Pricing Asian Options by the Method of Moments Matching

by

Pak Keung Chan

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Quantitative Finance
in
Mathematics

Waterloo, Ontario, Canada, 2015

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

This Master's Thesis explores the method of moments matching for pricing Asian options. In this thesis, the underlying asset is assumed to be non-dividend paying and its price process either follows the standard geometric Brownian motion (GBM) or the more advanced Heston volatility model. The average price process is either discretely monitored or continuously monitored.

The thesis is organized as follow. In Chapter 1, we give a brief introduction to Asian options, including their history, their advantage over the more classical European option, and various methods of pricing. In Chapter 2, we outline the mathematical framework for the equity price process, average process, and the risk-neutral price of a general Asian option. In Chapter 3, we introduce the standard Black-Scholes model and the Heston model. Then, for each model, we compute the first few moments of the average price at maturity. In Chapter 4, we explore various distributions which are used to fit the distribution of the average process at the maturity. In Chapter 5, based on moments fitting, we derive the pricing formulae for fixed-strike Asian options. In Chapter 6, we present the numerical results obtained by Monte Carlo simulation and by methods of moments fitting. The latter results are then compared against those obtained by Monte Carlo simulation. In Chapter 7, we draw some conclusions about various methods of moments fitting.

Acknowledgements

I am deeply indebted to my supervisor Professor Tony Wirjanto for his immeasurable guidance and valuable advice in the program. Without his timely motivation and encouragement, this thesis would have been impossible.

I would like to thank Professors Adam Kolkiewicz and Andrew Heunis for their wonderful courses in financial mathematics and stochastic calculus which provide me a solid foundation for future research.

Table of Contents

List of Tables

Chapter 1

Introduction

Asian options are derivatives whose payoff is determined by the average value of the underlying asset over some pre-set period of time. They were introduced in the oil market in the late 1970s and are nowadays traded in over-the-counter (OTC) markets all over the world. Asian options are path-dependent due to the averaging and they are often classified as exotic options.

Compared to standard European options, Asian options have several advantages. First of all, due to averaging, the volatility of the average value is generally lower than that of the equity's price. Asian options, in general, are cheaper than the European counterpart and hence are better suited for hedging purposes. Secondly, since the average price is computed by sampling the underlying equity's price over a long period of time, Asian options reduce the risk of price manipulation near the maturity date, especially when the underlying is a thinly traded asset or commodity.

However, one major drawback of Asian options is that, except some rare cases (for example, when the strike $K = 0$), in general, there is no known analytical formula for computing its risk-neutral price. The difficulty comes from the fact that even under the standard Black-Scholes model (in which the stock price follows the GBM model with constant drift and constant volatility, and the short rate process is assumed to be a deterministic constant) the average price at the maturity A_T is not log-normally distributed. In order to obtain the risk-neutral price, we have to employ some approximation schemes. Currently, we have the following three different approaches of approximation.

The first approach is to approximate the distribution of the random variable A_T by a known distribution. The methodology of this approach is to choose suitable parameters for a known distribution such that the first few moments of that distribution match with those of the average price A_T. Then we approximate the expectation of the payoff, say, in the case of fixed-strike Asian call, $E^Q[\max(A_T - K, 0)]$ by $\int \max(x - K, 0) d\mu(x)$, where μ is the distribution measure of a known distribution. We hope that the known distribution is simple enough, so the integral can be computed relatively quickly by a numerical quadrature method, and even calculated analytically.

The second approach is to price Asian options by a PDE method. For the case of continuous averaging, the pricing problem reduces to solving a three dimensional partial differential equations, that is, there are two spatial variables and a time variable. For both fixed-strike and floating-strike options, the spatial dimension can be reduced by one by utilizing some special techniques. For the case of discretely monitored averaging, we notice that one spatial variable is missing between any two successive observation times. Therefore, the problem reduces to a pseudo one-dimensional problem.

The last approach is to price Asian options by Monte Carlo simulation. The advantage of this approach is that it is intuitive and simple. However, due to the nature of Monte Carlo simulation, this approach is rather slow.

The thesis is organized as follow. In Chapter 2, we lay down the mathematical setting, review relevant concepts such as filtration, conditional expectation, martingale, and define the average process, the payoff of an Asian option and its risk-neutral price. In Chapter 3, we compute the first few moments of the average price A_T, where the equity price process either follows the GBM model or the Heston model. Both discrete averaging and continuous averaging are considered. In Chapter 4, we explore various distributions which will be used to fit moments of the average price A_T. For each distribution, its probability density function will be given, its first few moments will be derived and expressed in terms of the parameters of that distribution. Lastly, the parameters are solved and expressed in terms of moments, either in closed-form or by Newton's method. In Chapter 5, we derive the pricing formulae for fixed-strike Asian options, either in closed-form or as an integral which is then calculated by Simpson rule. In Chapter 6, we show the numerical results of pricing fixed-strike Asian call options. The numerical results obtained by various moments fitting methods are then compared against those obtained by Monte Carlo simulation, which serves as a benchmark. In Chapter 7, we draw some conclusions about various methods of moments fitting.

Chapter 2

The Mathematical Framework for Asian Options

In this chapter, we outline the mathematical setting and define the payoff, which is the risk-neutral price of an Asian option. Throughout the thesis, the triplet $(\Omega, \mathcal{F}, Q)$ denotes a complete probability space and $(\mathcal{F}_t)_{t\in[0,\infty)}$ denotes a filtration satisfying the usual conditions. That is, for each $t \in [0, \infty)$, $\mathcal{F}_t$ is a σ-algebra on Ω such that for any $s, t \in [0, \infty)$ with $s < t$, $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$, and the following conditions are satisfied:

- completeness of Q: For any $A \subseteq \Omega$, $A \in \mathcal{F}$ whenever there exists $B \in \mathcal{F}$ such that $A \subseteq B$ and $Q(B) = 0$.
- richness of $\mathcal{F}_0$: For any $A \in \mathcal{F}$, if $Q(A) = 0$, then $A \in \mathcal{F}_0$.
- right-continuity of filtration: For any $t \in [0, \infty)$, $\mathcal{F}_t = \cap\{\mathcal{F}_u \mid u \in (t, \infty)\}$.

Given a σ-algebra $\mathcal{G}$ with $\mathcal{G} \subseteq \mathcal{F}$ and an integrable random variable X, there always exists a unique (up to a Q-measure zero set, i.e. Q-a.e.) $\mathcal{G}$-measurable integrable random variable Y such that for any $A \in \mathcal{G}$, $\int_A Y dQ = \int_A X dQ$. Y is called the conditional expectation of X given $\mathcal{G}$, denoted by $Y = E[X \mid \mathcal{G}]$. By a random process $(X_t)_{t\in I}$, we mean a family of random variables indexed by a sub-interval I of $[0, \infty)$, i.e., I is a sub-interval of $[0, \infty)$ such that for each $t \in I$, $X_t : \Omega \to \mathbb{R}$ is a $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable mapping. We say that the random process $(X_t)_{t\in I}$ is $(\mathcal{F}_t)_{t\in I}$-adapted if for each $t \in I$, X_t is $\mathcal{F}_t/\mathcal{B}(\mathbb{R})$-measurable. In this case, we say that $(X_t, \mathcal{F}_t)_{t\in I}$ is an adapted process. If the index set I is clear from the context, we sometime simplify the notation and simply denote the adapted process by

$(X_t, \mathcal{F}_t)_t$ or even $(X_t)_t$ if the filtration is understood. An adapted process $(X_t, \mathcal{F}_t)_{t \in I}$ is martingale if for each $t \in I$, $E[|X_t|] < \infty$ and $X_s = E[X_t \mid \mathcal{F}_s]$ whenever $s, t \in I$ with $s < t$.

Throughout the thesis, we fix a time horizon $T > 0$. Let $(r_t, \mathcal{F}_t)_{t \in [0,T]}$ be the short rate process, which is assumed to be sufficiently integrable. Let $(B_t)_{t \in [0,T]}$ be a risk-free bank account, defined by $B_t = \exp(\int_0^t r_u du)$. Here, we suppose that Q is a martingale measure in the following sense: If $(\Pi_t, \mathcal{F}_t)_{t \in [0,T]}$ is the price process of a traded asset in the market, the discounted price process, namely $(\frac{\Pi_t}{B_t}, \mathcal{F}_t)_{t \in [0,T]}$ is a martingale with respect to the Q-measure. Let $(S_t, \mathcal{F}_t)_{t \in [0,T]}$ be the price process of a non-dividend paying stock. In this thesis, $(S_t)_t$ is assumed to follow either a standard geometric Brownian motion (GBM) or Heston dynamics, which will be discussed in more detail at a later time. Let $(A_t, \mathcal{F}_t)_{t \in [0,T]}$ be the average process associated with $(S_t, \mathcal{F}_t)_t$, defined as follow. For the case of continuous averaging,

$$A_t = \begin{cases} S_0, & \text{if } t = 0, \\ \frac{1}{t} \int_0^t S_u du, & \text{if } t \in (0, T]. \end{cases}$$

For the case of discrete averaging, we fix a sequence $(t_i)_{i=0}^N$, with $0 = t_0 \leq t_1 < \ldots < t_N \leq T$. Then we define

$$A_t = \begin{cases} 0, & \text{if } t \in [0, t_1), \\ \frac{1}{k} \sum_{i=1}^k S_{t_i}, & \text{if } t \in [t_k, t_{k+1}), \text{ for } k = 1, \ldots, N-1, \\ \frac{1}{N} \sum_{i=1}^N S_{t_i}, & \text{if } t \in [t_N, T]. \end{cases}$$

Now, we are ready to introduce Asian options mathematically. An Asian option with strike K and maturity T is a contingent T-cliam whose payoff χ at the maturity T is:

- Fixed-strike Call: $\chi = \max(A_T - K, 0)$,
- Fixed-strike Put: $\chi = \max(K - A_T, 0)$,
- Floating-strike Call: $\chi = \max(S_T - KA_T, 0)$,
- Floating-strike Put: $\chi = \max(KA_T - S_T, 0)$.

By the general pricing theory, for each $t \in [0, T]$, we have

$$\frac{\Pi_t}{B_t} = E^Q \left[\frac{\Pi_T}{B_T} \mid \mathcal{F}_t \right] = E^Q \left[\frac{\chi}{B_T} \mid \mathcal{F}_t \right],$$

and hence $\Pi_t = B_t \cdot E^Q \left[\frac{\Pi_t}{B_t} \mid \mathcal{F}_t \right]$, ($Q$-a.e.). In this thesis, we will only consider the case of fixed-strike Asian options.

Chapter 3

Moments of the Average Price A_T

In this chapter, we give an introduction to the GBM model and the Heston model. Under each of the models, we compute the first few moments of the average price A_T. These results will be used in subsequent chapters for moments fitting to various distributions.

3.1 The Geometric Brownian Motion

The Geometric Brownian Motion (GBM) model, also known as the standard Black-Scholes model, is one the oldest mathematical model used for pricing derivatives. It was first published by Fischer Black and Myron Scholes in [3]. The main advantage of this model is its simplicity. For example, under the Black-Scholes model, we have closed-form formulae for the risk-neutral price of European options and continuously monitored barrier options. However, it is well-known that Black-Scholes model is over simplified and cannot capture the reality of the market. For example, the implied volatility of European options observed from the market is not constant among different strikes and maturities. Therefore, more advanced models like a local volatility model, a Heston model, etc. have been proposed and studied in the literature. In this section, we define the GBM model and derive first few moments of the average price A_T.

In the GBM model, the short rate process $(r_t)_t$ is assumed to be a deterministic constant, i.e., there is a constant r such that $r_t(\omega) = r$ for any $(t, \omega) \in [0, T] \times \Omega$. Let $(W_t, \mathcal{F}_t)_{t\in[0,\infty)}$ be a standard 1-dimensional Wiener process. The equity price process $(S_t)_{t\in[0,T]}$ is assumed to follow the dynamics:

$$dS_t = S_t \left(r dt + \sigma dW_t \right),$$

where the initial stock price $S_0 > 0$ and the volatility $\sigma \neq 0$ are deterministic constants.

In the following, we consider two modes of averaging, namely, continuous averaging and discrete averaging.

3.1.1 The case of continuous averaging

In this case, we can compute recursively the moments of A_T to an arbitrary order. However, for our moments fitting purpose, only the first three moments are needed. Define an auxiliary integral process $(I_t)_t$ by $I_t = \int_0^t S_u du$. In order to compute the moments of I_T, we need the following lemma:

Lemma 3.1.1. *Using the above notations, let $0 \le t_1 \le t_2 \le \ldots \le t_n \le T$. Then we have*

$$E\left[S_{t_1} S_{t_2} \cdots S_{t_n}\right] = S_0^n \exp\left(r(t_1 + t_2 + \ldots + t_n) + \sigma^2\left[(n-1)t_1 + (n-2)t_2 + \ldots + t_{n-1}\right]\right).$$

Proof. By Ito's lemma, we have

$$\begin{aligned} d\ln S_t &= \frac{1}{S_t} dS_t + \frac{1}{2} \cdot \frac{-1}{S_t^2} dS_t dS_t \\ &= (r dt + \sigma dW_t) - \frac{1}{2}\sigma^2 dt \\ &= \left(r - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t, \end{aligned}$$

so for $0 \le s \le t$, $k \in \mathbb{N}$,

$$\begin{aligned} \ln S_t - \ln S_s &= \int_s^t \left(r - \frac{1}{2}\sigma^2\right) du + \int_s^t \sigma dW_u, \\ S_t^k &= S_s^k \exp\left\{k\left(r - \frac{1}{2}\sigma^2\right)(t-s) + k\sigma\left(W_t - W_s\right)\right\}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} E\left[S_t^k \mid \mathcal{F}_s\right] &= S_s^k \cdot \exp\left\{k\left(r - \frac{1}{2}\sigma^2\right)(t-s) + \frac{1}{2}k^2\sigma^2(t-s)\right\} \\ &= S_s^k \cdot \exp\left\{kr(t-s) + \frac{k(k-1)}{2}\sigma^2(t-s)\right\}. \end{aligned}$$

Now, by the tower property of conditional expectation, we obtain

$$\begin{aligned}
E\left[S_{t_1}S_{t_2}\cdots S_{t_n}\right] &= E\left[E\left[S_{t_1}S_{t_2}\cdots S_{t_n} \mid \mathcal{F}_{t_{n-1}}\right]\right] \\
&= E\left[S_{t_1}S_{t_2}\cdots S_{t_{n-1}}\cdot S_{t_{n-1}}\exp\left\{r(t_n-t_{n-1})\right\}\right] \\
&= E\left[E\left[S_{t_1}S_{t_2}\cdots S_{t_{n-1}}\cdot S_{t_{n-1}}\exp\left\{r(t_n-t_{n-1})\right\} \mid \mathcal{F}_{t_{n-2}}\right]\right] \\
&= E\left[S_{t_1}S_{t_2}\cdots S_{t_{n-2}}\exp\left\{2r(t_{n-1}-t_{n-2})+\sigma^2(t_{n-1}-t_{n-2})+r(t_n-t_{n-1})\right\}\right] \\
&= \cdots \\
&= S_0^n\exp\left\{r(t_1+t_2+\cdots+t_n)+\sigma^2\cdot\sum_{k=1}^{n-1}(n-k)t_k\right\}.
\end{aligned}$$

□

Next we compute moments of I_T.

Proposition 3.1.1. *Using the above notations, the first three moments of I_T are*

$$\begin{aligned}
E\left[I_T\right] &= \frac{S_0}{r}\left\{\exp(rT)-1\right\}, \\
E[I_T^2] &= \frac{2S_0^2}{r(r+\sigma^2)(2r+\sigma^2)}\left\{r\exp\left((2r+\sigma^2)T\right)-(2r+\sigma^2)\exp(rT)+r+\sigma^2\right\}, \\
E\left[I_T^3\right] &= \frac{6S_0^3}{r+2\sigma^2}\left\{\frac{1}{2r+3\sigma^2}\cdot\frac{1}{3(r+\sigma^2)}\exp\left(3(r+\sigma^2)T\right)-\frac{1}{r+\sigma^2}\cdot\frac{1}{2r+\sigma^2}\exp\left(\left(2r+\sigma^2\right)T\right)\right. \\
&\quad+\left(\frac{1}{r(r+\sigma^2)}-\frac{1}{r(2r+3\sigma^2)}\right)\exp(rT)+\frac{1}{r(2r+3\sigma^2)}+\frac{1}{(r+\sigma^2)(2r+\sigma^2)} \\
&\quad\left.-\frac{1}{3(r+\sigma^2)(2r+3\sigma^2)}-\frac{1}{r(r+\sigma^2)}\right\}.
\end{aligned}$$

Proof. Fix $n\in\mathbb{N}$. Observe that the cube $[0,T]^n$ can be represented as a disjoint union of sets of the form $\{(t_1,\ldots,t_n)\,|\,0<t_1<\ldots<t_n<T\}$ and a set of Lebesgue measure zero, namely

$$[0,T]^n=\cup_\tau\{(t_{\tau_1},\ldots,t_{\tau_n})\,|\,0<t_{\tau_1}<\ldots<t_{\tau_n}<T\}\cup A,$$

where A is a subset of $\mathbb{R}^n$ with Lebesgue measure zero and the union is taken over all permuations τ on the set $\{1,2,\ldots,n\}$. For each permutation τ on the set $\{1,2,\ldots,n\}$, let

$$A_\tau=\{(t_{\tau_1},t_{\tau_2},\ldots,t_{\tau_n})\,|\,0<t_{\tau_1}<t_{\tau_2}<\ldots<t_{\tau_n}<T\}$$

and let

$$A' = \{(t_1, t_2, \cdots, t_n) \,|\, 0 < t_1 < t_2 < \ldots < t_n < T\}.$$

By symmetry,

$$\begin{aligned} E^Q\left[(I_T)^n\right] &= \sum_{\tau} E^Q\left[\int_{A_\tau} S_{t_{\tau_1}} S_{t_{\tau_2}} \cdots S_{t_{\tau_n}} d\lambda^n\right] \\ &= (n!)\int_{A'} E^Q\left[S_{t_1} S_{t_2} \ldots S_{t_n}\right] d\lambda^n. \end{aligned} \tag{3.1}$$

For small n, the integral in (3.1) can be computed directly. For example, we have

$$E^Q\left[I_T\right] = \int_0^T E^Q\left[S_u\right] du = \int_0^T S_0 \exp(ru) du = \frac{S_0}{r}\left\{\exp(rT) - 1\right\}.$$

