Static and Dynamic Modelling of Credit Default Risk: Tails, Moments, and Calibration

by

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A thesis
presented to the University of Waterloo
in fulfilment of the
thesis requirement for the degree of
Master of Quantitative Finance

Waterloo, Ontario, Canada, 2014

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Credit risk modelling can take many different approaches. Each method has its strengths and weaknesses and studying a variety of them can help find new ways of performing credit risk analysis. We present here three different models, each classified either as static or dynamic, and structural or reduced-form. The static structural model from Lucas et al. (2000) helps us derive a moment behaviour theorem within the dynamic structural setting of Bush et al. (2011). For comparison, we also present the dynamic reduced-form model of Giesecke et al. (2012). A calibration exercise of the dynamic structural model is implemented and we study its performance through changing financial environment. This highlights the horse race between simplicity and efficiency of a model that still needs to be adequately addressed, as the results from the calibration show the difficulty of capturing the key financial environment's aspects.

Acknowledgements

I would like to thank all the people who made this thesis possible, especially my supervisor David Saunders and my reading committee Adam Kolkiewicz and Tony Wirjanto. The amount of effort they have put to help me is incomparable. Special thoughts go as well to those who delayed its completion, but made it so much more enjoyable.

Dedication

À ceux qui m'ont transmis leur curiosité.

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Chapter 1

Introduction

There is not one day that passes without hearing about the ups and downs of the economy, the surprises of analysts, or the scepticism of investors. We hear so much about these news because our way of living is intrinsically connected to our economic system. As soon as humans got together as a society, they had to develop a way of exchanging skills and goods to make their lives easier. The "I.O.U." is the first and easiest way to develop an economic system within a community. Indeed, two parties agreeing to help each other in exchange for future services is the most basic form of contract in human history. It was the birth of debt and credit.

Everyone taking part in a community is involved in the meanders of debt and credit. Be it from a family borrowing money to buy a house, an entrepreneur asking funds to develop a new technology, a bank managing clients' deposits and lending to others, or a big firm trying to diversify its cash by investing it in new products, everybody gives or owns an "I.O.U.". On top of that, the world being as interconnected as it has become, the range of parties with which one might be linked is extremely broad. Banks acting on the international level, a firm in China might be much closer to an investor in Canada than meets the eye. This wide web of debts and credits forms what is called the credit market.

The credit market is broad, and extremely complicated. All these parties act on their own, but affect the situation and actions of their interconnected counterparties. This web of counterparties creates a web of risk. Each string reinforces the web, but could also make it collapse. Every time a party lends to another one, it takes the risk of not seeing its money again. Through bankruptcy, poor accountancy, felony, or any other means, the debtor might not repay the creditor. Because of this risk, and because credit is ubiquitous in our economic system, it is very important to understand it.

1.1 Understanding Credit Risk

One of the significant events related to the credit market's risk is the 2008 global financial crisis. It clearly showed how important it is to understand the credit market and its risks. During that crisis, it was realized that finance's traditional tools had failed to prevent the worst outcomes. We witnessed an overall systemic breakdown of the financial system, leading into the worst economic crisis since the Great Depression. A lesson learned from this crisis is that there is a need for more flexible and robust tools to manage the risk and face the new challenges that credit brings to us. This thesis will focus on the analysis of such tools.

The types of risks associated to credit, called credit risk, can in fact come from many different situations. Either from default of the debtor, from concentration of the creditor's exposures, or from the country in which we execute the contract, credit risk is important to manage and quite broad. In this thesis, we will define credit risk as credit default risk, as it is the most common and unavoidable type to take into account when managing credit.

Assessing the risk of an investment is a difficult task. It requires an understanding of the investment you made, both its nature, and the market it takes part in. One type of investment that is relevant to this thesis is the now infamous Collateralized Debt Obligation (CDO). A CDO is a financial product created by securitization. One would exchange money today for a promise of future cash flows in a predetermined sequence. The amount paid in each period of the cash flow is determined by the money the CDO is able to collect from the pool of assets constituting its underlying base.

For example, a bank lends money to a firm. By lending this money, it exchanges its money now for a promise of repayments, with interest. By doing this, it takes a risky position, since the firm might not repay the loan. The idea for the bank is then to turn around and sell this promise of cash flows to risk taking investors. Before doing this however, it pools some debts together, and separates the pool in different risk layers, called tranches. This process is known as securitization. It makes the product less vulnerable to individual actions, treating in effect the CDO as an insurance policy. Therefore, even though the investors may be willing to bear risk, the risk they will bear will be smaller than what the bank bears before creating the CDO.

There are many reasons to explain the popularity of CDO's. They can be created from virtually anything the investor wishes (loans, bonds, mortgages, etc.), as long as the payment structure defined above works. Since they can be tailor-made, they serve as a good component to hedging strategies and risk transfer. Also, CDOs allow some funding benefits due to their flexible structure. This flexible structure makes them very interesting to study since the theory developed can be applied to many other situations.

But even after we have an investment tool at our disposal, that does not mean that we can sit back and enjoy the money flowing in. This activity entails the notion that we are actually willing to take on some risks, which is at the core of the money-making game. But since we can not take into account everything that happens in the economy and identify every single force at work in the economy, we content to rely on an approximate reality. This is where the use of models becomes handy. We use models in order to replicate as closely as possible a particular aspect of reality, while keeping them easy to manipulate and work with.

1.2 Credit Risk Models

When someone is looking into buying good or service, the price is a critical factor to contend with. One would want to have a fair price and pay for what it is receiving. The same thing applies to the credit market. We want to be able to find the fair price for an instrument. The difference is that when buying good, you know what you are getting and can use it. In finance, you are exchanging money now for money in the future, hence there is some unknown inherent in the transaction and you might not even be able to recover its value. Similarly with goods, when investing, you take the risk of the investment losing its value before you retrieve the full benefits. Therefore, the price you pay must reflect the different possibilities you might encounter while your investment comes to maturity.

To reflect these possibilities, we use models and assume that a number of parameters will be influencing the value of our investment. Then, making sure that we do not allow for arbitrage among other instruments, we give the price that reflects the fair value of the investment relative to its future outcomes. The models we use to do so come in various forms. Each allows us to control different parameters and each has its own strengths and weaknesses. Each also tries to replicate the market in which they are applied in order to understand the risk implied by the market. This enables us to take action to mitigate it.

In the case of credit risk, we want to be able to replicate the credit market, with its time of defaults, size of defaults, losses and gains, etc. There is no consensus in the literature as to what technique is best for this purpose, but two of the main approaches used for this purpose are structural models and reduced-form models (Elizalde, 2012a).

Structural models try to explain reality by hypothesizing about the structure of the market in which the portfolio evolves. They use asset and debt value as indicators to determine time of default (Elizalde, 2012b) of a company included in the portfolio. They take their explanatory value through the economics of the financial portfolio. The way to use these models is to assume a certain structure of the markets, with the assumed relevant factors influencing the value of the company, then incorporate the other characteristics of our setting before running the simulation. The assumed structure influences the results of our simulations. These models are very intuitive because we can identify the economic meaning of each variable. Hence, they can be very useful for risk management and investment because we can see the effect of real-world factors on our portfolio.

Reduced-form models do not consider the relation between firm's value and their default time (Elizalde, 2012a). They rely mainly on the information given by the market and model default as a random variable, for example using an intensity model where the first jump of an exogenous jump process defines a default. The parameters of this jump process are directly inferred from the market data. Because the explanatory variables in the models do not have a specific economic interpretation, they tend to be more malleable, but also harder to understand and explain. To an investor, these models tend to replicate well market data and have been very popular in more quantitative analyses and valuations. They constitute an important part of the literature in finance.

Once we have defined how the source of randomness in the market, we need to decide how to model the credit market. We may chose to focus on a single period or the inter-temporal nature of the model. The single-period model, as the name suggests, studies the market within one period of time. It is also called a static model since it does not take into account the way the market changes over time.

On the other hand, a dynamic model will try to model the impacts of future behaviours based on today's price. According to Schönbucher (2003), every good model should fill certain properties in order to be considered reliable. Such properties include:

1. able to produce default correlations of realistic magnitudes;
2. able to keep the number of parameters in the model as few as possible (i.e. not to let it grow too fast when adding more names to our portfolio);
3. dynamic, and able to count the number of defaults and their timing accurately;
4. able to reproduce clusters of time of defaults;

5. easy to calibrate under different scenarios.

Thus, a dynamic model should be our preferred type when modelling credit risk. Unfortunately, dynamic models require much more computing power and are sometimes very costly to run, due to the great deal of information they try to simulate. Hence, it is important to control the number of parameters and the ease of calibration. Otherwise, such models, even though being able to produce better results, might computationally be too cumbersome and, moreover, give obsolete information when we need it most.

1.3 Thesis Structure

The rest of the thesis will be structured as follows:

In Chapter 2, we give an introduction to credit risk models. We will guide the reader through the different models that have been studied in the literature in order to position our contribution. We will thus introduce the building blocks required to understand the work of this thesis. This includes an introduction to credit portfolio analysis and a more detailed explanation of CDOs. Then the key aspects of structural and reduced-form models will be laid out in the presentation of this chapter. Each will constitute a stepping stone towards next, more refined models, until we reach the ones that are the focus of this thesis.

In Chapter 3, we will present models addressing the need for simplicity. We will outline the theory underlying these powerful tools. Proofs and key concepts of risk management will be provided, as well as the strengths and weaknesses of each model. By studying different types of models, we will be able to take key elements from each model into the model studied in this thesis and lead us to the development of a new theorem for the moments of the loss distribution and its tail behaviour. We will prove that all the moments of the loss distribution in a factor model can be determined by the joint distribution of the factors and of the idiosyncratic risk imparted to our asset.

In Chapter 4, we will test our results from the previous chapter and provide insight on implementation of the model. We will numerically implement and test the dynamic structural model discussed earlier. We will provide explanation for the steps involved in the implementation before proceeding with the replication of the results of Bush et al. (2011). Moreover, testing with different data sets and through time series analysis will lead us to some conclusions about the robustness of the model. It will also serve to provide an avenue for future research.

