Quantum Reference Frames
and
The Poincaré Symmetry

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The concept of a reference frame has been a part of the physical formalism since the early days of physics. The reference frame was always understood as a physical system with respect to which states of other physical systems are described. With the adoption of quantum theory, our understanding of physical systems has fundamentally changed, yet we still rely on classical systems for the definition of reference frames. If we acknowledge the fact that all systems are quantum mechanical, including reference frames, then we have to be able to describe quantum systems relative to other quantum systems.

In this thesis we develop a framework for integrating quantum reference frames into the formalism of quantum mechanics. We show how the description of states and measurements has to be refined in order to account for the quantum nature of the reference frame. Initially, we study the implications of the new description with a quantum reference frame associated with a generic compact group. Then, we apply the same approach to the study of relativistic quantum reference frames associated with the Poincaré group, which is not compact.

Our findings for the generic compact group include an analysis of how well a quantum system defines a frame of reference for other systems. Also in the generic case we analyze the effects of a measurement on a quantum reference frame. In the relativistic case we first find how the massive representations of the Poincaré group decompose into irreducible representations. This is a key problem in the analysis of quantum reference frames. Finally, we analyze a relativistic quantum reference frame that is used for measuring the total momentum. We will show how the relativistic measurements of total momentum should be described with respect to a quantum reference frame. We will also show how these measurements affect the quantum reference frame.

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"The medium is the message"
– Marshall McLuhan

1 Introduction

1.1 Background and motivation

It was argued by R. Landauer (among others) that "*Information is physical*" [1]. This statement asserts the idea that information only exists as a state of the physical medium that carries it. A vivid way to illustrate this assertion is with the following communication problem.

Consider an explorer named Bob who got lost in the ocean. Bob has a radio and he contacts his friend Alice to guide him to safety. Even though Alice knows the location of a safe harbor, and she can send Bob an unlimited amount of information, she cannot help unless they both have access to a common frame of reference. The reason for that is because one cannot encode in speech or in a bit-string alone, information specifying a direction or location in space. In order to encode this information, Alice needs additional communication resource, something that provides a common frame of reference. Such resource can be the magnetic field of the Earth, distant stars, or a way to parallel transport a gyroscope to Bob. Information that can only be encoded in a state of a certain system, and cannot be expressed as an abstract bit-string, is called *unspeakable information*[1], a term coined by A. Peres et al [2]. The fact that not all information can be encoded in an arbitrary medium is what makes information physical.

The fundamental concept that allows us to describe physical reality with abstract information is the concept of a *reference frame* (RF). Once Alice communicates a reference frame to Bob, she can encode directions and distances abstractly with numbers that obtain meaning in that RF. When describing physical systems, reference frames are the objects to which we assign all the unspeakable information so that we can describe everything else abstractly, that is with *speakable information*.

In everyday life, reference frames are abundant and it is easy to take them for granted. We are encoding velocities, locations and direction with respect to a common RF that is the earth, the sun and other celestial bodies. We are encoding time using a frequency of some naturally oscillating systems such as the solar system or an atom. Almost every description of the physical reality that we have, be it a clock or a textbook on particle physics, implicitly invokes a reference frame but then neglects to include it in the description. Of course in daily life this omission makes no difference

[1] One can argue that unspeakable information is in fact quantum information. After all, all systems are ultimately quantum so information that can only be encoded in a physical system is quantum information. That is true, yet unspeakable information has a more general meaning. It specifies information encoded in a state of a physical system regardless of whether the physics is quantum, classical or otherwise.

as long as we know what the reference frames are up to an acceptable degree of accuracy. If Bob wants to navigate the ocean then a compass that resolves direction of the magnetic pole up to 1000th of a degree should be good enough. Nevertheless, Bob can not expect to have an arbitrary resolution. The reason is not only because his measurement apparatus, the compass, has a finite resolution, but also because his reference frame, the magnetic field, has an inherent uncertainty in the microscopic limit. Neglecting to integrate the reference frames into our description of reality, gives us an effective picture that is accurate for many purposes, but one that is fundamentally incomplete.

In order to integrate reference frames into the physical formalism we have to put them on the same footing as every other system in our description. Arguably the most interesting consequences of this integration arise from the quantum nature of all systems, including the reference frames. If Alice were to send Bob a quantum version of a gyroscope, a spin system, then Bob's ability to infer the direction specified by that system would greatly depend on that system's total spin. In the macroscopic limit a spin system can be described in classical terms as a magnet which makes it a valid RF of direction. On the other hand, if it was a spin $1/2$ system, then even with the best state estimation techniques, Bob would not be able to infer with high probability the direction that Alice encoded in the state of that spin. This demonstrates the need to extend the notion of a reference frame to a new concept, the *quantum reference frame* (QRF).

Quantum reference frames are reference frames represented by quantum systems. This concept fills a gap in the physical formalism between the classical notions of having a reference frame, and not having one. Consider again Bob with the spin system that Alice has sent him. On one hand, in the macroscopic limit, having such a QRF is equivalent to having a RF. Then again, the deeper we go into the microscopic regime, the more uncertainty a QRF introduces compared to a RF, and the closer Bob gets to a situation of a complete lack of RF.

Accounting for an inherent uncertainty in the RFs is not the only reason QRFs have to be introduced. With QRFs we can also account for the effects of a *back-action*. These effects were first studied in the context of non-relativistic RFs of position by Y. Aharonov in [4]. Back-action is not a purely quantum effect though, it just that the microscopic regime is where its effects are unavoidable. If for example Bob had a compass big enough to create a magnetic field comparable to that of the earth, then at least in principle he could unintentionally rotate the magnetic poles just by trying to measure their orientation. Bob of course can avoid that by using a “small” compass. In the microscopic regime, when the RF itself is microscopic and comparable in size to the systems it is used to measure, back action is not only unavoidable but quantum mechanical in nature.

According to the von Neumann measurement scheme, for a measurement to take place an interaction has to occur. This interaction is known to disturb the measured system but what is usually not accounted for is the disturbance of a QRF with respect to which the measurement took place. This disturbance is the back-action. QRFs are affected by the measurement interaction in the same way the measured systems are. In general, depending on the interaction and the measured system, the effects of the back-action are a combination of decoherence (increasing uncertainty of the QRF) and unitary evolution (changing the QRF). Both of these effects diminish the potency of a RF and have to be accounted for in order to produce a complete picture of the underlying physics.

Understanding the implications of having a QRF is particularly interesting in the context of quantum information and quantum computing. In fact, most of the research on QRFs is performed in the context of these fields. Perhaps the most profound result of this research, first shown in [8], is the explanation of superselection rules as a consequence of not having an appropriate RF. Other examples of QRF related studies in these fields include: communication complexity of sending a RF; precision and degradation of quantum gates; cryptography with QRF. The different topics are too numerous to mention here so we refer the reader to [9] for a comprehensive review.

Integration of RFs into the physical formalism is of course not a new idea. Einstein's theory of relativity is the result of abandoning the notion of an absolute reference frame in favor of the relativity principle. Consequently we got more than just a new theory of gravity that is more accurate in the strong gravity regimes. We achieved a paradigm shift in our understanding of gravity and space-time, which led to a revolution in modern physics. Successful integration of QRFs into the formalism of quantum mechanics can also be expected to have consequence beyond an improved theory of physics in the microscopic regime.

Ultimately we seek a relational formulation of quantum mechanics. That is a formulation that does not rely on any external RF in the background; in particular on the external parameter of time. A toy model by D. Poulin [12] demonstrates how a relational quantum theory can be formulated by including the QRFs in the formalism. The need for such formulation is naturally motivated in the framework of Quantum Gravity theories [5], [6]. As argued by C. Rovelli in [7], observer independent states of a system do not exist. This statement is a call for a quantum theory that incorporates the observer and that is what the program of QRFs attempts to address.

1.2 Preliminary discussion

Before we begin the discussion of QRFs we have to clarify the nomenclature. We will use the name "QRF" to refer to a quantum system designated to be the physical representative of the RF. QRFs are the quantum rulers and clocks. We will use the name "ERF" – *external reference frame* – to refer to a non-physical notion of a RF. ERF is what the standard physical formalism usually means by a RF. The name "RF" itself will be used to refer to the general concept regardless of its representation.

The immediate goal of the study of QRFs is to develop a quantitative understanding of physics without referring to an ERF. In order to see how this can be achieved consider a concrete physical system S in a state ρ_S. The state ρ_S is defined with respect to an ERF that is not part of S so the state of the ERF itself is not specified. In order to address this ambiguity we explicitly introduce the RF as a system R in a state ρ_R so that the composite system RS is in the state $\rho_R \otimes \rho_S$. This step by itself does not address the issue as now both of the states ρ_R, ρ_S are defined with respect to an ERF. In order to describe the state of the system RS without referring to an ERF, we have to assume complete ignorance about the ERF. Operationally this means that the state $\rho_R \otimes \rho_S$ has to be transformed (by a changing the ERF) to all possible ERFs with equal probability. This transformation results in a state of RS that is completely independent from the ERF, rendering the latter irrelevant. The transformation that produces such ERF-independent states will be called a *twirl* and denoted $\mathcal{G}$ which we will formally define and study in Section 2. The state $\mathcal{G}\left(\rho_R \otimes \rho_S\right)$ retains only the information encoded in the relative degrees of freedom between the systems R and S. All other information that was encoded in the original state $\rho_R \otimes \rho_S$ is destroyed. Studying the states $\mathcal{G}\left(\rho_R \otimes \rho_S\right)$ will allow us to understand how the physical predictions change as a result of abandoning the ERF in favor of the QRF R.

The two key questions that this analysis will allow us to address are:

1. How do the physical predictions about the system S change as a result of substituting the ERF with the QRF R?

2. How do the physical predictions about the RF change if the RF is represented by the QRF R?

In particular, we want to understand how the answers to these questions scale with the "size" of the systems R (by size we mean some parameter that determines how microscopic or macroscopic the system is). The expectation is that the closer the system R is to being macroscopic, the closer the predictions are to the ones given with the ERFs, and in the limit, ERFs are just macroscopic QRFs.

The mathematical structure behind the notion of a reference frame is that of a group. This follows from the observation that the transformations induced by a change of RF obey the axioms of a group. This makes the group G a core property of a RF. In fact, that is the only property we can attribute to a non-physical ERF. This observation also means that the systems R and S carry a representation of the group G, and the composite system RS carries a product of these representations. The relative degrees of freedom between the systems R and S are defined as the invariants (constants) of the RF transformations. Now we can also identify them as the invariants of the group G. These invariants are the only physical degrees of freedom that specify the state of the system S relative to the system R (or vice versa). Finding these invariants requires a decomposition of the representation given by RS into a direct sum of irreducible representations (irreps). Understanding how the reducible representations decompose turns out to be a key problem in the QRF analysis.

Most of the work on QRFs has focused on the degrees of freedom associated with the groups $SU(2)$ and $U(1)$. These degrees of freedom are the spin and the phase, which are the principal degrees of freedom in many systems of interest. Such systems are very common in the study of quantum information and its applications, which is where most of the interest in QRFs is concentrated. However, the $SU(2)$ and $U(1)$ degrees of freedom provide very restricted description of the real physical systems, elementary or otherwise. In order to see more fundamental implications of abandoning the ERFs, especially in the relativistic framework of quantum gravity, we will have to study a more fundamental group of space-time symmetry.

The Poincaré group is a natural candidate for the QRF study program. The irreducible representations of the Poincaré group are carried by all of the elementary particles, which were classified by E. Wigner [22] according to their mass and total spin. The degrees of freedom within the representations are usually given by the relativistic 4-momentum and a component of spin. The component of spin is not independent though, its transformations between the different RFs depend on the 4-momentum. This dependence is not accounted for in the treatment with $SU(2)$. A. Peres et al have shown in [3] that by not accounting for the momentum degrees of freedom we are effectively introducing decoherence to the spin system. Therefore, one consequence of extending the QRF analysis to the Poincaré group will be a better understanding of the QRFs of spin, particularly in the relativistic regimes where the momentum is not negligible. In addition, we can focus on the QRFs of the 4-momentum itself. The momentum degrees of freedom are common to all systems but their QRFs have not been studied at all. Ultimately of course, we would like to study the full QRF of space-time, one that does not separate the momentum from the spin, as the Poincaré symmetry suggests.

The study of QRFs associated with the Poincaré group is expected to be mathematically challenging. The groups that were studied previously were all compact groups. The Poincaré group is not, which leads to unitary irreps that are infinite-dimensional. This suggests that the techniques that were used in the analysis of the QRFs of compact groups will either have to be extended to infinite dimensions or abandoned. Another challenge is the decomposition of reducible representations. The parameter space of the irreducible representations of the Poincaré group is 2-dimensional (mass and total spin). Furthermore, one of the irrep parameters is continuous (mass) while the other is discrete (total spin). This makes the decomposition of representations of this group, which are infinite dimensional, a much more complicated task than in the case of $SU(2)$.

1.3 Outline

In this thesis we consider the QRFs associated with the Poincaré symmetry group. Before we carry out the analysis for this qualitatively different case we want to take a broader perspective and generalize the $SU(2)$ results to a generic compact group. By studying the generic case we will see how the QRFs fit into the general formalism and identify the quantities that are important to the analysis. Section 2 is dedicated to developing the generic results. We find that a natural setting for introducing a QRF is through solving a communication problem between two parties that lack a common RF. The solution to this problem yields a "transmission" map $\mathcal{T}$ that maps the states of a system S to what they become if the ERF is replaced with a QRF R. This map will allow us to study the quality of a QRF by quantifying its ability to approximate the ERF using the standard tools of quantum information. The same map $\mathcal{T}$ will also be used to relate an ERF-*dependent* measurement on the system S to an ERF-*independent* measurement on the system RS. This will allow us to see how the QRF enters the formalism from the perspective of the measurement. This perspective will be used to simplify the analysis by restricting their scope to a limited class of measurements. We will also use this perspective to study how QRFs are affected by the measurements. This will lead us to a derivation of the "frame degradation" map $\mathcal{F}$ that acts on the QRF system R by applying the effects of the back-action from a single measurement. Our study of the map $\mathcal{F}$ in the generic case will focus on its fixed points that determine where the repeated frame degradation process will end up.

The key problem in the concrete case analysis of QRFs is the decomposition of reducible representations of the underlying symmetry group. Before we address the decomposition of representations, we have to identify the irreducible representations of the Poincaré group. The two physically significant classes of unitary irreps of the Poincaré group are associated with massive and massless particles. In this thesis we will focus on the irreps of massive particles only. Section 3 is dedicated to the study of these representations. There, we will define the Poincaré group and identify a complete set of commuting operators in its Lie algebra. This will allow us to construct the Hilbert spaces of the unitary irreps of the group. In Section 4 we will take a tensor product of two of these irreps and show their decomposition. The solution that we find presents the decomposition as a mapping from a tensor product of two spaces carrying the irreps to a direct sum of multiple spaces carrying the irreps. This solution differs from the usual approach, in that it produces this mapping directly and not in the form of the Clebsch-Gordan coefficients from which this mapping has to be constructed. In the solution of the decomposition we further restrict our scope to the irreps with no spin (generalizing this result to an arbitrary spin should be a relatively straightforward task involving spin-orbit coupling in the framework of $SU(2)$).

In Section 5 we will present the analysis of a QRF of total momentum. This is a restricted case of the full QRF associated with the Poincaré group. Analysis of the quality, that is how well such QRF approximates an ERF, yields a partial result. However, it does provide an insight into how this approximation depends on the mass of the QRF. Analysis of the degradation (resulting from the measurement back-action) of this QRF, shows a qualitatively different behavior than what was shown for the QRFs of spin. Specifically, the measurements are shown to be insensitive to the frame degradation that they cause.

2 Quantum reference frames

In this section we introduce the formalism of quantum reference frames from a general perspective applicable to any compact group. Although the Poincaré group is not compact, the general approach taken here will be a useful even for the non-compact case. The way we choose to present the formalism is through a communication problem in which the need to introduce a QRF naturally arises. In this framework a RF needs to be communicated which forces us to represent it as a physical system and deal with its physical limitations. The primary tool that we derive in this framework is the transmission map $\mathcal{T}$ that relates the states of a system given with respect to an ERF, to the same states as described relative to a QRF. With this map we will analyze how QRFs compare to ERFs in the general case of an arbitrary compact group.

In Section 2.1 the notation is introduced and some basic definitions and derivations are provided. In Section 2.2 we introduce the formalism of QRFs as a solution to a communication problem between parties that do not share a RF. This introduction is based on the analysis of [10] restricted to QRFs that are unitary irreducible representations. We refine the formalism of [10] with more extensive use of the superoperator algebra and with the generalization to QRF states that are not necessarily pure. We also incorporate measurements into the formalism. We show how to relate the POVMs defined with respect to an ERFs, to the POVMs measuring the same degrees of freedom relative to QRFs. In Section 2.3, we analyze how well a QRF approximates the ERF. The main result of this section is the Theorem 2.2 that presents a general measure of quality for a QRF. A similar analysis has been performed previously in [11] for the irreps of $SU(2)$. We generalize their result to an arbitrary compact group that admits multiplicity free decomposition. In Section 2.4 we will present the frame degradation map $\mathcal{F}$. The main result of this section are the Theorems 2.4 and 2.5. These theorems indicate the possible points of convergence of a degradation process in the case of an arbitrary compact group with multiplicity free decomposition. These theorems are shown to be consistent with the results of a concrete case studies of $SU(2)$ in [11] and [13].

2.1 Basic definitions, notation and identities

We will work with generic physical systems assumed to carry representations of an arbitrary compact group G. Irreducible unitary representations of G will be labeled with Greek letters such as α, β and placed as a superscripts on operators and states. In general operators acting on the irrep α will be specified with capital Roman letters such as A^{α}, and density operators with a lower case Greek ρ^{α}. One exception to this convention will be projection operators which we will denote by Π^{α} to avoid confusion with the momentum operator P. The identity operator I^{α} on an irrep of dimension d_{α} obey $tr\left[I^{\alpha}\right] = d_{\alpha}$. The group action of an element $g \in G$ on irrep α will be given by an unitary $U^{\alpha}\left(g\right)$. States and operators belonging to a tensor products of two irreps we will label with both irreps such as $E^{\alpha\beta}$, $\rho^{\alpha\beta}$ and for the group action $U^{\alpha\beta}(g) := U^{\alpha}(g) \otimes U^{\beta}(g)$. Superoperators (linear maps on operators) will be used extensively and will be specified with capital Roman letters in calligraphic font such as $\mathcal{E}$,$\mathcal{P}$. Once again the one exception to this are Hilbert spaces which will be denoted $\mathcal{H}^{\alpha}$. To avoid overloaded notation we will not use superscripts specifying the irreps on superoperators unless the superoperator is defined for an arbitrary representation, in that case we will use the superscript to differentiate between the different versions of it.

Some expressions will involve compositions of superoperators specified explicitly. For example for some $\mathcal{E} : \mathcal{B}\left(\mathcal{H}^{\alpha\beta}\right) \longrightarrow \mathcal{B}\left(\mathcal{H}^{\alpha\beta}\right)$ ($\mathcal{B}$ is a set of bounded operators) we can define

$$\begin{aligned} \mathcal{T} : \mathcal{B}\left(\mathcal{H}^{\beta}\right) &\longrightarrow \mathcal{B}\left(\mathcal{H}^{\alpha\beta}\right) \\ B^{\beta} &\longmapsto \mathcal{E}\left(A^{\alpha} \otimes B^{\beta}\right) \end{aligned}$$

which we will specify concisely as

$$\mathcal{T} := \mathcal{E} \circ \left(A^{\alpha} \otimes \cdot\right). \tag{2.1}$$

The superoperator $\left(A^{\alpha} \otimes \cdot\right)$ appears here explicitly. Another example is for some $\mathcal{D} : \mathcal{B}\left(\mathcal{H}^{\alpha}\right) \longrightarrow \mathcal{B}\left(\mathcal{H}^{\alpha}\right)$ we define

$$\begin{aligned} \mathcal{T} : \mathcal{B}\left(\mathcal{H}^{\alpha}\right) &\longrightarrow \mathcal{B}\left(\mathcal{H}^{\alpha\beta}\right) \\ A^{\alpha} &\longmapsto \mathcal{D}\left(A^{\alpha}\right) \otimes B^{\beta} \end{aligned}$$

which is simply

$$\mathcal{T} := \left(\cdot \otimes B^{\beta}\right) \circ \mathcal{D}.$$

Note that the dot $\cdot$ is a part of an explicit specification of the map $\left(\cdot \otimes B^{\beta}\right)$ and not where the initial input of $\mathcal{T}$ goes. On the other hand if we do specify a dot on the left hand side like so $\mathcal{T}(\cdot)$ then the initial input is intended to go to the corresponding dot on the right hand side of the expression, for example

$$\mathcal{T}(\cdot) := B(\cdot)B^{\dagger}.$$

The adjoint $\mathcal{E}^{\dagger}$ of a superoperator $\mathcal{E}$ is defined with respect to the Hilbert-Schmidt inner product according to which it satisfies

$$tr\left(\mathcal{E}^{\dagger}(A)B\right) \equiv tr\left(A\mathcal{E}(B)\right). \tag{2.2}$$

The adjoint of a composition of superoperators $\mathcal{R}\circ\mathcal{E}$ can then be derived by applying the definition

$$tr\left((\mathcal{R}\circ\mathcal{E})^{\dagger}(A)B\right) = tr\left(A(\mathcal{R}\circ\mathcal{E})(B)\right) = tr\left(\left(\mathcal{E}^{\dagger}\circ\mathcal{R}^{\dagger}\right)(A)B\right)$$

therefore

$$(\mathcal{R}\circ\mathcal{E})^{\dagger}\equiv\mathcal{E}^{\dagger}\circ\mathcal{R}^{\dagger}. \tag{2.3}$$

The superoperators $\mathcal{U}_g^{\alpha}$ associated with the group action, and their adjoints, are

$$\mathcal{U}_g^{\alpha}(\cdot):=U^{\alpha}(g)(\cdot)U^{\alpha\dagger}(g),$$

$$\mathcal{U}_g^{\alpha\dagger}(\cdot):=U^{\alpha\dagger}(g)(\cdot)U^{\alpha}(g).$$

The superoperators $\mathcal{U}_g^{\alpha\beta}$ acting on the product of irreps are similarly defined with $U^{\alpha\beta}(g) = U^{\alpha}(g)\otimes U^{\beta}(g)$. Note that from unitarity we have $\mathcal{U}_g^{\alpha\dagger} = \mathcal{U}_{g^{-1}}^{\alpha}$ and the composition rule is ${\mathcal{U}^{\alpha}}_{g'}\circ\mathcal{U}_g^{\alpha}=\mathcal{U}_{g'g}^{\alpha}$.

Since the group G is compact it admits a Haar measure which we assume to be normalized such that $\int_G dg=1$ (we will omit G from the integral over the group unless the domain of integration is not the whole group). The *twirl* over G is then defined as the superoperators

$$\mathcal{G}^{\alpha}:=\int dg\,\mathcal{U}_g^{\alpha},$$

and $\mathcal{G}^{\alpha\beta}$ are similarly defined with $\mathcal{U}_g^{\alpha\beta}$. From the definition of the adjoint we see that $\mathcal{G}^{\alpha\dagger}:=\int dg\,\mathcal{U}_g^{\alpha\dagger}$ but we can show that it is self adjoint. Using the invariance of the measure dg' and its normalization we see that

$$\mathcal{G}^{\alpha\dagger}\circ\mathcal{G}^{\alpha}=\int dg\int dg'\,\mathcal{U}_{g^{-1}g'}^{\alpha}=\int dg\int dg'\,\mathcal{U}_{g'}^{\alpha}=\mathcal{G}^{\alpha},$$

but we can also use the invariance of dg instead of dg', hence

$$\mathcal{G}^{\alpha\dagger}\circ\mathcal{G}^{\alpha}=\int dg\int dg'\,\mathcal{U}_{g^{-1}g'}^{\alpha}=\int dg\int dg'\,\mathcal{U}_{g^{-1}}^{\alpha}=\mathcal{G}^{\alpha\dagger}.$$

Comparing the two we get the projection property, and self adjointness of $\mathcal{G}^{\alpha}$

$$\mathcal{G}^{\alpha}\circ\mathcal{G}^{\alpha}=\mathcal{G}^{\alpha}=\mathcal{G}^{\alpha\dagger}. \tag{2.4}$$

Similarly we can show that for all $g\in G$

$$\mathcal{U}_g^{\alpha}\circ\mathcal{G}^{\alpha}=\mathcal{G}^{\alpha}\circ\mathcal{U}_g^{\alpha}=\mathcal{G}^{\alpha}. \tag{2.5}$$

This shows that for any operator $A^{\alpha}\in\mathcal{B}(\mathcal{H}^{\alpha})$, the operator $\mathcal{G}^{\alpha}(A^{\alpha})$ is invariant under the group action.

