

Two-Class $M/M/1$ Make-to-Stock Queueing Systems with Both Backlogs and Lost Sales

by

Meisam Soltani-koopa

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

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Abstract

We introduce a new simple allocation policy which is a very good approximation of the optimal allocation policy in an inventory system with a single product and two priority classes of customers. A production facility produces new items with exponentially distributed production times as long as the inventory level is below a base-stock level of inventory. We assume that customers arrive to the system according to a Poisson process. They may be satisfied, backlogged, or rejected depending on their priority, the inventory level upon their arrivals, availability of products in stock, and availability of a finite waiting area.

We define a categorized cost function to investigate the efficiency of the new allocation policy and several known allocation policies in the literature. The system is modeled as a combination of one- and two-dimensional Birth-and-Death processes under four different allocation policies: Sharing with Minimum Allocation (SMA) policy, Complete Partitioning (CP) policy, Multilevel Rationing (MR) policy, and Lost Sales (LS) policy. By solving the model and deriving the relevant probabilities, we calculate the relative gap between each policy and the optimal policy.

Based on the numerical results, we find that the SMA policy provides a very good approximation of the optimal policy, and is applicable in practical problems with high dimensions and static levels of inventory and waiting areas. We show that the MR and LS policies are special cases of the SMA policy. Therefore, their performances can be evaluated using the results obtained under the SMA allocation policy.

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Chapter 1

1. Introduction

An inventory management system with customer differentiation is a critical factor to serve customers efficiently and obtain a good level of satisfaction in different industries. For example, in the health care system, hospitals manage blood product inventory so that some units of blood are always available to be transfused to patients who need blood immediately. To keep some blood for urgent patient arrivals, hospitals may reschedule some elective surgeries or they may place some emergency orders to a blood bank if the inventory level is very low. The former can be considered as backlogging some surgeries until there are enough blood products in the hospital. The latter can be modelled as losing the opportunity of placing a routine order to the blood bank, which is much less expensive than the emergency order.

The inventory problems in the literature are mostly pure backlog or pure lost sales. In pure backlog problems, arriving customers are always served or backlogged. In pure lost sales problems, arriving customers are always served or rejected. However, in the real world, keeping customers waiting for long times may have negative effects on the total profit of the system. Therefore, it may be beneficial to backlog customers as long as it is possible to serve them shortly after their arrivals, and reject new arriving customers as soon as we realize the waiting time is too long.

In this thesis, we consider an inventory system managed under a central decision maker to examine how considering both backlog and lost sales may affect the known results in the literature of inventory systems with backlogs or lost sales. We assume customers arrive to the system according

to a Poisson process and production times are exponentially distributed. Customers with different priorities can be served, backlogged, or rejected, and products can be used to serve customers or be kept in stock.

In this research, we consider three known allocation policies in the literature defined in Chapter 4: Multilevel Rationing (MR) policy, which is a pure backlog policy; Lost Sales (LS) policy, which is a pure rejection policy without any backlog and Complete Partitioning (CP) policy which considers both backlogging and lost sales. We also introduce a new policy called Sharing with Minimum Allocation (SMA) policy which allows both backlogging and lost sales. We model the problem under the SMA and CP policies using the Birth-and-Death (B&D) process and show that the MR and LS policies are special cases of the SMA policy. To analyze the model and determine the optimal controls of the systems under the SMA and CP policies, we break down the system to three stages, and analyze each stage separately using the queueing model developed by Bondi (1989). Then, to analyze the model, we connect the stages so that they together represent the original system.

Benjaafar et al. (2010) characterize the optimal policy and obtain the optimal controls of the system under the CP policy using Dynamic Programming (DP). They show that the optimal policy is state-dependent. The model we present minimizes the cost with optimal static levels, which are independent of the system's state and more applicable in real world inventory problems. Moreover, DP modeling has the curse of dimensionality, but our model does not have any difficulty solving problems with high dimensions. Furthermore, our results under the CP and SMA policies are almost the same as the DP modeling under the CP and the optimal policy, respectively. This demonstrates that the SMA policy introduced in this thesis is a very good approximation for the optimal policy of the problem characterized in Benjaafar et al. (2010).

In the remainder of this thesis, we review the literature in Chapter 2. The problem is defined in detail in Chapter 3. Different allocation policies are presented in Chapter 4 and the problem is modeled under different policies in Chapter 5. Then, numerical results, including percentage of cost differences with respect to the optimal policy, are discussed in Chapter 7. Finally, conclusions, recommendations, and future works are presented in Chapter 8.

Chapter 2

2. Literature Review

Make-to-stock inventory systems with different characteristics have been studied in the literature. These characteristics can be seen as stock allocation policies, number of priority classes, number of servers, and capacity of the buffers. From the point of view of the buffer allocation, the problem can be converted into a queueing problem, which has also been studied in many articles in electrical and computer engineering. In this chapter, we review two categories of related papers in literature: Operation Research, and Electrical and Computer Engineering.

Ha (1997) studies the inventory rationing in a make-to-stock production system with two classes of customers and Lost Sales (LS). In Ha's study, there is no waiting area for backlogged customers when there is not any product in stock. He presents the optimal rationing levels for each class of customer and compares the cost of this system with the one managed under the First-Come First-Served (FCFS) policy.

Considering a buffer area, de Vericourt et al. (2001) assess the benefits of different stock-allocation policies for a make-to-stock $M/M/1$ production system. They model the system under the FCFS, SP, and MR policies. de Vericourt et al. (2001) determine the rationing levels for n classes of customers and calculate the cost. de Vericourt et al. (2000) show that the MR policy is the optimal policy among the other policies with infinite buffer spaces.

Abouee-Mehrizi et al. (2012) analyze a centralized single product multi-class $M/G/1$ make-to-stock queue. Their focus is on the customer composition. Dynamic programming, which normally is used

for analyzing queueing problems, is not practical in this case. Because of this, they model the system by considering a series of two-priority $M/G/1$ queues with high and low priority customers. Based on their analysis the optimal cost and the rationing levels have been derived. Here, the buffer area is assumed to be infinite.

There are some other studies on multi-class systems with rationing levels. Topkis (1968) presented an optimal rationing policy for a dynamic inventory model with n classes of customers. His model does not include lost sales.

In the above studies, the inventory systems have only the pure backlog or rejection mechanism. Janakariam (2007) compared two systems, pure backlog and pure lost sales. He shows the conditions under which one system or the other has a lower cost.

We can find some studies in the literature with both backlog and rejection. For example, Rabinowitz (1995) considers a system with a limited buffer area with Poisson customer arrivals. When the waiting area is full, the arriving customers are rejected. He presents the optimal level and the cost.

Cohen et al. (1988) present the model of an (s, S) inventory system for two priority customers with a rationing level. They present a heuristic that is a good approximation for the system with lost sales for excess customers. Frank et al. (2003) also study an optimal policy for an inventory system with priority. Deshpande et al. (2003) present another solution algorithm with a static threshold-base rationing policy and a buffer.

Our problem is the same as the problem that Benjaafar et al. (2010) analyze using DP. They consider a production-inventory system with both backlogged and lost sales for two classes of customers. They apply DP to analyze the system and find the optimal policy. They show that the

optimal policy can be characterized by three state-dependent thresholds: one production base-stock level and two admission levels, one for either class. They present the CP policy which assigns a separate area for each class of customers. By obtaining the cost of the optimal and CP policies, they show the advantage of the optimal policy. Additionally, Benjaafar et al. (2010) compare the optimal policy with other policies including the MR and LS policies numerically.

There are other articles in Electrical and Computer Engineering that concern queuing systems and are related to our research. These papers concentrate on transferring data and signals through channels, the rate of data transfer and the capacity of the channel result in creating queues. Because of the cost of a buffer, those authors try to optimize the cost by allocating minimum required spaces as a buffer. Because of arriving signals with different priorities, priority queues need to be analyzed.

Priority and capacitated queues have been studied in different ways in the literature since 1967, (Basharrin, 1967). Kobotushkin and Mikhalev (1969) analyze finite capacity queues with non-preemptive priorities under different allocation policies. Kapadia et al. (1984) study the finite capacity a non-preemptive priority queue with replacement of the high priority arrival with several service centres. They present another analysis (Kapadia et al., 1985) with application to health care systems. Kao and Wilson (1999) analyze a non-preemptive priority queue with multiple servers and two priority classes.

A queue with two classes of customers under complete sharing policy with a class-dependent service time is analysed by Wagner and Krieger (1999).

Bondi (1989) presents an analysis of finite capacity queues $M/M/1/K$ with priority under three different buffer allocation policies for two classes of customers. Specifically, he models a queueing system under the Complete Sharing (CS), CP, and SMA policies.

We use Bondi's queueing model to analyze the some parts of our queueing system under the CP and SMA policies.

Chapter 3

3. Problem Definition

We consider a centralized 2-class $M/M/1$ make-to-stock system with both backlogs and lost sales where customers of class 1 have priority over customers of class 2. Customers thus have two levels of priority. They arrive in the system according to a Poisson process with rates λ_1 and λ_2 . The production times are exponentially distributed with a rate μ . A manufacturing plant produces products until it satisfies all backlogged customers and fulfills the base-stock level of S with a holding cost of h per unit time. Each customer needs one unit of the product to be satisfied.

The manager should decide whether to satisfy arriving customers of each class, backlog them, or reject them. These decisions will be made based on the inventory level, and number of backlogs of each class of customers. Waiting in the backlog queue has a cost of b_i per unit time for class $i = 1, 2$ customers. Based on the customers' priorities, we assume that $b_1 \geq b_2$. Customers of class 1 will be satisfied as long as inventory level is not zero (i.e. there is some stock), while class 2 customers will be backlogged when the level of inventory is less than or equal to the specific level, R_2 . Due to space constraints, not all arriving customers can be backlogged; thus, the manager rejects new arriving customers when there are specific numbers of each class of customers in the backlog queue. Each lost customer has a cost of c_i per unit for class $i = 1, 2$ customers. We assume that $c_1 \geq c_2$ since class 1 customers have higher priority than class 2 customers to receive the product.

The objective is to minimize the total holding, backlog, and lost sales costs by controlling the base-stock level as well as the rationing levels for backlogs and lost sales.

Chapter 4

4. Stock Allocation Policies

To serve customers, we consider different allocation policies. These policies determine how we assign products; and backlog and reject customers of each class. We denote the stock level at time t by $I(t)$, which is non-negative. We find the cost of implementing each policy to determine the best one.

4.1. First-Come First-Served Policy

Under the FCFS policy, customers are served in the sequence of their arrival times. No priority is considered under this policy. The order of service is the order of arrival. In the case of backlog, when $I(t) = 0$, the customer with the longest waiting time will be served first. The buffer size here is infinite; we have enough spaces to backlog all arriving customers. When there is not any backlogged customer, a new entering product is stored until the stock level reaches S . At this point, production is stopped.

4.2. Strict Priority Policy

Under the Strict Priority (SP) policy, as long as the stock level is positive, arriving customers are served via the FCFS policy. When the stock level is zero, arriving customers are backlogged. When a new product enters to the system, backlogged customers are served depending on their priority. In other words, backlogged customers of class 1 will be served using the FCFS policy; otherwise, existing backlogged customers of class 2 will be served with the FCFS policy. In the case of no

backlogged customers, new products are stored until the stock level reaches S . At this point, the production ends. Note that under the SP policy, all customers who find the system out of stock are backlogged. The following table shows how customers are served under the FCFS and SP policies:

Stock level	FCFS		SP	
	Class 1	Class 2	Class 1	Class 2
$0 < I(t) \leq S$	<i>Satisfied</i>		<i>Satisfied</i>	
$I(t) = 0$	<i>Backlogged (no priority)</i>		<i>Backlogged</i> <i>(Priority 1)</i>	<i>Backlogged</i> <i>(Priority 2)</i>

Figure 1: Serving customers under the FCFS and SP policies.

4.3. Multilevel Rationing Policy

Under the Multilevel Rationing (MR) policy, the system uses a non-negative rationing level, R_2 , to decide whether customers of class 2 should be served or be backlogged. When the stock level, $I(t)$, is greater than R_2 , all customers are served under the FCFS policy. As the stock level drops to R_2 or lower, customers of class 2 are backlogged, but customers of class 1 are served as long as the stock level is positive; otherwise, they are also backlogged. With the newly-produced product, backlogged class 1 customers are served on a FCFS basis; otherwise, the new products are stored until the stock level increases to R_2 . Afterward, any backlogged customer of class 2 and arriving class 1 customers are served based on their priority. In the case of no backlogged customers, new products are stored until the stock level meets S . Then, the production is stopped. Figure 2 shows how the MR policy serves customers.

The SP policy can be seen as a special case of the MR policy in which $R_2 = 0$. The MR policy is the optimal policy, in the $M/M/1$ centralized make-to-stock systems with infinite buffer capacity (de Vericourt et al., 2001).

4.4. Sharing with Minimum Allocation Policy

Under the SMA policy, we define a rationing level R_2 to decide whether class 2 customers should be served. Moreover, we assign a finite waiting area of B to the system as a buffer for backlogged customers.

Both classes of arriving customers are served on a FCFS basis as long as the stock level, $I(t)$, is greater than the rationing level, R_2 . When the stock level is less than or equal to R_2 , customers of class 2 are backlogged, and only customers of class 1 are served, until the stock level drops to zero. Afterward, customers of class 1 are also backlogged. Let $B_1(t)$ and $B_2(t)$ denote the number of backlogged class 1 and 2 customers at time t , respectively.

Under this policy, the buffer is split to two parts. One part of the buffer with size B^c is shared between both classes of customers and can be allocated to backlogged customers of either class. Another part of the buffer with size B^1 is reserved only for class 1 customers and used in certain circumstances: First, all the shared area spaces are allocated to customers of both classes on a FCFS basis, but when this shared area is full, $B_1(t) + B_2(t) = B^c$, class 2 customers are rejected and only arriving class 1 customers are backlogged in B^1 . When B^c and B^1 are full, $B_1(t) + B_2(t) = B^c + B^1$, class 1 customers are also rejected.