By Lemma 3.1.1 , we also have

$$\begin{aligned} E^Q\left[(I_T)^2\right] &= 2!\int_{A'} E^Q\left[S_{t_1} S_{t_2}\right] d\lambda^2 \\ &= 2\int_{A'} S_0^2 \exp\left\{r(t_1+t_2) + \sigma^2 t_1\right\} d\lambda^2 \\ &= 2\int_0^T \int_0^{t_2} S_0^2 \exp\left\{r(t_1+t_2) + \sigma^2 t_1\right\} dt_1 dt_2 \\ &= \frac{2S_0^2}{r+\sigma^2}\int_0^T \exp(rt_2)\left\{\exp\left((r+\sigma^2)t_2\right) - 1\right\} dt_2 \\ &= \frac{2S_0^2}{r+\sigma^2}\left\{\frac{1}{2r+\sigma^2}\left(e^{(2r+\sigma^2)T} - 1\right) - \frac{1}{r}\left(e^{rT} - 1\right)\right\} \\ &= \frac{2S_0^2}{r(r+\sigma^2)(2r+\sigma^2)}\left\{re^{(2r+\sigma^2)T} - (2r+\sigma^2)e^{rT} + r + \sigma^2\right\}. \end{aligned}$$

Finally, we have

$$
\begin{aligned}
E^Q\left[(I_T)^3\right] &= 3!\int_{A'} E^Q\left[S_{t_1}S_{t_2}S_{t3}\right]d\lambda^3 \\
&= 6S_0^3\int_0^T\int_0^{t_3}\int_0^{t_2}\exp\left\{r(t_1+t_2+t_3)+\sigma^2(2t_1+t_2)\right\}dt_1dt_2dt_3 \\
&= \frac{6S_0^3}{r+2\sigma^2}\int_0^T\int_0^{t_3}\exp\left\{r(t_2+t_3)+\sigma^2t_2\right\}\left\{\exp\left((r+2\sigma^2)t_2\right)-1\right\}dt_2dt_3 \\
&= \frac{6S_0^3}{r+2\sigma^2}\int_0^T e^{rt_3}\left\{\frac{1}{2r+3\sigma^2}\left(e^{t_3(2r+3\sigma^2)}-1\right)-\frac{1}{r+\sigma^2}\left(e^{t_3(r+\sigma^2)}-1\right)\right\}dt_3 \\
&= \frac{6S_0^3}{r+2\sigma^2}\int_0^T\left\{\frac{1}{2r+3\sigma^2}\left(e^{3(r+\sigma^2)t_3}-e^{rt_3}\right)-\frac{1}{r+\sigma^2}\left(e^{(2r+\sigma^2)t_3}-e^{rt_3}\right)\right\}dt_3 \\
&= \frac{6S_0^3}{r+2\sigma^2}\left\{\frac{1}{2r+3\sigma^2}\left[\frac{1}{3(r+\sigma^2)}\left(e^{3(r+\sigma^2)T}-1\right)-\frac{1}{r}\left(e^{rT}-1\right)\right]-\right. \\
&\quad\left.\frac{1}{r+\sigma^2}\left[\frac{1}{2r+\sigma^2}\left(e^{(2r+\sigma^2)T}-1\right)-\frac{1}{r}\left(e^{rT}-1\right)\right]\right\} \\
&= \frac{6S_0^3}{r+2\sigma^2}\left\{\frac{1}{2r+3\sigma^2}\cdot\frac{1}{3(r+\sigma^2)}e^{3(r+\sigma^2)T}-\frac{1}{r+\sigma^2}\cdot\frac{1}{2r+\sigma^2}e^{(2r+\sigma^2)T}+\right. \\
&\quad\left(\frac{1}{r(r+\sigma^2)}-\frac{1}{r(2r+3\sigma^2)}\right)e^{rT}+\frac{1}{r(2r+3\sigma^2)}+\frac{1}{(r+\sigma^2)(2r+\sigma^2)}- \\
&\quad\left.\frac{1}{3(r+\sigma^2)(2r+3\sigma^2)}-\frac{1}{r(r+\sigma^2)}\right\}.
\end{aligned}
$$

□

Remark 3.1.1. *Since* $A_T = \frac{1}{T}I_T$*, the first three moments of* A_T *can be written down* *immediately as*

$$
\begin{aligned}
E[A_T] &= \frac{S_0}{rT}\left\{\exp(rT)-1\right\}, \\
E[A_T^2] &= \frac{2S_0^2}{r(r+\sigma^2)(2r+\sigma^2)T^2}\left\{r\exp\big((2r+\sigma^2)T\big)-(2r+\sigma^2)\exp(rT)+r+\sigma^2\right\}, \\
E[A_T^3] &= \frac{6S_0^3}{(r+2\sigma^2)T^3}\left\{\frac{1}{2r+3\sigma^2}\cdot\frac{1}{3(r+\sigma^2)}\exp\big(3(r+\sigma^2)T\big)-\frac{1}{r+\sigma^2}\cdot\frac{1}{2r+\sigma^2}\exp\big((2r+\sigma^2)T\big)\right. \\
&\quad +\left(\frac{1}{r(r+\sigma^2)}-\frac{1}{r(2r+3\sigma^2)}\right)\exp(rT)+\frac{1}{r(2r+3\sigma^2)}+\frac{1}{(r+\sigma^2)(2r+\sigma^2)} \\
&\quad \left.-\frac{1}{3(r+\sigma^2)(2r+3\sigma^2)}-\frac{1}{r(r+\sigma^2)}\right\}.
\end{aligned}
$$

3.1.2 The case of discrete averaging

Let T be the maturity time and let $t_1, t_2, \ldots, t_n$ be discrete observing times, satisfying $0 \le t_1 < t_2 < \ldots < t_n \le T$. Define $I_T = \sum_{k=1}^n S_{t_k}$ and $A_T = \frac{1}{n}\sum_{k=1}^n S_{t_k}$. We have the following proposition.

Proposition 3.1.2. *Using the above notations, we have:*

$$
\begin{aligned}
E[I_T] &= S_0\sum_{k=1}^n \exp(rt_k), \\
E[I_T^2] &= 2S_0^2\sum_{i<j}\exp(r(t_i+t_j)+\sigma^2 t_i)+S_0^2\sum_{i=1}^n\exp(2rt_i+\sigma^2 t_i), \\
E[I_T^3] &= 6S_0^3\sum_{i<j<k}\exp\big(r(t_i+t_j+t_k)+\sigma^2(2t_i+t_j)\big)+ \\
&\quad 3S_0^3\sum_{i<j}\exp\big(r(2t_i+t_j)+\sigma^2(3t_i)\big)+ \\
&\quad 3S_0^3\sum_{i<j}\exp\big(r(t_i+2t_j)+\sigma^2(2t_i+t_j)\big)+ \\
&\quad S_0^3\sum_{i=1}^n\exp\big(3rt_i+3\sigma^2 t_i\big).
\end{aligned}
$$

Proof. By Lemma 3.1.1,

$$E[I_T] = \sum_{k=1}^{n} E[S_k] = S_0 \sum_{k=1}^{n} \exp{(rt_k)}\,.$$

$$\begin{aligned} E[I_T^2] &= \sum_{i,j=1}^{n} E[S_{t_i} S_{t_j}] \\ &= 2\sum_{i<j} E[S_{t_i} S_{t_j}] + \sum_{i=1}^{n} E[S_{t_i}^2] \\ &= 2S_0^2 \sum_{i<j} \exp\left(r(t_i + t_j) + \sigma^2 t_i\right) + S_0^2 \sum_{i=1}^{n} \exp\left(2rt_i + \sigma^2 t_i\right). \end{aligned}$$

$$\begin{aligned} E[I_T^3] &= \sum_{i,j,k=1}^{n} E[S_{t_i} S_{t_j} S_{t_k}] \\ &= 3! \sum_{i<j<k} E[S_{t_i} S_{t_j} S_{t_k}] +_3 C_1 \sum_{i<j} E[S_{t_i}^2 S_{t_j}] \\ &\quad +_3 C_1 \sum_{i<j} E[S_{t_i} S_{t_j}^2] + \sum_{i=1}^{n} E[S_{t_i}^3] \\ &= 6S_0^3 \sum_{i<j<k} \exp\left(r(t_i + t_j + t_k) + \sigma^2(2t_i + t_j)\right) + \\ &\quad 3S_0^3 \sum_{i<j} \exp\left(r(2t_i + t_j) + \sigma^2(3t_i)\right) + \\ &\quad 3S_0^3 \sum_{i<j} \exp\left(r(t_i + 2t_j) + \sigma^2(2t_i + t_j)\right) + \\ &\quad S_0^3 \sum_{i=1}^{n} \exp\left(3rt_i + 3\sigma^2 t_i\right). \end{aligned}$$

□

Notice that if the observation times t_i are evenly spaced and $t_n = T$, the moments in Proposition 3.1.2 can be simplified as follows.

Proposition 3.1.3. *Using the above notations and further assuming that* $t_i = ih$, *where* $h = \frac{T}{n}$, $i = 1, \ldots, n$, *we have:*

$$\begin{aligned}
E[I_T] &= S_0 e^{rh} \cdot \frac{1-e^{rhn}}{1-e^{rh}}, \\
E[I_T^2] &= S_0^2 \Bigg\{ \frac{2e^{(r+\sigma^2)h}}{1-e^{(r+\sigma^2)h}} \cdot \frac{e^{2rh}-e^{rh(n+1)}}{1-e^{rh}} \\
&\quad -2 \cdot \frac{e^{2(2r+\sigma^2)h}-e^{(2r+\sigma^2)h(n+1)}}{\left(1-e^{(r+\sigma^2)h}\right)\left(1-e^{(2r+\sigma^2)h}\right)} \\
&\quad + \frac{1-e^{(2r+\sigma^2)hn}}{1-e^{(2r+\sigma^2)h}} e^{(2r+\sigma^2)h} \Bigg\},
\end{aligned}$$

$$\begin{aligned}
E[I_T^3] &= S_0^3 \Bigg\{ \frac{6e^{(6r+4\sigma^2)h}}{\left(1-e^{(r+2\sigma^2)h}\right)\left(1-e^{(r+\sigma^2)h}\right)} \cdot \frac{1-e^{rh(n-2)}}{1-e^{rh}} \\
&\quad - \frac{6e^{(7r+5\sigma^2)h}}{\left(1-e^{(r+2\sigma^2)h}\right)\left(1-e^{(r+\sigma^2)h}\right)} \cdot \frac{1-e^{(2r+\sigma^2)h(n-2)}}{1-e^{(2r+\sigma^2)h}} \\
&\quad - \frac{6e^{(7r+6\sigma^2)h}}{\left(1-e^{(r+2\sigma^2)h}\right)\left(1-e^{(2r+3\sigma^2)h}\right)} \cdot \frac{1-e^{rh(n-2)}}{1-e^{rh}} \\
&\quad + \frac{6e^{9(r+\sigma^2)h}}{\left(1-e^{(r+2\sigma^2)h}\right)\left(1-e^{(2r+3\sigma^2)h}\right)} \cdot \frac{1-e^{3(r+\sigma^2)h(n-2)}}{1-e^{3(r+\sigma^2)h}} \\
&\quad + \frac{3e^{(4r+3\sigma^2)h}\left(1-e^{rh(n-1)}\right)}{\left(1-e^{(2r+3\sigma^2)h}\right)\left(1-e^{rh}\right)} - \frac{3e^{6(r+\sigma^2)h}\left(1-e^{3(r+\sigma^2)h(n-1)}\right)}{\left(1-e^{(2r+3\sigma^2)h}\right)\left(1-e^{3(r+\sigma^2)h}\right)} \\
&\quad + \frac{3e^{(5r+4\sigma^2)h}\left(1-e^{(2r+\sigma^2)h(n-1)}\right)}{\left(1-e^{(r+2\sigma^2)h}\right)\left(1-e^{(2r+\sigma^2)h}\right)} - \frac{3e^{6(r+\sigma^2)h}\left(1-e^{3(r+\sigma^2)h(n-1)}\right)}{\left(1-e^{(r+2\sigma^2)h}\right)\left(1-e^{3(r+\sigma^2)h}\right)} \\
&\quad + \frac{e^{3(r+\sigma^2)h}\left(1-e^{3(r+\sigma^2)hn}\right)}{1-e^{3(r+\sigma^2)h}} \Bigg\}.
\end{aligned}$$

Proof. By Proposition 3.1.2,

$$E[I_T] = S_0 \sum_{i=1}^{n} \exp(rt_i) = S_0 \sum_{i=1}^{n} \exp(irh) = S_0 e^{rh} \frac{1-e^{rhn}}{1-e^{rh}}.$$

Next, we compute $\sum_{i<j} \exp\left(r(t_i+t_j)+\sigma^2 t_i\right)$ and $\sum_{i=1}^{n} \exp(2rt_i+\sigma^2 t_i)$.

$$\begin{aligned}
&\sum_{i<j} \exp\Big(r(t_i+t_j)+\sigma^2 t_i\Big) \\
=\ & \sum_{j=2}^{n}\sum_{i=1}^{j-1} \exp\Big(rh(i+j)+\sigma^2 hi\Big) \\
=\ & \sum_{j=2}^{n} \exp(rhj)\cdot\exp\Big((r+\sigma^2)h\Big)\cdot\frac{1-e^{(r+\sigma^2)h(j-1)}}{1-e^{(r+\sigma^2)h}} \\
=\ & \frac{e^{(r+\sigma^2)h}}{1-e^{(r+\sigma^2)h}}\cdot\left\{e^{2rh}\cdot\frac{1-e^{rh(n-1)}}{1-e^{rh}} - e^{-(r+\sigma^2)h}\cdot e^{2h(2r+\sigma^2)}\cdot\frac{1-e^{(2r+\sigma^2)h(n-1)}}{1-e^{(2r+\sigma^2)h}}\right\} \\
=\ & \frac{e^{(r+\sigma^2)h}}{1-e^{(r+\sigma^2)h}}\cdot\frac{e^{2rh}-e^{rh(n+1)}}{1-e^{rh}} - \frac{e^{2(2r+\sigma^2)h}-e^{(2r+\sigma^2)h(n+1)}}{\left(1-e^{(r+\sigma^2)h}\right)\left(1-e^{(2r+\sigma^2)h}\right)}.
\end{aligned}$$

$$\sum_{i=1}^{n} \exp(2rt_i+\sigma^2 t_i) = \sum_{i=1}^{n} \exp(2rhi+\sigma^2 hi) = e^{(2r+\sigma^2)h}\cdot\frac{1-e^{(2r+\sigma^2)hn}}{1-e^{(2r+\sigma^2)h}}.$$

Hence, we have

$$\begin{aligned}
E[I_T^2] \ =\ & S_0^2\left\{\frac{2e^{(r+\sigma^2)h}}{1-e^{(r+\sigma^2)h}}\cdot\frac{e^{2rh}-e^{rh(n+1)}}{1-e^{rh}}\right. \\
& -2\cdot\frac{e^{2(2r+\sigma^2)h}-e^{(2r+\sigma^2)h(n+1)}}{\left(1-e^{(r+\sigma^2)h}\right)\left(1-e^{(2r+\sigma^2)h}\right)} \\
& \left.+\frac{1-e^{(2r+\sigma^2)hn}}{1-e^{(2r+\sigma^2)h}}e^{(2r+\sigma^2)h}\right\}.
\end{aligned}$$

To compute the third moment $E[I_T^3]$, we need $\sum_{i<j<k} \exp\left(r(t_i+t_j+t_k)+\sigma^2(2t_i+t_j)\right)$, $\sum_{i<j} \exp\left(r(2t_i+t_j)+\sigma^2(3t_i)\right)$, $\sum_{i<j} \exp\left(r(t_i+2t_j)+\sigma^2(2t_i+t_j)\right)$, and $\sum_{i=1}^{n} \exp\left(3rt_i+3\sigma^2 t_i\right)$.

$$
\begin{aligned}
&\sum_{i<j<k} \exp\big(r(t_i+t_j+t_k)+\sigma^2(2t_i+t_j)\big) \\
=\ & \sum_{k=3}^{n}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1} \exp\big(rh(j+k)+\sigma^2 hj\big)\exp\big((r+2\sigma^2)hi\big) \\
=\ & \sum_{k=3}^{n}\sum_{j=2}^{k-1} \exp\big((r+\sigma^2)hj+rhk\big)\cdot e^{(r+2\sigma^2)h}\cdot\frac{1-e^{(r+2\sigma^2)h(j-1)}}{1-e^{(r+2\sigma^2)h}} \\
=\ & \frac{e^{(r+2\sigma^2)h}}{1-e^{(r+2\sigma^2)h}}\sum_{k=3}^{n} e^{rhk}\sum_{j=2}^{k-1}\Big(e^{(r+\sigma^2)hj}-e^{-(r+2\sigma^2)h}e^{(2r+3\sigma^2)hj}\Big) \\
=\ & \frac{e^{(r+2\sigma^2)h}}{1-e^{(r+2\sigma^2)h}}\sum_{k=3}^{n} e^{rhk}\Bigg\{e^{2(r+\sigma^2)h}\cdot\frac{1-e^{(r+\sigma^2)h(k-2)}}{1-e^{(r+\sigma^2)h}} \\
& -e^{-(r+2\sigma^2)h}\cdot e^{(2r+3\sigma^2)\cdot 2h}\cdot\frac{1-e^{(2r+3\sigma^2)h(k-2)}}{1-e^{(2r+3\sigma^2)h}}\Bigg\} \\
=\ & \frac{e^{(3r+4\sigma^2)h}}{\big(1-e^{(r+2\sigma^2)h}\big)\big(1-e^{(r+\sigma^2)h}\big)}\Bigg\{e^{3rh}\cdot\frac{1-e^{rh(n-2)}}{1-e^{rh}} \\
& -e^{-2(r+\sigma^2)h}\cdot e^{(2r+\sigma^2)\cdot 3h}\cdot\frac{1-e^{(2r+\sigma^2)h(n-2)}}{1-e^{(2r+\sigma^2)h}}\Bigg\} \\
& -\frac{e^{(4r+6\sigma^2)h}}{\big(1-e^{(r+2\sigma^2)h}\big)\big(1-e^{(2r+3\sigma^2)h}\big)}\Bigg\{e^{3rh}\cdot\frac{1-e^{rh(n-2)}}{1-e^{rh}} \\
& -e^{-(4r+6\sigma^2)h}\cdot e^{9(r+\sigma^2)h}\cdot\frac{1-e^{3(r+\sigma^2)h(n-2)}}{1-e^{3(r+\sigma^2)h}}\Bigg\} \\
=\ & \frac{e^{(6r+4\sigma^2)h}}{\big(1-e^{(r+2\sigma^2)h}\big)\big(1-e^{(r+\sigma^2)h}\big)}\cdot\frac{1-e^{rh(n-2)}}{1-e^{rh}} \\
& -\frac{e^{(7r+5\sigma^2)h}}{\big(1-e^{(r+2\sigma^2)h}\big)\big(1-e^{(r+\sigma^2)h}\big)}\cdot\frac{1-e^{(2r+\sigma^2)h(n-2)}}{1-e^{(2r+\sigma^2)h}} \\
& -\frac{e^{(7r+6\sigma^2)h}}{\big(1-e^{(r+2\sigma^2)h}\big)\big(1-e^{(2r+3\sigma^2)h}\big)}\cdot\frac{1-e^{rh(n-2)}}{1-e^{rh}} \\
& +\frac{e^{9(r+\sigma^2)h}}{\big(1-e^{(r+2\sigma^2)h}\big)\big(1-e^{(2r+3\sigma^2)h}\big)}\cdot\frac{1-e^{3(r+\sigma^2)(n-2)h}}{1-e^{3(r+\sigma^2)h}}.
\end{aligned}
$$

