Coordinated path following: A nested invariant sets approach

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In this thesis we study a coordinated path following problem for multi-agent systems. Each agent is modelled by a smooth, nonlinear, autonomous, deterministic control-affine ordinary differential equation. Coordinated path following involves designing feedback controllers that make each agent's output approach and traverse a pre-assigned path while simultaneously coordinating its motion with the other agents. Coordinated motion along paths includes tasks like maintaining formations, traversing paths at a common speed and more general tasks like making the positions of some agents obey functional constraints that depend on the states of other agents.

The coordinated path following problem is viewed as a nested set stabilization problem. In the nested set stabilization approach, stabilization of the larger set corresponds to driving the agents to their assigned paths. This set, under suitable assumptions, is an embedded, controlled invariant, product submanifold and is called the multi-agent path following manifold. Stabilization of the nested set, contained in the multi-agent path following manifold, corresponds to meeting the coordination specification. Under appropriate assumptions, this set is also an embedded controlled invariant submanifold which we call the coordination set.

Our approach to locally solving nested set stabilization problems is based on feedback equivalence of control systems. We propose and solve two local feedback equivalence problems for nested invariant sets. The first, less restrictive, solution gives necessary and sufficient conditions for the dynamics of a system restricted to the larger submanifold and transversal to the smaller submanifold to be linear and controllable. This normal form facilitates designing controllers that locally stabilize the coordination set relative to the multi-agent path following manifold. The second, more restrictive, result additionally im-

poses that the transversal dynamics to the larger submanifold be linear and controllable. This result can simplify designing controllers to locally stabilize the multi-agent path following manifold. We propose sufficient conditions under which these normal forms can be used to locally solve the nested set stabilization problem.

To illustrate these ideas we consider a coordinated path following problem for a multi-agent system of dynamic unicycles. The multi-agent path following manifold is characterized for arbitrary paths. We show that each unicycle is feedback equivalent, in a neighbourhood of its assigned path, to a system whose transversal and tangential dynamics to the path following manifold are both double integrators. We provide sufficient conditions under which the coordination set is nonempty. The effectiveness of the proposed approach is demonstrated experimentally on two robots.

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I owe my deepest gratitude to my parents. Words cannot express how grateful I am to you for all of your love and support.

Dedication

To my love, Khojasteh Sadat

for her dedicated support

Table of Contents

List of Figures

List of notation

$\mathcal{N}(A)$ An open set containing A, page20.

$C^\infty(M)$ The ring of smooth real-valued functions on M, page138.

(U, φ) A coordinate chart on a manifold with domain U and mapping φ, page136.

$\mathrm{ann}(\mathscr{V})$ The annihilator of subspace $\mathscr{V}$, page132.

$\mathrm{inv}(D)$ Involutive closure of the distribution D, page144.

$[\cdot, \cdot]$ Lie bracket, page142.

$\mathbb{S}^1$ Unit circle, page4.

$\mathbb{Z}$ The set of integer numbers, page18.

$\mathscr{X}'$ The dual space of a vector space $\mathscr{X}$, page132.

$\|\cdot\|$ Euclidean norm, page19.

$\phi(t, x_0)$ The solution of the system $\dot{x} = f(x)$ with initial condition x_0, page20.

$\phi_t^Y(p)$ The integral curve through p generated by the vector field Y, page141.

$\mathbb{R}$ The set of real numbers, page18.

$f_\star$ The push-forward map, page140.

$L_Y\lambda$ Lie derivative of λ along Y, page142.

$L_Y^k\lambda$ iterated Lie derivatives, page142.

T_pM The tangent space to manifold M at the point p, page139.

TM Tangent bundle to a manifold M, page140.

arg Principle argument of a complex number, page18.

$\mathrm{dist}(x, A)$ Distance from the point x to the set A, page19.

$\mathsf{GL}\,(n, \mathbb{R})$ The group of nonsingular $n \times n$ matrices with real coefficients, page19.

$\mathbb{1}_A$ The identity map on a set A, page18.

$\langle \cdot,\ \cdot \rangle$ The standard inner product on $\mathbb{R}^n$, page19.

$\mathbb{C}$ The set of complex numbers, page18.

$\mathbb{N}$ The set of natural numbers, page18.

$\mathbb{T}^r$ r-torus, page93.

$\mathbf{k}$ The set of integers $\{0, \cdots, k-1\}$, page18.

$\mathbf{U}_n$ The $n \times n$ upper-triangular matrix, page19.

$\mathscr{V} \oplus \mathscr{W}$ The internal direct sum of independent vector spaces, page132.

$\mathscr{X}/\mathscr{X}$ Quotient space of the vector space $\mathscr{X}$ and $\mathscr{V}$, page134.

$\varnothing$ The empty set, page18.

$ad_f^k g$ Iterated Lie brackets of the vectors f and g, page142.

$B_r(A)$ The tubular neighbourhood of A with radius r, page20.

$B_r(x)$ The open ball of radius r centered at x, page20.

C^∞ The class of smooth functions, page19.

0_n The $n \times 1$ vector of zeros, page19.

1_n The $n \times 1$ vector of ones, page19.

$\mathbf{0}_n$ The $n \times n$ matrix of zeros, page19.

$\mathbf{I}_n$ The $n \times n$ identity matrix, page19.

Diff(U) The family of diffeomorphisms with domain U, page19.

Abbreviations

ASC autonomous surface craft (p. 9)

AUV autonomous underwater vehicle (p. 9)

IOFLP input-output feedback linearization problem (p. 15)

LTI linear time invariant (p. 15)

MIMO multi-input multi-output (p. 15)

PFLP partial feedback linearization problem (p. 15)

SEFLP state-space exact feedback linearization problem (p. 15)

SISO single-input single-output (p. 15)

TFL transverse feedback linearization (p. 16)

Chapter 1

Introduction and literature review

1.1 Motivation

In many situations team work helps people achieve their goals more efficiently than working alone; there are tasks that can only be completed through collaboration. Similarly, in the field of control systems, the use of cooperative control systems is unavoidable in many occasions and a large amount of research has concentrated on cooperative control systems [66]. A cooperative control system is a system in which a group of autonomous control systems work together to achieve a common goal. This thesis studies a cooperative control problem called coordinated path following [44]. Coordinated path following involves designing feedback controllers that make each agent's output approach and traverse a pre-assigned path while simultaneously coordinating its motion with the other agents.

As a motivating example for the study of coordinated path following consider the following mine sweeping scenario. Suppose that there is an area with explosive mines that must be detected and disarmed. It is a highly dangerous task for humans so it is natural to have robots perform the task. In this scenario a pair of mobile robots are employed to safely deactivate the mines in a cooperative manner. The mobile robots are assigned predefined paths in the field. For the sake of illustration we assume the first robot's path is an ellipse and the second robot's path is a circle. Once the mobile robots reach their paths they must have equal phase, see Figure 1.1. Control design specifications in this example

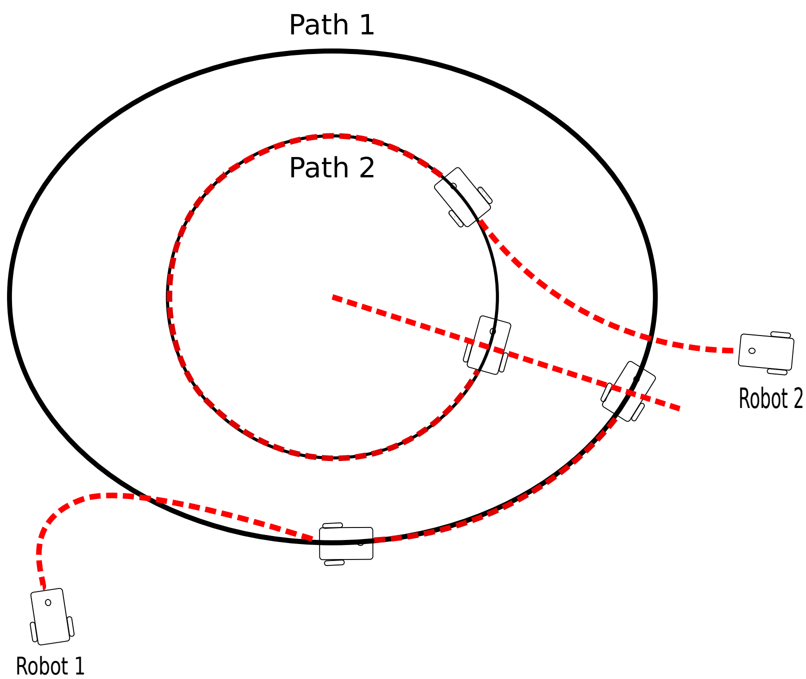


Figure 1.1: Schematic depiction of the mine field operation.

are:

1. Each mobile robot must converge to its assigned path and move along the path.

2. Once the mobile robots have reached their assigned paths they must remain there during the whole mission.

3. When the mobile robots are on their corresponding paths they must have the same

phase.

4. Once the robots become coordinated they must remain coordinated during the whole mission.

5. If the coordination task is changed, the robots should resynchronize but not leave their paths.

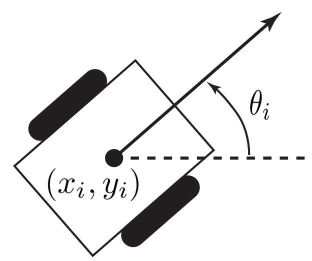
The coordination aspect of coordinated path following includes tasks like maintaining formations, traversing paths at a common speed and more general tasks like making the positions of some agents obey functional constraints that depend on the states of other agents. Coordinated path following is well-suited to applications in which accurate path traversal is vital and is motivated by applications in marine vehicle control [30, 61], mobile manipulation [24], search and rescue operations [16], and patrolling a pre-defined region [95].

Most studies on coordinated path following take an ad-hoc approach; they consider specific systems and devise system specific solutions. One motivation for this research is to take a more structural approach to coordinated path following that can be applied to a large class of systems. A second motivation is to consider applications in which invariance of the path of each agent as well as the invariance of the coordination specification are prioritized. This distinguishes this thesis from most studies in the literature. Finally, we formulate coordinated path following in such a way that many existing results on coordination of multi-agent systems can be incorporated into our solution to achieve decentralized coordination.

In Chapter 2 the coordinated path following problem is formulated as two set stabilization problems in the state space of a multi-agent system. It turns out that, under suitable assumptions, the sets are embedded controlled invariant[1] submanifolds and one set is nested inside the other. The simultaneous set stabilization problem is referred to as a nested set stabilization problem throughout this thesis. In the nested set stabilization problem stabilization of the larger set corresponds to driving the agents to their assigned paths, and stabilization of the nested set corresponds to meeting the coordination specification.

Example 1.1.1. Recall the previously described mine sweeping application. Each robot is modeled as a dynamic unicycle

$$\begin{aligned}\dot{x}_i &= v_i \cos(\theta_i)\\ \dot{y}_i &= v_i \sin(\theta_i)\\ \dot{\theta}_i &= u_{i,2}\\ \dot{v}_i &= u_{i,1}\end{aligned}$$



where (x_i, y_i) denotes the position of the unicycle in the plane, θ_i is the heading angle, and v_i is the forward velocity of the unicycle. The control inputs $u_{i,1}$ and $u_{i,2}$ are, respectively, the forward acceleration and angular velocity. Let $\mathbf{x}_i \coloneqq (x_i, y_i, \theta_i, v_i) \in \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R}$ denote the state space of each unicycle. We take the position of the unicycle i as its output $\mathbf{y}_i = h_i(\mathbf{x}_i) \coloneqq (x_i, y_i)$. The state of the multi-agent system of two unicycles is $\mathbf{x} \coloneqq (\mathbf{x}_1, \mathbf{x}_2) \in (\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R})^2$. The assigned paths are

$$\begin{aligned}\gamma_1 &= \left\{\mathbf{y}_1 \in \mathbb{R}^2 : s_1(\mathbf{y}_1) = \frac{y_{1,1}^2}{a^2} + \frac{y_{1,2}^2}{b^2} - 1 = 0\right\}\\ \gamma_2 &= \left\{\mathbf{y}_2 \in \mathbb{R}^2 : s_2(\mathbf{y}_2) = y_{2,1}^2 + y_{2,2}^2 - r_2^2 = 0\right\}\end{aligned}$$

where $a, b, r_2 \in \mathbb{R}$. The paths give rise to the following sets in the state spaces of each

[1]Refer to Appendix D and Definition D.1 for the definition of a controlled invariant submanifold.

robot

$$\Gamma_1 = \left\{ \mathbf{x}_1 \in \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R} : s_1(\mathbf{x}_1) = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 = 0 \right\}$$

$$\Gamma_2 = \left\{ \mathbf{x}_2 \in \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R} : s_2(\mathbf{x}_2) = x_2^2 + y_2^2 - r_2^2 = 0 \right\}.$$

If the state of robot i, $i \in \{1, 2\}$, reach the set Γ_i its output reaches path γ_i. However, there are points in this set at which the heading angle θ_i is not tangent to the path in the output space. Thus, if the robot is initialized at such a point it leaves the path instantaneously. In order to ensure that robots do not leave their corresponding paths we consider a subset $\Gamma_i^\star \subset \Gamma_i$ where there do not exist such points. If we define $\Gamma^\star := \Gamma_1^\star \times \Gamma_2^\star$, then the path following specification for the multi-agent system of two robots is accomplished if and only if this set stabilized.

The coordination specification can be cast as the stabilization of the set

$$\mathcal{C} = \{ \mathbf{x} \in \Gamma^\star : \arg(x_1 + jy_1) - \arg(x_2 + jy_2) = 0 \}.$$

Similarly, $\mathcal{C}$ contains points at which the angular velocity of the robots are not the same. Thus, if the robots are initialized on the paths and in coordination but with different angular velocities the coordination set will be left instantaneously. In order to make sure this situation is avoided we consider the largest controlled-invariant subset $\mathcal{C}^\star \subset \mathcal{C}$. The coordination specification for the two robots is accomplished if and only if $\mathcal{C}^\star$ is stabilized.

$\triangle$

In this thesis we (1) stabilize the larger set in the state space of the multi-agent system and (2) stabilize the nested set for the states of the multi-agent system contained in the larger set. We employ feedback equivalence of control systems to accomplish (1) and (2) locally. Two control systems are said to be feedback equivalent if there exists a coordinate transformation and a feedback transformation that maps trajectories of one control system to the other one. Given a control system and two nested sets, instead of designing controllers to accomplish (1) and (2) directly, we first seek a coordinate and feedback transformation that brings the control system to a form which is specially suitable for designing stabilizing controllers.

1.2 Coordinated path following

In this section we, somewhat informally, state a general coordinated path following problem which is the main subject of study in this thesis. Consider a multi-agent system consisting of N heterogeneous agents. Each agent i is modeled by

$$\begin{aligned} \dot{\mathbf{x}}_i &= f_i(\mathbf{x}_i) + \sum_{j=1}^{m_i} g_{i,j}(\mathbf{x}_i)u_{i,j} := f_i(\mathbf{x}_i) + g_i(\mathbf{x}_i)\mathbf{u}_i \\ \mathbf{y}_i &= h_i(\mathbf{x}_i), \end{aligned} \qquad i \in \{1, \cdots, N\} \tag{1.1}$$

where $\mathbf{x}_i \in \mathbb{R}^{n_i}$ denotes the states, $\mathbf{u}_i \in \mathbb{R}^{m_i}$ the control inputs, and $\mathbf{y}_i \in \mathbb{R}^{p_i}$ the outputs of the agent i. In equation (1.1) $f_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$, $h_i : \mathbb{R}^{n_i} \to \mathbb{R}^{p_i}$, and $g_{i,j} : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ for $j \in \{1, \cdots m_j\}, i \in \{1, \cdots, N\}$, are smooth functions.

We define the state of the overall multi-agent system as $\mathbf{x} := (\mathbf{x}_1, \cdots, \mathbf{x}_N) \in \mathbb{R}^n$ with $n := n_1 + \cdots + n_N$, the control input of the overall multi-agent system is $\mathbf{u} := (\mathbf{u}_1, \cdots, \mathbf{u}_N) \in \mathbb{R}^m$ with $m := m_1 + \cdots + m_N$, and the output of the overall multi-agent system is $\mathbf{y} := (\mathbf{y}_1, \cdots, \mathbf{y}_N) \in \mathbb{R}^p$ with $p := p_1 + \cdots + p_N$. With these definitions, the dynamics of the overall multi-agent system are compactly expressed as

$$\begin{aligned}\dot{\mathbf{x}} &= f(\mathbf{x}) + g(\mathbf{x})\mathbf{u} \\ \mathbf{y} &= h(\mathbf{x})\end{aligned} \tag{1.2}$$

where

$$f \coloneqq \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix}, \qquad g \coloneqq \begin{bmatrix} g_1 & 0 & \cdots & 0 \\ 0 & g_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g_N \end{bmatrix}, \qquad h \coloneqq \begin{bmatrix} h_1 \\ \vdots \\ h_N \end{bmatrix}.$$

1.2.1 Allowable paths

Suppose that each agent is assigned a path γ_i in its output space which is the image of a smooth, regular, map

$$\sigma_i \ : \ \mathbb{R} \longrightarrow \mathbb{R}^{p_i} \quad i \in \{1, \cdots, N\}. \tag{1.3}$$

We denote by $\gamma_i \coloneqq \sigma_i(\mathbb{R})$ the path of agent i. Since each curve σ_i is regular, we assume, without loss of generality, that it is unit-speed parameterized, i.e., for each $\lambda \in \mathbb{R}$, $\|\sigma_i'\| \equiv 1$. We henceforth assume that each path satisfies the following.

Assumption 1.2.1. For $i \in \{1, \cdots, N\}$, the path $\gamma_i \subset \mathbb{R}^{p_i}$ is a one-dimensional embedded submanifold[2]. There exists a smooth map $s_i : \mathbb{R}^{p_i} \to \mathbb{R}^{p_i - 1}$ such that $\gamma_i = s_i^{-1}(0)$ and $\mathrm{d}s_i(\mathbf{y}_i) \neq 0$ for all $\mathbf{y}_i \in \gamma_i$. Moreover, there exist $2N$ class-$\mathcal{K}_\infty$ functions $\rho_{i,1}, \rho_{i,2} : [0, \infty) \to \mathbb{R}_+$ such that

$$(\forall \mathbf{y}_i \in \mathbb{R}^{p_i}) \ \ \rho_{i,1}(\mathrm{dist}(\mathbf{y}_i, \gamma_i)) \leq \|s_i(\mathbf{y}_i)\| \leq \rho_{i,1}(\mathrm{dist}(\mathbf{y}_i, \gamma_i)). \tag{1.4}$$

[2]Refer to Appendix C.2 Definition C.12 for the definition of an embedded submanifold

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Assumption 1.2.1 requires that the entire path can be represented as the zero level set of the function $s_i : \mathbb{R}^{p_i} \to \mathbb{R}^{p_i-1}$ so that its Jacobian has full rank $p_i - 1$ at each point of the path. While the curve γ_i may be unbounded, condition (1.4) in Assumption 1.2.1 ensures that $\text{dist}(\mathbf{y}_i, \gamma_i) \to 0$ if and only if $s(\mathbf{y}_i) \to 0$.

1.2.2 Control design objectives

A coordination specification is described with a set of constraints on the position and velocity of agents along their paths. In order for the agents to coordinate their motions they must exchange state information. However, in general, communication constraints might be present so that some agents cannot access some other agents' states.

Given the multi-agent system (1.1) or (1.2), paths γ_i, $i \in \{1, \cdots, N\}$, assigned to each agent, a set of coordination constraints, and communication restrictions, the coordinated path following problem entails designing a state feedback controller $\mathbf{u} : \mathbb{R}^n \to \mathbb{R}^m$ such that the closed loop satisfies

PF1 For each initial condition the corresponding solution is defined for all $t \geq 0$ and all solutions are such that $\mathbf{y}_i \to \gamma_i$ as $t \to \infty$.

PF2 The assigned path γ_i for each agent is invariant in the sense that, if each agent is appropriately initialized on its corresponding path it remains there for all future time.

C1 When restricted to their assigned paths, the evolution of the agents is such that they approach coordination.

C2 The coordination specification must be invariant in the sense that, if all the agents are appropriately initialized on their corresponding path and in coordination, they maintain coordination for all $t \geq 0$.

C3 The state feedback $\mathbf{u}$ must be decentralized in the sense that it only uses state information that satisfy all inter-agent communication constraints.

A1 Once all the agents are on their corresponding paths and coordinated the overall multi-agent system must satisfy application specific specifications, e.g., boundedness, tracking, etc.

1.3 Literature review

In this section we review the literature on coordinated path following problem, nested set stabilization problem, and feedback equivalence.

1.3.1 Coordinated path following

Early studies on coordinated path following investigated the problem for two agents. For instance in [61] coordinated path following for two underwater vehicles is studied. A nonlinear control law based on Lyapunov theory is designed to steer two vehicles toward their paths. While they are on the paths one of them is selected as the leader and it travels with a desired velocity profile. The other one is selected as follower and adapts its distance with the leader. In [30] the coordinated path following problem between an autonomous surface craft (ASC) and an autonomous underwater vehicle (AUV) is considered. The ASC is assumed to be the leader and is launched to follow a path. The AUV is required to follow the ASC in the $x - y$ plane at a certain depth from the ASC.

Predominantly, two distinct approaches has been employed to solve a coordinated path following problem; namely, decoupling method [38] and curve extension method [13]. While the path following portion of the coordinated path following is invariably solved using only each agent's own information the specific manner in which the path following portion of the problem is solved distinguishes these studies.

The key idea in the decoupling method is to separate controller design for the path following and the coordination portions of the coordinated path following problem. In order to solve the path following portion each path is parameterized, the parameterization is utilized as a reference trajectory, and the evolution of the path parameter is treated as an additional control input. This approach, and variations on its theme, is popular and the subject of a considerable amount of work, see [86, 1, 2, 15, 44] among others. In this approach, path following is accomplished when the error between the output of an agent and the reference point on the path defined by the path parameter is asymptotically driven to zero. The point on the path defined by the controlled path parameter can be viewed as a virtual target in this approach. In [37] linearization and gain scheduling are utilized to stabilize the error dynamics for a multi-agent system of wheeled mobile robots. Lyapunov-based methods are employed in [35, 93] to stabilize the error dynamics for the multi-agent system of mobile wheeled robots, unicycles, and autonomous underwater vehicles, respectively. In [39] a multi-agent system of autonomous surface vessels is considered and adaptive back-stepping is employed to stabilize the error dynamics. In [91] a multi-agent system of marine surface vehicles is investigated and a neural network adaptive technique is employed to accomplish path following in the presence of disturbances and model uncertainties.

As far as the coordination portion is concerned, in the decoupling method it is typically enforced through clever re-parameterization of each system's assigned path. In [61, 35] each path is parameterized so that the desired formation corresponds to having each system's path parameter approach a common value. In [74, 44, 16, 97] a method called formation reference point is suggested to re-parameterize the assigned paths for each agent. The desired formation, which can change over time, is treated as a virtual geometric structure and a desired reference path is assigned to the centroid of the virtual structure. The reference path of the centroid determines the movement of the whole multi-agent system. The path of each agent is then re-parameterized according to its position in the virtual structure.

As opposed to the decoupling method, in the curve extension method the path following and coordination controllers are designed simultaneously. In addition, path following is viewed as a set stabilization problem. That is, a smooth function is employed for each path such that the zero level set of this function is the desired path. Convergence to the path is achieved when the value of the smooth function reaches zero. The process of finding a smooth function for a path is called curve extension. All the studies on the curve extension method are limited to closed paths. In order to drive each agent to the zero level set of the smooth function an error between the velocity vector of an agent and the tangent vector to the path is considered and path following is achieved if this error reaches zero. In [94, 13] coordinated path following of differential drive robots is solved using the curve extension method. A novel curve extension method is proposed which is effective for curves like circles, ellipses, and rounded parallelograms. In [96, 72] coordinated path following of unit speed particles in the plane along closed curves is studied. The control laws are designed to steer the particles to a pattern. In [95] it is assumed that a group of autonomous underwater vehicles are assigned to collect oceanographic information. They are required to traverse closed curves and to achieve patterns under which the spacing between neighbour vehicles are constant. In the curve extension method coordination is implemented by forcing the relative arc-length between each pair of vehicles to a constant value.

In some studies a different notion of coordinated path following is investigated in which all the agents are required to reach a common path. In [60] a group of unicycles is tasked to approach a common path and to achieve a desired inter-vehicle formation. It is assumed that each unicycle can measure its distance to the path and the heading error as well as the curvature of the path segment it can sense. Another distinct assumption made in this paper is that no global coordinates exist and each vehicle can only sense a part of the path locally. In this study it is required that the vehicles be located close enough to the path. A hybrid control approach is employed to solve the described coordinated path following problem in which each vehicle either runs a coordination algorithm or a single-vehicle algorithm. The main stability result states that if a vehicle is running coordination algorithm, it will not switch to the single-agent algorithm. And if they are running single-agent algorithm they will get to run coordination algorithm after a finite time.

In practice there are some obstacles to implementation such as bandwidth limitations, time delays, transmission noise, and communication failures. Some studies have investigated the impact of different imperfections in the communication topology on the stability of closed loop systems. Because of bandwidth limitations in [5] it is assumed that the information is exchanged in discrete-time instants. It is further assumed that the communication is directed and no delay and packet collisions exists. Under mild assumptions on the connectivity of the graph and assuming periodic communications, stability of the control laws are shown. In [36] the coordinated path following problem for a group of vehicles is considered and it is assumed that the communication topology suffers from time delays and losses. Two cases considered for the communication topology. In the first case it is assumed that there are brief connectivity losses. The second case assumes that the union of the communication graphs over uniform intervals of time remains connected.

The first drawback of the studies on coordinated path following is that a particular multi-agent systems with homogeneous agents is considered; thus application to other examples is not straightforward. The second drawback of the aforementioned approaches is that invariance of the paths is not guaranteed independently of the coordination task. This is important because it ensures that even if coordination fails due to, say, communication errors, or the coordination task changes the individual robots remain on their paths. The third drawback is that the coordination is not invariant; thus even if the agents are initially coordinated they might leave the coordination. Finally, most previous studies only consider position coordination, i.e., formation control, along the desired paths.

1.3.2 Hierarchical control design approach

A common practice for approaching a sophisticated control problem is to split the problem into prioritized sub-problems and solve them separately. This method is known as hierarchical control design. In [28] the hierarchical control design problem is viewed as the simultaneous stabilization of a chain of closed nested, controlled invariant, sets $S_1 \supset S_2 \supset \cdots \supset S_n$ in which set S_i represents the sub-problem i. Set S_i being nested in the set S_{i-1} indicates that the sub-problem $i + 1$ is solved only if the sub-problem i is solved. For instance the hierarchical control design approach is employed to formulate the position control problem of thrust-propelled vehicles in [78]. Other applications using hierarchical control design viewpoint include the circular formation stabilization problem for kinematic unicycles in [25, 83, 84], and the three-dimensional circular formation stabilization problem for kinematic particles in [26].

The objective of hierarchical control design problem is twofold : to design control laws solving each sub-problem independently and to investigate conditions under which the designed control laws solve the main problem when working together. The solution to the second aspect is closely related to the so-called reduction problem. The reduction problem was initially formulated in [80, 81]. There, a dynamical system with two closed invariant sets $S_1 \supset S_2$ in its state space is considered, and it is assumed that S_1 is asymptotically stable and S_2 is asymptotically stable relative to S_1. The reduction problem seeks conditions required to guarantee S_2 is asymptotically stable. The reduction theorems for stability and asymptotic stability of compact sets are proven in [82]. The extension to non-compact sets as well as the reduction theorem for attractivity are proved in [28].

In this thesis coordinated path following problem is cast as an instance of hierarchical control design and the first aspect of the hierarchical control design problem is investigated.

1.3.3 Feedback equivalence

Feedback equivalence problems have been extensively studied in the last thirty five years and their solutions have been a valuable tool in making many nonlinear control design problems tractable. Two control systems are said to be feedback equivalent if there exists a coordinate transformation and a feedback transformation mapping trajectories of two control systems to each other. A large portion of the control literature on feedback equivalence is dedicated to finite-dimensional, autonomous, deterministic, nonlinear control-affine systems and the feedback equivalence is local, i.e., valid in a neighbourhood of a point in the system's state space. Most of the studies on this field are heavily influenced by the seminal works of Poincaré [73] and Cartan [11]. Poincaré found sufficient conditions for a dynamic system to be locally equivalent to a linear one by means of an analytic transformation. In Cartan's method of equivalence a Pfaffian system of differential forms is generated by a dynamical or control system. The feedback equivalence problem of control systems is analyzed by studying the feedback equivalence of the corresponding Pfaffian systems. Many studies have built up on Cartan's method of equivalence the most prominent of which are [32, 33, 34, 31, 40, 42, 43, 49, 85]. A profound survey of different studies on feedback equivalence problem can be found in [77] and references therein.

A distinguished subdivision of studies on feedback equivalence is called feedback linearization in which feedback equivalence of a nonlinear control system to a controllable linear, or partially linear, system is sought. The motivation for studying this special case is clear. Rather than designing a feedback controller for the nonlinear control system, a potentially difficult task, the designer first finds the feedback equivalent linear system. The controller is designed for the linear equivalent system using the rich set of design tools for this class of system and then implemented on the nonlinear plant. Feedback equivalence to a complete linear time invariant (LTI) control system, known as state-space exact feedback linearization problem (SEFLP), was initially introduced in [53]. In [9] this problem was solved for single-input single-output (SISO) nonlinear control systems. Extension to the multi-input multi-output (MIMO) case was investigated in [87, 88]. In [47] the feedback equivalence of a control system to a partially linear control system, known as partial feedback linearization problem (PFLP), was investigated. In [55] PFLP yielding a linear subsystem of maximal size is investigated for SISO systems. In [64, 65] the MIMO case is considered.

Frequently, the input of a control system is employed to control its output. Thus, a natural feedback equivalence problem is to find, if possible, a coordinate and feedback transformation linearizing the input-output dynamics. This problem is referred to as input-output feedback linearization problem (IOFLP) and in [47, 48] is investigated for SISO control systems. Similar results for MIMO control systems are discussed in [45]. Since a few control systems are feedback linearizable, the concept of approximate feedback linearization was first raised by [54] and later different aspects of which was studied in [57, 56, 58, 51]. IOFLP is solvable if and only if the system possesses a well-defined relative degree. It turns out that the SEFLP and PFLP are closely related to IOFLP. That is, because solving the SEFLP or PFLP amounts to finding a virtual output which yields a well-defined relative degree.

In [6] transverse feedback linearization is proposed for stabilizing the periodic orbits of SISO control-affine systems. The term transverse feedback linearization (TFL) was coined in this study and it is motivated by the fact that in their method the dynamics transversal to the orbit is made linear and controllable. In [70] the results are generalized to MIMO control-affine systems and the target set is allowed to be an arbitrary embedded, controlled-invariant, submanifold of the state-space. Our solution to the nested set stabilization problem is based on this study.

1.4 Organization and contributions of the thesis

This thesis is organized as follows. In Chapter 2 we formalize the problem described in Section 1.2. The outline of Chapters 3 and 4 alongside their main contributions are listed below.

1.4.1 Chapter 3

- In Theorem 3.4.2 necessary and sufficient conditions are provided for the solvability of the main problem of Chapter 3 which is introduced as Problem 1. Given a control system and two nested sets the main problem asks to feedback linearize the dynamics with respect to the nested set when restricted to the larger set.

- In Section 3.3 we present results which are needed to solve Problem 1. The problem of restricting a control-affine system to nested, controlled invariant, embedded submanifolds is introduced in Problem 3. In order to provide the solution to this problem two results in Lemma 3.3.8 and Lemma 3.3.11 are produced in which, respectively, slice coordinates are generalized to two nested sets and the conditions under which a mutual friend of two controlled invariant submanifolds exists are provided. Finally, the necessary and sufficient conditions under which Problem 3 is solvable are presented in Theorem 3.3.13.

- In Problem 4, an extension of Problem 1 is introduced and in Theorem 3.5.1 necessary and sufficient conditions under which Problem 4 is solvable are presented. Given a control system and two nested sets Problem 4 asks to feedback linearize the dynamics with respect to the larger set as well as to feedback linearize the dynamics with respect to the nested set when restricted to the larger set.

- In Section 3.6 a local version of the nested set stabilization problem is introduced in Problem 5. In Theorem 3.6.1 we present sufficient conditions under which Problem 5 is solvable.

1.4.2 Chapter 4

- Coordinated path following problem for a multi-agent system of dynamics unicycles is considered. In Section 4.3 the multi-agent path following submanifold is globally characterized and in Proposition 4.3.1 it is shown that it consists of four disjoint components.

- In Section 4.3.2 it is shown that dynamics transversal to the multi-agent path following can always be transversally feedback linearized in a tubular neighbourhood of each component of the multi-agent path following manifold and in Lemma 4.3.3 the corresponding diffeomorphism is presented.

- In Section 4.6 experimental results are provided to demonstrate the effectiveness of the proposed control algorithms.

1.5 Notation

In this thesis, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers. If k is a positive integer, $\mathbf{k}$ denotes the set of integers $\{0, \dots, k-1\}$. Let $j \coloneqq \sqrt{-1}$ and let $\arg : \mathbb{C} \to (-\pi, \pi]$ map a complex number to its principle argument. Let $\varnothing$ denote empty set. The identity map on a set A is $\mathbb{1}_A$.

Let $\mathbb{R}^k$, $k \in \mathbb{N}$, denote the k-fold Cartesian product $\mathbb{R} \times \cdots \times \mathbb{R}$. Elements of $\mathbb{R}^k$ are ordered k-tuples of real numbers $(x_1, \cdots, x_k)$ and are called vectors. If $x \in \mathbb{R}^k$, we denote by x_i the ith component of x.

We treat $\mathbb{R}^n$ as an inner product space with the standard inner product

$$\langle x,\ y \rangle \coloneqq \sum_{i=1}^{n} x_i y_i$$

which induces the Euclidean norm

$$\|x\| := \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}.$$

Given a nonempty set $A \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, the point-to-set distance is defined as

$$\operatorname{dist}(x, A) := \inf\left\{\|x - y\| : y \in A\right\}.$$

The symbols $\mathbf{I}_n$ and $\mathbf{0}_n$ represent, respectively, the $n \times n$ identity matrix and matrix of zeros while 1_n and 0_n represent the $n \times 1$ vector of ones and vector of zeros. Let $\mathbf{U}_n$ denote an $n \times n$ upper-triangular matrix with $u_{ij} = 1$, $i \leq j$, $u_{ij} = 0$, $i > j$. Let

$$\mathsf{GL}\left(n, \mathbb{R}\right) := \left\{M \in \mathbb{R}^{n \times n} : \det M \neq 0\right\}$$

denote the set of nonsingular $n \times n$ matrices with real coefficients. This set has the algebraic structure of group and is called the general linear group.

Let f be a scalar-valued function from an open set $U \subseteq \mathbb{R}^n$ into $\mathbb{R}$ we denote by $\partial_{x_i} f$ its partial derivative with respect to x_i. The function is said to be smooth at $p \in U$ if it possesses continuous partial derivatives. If f is continuous at every $p \in U$ then we say it is smooth. We denote by C^∞ the class of smooth scalar-valued functions defined on an open set $U \subseteq \mathbb{R}^n$. A map $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ is smooth if each of its component scalar functions is smooth. Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^n$ be two open and connected sets, i.e., domains. A map $f : U \to V$ is a diffeomorphism if it is bijective and both f and f^{-1} are of class C^∞. Let U be an open and connected subset of $\mathbb{R}^n$ and denote by $\operatorname{Diff}(U)$ the collection of diffeomorphisms onto their images with domain U. Two sets U and V are diffeomorphic is there exists a diffeomorphism between them. The Jacobian of a C^1 map

$f : \mathbb{R}^n \to \mathbb{R}^m$ if evaluated at $p \in \mathbb{R}^n$ is written $\mathrm{d}f(p)$. A vector field on an open set $U \subseteq \mathbb{R}^n$ is a continuous map from U to $\mathbb{R}^n$. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector field and $\lambda : \mathbb{R}^n \to \mathbb{R}$ is a smooth map then $L_f\lambda(x) \coloneqq \langle \mathrm{d}\lambda(x),\ f(x)\rangle$. If f and g are smooth vector fields, then the Lie bracket of f and g is defined by the relation

$$(\forall \lambda \in C^\infty) \quad L_{[f,g]}\lambda = L_f(L_g\lambda) - L_g(L_f\lambda).$$

The following standard notation is used for iterated Lie derivatives and Lie brackets:

$$L_gL_f\lambda \coloneqq L_g(L_f\lambda),$$

$$L_g^0\lambda \coloneqq \lambda, \quad L_g^k\lambda \coloneqq L_g(L_f^{k-1}\lambda),$$

$$ad_f^0 g \coloneqq g, \quad ad_f^k g \coloneqq \left[f, ad_f^{k-1} g\right], \quad k \geq 1.$$

If $F : M \to N$ is a diffeomorphism between two manifolds, and if v is a vector field on M, then the differential of F can be used to define a vector field on N by means of the push-forward map $F_\star$, defined as $F_\star v(q) = (dF_p v(p))|_{p=F^{-1}(q)}$. If D is a non-singular distribution on a manifold M, $D^\perp$ is the orthogonal complement of D obtained from the orthogonal structure on the tangent bundle TM. The non-singular distribution $D^\perp$ is a subbundle of TM and satisfies, for each $p \in M$, $T_pM = D(p) \oplus D^\perp(p)$. Let $\mathrm{inv}(D)$ denote the involutive closure of D.

Given $r > 0$, $x \in \mathbb{R}^n$, and $A \subset \mathbb{R}^n$ then $B_r(x) = \{y \in \mathbb{R}^n : \|y - x\| < r\}$ is the open ball of radius r centered at x and $B_r(A) = \{x \in \mathbb{R}^n : \mathrm{dist}(x, A) < r\}$ is the tubular neighbourhood of A. We denote by $\mathcal{N}(A)$ an open set containing A. Given $\mathcal{N}(A)$ there does not necessarily exist an $r > 0$ so that $B_r(A) \subset \mathcal{N}(A)$. By $\phi(t, x_0)$ we denote the solution of the system $\dot{x} = f(x)$ at time t with initial condition x_0.

Chapter 2

Coordinated path following as a nested set stabilization problem

In this chapter, the coordinated path following problem from Section 1.2 is re-formulated as an instance of a nested set stabilization problem. A set called the multi-agent path following manifold is introduced with the property that if it is stabilized then control objectives **PF1** and **PF2** of Section 1.2.2 are met. A second set, nested in the multi-agent path following manifold, called the coordination set is introduced. If the coordination set is stabilized then the control objectives **C1** and **C2** of Section 1.2.2 are accomplished.

2.1 Path following

Control objective **PF1** of Section 1.2.2 requires each agent's closed-loop output approach its corresponding path γ_i. Moreover, **PF2** requires that γ_i be output invariant. Let

$$\Gamma_i \coloneqq \{\mathbf{x}_i \in \mathbb{R}^{n_i} : \alpha_i(\mathbf{x}_i) \coloneqq s_i \circ h_i(\mathbf{x}_i) = 0\}. \tag{2.1}$$

and $\Gamma \coloneqq \Gamma_1 \times \cdots \times \Gamma_N$. Driving $\mathbf{x}_i$ towards the set Γ_i corresponds to sending the output $\mathbf{y}_i$ of agent i to its desired path. However, generally Γ_i cannot be made invariant under the dynamics of (1.1) via a smooth feedback. Therefore, we seek to stabilize the largest controlled-invariant subset of Γ_i, which we denote by $\Gamma_i^\star$. Intuitively, the set $\Gamma_i^\star$ is the collection of all those motions of agent i whose associated output signals can be made to lie in γ_i for all time by a suitable choice of input signal. The set $\Gamma_i^\star$ is not necessarily connected. Moreover, if it is disjoint the components might have different dimensions.