In Chapter 5, we will conclude this thesis by providing a summary of our findings, and suggest avenues for future research.

Chapter 2

Credit Risk

A casual internet search on the topics of credit market will show us the size of this market and its importance to a well-functioning economy. Unfortunately, the credit market is a complex world, as it has evolved from the simple "I.O.U." system to a present system characterized by jargon, complex instruments, and different approaches to analyse it. Therefore when dealing with the credit market, we need to have a good understanding of these terms, instruments, and approaches.

To manage our risk profile in the credit market, there are certain important factors to be aware of. It is also important to realize how the market works and how instruments are priced. Finally, we will want to know why the models we use are not performing as well as they were intended to and why we need new and improved models. We will want to understand the limits of the existing models and investigate potential improvements to the models to better reflect the reality of today‘s market. To this end, a quick survey on the history of financial models for credit risk will point towards desirable characteristics for an optimal model to possess.

Therefore, we will examine broad aspects of Credit Portfolio Analysis. Then, we will explain one particular tool we will use in this thesis to study credit risk, namely the CDO, and understand why it is an interesting instrument to study. Finally, we will present the evolution of the models over time, and highlight the particular characteristics upon which our analysis will be built. These models will be of structural and reduced-form types. In each section, we will highlight the building blocks used to explain our credit risk models.

2.1 Credit Portfolio Analysis

When managing credit risk, one must be able to understand the economics and mathematics underlying the credit risk. The easiest way to do this is to start with single-name credit risk, which is the credit risk associated with a single company. We will, in this section, introduce the relevant tools and concepts used in credit risk management, and discuss the challenges faced by the credit market.

In analysing a single name before a portfolio, we want to define the risk associated with this company's credit. As mentioned before, the risk is defined as failure to recover the full benefits of our financial position. Therefore, we are interested in the probability of default of the firm representing the single-name credit product. It quantifies the likelihood of the company to default on its debt repayment at a given time. This probability of default (PD) is expressed through a rank of credit worthiness, given by rating firms like Moody's, S&P, Fitch or an internal rating. This rank is computed by the firms according to their internal model, usually using data and assumptions known only to them, to indicate the likelihood of default. This gives the PD within a predetermined period of time, usually the next calendar year.

One way of modelling credit in portfolio models is using a Creadit Worthiness Index (CWI), defined as

$$\mathrm{CWI}_i^{(t)} = \rho_i \Phi_i^{(t)} + \sqrt{1-\rho}\epsilon_i^{(t)} \tag{2.1}$$

where

$$\Phi_i^{(t)} = \sum_{n=1}^{N} w_{i,n} \Psi_n^{(t)}$$

where Ψ_n are market-wide risk factors such as country, industry, etc. and the $w_{i,n}$'s are weights of each factor of the i-th firm. The Ψ's, and therefore the Φ's are random variables, and assumed to be independent of the ϵ's. One can easily see that this model resembles a linear regression model where one would want to estimate the ρ's, corresponding to the influence of the market on the underlying's CWI. For example, the popular Capital Asset Pricing Model (CAPM) uses the same kind of regression model to estimate the expected rate of returns off of the market's over performance with respect to the risk-free rate. One major difficulty in the CWI regression is to choose which factors influence the underlying, as well as the weight to put to each before performing the regression. Nevertheless, the information about the PD inferred from the CWI can be important in assessing the risk of a potential credit investment.

The PD can be represented as the probability that the i-th firm's CWI value at time T

falls below a default threshold K_i, given the current knowledge of the CWI at time t, i.e.,

$$\mathrm{PD}_i = \mathbb{P}[\mathrm{CWI}_i^{(T)} < D_i | \mathrm{CWI}_i^{(t)}]$$

Once we know the likelihood of a default occurring, we want to know the impact this default would have on an investor. This impact can be measured by the *Exposure at Default* (EAD) and by *Loss Given Default* (LGD). The EAD represents all the principals and the interest streams that could be lost if a default occurred. It takes into account all the money, and is sometimes referred to as the outstanding notional exposure, or the potential exposure.

On the other hand, LGD represents the portion of how much we would lose after recovering some value from the default, either by liquidation at market value (MV) or other means of recovery. Liquidation at market value means selling the remaining assets at the best value we can get in the market. When estimating LGD, we consider the current trading price of an asset to be the value at which we would be able to liquidate, even though it might not be the same price at the time when we need to liquidate it. Also, we might hold some defaultable assets, but ask for collateral. Collateral covers part of the loss in case of a default. It is a way to protect oneself from bad events and to minimize their gravity. Therefore, the loss incurred is not as big as our exposure. Thus, the expectation of realized loss, conditional on the loss actually occurring, is given by

$$\mathrm{Loss} = \max[\mathrm{Exposure} - \mathrm{Collateral}, 0].$$

then, for the LGD, we compute the loss we would incur, then take the ratio of the Loss to our exposure to have an idea of how risky our investment is. LGD implies that we know how much value we would be able to save from the loss with the help of the existing collateral. Then the LGD, representing the severity of the loss as a percentage of the loss quote, is defined as

$$\mathrm{LGD} = \frac{\mathrm{Loss}}{\mathrm{EAD}}.$$

Therefore, the LGD is the ratio of what we actually lose during a default against what we could have lost. From this, we can extract the Expected Loss (EL) as

$$\mathrm{EL} = \mathrm{PD} \times \mathrm{EAD} \times \mathrm{LGD}$$

When modelling portfolio credit risk, these quantities serve as the basic variables since they are economically meaningful and can be interpreted easily.

It is important to be able to model credit risk at the portfolio level because credit portfolios are a major type of investment in the credit market. Correlation in the models allows us to understand better what is involved in trading credit risk. Trading credit risk could be epitomized by trading correlation between underlying assets. The models used for credit risk can also be used to measure the risk associated with loan portfolios losses. This is an important part of retail banking. Finally, following the Basel III Accords, modelling portfolio credit risk has become prominent in modelling counter-party credit risk losses (Basel Comittee on Banking Supervision, 2011).

As mentioned earlier, one could model the CWI by regression on some factors in order to find the probability of default. In fact, this would be giving us an estimate of the real-world probability of default, as the required numbers could be obtained directly from firms' fundamentals. However, we will use risk-neutral PDs as they are consistent with pricing assumptions of no arbitrage, and allowing pricing by replication. Indeed, risk-neutral PDs are changed in a way that every agent would be considered to be risk-neutral, therefore eliminating discrepancies in prices. This type of risk-neutral measure exists in any complete market with no arbitrage opportunity. Therefore, another way to bootstrap the PD is to use market data on Credit Default Swaps (CDS). As we assume that the CDS market is complete and does not allow for arbitrage, we are able to retrieve the PD that would reflect the market's assessment on the probability that company X will default. As we mentioned in Chapter 1 that CDOs could be built from many different types of underlyings, the financial instruments we will use to build CDOs are based on CDS. We turn to this next.

A CDS is an agreement between two parties to insure the default of a certain underlying. Similar to a CDO, party **B** agrees to pay a certain amount to party **A** in the event that a default occurs. This amount is computed to replace the loss that party **A** would've incurred if he had held the underlying without a CDS contract. If there is no default, party **B** does not pay anything. On the other hand, upon entering the contract until its maturity, party **A** pays a protection fee to party **B**. This fee is computed to equate the risk-free expected value of both sides of the contract.

Each side is also called a leg. In our example, party **A** holds a fixed, or *fee-payment*, leg, and part **B** holds a floating, or *default-insurance* leg. Using the notation and formulas in Schönbucher (2003), we present below how the fee, also called spread, is computed. Then we show how to extract the probability of default from this.

The spread s of a CDS is calculated by equating the values of the two legs. By letting the number of payment dates be N, the difference between two payment dates be δ_n, and assuming the protection buyer pays $s \cdot \delta_{n-1}$ at T_{k_n} if no default until T_{k_n}, the value of the fixed leg is given by

$$V^{\text{fixed}} = s \sum_{n=1}^{N} \delta_{n-1} \overline{B}(0, T_{k_n}),$$

with $\overline{B}(0, T_k)$ being the price of a defaultable zero coupon bond with maturity T_k at time 0. In other words, $\overline{B}(0, T_k) = B(0, T_{k_n})P(0, T_k)$, where $P(0, T_k)$ is the survival probability until T_k and $B(0, T_{k_n})$ is the price of a risk-free bond, which is usually proxied by short-term government bills.

With a payment of $(1 - R)$ at T_k if default occurs in $]T_{k-1}, T_k]$, the value of the floating leg is

$$V^{\text{float}} = (1 - R) \sum_{k=1}^{k_N} \delta_{k-1} H(0, T_{k-1}, T_k) \overline{B}(0, T_k),$$

where $H(0, T_{k-1}, T_k)$ is the implied hazard rate, i.e.,

$$H(0, T_{k-1}, T_k) = \frac{1}{\delta} \frac{P^{\text{def}}(0, T_{k-1}, T_k)}{P(0, T_{k-1}, T_k)},$$

for $\delta = T_k - T_{k-1}$. Moreover, P^{def} is the probability of default of the underlying and is what we are interested in extracting. Therefore, the value of both legs must be equal at the time we enter the contract, in order to mitigate any arbitrage opportunities. This means that the spread we observe in the market gives us the value of the ratio of the two legs.

$$s = \frac{V^{\text{float}}}{V^{\text{fixed}}}$$

By assuming a constant hazard rate and default observation on the tenor dates, we can bootstrap the probability of default of the underlying as

$$
\begin{aligned}
s &= \frac{V^{\text{float}}}{V^{\text{fixed}}} \\
&= (1-R)\frac{\sum_{k=1}^{k_N}\delta H(0,T_{k-1},T_k)\overline{B}(0,T_k)}{\sum_{n=1}^{N}\delta\overline{B}(0,T_{k_n})} \\
&= (1-R)H(0,T,T+\delta)\frac{\sum_{k=1}^{N}\delta\overline{B}(0,T_k)}{\sum_{n=1}^{N}\delta\overline{B}(0,T_{k_n})} \\
&= (1-R)H(0,T,T+\delta) \\
&= \frac{(1-R)}{\delta}\frac{P^{\text{def}}(0,T_{i-1},T_i)}{P(0,T_{i-1},T_i)} \\
s &= \frac{(1-R)}{\delta}\frac{P^{\text{def}}(0,T_{i-1},T_i)}{1-P^{\text{def}}(0,T_{i-1},T_i)} \\
\frac{\delta}{(1-R)}\left[1-P^{\text{def}}(0,T_{i-1},T_i)\right]s &= P^{\text{def}}(0,T_{i-1},T_i) \\
\frac{\delta s}{1-R} &= P^{\text{def}}(0,T_{i-1},T_i)\left(1+\frac{\delta s}{1-R}\right) \\
P^{\text{def}}(0,T_{i-1},T_i) &= \frac{\delta s}{(1-R)+\delta s}. \qquad (2.2)
\end{aligned}
$$

Therefore, the conditional probability of defaulting until the maturity, given that the underlying has not defaulted as of $T_0 = 0$, is given by

$$
P^{\text{def}}(0,0,\text{Maturity}) = \frac{\text{Maturity}\times s}{(1-R)+\text{Maturity}\times s}. \qquad (2.3)
$$

With this method, it more common to use either the most liquid maturity or piecewise constant hazard rates to calibrate to the entire CDS term structure. In our pricing, we will be using the most liquid maturity available, which is the 5 years maturity.