Next, we compute the quantity $\sum_{i<j} \exp\left(r(2t_i + t_j) + \sigma^2(3t_i)\right)$ as follow.

$$\begin{aligned}
& \sum_{i<j} \exp\Big(r(2t_i + t_j) + \sigma^2(3t_i)\Big) \\
= \quad & \sum_{j=2}^{n} \sum_{i=1}^{j-1} \exp\Big((2r + 3\sigma^2)hi + rhj\Big) \\
= \quad & \sum_{j=2}^{n} e^{rhj} \cdot \sum_{i=1}^{j-1} e^{(2r+3\sigma^2)hi} \\
= \quad & \sum_{j=2}^{n} e^{rhj} \cdot e^{(2r+3\sigma^2)h} \cdot \frac{1 - e^{(2r+3\sigma^2)h(j-1)}}{1 - e^{(2r+3\sigma^2)h}} \\
= \quad & \frac{e^{(2r+3\sigma^2)h}}{1 - e^{(2r+3\sigma^2)h}} \left[\sum_{j=2}^{n} e^{rhj} - \sum_{j=2}^{n} e^{3(r+\sigma^2)hj} \cdot e^{-(2r+3\sigma^2)h} \right] \\
= \quad & \frac{e^{(2r+3\sigma^2)h}}{1 - e^{(2r+3\sigma^2)h}} \cdot e^{2rh} \cdot \frac{1 - e^{rh(n-1)}}{1 - e^{rh}} \\
& - \frac{1}{1 - e^{(2r+3\sigma^2)h}} \cdot e^{6(r+\sigma^2)h} \cdot \frac{1 - e^{3(r+\sigma^2)h(n-1)}}{1 - e^{3(r+\sigma^2)h}} \\
= \quad & \frac{e^{(4r+3\sigma^2)h}\left(1 - e^{rh(n-1)}\right)}{\left(1 - e^{(2r+3\sigma^2)h}\right)\left(1 - e^{rh}\right)} - \frac{e^{6(r+\sigma^2)h}\left(1 - e^{3(r+\sigma^2)h(n-1)}\right)}{\left(1 - e^{(2r+3\sigma^2)h}\right)\left(1 - e^{3(r+\sigma^2)h}\right)}.
\end{aligned}$$

For $\sum_{i<j} \exp\left(r(t_i + 2t_j) + \sigma^2(2t_i + t_j)\right)$, we have

$$\begin{aligned}
&\sum_{i<j} \exp\Big(r(t_i + 2t_j) + \sigma^2(2t_i + t_j)\Big) \\
= \;& \sum_{j=2}^{n} \sum_{i=1}^{j-1} \exp\Big((2r + \sigma^2)hj + (r + 2\sigma^2)hi\Big) \\
= \;& \sum_{j=2}^{n} \exp\Big((2r + \sigma^2)hj\Big) \cdot e^{(r+2\sigma^2)h} \cdot \frac{1 - e^{(r+2\sigma^2)h(j-1)}}{1 - e^{(r+2\sigma^2)h}} \\
= \;& \frac{e^{(r+2\sigma^2)h}}{1 - e^{(r+2\sigma^2)h}} \cdot \Bigg\{ e^{2(2r+\sigma^2)h} \cdot \frac{1 - e^{(2r+\sigma^2)h(n-1)}}{1 - e^{(2r+\sigma^2)h}} \\
& - e^{-(r+2\sigma^2)h} \cdot e^{6(r+\sigma^2)h} \cdot \frac{1 - e^{3(r+\sigma^2)h(n-1)}}{1 - e^{3(r+\sigma^2)h}} \Bigg\} \\
= \;& \frac{e^{(5r+4\sigma^2)h}}{1 - e^{(r+2\sigma^2)h}} \cdot \frac{1 - e^{(2r+\sigma^2)h(n-1)}}{1 - e^{(2r+\sigma^2)h}} - \frac{e^{6(r+\sigma^2)h}}{1 - e^{(r+2\sigma^2)h}} \cdot \frac{1 - e^{3(r+\sigma^2)h(n-1)}}{1 - e^{3(r+\sigma^2)h}}.
\end{aligned}$$

Lastly,

$$\sum_{i=1}^{n} \exp\Big(3rt_i + 3\sigma^2 t_i\Big) = \sum_{i=1}^{n} e^{3(r+\sigma^2)hi} = e^{3(r+\sigma^2)h} \cdot \frac{1 - e^{3(r+\sigma^2)hn}}{1 - e^{3(r+\sigma^2)h}}.$$

Therefore, we have

$$
\begin{aligned}
E[I_T^3] \quad = \quad & S_0^3 \Bigg\{ \frac{6e^{(6r+4\sigma^2)h}}{\left(1-e^{(r+2\sigma^2)h}\right)\left(1-e^{(r+\sigma^2)h}\right)} \cdot \frac{1-e^{rh(n-2)}}{1-e^{rh}} \\
& - \frac{6e^{(7r+5\sigma^2)h}}{\left(1-e^{(r+2\sigma^2)h}\right)\left(1-e^{(r+\sigma^2)h}\right)} \cdot \frac{1-e^{(2r+\sigma^2)h(n-2)}}{1-e^{(2r+\sigma^2)h}} \\
& - \frac{6e^{(7r+6\sigma^2)h}}{\left(1-e^{(r+2\sigma^2)h}\right)\left(1-e^{(2r+3\sigma^2)h}\right)} \cdot \frac{1-e^{rh(n-2)}}{1-e^{rh}} \\
& + \frac{6e^{9(r+\sigma^2)h}}{\left(1-e^{(r+2\sigma^2)h}\right)\left(1-e^{(2r+3\sigma^2)h}\right)} \cdot \frac{1-e^{3(r+\sigma^2)h(n-2)}}{1-e^{3(r+\sigma^2)h}} \\
& + \frac{3e^{(4r+3\sigma^2)h}\left(1-e^{rh(n-1)}\right)}{\left(1-e^{(2r+3\sigma^2)h}\right)\left(1-e^{rh}\right)} - \frac{3e^{6(r+\sigma^2)h}\left(1-e^{3(r+\sigma^2)h(n-1)}\right)}{\left(1-e^{(2r+3\sigma^2)h}\right)\left(1-e^{3(r+\sigma^2)h}\right)} \\
& + \frac{3e^{(5r+4\sigma^2)h}\left(1-e^{(2r+\sigma^2)h(n-1)}\right)}{\left(1-e^{(r+2\sigma^2)h}\right)\left(1-e^{(2r+\sigma^2)h}\right)} - \frac{3e^{6(r+\sigma^2)h}\left(1-e^{3(r+\sigma^2)h(n-1)}\right)}{\left(1-e^{(r+2\sigma^2)h}\right)\left(1-e^{3(r+\sigma^2)h}\right)} \\
& + \frac{e^{3(r+\sigma^2)h}\left(1-e^{3(r+\sigma^2)hn}\right)}{1-e^{3(r+\sigma^2)h}} \Bigg\}.
\end{aligned}
$$

□

Remark 3.1.2. *Consider the case when $n \to \infty$, or equivalently $h \to 0$. As expected, we will see that moments of A_T approach those where A_T is defined by continuous averaging, i.e., $A_T = \frac{1}{T}\int_0^T S_u du$. Recall that in the discrete case, $A_T^{(n)} = \frac{1}{n}\sum_{i=1}^n S_{t_i}$. Therefore, the limits of moments are given as follow:*

$$
\begin{aligned}
\lim_{h\to 0} E[A_T^{(n)}] &= \lim_{h\to 0} \frac{S_0}{\frac{T}{h}} e^{rh} \cdot \frac{1-e^{rhn}}{1-e^{rh}} \\
&= \frac{S_0}{T}(1-e^{rT}) \lim_{h\to 0} \frac{h}{1-e^{rh}} \\
&= \frac{S_0}{rT}(e^{rT}-1).
\end{aligned}
$$

In the above, we evaluate the limit by L'Hospital rule: $\lim_{x\to 0+} \frac{x}{1-e^{rx}} = \lim_{x\to 0+} \frac{1}{-re^{rx}} = -\frac{1}{r}$.

Similarly, we have:

$$\begin{aligned}
&\lim_{h\to 0} E[A_T^{(n)2}] \\
= \;& \frac{S_0^2}{T^2}\left\{2\lim_{h\to 0}\frac{h}{1-e^{(r+\sigma^2)h}}\cdot\frac{h(1-e^{rT})}{1-e^{rh}} - 2(1-e^{(2r+\sigma^2)T})\lim_{h\to 0}\frac{h}{1-e^{(r+\sigma^2)h}}\cdot\frac{h}{1-e^{(2r+\sigma^2)h}}\right\} \\
= \;& \frac{2S_0^2}{T^2}\left\{\frac{1}{-(r+\sigma^2)}\cdot\frac{1-e^{rT}}{-r} - (1-e^{(2r+\sigma^2)T})\cdot\frac{1}{-(r+\sigma^2)}\cdot\frac{1}{-(2r+\sigma^2)}\right\} \\
= \;& \frac{2S_0^2}{T^2 r(r+\sigma^2)(2r+\sigma^2)}\left\{(1-e^{rT})(2r+\sigma^2) - (1-e^{(2r+\sigma^2)T})r\right\} \\
= \;& \frac{2S_0^2}{T^2 r(r+\sigma^2)(2r+\sigma^2)}\left\{r+\sigma^2-(2r+\sigma^2)e^{rT}+re^{(2r+\sigma^2)T}\right\},
\end{aligned}$$

and

$$
\begin{aligned}
& \lim_{h\to 0} E[A_T^{(n)3}] \\
= \; & \frac{S_0^3}{T^3}\Bigg\{6\left(1-e^{rT}\right)\cdot \lim_{h\to 0}\frac{h}{1-e^{(r+2\sigma^2)h}}\cdot\frac{h}{1-e^{(r+\sigma^2)h}}\cdot\frac{h}{1-e^{rh}} \\
& -6\left(1-e^{(2r+\sigma^2)T}\right)\cdot \lim_{h\to 0}\frac{h}{1-e^{(r+2\sigma^2)h}}\cdot\frac{h}{1-e^{(r+\sigma^2)h}}\cdot\frac{h}{1-e^{(2r+\sigma^2)h}} \\
& -6\left(1-e^{rT}\right)\cdot \lim_{h\to 0}\frac{h}{1-e^{(r+2\sigma^2)h}}\cdot\frac{h}{1-e^{(2r+3\sigma^2)h}}\cdot\frac{h}{1-e^{rh}} \\
& +6\left(1-e^{3(r+\sigma^2)T}\right)\cdot \lim_{h\to 0}\frac{h}{1-e^{(r+2\sigma^2)h}}\cdot\frac{h}{1-e^{(2r+3\sigma^2)h}}\cdot\frac{h}{1-e^{3(r+\sigma^2)h}}\Bigg\} \\
= \; & \frac{6S_0^3}{T^3}\Bigg\{\frac{-\left(1-e^{rT}\right)}{(r+2\sigma^2)(r+\sigma^2)r}+\frac{1-e^{(2r+\sigma^2)T}}{(r+2\sigma^2)(r+\sigma^2)(2r+\sigma^2)} \\
& +\frac{1-e^{rT}}{(r+2\sigma^2)(2r+3\sigma^2)r}-\frac{1-e^{3(r+\sigma^2)T}}{(r+2\sigma^2)(2r+3\sigma^2)\cdot 3(r+\sigma^2)}\Bigg\} \\
= \; & \frac{6S_0^3}{T^3(r+2\sigma^2)}\Bigg\{\left[\frac{-1}{r(r+\sigma^2)}+\frac{1}{r(2r+3\sigma^2)}\right]\left(1-e^{rT}\right)+\frac{1-e^{(2r+\sigma^2)T}}{(r+\sigma^2)(2r+\sigma^2)} \\
& -\frac{1-e^{3(r+\sigma^2)T}}{3(2r+3\sigma^2)(r+\sigma^2)}\Bigg\} \\
= \; & \frac{6S_0^3}{(r+2\sigma^2)T^3}\Bigg\{\frac{e^{3(r+\sigma^2)T}}{3(r+\sigma^2)(2r+3\sigma^2)}-\frac{e^{(2r+\sigma^2)T}}{(r+\sigma^2)(2r+\sigma^2)} \\
& +\left[\frac{1}{r(r+\sigma^2)}-\frac{1}{r(2r+3\sigma^2)}\right]e^{rT}+\frac{1}{r(2r+3\sigma^2)}+\frac{1}{(r+\sigma^2)(2r+\sigma^2)} \\
& -\frac{1}{3(r+\sigma^2)(2r+3\sigma^2)}-\frac{1}{r(r+\sigma^2)}\Bigg\}.
\end{aligned}
$$

3.2 The Heston Model

The Heston Model, proposed by Steven Heston [5], is a more advanced mathematical model for modeling the evolution of the price of an asset. Under the Heston model assumption, the volatility of the price process of an asset is no longer deterministic. Rather, it is stochastic and follows the Cox-Ingersoll-Ross model (CIR-model), first studied in [4]. The main advantage of the Heston model over the GBM model is that the Heston model is more flexible and can capture the reality of the market better. However, due to its complexity, in general, it is much more difficult to obtain closed form formulae for pricing derivatives. In the following, we define the Heston model, derive the first moment and approximate the second moment of the average price A_T, where the average price is computed in continuous basis.

In the Heston model, the short rate process $(r_t)_t$ is assumed to be a deterministic constant, i.e., there is a constant $r \in \mathbb{R}$ such that $r_t(\omega) = r$, for any $(t, \omega) \in [0, \infty) \times \Omega$. Let $(W_t, \mathcal{F}_t)_{t\in[0,\infty)}$ be a correlated 2-dimensional Wiener process (under the Q-measure), i.e., $W_t = (W_t^1, W_t^2)$, with a quadratic covariance process $[W^1, W^2]_t = \rho t$, where $\rho \in [-1, 1]$, known as the correlation (sometime denoted by $dW_t^1 dW_t^2 = \rho dt$.). The variance process $(V_t)_{t\in[0,\infty)}$ is a $(\mathcal{F}_t)_{t\in[0,\infty)}$-adapted process, satisfying the following conditions:

- V_0 is deterministic and $V_0 > 0$, and
- $dV_t = a(b - V_t)dt + c\sqrt{V_t}dW_t^2$, where $a, b, c \in \mathbb{R}$ are constants, with $a > 0$, $b > 0$, $c \neq 0$, $2ab > c^2$.

The constants a, b, c are known as the mean reversion rate, the anchor, and the vol-vol respectively. Moreover, the condition $2ab > c^2$ is known as the Feller's condition which guarantees the existence of a strictly positive solution of the SDE. The stock price process $(S_t)_{t\in[0,\infty)}$ is $(\mathcal{F}_t)_{t\in[0,\infty)}$-adapted, satisfying the SDE: $dS_t = S_t\left(rdt + \sqrt{V_t}dW_t^1\right)$, and the initial condition that S_0 is deterministic and $S_0 > 0$. We define the integral process $(I_t)_t$ by $I_t = \int_0^t S(u)du$.

We follow [2] and consider the transformation $X_t = \ln(S_t) - rt$. By Ito's lemma, we have that

$$\begin{aligned} dX_t &= \frac{1}{S_t}dS_t + \frac{1}{2}\left(\frac{-1}{S_t^2}\right)dS_t dS_t - rdt \\ &= \left(rdt + \sqrt{V_t}dW_t^1\right) - \frac{1}{2}V_t dt - rdt \\ &= -\frac{1}{2}V_t dt + \sqrt{V_t}dW_t^1. \end{aligned}$$

Hence, the system of SDEs becomes

$$\begin{aligned} dX_t &= -\frac{1}{2}V_t dt + \sqrt{V_t}dW_t^1, \\ dV_t &= a(b - V_t)dt + c\sqrt{V_t}dW_t^2, \end{aligned}$$

with initial conditions $X_0 = x_0 = \ln(S_0)$ and $V_0 = v_0$. Fix $t > 0$. By page 4 of [2], there exists an open neighborhood U of 0 such that for each $u \in U$, we have that

$$E^Q\left[e^{uX_t}\right] = e^{ux_0}A(t)B(t),$$

where

$$A(t) = \left(\frac{e^{(a-c\rho u)\frac{t}{2}}}{\cosh\left(\frac{t}{2}P(u)\right) + \frac{a-c\rho u}{P(u)}\sinh\left(\frac{t}{2}P(u)\right)}\right)^{\frac{2ab}{c^2}},$$

$$B(t) = \exp\left\{-v_0\frac{\frac{u-u^2}{P(u)}\sinh\left(\frac{t}{2}P(u)\right)}{\cosh\left(\frac{t}{2}P(u)\right) + \frac{a-c\rho u}{P(u)}\sinh\left(\frac{t}{2}P(u)\right)}\right\},$$

and

$$P(u) = \sqrt{(a - c\rho u)^2 + c^2(u - u^2)}.$$

Using this formula, we are able to compute moments of S_t, the first moment of I_t and an approximation of the second moment of I_t. Notice that $S_t^u = e^{rut} \cdot e^{uX_t}$, and hence $E^Q\left[S_t^u\right] = e^{rut} \cdot E^Q\left[e^{uX_t}\right]$.

We follow the idea of [1] and have the following.

Proposition 3.2.1. *Using the notations as above, for $t \in [0, \infty)$, we have that*

$$\begin{aligned} E^Q\left[I_t\right] &= \frac{S_0}{r}\left(e^{rt} - 1\right), \\ E^Q\left[I_t^2\right] &\approx 2S_0^2\left\{\frac{t^2}{2} + \frac{3r + v_0}{6}t^3 + \frac{1}{24}\right. \\ &\qquad \left.\left[r(7r + 5v_0) + ab + v_0(2c\rho - a) + v_0^2\right]t^4\right\}, \end{aligned}$$

where the right hand side of the second expression is the Taylor expansion of $E^Q[I_t^2]$ about 0 up to order 4.

