K-Theory for C*-Algebras and for Topological Spaces
by
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Author’s Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.
Abstract

K-theory is the study of a collection of abelian groups that are invariant to C*-algebras or to locally compact Hausdorff spaces. These groups are useful for distinguishing C*-algebras and topological spaces, and they are used in classification programs. In the thesis we will focus attention on the abelian groups $K_0(A)$ and $K^0(X)$ for a C*-algebra $A$ and for a locally compact Hausdorff space $X$. The group $K_0(C(X))$ is naturally isomorphic to $K^0(X)$ whenever $X$ is a locally compact Hausdorff space. The maps $K_0$ and $K^0$ are covariant and contravariant functors respectively, they satisfy some functorial properties that are useful for computation.
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1 Introduction

The K-theory of C*-algebras is the study of a collection of abelian groups $K_n(A)$ that are invariants of a C*-algebra $A$ for $n \in \mathbb{N}$. In this paper we will focus on the group $K_0(A)$. The map $K_0$ taking a C*-algebra to an abelian group can be viewed as a covariant functor from the category of C*-algebras to the category of abelian groups with some additional properties. We will follow [4] for this part of the theory.

The K-theory is useful in distinguishing C*-algebras. The class of AF-algebras is completely classified by their $K_0$ groups. In general, the $K_0$ group is not a complete invariant for all C*-algebras, but it is an important part of the classification program of C*-algebras.

Topological K-theory is the “original version” of K-theory, introduced by Sir Michael Atiyah. We will follow his classical text [1]. Topological K-theory is the study of a collection of abelian groups $K^n(X)$ that are invariants of a locally compact Hausdorff space $X$. Unlike the case of C*-algebras, the map $K^0$ is a contravariant functor from the category of locally compact Hausdorff spaces to the category of abelian groups.

It is well-known that there is a contravariant functor mapping the category of unital C*-algebras bijectively onto the category of compact Hausdorff spaces that reverses the direction of morphisms. We will see that $K^0(X) \cong K_0(C(X))$ for every compact Hausdorff space $X$. Furthermore, the functors $K_0$ and $K^0$ preserve morphisms by reversing their directions. This result can be extended to non-unital C*-algebras and locally compact Hausdorff spaces, where $K^0(X) \cong K_0(C_0(X))$ for every locally compact Hausdorff space $X$. This correspondence is explained in [6].

The reader is assumed to be familiar with the basics of C*-algebras and topological bundles. If one needs a review on these subjects, we recommend [2] for C*-algebras and the introductory chapter of [6] for vector bundles.
2 K-theory of C*-algebras

**Definition 2.1.** Let $A$ be a C*-algebra. For $n, m \in \mathbb{N}$, let $M_{m,n}(A)$ be the set of all $m \times n$ matrices with entries in $A$. If $m = n$, write $M_{n,n}(A) = M_n(A)$, then $M_n(A)$ is a C*-algebra with the involution $(a^*)_{ij} = (a_{ji})^*$.

**Definition 2.2.** Let $A$ be a C*-algebra. For $n \in \mathbb{N}$ we define $P_n(A)$ to be the set of all projections in $M_n(A)$. For $n \leq m$, there is a natural embedding of $P_n(A)$ into $P_m(A)$ given by

$$p \mapsto \text{Diag}(p, 0_{m-n}) = p \oplus 0_{m-n}.$$ 

Define $P_\infty(A) = \lim_{\rightarrow n} P_n(A)$ as the direct limit of this inclusion. We can also think of it as $P_\infty(A) = \bigcup_{n=1}^{\infty} P_n(A)$.

**Note 2.3.** It might be more notationally clear to write $p$ as an element in $P_n(A)$ for $n \in \mathbb{N}$, and let $[p]$ denote its equivalence class in the direct limit $P_\infty(A)$. But there are two more equivalence relations to be quotiented by later, and to save ourselves from the nested square brackets, $p$ will denote a finite matrix as well as its equivalence class in $P_\infty(A)$, or, an $\aleph_0 \times \aleph_0$ matrix with finitely many non-zero entries.

**Definition 2.4.** Let $\sim_0$ be the relation on $P_\infty(A)$ given by the following: for $p \in P_n(A)$ and $q \in P_m(A)$, we say $p \sim_0 q$ if there exists $v \in M_{m,n}(A)$ such that $v^*v = p$ and $vv^* = q$. The relation $\sim_0$ is called the Murray - von Neumann equivalence.

**Remark 2.5.** A matrix $v \in M_{m,n}(A)$ for some $m, n \in \mathbb{N}$ such that $v^*v$ and $vv^*$ are both projections is called a partial isometry. In the special case that $A = B(H)$ for some Hilbert space $H$, then $v$ is a partial isometry if and only if it maps $(\ker v)^\perp$ isometrically onto $\text{im} v$. If $T$ is a partial isometry in $B(H)$, then $TT^*$ is the projection onto $\text{im} T$ and $T^*T$ is the projection onto $(\ker T)^\perp$.

**Example 2.6.** Let $H$ be an infinite dimensional Hilbert space. Since $H \cong H \oplus H$, there exists some $T \in B(H \oplus H)$ such that $T|_{H \oplus 0}$ is an isometry from $H \oplus 0$ onto $H \oplus H$, and $T|_{0 \oplus H} = 0$. Then $TT^* = I_{H \oplus H}$ and $T^*T = P_{H \oplus 0}$. Note that $T$ can be considered as an element in $B(H \oplus H)$ as well as an element in $M_2(B(H))$. In the latter case

$$T = \begin{bmatrix} T_1 & 0 \\ T_2 & 0 \end{bmatrix}$$
for some $T_1, T_2 \in B(H)$. If we let $S = [T_1 \ T_2]$, then $SS^* = I_1 \in M_1(B(H))$ and $S^*S = I_2 \in M_2(B(H))$. So $I_2 \sim_0 I_1$.

**Lemma 2.7.** Let $A$ be a $C^*$-algebra, let $p \in P_n(A)$ and $q \in P_m(A)$ for some $n, m \in \mathbb{N}$, and suppose there exists $v \in M_{m,n}(A)$ for which $v^*v = p$ and $vv^* = q$. Then $v = qv = vp = qvp$.

**Proof.** Let $w = (1 - q)v$, then

$$w^*w = v^*(1 - q)(1 - q)v = v^*v - v^*vv^*v = p - pp = 0.$$ 

However $\|w\|^2 = \|w^*w\| = 0$, which implies that $w = 0$. So $0 = w = v - qv$. This implies that $v = qv$. The case $v = pv$ is proved similarly. Lastly,

$$qvp = (qv)p = vp = v.$$ 

**Proposition 2.8.** The relation $\sim_0$ is an equivalence relation on $P_\infty(A)$.

**Proof.** It is not yet clear that $\sim_0$ is well-defined on $P_\infty(A)$, since $P_\infty(A)$ is a direct limit, where $p \in P_n(A)$ can also be represented by $p \oplus 0_k$ in $P_\infty(A)$, for any $k \geq 0$. We will show that $\sim_0$ is an equivalence relation on $\bigsqcup_{r=1}^{\infty} P_r(A)$, and also satisfies $p \sim_0 p \oplus 0_k$ for $p \in P_n(A)$, $n \geq 1$ and $k \geq 0$. Then for any $p \in P_n(A)$, $q \in P_m(A)$ and $k, k' \geq 0$, have $p \sim_0 q$ if and only if

$$p \oplus 0_k \sim_0 p \sim_0 q \sim_0 q \oplus 0_{k'}.$$ 

So $\sim_0$ descends to an equivalence relation on $P_\infty(A)$. To this end, let $p \in P_n(A)$, $q \in P_m(A)$ and $r \in P_l(A)$ for some $l, m, n \geq 1$.

To show $p \sim_0 p \oplus 0_k$, let $v = [p \ 0_{n \times k}]$, then $v^*v = p$ and $vv^* = p \oplus 0_k$. The special case with $k = 0$ verifies reflexivity.

Suppose there exists $v \in M_{m,n}(A)$ such that $v^*v = p$ and $vv^* = q$. Let $w = v^* \in M_{n,m}(A)$. We have

$$w^*w = q \text{ and } ww^* = p.$$ 

So $\sim_0$ is symmetric.

Suppose $p \sim_0 q$ and $q \sim_0 r$. Then there exists some $v \in M_{m,n}(A)$ and $u \in M_{l,m}(A)$ for which

$$v^*v = p, \quad vv^* = q, \quad u^*u = q \quad \text{and} \quad uu^* = r$$

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hold. Let $z = uv$. Using Lemma 2.7, the following computations hold.

\[ z^*z = v^*u^*uv = v^*qv = v^*v = p, \]
\[ zz^* = uvv^*u^* = uqu^* = r. \]

Thus $p \sim_0 r$, which proves transitivity. □

**Definition 2.9.** Let $A$ be a C*-algebra and $p, q$ projections in $\mathcal{P}_\infty(A)$. We say that $p$ and $q$ are **mutually orthogonal** if $pq = 0$, written $p \perp q$.

**Remark 2.10.** If $p \perp q$ then

\[ qp = q^*p^* = (pq)^* = 0^* = 0, \]

so $q \perp p$. And also,

\[ (p + q)^* = p^* + q^* = p + q \]
\[ (p + q)(p + q) = pp + pq + qp + qq = pp + qq = p + q. \]

So $p + q$ is also a projection in $A$.

In the special case that $A = B(H)$ for some Hilbert space $H$ and $P, Q \in B(H)$ are projections, we have $P \perp Q$ if and only if ran $P \perp$ ran $Q$.

**Proposition 2.11.** Let $p, p', q, q' \in \mathcal{P}_\infty(A)$. Then

1. $p \oplus q \sim_0 q \oplus p$.
2. $p \sim_0 p'$ and $q \sim_0 q'$ implies $p \oplus q \sim_0 p' \oplus q'$.
3. $(p \oplus q) \oplus r = p \oplus (q \oplus r)$.
4. Suppose $p$ and $q$ are represented by matrices of the same size, and $p \perp q$, then $p + q \sim_0 p \oplus q$.

**Proof.**

1. Suppose $p$ is $n \times n$ and $q$ is $m \times m$. Let $v = \begin{bmatrix} 0_{n \times m} & p \\ q & 0_{m \times n} \end{bmatrix}$. Then

\[ v^*v = \begin{bmatrix} 0_{n \times n} & p^* \\ p & 0_{n \times m} \end{bmatrix} \begin{bmatrix} 0_{n \times m} & p \\ q & 0_{m \times n} \end{bmatrix} = \begin{bmatrix} q^*q & 0_{n \times n} \\ 0_{m \times n} & p^*p \end{bmatrix} = q \oplus p; \]
\[ vv^* = \begin{bmatrix} 0_{n \times m} & p \\ q & 0_{m \times n} \end{bmatrix} \begin{bmatrix} 0_{m \times n} & q^* \\ p^* & 0_{n \times m} \end{bmatrix} = \begin{bmatrix} pp^* & 0_{n \times m} \\ 0_{m \times n} & qq^* \end{bmatrix} = q \oplus p. \]
So \( q \oplus p \sim_0 p \oplus q \).

2. Suppose \( v^*v = p \), \( vv^* = p' \), \( w^*w = q \) and \( ww^* = q' \), then

\[(v \oplus w)^*(v \oplus w) = p \oplus q \]

and

\[(v \oplus w)(v \oplus w)^* = p' \oplus q'. \]

So \( p \oplus q \sim_0 p' \oplus q' \).

3. This is by definition.

4. Suppose \( p \) and \( q \) are of the same size and \( pq = 0 \). Let \( v = \begin{bmatrix} p & q \end{bmatrix} \), then

\[ vv^* = \begin{bmatrix} p & q \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = pp + qq = p + q, \]

\[ v^*v = \begin{bmatrix} p \\ q \end{bmatrix} \begin{bmatrix} p & q \end{bmatrix} = \begin{bmatrix} pp & pq \\ qp & qq \end{bmatrix} = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix} = p \oplus q. \]

So \( p + q \sim_0 p \oplus q \). ■

**Definition 2.12.** Let \( A \) be a C*-algebra. Define \( \mathcal{D}(A) = \mathcal{P}_\infty(A)/ \sim_0 \). The equivalence class of \( p \) in \( \mathcal{D}(A) \) is written \( [p]_\mathcal{D} \). Equip \( \mathcal{D}(A) \) with an operation \( + \) by \( [p]_\mathcal{D} + [q]_\mathcal{D} = [p \oplus q]_\mathcal{D} \).

**Proposition 2.13.** \( (\mathcal{D}(A), +) \) is an abelian monoid.

**Proof.** This is mostly a consequence of Proposition 2.11. Point 2 implies that the operation \( + \) is well-defined after quotienting by \( \sim_0 \). Point 3 implies that \( + \) is associative. Point 1 implies that it is commutative. So \( (\mathcal{D}(A), +) \) is an abelian semigroup. Now we claim that \( [0_1]_\mathcal{D} \) is the identity element (note that \( 0_n \sim_0 0_m \) for all \( n, m \in \mathbb{N} \) by Proposition 2.8). To this end, take any \( p \in \mathcal{P}_\infty(A) \). By point 1 of Proposition 2.11 and Proposition 2.8,

\[ 0_1 \oplus p \sim_0 p \oplus 0_1 \sim_0 p, \]

so

\[ [0_1]_\mathcal{D} + [p]_\mathcal{D} = [p]_\mathcal{D} + [0_1]_\mathcal{D} = [p]_\mathcal{D}. \]

From the abelian monoid \( \mathcal{D}(A) \) we will construct an abelian group, by a construction called the **Grothendieck completion**.
Definition 2.14. Let \((S, +)\) be an abelian semigroup, then \(S \times S\) is also naturally a semigroup. Let \(\sim\) be a relation on \(S \times S\) given by \((a_1, b_1) \sim (a_2, b_2)\) if there exists \(x \in S\) so that
\[
a_1 + b_2 + x = a_2 + b_1 + x.
\]
Define \(G(S) = (S \times S)/\sim\), and equip it with the operation \(+\) by
\[
[(a, b)] + [(c, d)] = [(a + c, b + d)].
\]

Proposition 2.15. The above construction is well-defined, and \(G(S)\) is an abelian group. Furthermore, if \(S\) is an abelian monoid with identity element \(0\), then \(\varphi : S \to G(S)\) by \(\varphi(s) = [(s, 0)]\) is a monoid homomorphism.

Proof. It is easy to see that \(\sim\) is an equivalence relation on \(S \times S\). To see that \(+\) is well-defined on \(G(S)\), let \(a_i, b_i, c_i, d_i \in S\) for \(i = 1, 2\), and suppose that \((a_1, b_1) \sim (a_2, b_2)\) and \((c_1, d_1) \sim (c_2, d_2)\). Then there exists \(x, y \in S\) such that
\[
a_1 + b_2 + x = a_2 + b_1 + x \quad \text{and} \quad c_1 + d_2 + y = c_2 + d_1 + y.
\]
Then
\[
(a_1 + c_1) + (b_2 + d_2) + (x + y) = (a_2 + c_2) + (b_2 + d_2) + (x + y),
\]
so \([(a_1 + c_1, b_1 + d_1)] = [(a_2 + c_2, b_2 + d_2)]\).

Since \(+\) is associative and commutative on \(S\), the addition induced on \(G(S)\) is associative and commutative as well. For \(a, b, c, d \in S\), it is clear that \([(a, a)] = [(b, b)]\). Furthermore
\[
[(c, d)] + [(a, a)] = [(c + a, d + a)] = [(c, d)].
\]
So \((a, a)\) is the identity element of \(G(S)\). Also,
\[
[(a, b)] + [(b, a)] = [(a + b, a + b)],
\]
so \([(b, a)]\) is the inverse of \([(a, b)]\). Hence \(G(S)\) is indeed an abelian group.

Now suppose that \(S\) is an abelian monoid with \(0\), and \(\varphi : S \to G(S)\) by \(\varphi(s) = [(s, 0)]\). Then it is clear that \(\varphi(a + b) = \varphi(a) + \varphi(b)\) and that \(\varphi(0)\) is the identity element of \(G(S)\). \(\blacksquare\)

It is convenient to think of \([(a, b)] \in G(S)\) as “\(a - b\)”. 

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Example 2.16. 1. $S = \mathbb{N}$. Then $G(\mathbb{N}) = \mathbb{Z}$. This is the standard construction of $\mathbb{Z}$.

2. $S = \mathbb{N} \cup \{\infty\}$. For any $a, b, c, d \in \mathbb{N} \cup \{\infty\}$,

$$a + c + \infty = \infty = b + d + \infty,$$

so $[(a, b)] = [(c, d)]$. Hence $G(S) \cong \{0\}$. This example demonstrates why we required the $x$ in defining $\sim$ in Definition 2.14, where $(a_1, b_1) \sim (a_2, b_2)$ if and only if there exists $x$ for which $a_1 + b_2 + x = a_2 + b_1 + x$. Suppose for instance we define another relation $\sim_{\text{bad}}$ on $S$ by $(a_1, b_1) \sim_{\text{bad}} (a_2, b_2)$ if $a_1 + b_2 = a_2 + b_1$. For any $a, b \in S$, we have

$$\infty + a = \infty = b + \infty,$$

so $(\infty, \infty) \sim_{\text{bad}} (a, b)$. In particular, $(1, 1) \sim_{\text{bad}} (\infty, \infty) \sim_{\text{bad}} (1, 2)$, but clearly $(1, 1) \not\sim_{\text{bad}} (1, 2)$, which shows that $\sim_{\text{bad}}$ is not an equivalence relation! This is the same problem that one runs into when asking “Surely $\infty + \infty = \infty$, but what is $\infty - \infty$?”

Now we are ready to give the definition of the $K_0$ group of a unital C*-algebra.

Definition 2.17. Let $A$ be a unital C*-algebra. Define $K_0(A) = G(D(A))$.

Define the map $[\cdot]_0 : \mathcal{P}_\infty(A) \to K_0(A)$ by $[p]_0 = \varphi([p]D)$ where $\varphi : D(A) \to G(D(A))$ is the monoid homomorphism defined in Proposition 2.15.

Example 2.18. 1. Let $A = \mathbb{C}$. All projections in $\mathcal{P}_\infty(\mathbb{C})$ are projection matrices. Take $p, q \in \mathcal{P}_\infty(\mathbb{C})$. We may assume that $p$ and $q$ are both $n \times n$. Suppose $p$ and $q$ have the same rank $k \leq n$, and let $\{z_1, \ldots, z_k\}$ be an orthonormal basis of ran $p$ and extend it to an orthonormal basis $\{z_1, \ldots, z_n\}$ of $\mathbb{C}^n$; let $\{w_1, \ldots, w_k\}$ be an orthonormal basis of ran $q$. Let $v \in M_n(\mathbb{C})$ be the matrix that takes $z_j$ to $w_j$ for $j = 1, \ldots, k$, and takes $z_j$ to $0$ for all $j = k + 1, \ldots, n$. Then

$$v^*vz_j = \begin{cases} v^*w_j = z_j & : j = 1, \ldots, k \\ v^*0 = 0 & : j = k + 1, \ldots, n \end{cases}.$$ 

So $v^*v$ is the projection onto ran $p$, hence $v^*v = p$. Similarly, $vv^* = q$, so $p \sim_0 q$. 