Assumption 2.1.1. For each $i \in \{1, \cdots, N\}$ the largest controlled-invariant submanifold of Γ_i is non-empty and has a closed embedded component whose dimension is $n_i^\star > 0$. $\triangleleft$

Remark 2.1.2. Assumption 2.1.1 requires the global characterization of a component of $\Gamma_i^\star$. However, global characterization of $\Gamma_i^\star$ is, in general, an open problem. One can utilize the zero dynamics algorithm [45] or the constrained dynamics algorithm [71] to generate a local characterization of a connected component of $\Gamma_i^\star$ that contains the initialization point of the algorithm.

Definition 2.1.3. ([68]). The **path following manifold** $\Gamma_i^\star$ of γ_i with respect to (1.1) is a connected component of the largest controlled-invariant submanifold contained in Γ_i.

•

Definition 2.1.4. Let $\hat{\mathbf{u}}_i : \mathbb{R}^{n_i} \to \mathbb{R}^{m_i}$ be a smooth feedback and let $\Gamma_i^\star$ be the path following manifold of γ_i. The path γ_i is **output invariant** under the closed-loop vector field $f_i + g_i\hat{\mathbf{u}}_i$ if $\Gamma_i^\star$ is invariant under $f_i + g_i\hat{\mathbf{u}}_i$. •

Definition 2.1.5. The **multi-agent path following manifold** for N paths $\gamma_1, \ldots, \gamma_N$ that satisfy Assumption 1.2.1 is

$$\Gamma^\star \coloneqq \Gamma_1^\star \times \cdots \times \Gamma_{\mathrm{N}}^\star \tag{2.2}$$

and its dimension is $n^\star \coloneqq \sum n_i^\star \geq N$. •

Figure 2.1 summarizes the construction of the path following manifold of each agent and multi-agent path following manifold.

$$\dot{\mathbf{x}}_1 = f_1(\mathbf{x}_1) + g_1(\mathbf{x}_1)\mathbf{u}_1 \qquad \cdots \qquad \dot{\mathbf{x}}_N = f_N(\mathbf{x}_N) + g_N(\mathbf{x}_N)\mathbf{u}_N$$

$$\begin{array}{ccc}
\mathbf{x}_1 \in \mathbb{R}^{n_1} & \times \cdots \times & \mathbf{x}_N \in \mathbb{R}^{n_N} \\
\cup & & \cup \\
\Gamma_1 & \times \cdots \times & \Gamma_N \\
\cup & & \cup \\
\Gamma_1^\star & \times \cdots \times & \Gamma_N^\star \\
 & \parallel & \\
 & \Gamma^\star &
\end{array}$$

Figure 2.1: An illustration of the construction of the multi-agent path following manifold.

2.2 Coordination

A coordination specification is viewed as a set of constraints on the motions of agents along their paths. Also, as discussed in the previous section, when the state of a multi-agent system lies in the multi-agent path following manifold, the output of each individual agent is on its assigned path. Therefore, the coordination specification along the paths can be alternatively modeled as a constraint on the allowable motions on the multi-agent path following manifold. To this end, we model a coordination specification as a smooth constraint map $\beta : \Gamma^\star \to \mathbb{R}^c$ with $c \leq n^\star$.

Definition 2.2.1. Consider a multi-agent system with its multi-agent path following manifold $\Gamma^\star$ which has dimension $n^\star$. A **coordination function** is a smooth map

$$\beta : \Gamma^\star \to \mathbb{R}^c$$

such that $c \leq n^\star$ and $\mathrm{d}\beta$ has rank c at each point on $\Gamma^\star$. •

Let the restriction of (1.2) to $\Gamma^\star$ be

$$\dot{\mathbf{x}} = \bar{f}(\mathbf{x}) + \bar{g}(\mathbf{x})\mathbf{v} \tag{2.3}$$

where $\bar{f}\big|_{\Gamma^\star}$ and $\bar{g}|_{\Gamma^\star}$ are tangent to $\Gamma^\star$ and $\mathbf{v} \in \mathbb{R}^r$ are control inputs restricted to $\Gamma^\star$. In Chapter 3 we find necessary and sufficient conditions under which such a restriction is well-defined.

Let $\beta : \Gamma^\star \to \mathbb{R}^c$ be a coordination function and consider the following subset of $\Gamma^\star$

$$\mathcal{C} := \{\mathbf{x} \in \Gamma^\star : \beta(\mathbf{x}) = 0\}. \tag{2.4}$$

Definition 2.2.1 and the constant-rank level set theorem [62, Theorem 8.8] imply that the not necessarily bounded set $\mathcal{C}$ is a closed embedded submanifold of $\Gamma^\star$ of dimension $n^\star - c$. We take the view that stabilizing $\mathcal{C}$ corresponds to achieving coordination. This motivates us to characterize the largest controlled-invariant subset of $\mathcal{C}$. Just as for the multi-agent path following manifold, the largest controlled-invariant subset of $\mathcal{C}$ is not necessarily connected and if it is disjoint the components do not have necessarily the same dimension. Furthermore, any smooth feedback that makes $\mathcal{C}^\star$ invariant must satisfy the communication constraint in the sense that the control laws of each agent only use the state information of agents it can communicate with.

Throughout this thesis we model communication between agents of a multi-agent system using a weighted directed graph $\mathscr{G}$ called a **communication graph**[1].

Definition 2.2.2. Let $\mathscr{G}$ be a communication graph. The **coordination set** $\mathcal{C}^\star$ associated to a coordination function $\beta : \Gamma^\star \to \mathbb{R}^c$ is a connected component of the largest, controlled-invariant subset of $\mathcal{C}$ and the control laws that make $\mathcal{C}^\star$ invariant satisfy the communication constraints defined by $\mathscr{G}$. •

The set $\mathcal{C}^\star$ consists of all the trajectories of the multi-agent system for which the output trajectory is both on the path and satisfies the coordination constraint. In general, obtaining a global characterization of $\mathcal{C}^\star$, even in the absence of communication constraints, is an open problem.

Definition 2.2.3. Let $\mathscr{G}$ be a communication graph. A coordination specification is called **feasible** if the largest controlled-invariant subset of $\mathcal{C}$ is a non-empty, closed, embedded submanifold with dimension $c^\star$ and the control laws that make $\mathcal{C}^\star$ invariant satisfy the communication constraints defined by $\mathscr{G}$. •

[1] Refer to Appendix A for the definition of a weighted directed graph

Definition 2.2.4. Let $\mathscr{G}$ be a communication graph. Let $\hat{\mathbf{v}} : \mathbb{R}^n \longrightarrow \mathbb{R}^r$ be a smooth feedback respecting $\mathscr{G}$ and let $\mathcal{C}^\star$ be the coordination set. A coordination specification is **invariant** if $\mathcal{C}^\star$ is invariant under the closed-loop system $\bar{f} + \bar{g}\hat{\mathbf{v}}$. •

Example 2.2.5. Suppose that, once each agent reaches its path it must approach a particular point on its path. In this case we can model the coordination task using a function $\beta : \Gamma^\star \to \mathbb{R}^c$ with $c = n^\star$, i.e., $\dim(\mathcal{C}) = 0$, a point on $\Gamma^\star$. Such a coordination task does not require any interaction between agents. In such cases the coordination function $\beta = (\beta_1, \cdots, \beta_{n^\star})$ naturally decomposes into the constituent parts of the product manifold $\Gamma^\star$ with $\beta_i : \Gamma_i^\star \to \mathbb{R}^{n_i^\star}$ for $i \in \{1, \cdots, N\}$.

In coordinates, the decentralized nature of this coordination task is evident in the derivative $\mathrm{d}\beta$. Specifically, if we choose local coordinates charts for $\Gamma^\star$ using its product structure, i.e., local chart on $\Gamma^\star$ are the product of the individual coordinate charts of $\Gamma_i^\star$, then the derivative of $\mathrm{d}\beta$ in coordinates takes a block diagonal structure

$$\mathrm{d}\beta = \begin{bmatrix} \mathrm{d}\beta_1 & 0 & \cdots & 0 \\ 0 & \mathrm{d}\beta_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & \mathrm{d}\beta_{n^\star} \end{bmatrix}.$$

△

Figure 2.2 illustrates the relationship between the sets Γ, $\Gamma^\star$, $\mathcal{C}$, and $\mathcal{C}^\star$.

$$
\begin{array}{ccc}
\dot{\mathbf{x}}_1 = f_1(\mathbf{x}_1) + g_1(\mathbf{x}_1)\mathbf{u}_1 & \cdots & \dot{\mathbf{x}}_N = f_N(\mathbf{x}_N) + g_N(\mathbf{x}_N)\mathbf{u}_N \\
\\
\mathbf{x}_1 \in \mathbb{R}^{n_1} & \times \cdots \times & \mathbf{x}_N \in \mathbb{R}^{n_N} \\
\cup & & \cup \\
\Gamma_1 & \times \cdots \times & \Gamma_N \\
\cup & & \cup \\
\Gamma_1^\star & \times \cdots \times & \Gamma_N^\star \\
 & \cup & \\
 & \mathcal{C} = \beta^{-1}(0) & \\
 & \cup & \\
 & \mathcal{C}^\star &
\end{array}
$$

Figure 2.2: Inclusion diagram for the multi-agent path following manifold and the coordination set.

2.3 Nested set stabilization

In this section we reformulate the coordinated path following problem in Section 1.2.2 as a nested set stabilization problem. Consider a multi-agent system of the form (1.2) with paths γ_i, $i \in \{1, \cdots, N\}$, that satisfy Assumption 1.2.1. Let the inter-agent communication be modelled by a weighted, directed graph $\mathscr{G}$. Suppose, the multi-agent path following manifold of the paths $\gamma_1, \cdots, \gamma_N$, $\Gamma^\star$ is characterized. Consider a coordination specification which is expressed by coordination function β. Suppose the coordination set, $\mathcal{C}^\star$, is characterized. The coordinated path following control design problem entails designing feedback control laws such that for each initial condition $\mathbf{x}(0)$ in a neighbourhood of $\Gamma^\star$, the corresponding solution $\phi(t, \mathbf{x}(0))$ is defined for all $t \geq 0$ and the closed-loop multi-agent system meets the following objectives.

S1 For each initial condition $\mathbf{x}(0)$ in a neighbourhood of $\Gamma^\star$, the corresponding solution $\phi(t, \mathbf{x}(0))$ is defined for all $t \geq 0$ and $\Gamma^\star$ is asymptotically stable.

S2 For each initial condition $\mathbf{x}(0)$ in a neighbourhood of $\mathcal{C}^\star$, with $\mathbf{x}(0) \in \Gamma^\star$, the corresponding solution $\phi(t, \mathbf{x}(0)) \in \Gamma^\star$ is defined for all $t \geq 0$ and $\mathcal{C}^\star$ is asymptotically stable relative[2] to $\Gamma^\star$.

S3 The proposed control laws must satisfy the communication constraints defined by $\mathscr{G}$.

S4 The dynamics of the multi-agent system restricted to $\mathcal{C}^\star$ satisfy application specific specifications, e.g., boundedness, tracking, etc.

We have cast the coordinated path following problem as two set stabilization problems; namely the stabilization of $\Gamma^\star$ and $\mathcal{C}^\star$ relative to $\Gamma^\star$. This way, path following control design and coordination control design are performed separately. Although one could, in principle, achieve coordinated path following by directly stabilizing $\mathcal{C}^\star$, we take a nested set stabilization approach. This approach has three distinct advantages. First, this view point enables us to decouple the design of path following controllers from coordination controllers. Second, this approach ensures that if coordination fails due to, say, communication errors, the individual agents remain on their paths. Third, the nested set stabilization approach allows one to change the coordination specification without causing the agents to leave their paths.

Assume that one has designed a feedback control $\hat{\mathbf{u}}(\mathbf{x})$ which accomplishes **S1** and **S2**. In other words, those solutions of $\dot{\mathbf{x}} = f + g\hat{\mathbf{u}}$ starting in a neighbourhood of $\Gamma^\star$ asymptotically approach $\Gamma^\star$ and those solution of $\dot{\mathbf{x}} = f + g\hat{\mathbf{u}}$ starting from a neighbourhood of $\mathcal{C}^\star$ contained in $\Gamma^\star$ asymptotically approach $\mathcal{C}^\star$. There is no guarantee that the solutions

[2]Refer to Appendix E and Definition E.2 for relative stability.

of $\dot{\mathbf{x}} = f + g\hat{\mathbf{u}}$ starting in a neighbourhood of $\mathcal{C}^\star$ but not contained in $\Gamma^\star$ asymptotically approach $\mathcal{C}^\star$. In Section 3.6, sufficient conditions are provided for a local version of the nested set stabilization problem under which the asymptotic stability of the nested set is guaranteed.

Chapter 3

Local nested transverse feedback linearization

In this chapter we study two local feedback equivalence problems for a nonlinear control-affine system with two nested, controlled invariant, embedded submanifolds in its state space. We do not assume the larger submanifold to be a product manifold. The first, less restrictive, result gives necessary and sufficient conditions for the dynamics of the system restricted to the larger submanifold and transversal to the smaller submanifold to be linear and controllable. This normal form facilitates designing controllers that locally stabilize the smaller set relative to the larger set. The second, more restrictive, result additionally imposes that the transversal dynamics to the larger set be linear and controllable. This result can simplify designing controllers to locally stabilize the larger submanifold. This is illustrated by sufficient conditions under which these normal forms can be used to locally solve a nested set stabilization problem. Portions of this chapter have been submitted for publication in [20, 21].

3.1 Partial local nested transversal feedback linearization

Consider a control-affine system

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i =: f(x) + g(x)u \tag{3.1}$$

where $x \in \mathbb{R}^n$ denotes the state, $u = (u_1, \cdots, u_m) \in \mathbb{R}^m$ is the control input, and $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g_i : \mathbb{R}^n \to \mathbb{R}^n$, $i \in \{1, \cdots, m\}$, are smooth. To (3.1) we associate the family of distributions

$$G_i := \text{span}\left\{ad_f^j g_k \ : \ 0 \leq j \leq i, \ 1 \leq k \leq m\right\}. \tag{3.2}$$

The vectors $g_1(x), \ldots, g_m(x)$ are assumed to be linearly independent at each $x \in \mathbb{R}^n$, i.e., $\dim(G_0(x)) = m$. Along with (3.1), we are also given two embedded submanifolds $S_1 \subset \mathbb{R}^n$ and $S_2 \subset \mathbb{R}^n$ with $s_1 := \dim(S_1)$, $s_2 := \dim(S_2)$. The following assumption is made throughout this chapter.

Assumption 3.1.1. The sets S_1 and S_2 are controlled-invariant embedded submanifolds for (3.1) and $S_1 \supset S_2$. $\triangleleft$

The main problem investigated in this chapter, Problem 1, seeks a decomposition of (3.1) into three subsystems modelling its evolution on (i) S_2 (ii) $S_1 \backslash S_2$ and (iii) $\mathbb{R}^n \backslash S_1$, with the essential requirement that the dynamics on $S_1 \backslash S_2$ be linear and controllable.

Problem 1. (Partial local nested transversal feedback linearization): Given (3.1), nested sets $S_1 \supset S_2$ satisfying Assumption 3.1.1 and a point $\bar{x} \in S_2$, find, if possible, a diffeomorphism $\Xi : U \to \Xi(U) \subset \mathbb{R}^{s_2} \times \mathbb{R}^{s_1 - s_2} \times \mathbb{R}^{n-s_1}$, $x \mapsto (\zeta, \mu, \xi)$, and a regular feedback transformation (α, β) valid in a neighbourhood $U \subseteq \mathbb{R}^n$ of $\bar{x}$, such that (3.1) is feedback equivalent to

$$\begin{aligned}
\dot{\zeta} &= f_1(\zeta,\mu,\xi) + g_{11}(\zeta,\mu,\xi)v^{\parallel} + g_{12}(\zeta,\mu,\xi)v^{\parallel,\pitchfork} + g_{13}(\zeta,\mu,\xi)v^{\pitchfork} \\
\dot{\mu} &= A\mu + Bv^{\parallel,\pitchfork} + f_2(\zeta,\mu,\xi) + g_{21}(\zeta,\mu,\xi)v^{\parallel} + g_{22}(\zeta,\mu,\xi)v^{\parallel,\pitchfork} + g_{23}(\zeta,\mu,\xi)v^{\pitchfork} \\
\dot{\xi} &= f_3(\zeta,\mu,\xi) + g_{31}(\zeta,\mu,\xi)v^{\parallel} + g_{32}(\zeta,\mu,\xi)v^{\parallel,\pitchfork} + g_{33}(\zeta,\mu,\xi)v^{\pitchfork}
\end{aligned} \tag{3.3}$$

where

$$\Xi(S_1 \cap U) = \{(\zeta,\mu,\xi) \in \Xi(U) : \xi = 0\}\,, \tag{3.4a}$$

$$\Xi(S_2 \cap U) = \{(\zeta,\mu,\xi) \in \Xi(U) : \xi = 0,\ \mu = 0\}\,, \tag{3.4b}$$

$f_3(\zeta,\mu,0) = 0$, $g_{31}(\zeta,\mu,0) = 0$, $g_{32}(\zeta,\mu,0) = 0$, $f_2(\zeta,\mu,0) = 0$, $g_{21}(\zeta,\mu,0) = 0$, $g_{22}(\zeta,\mu,0) = 0$, the pair (A, B) is controllable, and B is full rank. ▲

Problem 1 seeks a coordinate and feedback transformation valid in a neighbourhood of $\bar{x}$ which generates a normal form with two types of decompositions. First the dynamics are decomposed into three subsystems; namely the ξ-, μ-, and ζ-subsystems. We call the ξ-subsystem the transversal dynamics to S_1. This is motivated by the fact that, in the light of (3.4a), stabilizing $S_1 \cap U$ is equivalent, under mild assumptions, to stabilizing the origin of the ξ-subsystem. We call the μ-subsystem the transversal dynamics of S_2, restricted to S_1. Similarly, this name is motivated by the fact that, in the light of (3.4b), stabilizing $S_2 \cap U$ relative to $S_1 \cap U$ is equivalent, under mild assumptions, to stabilizing the μ-subsystem when $\xi = 0$. The ζ-subsystem is called the tangential dynamics to S_2 because when $\xi = 0$ and $\mu = 0$, the ζ dynamics govern the system's evolution on $\Xi(S_2 \cap U)$.

The second type of decomposition is in the original m inputs. They are partitioned into three groups : $v^{\parallel}$, $v^{\parallel,\pitchfork}$, and $v^{\pitchfork}$. The restrictions imposed on f_2, f_3, g_{31}, g_{32}, g_{21}, and

g_{22} after (3.4) imply that $\left(v^{\parallel}, v^{\parallel,\pitchfork}, v^{\pitchfork}\right) = (\star, \star, 0)$, where $\star$ represents smooth functions for which the closed-loop system has solutions, renders $\Xi(S_1 \cap U)$ locally invariant, i.e., $(\star, \star, 0)$ is a friend of $\Xi(S_1 \cap U)$. Substituting $\xi = 0$ and $v^{\pitchfork} = 0$ in (3.3) the dynamics of (3.1) restricted to $\Xi(S_1 \cap U)$ are

$$\begin{aligned} \dot{\zeta} &= f_1(\zeta, \mu, 0) + g_{11}(\zeta, \mu, 0)v^{\parallel} + g_{12}(\zeta, \mu, 0)v^{\parallel,\pitchfork} \\ \dot{\mu} &= A\mu + Bv^{\parallel,\pitchfork}. \end{aligned} \tag{3.5}$$

The μ-subsystem in (3.5) is linear and controllable and represents the dynamics of (3.1) restricted to $\Xi(S_1 \cap U)$ and transversal to $\Xi(S_2 \cap U)$. The control input $v^{\parallel,\pitchfork}$ can effectively be used to stabilize $S_2 \cap U$ relative to $S_1 \cap U$. Finally $\Xi(S_2 \cap U)$ is controlled-invariant with friend $\left(v^{\parallel}, v^{\parallel,\pitchfork}, v^{\pitchfork}\right) = (\star, 0, 0)$. The dynamics of (3.1) restricted to $\Xi(S_2 \cap U)$ are

$$\dot{\zeta} = f_1(\zeta, 0, 0) + g_{11}(\zeta, 0, 0)v^{\parallel}. \tag{3.6}$$

Remark 3.1.2. In (3.3) the μ-subsystem is not feedback linearized. It only becomes linear when it evolves on $\Xi(S_1 \cap U)$. Thus (3.3) is less restrictive compared to a normal form in which the μ-subsystem is linear off the set $\Xi(S_1 \cap U)$.

The normal form (3.3) finds application in the stabilization of $S_2 \cap U$ relative to $S_1 \cap U$. The main result of this chapter, Theorem 3.4.2, provides necessary and sufficient conditions for Problem 1 to be solvable.

3.2 Linear time invariant systems

To facilitate understanding of the results of this chapter, we begin with LTI control systems. Consider

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{3.7}$$

where $x \in \mathscr{X}$, with $\mathscr{X}$, an n-dimensional vector space and $u \in \mathscr{U}$, with $\mathscr{U}$ an m-dimensional vector space. The maps $A : \mathscr{X} \to \mathscr{X}$ and $B : \mathscr{U} \to \mathscr{X}$ are linear. System (3.7) is a special case of (3.1) in which $f(x) = Ax$ and $g(x) = B$. As such, the distributions introduced in (3.2) for LTI system (3.7) take the following familiar form

$$G_i = \operatorname{Im} \begin{bmatrix} B & \cdots & A^i B \end{bmatrix}.$$

Similarly, throughout this section it is assumed that B is full rank. In the linear setting $\mathscr{S}_1 \subset \mathscr{X}$ and $\mathscr{S}_2 \subset \mathscr{X}$ are (A, B)-invariant[1] subspaces with dimensions s_1 and s_2, respectively, and $\mathscr{S}_1 \supset \mathscr{S}_2$.

Problem 2. (Nested cascade connected LTI control system): Given (3.7), nested (A, B)-invariant subspaces $\mathscr{S}_1 \supset \mathscr{S}_2$, find, if possible, an isomorphism $\Xi : \mathscr{X} \to \mathscr{X}^{\parallel} \times \mathscr{X}^{\parallel,\pitchfork} \times \mathscr{X}^{\pitchfork}, x \mapsto (\zeta, \mu, \xi)$, with $\dim \mathscr{X}^{\parallel} = s_2$, $\dim \mathscr{X}^{\parallel,\pitchfork} = s_1 - s_2$, and $\dim \mathscr{X}^{\pitchfork} = n - s_1$, and a linear feedback transformation $u = Fx + Hv$ such that (3.7) is feedback equivalent to

$$\begin{bmatrix} \dot{\zeta} \\ \dot{\mu} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \begin{bmatrix} \zeta \\ \mu \\ \xi \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ 0 & B_{22} & B_{23} \\ 0 & 0 & B_{33} \end{bmatrix} \begin{bmatrix} v^{\parallel} \\ v^{\parallel,\pitchfork} \\ v^{\pitchfork} \end{bmatrix} \tag{3.8}$$

[1] Refer to Definition D.1 for the definition of an (A, B)-invariant subspace.

where

$$\Xi(\mathscr{S}_1) = \{(\zeta, \mu, \xi) : \xi = 0\}\,, \tag{3.9a}$$

$$\Xi(\mathscr{S}_2) = \{(\zeta, \mu, \xi) : \xi = 0,\ \mu = 0\}\,, \tag{3.9b}$$

the pairs (A_{22}, B_{22}) and (A_{33}, B_{33}) are controllable, and B_{22} and B_{33} are full rank. ▲

The control input $\left(v^{\parallel}, v^{\parallel,\pitchfork}, v^{\pitchfork}\right) = (\star, \star, 0)$, where $\star$ represents smooth functions for which the closed-loop system has solutions, is a friend of $\Xi(\mathscr{S}_1)$. Thus, substituting $\xi = 0$ and $v^{\pitchfork} = 0$ in (3.3) the dynamics of (3.7) restricted to $\Xi(\mathscr{S}_1)$ are

$$\begin{bmatrix} \dot{\zeta} \\ \dot{\mu} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} \zeta \\ \mu \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix} \begin{bmatrix} v^{\parallel} \\ v^{\parallel,\pitchfork} \end{bmatrix} \tag{3.10}$$

The control input $\left(v^{\parallel}, v^{\parallel,\pitchfork}, v^{\pitchfork}\right) = (\star, 0, 0)$ is a friend of $\Xi(\mathscr{S}_2)$. Thus, the dynamics of (3.7) restricted to $\Xi(\mathscr{S}_2)$ are

$$\dot{\zeta} = A_{11}\zeta + B_{11}v^{\parallel}. \tag{3.11}$$

Provided $\mathscr{S}_1$ and $\mathscr{S}_2$ are (A, B)-invariant it is always possible to find a coordinate and feedback transformation bringing the system (3.7) to the normal form (3.8). However, there is no guarantee that the pairs (A_{22}, B_{22}) and (A_{33}, B_{33}) are controllable. Thus, we aim to determine checkable necessary and sufficient geometric conditions under which the controllability requirements in Problem 2 are satisfied.

3.2.1 Linear coordinate and feedback transformation

In this section we review how to construct an isomorphism and a linear feedback transformation bringing (3.7) to the normal form (3.8). We emphasize that in order to put (3.7) in the form (3.8), it is necessary and sufficient that $\mathscr{S}_1 \supset \mathscr{S}_2$ be (A, B)-invariant. This does not, however, guarantee that the matrix pairs outlined in Problem 2 are controllable.

Fix the basis on $\mathscr{X}$ as, without loss of generality, the natural basis. Let $\mathscr{T} \subseteq \mathscr{X}$ be any subspace such that $\mathscr{S}_1 \oplus \mathscr{T} = \mathscr{X}$. Let $\mathscr{S} \subset \mathscr{S}_1$ be any subspace such that $\mathscr{S}_2 \oplus \mathscr{S} = \mathscr{S}_1$. Take as an alternate basis for $\mathscr{X}$ the union of a basis $\{\zeta_1, \cdots, \zeta_{s_2}\}$ for $\mathscr{S}_2$, a basis $\{\mu_{s_2+1}, \cdots, \mu_{s_1}\}$ for $\mathscr{S}$, and a basis $\{\xi_{s_1+1}, \cdots, \xi_n\}$ for $\mathscr{T}$. Let $\Xi : \mathscr{X} \to \mathscr{X}$ be the isomorphism representing the change of basis form the natural basis to the one described above.

By [92, Lemma 5.7] there exists a mutual friend F of $\mathscr{S}_1$ and $\mathscr{S}_2$. Set $u = Fx + v$ where v is an external input. In the new basis and with this choice for control, system (3.7) takes the form

$$\begin{bmatrix} \dot{\zeta} \\ \dot{\mu} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \begin{bmatrix} \zeta \\ \mu \\ \xi \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} v \tag{3.12}$$

where $A_{11} \in \mathbb{R}^{s_2 \times s_2}$, $A_{12} \in \mathbb{R}^{s_2 \times s_1 - s_2}$, $A_{13} \in \mathbb{R}^{s_2 \times s_1 - s_2}$, $A_{22} \in \mathbb{R}^{s_1 - s_2 \times s_1 - s_2}$, $A_{23} \in \mathbb{R}^{s_1 - s_2 \times n - s_1}$, $A_{33} \in \mathbb{R}^{n - s_1 \times n - s_1}$, $B_1 \in \mathbb{R}^{s_2 \times m}$, $B_2 \in \mathbb{R}^{s_1 - s_2 \times m}$, and $B_3 \in \mathbb{R}^{n - s_1 \times m}$.

In order to impose the desired decomposition of control inputs in Problem 2 we define the following subspaces

$$\begin{aligned} \mathscr{B}^{\parallel} &:= \mathscr{B} \cap \mathscr{S}_2 \\ \mathscr{B}^{\parallel, \pitchfork} &:= [\mathscr{B} \cap \mathscr{S}_2]^{\perp} \cap [\mathscr{B} \cap \mathscr{S}_1] \\ \mathscr{B}^{\pitchfork} &:= [\mathscr{B} \cap \mathscr{S}_1]^{\perp} \cap \mathscr{B}, \end{aligned} \tag{3.13}$$

and integers

$$
\begin{aligned}
\nu &\coloneqq \dim(\mathscr{S}_2 \cap \mathscr{B}) \\
\rho &\coloneqq \dim\left((\mathscr{S}_1 \cap \mathscr{B})/(\mathscr{S}_2 \cap \mathscr{B})\right) \\
\sigma &\coloneqq \dim\left((\mathscr{S}_1 + \mathscr{B})/\mathscr{S}_1\right),
\end{aligned} \tag{3.14}
$$

where $\mathscr{B} = \operatorname{Im} B$. It is immediate that $\dim \mathscr{B}^{\|} = \nu$. We calculate

$$
\begin{aligned}
\dim \mathscr{B}^{\|,\pitchfork} &= n - \dim\left(\mathscr{B} \cap \mathscr{S}_2\right) + \dim\left(\mathscr{B} \cap \mathscr{S}_1\right) - \dim\left([\mathscr{B} \cap \mathscr{S}_2]^{\perp} + [\mathscr{B} + \mathscr{S}_1]\right) \\
&= \dim\left(\mathscr{B} \cap \mathscr{S}_1\right) - \dim\left(\mathscr{B} \cap \mathscr{S}_2\right) = \rho
\end{aligned}
$$

and

$$
\begin{aligned}
\dim \mathscr{B}^{\pitchfork} &= n - \dim\left(\mathscr{B} \cap \mathscr{S}_1\right) + \dim\left(\mathscr{B}\right) - \dim\left([\mathscr{B} \cap \mathscr{S}_1]^{\perp} + \mathscr{B}\right) \\
&= \dim\left(\mathscr{B}\right) - \dim\left(\mathscr{B} \cap \mathscr{S}_1\right) = \sigma.
\end{aligned}
$$

The sum of the integers in (3.14) is m since

$$
\begin{aligned}
\nu + \rho + \sigma &= \dim(\mathscr{S}_2 \cap \mathscr{B}) + \dim(\mathscr{S}_1 \cap \mathscr{B}) - \dim(\mathscr{S}_2 \cap \mathscr{B}) + \dim(\mathscr{S}_1 + \mathscr{B}) - \dim \mathscr{S}_1 \\
&= \dim(\mathscr{S}_1 \cap \mathscr{B}) + \dim(\mathscr{S}_1 + \mathscr{B}) - \dim \mathscr{S}_1 \\
&= \dim \mathscr{S}_1 + \dim \mathscr{B} - \dim(\mathscr{S}_1 + \mathscr{B}) + \dim(\mathscr{S}_1 + \mathscr{B}) - \dim \mathscr{S}_1 = \dim \mathscr{B} = m.
\end{aligned}
$$

It can be readily verified that the subspaces in (3.13) are independent, thus $\mathscr{B} = \mathscr{B}^{\|} \oplus \mathscr{B}^{\|,\pitchfork} \oplus \mathscr{B}^{\pitchfork}$. We have that $\mathscr{B}^{\|}$ is a subspace of $\mathscr{S}_2$, $\mathscr{B}^{\|,\pitchfork}$ is a subspace of $\mathscr{S}_1$ but not $\mathscr{S}_2$, and $\mathscr{B}^{\pitchfork}$ is not a subspace of $\mathscr{S}_1$.

Let $H = (H_1, H_2, H_3) : \mathscr{U} \to \mathscr{U}$ be defined as a map such that $\operatorname{Im}(BH_1) = \mathscr{B}^{\|}$, $\operatorname{Im}(BH_2) = \mathscr{B}^{\|,\pitchfork}$, and $\operatorname{Im}(BH_3) = \mathscr{B}^{\pitchfork}$. Thus, $u = Fx + Hv$ is the desired linear feedback transformation yielding the normal form (3.8).

3.2.2 Solution to Problem 2

Now, we turn our attention to finding geometric conditions under which the pairs (A_{22}, B_{22}) and (A_{33}, B_{33}) are controllable. Since $\mathscr{S}_1$ is $(A+BF)$-invariant one can extract the dynamics transversal to $\mathscr{S}_1$ and tangential to $\mathscr{S}_1$. Let $P_1 : \mathscr{X} \to \mathscr{X}/\mathscr{S}_1$ be the canonical projection introduced in Definition B.3. The map $\bar{A} : \mathscr{X}/\mathscr{S}_1 \to \mathscr{X}/\mathscr{S}_1$ is the unique solution of $P_1(A+BF) = \bar{A}P_1$. This map is called the induced map by $(A+BF)$ and is well-defined. Letting $\bar{B} = P_1 B$ the following diagram commutes

$$\begin{tikzcd}
 & \mathscr{X} \arrow[r, "(A+BF)"] \arrow[dd, "P_1"] & \mathscr{X} \arrow[dd, "P_1"] \\
\mathscr{U} \arrow[ur, "B"] \arrow[dr, "\bar{B}"'] & & \\
 & \mathscr{X}/\mathscr{S}_1 \arrow[r, "\bar{A}"] & \mathscr{X}/\mathscr{S}_1.
\end{tikzcd}$$

The pair $(\bar{A}, \bar{B})$, called the quotient system, isolates the dynamics transversal to $\mathscr{S}_1$. In the coordinates of (3.8) their matrix representation is (A_{33}, B_{33}). Thus, the eigenvalues of the pair (A_{33}, B_{33}) is controllable if and only if $(\bar{A}, \bar{B})$ is controllable, i.e.,

$$\mathscr{X}/\mathscr{S}_1 = \bar{\mathscr{R}}_{n-s_1-1} \tag{3.15}$$

where $\bar{\mathscr{R}}_{n-s_1-1} := \bar{\mathscr{B}} + \bar{A}\bar{\mathscr{B}} + \cdots + \bar{A}^{n-s_1-1}\bar{\mathscr{B}}$ and $\bar{\mathscr{B}} = \operatorname{Im} \bar{B}$. Condition 3.15 is necessary and sufficient for the pair (A_{33}, B_{33}) to be controllable as required in Problem 2.

Let $Q : \mathscr{X} \to \mathscr{S}_1$ be the natural projection on $\mathscr{S}_1$ along $\mathscr{T}$ with $\mathscr{T}$ being such that $\mathscr{T} \oplus \mathscr{S}_1 = \mathscr{X}$. The map $A_{\mathscr{S}_1} : \mathscr{S}_1 \to \mathscr{S}_1$ is the unique solution of $Q(A+BF) = A_{\mathscr{S}_1}Q$ and has the action of $(A+BF)$ on $\mathscr{S}_1$ with codomain $\mathscr{S}_1$. Since $\mathscr{S}_2$ is invariant for $(A+BF)$ it is also invariant for its restriction $A_{\mathscr{S}_1}$. Define $B_{\mathscr{S}_1} := QB$. With this construction, the following diagram commutes

$$
\begin{tikzcd}
 & \mathscr{X} \arrow[r, "(A+BF)"] \arrow[d, "Q"] & \mathscr{X} \arrow[d, "Q"] \\
\mathscr{U} \arrow[ur, "B"] \arrow[dr, "B_{\mathscr{S}_1}"'] & & \\
 & \mathscr{S}_1 \arrow[r, "A_{\mathscr{S}_1}"] & \mathscr{S}_1.
\end{tikzcd}
$$

The pair $(A_{\mathscr{S}_1}, B_{\mathscr{S}_1})$ describes the dynamics of (3.7) restricted to $\mathscr{S}_1$. In the coordinates of (3.8) their matrix representation is

$$
\left(\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix} \right).
$$

Let $P_2 : \mathscr{S}_1 \to \mathscr{S}_1/\mathscr{S}_2$ be the canonical projection. The map $\bar{A}_{\mathscr{S}_1} : \mathscr{S}_1/\mathscr{S}_2 \to \mathscr{S}_1/\mathscr{S}_2$ which is the unique solution of $P_2 A_{\mathscr{S}_1} = \bar{A}_{\mathscr{S}_1} P_2$ is the induced map by $A_{\mathscr{S}_1}$ and is well-defined. Letting $\bar{B}_{\mathscr{S}_1} \coloneqq P_2 B_{\mathscr{S}_1}$ we have that the following commutative diagram

$$
\begin{tikzcd}
 & \mathscr{S}_1 \arrow[r, "A_{\mathscr{S}_1}"] \arrow[d, "P_1"] & \mathscr{S}_1 \arrow[d, "P_1"] \\
\mathscr{U} \arrow[ur, "B_{\mathscr{S}_1}"] \arrow[dr, "\bar{B}_{\mathscr{S}_1}"'] & & \\
 & \mathscr{S}_1/\mathscr{S}_2 \arrow[r, "\bar{A}_{\mathscr{S}_1}"] & \mathscr{S}_1/\mathscr{S}_2.
\end{tikzcd}
$$

The matrix representation of the pair $(\bar{A}_{\mathscr{S}_1}, \bar{B}_{\mathscr{S}_1})$ in the coordinates of (3.8) is (A_{22}, B_{22}). Thus, the pair (A_{22}, B_{22}) is controllable if and only if the pair $(\bar{A}_{\mathscr{S}_1}, \bar{B}_{\mathscr{S}_1})$ is controllable, i.e.

$$
\mathscr{S}_1/\mathscr{S}_2 = \bar{\mathscr{R}}_{s_1 - s_2 - 1}, \tag{3.16}
$$

where $\bar{\mathscr{R}}_{s_1-s_2-1} = \bar{\mathscr{B}}_{\mathscr{S}_1} + \bar{A}_{\mathscr{S}_1}\bar{\mathscr{B}}_{\mathscr{S}_1} + \cdots + \bar{A}_{\mathscr{S}_1}^{s_1-s_2-1}\bar{\mathscr{B}}_{\mathscr{S}_1}$ and $\bar{\mathscr{B}}_{\mathscr{S}_1} = \operatorname{Im}\bar{B}_{\mathscr{S}_1}$.

Theorem 3.2.1. *Let $F : \Xi \to \mathscr{U}$ be a mutual friend of $\mathscr{S}_1$ and $\mathscr{S}_2$, $\mathscr{B} = \operatorname{Im} B$, $\mathscr{B}_{\mathscr{S}_1} = \operatorname{Im} B_{\mathscr{S}_1}$, $\mathscr{R}_{n-s_1-1} = \mathscr{B} + (A+BF)\mathscr{B} + \cdots + (A+BF)^{n-s_1-1}\mathscr{B}$, and $\mathscr{R}_{s_1-s_2-1} = \mathscr{B}_{\mathscr{S}_1} + A_{\mathscr{S}_1}\mathscr{B}_{\mathscr{S}_1} + \cdots + A_{\mathscr{S}_1}^{s_1-s_2-1}\mathscr{B}_{\mathscr{S}_1}$. Problem 2 is solvable if and only if*

(a) $\mathscr{X} = \mathscr{S}_1 + \mathscr{R}_{n-s_1-1}$

(b) $\mathscr{S}_1 = \mathscr{S}_2 + \mathscr{R}_{s_1-s_2-1}$.