Proof. Observe that $\left(\frac{S_t}{B_t}\right)_t$ is a martingale under the measure Q. In particular, for any $u \in [0, \infty)$,

$$\begin{aligned} E^Q[S_u] &= E^Q\left[E^Q\left[S_u \middle| \mathcal{F}_0\right]\right] \\ &= E^Q\left[E^Q\left[\frac{S_u}{B_u}\middle| \mathcal{F}_0\right] B_u\right] \\ &= E^Q\left[\frac{S_0}{B_0} B_u\right] \\ &= e^{ru} S_0. \end{aligned}$$

Therefore, we have

$$E^Q[I_t] = \int_0^t E^Q[S_u]\, du = \int_0^t S_0 e^{ru} du = \frac{S_0}{r}\left(e^{rt} - 1\right).$$

Next, we compute the second moment.

$$\begin{aligned} E^Q\left[I_t^2\right] &= E^Q\left[\int_0^t S_u du \cdot \int_0^t S_v dv\right] \\ &= \int_0^t \int_0^t E^Q[S_u S_v]\, du dv \\ &= 2 \iint_{\{(u,v)\,|\,0 \le u \le v \le t\}} E^Q[S_u S_v]\, d\lambda^2(u, v) \\ &= 2 \int_0^t \int_0^v E^Q[S_u S_v]\, du dv \\ &= 2 \int_0^t \int_0^v E^Q\left[E^Q\left[S_u S_v \middle| \mathcal{F}_u\right]\right] du dv \\ &= 2 \int_0^t \int_0^v E^Q\left[e^{r(v-u)} S_u^2\right] du dv \\ &= 2 \int_0^t e^{rv} \int_0^v e^{-ru} E^Q\left[S_u^2\right] du dv. \end{aligned}$$

For simplicity, we denote $\theta(t) = E^Q[I_t^2]$ and compute the derivatives of θ at $t = 0$. Clearly,

$\theta(0) = 0$. $\theta'(t) = 2e^{rt}\int_0^t e^{-ru}E^Q\left[S_u^2\right]du$, so $\theta'(0) = 0$.

$$\begin{aligned}\theta''(t) &= 2re^{rt}\int_0^t e^{-ru}E^Q\left[S_u^2\right]du + 2e^{rt}e^{-rt}E^Q\left[S_t^2\right] \\ &= r\theta'(t) + 2E^Q\left[S_t^2\right].\end{aligned}$$

Therefore, we have $\theta''(0) = 2S_0^2$.

$$\begin{aligned}\theta'''(t) &= r\theta''(t) + 2\frac{d}{dt}\left\{e^{2rt}e^{2x_0}A(t)B(t)\right\} \\ &= r\theta''(t) + 4rE^Q\left[S_t^2\right] + 2S_0^2 e^{2rt}\left\{A'(t)B(t) + A(t)B'(t)\right\},\end{aligned}$$

where $A(t)$, $B(t)$ are the functions defined in above with $u = 2$. Now,

$$\begin{aligned}A'(t) &= \frac{2ab}{c^2}\left(\frac{e^{(a-2c\rho)\frac{t}{2}}}{\cosh(\frac{t}{2}P(2)) + \frac{a-2c\rho}{P(2)}\sinh\left(\frac{t}{2}P(2)\right)}\right)^{\frac{2ab}{c^2}-1}\cdot \\ &\quad \left\{\frac{a-2c\rho}{2}e^{\frac{a-2c\rho}{2}t}\left[\cosh(\frac{t}{2}P(2)) + \frac{a-2c\rho}{P(2)}\sinh\left(\frac{t}{2}P(2)\right)\right]\right. \\ &\quad \left.-e^{\frac{a-2c\rho}{2}t}\left[\frac{P(2)}{2}\sinh\left(\frac{t}{2}P(2)\right) + \frac{a-2c\rho}{2}\cosh\left(\frac{t}{2}P(2)\right)\right]\right\}/ \\ &\quad \left[\cosh\left(\frac{t}{2}P(2)\right) + \frac{a-2c\rho}{P(2)}\sinh\left(\frac{t}{2}P(2)\right)\right]^2 \\ &= \frac{2ab}{c^2}\left(e^{\frac{a-2c\rho}{2}t}\right)^{\frac{2ab}{c^2}}\cdot\frac{c^2}{P(2)}\cdot\frac{\sinh\left(\frac{t}{2}P(2)\right)}{\left[\cosh\left(\frac{t}{2}P(2)\right) + \frac{a-2c\rho}{P(2)}\sinh\left(\frac{t}{2}P(2)\right)\right]^{\frac{2ab}{c^2}+1}} \\ &= \frac{2ab}{P(2)}e^{(a-2c\rho)\frac{ab}{c^2}t}\cdot\frac{\sinh\left(\frac{t}{2}P(2)\right)}{\left[\cosh\left(\frac{t}{2}P(2)\right) + \frac{a-2c\rho}{P(2)}\sinh\left(\frac{t}{2}P(2)\right)\right]^{\frac{2ab}{c^2}+1}},\end{aligned}$$

and

$$
\begin{aligned}
B'(t) &= B(t)\cdot\frac{2v_0}{P(2)}\cdot\left\{\frac{P(2)}{2}\cosh\left(\frac{t}{2}P(2)\right)\left[\cosh\left(\frac{t}{2}P(2)\right)+\frac{a-2c\rho}{P(2)}\sinh\left(\frac{t}{2}P(2)\right)\right]\right.\\
&\quad\left.-\sinh\left(\frac{t}{2}P(2)\right)\left[\frac{P(2)}{2}\sinh\left(\frac{t}{2}P(2)\right)+\frac{a-2c\rho}{2}\cosh\left(\frac{t}{2}P(2)\right)\right]\right\}/\\
&\quad\left[\cosh\left(\frac{t}{2}P(2)\right)+\frac{a-2c\rho}{P(2)}\sinh\left(\frac{t}{2}P(2)\right)\right]^2\\
&= B(t)\cdot\frac{2v_0}{P(2)}\cdot\frac{\frac{P(2)}{2}\left[\cosh^2\left(\frac{t}{2}P(2)\right)-\sinh^2\left(\frac{t}{2}P(2)\right)\right]}{\left[\cosh\left(\frac{t}{2}P(2)\right)+\frac{a-2c\rho}{P(2)}\sinh\left(\frac{t}{2}P(2)\right)\right]^2}\\
&= \frac{v_0B(t)}{\left[\cosh\left(\frac{t}{2}P(2)\right)+\frac{a-2c\rho}{2}\sinh\left(\frac{t}{2}P(2)\right)\right]^2}.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\theta'''(t) &= r\theta''(t)+4rE^Q\left[S_t^2\right]+2S_0^2e^{2rt}B(t)\left\{\frac{2ab}{P(2)}e^{(a-2c\rho)\cdot\frac{ab}{c^2}t}\cdot\right.\\
&\quad\frac{\sinh\left(\frac{t}{2}P(2)\right)}{\left[\cosh\left(\frac{t}{2}P(2)\right)+\frac{a-2c\rho}{P(2)}\sinh\left(\frac{t}{2}P(2)\right)\right]^{\frac{2ab}{c^2}+1}}+\\
&\quad\left. v_0\cdot\frac{e^{(a-2c\rho)\cdot\frac{ab}{c^2}t}}{\left[\cosh\left(\frac{t}{2}P(2)\right)+\frac{a-2c\rho}{P(2)}\sinh\left(\frac{t}{2}P(2)\right)\right]^{\frac{2ab}{c^2}+2}}\right\}\\
&= r\theta''(t)+4rE^Q\left[S_t^2\right]+2S_0^2e^{(2r+(a-2c\rho)\cdot\frac{ab}{c^2})t}\cdot B(t)\cdot\\
&\quad\left\{\frac{\frac{2ab}{P(2)}\sinh\left(\frac{t}{2}P(2)\right)\left[\cosh\left(\frac{t}{2}P(2)\right)+\frac{a-2c\rho}{P(2)}\sinh\left(\frac{t}{2}P(2)\right)\right]+v_0}{\left[\cosh\left(\frac{t}{2}P(2)\right)+\frac{a-2c\rho}{P(2)}\sinh\left(\frac{t}{2}P(2)\right)\right]^{\frac{2ab}{c^2}+2}}\right\}.
\end{aligned}
$$

In particular, we have

$$
\begin{aligned}
\theta'''(0) &= 2rS_0^2+4rS_0^2+2S_0^2v_0\\
&= \left(6r+2v_0\right)S_0^2.
\end{aligned}
$$

Lastly, we compute the quantity $\theta^{(4)}(0)$. Denote

$$
C(t)=\exp\left\{\left(2r+(a-2c\rho)\cdot\frac{ab}{c^2}\right)t\right\}
$$

and

$$D(t) = \frac{\frac{2ab}{P(2)} \sinh\left(\frac{t}{2}P(2)\right)\left[\cosh\left(\frac{t}{2}P(2)\right) + \frac{a-2c\rho}{P(2)}\sinh\left(\frac{t}{2}P(2)\right)\right] + v_0}{\left[\cosh\left(\frac{t}{2}P(2)\right) + \frac{a-2c\rho}{P(2)}\sinh\left(\frac{t}{2}P(2)\right)\right]^{\frac{2ab}{c^2}+2}}.$$

We now compute the quantities $C'(0)$ and $D'(0)$. We have that

$$C'(t) = \left(2r + (a-2c\rho)\cdot\frac{ab}{c^2}\right)\cdot e^{\left(2r+(a-2c\rho)\cdot\frac{ab}{c^2}\right)t},$$

so $C'(0) = 2r + (a-2c\rho)\cdot\frac{ab}{c^2}$. We further write $D(t) = \frac{f(t)}{g(t)}$, where

$$f(t) = \frac{2ab}{P(2)}\cdot\sinh\left(\frac{t}{2}P(2)\right)\cdot\left[\cosh\left(\frac{t}{2}P(2)\right) + \frac{a-2c\rho}{P(2)}\sinh\left(\frac{t}{2}P(2)\right)\right] + v_0,$$

and

$$g(t) = \left[\cosh\left(\frac{t}{2}P(2)\right) + \frac{a-2c\rho}{P(2)}\sinh\left(\frac{t}{2}P(2)\right)\right]^{\frac{2ab}{c^2}+2}.$$

Now

$$\begin{aligned} f'(t) \;&=\; ab\cosh\left(\frac{t}{2}P(2)\right)\cdot\left[\cosh\left(\frac{t}{2}P(2)\right) + \frac{a-2c\rho}{P(2)}\sinh\left(\frac{t}{2}P(2)\right)\right] + \\ &\quad\; ab\sinh\left(\frac{t}{2}P(2)\right)\cdot\left[\sinh\left(\frac{t}{2}P(2)\right) + \frac{a-2c\rho}{P(2)}\cosh\left(\frac{t}{2}P(2)\right)\right] \end{aligned}$$

and hence $f'(0) = ab$.

$$\begin{aligned} g'(t) \;&=\; \left(\frac{2ab}{c^2}+2\right)\cdot\left[\cosh\left(\frac{t}{2}P(2)\right) + \frac{a-2c\rho}{P(2)}\sinh\left(\frac{t}{2}P(2)\right)\right]^{\frac{2ab}{c^2}+1} \\ &\quad\cdot\left[\frac{P(2)}{2}\sinh\left(\frac{t}{2}P(2)\right) + \frac{a-2c\rho}{2}\cosh\left(\frac{t}{2}P(2)\right)\right], \end{aligned}$$

so, we have

$$\begin{aligned} g'(0) \;&=\; \left(\frac{2ab}{c^2}+2\right)\cdot 1\cdot\frac{a-2c\rho}{2} \\ &=\; \left(\frac{ab}{c^2}+1\right)(a-2c\rho). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} D'(0) &= \frac{f'(0)g(0) - f(0)g'(0)}{[g(0)]^2} \\ &= \frac{ab \cdot 1 - v_0 \cdot \left(\frac{ab}{c^2} + 1\right)(a - 2c\rho)}{1} \\ &= ab - v_0 \left(\frac{ab}{c^2} + 1\right)(a - 2c\rho). \end{aligned}$$

Finally,

$$\begin{aligned} \theta^{(4)}(0) &= r \cdot (6r + 2v_0)S_0^2 + 4r \cdot (2r + v_0)S_0^2 + 2S_0^2 \Bigg\{ \left[2r + (a - 2c\rho) \cdot \frac{ab}{c^2}\right] \\ &\quad \cdot 1 \cdot v_0 + 1 \cdot v_0 \cdot v_0 + 1 \cdot 1 \cdot \left[ab - v_0 \cdot (\frac{ab}{c^2} + 1) \cdot (a - 2c\rho)\right] \Bigg\} \\ &= rS_0^2 \cdot (14r + 6v_0) + 2S_0^2 \cdot \Bigg\{ v_0 \left[2r + (a - 2c\rho) \cdot \frac{ab}{c^2}\right] + v_0^2 \\ &\quad + ab - v_0 \cdot \left(\frac{ab}{c^2} + 1\right)(a - 2c\rho) \Bigg\} \\ &= rS_0^2 \cdot (14r + 6v_0) + 2S_0^2 \left\{ab + v_0 (2r + 2c\rho - a) + v_0^2\right\} \\ &= rS_0^2 \cdot (14r + 10v_0) + 2S_0^2 \left\{ab + v_0(2c\rho - a) + v_0^2\right\}, \end{aligned}$$

and hence we have obtained the following approximation

$$\begin{aligned} \theta(t) &\approx \theta(0) + \theta'(0)t + \frac{1}{2!}\theta''(0)t^2 + \frac{1}{3!}\theta'''(0)t^3 + \frac{1}{4!}\theta^{(4)}(0)t^4 \\ &= 2S_0^2 \left\{ \frac{1}{2}t^2 + \frac{1}{6}(3r + v_0)t^3 + \frac{1}{24}\left[r(7r + 5v_0) + ab + v_0(2c\rho - a) + v_0^2\right] t^4 \right\}. \end{aligned}$$

□

Chapter 4

Distributions and Moments Fitting

In this chapter, we explore several distributions, namely, log-normal distributions, shifted log-normal distributions, skew-normal distributions, and log-skew-normal distributions. We also compute their first few moments and express the parameters in terms of moments. These results will be used in the next chapter for pricing Asian options.

4.1 Log-Normal Distributions

In this section, we firstly define a log-normal distribution, then we derive the formulae for moments fitting of the average stock price.

Log-normal distribution is a very common distribution used in the financial industry. In fact, if the stock price $(S_t)_t$ process follows the GBM model, the stock price S_T at the time horizon T is log-normally distributed. One of the advantages of the log-normal distribution is its simplicity and its nice behavior. For example, for this distribution all moments exist and have closed-form formulae. Another property of the log-normal distribution is that a log-normally distributed random variable is strictly positive which is a nice feature to model the price of a stock.

Definition 4.1.1. *Let X be a random variable. We say that X follows a log-normal distribution with parameters μ and σ if $\ln(X)$ is normally distributed with mean μ and variance σ^2.*

Proposition 4.1.1. *Let X be a log-normally distributed random variable with parameters μ and σ. Define $Z = \ln(X)$ and let ϕ_X, ϕ_Z be the probability density functions of X*

and Z respectively. Then ϕ_X and ϕ_Z are related as follow. $\phi_X(x) = 0$, if $x \leq 0$ and $\phi_X(x) = \frac{1}{x}\phi_Z(\ln x)$, if $x > 0$. $\phi_Z(z) = \phi_X(e^z)e^z$.

Proof. By definition, since $\ln(X)$ is well-defined, $X > 0$ everywhere. Therefore $P[X \leq 0] = 0$. In particular, $\phi_X(x) = 0$ if $x \leq 0$. For $x > 0$, $P[X \leq x] = P[Z \leq \ln(x)]$. Differentiating both sides with respect to x yields $\phi_X(x) = \phi_Z(\ln(x)) \cdot \frac{1}{x}$. Let $z \in \mathbb{R}$, then $P[Z \leq z] = P[X \leq e^z]$. Differentiating both sides with respect to z, we obtain $\phi_Z(z) = \phi_X(e^z)e^z$. □

Proposition 4.1.2. *Let X be a log-normally distributed random variable with parameters μ and σ. Then the first two moments of X are given by $E[X] = e^{\mu+\frac{1}{2}\sigma^2}$ and $E[X^2] = e^{2\mu+2\sigma^2}$.*

Proof. Let $Z = ln(X)$ distributed normally with mean μ and variance σ^2, then

$$\begin{aligned} E[X] &= E\left[e^Z\right] \\ &= \int_{-\infty}^{\infty} e^t \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(t-\mu)^2} dt \\ &= \int_{-\infty}^{\infty} e^{\mu+\frac{1}{2}\sigma^2} \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}\left[t-(\mu+\sigma^2)\right]^2} dt \\ &= e^{\mu+\frac{1}{2}\sigma^2}. \end{aligned}$$

Note that $2Z \sim N(2\mu, 4\sigma^2)$, so by the previous result, we have

$$E\left[X^2\right] = E\left[e^{2Z}\right] = e^{2\mu+\frac{1}{2}\cdot 4\sigma^2} = e^{2\mu+2\sigma^2}.$$

□

We are now ready to state the general result for moments fitting by the log-normal distribution.

Proposition 4.1.3. *Let F be a distribution such that its first two moments exist. Let $m_1 = \int_{-\infty}^{\infty} x dF(x)$ and $m_2 = \int_{-\infty}^{\infty} x^2 dF(x)$. Suppose further that $m_1 > 0$ and $m_2 > 0$. Then there exists a log-normally distributed random variable X with parameters μ and σ such that $E[X] = m_1$ and $E[X^2] = m_2$. Moreover, μ and σ are given by*

$$\mu = \ln\left(\frac{m_1^2}{\sqrt{m_2}}\right), \text{ and } \sigma = \sqrt{\ln\left(\frac{m_2}{m_1^2}\right)}.$$

Proof. Let X be a log-normally distributed random variable with parameters μ and σ. By the previous proposition, we have that $E[X] = e^{\mu+\frac{1}{2}\sigma^2}$ and $E[X^2] = e^{2\mu+2\sigma^2}$. Suppose that $E[X] = m_1$ and $E[X^2] = m_2$, then

$$\begin{cases} e^{\mu+\frac{1}{2}\sigma^2} & = & m_1, \\ e^{2\mu+2\sigma^2} & = & m_2. \end{cases}$$

Solving the above simultaneous equation leads to the desired result. Conversely, let X be a log-normally distributed random variable with parameters $\mu = \ln\left(\frac{m_1^2}{\sqrt{m_2}}\right)$ and $\sigma = \sqrt{\ln\left(\frac{m_2}{m_1^2}\right)}$, then by a direct calculation, we have $E[X] = m_1$ and $E[X^2] = m_2$. □

4.2 Shifted-Log-Normal Distributions

Log-normal distribution has a drawback that it can only fit the first two moments because there are only two parameters. We slightly modify it and define the notion of a shifted log-normal distribution.

