amplitudes in NLSM and YMS (with $m=0,-1$) we obtain

$$\begin{aligned} M_n[1,2,\ldots,n] = &-\int d\mu_{n-2}\,I_{n-2}[1,2,\ldots,n-2]\oint\frac{d\rho}{2\pi i}\,\frac{\xi_1^2/p\cdot q}{\tau\sum_{a=1}^{n-2}\frac{k_a\cdot(p+q)}{\rho-\sigma_a}} \\ &\times\frac{\sigma_{n-2}-\sigma_1}{(\sigma_{n-2}-\rho)(-\tau\,\xi_1)(\rho-\sigma_1)}\left(\frac{1}{\tau\xi_1}\right)^{-m}\left(\frac{\tau\,p\cdot q}{\xi_1}\right)^{2+m}+\mathcal{O}(\tau^{2m+1}) \\ = &\int d\mu_{n-2}\,I_{n-2}[1,2,\ldots,n-2] \\ &\times\frac{(\tau^2 p\cdot q)^m}{2}\oint\frac{d\rho}{2\pi i}\,\frac{\sum_{a=1}^{n-2}\frac{k_a\cdot(p-q)}{\rho-\sigma_a}}{\sum_{a=1}^{n-2}\frac{k_a\cdot(p+q)}{\rho-\sigma_a}}\,\frac{\sigma_{n-2}-\sigma_1}{(\sigma_{n-2}-\rho)\,(\rho-\sigma_1)}+\mathcal{O}(\tau^{2m+1})\,, \end{aligned}$$

(6.79)

Obviously there is no pole at $\rho = \infty$, and we pick up two poles at $\rho = \sigma_1$ and $\rho = \sigma_n$, which then leads to

$$\begin{aligned} M_n[1,2,\ldots,n] =& \frac{(\tau^2 p\cdot q)^m}{2}\left(\frac{k_{n-2}\cdot(p-q)}{k_{n-2}\cdot(p+q)}+\frac{k_1\cdot(q-p)}{k_1\cdot(q+p)}\right)M_{n-2}[1,2,\ldots,n-2] \\ &+\mathcal{O}(\tau^{2m+1})\,. \end{aligned} \tag{6.80}$$

The derivation of the double soft theorem at the sub-leading order is more involved, and we refer interested readers to [63] for details.

6.3.3 *More General Cases*

So far all what we have studied are scalar amplitudes only, even when the theory may involve other types of particles. In this section, we extend the discussion in two directions. Firstly we are going to show that the scalar double soft theorem (6.60) is still valid in DBI when external photons are present (the only modification, as usual, is to regard $J_a^{\mu\nu}$ as the full angular momentum operator, i.e., including both the orbital and the spin part). And secondly, we are going to show that there is also a similar leading-order theorem for simultaneous emission of two soft photons in photon amplitudes in Born–Infeld (BI) and Einstein–Maxwell (EM).

DBI Amplitudes with Mixed External States

For a DBI amplitudes with both scalar and photon external states, with two soft scalars the soft theorem is still in the form of (6.60). Here we sketch the proof still only at the leading order. The main task is to study the behavior of the new building block $\text{Pf}'[\Psi_n]_{:\hat{\mathsf{s}}}$ in the double soft limit. When evaluated on the degenerate solution the detailed structure of the matrix $[\Psi_n]_{:\hat{\mathsf{s}}}$ is

$$[\Psi_n]_{:\hat{\mathsf{s}}} \approx \left(\begin{array}{c:cc|c} (A_{n-2})_{a,b} & \frac{\tau\, k_a\cdot p}{\sigma_a-\rho} & \frac{\tau\, k_a\cdot q}{\sigma_a-\rho} & (-C^{\mathrm{T}}_{n-2})_{a,d} \\ \hdashline \frac{\tau\, p\cdot k_b}{\rho-\sigma_b} & 0 & \frac{\tau\, p\cdot q}{-\xi_1} & \frac{\tau\, p\cdot\epsilon_d}{\rho-\sigma_d} \\ \frac{\tau\, q\cdot k_b}{\rho-\sigma_b} & \frac{\tau\, q\cdot p}{\xi_1} & 0 & \frac{\tau\, q\cdot\epsilon_d}{\rho-\sigma_d} \\ \hline (C_{n-2})_{c,b} & \frac{\tau\,\epsilon_c\cdot p}{\sigma_c-\rho} & \frac{\tau\,\epsilon_c\cdot q}{\sigma_c-\rho} & (B_{n-2})_{c,d} \end{array}\right), \tag{6.81}$$

where $a, b \in \{1, 2, \ldots, n-2\}$ and $c, d \in \gamma$ (we assume the last two labels denote scalars). Since the four blocks at the corner are completely finite we do not bother to write them explicitly. This is very similar to the case of matrix A_n, and we immediately know

$$\mathrm{Pf}'[\Psi_n]_{:\hat{s}} = \frac{\tau\, q \cdot p}{-\xi_1}\,\mathrm{Pf}'[\Psi_{n-2}]_{:\hat{s}} + \mathcal{O}(\tau^2)\,. \tag{6.82}$$

Recalling (6.76) we see that when we replace the integrand for scalar amplitude $\mathrm{Pf}X_n\,(\mathrm{Pf}'A_n)^3$ to that for mixed amplitude $\mathrm{Pf}[X_n]_{\hat{\gamma}}\,\mathrm{Pf}'[\Psi_n]_{:\hat{s}}\,(\mathrm{Pf}'A_n)^2$, the expression (6.77) stays the same, and hence we have exactly the same $S^{(0)}$.

Double Soft Photons Emission in Born–Infeld and Einstein–Maxwell

We go on to investigate photon amplitudes in BI and EM, with two soft photons. We can use a single integrand for this class of amplitudes

$$I_n = (\mathrm{Pf}X_n)^{-m}\,(\mathrm{Pf}'A_n)^{2+m}\,\mathrm{Pf}'\Psi_n\,, \tag{6.83}$$

where $m = 0, -1$ denotes BI and EM respectively. The prescription of the limit is still the same as that for the soft scalars. Again the only thing we need the check is the behavior of the matrix Ψ_n since now the second block of labels also includes $n-1$ and n.