Proof. Assume that Problem 2 is solvable. As shown in the preceding discussion, this means that conditions (3.15) and (3.16) hold. Consider the identity

$$\dim\left(P_1(\mathscr{S}_1 + \mathscr{R}_{n-s_1-1})\right) = \dim\left(\mathscr{S}_1 + \mathscr{R}_{n-s_1-1}\right) - \dim\left((\mathscr{S}_1 + \mathscr{R}_{n-s_1-1}) \cap \operatorname{Ker}(P_1)\right).$$

We have that

$$\dim\left(P_1\left(\mathscr{S}_1 + \mathscr{R}_{n-s_1-1}\right)\right) = \dim\left(P_1\mathscr{R}_{n-s_1-1}\right) = \dim\left(\bar{\mathscr{R}}_{n-s_1-1}\right).$$

But, by (3.15) $\dim\left(\bar{\mathscr{R}}_{n-s_1-1}\right) = n - s_1$. Moreover, since $\operatorname{Ker}(P_1) = \mathscr{S}_1$

$$\dim\left((\mathscr{S}_1 + \mathscr{R}_{n-s_1-1}) \cap \operatorname{Ker}(P_1)\right) = \dim\left(\operatorname{Ker}(P_1)\right) = s_1.$$

Thus, we conclude that $\dim(\mathscr{S}_1 + \mathscr{R}_{n-s_1-1}) = \dim \mathscr{X} = n$ and condition (a) holds. Assuming condition (3.16) one can prove that condition (b) holds in the same manner.

Conversely, suppose condition (a) holds. Then

$$\mathscr{X}/\mathscr{S}_1 = P_1\mathscr{X} = P_1\mathscr{S}_1 + P_1\mathscr{R}_{n-s_1-1} = P_1\mathscr{B} + P_1(A+BF)\mathscr{B} + \cdots + P_1(A+BF)^{n-s_1-1}\mathscr{B}$$

Since, $P_1(A+BF) = \bar{A}P_1$ and $\bar{B} = P_1 B$

$$\begin{aligned}\mathscr{X}/\mathscr{S}_1 &= \bar{\mathscr{B}} + \bar{A}P_1\mathscr{B} + \cdots + \bar{A}P_1(A+BF)^{n-s_1-2}\mathscr{B} = \bar{\mathscr{B}} + \bar{A}\bar{\mathscr{B}} + \cdots + \left(\bar{A}\right)^{n-s_1-1}\bar{\mathscr{B}} \\ &= \bar{\mathscr{R}}_{n-s_1-1}.\end{aligned}$$

Thus (3.15) holds. Assuming condition (b) holds one can show that condition (3.16) holds in the same manner. Thus, Problem 2 is solvable. $\square$

Conditions (a) and (b) of Theorem 3.2.1 are easier to verify in comparison with conditions (3.15) and (3.16) because we only need to find the dynamics restricted to $\mathscr{S}_1$ instead of finding the quotient systems $(\bar{A}, \bar{B})$ and $(\bar{A}_{\mathscr{S}_1}, \bar{B}_{\mathscr{S}_1})$. The following theorem shows that one can replace condition (b) of Theorem 3.2.1 with a condition that can be checked using the original pair (A, B) and avoid computing restricted dynamics.

Theorem 3.2.2. *Let $F : \Xi \to \mathscr{U}$ be a mutual friend of $\mathscr{S}_1$ and $\mathscr{S}_2$, $\mathscr{B} = \operatorname{Im} B$, $\mathscr{B}_{\mathscr{S}_1} = \operatorname{Im} B_{\mathscr{S}_1}$, and $\mathscr{R}_{s_1-s_2-1} = \mathscr{B}_{\mathscr{S}_1} + A_{\mathscr{S}_1}\mathscr{B}_{\mathscr{S}_1} + \cdots + A_{\mathscr{S}_1}^{s_1-s_2-1}\mathscr{B}_{\mathscr{S}_1}$. Let $\mathscr{R}^\star := \sup \mathfrak{C}(\mathscr{S}_1)$ be the largest controllability subspace[2] contained in $\mathscr{S}_1$. Problem 2 is solvable if and only if*

(a) $\mathscr{X} = \mathscr{S}_1 + \mathscr{R}_{n-s_1-1}$

(b) $\mathscr{S}_1 = \mathscr{R}^\star + \mathscr{S}_2$.

Proof. We must show condition (b) of this theorem is equivalent to condition (b) of Theorem 3.2.1. Assume that condition (b) of this theorem holds. Then

$$Q\mathscr{S}_1 = Q\left(\mathscr{R}^\star + \mathscr{S}_2\right) = Q\mathscr{R}^\star + Q\mathscr{S}_2$$

[2]See Appendix D and Definition D.6 for the definition of a controllability subspace

By [92, Theorem 5.5], $\mathscr{R}^\star = (\mathscr{B} \cap \mathscr{S}_1) + (A+BF)(\mathscr{B} \cap \mathscr{S}_1) + \cdots + (A+BF)^{n-1}(\mathscr{B} \cap \mathscr{S}_1)$, thus

$$Q\mathscr{S}_1 = Q(\mathscr{B} \cap \mathscr{S}_1) + Q(A+BF)(\mathscr{B} \cap \mathscr{S}_1) + \cdots + Q(A+BF)^{n-1}(\mathscr{B} \cap \mathscr{S}_1) + \mathscr{S}_2$$

where $\mathscr{S}_2$ is a subset of $\mathscr{S}_1$ and not $\mathscr{X}$. Since $Q(A+BF) = A_{\mathscr{S}_1}Q$ and $Q(\mathscr{B} \cap \mathscr{S}_1) = \mathscr{B}_{\mathscr{S}_1}$

$$Q\mathscr{S}_1 = \mathscr{B}_{\mathscr{S}_1} + A_{\mathscr{S}_1}\mathscr{B}_{\mathscr{S}_1} + \cdots + (A_{\mathscr{S}_1})^{n-1}\mathscr{B}_{\mathscr{S}_1} + \mathscr{S}_2.$$

By Cayley-Hamilton, for $i > s_1 - s_2 - 1$, we have that $(A_{\mathscr{S}_1})^i \mathscr{B}_{\mathscr{S}_1} \subset \mathscr{R}_{s_1-s_2-1}$. Thus

$$\mathscr{B}_{\mathscr{S}_1} + A_{\mathscr{S}_1}\mathscr{B}_{\mathscr{S}_1} + \cdots + (A_{\mathscr{S}_1})^{n-1}\mathscr{B}_{\mathscr{S}_1} = \mathscr{R}_{s_1-s_2-1},$$

and

$$Q\mathscr{S}_1 = \mathscr{R}_{s_1-s_2-1} + \mathscr{S}_2.$$

We have that $Q\mathscr{S}_1 = \mathscr{S}_1$. Thus the condition (b) of Theorem 3.2.1 hold.

Conversely, suppose that the condition (b) of Theorem 3.2.1 hold. Let $S : \mathscr{S}_1 \to \mathscr{X}$ be the insertion map. Thus, $S\mathscr{S}_1 = S\mathscr{S}_2 + S\mathscr{R}_{s_1-s_2-1}$ and

$$S\mathscr{R}_{s_1-s_2-1} = S\mathscr{B}_{\mathscr{S}_1} + SA_{\mathscr{S}_1}\mathscr{B}_{\mathscr{S}_1} + \cdots + SA_{\mathscr{S}_1}^{s_1-s_2-1}\mathscr{B}_{\mathscr{S}_1}$$

Since, $(A+BF)S = A_{\mathscr{S}_1}S$ and $S\mathscr{B}_{\mathscr{S}_1} = (\mathscr{B} \cap \mathscr{S}_1)$

$$\begin{aligned} S\mathscr{R}_{s_1-s_2-1} &= (\mathscr{B} \cap \mathscr{S}_1) + (A+BF)S\mathscr{B}_{\mathscr{S}_1} + \cdots + (A+BF)SA_{\mathscr{S}_1}^{s_1-s_2-2}\mathscr{B}_{\mathscr{S}_1} \\ &= (\mathscr{B} \cap \mathscr{S}_1) + (A+BF)(\mathscr{B} \cap \mathscr{S}_1) + \cdots + (A+BF)^{s_1-s_2-1}(\mathscr{B} \cap \mathscr{S}_1). \end{aligned}$$

Thus, $\dim S\mathscr{R}_{s_1-s_2-1} \leq \dim \mathscr{R}^\star \leq s_1$. Thus condition (b) of this theorem holds. $\square$

Lemma 3.2.3. *Consider the LTI control system (3.7) with given subspaces $\mathscr{S}_1 \supset \mathscr{S}_2$. Suppose that Problem 2 is solvable. Let $v^{\pitchfork} = K_1\xi$ and $v^{\parallel,\pitchfork} = K_2\mu$ such that $A_{33} + B_{33}K_1$ and $A_{22} + B_{22}K_2$ are Hurwitz. Then $(\xi, \mu) = (0, 0)$ is globally exponentially stable for the closed-loop system.*

Proof. After applying $v^{\pitchfork} = K_1\xi$ and $v^{\parallel,\pitchfork} = K_2\mu$ to (3.8) we obtain

$$\begin{bmatrix} \dot{\zeta} \\ \dot{\mu} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} + B_{12}K_2 & A_{13} + B_{13}K_1 \\ 0 & A_{22} + B_{22}K_2 & A_{23} + B_{23}K_1 \\ 0 & 0 & A_{33} + B_{33}K_1 \end{bmatrix} \begin{bmatrix} \zeta \\ \mu \\ \xi \end{bmatrix} + \begin{bmatrix} B_{11} \\ 0 \\ 0 \end{bmatrix} v^{\parallel}$$

The eigenvalues of the closed-loop system is the union of the eigenvalues of A_{11}, $A_{22}+B_{22}K_2$, and $A_{33} + B_{33}K_1$ and all the eigenvalues of matrices $A_{22} + B_{22}K_2$, and $A_{33} + B_{33}K_1$ belong to $\mathbb{C}^-$. Therefore, $(\xi, \mu) = (0, 0)$ is globally exponentially stable. $\square$

Example 3.2.4. Consider the LTI system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 2 & 0 \\ -2 & 1 & 0 & -2 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & -1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 4 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} u$$

and the nested (A,B)-invariant subspaces

$$\mathscr{S}_1 = \operatorname{span}\left\{\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\1\end{bmatrix}\right\}, \qquad \mathscr{S}_2 = \operatorname{span}\left\{\begin{bmatrix}0\\1\\1\\1\end{bmatrix}\begin{bmatrix}0\\-1\\1\\1\end{bmatrix}\right\},$$

with $s_1 = \dim \mathscr{S}_1 = 3$ and $s_2 = \dim \mathscr{S}_2 = 2$. In order to determine the solvability of Problem 2 we check conditions of Theorem 3.2.2. Condition (a) holds since

$$\mathscr{X} = \mathscr{S}_1 + \mathscr{R}_0 = \mathscr{S}_1 + \mathscr{B} = \operatorname{rank}\begin{bmatrix}1 & 0 & 1 & 1 & 4 & 1\\ 1 & 1 & 0 & 1 & 0 & 1\\ 0 & 1 & 1 & 0 & 2 & 1\\ 0 & 1 & 1 & 1 & 1 & 1\end{bmatrix} = 4.$$

Condition (b) holds since

$$\mathscr{S}_1 = \mathscr{S}_2 + \mathscr{R}^\star = \operatorname{rank}\begin{bmatrix}0 & 0 & 1 & 7 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & -1 & 5 & 3 & -38 & -42 & 244 & 236 & -1616 & -1584\\ 1 & 1 & 4 & 4 & -23 & -17 & 160 & 160 & -1052 & -1028\\ 1 & 1 & 4 & 4 & -23 & -17 & 160 & 160 & -1052 & -1028\end{bmatrix} = 3.$$

To find the normal form (3.8) we find the coordinate transformation Ξ and feedback transformation $u = Fx + Hv$. The coordinate transformation is found by finding a new basis for $\mathscr{X}$

$$\mathscr{X} = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \right\}$$

Then, the coordinate transformation is the isomorphism Ξ transferring to the new basis

$$\Xi = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

In order to find the feedback transformation $u = Fx + Hv$ we first find a mutual friend of $\mathscr{S}_1$ and $\mathscr{S}_2$. Following Lemma D.5 we find the following mutual friend

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & -6 & -3 & -1 \end{bmatrix}.$$

In order to find the matrix H we first calculate the integers in (3.14)

$$\nu = \operatorname{rank}\begin{bmatrix} 0 \\ 4 \\ 3 \\ 3 \end{bmatrix} = 1, \quad \rho = \operatorname{rank}\begin{bmatrix} 5 & 1 \\ 1 & 1 \\ 2 & 1 \\ 2 & 1 \end{bmatrix} - \nu = 1, \quad \sigma = \operatorname{rank}\begin{bmatrix} 1 & 0 & 1 & 1 & 4 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} - s_1 = 1.$$

Simple matrix calculations yield the subspaces defined in (3.13)

$$\mathscr{B}^{\parallel} = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 4 \\ 3 \\ 3 \end{bmatrix}\right\}, \quad \mathscr{B}^{\parallel,\pitchfork} = \operatorname{span}\left\{\begin{bmatrix} 68 \\ -12 \\ 8 \\ 8 \end{bmatrix}\right\}, \quad \mathscr{B}^{\pitchfork} = \operatorname{span}\left\{\begin{bmatrix} -2 \\ -6 \\ 13 \\ -5 \end{bmatrix}\right\}.$$

We calculate $H : \mathscr{U} \to \mathscr{U}$ to be

$$H = \begin{bmatrix} -1 & 20 & -17 \\ -1 & 20 & 1 \\ 5 & -32 & 11 \end{bmatrix}.$$

After applying the coordinate and feedback transformations the LTI system reads

$$\begin{bmatrix} \dot{\zeta}_1 \\ \dot{\zeta}_2 \\ \dot{\mu} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} -\frac{13}{2} & \frac{1}{2} & \frac{5}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \mu \\ \xi \end{bmatrix} + \begin{bmatrix} \frac{7}{2} & 32 & \frac{5}{2} \\ -\frac{1}{2} & -24 & \frac{21}{2} \\ 0 & 68 & -2 \\ 0 & 0 & -18 \end{bmatrix} \begin{bmatrix} v^{\parallel} \\ v^{\parallel,\pitchfork} \\ v^{\pitchfork}\dot{\xi} \end{bmatrix}$$

△

3.3 Preliminary results

We now return to the nonlinear setting. The results in this section lay the foundation for our solution to Problem 1. We investigate the problem of restricting the control-affine system (3.1) to nested sets $S_1 \supset S_2$ satisfying Assumption 3.1.1.

Problem 3. (Restricting control-affine systems to nested sets): Given the control system (3.1), nested sets $S_1 \supset S_2$ satisfying Assumption 3.1.1, and a point $\bar{x} \in S_2$, find, if possible, a diffeomorphism $\Xi : U \to \Xi(U) \subseteq \mathbb{R}^{s_2} \times \mathbb{R}^{s_1 - s_2} \times \mathbb{R}^{n-s_1}$, $x \mapsto (\zeta, \mu, \xi)$, and a regular feedback transformation (α, β) valid in a neighbourhood $U \subseteq \mathbb{R}^n$ of $\bar{x}$, such that (3.1) is feedback equivalent to

$$\begin{aligned}
\dot{\zeta} &= f_1(\zeta,\mu,\xi) + g_{11}(\zeta,\mu,\xi)v^{\parallel} + g_{12}(\zeta,\mu,\xi)v^{\parallel,\pitchfork} + g_{13}(\zeta,\mu,\xi)v^{\pitchfork} \\
\dot{\mu} &= f_2(\zeta,\mu,\xi) + g_{21}(\zeta,\mu,\xi)v^{\parallel} + g_{22}(\zeta,\mu,\xi)v^{\parallel,\pitchfork} + g_{23}(\zeta,\mu,\xi)v^{\pitchfork} \\
\dot{\xi} &= f_3(\zeta,\mu,\xi) + g_{31}(\zeta,\mu,\xi)v^{\parallel} + g_{32}(\zeta,\mu,\xi)v^{\parallel,\pitchfork} + g_{33}(\zeta,\mu,\xi)v^{\pitchfork}
\end{aligned} \tag{3.17}$$

where

$$\Xi(S_1 \cap U) = \{(\zeta,\mu,\xi) \in \Xi(U) : \xi = 0\}, \tag{3.18a}$$

$$\Xi(S_2 \cap U) = \{(\zeta,\mu,\xi) \in \Xi(U) : \xi = 0,\ \mu = 0\}, \tag{3.18b}$$

$f_3(\zeta,\mu,0) = 0$, $g_{31}(\zeta,\mu,0) = 0$, $g_{32}(\zeta,\mu,0) = 0$, $f_2(\zeta,0,0) = 0$, and $g_{21}(\zeta,0,0) = 0$. ▲

The normal form (3.17) features two types of decomposition similar to those in (3.3). However, unlike (3.3), we do not require the transversal dynamics to S_2, restricted to S_1 be linear and controllable. The normal form (3.17) is useful for understanding the interplay between the control vector fields $g_1, \cdots, g_m$ of (3.1) and the nested sets $S_1 \supset S_2$. That is, g is partitioned into three sub-matrices corresponding to $v^{\parallel}$, $v^{\parallel,\pitchfork}$, and $v^{\pitchfork}$. The impositions on g_{21} and g_{31} mean that the columns of the matrix $\Xi_\star(g\beta)$ corresponding to $v^{\parallel}$ are tangent to both $\Xi(S_1 \cap U)$ and $\Xi(S_2 \cap U)$. The requirement on g_{32} implies that the columns corresponding to $v^{\parallel,\pitchfork}$ are tangent to $\Xi(S_1 \cap U)$ but not $\Xi(S_2 \cap U)$. Finally, the requirements on f_2, f_3, imply that the vector field $\Xi_\star(f + g\alpha) = (f_1, f_2, f_3)$ is tangent to both $\Xi(S_1 \cap U)$ and $\Xi(S_2 \cap U)$. As with the normal form (3.3), substituting $\xi = 0$ and $v^{\pitchfork} = 0$, the dynamics of (3.1) restricted to $\Xi(S_1 \cap U)$ are

$$\begin{aligned}\dot{\zeta} &= f_1(\zeta, \mu, 0) + g_{11}(\zeta, \mu, 0)v^{\parallel} + g_{12}(\zeta, \mu, 0)v^{\parallel,\pitchfork} \\ \dot{\mu} &= f_2(\zeta, \mu, 0) + g_{21}(\zeta, \mu, 0)v^{\parallel} + g_{22}(\zeta, \mu, 0)v^{\parallel,\pitchfork}.\end{aligned} \tag{3.19}$$

The tangential dynamics on S_2 are the same as (3.6). In principle, the normal form (3.17) may facilitate the design of control laws to stabilize $\xi = 0$ and $\mu = 0$. However, a drawback of (3.17) is that the dynamics remain nonlinear and it may not be clear how to proceed with control design. The aforementioned partition of g is closely related to the properties of the distributions

$$\begin{aligned}P &:= G_0 \cap TS_2 \\ Q &:= [G_0 \cap TS_2]^{\perp} \cap [G_0 \cap TS_1] \\ R &:= [G_0 \cap TS_1]^{\perp} \cap G_0\end{aligned} \tag{3.20}$$

and the integer-valued functions $\nu, \rho : S_2 \to \mathbb{Z}$, $\sigma : S_1 \to \mathbb{Z}$

$$\begin{aligned}\nu(x) &:= \dim(T_x S_2 \cap G_0(x)) \\ \rho(x) &:= \dim(T_x S_1 \cap G_0(x)) - \nu(x)\end{aligned} \tag{3.21a}$$

$$\sigma(x) := \dim(T_x S_1 + G_0(x)) - s_1. \tag{3.21b}$$

The values of (3.21) equal the dimensions of the distributions (3.20) and the sizes of the sub-matrices corresponding to $v^{\parallel}$, $v^{\parallel,\pitchfork}$, and $v^{\pitchfork}$ in (3.17). In Figure 3.1 the distributions in (3.20) are illustrated in an example.

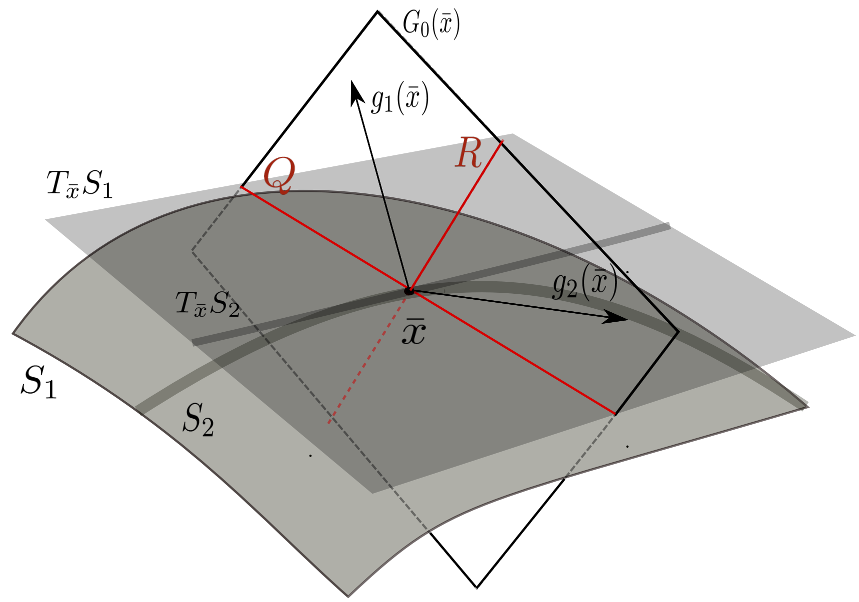


Figure 3.1: An illustration of the distributions in (3.20). In this figure $\nu(\bar{x}) = 0$, $\rho(\bar{x}) = 1$, and $\sigma(\bar{x}) = 1$.

Proposition 3.3.1. *For all $p \in S_1$, $q \in S_2$, $\dim(P(q)) = \nu(q)$, $\dim(Q(q)) = \rho(q)$ and $\dim(R(p)) = \sigma(p)$.*

Proof. The proof that $\dim(P(q)) = \nu(q)$ is obvious from their definitions and is omitted. Next we have

$$\begin{aligned}
&\dim(Q(q)) \\
&= n - \dim(G_0(q) \cap T_qS_2) + \dim(G_0(q) \cap T_qS_1) - \dim\Big([G_0(q) \cap T_qS_2]^\perp + [G_0(q) \cap T_qS_1]\Big) \\
&= \dim(G_0(q) \cap T_qS_1) - \dim(G_0(q) \cap T_qS_2) \\
&= \rho(q).
\end{aligned}$$

Similar computations yield $\dim(R(p)) = \sigma(p)$ on S_1. $\square$

Proposition 3.3.1 motivates Definition 3.3.2.

Definition 3.3.2. A point $\bar{x} \in S_2$ is a **regular point** of the distributions (3.20) if there exists an open set $V_1 \subseteq S_1$ containing $\bar{x}$ such that for all $p \in V_1$, $q \in V_1 \cap S_2$, the functions $\sigma(p)$, $\nu(q)$, $\rho(q)$ are constant. •

Remark 3.3.3. Under Assumption 3.1.1 the topology of S_1 is its subspace topology as a subset of $\mathbb{R}^n$. Thus for each open set $V_1 \subseteq S_1$ there exists an open set $U \subseteq \mathbb{R}^n$ such that $V_1 = U \cap S_1$.

The next proposition provides a computationally tractable way of checking the regularity of the distributions (3.20).

Proposition 3.3.4. *A point $\bar{x} \in S_2$ is a regular point of (3.20) if and only if* $\dim(T_x S_1 \cap G_0(x))$ *and* $\dim(T_x S_2 \cap G_0(x))$ *are constant in, respectively, open sets $V_1 \subseteq S_1$, $V_2 \subseteq S_2$ containing $\bar{x}$.*

Proof. Let $U \subseteq \mathbb{R}^n$ be an open set containing $\bar{x}$ and set $V_1 = S_1 \cap U$. If $\dim(T_x S_1 \cap G_0(x))$ is constant on V_1 then, since $\dim(T_x S_1)$ and $\dim(G_0(x))$ are constant on V_1, the function $\sigma(x)$ in (3.21) is constant on V_1. If both $\dim(T_x S_2 \cap G_0(x))$ and $\dim(T_x S_1 \cap G_0(x))$ are constant on $V_2 = S_2 \cap U$ then the functions ν and ρ in (3.21) are constant on V_2.

Conversely, if the function σ is constant on an open set $V_1 \subset S_1$ with $\bar{x} \in V_1$, then since $T_x S_1$ and $G_0(x)$ are constant dimensional and from the definition of σ it follows that $\dim(T_x S_1 \cap G_0(x))$ is constant on V_1. If ν, ρ are constant on an open set $V_2 \subset S_2$ with $\bar{x} \in V_2$ then from their definitions it follows that $\dim(T_x S_2 \cap G_0(x))$ is constant on V_2. □

Remark 3.3.5. When $\bar{x}$ is a regular point of (3.20), $TS_1 \cap G_0$ and $TS_2 \cap G_0$ can be viewed as vector bundles over the base spaces V_1 and V_2, receptively. In the remainder of this chapter we forgo this formality and refer to them as distributions. It is easy to show that if any two of the functions in (3.21) are constant in an open subset of S_2, then the remaining function is also constant on this set. Furthermore, if x is a regular point of (3.21) then $\nu(x) + \rho(x) + \sigma(x) = m$.

Proposition 3.3.6. *A point $\bar{x} \in S_2$ is a regular point of (3.20) if and only if there exists an open set $V_1 \subseteq S_1$ containing $\bar{x}$ such that the distributions (3.20) are smooth and non-singular in V_1 and $V_1 \cap S_2$.*

Proof. Let $\bar{x} \in S_2$ be a regular point of the distributions (3.20). Then by Proposition 3.3.4 and Definition 3.3.2 P is non-singular in a neighbourhood $V_2 = V_1 \cap S_2$, with $V_1 \subseteq S_1$ and containing $\bar{x}$. Lemma C.29 proves that P is also smooth in a neighbourhood of $\bar{x}$, without loss of generality, V_2. Proposition 3.3.1 shows that Q is non-singular on V_2 and R is non-singular on V_1. Furthermore, by Proposition 3.3.4, the assumed non-singularity of G_0 and Lemma C.29 we have, by possibly shrinking V_1, and hence V_2, that $G_0 \cap TS_1$ and $[G_0 \cap TS_1]^{\perp}$ are smooth on V_1 and $[G_0(x) \cap TS_2]^{\perp}$ is smooth on V_2. Therefore Q and R are the non-singular intersection of smooth non-singular distributions and by [45, Lemma 1.3.5] they are smooth themselves.

Conversely, suppose that the distribution R in (3.20) is smooth and non-singular in a neighbourhood $V_1 \subseteq S_1$ containing $\bar{x}$ and distributions P and Q in (3.20) are smooth and non-singular in $V_2 = V_1 \cap S_2$. By Proposition 3.3.1 and Definition 3.3.2 $\bar{x}$ is a regular point of (3.21). ☐

Lemma 3.3.7. *The values of the functions (3.21) are invariant under coordinate and feedback transformation.*

Proof. Let $x \in S_2$ be fixed but arbitrary and let $\Xi \in \mathrm{Diff}(U)$ be a diffeomorphism onto its image with domain U containing x. Let (α, β) be a regular feedback transformation also defined on U and let $\tilde{g}(x) := g(x)\beta(x)$, $\tilde{G}_0(x) := \operatorname{span}\{\tilde{g}_1(x), \cdots, \tilde{g}_m(x)\}$. Since each $\tilde{g}_i(x)$ is a linear combination of $g_1(x), \ldots, g_m(x)$, $\tilde{G}_0(x) \subseteq G_0(x)$. Furthermore, since $\beta : U \subseteq \mathbb{R}^n \to \mathsf{GL}(m, \mathbb{R})$ is non-singular, $\tilde{G}_0(x) = G_0(x)$ and therefore

$$\nu(x) = \dim(T_x S_2 \cap G_0(x)) = \dim(T_x S_2 \cap \tilde{G}_0(x)).$$

Next, let $\hat{g} := \Xi_\star(g\beta) = \Xi_\star(\tilde{g})$ and $\hat{G}_0 := \operatorname{span}\{\hat{g}_1, \cdots, \hat{g}_m\}$. Since $\mathrm{d}\Xi_x$ is an isomorphism at each $x \in U$, we have

$$\begin{aligned}\dim(T_x S_2 \cap \tilde{G}_0(x)) &= \dim\left(\mathrm{d}\Xi_x\left(T_x S_2 \cap \tilde{G}_0(x)\right)\right) = \dim\left(\mathrm{d}\Xi_x\left(T_x S_2\right) \cap \mathrm{d}\Xi_x\left(\tilde{G}_0(x)\right)\right) \\ &= \dim\left(T_{\Xi(x)}\Xi(S_2 \cap U) \cap \hat{G}_0(\Xi(x))\right)\end{aligned}$$

where the next to last equality comes from the fact that $\operatorname{Ker}(\mathrm{d}\Xi_x) = \{0\}$. From this it follows that the value $\nu(x)$ is unchanged under coordinate and feedback transformations. The same arguments hold for the other functions in (3.21). $\square$

The following lemma generalizes slice coordinates for two nested embedded submanifolds.

Lemma 3.3.8. *Let $S_1 \supset S_2$ be two smooth embedded submanifolds of $\mathbb{R}^n$. For all $\bar{x} \in S_2$ there exists an open set $U \subseteq \mathbb{R}^n$ such that S_1 and S_2 are, respectively, s_1-slices and s_2-slices of U.*

Proof. Let $\bar{x} \in S_2$ be arbitrary. Since $S_1 \subseteq \mathbb{R}^n$ is an embedded submanifold there exist slice coordinates (V_1, ψ) for $\mathbb{R}^n$ with $\bar{x} \in V_1$ such that

$$\psi(S_1 \cap V_1) = \{x \in V_1 : \psi_{s_1+1}(x) = c_{s_1+1}, \cdots, \psi_n(x) = c_n\}$$

where, without loss of generality, we take the constants c_i to be zero. Let $\pi_1 : \mathbb{R}^n \to \mathbb{R}^{n-s_1}$ denote the projection onto the last $n - s_1$ factors, i.e, $\pi_1(x) = (x_{s_1+1}, \cdots, x_n)$. Define $\Phi_1 : V_1 \to \mathbb{R}^{n-s_1}, x \mapsto \pi_1 \circ \psi(x)$. Then Φ_1 is a submersion and

$$\psi(S_1 \cap V_1) = \{x \in V_1 : \Phi_1(x) = 0\}.$$

This construction is summarized in the following commutative diagram

$$\begin{tikzcd}
V_1 \subseteq \mathbb{R}^n \arrow[d, "\psi"] \arrow[dr, dashed, "\Phi_1"] & \\
\psi(V_1) \subseteq \mathbb{R}^n \arrow[r, "\pi_1"] & \mathbb{R}^{n-s_1}.
\end{tikzcd}$$

We now apply a similar construction to S_2. Let (V_2, φ) be slice coordinates for $\mathbb{R}^n$ with $\bar{x} \in V_2$ and let $\pi_2 : \mathbb{R}^n \to \mathbb{R}^{n-s_2}$ be the projection onto the last $n - s_2$ factors. Then, letting $\Phi_2 \coloneqq \pi_2 \circ \phi$ we have

$$\varphi(S_2 \cap V_2) = \{x \in V_2 : \Phi_2(x) = 0\}.$$

and the commutative diagram

$$\begin{tikzcd}
V_2 \subseteq \mathbb{R}^n \arrow[d, "\varphi"] \arrow[dr, dashed, "\Phi_2"] & \\
\varphi(V_2) \subseteq \mathbb{R}^n \arrow[r, "\pi_2"] & \mathbb{R}^{n-s_2}.
\end{tikzcd}$$

Let $U \coloneqq V_1 \cap V_2$ and note that $\bar{x} \in U$. Since Φ_1 and Φ_2 are submersions we have that, for all $x \in U$, $\operatorname{rank}(\mathrm{d}\Phi_1) = n - s_1$ and $\operatorname{rank}(\mathrm{d}\Phi_2) = n - s_2$. Furthermore, by [62, Lemma 8.15], for all $x \in S_2 \cap U$, $\operatorname{Ker} \mathrm{d}\Phi_{1,x} = T_x S_1$ and $\operatorname{Ker} \mathrm{d}\Phi_{2,x} = T_x S_2$. Therefore $\operatorname{Ker} \mathrm{d}\Phi_{2,x} \subset \operatorname{Ker} \mathrm{d}\Phi_{1,x}$

and

$$\operatorname{rank}\begin{bmatrix} \mathrm{d}\Phi_{1,x} \\ \mathrm{d}\Phi_{2,x} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \mathrm{d}\Phi_{2,x} \end{bmatrix} = n - s_2. \tag{3.22}$$

This allows us to construct a submersion $\Phi : U \to \mathbb{R}^{n-s_2}$. We take the last $n - s_1$ components of Φ to be the function Φ_1. From (3.22) we conclude that in the set $\Phi_2 = \{\varphi_{s_2+1}, \cdots, \varphi_n\}$ it is possible to find $s_1 - s_2$ functions, without of loss of generality $\{\varphi_{s_2+1}, \cdots, \varphi_{s_1}\} =: \bar{\Phi}_2$, with the property that the $n - s_2$ differentials $\mathrm{d}\varphi_{s_2+1}, \cdots, \mathrm{d}\varphi_{s_1}, \mathrm{d}\psi_{s_1+1}, \cdots, \mathrm{d}\psi_n$ are linearly independent at $\bar{x}$. Let $\Phi := \left(\bar{\Phi}_2, \Phi_1\right)$.

Since, $\mathrm{d}\Phi(\bar{x})$ has rank $n - s_2$ it has some $(n - s_2) \times (n - s_2)$ minor with non-zero determinant. By re-ordering the coordinates we assume that it is the minor corresponding to the first $n - s_2$ rows and columns of $\mathrm{d}\Phi(\bar{x})$. Relabel the coordinates as $(y, z) = (x_1, \cdots, x_{n-s_2}, x_{n-s_2+1}, \cdots, x_n)$ in $\mathbb{R}^n$. Define $\Xi : U \to \mathbb{R}^n$ by $\Xi(y, z) := (z, \Phi(y, z))$. Its total derivative at $\bar{x}$ is

$$\mathrm{d}\Xi(\bar{x}) = \begin{bmatrix} 0 & I_{s_2} \\ \frac{\partial \Phi_i}{\partial y_j} & \frac{\partial \Phi_i}{\partial z_j} \end{bmatrix},$$

which is non-singular because its columns are independent. Therefore, by the inverse function theorem [62, Theorem 7.6], by possibly shrinking U, $\Xi \in \mathrm{Diff}(U)$. In the chart (U, Ξ) of $\mathbb{R}^n$ we have

$$\Xi(S_1 \cap U) = \{x \in U : \Xi_{s_1+1}(x) = \cdots = \Xi_n(x) = 0\}$$

and

$$\Xi(S_2 \cap U) = \{x \in U : \Xi_{s_2+1}(x) = \cdots = \Xi_n(x) = 0\}.$$

□

Definition 3.3.9. **([62]).** A **retraction** of a topological space X onto a subspace $A \subset M$ is a continuous map $r : X \to A$ such that $r|_A$ is the identity map of A. $\bullet$

The tubular neighbourhood theorem [62, Theorem 10.19] states that every embedded submanifold M of $\mathbb{R}^n$ has a tubular neighbourhood $\mathcal{N}(M)$. It follows [62, Proposition 10.20] that if $\mathcal{N}(M)$ is a tubular neighbourhood of an embedded submanifold $M \subset \mathbb{R}^n$, there exists a smooth retraction of $\mathcal{N}(M)$ onto M. In this paper we use a simpler, local version of these ideas.

Lemma 3.3.10. *Let $M \subset \mathbb{R}^n$ be an m-dimensional embedded submanifold of $\mathbb{R}^n$. Then, for every $x \in M$ there exist a neighbourhood U of x in $\mathbb{R}^n$ and a smooth retraction $r : U \to M \cap U$.*

Proof. Let $\mathcal{N}(M)$ be a tubular neighbourhood of M. By [62, Proposition 10.20] there exists a smooth retraction $r : \mathcal{N}(M) \to M$. Let $U \subseteq \mathcal{N}(M)$ be an open set containing x. Then restriction $r|_U$ is a smooth retraction of U to $M \cap U$. $\square$

Lemma 3.3.11. *Consider two sets S_1 and S_2 satisfying Assumption 3.1.1 and let $\bar{x} \in S_2$ be a regular point of (3.20). There exists an open set $U \subseteq \mathbb{R}^n$ containing $\bar{x}$ and a smooth feedback $\alpha : U \to \mathbb{R}^m$ such that $(f + g\alpha)|_{S_1 \cap U}$ is tangent to $S_1 \cap U$ and $(f + g\alpha)|_{S_2 \cap U}$ is tangent to $S_2 \cap U$.*

Proof. Apply Lemma 3.3.8 to obtain an open set $U \subseteq \mathbb{R}^n$ containing $\bar{x}$ and maps Φ_1 and $\bar{\Phi}_2$ such that $V_1 = \Phi_1^{-1}(0)$ and $V_2 = \left(\bar{\Phi}_2, \Phi_1\right)^{-1}(0)$ where $V_1 \coloneqq S_1 \cap U$ and $V_2 \coloneqq S_2 \cap U$. Since S_1 is a controlled-invariant submanifold there exists a smooth state feedback $\alpha_1 : V_1 \to \mathbb{R}^m$ such that

$$(\forall x \in V_1) \quad \mathrm{d}\Phi_1(x)\left(f(x) + g(x)\alpha_1(x)\right) = 0. \tag{3.23}$$

Similarly, since S_2 is a controlled-invariant submanifold there exists a smooth state feedback $\alpha_2 : V_2 \to \mathbb{R}^m$ such that

$$(\forall x \in V_2) \quad \begin{bmatrix} \mathrm{d}\bar{\Phi}_2(x) \\ \mathrm{d}\Phi_1(x) \end{bmatrix} (f(x) + g(x)\alpha_2(x)) = 0. \tag{3.24}$$

We now modify α_1 so that the resulting state feedback simultaneously satisfies (3.23) and (3.24). We have that

$$\begin{aligned}(\forall x \in V_2) \quad & \mathrm{d}\Phi_1(x)\left(f(x) + g(x)\alpha_2(x)\right)\big|_{V_2} - \mathrm{d}\Phi_1(x)\left(f(x) + g(x)\alpha_1(x)\right)\big|_{V_2} = 0 \\ & \Rightarrow \mathrm{d}\Phi_1(x)g(x)\left(\alpha_2(x) - \alpha_1(x)\right)\big|_{V_2} = 0.\end{aligned}$$

Since α_1 and α_2 are both smooth, there exists a smooth $\hat{v}(x) \in \mathrm{Ker}(\mathrm{d}\Phi_1(x)g(x)|_{V_2})$ such that, for all $x \in V_2$, $\alpha_2(x) = \alpha_1(x)|_{V_2} + \hat{v}(x)$. We have that

$$\begin{aligned}(\forall x \in V_1) \quad \mathrm{rank}(\mathrm{d}\Phi_1(x)g(x)) &= \mathrm{rank}\, g(x) - \dim(\mathrm{Ker}\, \mathrm{d}\Phi_1(x) \cap \mathrm{Im}\, g(x)) \\ &= \dim(G_0(x)) - \dim(T_x S_1 \cap G_0(x)).\end{aligned}$$

By hypothesis, $\bar{x}$ is a regular point of (3.20) and by Proposition 3.3.4, by possibly shrinking V_1, $\dim(T_x S_1 \cap G_0(x))$ is constant and $\dim G_0(x)$ is constant. Thus, $\mathrm{rank}(\mathrm{d}\Phi_1(x)g(x))$ is constant on V_1. It implies that $\dim\left(\mathrm{Ker}(\mathrm{d}\Phi_1(x)g(x))\right)$ is also constant on V_1. Assume that $\dim\left(\mathrm{Ker}(\mathrm{d}\Phi_1(x)g(x))\right) = q$. By [45, Lemma 1.3.1], there exists a set $\{v_1, \cdots, v_q\}$ of smooth vector fields defined on V_1 such that at each $x \in V_1$, the vectors $v_1(x), \cdots, v_q(x)$ are linearly independent and

$$(\forall x \in V_1) \quad \mathrm{Ker}(\mathrm{d}\Phi_1(x)g(x)) = \mathrm{span}\{v_1(x), \cdots, v_q(x)\}.$$

Thus we can write

$$\hat{v}(x) = \sum_{i=1}^{q} \hat{c}_i(x) v_i(x).$$

where $\hat{c}_i : V_2 \to \mathbb{R}$ are smooth real-valued functions. Apply Lemma 3.3.10 and, by possibly shrinking U, introduce a retraction $r_1 : V_1 \to V_2$ of V_1 onto V_2 and define

$$\begin{aligned} c_i : & V_1 \to \mathbb{R} \\ & x \mapsto \hat{c}_i \circ r_1(x). \end{aligned}$$

and

$$v(x) = \sum_{i=1}^{q} c_i(x) v_i(x).$$

Let $\alpha' \coloneqq \alpha_1 + v$. It solves equation (3.23) since

$$(\forall x \in V_1)\ \mathrm{d}\Phi_1(x)\left(f(x) + g(x)\alpha'(x)\right) = \mathrm{d}\Phi_1(x)\left(f(x) + g(x)\alpha_1(x)\right) + \mathrm{d}\Phi_1(x) g(x) v(x) = 0.$$

Similarly it can be verified that it solves equation (3.24). Again, applying Lemma 3.3.10 we introduce a retraction $r_2 : U \to V_1$ of U into V_1 and define

$$\begin{aligned} \alpha : & U \to \mathbb{R}^m \\ & x \mapsto \alpha' \circ r_2(x). \end{aligned}$$

The state feedback α has the desired property. $\square$

Remark 3.3.12. For LTI control systems [92, Lemma 5.7] asserts that, for nested (A, B)-invariant subspaces $\mathscr{S}_1 \supset \mathscr{S}_2$, if F_0 is a friend of $\mathscr{S}_2$ there always exists a mutual friend F such that $F|_{\mathscr{S}_2} = F_0|_{\mathscr{S}_2}$. Lemma 3.3.11 recovers this result because the integers in (3.14) are constant, and the regularity assumption always holds for LTI systems.