Definition 4.2.1. *A random variable X is said to be shifted-log-normally distributed if there exists $\alpha \in \mathbb{R}$, and a normally distributed random variable $Z \sim N(\mu, \sigma^2)$, where $\mu \in \mathbb{R}$ and $\sigma > 0$, such that $X = \alpha + \exp(Z)$.*

A shifted-log-normal distribution admits a probability density function which can be calculated as follows.

Proposition 4.2.1. *Let $\alpha \in \mathbb{R}$, $\mu \in \mathbb{R}$, $\sigma > 0$, and let Z be a normally distributed random variable with $Z \sim N(\mu, \sigma^2)$. Let $X = \alpha + \exp(Z)$, then X admits a probability density function ϕ_X. Moreover, ϕ_X, and ϕ_Z, the probability density function of Z, are related through*

$$\phi_X(x) = \begin{cases} 0, & \text{if } x \in (-\infty, \alpha] \\ \phi_Z(\ln(x-\alpha)) \cdot \frac{1}{x-\alpha}, & \text{if } x \in (\alpha, \infty), \end{cases}$$

and $\phi_Z(z) = \phi_X(e^z + \alpha)e^z$.

Next, we consider the problem of matching the first three moments of a given distribution by a shifted-log-normal distribution. Before we state and prove the main theorem, we need the following lemmas.

Lemma 4.2.1. *Let $a \in \mathbb{R}$ be a constant, then the cubic equation $x^3 - 3x + a = 0$ has at most one root (counting multiplicity) in $[2, \infty)$.*

Proof. Define a function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3 - 3x + a$. We prove by contradiction and suppose to the contrary that the equation has at least two roots in $[2, \infty)$. If $x_0 \in [2, \infty)$ is a repeated root, $f'(x_0) = 0$. However, $f'(x_0) = 3x_0^2 - 3 > 0$ which is a contradiction. Suppose that the equation has roots $x_1, x_2 \in [2, \infty)$ with with $x_1 \neq x_2$. By Rolle's theorem, there exists ξ strictly between x_1 and x_2 such that $f'(\xi) = 0$. However, since $\xi > 2$, we have that $f'(\xi) = 3\xi^2 - 3 > 0$ which is again a contradiction. □

Lemma 4.2.2. *Let $a > 0$, $b > 0$ such that $ab = 1$. Then $a + b \geq 2$ and the equality holds if and only if $a = b = 1$.*

Proof. Observe that $(\sqrt{a} - \sqrt{b})^2 \geq 0$, so $a + b - 2\sqrt{ab} \geq 0$ and hence $a + b \geq 2$. If $a = b = 1$, we clearly have $a + b = 2$. Conversely, suppose that $a + b = 2$, then $(\sqrt{a} - \sqrt{b})^2 = 0$. It follows that $\sqrt{a} = \sqrt{b}$ and hence $a = b = 1$. □

Now, we are ready to state and prove the main result.

Theorem 4.2.1. *Let F be a distribution. Suppose that the first three moments of F, i.e. $\int_{-\infty}^{\infty} x dF(x)$, $\int_{-\infty}^{\infty} x^2 dF(x)$, $\int_{-\infty}^{\infty} x^3 dF(x)$ exist and the centered second and third moments are non-zero, i.e., $\int_{-\infty}^{\infty} (x - x_0)^2 dF(x) \neq 0$ and $\int_{-\infty}^{\infty} (x - x_0)^3 dF(x) \neq 0$, where $x_0 = \int_{-\infty}^{\infty} x dF(x)$. Then there exist unique $\alpha, \mu, \sigma \in \mathbb{R}$ with $\sigma > 0$ such that if a random variable Z is normally distributed with $Z \sim N(\mu, \sigma^2)$ and $X = \alpha + e^Z$, then $E(X) = \int_{-\infty}^{\infty} x dF(x)$, $E(X^2) = \int_{-\infty}^{\infty} x^2 dF(x)$ and $E(X^3) = \int_{-\infty}^{\infty} x^3 dF(x)$. Moreover, α, μ, σ are given explicitly as follows. Denote the centered second and the third moments by b and c respectively, i.e., $b = \int_{-\infty}^{\infty} (x - x_0)^2 dF(x)$ and $c = \int_{-\infty}^{\infty} (x - x_0)^3 dF(x)$ and let $d = \frac{c^2}{b^3} + 2$. Let x' be the unique root in $(2, \infty)$ of the cubic equation $x^3 - 3x - d = 0$. Then $\sigma = \sqrt{\ln(x' - 1)}$, $\mu = \ln\left(\frac{c}{b} \cdot \frac{1}{(x'-2)(x'+1)} \cdot \frac{1}{\sqrt{x'-1}}\right)$, and $\alpha = \int_{-\infty}^{\infty} x dF(x) - \frac{c}{b} \cdot \frac{1}{(x'-2)(x'+1)}$.*

Proof. Firstly, we prove the uniqueness part. Suppose that there exists $\alpha, \mu, \sigma \in \mathbb{R}$ with $\sigma > 0$ such that if $Z \sim N(\mu, \sigma^2)$ and $X = \alpha + e^Z$, then $E(X) = \int_{-\infty}^{\infty} x dF(x)$, $E(X^2) = \int_{-\infty}^{\infty} x^2 dF(x)$ and $\int_{-\infty}^{\infty} x^3 dF(x)$. For convenience, we denote the first three moments of F by m_1, m_2, and m_3 respectively, and denote the mean, the second and the third central moments by a, b, c respectively, i.e.

$$\begin{aligned}
m_1 &= \int_{-\infty}^{\infty} x dF(x), \\
m_2 &= \int_{-\infty}^{\infty} x^2 dF(x), \\
m_3 &= \int_{-\infty}^{\infty} x^3 dF(x), \\
a &= m_1, \\
b &= \int_{-\infty}^{\infty} (x - m_1)^2 dF(x), \\
c &= \int_{-\infty}^{\infty} (x - m_1)^3 dF(x).
\end{aligned}$$

We remark that a, b, and c can be expressed in terms of m_1, m_2 and m_3, namely, $a = m_1$, $b = m_2 - m_1^2$ and $c = m_3 - 3m_1m_2 + 2m_1^3$. Since $2Z \sim N(2\mu, 4\sigma^2)$ and $3Z \sim N(3\mu, 9\sigma^2)$, we have that $E(e^{2Z}) = \exp(2\mu + \frac{1}{2} \cdot 4\sigma^2) = \exp(2\mu + 2\sigma^2)$ and $E(e^{3Z}) = \exp(3\mu + \frac{1}{2} \cdot 9\sigma^2)$. It follows that $E(X^2) = \alpha^2 + 2\alpha e^{\mu + \frac{1}{2}\sigma^2} + e^{2\mu + 2\sigma^2}$ and $E(X^3) = \alpha^3 + 3\alpha^2 e^{\mu + \frac{1}{2}\sigma^2} + 3\alpha e^{2\mu + 2\sigma^2} + e^{3\mu + \frac{9}{2}\sigma^2}$. Now, we have

$$b = m_2 - m_1^2 = e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2} = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1),$$

and

$$\begin{aligned}
c &= m_3 - 3m_1m_2 + 2m_1^3 \\
&= \left(\alpha^3 + 3\alpha^2 e^{\mu + \frac{1}{2}\sigma^2} + 3\alpha e^{2\mu + 2\sigma^2} + e^{3\mu + \frac{9}{2}\sigma^2}\right) \\
&\quad -3\left(\alpha + e^{\mu + \frac{1}{2}\sigma^2}\right)\left(\alpha^2 + 2\alpha e^{\mu + \frac{1}{2}\sigma^2} + e^{2\mu + 2\sigma^2}\right) \\
&\quad +2\left(\alpha^3 + 3\alpha^2 e^{\mu + \frac{1}{2}\sigma^2} + 3\alpha e^{2\mu + \sigma^2} + e^{3\mu + \frac{3}{2}\sigma^2}\right) \\
&= e^{3\mu + \frac{9}{2}\sigma^2} - 3e^{3\mu + \frac{5}{2}\sigma^2} + 2e^{3\mu + \frac{3}{2}\sigma^2} \\
&= \left(e^{\mu + \frac{1}{2}\sigma^2}\right)^3 \left(e^{3\sigma^2} - 3e^{\sigma^2} + 2\right).
\end{aligned}$$

Define $\phi = e^{\mu + \frac{1}{2}\sigma^2}$, $\eta = e^{\sigma^2}$, then we have

$$a = \alpha + \phi, \tag{4.1}$$

$$b = \phi^2(\eta - 1), \tag{4.2}$$

$$c = \phi^3(\eta - 1)^2(\eta + 2). \tag{4.3}$$

By assumption, $b > 0$, so $\frac{c^2}{b^3}$ is well-defined. By (4.2) and (4.3), $\frac{c^2}{b^3} = (\eta - 1)(\eta + 2)^2$. Therefore η satisfies a cubic equation. We try to get rid of the square term by considering the Tschirnhaus transformation $x' = \eta + k$, where k will be determined later. We have that $(\eta - 1)(\eta + 2)^2 = x'^3 + [-(k+1) - 2(k-2)]\, x'^2 + \cdots$, so we set $-(k+1) - 2(k-2) = 0$, i.e. $k = 1$. Formally, we define $x' = \eta + 1$, then

$$\frac{c^2}{b^3} = (x' - 2)(x' + 1)^2 = x'^3 - 3x' - 2.$$

Define $d = \frac{c^2}{b^3} + 2$, then $x'^3 - 3x' - d = 0$. Observe that $x' = \eta + 1 > 2$, so by Lemma 4.2.1, x' is uniquely determined by d. Now $\sigma = \sqrt{\ln(\eta)} = \sqrt{\ln(x' - 1)}$. By considering (4.3)/(4.2), we obtain $\frac{c}{b} = \phi(\eta - 1)(\eta + 2)$ and hence $\phi = \frac{c}{b} \cdot \frac{1}{(\eta - 1)(\eta + 2)}$. As $\phi = e^{\mu}\sqrt{\eta}$, we have

$$\mu = \ln\left(\frac{\phi}{\sqrt{\eta}}\right) = \ln\left(\frac{c}{b} \cdot \frac{1}{(x' - 2)(x' + 1)} \cdot \frac{1}{\sqrt{x' - 1}}\right).$$

Lastly, $\alpha = a - \phi = a - \frac{c}{b} \cdot \frac{1}{(x' - 2)(x' + 1)}$. This shows that α, μ, σ are uniquely determined by m_1, m_2, m_3.

Secondly, we prove the existence part. By the previous discusssion, it prompts us to consider the cubic equation $x^3 - 3x = d$, where $a = m_1$, $b = m_2 - m_1^2$, $c = m_3 - 3m_1m_2 + 2m_1^3$, and $d = \frac{c^2}{b^3} + 2$. We show that the equation indeed has a solution in $(2, \infty)$ and can be solved explicitly by Cardano's method. Write $x = u + v$, where u and v will be determined later, then we have:

$$\begin{aligned} x^3 - 3x &= u^3 + v^3 + 3uv(u + v) - 3(u + v) \\ &= u^3 + v^3 + 3(u + v)(uv - 1). \end{aligned}$$

Now we impose another condition on u and v, namely, $uv = 1$, and try to solve the following simultaneous equations

$$\begin{cases} u^3 + v^3 = d, \\ uv = 1. \end{cases}$$

Clearly, $u^3 = \frac{d \pm \sqrt{d^2 - 4}}{2}$. Notice that $d > 2$, so the roots are distinct and positive. If we choose $u^3 = \frac{d + \sqrt{d^2 - 4}}{2}$, then $v^3 = \frac{1}{u^3} = \frac{d - \sqrt{d^2 - 4}}{2}$, and vice versa. Formally, we define

$$\begin{aligned} u &= \sqrt[3]{\frac{d + \sqrt{d^2 - 4}}{2}}, \\ v &= \sqrt[3]{\frac{d - \sqrt{d^2 - 4}}{2}} \end{aligned}$$

and define $x_0 = u + v$. By direct calculation, $u^3v^3 = 1$ and hence $uv = 1$. Clearly, $u \neq v$, so by Lemma 4.2.2, $x_0 > 2$. Lastly,

$$\begin{aligned} x_0^3 - 3x_0 &= (u+v)^3 - 3(u+v) \\ &= u^3 + v^3 - 3(u+v)(uv-1) \\ &= u^3 + v^3 \\ &= d. \end{aligned}$$

This demonstrates that the cubic equation $x^3 - 3x = d$ has a root $x_0 \in (2, \infty)$. Lastly, we define $\sigma = \sqrt{\ln(x_0 - 1)} > 0$, $\mu = \ln\left(\frac{c}{b} \cdot \frac{1}{(x_0-2)(x_0+1)} \cdot \frac{1}{\sqrt{x_0-1}}\right)$ and $\alpha = a - \frac{c}{b} \cdot \frac{1}{(x_0-2)(x_0+1)}$. It is now a routine to check that if $Z \sim N(\mu, \sigma^2)$ and $X = \alpha + e^Z$, then $E(X) = m_1$, $E(X^2) = m_2$ and $E(X^3) = m_3$. ☐

4.3 Skew-Normal Distributions and Log-Skew-Normal Distributions

4.3.1 Standard Skew-Normal Distributions

In this section, we firstly define a standard skew-normal distribution, then we compute the first three moments of the standard skew-normal distribution. In the second half of this section, we consider affine transformation of the standard skew-normal distribution.

Proposition 4.3.1. *Let $\phi : \mathbb{R} \to \mathbb{R}$, $\Phi : \mathbb{R} \to \mathbb{R}$ be the probability density function and the culmulative distribution function of the standard normal distribution, that is, $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ and $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2} dt$. For each $\alpha \in \mathbb{R}$, define $f_\alpha : \mathbb{R} \to \mathbb{R}$ by $f_\alpha(x) = 2\phi(x)\Phi(\alpha x)$. Then $f_\alpha(x) > 0$ for each $x \in \mathbb{R}$ and $\int_{-\infty}^{\infty} f_\alpha(x)dx = 1$. Hence, f_α is the probability density function of some distribution.*

Proof. Clearly $f_\alpha(x) > 0$ for each $x \in \mathbb{R}$. It remains to show that $\int_{-\infty}^{\infty} f_\alpha(x)dx = 1$. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(\alpha) = \int_{-\infty}^{\infty} \phi(x)\Phi(\alpha x)dx$. Note that the integrand is positive with $\phi(x)\Phi(\alpha x) \leq \phi(x)$ and ϕ is integrable, so $g(\alpha)$ is well-defined. Observe that

$$g'(\alpha) = \int_{-\infty}^{\infty} \phi(x)\phi(\alpha x)x dx = 0,$$

because the integrand is an odd function, $|\phi(x)\phi(\alpha x)x| \le |\phi(x)x|$, and the right hand side of the above first equation is integrable. Therefore, g is a constant function. In particular, we have

$$g(\alpha) = g(0) = \frac{1}{2}\int_{-\infty}^{\infty} \phi(x)dx = \frac{1}{2}.$$

It follows that $\int_{-\infty}^{\infty} f_\alpha(x)dx = 1$. □

Definition 4.3.1. *The standard skew-normal distribution with parameter α is the distribution with probability density function f_α defined in Proposition 4.3.1.*

Remark 4.3.1. *If $\alpha = 0$, the skew-normal distribution defined in above reduces to the standard normal distribution.*

Proposition 4.3.2. *Consider the standard skew-normal distribution with parameter α and probability density function f_α. Then:*

(a) Moment of any order exists,

(b) The first three moments are given by

$$\begin{aligned} m_1 &\triangleq \int_{-\infty}^{\infty} x f_\alpha(x)dx = \sqrt{\frac{2}{\pi}} \cdot \frac{\alpha}{\sqrt{1+\alpha^2}}, \\ m_2 &\triangleq \int_{-\infty}^{\infty} x^2 f_\alpha(x)dx = 1, \\ m_3 &\triangleq \int_{-\infty}^{\infty} x^3 f_\alpha(x)dx = \sqrt{\frac{2}{\pi}} \cdot \frac{\alpha}{\sqrt{1+\alpha^2}} \cdot \left\{3 - \frac{\alpha^2}{1+\alpha^2}\right\}. \end{aligned}$$

Proof. Let ϕ and Φ respectively be the probability density function and the culmulative distribution function of the standard normal distribution. For any non-negative integer n, we have that

$$\int_{-\infty}^{\infty} |x^n| f_\alpha(x)dx \le \int_{-\infty}^{\infty} |x^n| \cdot 2\phi(x)dx < \infty,$$

since the moment of any order exists for the standard normal distribution. Therefore the moment of any order exists for the standard skew-normal distribution with parameter α. Let m_1, m_2, m_3 be the first, the second, and the thrid moments of the skew-normal distribution with parameter α respectively. Define $g_1 : \mathbb{R} \to \mathbb{R}$ by $g_1(\alpha) = \int_{-\infty}^{\infty} x f_\alpha(x)dx$, then

$$\begin{aligned}
g_1'(\alpha) &= \int_{-\infty}^{\infty} x \cdot \frac{d}{d\alpha}\left[2\phi(x)\Phi(\alpha x)\right] dx \\
&= \int_{-\infty}^{\infty} x \cdot 2\phi(x)\phi(\alpha x) x dx \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} x^2 \cdot e^{-\frac{1}{2}(1+\alpha^2)x^2} dx \\
&= \frac{1}{\pi}\left\{\left[-\frac{x}{1+\alpha^2} e^{-\frac{1}{2}(1+\alpha^2)x^2}\right]_{-\infty}^{\infty} + \frac{1}{1+\alpha^2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(1+\alpha^2)x^2} dx\right\} \\
&= \frac{1}{\pi} \cdot \frac{1}{1+\alpha^2}\int_{-\infty}^{\infty} \frac{1}{\sqrt{1+\alpha^2}} e^{-\frac{1}{2}y^2} dy \\
&= \frac{1}{\pi} \cdot \left(\frac{1}{1+\alpha^2}\right)^{\frac{3}{2}} \cdot \sqrt{2\pi} \\
&= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{(1+\alpha^2)^{\frac{3}{2}}}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
g_1(\alpha) &= \sqrt{\frac{2}{\pi}} \int \frac{d\alpha}{(1+\alpha^2)^{\frac{3}{2}}} \\
&= \sqrt{\frac{2}{\pi}} \int \frac{\sec^2\theta d\theta}{\sec^3\theta} \text{ (let } \alpha{=}\tan\theta \text{)} \\
&= \sqrt{\frac{2}{\pi}} (\sin\theta + C) \\
&= \sqrt{\frac{2}{\pi}} \left(\frac{\alpha}{\sqrt{1+\alpha^2}} + C\right).
\end{aligned}$$

Observe that since the function $x\phi(x)$ is odd, we have

$$g_1(0) = \int_{-\infty}^{\infty} x\phi(x) dx = 0.$$

It follows that $C = 0$. Hence $m_1 = g_1(\alpha) = \sqrt{\frac{2}{\pi}} \cdot \frac{\alpha}{\sqrt{1+\alpha^2}}$.