Conversely suppose $p \sim_0 q$. Then there exists a matrix $v$ for which $v^*v = p$ and $vv^* = q$. Since row rank and column rank coincide, we have

$$\text{rank } p = \text{rank } v^*v = \text{rank } vv^* = \text{rank } q.$$ 

Hence $p \sim_0 q$ if and only if $p$ and $q$ have the same rank. Furthermore it is clear that $\text{rank } p + \text{rank } q = \text{rank } (p \oplus q)$. Thus $\mathcal{D}(\mathbb{C}) \cong \mathbb{N}$. Therefore $K_0(\mathbb{C}) \cong G(\mathbb{N}) = \mathbb{Z}$.

2. Let $A = M_m(\mathbb{C})$ for some $m \in \mathbb{N}$. Then for $n \in \mathbb{N}$, the $C^*$-algebra $M_n(A)$ is naturally a subalgebra of $M_{mn}(\mathbb{C})$, and the rank argument from above works just as well. Hence $K_0(M_m(\mathbb{C})) \cong \mathbb{Z}$.

3. Let $A = \mathcal{B}(\mathcal{H})$ for $\mathcal{H}$ an infinite dimensional Hilbert space. The same rank argument works since every two Hilbert spaces of the same dimension are isometric. So projections in $\mathcal{P}_\infty(A)$ are once again determined up to Murray - von Neumann equivalence by their dimensions, and $\mathcal{D}(A) \cong \{\dim p : p \in \mathcal{P}_\infty(A)\}$. Since $\mathcal{H}$ is infinite dimensional, $\mathcal{D}(A)$ has a largest element $\alpha_0 = \dim \mathcal{H}$ since $\dim(\mathcal{H}^n) = \dim \mathcal{H}$ for all finite $n$, and $\alpha_0 + \alpha = \alpha_0$ for all $\alpha \in \mathcal{D}(A)$. So by the same argument in part 2 of Example 2.16, have $K_0(\mathcal{B}(\mathcal{H})) = G(\mathcal{D}(\mathcal{B}(\mathcal{H}))) = 0$.

To summarize,

$$K_0(\mathcal{B}(\mathcal{H})) \cong \begin{cases} \mathbb{Z} & : \dim \mathcal{H} < \aleph_0 \\ 0 & : \dim \mathcal{H} \geq \aleph_0 \end{cases}$$
3 Unitaries and projections

In this section we develop some properties of unitary and projection elements in a C*-algebra. These will be necessary for exploring meaningful properties of the $K_0$-group of C*-algebras.

From here on $\tilde{A}$ denotes the unitization of the C*-algebra $A$. For more information on unitization, see [2].

Definition 3.1. Let $X$ be a topological space and $x, y \in X$. Say $x$ and $y$ are homotopy equivalent in $X$, written $x \sim_h y$, if there exists a continuous path $\alpha : [0, 1] \to X$ such that $\alpha(0) = x$ and $\alpha(1) = y$.

Definition 3.2. Let $A$ be a C*-algebra, and $a, b \in A$. We say $a$ is unitarily equivalent to $b$, written $a \sim_u b$, if there exists a unitary $u \in \tilde{A}$ such that $uau^* = b$. It is clear that these are equivalence relations.

Definition 3.3. Let $A$ be a unital C*-algebra, define $U(A)$ to be the group of unitary elements in $A$, and define $U_0(A)$ to be all $u \in U(A)$ such that $u \sim_h 1$. That is, $U_0(A)$ is the path-connected component of 1 in $U(A)$.

Definition 3.4. Let $A$ be a unital C*-algebra and let $a \in A$. The spectrum $\sigma(a)$ of $a$ is defined to be

$$\sigma(a) := \{ \lambda \in \mathbb{C} : a - \lambda 1 \text{ is not invertible in } A \}.$$ 

The general theory of spectrum and of continuous functional calculus can be found in [2].

Lemma 3.5. Let $A$ be a unital C*-algebra and $u \in U(A)$. If $\sigma(u) \neq \mathbb{T}$, then $u \in U_0(A)$.

Proof. Suppose $\sigma(u) \neq \mathbb{T}$. Let $w \in \mathbb{T} \setminus \sigma(u)$ and let $\log_w : \mathbb{C} \setminus [0, \infty) \to \mathbb{C}$ be the branch of logarithm that avoids the ray containing $w$. Then $\exp(\log_w(z)) = z$ for all $z \in \mathbb{T} \setminus \{w\} \supseteq \sigma(u)$, so $\exp(\log_w(u)) = u$. Let $h = \log_w(u)$, then $\sigma(h) \subseteq \log_w(\mathbb{T} \setminus w) \subseteq i\mathbb{R}$. 

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For \( t \in [0, 1] \), let \( h_t = th \). Clearly \( \sigma(th) \subseteq i\mathbb{R} \) for all \( t \in [0, 1] \), so \( \sigma(\exp(th)) \subseteq \mathbb{T} \) for all \( t \in [0, 1] \), which implies that \( \exp(th) \) is unitary for any \( t \in [0, 1] \). Furthermore the map \( \beta : [0, 1] \to \mathcal{U}(A) \) mapping \( \beta(t) = \exp(th) \) is a continuous path of unitaries from \( 1_A \in A \) to \( u \in A \). Hence \( u \in \mathcal{U}_0(A) \). ■

**Lemma 3.6** (Whitehead). Let \( A \) be a unital \( C^* \)-algebra and let \( u, v \in \mathcal{U}(A) \). Then

\[
\begin{bmatrix}
  u & 0 \\
  0 & v \\
\end{bmatrix}
\sim_h
\begin{bmatrix}
  uv & 0 \\
  0 & 1 \\
\end{bmatrix}
\sim_h
\begin{bmatrix}
  vu & 0 \\
  0 & 1 \\
\end{bmatrix}
\sim_h
\begin{bmatrix}
  v & 0 \\
  0 & u \\
\end{bmatrix}
\text{ in } \mathcal{U}(M_2(A)).
\]

**Proof.** Since

\[
\begin{bmatrix}
  0 & 1 \\
  1 & 0 \\
\end{bmatrix}
\] has spectrum \( \{ \pm 1 \} \), by Lemma 3.5 have

\[
\begin{bmatrix}
  0 & 1 \\
  1 & 0 \\
\end{bmatrix}
\sim_h
\begin{bmatrix}
  1 & 0 \\
  0 & 1 \\
\end{bmatrix}.
\]

Let \( \alpha : [0, 1] \to \mathcal{U}_0(M_2(A)) \) be a path from \( \begin{bmatrix}
  0 & 1 \\
  1 & 0 \\
\end{bmatrix} \) to \( \begin{bmatrix}
  1 & 0 \\
  0 & 1 \\
\end{bmatrix} \). Define \( \beta : [0, 1] \to M_2(A) \) by

\[
\beta(t) = \begin{bmatrix}
  u & 0 \\
  0 & 1 \\
\end{bmatrix} \alpha(t) \begin{bmatrix}
  v & 0 \\
  0 & 1 \\
\end{bmatrix} \alpha(t).
\]

Since for all \( t \in [0, 1] \), \( \beta(t) \) is the product of four unitaries, so \( \beta \) is in fact a path in \( \mathcal{U}(M_2(A)) \). Further,

\[
\beta(0) = \begin{bmatrix}
  u & 0 \\
  0 & 1 \\
\end{bmatrix} \begin{bmatrix}
  0 & 1 \\
  1 & 0 \\
\end{bmatrix} \begin{bmatrix}
  v & 0 \\
  0 & 1 \\
\end{bmatrix} \begin{bmatrix}
  0 & 1 \\
  1 & 0 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  0 & u \\
  1 & 0 \\
\end{bmatrix} \begin{bmatrix}
  0 & v \\
  1 & 0 \\
\end{bmatrix} = \begin{bmatrix}
  u & 0 \\
  0 & v \\
\end{bmatrix},
\]

and

\[
\beta(1) = \begin{bmatrix}
  u & 0 \\
  0 & 1 \\
\end{bmatrix} \begin{bmatrix}
  v & 0 \\
  0 & 1 \\
\end{bmatrix} = \begin{bmatrix}
  uv & 0 \\
  0 & 1 \\
\end{bmatrix}.
\]

So

\[
\begin{bmatrix}
  u & 0 \\
  0 & 1 \\
\end{bmatrix}
\sim_h
\begin{bmatrix}
  uv & 0 \\
  0 & 1 \\
\end{bmatrix}.
\]
By symmetry and transitivity, it is only left to prove that
\[
\begin{bmatrix}
u & 0 \\0 & v
\end{bmatrix} \sim_h \begin{bmatrix}
u & 0 \\0 & u
\end{bmatrix}.
\]
This can be accomplished by defining the path
\[
\gamma(t) = \alpha(t) \begin{bmatrix}
u & 0 \\0 & v
\end{bmatrix} \alpha(t).
\]

**Corollary 3.7.** Let $A$ be a unital C*-algebra, $u \in U(A)$, then
\[
\begin{bmatrix}
u & 0 \\0 & u^*
\end{bmatrix} \in U_0(M_2(A)).
\]

**Proof.** By Lemma 3.6,
\[
\begin{bmatrix}
u & 0 \\0 & u^*
\end{bmatrix} \sim_h \begin{bmatrix}
u u^* & 0 \\0 & 1
\end{bmatrix} = \begin{bmatrix}1 & 0 \\0 & 1
\end{bmatrix}.
\]

**Lemma 3.8.** Let $A$ be a unital C*-algebra and $u \in U(A)$. If $\|u - 1\| < 2$ then $u = \exp(\text{i}h)$ for some self-adjoint element $h \in A$.

**Proof.** If $\|u - 1\| < 2$ then $\sigma(u - 1) \subseteq B_2(0)$, in particular $-2 \not\in \sigma(u - 1)$, so $-1 \not\in \sigma(u)$. Since $\sigma(u) \neq \mathbb{T}$, by the proof of Lemma 3.5, $u = \exp(s)$ for some $s \in A$ with $\sigma(s) \in i\mathbb{R}$. Let $h = -is$, then $h$ is self-adjoint and $\exp(\text{i}h) = \exp(s) = u$.

**Proposition 3.9.** Let $A$ be a unital C*-algebra. Then
\[
U_0(A) = \{\exp(\text{i}h_1)\ldots\exp(\text{i}h_l) : l \in \mathbb{N}, h_j \in A \text{ self-adjoint}\}.
\]

**Proof.** Let $u \in U_0(A)$. A continuous path from $u$ to 1 can be partitioned into segments
\[
u = u_0 \sim_h u_1 \sim_h \ldots \sim_h u_k = 1
\]
where $\|u_{j-1} - u_j\| < 2$ for $j = 1, \ldots, k$. Now apply induction on $k$. For $k = 1$, $\|u - 1\| < 2$, and the result follows Lemma 3.8. Suppose the result is true for $k = n - 1$, and the inductive step for $n$ has been completed. Then $u_1 = \exp(\text{i}h_1)\ldots\exp(\text{i}h_l)$ for some $l \in \mathbb{N}$ and $h_j$ self-adjoint. Because $\|u - u_1\| < 2$, so
\[
\|uu_1^* - 1\| = \|(u - u_1)u_1^*\| = \|u - u_1\| < 2.
\]
By Lemma 3.8, there exists a self-adjoint element $h_0 \in A$ such that $uu^*_1 = \exp(ih_0)$. Then

$$u = \exp(ih_0) u_1 = \exp(ih_0) \exp(ih_1) \ldots \exp(ih_l).$$

This completes the induction.

Conversely if $h$ is self-adjoint, the proof of Lemma 3.5 implies that $\exp(ih) \in U_0(A)$. The product of such unitaries is also homotopic to the identity. Thus all elements in $U_0(A)$ are indeed equal to finite products as in the claim. ■

**Proposition 3.10.** Let $A, B$ be unital C*-algebras, $\varphi : A \to B$ a surjective *-homomorphism. Then

1. $\varphi(U_0(A)) = U_0(B)$
2. For any $u \in U(B)$, there exists $v \in U_0(M_2(A))$ such that

$$\varphi(v) = \begin{bmatrix} u & 0 \\ 0 & u^* \end{bmatrix}$$

**Proof.** 1. Since $\varphi$ takes unitaries to unitaries, $\varphi(U_0(A)) \subseteq U_0(B)$. The converse requires some work. Let $u \in U_0(B)$. By Proposition 3.9, there exists hermitian elements $h_1, \ldots, h_l \in B$ such that

$$u = \exp(ih_1) \exp(ih_2) \ldots \exp(ih_l).$$

Let $t_1, \ldots, t_l \in A$ such that $\varphi(t_j) = h_j$ for $j = 1, \ldots, l$, and let $\tilde{t}_j = \frac{1}{2}(t_j + t_j^*)$ for $j = 1, \ldots, l$. Then $\tilde{t}_j$ are self-adjoint, and

$$\varphi(\tilde{t}_j) = \frac{1}{2}(\varphi(t_j) + \varphi(t_j)^*) = \frac{1}{2}(h_j + h_j) = h_j.$$

Let

$$v = \exp(i\tilde{t}_1) \ldots \exp(i\tilde{t}_l).$$

The proof of Lemma 3.5 implies that $v \in U_0(A)$. And happily, $\varphi(v) = u$.

2. Let $u \in U(B)$. By Corollary 3.7 $\begin{bmatrix} u & 0 \\ 0 & u^* \end{bmatrix} \in U_0(M_2(B))$. Then by part 1 there exists some $v \in U_0(M_2(A))$ such that $\varphi(v) = u \oplus u^*$. ■
Definition 3.11. Let $A$ be a unital C*-algebra and $a \in A$. Then $\sigma(a^*a) \subseteq \mathbb{R}_{\geq 0}$, where the square root function is defined. So we may define $|a| = (a^*a)^{1/2}$.

Proposition 3.12. Let $A$ be a unital C*-algebra.

1. If $z \in GL(A)$, then $|z| \in GL(A)$, and $w(z) := z|z|^{-1} \in U(A)$.

2. The map $w : GL(A) \to U(A)$ defined in 1. is continuous. And $w(u) = u$ for all $u \in U(A)$.

3. If $a, b \in GL(A)$ with $a \sim_h b$ in $GL(A)$, then $w(a) \sim_h w(b)$ in $U(A)$.

Proof. 1. Suppose $z$ is invertible. Then $z^*$ is also invertible, so $z^*z \in GL(A)$. It follows that

$$
\sigma(|z|) = \sigma((z^*z)^{1/2}) = \{t^{1/2} : t \in \sigma(z^*z)\} \neq 0.
$$

Thus $|z|$ is invertible.

Furthermore,

$$
w(z)w(z)^* = z|z|^{-1}(z|z|^{-1})^* = z|z|^{-1}|z|^{-1}z^* = z(z^*z)^{-1}z^* = zz^{-1}(z^*)^{-1}z^* = 1,
$$

and similarly $w(z)^*w(z) = 1$. So $w(z) \in U(A)$.

2. The map $a \mapsto a^12$ is continuous. Also inversion and multiplication are continuous in $GL(A)$. So to prove the claim it is sufficient to prove that $a \mapsto a^{1/2}$ is continuous on $A_{\geq 0}$, where $A_{\geq 0}$ is the set of normal elements in $A$ with spectrum contained in $[0, \infty)$.

Suppose we fix $a \in A_{\geq 0}$ and let $U$ be a bounded open neighbourhood containing $\sigma(a)$. The upper-semicontinuity of spectra [5] implies that there is some $d > 0$ such that if $b \in A$ and $\|b - a\| < d$ then $\sigma(b) \subseteq U$. Thus the problem reduces to proving that the square root map is continuous on $\Omega_r \subseteq A_{\geq 0}$ where

$$
\Omega_r = \{a \in A : a^*a = aa^*, \ \sigma(a) \subseteq [0, r]\}.
$$

Let $f$ denote the square root function and let $\varepsilon > 0$ be given. By the Stone-Weierstrass theorem, there exists a complex polynomial $g$ such that
\[\|g - f\|_\infty < \varepsilon/3\] on \([0, r].\) For \(c \in \Omega,\)
\[
\|f(c) - g(c)\| = \|(f - g)(c)\|
= \sup\{|(f - g)(z)| : z \in \sigma(c)\}
\leq \|f - g\|_\infty < \varepsilon/3.
\]

Therefore \(g\) is continuous on \(\Omega\) since \(a \mapsto a^n\) is continuous. So there exists \(\delta > 0\) such that \(\|g(a) - g(b)\| < \varepsilon/3\) whenever \(a, b \in A\) with \(\|a - b\| < \delta.\) Thus when \(a, b \in \Omega\) with \(\|a - b\| < \delta\), have \(\|f(a) - f(b)\| < \varepsilon.\)

3. Let \(\alpha : [0, 1] \to GL(A)\) be a continuous path from \(a\) to \(b.\) Then by part 2, \(w \circ \alpha : [0, 1] \to U(A)\) is a continuous path from \(w(a)\) to \(w(b).\) \(\blacksquare\)

For an element \(z \in A,\) the form \(z = w(z)|z|\) is called the polar decomposition of \(z.\)

Definition 3.13. The relations \(\sim_u\) and \(\sim_h\) induce equivalence relations on \(\mathcal{P}_\infty(A)\) as follows: \(p \sim_u q,\) if by representing \(p\) and \(q\) both as \(n \times n\) matrices for some \(n \in \mathbb{N},\) there exists a unitary element \(u \in M_n(A)\) such that \(u^*pu = q.\) We say that \(p \sim_h q\) if by representing \(p\) and \(q\) both as \(n \times n\) matrices for some \(n \in \mathbb{N},\) there exists a path \(\alpha(t)\) in \(\mathcal{P}_n(A)\) such that \(\alpha(0) = p\) and \(\alpha(1) = q.\)

Proposition 3.14. Let \(A\) be a unital C*-algebra, \(a, b \in A\) self-adjoint elements, \(z \in GL(A)\) and \(z = u|z|\) the polar decomposition of \(z.\) If \(za = bz\) then \(ua = bu.\)

Proof. Since \(a\) and \(b\) are self-adjoint, take the adjoint of the equality to have \(az^* = z^*b.\) Then
\[
|z|^2a = z^*za = z^*bz = az^*z = a|z|^2.
\]
So \(a\) commutes with \(|z|^2.\) Consequently \(a\) commutes with \(g(|z|^2)\) for all complex polynomials \(g.\) By Stone-Weierstrass theorem, the element \(|z|^{-1} = ((|z|^2)^{1/2})^{-1}\) is the limit of a sequence of polynomials in \(|z|^2.\) Hence \(a\) commutes with \(|z|^{-1}.\) It follows that
\[
ua^* = z|z|^{-1}au^* = za|z|^{-1}u^* = bz|z|^{-1}u^* = buu^* = b. \]

Proposition 3.15. Let \(n \in \mathbb{N}_{\geq 1},\) and \(p, q \in \mathcal{P}_n(A).\) Then
1. \( p \sim_h q \) implies \( p \sim_u q \).