3.3.1 Solution to Problem 3

Theorem 3.3.13. *Problem 3 is solvable at $\bar{x} \in S_2$ if and only if $\bar{x}$ is a regular point of* (3.20)*.*

Proof. Assume that Problem 3 is solvable at $\bar{x} \in S_2$. Then there exists a neighbourhood $U \subseteq \mathbb{R}^n$ containing $\bar{x}$, a feedback transformation (α, β) defined on U, and a diffeomorphism $\Xi \in \text{Diff}(U)$ such that (3.1) is locally feedback equivalent to (3.17). Let $V_2 := S_2 \cap U$, $V_1 := S_1 \cap U$, denote by $(\zeta, \mu, 0) = \Xi(x)$ the image of a point $x \in V_1$ and by $(\zeta, 0, 0) = \Xi(x)$ the image of a point $x \in V_2$ under the map Ξ, and let $\hat{g} := \Xi_\star(g\beta)$, $\hat{G}_0 := \text{span}\,\{\hat{g}_1, \cdots, \hat{g}_m\}$. In (ζ, μ, ξ)-coordinates the value of σ in (3.21b) at an arbitrary point $(\zeta, \mu, 0) \in \Xi(V_1)$ equals

$$\sigma(\zeta, \mu, 0) = \dim\left(\text{Im}\begin{bmatrix} I_{s_1} & \star \\ 0 & g_{33}(\zeta, \mu, 0) \end{bmatrix}\right) - s_1 = \text{rank}\,(g_{33}(\zeta, \mu, 0)).$$

We now argue that g_{33} has full column rank. The equality above implies that the number of columns in $g_{33}(\zeta, \mu, 0)$ is greater than or equal to $\sigma(\zeta, \mu, 0)$. Suppose, by way of contradiction, that g_{33} has $\sigma(\zeta, \mu, 0) + 1$ columns. Then, since there are m inputs

$$\text{rank}\begin{bmatrix} g_{11}(\zeta, \mu, 0) & g_{12}(\zeta, \mu, 0) \\ g_{21}(\zeta, \mu, 0) & g_{22}(\zeta, \mu, 0) \end{bmatrix} \leq m - \sigma(\zeta, \mu, 0) - 1.$$

But this means that $\dim\,(\hat{G}_0(\zeta, \mu, 0)) \leq m - 1$ which is a contradiction since (α, β) is a regular feedback transformation and $\dim\,(G_0(x)) = m$. Thus $g_{33}(\zeta, \mu, 0)$ has full column rank. This shows that at an arbitrary point $(\zeta, \mu, 0)$, the integer function σ is equal to the number of columns in g_{33}. Since $(\zeta, \mu, 0)$ is arbitrary, we conclude that σ is constant on $\Xi(V_1)$.

Having shown that $g_{33}(\zeta,\mu,0)$ has full column rank on $\Xi(V_1)$ it follows that $\nu(\zeta,0,0)$ equals

$$\dim\left(T_{(\zeta,0,0)}\Xi(V_2)\cap\hat{G}_0(\zeta,0,0\right)=\dim\left(\operatorname{Im}\begin{bmatrix} I_{s_2}\\ 0\end{bmatrix}\cap\operatorname{Im}\begin{bmatrix} g_{11}(\zeta,0,0) & g_{12}(\zeta,0,0)\\ 0 & g_{22}(\zeta,0,0)\end{bmatrix}\right).$$

Additionally, since

$$\dim\left(\begin{bmatrix} I_{s_2}\\ 0\end{bmatrix}+\operatorname{Im}\begin{bmatrix} g_{11}(\zeta,0,0) & g_{12}(\zeta,0,0)\\ 0 & g_{22}(\zeta,0,0)\end{bmatrix}\right)=\operatorname{rank}\begin{bmatrix} I_{s_2} & \star\\ 0 & g_{22}(\zeta,0,0)\end{bmatrix}$$

we get that

$$\nu(\zeta,0,0)=s_2+m-\sigma(\zeta,0,0)-(\operatorname{rank}g_{22}(\zeta,0,0)+s_2).$$

Using the above expression for ν and the identity, see Remark 3.3.5, $\nu+\rho+\sigma=m$ at $(\zeta,0,0)$ we obtain $\rho(\zeta,0,0)=\operatorname{rank}g_{22}(\zeta,0,0)$. Using the same reasoning as earlier, one can show $g_{22}(\zeta,0,0)$ has full column rank. Thus, at $(\zeta,0,0)$, the integer function ρ is equal to the number of columns in g_{22} which is constant at any point $(\zeta,0,0)\in\Xi(V_2)$. Finally, if two of the functions in (3.21) are constant on $\Xi(V_2)$, then so is the third. By Lemma 3.3.7 the values of (3.21) are invariant under feedback and coordinate transformation which shows that $\bar{x}$ is a regular point of (3.20).

Assume that $\bar{x}\in S_2$ is a regular point of (3.20). By Proposition 3.3.6 the distributions R in (3.20) is smooth and non-singular in a neighbourhood $V_1\subseteq S_1$ containing $\bar{x}$ and the distributions P and Q in (3.20) are smooth and non-singular in $V_2=V_1\cap S_2$. As a result, there exist local generators $\hat{p}_i:V_2\to\mathbb{R}^n$, $i\in\{1,\cdots,\nu\}$, $\hat{q}_i:V_2\to\mathbb{R}^n$, $i\in\{1,\cdots,\rho\}$, and $\hat{r}_i:V_1\to\mathbb{R}^n$, $i\in\{1,\cdots,\sigma\}$ such that, for all $x\in V_2$ $P(x)=\operatorname{span}\{\hat{p}_1,\cdots,\hat{p}_\nu\}(x)$ and

$Q(x) = \operatorname{span}\{\hat{q}_1, \cdots, \hat{q}_\rho\}(x)$ and for all $x \in V_1$ $R(x) = \operatorname{span}\{\hat{r}_1, \cdots, \hat{r}_\sigma\}(x)$.

Next, applying Lemma 3.3.10 we introduce a retraction $r_1 : U \to V_1$ of an open set $U \subseteq \mathbb{R}^n$, $\bar{x} \in U$, onto V_1 and a retraction $r_2 : U \to V_2$ of an open set $U \subseteq \mathbb{R}^n$, $\bar{x} \in U$, onto V_2 and define

$$\begin{aligned} p_i &: U \to \mathbb{R}^n \\ x &\mapsto \hat{p}_i \circ r_2(x) \end{aligned} \qquad i \in \{1, \cdots, \nu\}$$

$$\begin{aligned} q_i &: U \to \mathbb{R}^n \\ x &\mapsto \hat{q}_i \circ r_2(x) \end{aligned} \qquad i \in \{1, \cdots, \rho\}$$

$$\begin{aligned} r_i &: U \to \mathbb{R}^n \\ x &\mapsto \hat{r}_i \circ r_1(x) \end{aligned} \qquad i \in \{1, \cdots, \sigma\}$$

so that the local generators of $P(x)$, $Q(x)$, and $R(x)$ are now defined on U. We set up the following equations

$$\begin{bmatrix} p_1 & \cdots & p_\nu \end{bmatrix} = \begin{bmatrix} g_1 & \cdots & g_m \end{bmatrix} \beta_1 \tag{3.25a}$$

$$\begin{bmatrix} q_1 & \cdots & q_\rho \end{bmatrix} = \begin{bmatrix} g_1 & \cdots & g_m \end{bmatrix} \beta_2 \tag{3.25b}$$

$$\begin{bmatrix} r_1 & \cdots & r_\sigma \end{bmatrix} = \begin{bmatrix} g_1 & \cdots & g_m \end{bmatrix} \beta_3 \tag{3.25c}$$

where $\beta_1 : U \to \mathbb{R}^{m\times\nu}$, $\beta_2 : U \to \mathbb{R}^{m\times\rho}$, and $\beta_3 : U \to \mathbb{R}^{m\times\sigma}$ are unknown matrices. Since, $P \subseteq G_0$ and both are constant dimensional, by possibly shrinking U, there exists a unique smooth solution β_1 to (3.25a). Similarly, by shrinking U, we can find β_2 and β_3 in equations (3.25b) and (3.25c), respectively. Define $[\beta_1\ \beta_2\ \beta_3] =: \beta : U \to \mathsf{GL}(m, \mathbb{R})$. Since $P(x)$, $Q(x)$, and $R(x)$ span independent subspaces for each $x \in U$, the matrix β is non-singular.

By Lemma 3.3.11 there exists a feedback $\alpha : U \to \mathbb{R}^m$ defined, without loss of generality, on U such that $(f+g\alpha)|_{S_1\cap U}$ is tangent to $V_1 \coloneqq S_1 \cap U$ and $(f+g\alpha)|_{S_2\cap U}$ is tangent to $V_2 \coloneqq S_2 \cap U$. The pair (α, β) is the desired feedback transformation. Applying it to (3.1) yields

$$\dot{x} = f(x) + g(x)\alpha(x) + g(x)\beta_1(x)v^{\parallel} + g(x)\beta_2(x)v^{\parallel,\pitchfork} + g(x)\beta_3(x)v^{\pitchfork} \tag{3.26}$$

where $v^{\parallel} \in \mathbb{R}^{\nu}$, $v^{\parallel,\pitchfork} \in \mathbb{R}^{\rho}$, and $v^{\pitchfork} \in \mathbb{R}^{\sigma}$.

By Lemma 3.3.11 the vector field $(f(x)+g(x)\alpha(x))|_{V_2}$ is tangent to both V_1 and V_2. Columns of $g\beta_1|_{V_2}$ are the local generators of P thus are tangent to V_2. the columns of $g\beta_2|_{V_2}$ are local generators of Q, so are tangent to V_1 and not V_2. Finally, columns of $g\beta_3|_{V_2}$ are local generators of R, so are tangent to neither V_1 nor V_2. Select Ξ to be the diffeomorphism from Lemma 3.3.8. Applying the coordinate transformation Ξ to (3.26) yielding the desired normal form (3.17). $\square$

The following example is intended to shed light on the concepts discussed in this section.

Example 3.3.14. Consider the control system

$$\dot{x} = \begin{bmatrix} -x_2 \\ x_1 - (x_1^2 + x_2^2 - 1)(x_3^2 + x_4^2 - 1) \\ -x_4 \\ x_3 - x_1^2 - x_2^2 - x_3^2 - x_4^2 + 2 \end{bmatrix} + \begin{bmatrix} -x_2 \\ x_1 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} -{x_2}^2 \\ x_1\, x_2 \\ -x_4 \\ x_3 \end{bmatrix} u_2 + \begin{bmatrix} 0 \\ 0 \\ x_2 \\ 0 \end{bmatrix} u_3, \tag{3.27}$$

and two nested sets

$$S_1 \coloneqq \left\{x \in \mathbb{R}^4 : {x_1}^2 + {x_2}^2 - 1 = 0\right\},\ S_2 \coloneqq \left\{x \in S_1 : x_3^2 + x_4^2 - 1 = 0\right\},$$

and the point $\bar{x} = (0, 1, 1, 0) \in S_2$. The objective is to solve Problem 3 at $\bar{x}$. We first check the conditions of Theorem 3.3.13 which requires $\bar{x}$ to be a regular point of (3.20). Since S_1 and S_2 are embedded in $\mathbb{R}^4$ as the zero level sets of smooth functions, it is easy to show that

$$T_xS_1 = \text{span}\left\{ \begin{bmatrix} -x_2 \\ x_1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \qquad T_xS_2 = \text{span}\left\{ \begin{bmatrix} -x_2 \\ x_1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -x_4 \\ x_3 \end{bmatrix} \right\}.$$

We compute $\nu(x)$ and $\rho(x)$ as follows

$$\nu(x) = \text{rank} \begin{bmatrix} -x_2 & -x_2^2 \\ x_1 & x_1x_2 \\ 0 & -x_4 \\ 0 & x_3 \end{bmatrix}, \qquad \rho(x) = \text{rank} \begin{bmatrix} -x_2 & -{}^2_2 & 0 \\ x_1 & x_1x_2 & 0 \\ 0 & -x_4 & x_2 \\ 0 & x_3 & 0 \end{bmatrix} - \nu(x)$$

Let $U = \{x \in \mathbb{R}^4 \; : \; x_2 \neq 0 \; \wedge \; x_3 \neq 0\}$ be a neighbourhood of $\bar{x} = (0, 1, 1, 0)$ where, for all $x \in U$, $\nu(x) = 2$ and $\rho(x) = 1$. Since the constancy of any two functions in (3.21) implies the constancy of the third, $\bar{x}$ is a regular point of (3.20) and by Theorem 3.3.13 Problem 3 is solvable there.

First, we find the feedback transformation (α, β). The function $\alpha : U \to \mathbb{R}^3$ is a mutual friend of $S_1 \cap U$ and $S_2 \cap U$. Following the procedure in the proof of Lemma 3.3.11 we obtain $\alpha(x) = 0$. In order to find β we compute the distributions (3.20) of Proposition 3.3.6

$$P(x) = \operatorname{span}\left\{ \begin{bmatrix} -x_2 \\ x_1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -x_1^2 \\ x_1 x_2 \\ -x_4 \\ x_3 \end{bmatrix} \right\},\ Q(x) = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} \right\},\ R(x) = 0.$$

Even though, the distributions (3.20) are only defined on $S_1 \cap U$ and $S_2 \cap U$, the calculated distributions are valid on the entire set U eliminating the need for the retractions in the proof of Theorem 3.3.13. Solving equation (3.25) yields

$$\beta : U \to \mathsf{GL}(3, \mathbb{R}),\ x \mapsto \begin{bmatrix} 1 & 0 & \frac{-x_2 x_4}{x_3} \\ 0 & 1 & \frac{x_4}{x_3} \\ 0 & 0 & \frac{x_3^2 + x_4^2}{x_2 x_3} \end{bmatrix}$$

We follow the proof of Lemma 3.3.8 to find the coordinate transformation $\Xi \in \mathrm{Diff}(U)$ to be defined by $x \mapsto (x_1, x_3, x_3^2 + x_4^2 - 1, {x_1}^2 + {x_2}^2 - 1)$. Applying the feedback transformation (α, β) and coordinate transformation Ξ to (3.27) we obtain

$$\begin{aligned} \dot{\zeta}_1 &= (1 + \xi - \zeta_1^2)^{\frac{1}{2}} + (1 + \xi - \zeta_1^2)^{\frac{1}{2}} v_1^{\parallel} - (1 + \xi - \zeta_1^2) v_2^{\parallel} \\ \dot{\zeta}_2 &= (1 + \mu - \zeta_2^2)^{\frac{1}{2}} + (1 + \mu - \zeta_2^2)^{\frac{1}{2}} v_2^{\parallel} + \zeta_2 v^{\parallel, \pitchfork} \\ \dot{\mu} &= 2(\xi + \mu)(1 + \mu - \zeta_2^2)^{\frac{1}{2}} + 2(\mu + 1) v^{\parallel, \pitchfork} \\ \dot{\xi} &= 2\xi\mu(1 + \xi - \zeta_1^2)^{\frac{1}{2}}. \end{aligned} \tag{3.28}$$

Note that when $\xi = 0$, the term $f_3(\zeta, \mu, \xi) = 2\xi\mu(1 + \xi - \zeta_1^2)^{\frac{1}{2}}$ vanishes and g_{31} and g_{32} are identically zero. Also, when $\xi = 0$ and $\mu = 0$ the terms $f_2(\zeta, \mu, \xi) = 2(\xi + \mu)(1 + \mu - \zeta_2^2)^{\frac{1}{2}}$

vanishes and $g_{21}(\zeta, \mu, \xi)$ is identically zero. Thus, the requirements on normal form (3.17) are satisfied. The dynamics restricted to $S_1 \cap U$ are obtained by substituting $\xi = 0$ in (3.28) and the dynamics restricted to $S_2 \cap U$ is obtained by substituting $\xi = 0$ and $\mu = 0$ in (3.28).

$\triangle$

3.4 Solution to partial local nested transverse feedback linearization problem

We are now ready to present the main result of this chapter, necessary and sufficient conditions for Problem 1 to be solvable. It is evident that (3.3) is a refinement of (3.17) and thus the solvability of Problem 3 is a necessary condition for Problem 1 to be solvable. Thus throughout this section we make the following assumption.

Assumption 3.4.1. The point $\bar{x} \in S_2$ is a regular point of (3.20). $\triangleleft$

Assumption 3.4.1 implies, by Theorem 3.3.13, that Problem 3 is solvable at $\bar{x}$. Therefore, there exists a regular feedback transformation (α, β) such that, control system (3.1) on a neighbourhood $U \subset \mathbb{R}^n$ of $\bar{x}$ writes as (3.26), re-written here for convenience,

$$\dot{x} = f(x) + g(x)\alpha(x) + g(x)\beta_1(x)v^{\parallel} + g(x)\beta_2(x)v^{\parallel,\pitchfork} + g(x)\beta_3(x)v^{\pitchfork}$$

where $\left(v^{\parallel}, v^{\parallel,\pitchfork}, v^{\pitchfork}\right) \in \mathbb{R}^{\nu} \times \mathbb{R}^{\rho} \times \mathbb{R}^{\sigma}$. Recall that $v^{\pitchfork} = 0$ renders $S_1 \cap U$ invariant; and the vector field $f + g\alpha$ and columns of $g\beta_1$ and columns of $g\beta_2$ are tangent to $S_1 \cap U$. Thus, the restriction of (3.1) with $v^{\pitchfork} = 0$ to $S_1 \cap U$ is well-defined. We introduce the following short hand notation for the restriction

$$f_{S_1} := (f + g\alpha)|_{S_1 \cap U}\,, \quad g_{S_1} := [g\beta_1\; g\beta_2]|_{S_1 \cap U} \quad v^{\parallel}_{S_1} := (v^{\parallel}, v^{\parallel,\pitchfork}).$$

Then, the dynamics restricted to $S_1 \cap U$ are

$$\dot{x} = f_{S_1}(x) + g_{S_1}(x) v^{\parallel}_{S_1}. \tag{3.29}$$

Similar to (3.2), we associate to (3.29) a family of distribution $G^{\parallel}_i : S_1 \cap U \to T(S_1 \cap U) \subseteq \mathbb{R}^n$

$$G^{\parallel}_i(x) := \operatorname{span}\left\{ ad^j_{f_{S_1}} g_{S_1,k}(x) \ : \ 0 \le j \le i,\ 1 \le k \le \nu + \rho \right\}. \tag{3.30}$$

Theorem 3.4.2 (Main Result)**.** *Consider control system* (3.1) *and nested sets* $S_1 \supset S_2$ *satisfying Assumption 3.1.1. Let* $\bar{x} \in S_2$ *and suppose that* $\operatorname{inv}(G^{\parallel}_i), i \in \mathbf{s_1} - \mathbf{s_2} - \mathbf{1}$ *are regular at* $\bar{x} \in S_2$*. Then, Problem 1 is solvable if and only if*

(a) $\bar{x}$ *is a regular point of* (3.20)

(b) $\dim\left(T_{\bar{x}} S_2 + G^{\parallel}_{s_1 - s_2 - 1}(\bar{x})\right) = s_1$

(c) There exists an open set $U \subseteq \mathbb{R}^n$ *containing* $\bar{x}$ *such that, for all* $i \in \mathbf{s_1} - \mathbf{s_2} - \mathbf{1}$*, for all* $x \in S_2 \cap U$

$$\dim\left(T_x S_2 + G^{\parallel}_i(x)\right) = \left(T_x S_2 + \operatorname{inv}(G^{\parallel}_i(x))\right) = \textit{constant.}$$

Proof. Suppose that Problem 1 is solvable at $\bar{x} \in S_2$. Then Problem 3 is solvable since the normal form (3.3) is a refinement of the normal form (3.17). Thus, $\bar{x}$ is a regular point of (3.20) and condition (a) holds. As a result, the assumption requiring $\operatorname{inv}(G^{\parallel}_i), i \in$

$\mathbf{s_1 - s_2 - 1}$ be regular at $\bar{x}$ is a valid assumption. Moreover, since Problem 1 is solvable at $\bar{x} \in S_2$ there exists a neighbourhood $U \subseteq \mathbb{R}^n$ of $\bar{x}$, a coordinate transformation $\Xi \in \mathrm{Diff}(U)$, and a feedback transformation (α, β) such that (3.1) is feedback equivalent to (3.3) in U. Define $V_1 := \Xi(S_1 \cap U)$ and $V_2 := \Xi(S_2 \cap U)$. The system dynamics restricted to V_1 are given in (3.5), and any point in V_1 and V_2 is represented by (ζ, μ) and $(\zeta, 0)$, respectively. In transformed coordinates we have

$$(\forall (\zeta, 0) \in V_2)\,,\ (\forall i \in \mathbf{s_1 - s_2}),\ T_{(\zeta,0)}V_2 + G_i^{\parallel}(\zeta, 0) = \mathrm{Im}\left(\begin{bmatrix} I_{s_2} & \star & \star & \cdots & \star \\ 0_{r-s_2 \times s_2} & B & AB & \cdots & A^i B \end{bmatrix}\right)$$

which shows that the dimension of $T_{(\zeta,0)}V_2 + G_i^{\parallel}(\zeta,0)$ is $s_2 + \mathrm{rank}\left(\begin{bmatrix} B & \cdots & A^i B \end{bmatrix}\right)$. Since the pair (A, B) is controllable, $\mathrm{rank}\left(\begin{bmatrix} B & \cdots & A^{s_1 - s_2 - 1} B \end{bmatrix}\right) = s_1 - s_2$; thus we have that $\dim\left(T_{(\zeta,0)}V_2 + G_{s_1-s_2-1}^{\parallel}(\zeta, 0)\right) = s_1$. Since condition (a) is invariant under coordinate and feedback transformations it follows that condition (a) holds in original coordinates as well.

In V_1, consider the collection of constant distributions $\Delta_i^{\parallel}$, $i \in \mathbf{s_1 - s_2}$ given by

$$\Delta_i^{\parallel} := \mathrm{Im}\left(I_{s_2} \oplus \begin{bmatrix} B & \cdots & A^i B \end{bmatrix}\right).$$

At each $(\zeta, 0) \in V_2$, $\Delta_i^{\parallel}(\zeta, 0) = T_{(\zeta,0)}V_2 + G_i^{\parallel}(\zeta, 0)$; thus, $G_i^{\parallel}(\zeta, 0) \subseteq \Delta_i^{\parallel}$. Furthermore, since each $\Delta_i^{\parallel}$ is (trivially) involutive, it follows that $\mathrm{inv}(G_i^{\parallel}(\zeta, 0)) \subseteq \Delta_i^{\parallel}$. This shows that for all $i \in \mathbf{s_1 - s_2}$

$$TV_2 + \mathrm{inv}(G_i^{\parallel}(\zeta, 0)) \subseteq \Delta_i^{\parallel}(\zeta, 0) = TV_2 + G_i^{\parallel}(\zeta, 0).$$

On the other hand $TV_2 + G_i^{\parallel}(\zeta, 0) \subseteq TV_2 + \mathrm{inv}(G_i^{\parallel}(\zeta, 0))$ always holds which shows that $\Delta_i^{\parallel}(\zeta, 0) = TV_2 + G_i^{\parallel}(\zeta, 0) = TV_2 + \mathrm{inv}(G_i^{\parallel}(\zeta, 0))$. Condition (b) is invariant under coordinate and feedback transformations; thus it holds in original coordinates.

Conversely, assume that conditions (a), (b), and (c) hold. By condition (a) Problem 3 is solvable and there exists a neighbourhood $U \subseteq \mathbb{R}^n$ of $\bar{x}$, a coordinate transformation $\Xi_1 : U \to \Xi_1(U) \subseteq \mathbb{R}^{s_2} \times \mathbb{R}^{s_1 - s_2} \times \mathbb{R}^{n-s_1}$, $x \mapsto \left(\bar{\zeta}, \bar{\mu}, \bar{\xi}\right)$, and feedback transformation (α_1, β_1) such that (3.1) is feedback equivalent to (3.17) on U. Let $\pi_1 : \mathbb{R}^n \to \mathbb{R}^{s_1}$ be the projection to the first s_1 factors. Let $\bar{V}_1 \coloneqq \pi_1 \circ \Xi_1(S_1 \cap U) \subseteq \mathbb{R}^{s_1}$ and $\psi = \pi_1 \circ \Xi_1|_{S_1 \cap U} : S_1 \cap U \to \bar{V}_1$. By [62, Theorem 8.2] S_1 is a smooth manifold of dimension s_1 and $(S_1 \cap U, \psi)$ is a coordinate chart. Define $\bar{V}_2 \coloneqq \pi_1 \circ \Xi_1(S_2 \cap U)$. Since $\bar{V}_2 = \left\{\left(\bar{\zeta}, \bar{\mu}\right) \in \bar{V}_1 : \bar{\mu} = 0\right\}$ we conclude that $\bar{V}_2$ is an embedded submanifold of $\bar{V}_1$. In this coordinate chart (3.29) writes as

$$\begin{aligned}\dot{\bar{\zeta}} &= \bar{f}_1(\bar{\zeta}, \bar{\mu}, 0) + \bar{g}_{11}(\bar{\zeta}, \bar{\mu}, 0)\bar{v}^{\parallel} + \bar{g}_{12}(\bar{\zeta}, \bar{\mu}, 0)\bar{v}_2^{\pitchfork} \\ \dot{\bar{\mu}} &= \bar{f}_2(\bar{\zeta}, \bar{\mu}, 0) + \bar{g}_{21}(\bar{\zeta}, \bar{\mu}, 0)\bar{v}^{\parallel} + \bar{g}_{22}(\bar{\zeta}, \bar{\mu}, 0)\bar{v}_2^{\pitchfork}.\end{aligned} \tag{3.31}$$

Since condition (a) holds one can assume $\mathrm{inv}(G_i^{\parallel}), i \in \mathbf{s_1 - s_2 - 1}$ are regular at $\bar{x}$ and considering conditions (b) and (c) all the assumptions and conditions of [70, Theorem 3.2] for (3.31) with respect to $\bar{V}_2$ at $(\bar{\zeta}, 0) \coloneqq \pi_1 \circ \Xi_1(\bar{x})$ hold. Therefore, by possibly shrinking $\bar{V}_1$ (and hence U), there exist a coordinate transformation $\Xi_2 : \bar{V}_1 \to \Xi_2(\bar{V}_1) \subseteq \mathbb{R}^{s_2} \times \mathbb{R}^{s_1 - s_2}$, and a regular feedback transformation $\left(\bar{\alpha}_2, \bar{\beta}_2\right)$, with $\bar{\alpha}_2 : \bar{V}_1 \to \mathbb{R}^{\nu+\rho}$ and $\bar{\beta}_2 : \bar{V}_1 \to \mathsf{GL}\,(\nu + \rho, \mathbb{R})$, such that (3.31) is feedback equivalent to

$$\begin{aligned}\dot{\zeta} &= f_1(\zeta, \mu) + g_{11}(\zeta, \mu)v^{\parallel} + g_{12}(\zeta, \mu)v^{\parallel, \pitchfork} \\ \dot{\mu} &= A\mu + Bv_2^{\pitchfork}.\end{aligned} \tag{3.32}$$

Let $\pi_2 : \mathbb{R}^n \to \mathbb{R}^{n-s_1}$ be the projection to the last $n - s_1$ factors. We construct a function

$\Xi : U \subseteq \mathbb{R}^n \to \Xi(U) \subseteq \mathbb{R}^{s_2} \times \mathbb{R}^{s_1 - s_2} \times \mathbb{R}^{n-s_1}$ as follows

$$\Xi := (\Xi_2 \circ \pi_1 \circ \Xi_1) \times (\pi_2 \circ \Xi_1) = \begin{bmatrix} \Xi_2 \circ \pi_1 \\ \pi_2 \end{bmatrix} \circ \Xi_1.$$

The following diagram illustrates our construction.

$$\begin{tikzcd}
U \subseteq \mathbb{R}^n \arrow[r, "\Xi_1"] & \Xi_1(U) \subseteq \mathbb{R}^{s_2} \times \mathbb{R}^{s_1-s_2} \times \mathbb{R}^{n-s_1} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \mathbb{R}^{n-s_1} \\
 & \bar{V}_1 \subseteq \mathbb{R}^{s_1} \arrow[r, "\Xi_2"] & \Xi_2(\bar{V}_1) \subseteq \mathbb{R}^{s_2} \times \mathbb{R}^{s_1-s_2}.
\end{tikzcd}$$

The function Ξ is a well-defined diffeomorphism since at $\bar{x}$

$$\det(\mathrm{d}\Xi) = \det\left(\begin{bmatrix} \mathrm{d}\Xi_2 & 0_{s_1 \times n-s_1} \\ 0_{n-s_1 \times s_1} & I_{n-s_1} \end{bmatrix}\right) \det(\mathrm{d}\Xi_1) \neq 0.$$

Therefore, by the inverse function theorem [62, Theorem 7.6], it is a valid coordinate transformation in a neighbourhood of $\bar{x}$, without loss of generality U. In order to construct the feedback transformation we define

$$\alpha_2 := \begin{bmatrix} \bar{\alpha}_2 \circ \pi_1 \circ \Xi_1 \\ 0_\sigma \end{bmatrix}, \quad \beta_2 := \begin{bmatrix} \bar{\beta}_2 \circ \pi_1 \circ \Xi_1 & 0_{(\nu+\rho)\times\sigma} \\ 0_{\sigma \times (\nu+\rho)} & I_\sigma \end{bmatrix},$$

where $\alpha_2 : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $\beta : U \subseteq\to \mathsf{GL}(m, \mathbb{R})$. The feedback transformation $(\alpha, \beta) := (\alpha_1 + \beta_1\alpha_2, \beta_1\beta_2)$ and $\Xi \in \mathrm{Diff}(U)$ solve Problem 1. $\square$

Remark 3.4.3. If the conditions of Theorem 3.4.2 hold, then following [70, Theorem 3.1] one can find ρ smooth $\mathbb{R}$-valued functions $\lambda_1(\zeta, \mu), \cdots, \lambda_\rho(\zeta, \mu)$ defined on $\bar{V}_1$, where ρ is given in (3.21a), such that (a) $\bar{V}_2 \subset \left\{(\zeta, \mu) \in \bar{V}_1 : \lambda_i(\zeta, \mu) = 0,\ i \in 1, \cdots, \rho\right\}$ (b) the system (3.31) with output $y \coloneqq (\lambda_1(\zeta,\mu), \cdots, \lambda_\rho(\zeta,\mu))$ has vector relative degree $\{k_1 \cdots, k_\rho\}$ with $k_1 + \cdots + k_\rho = s_1 - s_2$ at $(\bar{\zeta}, 0)$. Thus, the nested local transverse feedback linearization problem is equivalent to a zero dynamics assignment with well-defined relative degree problem. A semi-constructive procedure to find such functions is presented in the proof of [70, Theorem 3.1].

3.5 Extension of the main result

We now outline an extension of the solution to Problem 1. In the extension we seek that the dynamics transversal to the larger set S_1 also be feedback linearizable. The resulting normal form facilitates the design of controllers to locally stabilize S_1.

Problem 4. (Local nested transversal feedback linearization): Find, if possible, a solution to Problem 1 in which the normal form (3.3) is replaced by

$$\begin{aligned}
\dot{\zeta} &= f_1(\zeta,\mu,\xi) + g_{11}(\zeta,\mu,\xi)v^{\parallel} + g_{12}(\zeta,\mu,\xi)v^{\parallel,\pitchfork} + g_{13}(\zeta,\mu,\xi)v^{\pitchfork} \\
\dot{\mu} &= A\mu + Bv^{\parallel,\pitchfork} + f_2(\zeta,\mu,\xi) + g_{21}(\zeta,\mu,\xi)v^{\parallel} + g_{22}(\zeta,\mu,\xi)v^{\parallel,\pitchfork} + g_{23}(\zeta,\mu,\xi)v^{\pitchfork} \\
\dot{\xi} &= E\xi + Fv^{\pitchfork}
\end{aligned} \tag{3.33}$$

where the pair (E,F) is controllable and F is full rank. ▲

In Problem 4 the normal form (3.3) has been refined because the ξ-subsystem is now linear, controllable and decoupled from the ζ- and μ-subsystems.

Theorem 3.5.1. *Consider control system (3.1) and nested sets $S_1 \supset S_2$ satisfying Assumption 3.1.1. Let $\bar{x} \in S_2$ and assume that the distributions* $\mathrm{inv}\,(G_i^{\parallel})$, $\mathrm{inv}(G_j)$, $i \in \mathbf{n} - \mathbf{s_1} - \mathbf{1}$, $j \in \mathbf{s_1} - \mathbf{s_2} - \mathbf{1}$ *are regular at $\bar{x} \in S_2$. Then, Problem 4 is solvable if and only if* *(a) Problem 1 is solvable.*

(b) $\dim\left(T_{\bar{x}}S_1 + G_{n-s_1-1}(\bar{x})\right) = n$.

(c) There exist a neighbourhood U of $\bar{x}$ in $\mathbb{R}^n$ such for all $i \in \mathbf{n-s_1-1}$, for all $(x \in S_1 \cap U)$,

$$\dim\left(T_xS_1 + G_i(x)\right) = \left(T_xS_1 + \operatorname{inv}(G_i)(x)\right) = \textit{constant}.$$

Proof. Assume that Problem 4 is solvable at $\bar{x} \in S_2$. The normal form (3.33) is a refinement of the normal forms (3.3) and (3.17). Thus, if Problem 4 is solvable Problems 1 and 3 are solvable. Thus, condition (a) holds. Besides, if Problem 3 is solvable $\bar{x}$ must be a regular point of (3.20) which implies that it is valid to assume $\operatorname{inv}(G_i^{\|}), i \in \mathbf{n-s_1-1}$ are regular at $\bar{x}$. The proof of the necessity of conditions (b) and (c) is easily checked in transformed coordinates using arguments analogous to those in the proof of Theorem 3.4.2.

Conversely, assume that conditions (a), (b) and (c) hold. By [70, Theorem 3.2], since conditions (b) and (c) hold, there exist a neighbourhood U of the point $\bar{x}$, a diffeomorphism $\Xi_1 : U \to \Xi_1(U) \subset \mathbb{R}^{s_1} \times \mathbb{R}^{n-s_1}$ and a regular feedback transformation (α_1, β_1) such that system (3.1), on U, is feedback equivalent to

$$\begin{aligned} \dot{\eta} &= f_0(\eta,\xi) + g^{\|}(\eta,\xi)v_1 + g^{\pitchfork}(\eta,\xi)v_2 \\ \dot{\xi} &= E\xi + Fv_2 \end{aligned} \tag{3.34}$$

where the pair (E, F) is controllable and $\Xi_1(S_1 \cap U) = \{(\eta,\xi) \in \Xi_1(U) : \xi = 0\}$. Since Problem 1 is solvable Assumption 3.4.1 must hold. Thus, $v_1 \in \mathbb{R}^{\nu+\rho}$, $v_2 \in \mathbb{R}^{\sigma}$.

Let $\pi_1 : \mathbb{R}^n \to \mathbb{R}^{s_1}$ be the projection to the first s_1 factors. Let $\bar{V}_1 \coloneqq \pi_1 \circ \Xi_1(S_1 \cap U)$ and $\psi = \pi_1 \circ \Xi_1|_{S_1\cap U} : S_1 \cap U \to \bar{V}_1$. By [62, Theorem 8.2] S_1 is a smooth manifold of dimension s_1 and $(S_1 \cap U, \psi)$ is a coordinate chart. Define $\bar{V}_2 \coloneqq \pi_1 \circ \Xi_1(S_2 \cap U)$. Since $\bar{V}_2$ is diffeomorphic to $S_2 \cap U$ it is an embedded submanifold of $\bar{V}_1$. In this coordinate chart the restricted dynamics in (3.29) writes as

$$\dot{\eta} = f_0(\eta, 0) + g^{\parallel}(\eta, 0)v_1. \tag{3.35}$$

By condition (a) Problem 1 is solvable and the conditions of Theorem 3.4.2 hold. Therefore the assumption that $\operatorname{inv}(G_i^{\parallel}), i \in \boldsymbol{n - s_1 - 1}$ are regular at $\bar{x}$ is well-posed. Thus, all the assumptions and conditions of [70, Theorem 3.2] for (3.35) with respect to $\bar{V}_2$ at $\bar{\eta} \coloneqq \pi_1 \circ \Xi_1(\bar{x})$ hold. Therefore, by possibly shrinking $\bar{V}_1$ (and hence U), there exists a coordinate transformation $\Xi_2 : \bar{V}_1 \to \Xi_2(\bar{V}_1) \subseteq \mathbb{R}^{s_2} \times \mathbb{R}^{s_1 - s_2}$, and a regular feedback transformation $(\bar{\alpha}_2, \bar{\beta}_2)$, with $\bar{\alpha}_2 : \bar{V}_1 \to \mathbb{R}^{\nu+\rho}$ and $\bar{\beta}_2 : \bar{V}_1 \to \mathsf{GL}(\nu + \rho, \mathbb{R})$, such that (3.35) is feedback equivalent to

$$\begin{aligned} \dot{\zeta} &= f_1(\zeta, \mu) + g_{11}(\zeta, \mu)v^{\parallel} + g_{12}(\zeta, \mu)v^{\parallel,\pitchfork} \\ \dot{\mu} &= A\mu + Bv^{\parallel,\pitchfork}. \end{aligned} \tag{3.36}$$

The desired diffeomorphism Ξ is constructed from Ξ_1 and Ξ_2 in the same manner as in the proof of Theorem 3.4.2. The feedback transformation (α, β) is also constructed from (α_1, β_1) and $(\bar{\alpha}_2, \bar{\beta}_2)$ in the same way as in the proof of Theorem 3.4.2.