To the order sufficient for our interest the structure of matrix Ψ_n is

$$\Psi_n \approx \left(\begin{array}{c:cc|c:cc}
(A_{n-2})_{a,b} & \frac{\tau\,k_a\cdot p}{\sigma_a-\rho} & \frac{\tau\,k_a\cdot q}{\sigma_a-\rho} & (-C_n^T)_{a,d} & \frac{k_a\cdot\epsilon_{n-1}}{\sigma_a-\rho} & \frac{k_a\cdot\epsilon_n}{\sigma_a-\rho} \\ \hdashline
\frac{\tau\,p\cdot k_b}{\rho-\sigma_b} & 0 & \frac{\tau\,p\cdot q}{-\xi_1} & \frac{\tau\,p\cdot\epsilon_d}{\rho-\sigma_d} & (-C_n^T)_{n-1,n-1} & \frac{p\cdot\epsilon_n}{-\xi_1} \\
\frac{\tau\,q\cdot k_b}{\rho-\sigma_b} & \frac{\tau\,q\cdot p}{\xi_1} & 0 & \frac{\tau\,q\cdot\epsilon_d}{\rho-\sigma_d} & \frac{q\cdot\epsilon_{n-1}}{\xi_1} & (-C_n^T)_{n,n} \\ \hline
(C_n)_{c,b} & \frac{\tau\,\epsilon_c\cdot p}{\sigma_c-\rho} & \frac{\tau\,\epsilon_c\cdot q}{\sigma_c-\rho} & (B_{n-2})_{c,d} & \frac{\epsilon_c\cdot\epsilon_{n-1}}{\sigma_c-\rho} & \frac{\epsilon_c\cdot\epsilon_n}{\sigma_c-\rho} \\ \hdashline
\frac{\epsilon_{n-1}\cdot k_a}{\rho-\sigma_a} & (C_n)_{n-1,n-1} & \frac{\epsilon_{n-1}\cdot q}{-\xi_1} & \frac{\epsilon_{n-1}\cdot\epsilon_d}{\rho-\sigma_d} & 0 & \frac{\epsilon_{n-1}\cdot\epsilon_n}{-\tau\,\xi_1} \\
\frac{\epsilon_n\cdot k_a}{\rho-\sigma_a} & \frac{\epsilon_n\cdot p}{\xi_1} & (C_n)_{n,n} & \frac{\epsilon_n\cdot\epsilon_d}{\rho-\sigma_d} & \frac{\epsilon_n\cdot\epsilon_{n-1}}{\tau\,\xi_1} & 0
\end{array}\right), \tag{6.84}$$

where the two extra diagonal terms of matrix C_n are approximated as

$$(C_n)_{n-1,n-1} = \sum_{i=1}^{n-2}\frac{\epsilon_{n-1}\cdot k_i}{\rho-\sigma_i} - \frac{\epsilon_{n-1}\cdot q}{\xi_1} + \mathcal{O}(\tau) = \sum_{i=1}^{n-2}\frac{\epsilon_{n-1}\cdot p_i^{\perp}}{\rho-\sigma_i} + \mathcal{O}(\tau)\,, \tag{6.85}$$

with $p_i^{\perp} := k_i - \frac{p\cdot k_i}{p\cdot q}\,q$ (We denote this new vector as $p_i^{\perp}$ because $p\cdot p_i^{\perp} = 0$). In the second equality above we applied the scattering equation labeled by $n-1$ to get rid of ξ_1. Similarly, we have

$$(C_n)_{n,n} = \sum_{i=1}^{n-2}\frac{\epsilon_n\cdot q_i^{\perp}}{\rho-\sigma_i} + \mathcal{O}(\tau)\,, \tag{6.86}$$

with $q_i^\perp := k_i - \frac{q\cdot k_i}{q\cdot p}\, p$. By studying carefully the scaling of each entry in (6.84) we observe that

$$\mathrm{Pf}'\Psi_n = \mathrm{Pf}[\Psi_n]_{n-1,n:n-1,n}\,\mathrm{Pf}'\Psi_{n-2} + \mathcal{O}(\tau^2)\,, \tag{6.87}$$

where $[\Psi_n]_{n-1,n:n-1,n}$ is the minor of Ψ_n with entries $\{n-1, n : n-1, n\}$ only. we obtain

$$\begin{aligned} S^{(0)} &= \oint \frac{d\rho}{2\pi i}\, \frac{1}{\sum_{b=1}^{n-2} \frac{k_b\cdot(p+q)}{\rho-\sigma_b}}\, \frac{-\xi_1^2}{\tau\, p\cdot q} \left(\frac{-1}{\tau\,\xi_1}\right)^{-m} \left(\frac{\tau\, p\cdot q}{-\xi_1}\right)^{2+m} \\ &\quad \times \left(\frac{p\cdot q\;\epsilon_{n-1}\cdot\epsilon_n}{\xi_1^2} - \sum_{i,j=1}^{n-2} \frac{\epsilon_{n-1}\cdot p_i^\perp\;\epsilon_n\cdot q_i^\perp}{(\rho-\sigma_i)\,(\rho-\sigma_j)} - \frac{p\cdot\epsilon_n\; q\cdot\epsilon_{n-1}}{\xi_1^2} \right) \\ &= \oint \frac{d\rho}{2\pi i}\, \frac{(\tau^2\, p\cdot q)^{1+m}}{-\tau \sum_{b=1}^{n-2} \frac{k_b\cdot(p+q)}{\rho-\sigma_b}} \left(\frac{p\cdot q\;\epsilon_{n-1}\cdot\epsilon_n - p\cdot\epsilon_n\; q\cdot\epsilon_{n-1}}{\xi_1^2} - \sum_{i,j=1}^{n-2} \frac{\epsilon_{n-1}\cdot p_i^\perp\;\epsilon_n\cdot q_i^\perp}{(\rho-\sigma_i)\,(\rho-\sigma_j)} \right). \end{aligned} \tag{6.88}$$