2. \( p \sim_u q \) implies \( p \sim_0 q \).

3. \( p \sim_0 q \) implies \( p \oplus 0_n \sim_u q \oplus 0_n \).

4. \( p \sim_u q \) implies \( p \oplus 0_n \sim_h q \oplus 0_n \).

**Proof.** 1. Let \( \alpha(t) \) be a path in \( \mathcal{P}_n(A) \) that connects \( p \) to \( q \), then we can partition the path into segments of length less than \( 1/2 \). It is now sufficient to prove that if \( \|p - q\| < 1/2 \) then \( p \sim q \). Let \( z = pq + (I - p)(I - q) \in \tilde{A} \), and \( pz = pq = qz \). Also

\[
\|z - I\| = \|pq + (I - p)(I - q) - I\|
\]

\[
= \|pq + (I - p)(I - q) - p - (I - p)\|
\]

\[
= \|p(q - p) + (I - p)((I - q) - (I - p))\|
\]

\[
= \|p(q - p) + (I - p)(p - q)\|
\]

\[
\leq \|p\|\|(q - p)\| + \|I - p\|\|p - q\|
\]

\[
\leq 2\|p - q\| < 1.
\]

Hence \( z \in GL(A) \). Let \( z = u|z| \) be the polar decomposition of \( z \). By Proposition 3.14, \( pu = uq \).

2. Suppose \( p \sim_u q \). Then there exists some unitary \( u \in \tilde{M}_n(A) \) such that \( u^*pu = q \). Let \( v = u^*p \), then \( vv^* = u^*ppu = q \) and \( v^*v = puu^*p = pp = p \). Also note that \( v = u^*p \in M_n(A) \) since \( M_n(A) \) is an ideal in \( M_n(A) \). Hence \( p \sim_0 q \).

3. Suppose there exists \( v \in M_n(A) \) such that \( vv^* = q \) and \( v^*v = p \). Define

\[
u = \begin{bmatrix} v & 1-q \\ 1-p & v^* \end{bmatrix}
\]

and \( w = \begin{bmatrix} q & 1-q \\ 1-q & q \end{bmatrix} \).

Then

\[
u^*u = \begin{bmatrix} v & I_n - q \\ I_n - p & v^* \end{bmatrix} \begin{bmatrix} v^* & I_n - p \\ I_n - q & v \end{bmatrix}
\]

\[
= \begin{bmatrix} vv^* + (I_n - q) & v - vp + v - qv \\ v^* - pv^* + v^* - v^*q & (I_n - p) + v^*v \end{bmatrix}
\]

\[
= \begin{bmatrix} I_n + q - q & v - v + vv^*v - vv^*v \\ v^* - v^* + v^*v^* - v^*v^* & I_n - v^*v + v^*v \end{bmatrix}
\]

\[
= I_{2n}
\]

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Lemma 2.7 is used to equate the second line to the third in the above equation. Similar computations show that
\[ uu^* = w^* w = w w^* = I_{2n}. \]
So \( u, w, wu \in \mathcal{U}_{2n}(A) \). And
\[
w u = \begin{bmatrix} q & I - q \\ I - q & q \end{bmatrix} \begin{bmatrix} v & I - q \\ I - p & v^* \end{bmatrix} = \begin{bmatrix} q v + (I - q)(I - p) & q - qq + v^* - qv^* \\ v - qv + q - qp & (I - q)(I - q) + qv^* \end{bmatrix} = \begin{bmatrix} v + (I - q)(I - p) & (I - q)v^* \\ q(I - p) & (I - q) + qv^* \end{bmatrix}
\]
is an element of \( \tilde{M}_{2n}(A) \). Now,
\[
w u(p \oplus 0_n)(wu)^* = \begin{bmatrix} v + (I - q)(I - p) & 0 \\ q - qp & I - q + v^* \end{bmatrix} \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} = \begin{bmatrix} v + (I - p)(I - q) & q - pq \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} v + (I - q)(I - p) & (I - q)v^* \\ q(I - p) & (I - q) + qv^* \end{bmatrix}
\]
noting that
\[ v(I - p)(I - q) = (v - vv^* v)(I - q) = 0 \]
and
\[ vq - vpq = vv^* - vv^* v = vv^* v - vv^* = 0 \]
by Lemma 2.7.

4. Suppose \( p \sim_u q \). Then there exists unitary \( u \in \tilde{M}_{2n}(A) \) such that \( upu^* = q \). By Lemma 3.6 there exists a path \( t \mapsto w_t \) in \( \mathcal{U}(M_{2n}(A)) \) such that
\[
w_0 = \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix} \quad \text{and} \quad w_1 = \begin{bmatrix} u & 0 \\ 0 & u^* \end{bmatrix}.
\]
Let \( p_t = w_t \text{Diag}(p, 0_n)w_t^* \). Then \( p_t \in \mathcal{P}_{2n}(A) \) for each \( t \in [0, 1] \). Furthermore,
\[
p_0 = \text{Diag}(p, 0_n) \quad \text{and} \quad p_1 = \begin{bmatrix} upu^* & 0 \\ 0 & 0 \end{bmatrix} = \text{Diag}(q, 0_n).
\]
Therefore \( p \oplus 0_n \sim_h q \oplus 0_n \).
4  $K_0$ as a functor

We will see that $K_0$ is a contravariant functor from the category of C*-algebras to the category of abelian groups, and that it enjoys many useful properties. Before starting the functoriality, we will first need a way to induce group homomorphisms from semigroups homomorphisms in the Grothendieck completion.

Proposition 4.1. Let $S$ be an abelian semigroup. For any abelian group $H$ and any semigroup homomorphism $\rho : S \rightarrow H$, the map $\rho_G : G(S) \rightarrow H$ given by $\rho_G([(s,t)]_G) = \rho(s) - \rho(t)$ for all $(s,t) \in S \times S$ is a well-defined group homomorphism.

Proof. Let $\rho_G$ be as defined above and let $s_1, s_2, t_1, t_2 \in S$. To see that $\rho_G$ is well-defined, suppose that $[(s_1,t_1)]_0 = [(s_2,t_2)]_0$. Then there exists $r \in S$ such that $s_1 + t_2 + r = s_2 + t_1 + r$, which implies that

$$\rho(s_1) + \rho(t_2) + \rho(r) = \rho(s_2) + \rho(t_1) + \rho(r).$$

But $H$ is a group, where all elements are invertible. So

$$\rho_G([(s_1,t_1)]_G) = \rho(s_1) - \rho(t_1) = \rho(s_2) - \rho(t_2) = \rho_G([(s_2,t_2)]_G).$$

Hence $\rho_G$ is well-defined. Now to check that $\rho_G$ is a homomorphism:

$$\rho_G([(s_1,t_1)]_G + [(s_2,t_2)]_G) = \rho_G([(s_1+s_2,t_1+t_2)]_G)$$
$$= \rho(s_1 + s_2) - \rho(t_1 + t_2)$$
$$= (\rho(s_1) - \rho(t_1)) + (\rho(s_2) - \rho(t_2))$$
$$= \rho_G([(s_1,t_1)]_0) + \rho_G([(s_2,t_2)]_0).$$

If $A$ and $B$ are C*-algebras, with $\varphi : A \rightarrow B$ a continuous *-homorphism, then $\varphi$ extends naturally to a *-homomorphism $M_n(A) \rightarrow M_n(B)$ for all $n \in \mathbb{N}$ by applying $\varphi$ entry-wise to matrix entries, i.e. $\varphi(T)_{ij} = \varphi(T_{ij})$. This map clearly respects matrix multiplication and involution. In the same way, $\varphi$ extends entry-wise to $\mathcal{P}_\infty(A)$ and respects direct sum, and is thus a monoid homomorphism $\mathcal{P}_\infty(A) \rightarrow \mathcal{P}_\infty(B)$. Let $\pi : \mathcal{P}_\infty(B) \rightarrow \mathcal{P}_\infty(B)/\sim_0$ be the quotient map. Then $\pi \circ \varphi$ is a monoid homomorphism $\mathcal{P}_\infty(A) \rightarrow \mathcal{P}_\infty(B)/\sim_0$. If $p, q \in \mathcal{P}_\infty(A)$ with $p \sim_0 q$, there exists some matrix $v$ with entries in $A$
such that $vv^* = p$ and $v^*v = q$. Hence
\[
\pi \circ \varphi(p) = \pi(\varphi(vv^*)) = \pi(\varphi(v)v^*) = \pi(\varphi(v^*)\varphi(v)) = \pi(\varphi(v^*v)) = \pi \circ \varphi(q)
\]

So $\pi \circ \varphi(p)$ factors into a monoid homomorphism $\tilde{\varphi} : \mathcal{P}_\infty(A)/\sim_0 \to \mathcal{P}_\infty(B)/\sim_0$ by $\tilde{\varphi}([p]) = \pi \circ \varphi(p)(= [\varphi(p)])$.

**Proposition 4.2.** Let $A$ and $B$ be $C^*$-algebras and $\varphi : A \to B$ a continuous $\ast$-homomorphism. Then there exists a group homomorphism $K_0(\varphi) : A \to B$ satisfying $K_0(\varphi)([p]_0) = [\varphi(p)]_0$ for all $p \in \mathcal{P}_\infty(A)$.

**Proof.** Recall that $K_0(A) = G(\mathcal{P}_\infty(A)/\sim_0)$, where there is a monoid homomorphism $[\cdot]_0 : A \to K_0(A)$. By the previous paragraph, we have a monoid homomorphism
\[
\tilde{\varphi} : \mathcal{P}_\infty(A)/\sim_0 \to \mathcal{P}_\infty(B)/\sim_0.
\]

By Proposition 4.1, let $K_0 = \tilde{\varphi}_G$, and let $\iota_A, \iota_B$ be the “inclusion” from $\mathcal{D}(A) \to K_0(A)$ and $\mathcal{D}(B) \to K_0(B)$ respectively, as in Proposition 2.14. Then
\[
K_0(\varphi)([p]_0) = K_0(\varphi)(\iota_A([p]_\mathcal{D})) = \tilde{\varphi}_G([([p]_\mathcal{D}, [0]_\mathcal{D}])_G)
= [(\tilde{\varphi}(p)_\mathcal{D}), \tilde{\varphi}(0)_\mathcal{D})]_G = \iota_B \circ \tilde{\varphi}([p]_\mathcal{D})
= \iota_B([\varphi(p)]_\mathcal{D}) = [\varphi(p)]_0
\]

**Proposition 4.3.** Let $A$ be a unital $C^*$-algebra, then $K_0(A) = \{[p]_0 - [q]_0 : p, q \in \mathcal{P}_\infty(A)\}$, and $[0]_0 = 0$.

**Proof.** Every element of $K_0(A)$ can be written as $[[p]_\mathcal{D}, [q]_\mathcal{D}]_G$ for some $p, q \in \mathcal{P}_\infty(A)$, and
\[
[[p]_\mathcal{D}, [q]_\mathcal{D}]_G = [[p]_\mathcal{D}, 0]_G + [[0, [q]_\mathcal{D}]_G
= [[p]_\mathcal{D}, 0]_G - [[q]_\mathcal{D}, 0]_G.
\]

Also,
\[
[0]_0 = [[0]_\mathcal{D}, 0]_G = [(0, 0)]_G = 0.
\]

**Proposition 4.4.** Let $A, B$ and $C$ be $C^*$-algebras, let $\varphi : A \to B$ and $\psi : B \to C$ be continuous $\ast$-homomorphisms. Then $K_0(\psi) \circ K_0(\varphi) = K_0(\psi \circ \varphi)$. Also, let 0 denote the zero map between any two $C^*$-algebras, then $K_0(0) = 0$, the zero group map.
Proof. By Proposition 4.3, every element in $K_0(A)$ is of the form $[p]_0 - [q]_0$ for some $p, q \in \mathcal{P}_\infty(A)$. Computing using Proposition 4.2,

\[
K_0(\psi) \circ K_0(\varphi)([p]_0 - [q]_0) = K_0(\psi) (K_0(\varphi)([p]_0) - K_0(\varphi)([q]_0)) \\
= K_0(\psi) ([\varphi(p)]_0 - [\varphi(q)]_0) \\
= [\psi \circ \varphi(p)]_0 - [\psi \circ \varphi(q)]_0 \\
= K_0(\psi \circ \varphi)([p]_0 - [q]_0).
\]

Moreover,

\[
K_0(0)([p]_0 - [q]_0) = [0(p)]_0 - [0(q)]_0 = 0 - 0 = 0. \quad \blacksquare
\]

**Corollary 4.5.** The map $K_0$ is a (covariant) functor, with $K_0$ on $C^*$-algebras defined as in Definition 2.17 and $K_0$ on continuous $*$-morphisms defined as in Proposition 4.2.

**Proof.** Simply collect the results from Propositions 4.2 and 4.4. \quad \blacksquare
5 \( K_0 \) of general C*-algebras

Let \( A \) be a C*-algebra, possibly non-unital. Let \( \widetilde{A} \) denote the unitization of \( A \). Then \( \widetilde{A} = A \oplus \mathbb{C}I \) as a vector space, and \( A \) is an ideal in \( \widetilde{A} \). Let \( \iota_I, \iota_A \) be the inclusion maps from \( \mathbb{C}I \) and \( A \) into \( \widetilde{A} \) respectively, and let \( \pi_I \) and \( \pi_A \) be the natural quotient maps from \( \widetilde{A} \) onto \( \mathbb{C}I \) and \( A \) respectively. Both \( A \) and \( \mathbb{C}I \) are unital C*-algebras. Their \( K_0 \) groups are defined as in the first section. Also, the inclusion \( \iota_I \) induces a group homomorphism \( K_0(\iota_I) : K_0(\mathbb{C}I) = \mathbb{Z} \to \widetilde{A} \).

**Definition 5.1.** Let \( A \) be a C*-algebra. Define \( K_0(A) = \ker K_0(\pi_I) \).

**Proposition 5.2.** Let \( A \) be a C*-algebra. Then

\[
K_0(A) = \{ [p]_0 - [q]_0 : p, q \in \mathcal{P}_\infty(\widetilde{A}), \pi_I(p) \sim_0 \pi_I(q) \} =: S_1
\]

\[
= \{ ([p]_0 - [q]_0) - ([\pi_I]_0(p) - [\pi_I]_0(q)) : p, q \in \mathcal{P}_\infty(\widetilde{A}) \} =: S_2
\]

\[
= \{ [p]_0 - [\pi_I(p)]_0 : p \in \mathcal{P}_\infty(\widetilde{A}) \} =: S_3
\]

**Proof.** Let \( g \in K_0(\widetilde{A}) \) and \( g \in \ker K_0(\pi_I) \). Then there exists some \( n \in \mathbb{N} \) and \( p, q \in \mathcal{P}_\infty(\widetilde{A}) \) such that \( g = [p]_0 - [q]_0 \), and that

\[
0 = K_0(\pi_I)([p]_0 - [q]_0) = [\pi_I(p)]_0 - [\pi_I(q)]_0.
\]

So \( \pi_I(p) \sim_0 \pi_I(q) \). Conversely suppose \( \pi_I(p) \sim_0 \pi_I(q) \), then

\[
K_0(\pi_I)([p]_0 - [q]_0) = [\pi_I(p)]_0 - [\pi_I(q)]_0.
\]

This proves the first equality.

With the first equality in mind, suppose \( \pi_I(p) \sim_0 \pi_I(q) \). Then

\[
[p]_0 - [q]_0 = ([p]_0 - [q]_0) - ([\pi_I(p)]_0 - [\pi_I(q)]_0) \in S_2.
\]

So \( K_0(A) = S_1 \subseteq S_2 \). And

\[
K_0(\pi_I) \left( ([p]_0 - [q]_0) - ([\pi_I(p)]_0 - [\pi_I(q)]_0) \right)
\]

\[
= ([\pi_I(p)]_0 - [\pi_I(q)]_0) - ([\pi_I \circ \pi_I(p)]_0 - [\pi_I \circ \pi_I(q)]_0)
\]

\[
= ([\pi_I(p)]_0 - [\pi_I(q)]_0) - ([\pi_I(p)]_0 - [\pi_I(q)]_0)
\]

\[
= 0
\]

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So $S_2 \subseteq \overline{K}_0(A)$, this proves the second equality.

Clearly $S_3 \subseteq S_2$. Take

$$g = ([p]_0 - [q]_0) - ([\pi_I(p)]_0 - [\pi_I(q)]_0) \in S_2.$$ 

Suppose $q$ is $n \times n$, and let $p' = p \oplus (I_n - q)$. Then

$$[p']_0 = [p]_0 - [q]_0 + [I_n]_0.$$

Also

$$\pi_I(p') = \pi_I(p) \oplus (I_n - \pi_I(q)),$$

so

$$[\pi_I(p')]_0 = [\pi_I(p)]_0 - [\pi_I(q)]_0 + [I_n]_n.$$

Thus $[p']_0 - [\pi_I(p)]_0 = g$, this proves $S_2 = S_3$. 

The above gives a definition for the $K_0$ group of non-unital $C^*$-algebras, and defines another abelian group for a unital $C^*$-algebra. We need to verify that it coincides with the previous definition for the unital case.

**Lemma 5.3.** Let $A$ be a unital $C^*$-algebra. Let $1_A$ denote the identity of $A$, and let $\tilde{A} = A \oplus \mathbb{C}I$ as vector space. Then $\tilde{A} \cong A \oplus \mathbb{C}J$. The $C^*$-algebra $A \oplus \mathbb{C}J$ is defined with norm $\|a + zJ\| = \max(\|a\|, |z|)$ and involution $(a + zJ)^* = a^* + \bar{z}J$.

**Proof.** Define $\tau : A \oplus \mathbb{C}J \rightarrow \tilde{A}$ by $a \oplus zJ \mapsto a + z(I - 1_A)$. This is clear a vector space isomorphism and respects the involution. Lastly,

$$\tau(a + zJ)\tau(b \oplus wJ) = (a + z(I - 1_A))(b + w(I - 1_A))$$

$$= ab + w(aI - a1_A) + z(1b - 1_Ab) + zw(I - I1_A - 1_AI + 1_A1_A)$$

$$= ab + w(a - a) + z(b - b) + zw(I - 1_A - 1_A + 1_A)$$

$$= ab + zw(I - 1_A)$$

$$= \tau(ab \oplus zwJ).$$

So $\tau$ is an isomorphism. 