This gives a brief overview of the key elements needed to understand our study of credit default risk and instruments used in this thesis. It is worth noting that the structure and construction of one specific type of instruments is key to the reader's understanding of our motivation, and therefore deserves a section of itself.

2.2 Collateralized Debt Obligations

We have briefly mentioned CDOs, but the short description we gave in Chapter 1 does not give the whole picture of the instrument. We want to understand what type of investment would occur if the investor considers a CDO. We also want to give a motivation to the use of CDOs in this thesis.

It is important to emphasize the flexibility of the CDO. Since one can be developed by using a variety of underlying instruments, it is an tool of choice in studying credit risk, capturing various aspects of the credit market. It is thus important to understand their structure and their relation to the credit market in order to grasp how they represent credit risk. We will do this by focussing on their pricing practices, which highlights the heavy reliance of our analysis on models.

First, consider the structure of the instrument. A CDO consists of two parts: the CDO itself, fulfilling the contract, and the underlying reference pool, driving the performance of the contract. The CDO contract acts as the liability side and is tranched and linked to the performance of the asset side. The asset part is the reference pool that is securitized. A CDO can be created with different types of credit products, either loans, bonds, mortgages, Credit Default Swaps (CDS), or even other CDO tranches (then called CDO-squared). To categorize CDOs, we can divide them by type of underlying, by type of cash flows, etc.

Once the contract has been agreed upon and the type of CDO has been determined, we set up the two CDO legs: a fixed, or protection leg, and a floating, or fee leg. Similar to the CDS, the floating leg is the payments the holders make if there is a default. It is called floating because this payment is contingent on the default of the underlying. The fixed leg constitutes the payment tranche holders receive; it is called fixed because it is predetermined and known at the time the CDO contract is created. The cash flows are accounted as positive cash flows.

When describing how a CDO works, we need to define the tranche loss and the outstanding tranche notional. Tranching occurs when we allocate cash flows to different components of the investment. For example, as is done in figure 2.1, by picturing the whole portfolio as being 100%, senior tranches will usually occupy the top 60%, the mezzanine tranches will range from 10% to 40%, and the equity ones will occupy the bottom 10%. The investor chooses which tranche she wants to invest by choosing the attachment and detachment points which, respectively are the lower and upper bounds of her tranche. After that, the good cash flows (interest, repayments, etc.) go top-down, senior having priority over the other, and each receiving a predetermined share of them. The bad cash flows (losses), on the other hand, will go bottom-up, with the equity tranche being affected first. Thus their

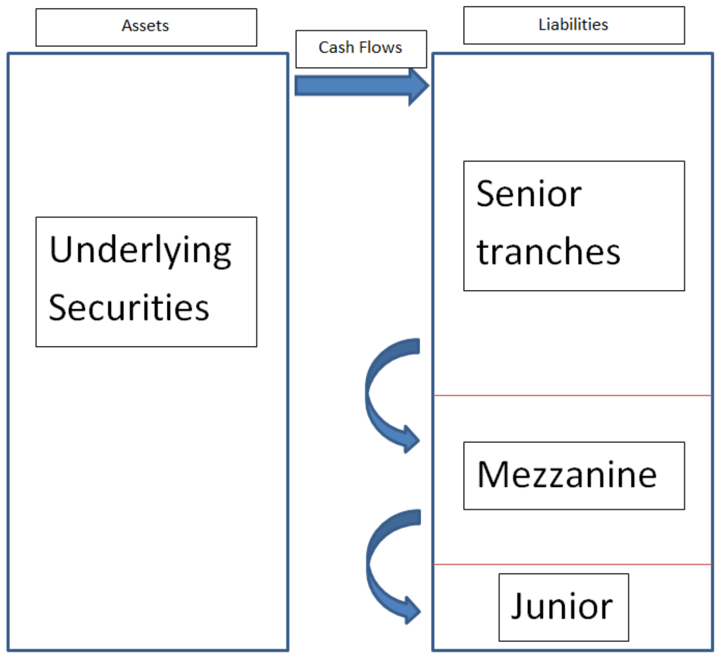


Figure 2.1: Example of a basic structure of CDO

name *first loss piece*.

The risk transfer is made possible because an investor takes risk in exchange of a premium. Tranching can be considered as a process of partitioning the portfolio into slices corresponding to certain percentage loss. By noting a certain tranche T with attachment point a and detachment point d as $T_{a,d}$, and the cumulative loss as L_t, the relative tranche loss Λ is

$$\Lambda_{a,d}(L_{(t)}) = \Lambda(L_t; a, d) = \frac{1}{d-a}\min[d-a, \max[0, L_t - a]]$$

where the loss of tranche $T_{a,d}$, $\Lambda_{a,d}(L_t)$ is normalized for size. This illustrates how choosing a and d allows us to choose our risk position.

To define the evolution of L_t, we must take into account every name in our reference pool, and their importance in value. The value of each name in our portfolio is called notional. We will refer to the tranche notional as the value of the portion of the portfolio in that tranche, and to the outstanding tranche notional at time $T = t$ as the portion of the tranche's notional that is still receiving positive cash flows after a certain period of time. Suppose that we have N entities in our reference portfolio of underlying assets, each with notional N_0. We define the total loss L_t on the portfolio as

$$L_t = \sum_{i=1}^{N} L_i \mathbb{1}_{\{\tau_i \leq t\}},$$

where $L_i = N_0(1 - R_i)$, $R_i = 1 - \mathrm{LGD}_i$ the recovery rate, and τ_i is the default time of the i-th entity. The loss within a tranche is adjusted for the recovery rate. This is the portion of the notional which can be recovered should a default occur. The outstanding tranche notional, Z_t, of a single tranche within a CDO is then given by

$$Z_t = [d - L_t]^+ - [a - L_t]^+, \tag{2.4}$$

and the absolute tranche loss Y_t as

$$Y_t = [L_t - a]^+ - [L_t - d]^+,$$

for a the tranche's attachment point and d its detachment point. Therefore, we have the cases

$$\begin{array}{rcll} L_t < a & \Rightarrow & Z_t = d - a & Y_t = 0 \\ a < L_t < d & \Rightarrow & Z_t = d - L_t & Y_t = L_t - a \\ L_t > d & \Rightarrow & Z_t = 0 & Y_t = d - a. \end{array}$$

To set up the payment the protection buyer needs to pay to the holder of the fee leg, we must ensure that we do not create an arbitrage opportunity with the CDO. Therefore, we should make sure that the risk-neutral expected present value of the two legs are equal, since the initial price of a CDO is 0 in the risk-neutral environment. Since it relies on the random events that cause some firms to default, we assume that the value of the fee leg will be equal to the *discounted expected outstanding tranche notional at each payment date*. The value of the protection leg will be given by the discounted expected *changes* in the tranche notionals between payment dates. The payment, or spread, is paid on the outstanding tranche notional and is calculated to make the initial value equal to zero. Note that all expectations are computed with respect to the risk-neutral measure.

By denoting the payment dates by T_i, $1 \leq i \leq n$, the intervals between each payment date by $\delta_i = T_i - T_{i-1}$ and the value of a risk-free investment of \$1 at time t by $b(t)$, the value of the fee leg is given by

$$sV^{\text{fee}} \stackrel{\text{def}}{=} s \sum_{i=1}^{n} \frac{\delta_i}{b(T_i)} \mathbb{E}[Z_{T_i}], \tag{2.5}$$

where Z_{T_i} represents the outstanding tranche notional. Notice the spread s multiplying both sides of the equation. As mentioned previously, the spread is taken as a proportion of the insured value. Therefore, the spread $0 \leq s \leq 1$ is multiplied by the expected present value of the fee leg V^{fee}. We use this notation as it is consistent with the model developed by Bush et al. (2011) which will be discussed in more details in Section 3.2.

In a similar fashion, we can express the value of the protection leg in terms of the outstanding tranche notional Z_t as

$$V^{\text{prot}} = \sum_{i=1}^{n} \frac{1}{b(T_i)} \mathbb{E}[Z_{T_{i-1}} - Z_{T_i}], \tag{2.6}$$

assuming that the losses are paid at the coupon dates T_i. Once the values of the protection and the fee legs are known, the spread is computed as

$$s = \frac{V^{\text{prot}}}{V^{\text{fee}}}. \tag{2.7}$$

This spread is computed for each tranche.

Therefore understanding how the study of CDOs and the different strategies they offer can help us understand the credit market. Most importantly, CDOs' structure stresses the use of reliable and efficient models for credit events in order to make wise decisions. Moreover, this theory can be adapted to many other situations, as CDOs are flexible and can serve many different investment strategies.