2.2 Communication using quantum reference frames

Consider a state ρ_S^β of some system S carrying an irrep β of G. Alice encodes a message in the state of S with respect to her ERF and sends it to Bob. If $g_{AB} \in G$ is the symmetry transformation that transforms states from Alice's ERF to Bob's then the system received by Bob will be described in his ERF as $\mathcal{U}_{g_{AB}}^{\beta\,\dagger}\left(\rho_S^\beta\right)$ and he needs to apply $\mathcal{U}_{g_{AB}}^\beta$ before he reads it. If Bob has no prior knowledge about g_{AB}, then the state he receives must be described in his ERF by applying all $\mathcal{U}_g^{\beta\,\dagger}$ uniformly distributed over G, resulting in a twirl

$$\rho_S^\beta \longmapsto \mathcal{G}^\beta\left(\rho_S^\beta\right).$$

Since $\mathcal{G}^\beta\left(\rho_S^\beta\right)$ is invariant under G (see Eq. (2.5)) and β is an irrep, we know from Schur's lemma that $\mathcal{G}^\beta\left(\rho_S^\beta\right) \propto I^\beta$, and comparing the trace on both sides requires $\mathcal{G}^\beta\left(\rho_S^\beta\right) = \frac{1}{d_\beta} I^\beta$. Therefore no information about the state ρ_S^β is preserved and Bob cannot read Alice's messages.

2.2.1 The transmission map

In order to enable encoding of information, Alice introduces a QRF system R, carrying an irrep α of G and prepared in the state ρ_R^α. Keeping in mind that the "reference" is in the α irrep and the "system" is in the β irrep we will omit the superscripts wherever it is clear from the context. The state ρ_R is aligned with Alice's ERF making it the physical token that represents it. It may not be clear at this point how well ρ_R can do the job but at the very least we will assume that it transforms non-trivially under all elements of the group; that is $g \in G : \mathcal{U}_g^\alpha\left(\rho_R\right) = \rho_R$ if and only if $g = e$ (e is the group identity). Sending the product state $\rho_R \otimes \rho_S$ to Bob and repeating the argument of how this state is described in Bob's ERF we again get the twirl

$$\rho_R \otimes \rho_S \longmapsto \mathcal{G}^{\alpha\beta}\left(\rho_R \otimes \rho_S\right).$$

As a result we define the *encoding map* (see Eq. (2.1) for clarification of this notation)

$$\mathcal{E}_{\rho_R} := \mathcal{G}^{\alpha\beta} \circ \left(\rho_R \otimes \cdot\right) \tag{2.6}$$

which depends on ρ_R and maps the states ρ_S, from Alice to Bob, while adding the QRF R. Using equations (2.5) and (2.4) we can also derive two useful identities

$$\mathcal{E}_{\mathcal{U}_g^\alpha(\rho_R)} = \mathcal{G}^{\alpha\beta} \circ \left(\mathcal{U}_g^\alpha\left(\rho_R\right) \otimes \cdot\right) = \mathcal{G}^{\alpha\beta} \circ \mathcal{U}_g^{\alpha\beta} \circ \left(\rho_R \otimes \mathcal{U}_g^{\beta\,\dagger}\left(\cdot\right)\right) = \mathcal{E}_{\rho_R} \circ \mathcal{U}_g^{\beta\,\dagger}\ , \tag{2.7}$$

$$\mathcal{E}_{\mathcal{G}^\alpha(\rho_R)} = \mathcal{G}^{\alpha\beta} \circ \left(\int dg\, \mathcal{U}_g^\alpha\left(\rho_R\right) \otimes \cdot\right) = \mathcal{E}_{\rho_R} \circ \int dg\, \mathcal{U}_g^{\beta\,\dagger} = \mathcal{E}_{\rho_R} \circ \mathcal{G}^\beta. \tag{2.8}$$

This means that action of $g \in G$ on the QRF that is used in the encoding, is equivalent to the action of g^{-1} on the system S and then encoding with the original QRF. Having the encoding map above we can now simply define Bob's task of recovering the state ρ_S as finding the best way to invert the encoding $\mathcal{E}_{\rho_R}$. As proposed by [10] based on the work of [14] the adjoint of $\mathcal{E}_{\rho_R}$ is a good approximation of the optimal inverse as it is no more then twice as bad as the optimal[2]. With definition (2.2) and the definition of the partial trace we compute the adjoint of $(\rho_R \otimes \cdot)$ (note that it is a map between different Hilbert spaces)

$$tr_{\alpha\beta}\left(A^{\alpha\beta}\left(\rho_R^\alpha \otimes B^\beta\right)\right) = tr_{\alpha\beta}\left(A^{\alpha\beta}\left(\rho_R^\alpha \otimes I^\beta\right)\left(I^\alpha \otimes B^\beta\right)\right) = tr_\beta\left(tr_\alpha\left[A^{\alpha\beta}\left(\rho_R^\alpha \otimes I^\beta\right)\right]B^\beta\right)$$

thus

$$(\rho_R^\alpha \otimes \cdot)^\dagger = tr_\alpha\left[(\cdot)\left(\rho_R^\alpha \otimes I^\beta\right)\right].$$

With this and the equations (2.3) and (2.4) we find the adjoint of $\mathcal{E}_{\rho_R}$,

$$\mathcal{E}_{\rho_R}^\dagger = \left(\mathcal{G}^{\alpha\beta} \circ (\rho_R \otimes \cdot)\right)^\dagger = (\rho_R \otimes \cdot)^\dagger \circ \mathcal{G}^{\alpha\beta} = tr_\alpha\left[(\cdot)\left(\rho_R \otimes I^\beta\right)\right] \circ \mathcal{G}^{\alpha\beta}. \tag{2.9}$$

Now we define the *recovery map*

$$\mathcal{R}_{\rho_R} := d_\alpha \mathcal{E}_{\rho_R}^\dagger = d_\alpha tr_\alpha\left[(\cdot)\left(\rho_R \otimes I^\beta\right)\right] \circ \mathcal{G}^{\alpha\beta} \tag{2.10}$$

where the dimension factor d_α comes in to keep $\mathcal{R}_{\rho_R}$ trace preserving. Note that this definition, property (2.3) and equations (2.7), (2.8) immediately give us

$$\mathcal{R}_{\mathcal{U}_g^\alpha(\rho_R)} = \mathcal{U}_g^\beta \circ \mathcal{R}_{\rho_R}, \tag{2.11}$$

$$\mathcal{R}_{\mathcal{G}^\alpha(\rho_R)} = \mathcal{G}^\beta \circ \mathcal{R}_{\rho_R}. \tag{2.12}$$

Both encoding and recovery maps depend on a choice of the state ρ_R of the QRF, but there is no reason to assume that both maps use the same ρ_R. In fact we assume that Bob does not know the state ρ_R that is aligned with Alice's ERF, and is related to Bob's ERF by the transformation g_{AB}. Consider a recovery map that Bob implements using the same reference system as Alice but aligned with his ERF and is described by Alice as $\tilde{\rho}_R := \mathcal{U}_{g_{AB}}^\alpha(\rho_R)$. Such recovery map then decomposes using (2.11) to

$$\mathcal{R}_{\tilde{\rho}_R} = \mathcal{U}_{g_{AB}}^\beta \circ \mathcal{R}_{\rho_R}, \tag{2.13}$$

where $\mathcal{U}_{g_{AB}}^\beta$ is the desired part of the recovery that transforms the received state from Alice's ERF to Bob's, while $\mathcal{R}_{\rho_R}$ is exactly the adjoint (up to factor d_α) of the encoding $\mathcal{E}_{\rho_R}$ that Alice performed.

[2] Using the entanglement fidelity as the figure of merit.

Finally we can define the *transmission map*

$$\mathcal{T}_{\rho_R} := \mathcal{R}_{\tilde{\rho}_R} \circ \mathcal{E}_{\rho_R}, \tag{2.14}$$

that maps the states ρ_S from Alice to Bob following encoding and recovery (see Fig. 2.1). It is instructive to introduce the *decoherence map*

$$\mathcal{D}_{\rho_R} := \mathcal{R}_{\rho_R} \circ \mathcal{E}_{\rho_R}$$

and express the transmission map with it. Substituting Eq. (2.13) into (2.14) we can see that

$$\mathcal{T}_{\rho_R} = \mathcal{U}^{\beta}_{g_{AB}} \circ \mathcal{D}_{\rho_R}. \tag{2.15}$$

To justify its name we express $\mathcal{D}_{\rho_R}$ explicitly with Eq. 2.6 and (2.10), and use the identity (2.4)

$$\begin{aligned} \mathcal{D}_{\rho_R} &= d_\alpha \mathcal{E}^{\dagger}_{\rho_R} \circ \mathcal{E}_{\rho_R} \\ &= d_\alpha tr_\alpha \left[(\cdot) \left(\rho_R \otimes I^{\beta} \right) \right] \circ \mathcal{G}^{\alpha\beta} \circ \mathcal{G}^{\alpha\beta} \circ (\rho_R \otimes \cdot) \\ &= d_\alpha tr_\alpha \left[\mathcal{G}^{\alpha\beta} \circ (\rho_R \otimes \cdot) \left(\rho_R \otimes I^{\beta} \right) \right] \\ &= \int dg\, d_\alpha tr_\alpha \left[\mathcal{U}^{\alpha}_{g} (\rho_R)\, \rho_R \right] \mathcal{U}^{\beta}_{g}. \end{aligned} \tag{2.16}$$

The decoherence map $\mathcal{D}_{\rho_R}$ has the form of a weighted twirl map with the weights $d_\alpha tr_\alpha \left[\mathcal{U}^{\alpha}_{g} (\rho_R)\, \rho_R \right]$ for $g \in G$. Indeed by Schur's lemma we have $d_\alpha \int dg \mathcal{U}^{\alpha}_{g} (\rho_R) = I^\alpha$, therefore

$$\int dg\, d_\alpha tr_\alpha \left[\mathcal{U}^{\alpha}_{g} (\rho_R)\, \rho_R \right] = tr_\alpha \left[I^\alpha \rho_R \right] = 1,$$

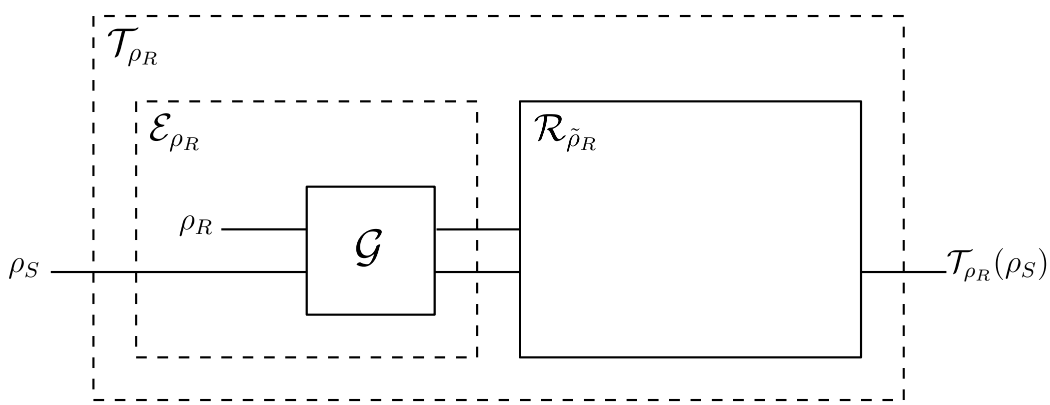

Figure 2.1: The Transmission map $\mathcal{T}_{\rho_R}$ maps the states ρ_S from Alice to Bob using the QRF ρ_R.

and from non-negativity of density operators we have $tr_\alpha\left[\mathcal{U}_g^\alpha\left(\rho_R\right)\rho_R\right]\geq 0$. This means that we can interpret the weights as probability density over the group

$$p\left(g\right):=d_\alpha tr_\alpha\left[\mathcal{U}_g^\alpha\left(\rho_R\right)\rho_R\right]$$

with the measure dg. Substituting the above into (2.15) we get a simple expression for the transmission map

$$\mathcal{T}_{\rho_R}=\mathcal{U}_{g_{AB}}^\beta\circ\int dg\,p\left(g\right)\mathcal{U}_g^\beta=\int dg\,p\left(g_{AB}^{-1}\,g\right)\mathcal{U}_g^\beta. \tag{2.17}$$

Recall that when Bob had no prior knowledge about g_{AB} we had to describe the state ρ_S in his ERF by applying $\mathcal{U}_g^\beta$ for a uniformly distributed $g\in G$. Now we can see that by introducing ρ_R we provide prior knowledge about g_{AB} in the form of the distribution $p\left(g\right)$ that depends on ρ_R. The more peaked $p\left(g\right)$ at the group identity element, the better job the transmission map does as $p\left(g_{AB}^{-1}\,g\right)$ will be peaked around g_{AB}. We will examine how ρ_R affects the effectiveness of the transmission map in Section 2.3.

2.2.2 Invariant measurements

If Bob intends to read the message that Alice has sent him he needs to perform a measurement on the transmitted state. Consider an arbitrary POVM $\left\{E_i^\beta\right\}$ with outcome label i. The probability for outcome i to be measured on the transmitted state is $tr\left[E_i^\beta\mathcal{T}_{\rho_R}\left(\rho_S\right)\right]$. Using the definition of the adjoint (2.2) and the transmission map (2.14) we can rewrite this as

$$tr\left[E_i^\beta\mathcal{T}_{\rho_R}\left(\rho_S\right)\right]=tr\left[\mathcal{R}_{\tilde{\rho}_R}^\dagger\left(E_i^\beta\right)\mathcal{G}^{\alpha\beta}\left(\rho_R\otimes\rho_S\right)\right]=tr\left[\mathcal{R}_{\tilde{\rho}_R}^\dagger\left(E_i^\beta\right)\rho_R\otimes\rho_S\right]. \tag{2.18}$$

In the last step we eliminated $\mathcal{G}^{\alpha\beta}$ by applying its adjoint on

$$\mathcal{R}_{\tilde{\rho}_R}^\dagger\left(E_i^\beta\right)=d_\alpha\mathcal{E}_{\tilde{\rho}_R}\left(E_i^\beta\right)=d_\alpha\mathcal{G}^{\alpha\beta}\left(\tilde{\rho}_R\otimes E_i^\beta\right)$$

which already does a twirl so it has no additional effect. Recognizing a new POVM we define

Definition 2.1. *The invariant POVM* $\left\{E_i^{\alpha\beta}\right\}$ on the joint system RS induced by the POVM $\left\{E_i^\beta\right\}$ is

$$E_i^{\alpha\beta}:=\mathcal{R}_{\tilde{\rho}_R}^\dagger\left(E_i^\beta\right)=d_\alpha\,\mathcal{G}\left(\tilde{\rho}_R\otimes E_i^\beta\right) \tag{2.19}$$

where $\tilde{\rho}_R$ is a state of R aligned with Bob's RF.

The POVM $\left\{E_i^{\alpha\beta}\right\}$ acts on the product of Hilbert spaces, it is invariant under G (meaning $\mathcal{G}\left(E_i^{\alpha\beta}\right)=E_i^{\alpha\beta}$) and it produces the same probability distribution over the outcomes i when applied on $\rho_R\otimes\rho_S$ as the POVM $\left\{E_i^\beta\right\}$ when applied on $\mathcal{T}_{\rho_R}\left(\rho_S\right)$ (see Fig. 2.2). Consequently

Definition 2.1 can be viewed as a prescription for constructing an invariant measurement on the product system $\rho_R \otimes \rho_S$ that implements a measurement on a single system ρ_S, relative to a QRF defined by ρ_R. The POVM $\left\{E_i^{\alpha\beta}\right\}$ incorporates in a single operation both the recovery map $\mathcal{R}_{\tilde{\rho}_R}$, and the measurement $\left\{E_i^{\beta}\right\}$; it approximates the measurement $\left\{E_i^{\beta}\right\}$ on ρ_S as much as the state $\mathcal{T}_{\rho_R}(\rho_S)$ approximates $\mathcal{U}^{\beta}_{g_{AB}}(\rho_S)$.

If we are willing to compromise on what we want to know about $\mathcal{T}_{\rho_R}(\rho_S)$ we can simplify the descriptions of the transmitted state and of the invariant POVM. The compromise comes in a form of a subgroup $H \subseteq G$ such that for all $h \in H$ we have

$$\mathcal{U}_h^{\beta}\left(E_i^{\beta}\right) = E_i^{\beta}$$

so E_i^{β} is invariant under H. This invariance restricts the amount of information we can extract from the system and in the extreme case of $H = G$ we have $E_i^{\beta} \propto I^{\beta}$ making it a measurement with no information gain. Given a subgroup H, the twirl map can be split according to [3]

$$\mathcal{G}^{\alpha} = \int_G dg\, \mathcal{U}_g^{\alpha} = \int_{G/H} dc \int_H dh\, \mathcal{U}_{ch}^{\alpha} = \int_{G/H} dc\, \mathcal{U}_c^{\alpha} \circ \int_H dh\, \mathcal{U}_h^{\alpha} = \mathcal{G}_{G/H}^{\alpha} \circ \mathcal{G}_H^{\alpha}$$

where we have introduced the coset space G/H and an invariant measure dc over it. The maps $\mathcal{G}_{G/H}^{\alpha}$, $\mathcal{G}_H^{\alpha}$ will be called a *partial twirl*. According to Eq. (2.19) the invariant POVM on the joint system is then

$$E_i^{\alpha\beta} = d_{\alpha} \mathcal{G}_{G/H}^{\alpha\beta} \circ \mathcal{G}_H^{\alpha\beta}\left(\tilde{\rho}_R \otimes E_i^{\beta}\right) = d_{\alpha}\, \mathcal{G}_{G/H}^{\alpha\beta}\left(\mathcal{G}_H^{\alpha}(\tilde{\rho}_R) \otimes E_i^{\beta}\right). \tag{2.20}$$

This simplifies the construction of $E_i^{\alpha\beta}$ since we use the states $\mathcal{G}_H^{\alpha}(\tilde{\rho}_R)$ that now have H symmetry (states with additional symmetry are usually simpler) and we twirl only over the cosets G/H.

Reverting the reasoning that led to a definition of the invariant POVM, we can also simplify the

[3]We do not claim here that splitting the integral $\int_G dg = \int_{G/H} dc \int_H dh$ is trivial or even possible for any group G with a subgroup H. We are simply considering a case when it can be done.

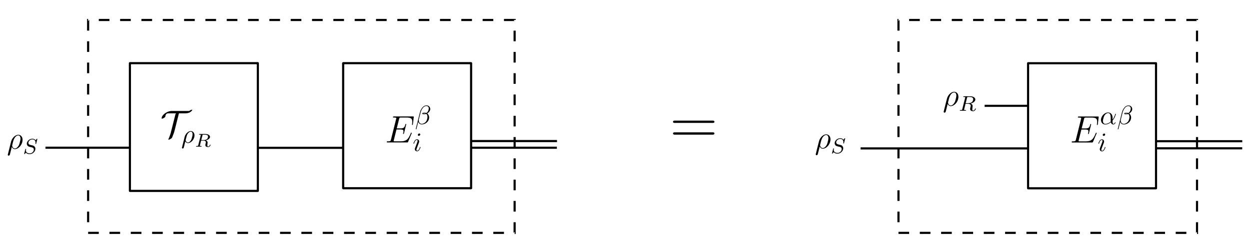

Figure 2.2: Equivalence of measurements E_i^{β} and $E_i^{\alpha\beta}$.

transmission map. Substituting (2.20) into (2.18) and applying the adjoint of the map (2.20) we get

$$tr\left[E_i^{\beta}\mathcal{T}_{\rho_R}\left(\rho_S\right)\right] = tr\left[E_i^{\alpha\beta}\rho_R\otimes\rho_S\right] = tr\left[E_i^{\beta}tr_{\alpha}\left[d_{\alpha}\mathcal{G}_{G/H}^{\alpha\beta}\left(\rho_R\otimes\rho_S\right)\left(\mathcal{G}_H^{\alpha}\left(\tilde{\rho}_R\right)\otimes I^{\beta}\right)\right]\right]. \quad (2.21)$$

This map on ρ_S may seem complicated but it actually is a simpler version of the transmission map. If we compare it to the original transmission map

$$\mathcal{T}_{\rho_R}\left(\rho_S\right) = \mathcal{R}_{\tilde{\rho}_R}\circ\mathcal{E}_{\rho_R}\left(\rho_S\right) = tr_{\alpha}\left[d_{\alpha}\mathcal{G}^{\alpha\beta}\left(\rho_R\otimes\rho_S\right)\left(\tilde{\rho}_R\otimes I^{\beta}\right)\right]$$

we see a reduction of a twirl to a partial twirl and the use of H-symmetrical state $\mathcal{G}_H^{\alpha}\left(\tilde{\rho}_R\right)$ in the recovery. Expressing it in the same way as we did in Eq. (2.17) we get a simpler version of the transmission map

$$\mathcal{T}_{\rho_R} := \int_{G/H} dc\, p\left(c\right)\mathcal{U}_c^{\beta} \quad (2.22)$$

$$p\left(c\right) := d_{\alpha}\, tr_{\alpha}\left[\mathcal{U}_c^{\alpha}\left(\rho_R\right)\mathcal{G}_H^{\alpha}\left(\tilde{\rho}_R\right)\right]$$

where $p\left(c\right)$ is a probability density over the coset elements.

To conclude this subsection we would like to emphasize a more general interpretation of this communication problem. Consider Alice as an experimenter and Bob as a measurement apparatus or a quantum computer. Bob's RF with respect to which he measures or performs computations is a physical QRF R that is part of Bob. Alice may calibrate the state of R to match her own notion of RF which in the above formalism was described as sending the state ρ_R to Bob. Any measurement or computation that Bob implements on the system S is performed with respect to this ρ_R and is limited by its ability to represent a RF. Whether we are describing a frameless communication, a computation, or a measurement there is always a physical system R that is part of the full description of the problem and the above formalism is equally valid for all of them. Understanding how well this R performs as a RF is the subject of the next section.

2.3 Quality of quantum reference frames

A quantum reference frame can in principle be any system that transforms non trivially under the action of the group; in other words it has to carry a non trivial representation. For reducible representations, the question of which states within a given representation do a better job as a RF, is analyzed thoroughly in [10]. Here we would like to answer a different question of which *irreducible* representations do a better job as a RF. We will not be concerned with the different states in the same irrep but we will compare the best states from different irreps. Similar analysis for the case of $SU(2)$ have been performed in [11], here we present a generalization of that to an arbitrary compact group that admit multiplicity free decomposition.