Next, we define $g_2 : \mathbb{R} \to \mathbb{R}$, $g_2(\alpha) = \int_{-\infty}^{\infty} x^2 f_\alpha(x) dx$. We have that

$$g_2'(\alpha) = \int_{-\infty}^{\infty} 2x^2 \cdot \phi(x)\phi(\alpha x) x dx = 0,$$

since the integrand is an odd function. Therefore g_2 is a constant function. In particular,

$$g_2(\alpha) = g_2(0) = \int_{-\infty}^{\infty} x^2\phi(x)dx = 1.$$

Hence $m_2 = 1$.

Lastly, we define $g_3 : \mathbb{R} \to \mathbb{R}$, $g_3(\alpha) = \int_{-\infty}^{\infty} x^3 f_\alpha(x)dx$, then

$$\begin{aligned}
g_3'(\alpha) &= \int_{-\infty}^{\infty} x^3 \cdot 2\phi(x)\phi(\alpha x)x dx \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} x^4 \cdot e^{-\frac{1}{2}(1+\alpha^2)x^2} dx \\
&= \frac{1}{\pi}\left\{\left[\frac{-x^3}{1+\alpha^2}e^{-\frac{1}{2}(1+\alpha^2)x^2}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-\frac{1}{2}(1+\alpha^2)x^2} \cdot \frac{3x^2}{1+\alpha^2}dx\right\} \\
&= \frac{1}{\pi} \cdot \frac{3}{1+\alpha^2} \cdot \int_{-\infty}^{\infty} x^2 e^{-\frac{1}{2}(1+\alpha^2)x^2} dx \\
&= \frac{1}{\pi} \cdot \frac{3}{1+\alpha^2} \cdot \left(\frac{1}{1+\alpha^2}\right)^{\frac{3}{2}} \cdot \sqrt{2\pi} \\
&= 3 \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{1}{(1+\alpha^2)^{\frac{5}{2}}}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
g_3(\alpha) &= 3 \cdot \sqrt{\frac{2}{\pi}} \cdot \int \frac{d\alpha}{(1+\alpha^2)^{\frac{5}{2}}} \\
&= 3 \cdot \sqrt{\frac{2}{\pi}} \cdot \int \frac{\sec^2\theta d\theta}{\sec^5\theta} \text{ (let } \alpha{=}\tan\theta \text{)} \\
&= 3 \cdot \sqrt{\frac{2}{\pi}} \cdot \int \cos^3\theta d\theta \\
&= 3 \cdot \sqrt{\frac{2}{\pi}} \left\{\sin\theta - \frac{1}{3}\sin^3\theta\right\} + C \\
&= 3 \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{\alpha}{\sqrt{1+\alpha^2}} \left\{1 - \frac{\alpha^2}{3(1+\alpha^2)}\right\} + C.
\end{aligned}$$

Note that $x^3\phi(x)$ is an odd fuction, so $g_3(0) = \int_{-\infty}^{\infty} x^3\phi(x)dx = 0$. It follows that $C = 0$ and hence $m_3 = \sqrt{\frac{2}{\pi}} \cdot \frac{\alpha}{\sqrt{1+\alpha^2}} \left\{3 - \frac{\alpha^2}{1+\alpha^2}\right\}$. □

We can also compute the mean, the variance, and the skewness of the standard skew-normal distribution.

Proposition 4.3.3. *The mean μ_{SNK}, the variance σ^2_{SNK}, and the skewness of the standard skew-normal distribution with parameter α are respectively given by:*

$$\begin{aligned}
\mu_{SNK} &= \sqrt{\frac{2}{\pi}} \cdot \frac{\alpha}{\sqrt{1+\alpha^2}}, \\
\sigma^2_{SNK} &= 1 - \frac{2}{\pi} \cdot \frac{\alpha^2}{1+\alpha^2}, \\
\textit{skewness} &= \frac{1}{\sigma^3} \cdot \sqrt{\frac{2}{\pi}} \cdot \left(\frac{4}{\pi} - 1\right) \cdot \left(\frac{\alpha}{\sqrt{1+\alpha^2}}\right)^3.
\end{aligned}$$

Proof. The mean is just the first moment, and hence

$$\mu = \sqrt{\frac{2}{\pi}} \cdot \frac{\alpha}{\sqrt{1+\alpha^2}}.$$

For the variance, we have that

$$\begin{aligned}
\text{variance } \sigma^2 &= \text{second moment} - \text{mean}^2 \\
&= 1 - \frac{2}{\pi} \cdot \frac{\alpha^2}{1+\alpha^2}.
\end{aligned}$$

Denote the probability density function of the standard skew-normal distribution with parameter α by f_α, and denote the first, the second, and the third moments by m_1, m_2, and m_3 respectively. We have that

$$\begin{aligned}
\text{skewness} &= \frac{\int_{-\infty}^{\infty} (x-\mu)^3 f_\alpha(x) dx}{\sigma^3} \\
&= \frac{1}{\sigma^3} \left\{ m_3 - 3\mu m_2 + 3\mu^2 m_1 - \mu^3 \right\} \\
&= \frac{1}{\sigma^3} \left\{ \sqrt{\frac{2}{\pi}} \cdot \frac{\alpha}{\sqrt{1+\alpha^2}} \left(3 - \frac{\alpha^2}{1+\alpha^2}\right) - 3\sqrt{\frac{2}{\pi}} \cdot \frac{\alpha}{\sqrt{1+\alpha^2}} \cdot 1 + 2 \cdot \left(\sqrt{\frac{2}{\pi}} \cdot \frac{\alpha}{\sqrt{1+\alpha^2}}\right)^3 \right\} \\
&= \frac{1}{\sigma^3} \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{\alpha}{\sqrt{1+\alpha^2}} \left\{ 3 - \frac{\alpha^2}{1+\alpha^2} - 3 + \frac{4}{\pi} \cdot \frac{\alpha^2}{1+\alpha^2} \right\} \\
&= \frac{1}{\sigma^3} \cdot \sqrt{\frac{2}{\pi}} \cdot \left(\frac{4}{\pi} - 1\right) \cdot \left(\frac{\alpha}{\sqrt{1+\alpha^2}}\right)^3.
\end{aligned}$$

□

4.3.2 Affine Transformation of Skew-Normal Distributions

Since a standard skew-normal distribution is governed by one variable α only, it is quite restrictive. We increase the degree of freedom by considering an affine transformation of it.

Definition 4.3.2. *Let X be a random variable, distributed standard skew-normally with parameter α. Let $\xi \in \mathbb{R}$ and $\omega > 0$. Define $Y = \omega X + \xi$. We say that Y is a random variable, distributed skew-normally with parameters α, ω, and ξ. ω is called the scale parameter while ξ is called the location parameter. We are ready to derive the mean, the variance, and the skewness of Y.*

Proposition 4.3.4. *Let Y be a random variable distributed skew-normally with parameters α, ω, and ξ. Let ϕ and Φ be the probability density function and the culmulative distribution function of the standard normal distribution. Then the mean, the variance, the skewness, and the probability density function ϕ_Y of Y are respectively given by:*

$$\begin{aligned}
\textit{mean} &= \sqrt{\frac{2}{\pi}} \cdot \frac{\alpha}{\sqrt{1+\alpha^2}} \cdot \omega + \xi, \\
\textit{variance } \sigma^2 &= \omega^2 \left(1 - \frac{2}{\pi} \cdot \frac{\alpha^2}{1+\alpha^2}\right), \\
\textit{skewness} &= \frac{\sqrt{\frac{2}{\pi}} \cdot \left(\frac{4}{\pi} - 1\right) \cdot \left(\frac{\alpha}{\sqrt{1+\alpha^2}}\right)^3}{\left(1 - \frac{2}{\pi} \cdot \frac{\alpha^2}{1+\alpha^2}\right)^{\frac{3}{2}}}, \\
\phi_Y(y) &= \frac{1}{\omega} \cdot 2\phi\left(\frac{y-\xi}{\omega}\right) \cdot \Phi\left(\alpha \cdot \frac{y-\xi}{\omega}\right).
\end{aligned}$$

Proof. Define $X = (Y - \xi)/\omega$, then X is distributed with the standard skew-normal distribution with parameter α. That the mean, the variance, and the skewness of Y are given as above follows immediately from the general theory of probability and the results in the previous proposition. For example, the variance of Y is ω^2 times the variance of X, and the skewness of Y is the same as the skewness of X. Denote the probability density function of X by ϕ_X. We have that

$$P\left[Y \le y\right] = P\left[\omega X + \xi \le y\right] = P\left[X \le \frac{y-\xi}{\omega}\right].$$

Differentiating both sides with respect to y, we obtain

$$\phi_Y(y) = \phi_X\left(\frac{y-\xi}{\omega}\right) \cdot \frac{1}{\omega} = \frac{1}{\omega} \cdot 2\phi\left(\frac{y-\xi}{\omega}\right) \cdot \Phi\left(\alpha \cdot \frac{y-\xi}{\omega}\right).$$

$\square$

Remark 4.3.2. *We can express the parameters* α*,* ω*, and* ξ *in terms of the moments of* Y*. More precisely, let* $m_k = E\left[Y^k\right]$*, for* $k = 1, 2, 3$*, be the first three moments of* Y*, then we write the mean, the standard deviation* σ*, and the skewness in terms of the moments. For example, mean* $= m_1$*, standard deviation* $\sigma = \sqrt{m_2 - m_1^2}$*, and skewness* $= (m_3 - 3m_1m_2 + 2m_1^3)/\sigma^3$*. Now denote the skewness by* c*, and solve for* α *in the following equation:*

$$\sqrt{\frac{2}{\pi}}\left(\frac{4}{\pi}-1\right)\left(\frac{\alpha}{\sqrt{1+\alpha^2}}\right)^3 = c\left(1-\frac{2}{\pi}\cdot\frac{\alpha^2}{1+\alpha^2}\right)^{\frac{3}{2}}.$$

If $c \neq 0$*, we let* $\beta = \frac{\alpha}{\sqrt{1+\alpha^2}}$ *and* $a = \left(\frac{1}{c}\cdot\sqrt{\frac{2}{\pi}}\cdot\left(\frac{4}{\pi}-1\right)\right)^{\frac{1}{3}}$*, then the equation is simplified to* $a\beta = \sqrt{1-\frac{2}{\pi}\beta^2}$*. Therefore* $\beta^2 = \frac{1}{a^2+\frac{2}{\pi}}$*. Notice that* a *and* β *are of the same sign and* a *and* c *are of the same sign, so* β *and* c *are of the same sign. Hence we have:*

$$\beta = \begin{cases} \frac{1}{\sqrt{a^2+\frac{2}{\pi}}}, & \text{if the given skewness } c > 0, \\ -\frac{1}{\sqrt{a^2+\frac{2}{\pi}}}, & \text{if the given skewness } c < 0. \end{cases}$$

Note that α *and* β *are of the same sign, so* $\alpha = \frac{\beta}{\sqrt{1-\beta^2}}$*. If* $c = 0$*, we clearly have* $\alpha = 0$*. Once we obtain* α*,* ω *is given by*

$$\omega = \frac{\sigma}{\sqrt{1-\frac{2}{\pi}\cdot\frac{\alpha^2}{1+\alpha^2}}}.$$

Finally, ξ *is given by*

$$\xi = m_1 - \sqrt{\frac{2}{\pi}}\cdot\frac{\alpha\omega}{\sqrt{1+\alpha^2}}.$$

Lastly, we derive the moment generating function of the standard skew-normal distribution with parameter α. This result permits us to study the moments of log-skew-normal distribution in the next section.

Proposition 4.3.5. *Let* f_α *be the probability density function of the standard skew-normal distribution with parameter* α*. Denote the probability density function and the cumulative distribution function of the standard normal distribution by* ϕ *and* Φ *respectively. Then the moment generating function of this distribution is given by*

$$\theta(u) \triangleq \int_{-\infty}^{\infty} e^{ux} f_\alpha(x)dx = 2e^{\frac{u^2}{2}}\Phi\left(\frac{\alpha u}{\sqrt{1+\alpha^2}}\right).$$

Proof. Fix u and define $g(\alpha) = \int_{-\infty}^{\infty} e^{ux} f_\alpha(x)dx$, then we have

$$\begin{aligned}
g'(\alpha) &= \int_{-\infty}^{\infty} e^{ux} \cdot 2x\phi(x)\phi(\alpha x)dx \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} xe^{ux}e^{-\frac{1}{2}(1+\alpha^2)x^2}dx \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} xe^{-\frac{1}{2}(1+\alpha^2)(x-\frac{u}{1+\alpha^2})^2+\frac{1}{2}\cdot\frac{u^2}{1+\alpha^2}}dx \\
&= \frac{1}{\pi}e^{\frac{1}{2}\cdot\frac{u^2}{1+\alpha^2}}\int_{-\infty}^{\infty} xe^{-\frac{1}{2}(1+\alpha^2)(x-\frac{u}{1+\alpha^2})^2}dx.
\end{aligned}$$

Now let $y = \sqrt{1+\alpha^2}(x - \frac{u}{1+\alpha^2})$, then the integral becomes:

$$\begin{aligned}
&\int_{-\infty}^{\infty} xe^{-\frac{1}{2}(1+\alpha^2)(x-\frac{u}{1+\alpha^2})^2}dx \\
=& \int_{-\infty}^{\infty}\left(\frac{y}{\sqrt{1+\alpha^2}}+\frac{u}{1+\alpha^2}\right)e^{-\frac{1}{2}y^2}\cdot\frac{1}{\sqrt{1+\alpha^2}}dy \\
=& \ 0+\frac{u}{(1+\alpha^2)^{\frac{3}{2}}}\sqrt{2\pi}.
\end{aligned}$$

Hence, we obtain

$$g'(\alpha) = \sqrt{\frac{2}{\pi}}\cdot\frac{u}{(1+\alpha^2)^{\frac{3}{2}}}\cdot e^{\frac{1}{2}\cdot\frac{u^2}{1+\alpha^2}}.$$

Next, we compute the indefinite integral $\int \frac{u}{(1+\alpha^2)^{\frac{3}{2}}}\cdot e^{\frac{1}{2}\cdot\frac{u^2}{1+\alpha^2}}d\alpha$. Let $y = \frac{\alpha u}{\sqrt{1+\alpha^2}}$, then

$$dy = u\cdot\frac{1\cdot\sqrt{1+\alpha^2}-\alpha\cdot\frac{\alpha}{\sqrt{1+\alpha^2}}}{1+\alpha^2}d\alpha = \frac{ud\alpha}{(1+\alpha^2)^{\frac{3}{2}}}.$$

Also, $y^2 = \frac{\alpha^2u^2}{1+\alpha^2}$ which implies $\alpha^2 = \frac{y^2}{u^2-y^2}$. It follows that $\frac{1}{2}\cdot\frac{u^2}{1+\alpha^2} = \frac{u^2-y^2}{2}$. Therefore, we have

$$\int \frac{u}{(1+\alpha^2)^{\frac{3}{2}}}\cdot e^{\frac{1}{2}\cdot\frac{u^2}{1+\alpha^2}}d\alpha = \int e^{\frac{u^2-y^2}{2}}dy = e^{\frac{u^2}{2}}\sqrt{2\pi}\Phi(y)+C'$$

and hence, we have

$$g(\alpha) = 2e^{\frac{u^2}{2}}\Phi(y)+C = 2e^{\frac{u^2}{2}}\Phi\left(\frac{\alpha u}{\sqrt{1+\alpha^2}}\right)+C.$$

On the other hand,

$$
\begin{aligned}
g(0) &= \int_{-\infty}^{\infty} e^{ux}\phi(x)dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{ux} e^{-\frac{1}{2}x^2 dx} \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-u)^2+\frac{u^2}{2}} dx \\
&= e^{\frac{u^2}{2}}.
\end{aligned}
$$

It follows that $C = 0$ and hence $\theta(u) = g(\alpha) = 2e^{\frac{u^2}{2}}\Phi\left(\frac{\alpha u}{\sqrt{1+\alpha^2}}\right)$. □

4.3.3 Log-Skew-Normal Distributions

One drawback of the skew-normal distributions is that the probability of the skew-normally distributed random variable being negative is positive. This phenomenon makes them inappropriate to model the average stock price since the latter one is always positive. One way to overcome this difficulty is to consider the exponential of a skew-normal distribution.

Definition 4.3.3. *We say that a random variable X is log-skew-normally distributed if $\ln(X)$ is skew-normally distributed.*

Proposition 4.3.6. *Let X be a log-skew-normally distributed random variable with $\ln(X)$ distributed skew-normally with parameters α, ω, ξ. Then the first three moments of X are respectively given by*

$$
\begin{aligned}
E[X] &= 2\exp\left(\xi + \frac{\omega^2}{2}\right)\Phi\left(\frac{\omega\alpha}{\sqrt{1+\alpha^2}}\right), \\
E\left[X^2\right] &= 2\exp\left(2\xi + 2\omega^2\right)\Phi\left(\frac{2\omega\alpha}{\sqrt{1+\alpha^2}}\right), \\
E\left[X^3\right] &= 2\exp\left(3\xi + \frac{9}{2}\omega^2\right)\Phi\left(\frac{3\omega\alpha}{\sqrt{1+\alpha^2}}\right),
\end{aligned}
$$

where Φ is the culmulative distribution function of the standard normal distribution.