As long as $I(t) = 0$, only backlogged customers of class 1 are served on a FCFS basis by new entering products; otherwise, entering new products are stored until $I(t) = R_2$. Afterward, any

backlogged customer of class 2 is served based on a FCFS basis. It is important to consider that, when one backlogged customer of class 1 in B^c is served, any backlogged customer of class 1 in B^1 , should be moved to B^c . This is due to the fact that B^1 should be kept empty as long as the number of backlogs in the system is less than B^c .

The MR policy can be considered as a special case of the SMA policy in which B^c approaches infinity.

4.5. Complete Partitioning Policy

Under the CP policy, a finite waiting area, B^c , is split into two separate areas, each reserved for a class of customers. Waiting areas with sizes B^1 and B^2 are assigned to class 1 and 2 customers, respectively.

Customers of class 2 are served only when $R_2 < I(t)$ on a FCFS basis with class 1 customers; otherwise, class 2 customers are backlogged in B^2 . As long as $0 < I(t)$, customers of class 1 are served; otherwise, they are backlogged in B^1 . Backlogged customers of each class wait in their reserved area until the area becomes full. Afterward, arriving customers are rejected. This means that if the waiting area of class 1 customers is full, arriving class 1 customers are rejected, even if there is a space in the class 2 waiting area. When a new product enters the system, only backlogged customers of class 1 are served, or the product is stored until the stock level reaches R_2 . Afterward, backlogged customers of class 2 are served. If there is no backlogged class 2 customers the new product is stored until the stock level reaches the base-stock level, S . At this point, production is stopped.

4.6. Lost Sales Policy

Under the Lost Sales (LS) policy, there is no space to backlog customers of either class. In other words, all arriving customers are served or rejected. When the stock level $I(t)$ is greater than the rationing level R_2 , both classes of customers are satisfied on a FCFS basis; otherwise, if there is any product in stock, only class 1 customers are satisfied, and class 2 customers are rejected. In the case of no product in stock that is $I(t) = 0$, both classes of customers are rejected. This policy can be considered as a special case of the SMA policy in which both B^c and B^1 are zero.

Figure 2¹ presents all four policies, with a rationing level of R_2 .

¹ State Indicators in Figure 2 are discussed in Chapter 6

Stock Level	MR		SMA		CP		LS		Stages	State Indicator
	Class 1	Class 2	Class 1	Class 2	Class 1	Class 2	Class 1	Class 2		
$I(t) = S$	Satisfied	Satisfied	Satisfied	Satisfied	Satisfied	Satisfied	Satisfied	Satisfied	Stage 1	$I(t)$
$R_2 < I(t) \leq S$	Satisfied	Satisfied	Satisfied	Satisfied	Satisfied	Satisfied	Satisfied	Satisfied		
$I(t) = R_2$	Satisfied	Backlogged	Satisfied	Backlogged	Satisfied	Backlogged	Satisfied	Rejected		
$0 < I(t) \leq R_2$	Satisfied	Backlogged	Satisfied	Backlogged	Backlogged	Backlogged	Backlogged	Rejected	Stage 2	i, j
$I(t) = 0$	Backlogge	Backlogged	Backlogged	Backlogged	Backlogged	B^1	B^2			
			Backlogged	Rejected	Rejected		Rejected		Stage 3	$\tilde{B}_1(t)$
			B^1							
			Rejected							

Figure 2: Serving customers under the MR, SMA, CP, and LS policies.

Chapter 5

5. Cost Function

To determine which policy is more effective, a cost function needs to be developed. This cost function should cover all holding, backlog, and lost sales costs. These costs depend on the base-stock level S , rationing level R_2 , and buffer spaces B^c, B^1 and B^2 under the different policies.

A general formula for the cost function is:

$$\begin{aligned} C^\bullet(S, R_2, B^c, B^1, B^2) = & \\ & h \cdot E^\bullet(\text{number of product in stock}) + b_1 \cdot E^\bullet(\text{number of backlogged class 1 customers}) + \\ & b_2 \cdot E^\bullet(\text{number of backlogged class 2 customers}) + \\ & c_1 \cdot E^\bullet(\text{number of rejected class 1 customers per unit time}) + \\ & c_2 \cdot E^\bullet(\text{number of rejected class 2 customers per unit time}) = \\ & h \sum_{i=0}^S i \cdot Ph_i^\bullet(S, R_2, B^c, B^1, B^2) + b_1 \sum_{i=0}^{B^c+B^1} i \cdot Pb_i^\bullet(S, R_2, B^c, B^1, B^2) + \\ & b_2 \sum_{i=0}^{B^c+B^2} i \cdot \overline{Pb}_i^\bullet(S, R_2, B^c, B^1, B^2) + c_1 \lambda_1 Pl^\bullet(S, R_2, B^c, B^1, B^2) + c_2 \lambda_2 \overline{Pl}^\bullet(S, R_2, B^c, B^1, B^2), \end{aligned}$$

where “ \bullet ” specifies the cost and the probabilities for a specific policy and $E^\bullet(\cdot)$ denotes the expected value under a specific policy, “ \bullet ”. The other notations are defined as follows:

- Ph_i^\bullet : Probability of i units of the product in stock.
- Pb_i^\bullet and \overline{Pb}_i^\bullet : Probability of i backlogged class 1 and 2 customers, respectively.
- Pl^\bullet and \overline{Pl}^\bullet : Probabilities of rejecting arriving customers of class 1 and 2, respectively

The vector (S, R_2, B^c, B^1, B^2) varies under different policies as follows:

- *MR policy*: $(S, R_2, B^c, B^1, B^2) \rightarrow (S, R_2, \infty, 0, 0)$
- *SMA policy*: $(S, R_2, B^c, B^1, B^2) \rightarrow (S, R_2, B^c, B^1, 0)$
- *CP policy*: $(S, R_2, B^c, B^1, B^2) \rightarrow (S, R_2, 0, B^1, B^2)$
- *LS policy*: $(S, R_2, B^c, B^1, B^2) \rightarrow (S, R_2, 0, 0, 0)$

In order to calculate the cost, we need to analyze the system under each policy to find the probabilities required to obtain the cost function. To analyze, we break down the system into three stages, as presented in Figure 2. Then, we model and analyze each stage separately. Finally, the required probabilities are obtained by connecting the stages such that they together represent the original system.

Chapter 6

6. Modeling the Problem

In this chapter, we model the problem under different policies. Based on the models, we obtain the probabilities to calculate the cost of implementing each policy, and find the best choice among different policies discussed in Chapter 4.

As discussed in Chapter 4, the MR and LS policies are special cases of the SMA policy. Therefore, we focus here on SMA and CP policies. In order to model the problem, we break the system down under SMA policy into three stages, and under CP policy into two stages, as shown in Figure 2. The assumption of exponential production times and Poisson arrivals allow us to define steady state diagrams (Birth-and-Death processes, B&D) for the SMA and CP policies. We develop combined one- and two-dimensional B&D processes to model the system.

6.1. Sharing with Minimum Allocation Policy

To analyze the SMA policy, each stage of the system shown in Figure 2 is modeled as a B&D process. Then, the stages are connected to one another. These stages are defined as follows under the SMA policy:

6.1.1. Modeling Stage 1

In stage 1, $R_2 < I(t) \leq S$ and customers are served on a FCFS basis. In this stage, there are no backlogged customers, and the only change in the state of the system is the inventory level, so $I(t)$ is used as a state indicator. Because no priority differences exist between customers in this stage,

the system moves from a state I to $(I-1)$, Birth, with the rate $\lambda = \lambda_1 + \lambda_2$. This movement represents serving an arriving customer with one product from the stock, which places a new order to the manufacturer. Death happens with the rate μ , which represents producing one unit of the product. This the rate at which the number of orders in the manufacturer reduces by one. We denote the probability of $I(t)=i$ by P_i^{SMA} . Therefore, to find $P_i^{SMA}, i = R_2 + 1, \dots, S - 1, S$, we use the B&D process illustrated in Figure 3. All the steady state probabilities in this stage can be obtained in terms of the probability of $(I(t) = R_2, B_2(t) = 0)$. Here this state is called the “reference state” denoted by r and its probability is denoted by P_r^{SMA} . This state is the connection point to stage 2.

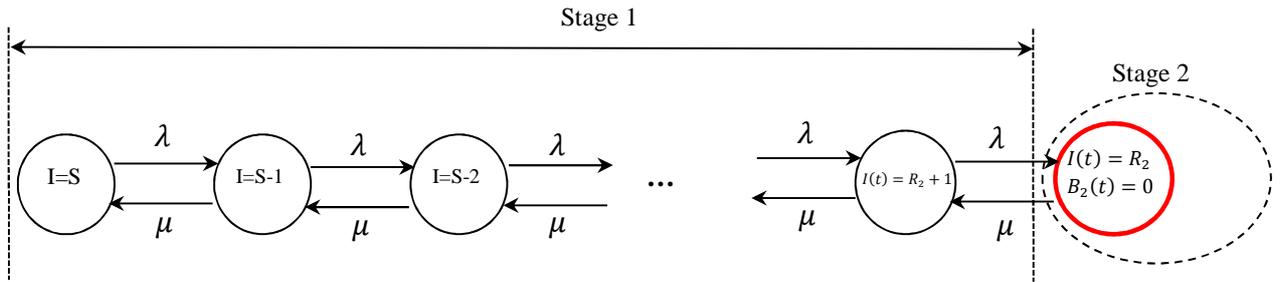


Figure 3: The steady state transition diagram for stage 1 under the SMA policy.

Based on the diagram in

Figure 3, steady state equations can be written as:

$$\lambda P_S^{SMA} = \mu P_{S-1}^{SMA}, \quad (1)$$

$$(\mu + \lambda) P_i^{SMA} = \lambda P_{i+1}^{SMA} + \mu P_{i-1}^{SMA}, \quad i = R_2 + 2, \dots, S - 1, \quad (2)$$

$$(\mu + \lambda) P_{R_2+1}^{SMA} = \lambda P_{R_2+2}^{SMA} + \mu P_r^{SMA}. \quad (3)$$

Then, according to equation (1),

$$P_S^{SMA} = \frac{\mu}{\lambda} P_{S-1}^{SMA} = \frac{1}{\rho} P_{S-1}^{SMA}, \quad (4)$$

where $\rho = \frac{\lambda}{\mu}$. By extending the equation (2) to $i=S-I$, and inserting it to equation (1), we obtain the same equation for P_{S-1}^{SMA} as $P_{S-1}^{SMA} = \frac{1}{\rho} P_{S-2}^{SMA}$ and in general:

$$P_{S-j}^{SMA} = \frac{1}{\rho} P_{S-j-1}^{SMA}, \quad j = 0, 1, \dots, S - (R_2 + 2). \quad (5)$$

Then, for $j = S - (R_2 + 2)$, we have $P_{R_2+2}^{SMA} = \frac{1}{\rho} P_{R_2+1}^{SMA}$, and by substituting $P_{R_2+2}^{SMA}$ into equation (3), we obtain $P_{R_2+1}^{SMA} = \left(\frac{1}{\rho}\right)^1 P_r^{SMA}$. Therefore, we have $P_{R_2+2}^{SMA} = \frac{1}{\rho} P_{R_2+1}^{SMA} = \left(\frac{1}{\rho}\right)^2 P_r^{SMA}$. By substituting $P_{R_2+2}^{SMA}$ into equation (5) for the rest of values of j , we derive all steady state probabilities in terms of P_r^{SMA} as

$$P_{R_2+i}^{SMA} = \left(\frac{1}{\rho}\right)^i P_r^{SMA}, \quad i = 1, 2, \dots, S - R_2. \quad (6)$$

Let $P_{stage\ 1}^{SMA}$ denote the probability of being in stage 1. Then,

$$P_{stage\ 1}^{SMA} = \sum_{i=R_2+1}^S P_i^{SMA} = P_r^{SMA} \sum_{i=R_2+1}^S \left(\frac{1}{\rho}\right)^i. \quad (7)$$

6.1.2. Modeling Stage 2

In stage 2, $0 \leq I(t) \leq R_2$ and $B_1(t) + B_2(t) \leq B^c$. In this stage, customers of class 2 are served with a new entering product if and only if $B_1(t) = 0$ and $I(t) = R_2$; otherwise, the product is used to serve backlogged class 1 customers or stored. The B&D process that presents this stage is two dimensional. One dimension, i , is the number of products needed to serve all current backlogged class 1 customers and increase the inventory to the stock level R_2 . In other words, as long as we are in Stage 2, $i = B_1(t) + (R_2 - I(t))$ and the other dimension, j , is the number of orders needed to

serve all backlogged class 2 customers. Thus, $j = B_2(t)$. As long as $i \leq R_2$, which means that $B_1(t) = 0$, the first dimension, i , represents the differences between the stock level and the rationing level of R_2 , $i = R_2 - I(t)$. When $i \geq R_2$, i represents $B_1(t) + R_2$. The maximum value for the j dimension is B^c , a value that happens when all spaces in B^c are allocated to class 2 customers, which means $B_2(t) = B^c$ and $B_1(t) = 0$. Furthermore, the maximum value for the i dimension is $B^c + R_2$, which occurs when $B_1(t) = B^c$, $I(t) = 0$, and $B_2(t) = 0$. Figure 4 shows the steady state diagram of Stage 2.