As discussed in Section 2.2 the quality of the transmission map and therefore the QRF depends on how peaked the probability distribution $p\left(g\right)=d_{\alpha}tr_{\alpha}\left[\mathcal{U}_{g}^{\alpha}\left(\rho_{R}\right)\rho_{R}\right]$ is at the identity e. The simple observation

$$p\left(e\right)=d_{\alpha}tr_{\alpha}\left[\left(\rho_{R}\right)^{2}\right]=d_{\alpha}\mathrm{Purity}\left(\rho_{R}\right) \tag{2.23}$$

has two implications: one is that pure states will make $p\left(g\right)$ more peaked than mixed states; and the second is that the dimension d_{α} of the irrep bounds the maximal value of $p\left(\cdot\right)$. For a more thorough analysis we need to ask how well does the transmission map preserves information. According to Eq. (2.15) the transmission map can be decomposed into unitary map and a decohering map given by

$$\mathcal{D}_{\rho_{R}}=\int dg\,p\left(g\right)\mathcal{U}_{g}^{\beta},$$

which is the part that looses information. Ideally $\mathcal{D}_{\rho_{R}}\equiv\mathcal{I}^{\beta}$ that can only happen if $p\left(g\right)=\delta\left(g\right)$ where the delta is by definition such that $\int dg\,\delta\left(g\right)f\left(g\right)=f\left(e\right)$. Since $p\left(g\right)$ is bounded by d_{α}, we can not have $\int dg\,p\left(g\right)f\left(g\right)=f\left(e\right)$ for any finite dimensional representation. A simple notion for how well a quantum channel preserves information can be given with the fidelity between the input and the output. The fidelity for the general states is defined as $F\left(\rho,\sigma\right):=tr\left[\sqrt{\rho^{1/2}\sigma\rho^{1/2}}\right]$. For pure states this reduces to

$$F\left(\left|\psi\right\rangle ,\sigma\right):=\sqrt{\left\langle \psi\right|\sigma\left|\psi\right\rangle }. \tag{2.24}$$

For some representative choice of ρ_{S} we can quantify the quality of a QRF in the state ρ_{R} with the quantity

$$Q_{\rho_{S}}\left(\rho_{R}\right):=F\left(\rho_{S},\mathcal{D}_{\rho_{R}}\left(\rho_{S}\right)\right)^{2}. \tag{2.25}$$

Furthermore, for pure states we can express $Q_{\psi_{S}}$ more explicitly. This is presented in the following theorem.

Theorem 2.2. *Let $\mathcal{H}^{\alpha}$, $\mathcal{H}^{\beta}$ be two irreps with the property that their product $\mathcal{H}^{\alpha}\otimes\mathcal{H}^{\beta}$ decomposes into a direct sum $\bigoplus\limits_{\lambda}\mathcal{H}^{\lambda}$ of irreps such that each $\mathcal{H}^{\lambda}$ in the sum appears at most once ($\mathcal{H}^{\alpha}\otimes\mathcal{H}^{\beta}$ admits multiplicity free decomposition). Let Π^{λ} be a projection on the d_{λ}-dimensional irreducible subspace $\mathcal{H}^{\lambda}$, and $\left|\psi_{R}\right\rangle \in\mathcal{H}^{\alpha}$, $\left|\psi_{S}\right\rangle \in\mathcal{H}^{\beta}$ such that $\left|a_{\lambda}\right|^{2}:=\left\langle \psi_{R}\right|\otimes\left\langle \psi_{S}\right|\Pi^{\lambda}\left|\psi_{R}\right\rangle \otimes\left|\psi_{S}\right\rangle$, then*

$$Q_{\psi_{S}}\left(\left|\psi_{R}\right\rangle \right)=d_{\alpha}\sum_{\lambda}\frac{\left|a_{\lambda}\right|^{4}}{d_{\lambda}}. \tag{2.26}$$

Proof. Let $\mathcal{P}^{\lambda}\left(\cdot\right):=\Pi^{\lambda}\left(\cdot\right)\Pi^{\lambda}$ be the superoperator counterpart of the projection Π^{λ}. The fact that the decomposition of $\mathcal{H}^{\alpha}\otimes\mathcal{H}^{\beta}$ is multiplicity-free, implies a decomposition of the group action $U^{\alpha\beta}\left(g\right)=\bigoplus\limits_{\lambda}U^{\lambda}\left(g\right)=\sum\limits_{\lambda}U^{\lambda}\left(g\right)\Pi^{\lambda}$ which leads to the decomposition of a twirl $\mathcal{G}^{\alpha\beta}=\bigoplus\limits_{\lambda}\mathcal{G}^{\lambda}=$

$\sum_{\lambda}\mathcal{G}^{\lambda}\circ\mathcal{P}^{\lambda}$. With Schur's lemma we can show that $\left(\mathcal{G}^{\lambda}\circ\mathcal{P}^{\lambda}\right)\left(|\psi_R,\psi_S\rangle\langle\psi_R,\psi_S|\right)=\frac{|a_\lambda|^2}{d_\lambda}\Pi^{\lambda}$ where we have introduced the notation $|\psi_R,\psi_S\rangle:=|\psi_R\rangle\otimes|\psi_S\rangle$. Equipped with this expressions and using the definition (2.24) of fidelity and Eq. (2.16) for $\mathcal{D}_{\psi_R}$, we derive

$$
\begin{aligned}
Q_{\psi_S}\left(|\psi_R\rangle\right) &= \langle\psi_S|\,\mathcal{D}_{\psi_R}\left(|\psi_S\rangle\langle\psi_S|\right)|\psi_S\rangle\\
&= d_\alpha\int dg\,\langle\psi_R\psi_S|\,\mathcal{U}_g^{\alpha\beta}\left(|\psi_R\psi_S\rangle\langle\psi_R\psi_S|\right)|\psi_R\psi_S\rangle\\
&= d_\alpha\,\langle\psi_R\psi_S|\,\mathcal{G}^{\alpha\beta}\left(|\psi_R\psi_S\rangle\langle\psi_R\psi_S|\right)|\psi_R\psi_S\rangle\\
&= d_\alpha\sum_{\lambda}\frac{|a_\lambda|^2}{d_\lambda}\langle\psi_R\psi_S|\,\Pi^{\lambda}\,|\psi_R\psi_S\rangle=d_\alpha\sum_{\lambda}\frac{|a_\lambda|^4}{d_\lambda}
\end{aligned}
$$

□

For a more detailed analysis one needs to know the possible values of a_λ which can be computed by decomposing the product representation to irreps.

Consider for example the product of $SU(2)$ irreps of spin l (which is the reference) and spin $\frac{1}{2}$. In this case the product of Hilbert spaces carrying the irreps decomposes to a direct sum of Hilbert spaces carrying the irreps according to

$$\mathcal{H}^{l}\otimes\mathcal{H}^{1/2}=\mathcal{H}^{l+1/2}\oplus\mathcal{H}^{l-1/2}.$$

Fixing the state of R to be $|\psi_R\rangle:=|l,l\rangle$ we consider an arbitrary pure state of S such that the joint state is

$$|\psi_R,\psi_S\rangle:=|l,l\rangle\otimes\left(c_+\left|\frac{1}{2},\frac{1}{2}\right\rangle+c_-\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right).$$

Explicit computation using Clebsch-Gordan coefficients of $SU(2)$ [16] yields

$$
\begin{aligned}
\left|a_{l+1/2}\right|^2:=\langle\psi_R\psi_S|\,\Pi^{l+1/2}\,|\psi_R\psi_S\rangle &= |c_+|^2+|c_-|^2\,\frac{1}{d}\\
\left|a_{l-1/2}\right|^2:=\langle\psi_R\psi_S|\,\Pi^{l-1/2}\,|\psi_R\psi_S\rangle &= |c_-|^2\,\frac{d-1}{d}
\end{aligned}
$$

where $\Pi^{l\pm1/2}$ are projectors on the subspaces $\mathcal{H}^{l\pm1/2}$ and $d:=2l+1$. The numbers $\left|a_{l\pm1/2}\right|^2$ depend only on the orientation of S relative to R which is specified by $c_\pm$. With Eq. (2.26) we can now compute

$$Q_{\psi_S}\left(|l,l\rangle\right)=d\left[\frac{\left|a_{l+1/2}\right|^4}{d+1}+\frac{\left|a_{l-1/2}\right|^4}{d-1}\right]\;=\;\frac{d}{d+1}-2\left(\frac{d-1}{d+1}\right)|c_+|^2\,|c_-|^2\,.$$

The maximal/minimal values of Q_{ψ_S} and the corresponding states of S are then

$$\begin{array}{ll} Q_{\max}\left(|l,l\rangle\right)=\frac{d}{d+1} & |\psi_S\rangle := \left|\frac{1}{2},\pm\frac{1}{2}\right\rangle \\ Q_{\min}\left(|l,l\rangle\right)=\frac{1}{2} & |\psi_S\rangle := \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2},\frac{1}{2}\right\rangle + e^{i\phi}\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right) . \end{array}$$

The fact that $Q_{\min} = \frac{1}{2}$ should not be surprising as we know that even with a classical RF we cannot distinguish between the states of spin that are polarized orthogonally to our RF. On the other hand we expect to have perfect distinguishability when measuring relative to a classical RF that is aligned with the spin states. Here we see that this corresponds to the maximal case only now the best attainable distinguishability, as reflected by $Q_{\max}$, goes to 1 as d goes to infinity. In [11] only the maximal case was analyzed using an operational measure of quality. The measure of quality that they defined was a probability of successfully identifying the spin up/down state of a particle using the PVM $\left\{\Pi^{l\pm 1/2}\right\}$ and the QRF in the state $|l,l\rangle$. The resulting probability of success was computed to be $P_{success} = 1-(2d)^{-1}$ which is very similar to our result $Q_{\max} = 1-(d+1)^{-1}$. We cannot compare these results directly as these are different measures of quality. Nevertheless they predict the same scaling of quality of a QRF with d.

2.4 Degradation of quantum reference frames

In Sections 2.2 and 2.3 we have focused on the description and the measurement of the system ρ_S relative to ρ_R. In this section we will discuss how ρ_R is affected by the measurement.

If the same reference system is to be used multiple times in the joint measurement of $\rho_R \otimes \rho_S$ (each time with a new system S in the same state ρ_S) its state needs to be updated after each measurement. The update rule depends on the POVM that is being measured which encapsulates the underlying dynamics of the measurement process. Before we introduce the update rule we have to identify the relevant POVMs.

Definition (2.19) provides a constructive introduction to the invariant POVM $\left\{E_i^{\alpha\beta}\right\}$ that extracts information from the joint system $\rho_R \otimes \rho_S$; it was motivated by and produced from the actual POVM $\left\{E_i^{\beta}\right\}$ that we wanted to implement on ρ_S. The reason that Bob can implement $\left\{E_i^{\alpha\beta}\right\}$ on $\rho_R \otimes \rho_S$, even though he does not share Alice's ERF, is solely because $E_i^{\alpha\beta}$ is G-invariant. In principle any invariant POVM on the joint system can be implemented by Bob. Invariance of a POVM implies

$$E_i^{\alpha\beta} = \mathcal{G}^{\alpha\beta}\left(E_i^{\alpha\beta}\right) = \sum_\lambda \mathcal{G}^{\lambda} \circ \mathcal{P}^{\lambda}\left(E_i^{\alpha\beta}\right) = \sum_\lambda a_{i\lambda}\Pi^{\lambda} \tag{2.27}$$

where we assumed again a multiplicity-free decomposition of $\alpha \otimes \beta$ into irreps and used Schur's lemma. The above equation demonstrates that any invariant POVM can be constructed from a linear sum of projectors on the invariant subspaces with coefficients $a_{i\lambda} := tr\left[\mathcal{P}^{\lambda}\left(E_i^{\alpha\beta}\right)\right]/d_\lambda$

derived by tracing both sides of $\sum_{\lambda} \mathcal{G}^{\lambda} \circ \mathcal{P}^{\lambda}\left(E_i^{\alpha\beta}\right) = \sum_{\lambda} a_{i\lambda} \Pi^{\lambda}$. Because of

$$tr\left[E_i^{\alpha\beta}\left(\rho_R \otimes \rho_S\right)\right] = \sum_{\lambda} a_{i\lambda} tr\left[\Pi^{\lambda}\left(\rho_R \otimes \rho_S\right)\right]$$

the probability of outcomes for any invariant POVM $\left\{E_i^{\alpha\beta}\right\}$ can be calculated from the probability of outcomes for the PVM $\left\{\Pi^{\lambda}\right\}$ which makes it the universal invariant measurement one can perform on $\rho_R \otimes \rho_S$. For this reason the PVMs $\left\{\Pi^{\lambda}\right\}$ are the most representative of all invariant measurements on the joint system.

Given the outcome λ of the measurement $\left\{\Pi^{\lambda}\right\}$ on $\rho_R \otimes \rho_S$, the post-measurement *outcome state* of the system is [15]

$$\rho_R \otimes \rho_S \longmapsto \frac{\Pi^{\lambda}\left(\rho_R \otimes \rho_S\right)\Pi^{\lambda}}{p\left(\lambda\right)}$$

where $p\left(\lambda\right) := tr\left[\Pi^{\lambda}\left(\rho_R \otimes \rho_S\right)\right]$ is the probability of the outcome λ. We consider the scenario in which separate measurements do not affect each other and the outcome of each measurement is not carried on to the next. In this case before the next measurement the system can be in any of the outcome states with probability $p\left(\lambda\right)$ [15]

$$\rho_R \otimes \rho_S \longmapsto \sum_{\lambda} p\left(\lambda\right) \frac{\Pi^{\lambda}\left(\rho_R \otimes \rho_S\right)\Pi^{\lambda}}{p\left(\lambda\right)} = \sum_{\lambda} \Pi^{\lambda}\left(\rho_R \otimes \rho_S\right)\Pi^{\lambda}.$$

The state of the reference system ρ_R before the next measurement is then described by tracing out the previous system S in the above state. Consequently we have the following map on ρ_R

Definition 2.3. *The frame degradation map* is

$$\mathcal{F}_{\rho_S}\left(\cdot\right) := tr_{\beta}\left[\sum_{\lambda} \Pi^{\lambda}\left(\cdot \otimes \rho_S\right)\Pi^{\lambda}\right] = tr_{\beta} \circ \sum_{\lambda} \mathcal{P}^{\lambda} \circ \left(\cdot \otimes \rho_S\right). \tag{2.28}$$

This map defines a discrete dynamical rule that acts on the QRF with each use. This dynamical rule defines how the QRF degrades as a result of past correlations induced by the measurement. The degradation depends on the states ρ_S that are being measured, and on the action of the irrep projectors Π^{λ} given by the decomposition of the product representation to irreps. The process that we are interested in, is the evolution of the reference state ρ_R as a result of a repeated application of $\mathcal{F}_{\rho_S}$ on the system R, which propagates this state in the Hilbert space. With some notion of quality of a QRF we may ask how many applications of $\mathcal{F}_{\rho_S}$ it takes to reach some threshold of quality. We call that the *longevity* of a QRF. As was shown in [11] and [13] calculation of longevity for QRFs of $SU(2)$ is not a simple task and for the general case it is probably not possible to find analytical expressions without some further assumptions on the decomposition to irreps. We will not attempt that here, however we would like to offer a few observations about the fixed points of $\mathcal{F}_{\rho_S}$.

A fixed point of $\mathcal{F}_{\rho_S}$, is a state ρ_R such that $\mathcal{F}_{\rho_S}(\rho_R) = \rho_R$. The reason that the fixed points are of interest to us, is because these are the states we should expect ρ_R to converge to with repeated application of $\mathcal{F}_{\rho_S}$. Strictly speaking, existence of fixed points of a map does not guarantee convergence to these points under repeated application of the map. Even though we will not deal with the issues of convergence here, we do know from the analysis of $SU(2)$, that the reference system there does converge to a unique state. Therefore we expect that at least in some cases the fixed points play a key role in the dynamics generated by $\mathcal{F}_{\rho_S}$.

Theorem 2.4. *Let $\rho_S = \frac{1}{d_\beta} I^\beta$ be a completely mixed state in an irrep β of G . Let $\mathcal{F}_{\rho_S}$ be a frame degradation map acting on $\mathcal{B}(\mathcal{H}^\alpha)$ according to the Definition 2.3. Then ρ_R is a fixed point of $\mathcal{F}_{\rho_S}$ if and only if all the states in the orbit of ρ_R are fixed points of $\mathcal{F}_{\rho_S}$. In addition, if $\mathcal{F}_{\rho_S}$ has a unique fixed point ρ_R, then $\rho_R = \frac{1}{d_\alpha} I^\alpha$.*

Proof. Both superoperators tr_β and $\sum_\lambda \mathcal{P}^\lambda$ are covariant with the group action $\mathcal{U}_g^\alpha$ in the sense that

$$tr_\beta \circ \mathcal{U}_g^{\alpha\beta} = \mathcal{U}_g^\alpha \circ tr_\beta,$$

$$\sum_\lambda \mathcal{P}^\lambda \circ \mathcal{U}_g^{\alpha\beta} = \mathcal{U}_g^{\alpha\beta} \circ \sum_\lambda \mathcal{P}^\lambda.$$

On the other hand the superoperator $(\cdot \otimes \rho_S)$, does not have this property so

$$(\cdot \otimes \rho_S) \circ \mathcal{U}_g^\alpha = \left(\mathcal{U}_g^\alpha(\cdot) \otimes \rho_S\right) = \mathcal{U}_g^{\alpha\beta} \circ \left(\cdot \otimes \mathcal{U}_g^{\beta\,\dagger}(\rho_S)\right).$$

Using these properties and Definition 2.3, we get the following commutation relation of $\mathcal{F}_{\rho_S}$ with the group action $\mathcal{U}_g^\alpha$

$$\mathcal{F}_{\rho_S} \circ \mathcal{U}_g^\alpha = \mathcal{U}_g^\alpha \circ \mathcal{F}_{\mathcal{U}_g^{\beta\,\dagger}(\rho_S)}. \tag{2.29}$$

Consider a state ρ_R which is a fixed point of $\mathcal{F}_{\rho_S}$ (that is $\mathcal{F}_{\rho_S}(\rho_R) = \rho_R$). Using the fact that $\rho_S = \frac{1}{d_\beta} I^\beta$ and Eq. (2.29) we derive

$$\mathcal{U}_g^\alpha(\rho_R) = \mathcal{U}_g^\alpha(\mathcal{F}_{\rho_S}(\rho_R)) = \mathcal{F}_{\mathcal{U}_g^\beta(\rho_S)}\left(\mathcal{U}_g^\alpha(\rho_R)\right) = \mathcal{F}_{\rho_S}\left(\mathcal{U}_g^\alpha(\rho_R)\right)$$

so $\mathcal{U}_g^\alpha(\rho_R)$ is also a fixed point of $\mathcal{F}_{\rho_S}$ for all $g \in G$. Therefore if ρ_R is a fixed point of $\mathcal{F}_{\rho_S}$ then all the states in the orbit of ρ_R are fixed points of $\mathcal{F}_{\rho_S}$. The converse follows from the fact that every state belongs to its own orbit.

If $\rho_R \neq \frac{1}{d_\alpha} I^\alpha$ is a unique fixed point of $\mathcal{F}_{\rho_S}$ then for any $g \in G$ such that $g \neq e$ the state $\mathcal{U}_g^\alpha(\rho_R) \neq \rho_R$ is also a fixed point of $\mathcal{F}_{\rho_S}$ which contradicts the uniqueness. Therefore $\rho_R = \frac{1}{d_\alpha} I^\alpha$ is the only possible unique fixed point of $\mathcal{F}_{\rho_S}$. □

The Theorem 2.4 is consistent with the $SU(2)$ analysis performed in [11] and [13]. They have showed that the frame degradation process induced by the measurements of a completely mixed state of spin, generates an evolution of the reference state toward a completely mixed state.

For frame degradation maps with an arbitrary ρ_S, we offer the following theorem

Theorem 2.5. *Let Π^λ be a projector on the irrep subspace $\mathcal{H}^\lambda$ of the Hilbert space $\mathcal{H}^\alpha \otimes \mathcal{H}^\beta$ for some irreps α, β, λ of G. Let $\mathcal{F}_{\rho_S}$ be a frame degradation map for some $\rho_S \in \mathcal{B}\left(\mathcal{H}^\beta\right)$ acting on $\mathcal{B}\left(\mathcal{H}^\alpha\right)$ according to the Definition 2.3. Then ρ_R is a fixed point of $\mathcal{F}_{\rho_S}$ if*

$$\sum_{\lambda \neq \lambda'} \Pi^\lambda \rho_R \otimes \rho_S \Pi^{\lambda'} \equiv 0. \tag{2.30}$$

In particular, $|\psi_R\rangle\langle\psi_R|$ ($|\psi_R\rangle \in \mathcal{H}^\alpha$) is a fixed point of $\mathcal{F}_{|\psi_S\rangle\langle\psi_S|}$ ($|\psi_S\rangle \in \mathcal{H}^\beta$) if

$$|\psi_R\rangle \otimes |\psi_S\rangle \in \mathcal{H}^\lambda.$$

Proof. From Definition 2.3 we can immediately see that ρ_R is a fixed point of $\mathcal{F}_{\rho_S}$ if $\rho_R \otimes \rho_S$ is a fixed point of $\sum_\lambda \mathcal{P}^\lambda$. Using the fact that $\sum_\lambda \Pi^\lambda = I^{\alpha\beta}$ we can express $\sum_\lambda \mathcal{P}^\lambda$ as

$$\sum_\lambda \mathcal{P}^\lambda(\cdot) = \sum_\lambda \Pi^\lambda(\cdot)\Pi^\lambda = \sum_{\lambda\lambda'} \Pi^\lambda(\cdot)\Pi^{\lambda'} - \sum_{\lambda\neq\lambda'} \Pi^\lambda(\cdot)\Pi^{\lambda'} = \mathcal{I}^{\alpha\beta}(\cdot) - \sum_{\lambda\neq\lambda'} \Pi^\lambda(\cdot)\Pi^{\lambda'}.$$

This shows that the state $\rho_R \otimes \rho_S$ is a fixed point of $\sum_\lambda \mathcal{P}^\lambda$ if

$$\sum_{\lambda\neq\lambda'} \Pi^\lambda(\rho_R \otimes \rho_S)\Pi^{\lambda'} \equiv 0.$$

This proves the general case. In particular, If $|\psi_R\rangle \otimes |\psi_S\rangle \in \mathcal{H}^\lambda$ then

$$\Pi^{\lambda'}(|\psi_R\rangle \otimes |\psi_S\rangle) = \delta_{\lambda\lambda'}|\psi_R\rangle \otimes |\psi_S\rangle$$

and Eq. (2.30) holds. □

In [13] it was shown that the frame degradation process induced by the measurements of a pure state of spin, results in the evolution of a reference state toward a pure state aligned with the measured spin. This means that the point of convergence of this process is the state $|l,l\rangle \otimes |s,s\rangle$. This result is consistent with the Theorem 2.5 as $|l,l\rangle \otimes |s,s\rangle$ is entirely in the irrep $\mathcal{H}^{l+s}$ of the sum of spins. It is curious to note that the same state $|l,l\rangle \otimes |s,s\rangle$ also maximizes the measure of the quality Q_{ψ_S} of a QRF (see Section 2.3). Further investigation into this correspondence may reveal a general principle that relates the fixed points of the frame degradation map to the quantity Q_{ψ_S}.

The Theorems 2.4 and 2.5 provide us with the possible states to which the reference frame may converges under repeat measurement. The theorems do not indicate uniqueness of the fixed points, and they do not promise convergence to them. Nevertheless, the analysis of the $SU(2)$ case show convergence to unique fixed points consistent with these theorems. That provides some evidence of their value.

3 The Poincaré Group and its unitary irreps with mass and spin

The Poincaré group is a group of symmetries of space-time. Representations of the Poincaré group are at the core of relativistic physics and appear in the description of many physical systems. The goal of this section is to introduce the Poincaré group and its massive unitary irreps in order to study the QRF associated with it. This section contains no original results and it is mostly based on the works of [17], [18], [19] and [20]. Most of the results and conventions presented here will serve us in the next section where we attempt to decompose a product of two irreps of the Poincaré group.

3.1 Basic definitions and notation

Definition 3.1. The *Minkowski space* is a 4-dimensional inner product space with respect to the metric

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

Elements of the Minkowski space, called *4-vectors,* will be denoted with the lower case Roman letters such as x, y, p, q while their 3-vector part will be denoted with the bold version $\mathbf{p}$, $\mathbf{q}$. Unit 3-vectors corresponding to $\mathbf{p}$ or p are specified as $\hat{\mathbf{p}}$. Also note that definition of the norm $\|\cdot\|$ depends on the context, that is $\|p\|$ is the Minkowski space norm, while $\|\mathbf{p}\|$ is the Euclidean norm.

Components of the 4-vectors will be specified with the Greek letter indices such as μ and ν that run over $0, 1, 2, 3$; indices specified with the Roman letters i,j run over $1, 2, 3$. Repeated upper and lower indices in expression are assumed to be summed over according to the Einstein convention. We choose the speed of light to be $c = 1$ so the energy and the momentum have the same units.