$\square$

The following example concerning the system from Example 3.3.14 illustrates a case in which Problem 1 is solvable while Problem 4 is not.

Example 3.5.2. Recall the system, nested sets $S_1 \supset S_2$, and point $\bar{x} = (0, 1, 1, 0)$ from Example 3.3.14. Since Problem 3 is solvable at $\bar{x}$ in $U = \{x \in \mathbb{R}^4 : \ x_2 \neq 0 \text{ and } x_3 \neq 0\}$ we can easily compute the dynamics restricted to $S_1 \cap U$ in (3.29) as

$$\dot{x} = \begin{bmatrix} -x_2 \\ x_1 \\ -x_4 \\ x_3 - x_3^2 - x_4^2 + 1 \end{bmatrix} + \begin{bmatrix} -x_2 \\ x_1 \\ 0 \\ 0 \end{bmatrix} v_1^{\parallel} + \begin{bmatrix} -{x_2}^2 \\ x_1\, x_2 \\ -x_4 \\ x_3 \end{bmatrix} v_2^{\parallel} + \begin{bmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} v^{\parallel,\pitchfork}.$$

Condition (a) of Theorem 3.4.2 is satisfied since

$$\dim\left(T_{\bar{x}} S_2 + G_0^{\parallel}(\bar{x})\right) = \operatorname{rank} \begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} = 3$$

Moreover, since

$$G_0^{\parallel} = \operatorname{span}\left\{ \begin{bmatrix} -x_2 \\ x_1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -x_2^2 \\ x_1 x_2 \\ -x_4 \\ x_3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} \right\}$$

is involutive, condition (b) of Theorem 3.4.2 holds. Therefore Problem 1 is solvable at $\bar{x}$.

However, since

$$\dim\left(T_{\bar{x}}S_1 + G_0(\bar{x})\right) = \operatorname{rank}\begin{bmatrix} -1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \neq 4$$

condition (a) of Theorem 3.5.1 is not satisfied and Problem 4 is cannot be solved at $\bar{x}$. We proceed to find the normal form (3.3) of Problem 1.

In order to find the desired feedback transformation and coordinate transformation we follow the construction in the proof of Theorem 3.4.2. The feedback transformation (α_1, β_1) and the coordinate transformation Ξ_1 were already found in Example 3.3.14. Letting $\xi = 0$ in (3.28), the dynamics (3.31) are

$$\begin{aligned}
\dot{\bar{\zeta}}_1 &= \left(1 - \bar{\zeta}_1^2\right)^{\frac{1}{2}} + \left(1 - \bar{\zeta}_1^2\right)^{\frac{1}{2}} v_1^{\|} - \left(1 - \bar{\zeta}_1^2\right) v_2^{\|} \\
\dot{\bar{\zeta}}_2 &= \left(1 + \bar{\mu} - \bar{\zeta}_2^2\right)^{\frac{1}{2}} + \left(1 + \bar{\mu} - \bar{\zeta}_2^2\right)^{\frac{1}{2}} v_2^{\|} + \bar{\zeta}_2 v^{\|,\pitchfork} \\
\dot{\bar{\mu}} &= 2\bar{\mu}(1 + \bar{\mu} - \bar{\zeta}_2^2)^{\frac{1}{2}} + 2(\bar{\mu} + 1)v^{\|,\pitchfork}.
\end{aligned}$$

We employ the results of [70, Theorem 3.2] to find

$$\Xi_2 = \mathbb{1}_3, \qquad \left(\bar{\alpha}_2, \bar{\beta}_2\right) = \left(\begin{bmatrix} 0 \\ 0 \\ -\frac{-2\bar{\mu}(1+\bar{\mu}-\bar{\zeta}_2^2)^{\frac{1}{2}}}{2(\bar{\mu}+1)} \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2(\bar{\mu}+1)} \end{bmatrix}\right)$$

where $\mathbb{1}_3$ is the identity map. It should be noted that $\bar{\mu} + 1 = x_3^2 + x_4^4$. Since, $x_3 \neq 0$ on U it follows that $\bar{\mu} + 1 \neq 0$ on $\Xi_1(U)$; thus it is not required to shrink the neighbourhood U.

Next, we find $\pi_1 = \mathbb{R}^4 \to \mathbb{R}^3$, $(\bar{\zeta}_1, \bar{\zeta}_2, \bar{\mu}, \bar{\xi}) \mapsto (\bar{\zeta}_1, \bar{\zeta}_2, \bar{\mu})$, $\pi_2 = \mathbb{R}^4 \to \mathbb{R}$, $(\bar{\zeta}_1, \bar{\zeta}_2, \bar{\mu}, \bar{\xi}) \mapsto \bar{\xi}$, introduced in the proof of Theorem 3.4.2. Therefore,

$$\Xi = (\Xi_2 \circ \pi_1 \circ \Xi_1) \times (\pi_2 \circ \Xi_1) = (x_1, x_3, x_3^2 + x_4^2 - 1, x_1^2 + x_2^2 - 1)$$

and

$$\alpha_2 = \begin{bmatrix} 0 \\ 0 \\ \frac{(x_3^2+x_4^2-1)x_4}{x_3^2+x_4^2} \end{bmatrix}, \quad \beta_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2(x_3^2+x_4^2)} \end{bmatrix}.$$

And finally the feedback transformation (α, β) is

$$(\alpha, \beta) = (\alpha_1 + \beta_1\alpha_2, \beta_1\beta_2) = \left(\begin{bmatrix} \frac{-x_2x_4^2(x_3^2+x_4^2-1)}{x_3(x_3^2+x_4^2)} \\ \frac{x_4^2(x_3^2+x_4^2-1)}{x_3(x_3^2+x_4^2)} \\ \frac{(x_3^2+x_4^2-1)}{x_2x_3} \end{bmatrix}, \begin{bmatrix} 1 & 0 & \frac{-x_2x_4}{2x_3(x_3^2+x_4^2)} \\ 0 & 1 & \frac{x_4}{2x_3(x_3^2+x_4^2)} \\ 0 & 0 & \frac{1}{x_2x_3} \end{bmatrix} \right).$$

Applying the feedback transformation (α, β) and the coordinate transformation Ξ the control system is feedback equivalent to

$$\begin{aligned}
\dot{\zeta}_1 &= (1 + \xi - \zeta_1^2)^{\frac{1}{2}} + (1 + \xi - \zeta_1^2)v_1^{\|} - (1 + \xi - \zeta_1^2)v_2^{\|} \\
\dot{\zeta}_2 &= (1 + \mu - \zeta_2^2)^{\frac{1}{2}} + 1 + \mu - \zeta_2^{2\,\frac{1}{2}} v_2^{\|} + \frac{\zeta_2}{2(\mu + 1)} v^{\|,\pitchfork} \\
\dot{\mu} &= 2\xi(1 + \mu - \zeta_2^2)^{\frac{1}{2}} + v^{\|,\pitchfork} \\
\dot{\xi} &= 2\xi\mu(1 + \xi - \zeta_1^2)^{\frac{1}{2}}.
\end{aligned}$$

when $\xi = 0$ the μ-subsystem is $\dot{\mu} = v^{\|,\pitchfork}$ which is linear and controllable as desired. $\triangle$

The following example illustrates a case in which Problem 4 is solvable

Example 3.5.3. Consider the control system

$$\dot{x} = \begin{bmatrix} -x_1 \\ -x_2 - x_5 - x_5\left(-{x_1}^2 + x_2 + x_3\right) \\ x_2 + x_5 + x_5\left(-{x_1}^2 + x_2 + x_3\right) - 2{x_1}^2 \\ x_2 + x_3 - x_4 + x_5 + x_5\left(-{x_1}^2 + x_2 + x_3\right) - 2{x_1}^2 \\ x_4 - x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \\ x_2 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ -1 \\ -2 \\ 0 \end{bmatrix} u_2,$$

sets $S_1 = \{x \in \mathbb{R}^5 : x_1^2 - x_2 - x_3 = 0\}$ and $S_2 = \{x \in S_1 : x_2 = x_3 - x_4 = x_5 = 0\}$, and a point $\bar{x} = (1, 0, 1, 0, 1)$. It is easy to show that

$$T_x S_1 = \operatorname{Im} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2\,x_1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_x S_2 = \operatorname{Im} \begin{bmatrix} 1 \\ 0 \\ 2\,x_1 \\ 2\,x_1 \\ 0 \end{bmatrix}.$$

We check the conditions of Theorem 3.5.1. Condition (a) of Theorem 3.5.1 requires that Problem 1 be solvable. Thus we first check the conditions of Theorem 3.4.2.

We compute for all $x \in \mathbb{R}^n$ $\nu(x) = 0$ and $\rho(x) = 1$. Since the constancy of any two functions in (3.21) implies the constancy of the third, $\bar{x}$ is a regular point of (3.20) and condition (a) of Theorem 3.4.2 hold. In order to check conditions (b) and (c) of Theorem 3.4.2 we need to find (3.29). Following the proof of Theorem 3.3.13 we find the restricted dynamics as

$$\dot{x} = \begin{bmatrix} -x_1 \\ -x_2 - x_5 \\ x_2 + x_5 - 2\,{x_1}^2 \\ x_2 + x_3 - x_4 + x_5 - 2\,{x_1}^2 \\ x_4 - x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -1 \\ -2 \\ 0 \end{bmatrix} v^{\parallel,\pitchfork}.$$

Condition (b) of Theorem 3.4.2 is satisfied since at point $\bar{x}$

$$\dim(T_{\bar{x}}S_2 + G_2^{\parallel}(\bar{x})) = \operatorname{rank} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 2 & -1 & -1 & -2 \\ 2 & -2 & -2 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} = 4.$$

Moreover, since $G_0^{\parallel}$ contains a single vector it is involutive and condition (e) of Theorem 3.4.2 hold.

Condition (b) of Theorem 3.5.1 holds since

$$\dim(T_{\bar{x}}S_1 + G_0(\bar{x})) = \operatorname{rank} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} = 5.$$

Condition (c) of Theorem 3.5.1 holds since $\dim(T_xS_1 + G_0(x)) = \dim(T_xS_1 + \operatorname{inv}(G_0(x))) = 5$

for all $x \in \mathbb{R}^5$. Thus Problem 4 is solvable. Following the proof of Theorem 3.5.1 we find the following feedback transformation and coordinate transformation

$$(\alpha, \beta) = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right),$$

$$\Xi = (x_1, x_2, x_5, x_3 - x_4, x_1^2 - x_2 - x_3).$$

The given control system is feedback equivalent on $\mathbb{R}^5$ to

$$\begin{aligned}
\dot{\zeta} &= -\zeta \\
\dot{\mu}_1 &= -\mu_1 - \mu_2 + \mu_2 \xi + v^{\parallel,\pitchfork} \\
\dot{\mu}_2 &= -\mu_3 + \mu_1 v^{\pitchfork} \\
\dot{\mu}_3 &= -\mu_3 + v^{\parallel,\pitchfork} \\
\dot{\xi} &= v^{\pitchfork}.
\end{aligned}$$

One can verify the second condition of [70, Theorem 3.2] does not hold for S_2 since for all $x \in \mathbb{R}^n$ $\dim(T_x S_2 + G_0) \neq \dim(T_x S_2 + \mathrm{inv}(G_0))$. This implies that one cannot make the μ-subsystem linear and decoupled. Thus the class of systems which is feedback equivalent to (3.3) is strictly larger than the class of systems for which the dynamics transversal to both S_1 and S_2 can be transversally feedback linearized. $\triangle$

3.6 Control design for a nested set stabilization problem

The normal form (3.3) finds application in the stabilization of S_2 relative to S_1 locally. For, if $v^{\parallel,\pitchfork}$ is designed to stabilize $\mu = 0$ and the trajectories of the closed-loop system are bounded, then the controller locally stabilizes S_2 relative to S_1 in original coordinates. If, on the other hand, the trajectories of the closed-loop system are not bounded, then the stabilization of $\mu = 0$ implies the stabilization of S_2 relative to S_1 if and only if the necessary and sufficient conditions of [29, Theorem IV.1] hold. Similarly, the refined normal form (3.33) can simplify the problem of designing controllers to locally stabilize S_1. If $v^{\pitchfork}$ is designed such that $\xi = 0$ is asymptotically stable and the trajectories of the closed-loop system are bounded, then the controller renders $S_1 \cap U$ asymptotically stable. If, on the other hand, the trajectories of the closed-loop system are not all bounded, then the stabilization of $\xi = 0$ implies the local stabilization of S_1 under necessary and sufficient conditions of [29, Theorem IV.1]. We now present a local solution to a nested set stabilization problem using the results of Section 3.4.

Problem 5. (Local nested set stabilization problem): Given the control system (3.1), two nested sets $S_1 \supset S_2$ satisfying Assumption 3.1.1, a point $\bar{x} \in S_2$, and an open set U containing $\bar{x}$ such that $S_1 \cap U$ and $S_2 \cap U$ are controlled invariant, find, if possible, a smooth feedback control law such that the closed-loop system meets the following three specifications.

- **S1** The set $S_1 \cap U$ is asymptotically stable.
- **S2** The set $S_2 \cap U$ is asymptotically stable relative to $S_1 \cap U$.

S3 The set $S_2 \cap U$ is asymptotically stable.

▲

Notice the requirement that $S_1 \cap U$ and $S_2 \cap U$ be controlled invariant is based on the fact that invariance of a set is a necessary condition for its stability [8, Theorem 1.6.6]

3.6.1 Stabilizing S_1

When Problem 1, 3 or 4 is solvable, the submanifold $S_1 \cap U$ in transformed coordinates is $\Xi(S_1 \cap U) = \{(\zeta, \mu, \xi) : \xi = 0\}$. It may be possible to utilize normal form (3.3) or (3.17) to design a feedback law $v^{\pitchfork}(\zeta, \mu, \xi)$ that stabilizes the origin of the ξ-subsystem. However, if it happens that all the conditions of Theorem 3.5.1 hold one can solve Problem 4 which has a decoupled, linear and controllable ξ-subsystem. Thus, designing $v^{\pitchfork}$ is considerably simplified. We select the simplest controller

$$v^{\pitchfork} = K_1 \xi \tag{3.37}$$

with $K_1 \in \mathbb{R}^{\sigma \times n - s_1}$ such that $E + FK_1$ is Hurwitz. For fast convergence to S_1 one typically chooses the matrix K_1 so that the eigenvalues of $E + FK_1$ are far in the open left-half complex plane. With the above choice the origin of the ξ-subsystem in (3.33) is rendered exponentially stable and under necessary and sufficient conditions of [29, Theorem IV.1] $\xi \to 0$ if and only if $x \to S_1 \cap U$.

3.6.2 Stabilizing S_2 relative to S_1

In the normal forms of both Problems 1 and 4 the dynamics restricted to $\Xi(S_1 \cap U)$ are

$$
\begin{aligned}
\dot{\zeta} &= f_1(\zeta, \mu, 0) + g_{11}(\zeta, \mu, 0)v^{\parallel} + g_{12}(\zeta, \mu, 0)v^{\parallel,\pitchfork} \\
\dot{\mu} &= A\mu + Bv^{\parallel,\pitchfork}.
\end{aligned}
\tag{3.38}
$$

Since (A, B) is controllable, there exists a linear feedback

$$
v^{\parallel,\pitchfork} = K_2\mu \tag{3.39}
$$

with $K_2 \in \mathbb{R}^{\rho \times s_1 - s_2}$, such that $A + BK_2$ is Hurwitz; thus control law (3.39) exponentially stabilizes the origin of the μ-subsystem restricted to $\xi(S_1 \cap U)$ and under necessary and sufficient conditions of [29, Theorem IV.1], $\mu \to 0$ if and only if $x \to S_2 \cap U$

The ζ-subsystem describes the dynamics tangent to both S_1 and S_2. When restricted to S_2, control system (3.1) evolves according to (3.6). In some cases it may be possible to utilize the remaining control inputs $v^{\parallel}$ to control dynamics (3.6) to accomplish application specific specifications such as boundedness or tracking.

3.6.3 Stability analysis

When Problem 4 is solvable the following theorem presents sufficient conditions under which Problem 5 is solvable. Given a continuous signal u let $\|u\|_\infty := \sup_{t \geq 0} \|u(t)\|$ where the norm on the left is a function norm and the norm on the right is a vector norm.

Theorem 3.6.1. *Assume Problem 4 is solvable at $\bar{x} \in S_2$. If* *(a)* *The feedback laws* $v^{\pitchfork} = K_1\xi$ *and* $v^{\|,\pitchfork} = K_2\mu$ *are such that* $(E + FK_1)$ *and* $(A + BK_2)$ *are Hurwitz*

(b) *The control signal* $v^{\|}$ *is such that*

(i) $(\forall x(0) \in U)(\forall\, t \geq 0)\ x(t) \in U$.

(ii) $(\forall x(0) \in U)(\exists M > 0)\|(\zeta, \mu, \xi)\|_\infty < M$.

Then Problem 5 is solvable.

Proof. Since Problem 4 is solvable there exists a neighbourhood U, a coordinate transformation $\Xi \in \mathrm{Diff}(U)$, and a feedback transformation (α, β) such that (3.1) is locally feedback equivalent to (3.33). By (a) the closed-loop system is given by

$$\begin{aligned}
\dot{\zeta} &= \tilde{f}_1(\zeta, \mu, \xi) + g_{11}(\zeta, \mu, \xi)v^{\|} \\
\dot{\mu} &= (A + BK_2)\mu + \tilde{f}_2(\zeta, \mu, \xi) + g_{21}(\zeta, \mu, \xi)v^{\|} \\
\dot{\xi} &= (E + FK_1)\xi
\end{aligned} \tag{3.40}$$

where $\tilde{f}_1(\zeta, \mu, \xi) := f_1(\zeta, \mu, \xi) + g_{12}(\zeta, \mu, \xi)K_2\mu + g_{13}(\zeta, \mu, \xi)K_1\xi$ and $\tilde{f}_2 = f_2(\zeta, \mu, \xi) + g_{22}(\zeta, \mu, \xi)K_2\mu + g_{23}(\zeta, \mu, \xi)K_1\xi$. By conditions (a) and (b), if $x(0) \in U$, then (3.1) is feedback equivalent to (3.40) for all $t \geq 0$ and (ζ, μ, ξ) is bounded.

Let $V_\zeta \times V_\mu \times V_\xi := \Xi(U)$. The ξ-subsystem is decoupled from the other subsystems and $\xi = 0$ is exponentially stable by (a). Therefore, by (b), for any $x(0) \in U$, $\xi \to 0$. In particular

1. $(\forall \epsilon_1 > 0)(\exists \delta_1 > 0)(\forall \xi(0) \in B_{\delta_1}(0)(\forall t \geq 0)\ \xi(t) \in B_{\epsilon_1}(0)$.

2. $\xi(t) \to 0$ as $t \to \infty$ at an exponential rate.

By continuity $\Xi^{-1}(V_\zeta \times V_\mu \times B_{\epsilon_1}) =: \mathcal{N}_1(S_1 \cap U)$ and $\Xi^{-1}(V_\zeta \times V_\mu \times B_{\delta_1}) =: \mathcal{N}_2(S_1 \cap U)$ are neighbourhoods of $S_1 \cap U$. Thus, we conclude that

$$(\forall x(0) \in \mathcal{N}_2(S_1 \cap U))(\forall t \geq 0)\ x(t) \in \mathcal{N}_1(S_1 \cap U)$$

which means that $S_1 \cap U$ is stable. Moreover, since $\xi(t)$ is bounded, $\xi \to 0$ if and only if $x \to S_1 \cap U$ which implies that $S_1 \cap U$ is also attractive for all $x_0 \in U$. Thus, $S_1 \cap U$ is asymptotically stable and **S1** of Problem 5 holds.

When $\xi = 0$, the μ-subsystem becomes $\dot{\mu} = (A + BK_2)\mu$ and $\mu = 0$ is exponentially stable, by (a), for all initial conditions in $V_\zeta \times V_\mu \times \{0\}$. Thus, when $\xi = 0$

1. $(\forall \epsilon_2 > 0)(\exists \delta_2 > 0)(\forall \mu(0) \in B_{\delta_2}(0))(\forall t \geq 0)\ \mu(t) \in B_{\epsilon_2}(0)$.
2. $\mu(t) \to 0$ as $t \to \infty$ at an exponential rate.

By continuity of Ξ restricted to $S_1 \cap U$ the sets $\Xi^{-1}(V_\zeta \times B_{\epsilon_2} \times \{0\}) =: \mathcal{N}_3(S_2 \cap U)$, $\Xi^{-1}(V_\zeta \times B_{\delta_2} \times \{0\}) =: \mathcal{N}_4(S_2 \cap U)$ are open neighbourhoods of $S_2 \cap U$ in the topology of $S_1 \cap U$. We emphasize that $\mathcal{N}_3, \mathcal{N}_4 \subseteq S_1 \cap U$. These considerations yield

$$(\forall x(0) \in \mathcal{N}_4(S_2 \cap U))(\forall t \geq 0)\ x(t) \in \mathcal{N}_3(S_2 \cap U)$$

which means that $S_2 \cap U$ is stable relative to $S_1 \cap U$. Moreover, since $\mu(t)$ is bounded, $\mu \to 0$ if and only if $x|_{S_1 \cap U} \to S_2 \cap U$ which implies that $S_2 \cap U$ is attractive relative to $S_1 \cap U$. Thus $S_2 \cap U$ is asymptotically stable relative to $S_1 \cap U$ and **S2** of Problem 5 holds.

Since : (1) $\xi = 0$ is exponentially stable, (2) when $\xi = 0$, $\mu = 0$ is exponentially stable and (3) by hypothesis (b) $(\mu(t), \xi(t))$ is bounded all the conditions of [46, Corollary 10.3.3] hold. Therefore $(\mu, \xi) = (0, 0)$ is asymptotically stable. That is

1. $(\forall \epsilon_3 > 0)(\exists \delta_3 > 0)(\forall(\mu(0), \xi(0)) \in B_{\delta_3}(0))(\forall t \geq 0)\ (\mu(t), \xi(t)) \in B_{\epsilon_3}(0)$.

2. $(\mu(t), \xi(t)) \to 0$ as $t \to \infty$.

By continuity $\Xi^{-1}(V_\zeta \times B_{\epsilon_3}) =: \mathcal{N}_5(S_2 \cap U)$ and $\Xi^{-1}(V_\zeta \times B_{\delta_3}) =: \mathcal{N}_6(S_2 \cap U)$ are neighbourhoods of $S_2 \cap U$ in $U \subseteq \mathbb{R}^n$. Therefore

$$(\forall x(0) \in \mathcal{N}_6(S_2 \cap U))(\forall t \geq 0)\ x(t) \in \mathcal{N}_5(S_2 \cap U)$$

which means that $S_2 \cap U$ is stable. Moreover, since $(\mu(t), \xi(t))$ is bounded we have that $(\mu, \xi) \to 0$ if and only if $x \to S_2 \cap U$ which implies that $S_2 \cap U$ is also attractive for all $x_0 \in U$. Thus, $S_2 \cap U$ is asymptotically stable and **S3** of Problem 5 holds. □

Remark 3.6.2. The rate of convergence to $\xi = 0$ and $\mu = 0$ is proportional to the rate at which the trajectories of the closed-loop system approach sets S_1 and S_2, respectively.

Chapter 4

Coordinated path following of dynamic unicycles

In this chapter we consider the coordinated path following problem for N unicycle mobile robots, and solve it as a nested set stabilization problem. Experimental results are also presented. Portions of this chapter have appeared in [18] and have been accepted for publication in [19].

4.1 Introduction

Following Chapter 2 the coordinated path following problem is formulated as a nested set stabilization problem. Given arbitrary paths for unicycles we characterize the path following manifold of each agent. We show that each unicycle is feedback equivalent, in a neighbourhood of its path following manifold, to a system whose transversal and tangential dynamics to the path following manifold are both double integrators. While our results are local, valid in a neighbourhood of each unicycle's path, it is possible to extend the region of attraction of the proposed controllers using switched controllers [89, 63]

As discussed in Chapter 2, we model the coordination task as an embedded submanifold of the multi-agent path following manifold. In the case where the communication graph is complete we present sufficient conditions under which coordination specifications are feasible in the sense of Definition 2.2.3. For arbitrary coordination tasks we utilize feedback linearization to stabilize the nested set in a centralized manner. In the special case in which coordination entails making the unicycles maintain a formation along their paths, we propose semi-distributed control law under less restrictive communication assumptions.

When all of the unicycles are assigned simple closed curves, our coordination problem becomes closely related to the problem of oscillator synchronization [22]. In this case each unicycle's path following manifold is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$. An oscillator can be modeled as a double-integrator with state space $\mathbb{S}^1 \times \mathbb{R}$. In this case coordinating the unicycle's velocities can be viewed as frequency synchronization of oscillators [59], [23]. When all the unicycles must have the same position along their paths, coordination can be viewed as a phase synchronization problem [79]. When all the unicycles are asked to maximally spread themselves along their closed-paths, the coordination task can be viewed as phase balancing [83]. In these application scenarios, existing distributed control laws can be employed in conjunction with our path following control laws in a modular manner.

Similarly, when all unicycles are assigned non-closed paths, coordination is closely related to the consensus problem for double integrator dynamics. In this case each unicycle's path following manifold is diffeomorphic to $\mathbb{R}^2$. Consequently, in this special case, our approach allows one to use control laws from the literature that achieve consensus for double-integrators for our coordination tasks. For example, when coordination entails reaching consensus along paths, the results in [75] provide minimal connectedness conditions on the communication graph. When coordination involves reaching a common velocity along the paths the results in [12] provide control laws respecting a switching communication graph.

4.2 The multi-agent system of dynamic unicycles

We consider a multi-agent system consisting of N unicycles. Following [25], the model of unicycle i, $i \in \{1, \cdots, N\}$, is

$$\begin{aligned}\dot{x}_i &= v_i \cos(\theta_i)\\ \dot{y}_i &= v_i \sin(\theta_i)\\ \dot{\theta}_i &= u_{i,2}\\ \dot{v}_i &= u_{i,1}\end{aligned} \tag{4.1}$$

where (x_i, y_i) denotes the position of the unicycle in the plane, θ_i is the heading angle, and v_i is the forward velocity of the unicycle. The control inputs $u_{i,1}$ and $u_{i,2}$ are, respectively, the forward acceleration and angular velocity. Let $\mathbf{x}_i \coloneqq (x_i, y_i, \theta_i, v_i) \in \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R}$. Let $\tau(\theta_i) \coloneqq (\cos(\theta_i), \sin(\theta_i))$ denote the unicycle's heading. We take the position of the unicycle i as its output $\mathbf{y}_i = h_i(\mathbf{x}_i) \coloneqq (x_i, y_i)$. The state of the multi-agent system is $\mathbf{x} \coloneqq (\mathbf{x}_1, \ldots, \mathbf{x}_N) \in (\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R})^N$. Unless otherwise stated, we make the following simplifying assumption throughout this chapter.

Assumption 4.2.1. The communication graph $\mathscr{G}$ of the multi-agent unicycle system is complete. $\triangleleft$

Assumption 4.2.1 is unnecessarily restrictive for implementing the proposed control laws. Characterizing the minimal communication requirements needed for implementation is an open problem.

4.3 The multi-agent path following manifold

4.3.1 Characterization of the multi-agent path following manifold

Each unicycle is assigned a path $\gamma_i \subset \mathbb{R}^2$ in its output space. The parameterization of path i is $\sigma_i : \mathbb{R} \to \mathbb{R}^2$ with $\gamma_i = \sigma_i(\mathbb{R})$. If path i is closed then the domain of σ_i is $\mathbb{R} \bmod L_i$ where $L_i > 0$ is the length of the curve. Let $\varphi_i : \mathbb{R} \to \mathbb{S}^1$ be the map associating to each $\lambda \in \mathbb{R}$ the angle of the tangent vector $\sigma_i'(\lambda)$ to γ_i at $\sigma_i(\lambda)$. As discussed in Section 1.2.1 we assume that each path satisfies Assumption 1.2.1. Let

$$\Gamma_i \coloneqq \{\mathbf{x}_i \in \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R} \ : \alpha_i(\mathbf{x}_i) \coloneqq s_i \circ h_i(\mathbf{x}_i) = 0\}$$

and $\Gamma \coloneqq \Gamma_1 \times \cdots \times \Gamma_N$. The largest-controlled invariant subset of Γ_i is [69]

$$\Gamma_i^\star = \{\mathbf{x}_i : \alpha_i(\mathbf{x}_i) = \langle \mathrm{d}s_i(h(\mathbf{x}_i)),\ \tau(\theta_i)\rangle = 0\}\,. \tag{4.2}$$

This set has dimension $n_i^\star = 2$. By Definition 2.1.5, the multi-agent path following manifold is the product of each agent's individual path following manifold.

Proposition 4.3.1. *For $i \in \{1, \cdots, N\}$ the set $\Gamma_i^\star \backslash \{v_i = 0\}$, where $\Gamma_i^\star$ is given by (4.2), consists of four disconnected components.*

Proof. Fix $i \in \{1, \cdots, N\}$ and consider the set $\Gamma_{i,\bullet}^{\star,\bullet} \coloneqq \Gamma_{i,+}^{\star,\mathrm{f}} \cup \Gamma_{i,+}^{\star,\mathrm{r}} \cup \Gamma_{i,-}^{\star,\mathrm{f}} \cup \Gamma_{i,-}^{\star,\mathrm{r}}$ where

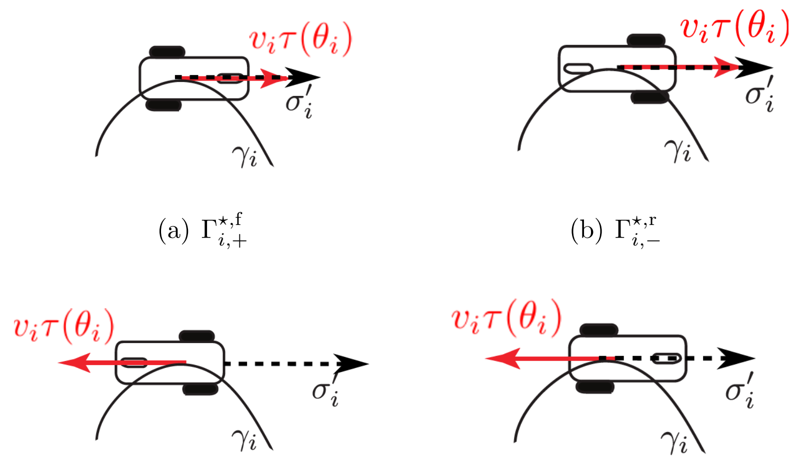
$$\begin{aligned}
\Gamma_{i,+}^{\star,\mathrm{f}} &\coloneqq \{\mathbf{x}_i \in \Gamma_i^\star : v_i > 0, \langle \sigma_i'(\lambda),\ \tau(\theta_i)\rangle = 1, \lambda \in \mathbb{R}\} \\
\Gamma_{i,+}^{\star,\mathrm{r}} &\coloneqq \{\mathbf{x}_i \in \Gamma_i^\star : v_i < 0, \langle \sigma_i'(\lambda),\ \tau(\theta_i)\rangle = 1, \lambda \in \mathbb{R}\} \\
\Gamma_{i,-}^{\star,\mathrm{f}} &\coloneqq \{\mathbf{x}_i \in \Gamma_i^\star : v_i > 0, \langle \sigma_i'(\lambda),\ \tau(\theta_i)\rangle = -1, \lambda \in \mathbb{R}\} \\
\Gamma_{i,-}^{\star,\mathrm{r}} &\coloneqq \{\mathbf{x}_i \in \Gamma_i^\star : v_i < 0, \langle \sigma_i'(\lambda),\ \tau(\theta_i)\rangle = -1, \lambda \in \mathbb{R}\}\,.
\end{aligned} \tag{4.3}$$

We first show that $\Gamma_{i,\bullet}^{\star,\bullet}$ has four disconnected components, namely the sets (4.3). The sets $\Gamma_{i,+}^{\star,\mathrm{f}} \cup \Gamma_{i,-}^{\star,\mathrm{f}}$ and $\Gamma_{i,+}^{\star,\mathrm{r}} \cup \Gamma_{i,-}^{\star,\mathrm{r}}$ are disjoint because any path in the state space connecting these sets must pass through a point at which $v_i = 0$.

Next assume, without loss of generality, that $\bar{\mathbf{x}}_i \in \Gamma_{i,+}^{\star,\mathrm{f}}$. To connect $\bar{\mathbf{x}}_i$ to another point $\bar{\bar{\mathbf{x}}}_i \in \Gamma_{i,-}^{\star,\mathrm{f}}$ it has to pass through a point corresponding to $\langle \sigma_i'(\lambda),\ \tau(\theta_i)\rangle = 0$. This shows that $\Gamma_{i,+}^{\star,\mathrm{f}}$ and $\Gamma_{i,-}^{\star,\mathrm{f}}$ are not path connected. A similar argument holds for $\Gamma_{i,+}^{\star,\mathrm{r}}$ and $P_{i,-}^{\star,\mathrm{r}}$. Together, these facts show that $\Gamma_{i,\bullet}^{\star,\bullet}$ is not path connected. By, Proposition [62, Proposition 1.8] a topological manifold is connected if and only it is path connected. Thus, $\Gamma_{i,\bullet}^{\star,\bullet}$ is not connected.

Lastly, we show that $\Gamma_{i,\bullet}^{\star,\bullet} = \Gamma_i^\star \backslash \{v_i = 0\}$. By definition we have that $\Gamma_{i,\bullet}^{\star,\bullet} \subseteq \Gamma_i^\star$. Conversely, let $\mathbf{x}_i = (x_i, y_i, \theta_i, v_i) \in \Gamma_i^\star$ with $v_i \neq 0$. Since $\Gamma_i^\star$ is an invariant set contained in Γ_i, the unicycle's heading must be tangent to the path for, otherwise, it would leave the path for some time and hence leave the set Γ_i. This implies that $|\langle \sigma_i'(\lambda),\ \tau(\theta_i)\rangle| = 1$ where $\lambda \in \mathbb{R}$ satisfies $\mathbf{y}_i = \sigma_i(\lambda)$. This shows that $\Gamma_i^\star \subseteq \Gamma_{i,\bullet}^{\star,\bullet}$. □

The notation in (4.3) is evocative of the physical interpretation of these sets. The superscript f stands for forward direction, the superscript r stands for reverse direction, the subscript + indicates the unicycle is moving in the same direction as curve's orientation, and subscript $-$ indicates the unicycle is moving opposite to the curve's orientation. In Figure 4.1 the four different types of motion corresponding to the four components of $\Gamma_i^\star \backslash \{v_i = 0\}$ are illustrated.



(a) $\Gamma_{i,+}^{\star,\mathrm{f}}$ (b) $\Gamma_{i,-}^{\star,\mathrm{r}}$

(c) $\Gamma_{i,-}^{\star,\mathrm{f}}$ (d) $\Gamma_{i,+}^{\star,\mathrm{r}}$

Figure 4.1: The motion of unicycle i restricted to the four components of $\Gamma_i^\star \backslash \{v_i = 0\}$.

4.3.2 Unicycle normal form

We transform the model of unicycle i to a convenient normal form. Among other useful properties, the normal form suggests local coordinates on $\Gamma_i^\star$ that simplify finding the coordination set. It also facilitates the design of decentralized control laws to stabilize $\Gamma^\star$.

Inspired by [14], we introduce a projection in the output space of the unicycle that associates to each point $\mathbf{y}_i$ sufficiently close to the path γ_i a number in $\mathbb{R}$. Let

$$\begin{aligned} \varpi_i \ : \ & \mathcal{N}(\gamma_i) \to \mathbb{R} \\ & \mathbf{y}_i \mapsto \arg\inf_{\lambda\in\mathbb{R}} \|\mathbf{y}_i - \sigma_i(\lambda)\| \end{aligned} \tag{4.4}$$

where $\mathcal{N}(\gamma_i)$ is a neighbourhood of γ_i. The open set $\mathcal{N}(\gamma_i)$ is such that, for all $\mathbf{y}_i \in \mathcal{N}(\gamma_i)$, there exists a unique $\mathbf{y}_i{}^\star \in \gamma_i$ closest to $\mathbf{y}_i$, so ϖ_i is well-defined. The higher the curvature of the path γ_i, the smaller the domain $\mathcal{N}(\gamma_i)$ of (4.4). Using (4.4) define $\pi_i(\mathbf{x}_i) \coloneqq \varpi_i \circ h_i(\mathbf{x}_i)$. Finally, define the path following output function

$$\hat{\mathbf{y}}_i \coloneqq (\pi_i(\mathbf{x}_i), \alpha_i(\mathbf{x}_i)) \,. \tag{4.5}$$

Lemma 4.3.2. *The unicycle* (4.1) *with output* (4.5) *yields a well-defined vector relative degree of* $\{2, 2\}$ *at each* $\mathbf{x}_i \in \Gamma_i^\star \backslash \{v_i = 0\}$.

The proof of Lemma 4.3.2 is omitted because it is similar to [3, Lemma 4.1]. The next lemma defines a coordinate transformation valid in a neighbourhood of each component (4.3) of $\Gamma_i^\star \backslash \{v_i = 0\}$. The lemma explicitly addresses the component $\Gamma_{i,+}^{\star,\mathrm{f}}$ but a similar result can be obtained for the remaining three components of $\Gamma_i^\star$.

Lemma 4.3.3. *There exists an open set* $U_{i,+}^{\mathrm{f}} \subseteq \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R}$, *with* $\Gamma_{i,+}^{\star,\mathrm{f}} \subset U_{i,+}^{\mathrm{f}}$ *such that* $T_i : U_{i,+}^{\mathrm{f}} \to T_i(U_{i,+}^{\mathrm{f}})$, $\mathbf{x}_i \mapsto (\eta_{i,1}, \eta_{i,2}, \xi_{i,1}, \xi_{i,2}) = (\pi_i(\mathbf{x}_i), L_f\pi_i(\mathbf{x}_i), \alpha_i(\mathbf{x}_i), L_f\alpha_i(\mathbf{x}_i))$ *is a diffeomorphism onto its image.*

Proof. The generalized inverse function theorem [41, p.56] is employed to prove this result. We must show that

1. for all $\mathbf{x}_i \in \Gamma_{i,+}^{\star,\mathrm{f}}$, $\mathrm{d}T_i(\mathbf{x}_i)$ is an isomorphism

2. $T_i|_{\Gamma_{i,+}^{\star,\mathrm{f}}}$ is a diffeomorphism.

To show that (1) holds, observe that $\det(\mathrm{d}T_i)|_{\Gamma_{i,+}^{\star,\mathrm{f}}} = -v_i(\partial_{x_i}\pi_i\partial_{y_i}\alpha_i - \partial_{x_i}\alpha_i\partial_{y_i}\pi_i)^2$. On $\Gamma_{i,+}^{\star,\mathrm{f}}$, $v_i \neq 0$. Using arguments analogous to those in [14, Lemma 3.2], $\mathrm{d}\alpha_i = [\ \partial_{x_i}\alpha_i \quad \partial_{y_i}\alpha_i \quad 0 \quad 0\ ]$ and $\mathrm{d}\pi_i = [\ \partial_{x_i}\pi_i \quad \partial_{y_i}\pi_i \quad 0 \quad 0\ ]$ are orthogonal on $\Gamma_{i,+}^{\star,\mathrm{f}}$ and so $(\partial_{x_i}\pi_i\partial_{y_i}\alpha_i - \partial_{x_i}\alpha_i\partial_{y_i}\pi_i) \neq 0$.