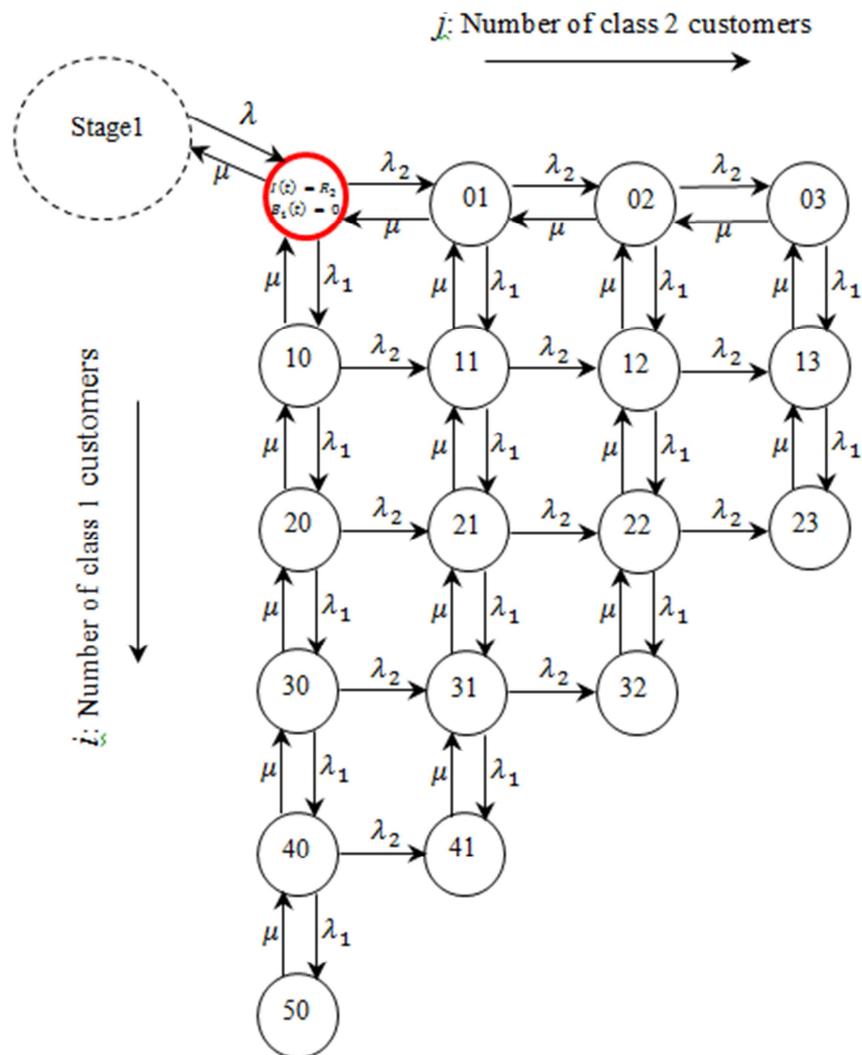


Figure 4: The steady state transition diagram for stage 2 under the SMA policy with $R_2=2$ and $B^c = 3$.

Then, we need to find the two dimensional probabilities of $P_{i,j}^{SMA}$, $i = 0, \dots, H(j)$ and $j = 0, 1, \dots, B^c$ for state (i, j) where $H(j)$ is the maximum value for i when the number of backlogged class 2 customers in the system is j . $H(j)$ is defined under the SMA policy as:

$$\max_j i = H(j) \triangleq R_2 + B^c - j .$$

The state $(i=0, j=0)$ represents the situation when $I(t) = R_2$, and there are no backlogged customers of class 2 in the system, $B_2(t) = 0$. This state is the reference state denoted with red in Figure 3, and is the connection point to stage 1. All probabilities in stage 2 are calculated in terms of $P_r^{SMA} = P_{0,0}^{SMA}$ as we did in stage 1. By applying the steady state equations of the B&D process in Figure 4 and using the method developed by Bondi (1989), we can find all steady state probabilities of stage 2, in terms of the probability of the reference state, P_r^{SMA} . We start with the first column in the next section, and then the other columns before the last column is analyzed in Section 6.1.2.2. That last column has a different pattern to derive its probabilities. This column comes separately.

6.1.2.1. Solving for the First Column

When we go through the first column in the steady state diagram of stage 2 to find P_{i0}^{SMA} , the following steady state equations can be derived:

For the state of $(i = 0, j = 0)$:

$$(\lambda_1 + \lambda_2 + \mu)P_r^{SMA} = \mu P_{01}^{SMA} + \mu P_{10}^{SMA} + \lambda P_{R_2+1}^{SMA}. \quad (8)$$

For the rest of states in column 1 except the last state at the bottom we have

$$(\lambda_1 + \lambda_2 + \mu)P_{i0}^{SMA} = \lambda_1 P_{i-1,0}^{SMA} + \mu P_{i+1,0}^{SMA} \quad \text{or} \quad (\rho_1 + \rho_2 + 1)P_{i0}^{SMA} = \rho_1 P_{i-1,0}^{SMA} + P_{i+1,0}^{SMA},$$

for $i = 1, 2, \dots, H(0) - 1$, (9)

where $\rho_1 = \frac{\lambda_1}{\mu}$, $\rho_2 = \frac{\lambda_2}{\mu}$.

The steady state equation for the last state at the bottom of the column is

$$\mu P_{H(0),0}^{SMA} = \lambda_1 P_{H(0)-1,0}^{SMA}. \quad (10)$$

Therefore,

$$P_{H(0),0}^{SMA} = \frac{\lambda_1}{\mu} P_{H(0)-1,0}^{SMA} = \rho_1 P_{H(0)-1,0}^{SMA} = A_0 P_{H(0)-1,0}^{SMA}, \quad (11)$$

where $A_0 = \rho_1$.

By substituting the equation (11) into equation (9), we obtain the following equations for P_{i_0} 's.

When $i = H(0) - 1$, we have $(\rho_1 + \rho_2 + 1)P_{H(0)-1,0}^{SMA} = \rho_1 P_{H(0)-2,0}^{SMA} + P_{H(0),0}^{SMA}$. By substituting equation (11) into this equation and considering $\rho = \rho_1 + \rho_2$, we obtain the following equation:

$$(\rho_1 + \rho_2 + 1)P_{H(0)-1,0}^{SMA} = \rho_1 P_{H(0)-2,0}^{SMA} + \rho_1 P_{H(0)-1,0}^{SMA},$$

so,
$$(\rho_2 + 1)P_{H(0)-1,0}^{SMA} = \rho_1 P_{H(0)-2,0}^{SMA},$$

and then,
$$P_{H(0)-1,0}^{SMA} = \frac{\rho_1}{\rho+1-\rho_1} P_{H(0)-2,0}^{SMA} = A_1 P_{H(0)-2,0}^{SMA}, \quad (12)$$

where, $A_1 = \frac{\rho_1}{\rho+1-\rho_1}$.

Similar to Bondi (1989), we continue this method for the other values of i . When $i = H(0) - 2$ in equation (9), we have:

$$(\rho_1 + \rho_2 + 1)P_{H(0)-2,0}^{SMA} = \rho_1 P_{H(0)-3,0}^{SMA} + P_{H(0)-1,0}^{SMA}, \quad (13)$$

by substituting $P_{H(0)-1,0}^{SMA}$ from equation (12) into equation (13), we obtain:

$$(\rho + 1)P_{H(0)-2,0}^{SMA} = \rho_1 P_{H(0)-3,0}^{SMA} + A_1 P_{H(0)-2,0}^{SMA}.$$

So,

$$(\rho + 1 - A_1)P_{H(0)-2,0}^{SMA} = \rho_1 P_{H(0)-3,0}^{SMA} \Rightarrow P_{H(0)-2,0}^{SMA} = \frac{\rho_1}{\rho+1-A_1} P_{H(0)-3,0}^{SMA} = A_2 P_{H(0)-3,0}^{SMA}, \quad (14)$$

where $A_2 = \frac{\rho_1}{\rho+1-A_1}$.

For $i = H(0) - 3$ in equation (9), we have $(\rho_1 + \rho_2 + 1)P_{H(0)-3,0}^{SMA} = \rho_1 P_{H(0)-4,0}^{SMA} + P_{H(0)-2,0}^{SMA}$. By substituting $P_{H(0)-2,0}^{SMA}$ from equation (14), we obtain $P_{H(0)-3,0}^{SMA}$ as follows:

$$\begin{aligned} (\rho_1 + \rho_2 + 1)P_{H(0)-3,0}^{SMA} &= \rho_1 P_{H(0)-4,0}^{SMA} + A_2 P_{H(0)-3,0}^{SMA}, \\ P_{H(0)-3,0}^{SMA} &= \frac{\rho_1}{(\rho+1-A_2)} P_{H(0)-4,0}^{SMA} = A_3 P_{H(0)-4,0}^{SMA}, \end{aligned} \quad (15)$$

where, $A_3 = \frac{\rho_1}{(\rho+1-A_2)}$.

We can apply induction to prove that by using equation (9), in general we get:

$$P_{H(0)-n,0}^{SMA} = A_n P_{H(0)-n-1,0}^{SMA}, \text{ where } A_n = \frac{\rho_1}{(\rho+1-A_{n-1})} \text{ and } n = 1, 2, \dots, H(0) - 1, \quad (16)$$

which is similar to equation (A19) in Bondi (1989).

Therefore, for $n = H(0) - 1$, we have:

$$P_{H(0)-(H(0)-1),0}^{SMA} = \frac{\rho_1}{(\rho+1-A_{(H(0)-2)})} P_{H(0)-(H(0)-1)-1,0}^{SMA} \text{ or } P_{1,0}^{SMA} = A_{H(0)-1} P_{0,0}^{SMA} = P_r^{SMA}. \quad (17)$$

As seen in equation (17), $P_{1,0}^{SMA}$ has been calculated in terms of the probability of the reference state, $(0,0)$. As the probability of state $(i,0)$ has been presented in terms of the probability of state $(i-1,0)$, by substituting back the probability of state $(i-1,0)$ into the probability of state $(i,0)$ for $i = 1, 2, \dots, H(0) - 1$, similar to equation (A18) in Bondi (1989), we obtain the following equations for probability of state $(i,0)$, which can be proved by induction:

$$P_{i0}^{SMA} = P_r^{SMA} \prod_{n=H(0)-i+1}^{H(0)} A_{n-1}, \quad i = 1, 2, \dots, H(0). \quad (18)$$

Therefore, all the probabilities of the states in the first column have been presented in terms of P_r^{SMA} , the probability of “reference state”.

6.1.2.2. Solving for Middle Columns

In order to find the probabilities of states in the middle columns, $P_{ij}^{SMA} (0 < j < B^c - 1)$, in terms of P_r^{SMA} , as in Bondi (1989), we start with the second column and move to the next columns in order, until we reach the last column. Then, a common pattern is used for columns between the first column and the last column. The last column is analysed in the next section.

In each column in this section, we start with the last state, $(H(j), j)$, for $0 < j < B^c$. By writing the steady state equation for the last state, similar to equation (A17) in Bondi (1989), we can derive $P_{H(j),j}^{SMA}$ in terms of the probabilities of states that are connected to this state, $P_{H(j-1)-1,j-1}^{SMA}$ and $P_{H(j)-1,j}^{SMA}$, as follows:

$$\mu P_{H(j),j}^{SMA} = \lambda_1 P_{H(j)-1,j}^{SMA} + \lambda_2 P_{H(j),j-1}^{SMA}, \quad j = 1, 2, \dots, B^c - 1, \quad (19)$$

or
$$P_{H(j),j}^{SMA} = \rho_1 P_{H(j)-1,j}^{SMA} + \rho_2 P_{H(j),j-1}^{SMA}, \quad j = 1, 2, \dots, B^c - 1.$$

As we analyze columns in order, all probabilities of the previous column, $P_{H(j),j-1}^{SMA}$, are available in terms of P_r^{SMA} , when we are analyzing column j . The only unknown probability is $P_{H(j)-1,j}^{SMA}$. By writing steady state equations for the rest of the states in column j , from state $(H(j)-1, j)$ until $(1, j)$, Bondi (1989) has derived the following recursive equation:

$$P_{H(j)-i,j}^{SMA} = \frac{\rho_1 P_{H(j)-(i+1),j}^{SMA}}{B_i} + \rho_2 \sum_{n=H(j)-i}^{H(j)} \left\{ \frac{P_{n,j-1}^{SMA}}{\prod_{k=H(j)-n+1}^{i+1} B_{k-1}} \right\}, \quad (20)$$

$$i = 0, 1, 2, \dots, H(m) - 1; j = 1, 2, \dots, B^c - 1,$$

$$\text{where} \quad B_0 = 1, \quad (21)$$

$$B_1 = \rho_2 + 1,$$

$$B_n = \rho + 1 - \frac{\rho_1}{B_{n-1}}, n = 1, 2, 3, \dots$$

As seen in equation (20), when we are calculating probabilities for column j , the second term gives us probabilities of states in the previous column which have already been derived in terms of P_r^{SMA} , and the first term is the probability of the previous state in the same column and recursively this term is calculated in terms of P_{0j}^{SMA} .

In order to derive the probability of the first state in column j , P_{0j}^{SMA} , we consider the steady state equation for the first state in the previous column, $(0, j-1)$, similar to Section A.3 in Bondi (1989).