Definition 3.2. The 4-vector p will be called *time-like, null-like, or space-like* if its norm satisfies $\|p\| > 0,\ \|p\| = 0, \text{ or } \|p\| < 0$ respectively.

3.2 The Poincaré group

In this section we introduce the Poincaré group and review some of the algebraic results that will be relevant to the construction of irreducible representations. This section is intended to be an overview presenting the derivations of [17] and [19].

3.2.1 Definition of the Lorentz and the Poincaré groups

Definition 3.3. The *Proper Lorentz group* is denoted $SO^{+}(1,3)$ and is a matrix group of 4×4 matrices Λ with the properties: (i) $\det[\Lambda] = 1,$ (ii) $\Lambda^{0}{}_{0} \geq 1,$ (iii) $\Lambda^{\rho}{}_{\mu}\Lambda^{\sigma}{}_{\nu}\eta_{\rho\sigma} = \eta_{\mu\nu}.$ The elements $\Lambda \in SO^{+}(1,3)$ are called *Lorentz transformations.*

With this definition one can show that the proper Lorentz group is a subgroup of orthogonal transformation on the Minkowski space that are continuously connected to the identity. This means that these are all the inner product preserving transformations that are not space reflections or time reversals (or both). The key property of Lorentz transformations is the invariance of the metric expressed in (iii) which insures the invariance of the inner products

$$x \cdot y = x^{\mu}y^{\nu}\eta_{\mu\nu} = x^{\mu}y^{\nu}\Lambda^{\rho}{}_{\mu}\Lambda^{\sigma}{}_{\nu}\eta_{\rho\sigma} = (\Lambda x)\cdot(\Lambda y)\,.$$

Lemma 3.4. $SO(3)$ *is a subgroup of* $SO^{+}(1,3)$

Proof. Extending any $SO(3)$ matrix R to 4×4 according to

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix}$$

we can see that it has all the properties of the proper Lorentz matrix: (i) $\det[R] = 1$ is a defining property of $SO(3)$ matrices, (ii) $R^{0}{}_{0} = 1$ by the extension to 4×4 (iii) $R^{\rho}{}_{\mu}R^{\sigma}{}_{\nu}\eta_{\rho\sigma} = \eta_{\mu\nu}$ by the orthogonality of $SO(3)$ matrices. □

Definition 3.5. *The Poincaré group* is the semi-direct product $\mathcal{P}^{+} := SO^{+}(1,3) \ltimes \mathbb{R}^{4}$ consisting of the pairs (Λ, a) where $\Lambda \in SO^{+}(1,3)$ and $a \in \mathbb{R}^{4}$ with the multiplication given by $(\Lambda, a)(\Lambda', a') = (\Lambda\Lambda', \Lambda a' + a)$.

With this definition it is easy to verify that it is a group by checking that the multiplication is associative, the identity element is $(I, 0)$, and the inverse of (Λ, a) is $(\Lambda^{-1}, -\Lambda^{-1}a)$.

Lemma 3.6. *The set of transformations* $x \longmapsto \Lambda x + a$ *with* $\Lambda \in SO^{+}(1,3)$ *and* $x, a \in \mathbb{R}^{4}$*, is a representation of the Poincaré group on the Minkowski space.*

See [17] and [19] for the proof.

It is also easy to see that the subset of all the elements of the form $(\Lambda, 0)$ is isomorphic to the proper Lorentz group which makes $SO^{+}(1,3)$ a subgroup of the Poincaré group. Similarly the subset of all elements of the form (I, a) is isomorphic to $\mathbb{R}^{4}$ which is another subgroup. We will call the elements $(\Lambda, 0)$ *Lorentz transformations* and the elements (I, a) *translations* because of how they act on 4-vectors.

3.2.2 Lie algebra and Casimir operators of the Poincaré group

The Lie algebra and the Casimirs of the Poincaré group are derived in [17]. Here we present the results as definitions.

Definition 3.7. *The Lie algebra of the Poincaré group* consist of the 4 operators P_μ, the 3 operators J_i, and the 3 operators K_i with the commutation relations

$$\begin{array}{lll} [P_0, P_\nu] = 0 & [P_0, J_j] = 0 & [P_0, K_j] = -iP_i \\ [P_i, P_\nu] = 0 & [P_i, J_j] = i\epsilon_{ijk}P_k & [P_i, K_j] = -iP_0\delta_{ij} \\ & [J_i, J_j] = i\epsilon_{ijk}J_k & [J_i, K_j] = i\epsilon_{ijk}K_k \\ & & [K_i, K_j] = -i\epsilon_{ijk}J_k \end{array}$$

Definition 3.8. *The Pauli-Lubanski operator* is the 4-component operator defined by

$$W := \begin{pmatrix} \mathbf{P} \cdot \mathbf{J} \\ P_0 \, \mathbf{J} - \mathbf{P} \times \mathbf{K} \end{pmatrix}$$

where the bold $\mathbf{P}$, $\mathbf{J}$, and $\mathbf{K}$ are the 3-component operators $\mathbf{P} := (P_1, P_2, P_3)^T$, $\mathbf{J} := (J_1, J_2, J_3)^T$ and $\mathbf{K} := (K_1, K_2, K_3)^T$.

Lemma 3.9. *The operators $P^2 := P^\mu P_\mu$ and $W^2 := -W^\mu W_\mu$ commute with all the generators of the Lie algebra of the Poincaré group.*

The proof can be found in [17]. Since the Casimir operators P^2 and W^2 commute with all the operators in the Lie algebra, they commute with all the elements of the group which makes them essential to the construction of irreducible representations.

3.2.3 Standard boosts and Wigner rotations

Consider a Lorentz transformation $L(p) \in SO^+(1,3)$ that for any time-like 4-vector p acts on a the 4-vector $k = (\|p\|, 0, 0, 0)^T$ according to

$$L(p)\, k = p. \tag{3.1}$$

We will call such a $L(p)$ a *boost* and we will say that $L^{-1}(p)$ takes p to rest. The property $L(p)\, k = p$ does not uniquely define the transformation $L(p)$ since for any rotation $R \in SO(3)$, $Rk = k$ so the transformation $(L(p)\,R)$ also acts in the same way

$$(L(p)\,R)\,k = p.$$

In order to standardize the way that we transform to the rest frame we need to fix this freedom of rotation.

Definition 3.10. *The Standard boost* $L(p)$ is a Lorentz transformation such that for any time-like 4-vector p, for any rotation R and for $k = (\|p\|, 0, 0, 0)^T$

$$\begin{aligned} (i) \qquad & L(p)\,k = p, \\ (ii) \qquad & L(Rp) = RL(p)\,R^{-1}. \end{aligned}$$

One can show that the unique Lorentz transformations that satisfy these properties are of the form

$$L(p) = R(\hat{\mathbf{p}})\,B_3(p_0/\|p\|)\,R(\hat{\mathbf{p}})^{-1}$$

where $R(\hat{\mathbf{p}}) := R_3(\phi)\,R_2(\theta)$ is a rotation around the 2-axis followed by a rotation around the 3-axis with the angles θ, ϕ specified by $\hat{\mathbf{p}}$. The *pure boost* $B_3(\gamma)$ is

$$B_3(\gamma) := \begin{pmatrix} \gamma & 0 & 0 & \sqrt{\gamma^2 - 1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sqrt{\gamma^2 - 1} & 0 & 0 & \gamma \end{pmatrix}.$$

Note that for a given p, both $\hat{\mathbf{p}}$ and $\gamma = p_0/\|p\|$ do not depend on rescaling of norm $\|p\|$. Therefore the standard boost $L(p)$ depends only on the 4-velocity $v := p/\|p\|$, but we will still specify them with p for convenience.

The significance of Definition 3.10 is not very clear in the context of the defining representation of the Lorentz group but it is essential for representations with spin. At this point we will only note that by rearranging (ii) we have

$$R = L(Rp)\,RL^{-1}(p)\,, \tag{3.2}$$

and by applying the right hand side to p

$$\left(L(Rp)\,RL^{-1}(p)\right)p = L(Rp)\,Rk = L(Rp)\,k = Rp.$$

Obviously it eventually takes p to Rp but in between it acts with R on the rest vector k which is invariant in the defining representation. In spin representations the rest vector will not be invariant under R so the above definition of $L(p)$ also defines how the global rotation R acts on the local (rest frame) degrees of freedom which will be the spin components.

Having defined the relation $L(Rp) = RL(p)R^{-1}$ we may also ask what is the analogue relation for $L(\Lambda p)$ with a general Lorentz transformation Λ. From the property (3.1) we see that

$$L(\Lambda p)\,k = \Lambda p = \Lambda L(p)\,k.$$

This implies that $L(\Lambda p) = \Lambda L(p)\,R$ for some unknown rotation R.

Definition 3.11. *The Wigner rotation* [4] is the Lorentz transformation

$$W(\Lambda, p) := L^{-1}(\Lambda p)\,\Lambda L(p) \tag{3.3}$$

specified by any $\Lambda \in SO^{+}(1,3)$ and time-like 4-vector p.

As a result $L(\Lambda p) = \Lambda L(p)\,W^{-1}(\Lambda, p)$ which implies that $W(\Lambda, p)$, as defined above, must be in $SO(3)$. Explicit computation of $W(\Lambda, p)$ as a function of Λ and p is possible but is quite complicated, see for example [21]. For our purposes it suffices to know the definition and the fact that it is indeed a rotation. In addition we will require the following observations.

First of all, note that from the property (ii) of the standard boost and by the definition of Wigner rotation we immediately have

$$W(R, p) = R. \tag{3.4}$$

Rearranging the definition we can also see that

$$\Lambda = L(\Lambda p)\,W(\Lambda, p)\,L^{-1}(p)\,, \tag{3.5}$$

so the identity $W(R, p) = R$ implies that equation (3.5) is a generalization of (3.2). Following the same interpretation that was given for (3.2), we conclude that the Wigner rotation is the transformation on the local degrees of freedom such as spin, induced by the global transformation Λ. The difference between (3.2) and (3.5) is that in general this local transformation does depend on the location specified by p.

4

Definition 3.12. We will not confuse the Wigner rotation with the Pauli-Lubanski operator as we will always specify the dependence on (Λ, p) for the former.

3.3 Unitary irreps with mass and spin

The irreducible unitary representations of the Poincaré group were first classified by Wigner [22] and further developed by Bergmann [23] and Mackey [24, 25]. The goal of this section is to review the representations of the Poincaré group that act on the Hilbert space of a single particle with spin and mass. We will present the results discussed in [18], [19] and [20] omitting some details and derivations. Note that different authors use different choices of bases, standard boosts and normalization conventions.

3.3.1 A complete set of commuting operators

Irreducible unitary representations of the Poincaré group are constructed from the simultaneous eigenvectors of a complete set of commuting operators (CSCO). We will begin by reviewing the operators that will form our CSCO.

The operators P_μ in the Poincaré Lie algebra generate the subgroup of translations and therefore we associate their eigenvalues $p_\mu \in \mathbb{R}$ with 4-momentum. The Casimir operator $P^2 = P^\mu P_\mu$ has the eigenvalues $p^\mu p_\mu$ which must be the same in all reference frames. In particular in the rest reference frame $p^i p_i = 0$ therefore the eigenvalue of P^2 is $(p^0)^2$ i.e. the rest energy squared $(c = 1)$. Identifying the energy at rest as the mass we denote the eigenvalues of P^2 with m^2.

The Pauli-Lubanski operator W_μ acting on an eigenvector of P_μ can be expressed as

$$W_\mu = \begin{pmatrix} \mathbf{p} \cdot \mathbf{J} \\ p_0\, \mathbf{J} - \mathbf{p} \times \mathbf{K} \end{pmatrix}$$

where we have substituted the operators P_μ with their eigenvalues. In particular at rest we have $p = (m, 0, 0, 0)^T$ so the operator reduces to

$$W_\mu = m \begin{pmatrix} 0 \\ \mathbf{J} \end{pmatrix}. \tag{3.6}$$

The Casimir operator $W^2 = W^\mu W_\mu$ has the same eigenvalues in all reference frames so by taking the ones at rest we get the eigenvalues of $\mathbf{J}^2$ times m^2. The commutation relation $[J_i, J_j] = i\epsilon_{ijk} J_k$ implies that J_k are the generators of rotations so the eigenvalues of $\mathbf{J}^2$ are associated with the total angular momentum. Since the eigenvalues of W^2 are the total angular momentum at rest (times m^2) we will call them the total spin and denote with $m^2\, s(s+1)$ where s is a half-integer, integer or 0 as is well known for representations of $SO\,(3)$.

If spin is the angular momentum at rest then the Pauli-Lubanski operator reduces to the spin operator (times m) when acting on the eigenvectors of P_μ having the eigenvalue $p = (m, 0, 0, 0)^T$. In order to generalize the spin operators for an arbitrary eigenvector of P_μ we need to transform the Pauli-Lubanski operator to an arbitrary rest frame.

Definition 3.13. *The spin operators* S_i are defined for $m > 0$ as

$$S_i := \frac{1}{m}\left(L^{-1}(P)\,W\right)_i$$

where $L^{-1}(P)$ are the 4×4 matrices of the standard boost depending on the 4 components of the operator P. Using the explicit form of the matrices $L^{-1}(P)$ one can show that

$$S_i = \frac{1}{m}\left(W_i - \frac{W_0 P_i}{m+P_0}\right)$$

Lemma 3.14. *The spin operators* $S_i = \frac{1}{m}\left(W_i - \frac{W_0 P_i}{m+P_0}\right)$ *are the unique linear combinations of* W_μ *with the coefficients being the functions of* P_μ *such that the following conditions hold:*
(i) $[S_i, S_j] = i\epsilon_{ijk}S_k$, *(ii)* $[J_i, S_j] = i\epsilon_{ijk}S_k$, *(iii)* $S_i \equiv \frac{1}{m}W_i$ *when* $p_i = 0$ *.*

The proof of the lemma can be found in [18]. Conditions *(i)* and *(ii)* are necessary properties of the spin operators. Condition *(iii)* makes sure that $S^2 := S_i S^i = m^{-2}W_i W^i$ when $p_i = 0$. Furthermore when $p_i = 0$, according to Eq. (3.6), $W_0 = 0$ and so $S^2 = m^{-2}W_\mu W^\mu$, but since $W_\mu W^\mu$ is an invariant of the group so is S^2, and $S^2 = m^{-2}W_\mu W^\mu$ holds for all values of p_i. Thus condition *(iii)* makes sure that S^2 is an invariant of the group that is equal to $S_i S^i$ in the rest frame.

With lemma 3.14 we can be sure that the operators in the Definition 3.13 are the unique operators of spin such that the total spin is an invariant quantity. Note that the definition of S_i depends on the standard boosts $L(P)$. The choice of the standard boost that we have made in Section 3.2.3 is what resulted in operators $S_i = \frac{1}{m}\left(L^{-1}(P)\,W\right)_i$ having the properties of spin. Another common choice of the standard boosts is such that property (ii) in the Definition 3.10 reads $L(Rp) = RL(p)$ which leads to the operators of *helicity* instead of spin (see for example [20]).

Finally the CSCO we choose to work with is

$$\left\{P^2,\, S^2,\, \mathbf{P},\, S_3\right\}$$

where $\mathbf{P}$ are the 3 momentum operators P_i and $S^2 = m^{-2}W^2$. Next we will examine their simultaneous eigenvectors.

3.3.2 The momentum basis

Definition 3.15. *The momentum basis* consist of the eigenvectors of the CSCO $\{P^2,\, S^2,\, \mathbf{P},\, S_3\}$ specified with the kets $|\mathbf{p}, \sigma, m, s\rangle$. They are orthogonal

$$\langle \mathbf{p}', \sigma', m, s\,|\mathbf{p}, \sigma, m, s\rangle = N_{\mathbf{p}',m}\,\delta^3\left(\mathbf{p} - \mathbf{p}'\right)\delta_{\sigma\sigma'} \tag{3.7}$$

(the normalization factor $N_{\mathbf{p},m}$ will be chosen later) and their eigenvalues are

$$\begin{aligned}
P^2 \left|\mathbf{p}, \sigma, m, s\right\rangle &= m^2 \left|\mathbf{p}, \sigma, m, s\right\rangle , \\
S^2 \left|\mathbf{p}, \sigma, m, s\right\rangle &= s(s+1) \left|\mathbf{p}, \sigma, m, s\right\rangle , \\
P_i \left|\mathbf{p}, \sigma, m, s\right\rangle &= p_i \left|\mathbf{p}, \sigma, m, s\right\rangle , \\
S_3 \left|\mathbf{p}, \sigma, m, s\right\rangle &= \sigma \left|\mathbf{p}, \sigma, m, s\right\rangle .
\end{aligned}$$

Even though we label the kets with 3-momentum $\mathbf{p}$ it is trivial to derive the 4-momentum p according to $m^2 = p^\mu p_\mu$ thus $\left(p^0\right)^2 = m^2 + \|\mathbf{p}\|^2$ so by knowing m we can always assume the knowledge of the full p. Since p^0 is associated with the energy we will use the notation

$$E^m\left(\|\mathbf{p}\|\right) := \sqrt{m^2 + \|\mathbf{p}\|^2}$$

so that

$$p = \begin{pmatrix} E^m\left(\|\mathbf{p}\|\right) \\ \mathbf{p} \end{pmatrix} . \tag{3.8}$$

Note that for a fixed m the 4-vector $p = \left(E^m\left(\|\mathbf{p}\|\right), \mathbf{p}\right)^T$ lies on a 3-dimensional surface in Minkowski space.

Definition 3.16. For $m > 0$ *the positive mass shell* is the subset H_m^+ in Minkowski space such that $\mathsf{H}_m^+ := \{p \,|\, p^2 = m^2;\ p_0 \geq 0\}$.

Since Lorentz transformations leave the norm $p^2 = m^2$ invariant, positive mass shells H_m^+ are in fact orbits of time-like p under the Lorentz group. Geometrically speaking the surface H_m^+ is a 3-dimensional hyperboloid which in Minkowski space is the surface of equidistant points from the origin just like the sphere is the surface of equidistant points in Euclidean space. In this picture the mass m is the “radius” of the surface. According to Eq. (3.8) the 3-vectors $\mathbf{p} \in \mathbb{R}^3$ provide a parametrization of this surface. We can use this to construct an alternative coordinate system in Minkowski space. Mapping $(m, \mathbf{p}) \longmapsto p$ according to $p^0 = E^m\left(\|\mathbf{p}\|\right)$ and $p^i = \mathbf{p}^i$ we can specify a 4-momentum p by its mass shell and its 3-momentum part.

Another notation that we will use is for the 4×4 Lorentz transformation matrices. We will denote with the bold $\mathbf{\Lambda}$ (or $\mathbf{L}(p)$ for standard boosts) the lower 3×4 part of Λ so that $\mathbf{\Lambda} p$ is the 3-momentum part of the 4-momentum Λp. With this notation in mind we present the following lemma.

Lemma 3.17. *The momentum basis* $\left|\mathbf{p}, \sigma, m, s\right\rangle$ *transform under the Poincaré group action according to*

$$U\left(\Lambda, a\right) \left|\mathbf{p}, \sigma, m, s\right\rangle = e^{i(\Lambda p)\cdot a} \sum_{\sigma' = -s \ldots s} D^{(s)}_{\sigma'\sigma}\left[W\left(\Lambda, p\right)\right] \left|\mathbf{\Lambda} p, \sigma', m, s\right\rangle \tag{3.9}$$

where $D^{(s)}_{\sigma'\sigma}\left[W\left(\Lambda, p\right)\right]$ *is the spin* s *representation of* $SU(2)$.

For the proof see [19] but note the difference in normalization convention - we do not normalize the transformations, instead we normalize the orthogonality relation (3.7).

3.3.3 The Hilbert space of irreducible unitary representation

As is often the case when operators with continuous spectrum are involved, the Dirac's bra-ket formalism is mathematically ill defined unless a rigged Hilbert space is introduced [26]. We will avoid this complication by following [20] and define our irreducible representations as proper L^2 function spaces. Nevertheless, for convenience purposes, we will keep on using the bras and the kets of the momentum basis to span the irrep Hilbert space. However, we need to remember that they are not physical states and are not elements of the Hilbert space.

The operators P^2 and S^2 are the Casimirs which on irreducible representations are proportional to the identity (by Schur's lemma). This makes the quantities m and s, which specify their eigenvalues, the natural labels for inequivalent irreducible representations. We define the states $|\psi^{m,s}\rangle$ in the irrep of mass m and spin s as superpositions of elements in the momentum basis

$$|\psi^{m,s}\rangle \quad = \quad \sum_{\sigma=-s\dots s} \int_{\mathbb{R}^3} d\mu(\mathbf{p})|\mathbf{p},\sigma,m,s\rangle\psi_\sigma^{m,s}(\mathbf{p}) \tag{3.10}$$

where $\psi_\sigma^{m,s}$ are for now arbitrary complex valued functions over $\mathbb{R}^3$ and $d\mu$ is some measure. In order to determine the normalization $N_{\mathbf{p},m}$ in (3.7) and the measure $d\mu$ we require consistency of the orthogonality relation (3.7) with the inner product properties

$$\langle\psi^{m,s}|\psi^{m,s}\rangle = \sum_{\sigma=-s\dots s} \int_{\mathbb{R}^3} d\mu(\mathbf{p})\,|\psi_\sigma^{m,s}(\mathbf{p})|^2\,, \tag{3.11}$$

$$\langle\psi^{m,s}|\psi^{m,s}\rangle = \left\langle\psi^{m,s}|U^\dagger(\Lambda,a)\,U(\Lambda,a)\,|\psi^{m,s}\right\rangle. \tag{3.12}$$

Lemma 3.18. *With the choice of normalization $N_{\mathbf{p},m} := E^m(\|\mathbf{p}\|)$ and the measure $d\mu_m(\mathbf{p}) := \frac{d^3\mathbf{p}}{E^m(\|\mathbf{p}\|)}$ ($d^3\mathbf{p}$ is the Lebesgue measure on $\mathbb{R}^3$) the properties (3.11) and (3.12) of the inner product hold.*

Proof. Using (3.7) and $d\mu_m(\mathbf{p}) = \frac{d^3\mathbf{p}}{N_{\mathbf{p},m}}$ in the inner product of Eq. (3.10) with itself we have

$$\begin{aligned}\langle\psi^{m,s}|\psi^{m,s}\rangle &= \sum_{\sigma,\sigma'} \int_{\mathbb{R}^3} d\mu_m(\mathbf{p}) \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}'}{N_{\mathbf{p}',m}} N_{\mathbf{p}',m}\,\delta^3(\mathbf{p}-\mathbf{p}')\,\delta_{\sigma\sigma'}\,\psi_\sigma^{m,s}(\mathbf{p})\overline{\psi_{\sigma'}^{m,s}(\mathbf{p}')} \\ &= \sum_\sigma \int_{\mathbb{R}^3} d\mu_m(\mathbf{p})\,|\psi_\sigma^{m,s}(\mathbf{p})|^2\end{aligned}$$

which proves the property (3.11).