2.3 Structural Models

In order to understand the need for new models, and where the new models originate from, we need to present major building blocks in credit risk modelling. This section presents the evolution from the structural models perspective whereas the next section will focus on the reduced-form models. Both sections aim at developing basic understanding of the majors tools used from the beginning of credit market modelling up until the most recent and popular methods. We do not suggest which method might be more valid, but we clearly identify the strengths and weaknesses of each approach in order to underline the importance of innovation.

The first approach of modern finance to credit risk was made by Robert Merton (1974), who adapted the Black-Scholes option pricing theory developed earlier to the context of corporate debt. The setting of this approach is that the capital structure of a firm is composed of equity E and debt D, a zero-coupon bond with maturity T and face value D. He then compares the equity to an European call option with maturity T and strike price D, stating that, at time T, a firm defaults if its assets A are worth less than its debt. The debt is calculated as

$$\text{Debt} = \text{Asset} - \text{Equity}.$$

Thus,

$$E_T = \max[A_T - D, 0].$$

He stipulates that the firm's asset value follows a geometric Brownian motion (GBM) diffusion process, in the same way that Black and Scholes (1973) assume the stochastic process of stock prices. The value A of the firm follows

$$\mathrm{d}A_t = rA_t\mathrm{d}t + \sigma_A A_t \mathrm{d}W_t$$

and this yields the following pricing formula for the equity at time $t = 0$, derived from the Black-Scholes-Merton model:

$$\begin{aligned} E_0 &= \mathbb{E}\left[e^{-rT}(A_T - D)_+\right] \\ E_0 &= A_0\mathrm{N}(d_1) - D\mathrm{e}^{-rT}\mathrm{N}(d_2) \end{aligned}$$

where

$$d_1 = \frac{\ln\left(A_0\mathrm{e}^{rT}/D\right)}{\sigma_A\sqrt{T}} + \frac{1}{2}\sigma_A\sqrt{T};\ d_2 = d_1 - \sigma_A\sqrt{T}.$$

Here σ_A is the volatility of the asset value and r is the risk-free rate of interest, both assumed to be constant. Note that $\mathrm{N}(\cdot)$ is the cumulative distribution function of a standard normal random variable.

This model was used because of the simplicity involved in the implementation, but it suffers from several flaws. Firstly, Merton assumes that a default may only occur at maturity $t = T$. Secondly, it models the capital structure of a firm as a zero-coupon bond, which is too simplistic, considering the multiple debt structures that are possible, as presented in Geske (1977, 1979). Thirdly, the term structure of interest rates is assumed to be constant and flat, which is counter factual. Finally, just as with option pricing from the Black-Scholes model, as t approaches T, the predictability of a default increases significantly, rendering short-term spreads practically immaterial. These flaws serve to highlight the inherent trade-off between realistic assumptions and ease of implementation.

To address the time of default issue in the Merton model, Black and Cox (1976) developed a first-passage model (FPM) where a default threshold is defined, and default occurs as soon as it is hit. This threshold is either determined exogenously to act as safety covenant for bondholders, or endogenously, to maximize the firm's value by the managers. This model has been studied with stochastic or non-stochastic interest rates and other extensions to account for taxes, jumps, etc. However, every extension to the model is made at the expense of increased complexity.

The Black-Cox model has the advantage of allowing default to occur at any time, according to a threshold. Moreover, this threshold could also be stochastic. Consider a constant threshold $K > 0$. Then, building on Merton's firm's dynamics, the time of default τ is given by

$$\tau = \inf\{s \geq t | V_s \leq K\}.$$

Thus, using the reflection principle of the Brownian motion (Elizalde, 2012b), the default probability from t to T is

$$\mathbb{P}[\tau \leq T | \tau > t] = \Phi(h_1) + \exp\left\{2\left(r - \frac{\sigma_A^2}{2}\right)\ln\left(\frac{K}{A_t}\right)\frac{1}{\sigma_A^2}\right\}\Phi(h_2)$$

where

$$h_1 = \frac{\ln\left(\frac{K}{e^{r(T-t)}A_t}\right) + \frac{\sigma_A^2}{2}(T-t)}{\sigma_A\sqrt{T-t}}$$

$$h_2 = h_1 - \sigma_A\sqrt{T-t}$$

Some argue that only the ratio of A_t to K matters in the valuation process, and not the specific ways of choosing the threshold (Elizalde, 2012b). They thus model the ratio only as a stochastic process to price corporate bonds. Other modifications include a Liquidation Process Model (François and Morellec, 2004), or a State Dependent Model (Hackbarth et al., 2005), to take into account different characteristics of the reality, like the lapse of time it takes to liquidate assets, in the former, or business cycles or other states affecting cash flows in the latter.

The drawbacks of Black-Cox models are mainly due to its analytical complexity and failure to pass empirical testing (Elizalde, 2012b). Moreover, we find the same problem in this model as in the Merton model with the predictability of default, and predictability of recovery, since both model assume complete information about the value process A_t. This problem vanishes if we assume a stochastic and hidden threshold K. Also, empirical evidence suggests that default can occur for reasons other than the value of the assets falling below a threshold. For instance, a liquidity shortage or high funding costs could also put the firm in a defaulting position.

In order to address some of these problems, many researchers have considered incomplete information models, or tried to incorporate jumps and stochastic volatility in the firm's value dynamics (Fouque et al., 2006). There were attempts to introduce correlated Brownian motions in these dynamics (Giesecke, 2004). More recently, a contagion effect was added into the model in order to reproduce the observed concentration of default through time (Giesekce and Goldberg, 2004). Through either a copula function, or using an incomplete information model, or a common factor model with firm value being influenced by market factors and idiosyncratic factors, increased realism was introduced into the model. However, credible calibration results of these models remain inadequate with these extensions (Elizalde, 2012a).

A successful implementation of a structural credit model in a practical setting has been attempted on the so-called KMV model, named after Kealhofer, McQuown, and Vasicek (Grasselli and Hurd, 2010a). KMV departs slightly from a strict implementation of a structural model as it tries to circumvent the main difficulty of explicitly modelling dynamics of the firm's value A_t. Rather than trying to estimate A_t from the firm's balance sheet, it infers it from the value of debt and equity by exploiting the following equation

$$A_t = D_t + E_t.$$

The equity is defined as the market capitalization which is observed. Therefore we can infer A_t according to

$$E_t = \mathrm{BSCall}(A_t, T-t, r, \sigma, K)$$

where T is the maturity date representing the approximate time scale of the debt (e.g. the duration), and K is the default trigger.

K is determined from the structure of the firm's debt, placing the value somewhere between the face value of short term debt, and the face value of total debt. This is justified if we argue that the firm has to service its short term debt, but can be more flexible with its long term debt. Typically, K is given by the full short term debt plus half of the long term debt.

We then calibrate the model parameters to obtain $\hat{\mu}$ and $\hat{\sigma}$ to an observed time series $\{\hat{E}_{t_1}, \ldots, \hat{E}_{t_N}\}$ of market capitalization giving the time series $\{\hat{A}_{t_1}, \ldots, \hat{A}_{t_N}\}$. This allows us to compute the key credit score DD_t, called *distance to default*.

$$\mathrm{DD}_t = \frac{\log(A_t/K)}{\sigma}$$

This gives the amount by which $\log A_t$ exceeds K measured in standard deviations σ. It is called distance to default because it is to be interpreted as a measure of likelihood of default, which serves as a certain risk measure.

By following a strict interpretation of the structural model, the expected default frequency, EDF, i.e. the probability of observing the firm default within one year, should be equal to the normal probability EDF = N(DD). However, Moody's Analytics, using KMV, uses its large database of historical defaults to map DD to EDF using a proprietary function EDF = f(DD). This breaks the model but has the advantage of making it to produce more favourable empirical evidence. The function f is designed to give the actual fraction of all firms with the given DD that have been observed to default within one year. We should note that the DD_t is a firm-specific dynamic quantity that correlates strongly with credit spreads and observed historical default frequency.

In summary, structural models have he advantages of being economically intuitive and easy to interpret. However, they might be too rigid since after correcting the structural assumptions, we might have difficulties in calibrating to the market. This is why we turn to reduced-form models next.

2.4 Reduced-form Models

Reduced-form models, in contrast to the structural models, do not try to replicate the firms' asset dynamics and relate it to defaults. They simply observe that default occurs randomly in time, and try to match their outcome to reality, using only available market data. The most popular reduced-form models studied in the literature are intensity based models. In these models, a default is triggered by the first jump in an exogenous jump process. The parameters for this process (associated with the physical probability measure) are inferred from market data.

To begin the construction of our intensity-based model, we use a Poisson process to model the arrival risk of credit-defaults (Elizalde, 2012a). By considering an increasing sequence of stopping-times ($\tau_h < \tau_{h+1}$), we define the stochastic counting process N_t by

$$N_t = \sum_h \mathbb{1}_{\{\tau_h \leq t\}}$$

Then, a (homogeneous)Poisson process with intensity $\lambda > 0$ is a counting process whose increments are independent and satisfy

$$\mathbb{P}[N_t - N_s = n] = \frac{1}{n!}(t-s)^n \lambda^n \exp\left(-(t-s)\lambda\right),$$

for $0 \leq s \leq t$. Thus, $N_t - N_s$ are independent and have a Poisson distribution with parameter $\lambda(t-s)$ for $s \geq t$. By allowing for a time-dependent $\lambda_t = \lambda(t)$, we have a time-inhomogeneous Poisson process, and hence a time-inhomogeneous diffusion process

$$d\lambda_t = \mu(t, \lambda_t)dt + \sigma(t, \lambda_t)dW_t,$$

where W_t is a Brownian motion. This is a Cox process. A Cox process is sometimes called a doubly stochastic Poisson process (Grasselli and Hurd, 2010b).

Once a Cox process is defined, we can start modelling the single entity credit-default risk. The model is similar to the pricing model of a zero coupon bond. The only difference between the formulas for the survival probability $\mathbf{S}$ with intensity and price $\mathbf{P}$ of default-free zero coupon bonds lies in the discount rate, where λ is used instead of r. Therefore,

$$\mathbf{S}(0, t) = \mathbb{E}\left[\exp(-\int_0^t \lambda_s ds)\right]$$

is treated the same way as

$$\mathbf{P}(0, t) = \mathbb{E}\left[\exp(-\int_0^t r_s ds)\right]$$

so we end up using short-term rate models for defaults intensities. In this context, a default time would be defined by:

$$\tau = \inf\{t > 0 | \exp(-\int_0^t \lambda_s ds) \leq U\}$$

where $U \sim \text{Unif}(0, 1)$ is taken from a standard uniform distribution.