To show that (2) holds, note that the restriction of T_i to $\Gamma_{i,+}^{\star,\mathrm{f}}$ is given by $(\eta_{i,1}, \eta_{i,2}, \xi_{i,1}, \xi_{i,2}) = (\pi_i(\mathbf{x}_i), L_f\pi_i(\mathbf{x}_i), 0, 0)$. For $\mathbf{x}_i \in \Gamma_{i,+}^{\star,\mathrm{f}}$ we have that $\mathbf{y}_i = h(\mathbf{x}_i) \in \gamma_i$ and so by the definition of $\eta_{i,1} = \pi_i(\mathbf{x}_i)$, $(x_i, y_i) = \sigma_i(\eta_{i,1})$. The vector $\sigma'(\eta_{i,1})$ is tangent to the curve γ_i at $\sigma_i(\eta_{i,1})$. On the component $\Gamma_{i,+}^{\star,\mathrm{f}}$ of $\Gamma_i^\star$, $\tau(\theta_i) = \sigma'(\eta_{i,1})$ and therefore $\theta_i = \varphi_i(\eta_{i,1})$.

We are left to find an expression for v_i. Direct calculations using the expression $\eta_{i,2} = L_f\pi_i(\mathbf{x}_i)$ yield $v_i = \eta_{i,2}/(\partial_{x_i}\pi_i \cos\theta_i + \partial_{y_i}\pi_i \sin\theta_i)$.

By [14, Lemma 3.2], for all $\eta_{i,1} \in \mathbb{R}$, $\mathrm{d}\varpi_i(\sigma_i(\eta_{i,1})) = \sigma_i'(\eta_{i,1})$. Therefore, for $\mathbf{x}_i \in \Gamma_{i,+}^{\star,\mathrm{f}}$, $\mathrm{d}\pi_i(\mathbf{x}_i) = \begin{bmatrix} \sigma_i'(\eta_{i,1})^\top & 0 & 0 \end{bmatrix}$ and we can write $v_i = \frac{\eta_{i,2}}{\langle \sigma_i'(\eta_{i,1}),\ \tau(\theta_i)\rangle}$. On $\Gamma_{i,+}^{\star,\mathrm{f}}$, $\langle \sigma'(\eta_{i,1}),\ \tau(\theta_i)\rangle = 1$ so $v_i = \eta_{i,2}$. In summary, we have derived the inverse map $T_i^{-1}|_{\Gamma_{i,+}^{\star,\mathrm{f}}} = (\sigma_i(\eta_{i,1}), \varphi_i(\eta_{i,1}), \eta_{i,2})$ which shows that $T_i|_{\Gamma_{i,+}^{\star,\mathrm{f}}}$ is a diffeomorphism onto its image. □

Consider the regular feedback transformation

$$\begin{bmatrix} u_{i,1} \\ u_{i,2} \end{bmatrix} = \frac{1}{v_i^2} \begin{bmatrix} L_{g_{i,2}} L_f \alpha_i & -L_{g_{i,2}} L_f \pi_i \\ -L_{g_{i,1}} L_f \alpha_i & L_{g_{i,1}} L_f \pi_i \end{bmatrix} \begin{bmatrix} -L_f^2 \alpha + v_i^{\parallel} \\ -L_f^2 \pi + v_i^{\pitchfork} \end{bmatrix} \tag{4.6}$$

where $\left(v_i^{\parallel}, v_i^{\pitchfork}\right) \in \mathbb{R} \times \mathbb{R}$ are auxiliary control inputs. The elements in (4.6) can be readily computed [4, Section V]. By Lemma 4.3.2 this controller is well defined in a neighbourhood of $\mathcal{P}_i^\star \setminus \{v_i = 0\}$. Using the diffeomorphism T_i in Lemma 4.3.3, and feedback transformation (4.6), the dynamic unicycle (4.1) is feedback equivalent, in a neighbourhood of each component of $\mathcal{P}_i^\star \setminus \{v_i = 0\}$, to

$$\begin{aligned} \dot{\eta}_{i,1} &= \eta_{i,2} \\ \dot{\eta}_{i,2} &= v_i^{\parallel} \end{aligned} \tag{4.7a}$$

$$\begin{aligned} \dot{\xi}_{i,1} &= \xi_{i,2} \\ \dot{\xi}_{i,2} &= v_i^{\pitchfork}. \end{aligned} \tag{4.7b}$$

Remark 4.3.4. We stress that unicycle i is not globally feedback equivalent to (4.7). Furthermore, the equivalence does not hold when its translational velocity equals zero $v_i = 0$. The latter obstacle can be overcome using the switching scheme in [63].

We call the subsystem (4.7b) the transversal dynamics of unicycle i to each component of $\Gamma_i^\star$. This is because making each component of $\Gamma_i^\star$ attractive is equivalent, under Assumption 1.2.1 and in particular equation (1.4), to stabilizing the origin of (4.7b). The subsystem (4.7a) is called tangential dynamics of unicycle i with respect to each component of $\Gamma_i^\star$. The $\eta_{i,1}$ and $\eta_{i,2}$ states convey a strong physical meaning for coordinated path following. The state $\eta_{i,1}$ represents the position of unicycle i along the path and the state $\eta_{i,2} \in \mathbb{R}$ represents its velocity along the path. Let $\boldsymbol{\xi} := (\xi_{1,1}, \ldots, \xi_{N,1}, \xi_{1,2}, \ldots, \xi_{N,2})$ denote the transversal states of the entire multi-agent unicycle system and $\mathbf{v}^\pitchfork := (v_1^\pitchfork, \ldots, v_N^\pitchfork)$. Let $\boldsymbol{\eta}_1 := (\eta_{1,1}, \ldots, \eta_{N,1})$, $\boldsymbol{\eta}_2 := (\eta_{1,2}, \ldots, \eta_{N,2})$, $\boldsymbol{\eta} := (\boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$, and $\mathbf{v}^\parallel := (v_1^\parallel, \ldots, v_N^\parallel)$ then the overall dynamics of the multi-agent system can be compactly written as

$$\dot{\boldsymbol{\eta}} = A^\parallel \boldsymbol{\eta} + B^\parallel \mathbf{v}^\parallel \tag{4.8a}$$

$$\dot{\boldsymbol{\xi}} = A^\pitchfork \boldsymbol{\xi} + B^\pitchfork \mathbf{v}^\pitchfork \tag{4.8b}$$

where $(A^\pitchfork, B^\pitchfork)$, $(A^\parallel, B^\parallel)$ are controllable. The dynamics of the multi-agent system restricted to $\Gamma^\star$ are given by (4.8a) and these dynamics play a key role in achieving coordination.

Remark 4.3.5. The functions $\alpha_1(x), \cdots, \alpha_N(x)$ satisfy the conditions of [70, Theorem 3.1]. Thus, the multi-agent system dynamics transversal to the multi-agent path following manifold can be feedback linearized as in (4.8b). In turn, it implies that conditions (b) and (c) of Theorem 3.5.1 hold for the multi-agent system of unicycles.

4.3.3 Topology of multi-agent path following manifold

The tangential dynamics (4.7a) evolve on the set (4.2). When the curve γ_i is non-closed $\eta_{i,1} \in \mathbb{R}$ and $\eta_{i,2} \in \mathbb{R}$; thus each component of $\Gamma_i^\star \setminus \{v_i = 0\}$ is diffeomorphic to $\mathbb{R} \times \mathbb{R}$. When the path γ_i is closed then, by Assumption 1.2.1, it is a Jordan curve. In this case $\eta_{i,1} \in \mathbb{R} \bmod L_i \simeq \mathbb{S}^1$ and $\eta_{i,2} \in \mathbb{R}$; thus each component of $\Gamma_i^\star \setminus \{v_i = 0\}$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$.

Assume, without loss of generality, that γ_i is closed for $i \in \{1, \cdots, r\}$, $r \leq N$ and non-closed for $i \in \{r+1, \ldots, N\}$. Then each component of the multi-agent path following manifold is diffeomorphic to $\mathbb{T}^r \times \mathbb{R}^{N-r} \times \mathbb{R}^N$ where $\mathbb{T}^r$ is the r-torus. This shows that $\Gamma^\star$ is unbounded even if every curve is closed.

As shown in the proof of Lemma 4.3.3, the tangential states in (4.7a) represent local coordinates on each component of $\Gamma_i^\star \setminus \{v_i = 0\}$. When $\gamma_i \simeq \mathbb{R}$ then $(\Gamma_{i,+}^{\star,\mathrm{f}}, \psi_i)$ with $\psi_i := (\pi_i|_{\Gamma_i^\star}, L_f \pi_i|_{\Gamma_i^\star})$ is a global coordinate chart, i.e., a single chart that covers the entire set $\Gamma_{i,+}^{\star,\mathrm{f}}$. When γ_i is closed each component of $\Gamma_i^\star \setminus \{v_i = 0\}$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$ and cannot be covered with a single chart. Instead the coordinate chart (U_i, ψ_i) with $U_i := (\mathbb{R} \bmod L_i) \setminus \{0\} \times \mathbb{R} \subset \Gamma_{i,+}^{\star,\mathrm{f}}$ covers "most" of $\Gamma_{i,+}^{\star,\mathrm{f}}$. One could define another chart to cover the omitted region, but this complication is not needed [10].

Since $\Gamma^\star$ is a product manifold, (U, ψ) with $U := U_1 \times \cdots \times U_N$ and $\psi := \psi_1 \times \cdots \times \psi_N$ is a coordinate chart for $\Gamma^\star$. When all the paths are non-closed it covers the entire set $\Gamma^\star$, otherwise it covers "most" of $\Gamma^\star$.

4.4 Feasible coordination constraints

Let $\beta : \Gamma^\star \to \mathbb{R}^c$ be a coordination function, see Definition 2.2.1. As discussed in Section 4.3.3 each component of $\Gamma^\star$ is diffeomorphic to $\mathbb{T}^r \times \mathbb{R}^{N-r} \times \mathbb{R}^N$ and the coordinate chart (U, ψ) covers "most" of it. In this section we work with a local representation of the coordination function, $\hat{\beta} : \psi(U) \to \mathbb{R}^c$, defined by

$$\begin{tikzcd} U \subseteq \Gamma^\star \arrow[r, "\beta"] \arrow[d, "\psi"'] & \mathbb{R}^c \\ \psi(U) \subseteq \mathbb{R}^{2N} \arrow[ur, dashed, "\hat{\beta}"'] & \end{tikzcd}.$$

Remark 4.4.1. In order to define a coordination function globally, and avoid the use of charts on $\Gamma^\star$, one uniquely identifies smooth functions on the r-torus with smooth periodic functions on $\mathbb{R}^r$. In light of the discussion in Section 4.3.3 we can then treat $\hat{\beta}$ as a map $\mathbb{R}^{2N} \to \mathbb{R}^c$, L_i-periodic in its first r arguments [7].

For $\boldsymbol{\eta} \in \psi(U)$ the associated local coordination set introduced in Definition 2.2.2 is the largest controlled-invariant subset of

$$\mathcal{C} = \left\{ \boldsymbol{\eta} \in \psi(U) : \hat{\beta}(\boldsymbol{\eta}) = 0 \right\} \tag{4.9}$$

containing $\boldsymbol{\eta}$. A coordination specification is feasible if its corresponding coordination set is non-empty. We present sufficient conditions for both linear and nonlinear coordination specifications to be feasible under Assumption 4.2.1, i.e, all-to-all communication.

4.4.1 Linear-affine coordination

Consider a linear-affine coordination function

$$\mathcal{C} = \left\{ \boldsymbol{\eta} \in \psi(U) : \hat{\beta}(\boldsymbol{\eta}) = A\boldsymbol{\eta} + b = 0 \right\} \tag{4.10}$$

with $A \in \mathbb{R}^{c \times 2N}$, $\operatorname{rank}(A) = c$, and $b \in \operatorname{Im} A$. We write $A = [\ A_1 \quad A_2\ ]$ and $b = (b_1, b_2)$ in accordance with the partition $\boldsymbol{\eta} = (\boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$.

Proposition 4.4.2. *Consider the set (4.10) and let R be a full rank matrix satisfying $RA_2 = 0$. If*

1. $(\forall\, \boldsymbol{\eta}_2 \in \mathcal{C})\ RA_1\boldsymbol{\eta}_2 = 0$, *or*

2. $$\operatorname{Im} \begin{bmatrix} b \\ 0 \end{bmatrix} \in \operatorname{Im} \begin{bmatrix} A_1 & A_2 \\ 0 & RA_1 \end{bmatrix} \tag{4.11}$$

then, for each $\boldsymbol{\eta} \in \mathcal{C}$, the set $\mathcal{C}^\star$ is non-empty.

Proof. By the definition of (4.10), $b \in \operatorname{Im} A$, therefore the set $\mathcal{C}$ is non-empty. For the set $\mathcal{C}$ itself to be controlled invariant there must exist a control law $\mathbf{v}^\parallel$ such that the derivative of $A\boldsymbol{\eta} + b$ is identically zero. Taking the derivative of $\hat{\beta}$ along solutions of the system (4.8a) we obtain

$$A_1\boldsymbol{\eta}_2 + A_2\mathbf{v}^\parallel|_{\mathcal{C}} = 0. \tag{4.12}$$

Left multiply equation (4.12) by R to obtain that the equation (4.12) is solvable in $\mathbf{v}^\parallel$ if and only if $RA_1\boldsymbol{\eta}_2|_{\mathcal{C}} = 0$. In this case $\mathcal{C}^\star = \mathcal{C}$ and is non-empty.

If, on the other hand, there exists $\boldsymbol{\eta}_2 \in \mathcal{C}$ for which $RA_1\boldsymbol{\eta}_2 \neq 0$, then (4.12) is not solvable using $\mathbf{v}^{\|}$. In this case we add the constraint $RA_1\boldsymbol{\eta}_2 = 0$ to $\mathcal{C}$ and obtain a new set

$$\mathcal{C}_1 = \{\boldsymbol{\eta} \in \mathcal{C} : RA_1\boldsymbol{\eta}_2 = 0\}.$$

In order for the set $\mathcal{C}_1$ to be controlled invariant there must exist a control law $\mathbf{v}^{\|}$ such that the derivative of $RA_1\boldsymbol{\eta}_2$ along solutions of system (4.8a) is identically zero. Setting the derivative equal to zero we obtain

$$RA_1\mathbf{v}^{\|}|_{\mathcal{C}_1} = 0.$$

Any feedback control law that satisfies, for all $\boldsymbol{\eta} \in \mathcal{C}_1$, $\mathbf{v}^{\|}(\boldsymbol{\eta}) \in \operatorname{Ker}(RA_1)$, solves this equation, so the coordination set, $\mathcal{C}^\star$, equals $\mathcal{C}_1$. Condition (4.11) guarantees that $\mathcal{C}_1$ is non-empty. □

Example 4.4.3. Consider $N = 3$ unicycles. They are required to move along non-closed paths with constant velocity $v_d > 0$. That is $\beta_i = \eta_{i,2} - v_d$ for $i \in \{1, \ldots, 3\}$. Moreover, they are required to obtain a formation which is defined based on the relative distance between unicycles. For instance, consider the case when three parallel paths are given and the three unicycles are required to get on the paths and form a triangle as depicted in Figure 4.2. To accomplish this formation the $\eta_{i,1}$ for $i \in \{1, 2, 3\}$ need to satisfy

$$\begin{aligned} \eta_{1,1} - \eta_{2,1} &= b \\ \eta_{2,1} - \eta_{3,1} &= -b \end{aligned} \tag{4.13}$$

where $b \geq 0$. The above coordinated path following problem can be expressed in terms of

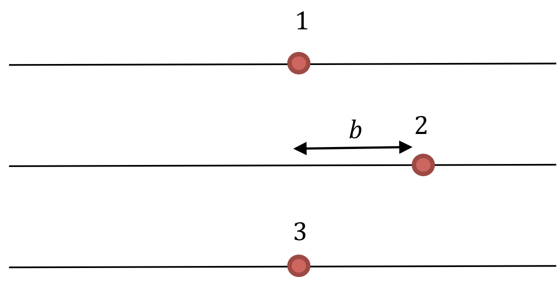


Figure 4.2: Triangle formation along parallel straight lines paths for three unicycles.

the matrices A_1, A_2, and b

$$A_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} b \\ -b \\ v_d \\ v_d \\ v_d \end{bmatrix}$$

Since $\boldsymbol{\eta}_2 = (v_d, v_d, v_d)$ and $R = \begin{bmatrix} \mathbf{1}_{2\times 2}, & \mathbf{0}_{3\times 3} \end{bmatrix}$, one can verify that $RA_1\boldsymbol{\eta}_2 = 0$; thus, using proposition (4.4.2) we conclude that $\mathcal{C}^\star = \mathcal{C}$ and is non-empty. △

We now present an example of an infeasible coordination task.

Example 4.4.4. Consider two unicycles. They are asked to be aligned, that is $\eta_{1,1} = \eta_{2,1}$. The velocity of the first unicycles is asked to be $\eta_{1,2} = v > 0$ and that of the second unicycle is asked to be $\eta_{2,2} = 2v$. Intuitively, we see that this coordination constraint does not make physical sense and should be infeasible. We verify our intuition using Proposition 4.4.2

.This coordination can be expressed in terms of the matrices A and b as

$$A_1 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} b = \begin{bmatrix} 0 \\ v \\ 2v \end{bmatrix} \tag{4.14}$$

We calculate $R = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ and $RA_1\boldsymbol{\eta}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix} \boldsymbol{\eta}_2$ which is nonzero on $\mathcal{C}$. We can see that

$$\operatorname{rank} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & v \\ 0 & 0 & 0 & 1 & 2v \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix} \neq \operatorname{rank} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \tag{4.15}$$

so based on Proposition 4.4.2 the set $\mathcal{C}^\star$ is empty as intuitively expected. $\triangle$

Nonlinear Coordination

Consider a coordination specification described by a nonlinear coordination function $\hat{\beta} : \psi(U) \subseteq \mathbb{R}^{2N} \to \mathbb{R}^c$. We emphasize that either (i) $\hat{\beta}$ is the local representation of a coordination function in a chart (U, ψ) or (ii) it is a global function $\hat{\beta} : \mathbb{R}^{2N} \to \mathbb{R}^c$ which is L_i-periodic in its first r arguments. By Definition 2.2.1 the set (4.9) is a smooth $(2N - c)$-dimensional embedded submanifold of $\psi(U)$. Consider the partition for $\mathrm{d}\hat{\beta}(\boldsymbol{\eta}) = \begin{bmatrix} \partial_{\boldsymbol{\eta}_1}\hat{\beta} & \partial_{\boldsymbol{\eta}_2}\hat{\beta} \end{bmatrix}$ in accordance with $\boldsymbol{\eta} = (\boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$. Propositions 4.4.5 and 4.4.6 give sufficient conditions for velocity coordination and position coordination constraints to be feasible.

Proposition 4.4.5 (Nonlinear velocity coordination)**.** *If the coordination function satisfies $\partial_{\boldsymbol{\eta}_1}\hat{\beta} \equiv 0$ then, for each $\boldsymbol{\eta} \in \mathcal{C}$, $\mathcal{C}^\star = \mathcal{C}$ and is non-empty.*

Proof. To check whether or not $\mathcal{C}$ is controlled-invariant we take the derivative of $\hat{\beta}$ along solutions of the system (4.8a) to obtain $\partial_{\boldsymbol{\eta}_2}\hat{\beta}\mathbf{v}^{\|} = 0$. Since $\mathbf{v}^{\|} = 0$ trivially solves this equation we have $\mathcal{C}^{\star} = \mathcal{C}$. □

Proposition 4.4.6 (Nonlinear position coordination)**.** *If the coordination function satisfies* $\partial_{\boldsymbol{\eta}_2}\hat{\beta} \equiv 0$ *and* $\boldsymbol{\eta} \in \mathcal{C}$ *is such that* $\partial_{\boldsymbol{\eta}_1}\left(\partial_{\boldsymbol{\eta}_1}\hat{\beta}\boldsymbol{\eta}_2\right)\boldsymbol{\eta}_2 \in \operatorname{Im}\partial_{\boldsymbol{\eta}_1}\hat{\beta}$ *then, the local coordination set* $\mathcal{C}^{\star}$ *is non-empty.*

Proof. By assumption, the derivative of $\hat{\beta}$ along solutions of (4.8a) equals $\partial_{\boldsymbol{\eta}_1}\hat{\beta}\boldsymbol{\eta}_2 = 0$. Since no control inputs appear we impose the additional constraint $\partial_{\boldsymbol{\eta}_1}\hat{\beta}\boldsymbol{\eta}_2 = 0$ to the set $\mathcal{C}$ and obtain $\mathcal{C}_1 = \{\boldsymbol{\eta} \in \mathcal{C} : \partial_{\boldsymbol{\eta}_1}\hat{\beta}\boldsymbol{\eta}_2 = 0\}$. This set is a closed-embedded submanifold of dimension $2N - 2c \geq 0$ because its defining constraints are a submersion

$$\operatorname{rank}\begin{bmatrix} \partial_{\boldsymbol{\eta}_1}\hat{\beta} & \mathbf{0}_N \\ \partial_{\boldsymbol{\eta}_1}\left(\partial_{\boldsymbol{\eta}_1}\hat{\beta}\boldsymbol{\eta}_2\right) & \partial_{\boldsymbol{\eta}_1}\hat{\beta} \end{bmatrix} = 2c \leq 2N.$$

In order to check controlled-invariance of $\mathcal{C}_1$ we take the derivative of the constraint $\partial_{\boldsymbol{\eta}_1}\hat{\beta}\boldsymbol{\eta}_2 = 0$ along solutions of (4.8a) to obtain $\partial_{\boldsymbol{\eta}_1}\left(\partial_{\boldsymbol{\eta}_1}\hat{\beta}\boldsymbol{\eta}_2\right)\boldsymbol{\eta}_2 + \partial_{\boldsymbol{\eta}_1}\hat{\beta}\mathbf{v}^{\|} = 0$. The condition $\partial_{\boldsymbol{\eta}_1}\hat{\beta}(\partial_{\boldsymbol{\eta}_1}\hat{\beta}\boldsymbol{\eta}_2)\boldsymbol{\eta}_2 \in \operatorname{Im}\partial_{\boldsymbol{\eta}_1}\beta$ guarantees that the above equation is solvable in $\mathbf{v}^{\|}$ for $\boldsymbol{\eta} \in \mathcal{C}_1 \subseteq \mathcal{C}$ and therefore $\mathcal{C}^{\star} = \mathcal{C}_1$. □

Example 4.4.7. Consider $N = 2$ unicycles that must coordinate their velocites. The first unicycle is asked to have its velocity as a function of the velocity of the second unicycle.

$$\beta(\boldsymbol{\eta}) = \eta_{1,2} - 2 - \sin(\eta_{2,2}). \tag{4.16}$$

Since $\partial_{\boldsymbol{\eta}_1}\beta \equiv 0$ by Proposition (4.4.5) we conclude that the coordination set is non-empty.

△

4.5 Control design

We now design feedback controllers to solve our coordinated path following problem.

4.5.1 Stabilizing the multi-agent path following manifold

To accomplish **PF** the multi-agent path following manifold $\Gamma^\star$ must be stabilized. As discussed in Section 4.3.2, this can be done by asking each unicycle to stabilize its own component $\Gamma_i^\star$. To stabilize $\Gamma_i^\star$ we select the simplest transversal controller for unicycle i

$$v_i^{\pitchfork}(\xi_i) = -k_{i,1}\xi_{i,1} - k_{i,2}\xi_{i,2} \tag{4.17}$$

with $k_{i,1}, k_{i,2} > 0$, $i \in \{1, \cdots, N\}$. For fast convergence to the path one typically chooses the gains so that the roots of the polynomial $s^2 + k_{i,2}s + k_{i,1}$ are far to left in the open left-half complex plane. Alternatively, optimal linear quadratic regulation or model predictive control can can be employed when actuator constraints are a concern. With the above choice the origin of each transversal subsystem is rendered exponentially stable.

Remark 4.5.1. If the trajectory of the unicycle is bounded, then stabilizing $(\xi_{i,1}, \xi_{i,2}) = 0$ is equivalent to stabilizing $\Gamma_i^\star$. When the path is non-closed γ_i itself is unbounded and so traversing the path results in unbounded trajectories for the unicycle. In that case, Assumption 1.2.1 and in particular (1.4) ensures that $(\xi_{i,1}, \xi_{i,2}) \to 0 \iff \mathbf{x}_i \to \Gamma_i^\star$ [27]. Intuitively, (1.4) requires that for all $c \neq 0$, the distance between any point in $s_i^{-1}(c)$ and any point in $s_i^{-1}(0)$ be bounded. The component of $\Gamma_i^\star$ that the unicycle approaches depends on initial conditions.

4.5.2 Centralized stabilization of the coordination set

Given a feasible local coordination set $\mathcal{C}^\star \subseteq \psi(U)$ and $\boldsymbol{\eta} \in \mathcal{C}^\star$ we seek to, under Assumption 4.2.1, feedback linearize that portion of the tangential dynamics (4.8a) that governs whether or not coordination is being achieved. This is equivalent to the following zero dynamics assignment problem [70] : Find a function $\tilde{\beta} : V \subseteq \psi(U) \to \mathbb{R}^c$, V is an open set containing $\boldsymbol{\eta}$ such that (i) $\tilde{\beta}$ yields a well-defined vector relative degree for the tangential dynamics (4.8a) at $\boldsymbol{\eta}$ and (ii) the associated zero dynamics manifold equals $\mathcal{C}^\star \cap V$. Following Remark 3.4.3 if conditions of Theorem 3.4.2 hold the zero dynamic assignment problem can be solved. If such a function exists then, in a neighbourhood of $\boldsymbol{\eta}$, the tangential dynamics (4.8a) are locally feedback equivalent to

$$\begin{aligned} \dot{\boldsymbol{\zeta}} &= f(\boldsymbol{\zeta}, \boldsymbol{\mu}) + g^{\pitchfork}(\boldsymbol{\zeta}, \boldsymbol{\mu})\boldsymbol{\tau}^{\pitchfork} + g^{\parallel}(\boldsymbol{\zeta}, \boldsymbol{\mu})\boldsymbol{\tau}^{\parallel} \\ \dot{\boldsymbol{\mu}} &= A\boldsymbol{\mu} + B\boldsymbol{\tau}^{\pitchfork} \end{aligned} \tag{4.18}$$

where $(\boldsymbol{\zeta}, \boldsymbol{\mu}) \in \mathbb{R}^{c^\star} \times \mathbb{R}^{2N-c^\star}$, $c^\star := \dim \mathcal{C}^\star$, (A, B) controllable, and $\mathcal{C}^\star$, expressed in $(\boldsymbol{\zeta}, \boldsymbol{\mu})-$coordinates, is given by $\{(\boldsymbol{\zeta}, \boldsymbol{\mu}) : \boldsymbol{\mu} = 0\}$. A natural candidate for the function $\tilde{\beta}$ is the coordination function $\hat{\beta}$ itself. We explore this possibility in Propositions 4.5.6 and 4.5.7.

In (4.18) the $\boldsymbol{\mu}$-subsystem describes the motion transversal to the set $\mathcal{C}^\star$ but tangential to $\Gamma^\star$. Since (A, B) is controllable, there exists a linear feedback $\boldsymbol{\tau}^{\pitchfork} = F\boldsymbol{\mu}$ that exponentially stabilizes the origin of the $\boldsymbol{\mu}$-subsystem. Then, because the set $\mathcal{C}^\star$ is not necessarily bounded, under similar caveats as those discussed in Remark 4.5.1 the set $\mathcal{C}^\star$ is rendered locally attractive and invariant and specification **S1** is achieved.

The $\boldsymbol{\zeta}$-subsystem in (4.18) describes the dynamics tangent to both $\mathcal{C}^\star$ and $\psi(U)$. When restricted to $\mathcal{C}^\star$, the multi-agent unicycle system evolves according to

$$\dot{\boldsymbol{\zeta}} = f(\boldsymbol{\zeta}, 0) + g^{\parallel}(\boldsymbol{\zeta}, 0)\boldsymbol{\tau}^{\parallel}. \tag{4.19}$$

System (4.19) models the group dynamics while restricted to evolve on the assigned paths and restricted to coordinated motion. In some cases it may be possible to use the remaining control inputs $\boldsymbol{\tau}^{\parallel}$ to satisfy **S2**, see Propositions 4.5.6 and 4.5.7. In Figure 4.3 the proposed control architecture is illustrated.

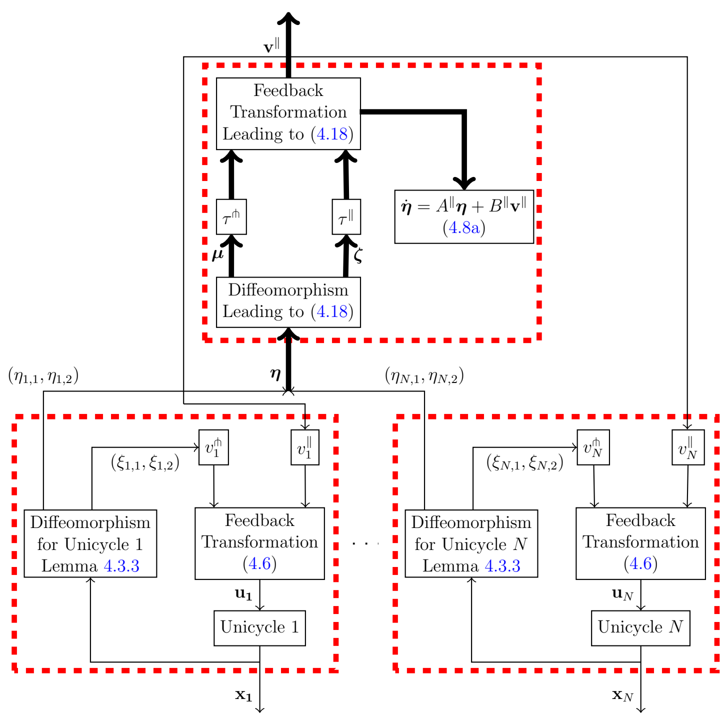


Figure 4.3: Block diagram of the closed-loop system.

Remark 4.5.2. It is worthwhile to compare the complexity of our proposed design to that of other studies. In our approach, instead of directly designing coordinated path following controllers, we first apply a coordinate and feedback transformation that brings the unicycles into the normal form (4.7) and then another that brings the multi-agent tangential dynamics (4.8a) into the form (4.18). The process of bringing the unicycles into these normal forms can involve complex, though straightforward, computations depending on the assigned paths and coordination task. However, once the normal form is obtained, controller design is greatly simplified. The main challenge in implementing our controllers is the computation of (4.4). In general this expression does not have a closed-form solution so numerical optimization algorithms must be employed [14]. A similar computation is required to implement methods relying on Frenet-Serret frames [36, 39]. In the curve extension method path following and coordination controller design are performed together. The implementation of the proposed controllers can involve complex computations depending on the paths. See, for example, [13, Equations (32),(52)].

As mentioned in Chapter 1, decoupling method and curve extension method are two distinct approaches employed to solve coordinated path following problem. In Table 4.1 a qualitative comparison between these methods and that of this thesis is provided

	Decoupling method [38]	**Curve extension method [13]**	**Our method**
Path following controller	Global	Local	Local
Coordination controller	Distributed	Distributed	Centralized
Path invariance	No	No	Yes
Coordination specification	Formation coordination	Formation coordination	General coordination
Coordination specification invariance	No	No	Yes

Table 4.1: Comparison of different approaches on coordinated path following problem

The next theorem explicitly addresses the component $\mathcal{P}_+^{\star,\mathrm{f}} \coloneqq \mathcal{P}_{1,+}^{\star,\mathrm{f}} \times \cdots \times \mathcal{P}_{N,+}^{\star,\mathrm{f}} \subset \mathcal{P}^\star$ but a similar result can be obtained for other components of $\mathcal{P}^\star$. The next theorem explicitly addresses the component $\Gamma_+^{\star,\mathrm{f}} \coloneqq \Gamma_{1,+}^{\star,\mathrm{f}} \times \cdots \times \Gamma_{N,+}^{\star,\mathrm{f}} \subset \Gamma^\star$ but a similar result can be obtained for other components of $\Gamma^\star$.

Theorem 4.5.3. *Suppose Assumptions 1.2.1 and 4.2.1 hold. Fix $\bar{\mathbf{x}} \in \Gamma_+^{\star,\mathrm{f}}$ and let $U \subset (\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R})^N$, $\Gamma_+^{\star,\mathrm{f}} \subset U$ be an open set on which the multi-unicycle system is feedback equivalent to (4.8). Suppose (4.8a) is feedback equivalent, in an open set $V \subseteq T(U)$ containing $T(\bar{\mathbf{x}})$ to (4.18) and there exist class-$\mathcal{K}_\infty$ functions $\rho_1, \rho_2 : [0,\infty) \to \mathbb{R}_+$ such that*

$$(\forall \boldsymbol{\eta} \in V) \;\; \rho_1\left(\mathrm{dist}(\boldsymbol{\eta}, \mathcal{C}^\star)\right) \leq \|\boldsymbol{\mu}(\boldsymbol{\eta})\| \leq \rho_2\left(\mathrm{dist}(\boldsymbol{\eta}, \mathcal{C}^\star)\right). \tag{4.20}$$

If each unicycle's transversal control is given by (4.17), $\boldsymbol{\tau}^{\pitchfork} = F\boldsymbol{\mu}$ is such that $A + BF$ is Hurwitz, and $\boldsymbol{\tau}^{\parallel}$ is such that for each $(\boldsymbol{\xi}(0), \boldsymbol{\eta}(0)) \in V$, $(\boldsymbol{\xi}(t), \boldsymbol{\eta}(t)) \in V$ for all $t \geq 0$, then $\Gamma^\star \cap T^{-1}(V)$ is asymptotically stable, $\mathcal{C}^\star \cap T^{-1}(V)$ is asymptotically stable relative to $\Gamma^\star \cap T^{-1}(V)$, and $\mathcal{C}^\star \cap T^{-1}(V)$ is asymptoticly stable.

Remark 4.5.4. We once again stress that the existence of the function $\tilde{\beta}$ only guarantees local equivalence between the tangential dynamics (4.8a) and (4.18). Intuitively this means that the controllers of Theorem 4.5.3 only solve the coordinated path following problem if the unicycles are not too far from coordination at $t = 0$.

Proof of Theorem 4.5.3. Let $\mathbf{v}^{\pitchfork} = F^{\pitchfork}\boldsymbol{\xi}$ be the overall transversal controller where $F^{\pitchfork}$ is an $N \times 2N$ matrix composed of the gains in equation (4.17). By hypothesis, the multi-agent tangential dynamics (4.8a) are locally feedback equivalent to (4.18). In $T^{-1}(V)$ the closed-loop system is feedback equivalent to

$$\begin{aligned}\dot{\boldsymbol{\zeta}} &= f(\boldsymbol{\zeta},\boldsymbol{\mu}) + g^{\pitchfork}(\boldsymbol{\zeta},\boldsymbol{\mu})F\boldsymbol{\mu} + g^{\parallel}(\boldsymbol{\zeta},\boldsymbol{\mu})\boldsymbol{\tau}^{\parallel}\\ \dot{\boldsymbol{\mu}} &= (A+BF)\boldsymbol{\mu}\\ \dot{\boldsymbol{\xi}} &= (A^{\pitchfork}+B^{\pitchfork}F^{\pitchfork})\boldsymbol{\xi}.\end{aligned}\tag{4.21}$$

By hypothesis, solutions starting in V remain in V for all $t \geq 0$, and $A^{\pitchfork}+B^{\pitchfork}F^{\pitchfork}$ and $A+BF$ are Hurwitz, therefore $(\boldsymbol{\mu},\boldsymbol{\xi}) = (0,0)$ is exponentially stable for the $(\boldsymbol{\mu},\boldsymbol{\xi})$-subsystem.

By hypothesis solutions starting in $T^{-1}(V)$ remain there. Under condition (1.4) $\Gamma^\star \cap T^{-1}(V)$ is attractive. Under condition (4.20) $\mathcal{C}^\star \cap T^{-1}(V)$ is attractive relative to $\Gamma^\star \cap T^{-1}(V)$. And under conditions (1.4) and (4.20), $\mathcal{C}^\star \cap T^{-1}(V)$ is attractive.

By exactly the same argument in the proof of Theorem 3.6.1 we conclude that $\Gamma^\star \cap T^{-1}(V)$ is stable, $\mathcal{C}^\star \cap T^{-1}(V)$ is stable relative to $\Gamma^\star \cap T^{-1}(V)$, and $\mathcal{C}^\star \cap T^{-1}(V)$ is stable.

Thus, $\Gamma^\star \cap T^{-1}(V)$ is asymptotically stable, $\mathcal{C}^\star \cap T^{-1}(V)$ is asymptotically stable relative to $\Gamma^\star \cap T^{-1}(V)$, and $\mathcal{C}^\star \cap T^{-1}(V)$ is asymptotically stable. □

Remark 4.5.5. In Section 3.6 sufficient conditions are presented under which Problem 5 is solvable. For unicycles, the previous result proves that Problem 5 is solvable under less restrictive sufficient condition where the boundedness assumption is removed provided that Assumption 4.2.1 holds.

Theorem 4.5.3 shows that under suitable assumptions, as opposed to more general scenarios [27, 17], for a multi-agent unicycle system the set $\mathcal{C}^\star$ can be rendered locally asymptotically stable via feedback regardless of whether it is initialized on $\Gamma^\star$.

4.5.3 Velocity and position coordination

In this section we investigate velocity and position coordinations as two special, but important, cases of coordination specifications.