In the second column, $j=1$, we can derive P_{01}^{SMA} in terms of P_r^{SMA} from equation (8):

$$\mu P_{01}^{SMA} = (\lambda_1 + \lambda_2 + \mu) P_r^{SMA} - \mu P_{10}^{SMA} - \lambda P_{R_2+1}^{SMA}. \quad (22)$$

As we have $P_{R_2+1}^{SMA}$ from equation (6) in stage 1 with $i=1$, and P_{10}^{SMA} from equation (18) with $i=1$, we rewrite equation (22) as $\mu P_{01}^{SMA} = (\lambda_1 + \lambda_2 + \mu)P_r^{SMA} - \mu P_r^{SMA} A_{H(0)-1} - \lambda(\frac{1}{\rho})^1 P_r^{SMA}$. Then, we can have P_{01}^{SMA} in terms of P_r^{SMA} as

$$P_{01}^{SMA} = (\rho + 1)P_r^{SMA} - P_r^{SMA} A_{H(0)-1} - \rho(\frac{1}{\rho})^1 P_r^{SMA} = [\rho - A_{H(0)-1}]P_r^{SMA}. \quad (23)$$

In order to derive the steady state probability of state $(0, j)$, $j = 2, 3, \dots, B^c - 1$, we write the steady state equation of state $(0, j-1)$ as follows:

$$(\lambda_1 + \lambda_2 + \mu)P_{0,j-1}^{SMA} = \mu P_{0j}^{SMA} + \mu P_{1,j-1}^{SMA} + \lambda_1 P_{0,j-2}^{SMA}, \quad J = 2, 3, \dots, B^c - 1. \quad (24)$$

Then, $\mu P_{0j}^{SMA} = (\lambda_1 + \lambda_2 + \mu)P_{0,j-1}^{SMA} - \mu P_{1,j-1}^{SMA} - \lambda_1 P_{0,j-2}^{SMA}$. Therefore,

$$P_{0j}^{SMA} = (\rho + 1)P_{0,j-1}^{SMA} - P_{1,j-1}^{SMA} - \rho_1 P_{0,j-2}^{SMA}, \quad J = 2, 3, \dots, B^c - 1. \quad (25)$$

Now, we have P_{0j}^{SMA} in terms of the probabilities of the states in previous columns that have already been derived in terms of the probability of the reference state, P_r^{SMA} . Therefore, we have P_{0j}^{SMA} in terms of P_r^{SMA} . Afterward, by substituting P_{0j}^{SMA} into the first term of equation (20) for state $(1, j)$ we can derive P_{1j}^{SMA} in terms of P_r^{SMA} . Then, by substituting P_{1j}^{SMA} into equation (20) and using the same pattern until $P_{H(j),j}^{SMA}$, we can derive all probabilities in equation (20) in terms of P_r^{SMA} , the probability of “reference state”.

Therefore, we have derived the steady state probabilities of P_{ij}^{SMA} for $i = 0, 1, \dots, H(j)$ and $j = 1, 2, \dots, B^c - 1$ in terms of P_r^{SMA} .

6.1.2.3. Solving for the Last Column

In the last column, we start to derive steady state probabilities from state $(0, B^c)$ until state $(H(B^c), B^c)$, $P_{iB^c}^{SMA}$ ($0 \leq i \leq H(B^c)$), in order, similar to Section A.2.3 in Bondi (1989).

By writing the steady state equation of state $(0, B^c)$, we derive P_{0,B^c}^{SMA} as:

$$(\lambda_1 + \lambda_2 + \mu)P_{0,B^c-1}^{SMA} = \mu P_{0,B^c}^{SMA} + \mu P_{1,B^c-1}^{SMA} + \lambda_1 P_{0,B^c-2}^{SMA}. \quad (26)$$

Therefore,

$$P_{0,B^c}^{SMA} = (\rho + 1)P_{0,B^c-1}^{SMA} - P_{1,B^c-1}^{SMA} + \rho_1 P_{0,B^c-2}^{SMA}. \quad (27)$$

So, we have P_{0,B^c}^{SMA} in terms of P_{0,B^c-1}^{SMA} , P_{1,B^c-1}^{SMA} , and P_{0,B^c-2}^{SMA} , all of which have been derived in the previous section.

By writing the steady state equation for state $(0, B^c)$, we can derive the steady state probability of P_{1,B^c}^{SMA} as follows:

$$(\lambda_1 + \mu)P_{0,B^c}^{SMA} = \mu P_{1,B^c}^{SMA} + \lambda_1 P_{0,B^c-1}^{SMA}. \quad (28)$$

Thus,
$$P_{1,B^c}^{SMA} = (\rho_1 + 1)P_{0,B^c}^{SMA} - \rho_1 P_{0,B^c-1}^{SMA}. \quad (29)$$

P_{0,B^c}^{SMA} has been presented in equation (27), and P_{0,B^c-1}^{SMA} , which is from the previous column, has been derived in the previous section.

The steady state equation for state $(i - 1, B^c)$, $i = 2, 3, \dots, H(B^c)$ can be presented by:

$$(\lambda_1 + \mu)P_{i-1,B^c}^{SMA} = \mu P_{i,B^c}^{SMA} + \lambda_2 P_{i-1,B^c-1}^{SMA} + \lambda_1 P_{i-2,B^c}^{SMA}, \quad i = 2, 3, \dots, H(B^c). \quad (30)$$

$$\text{Thus, } P_{i,B^c}^{SMA} = (\rho_1 + 1)P_{i-1,B^c}^{SMA} - \rho_2 P_{i-1,B^{c-1}}^{SMA} - \rho_1 P_{i-2,B^c}^{SMA}, \quad i = 2, 3, \dots, H(B^c), \quad (31)$$

where the second term is from the previous column and has been derived in terms of P_r^{SMA} . By using equation (27) and (29), in addition to using equation (31) recursively, we can derive all probabilities of P_{i,B^c}^{SMA} for $i = 2, 3, \dots, H(B^c)$ in terms of P_r^{SMA} .

Therefore, we have derived all probabilities of stage 2 in terms of P_r^{SMA} .

6.1.3. Modeling Stage 3

When $I(t) = 0$ and $B_1(t) + B_2(t) > B^c$, the system is in stage 3. In other words, no products are in stock, and all common waiting area spaces are full. In this situation, all arriving class 2 customers are rejected and arriving class 1 customers are backlogged or rejected depending on the availability of the waiting area spaces in B^1 , which is specified just for backlogged class 1 customers.

Under the SMA policy, backlogged class 1 customers in B^c and B^1 should be served on a FCFS basis. Afterward, new products are stored or used to serve any backlogged class 2 customers. In order to make the problem tractable, without any change to the probabilities of having different numbers of backlogged class 1 customers, we do not start serving backlogged class 1 customers in B^c if there is at least one backlogged class 1 customer in B^1 . Then, as long as the system is in stage 3 and there is any class 1 customer in B^1 , customers in the common waiting area, B^c , are not served. Moreover, backlogged class 1 customers in B^1 are served on a FCFS basis. Once there are zero backlogged class 1 customers in B^1 , we start serving the backlogged customers in B^c or store new entering products in stock according to the SMA policy. That is, backlogged customers in B^1 have higher priority than customers in B^c . We call this strategy as the “ B^1 strategy.”

According to the above discussion, in both the SMA policy and B^1 strategy, serving backlogged class 1 customers has higher priority than serving backlogged class 2 customers and storing new entering products in stock. Therefore, in terms of the number of existing backlogged class 1 customers in the system, both the SMA policy and B^1 strategy act the same, even though the order of serving customers of this class is different. Thus, the probabilities of having i backlogged class 1 customers in the system for both the SMA policy and B^1 strategy are the same. Therefore, we consider the B^1 strategy, which is tractable analytically, to obtain the probabilities of having different numbers of class 1 customers in the system when the system is in stage 3.

We specify each state in this stage by the number of backlogged class 1 customer in B^1 as $0 < \tilde{B}_1(t) \leq B^1$. Thus, the B&D process of this stage is one dimensional, as presented in Figure 5:

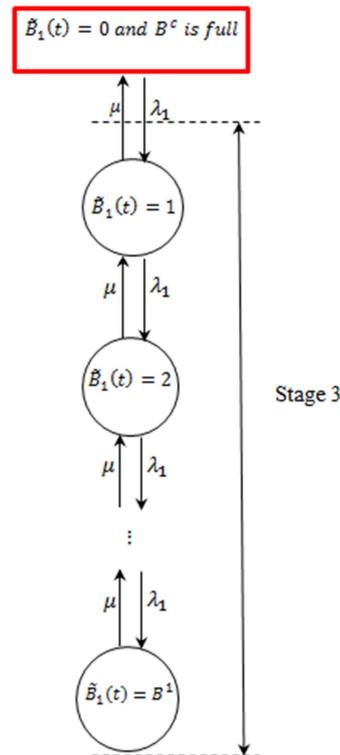


Figure 5: The steady state diagram for stage 3.

We denote the probability of the rectangular state by $P_{B^c}^{SMA}$. This state means that there is no customer in B^1 , and B^c is full. In other words, this state is the connection point to stage 2.

In stage 3, because of accepting only class 1 customers, the system moves from state $\tilde{B}_1(t) = i$ to state $\tilde{B}_1(t) = i + 1$, Birth, with the rate of arriving class 1 customers, λ_1 , and moves back from state $\tilde{B}_1(t) = i + 1$ to state $\tilde{B}_1(t) = i$, Death, with the production rate μ . Because of accepting only class 1 customers and rejecting class 2 customers when B^c is full and the system is in the rectangular state, moving from this state to state $\tilde{B}_1(t) = 1$ happens with the rate λ_1 , and moving back from state $\tilde{B}_1(t) = 1$ to the rectangular state happens with the rate μ in the B&D process. We denote the probability of being in state $\tilde{B}_1(t) = i$ by \tilde{P}_i^{SMA} . By writing steady state equations for each state of stage 3, we can derive the probabilities of the states in terms of $P_{B^c}^{SMA}$.

The steady state equation for state $\tilde{B}_1(t) = B^1$ and $\tilde{B}_1(t) = i$ can be written as:

$$\mu \tilde{P}_{B^1}^{SMA} = \lambda_1 \tilde{P}_{B^1-1}^{SMA}, \quad (32)$$

$$(\mu + \lambda_1) \tilde{P}_i^{SMA} = \mu \tilde{P}_{i+1}^{SMA} + \lambda_1 \tilde{P}_{i-1}^{SMA}, \quad i = 2, \dots, B^1 - 1, \quad (33)$$

$$(\mu + \lambda_1) \tilde{P}_1^{SMA} = \mu \tilde{P}_2^{SMA} + \lambda_1 P_{B^c}^{SMA}. \quad (34)$$

Then, according to equation (32),

$$\tilde{P}_{B^1}^{SMA} = \frac{\lambda_1}{\mu} \tilde{P}_{B^1-1}^{SMA} = \rho_1 \tilde{P}_{B^1-1}^{SMA}.$$

By writing the equation (33) for $i = B^1 - 1$, and adding it to equation (32), we obtain the same equation for $\tilde{P}_{B^1-1}^{SMA}$ as $\tilde{P}_{B^1-1}^{SMA} = \rho_1 \tilde{P}_{B^1-2}^{SMA}$. In general, we can prove the following formula by induction:

$$\tilde{P}_{B^1-k}^{SMA} = \rho_1 \tilde{P}_{B^1-(k+1)}^{SMA}, \quad k = 0, 1, \dots, B^1 - 2. \quad (35)$$

Then, for $k = B^1 - 2$, we have $\tilde{P}_2^{SMA} = \rho_1 \tilde{P}_1^{SMA}$. By substituting \tilde{P}_2^{SMA} in equation (34) we obtain $\tilde{P}_2^{SMA} = \rho_1 \tilde{P}_1^{SMA} = (\rho_1)^2 P_{B^c}^{SMA}$. By substituting back \tilde{P}_2^{SMA} into equation (35), for the rest of k , we can use the same method and derive all steady state probabilities in terms of $P_{B^c}^{SMA}$ as follows:

$$\tilde{P}_i^{SMA} = (\rho_1)^i P_{B^c}^{SMA}, \quad i = 1, \dots, B^1 - 1. \quad (36)$$

The rectangular state occurs when there is no customer of class 1 in B^1 and the common waiting area, B^c , is full. Having no spaces in the common waiting area happens with different combinations of backlogged customers in the area. For example, this can happen with no backlogged class 1 customer in the waiting area and all the area being occupied by only backlogged class 2 customers, or having one backlogged class 1 customer and the rest of the area being occupied by class 2 customers. In other words, when the system is in any one of the states with no spaces in the common waiting area in stage 2, the system is ready to move to stage 3 with arriving a new class 1 customer and backlogging the customer in the specified waiting area for class 1 customers, B^1 . Arriving class 2 customers at these states are rejected. These states can be denoted by $P_{H(j),j}^{SMA}$, which shows that we are in the last state in column j . Therefore, the probability of the rectangular state, $P_{B^c}^{SMA}$, can be expressed as:

$$P_{B^c}^{SMA} = \sum_{j=0}^{B^c} P_{H(j),j}^{SMA}, \quad (37)$$

where $P_{H(j),j}^{SMA}$ represents the probability of a state with j class 2 customers and $i=H(j)$, which means that there are $B^c - j$ class 1 customers in B^c . Figure 6 shows the rectangular state with its sub-states and also illustrates the connection between stages 1, 2, and 3.

Once all the probabilities for stage 2, including $P_{H(j),j}^{SMA}$, have been determined in terms of P_r^{SMA} , we use equation (37), P_B^{SMA} in terms of P_r^{SMA} , to obtain all the steady state probabilities for stage 3 in terms of P_r^{SMA} .

Now, we have all possible steady state probabilities in stages 1, 2, and 3 in terms of P_r^{SMA} , the probability of “reference state”. As the sum of all steady state probabilities is equal to one, and we have all probabilities as a function of P_r^{SMA} , we find P_r^{SMA} which helps us to calculate the preceding probabilities.