Recalling the solution (2.29) $\xi_1^{-1} = \frac{1}{2p\cdot q}\sum_{a=1}^{n-2} \frac{k_a\cdot(p-q)}{\rho-\sigma_a}$, although there seems to be a simple pole at $\rho=\infty$, as one can check its residue is proportional to

$$\frac{p\cdot q\;\epsilon_{n-1}\cdot\epsilon_n - p\cdot\epsilon_n\; q\cdot\epsilon_{n-1}}{4\,(p\cdot q)^2} \left(\sum_{b=1}^{n-2} k_b\cdot(p-q)\right)^2 - \sum_{i,j=1}^{n-2} \epsilon_{n-1}\cdot p_i^\perp\;\epsilon_n\cdot q_i^\perp\,, \tag{6.89}$$

which vanishes due to momentum conservation. Hence contour deformation again picks up a simple pole at every $\rho=\sigma_b$ with $b\in\{1,2,\dots,n-2\}$, which leads to the result

$$\sum_{b=1}^{n-2} \frac{(\tau^2\, p\cdot q)^{1+m}}{\tau\, k_b\cdot(p+q)} \left(\frac{p\cdot q\;\epsilon_{n-1}\cdot\epsilon_n - p\cdot\epsilon_n\; q\cdot\epsilon_{n-1}}{4\,(p\cdot q)^2}\,(k_b\cdot(p-q))^2 - \epsilon_{n-1}\cdot p_b^\perp\;\epsilon_n\cdot q_b^\perp \right). \tag{6.90}$$

Since the original expansion of the Pfaffian is not modified by the ρ integration, we observe that the soft factor can still be written as a Pfaffian, and so

$$M_n = \frac{\tau^3\,(\tau^2\, p\cdot q)^{m-1}}{4} \sum_{a=1}^{n-2} \left(\frac{(k_b\cdot(p-q))^2}{k_b\cdot(p+q)}\,\mathrm{Pf}\mathcal{S}_b \right) M_{n-2} + \mathcal{O}(\tau^{4+2m})\,. \tag{6.91}$$

where $\mathcal{S}_b$ is a 4×4 anti-symmetric matrix

$$\mathcal{S}_b = \left(\begin{array}{cc|cc} 0 & p\cdot q & \hat{p}_b^\perp\cdot\epsilon_{n-1} & p\cdot\epsilon_n \\ -q\cdot p & 0 & -q\cdot\epsilon_{n-1} & \hat{q}_b^\perp\cdot\epsilon_n \\ \hline -\epsilon_{n-1}\cdot\hat{p}_b^\perp & \epsilon_{n-1}\cdot q & 0 & \epsilon_{n-1}\cdot\epsilon_n \\ -\epsilon_n\cdot p & -\epsilon_n\cdot\hat{q}_b^\perp & -\epsilon_n\cdot\epsilon_{n-1} & 0 \end{array}\right), \tag{6.92}$$

where $\hat{p}_b^\perp := \frac{2\,p\cdot q}{k_b\cdot(p-q)}\, p_b^\perp$ and $\hat{q}_b^\perp := \frac{2\,p\cdot q}{k_b\cdot(p-q)}\, q_b^\perp$.

7

RELATIONS AMONG AMPLITUDES

7.1 RELATIONS AMONG AMPLITUDES IN YANG–MILLS AND GRAVITY

7.1.1 $U(1)$ *Decoupling Identities and Kleiss–Kuijf Relations*

In Section 1.2.2 of the introduction we introduced color decomposition, which breaks a full amplitude in a theory with non-trivial flavor/color group into smaller pieces called partial amplitudes. While these partial amplitudes can be determined individually, one may doubt whether this really simplifies computation, because we still have to determine all of them.

It has been known for a long time that the actual computational work needed is much less at least in the amplitudes of pure Yang–Mills, thanks to the many algebraic relations among the partial amplitudes. The simplest type of relations are called $U(1)$ decoupling identities

$$\sum_{a=1}^{n-1} M_n^{\text{YM}}[1,2,\ldots,a,n,a+1,a+2,\ldots,n-1] = 0\,. \tag{7.1}$$

Physically, this corresponds to the factor that if one considers the gauge group to be $U(1)$, then the amplitude where a photon scatters (in $U(1)$) with a set of gluons (in $SU(N)$) has to vanish. Apart from this, we also have the Kleiss–Kuijf (KK) relations [64, 65], which can be written as

$$M_n^{\text{YM}}[1,\alpha,n,\beta] = (-1)^{|\beta|} \sum_{\gamma\in\text{OP}(\alpha,\beta^{\text{T}})} M_n^{\text{YM}}[1,\gamma,n]\,, \tag{7.2}$$

where β^{T} denotes the reverse ordering of β, $|\beta|$ denotes the size of β, and $\text{OP}(\alpha,\beta^{\text{T}})$ denotes all possible shuffles of α and β, i.e., permutations of $\alpha\cup\beta$ with the ordering α and β^{T} preserve respectively. For example,

$$\begin{aligned} M_n^{\text{YM}}[1,2,3,6,4,5] = \\ M_n^{\text{YM}}[1,2,3,5,4,6] + M_n^{\text{YM}}[1,2,5,3,4,6] + M_n^{\text{YM}}[1,2,5,4,3,6] \\ +M_n^{\text{YM}}[1,5,2,4,3,6] + M_n^{\text{YM}}[1,5,4,2,3,6] + M_n^{\text{YM}}[1,5,2,3,4,6]\,. \end{aligned} \tag{7.3}$$

The $U(1)$ decoupling identities and the KK relations together reduce all the $(n-1)!$ partial amplitudes to $(n-2)!$ independent ones.

The CHY representation makes these relations particularly straightforward. Recall that in this formula

$$M_n^{\mathrm{YM}}[\alpha] = \int d\mu_n\, C_n[\alpha]\, \mathrm{Pf}'\Psi_n\,, \tag{7.4}$$

every part is permutation invariant except for $C_n[\alpha]$ which depends on the given ordering. As one can explicitly verify, if we substitute each $M_n^{\mathrm{YM}}[\alpha]$ by the corresponding $C_n[\alpha]$, (7.1) and (7.2) hold for generic σ's. So the above relations among amplitudes are direct consequences of the algebraic properties of the function $C_n[\alpha]$. In fact this has already been observed in the early proposals for formulas alternative to the CHY representation in four dimensions, commonly known as connected formulation.