Remark 5.4. To gain an intuitive idea of the above lemma, consider the case of where \( A = C(X) \) is the set of continuous functions from a compact Hausdorff space \( X \) into the complex numbers. The unitization \( C(X) \) is isomorphic to \( C(X \sqcup \{\ast\}) \) (see Proposition 9.9). Let \( 1_A \) denote the function that is constantly 1 on \( X \) and zero on \( \ast \). Let \( 1_* \) be the function that is 1 on \( \ast \) and constantly zero on \( X \). Then we have

\[
C(X \sqcup \{\ast\}) \cong C(X) \oplus C(\{\ast\}) \cong C(X) \oplus \mathbb{C}1_*,
\]

where \( 1_* = 1 - 1_A \). The proof of the lemma imitates this idea to prove it in the non-commutative case.

Proposition 5.5. Let \( A \) be a unital \( C^* \)-algebra, then \( K_0(A) \cong K_0(A) \).

Proof. By the lemma above, \( \widetilde{A} \cong A \oplus CJ \). Let \( \iota_A : A \to A \oplus CJ \) be the natural inclusion map and \( \pi_A : A \oplus CJ \to A \) the quotient map. The map \( \tau : A \oplus CJ \to \widetilde{A} \) is defined in the previous proof. Define \( \alpha : K_0(A) \to K_0(A) \) by

\[
[p]_0 - [q]_0 \mapsto [\tau(\iota_A(p))]_0 - [\tau(\iota_A(q))]_0.
\]

In other words, \( \alpha = K_0(\tau \circ \iota_A) \). Since \( \pi_I(\tau(\iota_A(p))) = \pi_I(\tau(\iota_A(q))) \), the image of \( \alpha \) is indeed in \( K_0(A) \). Let \( \beta = K_0(\pi_A \circ \tau^{-1}) : K_0(A) \to K_0(A) \). Then,

\[
\beta \circ \alpha = K_0(\pi_A \circ \tau^{-1} \circ \iota_A) = K_0(\pi_A \circ \iota_A) = K_0(\text{id}_A) = \text{id}_{K_0(A)}.
\]

For \( \widetilde{p}, \widetilde{q} \in \mathcal{P}_\infty(\widetilde{A}) \) with \( \pi_I(\widetilde{p}) = \pi_I(\widetilde{q}) \), let \( p_1 = \tau \circ \iota_A \circ \pi_A \circ \tau^{-1}(\widetilde{p}) \) and \( p_2 = \widetilde{p} - p_1 \). Then \( p_1 + p_2 = \widetilde{p} \) and \( p_1, p_2 \) are orthogonal projections. Write \( \widetilde{q} = q_1 + q_2 \) in the same way. Since \( \pi_I(\widetilde{p}) = \pi_I(\widetilde{q}) \), by the way that \( \tau \) is defined, we have that \( p_2 = q_2 \). So

\[
[p]_0 - [q]_0 = ([p_1]_0 + [p_2]_0) - ([q_1]_0 + [q_2]_0) = [p_1]_0 - [q_1]_0,
\]

and

\[
(\alpha \circ \beta)([p]_0 - [q]_0) = K_0(\tau \circ \iota_A \circ \pi_A \circ \tau^{-1})([p]_0 - [q]_0) = [p_1]_0 - [q_1]_0 = [\widetilde{p}]_0 - [\widetilde{q}]_0.
\]

Hence \( \alpha \) and \( \beta \) are mutual inverses. \( \blacksquare \)
Definition 5.6. Let $A$ be a non-unital $C^*$-algebra. Define $K_0(A) := \overline{K}_0(A)$.

Remark 5.7. By Proposition 5.5, we can safely write $K_0(A) = \overline{K}_0(A)$ for any unital $C^*$-algebras $A$.

The description $S_3$ in Proposition 5.2 is the one will be used most often. Next is a discussion of when two elements in such description are equivalent.

Lemma 5.8. Let $A$ be a $C^*$-algebra, $v \in M_{m,n}(A)$ and $w \in M_{n,k}(A)$ for some $k, m, n \in \mathbb{N}$. Then $\pi_I(vw) = \pi_I(v)\pi_I(w)$.

Proof. We compute $\pi_I(vw)$ to be

$\pi_I[(v - \pi_I(v))(w - \pi_I(w)) + \pi_I(v)(w - \pi_I(w)) + (v - \pi_I(v))w + \pi_I(v)\pi_I(w)]$

Since $A$ is an ideal in $\widetilde{A}$, all of $(v - \pi_I(v))(w - \pi_I(w))$, $\pi_I(v)(w - \pi_I(w))$ and $(v - \pi_I(v))w$ have entries in $A$, which are 0 when they are evaluated under $\pi_I$. So

$\pi_I(vw) = \pi_I(\pi_I(v)\pi_I(w)) = \pi_I(v)\pi_I(w)$

since $\pi_I(v)\pi_I(w) \in M_{k,l}(\mathbb{C}I)$.

Lemma 5.9. Let $A$ be a $C^*$-algebra, and let $p, q \in \mathcal{P}_\infty(\widetilde{A})$. Then $p \sim_0 q$ in $\mathcal{P}_\infty(\widetilde{A})$ implies $\pi_I(p) \sim_0 \pi_I(q)$.

Proof. There exists a matrix $v$ with entries in $\widetilde{A}$ such that $vv^* = p$ and $v^*v = q$. By Lemma 5.8,

$\pi_I(p) = \pi_I(vv^*) = \pi_I(v)\pi_I(v^*) \sim_0 \pi_I(v^*)\pi_I(v) = \pi_I(v^*v) = \pi_I(q)$.

Proposition 5.10. Let $A$ be a $C^*$-algebra, and $p, q \in \mathcal{P}_\infty(\widetilde{A})$. The following are equivalent

1. $[p]_0 - [\pi_I(p)]_0 = [q]_0 - [\pi_I(q)]_0$

2. there exists $r_1, r_2 \in \mathcal{P}_\infty(\widetilde{A})$ with $p \oplus r_1 \sim_0 q \oplus r_2$

3. there exists $k, l \in \mathbb{N}$ such that $p \oplus I_k \sim_0 q \oplus I_l$ in $\mathcal{P}_\infty(\widetilde{A})$
Proof. (1 $\implies$ 2) The equality $[p]_0 - [\pi_I(p)]_0 = [q]_0 - [\pi_I(q)]_0$ implies that

$$[p \oplus \pi_I(q)]_0 = [p]_0 + [\pi_I(q)]_0 = [q]_0 + [\pi_I(p)]_0 = [q \oplus [\pi_I(p)]]_0$$

So let $r_1 = \pi_I(q)$ and $r_2 = \pi_I(p)$. This satisfies 2.

(2 $\implies$ 3) Since $r_i = \pi_I(r_i)$ for $i = 1, 2$, we see that $r_1$ and $r_2$ can be considered as matrices in $M_n(\mathbb{C})$ and $M_n(\mathbb{C})$ respectively. Let $k = \text{rank } r_1 \leq n$. Let $\{z_1, \ldots, z_k\}$ be an orthonormal basis of $\text{Ran} r_1 \mathbb{C}^n$, and extend it to an orthonormal basis $\{z_1, \ldots, z_n\}$ of $\mathbb{C}^n$. Let $\{e_1, \ldots, e_n\}$ denote the standard basis of $\mathbb{C}^n$, and define $u \in M_n(\mathbb{C})$ by $uz_j = e_j$ for $j = 1, \ldots, n$. Then $u$ is unitary since it takes an orthonormal basis to another one, and

$$ur_1u^*e_j = ur_1z_j = \begin{cases} uz_j = e_j & : j = 1, \ldots, k \\ u0 = 0 & : j = k + 1, \ldots, n \end{cases}$$

So

$$r_1 \sim_0 ur_1u^* = I_k \oplus 0_{n-k} \sim_0 I_k.$$ 

By identifying $u$ as a unitary matrix in $M_k(\mathbb{C}I)$, this also holds true in $\mathcal{P}_\infty(\tilde{A})$.

Similarly, $r_2 \sim_0 I_l$ in $\mathcal{P}_\infty(\tilde{A})$ for $l = \text{rank } r_2$. So

$$p \oplus I_k \sim_0 p \oplus r_1 \sim_0 q \oplus r_2 \sim_0 q \oplus I_l.$$ 

(3 $\implies$ 1) We use Lemma 5.9 here and compute

$$[p]_0 - [\pi_I(p)]_0 = [p]_0 - [\pi_I(p)]_0 + [I_k]_0 - [I_k]_0$$

$$= [p \oplus I_k]_0 - [\pi_I(p) \oplus I_k]_0$$

$$= [p \oplus I_k]_0 - [\pi_I(p \oplus I_k)]_0$$

$$= [q \oplus I_l]_0 - [\pi_I(q \oplus I_l)]_0$$

$$= [q]_0 - [\pi_I(q)]_0. \blacksquare$$

The next natural step is to extend the functor $K_0$ to all $\ast$-homomorphisms on all $C^\ast$-algebras. Let $A, B$ be $C^\ast$-algebras. A $\ast$-homomorphism $\varphi : A \to B$ can be extended to a $\ast$-homomorphism $\tilde{\varphi} = A \oplus CI_A \to B = B \oplus CI_B$ by $\tilde{\varphi}|_A = \varphi$ and $\tilde{\varphi}(I_A) = I_B$.

**Definition 5.11.** Let $A, B$ be $C^\ast$-algebras, $\varphi : A \to B$ a $\ast$-homomorphism. Define $\overline{K}_0(\varphi) = K_0(\tilde{\varphi})|_{K_0(A)} : K_0(A) \to K_0(B)$. Then $\overline{K}_0(\varphi)$ is a well-defined group homomorphism.
Proof. Note that $\overline{K}_0(\varphi)$ is the restriction of $K_0(\tilde{\varphi})$ to $K_0(A)$. So it is a group homomorphism. $\pi_I(\tilde{\varphi}(p)) = \pi_I(\tilde{\varphi}(q))$ by the way $\tilde{\varphi}$ is defined. So the image of $\overline{K}_0(\varphi)$ is in $K_0(B)$. 

**Proposition 5.12.** Let $A, B$ be unital $C^*$-algebras, let $\alpha : K_0(A) \to \overline{K}_0(A)$ be the group isomorphism described in the proof of Proposition 5.5, and similarly let $\beta : K_0(B) \to \overline{K}_0(B)$ be such group isomorphism. Then for any group homomorphism $\varphi : A \to B$, we have

\[ \overline{K}_0(\varphi) \circ \alpha = \beta \circ K_0(\varphi). \]

**Proof.** We adopt all notation used in Proposition 5.5, where $\alpha = K_0(\tau_A \circ \iota_A)$ and $\beta = K_0(\tau_B \circ \iota_A)$. Then

\[
\beta \circ K_0(\varphi) = K_0(\tau_B \circ \iota_A) \circ K_0(\varphi) = K_0(\tau_B \circ \iota_B \circ \varphi)
\]

and

\[
\overline{K}_0(\varphi) \circ \alpha = K_0(\tilde{\varphi})|_{\overline{K}_0(A)} \circ K_0(\tau_A \circ \iota_A) = K_0(\tilde{\varphi} \circ \tau_A \circ \iota_A).
\]

For $a \in A,$

\[
\tau_B \circ \iota_B \circ \varphi(a) = \varphi(a) \oplus 0I_B = \tilde{\varphi} \circ \tau_A \circ \iota_A(a).
\]

So $\tau_B \circ \iota_B \circ \varphi = \tilde{\varphi} \circ \tau_A \circ \iota_A$ as maps $A \to \tilde{B}$, so applying $K_0$ they are the same as maps from $K_0(A)$ to $K_0(B)$ whose image lie in $\overline{K}_0(B)$. This concludes the proof. 

**Remark 5.13.** By the above proposition and Proposition 5.5, we can safely write $\overline{K}_0(\varphi) = K_0(\varphi)$ for any $*$-homomorphism $\varphi$.

**Proposition 5.14.** Let $A, B, C$ be $C^*$-algebras, and let $\varphi : A \to B$ and $\psi : B \to C$ be $*$-homomorphisms. Then $K_0(\psi) \circ K_0(\varphi) = K_0(\psi \circ \varphi)$. Also, $K_0(\text{id}_A) = \text{id}_{K_0(A)}$ and $K_0(0) = 0$ for 0 any zero map.

**Proof.** We compute:

\[
K_0(\psi) \circ K_0(\varphi) = K_0(\tilde{\psi})|_{K_0(B)} \circ K_0(\tilde{\varphi})|_{K_0(A)}
\]

\[
= K_0(\tilde{\psi} \circ \tilde{\varphi})|_{K_0(A)}
\]

\[
= K_0(\psi \circ \varphi)|_{K_0(A)}
\]

\[
= K_0(\psi \circ \varphi).
\]
Similarly,

\[ K_0(\text{id}_A) = K_0(\overline{\text{id}_A})|_{K_0(A)} \\
= K_0(\text{id}_A)|_{K_0(A)} \\
= \text{id}_{K_0(A)}|_{K_0(A)} \\
= \text{id}_{K_0(A)}. \]

Finally,

\[ K_0(0) = K_0(\overline{0})|_{K_0(A)} = K_0(\pi_I)|_{K_0(A)}. \]

But \( K_0(A) \) is exactly \( \ker K_0(\pi_I) \), so \( K_0(0) = 0 \). \( \blacksquare \)

Now we have a functor \( K_0 \) from the category of \( C^* \)-algebras to the category of abelian groups.
6 Functorial properties of $K_0$

The $K_0$-group of a C*-algebra can be difficult to compute even for most C*-algebras. With the functoriality of $K_0$ in hand, some useful properties of the functor $K_0$ will aid calculation. One might say this is similar to how exact sequences help the computation of cohomology groups. In fact, $K_0$ is an extraordinary cohomology functor, but this will not be discussed here.

In short summary, the most basic and important properties of the functor $K_0$ are homotopy invariance, half exactness and split exactness. Also, $K_0$ is a continuous functor, meaning that the inductive limit $K_0$-group is isomorphic to the $K_0$-group of inductive limits. Other useful tools for computing the $K_0$-groups include the higher $K$-groups, Bott periodicity, and the 6-term exact sequence. In this paper we will only prove the three basic functorial properties of $K_0$.

**Definition 6.1.** Let $A$ and $B$ be C*-algebras and $\varphi, \psi : A \to B$ be $*$-homomorphisms. We say $\varphi$ is **homotopic** to $\psi$, written $\varphi \sim_h \psi$, if there exists a family of continuous $*$-homomorphisms $\varphi_t : A \to B$ for $t \in [0, 1]$ such that $\varphi_0 = \varphi$ and $\varphi_1 = \psi$, and that for each $a \in A$, $t \mapsto \varphi_t(a)$ is a continuous map $[0, 1] \to B$. The family $\varphi_t$ is called a homotopy from $\varphi$ to $\psi$.

Let $A$ and $B$ be C*-algebras. We say $A$ is **homotopic** to $B$, written $A \sim_h B$, if there exists $\varphi : A \to B$ and $\psi : B \to A$ continuous $*$-homomorphisms such that $\varphi \circ \psi \sim_h \text{id}_A$ and $\psi \circ \varphi \sim_h \text{id}_B$.

**6.1 Homotopy invariance**

**Proposition 6.2.** Let $A$ and $B$ be C*-algebras, $\varphi, \psi : A \to B$ be continuous $*$-homomorphisms with $\varphi \sim_h \psi$, then $K_0(\varphi) = K_0(\psi)$. If $A \sim_h B$, then $K_0(A) \cong K_0(B)$.

**Proof.** Once again, a typical element in $K_0(A)$ is $[p]_0 - [q]_0$ for some $p, q \in \mathcal{P}_\infty(A)$. Hence it is sufficient to show that $K_0(\varphi)(p) = K_0(\psi)(p)$ for all $p \in \mathcal{P}_\infty$. Let $\varphi_t$ be a homotopy from $\varphi$ to $\psi$. The family $\varphi_t$ extends to a homotopy from $\varphi$ to $\psi$ on $M_n(A)$. The map $[0, 1] \to M_n(B)$ given by $t \mapsto \varphi_t(p)$ is continuous, and since each $\varphi_t$ is a $*$-homomorphism, $\varphi_t(p) \in \mathcal{P}_n(B)$, so $t \mapsto \varphi_t(p)$ is a homotopy of

$$\varphi(p) = \varphi_0(p) \sim_h \varphi_1(p) = \psi(p).$$

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But we know homotopic projections are equivalent in $\mathcal{D}(A)$, so
\[ K_0(\varphi)(p) = [\varphi(p)]_0 = [\psi(p)]_0 = K_0(\psi)(p). \]
Hence $K_0(\varphi) = K_0(\psi)$.

Suppose $A \sim_h B$. There exists continuous homomorphisms $\alpha : A \to B$ and $\beta : B \to A$ such that $\alpha \circ \beta \sim_h \text{id}_A$ and $\beta \circ \alpha \sim_h \text{id}_B$. Then using Proposition 4.4 and the first half of this proof,
\[ K_0(\alpha) \circ K_0(\beta) = K_0(\alpha \circ \beta) = K_0(\text{id}_A) = \text{id}_{K_0(A)}, \]
\[ K_0(\beta) \circ K_0(\alpha) = K_0(\beta \circ \alpha) = K_0(\text{id}_B) = \text{id}_{K_0(B)}. \]
Hence $K_0(\alpha) : K_0(A) \to K_0(B)$ is a group isomorphism, whose inverse is $K_0(\beta)$. ■

### 6.2 Half- and split-exactness

**Definition 6.3.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be a functor.

1. $\mathcal{F}$ is exact if whenever
\[
0 \longrightarrow A \overset{f}{\longrightarrow} B \overset{g}{\longrightarrow} C \longrightarrow 0
\]
is a short exact sequence in $\mathcal{C}$, then
\[
0 \longrightarrow \mathcal{F}(A) \overset{\mathcal{F}(f)}{\longrightarrow} \mathcal{F}(B) \overset{\mathcal{F}(g)}{\longrightarrow} \mathcal{F}(C) \longrightarrow 0
\]
is exact in $\mathcal{D}$.

2. $\mathcal{F}$ is half exact if whenever
\[
0 \longrightarrow A \overset{f}{\longrightarrow} B \overset{g}{\longrightarrow} C \longrightarrow 0
\]
is a short sequence in $\mathcal{C}$, then
\[
\mathcal{F}(A) \overset{\mathcal{F}(f)}{\longrightarrow} \mathcal{F}(B) \overset{\mathcal{F}(g)}{\longrightarrow} \mathcal{F}(C)
\]
is sequence in $\mathcal{D}$ that is exact at $\mathcal{F}(B)$. 28
3. \( \mathcal{F} \) is split exact if whenever

\[
0 \longrightarrow A \xrightarrow{f} B \xleftarrow{g} C \longrightarrow 0
\]

is a split exact sequence in \( \mathcal{C} \), then

\[
0 \longrightarrow \mathcal{F}(A) \xrightarrow{\mathcal{F}(f)} \mathcal{F}(B) \xleftarrow{\mathcal{F}(g)} \mathcal{F}(C) \longrightarrow 0
\]

is a split exact sequence in \( \mathcal{D} \).