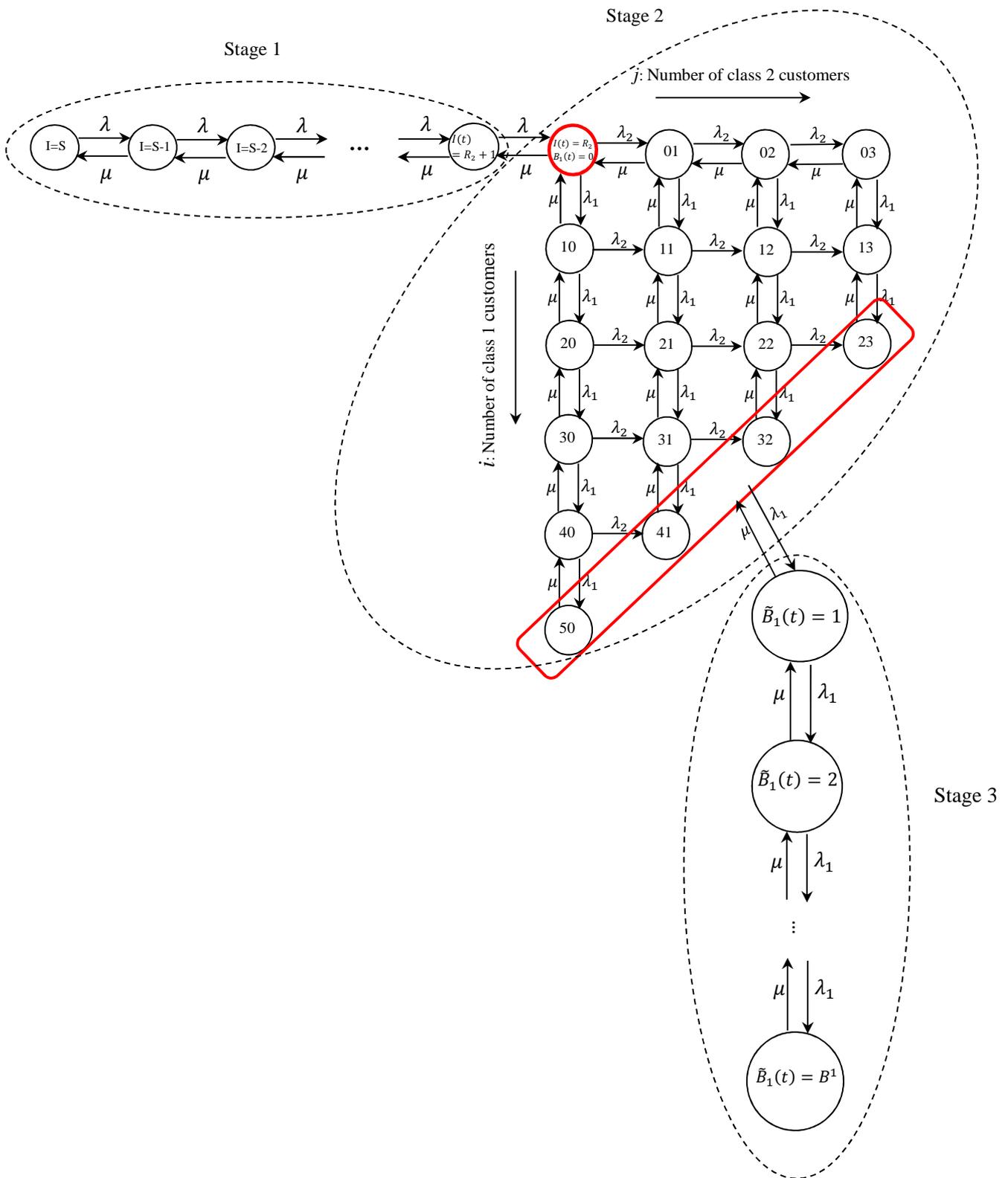


Figure 6: All three stages with their states and connections under the SMA policy.

6.1.4. Probabilities Required for the Cost Function

In this section, we derive the probabilities required to obtain the cost function under the SMA policy.

➤ *Holding cost-related probabilities:*

When $R_2 < I(t) \leq S$, the system is in stage 1. Therefore, we can obtain the probability that inventory level is $R_2 < I(t) \leq S$ using equation (6) as given in Ph_n^{SMA} , in equation (38), when $R_2 + 1 \leq n \leq S$.

Furthermore, when $0 \leq I(t) \leq R_2$, the system is in stage 2, and it is possible to have backlogged class 2 customers. In order to find Ph_n^{SMA} when $1 \leq n \leq R_2$, we derive $P_{R_2-n,j}^{SMA}$, in equation (38), using the probabilities from equations (18), (20), (29), and (31).

$$Ph_n^{SMA}(S, R_2, B^c, B^1, B^2) = \begin{cases} \sum_{j=0}^{B^c} P_{R_2-n,j}^{SMA}, & 1 \leq n \leq R_2 \\ P_n^{SMA} = \left(\frac{1}{\rho}\right)^{n-R_2} P_r^{SMA}, & R_2 + 1 \leq n \leq S \end{cases} \quad (38)$$

➤ *Class 2 backlogged customer-related probabilities:*

Backlogging of class 2 customers occurs when the system is in stage 2, as shown in Figure 2. Note that the second element of the state space (i, j) in stage 2 is the number of backlogs of class 2. Thus, we can present the probability of having n backlogged class 2 customers using the probabilities from equations (18), (20), (29), and (31) obtained for stage 2 as follows:

$$\overline{Pb}_n^{SMA}(S, R_2, B^c, B^1, B^2) = \sum_{i=0}^{H(n)} P_{i,n}^{SMA}, \quad n = 0, 1, \dots, B^c. \quad (39)$$

➤ *Class 1 backlogged customer-related probabilities:*

Backlogging of class 1 customers happens in stage 2 and 3. Equation (40) presents the probability of having n backlogged class 1 customers in the system.

Class 1 customers are backlogged in both waiting areas, B^c and B^1 . If there is enough space in B^c , class 1 customers are backlogged in B^c , when the system has no product in stock. Thus, the probability of having n backlogged class 1 customers when there is enough space in B^c is $\sum_{j=0}^{B^c} P_{n+R_2,j}^{SMA}$. Here, $\sum_{j=0}^{B^c} P_{n+R_2,j}^{SMA}$ is the sum of probabilities of states with n backlogged class 1 customers in B^c and the rest of B^c being occupied by class 2 customers. In the two dimensional probabilities, $i = n + R_2$ presents the number of products needed to satisfy n backlogged class 1 customers and fill the stock to meet the stock level of R_2 .

When there are not enough spaces in B^c to accommodate all n backlogged class 1 customers, only some of these customers can be backlogged in B^c and the remainders need to be backlogged in B^1 . Having n backlogged class 1 customers occurs with different combinations of backlogged class 1 customers in B^1 , k , and B^c , $n-k$, when k varies from 1 to $\min\{n, B^1\}$. $\min\{n, B^1\}$ indicates the maximum possible number of backlogged class 1 customers in B^1 when we have n backlogged class 1 customers in the system. Therefore, the probability of having n backlogged class 1 customers with $n-k$ backlogged class 1 customers in B^c , and k backlogged class 1 customers in B^1 can be presented as:

$\sum_{k=1}^{\min\{n, B^1\}} (P_{n-k+R_2, B^c-(n-k)}^{SMA} \cdot \tilde{P}_k^{SMA})$. This probability indicates that k of n backlogged class 1 customers are in B^1 , \tilde{P}_k^{SMA} , and $n-k$ class 1 customers in B^c , $P_{n-k+R_2, B^c-(n-k+R_2)}^{SMA}$. This is the probability that in the two-dimensional B&D process of stage 2 $i = n - k + R_2$, which means

$n - k$ backlogged class 1 customers in B^c , and $j = B^c - (n - k)$, which means $B^c - (n - k)$ backlogged class 2 customers have occupied the remaining of spaces in B^c . Since $i + j = (n - k + R_2) + B^c - (n - k) = B^c + R_2$, B^c is full.

In the case of $B^c < n \leq B^c + B^1$, we cannot backlog all n class 1 customers in B^c . Therefore, the first term, which indicates the probability of having n backlogged class 1 customers in B^c , has been removed in the second part of equation (40). In the second term, the lower bound of the summation indicates a situation in which B^c is full of class 1 customers and the rest of them, $n - B^c$, are backlogged in B^1 . Therefore,

$$Pb_n^{SMA}(S, R_2, B^c, B^1, B^2) = \begin{cases} \sum_{j=0}^{B^c} P_{n+R_2, j}^{SMA} + \sum_{k=1}^{\min\{n, B^1\}} (P_{n-k+R_2, B^c-(n-k+R_2)}^{SMA} \cdot \tilde{P}_k^{SMA}), & 0 < n \leq B^c \\ \sum_{k=n-B^c}^{\min\{n, B^1\}} (P_{n-k+R_2, B^c-(n-k+R_2)}^{SMA} \cdot \tilde{P}_k^{SMA}), & B^c < n \leq B^c + B^1 \end{cases} \quad (40)$$

➤ *Class 1 lost customers- related probabilities:*

When B^c and B^1 are full, arriving class 1 customers are rejected. This happens when the system is in stage 3 and $\tilde{B}_1(t) = B^1$. Furthermore, when $B^1 = 0$, rejection of class 1 customers occurs at any one of the rectangular states and the states of the last column in stage 2 shown in Figure 6, i.e., $B_1(t) + B_2(t) = B^c$. Therefore,

$$Pl^{SMA}(S, R_2, B^c, B^1, B^2) = \begin{cases} \tilde{P}_{B^1}^{SMA}, & B^1 > 0 \\ \sum_{j=0}^{B^c} P_{H(j), j}^{SMA} + \sum_{i=0}^{R_2-1} P_{i, B^c}^{SMA}, & B^1 = 0 \end{cases} \quad (41)$$

When $B^1 = 0$ in equation (41), the first term represents the probability of being in the rectangular states, and the second term represents the probability of being in one of the states in

the last column of the steady state diagram except state (R_2, B^c) . That state is considered in the first term as state $(H(B^c), B^c)$, since $H(B^c) = R_2$.

Because we do not have the number of rejected customers in our model, we use the rate of arriving class 1 customers, λ_1 , to find the class 1 customer rejection cost as $c_1 \lambda_1 Pl^{SMA}(S, R_2, B^c, B^1, B^2)$.

➤ *Class 2 lost customers-related probabilities:*

When the common waiting area, B^c , is full, arriving class 2 costumers are rejected. Thus, all states in the last column of the steady state diagram of Figure 6, (i, B^c) for $i=0, 1, \dots, R_2$, and rectangular states, $(H(j), j)$ for $j=0, 1, \dots, B^c$, represent the situation that arriving class 2 customers are rejected. Moreover, when the system is in any one of the states in stage 3, $\tilde{B}_1(t) = k$, for $k = 1, 2, \dots, B^1$, arriving class 2 customers are not accepted to be backlogged or satisfied. Therefore,

$$\bar{Pl}^{SMA}(S, R_2, B^c, B^1, B^2) = \sum_{j=0}^{B^c} P_{H(j), j}^{SMA} + \sum_{i=0}^{R_2-1} P_{i, B^c}^{SMA} + \sum_{k=1}^{B^1} \tilde{P}_k^{SMA}, \quad (42)$$

We should also consider state $(H(B^c), B^c) = (R_2, B^c)$ which is calculated in the first term of equation (42).

Based on probabilities presented in this section, we have all required probabilities to calculate the cost function defined in Chapter 5 under the SMA policy with any given values of S, R_2, B^c , and B^1 .

6.2. Complete Partitioning Policy

In order to analyze the CP policy, each stage of the system shown in Figure 2 is modeled as a B&D process, and then these stages are connected to one another. These stages are discussed in the following sections. We use the same approach that we used for the SMA policy to obtain the probability of each state.

6.2.1. Modeling Stage 1

In stage 1, $R_2 < I(t) \leq S$ and customers are served on a FCFS basis. In this stage, there are no backlogged customers, and the only change in the state of the system is the inventory level, so $I(t)$ is used as a state indicator. Because of no priority differences between customers in this stage, “Birth” happens with the rate $\lambda = \lambda_1 + \lambda_2$. Here, “Birth” represents serving an arriving customer with one product from the stock and sending a new order to manufacturer. “Death” occurs at the rate μ , which represents producing and storing one unit of product, and satisfying one manufacturer’s order. Therefore, to find the probability of having “ i ” product in stock, $P_i^{CP}, i = R_2 + 1, \dots, S - 1, S$, we use the following B&D process (Figure 7). All the steady state probabilities in this stage can be obtained in terms of probability of $(I(t) = R_2, B_2(t) = 0)$. This state is called of the CP policy and is denoted by r . Let denote the probability of the reference state by P_r^{CP} . This state is the connection point to stage 2.

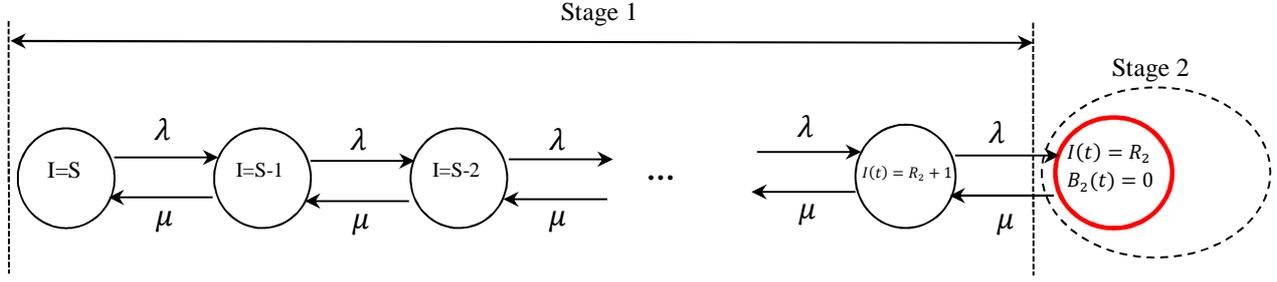


Figure 7: The steady state transition diagram for stage 1 under the CP policy.