7.1.2 *Bern–Carrasco–Johansson Relations*

It turned out the $(n-2)!$ basis for Yang–Mills partial amplitudes are still redundant, and the actual algebraically independent basis are of size $(n-3)!$. This is originally observed by the discovery of a set of new relations resulting from imposing algebraic relations among the kinematic part of the numerators in Feynman diagrams that resembles the Jacobi identity among the structure constants. These are called Bern–Carrasco–Johansson (BCJ) relations [66]. While the amount of such relations are huge, most of them are redundant and can be derived from the simplest ones known as the fundamental BCJ relations, which are

$$\sum_{a=1}^{n-1} \Big(\sum_{b=1}^{a} s_{n,b}\Big) M_n^{\mathrm{YM}}[1,2,\ldots,a,n,a+1,a+2,\ldots,n-1] = 0\,. \tag{7.5}$$

As was proven in [23], the building block $C_n[\alpha]$ is again solely responsible for these relations. To be precise, when we substitute $M_n[\alpha]$ by $C_n[\alpha]$ in (7.5), then these relations are valid on the support of the scattering equations. The fact that the scattering equations are needed is the reason why (7.5) are more non-trivial.

7.1.3 *Kawai–Lewellen–Tye Relations*

A practical way to study gravity amplitudes is to use the Kawai–Lewellen–Tye (KLT) relations [14] in the field-theory limit. These are originally relations that express a closed string amplitude as convolution of two copies of open string amplitudes. In the field theory limit, they reduce to the relations between gravity amplitudes and YM amplitudes. In the field theory context these relations were explicitly written down in [15]. When expressed in terms of the $(n-3)!$ fully independent basis as discussed before, they are commonly written as

$$M_n^{\mathrm{GR}} = \sum_{\alpha,\beta\in S_{n-3}} M_N^{\mathrm{YM}}[1,\alpha,n-1,n]\,\mathcal{K}_n[\alpha|\beta]\,M_n^{\mathrm{YM}}[1,\beta,n,n-1]\,, \tag{7.6}$$

where α,β are permutations of labels $\{2,3,\ldots,n-1\}$. $\mathcal{K}_n[\alpha|\beta]$ is called the KLT momentum kernel, which is a function of the Mandelstam variables only and depends on the two orderings α and β. Explicitly,

$$\mathcal{K}_n[\alpha|\beta] := \prod_{c=2}^{n-2}\left(s_{1,\alpha(c)} + \sum_{d=2}^{c-1}\theta^{(\beta)}_{\alpha(d),\alpha(c)}\,s_{\alpha(d),\alpha(c)}\right), \tag{7.7}$$

where $\theta^{(\beta)}_{a,b} = 1$ if a comes in front of b in the ordering β and zero otherwise [16].

From the experience in previous discussions one may wonder whether the KLT relation again is closely related to the Parke–Taylor factor in the context of the CHY representation. The answer is yes, and this is achieved by an identity called the *KLT orthogonality* [17].

7.2 KAWAI–LEWELLEN–TYE ORTHOGONALITY

7.2.1 *Proposition*

Let us first define an inner product

$$(i,j) := \sum_{\alpha,\beta\in S_{n-3}} C_n^{(i)}[1,\alpha,n-1,n]\,\mathcal{K}_n[\alpha|\beta]\,C_n^{(j)}[1,\beta,n,n-1]\,, \tag{7.8}$$

where the superscript (i) means to evaluate on the i^{th} solution to the scattering equations. With this definition, the Kawai–Lewellen–Tye orthogonality is stated as follows

Proposition 7.2.1. *The inner product (7.8) satisfies*

$$\frac{(i,j)}{(i,i)^{\frac{1}{2}}\,(j,j)^{\frac{1}{2}}} = \delta^{i,j}\,, \quad \forall i,j\in\{1,2,\ldots,(n-3)!\}\,. \tag{7.9}$$

For a proof, note that the LHS of (7.9) remains invariant under $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$, where one $SL(2,\mathbb{C})$ acts on $\sigma^{(i)}$'s while the other on $\sigma^{(j)}$'s. Hence it suffices to prove the proposition for a particular choice of gauge on $\sigma^{(i)}$'s and $\sigma^{(j)}$'s respectively. For convenience, we choose a partial fixing by setting $\{\sigma^{(i)}_{n-1}, \sigma^{(i)}_n\} = \{\infty, 1\}$ and $\{\sigma^{(j)}_{n-1}, \sigma^{(j)}_n\} = \{1, \infty\}$. Correspondingly, define the following object

$$K_n(\{\sigma\}, \{\tilde{\sigma}\}) := \sum_{\alpha,\beta \in S_{n-3}} \frac{1}{\sigma_{1,\alpha(2)} \cdots \sigma_{\alpha(n-3),\alpha(n-2)}} \mathcal{K}_n[\alpha|\beta] \frac{1}{\tilde{\sigma}_{1,\alpha(2)} \cdots \tilde{\sigma}_{\alpha(n-3),\alpha(n-2)}} . \quad (7.10)$$

The motivation for K_n is that it appears in the LHS of (7.9) after the above partial gauge fixing, i.e.,

$$\frac{(i,j)}{(i,i)^{\frac{1}{2}}\,(j,j)^{\frac{1}{2}}} = \frac{K_n(\{\sigma^{(i)}\}, \{\sigma^{(j)}\})}{K_n^{\frac{1}{2}}(\{\sigma^{(i)}\}, \{\sigma^{(i)}\})\, K_n^{\frac{1}{2}}(\{\sigma^{(j)}\}, \{\sigma^{(j)}\})} . \quad (7.11)$$

In addition, we define a $(n-2) \times (n-2)$ matrix $\widetilde{\varphi}_n$

$$(\widetilde{\varphi}_n)_{a,b} = \begin{cases} \frac{s_{a,b}}{\sigma_{a,b}\,\tilde{\sigma}_{a,b}} , & a \neq b , \\ -\sum_{\substack{c=1 \\ c\neq a}}^{n-2} (\widetilde{\varphi}_n)_{a,c} , & a = b . \end{cases} \quad (7.12)$$

This matrix has co-rank one, since all the rows/columns add up to zero. So an invariant quantity associated to this matrix is a reduced determinant $\det'\widetilde{\varphi}_n := \det[\widetilde{\varphi}_n]_{\hat{a}}$, where $[\widetilde{\varphi}_n]_{\hat{a}}$ is the minor of $\widetilde{\varphi}_n$ with the a^{th} row and column deleted. Obviously the reduced determinant is independent of the choice of a.