To prove (3.12) we first show the invariance of the measure under Lorentz transformations i.e. $d\mu_m(\mathbf{p}) = d\mu_m(\mathbf{\Lambda}p)$. Observe that the Lebesgue measure d^4p is an invariant measure in the Minkowski space with respect to the Lorentz group. Changing the coordinates to the mass shell coordinates $p \longmapsto (m, \mathbf{p})$ where $m = \sqrt{p^2}$ and $\mathbf{p}^i = p^i$ results in the Jacobian factor that is

$$d^4p = \frac{m}{E^m\left(\|\mathbf{p}\|\right)}\, dm\, d^3\mathbf{p} = m\, dm\, d\mu_m(\mathbf{p}). \tag{3.13}$$

The vector Λp maps to the new coordinates as $\Lambda p \longmapsto (m, \mathbf{\Lambda}p)$ and since $d^4p = d^4\left(\Lambda p\right)$ we have

$$m\, dm\, d\mu_m(\mathbf{\Lambda}p) = d^4\left(\Lambda p\right) = d^4p = m\, dm\, d\mu_m(\mathbf{p})$$

thus proving $d\mu_m(\mathbf{p}) = d\mu_m(\mathbf{\Lambda}p)$. Showing (3.12) is now a matter of explicit calculation applying the group action (3.9) and using the invariance of the measure. We show here the calculation for $s = 0$ (the choice of the measure is independent of the discrete degrees of freedom of spin).

$$\begin{aligned}
\left\langle \psi^m \middle| U^\dagger\left(\Lambda, a\right) U\left(\Lambda, a\right) \middle| \psi^m \right\rangle &= \int_{\mathbb{R}^3} d\mu_m(\mathbf{p}) \int_{\mathbb{R}^3} d\mu_m(\mathbf{p}') e^{i\Lambda(p-p')\cdot a} \langle \mathbf{\Lambda}p, m | \mathbf{\Lambda}p', m\rangle \psi^m(\mathbf{p})\overline{\psi^m(\mathbf{p}')} \\
&= \int_{\mathbb{R}^3} d\mu_m(\mathbf{p}) \int_{\mathbb{R}^3} d\mu_m(\mathbf{p}') e^{i(p-p')\cdot a} \langle \mathbf{p}, m | \mathbf{p}', m\rangle \psi^m(\mathbf{\Lambda}^{-1}p)\overline{\psi^m(\mathbf{\Lambda}^{-1}p')} \\
&= \int_{\mathbb{R}^3} d\mu_m(\mathbf{p}) \left|\psi^m(\mathbf{\Lambda}^{-1}p)\right|^2 = \int_{\mathbb{R}^3} d\mu_m(\mathbf{p}) \left|\psi^m(\mathbf{p})\right|^2 = \langle \psi^m | \psi^m \rangle
\end{aligned}$$

□

According to (3.11) for the inner product to be finite the functions $\psi_\sigma^{m,s}$ must be a part of the following function space:

Definition 3.19. The Hilbert space $L^2[\mathbb{R}^3, d\mu_m; \mathbb{C}^{2s+1}]$ is the space of vector valued functions

$$\begin{aligned}
\psi^{m,s}: \quad \mathbb{R}^3 &\longrightarrow \mathbb{C}^{2s+1} \\
\mathbf{p} &\longmapsto \psi_\sigma^{m,s}\left(\mathbf{p}\right)
\end{aligned}$$

with inner product

$$\langle \phi^{m,s}, \psi^{m,s} \rangle := \sum_{\sigma=-s\ldots s} \int_{\mathbb{R}^3} d\mu_m(\mathbf{p}) \overline{\phi_\sigma^{m,s}(\mathbf{p})} \psi_\sigma^{m,s}(\mathbf{p})$$

such that $\langle \phi^{m,s}, \psi^{m,s} \rangle < \infty$ for all $\phi^{m,s}, \psi^{m,s}$ in the space .

Observe that by using the invariance of the measure $d\mu_m$ we can lift the group action from the momentum basis to the functions in $L^2[\mathbb{R}^3, d\mu_m; \mathbb{C}^{2s+1}]$

$$\begin{aligned} U(\Lambda, a)\,|\psi^{m,s}\rangle &= \sum_{\sigma} \int_{\mathbb{R}^3} d\mu(\mathbf{p})\,\left(U(\Lambda, a)\,|\mathbf{p}, \sigma, m, s\rangle\right) \psi_{\sigma}^{m,s}(\mathbf{p}) \\ &= \sum_{\sigma} \int_{\mathbb{R}^3} d\mu_m(\mathbf{p}) \left(e^{i(\Lambda p)\cdot a} \sum_{\sigma'} D^{(s)}_{\sigma'\sigma}\left[W(\Lambda, p)\right] |\mathbf{\Lambda} p, \sigma', m, s\rangle \right) \psi_{\sigma}^{m,s}(\mathbf{p}) \\ &= \sum_{\sigma'} \int_{\mathbb{R}^3} d\mu_m(\mathbf{p}) |\mathbf{p}, \sigma', m, s\rangle \left(e^{ip\cdot a} \sum_{\sigma} D^{(s)}_{\sigma'\sigma}\left[W\left(\Lambda, \Lambda^{-1} p\right)\right] \psi_{\sigma}^{m,s}(\mathbf{\Lambda^{-1}} p) \right). \end{aligned}$$

The above observations lead us to the proposition from [20] that defines the irrep Hilbert space without referring to the momentum basis.

Proposition 3.20. *The Hilbert space $\mathcal{H}^{m,s} := L^2[\mathbb{R}^3, d\mu_m; \mathbb{C}^{2s+1}]$ carries an irreducible unitary representation of the Poincaré group given by the transformations*

$$\left(U(\Lambda, a)\,\psi^{m,s}\right)_{\sigma}(\mathbf{p}) := e^{ip\cdot a} \sum_{\sigma'} D^{(s)}_{\sigma\sigma'}\left[W\left(\Lambda, \Lambda^{-1} p\right)\right] \psi_{\sigma'}^{m,s}(\mathbf{\Lambda^{-1}} p). \tag{3.14}$$

For convenience we will specify the states in $\mathcal{H}^{m,s}$ either directly, in terms of the functions $\psi_{\sigma}^{m,s}(\mathbf{p})$, or use their ket form $|\psi^{m,s}\rangle$ as defined by Eq. (3.10). The domain $\mathbb{R}^3$ is a choice of parametrization of the mass shell H_m^+ through relation (3.8). The measure $d\mu_m$, according to Eq. (3.13), is the restriction of the invariant measure d^4p to the mass shell H_m^+. For this reason it is more natural to think of $\mathcal{H}^{m,s}$ as the space of functions over the mass shell H_m^+ with the restricted measure d^4p, rather then the space of functions over $\mathbb{R}^3$ with the measure $d\mu_m$.

4 Decomposition of a product of unitary irreps of the Poincaré group

A fundamental property of irreducible representations of a group is the way in which a product of irreps decomposes into a direct sum of irreps. The significance of this property to physics becomes clear in the analysis of composite systems consisting of subsystems each of which carry an irrep of the group. The most common example of such analysis is perhaps angular momentum addition (spin-orbit coupling as a special case) in which one derives the total angular momentum of a composite system consisting of irreps of $SU(2)$ [16]. In Section 2 we have seen that at the core of QRFs analysis we have a composite system consisting of a RF and a “system”. Also we have seen, that the projections of this composite system, on the subspaces of irreducible representations, play a key role in the analysis. Although the discussion in Section 2 was restricted to compact groups, it should be clear that one must understand the direct sum decomposition of the group (even if it is not compact) in order to analyze the QRFs associated with it. In this section we will present the decomposition of a product of two Poincaré irreps in the special case of massive irreps with no spin.

The decomposition of products of unitary irreps of the Poincaré group have been studied extensively, especially in the High-Energy community as part of scattering theory in relativistic regimes. One of the earliest papers on the subject is by Jacob and Wick [27]. They present the analysis for both massive and massless representations working with helicity states, as opposed to the spin states that we use here. Another work also in the context of particle collisions is by Macfarlane [28]. It gives a detailed overview of the decomposition focusing on massive representations and working with spin states. Later works attempt to cover all possible unitary irreps of the Poincaré group, including the space-like representations. One example of such study is by Whippman [29] which presents an original approach to the general decomposition. A rigorous mathematical approach to the general decomposition was presented by Schaaf [30]. A more modern account of the problem can be found in [31] where the Poincaré symmetry is extended with discrete symmetries of space-time (such as time reversal).

When dealing with finite-dimensional representations, decomposing a product of irreps is equivalent to finding the Clebsch-Gordan coefficients. These coefficients are the matrix elements of a similarity transformation that takes the product basis to a new basis. The desired property of the new basis is that it partitions the entire product Hilbert space into mutually orthogonal subspaces, each of which is an irrep. Once the Clebsch-Gordan coefficients are found, any vector in the product of irreps can be decomposed to its orthogonal components for each of the irreps in the decomposition. In the case of infinite-dimensional representations, like the ones given by $L^2[\mathbb{R}^3, d\mu_m; \mathbb{C}^{2s+1}]$ for the Poincaré group, Clebsch-Gordan coefficients are not numbers but normalized distributions which makes working with them as "coefficients" mathematically awkward. Nevertheless, the physics literature on this subject focuses exclusively on deriving this "coefficients" for the Poincaré group which are then used to decompose the states by integrating over the "coefficients" so that the distributions select the desired component of the state.

Realizing that CG coefficients are only a means to an end, which is decomposing the states, we propose a different approach here. We will circumvent the need for CG coefficients by directly showing the Hilbert space isomorphisms (which are a generalization of the similarity transformations to infinite-dimensional spaces) that map the product of Hilbert spaces into a direct sum of Hilbert spaces each of which is an irrep. The heuristic arguments and the "physics" that guide us through this process are of course similar to the ones that were used in the derivations of CG coefficients, but the mathematical structure is different.

4.1 Decomposition of a product of massive irreps with spin 0

We consider a spinless case setting $s = 0$ and focusing on the momentum degrees of freedom. The product of irreps with masses m_1 and m_2 is the space

$$\mathcal{H}^{m_1} \otimes \mathcal{H}^{m_2} = L^2[\mathbb{R}^3, d\mu_{m_1}; \mathbb{C}] \otimes L^2[\mathbb{R}^3, d\mu_{m_2}; \mathbb{C}].$$

Since irreps of the Poincaré group are a continuous set (in the m label) our goal is to find a Hilbert space isomorphism that maps this Hilbert space to a direct sum of a continuum of irreps, that is a *direct integral* of irreps. We will not define in advance the notion of direct integral, instead we will introduce a series of isomorphisms that eventually map the product space $\mathcal{H}^{m_1} \otimes \mathcal{H}^{m_2}$ to a new space in which the structure of a direct integral of irreps is easily identified. Each one of the Sections 4.1.1 through 4.1.4 presents a new Hilbert space which we show to be isomorphic to the one that came before, starting with $\mathcal{H}^{m_1} \otimes \mathcal{H}^{m_2}$. All of these isomorphisms composed result in the decomposition of the product of the two irreps. The final result is summarized in Section 4.1.5.

4.1.1 The Hilbert space $L^2[\mathbb{R}^3 \times \mathbb{R}^3, d\mu_{m_1} \times d\mu_{m_2}; \mathbb{C}]$

The first isomorphism we introduce is

$$\begin{array}{rcl} L^2[\mathbb{R}^3, d\mu_{m_1}; \mathbb{C}] \otimes L^2[\mathbb{R}^3, d\mu_{m_2}; \mathbb{C}] & \longrightarrow & L^2[\mathbb{R}^3 \times \mathbb{R}^3, d\mu_{m_1} \times d\mu_{m_2}; \mathbb{C}] \\ \psi^{m_1}(\mathbf{p}_1) \otimes \psi^{m_2}(\mathbf{p}_2) & \longmapsto & \psi^{m_1 m_2}(\mathbf{p}_1, \mathbf{p}_2) := \psi^{m_1}(\mathbf{p}_1)\, \psi^{m_2}(\mathbf{p}_2) \end{array}$$

This isomorphism is well known for L^2 spaces and it can be shown by definitions of the inner product on both spaces that

$$\langle \psi^{m_1} \otimes \psi^{m_2}, \phi^{m_1} \otimes \phi^{m_2} \rangle = \langle \psi^{m_1}, \phi^{m_1} \rangle \langle \psi^{m_2}, \phi^{m_2} \rangle = \langle \psi^{m_1 m_2}, \phi^{m_1 m_2} \rangle .$$

We can also express the group action in this space by acting on $\psi^{m_1} \otimes \psi^{m_2}$ and mapping it back to $\psi^{m_1 m_2}$

$$U(\Lambda, a)\, \psi^{m_1} \otimes U(\Lambda, a)\, \psi^{m_2} \longmapsto U(\Lambda, a)\, \psi^{m_1 m_2} := \left(U(\Lambda, a)\, \psi^{m_1}\right) \left(U(\Lambda, a)\, \psi^{m_2}\right).$$

Writing this explicitly using Eq. (3.14) we obtain the group action on the new space

$$\left(U(\Lambda, a)\, \psi^{m_1 m_2}\right)(\mathbf{p}_1, \mathbf{p}_2) = e^{i(p_1 + p_2)\cdot a} \psi^{m_1 m_2} \left(\mathbf{\Lambda}^{-1} p_1, \mathbf{\Lambda}^{-1} p_2\right).$$

4.1.2 The Hilbert space $L^2[\mathsf{H}^{\cup}_{m_1 m_2} \times \mathsf{S}^2, d\nu_{m_1 m_2} \times d\omega; \mathbb{C}]$

The main step in the decomposition of the product into irreps is the isomorphism induced by a change of the 6 variables of $\psi^{m_1 m_2}(\mathbf{p}_1, \mathbf{p}_2)$ to 6 new variables with new transformation properties under the group action. The goal is to find variables such that 2 of them are invariants corresponding to mass and spin, 3 of them transform like momentum, and 1 of them transforms like a spin component. Restriction of the function to a domain with fixed invariant variables will result in a function in 3 momentum like variables and 1 spin component which corresponds (if properly normalized) to a state in an irrep. This correspondence will allow us to identify the irreducible components of the function in the decomposition. Note that the 6 variables $\mathbf{p}_1, \mathbf{p}_2$ are continuous while 2 of the new variables we want are discrete (the spin ones). This implies that such a "change of variables" will involve an expansion of some of the continuous variables of the function in discrete series. Here we will derive intermediate continuous variables and in Section 4.1.3 we will identify and discretize the variable corresponding to spin.

Proposition 4.1. *The change of variables $(\mathbf{p}_1, \mathbf{p}_2) \longmapsto (p, \hat{\mathbf{q}})$ and its inverse as defined by the following functions, produce the variables that correspond to momentum and (later on) spin.*

$$\begin{aligned} \mathsf{p}(\mathbf{p}_1, \mathbf{p}_2) &:= p_1 + p_2 \\ \hat{\mathsf{q}}(\mathbf{p}_1, \mathbf{p}_2) &:= \frac{\mathbf{L^{-1}}(p_1+p_2)\,(p_1-p_2)}{\|\mathbf{L^{-1}}(p_1+p_2)\,(p_1-p_2)\|} \end{aligned} \tag{4.1}$$

$$\begin{aligned} \mathsf{p}_1(p, \hat{\mathbf{q}}) &:= \frac{\mathbf{p}}{2} + \mathbf{L}(p)\,\mathsf{q}(\|p\|, \hat{\mathbf{q}}) \\ \mathsf{p}_2(p, \hat{\mathbf{q}}) &:= \frac{\mathbf{p}}{2} - \mathbf{L}(p)\,\mathsf{q}(\|p\|, \hat{\mathbf{q}}) \end{aligned} \tag{4.2}$$

where the 4-vector valued function q *is*

$$\mathsf{q}(\|p\|, \hat{\mathbf{q}}) := \frac{1}{2\|p\|}\begin{pmatrix} m_1^2 - m_2^2 \\ \hat{\mathbf{q}}\,\kappa^{m_1 m_2}(\|p\|) \end{pmatrix}$$

and the scalar valued function $\kappa^{m_1 m_2}$ *is*

$$\kappa^{m_1 m_2}(\|p\|) := \sqrt{\|p\|^4 + m_1^4 + m_2^4 - 2(m_1^2 m_2^2 + m_1^2\|p\|^2 + m_2^2\|p\|^2)}. \tag{4.3}$$

Note that:

1. The script p, q (as opposed to p, q) is used to signify that these are *functions* that map the variables and are not variables themselves.

2. The masses m_1, m_2 implicitly participate in the change of variables through the mapping of $\mathbf{p}_1 \longmapsto p_1$ according to (3.8).

3. $\hat{\mathsf{q}}(\mathbf{p}_1, \mathbf{p}_2)$ is a unit 3-vector (note the bold $\mathbf{L}^{-1}$).

4. The proof that (4.2) is the inverse of (4.1) is a straightforward (but quite technical) application of the standard boosts in their explicit form.

The interpretation of $\mathsf{p}\,(\mathbf{p}_1, \mathbf{p}_2)$ is simple: it is the total 4-momentum of p_1 and p_2. The interpretation of $\hat{\mathsf{q}}(\mathbf{p}_1, \mathbf{p}_2)$ is a bit trickier. First note that the standard boost $L^{-1}(p_1+p_2)$ takes the total momentum $p_1 + p_2$ to rest, that is the reference frame where the total 3-momentum vanishes. This means that $\mathbf{L}^{-1}(p_1 + p_2)\,(p_1 + p_2) = \mathbf{0}$ and by linearity $\mathbf{L}^{-1}(p_1 + p_2)p_1 = -\mathbf{L}^{-1}(p_1 + p_2)p_2$. Using the last identity and linearity we arrive at

$$\mathbf{L}^{-1}(p_1 + p_2)\,(p_1 - p_2) = 2\mathbf{L}^{-1}(p_1 + p_2)p_1 = -2\mathbf{L}^{-1}(p_1 + p_2)p_2.$$

Taking this back to the definition (4.1) we can see that $\hat{\mathsf{q}}(\mathbf{p}_1, \mathbf{p}_2)$ is in the direction of $\mathbf{L}^{-1}(p_1+p_2)p_1$ and opposite to the direction of $\mathbf{L}^{-1}(p_1 + p_2)p_2$. The unit vector $\hat{\mathsf{q}}(\mathbf{p}_1, \mathbf{p}_2)$ selects the axis along which the 3-momenta are opposite to each other in the rest frame of the total momentum. This is the direction of relative momentum at rest.

In order to define a Hilbert space of functions in the new variables, we need to establish a few of their properties including how they transform, what the new domain is and what the new invariant measure is.

Lemma 4.2. *The functions in (4.1) have the following transformation properties under a Lorentz transformation Λ and for all $\mathbf{p}_1, \mathbf{p}_2$*

$$\begin{aligned} \mathsf{p}\,(\mathbf{\Lambda} p_1, \mathbf{\Lambda} p_2) &= \Lambda \mathsf{p}\,(\mathbf{p}_1, \mathbf{p}_2) \\ \hat{\mathsf{q}}(\mathbf{\Lambda} p_1, \mathbf{\Lambda} p_2) &= W\,(\Lambda, p_1 + p_2)\,\hat{\mathsf{q}}(\mathbf{p}_1, \mathbf{p}_2) \end{aligned} \tag{4.4}$$

where $W\,(\Lambda, p_1 + p_2)$ is the Wigner rotation defined in (3.3).

Proof. The first property is a trivial result of linearity

$$\mathsf{p}\,(\mathbf{\Lambda} p_1, \mathbf{\Lambda} p_2) = \Lambda p_1 + \Lambda p_2 = \Lambda\,(p_1 + p_2) = \Lambda \mathsf{p}\,(\mathbf{p}_1, \mathbf{p}_2)\,.$$

For the second property, recall Definition 3.11 of the Wigner rotation. By rearranging it, we can see that

$$L^{-1}\,(\Lambda\,(p_1 + p_2)) = W\,(\Lambda, p_1 + p_2)\,L^{-1}\,(p_1 + p_2)\,\Lambda^{-1},$$

therefore

$$L^{-1}(\Lambda p_1+\Lambda p_2)\,(\Lambda p_1-\Lambda p_2) \;=\; W\,(\Lambda,p_1+p_2)\,L^{-1}\,(p_1+p_2)\,(p_1-p_2)\,.$$

Substituting that into the definition of $\hat{\mathsf{q}}(\mathbf{p}_1,\mathbf{p}_2)$ and using invariance of the Euclidean norm under rotations, we get the desired result

$$\begin{aligned}\hat{\mathsf{q}}(\mathbf{\Lambda}p_1,\mathbf{\Lambda}p_2) &= \frac{W\,(\Lambda,p_1+p_2)\,\mathbf{L}^{-1}\,(p_1+p_2)\,(p_1-p_2)}{\|W\,(\Lambda,p_1+p_2)\,\mathbf{L}^{-1}\,(p_1+p_2)\,(p_1-p_2)\|}\\ &= W\,(\Lambda,p_1+p_2)\,\frac{\mathbf{L}^{-1}\,(p_1+p_2)\,(p_1-p_2)}{\|\mathbf{L}^{-1}\,(p_1+p_2)\,(p_1-p_2)\|} = W\,(\Lambda,p_1+p_2)\,\hat{\mathsf{q}}(\mathbf{p}_1,\mathbf{p}_2).\end{aligned}$$

□

Similarly to the proof of Lemma 4.2, one can show the transformation properties of the inverse change of variable

$$\mathbf{\Lambda}\mathsf{p}_{1,2}\,(p,\hat{\mathbf{q}}) = \mathsf{p}_{1,2}\,(\Lambda p, W\,(\Lambda,p)\,\hat{\mathbf{q}})$$

and also

$$\mathbf{\Lambda}^{-1}\mathsf{p}_{1,2}\,(p,\hat{\mathbf{q}}) \;=\; \mathsf{p}_{1,2}\left(\Lambda^{-1}p, W^{-1}\left(\Lambda,\Lambda^{-1}p\right)\hat{\mathbf{q}}\right). \tag{4.5}$$

Lemma 4.2 tells us that that the new variables $(p,\hat{\mathbf{q}})$ transform under Λ according to

$$(p,\hat{\mathbf{q}}) \overset{\Lambda}{\longmapsto} (\Lambda p, W\,(\Lambda,p)\,\hat{\mathbf{q}})\,. \tag{4.6}$$

Since p transforms like a 4-vector, its norm is an invariant quantity, associated with the rest energy of the total momentum. Recall that we have defined mass as the rest energy of the 4-momentum. We can now recognize the quantity $\|p\|$ as the new "mass" which will specify the different irreps in the decomposition and the 3-vector part $\mathbf{p}$ is the new 3-momentum variable. The transformation property of $\hat{\mathbf{q}}$ suggests that it is associated with the new "spin" because in irreps the spin degree of freedom is also being transformed by the Wigner rotation (see Eq. (3.14)). We will show this association in Section 4.1.3 when we expand the functions in spherical harmonics.

The next task is to determine the new domain for the variables $(p,\hat{\mathbf{q}})$

Lemma 4.3. *The functions in (4.1) have the following images*

$$\begin{aligned}\mathsf{p}\left(\mathbb{R}^3\times\mathbb{R}^3\right) &= \mathsf{H}^{\cup}_{m_1m_2} := \bigcup_{m\geq m_1+m_2}\mathsf{H}^{+}_{m}\\ \hat{\mathsf{q}}\left(\mathbb{R}^3\times\mathbb{R}^3\right) &= \mathsf{S}^2\end{aligned}$$

where H_m^+ *are the mass shells given by Definition 3.16 and* S^2 *is the unit 2-sphere in* $\mathbb{R}^3$*.*

Proof. Writing down the norm of p explicitly and using the fact that $\|p_i\| = m_i$ and that $p_1 \cdot p_2 = \|p_1\| \, \|p_2\| \cosh \eta$ where η is a *hyperbolic angle* between p_1 and p_2, we have

$$\begin{aligned} \|\mathsf{p}(\mathbf{p}_1, \mathbf{p}_2)\|^2 &= \|p_1 + p_2\|^2 \\ &= (m_1)^2 + (m_2)^2 + 2m_1 m_2 \cosh \eta \\ &\geq (m_1 + m_2)^2 . \end{aligned}$$

The bound is saturated for $\eta = 0$ which happens when $p_1 \propto p_2$. It is also easy to see that $\cosh \eta = \frac{p_1 \cdot p_2}{m_1 m_2}$ is unbounded, if we choose for example p_1, p_2 such that $\mathbf{p}_1 = -\mathbf{p}_2$ for arbitrarily large $\|\mathbf{p}_1\|$. Therefore

$$\|\mathsf{p}(\mathbf{p}_1, \mathbf{p}_2)\| \in [(m_1 + m_2), \infty) .$$

This implies that $\mathsf{p}(\mathbf{p}_1, \mathbf{p}_2) \in \mathsf{H}^{\cup}_{m_1 m_2}$ if $(\mathsf{p}(\mathbf{p}_1, \mathbf{p}_2))_0 \geq 0$ which is true because $(p_i)_0 \geq 0$. This shows that $\mathsf{p}(\mathbf{p}_1, \mathbf{p}_2)$ is a map into $\mathsf{H}^{\cup}_{m_1 m_2}$. On the other hand for any $p \in \mathsf{H}^{\cup}_{m_1 m_2}$, there are $(\mathbf{p}_1, \mathbf{p}_2) \in \mathbb{R}^3 \times \mathbb{R}^3$ such that $\|p\| = \|\mathsf{p}(\mathbf{p}_1, \mathbf{p}_2)\|$, so there must be a $\Lambda \in SO^+(1,3)$ such that $p = \Lambda \mathsf{p}(\mathbf{p}_1, \mathbf{p}_2) = \mathsf{p}(\mathbf{\Lambda} p_1, \mathbf{\Lambda} p_2)$. This shows that for any $p \in \mathsf{H}^{\cup}_{m_1 m_2}$ there are vectors $(\mathbf{\Lambda} p_1, \mathbf{\Lambda} p_2) \in \mathbb{R}^3 \times \mathbb{R}^3$ that map to it. This finished the proof that $\mathsf{H}^{\cup}_{m_1 m_2}$ is the image of p.