Proposition 4.5.6 (Velocity coordination)**.** *Given tangential dynamics* (4.8a)*, a coordination function* $\hat{\beta}$ *satisfying the hypotheses of Proposition 4.4.5, and a point* $\bar{\boldsymbol{\eta}} \in \mathcal{C}^{\star}$*, there exist a neighbourhood* $V \subseteq \psi(U)$ *containing* $\bar{\boldsymbol{\eta}}$ *and a function* $\tilde{\beta} : V \to \mathbb{R}^N$ *which satisfies* $\operatorname{rank} \mathrm{d}\tilde{\beta}_{\bar{\boldsymbol{\eta}}} = N$*. Moreover, the tangential dynamics* (4.8a) *with output* $\tilde{\beta}$ *yields a well-defined relative degree of* $\{1, \cdots, 1\}$ *at* $\bar{\boldsymbol{\eta}}$*.*

Proof. Since $\partial_{\boldsymbol{\eta}_1}\hat{\beta} \equiv 0$ it follows $\operatorname{rank} \mathrm{d}\hat{\beta} = \operatorname{rank} \partial_{\boldsymbol{\eta}_2}\hat{\beta} = c \leq N$. Let $\phi : V \subseteq \psi(U) \to \mathbb{R}^{N-c}$ be a function such that $\partial_{\boldsymbol{\eta}_1}\phi \equiv 0$ and $\operatorname{rank} \partial_{\boldsymbol{\eta}_2}\phi = N - c$. Define $\tilde{\beta} = (\phi, \hat{\beta})$. It is immediately evident that it has rank N at $\bar{\boldsymbol{\eta}}$. Direct calculations yield $L_{B^{\parallel}}\tilde{\beta}(\boldsymbol{\eta}) = (\partial_{\boldsymbol{\eta}_2}\phi(\boldsymbol{\eta}), \partial_{\boldsymbol{\eta}_2}\hat{\beta}(\boldsymbol{\eta}))$. Since $\operatorname{rank} L_{B^{\parallel}}\tilde{\beta}(\bar{\boldsymbol{\eta}}) = N$, the proof is complete. $\square$

Using the function $\tilde{\beta}$ of Proposition 4.5.6 and employing input-output feedback linearization, the tangential dynamics (4.8a) are locally feedback equivalent to

$$\begin{aligned}
\dot{\boldsymbol{\zeta}}_1 &= \hat{f}(\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, \boldsymbol{\mu}) + \hat{g}^{\pitchfork}(\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, \boldsymbol{\mu})\boldsymbol{\tau}^{\pitchfork} + \hat{g}^{\parallel}(\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, \boldsymbol{\mu})\tau^{\parallel} \\
\dot{\boldsymbol{\zeta}}_2 &= \boldsymbol{\tau}^{\parallel} \\
\dot{\boldsymbol{\mu}} &= \boldsymbol{\tau}^{\pitchfork}
\end{aligned} \tag{4.22}$$

where $(\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, \boldsymbol{\mu}) \in \mathbb{R}^N \times \mathbb{R}^{N-c} \times \mathbb{R}^c$, $(\boldsymbol{\tau}^{\pitchfork}, \boldsymbol{\tau}^{\parallel}) \in \mathbb{R}^c \times \mathbb{R}^{N-c}$, $\hat{f} : V \subseteq \mathbb{R}^{2N} \to \mathbb{R}^N$ is a smooth function, and $\hat{g}^{\parallel} : V \subseteq \mathbb{R}^{2N} \to \mathbb{R}^{N \times N-c}$ and $\hat{g}^{\pitchfork} : V \subseteq \mathbb{R}^{2N} \to \mathbb{R}^{N \times c}$ are smooth matrix-valued functions. In (4.22), the $\boldsymbol{\mu}$ and $\boldsymbol{\zeta}_2$-dynamics are decoupled which allows one to design the control laws $\boldsymbol{\tau}^{\pitchfork}$, $\boldsymbol{\tau}^{\parallel}$ separately, The input $\boldsymbol{\tau}^{\parallel}$ can be used to control the velocity of the coordinated unicycles along their assigned paths.

Proposition 4.5.7 (Position coordination)**.** *Given the tangential dynamics* (4.8a)*, a coordination function* $\hat{\beta}$ *satisfying conditions of Proposition 4.4.6, and a point* $\bar{\boldsymbol{\eta}} \in \mathcal{C}^\star$*, there exist a neighbourhood* $V \subseteq \psi(U)$ *containing* $\bar{\boldsymbol{\eta}}$ *and a function* $\tilde{\beta} : V \to \mathbb{R}^N$ *which satisfies* $\operatorname{rank} \mathrm{d}\tilde{\beta}_{\bar{\boldsymbol{\eta}}} = N$*. Moreover, the tangential dynamics* (4.8a) *with output* $\tilde{\beta}$ *yield a well-defined vector relative degree of* $\{2, \cdots, 2\}$ *at* $\bar{\boldsymbol{\eta}}$*.*

Proof. Let $\phi : V \subseteq \psi(U) \to \mathbb{R}^{N-c}$ be a function such that $\partial_{\boldsymbol{\eta}_2}\phi \equiv 0$ and $\operatorname{rank} \partial_{\boldsymbol{\eta}_1}\phi = N - c$. Define $\tilde{\beta} = (\phi, \beta)$. Since $\partial_{\boldsymbol{\eta}_2}\hat{\beta} \equiv 0$ it follows $\operatorname{rank} \mathrm{d}\hat{\beta} = \operatorname{rank} \partial_{\boldsymbol{\eta}_1}\hat{\beta} = c \leq N$. As a result $\tilde{\beta}$ has rank N at $\bar{\boldsymbol{\eta}}$. Simple calculations give $L_{B^\parallel}\tilde{\beta} = \mathbf{0}_N$ and $L_{B^\parallel}L_{A^\parallel\eta}\tilde{\beta}(\boldsymbol{\eta}) = (\partial_{\boldsymbol{\eta}_1}\phi(\boldsymbol{\eta}), \partial_{\boldsymbol{\eta}_1}\hat{\beta}(\boldsymbol{\eta}))$ which has rank N at $\bar{\boldsymbol{\eta}}$. □

The function $\tilde{\beta}$ of Proposition 4.5.7 can be used to feedback linearize the tangential dynamics (4.8a) to obtain

$$
\begin{aligned}
\dot{\boldsymbol{\zeta}}_1 &= \boldsymbol{\zeta}_2 \\
\dot{\boldsymbol{\zeta}}_2 &= \boldsymbol{\tau}^\parallel \\
\dot{\boldsymbol{\mu}}_1 &= \boldsymbol{\mu}_2 \\
\dot{\boldsymbol{\mu}}_2 &= \boldsymbol{\tau}^\pitchfork.
\end{aligned}
\tag{4.23}
$$

where $(\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) \in \mathbb{R}^{N-c} \times \mathbb{R}^{N-c} \times \mathbb{R}^c \times \mathbb{R}^c$, $(\boldsymbol{\tau}^\pitchfork, \boldsymbol{\tau}^\parallel) \in \mathbb{R}^c \times \mathbb{R}^{N-c}$. In (4.23), the $(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2)$ and $(\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2)-$dynamics are decoupled. As a result, the control inputs $\boldsymbol{\tau}^\pitchfork$ and $\boldsymbol{\tau}^\parallel$ can be designed separately. The input $\boldsymbol{\tau}^\parallel$ can be used to control the position and velocity of the coordinated unicycles along their assigned paths.

Example 4.5.8. Consider a multi-agent system of three unicycles which are required to follow the paths

$$\gamma_1 = y_{1,2} + 7 = 0$$

$$\gamma_2 = y_{2,1}^2 + y_{2,2}^2 - 4 = 0$$

$$\gamma_3 = y_{3,1}^2 + y_{3,2}^2 - 36 = 0.$$

The path following controller is designed by first feedback linearizing each unicycle to bring them into the normal form given (4.7). The transversal control input for unicycle i, $i \in \{1,2,3\}$ is designed as

$$v_i^{\pitchfork} = -k_{i,1}\xi_1 - k_{i,2}\xi_2$$

where $k_{i,1}, k_{i,2} > 0$.

While the unicycles are on their paths we ask that unicycles 1 and 3 traverse the same arc-length. Unicycle 2 is required to cover for unicycle 3 by oscillating about unicycle 3. Moreover, the unicycle 1 is required to move with a prescribed velocity $v_d = 3m/s$. Note that $\eta_{1,1} \in \mathbb{R}$ while $\eta_{3,1} \in [0, 12\pi)$ and they are arc-length. One choice of coordination function that encodes this coordination specification is

$$\beta = \begin{bmatrix} \eta_{3,1} - \eta_{1,1} \bmod 12\pi \\ \frac{\eta_{2,1}}{2} - \frac{\eta_{3,1}}{6} - c\sin(\frac{\eta_{3,1}}{6}) \end{bmatrix}$$

where $c \geq 0$ determines the range of coverage for unicycle 2. For this example we select $c = \frac{\pi}{8}$. One can verify that the coordination function satisfies the conditions of Proposition 4.4.6 thus $\mathcal{C}^{\star} = \mathcal{C} = \beta^{-1}(0)$ and is non-empty. Conditions of Proposition 4.5.7 hold and following its proof we find $\hat{\beta} = (\beta(\eta), \eta_{1,1})$. Thus, the tangential dynamics given by (4.8a) with $N = 3$

writes

$$
\begin{aligned}
\dot{\zeta}_1 &= \zeta_2 \\
\dot{\zeta}_2 &= \tau^{\parallel} \\
\dot{\mu}_1 &= \mu_2 \\
\dot{\mu}_2 &= \tau_1^{\pitchfork} \\
\dot{\mu}_3 &= \mu_4 \\
\dot{\mu}_4 &= \tau_2^{\pitchfork}.
\end{aligned}
$$

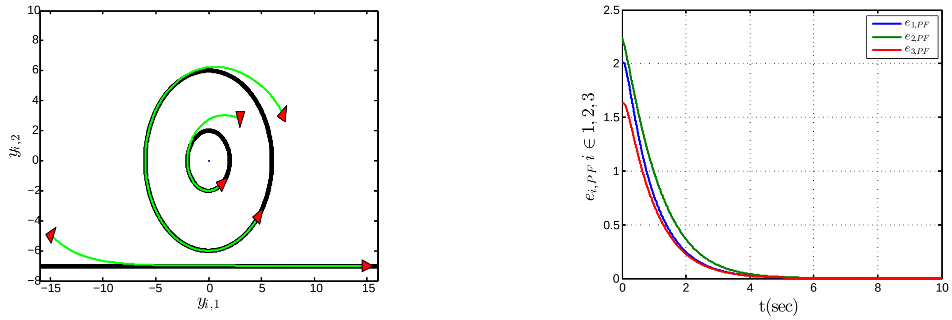
We design

$$
\tau_1^{\pitchfork} = -k_1 \sin(\mu_1) - k_2\mu_2, \qquad \tau_2^{\pitchfork} = -k_3 \sin(\mu_3) - k_4\mu_4
$$

with $k_1, k_2, k_3, k_4 > 0$ to stabilize the coordination set $\mathcal{C}^{\star}$. Since μ_1 and μ_3 are 2π-periodic we have added sinusoid functions to make $\tau_1^{\pitchfork}$ and $\tau_2^{\pitchfork}$ 2π-periodic. As a result, they are defined on multi-agent path following manifold globally. On $\mathcal{C}^{\star}$ we design $\tau^{\parallel}$ to make unicycle 1 move with constant velocity v_d. Let $\tau^{\parallel} = -k\,(\zeta_2 - 3)$ with $k > 0$. Figure 4.4(a) shows the position of unicycles along their paths. The path following errors for unicycles are computed as $e_{1,PF} = y_1 + 7$, $e_{2,PF} = \sqrt{x_2^2 + y_2^2} - 2$, $e_{3,PF} = \sqrt{x_3^2 + y_3^2} - 6$, and are depicted in Figure 4.4(b). Figure 4.5(a) shows that β_i, $i \in \{1, 2\}$ approaches zero. Figure 4.5(b) shows the velocities of the unicycles along their corresponding paths. It is evident that $\eta_{1,2}$ has reached $v_d = 3$ as desired. Note, since unicycles 1 and 3 must traverse the same arc-length they must have the same velocity.

$\triangle$

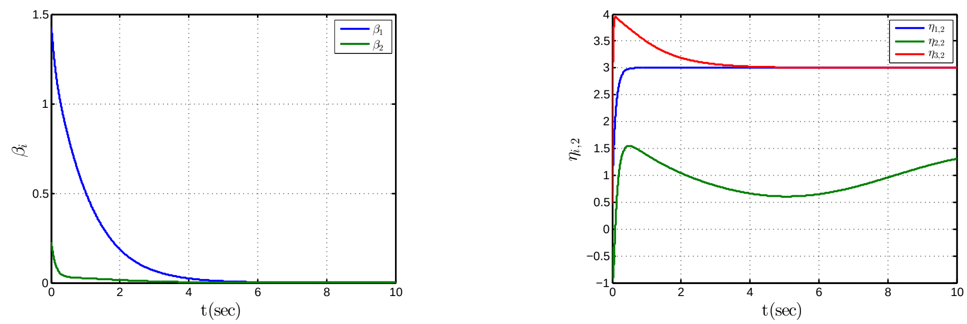
Feedback linearization is often criticized for being susceptible to modelling error and disturbances, in short, for not being robust. The following example, as well as the experimental results of Section 4.6 partially address these concerns.



(a) Unicycles following their corresponding paths

(b) Path following errors for robots 1 and 2

Figure 4.4: Path following of unicycles in Example 4.5.8.



(a) Coordination constraints β_i, for $i \in \{1, 2\}$

(b) Velocities of the unicycles $1, 2, 3$

Figure 4.5: Coordination of unicycles in Example 4.5.8.

Example 4.5.9. Consider a multi-agent system of three unicycles which are required to follow the paths

$$
\begin{aligned}
\gamma_1 &= y_{1,2} - \sin\left(\frac{1}{2\pi} y_{1,1}\right) = 0 \\
\gamma_2 &= y_{2,2} - 2 = 0 \\
\gamma_3 &= y_{3,2} + \sin\left(\frac{1}{2\pi} y_{3,1}\right) - 4 = 0.
\end{aligned}
$$

We assume that model of unicycle i, $i \in \{1, 2, 3\}$, is affected by disturbances as follows

$$\begin{aligned}
\dot{x}_i &= v_i \cos(\theta_i) + A_i \sin(\omega_i t + \phi_i) \\
\dot{y}_i &= v_i \sin(\theta_i) + A_i \sin(\omega_i t + \phi_i) \\
\dot{\theta}_i &= u_{i,2} + A_i \sin(\omega_i t + \phi_i) \\
\dot{v}_i &= u_{i,1} + 0.5 v_i,
\end{aligned}$$

where w_i, ω_i, and ϕ_i are given in Table 4.5.9.

	A_i	ω_i	ϕ_i
Unicycle 1	0.1	1	$\frac{\pi}{3}$
Unicycle 2	0.2	1.5	$-\frac{\pi}{3}$
Unicycle 3	0.3	2	0

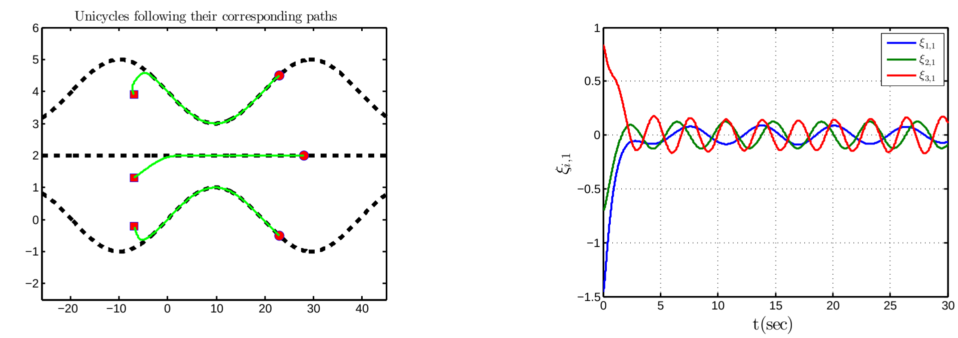
Table 4.2: Properties of disturbances

While the unicycles are on their paths we ask that all three unicycles traverse the same amount of arc-length at any moment in time. Moreover, the unicycles are required to move with a prescribed velocity $v_d = 1m/s$. Since $\eta_{i,1}$ specifies the arc-length of unicycle i along its path, one choice of coordination function that encodes this specification is

$$\beta = \begin{bmatrix} \eta_{3,1} - \eta_{1,1} \\ \eta_{2,1} - \eta_{3,1} \end{bmatrix}.$$

One can verify that the coordination function satisfies the conditions of Proposition 4.4.6 thus $\mathcal{C}^\star = \mathcal{C} = \beta^{-1}(0)$ and is non-empty. Following similar procedure as in Example 4.5.8 we design the path following and coordination control laws. Since the paths are all non-closed we do not use sinusoid functions.

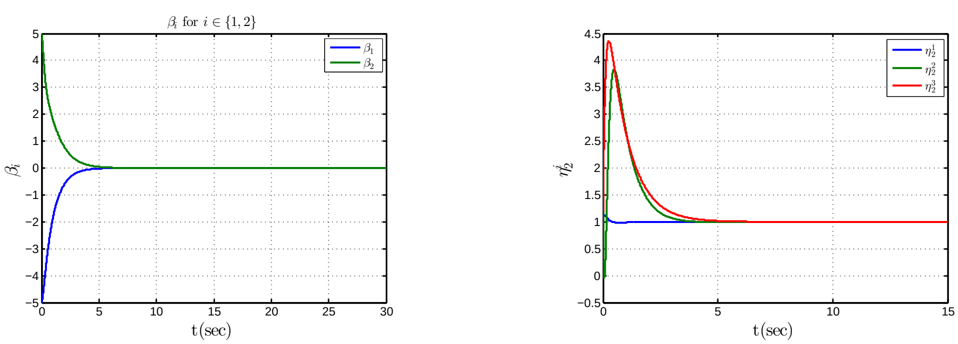
Figure 4.6(a) shows the position of unicycles along their paths. Figure 4.6(b) shows the $\xi_{i,1}$ as an indication of path following performance. Figure 4.7(a) shows the coordination function. Figure 4.7(a) depicts the velocity of unicycles along their paths. While the performance is clearly deteriorated, the simulations show that in this case the proposed approach still works reasonably well despite the presence of some unmodelled disturbances and modelling errors. $\triangle$



(a) Unicycles following their corresponding paths

(b) Transversal state $\xi_{i,1}$ for $i \in \{1, 2, 3\}$

Figure 4.6: Path following of unicycles in the presence of disturbances in Example 4.5.9.



(a) Coordination constraints β_i, for $i \in \{1, 2\}$

(b) Velocities of the unicycles $1, 2, 3$

Figure 4.7: Coordination of unicycles in the presence of disturbances in Example 4.5.9.

4.5.4 Semi-distributed stabilization of a linear-affine coordination set

Here we consider a specific, yet useful, choice of linear-affine coordination function $\hat{\beta}(\boldsymbol{\eta})$. Consider a coordination constraint in which every two consecutive unicycles must maintain a constant arc-length separation. i.e., $\eta_{i+1,1} - \eta_{i,1} - b_{i,1} = 0,\ i \in \{1, \cdots, N-1\}$ where $b_{i,1} \in \mathbb{R}$. Moreover, it is required that once all the agents are in formation they all move with a desired velocity $v_d > 0$, i.e., $\eta_{i,2} - v_d = 0,\ i \in \{1, \cdots, N\}$. This particular coordinated path following problem is the same as the formation control problem investigated in [75]. We can represent this special coordination function as a linear-affine function. In order to do so we add a redundant constraint $\eta_{N,1} - \eta_{1,1} - b_{N,1} = 0$ and define the coordination function $\hat{\beta}(\boldsymbol{\eta}) = A\boldsymbol{\eta} + b$ as

$$b_1 = \begin{bmatrix} b_{1,1} \\ \vdots \\ b_{N,1} \end{bmatrix},\ b_2 = -v_d 1_N,\ A_1 = \begin{bmatrix} A_{11} \\ \mathbf{0}_N \end{bmatrix},\ A_2 = \begin{bmatrix} \mathbf{0}_N \\ \mathbf{I}_N \end{bmatrix} \tag{4.24}$$

where $A_{11} \in \mathbb{R}^{N \times N}$ is the circulant matrix

$$A_{11} = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{4.25}$$

Assumption 4.5.10. The numbering assigned to the agents, the prescribed velocity v_d, the vector b_1, and a number $k > 0$ are known to each unicycle. $\triangleleft$

In particular, we no longer make Assumption 4.2.1 and instead make the following, less restrictive, assumption.

Assumption 4.5.11. The communication graph $\mathscr{G}$ of the multi-agent unicycle system is rooted out-branching. $\triangleleft$

Lemma 4.5.12. *Let $\mathscr{G}$ be a weighted and directed graph on N vertices and let L be its Laplacian matrix. If $H := -L\mathbf{U}_N$ then $HA_{11} = -L$ where A_{11} is given by (4.25).*

Proof. Since $H = -L\mathbf{U}_N$ we must show that $-L\mathbf{U}_N A_{11} = -L$. Let L_i denote the i^{th} column of $-L$, for $i \in \mathbf{N}$. Then $-L\mathbf{U}_N A_{11} = \begin{bmatrix} \sum_{i=2}^{N} -L_i & L_2 & \cdots & L_N \end{bmatrix}$. A well-know property of the Laplacian matrix is that its columns sum to zero, i.e, $L_1 + L_1 + \ldots + L_N = 0$ [76]. So, we can write the first column, L_1, in terms of other columns $L_1 = -(L_2 + L_3 + \ldots + L_N)$. As a result

$$-LUA_{11} = \begin{bmatrix} L_1 & L_2 & \cdots & L_N \end{bmatrix} = -L$$

$\square$

Proposition 4.5.13. *If each path γ_i, $i \in \{1, \cdots, N\}$ is non-closed and the communication graph $\mathscr{G}$ of the multi-unicycle system is rooted out-branching then the control law*

$$v_i^{\parallel} = \sum_{j=1}^{N} w_{ij} \left(\eta_{j,1} - \eta_{i,1} - \frac{(j-i)}{|j-i|} \sum_{p=\min\{i,j\}}^{\max\{i,j\}-1} b_{1,p} \right) - k\left(\eta_{i,2} - v_d\right) \tag{4.26}$$

where w_{ij} are entries of the adjacency matrix $W(\mathscr{G})$ and $k > 0$, renders $\mathcal{C}^\star$ globally asymptotically stable relative to $\Gamma^\star$.

Remark 4.5.14. The control law (4.26) is an extension of the control law given in [75, Control law 2]. Similar control laws can be found in swarming problems [50, Equation 1].

Proof. We view the coordination function as an error function $\mathbf{e} = A\boldsymbol{\eta} + b$. Partitioning $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2)$ in accordance with the linear coordination function we have

$$\begin{aligned} \mathbf{e}_1 &= A_{11}\boldsymbol{\eta}_1 + b_1 \\ \mathbf{e}_2 &= \boldsymbol{\eta}_2 + b_2. \end{aligned} \tag{4.27}$$

The closed-loop error dynamics resulting from applying the control law (4.26) to the dynamics on $\psi(U)$ given in (4.8a) are

$$\begin{bmatrix} \dot{\mathbf{e}}_1 \\ \dot{\mathbf{e}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0}_N & A_{11} \\ H & -k\mathbf{I}_N \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix} =: E\mathbf{e}. \tag{4.28}$$

An immediate result is that 0 is an eigenvalue of matrix E since A_{11} has rank $N - 1$. However, we do not yet know the algebraic multiplicity of the eigenvalue 0. In the following we find the remaining eigenvalues of the matrix E. Let λ be an eigenvalue of E with associated eigenvector $(x, y) \in \mathbb{C}^N \times \mathbb{C}^N$. The relation between eigenvalues and eigenvectors of the matrix E is

$$\begin{aligned} A_{11}y &= \lambda x \\ Hx - ky &= \lambda y. \end{aligned} \tag{4.29}$$

Suppose that $\lambda \neq 0$. Combining the equations in (4.29) and using the result from Lemma 4.5.12 we obtain

$$\frac{1}{\lambda}\underbrace{HA_{11}}_{-L}y - ky = \lambda y \Rightarrow -Ly = (k + \lambda)\lambda y. \tag{4.30}$$

The Laplacian L has eigenvalues $\{\nu_1, \ldots, \nu_N\}$, one of which is zero. From (4.30) we deduce that $(k+\lambda)\lambda$ is an eigenvalue of $-L$. This yields the following N equations

$$(k+\lambda_i)\lambda_i = \nu_i, \qquad i \in \{1, \cdots, N\}.$$

There are $2N$ solutions to the above equations

$$\lambda_i^{\pm} = \frac{-k \pm \sqrt{k^2 + 4\nu_i}}{2}, \qquad i \in \{1, \cdots, N\}.$$

Since we assumed that $\lambda \neq 0$ in (4.30), the solution $\lambda_1^+ = 0$ corresponding to $\nu_i = 0$ is not an allowable solution. However, since we already know that 0 is an eigenvalue of E; it's just not obtained from solving above equation. Since $\mathscr{G}$ is rooted out-branching, the algebraic and geometric multiplicity of $\nu_i = 0$ is 1. Therefore, all the $\lambda_i^{\pm}$'s obtained from above equation have negative real parts, since $\lambda_1^- = -k$ and rest of ν_i have negative real parts. As a result, the eigenvalue 0 has geometric and algebraic multiplicity of 1. Using standard spectral theory, there exists a $2N \times 2N$ matrix V such that the similarity transform $E \mapsto V^{-1}EV$ yields the Jordan form $E_{JF} = \mathrm{diag}(0, J_1, \ldots, J_r)$ where r is the number of distinct eigenvalues of E and each Jordan block J_i has the form $\lambda_i I + N_i$, where N_i is a nilpotent matrix in Jordan form and $\mathrm{Re}(\lambda_i) < 0$. We therefore have that

$$\begin{aligned} \lim_{t\to\infty} \mathbf{e}^{Et} &= \lim_{t\to\infty} V\mathbf{e}^{E_{JF}t}V^{-1} = V\,\mathrm{diag}(1, 0, \ldots, 0)V^{-1} \\ &= p_1 q_1 \end{aligned}$$

where p_1 is the first column of V and q_1 is the first row of V^{-1}. It is easy to see that p_1 and q_1 are, respectively, the right and left eigenvectors of E associated with the eigenvalue

0. Thus, for any $e(0) \in \mathbb{R}^{2N}$, the solution to (4.28) can be written.

$$\lim_{t\to\infty} \mathbf{e}(t) = (q_1\mathbf{e}(0))p_1. \tag{4.31}$$

Direct calculations reveal that the following are, respectively, the right and left eigenvectors of the matrix E associated with the zero eigenvalue

$$p_1 = (0_{N-1}, 1, 0_N) \qquad q_1^\top = (1_N, 0_N). \tag{4.32}$$

Therefore (4.31) is given by

$$\lim_{t\to\infty} \mathbf{e}(t) = (0_{N-1}, e_{1,1}(0) + \cdots + e_{N,1}(0), 0_N)$$

Since the errors are not independent, they satisfy, for all $t \geq 0$, $e_{1,1}(t) + \cdots + e_{N,1}(t) = 0 \quad \forall t \geq 0$. Thus $\lim_{t\to\infty} \mathbf{e}(t) = 0_{2N}$. □

We now consider the case when all paths are closed, i.e., $\eta_{i,1} \in \mathbb{R} \bmod L_i$. In order to define a similar formation coordination constraint in this case we define $\eta'_{i,1} := \frac{2\pi}{L_i}\eta_{i,1}$. The variable $\eta'_{i,1}$ belongs to $[0, 2\pi)$ so we can view it as an angular variable. Accordingly, we define $\eta'_{i,2} := \frac{2\pi}{L_i}\eta_{i,2}$. Thus, the coordination constraint (4.24) imposes that two consecutive unicycles maintain a constant angular separation, i.e., $\eta'_{i+1,1} - \eta'_{i,1} - b_{i,1} = 0$, $i \in \{1, \cdots, N-1\}$ where $b_{i,1} \in [0, 2\pi)$. Moreover, the formation must move with a common velocity $\omega_d > 0$, i.e., $\eta'_{i,2} - \omega_d = 0$, $i \in \{1, \cdots, N\}$.

Corollary 4.5.15. *Suppose γ_i, $i \in \{1, \cdots, N\}$ are closed paths and the communication graph $\mathscr{G}$ of the multi-unicycle system is rooted out-branching then the control law*

$$v_i^{\parallel} = \sum_{j=1}^{N} w_{ij} \sin\left(\eta'_{j,1} - \eta'_{i,1} - \frac{(j-i)}{|j-i|} \sum_{p=\min\{i,j\}}^{\max\{i,j\}-1} b_{i,p}\right) - k\left(\eta'_{i,2} - v_d\right) \tag{4.33}$$

where w_{ij} are the entries of $W(\mathscr{G})$ and $k > 0$, renders $\mathcal{C}^\star$ locally exponentially stable relative to $\Gamma^\star$.

Proof. The closed-loop error dynamics resulting from applying the control law (4.33) to the dynamics on $\Gamma^\star$ given in (4.8a) are

$$\begin{aligned}\dot{\mathbf{e}}_1 &= \mathbf{e}_2 \\ \dot{\mathbf{e}}_2 &= H \sin(\mathbf{e}_1) - k\mathbf{I}_N\mathbf{e}_2\end{aligned} \tag{4.34}$$

where $\sin(\mathbf{e}_1) \coloneqq (\sin(e_{1,1}), \cdots, \sin(e_{1,N}))$. Linearizing the above closed-loop about $(\mathbf{e}_1, \mathbf{e}_2) = (0, 0)$ results in error dynamics given in (4.28). Proposition 4.5.13 shows that $(0, 0)$ is globally exponentially stable for error dynamics (4.28). Therefore $(0, 0)$ is locally exponentially stable for (4.34). □

Remark 4.5.16. When all the paths are closed the multi-agent path following manifold is diffeomorphic to $\mathbb{T}^N \times \mathbb{R}^N$. Since $\mathbf{v}^\parallel$ is 2π-periodic in its N arguments it is a continuous function that is defined globally on the multi-agent path following manifold.

4.6 Experimental Implementation

In this section a coordinated path following problem for two unicycles is considered and the nested invariant set approach is employed to design controllers. The experimental goal is to implement the designed controllers on two robots to examine their performance in practice.

4.6.1 Experimental setup

We experimentally verify our results using two TurtleBots built by Clearpath Inc. The robots have a maximum translational speed of 65 cm/s and maximum rotational speed of π rad/s. Each robot is controlled using the Robot Operating System (R.O.S.) running on an Intel Atom Notebook with Linux. An Indoor Positioning System (I.P.S.) using NaturalPoint OptiTrack provides the states (x_i, y_i, θ_i) of robot i over WiFi at 100Hz. A practical consideration is that the I.P.S. system is sensitive to shiny objects and one needs to assure such objects are covered. The state v_i of robot i is obtained by integrating the control input $u_{i,1}$. In Figure 4.8 robots and I.P.S. system are shown.

(a) robot modified for use with IPS system

(b) Laboratory

Figure 4.8: Unicycle robot and experimental setup. IPS cameras provide the position and orientation of robots.

4.6.2 Coordination specification

The robots are assigned circular paths $\gamma_i = \{\mathbf{y}_i \in \mathbb{R}^2 : s_i(\mathbf{y}_i) = 0\}$, $i \in \{1, 2\}$, $s_i(\mathbf{y}_i) = \|\mathbf{y}_i\|_2^2 - r_i^2$, $r_1 = 1.1$, $r_2 = 0.75$ metres. These paths satisfy inequality (1.4) of Assumption 1.2.1 with $\rho_{i,1}$, $\rho_{i,2}$, taken as identity functions.

Since both robots can communicate with each other the communication graph is fully connected and **S4** is trivially satisfied. The coordination specification **S2** is that the two robots be on opposite sides of their respective circles. Such a coordination specification can arise in patrolling applications because it results in better coverage of an area. For **S3** we require that robots 1 and 2 move with a prescribed angular velocity $\omega_d = .03$ rad/s. Note that $\eta'_{i,1} = \frac{\eta_{i,1}}{r_i}, \eta'_{i,2} = \frac{\eta_{i,2}}{r_i}$ $i \in \{1, 2\}$. Thus the linear-affine coordination function (4.24) becomes

$$\beta(\boldsymbol{\eta}') = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \eta'_{1,1} \\ \eta'_{2,1} \\ \eta'_{1,2} \\ \eta'_{2,2} \end{bmatrix} - \begin{bmatrix} \pi \\ -\pi \\ \omega_d \\ \omega_d \end{bmatrix}.$$

Using Proposition 4.4.2, condition (1), we immediately find that $\mathcal{C} = \mathcal{C}^\star$ and $\dim \mathcal{C}^\star = 2$. After bringing each robot into the normal form (4.7), the transversal control laws are taken to be (4.17) with the high gains $k_{i,1} = 30, k_{i,2} = 20$, $i \in \{1, 2\}$ to make the robots approach their paths quickly. The coordination control laws are obtained using Corollary 4.5.15

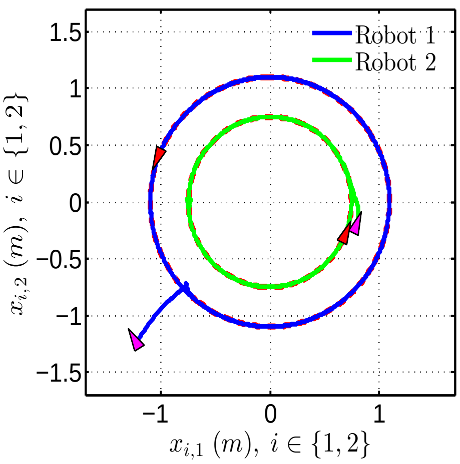
$$\begin{aligned} v_1^{\parallel} &= w_{12} \sin\left(\eta'_{2,1} - \eta'_{1,1} - \pi\right) - k\left(\eta'_{1,2} - \omega_d\right) \\ v_2^{\parallel} &= w_{21} \sin\left(\eta'_{1,1} - \eta'_{2,1} - \pi\right) - k\left(\eta'_{2,2} - \omega_d\right) \end{aligned} \tag{4.35}$$

where $w_{12}, w_{21}, k > 0$. The communication graph weights are treated as controller gains and taken to be $w_{21} = w_{12} = k = 10$. These gains are smaller, relatively, than the path following controller gains because we prioritize convergence to the paths over coordination.

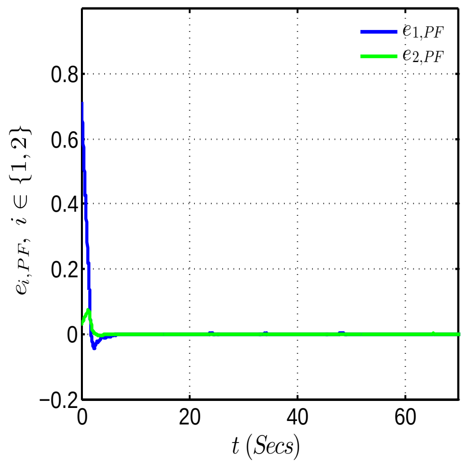
Experimental output trajectories are shown in Figure 4.9(a). The path error for each unicycle is computed as

$$e_{i,\mathrm{PF}} := \sqrt{x_i^2 + y_i^2} - r_i, \ i \in \{1, 2\} \tag{4.36}$$

and shown in Figure 4.9(b).



(a) Output trajectories: initial positions and final positions of robots are indicated, respectively, by magneta and red triangles



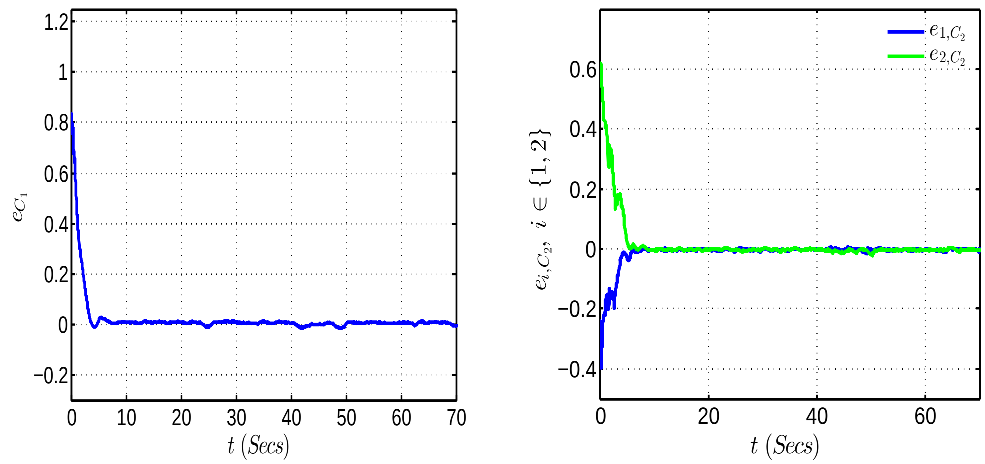
(b) Path following errors for robots 1 and 2

Figure 4.9: Experimental results: path following of robots while maintaining a phase difference of π.

Figure 4.10(a) displays the coordination error, $e_{\mathrm{C1}} := \eta'_{1,1} - \eta'_{2,1} - \pi$, expressed in radians, converging to zero. Figure 4.10(b) shows that each robot's angular velocity error, $e_{i,\mathrm{C2}} := \eta'_{i,2} - \omega_d, i \in \{1, 2\}$, converges to zero quickly. Figures 4.11(a) and 4.11(b) show the control effort required in these experiments.

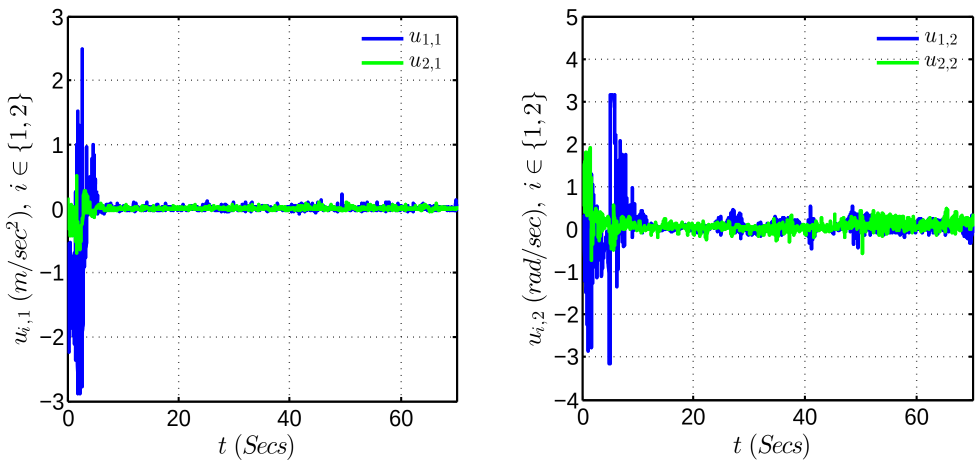
4.6.3 Switching coordination specifications

A distinguishing feature of the proposed controllers is that the paths γ_1 and γ_2 are invariant for robots 1 and 2. Therefore, if the coordination specification changes we expect the robots



(a) Coordination error (b) Angular velocity error

Figure 4.10: Experimental results: coordination and angular velocity error for robots while maintaining a phase difference of π.

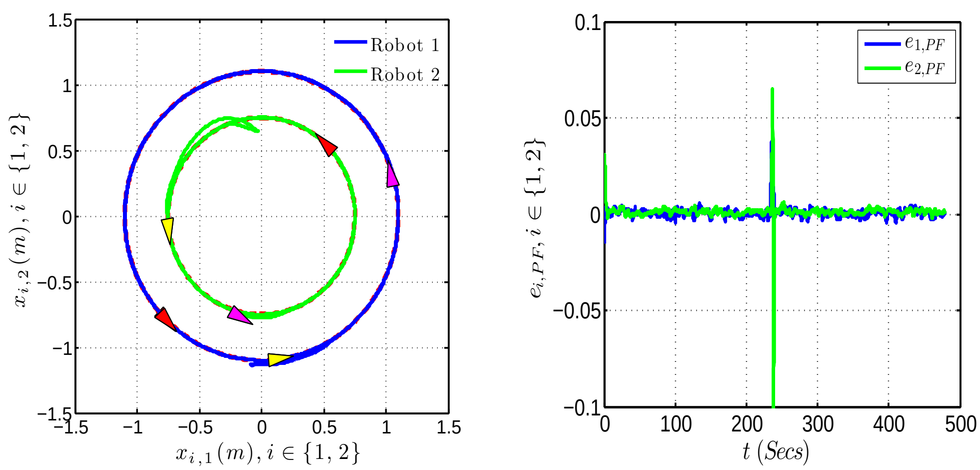


(a) Control signals $u_{i,1}$, $i \in \mathbf{2}$ (b) Control signals $u_{i,2}$, $i \in \mathbf{2}$

Figure 4.11: Experimental results: control signals $u_{i,j}$, $i, j \in \mathbf{2}$, while maintaining a phase difference of π.

to remain on their assigned paths, before eventually re-coordinating. In this experiment we ask that robot 1 initially be phase shifted by $\frac{\pi}{2}$ radians from robot 2 and after 240 seconds the phase difference change to π.