According to Schönbucher (2003), we should set up our models such that:

1. r_t and λ_t should be stochastic;
2. r_t and λ_t should have correlation between them;
3. r_t and λ_t should be greater than zero at all times;
4. the model should be as easy to price as possible.

Also, we recall Schönbucher's conditions for an optimal model, stated in Section 1.2.

The simplest model would be a Conditionally Independent Defaults (CID) model. In this model, we have a set of different firms, with different intensities and unknown default times. We observe the state of the market, X_t, and once these realizations are fixed, we assume that the intensities are independent. Thus, the intensity process follows

$$\lambda_{i,t} = a_{0,\lambda_i} + a_{1,\lambda_i} X_{1,t} + \ldots + a_{J,\lambda_i} X_{J,t} + \lambda^{\star}_{i,t}$$

where $X_{j,t},\ j = 1, \ldots, J$ are all state variables, a_i's are constant coefficients, and $\lambda^{\star}_{i,t}$ is a firm specific factor of stochasticity. Moreover,

$$\mathrm{d}\lambda^{\star}_{i,t} = \kappa_i(\theta_i - \lambda^{\star}_{i,t})\mathrm{d}t + \sigma_i\sqrt{\lambda^{\star}_{i,t}}\mathrm{d}W_{i,t}$$

where $W_{i,t}$ is a standard Brownian motion.

Unfortunately, CID models tend to produce too small default correlation between names due to the lack of sophistication in choosing our state variables. Some argue that this could be overcome by choosing more wisely the parameters representing latent variables, but this makes the model harder to calibrate (Yu, 2005). Also, since the model generates risk through dependence of the firm's intensity process to a set of state variables, once this variable is fixed and a realization is observed, the defaults become independent.

Other researchers have tried to extend the CID model by adding joint jumps in the firm's default processes, or by having common default events. This makes calibration and estimation even harder.

A second method used to model credit default risk is to introduce a contagion effect among firms. From empirical evidence, we observe that there exist periods in which credit risk increases simultaneously throughout the market. This leads us to believe that there is a contagion effect through some market factors (Davis and Lo, 1999) or through commercial or financial relationships (Jarrow and Yu, 2001).

Davis and Lo (1999) introduced a model where defaults occur infectiously. Each firm has an initial hazard rate $\lambda_{i,t}$, but when a default occurs, an enhancement factor $a > 1$ is applied to the remaining firms so their respective hazard rate becomes $a\lambda_{i,t}$. This enhanced rate remains active for an exponentially distributed period of time.

On the other hand, as for the contagion mechanism developed by Jarrow and Yu (2001), it is based on the propensity of firms to be influenced by others. It thus accounts for counterparty risk. It basically takes the CID models, but adds symmetric dependence. When one firm defaults, its counterparties have an increased chance of defaulting. Unfortunately, this can create looping defaults, and be unsuitable for simulations. Jarrow and Yu (2001) tried to implement asymmetric risk, with primary firms influencing secondary firms, but not getting influenced and therefore simply following a CID pattern. This created a serious problem for primary firms because of the low correlation of defaults observed in CID models.

The third method used in reduced-form models introduces correlation through copula functions. By estimating marginal distributions of default, and some joint distribution of default, a dependence structure is completely specified by a chosen copula function. This will not be studied further in this thesis, but it is worth mentioning since it is receives a lot of attention in current research.

This selected review of the existing models has led us to better appreciate the challenges faced by academics and practitioners in defining a suitable model for the credit market. With these building blocks in mind, we can set forth to study, in the next chapter, more recent developments in credit risk modelling.

Chapter 3

Modelling Credit Risk

After reviewing a wide range of credit market models that have been studied in the literature, we come to appreciate the need for new, more realistic and adaptable models. As a model developer, we are constantly being reminded that existing models would need to be improved to reflect the new reality of our market better. Even as an investor, we were reminded in 2008 that models could be far from reality. Therefore, there is a constant pressure to seek better ones. Of course, they should be mathematically sound, and provide a robust and viable alternative to the existing ones. Although as Schönbucher (2003) mentioned, an optimal model should be dynamic, we will still analyse a particularly interesting static model as it will become useful in the analysis of similar but dynamic models. We will then proceed to define and test the theory behind particular structural and reduced-form dynamic models.

Each section will include theory, heuristics of the main theorem, main qualities and worst flaws, as well as some possible avenues for improvement. We will then conclude this chapter with a summary of which comparison and a verdict on the model will be deemed most favourable in he context of our study.

3.1 Single Period Structural Model

As a first alternative to the industry-used model, we study a model relatively close to what is used by practitioners, known as a single-period structural model. It is defined in Lucas et al. (2000). The idea of this model is to simplify the credit portfolio modelling by reducing the amount of required simulations. It should have an edge over modelling every underlying individually, which would require more computing power, and hence be more costly. The authors derive an analytic approximation to the loss distribution when the portfolio contains a large number of exposures. It introduces a powerful Law of Large Numbers and allows us to study extreme events, which have become a main concern in risk management. This will lead us to the study of extreme value theory, and understanding what type of events can possibly be reproduced by this model. We will then extend the model, investigating the possibility of a different tail distribution, as well as the consequence of such result.

The authors use a simple structural model and assume that the assets underlying the bonds and loans in the portfolio follow a factor model defined as

$$S_n = \mu_n + \beta_n^\top f + \epsilon_n, \tag{3.1}$$

where the factors $f \in \mathbb{R}^m$ and innovations $\epsilon_n \in \mathbb{R}$ follow a certain probability distribution. Without loss of generality, we can set $\mu_n = 0$. In the context of Lucas et al. (2000), they assume that both random variables follow independent normal distributions with zero-mean and covariance matrix Ω_f and ω_n, respectively. We could however, use the theory developed by Schönbucher (2001) to allow for a more general distribution. The vector $\beta \in \mathbb{R}^m$ corresponds to the factor loadings of each element of f, representing the influence of each factor on the surplus variable S_n. This is the CWI model presented in Section 2.1. We will present in the following section the setting of Schönbucher's model, then use next the section to study the model and results from Lucas et al. (2000).

3.1.1 Single-factor model

The motivation of Schönbucher (2001) is to study default correlation through conditional default probabilities and joint default probabilities. Using Schönbucher's setting for a large uniform portfolio, i.e., $N \to \infty$, N the number of names in our portfolio, we call X the fraction of defaults that occurred until time T. The realized loss upon default is then

$$X(1 - R)L$$

where R is the recovery rate assumed to be $R_i \equiv R$, and L is the exposed notional, assumed equal for all names. The only thing we need to worry about in order to assess the distribution of the default losses is the distribution of the fraction X of defaults. Since X is a counting variable, it is based on N underlying processes of the firms' values S_n. The following assumption summarizes the context.

Assumption 3.1.1 (Generalised One Factor Model)**.** *The default of each obligor is triggered by the change of the value of the assets of its firm. We denote the value of the assets of the n-th obligor at time t by $S_n(t)$. We assume that the values of the assets of the obligors are driven by a common factor f which has a distribution function $G(y)$, and an idiosyncratic noise component ϵ_n which is distributed according to the distribution function $H(\epsilon)$. We thus have*

$$S_n(T) = \beta_n^\top \cdot f + \epsilon_n \quad \forall n \leq N, \tag{3.2}$$

where $Y \sim G$ and the ϵ_n, $n \leq N$ are i.i.d. $H(\epsilon)$-distributed. If the respective moments of Y and ϵ_n exist, we assume, without loss of generality that these random variables are centered and standardised.

Obligor n defaults if its firm's value falls below a pre-specified barrier $S_n(T) \leq K_n$.

Therefore, X_N is defined as

$$X_N = \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{\{S_n(T) \leq K_n\}}, \tag{3.3}$$

and $X_N \overset{a.s.}{\to} X$ as $N \to \infty$.

Using this approach, the values of the assets of two obligors n and $m \neq n$ are correlated with linear a correlation coefficient β. It is important to note that, conditional on the realisation of the systematic factor f, the firm's values and the defaults are independent. This is key to the large portfolio approximation. If we assume that all obligors have the same default barrier $K_n = K$ and the same exposure $L_n = 1$, we can obtain some intuition about the distribution of the portfolio's credit loss. In order to do so, we build up on the known facts about the different drivers. By fixing the market factor $f = y$, we first define the individual *conditional* default probability

$$\begin{aligned} p(y) &= \mathbb{P}\left[S_n(T) \leq K | f = y\right] \\ &= \mathbb{P}\left[\beta_n f + \epsilon_n \leq K | f = y\right] \\ &= \mathbb{P}\left[\epsilon_n \leq K - \beta_n f | f = y\right] \\ &= H\left(K - \beta_n y\right). \end{aligned} \tag{3.4}$$

Thus, conditional on the realisation y of f, the individual defaults occur independently from each other. Recall that X is the fraction of defaults in our portfolio, and is the limit as $N \to \infty$ of X_N defined in (3.3). Considering a large homogeneous portfolio approximation as $N \to \infty$ means that we can arrive to an almost sure relation between X and the conditional probability we have just derived. We have

$$\mathbb{P}\left[X = p(y) | f = y\right] = 1, \tag{3.5}$$

meaning that the fraction of obligors defaulting is almost surely equal to the individual default probability. We can now proceed to incorporate the distribution of the market factor Y in order to find the unconditional distribution function (df) of X. By invoking the iterated expectations, for $x \in [0, 1]$, we get

$$\begin{aligned}\mathbb{P}\left[X \leq x\right] &= \mathbb{E}\left[\mathbb{P}\left[X \leq x | f\right]\right] \\ &= \int_{-\infty}^{\infty} \mathbb{P}\left[X \leq x | f = y\right] \mathrm{d}G(y)\end{aligned}$$

and using (3.5)

$$\begin{aligned}\mathbb{P}\left[X \leq x\right] &= \int_{-\infty}^{\infty} \mathbb{P}\left[X = p(y) \leq x | f = y\right] \mathrm{d}G(y) \\ &= \int_{-\infty}^{\infty} \mathbb{1}_{\{p(y) \leq x\}} \mathrm{d}G(y) \\ &= \int_{y^\star}^{\infty} \mathrm{d}G(y)\end{aligned} \tag{3.6}$$

where $y^\star$ is chosen such that $p(-y^\star) = x$, and $p(y) \leq x \leq 1$ for $y > -y^\star$. Thus

$$y^\star = \frac{(H^{-1}(x) - K)}{\beta}.$$

By combining the results, we get that

$$F(x) := \mathbb{P}\left[X \leq x\right] = 1 - G\left(\frac{K - H^{-1}(x)}{\beta}\right). \tag{3.7}$$