Clearly an exact functor would be half-exact. In this section we will show that the functor \( K_0 \) is half-exact and split-exact. However, \( K_0 \) is not an exact functor. We will see a counterexample in a later section when we have developed more machinery.

**Lemma 6.4.** Let

\[
0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0
\]

be a short exact sequence of C*-algebras, and let \( n \in \mathbb{N} \). Let \( \tilde{\varphi} : M_n(\tilde{\mathcal{A}}) \rightarrow M_n(\tilde{\mathcal{B}}) \) and \( \tilde{\psi} : M_n(\tilde{\mathcal{B}}) \rightarrow M_n(\tilde{\mathcal{C}}) \) be the unital \(*\)-homomorphisms induced by \( \varphi \) and \( \psi \), respectively. Then,

1. The map \( \tilde{\varphi} : M_n(\tilde{\mathcal{A}}) \rightarrow M_n(\tilde{\mathcal{B}}) \) is injective.
2. An element \( a \in M_n(\tilde{\mathcal{B}}) \) belongs to the image of \( \tilde{\varphi} \) if and only if \( \tilde{\psi}(a) = \pi_I(\tilde{\psi}(a)) \).

**Proof.**

1. The map \( \tilde{\varphi} : A \oplus CI_A \rightarrow B \oplus CI_B \) is injective on both \( A \) and \( CI_A \). Therefore it is injective \( \tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{B}} \), and also the induced map \( \tilde{\varphi} : M_n(\tilde{\mathcal{A}}) \rightarrow M_n(\tilde{\mathcal{B}}) \) is continuous.

2. For \( a \in A \) and \( z \in \mathbb{C} \),

\[
\begin{align*}
\tilde{\psi} \circ \tilde{\varphi}(a + zI_A) &= \tilde{\psi}(\varphi(a) + zI_B) = \psi \circ \varphi(a) + zI_C = zI_C \\
&= \pi_I(\tilde{\psi} \circ \tilde{\varphi}(a + zI_A)).
\end{align*}
\]

Conversely, suppose \( b \in B \) and \( z \in \mathbb{C} \) with

\[
\psi(b) + zI_C = \tilde{\psi}(b + zI_B) = \pi_I(\tilde{\psi}(b + zI_B)) = zI_C.
\]

Then \( \psi(b) = 0 \). By exactness there exists \( a \in A \) such that \( \varphi(a) = b \), then \( b + zI_B = \tilde{\varphi}(a + zI_A) \).
Proposition 6.5. $K_0$ is half-exact.

Proof. Let $A, B$ and $C$ be C*-algebras with $*$-homomorphisms $\varphi : A \to B$ and $\psi : B \to C$, where $\varphi$ is injective, $\psi$ is surjective, and $\text{im}(\varphi) = \ker(\psi)$.

A typical element in $K_0(A)$ is $[p]_0 = [\pi_I(p)]_0$ for some $p \in P_\infty(A)$. By Lemma 6.4 the equation

$$\tilde{\psi} \circ \tilde{\varphi}(p) = \pi_I(\tilde{\psi} \circ \tilde{\varphi}(p)) = \tilde{\psi} \circ \tilde{\varphi}(\pi_I(p))$$

holds. So

$$K_0(\psi) \circ K_0(\varphi)([p]_0 - [\pi_I(p)]_0) = [\tilde{\psi} \circ \tilde{\varphi}(p)]_0 - [\tilde{\psi} \circ \tilde{\varphi}(\pi_I(p))]_0 = 0.$$ 

So $\text{im}(K_0(\varphi)) \subseteq \ker(K_0(\psi))$.

Conversely, let $[p]_0 - [\pi_I(p)]_0 \in K_0(B)$ be in the kernel of $K_0(\psi)$. Since $\tilde{\psi}(p) \sim_0 \tilde{\psi}((\pi_I(p))$ in $P_n(C)$ for some $n \in \mathbb{N}$, by Proposition 3.15 there exists a unitary element $u \in M_{2n}(C)$ such that

$$u(\tilde{\psi}(p) \oplus 0_n)u^* = \tilde{\psi}(\pi_I(p)) \oplus 0_n.$$ 

By Lemma 3.10 there exists a unitary $v \in M_{4n}(B)$ such that $\tilde{\psi}(v) = u \oplus u^*$. Let $p_1 = v(p \oplus 0_{3n})v^*$. Then

$$p \sim_0 p \oplus 0_{3n} \sim_0 p_1,$$

and similarly $\pi_I(p) \sim_0 \pi_I(p_1)$. Also,

$$\tilde{\psi}(p_1) = \begin{bmatrix} u & 0 \\ 0 & u^* \end{bmatrix} \begin{bmatrix} \tilde{\psi}(p) \oplus 0_n \\ 0 & 0_{2n} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ u^* & u \end{bmatrix} = \begin{bmatrix} u(\tilde{\psi}(p) \oplus 0_n)u^* & 0 \\ 0 & 0_{2n} \end{bmatrix} = \pi_I(\tilde{\psi}(p)) \oplus 0_{3n}.$$ 

It follows that $\tilde{\psi}(p_1) = \pi_I(\tilde{\psi}(p_1))$. By Lemma 6.4 there exists $e \in M_{3n}$ such that $\tilde{\varphi}(e) = p_1$. Also,

$$\tilde{\varphi}(ee) = \tilde{\varphi}(e)\tilde{\varphi}(e) = p_1p_1 = p_1,$$

$$\tilde{\varphi}(e^*) = p_1^* = p_1.$$
By Lemma 6.4, \( \tilde{\varphi} : M_{4n}(\tilde{A}) \to M_{4n}(\tilde{B}) \) is injective, which implies \( e = ee = e^* \), and hence \( e \) is a projection. Now

\[
K_0(\varphi)([e]_0 - [\pi_I(e)]_0) = [p_1]_0 - [\pi_I(p_1)]_0 = [p]_0 - [\pi_I(p)]_0.
\]

This shows that \( \ker K_0(\psi) \subseteq \text{im } K_0(\varphi) \). Therefore \( \ker K_0(\psi) = \text{im } K_0(\varphi) \).

**Proposition 6.6.** The functor \( K_0 \) is split-exact.

**Proof.** Suppose

\[
0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \xrightarrow{\lambda} 0
\]

is a split exact sequence of \( C^\ast \)-algebras. By the half-exactness just proved, the sequence

\[
K_0(A) \xrightarrow{K_0(\varphi)} K_0(B) \xrightarrow{K_0(\psi)} K_0(C)
\]

is exact. Also, since \( K_0 \) is a functor, we have

\[
K_0(\psi) \circ K_0(\lambda) = K_0(\psi \circ \lambda) = K_0(\text{id}_C) = \text{id}_{K_0(C)},
\]

so the sequence is also exact at \( K_0(C) \). It is left to show that \( K_0(\varphi) \) is injective.

Let \( g \in K_0(A) \) be in the kernel of \( K_0(\varphi) \). By the proof of Proposition 6.5, there exits some \( n \in \mathbb{N} \), \( p \in \mathcal{P}_n(\tilde{A}) \) and some unitary \( u \in M_n(\tilde{B}) \) such that \( g = [p]_0 - [\pi_I(p)]_0 \) and \( u \tilde{\varphi}(p) u^* = \pi_I(\tilde{\varphi}(p)) \). Let \( v = (\tilde{\lambda} \circ \tilde{\psi})(u^* u) \). Then

\[
v^* v = u^* (\tilde{\lambda} \circ \tilde{\psi}(u))(\tilde{\lambda} \circ \tilde{\psi}(u^*)) u = u^* I_n u = I_n,
\]

and

\[
v v^* = (\tilde{\lambda} \circ \tilde{\psi}(u^*)) uu^* (\tilde{\lambda} \circ \tilde{\psi}(u)) = I_n,
\]

and

\[
\tilde{\psi}(v) = (\tilde{\psi} \circ \tilde{\lambda} \circ \tilde{\psi}(u^*)) (\tilde{\psi}(u)) = \tilde{\psi}(u^*) \tilde{\psi}(u) = \tilde{\psi}(I_n) = I_n.
\]

Since \( \tilde{\psi}(v) = \pi_I(\tilde{\psi}(v)) \), by Lemma 6.4, there exists \( w \in M_n(\tilde{A}) \) such that \( \tilde{\varphi}(w) = v \). Since \( \tilde{\varphi} \) is injective and \( \tilde{\varphi}(w^* w) = I_n = \tilde{\varphi}(ww^*) \), have \( ww^* =
$I_n = w^*w$, so $w$ is unitary. Moreover,

$$
\varphi(wp^*) = v\varphi(p)v^* = (\tilde{\lambda} \circ \tilde{\psi})(u^*)u\varphi(p)u^*(\tilde{\lambda} \circ \tilde{\psi})(u) = (\tilde{\lambda} \circ \tilde{\psi})(u^*)\pi_I(\varphi(p))(\tilde{\lambda} \circ \tilde{\psi})(u) = (\tilde{\lambda} \circ \tilde{\psi})(u^*\pi_I(\varphi(p))u) = (\tilde{\lambda} \circ \tilde{\psi})(\varphi(p)) = \tilde{\lambda}((\tilde{\psi} \circ \varphi)(\pi_I(p))) = \varphi(\pi_I(p)).
$$

By the injectivity of $\tilde{\varphi}$ we can conclude that $\pi_I(p) = wp^*$. Hence $p \sim_0 \pi_I(p)$ in $\mathcal{P}_n(\tilde{A})$. Therefore $g = 0$. ■

**Corollary 6.7.** Let $A$ and $B$ be $C^*$-algebras. Then $K_0(A \oplus B) \cong K_0(A) \oplus K_0(B)$.

**Proof.** The sequence

$$
0 \longrightarrow A \longrightarrow A \oplus B \xrightarrow{\cong} B \longrightarrow 0
$$

is split-exact. Hence by the split-exactness of $K_0$, we have a split-exact sequence of abelian groups:

$$
0 \longrightarrow K_0(A) \longrightarrow K_0(A \oplus B) \xrightarrow{\cong} K_0(B) \longrightarrow 0.
$$

Therefore $K_0(A) \oplus K_0(B) \cong K_0(A \oplus B)$. ■
7 K-theory of compact Hausdorff spaces

**Definition 7.1.** Let $X$ be a Hausdorff topological space, $V$ and $W$ topological vector bundles over $X$. Define the map $\pi_V : V \to X$ by $\pi_V(v) = x$ if $v \in V_x$. We write $\pi = \pi_V$, when it is understood that $\pi$ has domain $V$. A map $\varphi : V \to W$ is a bundle homomorphism if $\varphi$ is continuous, $\varphi(v) \in \pi^{-1}_V(\pi_V(v))$ for all $v \in V$, and that $\varphi_x = \varphi|_{V_x} : V_x \to W_x$ is a linear homomorphism for all $x \in X$. We say $V$ is isomorphic to $W$ if there exists $\varphi : V \to W$ and $\psi : W \to V$ bundle homomorphisms such that $\varphi \circ \psi = \text{id}_V$ and $\psi \circ \varphi = \text{id}_W$.

**Definition 7.2.** Let $X$ be a Hausdorff space and let $n \in \mathbb{N}$. Define $\Theta^n(X)$ to be the rank-$n$ trivial bundle over $X$; specifically, $\Theta^n(X) = X \times \mathbb{C}^n$.

**Definition 7.3.** For $X$ a Hausdorff space, define $\text{Vect}(X)$ to be the set of all isomorphism classes of topological vector bundles on $X$.

**Definition 7.4.** Let $X$ be a Hausdorff space, define $C(X)$ to be the set of all continuous functions from $X$ to $\mathbb{C}$. If $X$ is compact, then $C(X)$ can be equipped with the sup-norm as the norm and with pointwise conjugation as its involution. This gives $C(X)$ a $\text{C}^*$-algebra structure.

**Remark 7.5.** Let $\mathcal{C}$ be the category of compact Hausdorff spaces and let $\mathcal{A}$ be the category of unital $\text{C}^*$-algebras. Define a contravariant functor $\mathcal{F} : \mathcal{C} \to \mathcal{A}$ as follows. If $X$ is a compact Hausdorff space, then $\mathcal{F}(X) = C(X)$. If $X, Y$ are compact Hausdorff spaces and $\varphi \in \text{Hom}(X, Y)$, then $\mathcal{F}(\varphi) = \varphi^* \in \text{Hom}(C(Y), C(X))$ where $\varphi^* f(x) = f(\varphi(x))$ for all $f \in C(Y)$ and $x \in X$, where $\text{Hom}(X, Y)$ is the set of continuous functions from $X$ to $Y$, and $\text{Hom}(C(Y), C(X))$ is the set of $*$-homomorphisms from $C(Y)$ to $C(X)$.

If $X$ is a Hausdorff space, not necessarily compact, then $C(X)$ is not necessarily a $\text{C}^*$-algebra since the sup-norm cannot be defined. However $C(X)$ is a ring, so for $m, n \in \mathbb{N}$, it makes sense to consider $M_{m,n}(C(X))$, all $m \times n$ matrices with entries in $C(X)$. Note that $M_{m,n}(C(X))$ is naturally isomorphic to $C(X, M_{m,n}(\mathbb{C}))$, by taking a matrix $F \in M_{m,n}(C(X))$ to $f \in C(X, M_{m,n}(\mathbb{C}))$, where $[f(x)]_{ij} = F_{ij}(x)$ for all $x \in X$.

**Lemma 7.6.** Let $X$ be a Hausdorff space, and let $m, n \in \mathbb{N}$. For every $f \in C(X, M_{m,n}(\mathbb{C}))$, define a bundle homomorphism $\Gamma(f) : \Theta^n(X) \to \Theta^m(X)$ by $\Gamma(f)(x, v) = (x, f(x)v)$. Then $\Gamma : f \mapsto \Gamma(f)$ is a bijection from $C(X, M_{m,n}(\mathbb{C}))$ to $\text{Hom}(\Theta^n(X), \Theta^m(X))$. In other words, we have a one-to-one correspondence between $\text{Hom}(\Theta^n(X), \Theta^m(X))$ and $C(X, M_{m,n}(\mathbb{C})) = M_{m,n}(C(X))$. 33
Proof. Suppose \( f, g \in M_{m,n}(C(X)) \) with \( f \neq g \). Pick \( x \in X \) for which \( f(x) \neq g(x) \). Then there exists \( v \in \mathbb{C}^n \) for which \( g(x)v \neq f(x)v \), which shows that \( \Gamma \) is injective. It is left to show that \( \Gamma \) is surjective.

Let \( \mathbb{C}^n \) and \( \mathbb{C}^m \) be equipped with their standard inner products. Define \( p : \Theta^n(X) \to \mathbb{C}^n \) by \( p(x,w) = w \). Suppose \( \varphi : \Theta^n(X) \to \Theta^m(X) \) is a bundle homomorphism. Define \( f : X \to M_{m,n}(\mathbb{C}) \) so that

\[
f(x)_{ij} = \langle p(\varphi(x,e_j)), e_i \rangle
\]

for all \( x \in X \). Clearly \( f \) is continuous. Moreover,

\[
\Gamma(f)(x,v) = (x, f(x)v)
\]

\[
= (x, \sum_{i=1}^m \sum_{j=1}^n f(x)_{ij}v_je_i)
\]

\[
= (x, \sum_{i=1}^m \sum_{j=1}^n \langle p(\varphi(x,e_j)), e_i \rangle v_je_i)
\]

\[
= (x, \sum_{i=1}^m \sum_{j=1}^n \langle p(\varphi(x,v_je_j)), e_i \rangle e_i)
\]

\[
= (x, \sum_{i=1}^m \langle p(\varphi(x,v)), e_i \rangle e_i)
\]

\[
= (x, \varphi(x,v))
\]

for all \((x,v) \in \Theta^n(X)\). Thus \( \Gamma(f) = \varphi \), and we conclude that \( \Gamma \) is surjective. \( \blacksquare \)

**Lemma 7.7.** Let \( V \) and \( W \) be vector bundles over a compact Hausdorff space \( X \), and suppose that \( \varphi : V \to W \) is a bundle homomorphism such that \( \varphi_x \) is a vector space isomorphism for every \( x \in X \). Then \( \varphi \) is a bundle isomorphism.

**Proof.** Let \( X_1, \ldots, X_k \) be the connected components of \( X \), let \( V_j = V|_{X_j} \) and \( W_j = W|_{X_j} \) for \( j = 1, \ldots, k \). If \( \varphi : V \to W \) is a bundle homomorphism such that \( \varphi|_{V_j} \) is an isomorphism from \( V_j \) onto \( W_j \), then \( \varphi \) is an isomorphism from \( V \) onto \( W \). Thus for the rest of the proof we may assume that \( X \) is connected.

By hypothesis \( \varphi \) is a bijection, so \( \varphi^{-1} \) is defined, with \( \varphi^{-1}|_x \) a vector space isomorphism. We need to check that \( \varphi^{-1} \) is continuous. Choose an open cover
\{U_1, \ldots, U_l\}$ for which $V|_{U_k}$ and $W|_{U_k}$ are trivial for $k = 1, \ldots, l$. For each $k$, let $\varphi_k = \varphi|_{V|_{U_k}}$. Then it is sufficient to show that $\varphi_k^{-1}$ is continuous.

Let $n$ be the rank of $V$ and $W$. We can identify $V|_{U_k}$ and $W|_{U_k}$ with $\Theta^n(U_k)$, and can consider $\varphi_k$ to be a bundle isomorphism from $\Theta^n(U_k)$ to itself. Apply Lemma 7.6 to obtain a continuous function $f_k : U_k \to M_n(\mathbb{C})$ such that $\varphi_k(x, v) = (x, f_k(x)v)$ for all $(x, v) \in \Theta^n(U_k)$. Since $\varphi_k(x)$ is an isomorphism for all $x \in U_k$, have $f_k(x) \in GL_n(\mathbb{C})$ for all $x \in U_k$.

Each $f_k$ is an element of $C(U_k, M_n(\mathbb{C}))$. The matrix $f_k(x)$ is invertible for every $x \in U_k$, since inversion is continuous, we have that $f_k^{-1}(x) \in C(U_k, M_n(\mathbb{C}))$. Apply the lemma again have $\varphi_k^{-1}$ is continuous.

**Proposition 7.8.** Let $V$ be a vector bundle over a compact Hausdorff space $X$. Then $V$ is isomorphic to a subbundle of the trivial bundle $\Theta^N(X)$ for some $N \in \mathbb{N}$.