Based on the diagram in Figure 7, steady state equations can be written as follows:

$$\lambda P_S^{CP} = \mu P_{S-1}^{CP}, \quad (43)$$

$$(\mu + \lambda) P_i^{CP} = \lambda P_{i+1}^{CP} + \mu P_{i-1}^{CP}, \quad i = R_2 + 1, R + 2, \dots, S - 1, \quad (44)$$

$$(\mu + \lambda) P_{R_2+1}^{CP} = \lambda P_{R_2+2}^{CP} + \mu P_r^{CP}. \quad (45)$$

Then, according to equation (43),

$$P_S^{CP} = \frac{\mu}{\lambda} P_{S-1}^{CP} = \frac{1}{\rho} P_{S-1}^{CP}. \quad (46)$$

By extending the equation (44) to $i=S-1$, and inserting it in equation (43), we can obtain the same equation for P_{S-1}^{CP} as $P_{S-1}^{CP} = \frac{1}{\rho} P_{S-2}^{CP}$ and in general:

$$P_{S-j}^{CP} = \frac{1}{\rho} P_{S-j-1}^{CP}, \quad j = 0, 1, \dots, S - (R_2 + 2). \quad (47)$$

Then, for $j = S - (R_2 + 2)$, we have $P_{R_2+2}^{CP} = \frac{1}{\rho} P_{R_2+1}^{CP}$, and by substituting $P_{R_2+2}^{CP}$ into equation (45), we obtain $P_{R_2+1}^{CP} = \left(\frac{1}{\rho}\right)^1 P_r^{CP}$. Therefore, we have $P_{R_2+2}^{CP} = \frac{1}{\rho} P_{R_2+1}^{CP} = \left(\frac{1}{\rho}\right)^2 P_r^{CP}$. By substituting

$P_{R_2+2}^{CP}$ into equation (45) for the rest of values of j , we derive all steady state probabilities in terms of P_r^{CP} as:

$$P_{R_2+i}^{CP} = \left(\frac{1}{\rho}\right)^i P_r^{CP}, \quad i = 1, 2, \dots, S - R_2. \quad (48)$$

Considering $P_{stage\ 1}^{CP}$ as a probability of being in stage 1, we have:

$$P_{stage\ 1}^{CP} = \sum_{i=R_2+1}^S P_i^{CP} = P_r^{CP} \sum_{i=R_2+1}^S \left(\frac{1}{\rho}\right)^i, \quad i = 1, 2, \dots, S - R_2. \quad (49)$$

6.2.2. Modeling Stage 2

In stage 2, $0 \leq I(t) \leq R_2$ and $B_1(t) + B_2(t) \leq B^1 + B^2$. In this stage, customers of class 2 are served with a new entering product if and only if $B_1(t) = 0$ and $I(t) = R_2$; otherwise, the product is used to serve backlogged class 1 customers or is stored. The B&D process that represents this stage is two dimensional. One dimension, i , is the number of orders needed to serve all current backlogged class 1 customers and increase the inventory to the stock level R_2 . As long as we are in stage 2, $i = B_1(t) + (R_2 - I(t))$ and the other dimension, j , is the number of orders needed to serve all backlogged class 2 customers. Thus, $j = B_2(t)$. As long as $i \leq R_2$, which means that $B_1(t) = 0$, the first dimension, i , represents the differences between the stock level and the rationing level of R_2 , $i = R_2 - I(t)$. When $i \geq R_2$, i represents $B_1(t) + R_2$. The maximum value for the j dimension is B^2 , a value that occurs when all spaces in B^2 are allocated to class 2 backlogs, which means $B_2(t) = B^2$. Under the CP policy, the number of spaces available for backlogging class 2 customers is independent of the backlogged class 1 customers. Furthermore, the maximum value for the i dimension is $B^1 + R_2$. This happens when $B_1(t) = B^1$, which means that $I(t) = 0$. Figure 8 shows the steady state diagram of stages 1 and 2.

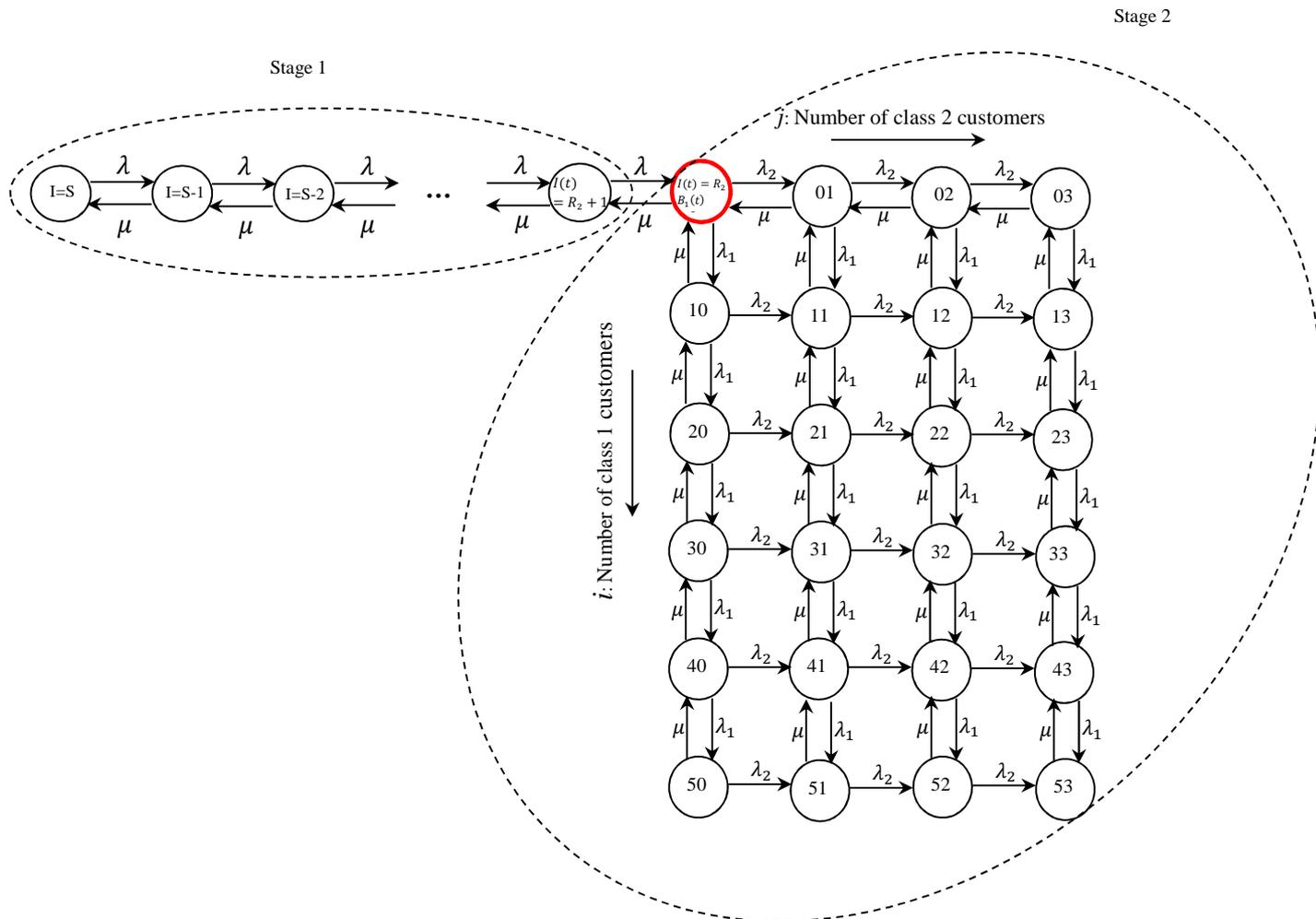


Figure 8: The steady state transition diagram for stages 1 and 2 under the CP policy with $R_2 = 2, B^1 = 3,$ and $B^2 = 3$.

According to the above discussion, in stage 2 we need to find the two dimensional probabilities of $P_{i,j}^{CP}, i = 0, \dots, H(j)$ and $j = 0, 1, \dots, B^c$. Based on the definition of $H(j)$ in Section 6.1.2, $H(j)$ is the maximum value for i when the number of backlogged class 2 customers in the system is j . But, because of the independence between the numbers of backlogged orders of class 1 and class 2 customers under the CP policy, $H(j)$ is constant. The maximum acceptable number of backlogged orders of class 1 customers in the CP policy is $H(j) = R_2 + B^1$.

The state $(i=0, j=0)$ represents the situation when $I(t) = R_2$, and there is no backlogged customer of class 2 in the system, $B_2(t) = 0$. This state is the reference state, denoted with red in Figures 7 and 8, and is the connection point to stage 1. All probabilities in stage 2 are calculated in terms of $P_r^{CP} = P_{0,0}^{CP}$ as we did in stage 1. By applying steady state equations of the B&D process in Figure 7 and using the method developed by Bondi (1989), we can find all steady state probabilities of stage 2, in terms of the probability of the reference state, P_r^{CP} . We start with the first column in the next section, and then the other columns before the last column are analyzed in Section 6.2.2.2. The last column has a different pattern. Derivation of the probabilities for that column comes separately.

6.2.2.1. Solving for the First Column

When we go through the first column in the steady state diagram for stage 2, similar to Bondi (1989), the following steady state equations are derived:

For the state of $(i=0, j=0)$:

$$(\lambda_1 + \lambda_2 + \mu)P_r^{CP} = \mu P_{01}^{CP} + \mu P_{10}^{CP} + \lambda P_{R_2+1}^{CP}. \quad (50)$$

For the rest of states in column 1 except the last state at the bottom:

$$(\lambda_1 + \lambda_2 + \mu)P_{i0}^{CP} = \lambda_1 P_{i-1,0}^{CP} + \mu P_{i+1,0}^{CP} \quad \text{or} \quad (\rho_1 + \rho_2 + 1)P_{i0}^{CP} = \rho_1 P_{i-1,0}^{CP} + P_{i+1,0}^{CP},$$

$$\text{for } i = 1, 2, \dots, H(0) - 1. \quad (51)$$

For the last state at the bottom of column 1, we have:

$$(\mu + \lambda_2)P_{H(0),0}^{CP} = \lambda_1 P_{H(0)-1,0}^{CP}. \quad (52)$$

Therefore,
$$P_{H(0),0}^{CP} = \frac{\lambda_1}{(\mu + \lambda_2)} P_{H(0)-1,0}^{CP} = \frac{\lambda_1}{(\mu + \lambda - \lambda_2)} P_{H(0)-1,0}^{CP}$$

$$= \frac{\lambda_1/\mu}{(\mu+\lambda-\lambda_1/\mu)} P_{H(0)-1,0}^{CP} = \frac{\rho_1}{1+\rho-\rho_1} P_{H(0)-1,0}^{CP} = A_1 P_{H(0)-1,0}^{CP}, \quad (53)$$

where $A_1 = \frac{\rho_1}{\rho+1-\rho_1}$.

By substituting equation (53) back into equation (51), we follow the next steps to obtain the P_{i_0}' 's.

When $i = H(0) - 1$, we have $(\rho_1 + \rho_2 + 1)P_{H(0)-1,0}^{CP} = \rho_1 P_{H(0)-2,0}^{CP} + P_{H(0),0}^{CP}$. By substituting equation (53) into this equation and considering $\rho = \rho_1 + \rho_2$, we obtain the following equation:

$$(\rho_1 + \rho_2 + 1)P_{H(0)-1,0}^{CP} = \rho_1 P_{H(0)-2,0}^{CP} + A_1 P_{H(0)-1,0}^{CP},$$

so, $(\rho + 1 - A_1)P_{H(0)-1,0}^{CP} = \rho_1 P_{H(0)-2,0}^{CP}$,

$$\text{and then, } P_{H(0)-1,0}^{CP} = \frac{\rho_1}{\rho+1-A_1} P_{H(0)-2,0}^{CP} = A_2 P_{H(0)-2,0}^{CP}, \quad (54)$$

where $A_2 = \frac{\rho_1}{\rho+1-A_1}$.

Similar to Bondi (1989), we continue this method for the other values of i . When $i = H(0) - 2$ in equation (44), we have:

$$(\rho_1 + \rho_2 + 1)P_{H(0)-2,0}^{CP} = \rho_1 P_{H(0)-3,0}^{CP} + P_{H(0)-1,0}^{CP}. \quad (55)$$

By substituting $P_{H(0)-1,0}^{CP}$ from equation (47) into equation (55), we obtain:

$$(\rho + 1)P_{H(0)-2,0}^{CP} = \rho_1 P_{H(0)-3,0}^{CP} + A_2 P_{H(0)-2,0}^{CP}.$$

$$\text{So, } (\rho + 1 - A_2)P_{H(0)-2,0}^{CP} = \rho_1 P_{H(0)-3,0}^{CP} \Rightarrow P_{H(0)-2,0}^{CP} = \frac{\rho_1}{\rho+1-A_2} P_{H(0)-3,0}^{CP} = A_3 P_{H(0)-3,0}^{CP}, \quad (56)$$

where $A_3 = \frac{\rho_1}{\rho+1-A_2}$.

We apply induction to prove that by using equation (51), in general we get:

$$P_{H(0)-n,0}^{CP} = A_{n+1}P_{H(0)-n-1,0}^{CP} \text{ where } A_n = \frac{\rho_1}{(\rho+1-A_{n-1})} \text{ and } n = 1, 2, \dots, H(0) - 1, \quad (57)$$

which is similar to equation (A18) in (Bondi, 1989).

Therefore, for $n = H(0) - 1$, we have:

$$P_{H(0)-(H(0)-1),0}^{CP} = \frac{\rho_1}{(\rho+1-A_{(H(0)-1+1)})} P_{H(0)-(H(0)-1)-1,0}^{CP} \text{ or } P_{1,0}^{CP} = A_{H(0)}P_{0,0}^{CP} = P_r^{CP}. \quad (58)$$

As seen in equation (58), $P_{1,0}^{CP}$ has been calculated in terms of the probability of the reference state, $(0,0)$. As the probability of state $(i,0)$ has been presented in terms of the probability of state $(i-1,0)$, by substituting the probability of the latter state $(i-1,0)$ into the probability of state $(i,0)$ for $i = 1, 2, \dots, H(0) - 1$, we obtain the following equations for the probability of state $(i,0)$, which can be proved by induction:

$$P_{i0}^{CP} = P_r^{CP} \prod_{n=H(0)-i+1}^{H(0)} A_n, \quad i = 1, 2, \dots, H(0) \quad . \quad (59)$$

Again, this is similar to equation (A18) in Bondi (1989).