We therefore have a form for the distribution function of a static one period single-factor model of a large portfolio containing $N \to \infty$ homogeneous underlying names. Having a single factor affecting our underlyings is a big simplification. We can view this single factor as capturing all of the exogenous randomness. Since we are allowed to put any distribution on G, this simplification is not overly restrictive. Using a single-factor also eases the presentation as it simplifies the algebra. Extending it to a multi-factor model would be similar, after accounting for the covariance between factors. In addition, this simple model will allow us to explore the extreme value theory and focus on the tail properties of the model. We will therefore present the refined model of Lucas et al. (2000) to introduce their tail index theorem and extend their conclusions to different tail behaviour.

3.1.2 Limiting distribution

We refine the Schönbucher (2003) model by assigning a particular form to G and H in order to obtain Lucas et al. (2000) model. Recall equation (3.1) where

$$S_n = \beta_n^\top \cdot f + \epsilon_n,$$

We will study the static model in a simplified two-state setting of *default* or *no default*. Let $l_n = 1, 2$ and $\pi(n, l_n)$ represent the end-of-period state and the credit loss of asset n, respectively. $l_n = 1$ is the state in which firm n is still alive and contributing to the portfolio. $l_n = 2$ occurs when the n-th firm's value falls below the default threshold, triggering a default. Firm n then becomes inactive and the firm's credit loss $\pi_n = (1 - R_n)L_n$ adds to the portfolio's credit loss C_N. Lucas et al. (2000) state that the firm's credit loss occurs if a firm defaults or its rating deteriorates. In our case, since we only assume two states and do not worry about credit ratings, π_n will refer to the LGD on firm n. We assume that all firms share a common Markovian transition matrix P between states. Finally, the portfolio credit loss is simply represented by the sum of the N individual credit losses:

$$C_N = \sum_{n=1}^{N} \pi(n, l_n). \tag{3.8}$$

In Schönbucher's notation, $C_N = NX(1 - R)L$, for homogeneous assets in our portfolio with common loss L and recovery rate R. We note that l_n is a stochastic variable since it is based on movements of the stochastic variable S_n. Therefore, π and C_N are also stochastic. We now take one more step towards studying the tail properties of the model by presenting a theorem for the distribution of the portfolio credit loss C_N for a large number of exposures. Before proceeding, we need the following assumption:

Assumption 3.1.2. $\sup_{n\geq 1} \mathbb{E}[\pi(n, k_n, l_n)^2 | f] < \infty \, (a.s.)$.

Assumption 3.1.2 assures that the conditional expectations of individual squared losses are almost surely bounded uniformly. It is not a strong assumption as most financial instruments technically satisfy it. Given this assumption, we can state Lucas et al. (2000) main theorem concerning the limit distribution law for two states.

Theorem 3.1.3. *Recall that Ω_f is the covariance matrix of the random factor vector f. Define*

$$R_n^2 = \frac{\beta_n^\top \Omega_f \beta_n}{\omega_n + \beta_n^\top \Omega_f \beta_n} \tag{3.9}$$

as the R^2 of the factor regression model (3.1), i.e., the squared correlation between S_n and its "fit" $\beta_n^\top f$. Moreover, let

$$v_j^\top = \frac{\beta_j^\top \Omega_f^{1/2}}{\sqrt{\beta_j^\top \Omega_f \beta_j}}, \tag{3.10}$$

such that $v_j^\top v_j = 1$. Define

$$B_N = \mathbb{E}[C_N|f] = \sum_{n=1}^{N} \mathbb{E}[\pi(n, l_n)|f] = \sum_{n=1}^{N} \Phi\left(\frac{s - \sqrt{R_j^2} v_j^\top \Omega^{-1/2} f}{\sqrt{1 - R_j^2}}\right). \tag{3.11}$$

Then, given Assumption 3.1.2 and the framework described earlier, we have

$$\frac{1}{N} C_N - \frac{1}{N} B_N \xrightarrow{a.s.} 0 \tag{3.12}$$

with C_N the portfolio credit loss as defined in (3.8) and $\xrightarrow{a.s.}$ denoting almost sure convergence.

This theorem states that the average credit loss C_N in (3.12) converges almost surely to the *conditional* (on f) expectation of credit losses B_N (see (3.11)). Therefore we average out all of the idiosyncratic risks by using B_N, since C_N depends on $(f, \epsilon_1, \epsilon_2, \ldots)$ whereas B_N only depends on f.

As we are interested in studying tail behaviour and extreme events, we use the simplified case of *default* or *no default* in order to derive simple but powerful conclusions. We assume that f and ϵ_n follow independent univariate normal distributions, where systematic risk is equal across all exposures ($R_n^2 \equiv \rho^2$, $\rho \geq 0$), and where $v_j \equiv 1$. Setting s to be the default threshold for all surplus variables S_n, we get

$$B_N = \sum_{n=1}^{N} \Phi\left(\frac{s - \rho \Omega_f f}{\sqrt{1 - \rho^2}}\right)$$

such that

$$\frac{C_N}{N} - \Phi\left(\frac{s - \rho \Omega_f f}{\sqrt{1 - \rho^2}}\right) \xrightarrow{a.s.} 0. \tag{3.13}$$

More details on these results and their derivations are given in Lucas et al. (2000). Below we discuss the tail properties which are one focus of study in this thesis. The discussion is preceded by a brief introduction to extreme value theory.

3.1.3 Extreme value theory

We want to study in this section the extreme tail behaviour of distributions. Lucas et al. (2000) state that their credit loss distribution has a tail expansion of the form

$$\overline{F}(c) = 1 - F(c) = (\bar{c} - c)^{\alpha} L[(\bar{c} - c)^{-1}], \tag{3.14}$$

where $\bar{c}$ is the maximum credit loss, $F(\cdot)$ the credit loss distribution, and $L(\cdot)$ is a slowly varying function. By slowly varying function, we mean $\lim_{x\to\infty} L(tx)/L(x) = 1$, for $t > 0$. In our case, $\bar{c} = 1$ as in 100% of our portfolio. The intuition for this is that the larger α is, the faster the tail decays to zero, therefore giving an indication about the shape of the tail of the credit loss distribution for large credit losses. Here, $F(x) \equiv \mathbb{P}[X \leq x]$ and $\overline{F}(x) \equiv \mathbb{P}[X > x] = 1 - F(x)$.

The function indicates that there are two important parts determining the tail behaviour. The first one is the rate of tail decay α. We can see that when $c \to \bar{c}$, the dominant factor of $\overline{F}$ becomes $(\bar{c} - c)^{\alpha}$. The other important factor of $\overline{F}$ is obviously the function $L(\cdot)$. We will focus our attention on the rate of tail decay α, but it is worth mentioning that different characterization of $L(\cdot)$ might lead to significantly different results, even for the same α.

The following theorem is from Lucas et al. (2000) and summarizes the tail behaviour of the setting presented earlier, and constitutes our main point of interest for this section:

Theorem 3.1.4 (Tail index)**.** *Consider the one-factor model and assume that a fraction $\lambda \in (0,1)$ of the firms has $R_n^2 = \hat{R}_1^2$, while the remaining firms have $R_n^2 = \hat{R}_2^2$. The credit loss distribution has a tail expansion as in (3.14) with a tail index*

$$\alpha = \max_{i\in\{1,2\}} \frac{1 - \hat{R}_i^2}{\hat{R}_i^2} \tag{3.15}$$

This theorem says that the loss distribution has algebraically declining tails instead of exponentially declining tails as could be expected. We notice that the exact proportion λ of names with a certain $\hat{R}^2$ does not matter, as only the ones with the highest factor will be determining the tail index. In the special case of a perfectly homogeneous portfolio, we have $\hat{R}_1^2 = \hat{R}_2^2 = \rho^2$, such that $\alpha = (1 - \rho^2)/\rho^2$. We then clearly see that increasing the degree of systematic risk (ρ^2) leads to a decrease in the tail index α, thus indicating a decrease in the rate of tail decay. By this simplistic illustration we can conclude that even if the model's exposures are driven by thin-tailed systematic and idiosyncratic shock distribution (in our case normally distributed), the portfolio credit loss distribution can still exhibit the empirical stylized fact of fat-tail shown by polynomially declining tails.

A second conclusion from this theorem is that the highest idiosyncratic risk component of an exposure, i.e., smallest $\hat{R}_i^2$, dominates the extreme tail behaviour. This implies a higher rate of tail decay. Intuitively, we would expect the worst tails to occur when all loans are very highly correlated, since there is more chance that all of our portfolio will default at once. Similarly, we can interpret the theorem by stating that when the exposure is exhibiting more idiosyncratic risk, it is less likely to have a joint default occurring. Because idiosyncratic sources of risk are independent from each other, the correlation is low, therefore one default does not attract other defaults. Our portfolio is thus less likely to reach the maximum credit loss possible. In order to have thin tails, we would need a high α, which would be created by having a small $\hat{R}_i^2$. Thus, a group within our portfolio with high idiosyncratic risk lowers the probability of extreme events. Equivalently, higher systematic risk implies higher probability of extreme event, as the tails get fatter. This argues in favour of diversification.