**Proof.** Let $X_1, \ldots, X_m$ be the distinct connected components of $X$. If $V|_{X_k}$ is a subbundle of $\Theta^{N_k}(X_k)$ for some $N_k \in \mathbb{N}$, then let $N = N_1 + N_2 + \cdots + N_m$, and $V$ is itself a subbundle of $\Theta^N(X)$. So for the rest of the proof we may assume that $X$ is connected.

Since $V$ is locally trivial, let $\mathcal{U} = \{U_1, \ldots, U_l\}$ be an open cover of $X$ such that $V|_{U_k} \cong \Theta^M(U_k)$ for some $M \in \mathbb{N}$. (Note that this $M$ is the same for all $k$ since $X$ is connected.) Let $\varphi_k : V|_{U_k} \to \Theta^M(U_k)$ be a bundle isomorphism. Define $q_k : \Theta^M(U_k) \to \mathbb{C}^M$ by $q_k(x, w) = w$ for $x \in U_k$ and $w \in \mathbb{C}^M$; also let $\pi : V \to X$ be projection onto the point in $X$ that an element $v \in V$ lies above. Choose a partition of unity $\{f_1, \ldots, f_l\}$ subordinate to the cover $\mathcal{U}$, and let $N = M \cdot l$. Then define $\Phi : V \to \bigoplus_{k=1}^l \mathbb{C}^M$ by

$$
\Phi(v) = (f_1(\pi(v))q_1(\varphi_1(v)) \oplus \cdots \oplus f_l(\pi(v))q_l(\varphi_l(v))).
$$

Then $\varphi(v) = (\pi(v), \Phi(v))$ defines a bundle homomorphism $V \to \Theta^N(X)$. Since $\varphi$ is injective, this is a bijective homomorphism onto a subbundle of $\Theta^N(X)$. By Lemma 7.7 this is indeed an isomorphism.

**Corollary 7.9.** Every vector bundle over a compact Hausdorff space admits a Hermitian metric.

**Proof.** It is clear that every trivial bundle naturally has a Hermitian metric, and since every bundle over a compact Hausdorff space is a subbundle of
some trivial bundle, then it inherits the restriction of the Hermitian metric.

Definition 7.10. Let $X$ be a Hausdorff space, and let $[V], [W] \in \text{Vect}(X)$. Define $[V \oplus W]$ to be the isomorphism class of bundles as follows. There exists $n, m \in \mathbb{N}$ such that $V$ is a subbundle of $\Theta^n(X)$ and $W$ is a subbundle of $\Theta^m(X)$. Let $Q$ be the subbundle of $\Theta^{n+m}(X)$ such that $Q_x = V_x \oplus W_x \subseteq \mathbb{C}^n \oplus \mathbb{C}^m$ for all $x \in X$. Define $[V \oplus W]$ to be $[Q]$.

Proposition 7.11. Let $X$ be a compact Hausdorff space, and let $V, W$ be vector bundles over $X$. Then $[V \oplus W]$ is well-defined and it is a vector bundle.

Proof. The proof is easy and is left as an exercise for the reader.

Remark 7.12. The vector bundle $V \oplus W$ is called the Whitney sum of $V$ and $W$. The general construction is more abstract and it may take some work to check the bundle definitions. Proposition 7.8 allows for a concrete description of the class $[V \oplus W]$. Also, in K-theory it is more helpful to think of a vector bundle as a subbundle of some trivial bundle, as we will see when we relate the topological K-theory to the C*-algebra K-theory.

Proposition 7.13. Let $X$ be a compact Hausdorff space. The set $\text{Vect}(X)$ equipped with the operation $[V] + [W] = [V \oplus W]$, is an abelian monoid.

Proof. The only non-trivial part is to verify that $[V] + [W] = [W] + [V]$. Suppose $V$ is a subbundle of $\Theta^n(X)$ and $W$ is a subbundle of $\Theta^m(X)$. We’ll write $V \oplus W$ and $W \oplus V$ as the corresponding subbundles of $\Theta^{n+m}(X)$. Let $\rho : V \oplus W \to W \oplus V$ be such that

$$\rho(x, v \oplus w) = \rho(x, w \oplus v)$$

for all $x \in X$ and $v \in V_x$, $w \in W_x$. Clearly $\rho|_x$ is a vector space isomorphism for all $x \in X$, so by Lemma 7.7 it is left to show that $\rho$ is continuous. For any $x \in X$, take an open neighbourhood $U$ of $x$ for which both $V|_U$ and $W|_U$ are trivial. There exists $k \leq n$ and $l \leq m$ for which there exists bundle isomorphisms

$$\varphi : V|_U \xrightarrow{\cong} \Theta^k(U); \quad \psi : W|_U \xrightarrow{\cong} \Theta^l(U).$$

Definition 7.14. Let $X$ be a compact Hausdorff space. Define $K^0(X) = G(\text{Vect}(X))$, where $G(\cdot)$ is the Grothendieck completion.
The following is a lemma that helps with computation of \( K^0 \)-groups.

**Lemma 7.15.** Let \( X \) be a compact Hausdorff space and let \( I \) denote the closed interval \([0, 1]\). If \( V \) is a vector bundle over \( X \times I \), then \( V|_{X \times \{0\}} \cong V|_{X \times \{1\}} \).

**Proof.** First we show that a bundle \( V \) over \( X \times [a, b] \) is trivial if there exists some \( c \in (a, b) \) such that \( V|_{X \times [a, c]} \) and \( V|_{X \times [c, b]} \) are trivial. To see this, let \( \varphi : V|_{X \times [a, c]} \to \Theta^n(X \times [a, c]) \) and \( \psi : V|_{X \times [c, b]} \to \Theta^n(X \times [c, b]) \) be bundle isomorphisms for some \( n \in \mathbb{N} \). There exists a function \( h : X \to GL_n(\mathbb{C}) \) such that \( \varphi(v) = h(\pi(v))\psi(v) \) for all \( v \in V|_X \). Then the map \( \Phi : V \to \Theta^n(X \times [a, b]) \) defined by

\[
\Phi(v) = \begin{cases} 
\varphi(v) & : a \leq t \leq c \\
\psi(v) & : c < t \leq b
\end{cases}
\]

is a bundle isomorphism.

Next, for every \( x \in X \) and \( t \in [0, 1] \) there exists some \( U_{x,t} \subseteq X \) a neighbourhood of \( x \) and some \( \delta_t > 0 \) such that \( V \) is trivial over

\[
U_{x,t} \times (t - \delta_t, t + \delta_t).
\]

Because \([0, 1]\) is compact, there exists a finite collection \( \{t_0, \ldots, t_k\} \subseteq [0, 1] \) such that

\[
\bigcup_{i=0}^k (t_i - \delta_{t_i}, t_i + \delta_{t_i}) \supseteq [0, 1].
\]

Let \( U_x = \bigcap_{i=0}^k U_{x, t_i} \). Then \( V \) is trivial over \( U_x \times (t_i - \delta_{t_i}, t_i + \delta_{t_i}) \) for all \( i = 0, \ldots, k \). Hence by observation from the previous paragraph, we see that \( V|_{U_x \times I} \) is trivial. Thus, since \( X \) is compact, there exists a finite cover \( \{U_1, \ldots, U_r\} \) of \( X \) such that \( V|_{U_j \times I} \) is trivial for all \( j = 1, \ldots, r \).

Let \( \{f_1, \ldots, f_r\} \) be a partition of unity subordinate to the cover \( \{U_1, \ldots, U_r\} \). For \( j = 0, \ldots, r \) let

\[
F_j = f_1 + \cdots + f_j.
\]

In particular \( F_0 = 0 \) and \( F_r = 1 \). Also define

\[
X_0 = \{(x, F_j(x)) : x \in X\}
\]
for \( j = 1, \ldots, r \). Because \( V|_{U_j \times I} \) is trivial, there exists a bundle isomorphism 
\( \Phi_j : V|_{U_j \times I} \to \Theta^n(U_j \times I) \). Define \( \Psi_j : V|_{X_{j-1}} \to V|_{X_j} \) by
\[
\Psi_j(v) = \begin{cases} 
  \Phi_j^{-1}(w) & : \pi(v) \notin U_j \times I \\
  v & : \pi(v) \in U_j \times I
\end{cases}
\]
where \( w = ((x, f_j(x)), u) \) if \( \Phi_j(v) = ((x, f_j-1(x)), u) \). Then \( \Psi_j \) is a bundle isomorphism. Thus we have
\[\text{Corollary 7.16. Every vector bundle over a contractible compact Hausdorff space is trivial.}\]

**Proof.** Let \( X \) be a contractible compact Hausdorff space. There exists a fixed point \( x_0 \in X \) and a continuous function \( \varphi : X \times [0, 1] \to X \) satisfying \( \varphi|_{X \times \{0\}}(x) = x \) for all \( x \in X \) and \( \varphi|_{X \times \{1\}}(x) = x_0 \) for all \( x \in X \). Suppose \( V \) is a vector bundle over \( X \). Then \( \varphi^* (V) \) is a bundle over \( X \times [0, 1] \) with
\[V \cong \varphi^*(V)|_{X \times \{0\}} \cong \varphi^*(V)|_{X \times \{1\}} \cong \Theta^{\text{rank} V}(X)\]
by Lemma 7.15.

**Example 7.17.** Consider the compact Hausdorff space \( S^1 = \{ z \in \mathbb{C} : |z| = 1 \} \). Let \( A = \{ e^{i\theta} : 0 \leq \theta \leq \pi \} \) be the closed upper half of \( S^1 \) and let \( B = \{ e^{i\theta} : \pi \leq \theta \leq 2\pi \} \) be the lower half of \( S^1 \). Fix a rank \( n \) complex vector bundle \( V \) over \( S^1 \). Because \( A \) and \( B \) are both contractible, by Corollary 7.16 \( V|_A \) and \( V|_B \) are trivial bundles. Let \( \varphi : V|_A \to \Theta^n(A) \) and \( \psi : V|_B \to \Theta^n(B) \) be bundle isomorphisms. Let \( g \in GL_n(\mathbb{C}) \) be the matrix that represents \( \varphi \circ \psi^{-1} \) at 1, and let \( h \) be the matrix that represents \( \varphi \circ \psi^{-1} \) at -1. The group \( GL_n(\mathbb{C}) \) is path connected, so let \( g_t \) and \( h_t \) be continuous paths from \( A \) and \( B \) respectively to the identity matrix.

Define a rank \( n \) bundle \( W \) over \( S^1 \times I \) as follows. The bundle \( W \) is trivial over \( A \times I \) and \( B \times I \), with trivializations \( \Phi : W|_{A \times I} \to \Theta^n(A \times I) \) and \( \Psi : W|_{B \times I} \to \Theta^n(B \times I) \). Furthermore, the transition function is defined to be
\[
\Psi^{-1}((1, t), u) = \Phi^{-1}((1, t), g_0 u) \quad \text{and} \quad \Psi^{-1}((-1, t), u) = \Phi^{-1}((-1, t), h_0 u)
\]
for \( \pm 1 \in S^1 \), \( t \in [0, 1] \) and \( u \in \mathbb{C}^n \). Finally, Lemma 7.15 implies that
\[V \cong W|_{S^1 \times \{0\}} \cong W|_{S^1 \times \{1\}} \cong \Theta^n(S^1)\].
Therefore equivalence classes of vector bundles over $S^1$ are characterized by ranks, and $K^0(S^1) \cong G(N) \cong \mathbb{Z}$. 
8 $K^0(X) \cong K_0(C(X))$

The main result of this section is the proof of the equivalence of K-theories. When $X$ is compact Hausdorff, then $C(X)$ is a unital C*-algebra, and it makes sense to ask if the two definitions of K-theories agree.

**Theorem 8.1.** Let $X$ be compact Hausdorff. Then $K_0(C(X)) \cong K^0(X)$ as abelian groups.

Now we will develop some results necessary to prove this theorem.

**Definition 8.2.** Let $X$ be a compact Hausdorff space. For $E \in \mathcal{P}_\infty(C(X))$, and $x \in X$, let $\text{Ran} E(x)$ be the image of $E(x)$. That is, if $E$ is $n \times n$, then $\text{Ran} E = E(x) \mathbb{C}^n$. Define $\text{Ran} E = \bigcup_{x \in X} \bigcup_{v \in \text{Ran} E(x)} (x, v)$.

**Proposition 8.3.** Let $X$ be a compact Hausdorff space, $n \in \mathbb{N}$ and $E \in \mathcal{P}_\infty(C(X))$. Then $\text{Ran} E$ is a vector bundle over $X$.

**Proof.** Fix $x_0 \in X$ and let

$$U = \{x \in X : \|E(x_0) - E(x)\|_{op} < 1\}$$

As $E$ and the operator norm are both continuous, the set $U$ is the pull back of $(-\infty, 1)$ through a continuous function, and is hence open. Observe that for any $x_1 \in X$, the element $I_n + E(x_0) - E(x_1)$ is within distance 1 from $I_n$, and as such is an invertible matrix. Also, for any $v \in \mathbb{C}^n$, we have

$$(I_n + E(x_0) - E(x_1))E(x_1)v = E(x_1)v + E(x_0)E(x_1)v - E(x_1)E(x_1)v$$

$$= E(x_1)v + E(x_0)E(x_1)v - E(x_1)v$$

$$= E(x_0)E(x_1)v$$

So $I_n + E(x_0) - E(x_1)$ maps $\text{Ran} E(x_1)$ into $\text{Ran} E(x_0)$, and since this is an invertible matrix, we have that $\dim \text{Ran} E(x_0) \geq \dim \text{Ran} E(x_1)$. A similar calculation shows that

$$(I_n - E(x_0) + E(x_1))(\text{Ran} E(x_0)) \subseteq \text{Ran} E(x_1))$$

Thus we see that $\text{Ran} E(x_0)$ and $\text{Ran} E(x_1)$ have the same dimension, and $I_n + E(x_0) - E(x_1)$ maps $\text{Ran} E(x_1)$ to $\text{Ran} E(x_0)$ isomorphically. Thus, the map

$$\varphi : \text{Ran} E|_U \to U \times \text{Ran} E(x_0)$$

$$(x, v) \mapsto (x, (I_n + E(x_0) - E(x_1))v)$$

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is a bundle isomorphism. So Ran $E$ is locally trivial, thus is a vector bundle. ■

**Proposition 8.4.** Let $X$ be a compact Hausdorff space, and let $E, F \in \mathcal{P}_\infty(C(X))$. Then Ran $E \cong$ Ran $F$ as bundles if and only if $E \sim_u F$.

**Proof.** Since Ran $Q \cong$ Ran $(\text{diag}(Q, 0_r))$ for any $Q \in \mathcal{P}_\infty(C(X))$ and $r \in \mathbb{N}$, we can take some $n \in \mathbb{N}$ large enough so that $E$ and $F$ are both in $M_n(C(X))$.

Suppose that $E \sim_u F$. Then we can find $U \in U_n(C(X))$ such that $UEU^* = F$. Define $\gamma : \text{Ran } E \to \text{Ran } F$ by

$$\gamma(x, E(x)v) = (x, U(x)E(x)v) = (x, F(x)U(x)v) \in \text{Ran } F(x),$$

for $x \in X$ and $v \in \mathbb{C}^n$. It has the inverse map

$$\gamma^{-1}(x, F(x)v) = (x, U^*(x)F(x)v) = (x, E(x)U^*(x)v).$$

So $\gamma$ is a bundle isomorphism between Ran $E$ and Ran $F$.

Conversely, suppose that Ran $E$ and Ran $F$ are isomorphic vector bundles. Let $\varphi : \text{Ran } E \to \text{Ran } F$ be a bundle isomorphism. We define matrices $A, B \in M_n(C(X))$ as follows. For $f \in (C(X))^n$, let $Af = \varphi(Ef)$ and $Bf = \varphi^{-1}(Ff)$. Then

$$ABf = A(\varphi^{-1}(Ff)) = \varphi(E(\varphi^{-1}(Ff))).$$

However $\varphi^{-1}(Ff)$ is a continuous section of Ran $E$, so

$$ABf = \varphi(E(\varphi^{-1}(Ff))) = \varphi(\varphi^{-1}(Ff)) = Ff$$

Which shows that $AB = F$. A similar computation shows that $BA = E$. Also,

$$EBf = E\varphi^{-1}(Ff) = \varphi^{-1}(Ff) = Bf$$

and

$$BFf = \varphi^{-1}(FFf) = \varphi^{-1}(Ff) = Bf.$$ 

So $EB = B = BF$. Similarly, $FA = A = AE$.

Now define

$$T = \begin{bmatrix} A & I_n - F \\ I_n - E & B \end{bmatrix} \in M_{2n}(C(X)).$$
With the observations above it is straightforward to check that $T$ is invertible, with inverse

$$T^{-1} = \begin{bmatrix} B & I_n - E \\ I_n - F & A \end{bmatrix}.$$ 

Then

$$T \text{diag}(E, 0_n) T^{-1} = \begin{bmatrix} A & I_n - F \\ I_n - E & B \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B & I_n - E \\ I_n - F & A \end{bmatrix}$$

$$= \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B & I_n - E \\ I_n - F & A \end{bmatrix}$$

$$= \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix} = \text{diag}(F, 0_n)$$

Thus $E$ is similar to $F$ through an invertible matrix $T$. Since $E$ and $F$ are normal and similar to each other, they are in fact unitarily equivalent by Proposition 3.14.

**Corollary 8.5.** Let $X$ be compact Hausdorff. The range map

$$\text{Ran} : \mathcal{P}_\infty(C(X)) / \sim_u \rightarrow \text{Vect}(X)$$

mapping

$$[E] \mapsto [\text{Ran} E]$$

is well-defined and injective.

**Proposition 8.6.** Let $X$ be a compact Hausdorff space, let $N \in \mathbb{N}$, and suppose that $V$ is a subbundle of $\Theta^N(X)$. Let $\Theta^N(X)$ be equipped with the standard Hermitian metric, and for $x \in X$, let $E(x)$ be the orthogonal projection of $\Theta^N(X)|_x$ onto $V|_x$. Then the map $E : x \mapsto E(x)$ defines an idempotent $E \in M_N(C(X))$.