Figure 4.12(a) shows experimental output trajectories of the robots. When the coordination specification is changed the robots are expected to stay on their paths. In this experiment the robots actually leave their paths because their forward velocities, v_i, pass through 0 which are singularities. The control signals remain bounded because we enforce actuator constraints $|u_{i,2}| \leq \frac{\pi}{2}$ and $|v_i| \leq 0.5$, once the robots' forward velocities become non-zero, the nominal controllers take over and drive them back to their paths as shown in Figure 4.12(b). Figure 4.13(a) plots the difference in phase $\eta'_{1,1} - \eta'_{2,1}$. It is initially $\pi/2$

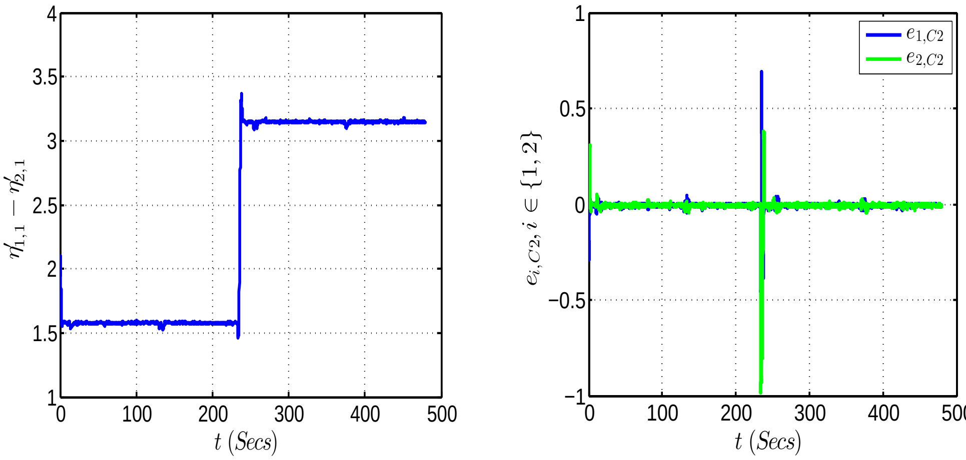


(a) Output trajectories for experiment 2: initial positions, positions before change of coordination task, and final positions of robots are indicated, respectively, by magneta, yellow, and red triangles

(b) Path following errors for robots 1 and 2

Figure 4.12: Experiment 2: coordinated path following with changing coordination task.

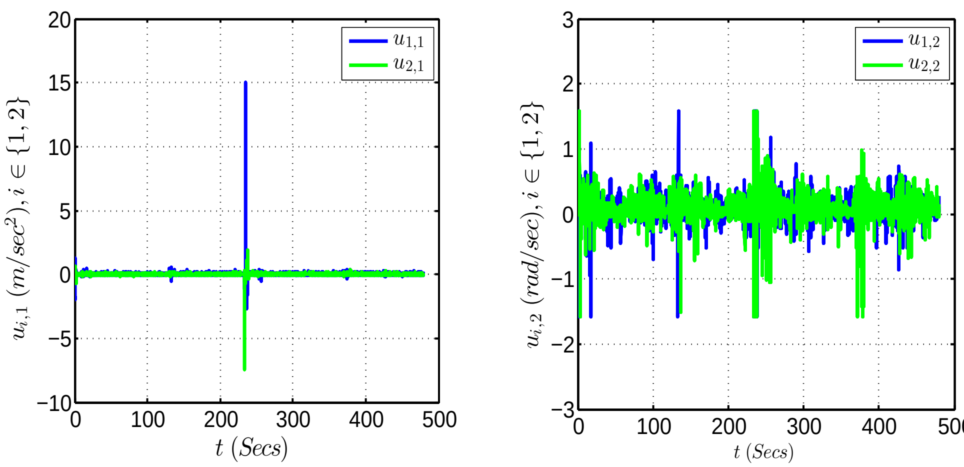
and at $t = 240$ it increases to π. Figure 4.13(b) shows that each robot's angular velocity error, $e_{i,C2}$, converges to zero quickly. Figures 4.14(a) and 4.14(b) illustrate control signals $u_{i,1}$ and $u_{i,2}$ for $i \in \{1, 2\}$, respectively.



(a) Phase difference of robots on circular paths

(b) Angular velocity error

Figure 4.13: Experiment 2: phase difference between robots and angular velocity error when the coordination specification changes.



(a) Control signals $u_{i,1}$, $i \in \mathbf{2}$

(b) Control signals $u_{i,2}$, $i \in \mathbf{2}$

Figure 4.14: Experiment 2: control signals $u_{i,j}$, $i, j \in \mathbf{2}$ when the coordination specification changes.

Chapter 5

Conclusions and future research

In this thesis we utilized a hierarchical control design approach to solve a coordinated path following problem for multi-agent systems. That is, the problem was split into the path following and the coordination sub-problems, and the path following sub-problem was prioritized over the coordination sub-problem. This viewpoint allowed us to design the path following and the coordination controllers separately.

In order to design the path following and the coordination controllers, we cast each sub-problem as a set stabilization problem. The path following sub-problem being prioritized over coordination sub-problem, implied that the set corresponding to coordination must be contained, nested, in the set corresponding to the path following sub-problem. This motivated us to call the two set stabilization problems a nested set stabilization problem. Under suitable assumption, the two sets turned out to be embedded, controlled invariant, submanifolds.

We proposed a method based on local feedback equivalence of control systems to design stabilizing control laws. The proposed method was not limited to the coordinated path following problem and may be employed to solve control specifications that can be broken down hierarchically.

We introduced normal forms (3.3) and (3.33) for a class of nonlinear control systems with two nested, controlled invariant, embedded submanifolds in its state space. Normal form (3.3) was particularly useful in designing controllers for stabilizing the nested set relative to the larger set. The normal form (3.33) was useful for designing controllers both for the stabilization of the larger set as well as the stabilization of the nested set relative to the larger set. Nevertheless, the class of control-affine systems that can be transformed to normal form (3.33) is strictly smaller than the class of systems that can be transformed to normal form (3.3). Moreover, the class of the system that can be transformed to a normal form in which ξ-subsystem and μ subsystem are decoupled is strictly smaller than the class of systems that can be transformed to normal form (3.33).

Whether or not a nonlinear control-affine system, with two nested, controlled invariant, embedded submanifolds in its state space, is locally feedback equivalent to normal form (3.3) was asked in Problem 1. In Theorem 3.4.2 we presented necessary and sufficient conditions under which Problem 1 is solvable. When Problem 1 was solvable, Problem 4 asked whether one can further refine normal form (3.3) and make the dynamics transversal to the larger set linear and controllable as well. Theorem 3.5.1, presented further necessary and sufficient conditions under which Problem 4 was solvable. In general, there is no guarantee that the controllers stabilizing the nested set relative to the larger set and those stabilizing the larger set work as desired when they work together. In Theorem 3.6.1 we presented sufficient conditions which address this stability concern.

To illustrate these ideas we considered a coordinated path following problem for a multi-agent system of dynamic unicycles in Chapter 4. The multi-agent path following manifold was characterized for an a large class paths. We showed that each unicycle was feedback equivalent, in a neighbourhood of its assigned path, to a system whose transversal and tangential dynamics to the path following manifold are both double integrators. We provided sufficient conditions under which the coordination set was nonempty. The effectiveness of the proposed approach was demonstrated experimentally on two robots.

5.1 Future research

The research in this thesis can be continued in different directions, some of which are highlighted below

5.1.1 Decentralized control laws

In order for the agents in a multi-agent system to accomplish a coordination specification they need to exchange state information. The main underlying assumption in this thesis is that the communication graph is fully connected. However, in some applications this assumption is not realistic and each agent can only communicate with certain agents called its neighbors. Future research entails taking into account communication constraints for general coordination constraints. There are open fundamental questions that need to be solved. For instance, even in the linear case, how does one characterize the largest invariant subspace for decentralized systems is an open problem?

5.1.2 Global results

One important issue is that the Problems 1, 3, and 4 are only valid in a neighbourhood of a point. Hence, a global version of Problems 1, 3, and 4 can be posed. Roughly speaking, one seeks a coordinate and feedback transformation such that (3.1) is feedback equivalent to (3.3), in Problem 1, to (3.17), in Problem 3, and to (3.33), in Problem 4, in a tubular neighbourhood of the larger set. Accordingly, the global version of Problem 4 can be employed to solve the global version of the nested set stabilization problem where the entire sets are considered.

In [67] it is shown that global transverse feedback linearization is restrictive because the target sets are required to be diffeomorphic to finite dimensional vector spaces. We conjecture that the global version of Problems 1, 3, and 4 would be as restrictive.

5.1.3 Relative coordinates

The control laws for coordinated path following of unicycles are expressed in terms of a global frame. However, in some applications the global coordinates of unicycles might not be available. Thus, it is of interest to develop control laws that depend on the local frames of unicycles. In other words, to develop control laws that are expressed in terms of relative distances and heading angles between unicycles.

5.1.4 Practical issues

As far as the multi-agent system of dynamic unicycles is concerned there are some practical issues. The first limitation is that unicycles cannot have zero forward velocity in our formulation. Hence, a future research direction is developing effective singularity avoidance methods alongside with proposed control laws. Another major limitation is that the possibility of collision is neglected in our formulation. However, it is not a safe assumption in reality. Hence, a future research direction entails employing collision avoidance techniques in conjunction with the results of this thesis. Finally, if the control laws use relative coordinates instead of the global coordinates their implementation becomes much easier. Another research direction involves designing distributed control laws for each agent based on the relative coordinates of the unicycles it can communicate with.

APPENDICES

In this chapter we review the mathematical concepts and supporting results used in this thesis. The materials are taken from [62, 41, 90, 67, 45, 52, 8]

A Graph theory

Throughout this thesis we model communication between agents of a multi-agent system using a weighted directed graph $\mathscr{G}$ called the *communication graph*. Let $\mathsf{V}(\mathscr{G}) = \{a_1, \ldots, a_N\}$ and $\mathsf{E}(\mathscr{G}) \subseteq \mathsf{V}(\mathscr{G}) \times \mathsf{V}(\mathscr{G})$ be, respectively, the vertex and edge set of $\mathscr{G}$. Each vertex represents an agent and an edge (a_i, a_j) indicates that agent j receives information from agent i. We denote by $w_{ij} \in \mathbb{R}$ the weight associated to edge (a_i, a_j). For each vertex we can define the *in-degree*, denoted $d_{\text{in}}(a_i)$, as

$$d_{\text{in}}(a_i) := \sum_{(a_j,a_i)\in \mathsf{E}} w_{ji}.$$

For each vertex we can also define the *out-degree*, denoted $d_{\text{out}}(v_i)$, as

$$d_{\text{out}}(a_i) := \sum_{(a_i,a_j)\in \mathsf{E}} w_{ji}.$$

When a graph is undirected the in-degree and out-degree of a vertex are equal. In the following we define matrices associated with a graph based on its in-degrees. It should be noted that the following matrices can be defined based on out-degrees in an analogous manner.

Definition A.1. The **in-degree matrix** of $\mathscr{G}$ is $\Delta(\mathscr{G}) \coloneqq \mathrm{diag}(d_{\mathrm{in}}(a_1), \ldots, d_{\mathrm{in}}(a_N))$. •

Definition A.2. The **in-degree adjacency matrix** of $\mathscr{G}$ is an $N \times N$ matrix whose ij-th element is given by

$$W(\mathscr{G})_{i,j} \coloneqq \begin{cases} w_{ji} & \text{if } (a_j, a_i) \in \mathsf{E}(\mathscr{G}) \\ 0 & \text{otherwise.} \end{cases}$$

•

Definition A.3. The **weighed graph Laplacian** of $\mathscr{G}$ is $L(\mathscr{G}) \coloneqq \Delta(\mathscr{G}) - W(\mathscr{G})$. •

Remark A.4. The sum of the columns of the Laplacian matrix $L(\mathscr{G})$ is 0 [76]; thus 0 is an eigenvalue of $L(\mathscr{G})$ with the associated eigenvector $\mathrm{col}(1, \cdots, 1)$.

Definition A.5. A directed graph $\mathscr{G}$ is **rooted out-branching** if it does not contain a directed cycle and there exists a vertex $a_r \in \mathsf{V}(\mathscr{G})$ such that for all $a_i \in \mathsf{V}(\mathscr{G})$ there is a directed path from a_r to a_i. •

B Linear algebra

This section gathers a few facts from linear algebra used throughout of this thesis, specially Section 3.2. All vector spaces in this thesis are finite-dimensional. For a given subspace $\mathscr{X}$ we define its **dual space**, written $\mathscr{X}'$, which is defined as the set of all linear functions $x' : \mathscr{X} \to \mathbb{R}$. For a subspace $\mathscr{V}$ of the vector space $\mathscr{X}$, we define its **annihilator**, written $\operatorname{ann}(\mathscr{V})$, to be the following set

$$\operatorname{ann}(\mathscr{V}) = \{x' \in \mathscr{X}' : x'\mathscr{V} = 0\}.$$

That is the set of all linear functionals on $\mathscr{X}$ annihilating $\mathscr{V}$. It should be noted that $\operatorname{ann}(\mathscr{V})$ is a subspace of $\mathscr{X}'$. When, $\mathscr{X}$ is an inner product vector space, we identify $\mathscr{X}$ with its dual, and hence consider $\operatorname{ann}(\mathscr{V})$ as a subspace of $\mathscr{X}$. That is,

$$\operatorname{ann}(\mathscr{V}) = \{x \in \mathscr{X} : \langle x,\ v\rangle = 0, \forall v \in \mathscr{V}\}.$$

Let $\mathscr{V}$ and $\mathscr{W}$ be two subspaces of the vector space $\mathscr{X}$. They are **independent** if $\mathscr{V} \cap \mathscr{W} = 0$. Their sum is called the **direct sum** and is denoted $\mathscr{V} \oplus \mathscr{W}$. If $\mathscr{X} = \mathscr{V} \oplus \mathscr{W}$ Then each $x \in \mathscr{X}$ has a unique representation $x = v + w$ with $v \in \mathscr{V}$ and $w \in \mathscr{W}$. The linear transformation

$$Q : \mathscr{X} \to \mathscr{V}$$

$$x \mapsto v$$

is called the **natural projection** on $\mathscr{V}$ along $\mathscr{W}$.

For a linear map $A : \mathscr{V} \to \mathscr{W}$ we define a linear map, written A', called **dual map**. The Dual map satisfies

$$A'(w') = w'A, \qquad w' \in \mathscr{W}'.$$

The matrix representation of the dual map A' is given by $A' = A^\top$.

Remark B.1. Let $\mathscr{V}$ and $\mathscr{W}$ be two vector spaces. Let $A : \mathscr{V} \to \mathscr{W}$ be a linear transformation with $\mathscr{A} \coloneqq \operatorname{Im} A \subseteq \mathscr{W}$ and $A' : \mathscr{V}' \to \mathscr{W}'$ its dual map. The annihilator $\operatorname{ann}(\mathscr{A})$ of $\mathscr{A}$ can be characterized as $\operatorname{ann}(\mathscr{A}) = \ker(A')$. If $\mathscr{K} = \ker A$. The annihilator of $\operatorname{ann}(\mathscr{K})$ of $\mathscr{K}$ can be charecterized as $\operatorname{ann}(\mathscr{K}) = \operatorname{Im}(A')$.

The following result is used in Chapter 3.

Lemma B.2. *Consider an LTI system*

$$\dot{x} = Ax + Bu$$

where $x \in \mathscr{X}$ is the state and $u \in \mathscr{U}$ is the control input. Let $\mathscr{K} \coloneqq \{x \in \mathscr{X} : Tx = 0\}$ be a subspace of $\mathscr{X}$ where $T : \mathscr{X} \to \mathscr{Z}$ is a surjective linear map. Then, $Tx \to 0$ if and only if $x \to \mathscr{K}$.

Proof. The solution to the LTI system with $x(0) = x_0$ is

$$x(t) = \mathrm{e}^{tA}x_0 + \int_0^t e^{(t-\tau)A}Bu(\tau)d\tau.$$

Assume $\lim_{t\longrightarrow\infty} Tx(t) \longrightarrow 0$. Let z' be an arbitrary vector in the dual space $\mathscr{Z}'$ of $\mathscr{Z}$. Then $\lim_{t\to\infty} z'Tx(t) \to 0$. Define $x' = z'T \in \operatorname{Im}(T')$. Thus, as discussed in Remark B.1, x' belongs to $\operatorname{ann}(\mathscr{K})$. By substituting x' for $z'T$ we obtain $\lim_{t\to\infty} x'x(t) \to 0$. Since, $x' \in \operatorname{ann}(\mathscr{K})$ we have that $\lim_{t\to\infty} x(t) \to \mathscr{K}$.

Conversely, Assume $\lim_{t\to\infty} x(t) \to \mathscr{K}$. Therefore, for an arbitrary $x' \in \operatorname{ann}(\mathscr{K}) \subseteq \mathscr{X}'$ we have that $\lim_{t\to\infty} x'x(t) \to 0$. Since T is surjective there exists $z' \in \mathscr{Z}'$ such that $x' = z'T$. Therefore, $\lim_{t\to\infty} z'Tx(t) \to 0$. Since x' is arbitrary so is z'. Thus, $\lim_{t\to\infty} Tx(t) \to 0$. $\square$

B.1 Quotient spaces

If $\mathscr{V}$ is a vector space and $\mathscr{W}$ is a subspace of $\mathscr{V}$, a **coset** of $\mathscr{W}$ in $\mathscr{V}$ is a subset of the form

$$\bar{v} \coloneqq \{v + w : w \in \mathscr{W}\}$$

for some $v \in \mathscr{V}$. Geometrically, $\bar{v}$ is the hyperplane passing through v obtained by parallel translation of $\mathscr{W}$. Two cosets $\bar{v}$ and $\bar{v'}$ are equal if and only if $v' - v \in \mathscr{W}$. This introduces an equivalence relation on $\mathscr{V}$ and a coset is an equivalence class. The set of all cosets of $\mathscr{W}$ in $\mathscr{V}$ is again a vector space called **quotient space** and is denoted by $\mathscr{V}/\mathscr{W}$. In the following we define a projection map between $\mathscr{V}$ and $\mathscr{V}/\mathscr{W}$.

Definition B.3. Let $\mathscr{V}$ be a vector space and $\mathscr{W}$ a subspace of $\mathscr{V}$. Let $\mathscr{V}/\mathscr{W}$ be the quotient space of $\mathscr{W}$. The map $P : \mathscr{V} \longrightarrow \mathscr{V}/\mathscr{W}$ which is surjective and $\operatorname{Ker} P = \mathscr{W}$ is called the **canonical projection** of $\mathscr{V}$ onto $\mathscr{V}/\mathscr{W}$. $\bullet$

C Differential geometry

In this section the notions and definitions from differential geometry along with supporting results are presented.

C.1 Smooth manifolds

Intuitively, a manifold is a generalization of curves and surfaces in $\mathbb{R}^3$ to higher dimensions. It is locally Euclidean in that every point has a neighbourhood homeomorphic to an open set of Euclidean space. We first recall a few basic definitions from topology.

Definition C.1. A **topology** on a set M is a collection τ of subsets of M, called **open sets**, satisfying

(i) M and $\varnothing$ are open.

(ii) The union of any family of open sets is open.

(iii) The intersection of any finite family of open sets is open.

•

Definition C.2. The pair $(M, \mathcal{T})$ is called a **topological space**. When, the topology is understood, we simply write M is a topological space. •

Definition C.3. Let $(M, \mathcal{T})$ be a topological space and $N \subset M$ be any subset of M. The **subspace topology** on N is

$$\mathcal{T}_N \coloneqq \{N \cap U : U \in \mathcal{T}\}.$$

•

Having introduced a topological space in the following we introduce topological spaces which locally around each point look like a Euclidean space.

Definition C.4. Suppose M is a topological space. We say that M is a **topological manifold** of dimension **m** or **topological m-manifold** if it has the following properties

(i) M is a **Hausdorff space**: for every pair of points $p, q \in M$, there are disjoint open subsets $U, V \subset M$ such that $p \in U$ and $q \in V$.

(ii) M is **second countable**: there exists a countable basis for the topology of M.

(iii) M is **locally Euclidean of dimension m**: every point of M has a neighbourhood that is homeomorphic to an open subset of $\mathbb{R}^m$.

•

A topological manifold is locally Euclidean; however, it is not clear how to do calculus on it. Thus, in th following we construct smooth structures for topological manifolds. The notion of coordinate charts is essential in our construction

Definition C.5. Let M be a topological m-manifold. A **coordinate chart** on M is a pair (U, φ), where U is an open subset of M and $\varphi : U \to \tilde{U}$ is a homeomorphism from U to an open subset $\tilde{U} = \varphi(U) \subseteq \mathbb{R}^m$. •

Let $\varphi_1, \cdots, \varphi_m$ denote the components of φ so that $\varphi = \mathrm{col}(\varphi_1, \cdots, \varphi_m)$. The functions $\varphi_i : W \to \mathbb{R}$ are called local coordinate functions and, for each point $p \in W$, the values $\varphi_1(p), \cdots, \varphi_m(p)$ are the local coordinates of p. Having defined a coordinate chart the next step is to define the notion of compatibility of two coordinate charts which, loosely speaking, guarantees that overlapping charts are related by a differential map.

Definition C.6. Let M be a topological m-manifold. If (U, φ) and (V, ψ) are two coordinate charts such that $U \cap V \neq \varnothing$, the composite map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is called the **transition map**. These two charts are said to be **smoothly compatible** if either $U \cap V = \varnothing$ or the transition map is a diffeomorphism. •

A collection of smoothly compatible coordinate charts that cover M is an **smooth atlas** denoted by $\mathscr{A}$. Generally, a topological manifold M may admit more than one atlases. For example, consider the following pair of atlases on $\mathbb{R}^n$

$$\mathscr{A}_1 = \{\mathbb{R}^n . \mathbb{1}_{\mathbb{R}^n}\}, \qquad \mathscr{A}_2 = \left\{(B_1(x), \mathbb{1}_{B_1(x)}) : \forall x \in \mathbb{R}^n\right\}.$$

the above atlases are however equivalent in the sense that their coordinate charts are smoothly compatible. One can define an equivalence relation between atlases of a topological manifold M as follows: $\mathscr{A}_1$ and $\mathscr{A}_2$ are equivalent if and only if their union is an atlas on M. This view point allows us to define a well-defined smooth structure

Definition C.7. A **smooth structure** on a topological m-manifold M is an equivalence class of equivalent atlases on M. •

Having introduced the smooth structure we formally define a smooth manifold as follows

Definition C.8. A **smooth manifold** is a pair $(M, \mathscr{A})$, where M is a topological m-manifold and $\mathscr{A}$ is a smooth structure on M. When the smooth structure is understood we just simply say M is a smooth manifold. •

The main reason for introducing smooth structures for topological manifolds is to enable one to carry over many concepts from Euclidean spaces to topological manifolds. In the following we define differential maps on manifolds

Definition C.9. Suppose that M and N are smooth manifolds. We say that a map $f : M \to N$ is of class C^k, $0 \leq k \leq \infty$, if for each $p \in M$ and each chart (V, ψ) of N with $f(p) \in V$, there exist a chart (U, φ) of M with $p \in U$ and $f(U) \subseteq V$, such that the **local representation of** f

$$\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$$

is of class C^k. •

Definition C.10. A map $f : M \to N$ between smooth manifolds M and N is a **diffeomorphism** if it is a smooth bijection and $f^{-1} : N \to M$ is smooth. If a diffeomorphism exists between two manifolds, they are called **diffeomorphic**. •

Definition C.11. Let M be a topological m-manifold. The symbol $C^\infty(M)$ denotes the ring of smooth real-valued functions $f : M \to \mathbb{R}$ on M. •

C.2 Submanifolds

Definition C.12. Let M be an m-dimensional smooth manifold. A subset $N \subset M$ is called an **embedded submanifold** of dimension $n \leq m$ if for each $p \in N$ there exists a coordinate chart (U, φ) of M, with $p \in U$, such that

$$N \cap U = \{q \in U : \varphi_{n+1}(q) = \cdots = \varphi_m(q) = 0\}.$$

Henceforth, we will refer to N simply as a submanifold of M. Submanifold N is also called n-**slice** of U. •

Theorem C.13 ([62], Theorem 8.2)**.** *Let $N \subset M$ be an embedded submanifold of dimension n. With the subspace topology, N is a topological manifold of dimension n, and has a unique smooth structure such that the inclusion map $i : N \to M$ is a smooth embedding.*

In the light of above theorem we introduce the smooth structure for N. Let $\pi : \mathbb{R}^m \to \mathbb{R}^n$ be the projection $\pi : (x_1. \cdots , x_m) \mapsto (x_1, \cdots , x_n)$ and let (U, φ) be a coordinate chart in the smooth structure of M. Then $(N \cap U, \pi \circ \varphi|_N)$ is a coordinate chart of N and these charts endow N with a smooth structure.

C.3 Tangent space

The notion of tangent space at a point of a manifold generalizes the concept of the tangent space plane at a point of a surface in $\mathbb{R}^3$. There are different equivalent ways to define a tangent space at a point of a manifold. We opt to use the notion of derivations at a point to define the tangent space at a point of a manifold.

Definition C.14. Let M be a smooth m-dimensional manifold and $p \in M$. A linear map $X : C^\infty(M) \to \mathbb{R}$ is called a **derivative at** p if it satisfies

$$X(fg) = f(p)Xg + g(p)Xf$$

for all $f, g \in C^\infty(M)$. The set of all derivations of $C^\infty(M)$ at p is a vector space of dimension m called the **tangent space** to M at p and is denoted by T_pM. An element of T_pM is called a **tangent vector at** p. •

a manifold can be approximated near a point by its tangent space at the point. Just as in $\mathbb{R}^n$, any smooth map can be approximated at a point of a manifold with a linear map defined in the following

Definition C.15. Let $f : M \to N$ be a smooth map between smooth manifolds M and N. We define a map $f_\star : T_pM \to T_{f(p)}N$, called the **push-forward** associated with f, by

$$(f_\star X)f = X(\phi \circ f)$$

for all $\phi \in C^\infty(M)$. •

C.4 Vector fields

Having introduced the notion of tangent spaces we would like to generalize the notion of vector fields to manifolds. the following definition is important in our generalization

Definition C.16. For any smooth m-dimensional manifold M we define the **tangent bundle** of M, denoted by TM, to be the disjoint union of the tangent spaces at all points of M:

$$TM := \bigcup_{p \in M} T_pM.$$

an element of TM can be taken to be a pair (p, X) with $p \in M$ and $X \in T_pM$. The map $\pi : TM \to M, (p, X) \mapsto p$, is called the **natural projection** of TM onto M. •

A vector field on a manifold M is the assignment of a tangent vector $X_p \in T_pM$ to each point $p \in M$. More formally,

Definition C.17. Let M be a smooth manifold. A **vector field** is a continuous map $Y : M \to TM$ with the property that

$$\pi \circ Y = \mathbb{1}_M.$$

•

The following result is known as inverse function theorem.

Theorem C.18 ([62], Theorem 7.10.)**.** *Suppose M and N are smooth manifolds, $p \in M$, and $f : M \to N$ is a smooth map such that $f_\star : T_pM \to T_{f(p)}N$ is bijective. Then, there exists a connected neighbourhood U_0 of p and V_0 of $f(p)$ such that $f|_{U_0} : U_0 \to V_0$ is a diffeomorphism.*

In this thesis we will need a generalization of the inverse function theorem as discussed in [41].

Theorem C.19. *Suppose that $f : M \to N$ is a smooth map between manifolds. Let $S \subset M$ be a submanifold of M and assume that*

(i) $f_\star : T_pM \to T_{f(p)}N$ is bijective for every $p \in S$

(ii) $f|_P$ maps S diffeomorphically onto $f(S)$.

Then, f amps a neighbourhood of S diffeomorphically onto a neighbourhood of $f(S)$.

Just as in $\mathbb{R}^n$ vector fields give rise to integral curves. Integral curves of a vector field in $\mathbb{R}^n$ are curves whose velocity at any point is the given vector field at the point. In the following we generalize the notion of integral curves to smooth manifolds

Definition C.20. Let Y be a smooth vector field on smooth manifold M. An **integral curve** of Y through a point $p \in M$ is a curve $c(t)$ at p such that the tangent vector at every point $q = c(t)$ coincides with $Y(q)$. •

Given a smooth vector field Y on a smooth manifold M there exist a unique **maximal integral curve** through p which we denote by $\phi_t^Y(p)$. The maximal integral curve $\phi_t^Y(p)$ is also called **flow** generated by the vector field Y.

In the following we generalize the Lie derivative and Lie bracket notions of $\mathbb{R}^n$ to smooth manifolds.

Definition C.21. If Y is a smooth vector field on M and $\lambda \in C^\infty(M)$ then the derivative of λ along Y is a function $L_Y\lambda$ defined by

$$L_Y\lambda(p) = \lim_{h\to 0} \frac{1}{h}\left[\lambda(\phi_h^Y(p)) - \lambda(p)\right]$$

and called **Lie derivative** of λ along Y at p. It is an element of $C^\infty(M)$. •

Definition C.22. If f and g are two smooth vector fields on a smooth manifold M, the **Lie bracket** of f and g is a smooth vector field $[f, g]$ defined by the relation

$$L_{[f,g]}\lambda = L_f(L_g\lambda) - L_g(L_f\lambda).$$

•

We will use the following standard notation for iterated Lie derivatives and Lie brackets

$$L_g L_f\lambda := L_g(L_f\lambda),$$

$$L_g^0\lambda := \lambda, \quad L_g^k\lambda := L_g(L_f^{k-1}\lambda),$$

$$ad_f^0 g := g, \quad ad_f^k g := \left[f, ad_f^{k-1}g\right], \quad k \geq 1.$$

C.5 Distributions

Distributions are generalization of vector fields on manifolds and are defined in the following

Definition C.23. Let M be a smooth manifold. A choice of k-dimensional linear subspace $D_p \subseteq T_pM$ at each point $p \in M$ is called a **distribution** on M. A distribution D is called smooth if for each $p \in M$, there exists a neighbourhood U of p and smooth vector fields $f_1, \cdots, f_k$ such that,

$$(\forall q \in U) \quad D(q) = \operatorname{span}\{f_1(q), \cdots, f_k(q)\}.$$

the vector fields $f_1, \cdots, f_k$ are called local **generators** around p. •

Definition C.24. A point $p \in M$ is a **regular point** of the distribution D if there exists a neighbourhood U containing p for which $\dim(D(q))$ is constant for all $q \in U$. In this case, D is said to be **nonsingular** on U. •

If p is a regular point of a distribution D with $\dim(D(p)) = k$, then there exist k linearly independent local generators around p, and we will write $D = \operatorname{span}\{f_1, \cdots, f_k\}$ on the domain of definition of the generators.

we defined integral curves associated to vector fields. Since, distributions are generalization of vector fields the natural question is whether there exist a submanifold whose tangent space at each point is the distribution at that point. In general, the answer is more complicated than the case of vector fields and the distribution must satisfy a nontrivial necessary condition, called involutivity, introduced in the following definition

Definition C.25. A distribution D on M is called **involutive** if the lie bracket of any pair of smooth vector fields Y_1 and Y_2 is a vector field in D, i.e.,

$$Y_1 \in D,\ Y_2 \in D \to [Y_1, Y_2] \in D.$$

•

The following lemma provides a tractable way of checking involutivity for a distribution

Lemma C.26. *A distribution D with local generators $f_1, \cdots, f_k$ is involutive if and only if*

$$[f_i, f_j] \in D, \qquad \forall i, j \in 1, \cdots, d.$$

The intersection of smooth distribution D_1 and D_2, $D_1 \cap D_2$, defined by

$$(D_1 \cap D_2)(p) \coloneqq D_1(p) \cap D_2(p),$$

may fail to be a smooth distribution. The next lemma guarantees conditions for smoothness.

Lemma C.27. *Let p be a reqular point of the smooth distributions D_1 and D_2. If p is also a regular point of $D_1 \cap D_2$ then there exist a neighbourhood U of p such that the restriction of $D_1 \cap D_2$ to U is smooth.*

Definition C.28. If D is a distribution, the **involutive closure** of D, written $\mathrm{inv}(D)$, is a distribution containing D with the property that if $\hat{D}$ is an involutive distribution containing D, then $\mathrm{inv}(D) \subseteq \hat{D}$ •

If D is a nonsingular distribution on a manifold M, $D^\perp$ is the orthogonal complement of D obtained from the orthogonal structure on the tangent bundle TM. The nonsingular distribution $D^\perp$ is a subbundle of TM and satisfies,

$$(\forall p \in M), \qquad T_pM = D(p) \oplus D^\perp(p).$$

If D is a distribution defined ona manifold M and $N \subset M$ is a submanifold we will consider the subbundles $TN + D$ and $TN \cap D$ of $TM|_M$ defined for each $p \in N$ by $T_pN + D(p)$ and $T_pN \cap D(p)$, respectively.

Lemma C.29 (Lemma 2.4.56, [67]). *Let $N \subset M$ be an n-dimensional submanifold of the m-dimensional manifold M. Let $p \in N$ be a regular point of a d-dimensional distribution D on M. Suppose there exists an open neighbourhood V of p in N such that $k = \dim(T_qN \cap D(q))$ is constant for all $q \in V$. Then, there exists a neighbourhood U of p in V such that $TN \cap D$ is smooth on U.*

D Control systems

Teh control systems considered in this thesis are control-affine system modeled by equations of the form

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i =: f(x) + g(x)u \tag{D.1}$$

where $x \in \mathbb{R}^n$ denotes the state, $u = (u_1, \cdots, u_m) \in \mathbb{R}^m$ is the control input, and $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g_i : \mathbb{R}^n \to \mathbb{R}^n$, $i \in \{1, \cdots, m\}$, are smooth.

Definition D.1. A closed connected submanifold $S \subseteq \mathbb{R}^n$ is **controlled-invariant** for (D.1) if there exists a smooth feedback $\bar{u} : S \to \mathbb{R}^m$ such that S is invariant for the closed-loop system $\dot{x} = f(x) + g(x)\bar{u}(x)$. •

Definition D.2. Given an open set $U \subseteq \mathbb{R}^n$, a **regular static feedback**, denoted (α, β), on U for control system (D.1) is a relation $u = \alpha(x) + \beta(x)v$ where $\alpha : U \to \mathbb{R}^m$ and $\beta : U \to \mathsf{GL}(m, \mathbb{R})$ are smooth mappings. •

Definition D.3. Two control systems, $\dot{x} = f + gu$ and $\dot{\hat{x}} = \hat{f} + \hat{g}\hat{u}$, are **feedback equivalent** on an open set $U \subseteq \mathbb{R}^n$ if there exist a regular static feedback (α, β) on U and a map $\Xi \in \mathrm{Diff}(U)$ such that $\hat{f} = \Xi_\star(f + g\alpha)$ and $\hat{g} = \Xi_\star(g\beta)$. •

Consider an LTI control system which is a special case of (D.1)

$$\dot{x} = Ax + Bu$$

where $x \in \mathscr{X}$ is the state of the system and $u \in \mathscr{U}$ is the control input. In the following we introduce two important vector spaces in its state space $\mathscr{X}$.

Definition D.4. Let $\mathscr{V}$ be a subspace of the state space $\mathscr{X}$. This subspace $\mathscr{V}$ is called (A, B)**-invariant** if there exist a state feedback $F : \mathscr{X} \to \mathscr{U}$ such that

$$(A + BF)\mathscr{V} \subseteq \mathscr{V}.$$

The state feedback F is called a **friend** of $\mathscr{V}$. •

Intuitively, a subspace $\mathscr{V}$ is (A, B)-invariant if one can find a state feedback control $u = Fx$ such that the solutions of $\dot{x} = (A + BF)x$ with $x_0 \in \mathscr{V}$ remains in $\mathscr{V}$ for all future time. The following result establishes a relation between the friend of two nested (A, B)-invariant subspaces.

Lemma D.5 ([62], Lemma 5.7)**.** *Let both $\mathscr{R}$ and $\mathscr{V}$ be (A, B)-invariant subspaces and suppose $\mathscr{R} \subset \mathscr{V}$. If F_0 is a friend of $\mathscr{R}$ there exists a mutual friend F of $\mathscr{R}$ and $\mathscr{V}$ such that*

$$F|_{\mathscr{R}} = F_0|_{\mathscr{R}}$$

An important subclass of (A, B)-invariant subspaces of an LTI control system is introduced in the following

Definition D.6. A subspace $\mathscr{R}$ is a **controllability subspace** of the pair (A, B) if there exist maps F and G such that

$$\mathscr{R} = \operatorname{Im} BG + (A + BF)\operatorname{Im} BG + \cdots + (A + BF)^{n-1}\operatorname{Im} BG. \tag{D.2}$$

•

If $\mathscr{R}$ is a controllability subspace the pair

$$((A + BF)|_{\mathscr{R}}, BG)$$

is controllable. The following result shows that the matrix G can be removed from the characterization of the controllability subspace $\mathscr{R}$. This result is used in the proof of Theorem 3.2.2.

Proposition D.7 (Proposition 5.2., [91])**.** *A subspace $\mathscr{R}$ is a controllability subspace if and only if there is an F such that*

$$\mathscr{R} = \operatorname{Im} B \cap \mathscr{R} + (A + BF) \operatorname{Im} B \cap \mathscr{R} + \cdots + (A + BF)^{n-1} \operatorname{Im} B \cap \mathscr{R}$$

Theorem D.8 ([92], Theorem 5.4)**.** *Consider an LTI system*

$$\dot{x} = Ax + Bu,$$

where $x \in \mathscr{X}$ is the state and $u \in \mathscr{U}$ is the control input. Every subspace $\mathscr{V} \subseteq \mathscr{X}$ contains a unique supremal controllability subspace denoted by $\mathscr{R}^\star$.

E Set stability

Here we review some basic notions of set stability.

Definition E.1.

1. A closed set S, invariant for $\dot{x} = f(x)$, is called **stable** if for all $\epsilon > 0$ there exists a neighbourhood $\mathcal{N}(S)$ such that for all $t \geq 0$, $\phi\left(t, \mathcal{N}(\Gamma_1)\right) \subset B_\epsilon(\Gamma_1)$.
2. A closed set S, invariant for $\dot{x} = f(x)$, is called an **attractor** for if there exists a neighbourhood $\mathcal{N}(S)$ such that $\lim_{t\to\infty} \|\phi(t, x_0)\|_S = 0$ for all $x_0 \in \mathcal{N}(S)$
3. A closed set S is asymptotically stable for $\dot{x} = f(x)$ if it is stable and attractive.

•

Definition E.2. ([28]).

1. The set S_2 is stable relative to S_1 for the dynamical system $\dot{x} = f(x)$ if, for all $\epsilon > 0$ there exists a neighbourhood $\mathcal{N}(S_2)$ such that for all $t \geq 0$, $\phi\left(t, \mathcal{N}(S_2) \cap S_1\right) \subset B_\epsilon(S_2) \cap S_1$.
2. The set S_2 is an attractor relative to S_1 for the dynamical system $\dot{x} = f(x)$ if there exists a neighbourhood $\mathcal{N}(S_2)$ such that $\text{dist}(\phi(t, x_0), S_2) \to 0$ as $t \to \infty$ for all $x_0 \in \mathcal{N}(S_2) \cap S_1$
3. The set S_2 is asymptotically stable relative to S_1 for (3.1) if it is stable and attractive relative to S_1 for the dynamical system $\dot{x} = f(x)$.

•

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