The fact that $\hat{\mathsf{q}} \in \mathsf{S}^2$ is self-evident from the defining Eq. (4.1). Recalling the transformation property (4.4) and the Wigner rotation property (3.4) we can write for any rotation $R \in SO(3)$

$$\hat{\mathsf{q}}(R\mathbf{p}_1, R\mathbf{p}_2) = W(R, p_1 + p_2)\, \hat{\mathsf{q}}(\mathbf{p}_1, \mathbf{p}_2) = R\hat{\mathsf{q}}(\mathbf{p}_1, \mathbf{p}_2).$$

For any $\hat{q} \in \mathsf{S}^2$ and for some $\hat{\mathsf{q}}(\mathbf{p}_1, \mathbf{p}_2)$ there is $R \in SO(3)$ such that $\hat{q} = R\hat{\mathsf{q}}(\mathbf{p}_1, \mathbf{p}_2) = \hat{\mathsf{q}}(R\mathbf{p}_1, R\mathbf{p}_2)$ thus there are $(R\mathbf{p}_1, R\mathbf{p}_2) \in \mathbb{R}^3 \times \mathbb{R}^3$ that can map to any $\hat{\mathsf{q}} \in \mathsf{S}^2$. This proves that S^2 is the image of $\hat{\mathsf{q}}$. □

This lemma shows that the domain of the functions in the new variables is $\mathsf{H}^{\cup}_{m_1 m_2} \times \mathsf{S}^2$.

The last item we need to establish in order to define the new Hilbert space is a measure on the new domain that will insure preservation of the inner product under the change of variables. The key property of the new measure $d\nu(p) \times d\omega(\hat{\mathbf{q}})$ that we demand is

$$\int\limits_{\mathsf{H}^{\cup}_{m_1 m_2}} d\nu(p) \int\limits_{\mathsf{S}^2} d\omega(\hat{\mathbf{q}})\; f(p, \hat{\mathbf{q}}) = \int\limits_{\mathbb{R}^3} d\mu_{m_1}(\mathbf{p}_1) \int\limits_{\mathbb{R}^3} d\mu_{m_2}(\mathbf{p}_2)\, f(\mathsf{p}(\mathbf{p}_1, \mathbf{p}_2), \hat{\mathsf{q}}(\mathbf{p}_1, \mathbf{p}_2)) \tag{4.7}$$

for any $f(p, \hat{\mathbf{q}})$. This ensures preservation of the L^2 inner product. Such a volume-preserving measure can in principle be derived by calculating the Jacobian of the change of variables (4.1). It turns out that this calculation is difficult so instead we offer the following ad hoc reasoning to get the result.

The new measure also has to be invariant under the Lorentz transformations (otherwise equation (4.7) will not hold if we transform f on both sides). According to Eq. (4.6) this means

$$d\nu\left(p\right)\times d\omega\left(\hat{\mathbf{q}}\right)=d\nu\left(\Lambda p\right)\times d\omega\left(W\left(\Lambda,p\right)\hat{\mathbf{q}}\right).$$

One measure with this property is $d^4p\times d^2\hat{\mathbf{q}}$ where d^4p is the Lebesgue measure and $d^2\hat{\mathbf{q}}$ is a rotationally invariant measure on S^2 such that $\int_{\mathsf{S}^2} d^2\hat{\mathbf{q}}=1$ (we will not use a coordinate system on S^2 so there is no need for an explicit expression of $d^2\hat{\mathbf{q}}$). This measure is not unique and if we multiply it with any function of Lorentz invariants it will remain an invariant measure. From this we conclude:

Proposition 4.4. *An invariant measure on $\mathsf{H}^{\cup}_{m_1m_2}\times\mathsf{S}^2$ with the property (4.7) is of the form*

$$\frac{1}{N^{m_1m_2}\left(\|p\|\right)}d^4p\times d^2\hat{\mathbf{q}}$$

up to a choice of normalization $N^{m_1m_2}\left(\|p\|\right)$.

In order to determine $N^{m_1m_2}\left(\|p\|\right)$ we choose $f\left(p,\hat{\mathbf{q}}\right):=\delta^4\left(p-k\right)\hat{\delta}^2\left(\hat{\mathbf{q}}-\hat{\mathbf{z}}\right)$ where $k=\left(\|k\|,\mathbf{0}\right)^T$ such that $\|k\|>m_1+m_2$ and $\hat{\delta}^2\left(\hat{\mathbf{q}}-\hat{\mathbf{n}}\right)$ is such that $\int_{\mathsf{S}^2} d^2\hat{\mathbf{q}}\,\hat{\delta}^2\left(\hat{\mathbf{q}}-\hat{\mathbf{n}}\right)g\left(\hat{\mathbf{q}}\right)=g\left(\hat{\mathbf{n}}\right)$ for any function g and $\hat{\mathbf{n}}\in\mathsf{S}^2$. Plugging this $f\left(p,\hat{\mathbf{q}}\right)$ into Eq. (4.7) and evaluating both sides yields

$$\begin{aligned}\frac{1}{N^{m_1m_2}\left(\|k\|\right)} &= \int_{\mathsf{H}^{\cup}_{m_1m_2}}\frac{1}{N^{m_1m_2}\left(\|p\|\right)}d^4p\int_{\mathsf{S}^2}d^2\hat{\mathbf{q}}\,\delta^4\left(p-k\right)\hat{\delta}^2\left(\hat{\mathbf{q}}-\hat{\mathbf{n}}\right)\\ &= \int_{\mathbb{R}^3}d\mu_{m_1}(\mathbf{p}_1)\int_{\mathbb{R}^3}d\mu_{m_2}(\mathbf{p}_2)\delta^4\left(\mathsf{p}\left(\mathbf{p}_1,\mathbf{p}_2\right)-k\right)\hat{\delta}^2\left(\hat{\mathsf{q}}(\mathbf{p}_1,\mathbf{p}_2)-\hat{\mathbf{n}}\right)\\ &= \frac{\kappa^{m_1m_2}\left(\|k\|\right)}{2\left\|k\right\|^2}\end{aligned}$$

with the factor $\kappa^{m_1m_2}$ as defined by Eq. (4.3). We omit the details of this calculation as it is long and uninsightful. Finally, the invariant measure is given by

$$d\nu_{m_1m_2}\left(p\right)\times d\omega\left(\hat{\mathbf{q}}\right)=\frac{\kappa^{m_1m_2}\left(\|p\|\right)}{2\left\|p\right\|^2}d^4p\times d^2\hat{\mathbf{q}}.$$

Consequently, the result of the change of variables is an isomorphism to a new Hilbert space given by the following proposition:

Proposition 4.5. *The change of variables (4.1), (4.2) defines the following isomorphism*

$$\begin{aligned} L^2[\mathbb{R}^3 \times \mathbb{R}^3, d\mu_{m_1} \times d\mu_{m_2}; \mathbb{C}] &\longrightarrow L^2[\mathsf{H}^{\cup}_{m_1 m_2} \times \mathsf{S}^2, d\nu_{m_1 m_2} \times d\omega; \mathbb{C}] \\ \psi^{m_1 m_2}(\mathbf{p}_1, \mathbf{p}_2) &\longmapsto \phi^{m_1 m_2}(p, \hat{\mathbf{q}}) := \psi^{m_1 m_2}(\mathsf{p}_1(p, \hat{\mathbf{q}}), \mathsf{p}_2(p, \hat{\mathbf{q}})) \end{aligned} \tag{4.8}$$

accompanied by the group action

$$(U(\Lambda, a)\,\phi^{m_1 m_2})(p, \hat{\mathbf{q}}) = e^{ip\cdot a}\phi^{m_1 m_2}\left(\Lambda^{-1}p, W^{-1}\left(\Lambda, \Lambda^{-1}p\right)\hat{\mathbf{q}}\right). \tag{4.9}$$

The proof follows from the preceding lemmas and discussion.

4.1.3 The Hilbert space $\bigoplus_l L^2[\mathsf{H}^{\cup}_{m_1 m_2}, d\nu_{m_1 m_2}; \mathbb{C}^{2l+1}]$

In order to complete the decomposition we need to identify the new spin variables. The heuristic argument is similar to that we used to identify the mass. If spin is the total angular momentum at rest and the individual particles have no spin then the new "spin" must be the orbital angular momentum at rest. By definition the variable $\hat{\mathbf{q}}$ specifies the direction of relative momentum of the individual particles in the rest frame of the total momentum. Functions of momentum can be expanded in spherical harmonics which are eigenfunctions of the orbital angular momentum operator [16]. Therefore if we expand the $\hat{\mathbf{q}}$ part of the state function $\phi^{m_1 m_2}(p, \hat{\mathbf{q}})$ in spherical harmonics we should have the components of the orbital angular momentum at rest.

Definition 4.6. *The spherical harmonics expansion coefficients* $\phi^{m_1 m_2}_{l\lambda}$ of a function $\phi^{m_1 m_2}(p, \hat{\mathbf{q}})$ are

$$\phi^{m_1 m_2}_{l\lambda}(p) := \int_{\mathsf{S}^2} d^2\hat{\mathbf{q}}\,\phi^{m_1 m_2}(p, \hat{\mathbf{q}})\overline{Y^{\lambda}_l(\hat{\mathbf{q}})} \tag{4.10}$$

where $Y^{\lambda}_l(\hat{\mathbf{q}})$ are the spherical harmonics. These coefficients are used in the expansion of $\phi^{m_1 m_2}(p, \hat{\mathbf{q}})$ in spherical harmonics according to

$$\phi^{m_1 m_2}(p, \hat{\mathbf{q}}) = \sum_{l=0}^{\infty} \sum_{\lambda=-l}^{l} \phi^{m_1 m_2}_{l\lambda}(p)\, Y^{\lambda}_l(\hat{\mathbf{q}}).$$

The coefficients $\phi^{m_1 m_2}_{l\lambda}$ and the expansion of $\phi^{m_1 m_2}$ are given here as a definition relying on the well known fact (see for example [32]) that the set of functions $\left\{Y^{\lambda}_l\right\}$ is a complete and orthonormal set in the space $L^2\left[\mathsf{S}^2, d^2\hat{\mathbf{q}}; \mathbb{C}\right]$ of square integrable functions on a sphere.

Lemma 4.7. *The expansion coefficients* $\phi^{m_1 m_2}_{l\lambda}(p)$ *transform under the Poincaré group according* *to*

$$\left(U\left(\Lambda,a\right)\phi^{m_1m_2}\left(p\right)\right)_{l\lambda}=e^{ip\cdot a}\sum_{\lambda'}D^{(l)}_{\lambda\lambda'}\left[W\left(\Lambda,\Lambda^{-1}p\right)\right]\phi^{m_1m_2}_{l\lambda'}\left(\Lambda^{-1}p\right)$$

where $D^{(l)}_{\lambda\lambda'}$ *is the spin* l *representation of* $SO(3)$ *and* $W\left(\Lambda,\Lambda^{-1}p\right)$ *is the Wigner rotation.*

Proof. By applying the group action (4.9) on the right hand side of definition (4.10) we have

$$\begin{aligned}\left(U\left(\Lambda,a\right)\phi^{m_1m_2}\right)_{l\lambda}\left(p\right) &= \int_{\mathsf{S}^2}d^2\hat{\mathbf{q}}\left(U\left(\Lambda,a\right)\phi^{m_1m_2}\right)\left(p,\hat{\mathbf{q}}\right)\overline{Y_l^{\lambda}\left(\hat{\mathbf{q}}\right)}\\ &= e^{ip\cdot a}\int_{\mathsf{S}^2}d^2\hat{\mathbf{q}}\phi^{m_1m_2}\left(\Lambda^{-1}p,W^{-1}\left(\Lambda,\Lambda^{-1}p\right)\hat{\mathbf{q}}\right)\overline{Y_l^{\lambda}\left(\hat{\mathbf{q}}\right)}.\end{aligned}$$

Using the invariance of the measure $d^2\hat{\mathbf{q}}$ we can lift the Wigner rotation from $\phi^{m_1m_2}$ to Y_l^{λ}

$$\left(U\left(\Lambda,a\right)\phi^{m_1m_2}\right)_{l\lambda}\left(p\right) = e^{ip\cdot a}\int_{\mathsf{S}^2}d^2\hat{\mathbf{q}}\phi^{m_1m_2}\left(\Lambda^{-1}p,\hat{\mathbf{q}}\right)\overline{Y_l^{\lambda}\left(W\left(\Lambda,\Lambda^{-1}p\right)\hat{\mathbf{q}}\right)}.$$

Finally with the spin representation identities

$$\begin{aligned}Y_l^{\lambda}\left(R^{-1}\hat{\mathbf{q}}\right) &= \sum_{\lambda'}Y_l^{\lambda'}\left(\hat{\mathbf{q}}\right)D^{(l)}_{\lambda'\lambda}\left[R\right],\\ D^{(l)}_{\lambda\lambda'}\left[R\right] &= \left(D^{(l)}_{\lambda\lambda'}\left[R^{-1}\right]\right)^{\dagger}=\overline{D^{(l)}_{\lambda'\lambda}\left[R^{-1}\right]}\end{aligned}$$

we get the desired result

$$\begin{aligned}\left(U\left(\Lambda,a\right)\phi^{m_1m_2}\right)_{l\lambda}\left(p\right) &= e^{ip\cdot a}\int_{\mathsf{S}^2}d^2\hat{\mathbf{q}}\phi^{m_1m_2}\left(\Lambda^{-1}p,\hat{\mathbf{q}}\right)\overline{\sum_{\lambda'}Y_l^{\lambda'}\left(\hat{\mathbf{q}}\right)D^{(l)}_{\lambda'\lambda}\left[W^{-1}\left(\Lambda,\Lambda^{-1}p\right)\right]}\\ &= e^{ip\cdot a}\sum_{\lambda'}D^{(l)}_{\lambda\lambda'}\left[W\left(\Lambda,\Lambda^{-1}p\right)\right]\phi^{m_1m_2}_{l\lambda'}\left(\Lambda^{-1}p\right).\end{aligned}$$

□

Recognizing the l and the λ as the new "spin" variables we construct the *spin states* $\phi_l^{m_1m_2}\left(p\right)\in\mathbb{C}^{2l+1}$ from the coefficients $\phi^{m_1m_2}_{l\lambda}\left(p\right)$

$$\left(\phi_l^{m_1m_2}\left(p\right)\right)_{\lambda}:=\phi^{m_1m_2}_{l\lambda}\left(p\right).$$

The elements $\phi_l^{m_1m_2}$ form the Hilbert space $L^2[\mathsf{H}^{\cup}_{m_1m_2},d\nu_{m_1m_2};\mathbb{C}^{2l+1}]$ which we use for a construction of the new isomorphic Hilbert space.

Proposition 4.8. *The expansion (4.10) defines the following isomorphism*

$$\begin{aligned} L^2[\mathsf{H}^{\cup}_{m_1m_2}\times\mathsf{S}^2, d\nu_{m_1m_2}\times d\omega;\mathbb{C}] &\longrightarrow \bigoplus_l L^2[\mathsf{H}^{\cup}_{m_1m_2}, d\nu_{m_1m_2};\mathbb{C}^{2l+1}] \\ \phi^{m_1m_2}(p,\hat{\mathbf{q}}) &\longmapsto \bigoplus_l \phi^{m_1m_2}_l(p) \end{aligned}$$

where

$$\left(\phi^{m_1m_2}_l(p)\right)_\lambda := \int_{\mathsf{S}^2} d^2\hat{\mathbf{q}}\,\phi^{m_1m_2}(p,\hat{\mathbf{q}})\overline{Y^\lambda_l(\hat{\mathbf{q}})}$$

and the group action

$$\left(U(\Lambda,a)\,\phi^{m_1m_2}_l\right)_\lambda(p) = e^{ip\cdot a}\sum_{\lambda'} D^{(l)}_{\lambda\lambda'}\left[W\left(\Lambda,\Lambda^{-1}p\right)\right]\phi^{m_1m_2}_{l\lambda'}\left(\Lambda^{-1}p\right). \tag{4.11}$$

The proof follows from orthogonality and completeness of spherical harmonics and the Lemma 4.7. The group action (4.11) implies that the orthogonal subspaces $L^2[\mathsf{H}^{\cup}_{m_1m_2}, d\nu_{m_1m_2};\mathbb{C}^{2l+1}]$ carry representations of spin l. We will have to further decompose them into representations of different masses.

4.1.4 The Hilbert space $\bigoplus_l \int_{\oplus\mathbb{R}_{m_1m_2}} d\xi_{m_1m_2}(m)\, L^2[\mathbb{R}^3, d\mu_m;\mathbb{C}^{2l+1}]$

Recall the change of variable $p\longmapsto(m,\mathbf{p})$ where $m=\|p\|$ and $\mathbf{p}^i=p^i$ and the measure calculated with the Jacobian

$$d^4p = m\,dm\,d\mu_m(\mathbf{p}).$$

This change of variable maps between the domains

$$\mathsf{H}^{\cup}_{m_1m_2}\longrightarrow\mathbb{R}_{m_1m_2}\times\mathbb{R}^3 := [(m_1+m_2),\infty)\times\mathbb{R}^3$$

and the measures

$$d\nu_{m_1m_2}(p)\longmapsto d\xi_{m_1m_2}(m)\times d\mu_m(\mathbf{p}) := \frac{\kappa^{m_1m_2}(m)}{2m}dm\times d\mu_m(\mathbf{p}).$$

This in turn induces the isomorphism

$$\begin{aligned} L^2[\mathsf{H}^{\cup}_{m_1m_2}, d\nu_{m_1m_2};\mathbb{C}^{2l+1}] &\longrightarrow L^2[\mathbb{R}_{m_1m_2}\times\mathbb{R}^3, d\xi_{m_1m_2}\times d\mu_m;\mathbb{C}^{2l+1}] \\ \phi^{m_1m_2}_l(p) &\longmapsto \phi^{m_1m_2}_l(m,\mathbf{p}) := \phi^{m_1m_2}_l\begin{pmatrix}E^m(\|\mathbf{p}\|)\\ \mathbf{p}\end{pmatrix} \end{aligned} \tag{4.12}$$

It is also easy to see that the transformation rule for $\phi_l^{m_1 m_2}(m, \mathbf{p})$ is essentially the same as for $\phi_l^{m_1 m_2}(p)$

$$\left(U(\Lambda, a)\, \phi_l^{m_1 m_2}\right)_\lambda (m, \mathbf{p}) = e^{ip\cdot a} \sum_{\lambda'} D^{(l)}_{\lambda\lambda'} \left[W\left(\Lambda, \Lambda^{-1} p\right)\right] \phi_{l\lambda'}^{m_1 m_2}\left(m, \mathbf{\Lambda}^{-1} p\right).$$

Definition 4.9. *The direct integral of Hilbert spaces* is the Hilbert space

$$\int_{\oplus X} d\xi(x)\, \mathcal{H}(x) := L^2\left[X, d\xi; \mathcal{H}(x)\right]$$

where X is a measurable set with the measure $d\xi(x)$ and $\mathcal{H}$ is a map that assigns to each $x \in X$ a Hilbert space $\mathcal{H}(x)$. The elements $f, g \in \int_{\oplus X} d\xi(x)\, \mathcal{H}(x)$ are vector/function valued functions

$$f, g : X \longrightarrow \mathcal{H}(x)$$

with summation and scalar multiplication defined point-wise: $(f+g)(x) = f(x) + g(x)$ and $(sf)(x) = sf(x)$. The inner product in $\int_{\oplus X} d\xi(x)\, \mathcal{H}(x)$ is defined in terms of the inner products in the individual Hilbert spaces $\mathcal{H}(x)$ according to

$$\langle f, g\rangle := \int_X \langle f(x), g(x)\rangle\, d\xi(x). \tag{4.13}$$

Lemma 4.10. *The Hilbert space* $L^2[\mathbb{R}_{m_1 m_2} \times \mathbb{R}^3, d\xi_{m_1 m_2} \times d\mu_m; \mathbb{C}^{2l+1}]$ *admits the following decomposition into a direct integral of Hilbert spaces*

$$L^2[\mathbb{R}_{m_1 m_2} \times \mathbb{R}^3, d\xi_{m_1 m_2} \times d\mu_m; \mathbb{C}^{2l+1}] \quad \longrightarrow \quad \int_{\oplus \mathbb{R}_{m_1 m_2}} d\xi_{m_1 m_2}(m)\, L^2[\mathbb{R}^3, d\mu_m; \mathbb{C}^{2l+1}]$$

Proof Outline: recognizing the function $\phi_l^{m_1 m_2}(m, \mathbf{p})$ as the map

$$\begin{aligned} \phi_l^{m_1 m_2} : \mathbb{R}_{m_1 m_2} \quad &\longrightarrow \quad L^2[\mathbb{R}^3, d\mu_m; \mathbb{C}^{2l+1}] \\ m \quad &\longmapsto \quad \phi_l^{m_1 m_2}(m, \mathbf{p}) \end{aligned}$$

implies that $\phi_l^{m_1 m_2} \in L^2\left[\mathbb{R}_{m_1 m_2}, d\xi; L^2[\mathbb{R}^3, d\mu_m; \mathbb{C}^{2l+1}]\right]$ for some measure $d\xi$ on $\mathbb{R}_{m_1 m_2}$. By Definition 4.9 that is a direct integral

$$L^2\left[\mathbb{R}_{m_1 m_2}, d\xi; L^2[\mathbb{R}^3, d\mu_m; \mathbb{C}^{2l+1}]\right] \equiv \int_{\oplus \mathbb{R}_{m_1 m_2}} d\xi_{m_1 m_2}(m)\, L^2[\mathbb{R}^3, d\mu_m; \mathbb{C}^{2l+1}]$$

Demanding equality of the inner product in

$$L^2[\mathbb{R}_{m_1m_2} \times \mathbb{R}^3, d\xi_{m_1m_2} \times d\mu_m; \mathbb{C}^{2l+1}]$$

to the inner product (4.13) as defined for direct integrals, fixes the measure $d\xi \equiv d\xi_{m_1m_2}$.

Applying the direct integral decomposition on the right hand side of (4.12) and taking a direct sum over l on both sides yields

$$\begin{aligned}\bigoplus_l L^2[\mathsf{H}^{\cup}_{m_1m_2}, d\nu_{m_1m_2}; \mathbb{C}^{2l+1}] &\longrightarrow \bigoplus_l \int_{\oplus\mathbb{R}_{m_1m_2}} d\xi_{m_1m_2}(m)\, \mathcal{H}^{l,m} \\ \bigoplus_l \phi_l^{m_1m_2}(p) &\longmapsto \bigoplus_l \int_{\oplus\mathbb{R}_{m_1m_2}} d\xi_{m_1m_2}(m)\, \phi^{l,m}(\mathbf{p})\end{aligned}$$

where

$$\phi^{l,m}(\mathbf{p}) := \phi_l^{m_1m_2}\left(\begin{pmatrix} E^m(\|\mathbf{p}\|) \\ \mathbf{p}\end{pmatrix}\right).$$

The group action on $\phi^{l,m}(\mathbf{p})$ is the same as on $\phi^{m_1m_2}(m,\mathbf{p})$ since we only changed the label convention (note that $\phi^{l,m}$ still implicitly depends on m_1, m_2 but we will not overload the notation) so

$$\left(U(\Lambda,a)\,\phi^{l,m}\right)_\lambda(\mathbf{p}) = e^{ip\cdot a}\sum_{\lambda'} D^{(l)}_{\lambda\lambda'}\left[W\left(\Lambda,\Lambda^{-1}p\right)\right]\phi^{l,m}_{\lambda'}\left(\mathbf{\Lambda}^{-1}p\right).$$

This concludes the decomposition and now we aggregate the results into a final form.