Proof. Let $Y = \frac{\ln(X)-\xi}{\omega}$ which is a random variable distributed standard skew-normally with parameter α. Let $\theta(u) = E\left[e^{uY}\right]$ be the moment generating function of Y. Observe that $X = \exp(\omega Y + \xi)$, and hence, by the previous proposition, we have

$$\begin{aligned}
E[X] &= e^{\xi} E\left[e^{\omega Y}\right] \\
&= e^{\xi}\theta(\omega) \\
&= e^{\xi} \cdot 2e^{\frac{\omega^2}{2}} \Phi\left(\frac{\alpha\omega}{\sqrt{1+\alpha^2}}\right) \\
&= 2\exp\left(\xi + \frac{\omega^2}{2}\right) \Phi\left(\frac{\omega\alpha}{\sqrt{1+\alpha^2}}\right), \\
E\left[X^2\right] &= e^{2\xi} E\left[e^{2\omega Y}\right] \\
&= e^{2\xi}\theta(2\omega) \\
&= e^{2\xi} \cdot 2e^{\frac{4\omega^2}{2}} \Phi\left(\frac{\alpha \cdot 2\omega}{\sqrt{1+\alpha^2}}\right) \\
&= 2\exp\left(2\xi + 2\omega^2\right) \Phi\left(\frac{2\omega\alpha}{\sqrt{1+\alpha^2}}\right), \\
E\left[X^3\right] &= e^{3\xi} E\left[e^{3\omega Y}\right] \\
&= e^{3\xi}\theta(3\omega) \\
&= e^{3\xi} \cdot 2e^{\frac{9\omega^2}{2}} \Phi\left(\frac{\alpha \cdot 3\omega}{\sqrt{1+\alpha^2}}\right) \\
&= 2\exp\left(3\xi + \frac{9\omega^2}{2}\right) \Phi\left(\frac{3\omega\alpha}{\sqrt{1+\alpha^2}}\right).
\end{aligned}$$

□

Remark 4.3.3. *Unlike the skew-normal case, it is very hard, if not impossible, to express the parameters α, ω, ξ in terms of the moments of X analytically. Here, we solve the problem numerically. Let $m_k = E[X^k]$, for $k = 1, 2, 3$, be the first three moments of X. Let*

$$F(\alpha, \omega, \xi) = \begin{pmatrix} F_1(\alpha, \omega, \xi) \\ F_2(\alpha, \omega, \xi) \\ F_3(\alpha, \omega, \xi) \end{pmatrix},$$

where

$$\begin{aligned}
F_1(\alpha, \omega, \xi) &= 2\exp\left(\xi + \frac{\omega^2}{2}\right) \Phi\left(\frac{\omega\alpha}{\sqrt{1+\alpha^2}}\right) - m_1, \\
F_2(\alpha, \omega, \xi) &= 2\exp\left(2\xi + 2\omega^2\right) \Phi\left(\frac{2\omega\alpha}{\sqrt{1+\alpha^2}}\right) - m_2, \\
F_3(\alpha, \omega, \xi) &= 2\exp\left(3\xi + \frac{9\omega^2}{2}\right) \Phi\left(\frac{3\omega\alpha}{\sqrt{1+\alpha^2}}\right) - m_3.
\end{aligned}$$

Then the Jacobi matrix of F at (α, ω, ξ) is

$$dF(\alpha, \omega, \xi) = \begin{pmatrix} \frac{\partial F_1}{\partial \alpha} & \frac{\partial F_1}{\partial \omega} & \frac{\partial F_1}{\partial \xi} \\ \frac{\partial F_2}{\partial \alpha} & \frac{\partial F_2}{\partial \omega} & \frac{\partial F_2}{\partial \xi} \\ \frac{\partial F_3}{\partial \alpha} & \frac{\partial F_3}{\partial \omega} & \frac{\partial F_3}{\partial \xi} \end{pmatrix},$$

where all partial derivatives are evaluated at the point (α, ω, ξ). We solve the equation $F(\alpha, \omega, \xi) = (0, 0, 0)^T$ by multi-dimensional Newton's method. Let $x_n = (\alpha_n, \omega_n, \xi_n)^T$, for $n = 0, 1, 2, \ldots$, where x_0 is an initial guess, and $x_{n+1}$is defined implicitly in terms of x_n by the equation

$$dF(x_n)(x_{n+1} - x_n) = -F(x_n).$$

The 3-by-3 equation in above is solved numerically. In the following, we work out the partial derivatives.

$$\begin{aligned}
\frac{\partial F_1}{\partial \alpha} &= 2e^{\xi + \frac{\omega^2}{2}} \phi\left(\frac{\omega\alpha}{\sqrt{1+\alpha^2}}\right) \cdot \frac{\omega}{(1+\alpha^2)^{\frac{3}{2}}}, \\
\frac{\partial F_2}{\partial \alpha} &= 4e^{2\xi + 2\omega^2} \phi\left(\frac{2\alpha\omega}{\sqrt{1+\alpha^2}}\right) \cdot \frac{\omega}{(1+\alpha^2)^{\frac{3}{2}}}, \\
\frac{\partial F_3}{\partial \alpha} &= 6e^{3\xi + \frac{9}{2}\omega^2} \phi\left(\frac{3\omega\alpha}{\sqrt{1+\alpha^2}}\right) \cdot \frac{\omega}{(1+\alpha^2)^{\frac{3}{2}}}, \\
\frac{\partial F_1}{\partial \xi} &= 2e^{\xi + \frac{\omega^2}{2}} \Phi\left(\frac{\omega\alpha}{\sqrt{1+\alpha^2}}\right), \\
\frac{\partial F_2}{\partial \xi} &= 4e^{2\xi + 2\omega^2} \Phi\left(\frac{2\omega\alpha}{\sqrt{1+\alpha^2}}\right), \\
\frac{\partial F_3}{\partial \xi} &= 6e^{3\xi + \frac{9}{2}\omega^2} \Phi\left(\frac{3\omega\alpha}{\sqrt{1+\alpha^2}}\right), \\
\frac{\partial F_1}{\partial \omega} &= 2e^{\xi + \frac{\omega^2}{2}} \left[\omega\Phi\left(\frac{\omega\alpha}{\sqrt{1+\alpha^2}}\right) + \phi\left(\frac{\omega\alpha}{\sqrt{1+\alpha^2}}\right) \cdot \frac{\alpha}{\sqrt{1+\alpha^2}}\right], \\
\frac{\partial F_2}{\partial \omega} &= 4e^{2\xi + 2\omega^2} \left[2\omega\Phi\left(\frac{2\omega\alpha}{\sqrt{1+\alpha^2}}\right) + \phi\left(\frac{2\omega\alpha}{\sqrt{1+\alpha^2}}\right) \cdot \frac{\alpha}{\sqrt{1+\alpha^2}}\right], \\
\frac{\partial F_3}{\partial \omega} &= 6e^{3\xi + \frac{9}{2}\omega^2} \left[3\omega\Phi\left(\frac{3\omega\alpha}{\sqrt{1+\alpha^2}}\right) + \phi\left(\frac{3\omega\alpha}{\sqrt{1+\alpha^2}}\right) \cdot \frac{\alpha}{\sqrt{1+\alpha^2}}\right].
\end{aligned}$$

Chapter 5

Pricing Asian Options by Moments Fitting

In this chapter, we derive the pricing formulae for Asian call and Asian put options, where the distribution of the average price at the maturity T, i.e., A_T, is approximated by the respective distributions discussed in the previous chapter. Here, we consider the cases where A_T is approximated by log-normal distributions, shifted-log-normal distributions, skew-normal distributions, and log-skew-normal distributions. Throughout the chapter, we denote the maturity and the strike of Asian options by T and K respectively.

5.1 Approximation by log-normal distributions

Suppose that we approximate the distribution of the average price A_T at the maturity T by a log-normal distribution, i.e., $A_T \approx e^Z$, where $Z \sim N(\mu, \sigma^2)$, in the sense that the first two moments of A_T and e^Z match. Then the pricing formulae for Asian call and Asian put options reduce to the usual Black-Schole formulae. Let $\phi(x)$ and $\Phi(x)$ be the probability density function and the cumulative distribution function of the standard normal distribution respectively. Then the price of Asian call at the time $t = 0$ is:

$$\begin{aligned}
\text{Price of Asian Call} &= E\left[\frac{(A_T - K)^+}{B_T}\right] \\
&= e^{-rT}E[(A_T - K)^+] \\
&\approx e^{-rT}E[(e^Z - K)^+] \\
&= e^{-rT}\int_{\ln K}^{\infty}(e^x - K)\cdot\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2\sigma^2}(x-\mu)^2}dx \\
&= e^{-rT}(I_1 - I_2),
\end{aligned}$$

where I_1 and I_2 are the integrals $\int_{\ln K}^{\infty} e^x \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2\sigma^2}(x-\mu)^2}dx$ and $\int_{\ln K}^{\infty} K\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2\sigma^2}(x-\mu)^2}dx$ respectively. For the first integral, we have

$$\begin{aligned}
I_1 &= \int_{\ln K}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2\sigma^2}[x-(\mu+\sigma^2)]^2}\cdot e^{\frac{1}{2}(\sigma^2+2\mu)}dx \\
&= \int_{\frac{\ln K-(\mu+\sigma^2)}{\sigma}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}dy\cdot e^{\frac{1}{2}(\sigma^2+2\mu)} \\
&= e^{\frac{1}{2}(\sigma^2+2\mu)}\left[1-\Phi\left(\frac{\ln K-(\mu+\sigma^2)}{\sigma}\right)\right] \\
&= e^{\frac{1}{2}(\sigma^2+2\mu)}\Phi(\frac{\mu-\ln K+\sigma^2}{\sigma}).
\end{aligned}$$

For the second integral, we have

$$\begin{aligned}
I_2 &= K\int_{\frac{\ln K-\mu}{\sigma}}^{\infty}\phi(y)dy \\
&= K\left[1-\Phi(\frac{\ln K-\mu}{\sigma})\right] \\
&= K\Phi(\frac{\mu-\ln K}{\sigma}).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\text{Price of Asian call} \\
\approx\ & e^{\mu-rT+\frac{1}{2}\sigma^2}\Phi(\frac{\mu-\ln K+\sigma^2}{\sigma}) - Ke^{-rT}\Phi(\frac{\mu-\ln K}{\sigma}).
\end{aligned}$$

Lastly, by observing that $\frac{(A_T-K)^+}{B_T} - \frac{(K-A_T)^+}{B_T} = \frac{A_T-K}{B_T}$ and taking expectation on both sides, we obtain

$$\begin{aligned}
& \text{Price of Asian put} \\
\approx\ & Ke^{-rT}\Phi(\frac{\ln K - \mu}{\sigma}) - e^{\mu - rT + \frac{1}{2}\sigma^2}\Phi(\frac{\ln K - \mu - \sigma^2}{\sigma}).
\end{aligned}$$

5.2 Approximation by shifted-log-normal distributions

The pricing formulae for Asian options, where A_T is approximated by a shifted-log-normal distribution are very similar to those in the previous section. Suppose that A_T is approximated by a shifted-log-normal distribution $A_T \approx \alpha + e^Z$, where $\alpha \in \mathbb{R}$ and $Z \sim N(\mu, \sigma^2)$, in the sense that the first three moments of A_T and $\alpha + e^Z$ match. Let $K_1 = K - \alpha$, then the price of Asian call option is

$$\begin{aligned}
& \text{Price of Asian call} \\
=\ & E\left[\frac{(A_T - K)^+}{B_T}\right] \\
\approx\ & e^{-rT}E\left[(e^Z - K_1)^+\right] \\
=\ & \begin{cases} e^{-rT}\left(e^{\mu + \frac{1}{2}\sigma^2} - K_1\right), & \text{if } K_1 \leq 0 \\ e^{\mu - rT + \frac{1}{2}\sigma^2}\Phi\left(\frac{\mu - \ln K_1 + \sigma^2}{\sigma}\right) - K_1 e^{-rT}\Phi\left(\frac{\mu - \ln K_1}{\sigma}\right), & \text{if } K_1 > 0 \end{cases}.
\end{aligned}$$

Similarly, the price of Asian put option is

$$\begin{aligned}
& \text{Price of Asian put} \\
=\ & E\left[\frac{(K - A_T)^+}{B_T}\right] \\
\approx\ & e^{-rT}E\left[(K_1 - e^Z)^+\right] \\
=\ & \begin{cases} 0, & \text{if } K_1 \leq 0 \\ K_1 e^{-rT}\Phi\left(\frac{\ln K_1 - \mu}{\sigma}\right) - e^{\mu - rT + \frac{1}{2}\sigma^2}\Phi\left(\frac{\ln K_1 - \mu - \sigma^2}{\sigma}\right), & \text{if } K_1 > 0 \end{cases}.
\end{aligned}$$

5.3 Approximation by skew-normal fitting

Suppose that we approximate A_T by a skew-normal distribution: $A_T \approx \omega X + \xi$, where $\omega > 0$, $\xi \in \mathbb{R}$, and X is standard skew-normally distributed with parameter α, in the sense that the first three moments of A_T and $\omega X + \xi$ match. Recall that the random variable $\omega X + \xi$ admits a probability density function $\varphi(x)$, given by $\varphi(x) = \frac{2}{\omega}\phi(\frac{x-\xi}{\omega})\Phi(\alpha \cdot \frac{x-\xi}{\omega})$, where ϕ and Φ are the probability density fucntion and the cumulative distribution function of the standard normal distribution respectively. Note that, due to the complexity of $\varphi(x)$, we are unable to obtain closed-form pricing formulae for Asian call nor Asian put options. Rather, we compute the expectation by a numerical method. For the price of Asian call we get

$$\begin{aligned} & \text{Price of Asian call} \\ = \; & E\left[\frac{(A_T - K)^+}{B_T}\right] \\ \approx \; & e^{-rT} E\left[(\omega X + \xi - K)^+\right] \\ = \; & e^{-rT} \int_K^\infty (x - K)\varphi(x)dx, \end{aligned}$$

where the integral is calculated numerically by Simpson's rule. Similarly, the price of Asian put is

$$\begin{aligned} & \text{Price of Asian put} \\ = \; & E\left[\frac{(K - A_T)^+}{B_T}\right] \\ \approx \; & e^{-rT} E\left[(K - (\omega X + \xi))^+\right] \\ = \; & e^{-rT} \int_{-\infty}^K (K - x)\varphi(x)dx, \end{aligned}$$

where again the integral is computed by Simpson's rule.

5.4 Approximation by log-skew-normal fitting

Suppose that we approximate A_T by a log-skew-normal distribution: $A_T \approx e^{\omega X + \xi}$, where $\omega > 0$, $\xi \in \mathbb{R}$, and X is standard skew-normally distributed with parameter α, in the sense that the first three moments of A_T and $e^{\omega X + \xi}$ match. Denote $Y = \omega X + \xi$ and $Z = e^Y$. Observe that for $z > 0$, $P[Z \leq z] = P[Y \leq \ln z]$. Differentiating both sides of the equation with respect to z, we obtain:

$$\begin{aligned} \frac{d}{dz} P[Z \leq z] &= \phi_Y(\ln z)\frac{1}{z} \\ &= \frac{2}{\omega}\phi(\frac{\ln z - \xi}{\omega})\Phi(\alpha \cdot \frac{\ln z - \xi}{\omega}) \cdot \frac{1}{z}. \end{aligned}$$

In another word, Z has a probability density function $\varphi(x)$, given by

$$\varphi(x) = \begin{cases} \frac{2}{\omega}\phi(\frac{\ln x - \xi}{\omega})\Phi(\alpha \cdot \frac{\ln x - \xi}{\omega}) \cdot \frac{1}{x}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}.$$

Now the prices of Asian call and Asian put can be computed numerically as:

$$\text{Price of Asian call} = e^{-rT} \int_K^{\infty} (x - K)\varphi(x)dx,$$

and

$$\text{Price of Asian put} = e^{-rT} \int_{-\infty}^{K} (K - x)\varphi(x)dx.$$

Chapter 6

Numerical Results

In this chapter, we present the prices of various Asian call options computed by Monte Carlo simulation and by moments matching approaches discussed in Chapter 5. The prices obtained by the Monte Carlo simulation act as benchmark values and are used to compare with those obtained by various moments matching methods.

For the GBM model, we sample the stock price 1000 times per year and compute the average; while for the Heston model, we sample the stock price 100 times per year and computer the average. More precisely, let T be the maturity of the option. For the GBM model, let $n = \max\{k \in \mathbb{Z} \mid k \geq 0 \text{ and } \frac{k}{1000} \leq T\}$, $t_k = \frac{k}{1000}$, for $k = 1, 2, \ldots, n$, and let $A_T = \frac{1}{n}\sum_{k=1}^{n} S_{t_k}$. For the Heston model, let $n = \max\{k \in \mathbb{Z} \mid k \geq 0 \text{ and } \frac{k}{100} \leq T\}$, $t_k = \frac{k}{100}$, for $k = 1, 2 \ldots, n$, and let $A_T = \frac{1}{n}\sum_{k=1}^{n} S_{t_k}$. In this way, these discretely averaging average prices can be considered good approximations of continuously averaged prices.

We employ Milstein's scheme for the Heston model. Recall that if $\mu = \mu(t, x)$, $\sigma = \sigma(t, x)$ are $C^{1,2}$ functions and $(X_t)_t$ is an Ito's process satisfying the SDE:

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t,$$

Milstein's scheme states that for $t \geq 0$ and small $h > 0$, we have:

$$X_{t+h} \approx X_t + \mu(t, X_t)h + \sigma(t, X_t)(W_{t+h} - W_t) + \frac{1}{2}\sigma(t, X_t)\sigma_x(t, X_t)\left[(W_{t+h} - W_t)^2 - h\right].$$

We recall the Heston model:

$$
\begin{aligned}
dS_t &= S_t\left(rdt + \sqrt{V_t}dW_t^1\right), \\
dV_t &= a\,(b - V_t)\,dt + c\sqrt{V_t}dW_t^2,
\end{aligned}
$$

where $(W_t^1, W_t^2)_t$ is a two dimensional Wiener process with $dW_t^1 dW_t^2 = \rho dt$. We apply Milstein's scheme to the variance process $(V_t)_t$. For $t \geq 0$ and small $h > 0$, we have:

$$
V_{t+h} \approx V_t + a(b - V_t)h + c\sqrt{V_t}\left(W_{t+h}^2 - W_t^2\right) + \frac{c^2}{4}\left[(W_{t+h}^2 - W_t^2)^2 - h\right].
$$

To be more precise, let $0 = t_0 < t_1 < \ldots < t_N$ be time points. Let $(z_1, z_2, \ldots, z_{2N})$ be a realization of a random vector $(Z_1, Z_2, \ldots, Z_{2N})$, where $Z_1, Z_2, \ldots, Z_{2N}$ are independent, identically distributed random variables with the standard normal $N(0,1)$ as the common distribution. Define $y_i^{(1)} = \sqrt{t_i - t_{i-1}}z_{2i-1}$ and $y_i^{(2)} = \rho\left(\sqrt{t_i - t_{i-1}}z_{2i-1}\right) + \sqrt{1-\rho^2}\left(\sqrt{t_i - t_{i-1}}z_{2i}\right)$, for $i = 1, 2, \ldots, N$. Suppose that we have already generated S_{t_i} and V_{t_i} for $i = 0, 1, \ldots, k-1$, and that $V_{t_i} > 0$ for $i = 0, 1, \ldots, k-1$. We define

$$
V_{t_k} := V_{t_{k-1}} + a(b - V_{t_{k-1}})(t_k - t_{k-1}) + c\sqrt{V_{t_{k-1}}}y_k^{(2)} + \frac{c^2}{4}\left[\left(y_k^{(2)}\right)^2 - (t_k - t_{k-1})\right],
$$

and

$$
S_{t_k} := S_{t_{k-1}} \exp\left((r - \frac{1}{2}V_{t_{k-1}})(t_k - t_{k-1}) + \sqrt{V_{t_{k-1}}}y_k^{(1)}\right).
$$

If $V_{t_k} < 0$, we discard all the results, choose another random sample $(z_1, z_2, \ldots, z_{2N})$, and repeat the whole procedure.