**Proof.** By using Lemma 7.7 again, we only need to show that each $x_0 \in X$ has an open neighbourhood for which $E|_U : x \mapsto E(x)$ is continuous on $U$. Fix $x_0$ and choose $U$ to be a connected open neighbourhood of $x_0$ over which $V$ is trivial. Let $n$ be the rank of $V$, and let $\varphi : \Theta^n(U) \rightarrow V|_U$ be a bundle isomorphism. For $k = 1, \ldots, n$, define $s_k : U \rightarrow \Theta^n(U)$ by $s_k(x) = (x, e_k)$, the $k^{th}$ standard basis vector lying above $x$. Then for each $x \in U$, the set

$$\{ \varphi(s_1(x)), \varphi(s_2(x)), \ldots, \varphi(s_n(x)) \}$$
is a vector space basis for $V_x$. Let $⟨.,.⟩$ be the standard Hermitian metric of $Θ^N(U)$ restricted to $V$. By the Gram-Schmidt process, we obtain a an orthogonal basis of $V_x$ by defining inductively

$$s'_k(x) = \varphi(s_k(x)) - \sum_{i=1}^{k-1} \frac{⟨\varphi(s_k(x)), s'_i(x)⟩}{⟨s'_i(x), s'_i(x)⟩} s'_i(x)$$

for $k = 1, \ldots, n$. Then the set

$$\left\{ \frac{s'_1(x)}{∥s'_1(x)∥}, \ldots, \frac{s'_n(x)}{∥s'_n(x)∥} \right\}$$

is an orthonormal basis for $V_x$ equipped with $⟨.,.⟩$, where $∥·∥$ denotes the norm induced by $⟨.,.⟩$. Moreover, the map $x \mapsto \frac{s'_1(x)}{∥s'_1(x)∥}$ is continuous. Finally, for $E$ the orthogonal projection as in the statement, we have

$$E(x)w = \sum_{k=1}^{n} \left< \varphi(x, w), \frac{s'_k(x)}{∥s'_k(x)∥} \right> \frac{s'_k(x)}{∥s'_k(x)∥}$$

and the above is jointly continuous in $x \in X$ and $w \in \mathbb{C}^n$. Therefore $x \mapsto E(x)$ is continuous.

**Corollary 8.7.** Let $V$ be a vector bundle over a compact Hausdorff space $X$. Then $V \cong \text{Ran } E$ for some $E \in \mathcal{P}_∞(C(X))$. Hence the map

$$\text{Ran} : \mathcal{P}_∞(C(X))/\sim_u \rightarrow \text{Vect}(X)$$

is surjective.

**Proof.** There exists $N \in \mathbb{N}$ such that $V$ is isomorphic to a subbundle of $Θ^N(X)$. So assume that $V$ is embedded in $Θ^N(X)$, and let $Θ^N(X)$ be equipped with the canonical metric. For each $x \in X$ let $E(x)$ be the orthogonal projection of $Θ^N(X)_x$ onto $V_x$. By Proposition 8.6 , $x \mapsto E(x)$ defines an element in $E \in \mathcal{P}_N(X)$, and Ran $E = V$.

**Corollary 8.8.** Let $V$ be a vector bundle over a compact Hausdorff space $X$. Then there exists another vector bundle $V^\perp$ over $X$ such that $V \oplus V^\perp \cong Θ^N(X)$ for some $N \in \mathbb{N}$. 

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Proof. We know that there exists some \( N \in \mathbb{N} \) such that \( V \) is isomorphic to a subbundle of \( \Theta^N(X) \). For each \( x \in X \), let \( E(x) \) be the orthogonal projection of \( \Theta^N(X)_x \) onto \( V_x \). By Proposition 8.6, this family of projections defines an element \( E \in \mathcal{P}_N(C(X)) \). Define \( V^\perp = \text{Ran} (I_N - E) \). Then

\[
V \oplus V^\perp \cong \text{Ran} E \oplus \text{Ran} (I_N - E) = \text{Ran} I_N = \Theta^N(X). \]

**Theorem 8.9.** Let \( X \) be a compact Hausdorff space. Then \( \mathcal{P}_\infty(C(X)) \) and \( \text{Vect}(X) \) are isomorphic as abelian monoids.

**Proof.** Define \( \Psi : \mathcal{P}_\infty(C(X)) \to \text{Vect}(X) \) by \( \Psi([E]) = [\text{Ran} E] \). By Corollaries 8.5 and 8.7, \( \Psi \) is well-defined, injective and surjective. It is left to show that it is a monoid homomorphism, i.e. \( \text{Ran} (E \oplus F) \cong \text{Ran} E \oplus \text{Ran} F \). But this is obvious, as they are not just isomorphic, but are in fact equal. \( \blacksquare \)

**Corollary 8.10.** Let \( X \) be a compact Hausdorff space. Then \( K^0(X) \cong K_0(C(X)) \) as abelian groups.

**Proof.** Apply the Grothendieck completion to the isomorphism obtained in Theorem 8.9 to obtain

\[
K^0(X) = G(\text{Vect}(X)) \cong G(\mathcal{P}_\infty(C(X))) = K_0(C(X)). \] \( \blacksquare \)

For \( X \) a compact Hausdorff space and \( V \) a topological vector bundle over \( X \), we write \([V]^0\) for the element in \( K^0(X) \) that is represented by \( V \).

**Proposition 8.11.** Let \( X \) be a compact Hausdorff space, then

\[
K^0(X) = \{[V]^0 - [W]^0 : V, W \text{ vector bundles over } X\}. \]

**Proof.** This follows from Corollary 8.10 and Proposition 4.3. \( \blacksquare \)

Now that we’ve shown that \( K^0(X) \) and \( K_0(C(X)) \) are isomorphic as abelian groups, we will verify that the associated morphisms are preserved by this identification.

**Definition 8.12.** Let \( X \) and \( Y \) be compact Hausdorff spaces, let \( f : X \to Y \) be a continuous map and let \( V \) be a rank \( r \) subbundle of some trivial bundle \( \Theta^n(Y) \) of \( Y \). (By Proposition 7.8 all vector bundles over \( Y \) are isomorphic to a bundle of this form). Define the pull-back of \( V \) via \( f \), written \( f^*(V) \), to be the rank \( r \) subbundle of \( \Theta^n(X) \) where the fibre at a point \( x \in X \) is \( (f^*(V))_x = V_{f(x)} \).
Proposition 8.13. Let \( X \) and \( Y \) be compact Hausdorff spaces, \( f : X \to Y \) continuous and \( V \) is a subbundle of \( \Theta^n(Y) \). Then \( f^*(V) \) is indeed a vector bundle on \( X \).

Proof. Take any \( x \in X \), let \( U \) be an open neighbourhood of \( f(x) \) in \( Y \) for which \( V|_U \) is trivial. Then \( f^{-1}(U) \) is an open neighbourhood of \( x \) and \( f^*(V)|_{f^{-1}(U)} = f^*(V|_U) \) is trivial. \( \blacksquare \)

Proposition 8.14. Let \( X \) and \( Y \) be compact Hausdorff spaces, let \( f : X \to Y \) be continuous, and \( E \in \mathcal{P}_\infty(C(Y)) \). Then \( f^*(E) \) is a projection in \( \mathcal{P}_\infty(C(X)) \), and \( f^*(\text{Ran} E) = \text{Ran} f^*(E) \).

Proof. For \( x \in X \),
\[
(E \circ f)(x) \cdot (E \circ f)(x) = E(f(x))E(f(x)) = EE(f(x)) = E \circ f(x)
\]
and
\[
(E \circ f)^*(x) = (E \circ f(x))^* = E^*(f(x)) = E \circ f(x).
\]
So \( E \circ f \) is a projection. Furthermore, suppose \( E \) is \( n \times n \). Then
\[
f^*(\text{Ran} E)_x = (\text{Ran} E)_{f(x)} = E(f(x))\mathbb{C}^n = (\text{Ran} f^*(E))_x.
\]
Therefore \( f^*(\text{Ran} E) = \text{Ran} f^*(E) \). \( \blacksquare \)

Definition 8.15. Let \( X \) and \( Y \) be compact Hausdorff spaces and let \( f : X \to Y \) be a continuous map. Then \( f^* \) is a *-homomorphism from \( C(Y) \) to \( C(X) \). Define \( K^0(f) : K^0(Y) \to K^0(X) \) by
\[
K^0(f)([V]_0 - [W]_0) = [f^*(V)]_0 - [f^*(W)]_0.
\]

Remark 8.16. According to Proposition 8.14, if \( f : X \to Y \) is a continuous map then by identifying \( K^0(Y) \) with \( K_0(C(Y)) \) and \( K^0(X) \) with \( K_0(C(X)) \), we conclude that \( K^0(f) \) and \( K_0(f^*) \) are the same map. To be precise, the diagram
\[
\begin{array}{ccc}
K^0(Y) & \xrightarrow{K^0(f)} & K^0(X) \\
\downarrow \cong & & \downarrow \cong \\
K_0(C(Y)) & \xleftarrow{K_0(f^*)} & K_0(C(X))
\end{array}
\]
commutes.
Proposition 8.17. The map $X \mapsto K^0(X)$ is a covariant functor from the category of compact Hausdorff spaces to the category of abelian groups.

Proof. Let $X, Y, Z$ be compact Hausdorff spaces, and let $f : X \to Y$ and $g : Y \to Z$ be continuous. Consider the commutative diagrams

\[
\begin{array}{ccc}
K^0(Z) & \xrightarrow{K^0(g)} & K^0(Y) \\
\downarrow{\cong} & & \downarrow{\cong} \\
K_0(C(Z)) & \xleftarrow{K_0(g^*)} & K_0(C(Y)) \\
\end{array}
\]

and

\[
\begin{array}{ccc}
K^0(Z) & \xrightarrow{K^0(f \circ g)} & K^0(X) \\
\downarrow{\cong} & & \downarrow{\cong} \\
K_0(C(Z)) & \xleftarrow{K_0((f \circ g)^*)} & K_0(C(X)) \\
\end{array}
\]

Since $K_0$ is a functor, we have

\[
K_0((f \circ g)^*) = K_0(g^* \circ f^*) = K_0(g^*) \circ K_0(f^*).
\]

Hence the first rows of the two diagrams imply that $K^0(f \circ g) = K^0(g) \circ K^0(f)$. The fact that $K^0(\text{id}_X) = \text{id}_{K^0(X)}$ also follows from the functoriality of $K_0$ and Remark 8.16 in a similar way. \square

Example 8.18. Let $X = \{\ast\}$ be a point. Then $C(X) \cong \mathbb{C}$. By Example 2.18 and Corollary 8.10, we see that $K^0(X) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$. 

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9 K-theory of locally compact spaces

The K-theory of locally compact spaces correspond to the K-theory of non-unital C*-algebras.

**Definition 9.1.** Let $X$ be a topological space. We say $X$ is locally compact if for every $x \in X$ there exists some open neighbourhood $U \subseteq X$ of $x$ such that the closure $\overline{U}$ of $U$ in $X$ is compact.

**Definition 9.2.** Let $X$ be a locally compact space. Define $X^+$ to be the set $X \sqcup \{\infty\}$ with the collection of open sets given by
\[
\mathcal{T}^+ := \{U \subseteq X : U \text{ open in } X\} \cup \{(X \setminus F) \cup \{\infty\} : F \text{ closed and compact in } X\}.
\]

**Proposition 9.3.** Let $X$ be a topological space, then $X^+$ is a compact topological space. Moreover, $X^+ \setminus \{\infty\}$ is homeomorphic to $X$ in the obvious way.

**Proof.** We first check that the collection of open sets $\mathcal{T}^+$ is a topology on $X^+$.

1. The empty set $\emptyset$ is open in $X$, so $\emptyset \in \mathcal{T}^+$. The empty set $\emptyset$ is obviously closed and compact, so $X^+ = (X \setminus \emptyset) \cup \{\infty\} \in \mathcal{T}^+$.

2. Define
\[
\mathcal{T}_0 := \{U : U \text{ open in } X\},
\]
\[
\mathcal{T}_1 := \{(X \setminus F) \cup \{\infty\} : F \text{ closed and compact in } X\}.
\]

Clearly $\mathcal{T}_0$ is closed under arbitrary union. Let $\{F_i : i \in I\}$ be an arbitrary collection of closed compact subsets of $X$. Then $F := \bigcap_{i \in I} F_i$ is clearly closed. Pick any $i_0 \in I$. Then $F$ is a closed subset of the compact set $F_{i_0}$, thus $F$ is also compact. Then
\[
\bigcup_{i \in I} (X \setminus F_i) \cup \{\infty\} = (X \setminus F) \cup \{\infty\} \in \mathcal{T}_1.
\]

So $\mathcal{T}_1$ is closed under arbitrary union. Finally, take $U \in \mathcal{T}_0$ and $(X \setminus F) \cup \{\infty\} \in \mathcal{T}_1$. We have
\[
U \cup (X \setminus F) \cup \{\infty\} = (X \setminus (X \setminus U)) \cup (X \setminus F) \cup \{\infty\} = (X \setminus ((X \setminus U) \cap F)) \cup \{\infty\} \in \mathcal{T}_1.
\]
because \((X \setminus U) \cap F\) is closed and compact (it is a closed subset of \(F\)). Therefore \(\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1\) is closed under arbitrary union.

3. Clearly \(\mathcal{T}_0\) is closed under finite intersection. A finite union of compact closed sets is also closed and compact, so \(\mathcal{T}_1\) is also closed under finite intersection. Lastly, suppose \(U\) is open and \(F\) is closed and compact, then

\[
U \cap ((X \setminus F) \cup \{\infty\}) = U \cap (X \setminus F) \in \mathcal{T}_1.
\]

Therefore \(\mathcal{T}\) is closed under finite intersection.

The above verifies that \(\mathcal{T}\) is a topology on \(X\). The subspace topology on \(X^+ \setminus \{\infty\}\) is \(\mathcal{T}_0\), which coincides with the topology on \(X\). Hence \(X^+ \setminus \{\infty\} \cong X\). Next we check that \(X^+\) is compact.

Let \(\{U_i\}_{i \in I}\) be a open cover for \(X^+\). Since this collection covers the point \(\infty\), there exists some \(i_0 \in I\) such that \(U_{i_0} \in \mathcal{T}_1\). Then \(X^+ \setminus U_{i_0}\) is a compact subset of \(X\), hence also a compact subset of \(X^+\), so there exists a finite subset \(J \subseteq I\) for which \(X^+ \setminus U_{i_0} \subseteq \bigcup_{i \in J} U_i\). Whence \(\{U_i : i \in J \cup \{i_0\}\}\) is a finite cover for \(X^+\). Therefore \(X^+\) is compact.

Remark 9.4. The space \(X^+\) is called the one point compactification of \(X\).

Proposition 9.5. Let \(X\) be a locally compact topological space. If \(X\) is Hausdorff then \(X^+\) is also Hausdorff.

Proof. Let \(\mathcal{T}_0\) be \(\mathcal{T}_1\) be as defined in the proof of Proposition 9.3. By Proposition 9.3 we know that \(X^+ \setminus \{\infty\} \cong X\) is Hausdorff. Fix \(x \in X^+ \setminus \{\infty\}\) and let \(U\) be an open neighbourhood of \(x\) where \(\overline{U}\) is compact in \(X\). Then \(V := X^+ \setminus \overline{U}\) is an open neighbourhood of \(\infty\), and \(U \cap V = \emptyset\). Therefore \(X^+\) is Hausdorff.

Proposition 9.6. Let \(X\) be a compact Hausdorff space, and let \(x_0 \in X\). The map \(f : X \to (X \setminus x_0)^+\) given by

\[
f(x) = \begin{cases} 
    x & : x \neq x_0 \\
    \infty & : x = x_0
\end{cases}
\]

is a homeomorphism.
Proof. It is clear that \( f \) is bijective. It is also clear that for any \( S \subseteq X \setminus \{x_0\} \), \( S \) is open in \( X \) if and only if \( f(S) \) is open in \((X \setminus \{x_0\})^+\).

Suppose \( U \subseteq X \) is an open neighbourhood of \( x_0 \). Let \( F = X \setminus U \). Since \( F \) is a closed subset of \( X \), it is compact. Also,

\[
U = ((X \setminus \{x_0\}) \setminus F) \cup \{x_0\}.
\]

On the other hand, suppose \( F \subseteq X \setminus \{x_0\} \) is closed and compact, then

\[
((X \setminus \{x_0\}) \setminus F) \cup \{\infty\} = X \setminus F
\]

is an open neighbourhood of \( x_0 \). Hence \( x_0 \in X \) and \( \infty \in (X \setminus \{x_0\})^+ \) have the “same” open neighbourhoods. It follows that a subset \( S \subseteq X \) containing \( x_0 \) is open if and only if \( f(S) \) is open. Therefore \( f \) is a homeomorphism.

**Definition 9.7.** Let \( X \) be a locally compact Hausdorff space. Define \( C_0(X) \) to be the set of all continuous functions \( f \in C(X) \) satisfying the following: for any \( \varepsilon > 0 \) there exists a compact subset \( F \subseteq X \) such that \(|f(x)| < \varepsilon\) for all \( x \in X \setminus F \).

**Proposition 9.8.** Let \( X \) be a locally compact Hausdorff space and let \( f \in C_0(X) \). Define \( \tilde{f} \) on \( X^+ \) to be

\[
\tilde{f} = \begin{cases} f(x) & : x \in X \\ 0 & : x = \infty \end{cases}.
\]

Then \( \tilde{f} \in C(X^+) \). If \( h \in C(X^+) \) satisfies \( h(\infty) = 0 \), then \( h|_X \in C_0(X) \) and \( \tilde{h}|_X = h \).

**Proof.** It is clear that \( \tilde{f} \) is continuous on \( X^+ \setminus \{\infty\} \), so we only need to check that \( \tilde{f} \) is continuous at \( \infty \). Given any \( \varepsilon > 0 \), by the definition of \( C_0(X) \), there exists a compact subset \( F \subseteq X \) such that \(|f(x)| < \varepsilon\) for all \( x \in X \setminus F \). But \( U := (X \setminus F) \cup \{\infty\} \) is an open neighbourhood of \( \infty \). We have \(|\tilde{f}(x) - \tilde{f}(\infty)| = |\tilde{f}(x)| < \varepsilon\) for all \( x \in U \). Therefore \( \tilde{f} \) is continuous.

The second part of the proof follows essentially the same proof.

**Proposition 9.9.** Let \( X \) be a locally compact Hausdorff space. Let \( I_X \) denote the identity element of \( C_0(X) \) and let \( 1_{X^+} \) denote the constant function 1 on \( X^+ \). Define \( \varphi : C_0(X) \to C(X^+) \) by \( \varphi(f) = \tilde{f} \) for all \( f \in C_0(X) \) and \( \varphi(I) = \varphi(1_{X^+}) \) and extend linearly. Then \( \varphi \) is a \( C^* \)-algebra isomorphism.
Proof. It is easy to see that $\varphi$ is a $\ast$-homomorphism. Suppose

$$0 = \varphi(f + z1_X) = \tilde{f} + z1_{X^+}$$

for some $f \in C_0(X)$ and $z \in \mathbb{C}$. Then

$$z = (\tilde{f} + z1_{X^+})(\infty) = 0.$$ 

It then follows that $\tilde{f}(x) = 0$ for all $x \in X$, so $f = 0$. Hence $\varphi$ is injective.

Take any $h \in C(X^+)$ and let $z = h(\infty)$. By Proposition 9.8 the function $(h - z1_{X^+})|X \in C_0(X)$. Also, $\varphi((h - z1_{X^+}) + z1_X) = h$. This shows that $\varphi$ is surjective. Therefore $\varphi$ is an isomorphism. \[\blacksquare\]

Definition 9.10. Let $X$ be a locally compact Hausdorff space, and let $\iota : \{\infty\} \to X^+$ be the inclusion map. Define $K^0(X) := \ker K^0(\iota) \subseteq K^0(X^+)$. 