4.1.5 Summary of the decomposition results

Composing all the isomorphisms yields a decomposition of the product Hilbert space

$$\mathcal{H}^{m_1}\otimes\mathcal{H}^{m_2} = \bigoplus_l \int_{\oplus\mathbb{R}_{m_1m_2}} d\xi_{m_1m_2}(m)\,\mathcal{H}^{m,l}$$

where $\mathbb{R}_{m_1m_2} \equiv [(m_1+m_2), \infty)$ and

$$\begin{aligned} d\xi_{m_1m_2}(m) &\equiv \frac{\kappa^{m_1m_2}(m)}{2m}dm, \\ \kappa^{m_1m_2}(m) &\equiv \sqrt{m^4+m_1^4+m_2^4-2(m_1^2m_2^2+m_1^2m^2+m_2^2m^2)}.\end{aligned}$$

See Definition 4.9 for the meaning of the direct integral $\int_\oplus$.

The isomorphisms decompose the product states to components in orthogonal subspaces according to the map

$$\begin{aligned} \psi^{m_1}(\mathbf{p}_1) \otimes \psi^{m_2}(\mathbf{p}_2) &\longmapsto \phi_\lambda^{l,m}(\mathbf{p}) \\ &\equiv \int\limits_{\mathsf{S}^2} d^2\hat{\mathbf{q}} \overline{Y_l^\lambda(\hat{\mathbf{q}})} \psi^{m_1}(\mathsf{p}_1(p,\hat{\mathbf{q}}))\, \psi^{m_2}(\mathsf{p}_2(p,\hat{\mathbf{q}})) \end{aligned} \tag{4.14}$$

where we used the convention $p = (E^m(\mathbf{p}), \mathbf{p})^T$ and $\mathsf{p}_{1,2}(p,\hat{\mathbf{q}})$ defined in (4.2).

Inverting the map recomposes the product state from the orthogonal components

$$\begin{aligned} \phi_\lambda^{l,m}(\mathbf{p}) &\longmapsto \psi^{m_1}(\mathbf{p}_1)\psi^{m_2}(\mathbf{p}_2) \\ &\equiv \sum_{l,\lambda} Y_l^\lambda(\hat{\mathsf{q}}(\mathbf{p}_1,\mathbf{p}_2))\, \phi_\lambda^{l,\|p_1+p_2\|}(\mathbf{p}_1+\mathbf{p}_2) \end{aligned} \tag{4.15}$$

with $\hat{\mathsf{q}}(\mathbf{p}_1,\mathbf{p}_2)$ defined in Eq. (4.1).

4.2 Total momentum basis and the Clebsch-Gordan coefficients

For some calculations Dirac's notation is a very convenient tool and it is worth making an effort to reformulate the decomposition in terms of bra-kets. A side effect of this reformulation is the Clebsch-Gordan coefficients that will naturally arise as the orthogonality relation between the different bases.

In Section 3.3.3 we have defined the states $|\psi^m\rangle$ by expansion (3.10) with respect to the momentum basis (specializing here to spin 0 case)

$$|\psi^m\rangle = \int\limits_{\mathbb{R}^3} d\mu_m(\mathbf{p})|\mathbf{p},m\rangle\psi^m(\mathbf{p}). \tag{4.16}$$

Extending this to the product of irreps we have

$$\begin{aligned} |\psi^{m_1},\psi^{m_2}\rangle &:= \int\limits_{\mathbb{R}^3} d\mu_{m_1}(\mathbf{p}_1)|\mathbf{p}_1,m_1\rangle\psi^{m_1}(\mathbf{p}_1) \otimes \int\limits_{\mathbb{R}^3} d\mu_{m_2}(\mathbf{p}_2)|\mathbf{p}_2,m_2\rangle\psi^{m_2}(\mathbf{p}_2) \\ &= \int\limits_{\mathbb{R}^3} d\mu_{m_1}(\mathbf{p}_1)\int\limits_{\mathbb{R}^3} d\mu_{m_2}(\mathbf{p}_2)|\mathbf{p}_1,\mathbf{p}_2\rangle\psi^{m_1}(\mathbf{p}_1)\psi^{m_2}(\mathbf{p}_2) \end{aligned}$$

where we have used $|\mathbf{p}_1,\mathbf{p}_2\rangle := |\mathbf{p}_1,m_1\rangle\otimes|\mathbf{p}_2,m_2\rangle$. If we substitute (4.15) into the above and then change the variables $(\mathbf{p}_1, \mathbf{p}_2) \longmapsto (p, \hat{\mathbf{q}}) \longmapsto (m, \mathbf{p}, \hat{\mathbf{q}})$ according to (4.8) and (4.12) we get

$$
\begin{aligned}
|\psi^{m_1}, \psi^{m_2}\rangle &= \int_{\mathbb{R}^3} d\mu_{m_1}(\mathbf{p}_1) \int_{\mathbb{R}^3} d\mu_{m_2}(\mathbf{p}_2) |\mathbf{p}_1, \mathbf{p}_2\rangle \sum_{l,\lambda} Y_l^\lambda \left(\hat{\mathsf{q}}\left(\mathbf{p}_1, \mathbf{p}_2\right)\right) \phi_\lambda^{l,\|p_1+p_2\|}\left(\mathbf{p}_1 + \mathbf{p}_2\right) \\
&= \sum_{l,\lambda} \int_{\mathsf{H}^{\cup}_{m_1 m_2}} d\nu_{m_1 m_2}(p)\, \phi_\lambda^{l,\|p\|}(\mathbf{p}) \left(\int_{\mathsf{S}^2} d^2\hat{\mathbf{q}}\, Y_l^\lambda(\hat{\mathbf{q}})\, \left|\mathsf{p}_1(p, \hat{\mathbf{q}}), \mathsf{p}_2(p, \hat{\mathbf{q}})\right\rangle \right) \\
&= \sum_{l,\lambda} \int_{\mathbb{R}_{m_1 m_2}} d\xi_{m_1 m_2}(m) \int_{\mathbb{R}^3} d\mu_m(\mathbf{p}) \phi_\lambda^{l,m}(\mathbf{p}) \left(\int_{\mathsf{S}^2} d^2\hat{\mathbf{q}}\, Y_l^\lambda(\hat{\mathbf{q}})\, \left|\mathsf{p}_1(p, \hat{\mathbf{q}}), \mathsf{p}_2(p, \hat{\mathbf{q}})\right\rangle \right).
\end{aligned}
$$

where we again used the convention $p = \left(E^m(\mathbf{p}), \mathbf{p}\right)^T$. Identifying in the above the new basis we define:

Definition 4.11. The *total momentum basis* $|\mathbf{p}, \lambda, l, m\rangle$ is constructed from the product basis $|\mathbf{p}_1, \mathbf{p}_2\rangle$ according to

$$
|\mathbf{p}, \lambda, l, m\rangle := \int_{\mathsf{S}^2} d^2\hat{\mathbf{q}}\, Y_l^\lambda(\hat{\mathbf{q}})\, \left|\mathsf{p}_1(p, \hat{\mathbf{q}}), \mathsf{p}_2(p, \hat{\mathbf{q}})\right\rangle \tag{4.17}
$$

with $\mathsf{p}_{1,2}(p, \hat{\mathbf{q}})$ defined in (4.2) and $p = \left(E^m(\mathbf{p}), \mathbf{p}\right)^T$.

With this definition we have the new expansion

$$
|\psi^{m_1}, \psi^{m_2}\rangle = \sum_{l,\lambda} \int_{\mathbb{R}_{m_1 m_2}} d\xi_{m_1 m_2}(m) \int_{\mathbb{R}^3} d\mu_m(\mathbf{p}) \phi_\lambda^{l,m}(\mathbf{p})\, |\mathbf{p}, \lambda, l, m\rangle. \tag{4.18}
$$

The orthogonality and normalization of $|\mathbf{p}, \lambda, l, m\rangle$ follows from the orthogonality and normalization of $|\mathbf{p}_1, \mathbf{p}_2\rangle$ and the spherical harmonics, yielding

$$
\langle \mathbf{p}, \lambda, l, m | \mathbf{p}', \lambda', l', m'\rangle = \frac{2m\, E^m(\|\mathbf{p}\|)}{\kappa^{m_1 m_2}(m)} \delta(m - m')\delta^3(\mathbf{p} - \mathbf{p}')\delta_{\lambda\lambda'}\delta_{ll'}. \tag{4.19}
$$

The normalization factor $\frac{2m\, E^m(\|\mathbf{p}\|)}{\kappa^{m_1 m_2}(m)}$ can be derived by observing that

$$
\langle \mathbf{p}_1, \mathbf{p}_2 | \psi^{m_1}, \psi^{m_2}\rangle = \psi^{m_1}(\mathbf{p}_1)\psi^{m_2}(\mathbf{p}_2)
$$

which is a consequence of the normalization of $\langle \mathbf{p}_1, \mathbf{p}_2 | \mathbf{p}_1', \mathbf{p}_2'\rangle$. Similarly we must have

$$
\langle \mathbf{p}, \lambda, l, m | \psi^{m_1}, \psi^{m_2}\rangle = \phi_\lambda^{l,m}(\mathbf{p}).
$$

Therefore the normalization of $\langle \mathbf{p}, \lambda, l, m | \mathbf{p}', \lambda', l', m'\rangle$ must be the inverse of the normalization of the measure in the expansion (4.18) which is

$$d\xi_{m_1m_2}(m)d\mu_m(\mathbf{p}) = \frac{\kappa^{m_1m_2}(m)}{2m}\frac{1}{E^m(\|\mathbf{p}\|)}dm\, d^3\mathbf{p}$$

hence the normalization factor.

By acting with $U^{m_1m_2}(\Lambda, a)$ on the definition of $|\mathbf{p}, \lambda, l, m\rangle$ and using the invariance of the measure and the spin-l representation of $SO(3)$ one can show that the group action on the total momentum basis is

$$U^{m_1m_2}(\Lambda, a)\,|\mathbf{p}, \lambda, l, m\rangle = e^{i(\Lambda p)\cdot a}\sum_{\lambda'} D^{(l)}_{\lambda'\lambda}\left[W(\Lambda, p)\right]|\mathbf{\Lambda}p, \lambda', l, m\rangle.$$

This shows that the total momentum basis in the product representation is equivalent to the momentum basis in the irreducible representations.

We can invert the relation (4.17) by starting with Eq. (4.18) and substituting (4.14) into it followed by the inverse change of variable resulting in

$$|\psi^{m_1}, \psi^{m_2}\rangle = \int_{\mathbb{R}^3} d\mu_{m_1}(\mathbf{p}_1)\int_{\mathbb{R}^3} d\mu_{m_2}(\mathbf{p}_2)\psi^{m_1}(\mathbf{p}_1)\,\psi^{m_2}(\mathbf{p}_2)\left(\sum_{l,\lambda}\overline{Y_l^{\lambda}(\hat{\mathrm{q}}(\mathbf{p}_1, \mathbf{p}_2))}\Big|(\mathbf{p}_1+\mathbf{p}_2), \lambda, l, \|p_1+p_2\|\Big\rangle\right).$$

Here we identify the product basis in terms of the total momentum basis

$$|\mathbf{p}_1, \mathbf{p}_2\rangle := \sum_{l,\lambda}\overline{Y_l^{\lambda}(\hat{\mathrm{q}}(\mathbf{p}_1, \mathbf{p}_2))}\Big|(\mathbf{p}_1+\mathbf{p}_2), \lambda, l, \|p_1+p_2\|\Big\rangle. \tag{4.20}$$

With this relation it is easy to show the resolution of identity in the total momentum basis by changing the variable again $(\mathbf{p}_1, \mathbf{p}_2) \longmapsto (m, \mathbf{p}, \hat{\mathbf{q}})$ so that

$$\begin{aligned} I^{m_1m_2} &\equiv \int_{\mathbb{R}^3} d\mu_{m_1}(\mathbf{p}_1)\int_{\mathbb{R}^3} d\mu_{m_2}(\mathbf{p}_2)|\mathbf{p}_1, \mathbf{p}_2\rangle\,\langle\mathbf{p}_1, \mathbf{p}_2| \\ &= \sum_{l,\lambda}\int_{\mathbb{R}_{m_1m_2}} d\xi_{m_1m_2}(m)\int_{\mathbb{R}^3} d\mu_m(\mathbf{p})\Big|\mathbf{p}, \lambda, l, m\Big\rangle\Big\langle\mathbf{p}, \lambda, l, m\Big|. \end{aligned}$$

Taking the inner product on both sides of Eq. (4.20) with $\langle\mathbf{p}, \lambda, l, m|$ and using the orthogonality (4.19) we get the result

$$\langle\mathbf{p}, \lambda, l, m|\mathbf{p}_1, \mathbf{p}_2\rangle = \frac{2m\,E^m(\|\mathbf{p}\|)}{\kappa^{m_1m_2}(m)}\overline{Y_l^{\lambda}(\hat{\mathrm{q}}(\mathbf{p}_1, \mathbf{p}_2))}\delta(m-\|p_1+p_2\|)\delta^3(\mathbf{p}-\mathbf{p}_1-\mathbf{p}_2).$$

This is a Clebsch-Gordan coefficient for the product of spin 0 irreps and our choice of bases.

5 Quantum reference frames of total momentum

This section presents the first results of the generalization of the QRFs formalism to a non-compact group case. We will focus on the analysis of QRFs of the total momentum in the framework of the Poincaré symmetry. Restricting our focus to this degree of freedom will allow us to simplify the calculations involved. As we have discussed in Section 2.2.2, the analysis can be simplified by considering measurements that are invariant under a subgroup of the full symmetry. In the case of the Poincaré group, the total momentum measurements are invariant under the subgroups of rotations and translations. The restriction to these measurements will restrict the integrals in the calculations of the transmission map $\mathcal{T}_{\rho_R}$, and the degradation map $\mathcal{F}_{\rho_S}$, to the cosets of these subgroups. These cosets are exactly the standard boosts.

In Section 5.1 we will present the PVM that measures the total momentum relative to an ERF and then use it to construct an invariant PVM that does the same relative to a QRF. In Sections 5.2 and 5.3 we will study the transmission and degradation maps that are simplified by the restriction to these measurements.

5.1 The invariant measurements of total momentum

Consider the projector $\Pi_{\Omega}^{m_S}$ acting on a spinless irrep Hilbert space $\mathcal{H}^{m_S}$ according to

$$\Pi_{\Omega}^{m_S} \quad := \quad \int_{\mathbb{R}^3} d\mu_{m_S}(\mathbf{p}_S)\, \chi_{\Omega}\left(\|\mathbf{p}_S\|\right) |\mathbf{p}_S\rangle \langle \mathbf{p}_S| \,. \tag{5.1}$$

The characteristic functions $\chi_{\Omega}(x)$ are defined for any $\Omega \subseteq \mathbb{R}$ as

$$\chi_{\Omega}(x) := \begin{cases} 1 & : x \in \Omega \\ 0 & : x \notin \Omega \end{cases} \quad . \tag{5.2}$$

We choose $\{\Omega_i\}$ to be a partition of $\mathbb{R}$ into finite disjoint intervals Ω_i of equal length for all i. It is easy to show that for any partition $\{\Omega_i\}$ the set $\left\{\Pi_{\Omega_i}^{m_S}\right\}$ is a PVM. The length of Ω_i defines the resolution of the measurements given by the PVM $\left\{\Pi_{\Omega_i}^{m_S}\right\}$. For any state $\rho_S \in \mathcal{B}\left(\mathcal{H}^{m_S}\right)$ this PVM assigns the probability of $\|\mathbf{p}_S\|_{\rho_S} \in \Omega_i$ according to

$$\Pr\left(\|\mathbf{p}_S\|_{\rho_S} \in \Omega_i\right) = tr\left[\Pi_{\Omega_i}^{m_S}\rho_S\right] = \int_{\mathbb{R}^3} d\mu_{m_S}(\mathbf{p}_S)\chi_{\Omega_i}\left(\|\mathbf{p}_S\|\right) \langle \mathbf{p}_S|\, \rho_S |\mathbf{p}_S\rangle. \tag{5.3}$$

The probability is equal to the weight of the distribution $d\mu_{m_S}(\mathbf{p}_S)\, \langle \mathbf{p}_S|\, \rho_S |\mathbf{p}_S\rangle$ in the subset of $\mathbb{R}^3$ for which $\|\mathbf{p}_S\| \in \Omega_i$. This is how the measurements of the total momentum are described in the usual formalism of quantum mechanics with respect to an ERF.

In order to construct the PVM that performs the same measurement with respect to a QRF, we recall the prescription given by Definition 2.1. This definition tells us that we need to introduce a reference system R, choose a state $\tilde{\rho}_R \in \mathcal{B}(\mathcal{H}^{m_R})$ (we assume system R to carry a spinless irrep m_R) and then twirl the joint operator $\tilde{\rho}_R \otimes \Pi_\Omega^{m_S}$ over the entire group. However, notice that $\Pi_\Omega^{m_S}$ is invariant under the action of $\mathcal{U}_{(R,a)}^{m_S}$ for any rotation $R \in SO(3)$ and translation $a \in \mathbb{R}^4$. This can be seen by applying the group action on the momentum basis in Eq. (5.1) and then by using the invariance of the measure to lift the rotation from the basis to $\chi_\Omega(\|\mathbf{p}_S\|)$, where it vanishes. The invariance of $\Pi_\Omega^{m_S}$ under the subgroups $SO(3)$ and $\mathbb{R}^4$ allows us to restrict the twirl to the cosets in the quotient of the Poincaré group with these subgroups. This result was presented in Eq. (2.20) and the preceding discussion. Quotienting the Poincaré group with $\mathbb{R}^4$ results in the Lorentz group. Quotienting the Lorentz group with $SO(3)$ results in the set (not a group) of the standard boosts. Thus, in order to construct the PVM $\left\{\Pi_{\Omega_i}^{m_R m_S}\right\}$ that measures the total momentum of the system S with respect to system R, we need to twirl the joint operator $\tilde{\rho}_R \otimes \Pi_\Omega^{m_S}$ over the standard boosts. In addition to that, as indicated by Eq. (2.20), the choice of $\tilde{\rho}_R$ is restricted to states that are invariant under $\mathcal{U}_{(R,a)}^{m_S}$ so we choose the state $\tilde{\rho}_R = |\mathbf{0}\rangle\langle\mathbf{0}|$ (where $|\mathbf{0}\rangle \equiv |\mathbf{p}_R = 0\rangle$). Therefore, the projectors of the invariant PVM, according to Eq. (2.20) are

$$\Pi_\Omega^{m_R m_S} := \int_{\mathbb{R}^3} d\mu_{m_R}(\mathbf{p}_R) \mathcal{U}_{L(p_R)}^{m_R m_S} \left(|\mathbf{0}\rangle\langle\mathbf{0}| \otimes \Pi_\Omega^{m_S}\right). \tag{5.4}$$

Here, the role of the measure $d\mu_{m_R}(\mathbf{p}_R)$ is two-fold: it serves as the invariant measure over the cosets $L(p_R)$; and it normalizes the projector $\Pi_\Omega^{m_R m_S}$ so that it acts as the identity on its eigenspace. For the latter role, in the finite-dimensional case, we had to multiply Eq. (2.20) with the dimension factor d_μ. In the infinite-dimensional case that is taken care of by the appropriately normalized measure.

Substituting Eq. (5.1) into (5.4) and lifting the group action from the momentum basis we get a more natural form of the invariant projectors

$$\Pi_\Omega^{m_R m_S} = \int_{\mathbb{R}^3} d\mu_{m_R}(\mathbf{p}_R) \int_{\mathbb{R}^3} d\mu_{m_S}(\mathbf{p}_S)\, \chi_\Omega\left(\left\|\mathbf{L}^{-1}(p_R)\, p_S\right\|\right) |\mathbf{p}_R, \mathbf{p}_S\rangle\langle\mathbf{p}_R, \mathbf{p}_S|. \tag{5.5}$$

Comparing it to the original measurement in Eq. (5.1), we see that now the characteristic function $\chi_\Omega(\|\mathbf{L}^{-1}(p_R)\, p_S\|)$ selects the basis element for which the norm of $\mathbf{p}_S$ in the rest frame of $\mathbf{p}_R$, is in the interval Ω. The PVM $\left\{\Pi_{\Omega_i}^{m_R m_S}\right\}$ measures the magnitude of the momentum of system S, not in the absolute way as $\left\{\Pi_{\Omega_i}^{m_S}\right\}$, but relative to the momentum of system R. How well the measurement of $\left\{\Pi_{\Omega_i}^{m_R m_S}\right\}$ on $\rho_R \otimes \rho_S$ approximates the measurement of $\left\{\Pi_{\Omega_i}^{m_S}\right\}$ on ρ_S depends on the state ρ_R of the QRF. In the idealized case where $\rho_R = |\mathbf{0}\rangle\langle\mathbf{0}|$ we have

$$\begin{aligned}
&\Pi_\Omega^{m_R m_S}\left(|\mathbf{0}\rangle\langle\mathbf{0}|\otimes\rho_S\right)\\
&=\int\limits_{\mathbb{R}^3} d\mu_{m_R}(\mathbf{p}_R)\int\limits_{\mathbb{R}^3} d\mu_{m_S}(\mathbf{p}_S)\,\chi_\Omega\left(\left\|\mathbf{L}^{-1}(p_R)\,p_S\right\|\right)|\mathbf{p}_R\rangle\langle\mathbf{p}_R|\,\mathbf{0}\rangle\langle\mathbf{0}|\otimes|\mathbf{p}_S\rangle\langle\mathbf{p}_S|\,\rho_S\\
&=|\mathbf{0}\rangle\langle\mathbf{0}|\otimes\int\limits_{\mathbb{R}^3} d\mu_{m_S}(\mathbf{p}_S)\,\chi_\Omega\left(\|p_S\|\right)|\mathbf{p}_S\rangle\langle\mathbf{p}_S|\,\rho_S \;=\; |\mathbf{0}\rangle\langle\mathbf{0}|\otimes\Pi_\Omega^{m_S}\rho_S
\end{aligned}$$

From this we can see that the measurement $\left\{\Pi_{\Omega_i}^{m_R m_S}\right\}$ reduces to $\left\{\Pi_{\Omega_i}^{m_S}\right\}$ when the QRF is $\rho_R = |\mathbf{0}\rangle\langle\mathbf{0}|$. The states given by $\rho_R = |\mathbf{p}\rangle\langle\mathbf{p}|$ are idealizations of the real physical states (remember that $|\mathbf{p}\rangle$ is not part of the irrep Hilbert space). Therefore the measurement given by $\left\{\Pi_{\Omega_i}^{m_S}\right\}$ is an idealization of the real physical measurements.