A crucial observation for the two objects defined above is that

$$K_n(\{\sigma\}, \{\tilde{\sigma}\}) = (-1)^n \det{}'\widetilde{\varphi}_n(\{\sigma\}, \{\tilde{\sigma}\}) , \quad (7.13)$$

for generic values of σ's and $\tilde{\sigma}$'s. Since this proof of (7.13) is technical, we refer interested reader to Appendix B of [11] for details. The basic idea here is that since both are functions of the unconstrained complex variables $\{\sigma\} \cup \{\tilde{\sigma}\}$, we can prove this identity by showing that they have the same set of poles (including a possible pole at infinity) and that the residues at each pole are identical.

Now the reduced determinant $\det'\widetilde{\varphi}_n$ has an interesting property. To illustrate it, let us further define an $n \times n$ matrix

$$(\widetilde{\Phi}_n)_{a,b} = \begin{cases} \frac{s_{a,b}}{\sigma_{a,b}\,\tilde{\sigma}_{a,b}} , & a \neq b , \\ -\sum_{\substack{c=1 \\ c\neq a}}^{n} (\widetilde{\Phi}_n)_{a,c} , & a = b . \end{cases} \quad (7.14)$$

Obviously we have the relation

$$\widetilde{\varphi}_n = \lim_{\substack{\sigma_{n-1}\to\infty \\ \tilde{\sigma}_n\to\infty}} [\widetilde{\Phi}_n]_{\widehat{n-1},\hat{n}} , \quad (7.15)$$

where the limit we take is consistent with the partial gauge fixing we did before. Note that when both $\{\sigma\}$ and $\{\tilde{\sigma}\}$ are certain solutions to the scattering equations the entries of $\widetilde{\Phi}_n$ satisfy

$$\sum_{b=1}^{n} (\widetilde{\Phi}_n)_{a,b}\, \sigma_b = -\sum_{b=1}^{n} (\widetilde{\Phi}_n)_{a,b}\, (\sigma_a - \sigma_b) + \sigma_a \sum_{b=1}^{n} (\widetilde{\Phi}_n)_{a,b} = -\sum_{b\neq a} \frac{s_{a,b}}{\tilde{\sigma}_{a,b}} = 0\,, \tag{7.16}$$

$$\sum_{b=1}^{n} (\widetilde{\Phi}_n)_{a,b}\, \sigma_a\, \tilde{\sigma}_b = \sum_{b=1}^{n} s_{a,b} - \sigma_a \sum_{b\neq a} \frac{s_{a,b}}{\tilde{\sigma}_{a,b}} - \tilde{\sigma}_a \sum_{b\neq a} \frac{s_{a,b}}{\sigma_{a,b}} + \sigma_a\, \tilde{\sigma}_a \sum_{b=1}^{n} (\widetilde{\Phi}_n)_{a,b} = 0\,, \tag{7.17}$$

which indicate that the matrix φ_n has a kernel that is spanned by the following four vectors

$$(\sigma_1^r\, \tilde{\sigma}_1^s, \sigma_2^r\, \tilde{\sigma}_2{}^s, \ldots, \sigma_n^r\, \tilde{\sigma}_n^s)^{\mathrm{T}}\,, \tag{7.18}$$

with $r, s \in \{0,1\}$. The situation divides into two cases: when $\{\sigma\}$ and $\{\tilde{\sigma}\}$ are two different solutions the kernel is four dimensional; but when $\{\sigma\}$ and $\{\tilde{\sigma}\}$ are evaluated on the same solution the kernel is instead three dimensional, because now the vector with $\{r,s\} = \{1,0\}$ becomes identical to that with $\{r,s\} = \{0,1\}$. Recalling (7.13), this means

$$K_n(\{\sigma^{(i)}\}, \{\sigma^{(j)}\}) \propto \det[\widetilde{\Phi}_n]_{\hat{a},\widehat{n-1},\hat{n}}(\{\sigma^{(i)}\}, \{\sigma^{(j)}\}) \propto \delta^{i,j}\,. \tag{7.19}$$

Applying this in (7.11) proves the KLT orthogonality (7.9).

In fact, by the form of the basis for the kernel (7.18), one can observe that when $\{\sigma\}$ and $\{\tilde{\sigma}\}$ are the same solution, we have the exact identity

$$K_n(\{\sigma\}, \{\sigma\}) = \frac{(-1)^n}{\begin{vmatrix} 1 & 1 & 1 \\ \sigma_a & \sigma_{n-1} & \sigma_n \\ \sigma_a^2 & \sigma_{n-1}^2 & \sigma_n^2 \end{vmatrix}} \det[\widetilde{\Phi}_n]_{\hat{a},\widehat{n-1},\hat{n}}(\{\sigma\}, \{\sigma\}) = J_n(\{\sigma\})\,, \tag{7.20}$$

where $J_n(\{\sigma\}) = \det'\Phi_n$ is exactly the Jacobian from solving the delta constraints in the CHY representation, as defined in (3.7), which is now evaluated on the solution $\{\sigma\}$. Using (7.20) and (7.11), the KLT orthogonality relation (7.9) can be equivalently written as

$$\sum_{\alpha,\beta\in S_{n-3}} C_n^{(i)}[1,\alpha,n-1,n]\, \mathcal{K}_n[\alpha|\beta]\, C_n^{(j)}[1,\beta,n,n-1] = \delta^{i,j}\, (\det'\Phi_n)^{(i)}\,, \tag{7.21}$$

which is what we have shown in (1.50) in the introduction.