When computing the prices of various Asian options by Monte Carlo simulation, we use 100 million paths for both the GBM model and the Heston model.

In the following, the percentage error is defined by

$$
\text{Percentage Error} = \frac{\text{Price by moments matching} - \text{Price by Monte Carlo simulation}}{\text{Price by Monte Carlo simulation}} \times 100\%.
$$

6.1 Pricing of Asian Call Options, for the GBM Model

In this section, we assume that the stock price follows the GBM model and show the prices of various Asian call options computed by Monte Carlo simulation, matching moments by log-normal distributions, matching moments by shifted-log-normal distributions, matching moments by skew-normal distributions, and matching moments by log-skew-normal distributions. In the following, the initial stock price S_0 for all cases is set to 100.00.

In Table 6.1, columns T, r, Sigma, Strike, MC, Std Err, LN, SLN, SN, LSN, and Err (%) represent maturity T, constant short rate r, volatility σ, strike K, prices by Monte Carlo simulation, standard error of the simulation, prices by log-normal fitting, prices by shifted-log-normal fitting, prices by skew-normal fitting, prices by log-skew-normal fitting, and percentage error respectively.

Table 6.1: Pricing of Asian call options, GBM model

T	r	Sigma	Strike	MC	Std Err	LN	Err (%)	SLN	Err (%)	SN	Err (%)	LSN	Err (%)
1.00	0.05	0.15	50	49.9790	8.52E-04	49.9821	0.01	49.9821	0.01	49.9821	0.01	49.9821	0.01
			75	26.1984	8.52E-04	26.2017	0.01	26.2016	0.01	26.2017	0.01	26.2016	0.01
			100	4.6868	6.06E-04	4.7019	0.32	4.6907	0.08	4.6817	-0.11	4.6887	0.04
			125	0.0429	5.51E-05	0.0377	-12.03	0.0429	0.04	0.0421	-1.75	0.0435	1.59
			150	0.0000	8.74E-07	0.0000	-50.79	0.0000	-4.09	0.0000	-66.04	0.0000	8.64
		0.20	50	49.9787	1.14E-03	49.9821	0.01	49.9821	0.01	49.9821	0.01	49.9821	0.01
			75	26.2056	1.14E-03	26.2122	0.03	26.2088	0.01	26.2109	0.02	26.2096	0.02
			100	5.7638	7.97E-04	5.7875	0.41	5.7690	0.09	5.7592	-0.08	5.7649	0.02
			125	0.2488	1.66E-04	0.2290	-7.95	0.2491	0.12	0.2496	0.30	0.2506	0.72
			150	0.0030	1.67E-05	0.0020	-34.39	0.0030	-1.96	0.0018	-40.87	0.0032	4.85
		0.25	50	49.9785	1.43E-03	49.9821	0.01	49.9821	0.01	49.9821	0.01	49.9821	0.01
			75	26.2510	1.42E-03	26.2703	0.07	26.2535	0.01	26.2543	0.01	26.2570	0.02
			100	6.8529	9.96E-04	6.8886	0.52	6.8598	0.10	6.8582	0.08	6.8528	0.00
			125	0.6669	3.21E-04	0.6285	-5.76	0.6682	0.20	0.6782	1.69	0.6692	0.34
			150	0.0324	6.71E-05	0.0243	-24.99	0.0320	-1.20	0.0240	-25.89	0.0333	2.65
	0.08	0.15	50	49.9480	8.43E-04	49.9526	0.01	49.9526	0.01	49.9526	0.01	49.9526	0.01
			75	26.8702	8.43E-04	26.8748	0.02	26.8748	0.02	26.8749	0.02	26.8748	0.02
			100	5.5127	6.42E-04	5.5323	0.36	5.5176	0.09	5.5085	-0.08	5.5162	0.06
			125	0.0686	7.03E-05	0.0618	-9.80	0.0687	0.18	0.0684	-0.17	0.0695	1.31
			150	0.0001	1.54E-06	0.0000	-46.40	0.0001	-3.16	0.0000	-58.68	0.0001	8.14
		0.20	50	49.9478	1.13E-03	49.9526	0.01	49.9526	0.01	49.9526	0.01	49.9526	0.01
			75	26.8747	1.13E-03	26.8818	0.03	26.8793	0.02	26.8811	0.02	26.8799	0.02
			100	6.5180	8.33E-04	6.5478	0.46	6.5240	0.09	6.5109	-0.11	6.5207	0.04
			125	0.3327	1.93E-04	0.3109	-6.53	0.3333	0.19	0.3362	1.08	0.3346	0.58
			150	0.0048	2.10E-05	0.0033	-31.18	0.0047	-1.59	0.0031	-34.87	0.0050	4.34
		0.25	50	49.9476	1.41E-03	49.9526	0.01	49.9526	0.01	49.9526	0.01	49.9526	0.01
			75	26.9083	1.40E-03	26.9262	0.07	26.9124	0.02	26.9138	0.02	26.9154	0.03
			100	7.5545	1.03E-03	7.5977	0.57	7.5620	0.10	7.5528	-0.02	7.5561	0.02
			125	0.8165	3.55E-04	0.7777	-4.74	0.8184	0.24	0.8339	2.14	0.8186	0.26
			150	0.0440	7.82E-05	0.0341	-22.55	0.0436	-0.91	0.0346	-21.45	0.0451	2.39

Continued on next page

Table 6.1 – Continued from previous page

T	r	Sigma	Strike	MC	Std Err	LN	Err (%)	SLN	Err (%)	SN	Err (%)	LSN	Err (%)
3.00	0.05	0.15	50	49.8250	1.43E-03	49.8283	0.01	49.8283	0.01	49.8283	0.01	49.8283	0.01
			75	28.3322	1.43E-03	28.3456	0.05	28.3346	0.01	28.3356	0.01	28.3371	0.02
			100	9.4629	1.13E-03	9.5180	0.58	9.4671	0.04	9.4435	-0.21	9.4637	0.01
			125	1.4130	4.76E-04	1.3736	-2.79	1.4151	0.15	1.4486	2.52	1.4126	-0.03
			150	0.1132	1.31E-04	0.0943	-16.67	0.1125	-0.63	0.1001	-11.54	0.1147	1.31
		0.20	50	49.8247	1.92E-03	49.8285	0.01	49.8283	0.01	49.8283	0.01	49.8284	0.01
			75	28.4953	1.89E-03	28.5503	0.19	28.4938	-0.01	28.4399	-0.19	28.5052	0.03
			100	11.0936	1.49E-03	11.1881	0.85	11.0999	0.06	11.1040	0.09	11.0918	-0.02
			125	2.8727	8.24E-04	2.8255	-1.64	2.8798	0.25	2.9962	4.30	2.8690	-0.13
			150	0.5691	3.70E-04	0.5085	-10.65	0.5688	-0.06	0.5519	-3.03	0.5717	0.45
		0.25	50	49.8264	2.42E-03	49.8345	0.02	49.8296	0.01	49.8283	0.00	49.8310	0.01
			75	28.8801	2.34E-03	29.0093	0.45	28.8705	-0.03	28.5178	-1.25	28.8952	0.05
			100	12.7795	1.87E-03	12.9293	1.17	12.7883	0.07	12.9393	1.25	12.7744	-0.04
			125	4.5346	1.21E-03	4.4997	-0.77	4.5500	0.34	4.8357	6.64	4.5270	-0.17
			150	1.4202	6.95E-04	1.3180	-7.19	1.4244	0.30	1.4590	2.74	1.4219	0.13
	0.08	0.15	50	49.5726	1.39E-03	49.5772	0.01	49.5772	0.01	49.5772	0.01	49.5772	0.01
			75	29.9160	1.38E-03	29.9256	0.03	29.9201	0.01	29.9214	0.02	29.9215	0.02
			100	11.7674	1.19E-03	11.8282	0.52	11.7713	0.03	11.7376	-0.25	11.7721	0.04
			125	2.2739	5.94E-04	2.2480	-1.14	2.2779	0.18	2.3219	2.11	2.2723	-0.07
			150	0.2387	1.89E-04	0.2115	-11.39	0.2382	-0.21	0.2292	-3.97	0.2405	0.75
		0.20	50	49.5724	1.86E-03	49.5773	0.01	49.5772	0.01	49.5772	0.01	49.5773	0.01
			75	30.0050	1.84E-03	30.0451	0.13	30.0059	0.00	29.9719	-0.11	30.0148	0.03
			100	13.0935	1.55E-03	13.1997	0.81	13.0980	0.03	13.0453	-0.37	13.0965	0.02
			125	3.9207	9.40E-04	3.9011	-0.50	3.9298	0.23	4.0658	3.70	3.9157	-0.13
			150	0.8981	4.58E-04	0.8331	-7.24	0.8993	0.14	0.9097	1.29	0.8999	0.20
		0.25	50	49.5731	2.35E-03	49.5807	0.02	49.5778	0.01	49.5772	0.01	49.5786	0.01
			75	30.2625	2.29E-03	30.3665	0.34	30.2555	-0.02	29.9636	-0.99	30.2782	0.05
			100	14.5467	1.92E-03	14.7126	1.14	14.5518	0.04	14.5906	0.30	14.5470	0.00
			125	5.6658	1.31E-03	5.6705	0.08	5.6824	0.29	5.9996	5.89	5.6570	-0.16
			150	1.9415	7.97E-04	1.8487	-4.78	1.9488	0.38	2.0478	5.48	1.9412	-0.01

In Table 6.1, we notice that if the option is deeply in-the-money, the price computed by any method of moments fitting is highly accurate. This phenomenon will be explained in the next chapter. The accuracy of all methods of moments fitting decreases as we move from deeply in-the-money towards deeply out-of-money. Since the method of moments fitting by the log-normal distribution only matches the first moments of A_T, it gives, as expected, the least accurate results. For the other three methods, all of them match the first three moments of A_T, so they yield more accurate results. For the method of moments fitting by the skew-normal distribution, we observe that the skew-normal distribution has a defect that it has a positive probability of being negative, so it gives less accurate results than the methods of fitting by the shifted-log-normal distribution or by the log-skew-normal distribution. This problem is especially serious when the option is deeply out-of-money. Regarding the method of moments fitting by the shifted-log-normal distribution and the method of moments fitting by the log-skew-normal distribution, for options with short maturity, the former one gives slightly better results. For options with long maturity, both methods give results with similar degree of accuracy.

6.2 Pricing of Asian Call Options, for the Heston Model

In this section, we assume that the stock price follows the Heston model and show the prices of various Asian call options computed by Monte Carlo, and matching moments by log-normal distribution. In the following, the constant short rate r, the initial stock price S_0, and the mean reversion rate a are set to 0.05, 100.00, and 1.0 respectively for all cases.

In Table 6.2, columns T, V_0, b, c, rho, strike, MC, Std Err, LN, Err (%) represent maturity T, initial variance V_0, anchor value b, vol-vol c, correlation of Wiener processes ρ, strike K, prices by Monte Carlo method, standard error of simulation, prices by log-normal fitting, and percentage error respectively.

Table 6.2: Pricing of Asian call options, Heston model

T	V_0	b	c	rho	strike	MC	Std Err	LN	Err (%)
1.00	0.0625	0.0625	0.0225	0.50	50	49.9770	1.43E-03	49.9797	0.01
					75	26.2444	1.42E-03	26.2677	0.09
					100	6.8477	1.00E-03	6.8846	0.54
					125	0.6903	3.33E-04	0.6271	-9.16
					150	0.0377	7.44E-05	0.0242	-35.81
				-0.50	50	49.9771	1.42E-03	49.9797	0.01
					75	26.2549	1.41E-03	26.2652	0.04
					100	6.8536	9.88E-04	6.8562	0.04
					125	0.6421	3.11E-04	0.6144	-4.32
					150	0.0277	6.03E-05	0.0231	-16.82
	0.16	0.16	0.1	0.50	50	49.9757	2.34E-03	49.9849	0.02
					75	26.8304	2.25E-03	27.0406	0.78
					100	10.1334	1.69E-03	10.2997	1.64
					125	2.9743	9.89E-04	2.8310	-4.82
					150	0.7883	5.23E-04	0.6232	-20.94
				-0.50	50	49.9784	2.29E-03	49.9836	0.01
					75	26.9515	2.18E-03	26.9714	0.07
					100	10.1225	1.60E-03	10.1043	-0.18
					125	2.7223	8.74E-04	2.6778	-1.63
					150	0.5857	4.03E-04	0.5596	-4.47
3.00	0.0625	0.0625	0.0225	0.50	50	49.8262	2.43E-03	49.8314	0.01
					75	28.8547	2.36E-03	28.9723	0.41
					100	12.7694	1.89E-03	12.8215	0.41
					125	4.5851	1.23E-03	4.3937	-4.17
					150	1.4809	7.22E-04	1.2586	-15.01
				-0.50	50	49.8274	2.41E-03	49.8306	0.01
					75	28.9062	2.32E-03	28.9358	0.10
					100	12.7868	1.85E-03	12.7073	-0.62
					125	4.4803	1.18E-03	4.2812	-4.45
					150	1.3603	6.61E-04	1.1964	-12.05
	0.16	0.16	0.1	0.50	50	49.9547	4.13E-03	50.1537	0.40
					75	31.0331	3.88E-03	31.6623	2.03
					100	17.9288	3.35E-03	18.3896	2.57
					125	10.1811	2.74E-03	10.1693	-0.12
					150	5.8516	2.20E-03	5.4922	-6.14
				-0.50	50	50.0368	3.90E-03	50.0655	0.06
					75	31.2507	3.62E-03	31.2259	-0.08
					100	17.8744	3.05E-03	17.6493	-1.26
					125	9.7521	2.41E-03	9.3808	-3.81
					150	5.2215	1.83E-03	4.8290	-7.52

In Table 6.2, we notice that the method of moments fitting by the log-normal distribution gives accurate results if the option is deeply in-the-money. However, the accuracy of the results decreases as we move from deeply in-the-money towards deeply out-of-money. Also, we observe that initial volatility V_0, the anchor value b and the correlation ρ have an impact on the accuracy. In general, the method of moments fitting by the log-normal distribution yields better result if the initial volatility V_0 and the anchor value b are small. Moreover, in general, for the Heston model, if the correlation ρ is negative, the average price A is more accurately approximated by the log-normal distribution, and hence it results in a more accurate pricing.

Remark 6.2.1. *In the above, we have tested only the pricing of Asian call options but not Asian put options. This is because the prices of Asian put options can be computed by put-call parity. More precisely, let T be the maturity, and let A_T be the average price at time T, i.e., $A_T = \frac{1}{T}\int_0^T S_u du$. Let r be a constant deterministic short rate and let $B_t = \exp(rt)$ be the risk-free banking account. Let K be the strike. Let χ_C and χ_P be the payoffs of an Asian call and an Asian put respectively, with maturity T and strike K. That is, $\chi_C = \max(A_T - K, 0)$ and $\chi_P = \max(K - A_T, 0)$. Denote the prices of the Asian call and the Asian put at the time 0 by $\Pi_C(0)$ and $\Pi_P(0)$ respectively. Observe that $\chi_C - \chi_P = A_T - K$, so $E^Q\left[\frac{\chi_C - \chi_P}{B_T}\right] = E^Q\left[\frac{A_T - K}{B_T}\right]$, and hence $\Pi_C(0) - \Pi_P(0) = E^Q\left[\frac{A_T}{B_T}\right] - Ke^{-rT}$. The term $E^Q\left[\frac{A_T}{B_T}\right]$ is computed by observing that $\left(\frac{S_t}{B_t}\right)_t$ is a martingale, namely*

$$E^Q\left[\frac{A_T}{B_T}\right] = \frac{e^{-rT}}{T}\int_0^T E^Q\left[\frac{S_t}{B_t}\right] e^{rt} dt = \frac{S_0 e^{-rT}}{rT}\left(e^{rT} - 1\right).$$

In the above, $E^Q[\cdots]$ denotes the expectation under the Q-measure.

Chapter 7

Conclusions

Firstly, we consider the pricing results where the stock price process follows the standard GBM model. By comparing the numerical results obtained from Monte Carlo simulation, matching moments by log-normal distributions, matching moments by shifted-log-normal distributions, matching moments by skew-normal distributions, and matching moments by log-skew-normal distributions, we conclude that, in general, the methods of matching moments by shifted-log-normal distributions and by log-skew-normal distributions give the most accurate results; while the method of matching moments by log-normal distributions gives the least accurate results. In general, if the option is deeply in-the-money, all the methods give very accurate results. This phenomenon can be explained as follow. Let T be the maturity, K the strike, A_T the average price at time T and r the constant deterministic short rate. If K is small compared to A_T, the payoff $\max(A_T - K, 0)$ of the Asian call option can be approximated by A_T well, and hence

$$\text{Price of the Asian call} = E\left[\frac{\max(A_T - K, 0)}{e^{rT}}\right] \approx E\left[\frac{A_T}{e^{rT}}\right],$$

where the expectation in the last line is model independent. Regarding the methods of matching moments by shifted-log-normal distributions and by log-skew-normal distributions, although both give fairly accurate results, fitting by shifted-log-normal distributions results in a closed-form pricing formula; while fitting by log-skew-normal distributions requires solving a system of 3×3 non-linear equations, and computing the prices requires a numerical integration. Due to the simplicity of the former one, that method is widely adopted in the industry.

Secondly, we consider the pricing results where the stock price process follows the Heston model. Due to the complexity of the model, we are only able to compute the first moment of A_T exactly and a good approximation of the second moment of A_T. Therefore, we consider the method of matching moments by the log-normal distributions only. Again, this method gives accurate results for in-the-money options but inaccurate results for out-of-money options.

Bibliography

[1] Martin Forde and Antoine Jacquier, (2010). *Robust Approximations for Pricing Asian Options and Volatility Swaps Under Stochastic Volatility*, Applied Mathematical Finance, Vol.17 No.3 241-259, July 2010.

[2] Sebastian del Bano Rollin, Albert Ferreiro-Castilla and Frederic Utzet (2009). *A New Look at the Heston Characteristic Function*, arXiv, arXiv:0902.2154.

[3] Fischer Black and Myron Scholes (1973). *The Pricing of Options and Corporate Liabilities*, The Journal of Political Economy, Vol. 81, No. 3 (May - Jun., 1973), P.637-654.

[4] John C. Cox, Jonathan E. Ingersoll and Stephen A. Ross. *A Theory of the Term Structure of Interest Rates*, Econometrica, Volume 53, Issue 2 (Mar., 1985), 385-408.

[5] Steven L. Heston. *A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options*, The Review of Financial Studies 6(2): 327-343.