Remark 9.11. Suppose $X$ is a locally compact Hausdorff space and $\iota : \{\infty\} \to X^+$ is the inclusion map. The induced $\ast$-homomorphism $\iota^* : C(X^+) \to C(\{\infty\})$ does the following:

$$\iota^*(\tilde{f}) = \tilde{f} \circ \iota = 0, \ \forall f \in C_0(X)$$

and

$$\iota^*(1_{X^+}) = 1_{X^+} \circ \iota = 1_{\{\infty\}}.$$ 

This means that $\iota : C(X^+) \to C(\{\infty\})$ is the projection onto the one dimensional subspace generated by the identity element and $\ker \iota = C_0(X)$. Whence in light of Remark 8.16 and Proposition 9.9, $K^0(X)$ is isomorphic to $K_0(C_0(X))$ in the expected way.

9.1 Relative and reduced K-theory

Definition 9.12. Let $X$ be a compact Hausdorff space, and let $A$ be a compact subset of $X$. Let $\iota : A \to X$ be the inclusion map. Then $K^0(\iota)$ is a group homomorphism $K^0(X) \to K^0(A)$. Define $K^0(X, A)$ to be $\ker(K^0(\iota))$. The group $K^0(X, A)$ is called the relative K-group of the compact pair $(X, A)$.

Proposition 9.13. Let $X$ be a locally compact Hausdorff space. Then $K^0(X) \cong K^0(X^+, \infty)$. 

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Proof. This is a consequence of Remark 9.11. ■

**Proposition 9.14.** Let \( X \) be a compact Hausdorff space and fix \( x_0 \in X \). Then \( K^0(X) \cong K^0(X, x_0) \oplus \mathbb{Z} \).

*Proof.* Let \( \iota : \{x_0\} \to X \) be the inclusion map, and let \( \lambda : X \to \{x_0\} \) be the only constant map. Consider the sequence

\[
0 \longrightarrow K^0(X, x_0) \longrightarrow K^0(X) \xrightarrow{K^0(\iota)} K^0(\{x_0\}) \longrightarrow 0.
\]

By the definition of \( K^0(X, x_0) \), this sequence is exact. Furthermore, \( \iota \circ \lambda = \text{id}_{\{x_0\}} \), then by the functoriality of \( K^0 \) we have that \( K^0(\lambda) \circ K^0(\iota) = K^0(\iota \circ \lambda) = K^0(\text{id}_{\{x_0\}}) = \text{id}_{K^0(\{x_0\})} \).

Hence the above is a split exact sequence of abelian groups. Therefore \( K^0(X) \cong K^0(X, x_0) \oplus K^0(\{x_0\}) \). Lastly, by Example 8.18 we have \( K^0(\{x_0\}) \cong \mathbb{Z} \).

**Remark 9.15.** Let \( X \) be a compact Hausdorff space. Let \( G_0 \) be the subgroup of \( K^0(X) \) generated by \( [\Theta^1(X)]_0 \). Since

\[
[\Theta^n(X)]_0 + [\Theta^m(X)]_0 = [\Theta^n(X) \oplus \Theta^m(X)]_0 = [\Theta^{n+m}(X)]_0,
\]

we have that \( G_0 = \{\pm [\Theta^n(X)]_0 : n \in \mathbb{N}_{\geq 0}\} \cong \mathbb{Z} \). Fix \( x_0 \in X \), and let \( \iota_{x_0} : \{x_0\} \to X \) be the inclusion map. Then

\[
K^0(\iota_{x_0})([\Theta^n(X)]_0) = [\iota_{x_0}^* (\Theta^n(X))]_0 = [\Theta^n(\{x_0\})]_0,
\]

which corresponds to \( n \in \mathbb{Z} \) in the isomorphism \( \mathbb{Z} \cong K^0(\{x_0\}) \). Hence \( K^0(\iota_{x_0})|_{G_0} \to K^0(\{x_0\}) \) is an isomorphism for any \( x_0 \in X \). Thus we have that \( K^0(X, x_0) \cong K^0(X)/G_0 \) for any \( x_0 \in X \). More importantly, we have that \( K^0(X, x_0) \cong K^0(X, x_1) \) for any \( x_0, x_1 \in X \).

**Definition 9.16.** Let \( X \) be a compact Hausdorff space. Define the **reduced K-group** of \( X \), denoted \( \tilde{K}^0(X) \), to be \( K^0(X, x_0) \) for any choice of \( x_0 \in X \).

**Remark 9.17.** Let \( X \) be a compact Hausdorff space and fix \( x_0 \in X \). By Proposition 9.13 we have \( \tilde{K}^0(X) \cong K^0(X, \{x_0\}) \). By Remark 9.15, the definition of \( \tilde{K}^0(X) \) is independent of the choice \( x_0 \in X \).
Proposition 10.4. Let $X$ and $Y$ be topological spaces and let $f, g : X \to Y$ be continuous maps. We say $f$ is homotopic to $g$ if there exists a continuous map $f_t : [0, 1] \times X \to Y$ mapping $(t, x) \mapsto f_t(x)$ such that $f_0(x) = f(x)$ and $f_1(x) = g(x)$ for all $x \in X$.

Definition 10.1. Let $X$ and $Y$ be topological spaces. We say $(\varphi, Y)$ and $(g, Y)$ are continuous maps. We say $(\varphi, Y)$ and $(g, Y)$ are homotopic if there exists a continuous family $(\varphi_t : X \to Y)_{t \in [0, 1]}$ such that $\varphi_0(x) = \varphi(x)$ and $\varphi_1(x) = g \circ \varphi(x)$ for all $x \in X$.

Definition 10.2. Let $X$ and $Y$ be topological spaces. Then $X$ is said to be homotopic to $Y$ if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is homotopic to the identity map $id_Y$ and $g \circ f$ is homotopic to the identity map $id_X$.

Lemma 10.3. Let $X$ and $Y$ be compact Hausdorff spaces, and let $\varphi : [0, 1] \times X \to Y$ mapping $(t, x) \mapsto \varphi_t(x)$ be continuous. Then the map $t \mapsto (\varphi_t)^*(f) = f \circ \varphi_t$ is continuous from $[0, 1]$ to $C(X)$ for any $f \in C(Y)$.

Proof. Let $f \in C(Y)$ and $\varepsilon > 0$ be given. Then $f \circ \varphi : [0, 1] \times X \to \mathbb{R}$ is a continuous function. By continuity, for any $t \in [0, 1]$ and $x \in X$, there exists $\delta_t > 0$ and an open neighbourhood $U_x \subseteq X$ of $x$ such that

$$|f \circ \varphi_t(y) - f \circ \varphi_t(x)| < \varepsilon$$

for every $s \in B_{\delta_t}(t) \cap [0, 1]$ and $y \in U_x$. By compactness, $X$ can be covered by a finite collection of open sets of the form $U_{x_1}, \ldots, U_{x_k}$. Let $\delta = \min\{\delta_{t_1}, \ldots, \delta_{t_k}\} > 0$. Then for any $x \in X$,

$$|(\varphi_s)^*(f)(x) - (\varphi_t)^*(f)(x)| = |f \circ \varphi_s(x) - f \circ \varphi_t(x)| < \varepsilon,$$

so $\|f(\varphi_s)^*(f) - (\varphi_t)^*(f)\|_{\infty} < \varepsilon$. ■

Proposition 10.4. Let $X$ and $Y$ be compact Hausdorff spaces. Let $f : X \to Y$ and $g : Y \to X$ be a homotopy between $X$ and $Y$. Then $f^* : C(Y) \to C(X)$ and $g^* : C(X) \to C(Y)$ give a homotopy between $C(X)$ and $C(Y)$.

Proof. By assumption $g \circ f$ is homotopic to the identity map $id_X$ on $X$. Hence there exists a continuous family $\varphi_t : X \to X$ for $t \in [0, 1]$ satisfying $\varphi_0 = id_X$ and $\varphi_1 = g \circ f$. By Lemma 10.3, $(\varphi_t)^*$ is a homotopy from $(\varphi_0)^* = (id_X)^* = id_{C(X)}$ to $(\varphi_1)^* = (g \circ f)^* = f^* \circ g^*$. Similarly $g^* \circ f^*$ is homotopic to $id_{C(Y)}$. ■
Corollary 10.5. Let $X$ and $Y$ be compact Hausdorff spaces and $f : X \to Y$ be a homotopy. Then $K^0(f) : K^0(Y) \to K^0(X)$ is a group isomorphism.

Proof. By Proposition 10.4 we see that $f^* : C(Y) \to C(X)$ is a homotopy. It follows by Proposition 6.2 that $K_0(f^*)$ is an isomorphism, whence Remark 8.16 gives us the conclusion that $K^0(f)$ is an isomorphism. ■

Example 10.6. Let $X = [0, 1]$. Then $X$ is homotopic to a point. Hence by Corollary 10.5 and Example 8.18, we have

$$K_0(C([0, 1])) \cong K_0([0, 1]) \cong K_0(*) \cong \mathbb{Z}.$$ 

Remark 10.7. The functor $K^0$ is not homotopy-invariant for locally compact Hausdorff spaces. In Example 7.17 we saw that $K^0(S^1) \cong \mathbb{Z}$. The unit circle $S^1$ is homeomorphic to the one point compactification of $\mathbb{R}$, and $\mathbb{R}$ is homotopic to a point. However, Proposition 9.14 says that $K^0(\mathbb{R}) \oplus \mathbb{Z} \cong K^0(S^1)$, which implies that $K^0(\mathbb{R}) \cong 0$. On the other hand, the $K^0$-group of a point is $\mathbb{Z}$, as shown in Example 10.6, which is not isomorphic to $K^0(\mathbb{R})$.

Example 10.8. We will now exhibit an example that shows $K_0$ is not an exact functor.

Consider the short exact sequence

$$0 \longrightarrow C_0((0, 1)) \overset{\iota}{\longrightarrow} C([0, 1]) \overset{\pi}{\longrightarrow} \mathbb{C} \oplus \mathbb{C} \longrightarrow 0.$$ 

Where

$$(\iota(f))(t) := \begin{cases} f(t) & : t \in (0, 1) \\ 0 & : t \in \{0, 1\} \end{cases}$$

for any $f \in C_0((0, 1))$ and $t \in [0, 1]$, and

$$\pi(g) := (g(0), g(1))$$

for any $g \in C([0, 1])$. It is left to the reader to check that this sequence is exact.

Corollary 6.7 and Example 8.18 give us the isomorphism

$$K_0(\mathbb{C} \oplus \mathbb{C}) \cong K_0(\mathbb{C}) \oplus K_0(\mathbb{C}) \cong \mathbb{Z} \oplus \mathbb{Z}.$$
On the other hand \( \mathbb{C}([0,1]) \cong \mathbb{Z} \) by Example 10.6. The map \( K_0(\pi) : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \) is not a surjection, since \( \mathbb{Z} \) is generated by one element but \( \mathbb{Z} \oplus \mathbb{Z} \) cannot be generated by one element. Therefore the functor \( K_0 \) does not take the short exact sequence in consideration to a short exact sequence of abelian groups.

### 10.2 Half-exactness of \( \tilde{K}^0 \)

**Proposition 10.9.** Let \( X \) be a compact Hausdorff space and let \( A \) be a closed subset of \( X \). Define \( I(A) \) to be all the continuous functions \( f \in C(X) \) that vanish on \( A \), i.e. \( f(A) = \{0\} \). Then the following are true

1. \( I(A) \) is a closed ideal of \( C(X) \).
2. \( I(A) \cong C_0(X \setminus A) \).
3. Let \([A]\) denote the point corresponding to \( A \) in the quotient \( X/A \). Then \((X/A) \setminus \{[A]\} \cong X \setminus A \) as locally compact Hausdorff spaces.
4. \( I(A) \cong C_0((X/A) \setminus \{[A]\}) \).
5. \( C(X)/I(A) \cong C(A) \).

**Proof.**

1. Let \( f \in I(A) \) and \( g \in C(X) \), then
   \[
   (f \cdot g)(a) = f(a)g(a) = 0g(a) = 0
   \]
   for all \( a \in A \), so \( f \cdot g \in I(A) \). Clearly if a convergent sequence of functions vanish on \( A \) then so does the limit. Hence \( I(A) \) is a closed ideal in \( C(X) \).
2. Let \( \varphi : C_0(X \setminus A) \to C(X) \) be defined by
   \[
   \varphi(f)(x) = \begin{cases} f(x) & : x \in X \setminus A \\ 0 & : x \in A \end{cases}
   \]
   for all \( f \in C_0(X \setminus A) \) and \( x \in X \). For each \( \varepsilon > 0 \), there exists an open neighbourhood \( U \subseteq X \) with \( A \subseteq U \) satisfying \( |\varphi(f)(x)| < \varepsilon \) for all \( x \in U \). Hence we see that \( \varphi(f) \in C(X) \) for all \( f \in C_0(X \setminus A) \). It is also clear from definition that the image of \( \varphi \) is contained in \( I(A) \). We also define a map \( \psi : I(A) \to C(X \setminus A) \) by
   \[
   \psi(g)(x) = g(x)
   \]
for all \( g \in I(A) \) and \( x \in X \setminus A \). Since \( g(A) = \{0\} \), then for every \( \varepsilon > 0 \) there exists an open neighbourhood \( U \supseteq A \) satisfying \( |g(x)| < \varepsilon \) for all \( x \in U \). Hence \( \psi(g) \in C_0(X \setminus A) \). It is easy to check that \( \varphi \) and \( \psi \) are mutual inverses. Therefore \( C_0(X \setminus A) \cong I(A) \).

3. This is obvious.
4. This is a consequence of 2 and 3.
5. Define \( \varphi : C(X)/I(A) \to C(A) \) by letting \( \varphi([f]) = f|_A \). If \( [f] = [g] \), then \( (f - g)|_A = 0 \), so \( \varphi([f]) = \varphi([g]) \). Hence \( \varphi \) is well-defined.

Define \( \psi : C(A) \to C(X)/I(A) \) as follows. Fix \( h \in C(A) \), by Tietze’s extension theorem [7] the function \( \tilde{h} \) extends to a continuous function \( \tilde{h} \in C(X) \). Let \( \psi(h) = [\tilde{h}] \). It is easy to check that \( \varphi \) and \( \psi \) are mutual inverses. Therefore

\[
C(A) \cong C(X)/I(A). \quad \blacksquare
\]

**Corollary 10.10.** Let \( X \) be a compact Hausdorff space and let \( A \) be a closed subset of \( X \). Under the identifications \( I(A) \cong C_0(X \setminus A) \) and \( C(A) \cong C(X)/I(A) \), the following sequence is exact:

\[
K_0(C_0((X/A) \setminus \{[A]\})) \longrightarrow K_0(C(X)) \longrightarrow K_0(C(A))
\]

**Proof.** Consider the following diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & I(A) & \longrightarrow & C(X) & \longrightarrow & C(X)/I(A) & \longrightarrow & 0 \\
& & \downarrow{\cong} & & \downarrow{=} & & \downarrow{\cong} & & \\
0 & \longrightarrow & C_0((X/A) \setminus \{[A]\}) & \longrightarrow & C(X) & \longrightarrow & C(A) & \longrightarrow & 0
\end{array}
\]

The upper row is clearly exact. The isomorphisms from the upper row to the lower row are given by Lemma 10.9. By the half-exactness of the functor \( K_0 \) 6.5, we obtain the exactness of the \( K_0 \)-groups. \( \blacksquare \)

**Corollary 10.11.** Let \( X \) be a compact Hausdorff space and let \( A \) be a closed subset of \( X \). Let \( \iota : A \to X \) be the inclusion map and let \( \pi : X \to X/A \) be the projection map. The following sequence is exact:

\[
\overline{K}^0(X/A) \xrightarrow{K^0(\iota)} K^0(X) \xrightarrow{K^0(\pi)} K^0(A).
\]
Proof. By Corollary 8.10, we know $K_0(X) \cong K_0(C(X))$ and $K_0(A) \cong K_0(C(A))$. By Remark 9.17 and Remark 9.11, we have that $K_0((X/A) \setminus \{[A]\}) \cong K_0((X/A) \setminus \{[A]\}) \cong \tilde{K}_0(X/A)$. To see that $K_0(\pi)$ and $K_0(\iota)$ are the maps in this exact sequence, one can take $\pi$ and $\iota$ and chase through the proofs in this section. ■

Remark 10.12. The functor $K^0$ is not half-exact. If $A$ is a compact subset of a compact Hausdorff space $X$ and we take the quotient $X/A$, the subspace $A$ is contracted to a point rather than deleted, and this point is not present in the corresponding $C^*$-algebra quotient. The point in $X/A$ representing $A$ detects the rank of the bundles, so we take the reduced $\tilde{K}^0$ to delete this extra information and make the sequence exact.

Proposition 10.13. Let $X$ and $Y$ be locally compact Hausdorff spaces. Then $K_0(X) \oplus K_0(Y) \cong K_0(X \sqcup Y)$.

Proof. It can be easily verified that $C(X) \oplus C(Y) \cong C(X \sqcup Y)$. By Corollaries 6.7 and 8.10 we have

$$K_0(X) \oplus K_0(Y) \cong K_0(C(X)) \oplus K_0(C(Y)) \cong K_0(C(X) \oplus C(Y)) \cong K_0(X \sqcup Y).$$
11 What's next

Computing the $K_0$ or $K^0$ group can be very difficult even with the machinery we have developed. The next step is to define the higher $K$-groups by $K_{n+1}(A) := K_n(SA)$ or $K^{n-1}(X) := K^n(SX)$, were $S$ denotes the suspension of the C*-algebra or the topological space. The isomorphism $K_n(C(X)) \cong K^{-n}(X)$ holds for all $n$. For a C*-algebra and a closed ideal $I$, there exist connecting maps for which the long sequence

$$\ldots \to K_2(A/I) \to K_1(I) \to K_1(A) \to K_1(A/I) \to K_0(I) \to K_0(A) \to K_0(A/I)$$

is exact. The corresponding sequence is exact for the reduced topological $K$-theory, with arrows pointed in the opposite direction.

The celebrated Bott Periodicity theorem says that $K_n(A) \cong K_{n+2}(A)$ (or $K^n(X) \cong K^{n+2}(X)$) for all $n$. This reduces the above sequence to a sequence with six elements. It also implies that if we know the $K_0$- and $K_1$-group of a C*-algebra then we can read off the $K$-groups of its suspensions. For example, to find the $K$-groups of spheres of any dimension, one only needs to compute $K^0$ and $K^1$ for the two pointed space $S^0$. The interested readers are referred to [1] and [4] for more details.
References