7.2.2 *Consequence*

With the KLT orthogonality in the form of (7.21), it is very easy to see how the KLT relations (7.6) work. Moreover, these relations can be set in a more general context.

Suppose we have two theories A and B, both of which assume non-trivial flavor/color groups such that we can define partial amplitudes as in (1.28). With this assumption the integrands for amplitudes in both theories have to contain a Parke–Taylor factor. Let us assume that these integrands have the form

$$I_n^{\mathrm{A}}[\alpha] = C_n[\alpha]\, \widetilde{I}_n^{\mathrm{A}}\,, \quad I_n^{\mathrm{B}}[\alpha] = C_n[\alpha]\, \widetilde{I}_n^{\mathrm{B}}\,, \tag{7.22}$$

where recall that $\widetilde{I}$ transforms as half of the full integrand. Then we can consider the following

$$\begin{aligned} &\sum_{\alpha,\beta\in S_{n-3}} M_n^{\mathrm{A}}[1,\alpha,n-1,n]\,\mathcal{K}_n[\alpha|\beta]\,M_n^{\mathrm{B}}[1,\beta,n,n-1] \\ &= \sum_{\alpha,\beta\in S_{n-3}} \sum_{i=1}^{(n-3)!} \frac{C_n^{(i)}[1,\alpha,n-1,n]\,\widetilde{I}_n^{\mathrm{A},(i)}}{J_n^{(i)}}\,\mathcal{K}_n[\alpha|\beta] \sum_{j=1}^{(n-3)!} \frac{C_n^{(j)}[1,\beta,n,n-1]\,\widetilde{I}_n^{\mathrm{B},(j)}}{J_n^{(j)}} \\ &= \sum_{i=1}^{(n-3)!} \frac{\widetilde{I}_n^{\mathrm{A},(i)}\,\widetilde{I}_n^{\mathrm{B},(j)}}{J_n^{(i)}} = \int d\mu_n\,\widetilde{I}_n^{\mathrm{A}}\,\widetilde{I}_n^{\mathrm{B}}\,, \end{aligned} \tag{7.23}$$

where in the first equality we used the CHY representation in terms of summing over the solutions (3.5). Obviously the combination $\widetilde{I}_n^{\mathrm{A}}\widetilde{I}_n^{\mathrm{B}}$ transforms correctly under $SL(2,\mathbb{C})$, and so we can regard it as the integrand for some theory C, and hence

$$M_n^{\mathrm{C}} = \sum_{\alpha,\beta\in S_{n-3}} M_n^{\mathrm{A}}[1,\alpha,n-1,n]\,\mathcal{K}_n[\alpha|\beta]\,M_n^{\mathrm{B}}[1,\beta,n,n-1]\,. \tag{7.24}$$

Let us abbreviate the relation (7.24) as $C = A \otimes_{\mathrm{KLT}} B$. When both $\widetilde{I}$'s are $\mathrm{Pf}'\Psi_n$ this is the well-known KLT construction of gravity amplitudes from Yang–Mills amplitudes $\mathrm{GR} = \mathrm{YM} \otimes_{\mathrm{KLT}} \mathrm{YM}$. But now we can also choose each $\widetilde{I}$ to be either $\mathrm{Pf}[\mathcal{X}_n]_S \mathrm{Pf}'[\Psi_n]_{:\hat{S}}$ or $\mathcal{C}_{\mathrm{tr}_1}\mathcal{C}_{\mathrm{tr}_2}\cdots\mathcal{C}_{\mathrm{tr}_t}\mathrm{Pf}'\Pi_n(\mathrm{tr}_1,\mathrm{tr}_2,\ldots,\mathrm{tr}_t)$ or $(\mathrm{Pf}'A_n)^2$. Hence we also have the following KLT relations

$$\mathrm{BI} = \mathrm{NLSM} \otimes_{\mathrm{KLT}} \mathrm{YM}\,, \tag{7.25}$$

$$\mathrm{sGal} = \mathrm{NKSM} \otimes_{\mathrm{KLT}} \mathrm{NLSM}\,, \tag{7.26}$$

$$\mathrm{EM} = \mathrm{YMS} \otimes_{\mathrm{KLT}} \mathrm{YM}\,, \tag{7.27}$$

$$\mathrm{DBI} = \mathrm{NLSM} \otimes_{\mathrm{KLT}} \mathrm{YMS}\,, \tag{7.28}$$

$$\mathrm{EYM} = \mathrm{gen.\,YMS} \otimes_{\mathrm{KLT}} \mathrm{YM}\,, \tag{7.29}$$

$$\mathrm{ext.\,DBI} = \mathrm{NLSM} \otimes_{\mathrm{KLT}} \mathrm{gen.\,YMS}\,. \tag{7.30}$$

The KLT construction for YM, YMS, gen. YMS and NLSM is trivial, because they are re-produced by the KLT of Φ^3 with themselves.

From the above results, it looks that the scalar amplitudes in Φ^3 are very special. The fact that $A = \Phi^3 \otimes_{\mathrm{KLT}} A$ for A being a theory in which flavor/color ordered partial amplitudes are defined suggests that the KLT momentum kernel is the inverse of $M_n^{\Phi^3}$, i.e., the trivalent massless scalar diagrams. This correspondence is made precise as follows.