Therefore, all the probabilities of states in the first column have been presented in terms of P_r^{CP} , the probability of the “reference state”.

6.2.2.2. Solving for the Middle Columns

In order to find the probabilities of states in the middle columns, P_{ij}^{CP} ($0 < j < B^2 - 1$), in terms of P_r^{CP} , similar to Bondi (1989), we start with the second column and move to the next columns in order, until we reach the last column. In this section, we use a common pattern for columns between the first column and the last column. The last column is analysed in the next section.

For each column in this section, we start with the last state, $(H(j), j)$, for $0 < j < B^2$. By writing the steady state equation for the last state, we can derive its probability, $P_{H(j),j}^{CP}$, in terms of the states that are connected to this state, $P_{H(j-1)-1,j-1}^{CP}$ and $P_{H(j)-1,j}^{CP}$ as follows:

$$(\mu + \lambda_2)P_{H(j),j}^{CP} = \lambda_1 P_{H(j)-1,j}^{CP} + \lambda_2 P_{H(j),j-1}^{CP}, \quad j = 1, 2, \dots, B^2 - 1, \quad (60)$$

or
$$P_{H(j),j}^{CP} = \frac{\lambda_1}{\mu + \lambda_2} P_{H(j)-1,j}^{CP} + \frac{\lambda_2}{\mu + \lambda_2} P_{H(j),j-1}^{CP}, \quad j = 1, 2, \dots, B^2 - 1,$$

which is similar to equation (A7) in Bondi (1989).

As we analyze columns in order, all probabilities of the previous column, $P_{H(j),j-1}^{CP}$, are available in terms of P_r^{CP} , when we are analyzing column j . The only unknown probability is $P_{H(j)-1,j}^{CP}$. By writing steady state equations for the rest of the states in column j , from state $(H(j)-1, j)$ until $(1, j)$, Bondi (1989) has derived the following recursive equation:

$$P_{H(j)-1,j}^{CP} = \frac{\rho_1 P_{H(j)-(i+1),j}^{CP}}{B_{i+1}} + \rho_2 \sum_{n=H(j)-i}^{H(j)} \left\{ \frac{P_{n,j-1}^{CP}}{\prod_{k=H(j)-n+1}^{i+1} B_k} \right\}, \quad (61)$$

$$i = 0, 1, 2, \dots, H(m) - 1; j = 1, 2, \dots, B^2 - 1,$$

where
$$B_0 = 1, \quad (62)$$

$$B_1 = \rho_2 + 1,$$

$$B_n = \rho + 1 - \frac{\rho_1}{B_{n-1}}, n = 1, 2, 3, \dots$$

As seen in equation (61), when we are calculating probabilities for column j , the second term gives us probabilities of states in the previous column which has already been derived in terms of P_r^{CP} .

The first term is the probability of the previous state in the same column and recursively this term is calculated in terms of P_{0j}^{CP} .

In order to derive the probability of the first state in column j , P_{0j}^{CP} , we consider the steady state equation for the first state in the previous column, $(0, j-1)$.

In the second column, $j=1$, we can derive P_{01}^{CP} in terms of P_r^{CP} from equation (50) as follows:

$$\mu P_{01}^{CP} = (\lambda_1 + \lambda_2 + \mu)P_r^{CP} - \mu P_{10}^{CP} - \lambda P_{R_2+1}^{CP}, \quad (63)$$

which is similar to equation (A2) in Bondi (1989).

As we have $P_{R_2+1}^{CP}$ from equation (41) in stage 1 with $i=1$, and P_{10}^{CP} from equation (48) with $i=1$, we rewrite this equation as $\mu P_{01}^{CP} = (\lambda_1 + \lambda_2 + \mu)P_r^{CP} - \mu P_r^{CP} A_{H(0)-1} - \lambda \left(\frac{1}{\rho}\right)^1 P_r^{CP}$. So, we have P_{01}^{CP} in terms of P_r^{CP} :

$$P_{01}^{CP} = (\rho + 1)P_r^{CP} - P_r^{CP} A_{H(0)-1} - \rho \left(\frac{1}{\rho}\right)^1 P_r^{CP} = [\rho - A_{H(0)-1}]P_r^{CP}. \quad (64)$$

In order to derive the steady state probability of state $(0, j)$, $j = 2, 3, \dots, B^2 - 1$, we write the steady state equation of state $(0, j-1)$ as:

$$(\lambda_1 + \lambda_2 + \mu)P_{0,j-1}^{CP} = \mu P_{0j}^{CP} + \mu P_{1,j-1}^{CP} + \lambda_1 P_{0,j-2}^{CP}, \quad J = 2, 3, \dots, B^2 - 1. \quad (65)$$

Then, $\mu P_{0j}^{CP} = (\lambda_1 + \lambda_2 + \mu)P_{0,j-1}^{CP} - \mu P_{1,j-1}^{CP} - \lambda_1 P_{0,j-2}^{CP}$. Therefore,

$$P_{0j}^{CP} = (\rho + 1)P_{0,j-1}^{CP} - P_{1,j-1}^{CP} - \rho_1 P_{0,j-2}^{CP}, \quad J = 2, 3, \dots, B^2 - 1. \quad (66)$$

Now, we have P_{0j}^{CP} in terms of the probabilities of the states in previous columns that have already been derived in terms of the probability of the reference state, P_r^{CP} . Therefore, we have P_{0j}^{CP} in

terms of P_r^{CP} . Afterward, by substituting P_{0j}^{CP} into the first term of equation (50) for state (l, j) we derive P_{1j}^{CP} in terms of P_r^{CP} . Then, by substituting P_{1j}^{CP} and using the same pattern until $P_{H(j),j}^{CP}$, we derive all probabilities in equation (50) in terms of P_r^{CP} .

Therefore, we have derived the steady state probabilities of P_{ij}^{CP} for $i=0, 1, \dots, H(j)$ and $j=1, 2, \dots, B^2 - 1$ in terms of P_r^{CP} , the probability of the reference state.

6.2.2.3. Solving for the Last Column

In the last column, we start to derive steady state probabilities from state $(0, B^2)$ until state $(H(B^2), B^2) = (R_2 + B^1, B^2)$ as P_{i,B^2}^{CP} ($0 \leq i \leq H(B^2)$), in order, similar to Bondi (1989).

By writing the steady state equation of state $(0, B^2)$, we derive P_{0,B^2}^{CP} :

$$(\lambda_1 + \lambda_2 + \mu)P_{0,B^2-1}^{CP} = \mu P_{0,B^2}^{CP} + \mu P_{1,B^2-1}^{CP} + \lambda_1 P_{0,B^2-2}^{CP}, \quad (67)$$

Therefore,

$$P_{0,B^2}^{CP} = (\rho + 1)P_{0,B^2-1}^{CP} - P_{1,B^2-1}^{CP} + \rho_1 P_{0,B^2-2}^{CP}, \quad (68)$$

So, we have P_{0,B^2}^{CP} in terms of P_{0,B^2-1}^{CP} , P_{1,B^2-1}^{CP} , and P_{0,B^2-2}^{CP} , all of which have been derived in the previous section.

By writing the steady state equation for state $(0, B^2)$, we can derive the steady state probability of P_{1,B^2}^{CP} as follows:

$$(\lambda_1 + \mu)P_{0,B^2}^{CP} = \mu P_{1,B^2}^{CP} + \lambda_1 P_{0,B^2-1}^{CP}. \quad (69)$$

Thus,

$$P_{1,B^2}^{CP} = (\rho_1 + 1)P_{0,B^2}^{CP} - \rho_1 P_{0,B^2-1}^{CP}. \quad (70)$$

P_{0,B^2}^{CP} has been presented in equation (52); P_{0,B^2-1}^{CP} , which is from the previous column, has been derived in the previous section.

The steady state equation for state $(i - 1, B^2)$, $i = 2, 3, \dots, H(B^2)$ can be presented as follows:

$$(\lambda_1 + \mu)P_{i-1,B^2}^{CP} = \mu P_{i,B^2}^{CP} + \lambda_2 P_{i-1,B^2-1}^{CP} + \lambda_1 P_{i-2,B^2}^{CP}, \quad i = 2, 3, \dots, H(B^2). \quad (71)$$

$$\text{Thus, } P_{i,B^2}^{CP} = (\rho_1 + 1)P_{i-1,B^2}^{CP} - \rho_2 P_{i-1,B^2-1}^{CP} - \rho_1 P_{i-2,B^2}^{CP}, \quad i = 2, 3, \dots, H(B^2), \quad (72)$$

where the second term is from the previous column and has been derived in terms of P_r^{CP} . By using equations (68) and (70), in addition to using equation (72) recursively, we derive all probabilities of P_{i,B^2}^{CP} for $i = 2, 3, \dots, H(B^2)$ in terms of P_r^{CP} .

Now, we have all probabilities under CP policy in terms of P_r^{CP} . As we considered all possible states, sum of all probabilities is equal to one. Then, we can obtain P_r^{CP} , and calculate probability of the states in stages 1 and 2.

6.2.3. Probabilities of the Cost Function

Now, we derive the probabilities required to obtain the cost function under the CP policy.

➤ *Holding cost-related probabilities:*

When $R_2 < I(t) \leq S$, the system is in stage 1. Therefore, we obtain the probability that inventory level $R_2 < I(t) \leq S$ using equation (48), as given in the second part of Ph_n^{CP} in equation (73).

Furthermore, when $0 \leq I(t) \leq R_2$, the system is in stage 2, and it is possible to have backlogged class 2 customers. Based on the two dimensional state space that we define in stage

2, the first part of Ph_n^{CP} is given in equation (73), using the marginal probabilities from equations (59), (61), (70), and (72).

$$Ph_n^{CP}(S, R_2, B^c, B^1, B^2) = \begin{cases} \sum_{j=0}^{B^c} P_{R_2-n,j}^{CP}, & 1 \leq n \leq R_2 \\ P_n^{CP} = \left(\frac{1}{\rho}\right)^{n-R_2} P_r^{CP}, & R_2 + 1 \leq n \leq S \end{cases} \quad (73)$$

➤ *Class 2 backlogged customer-related probabilities:*

Class 2 customers are backlogged when the system is in stage 2, Figure 2. Note that the second element of the state space (i, j) in stage 2 is the number of backlogged class 2 customers. Then, to obtain the probability of having n backlogged class 2 customers, we use the marginal probabilities from equations (59), (61), (70), and (72) obtained for stage 2 as follows:

$$\overline{Pb}_n^{CP}(S, R_2, B^c, B^1, B^2) = \sum_{i=0}^{H(n)} P_{i,n}^{CP}, \quad 0 \leq n \leq B^2, \quad (74)$$

where $H(n) = R_2 + B^1$ is the maximum of the i dimension under the CP policy as in Figure 8.

➤ *Class 1 backlogged customer-related probabilities:*

Backlogging of class 1 customers happens in stage 2. Because of the definition of the i dimension in Section 6.2.2, to consider states that present n class 1 customers in the system, we need to go through the row with $i = R_2 + n$. That is because having n backlogged class 1 customers in the class 1 customer waiting area occurs with different possible numbers of backlogged class 2 customers. Thus, equation (75) presents the probability of having n backlogged class 1 customers in the system.

$$Pb_n^{CP}(S, R_2, B^c, B^1, B^2) = \sum_{j=0}^{B^2} P_{n+R_2,j}^{CP}, \quad 1 \leq n \leq B^1. \quad (75)$$

➤ *Class 1 lost customer-related probabilities:*

When B^1 is full, arriving class 1 customers are rejected. This happens when the system is in stage 2 and $B_1(t) = B^1$. This situation corresponds to the lowest row in the state diagram given in Figure 8 under the CP policy. Under that policy, the waiting area for class 1 customers is isolated from the waiting area of class 2 customers. Then, any state in the row with $i = R_2 + B^1$ presents a situation that B^1 is full and arriving class 1 customers are rejected. Therefore,

$$Pl^{CP}(S, R_2, B^c, B^1, B^2) = \sum_{j=0}^{B^2} P_{H(j),j}^{CP}, \quad (76)$$

where $H(j) = R_2 + B^1$.

Because we do not have the number of rejected customers in our model, we use the rate of arriving class 1 customer, λ_1 , to find the class 1 customer rejection cost as $c_1 \lambda_1 Pl^{CP}(S, R_2, B^c, B^1, B^2)$.

➤ *Class 2 lost customer-related probabilities:*

When the waiting area of class 2 customers, B^2 , is full, arriving class 2 costumers are rejected. Thus, all states in the last column of the steady state diagram of Figure 8, (i, B^2) for $i=0, 1, \dots, H(B^1)$, represent situations that arriving class 2 customers are rejected. Therefore,

$$\bar{Pl}^{CP}(S, R_2, B^c, B^1, B^2) = \sum_{i=0}^{H(B^1)} P_{i,B^c}^{CP}, \quad (77)$$

where $H(B^1) = R_2 + B^1$.

Based on probabilities presented in this section, we have all the probabilities required to calculate the cost function defined in Chapter 5 under the CP policy, with given values of S, R_2, B^1 , and B^2 .