Before we finish this section we would like to examine the projectors $\Pi_\Omega^{m_R m_S}$ in a more general context. Using the results of the decomposition from Sections (4.1.5) and (4.2) we change Eq. (5.5) from the product basis $|\mathbf{p}_R, \mathbf{p}_S\rangle$ to the total momentum basis $|\mathbf{p}, \lambda, l, m\rangle$ along with the variables of integration and get

$$\Pi_\Omega^{m_R m_S} = \sum_{l,\lambda}\int\limits_{\mathbb{R}_{m_R m_S}} d\xi_{m_R m_S}(m)\int\limits_{\mathbb{R}^3} d\mu_m(\mathbf{p})\chi_\Omega\left(h_{m_R m_S}(m)\right)|\mathbf{p},\lambda,l,m\rangle\langle\mathbf{p},\lambda,l,m|$$

where we used the function

$$h_{m_R m_S}(m) := \sqrt{\left(\frac{m^2 - m_R^2 - m_S^2}{2m_R}\right)^2 - m_S^2}.$$

When the projector $\Pi_\Omega^{m_R m_S}$ acts on a composite state $\rho_R\otimes\rho_S$ it selects the total momentum basis elements for which the mass m is such that $h_{m_R m_S}(m)\in\Omega$ (the masses m_R, m_S are constant for all $\rho_R\otimes\rho_S$). Since $h_{m_R m_S}(m)$ is monotonic in m we can invert it and write

$$\chi_\Omega\left(h_{m_R m_S}(m)\right) = \chi_{h^{-1}_{m_R m_S}(\Omega)}(m),$$

hence

$$\Pi_\Omega^{m_R m_S} = \sum_{l,\lambda}\int\limits_{\mathbb{R}_{m_R m_S}} d\xi_{m_R m_S}(m)\int\limits_{\mathbb{R}^3} d\mu_m(\mathbf{p})\chi_{h^{-1}_{m_R m_S}(\Omega)}(m)\,|\mathbf{p},\lambda,l,m\rangle\langle\mathbf{p},\lambda,l,m|. \qquad (5.6)$$

Furthermore, with the basis $|\mathbf{p},\lambda,l,m\rangle$ we can easily construct the projectors

$$\Pi^{m_R m_S}_{l,m} := \sum_{\lambda}\int_{\mathbb{R}^3} d\mu_m(\mathbf{p})|\mathbf{p},\lambda,l,m\rangle\langle\mathbf{p},\lambda,l,m|$$

on the individual irreps l,m of the Poincaré group. Using $\Pi^{m_R m_S}_{l,m}$ we can expand $\Pi^{m_R m_S}_{\Omega}$ as

$$\Pi^{m_R m_S}_{\Omega} = \sum_{l}\int_{\mathbb{R}_{m_R m_S}} d\xi_{m_R m_S}(m)\chi_{h^{-1}_{m_R m_S}(\Omega)}(m)\,\Pi^{m_R m_S}_{l,m}.$$

This is consistent with our understanding of all invariant POVMs as a linear combination of the projectors on the irreps that appear in the decomposition (see Eq. (2.27) and the preceding discussion). The invariant measurements $\left\{\Pi^{m_R m_S}_{\Omega_i}\right\}$ are not the most informative as they choose to ignore the spin parameter of the irreps by summing over all l. These measurements focus on distinguishing between the different irreps of mass by restricting the integration of $\Pi^{m_R m_S}_{l,m}$ over m to $m \in h^{-1}_{m_R m_S}(\Omega)$.

We could have "guessed" the PVM $\left\{\Pi^{m_R m_S}_{\Omega_i}\right\}$ by realizing that the total momentum depends directly on the mass parameter of the irreps (remember $m \equiv \|p_R + p_S\|$) which would lead us to the form (5.6). It is not always easy to guess how the desired measurement depends on the irrep parameters (especially if there is more than one of them) which is why we demonstrated the derivation of $\left\{\Pi^{m_R m_S}_{\Omega_i}\right\}$ from the original PVM $\left\{\Pi^{m_S}_{\Omega_i}\right\}$. Another reason for this derivation is that in this way the resolution parameter Ω refers directly to the desired quantity $\|\mathbf{L}^{-1}(p_R)\,p_S\|$. When we express the same measurement with m, the resolution had to be rescaled as $h^{-1}_{m_R m_S}(\Omega)$ which is harder to guess a priori.

5.2 Quality of quantum reference frames of total momentum

In the analysis of the quality of QRFs, focusing on states and not on measurements, produces more general results as the states contain all the information about the measurements and their outcomes. In order to quantify how the states of system S are described relative to a QRF R, we have introduced the transmission map $\mathcal{T}$ given in Eq. (2.17). Since we are only interested in the measurements of the total momentum, and since such PVMs are invariant under the subgroups $SO(3)$ and $\mathbb{R}^4$, the calculation of the transmission map $\mathcal{T}$ can be simplified as was shown in Eq. (2.22). The transmission map in the current case is

$$\mathcal{T}_{\rho_R}(\rho_S) \quad = \quad \int_{\mathbb{R}^3} d\mu_{m_R}(\mathbf{p}_R)\,\langle\mathbf{p}_R|\,\rho_R|\mathbf{p}_R\rangle\,\mathcal{U}^{m_S}_{L^{-1}(p_R)}(\rho_S)\,. \tag{5.7}$$

The remarks about the measure $d\mu_{m_R}(\mathbf{p}_R)$ that followed the derivation of $\Pi_\Omega^{m_R m_S}$ in Eq. (5.4), apply here as well. The quantity $\langle \mathbf{p}_R | \rho_R | \mathbf{p}_R \rangle$ with the measure $d\mu_{m_R}(\mathbf{p}_R)$ is the probability distribution of the 3-momentum of the system R. As we can see in Eq. (5.7), this distribution is what determines the spread of the boosts $\mathcal{U}^{m_S}_{L^{-1}(p_R)}$ that act on the system S in the transmission map. The narrower the spread of R around the origin in the momentum space, the closer $\mathcal{T}_{\rho_R}$ is to the identity map, and the better system R approximates an ERF.

For the quantitative analysis of the quality of the QRF given by the system R, we need to analyze how well the map $\mathcal{T}$ preserves information. As we have discussed in Section 2.3, this can be achieved by the calculation of the fidelity squared (over pure states) between the input and the output states of the map $\mathcal{T}$. Using the identity

$$\langle \mathbf{p}_R | \psi_R \rangle \langle \psi_R | \mathbf{p}_R \rangle = \langle \mathbf{0} | \mathcal{U}^{m_R}_{L^{-1}(p_R)} \left(|\psi_R\rangle \langle \psi_R| \right) |\mathbf{0}\rangle$$

and Eq. (5.7) we calculate this fidelity for arbitrary pure states

$$\begin{aligned}
F\left(|\psi_S\rangle, \mathcal{T}_{|\psi_R\rangle}\left(|\psi_S\rangle\right) \right)^2 &= \int_{\mathbb{R}^3} d\mu_{m_R}(\mathbf{p}_R) \, \langle \mathbf{p}_R | \psi_R \rangle \langle \psi_R | \mathbf{p}_R \rangle \, \langle \psi_S | \mathcal{U}^{m_S}_{L^{-1}(p_R)} \left(|\psi_S\rangle \langle \psi_S| \right) |\psi_S\rangle \\
&= \langle \mathbf{0}, \psi_S | \left(\int_{\mathbb{R}^3} d\mu_{m_R}(\mathbf{p}_R) \mathcal{U}^{m_R m_S}_{L^{-1}(p_R)} \left(|\psi_R, \psi_S\rangle \langle \psi_R, \psi_S| \right) \right) |\mathbf{0}, \psi_S\rangle \\
&= \langle \mathbf{0}, \psi_S | \, \mathcal{G}_L \left(|\psi_R, \psi_S\rangle \langle \psi_R, \psi_S| \right) |\mathbf{0}, \psi_S\rangle \, .
\end{aligned}$$

The map $\mathcal{G}_L := \int_{\mathbb{R}^3} d\mu_{m_R}(\mathbf{p}_R) \mathcal{U}^{m_R m_S}_{L^{-1}(p_R)}$ is a partial twirl over the standard boosts. This form is analogous to what we have derived in the proof of Theorem 2.2 but with a partial twirl instead of the full twirl, and with the subgroup-invariant state $|\mathbf{0}\rangle$ instead of $|\psi_R\rangle$. Both of these differences are attributed to the use of the simplified transmission map resulting from the subgroup invariance. Unfortunately, the partial twirl does not allow the use of Schur's lemmas to simplify this result and derive a more explicit value of fidelity, as we did in the proof of Theorem 2.2. In order to address this issue more work needs to be done on the consequence of the subgroup-invariant measurements.

One thing we can do with this expression is derive a numerical formula for its calculation. Using the notation $\psi(\mathbf{p}) := \langle \mathbf{p} | \psi \rangle$ we derive

$$
\begin{aligned}
F\left(|\psi_S\rangle, \mathcal{T}_{|\psi_R\rangle}(|\psi_S\rangle)\right)^2 &= \int_{\mathbb{R}^3} d\mu_{m_R}(\mathbf{p}_R)\, \langle \mathbf{p}_R | \psi_R\rangle \langle \psi_R | \mathbf{p}_R\rangle \, \langle\psi_S| \mathcal{U}^{m_S}_{L^{-1}(p_R)}\left(|\psi_S\rangle\langle\psi_S|\right)|\psi_S\rangle \\
&= \int_{\mathbb{R}^3} d\mu_{m_R}(\mathbf{p}_R)\, |\psi_R(\mathbf{p}_R)|^2 \, |\langle\psi_S| U^{m_S}(L(p_R))|\psi_S\rangle|^2 \\
&= \int_{\mathbb{R}^3} d\mu_{m_R}(\mathbf{p}_R)\, |\psi_R(\mathbf{p}_R)|^2 \left| \int_{\mathbb{R}^3} d\mu_{m_S}(\mathbf{p}_S)\psi_S(\mathbf{p}_S)\, \overline{\psi_S(\mathbf{L}(p_R)\, p_S)} \right|^2
\end{aligned}
$$

In the last step we explicitly specified the elements $|\langle\psi_S| U^{m_S}(L(p_R))|\psi_S\rangle|^2$. This form is explicit in the functions $\psi(\mathbf{p})$ for both systems R and S, which makes it suitable for a numerical evaluation.

Quantitative analysis of the quality of this QRF requires an evaluation (analytical or numerical) of $F\left(|\psi_S\rangle, \mathcal{T}_{|\psi_R\rangle}(|\psi_S\rangle)\right)^2$, which we leave for further investigation.

5.3 Degradation of quantum reference frames of total momentum

According to the analysis of Section 2.4, repeated measurements with the PVM $\left\{\Pi^{m_R m_S}_{\Omega_i}\right\}$ of multiple copies of system S using the same QRF R, will result in the degradation of the state of R. The degradation map presented in the Definition 2.3 is

$$
\mathcal{F}_{\rho_S}(\rho_R) = tr_S\left[\sum_i \Pi^{m_R m_S}_{\Omega_i}(\rho_R \otimes \rho_S)\, \Pi^{m_R m_S}_{\Omega_i}\right].
$$

Substituting $\Pi^{m_R m_S}_{\Omega_i}$ from Eq. (5.5) into the degradation map, taking the trace over S and rearranging produces the form

$$
\mathcal{F}_{\rho_S}(\rho_R) = \int_{\mathbb{R}^3} d\mu_{m_R}(\mathbf{p}_R) \int_{\mathbb{R}^3} d\mu_{m_R}(\mathbf{p}'_R)\, |\mathbf{p}_R\rangle\langle\mathbf{p}'_R| \left(\langle\mathbf{p}_R|\rho_R|\mathbf{p}'_R\rangle\, f_{\rho_S}(\mathbf{p}_R, \mathbf{p}'_R)\right) \tag{5.8}
$$

where

$$
f_{\rho_S}(\mathbf{p}_R, \mathbf{p}'_R) := \int_{\mathbb{R}^3} d\mu_{m_S}(\mathbf{p}_S) \left(\sum_i \chi_{\Omega_i}\left(\left\|\mathbf{L}^{-1}(p_R)\, p_S\right\|\right) \chi_{\Omega_i}\left(\left\|\mathbf{L}^{-1}(p'_R)\, p_S\right\|\right)\right) \langle\mathbf{p}_S|\rho_S|\mathbf{p}_S\rangle. \tag{5.9}
$$

In order to see how $\mathcal{F}_{\rho_S}$ acts, we express the states ρ_R in the similar form

$$
\rho_R \equiv \int_{\mathbb{R}^3} d\mu_{m_R}(\mathbf{p}_R) \int_{\mathbb{R}^3} d\mu_{m_R}(\mathbf{p}'_R)\, |\mathbf{p}_R\rangle\langle\mathbf{p}'_R|\, \langle\mathbf{p}_R|\rho_R|\mathbf{p}'_R\rangle. \tag{5.10}
$$

Comparing Eq. (5.10) to (5.8) we see that $\mathcal{F}_{\rho_S}$ acts on ρ_R by multiplying the matrix elements $\langle \mathbf{p}_R| \rho_R |\mathbf{p}'_R\rangle$ with the coefficients $f_{\rho_S}(\mathbf{p}_R, \mathbf{p}'_R)$. Repeated application of $\mathcal{F}_{\rho_S}$ simply raises the power of the coefficients $f_{\rho_S}(\mathbf{p}_R, \mathbf{p}'_R)$, hence

$$\langle \mathbf{p}_R| \rho_R |\mathbf{p}'_R\rangle \overset{\mathcal{F}^{\circ n}_{\rho_S}}{\longmapsto} \langle \mathbf{p}_R| \rho_R |\mathbf{p}'_R\rangle \left(f_{\rho_S}(\mathbf{p}_R, \mathbf{p}'_R)\right)^n . \tag{5.11}$$

This suggests that in order to understand the dynamics generated by $\mathcal{F}_{\rho_S}$, we need to study the coefficient functions f_{ρ_S}.

Lemma 5.1. *The coefficient f_{ρ_S} given by Eq. (5.9) obey*

$$\begin{cases} f_{\rho_S}(\mathbf{p}_R, \mathbf{p}'_R) = 1 & \mathbf{p}_R = \mathbf{p}'_R \\ f_{\rho_S}(\mathbf{p}_R, \mathbf{p}'_R) < 1 & \mathbf{p}_R \neq \mathbf{p}'_R \end{cases}$$

Proof. Defining the quantity

$$\chi(p_R, p'_R, p_S) := \left(\sum_i \chi_{\Omega_i}\left(\left\|\mathbf{L}^{-1}(p_R)\, p_S\right\|\right)\chi_{\Omega_i}\left(\left\|\mathbf{L}^{-1}(p'_R)\, p_S\right\|\right)\right)$$

we can immediately see that for $p_R = p'_R$, and for all p_S we have $\chi(p_R, p_R, p_S) \equiv 1$. Substituting this into (5.9) we get

$$f_{\rho_S}(\mathbf{p}_R, \mathbf{p}_R) = \int_{\mathbb{R}^3} d\mu_{m_S}(\mathbf{p}_S)\ \langle \mathbf{p}_S| \rho_S |\mathbf{p}_S\rangle = tr(\rho_S) = 1.$$

For the case $p_R \neq p'_R$ we will use the identity $\|\mathbf{L}^{-1}(p_R)\, p_S\| = m_S \sinh\eta$, where η (η' resp.) is the hyperbolic angle between p_R (p'_R resp.) and p_S defined by the relation $\cosh\eta \equiv p_R \cdot p_S/(m_R m_S)$. The distance between $\|\mathbf{L}^{-1}(p_R)\, p_S\|$ and $\|\mathbf{L}^{-1}(p'_R)\, p_S\|$ can now be expressed as

$$\left|\left\|\mathbf{L}^{-1}(p_R)\, p_S\right\| - \left\|\mathbf{L}^{-1}(p'_R)\, p_S\right\|\right| = m_S\left|\sinh\eta - \sinh\eta'\right|.$$

If $p_R \neq p'_R$ this distance is unbounded when varying over all $\mathbf{p}_S \in \mathbb{R}^3$. This means that there are $\mathbf{p}_S \in \mathbb{R}^3$ such that $\|\mathbf{L}^{-1}(p_R)\, p_S\|$ and $\|\mathbf{L}^{-1}(p'_R)\, p_S\|$ are arbitrarily far from each other and thus cannot belong to the same finite length interval Ω_i. Therefore, if $p_R \neq p'_R$ there are $\mathbf{p}_S \in \mathbb{R}^3$ such that $\chi(p_R, p'_R, p_S) = 0$. Substituting this into (5.9) we get

$$f_{\rho_S}(\mathbf{p}_R, \mathbf{p}'_R) = \int_{\mathbb{R}^3} d\mu_{m_S}(\mathbf{p}_S)\chi(p_R, p'_R, p_S)\ \langle \mathbf{p}_S| \rho_S |\mathbf{p}_S\rangle < \int_{\mathbb{R}^3} d\mu_{m_S}(\mathbf{p}_S)\ \langle \mathbf{p}_S| \rho_S |\mathbf{p}_S\rangle = 1$$

□

Returning to the Eq. (5.11) we see that the degradation preserves all the diagonal matrix elements $\langle \mathbf{p}_R | \rho_R | \mathbf{p}_R \rangle$ since they are multiplied with powers of 1. All the off diagonal elements, on the other hand, are multiplied with the powers of the coefficients $f_{\rho_S}(\mathbf{p}_R, \mathbf{p}'_R) < 1$ that are smaller than 1. This means that the off diagonal elements $\langle \mathbf{p}_R | \rho_R | \mathbf{p}'_R \rangle$ are "killed" exponentially fast in the number of measurements, and the QRF evolves toward the diagonal state

$$\tilde{\rho}_R = \int_{\mathbb{R}^3} d\mu_{m_R}(\mathbf{p}_R) \, |\mathbf{p}_R\rangle \langle \mathbf{p}_R| \langle \mathbf{p}_R | \rho_R | \mathbf{p}_R \rangle \, .$$

It is interesting to note that in this measurement scheme, even though the state of the QRF degrades, the quality of the QRF does not. To see this observe that in Eq. (5.7), the transmission map depends only on the diagonal elements $\langle \mathbf{p}_R | \rho_R | \mathbf{p}_R \rangle$ which are unaffected by the degradation map. If the transmission map is not affected by the degradation of the QRF state, then the measurements are not affected either. Therefore these measurements are insensitive to the frame degradation that they themselves cause. This is in contrast to what was shown in [13] for the $SU(2)$ QRFs of spin. There the degradation process was shown to reduce the quality of a QRF used for the measurements that cause the degradation.

6 Summary and Outlook

In this thesis we have developed a framework for the analysis of quantum reference frames. Initially we have developed and studied the general framework in the context of compact groups. This allowed us to relate the states and measurements given by the traditional formalism of quantum mechanics to the same states and measurements given with respect to a QRF. Using this relation we have studied the quality of a general QRF by analyzing how well it approximates an ERF. Our main result about the quality of a QRF associated with a compact group is the formula for the input-output fidelity

$$F\left(\left|\psi_S\right\rangle, \mathcal{D}_{\psi_R}\left(\left|\psi_S\right\rangle\right)\right)^2 = d_\alpha \sum_\lambda \frac{\left|a_\lambda\right|^4}{d_\lambda}.$$

This result was also shown to be consistent with the analysis in [11, 13] of quality of QRFs for the special case of $SU(2)$.

Also in the context of general compact groups we have studied the frame degradation process of QRFs caused by repeated measurements. We have argued that the most informative measurement that one can do with respect to a QRF is a measurement with projectors Π^λ on the different irreps λ that appear in the decomposition of the joint system. The main insight from this analysis, given by Theorem 2.5, is that the degradation process caused by such measurements, tends to "align" the pure states of the QRF system, with the pure states of the measured system, so that their joint state will lie in a single irrep of the decomposition. This general insight is also consistent with the $SU(2)$ case analysis in [11, 13].

Taking a broader perspective we would like to offer an observation about our description of the dynamics in the frame degradation process. Essentially the frame degradation map

$$\mathcal{F}_{\rho_S}(\cdot) := tr_S\left[\sum_\lambda \Pi^\lambda\left(\cdot \otimes \rho_S\right)\Pi^\lambda\right]$$

defines a discrete dynamic rule that specifies the evolution of the system R after an interaction with the system S. What is interesting is that $\mathcal{F}_{\rho_S}$ does not depend on what the interaction between R and S was in order to specify the evolution of R. All it needs to know are the correlations that this interaction creates. The correlations are specified by the projectors Π^λ that project the composite system RS into the subspaces λ thus creating a correlation between them. Of course in the detailed description of this process we could specify the dynamics with an interaction Hamiltonian that creates these correlations. What the map $\mathcal{F}_{\rho_S}$ demonstrates is that Hamiltonians are not necessary for the specification of this dynamics, sufficient description is specified by the correlations alone. With the introduction of QRFs into the formalism we argue that all dynamics in physics can be specified with the structure of correlations generated between the systems. Maps such as $\mathcal{F}_{\rho_S}$

can provide an information-theoretic alternative to Hamiltonians in the description of the dynamics emphasizing the role of correlations in them.

The general formalism that we have developed for the compact groups we applied on the Poincaré group. The study of compact group QRFs has shown us that the structure of the decomposition of the group's reducible representations to its irreducible components is fundamental to the analysis of QRFs. Therefore, for the analysis of relativistic QRFs, we have studied the irreps of the Poincaré group, followed by the decomposition of its massive spinless representations. The solution that we have found for the decomposition problem provides an explicit formula (4.14) for computing the projections of a composite state on the subspaces of irreducible representations. This result was also reformulated in a more traditional form of the Clebsch-Gordan coefficients

$$\langle \mathbf{p}, \lambda, l, m | \mathbf{p}_1, \mathbf{p}_2 \rangle = \frac{2m\, E^m\left(\|\mathbf{p}\|\right)}{\kappa^{m_1 m_2}\left(m\right)} \overline{Y_l^{\lambda}\left(\hat{\mathsf{q}}\left(\mathbf{p}_1, \mathbf{p}_2\right)\right)} \delta(m - \|p_1 + p_2\|) \delta^3(\mathbf{p} - \mathbf{p}_1 - \mathbf{p}_2),$$

although the appearance of distributions makes them awkward to work with.

Finally we have studied the special case of a relativistic QRF of total momentum. We have demonstrated that the formalism which was developed for compact groups can be applied to the non-compact case as well. In particular, we have constructed an invariant POVM that measures the total momentum of one particle relative to another. The main result of this section is the analysis of the degradation process generated by the measurements of total momentum. It was shown that this process attenuates all the off-diagonal matrix elements of the QRF state, although it does not affect its ability to serve as a reference of total momentum. This is a qualitatively different result than what we saw in [13] for the case of $SU(2)$. In that case the degradation of a QRF has resulted in it being unusable (has fallen bellow a certain threshold of quality) for a reliable implementations of the measurements.

At the end of Section 5.1 we have presented the projectors

$$\Pi_{l,m}^{m_R m_S} = \sum_{\lambda} \int_{\mathbb{R}^3} d\mu_m(\mathbf{p}) |\mathbf{p}, \lambda, l, m\rangle \langle \mathbf{p}, \lambda, l, m|$$

that project the states of the composite system RS to the irrep l,m of the Poincaré group. These projectors can be used to construct any invariant POVM (see Eq. (2.27) and the preceding discussion). The most informative invariant measurement of the composite system RS is the one that perfectly resolves all the irreps and so is given by the PVM $\left\{\Pi_{l,m}^{m_R m_S}\right\}_{l,m}$. The total momentum measurements $\{\Pi_{\Omega}^{m_R m_S}\}$ that we have studied in Section 5 have resolved the irreps in the m parameter to an arbitrary (though finite) resolution but remained ignorant about the l parameter.

It will be interesting (and more challenging) to study the complementary case of measurements given by the projection

$$\Pi_l^{m_R m_S} = \int\limits_{\mathbb{R}_{m_R m_S}} d\xi_{m_R m_S}(m)\Pi_{l,m}^{m_R m_S}.$$

The PVM $\{\Pi_l^{m_R m_S}\}_l$ resolves the irreps of different l's but is ignorant about the m parameter as it sums over all m. Physically such PVMs measure the total angular momentum (AM) in the center of mass of the composite system RS. Without spin this is just the orbital AM resulting from the relative momentum of the two particles in their center of mass. Adding spin to the irreps R and S and measuring $\left\{\Pi_j^{m_R m_S}\right\}_j$ [5] will generalize the analysis of the spin QRFs [11, 13] that were performed in the context of $SU(2)$. With $\left\{\Pi_j^{m_R m_S}\right\}_j$ we will be measuring the total AM consisting from the contributions of the two spins and the orbital AM. In the context of $SU(2)$ it was also the total AM which was measured but it consisted only of the two spins. We expect that even if the two particles are relatively at rest, the non-zero spread in the momentum variable $\mathbf{p}$ will result in a non-zero orbital AM which will perturb the total AM from the sum of the two spins. The $SU(2)$ result should be recovered only in the limit of zero spread in the momentum space of both particles.

Ultimately the unrestricted case of the most resolving measurement $\left\{\Pi_{l,m}^{m_R m_S}\right\}$ should be considered but not before the restricted measurements of the total AM $\left\{\Pi_j^{m_R m_S}\right\}$ and the total momentum $\{\Pi_\Omega^{m_R m_S}\}$ are well understood.

[5] We changed the label l to j to reflect the fact that now the total AM is not just orbital AM.

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