First let us define two $(n-3)! \times (n-3)!$ matrices

$$\mathbf{U}^{(i)}{}_{\alpha} := \frac{C_n^{(i)}[1,\alpha,n-1,n]}{\sqrt{J_n(\{\sigma^{(i)}\})}}\,, \quad \mathbf{V}^{(i)}{}_{\alpha} := \frac{C_n^{(i)}[1,\alpha,n,n-1]}{\sqrt{J_n(\{\sigma^{(i)}\})}}\,, \tag{7.31}$$

and let $\mathbf{K}^{\alpha}{}_{\beta} := \mathcal{K}_n[\alpha|\beta]$. Then the KLT orthogonality (7.9) can be written in terms of these matrices as

$$\mathbf{U}\,\mathbf{K}\,\mathbf{V}^{\mathrm{T}} = \mathbf{1}\,, \tag{7.32}$$

where $\mathbf{1}$ is the identity in the solution space. Obviously both $\mathbf{U}$ and $\mathbf{V}$ are invertible. So let us multiply both sides of (7.32) by $\mathcal{V}^{\mathrm{T}}$ on the left and $(\mathbf{V}^{\mathrm{T}})^{-1}$ on the right, which yields

$$\mathbf{V}^{\mathrm{T}}\,\mathbf{U}\,\mathbf{K} = \mathbf{1}\,, \tag{7.33}$$

where now the $\mathbf{1}$ is the identity in the permutation space. Interestingly, note that

$$\begin{aligned} (\mathbf{V}^{\mathrm{T}}\,\mathbf{U})^{\alpha}{}_{\beta} &= \sum_{i=1}^{(n-3)!} \frac{C_n^{(i)}[1,\alpha,n,n-1]\,C_n^{(i)}[1,\beta,n,n-1]}{J_n(\{\sigma^{(i)}\})} \\ &= \int d\mu_n\, I_n^{\Phi^3}[1,\alpha,n,n-1|1,\beta,n-1,n]\,, \end{aligned} \tag{7.34}$$

which is exactly the double partial amplitude that we have defined in the Φ^3 theory flavored in $U(N) \times U(\tilde{N})$. Thus we observe that the KLT momentum kernel is nothing but the inverse of the cubic massless scalar diagrams [7]

$$\mathcal{K}_n[\alpha|\beta] = \big(m_n[1,\alpha,n,n-1|1,\beta,n-1,n]\big)^{-1}\,. \tag{7.35}$$

8 OUTLOOK

In this thesis we introduced the scattering equations, which provide a map from the kinematics data of an n-particle scattering to an auxiliary space $\mathfrak{M}_{0,n}$, the space of all inequivalent n-punctured Riemann spheres, with the purpose of modeling the tree-level S-matrix. The only assumption for these equations is that the momenta are conserved and massless on-shell. So it is expected that the scattering equations are universal, in the sense that their form remains the same for any type of massless particles and any interactions in a given theory.

The scattering equations lead to a very natural integral over $\mathfrak{M}_{0,n}$, which can be utilized to construct formulas for various tree-level amplitudes. By making simple proposals for the integrand therein, we found out several instances which generate physically sensible results, and these results are identified to tree-level amplitudes in certain theories. With the introduction of three operations acting on the integrand, we discovered other integrands that are more complicated but still well-defined, and they turned out to describe the tree-level scattering in several more theories. With the arguments from the study of a generic factorization channel and behaviors in the soft limits, we showed that all these formulas are closed (i.e., valid for all possible amplitudes to all multiplicities in the theory).

For the future explorations related to the scattering equations and the CHY representation, some of the most relevant ones are

1. Extension of the scattering equations and their applications to loop levels.
2. The CHY representation in fixed spacetime dimensions, in particular 4d and 6d.
3. The CHY representation in theories with fermions, and supersymmetries.
4. Scattering equations for massive particles.

In the following we briefly comment on these one by one, but they are also interrelated.

Conceptually there exists a very natural extension of the scattering equations to loop levels. Recall that in Chapter 2 we basically discovered these equations starting with a generic meromorphic form $\omega^\mu(z)$ on $\mathbb{CP}^1$ with only simple poles at the n marked points, and then imposing the requirement that it squares to zero everywhere on $\mathbb{CP}^1$, i.e., $\omega_\mu\omega^\mu \equiv 0$. Since the meromorphic forms exist for any Riemann surface, when we switch to amplitudes at the level of g loops, we can simply repeat the same analysis but on a Riemann surface of genus g [25, 27, 37]. Note that at $g > 0$ a Riemann surface allows for non-trivial holomorphic forms that can be freely added to a meromorphic form, which introduces more degrees of freedom. A straightforward interpretation is that for the scattering equations at loop orders, information about the loop momenta should also enter into the kinematics data, and so what the corresponding CHY representation (if it exists) computes is the integrand before integrating out the loop momenta (the *loop integrand*) instead of the final loop-level amplitude. It is an interesting fact that the number of independent loop momenta matches the dimension of the space of holomorphic forms. But it is difficult to solve the scattering equations even at one-loop and four-points, because elliptic functions start to enter (almost all concrete analysis regarding this up to now are consistency checks in various limits, in which the Riemann surface degenerates [27, 31]). Besides, it is puzzling how a loop integrand, which is expected to be rational, can come out of a formulation that contains elliptic functions. Nonetheless this is still a major direction worth exploring. And if this extension indeed works, we may hope to land on certain compact expressions as what we have already achieved at tree level.

Prior to the CHY representation, various alternative integral representations for the tree-level S-matrix in four dimensions were discovered. These are called connected formulations, and are particularly suitable for, e.g., $\mathcal{N} = 4$ super Yang–Mills and $\mathcal{N} = 8$ supergravity [10, 17, 41, 67–71]. As was shown in [10], one can obtain similar formulas for various theories in three dimensions, as a direct result of dimensional reduction from four dimensions. The availability of the spin-helicity formalism in 3d and 4d plays an important role here. A similar formalism also exist in six dimensions [72], but less is known about how an analogous connected formulation should work in six dimensions. It would be interesting to see how the formulas that we propose in this thesis can exactly reduce to existing formulas in various connected formulations when restricted to four dimensions. Understanding this may help us acquire intuition about how to reduce them to six dimensions.