In order to study the tail behaviour of the credit loss distribution in a more general setting, it is useful to consider the proof of theorem 3.1.4.

Proof of Theorem 3.1.4. Using the Von Mises condition for Weibull distributions (Corollary 3.3.13 of Embrechts et al. (1997)), it suffices to prove that

$$\lim_{C \uparrow 1} \frac{(1-C) \cdot f(C)}{1-F(C)} = \frac{1-\rho^2}{\rho^2} \tag{3.16}$$

with $F(\cdot)$ and $f(\cdot)$ the c.d.f. and p.d.f. of credit losses, respectively. Define $u_1 = 1 - C$. We can now rewrite (3.16) as

$$\lim_{u_1 \downarrow 0} \frac{u_1 \cdot f(1-u_1}{1-F(1-u_1)}. \tag{3.17}$$

Using (3.13), we can rewrite (3.17) as

$$\lim_{u_1 \downarrow 0} \frac{u_1 \cdot \sqrt{1-\rho^2} \cdot \phi\left(\frac{\Phi^{-1}(u_1)\sqrt{1-\rho^2}-C}{\rho}\right)}{\rho \cdot \Phi\left(\frac{\Phi^{-1}(u_1)\sqrt{1-\rho^2}-C}{\rho}\right) \cdot \phi(\Phi^{-1}(u_1))}, \tag{3.18}$$

with $\Phi(\cdot)$ and $\phi(\cdot)$ the standard normal c.d.f. and p.d.f., respectively. Using the substitution $u_2 = \Phi^{-1}(u_1) \Leftrightarrow u_1 = \Phi(u_2)$, (3.18) transforms into

$$\lim_{u_2 \to -\infty} \frac{\Phi(u_2) \cdot \sqrt{1-\rho^2} \cdot \phi\left(\frac{u_2 \cdot \sqrt{1-\rho^2}-C}{\rho}\right)}{\rho \cdot \Phi\left(\frac{u_2 \cdot \sqrt{1-\rho^2}-C}{\rho}\right) \cdot \phi(u_2)}. \tag{3.19}$$

Now from equation (26.2.13) of Abramowitz and Stegun (1970), we have that for large negative u_2

$$\Phi(u_2) = \frac{\phi(u_2)}{|u_2|}(1 + o(u_2^{-1})). \tag{3.20}$$

Applying this results to (3.19), we establish

$$\begin{aligned}
\lim_{u_2 \to -\infty} \frac{\Phi(u_2) \cdot \sqrt{1-\rho^2} \cdot \phi\left(\frac{u_2 \cdot \sqrt{1-\rho^2} - C}{\rho}\right)}{\rho \cdot \Phi\left(\frac{u_2 \cdot \sqrt{1-\rho^2} - C}{\rho}\right) \cdot \phi(u_2)} &= \\
\lim_{u_2 \to -\infty} \frac{\phi(u_2) \cdot \sqrt{1-\rho^2} \cdot \phi\left(\frac{u_2 \cdot \sqrt{1-\rho^2}}{\rho}\right) \left|\frac{u_2 \cdot \sqrt{1-\rho^2}}{\rho}\right|}{|u_2| \cdot \rho \cdot \phi\left(\frac{u_2 \cdot \sqrt{1-\rho^2}}{\rho}\right) \cdot \phi(u_2)} &= \frac{1-\rho^2}{\rho^2},
\end{aligned} \tag{3.21}$$

which proves the theorem (Lucas et al., 2000). □

Now we analyse the theory required for the proof of Theorem 3.1.4. It uses three results from other works, and the rest involves simple manipulations. It starts by using Corollary 3.3.13 of Embrechts et al. (1997), then uses equation (3.13) from its previous theorem 3.1.3 and ends by using a tail expansion for the normal distribution from Abramowitz and Stegun (1970). Let us see how and why we can use Corollary 3.3.13, as it is the basis for this proof. The other two results are specific cases to the setting of the particular paper by using the normal distribution for the factors and will not be of concern to us at the moment since we want to introduce the more general case.

From Embrechts et al. (1997), we can get that their Corollary 3.3.13, on page 136, states

Corollary 3.3.13 (from Embrechts et al. (1997), p.136)**.** *Let F be an absolutely continuous distribution function with density f which is positive on some finite interval (z, x_F). If*

$$\lim_{x \uparrow x_F} \frac{(x_F - x)f(x)}{\overline{F}(x)} = \alpha > 0, \tag{3.22}$$

then $F \in \mathrm{MDA}(\Psi_\alpha)$.

A Maximum Domain of Attraction (MDA) is defined as

Definition 3.1.5 (Maximum domain of attraction)**.** We say that the random variable X (the distribution function F of X, which is the distribution of X) belongs to the *maximum domain of attraction of the extreme value distribution* D if there exist constants $c_n > 0$, $d_n \in \mathbb{R}$, such that

$$c_n^{-1}(M_n - d_n) \xrightarrow{d} D \tag{3.23}$$

for $M_n \equiv \max(X_1, \ldots, X_n)$, the maximum value of the iid sequence (X_n), holds. We write $X \in \text{MDA}(D)\,(F \in \text{MDA}(D))$.

The Corollary thus says that as we progress in the tail of the distribution, if the ratio of its density to its right tail distribution function is constant, then the distribution belongs to the Weibull Maximum Domain of Attraction Ψ_α, α being the tail index of the distribution. In our context, this tail index happens to be the ratio of linear correlation to the market factor. We thus know that the probability of extreme events will change proportionally to the level of market correlation. The MDA Ψ_α is what is referred to as the Weibull df. The next theorem will illustrate why our situation falls into the Weibull distribution function. It will also give an explanation of the use of the parameter α, which is required to characterize the distribution functions and the thickness of the tail associated with each of them. To fully grasp this theorem, a definition of heavy-tails as well as an introduction to regular variation might be useful. We add details on those topics in Appendix B.

The following result, considered by Embrechts et al. (1997) and many others to be the basis of classical extreme value theory, identifies the only limit laws possible for maxima of i.i.d. r.v.'s.

Theorem 3.1.6 (Fisher-Tippett theorem, limit laws for maxima)**.** *Let (X_n) be a sequence of i.i.d. random variables. If there exist constants $c_n > 0$, $d_n \in \mathbb{R}$ and some non-degenerate distribution D such that*

$$c_n^{-1}(M_n - d_n) \xrightarrow{d} D, \tag{3.24}$$

Then D belongs to the type of one of the following three distribution functions:

Fréchet: $\Phi_\alpha(x) = \begin{cases} 0, & x \leq 0 \\ \exp\{-x^{-\alpha}\}, & x > 0 \end{cases} \quad \alpha > 0$

Weibull: $\Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^{\alpha}\}, & x \leq 0 \\ 1, & x > 0 \end{cases} \quad \alpha > 0$

Gumbel: $\Lambda(x) = \exp\{-\mathrm{e}^{-x}\}, \quad x \in \mathbb{R}.$

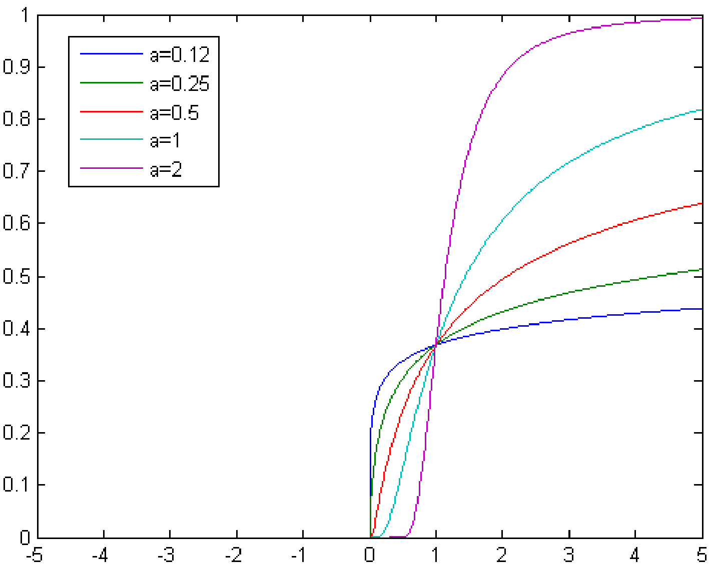


Figure 3.1: Fréchet MDA for various tail indices a

The functions for the three different MDAs are presented graphically in Graphs 3.1, 3.2, and 3.3. When possible, we plotted them for different values of α, represented by a in the legend, in order to show the variation of tail weight as the tail index changes. We observe that a smaller a implies fatter tails.

At this point, it is important to note that the distribution functions F in the Maximum Domain of Attraction of the Weibull distribution $\mathrm{MDA}(\Psi_\alpha)$ have a finite end point x_F. This is particularly relevant since we are concerned with loss distributions which have finite end points, namely a maximum loss of 100% of the initial portfolio. The following theorem characterizes the $\mathrm{MDA}(\Psi_\alpha)$ used in the proof of theorem 3.1.4.

Theorem 3.1.7 (Embrechts et al., 1997)**.** *The df F belongs to the maximum domain of attraction of Ψ_α, $\alpha > 0$, if and only if $x_F < \infty$ and $\overline{F}(x_F - x^{-1}) = x^{-\alpha}L(x)$ for some slowly varying function L.*

If $F \in \mathrm{MDA}(\Psi_\alpha)$, then

$$c_n^{-1}(M_n - x_F) \xrightarrow{d} \Psi_\alpha, \tag{3.25}$$

where the norming constants c_n can be chosen as $c_n = x_F - F^{\leftarrow}(1 - n^{-1})$ and $d_n = x_F$. Here $F^{\leftarrow}(p)$ represent the p-quantile of F.