6.3. Modeling Under the MR Policy

In this section, we analyze the system under the MR policy. Here, the buffer space for backlogged class 1 and 2 customers is infinite. Therefore, we do not have any rejection. As long as $R_2 < I(t) \leq S$, both classes of customers are served. When the inventory level is less than or equal to R_2 , arriving class 2 customers are backlogged, but class 1 customers are served as long as the inventory level is positive. In other words, we can look at the MR policy as a special case of the SMA policy when B^c approaches infinity, or a special case of the CP policy when B^1 and B^2 approach infinity.

de Vericourt et al. (2001) show that the MR policy is optimal among the other policies with infinite buffer spaces, to minimize the cost introduced in Chapter 5. Implementing that policy is thus more cost effective than the SP and FCFS policies. The MR policy has been modeled by de Vericourt et al. (2001) for n classes of customers with n rationing levels. They determine the optimal base-stock level and rationing levels for n classes of customers so that the cost of the system is minimized as follows:

$$S = \left[\frac{\ln \frac{\rho'_1(h+b_2)}{\rho'_1(h+b_1)+(1-\rho'_1)}}{\ln \rho'_1} \right], \quad (78)$$

$$R_2 = \left[\frac{\ln \frac{\rho'_2 h}{\rho'_2(h+b_2)+(1-\rho'_2)(g_1-(h+b_2)R_2)}}{\ln \rho'_2} \right], \quad (79)$$

where

$$g_1 = \left(R_2 - \frac{\rho'_1}{1-\rho'_1} \right) (h + b_2) + \left(\frac{\rho'_1}{1-\rho'_1} \right) (h + b_1) \rho_1^{R_2}, \quad \lambda'_k = \sum_{i=1}^k \lambda_i, \quad \text{and} \quad \rho'_k = \sum_{i=1}^k \rho_i.$$

Using the optimal S^* and R_2^* , de Vericourt et al. (2001) derive the minimum cost of the system

$$C^{MR} = \left(S - \frac{\rho_2'}{1-\rho_2'} \right) h + \left(g_1 - \left(R_2 - \frac{\rho_2'}{1-\rho_2'} \right) (h + b_2) \right) \rho_2^{S-R_2}. \quad (80)$$

Therefore, we can calculate the optimal base-stock level, S^* , the optimal rationing level, R_2^* , and the minimum cost of the system, C^{MR} , using equations (78), (79), and (80), respectively, with any specific holding cost, h , backlog costs, b_1 and b_2 , the average production time, $\frac{1}{\mu}$, and the rates of arrival, λ_1 and λ_2 .

6.4. Modeling Under the LS Policy

Under the LS policy, the system does not have any buffer spaces. All arriving customers are thus either served or rejected. Arriving class 1 customers are served as long as the inventory level is positive. To serve class 2 customers, the inventory level needs to be greater than the rationing level R_2 . Otherwise, they are rejected. Thus, we do not consider backlog cost in this section. As we mentioned in Chapter 5, the LS policy performs to the case of $(S, R_2, B^c, B^1, B^2) \rightarrow (S, R_2, 0, 0, 0)$.

Ha (1997) analyses the problem under the LS policy. He derives expected values of inventory level and the probabilities of rejecting class 1 and 2 customers, Pl^{CP} and $\bar{P}l^{CP}$, to minimize the total cost of the system, C^{LS} , as follows:

E^{LS} (number of production in stock)

$$\begin{aligned} &= S - \frac{\frac{\rho(1-\rho_1)}{1-\rho} \{1 - \rho^{S-R_2} - (1-\rho)(S-R_2)\rho^{S-R_2-1}\}}{(1-\rho_1)(1-\rho^{S-R_2}) + (1-\rho)\rho^{S-R_2}(1-\rho_1^{R_2+1})} \\ &\quad - \frac{\frac{\rho_1(1-\rho)}{1-\rho_1} \left(\frac{\rho}{\rho_1}\right)^{S-R_2} \{\rho_1^{S-R_2} - \rho_1^{S+1} + (1-\rho_1)[(S-R_2)\rho_1^{S-R_2-1} - (S+1)\rho_1^S]\}}{(1-\rho_1)(1-\rho^{S-R_2}) + (1-\rho)\rho^{S-R_2}(1-\rho_1^{R_2+1})}, \end{aligned}$$

$$Pl^{LS}(S, R_2, 0, 0, 0) = \frac{(1 - \rho)(1 - \rho_1)\rho^{S-R_2}\rho_1^{R_2}}{(1 - \rho_1)(1 - \rho^{S-R_2}) + (1 - \rho)\rho^{S-R_2}(1 - \rho_1^{R_2+1})},$$

$$\bar{Pl}^{LS}(S, R_2, 0, 0, 0) = \frac{(1 - \rho)\rho^{S-R_2}(1 - \rho_1^{R_2+1})}{(1 - \rho_1)(1 - \rho^{S-R_2}) + (1 - \rho)\rho^{S-R_2}(1 - \rho_1^{R_2+1})}.$$

Therefore, the total cost of the system under the LS policy for given S and R_2 can be obtained as:

$$C^{LS}(S, R_2, 0, 0, 0)$$

$$= hE^{LS}(\text{number of production in stock}) + \lambda_1 c_1 Pl^{LS}(S, R_2, 0, 0, 0) + \lambda_2 c_2 \bar{Pl}^{LS}(S, R_2, 0, 0, 0).$$

Now, we have all required information to calculate the cost function introduced in Chapter 5 under the SMA, CP, MR, and LS policies. In the next chapter, we present some numerical examples to compare those policies.

Chapter 7

7. Numerical Results

We now use the formulation of the cost function derived in previous chapters to investigate the performance of each policy by calculating the respective system cost of each. Then, we compare the numerical results of different policies to that of the optimal policy characterized by Benjaafar et al. (2010). Finally, we discuss the advantageous of our model and the SMA policy relative to the optimal policy.

Based on the cost function, $C^*(S, R_2, B^c, B^1, B^2)$, presented in Chapter 5 and the probabilities derived under the SMA and CP policies in Chapter 6, we have all requirements to calculate the cost of the system presented in Sections 6.1.4 and 6.2.3. Additionally, we have the exact solutions of the MR and LS policies in Sections 6.3 and 6.4, respectively.

We do a search on the independent variables, S, R_2, B^c, B^1 , and B^2 , of the cost function to find the minimum cost under each policy. We do not search on B^2 under the SMA policy nor on B^c under the CP policy. To obtain the optimal values of decision variables, we assume that the cost function is convex and vary all variables starting from zero. We use the model of the SMA policy to obtain numerical results of the MR and LS policies, as special cases of the SMA policy. Those numerical results will be the same as exact solutions from Sections 6.3 and 6.4, respectively.

As discussed in Chapter 2, Benjaafar et al. (2010) find the optimal policy of the problem considered in this research. They compare the optimal policy with other heuristics including the MR, LS, and CP policies numerically. They show that the optimal policy is state-dependent, i.e., the optimal

inventory and backlog levels depend on the state of the system. We calculate the relative percentage gap between the MR, LS, CP, and SMA policies and the optimal policy as follows:

$$\Delta^{\bullet} \triangleq \frac{C^{\bullet} - C^{Opt}}{C^{Opt}} \times 100.$$

We present the numerical results in four categories in Tables 1 to 4, similar to Benjaafar et al. (2010), by changing the holding cost, h ; the utilization of the system, $\hat{\rho} = \frac{\alpha(\lambda_1 + \lambda_2)}{\mu}$ when α varies; the rejection cost of class 1 customers, c_1 ; and the backlogged cost of class 1 customers, b_1 . Columns 2 and 3 show the relative gap percentage between the optimal policy and the MR and LS policies, respectively. Under the CP policy, the numerical results of DP from Benjaafar et al. (2010) and our B&D model have been presented.

Table 1: The relative gap % between heuristics and the optimal policy when h varies.

h	MR	LS	CP		SMA B&D
			DP	B&D	
0.1	24.1	11.19	0.18	0.18	0.00
0.2	27.22	14.07	0.29	0.29	0.00
0.3	29.11	16.43	0.21	0.21	0.00
0.4	30.44	18.92	0.24	0.24	0.00
0.5	31.41	20.5	0.3	0.3	0.00
1	33.99	29.04	0.29	0.29	0.00
1.5	35.20	36.45	0.35	0.35	0.00
2.0	35.77	43.39	0.32	0.32	0.00
3.0	36.37	55.92	0.32	0.32	0.00
5.0	36.83	78.39	0.41	0.41	0.00
10.0	36.82	124.29	0.46	0.46	0.00

$$\mu = 1, \lambda_1 = 0.4, \lambda_2 = 0.5, c_1 = 500, c_2 = 250, b_1 = 10, b_2 = 5.$$

Table 2: The relative gap % between heuristics and the optimal policy when $\hat{\rho}$ varies.

$\hat{\rho}$	MR	LS	CP		SMA
			DP	B&D	B&D
0.6	0	110.1	0	0	0.00
0.7	0.13	86.83	0.01	0.01	0.00
0.8	3.38	55.21	0.13	0.13	0.01
0.9	33.99	29.04	0.29	0.29	0.00
0.92	53.22	25.34	0.36	0.36	0.00
0.94	86.34	21.97	0.42	0.42	0.00
1.0	154.77	18.97	0.26	0.26	0.00
1.0	364.31	16.19	0.33	0.33	0.00
1.0	784.80	14.86	0.22	0.22	0.00

$$\mu = 1, \lambda_1 = 0.4, \lambda_2 = 0.5, c_1 = 500, c_2 = 250, h = 1, b_1 = 10, b_2 = 5.$$

Table 3: The relative gap % between heuristics and the optimal policy when c_1 varies.

c_1/c_2	MR	LS	CP		SMA
			DP	B&D	B&D
1	36.08	18.75	0.82	0.82	0.03
1.1	34.46	19.28	0.26	0.26	0.04
1.2	34.11	20.89	0.25	0.25	0.02
1.4	33.99	24.34	0.28	0.28	0.00
1.6	33.99	25.98	0.29	0.29	0.00
1.8	33.99	27.51	0.29	0.29	0.00
2.0	33.99	29.04	0.29	0.29	0.00
3.0	33.99	33.84	0.29	0.29	0.00
4.0	33.99	36.88	0.29	0.29	0.00
5.0	33.99	39.62	0.29	0.29	0.00
10.0	33.99	45.70	0.29	0.29	0.00

$$\mu = 1, \lambda_1 = 0.4, \lambda_2 = 0.5, c_2 = 250, h = 1, b_1 = 10, b_2 = 5.$$

Table 4: The relative gap % between heuristics and the optimal policy when b_1 varies.

b_1/b_2	MR	LS	CP		SMA
			DP	B&D	B&D
1	35.82	35.07	0.36	0.36	0.00
1.1	35.61	34.43	0.38	0.38	0.00
1.2	35.41	33.79	0.41	0.41	0.00
1.4	35.01	32.54	0.39	0.39	0.00
1.6	34.61	31.3	0.36	0.36	0.00
1.8	34.23	30.09	0.31	0.31	0.00
2.0	33.99	29.04	0.29	0.29	0.00
3.0	33.03	24.64	0.53	0.36	0.06
4.0	32.33	22.48	1.36	0.35	0.12
5.0	31.72	20.45	1.39	0.27	0.09
10.0	30.60	14.77	0.99	0.30	0.30

$$\mu = 1, \lambda_1 = 0.4, \lambda_2 = 0.5, c_1 = 500, c_2 = 250, h = 1, b_2 = 5.$$

Referring to the numerical results in Tables 1-4, the MR and LS policies perform poorly. The average relative gap under the MR and LS policies are 63.10% and 35.55%, respectively. These averages are much bigger than the ones under the CP/DP and SMA policies which are 0.38% and 0.02%, respectively. Since the MR and LS policies are pure backlog and pure lost policies, they do not get benefit from combination of backlogging and rejecting mechanisms to reduce the cost of the system. Performance of the MR policy deteriorates by increasing the backlog cost as it is shown in Table 4. From Table 3, performance of the LS policy gets worse when the rejection cost is higher.

According to the numerical results presented in Tables 1 to 4, under the CP policy, the relative gap of our B&D process is different from the DP model given in Benjaafar et al. (2010) with the maximum of 1.12%, in only four cases, and the average of 0.07%; these deviations would be due to numerical errors. The results of our modeling are the same as the ones obtained using simulation and manual calculations. In DP, the curse of dimensionality is an important factor that makes using

DP difficult for high-dimensional problems. However, our approach can be applied to those high-dimensional problems.

The minimum cost of the system under the SMA policy in 81% of cases is identical to the optimal cost of the system obtained by Benjaafar et al. (2010). The maximum relative gap is 0.3% with the average of 0.02%. Numerical results illustrate that the SMA policy is the closest to optimality, compared to the other policies discussed in this thesis. By considering the curse of dimensionality and dynamic levels of the optimal policy obtained by Benjaafar et al. (2010), we find that the SMA policy is a very good approximation of the optimal policy for the practical situations.

Chapter 8

8. Conclusion

In this research, we modeled a make-to-stock queueing system with a rationing level and two classes of customers who can be backlogged or rejected depending on the level of inventory and the number of backlogged customers. Based on the associated costs of the system including holding, backlogging, and rejecting, we defined a cost function for the system. Furthermore, we applied a combination of one- and two- dimensional B&D processes to model the system under the SMA, CP, MR, and LS allocation policies. Then, we derived the probabilities of the model's states to calculate the cost function. By minimizing that cost, we obtained the optimal base-stock and the rationing level, and the optimal capacity of the system buffers.

By comparing the numerical results of the cost function under each policy with the exact solutions and DP modeling of the CP and optimal policies, we demonstrated that the SMA policy with the static levels is a very good approximation to the optimal policy. The SMA policy, modeled using a B&D process, can be applied to high-dimensional problems without the curse of dimensionality.

Our numerical results show that the MR and LS policy perform poorly. The reason is that the MR policy does not get benefit from limiting backlog customers to reduce the cost of the system. This deteriorates the performance when the backlogging cost of one or both classes of customers are high. The LS policy is a pure lost policy. When the rejection cost is high, its performance is getting worse since this policy does not get benefit from backlogging customers.

Future research can extend this idea in different directions. Our problem can be modeled for more than two classes of customers. In this case, there will be several rationing levels and specified areas of the buffer for each class of customers. Furthermore, the distributions of arrivals and the production times can be general. For example, Abouee-Mehrzi et al. (2012) present a model for a Multiclass make-to-stock queue with general production times under the MR policy. That model can be extended to the SMA policy.

Considering transportation time from the manufacturing facility to the inventory location can be another avenue for future research. Going towards the real world applications, the model should be extended to the systems with several manufacturers and retailers. Moreover, modeling systems with batch arrivals could be another avenue for future work.

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