In this thesis we have fully restricted our attention to bosons. This does not mean that the current framework fails for fermions. As commented in Section 3.2, in principle any Feynman diagram acquires an integral formula based on the scattering equations (3.4). For instance, look at the simple process of Compton scattering $e^-\gamma \to e^-\gamma$ (assuming the energy is high enough so that the electron mass can be ignored), which contains two Feynman diagrams, leading to the following expression

$$-i\,e^2\left(\bar{u}_4\,\not{\epsilon}_3\,\frac{\not{k}_3+\not{k}_4}{(k_3+k_4)^2}\,\not{\epsilon}_1\,u_2+\bar{u}_4\,\not{\epsilon}_1\,\frac{\not{k}_2+\not{k}_3}{(k_2+k_3)^2}\,\not{\epsilon}_3\,u_2\right). \tag{8.1}$$

with the help of (3.14) this expression can be turned into

$$-\frac{i\,e^2}{2}\int d\mu_4\,\frac{1}{\sigma_{1,2}\,\sigma_{2,3}\,\sigma_{3,4}\,\sigma_{4,1}}\left(\frac{\bar{u}_4\,\not{\epsilon}_3\,(\not{k}_3+\not{k}_4)\,\not{\epsilon}_1\,u_2}{\sigma_{1,2}\,\sigma_{2,4}\,\sigma_{4,3}\,\sigma_{3,1}}+\frac{\bar{u}_4\,\not{\epsilon}_1\,(\not{k}_2+\not{k}_3)\,\not{\epsilon}_3\,u_2}{\sigma_{2,3}\,\sigma_{3,1}\,\sigma_{1,4}\,\sigma_{4,2}}\right). \tag{8.2}$$

Other examples of involving fermion are available in, e.g., [27, 35, 73]. The real problem, however, resides in whether the results like (8.2) can re-sum into something simple and compact, and at best, whether for a given theory involving fermions we can find a closed formula for its tree-level S-matrix, i.e., a CHY representation as those summarized in Table 1. This remains largely unclear.

More or less related to the above is the question of how to encode supersymmetries into the CHY representation. As mentioned before, in the connected formulations supersymmetries enter naturally with the help of spin-helicity formalism (mainly because we are able to introduce fermionic spinors to parametrize the on-shell superspace [74]), but this is absent in arbitrary dimensions, and in the current proposal of CHY representation polarization vectors/spinors are used instead. Since supersymmetric theories are very sensitive to spacetime dimensions, perhaps we should expect again to rely on certain techniques specialized to a chosen dimension.

There is another restriction to our discussions in this thesis: we require that the particles in the theory are all massless. This plays a fundamental role already in the proposal of the scattering equations, i.e., $Q \equiv 0$. So when we want to extend them to the case involving massive particles, it is natural to expect to modify these equations, or even the concept that leads to them. With some clever observations, modifications were worked out so as to allow for the description of scattering among a single type of massive scalars [38], and the scattering involving massive scalars and gauge bosons in the context of spontaneous symmetry breaking [39, 40]. But beyond these we are lacking a general concept of how to deal with massive particles.

Apart from these, it is also interesting and perhaps important to explore the connection of the CHY representation with string amplitudes. Note that a Riemann sphere is present in both the CHY representation and the ordinary string amplitudes (the worldsheet), but there is an essential difference between the two. In string amplitudes one sums over all possible histories of string propagation and thus integrates over the moduli space of the worldsheet where every point is equally important. And the boundaries of the moduli space single out to specific field-theory diagrams only in the infinite string tension limit. In addition, the moduli space depends on the external states and the theory. In contrast, in the CHY representation we are always dealing with Riemann spheres for whatever particles under consideration. And while it has the form of an integral, only at most a finite number of points in $\mathfrak{M}_{0,n}$ contribute to the amplitude, which highly depends on the kinematics. Despite these differences, there are still several remarkable similarities. It is well-known that the field-theory amplitude of gluon scattering comes as the infinite tension limit of the corresponding string amplitude. In a series of recent works [75], it was shown that the string amplitudes for gluons can be linearly expanded onto a basis in the form of KLT relations. This basis, the open string disk integral, depends on two orderings α,β, and is defined as

$$Z_n[\alpha|\beta] := \int_{z_{\alpha(1)}<\cdots<z_{\alpha(n)}} \prod_{a=1}^{n}{}' dz_a \prod_{a<b} |z_{a,b}|^{\alpha' k_a \cdot k_b} \frac{1}{z_{\beta(1),\beta(2)}\, z_{\beta(2),\beta(3)} \cdots z_{\beta(n),\beta(1)}}, \quad (8.3)$$

where all z's are real, and the prime in $\prod' dz$ is defined in the same way as (3.3). The factor $|z_{a,b}|^{\alpha' k_a \cdot k_b}$ is called the Koba–Nielsen factor. Note this has a remarkably similar form as the double partial amplitudes in Φ^3

$$\begin{aligned} m_n[\alpha|\beta] = \int \prod_{a=1}^{n}{}' d\sigma_a \prod_{a=1}^{n}{}' \delta\Big(\sum_{b\neq a} \frac{k_a \cdot k_b}{\sigma_{a,b}} \Big) \\ \times \frac{1}{\sigma_{\alpha(1),\alpha(2)}\, \sigma_{\alpha(2),\alpha(3)} \cdots \sigma_{\alpha(n),\alpha(1)}} \frac{1}{\sigma_{\beta(1),\beta(2)}\, \sigma_{\beta(2),\beta(3)} \cdots \sigma_{\beta(n),\beta(1)}}. \end{aligned} \quad (8.4)$$

Moreover, by expressing YM amplitudes as the KLT of Φ^3 amplitudes with themselves, one can find that $m_n[\alpha|\beta]$ is indeed exactly the $\alpha' \to 0$ limit of the disk integral $Z_n[\alpha|\beta]$. It would be interesting to find out whether this connection between (8.3) and (8.4) has to tell something deep. In particular, it is worth to mention that the scattering equations can also arise from the saddle point expansion of the Koba–Nielsen factor in (8.3) in the limit of high-energy fixed-angle scattering (the Gross–Mende limit), corresponding to $\alpha' \to \infty$ rather than $\alpha' \to 0$. Some results along this exploration were made in, e.g., [35].

Finally, the main open question is what is the space of all quantum field theories whose complete tree-level S-matrix admits a CHY representation (in the sense that a closed formula is available), and what is special about the theories that admit one. Once extended to loop level, one can ask whether the CHY representation is a hint for a different way of thinking about quantum field theory.

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