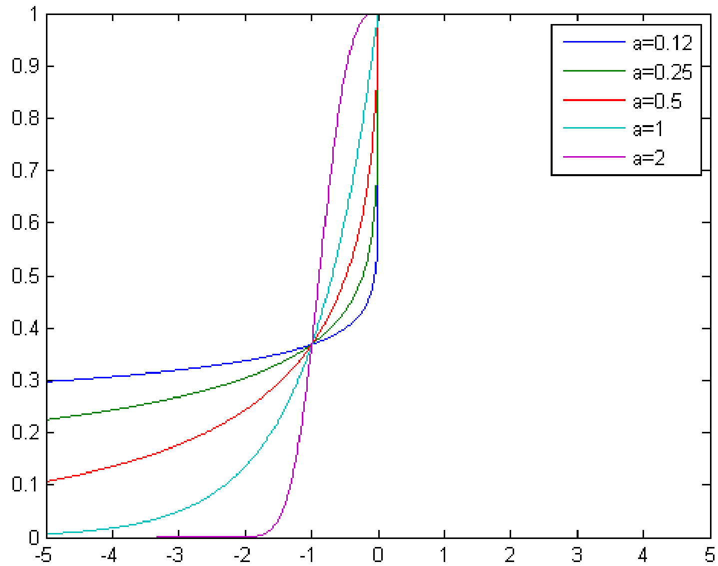


Figure 3.2: Weibull MDA for various tail indices a

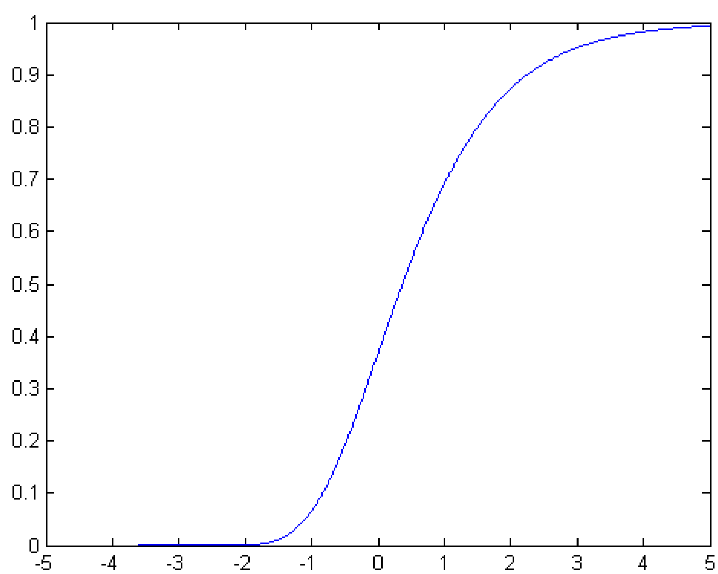


Figure 3.3: Gumbel MDA

This previous theorem is the one from which Corollary 3.3.13 mentioned earlier is derived. It is both easy and difficult at the same time to see why it makes sense for our loss distribution to be part of MDA(Ψ_α). It is easy to see because we know our losses are bounded. In our case, the rv for maximum credit loss C is bounded by $\bar{c} = 1$. MDA(Ψ_α) characterizes the distribution of the maximum of an i.i.d. sequence when the rv is bounded. For this reason, it doesn't make sense for C to be in MDA(Φ_α), since Φ_α is unbounded. This leads to the reason why it might be difficult to see why our loss distribution can fit in MDA(Ψ_α). Indeed, our model assumes that both f and ϵ are normally distributed, which are thin-tailed distributions. Yet, MDA(Ψ_α) exhibits heavy-tails characteristics. This is rather surprising and raises another important issue. We want to see if any distribution put in this model would exhibit heavy-tails or whether there is any reason that we should be careful when choosing our distributions. It is obvious that the same distribution function cannot be in two different MDA at once, but would we be able to find a distribution function for credit losses that would be in MDA(Λ). This would mean, by using Schönbucher's model for underyling assets of our credit portfolio (3.1.1), that Y and ϵ forge a rv S_n whose maxmimum's distribution is in MDA(Λ).

In order to do so, we need an equivalent to theorem 3.1.7 for Λ. The characterization of MDA(Λ) is less obvious than with MDA(Ψ_α), and is done mainly by defining Von Mises functions.

Theorem 3.1.8 (Von Mises function for MDA(Λ), Embrechts et al., 1997)**.** *The df F with right endpoint $x_F \leq \infty$ belongs to the maximum domain of attraction Λ if and only if there exists some $z < x_F$ such that F has representation*

$$\overline{F}(x) = c(x) \exp\left\{ -\int_z^x \frac{g(t)}{a(t)} \mathrm{d}t \right\}, \quad z < x < x_F, \tag{3.26}$$

where c and g are measurable functions satisfying $c(x) \to c > 0$, $g(x) \to 1$ as $x \uparrow x_f$, and $a(x)$ is a positive, absolutely continuous function (with respect to Lebesgue measure) with density $a'(x)$ such that $\lim_{x \uparrow x_F} a'(x) = 0$.

3.1.4 Analysis of the model

Lucas et al. (2000) mention that their model produces a random variable with MDA(Ψ_α), but we suggest that this is not the only possible characterization. This new result is presented in the derivation hereafter. This contribution raises awareness on the choice of the distribution functions for the factors in the initial setting of the model.

By recalling the previous section and using Schönbucher (2001)'s characterization for loss distribution, we try to find a function G and a function H for which $\overline{F}_X(x) = G\left(\frac{K-H^{-1}(x)}{\beta}\right)$ would be part of MDA(Λ). This means

$$G\left(\frac{K-H^{-1}(x)}{\beta}\right) = c(x)\exp\left(-\int_0^x \frac{g(t)}{a(t)}\mathrm{d}t\right)$$

Such a function can be found by assuming that $c(x) = 1$ and $g(x) = 1$. Also, by simply setting $G(y) = \exp(y)$, $y \in [-\infty, 0]$, we get

$$\exp\left(\frac{K-H^{-1}(x)}{\beta}\right) = \exp\left(-\int_0^x \frac{1}{a(t)}\mathrm{d}t\right)$$

Therefore, we need to see if we can find a function $a(\cdot)$ defined in Theorem 3.1.8 that would correctly define a distribution function H in our setting. In other words, we need

$$H^{-1}(x) = K + \beta\int_0^x \frac{1}{a(t)}\mathrm{d}t$$

In fact, by taking $a(x) = \frac{1}{\gamma}$, γ a constant, we have a function that satisfies the Von Mises functions condition 3.1.8 and that creates a distribution H which is well-defined within the model set up. Indeed, by setting a to be a constant function, we get

$$\begin{aligned} H^{-1}(x) &= K + \beta\gamma x \quad x \in [0,1] \\ &= \tilde{a} + \tilde{b}x \end{aligned} \tag{3.27}$$

where

$$\tilde{a} = K \quad \tilde{b} = \gamma\beta.$$

By checking that H is still a well-defined as a distribution function, we have that

$$\begin{aligned} H(y) &= \frac{y-\tilde{a}}{\tilde{b}} \\ &= \frac{y-K}{\gamma\beta} \Rightarrow y \geq K. \end{aligned}$$

Thus, H is a distribution function and we now know that the Scönbucher model we study allows for Gumbel and Weibull MDAs. This is important to take into consideration when working with a structural model of the type studied in Lucas et al. (2000) and

Schönbucher (2001), as it may have influence on the pricing and risk management of credit instruments.

A comment on the distribution is given here. Since the Gumbel distributions are often considered to be decaying faster than Weibull's, it is in fact not surprising that if the firms are influenced by a common factor exponentially distributed while the idiosyncratic shocks are uniform, we have a maximum credit loss that does not exhibit much thickness in the tails. We therefore see the importance of both the common and idiosyncratic factors for extreme events.

This was a simple way to verify the breadth of the single period factor model. We realized that it has a great analytical capacity as we are able to retrieve many useful behaviours of the random variables, especially their tail properties. On the other hand, this model's specification has to be carefully specified as a simple change in the definition of our factors and innovations can have a dramatic effect of rendering our simulation results far from being able to match market events.

3.2 Dynamic Structural Model

After studying the single-period structural model, we consider a closely related but improved model. This is a dynamic structural model introduced by Bush et al. (2011) which closely resembles the model of Lucas et al. (2000) in a dynamic setting.

This paper builds on static Conditionally Independent Factor (CIF) models and extends them to dynamic large portfolios. The model is obtained by taking the large portfolio limit of a multidimensional structural model, thus making the modelling of the value of a large basket of underlying assets possible via a stochastic partial differential equation. In this case, the key quantities of analysis are functions of the solutions of this SPDE.

The simple model developed by Bush et al. (2011) is considerably more useful because most of the other models are not dynamic. The copula and Conditionally Independent Factors (CIF) models, though the most popular ones, lack the specification of the evolution of their underlyings and only model expected default in one time period. Allowing the model to be dynamic increases the flexibility and puts within reach complex structured credit instruments such as forward starting tranches, options on tranches, Single Tranche CDO (STCDO), etc. We mention also Sircar and Zariphopoulou (2010), and Davis and Rodriguez (2007) as other large portfolio analysis models.

Bush et al. (2011) introduce a multi-dimensional structural model with the goal of modelling the empirical measure of the asset prices in a basket when the underlyings have dynamics obtained through a factor model. This means that the firms underlying our CDO are influenced by common market factors, and we use these factors' dynamics to model the asset prices and distance to default. By interpreting the dynamics of the structural variables, we can pull out an empirical measure of the portfolio. We then let the number of firms grow to infinity in order to obey a law of large numbers (LLN) and find the density of the limit, which satisfies a stochastic partial differential equation (SPDE)

The starting point of the SPDE model introduced by Bush et al. (2011) is similar to the model introduced in Hull et al. (2010). The assets have same constant volatility and are correlated via a single market factor. These are big simplifications but useful for the purpose of developing a simple, dynamic model. The assets follow a diffusion process under the risk-neutral measure $\mathbb{Q}$ such that

$$\mathrm{d}A_t^i = rA_t^i\mathrm{d}t + \sigma\sqrt{1-\rho}A_t^i\mathrm{d}W_t^i + \sigma\sqrt{\rho}A_t^i\mathrm{d}M_t,\ i = 1,\ldots,N \tag{3.28}$$

until they hit a constant barrier B^i, representing their default threshold, or the horizon time T. We assume that W_t^i and M_t are Brownian motions representing idiosyncratic fluctuations and common market fluctuations, respectively, and such that

$$\mathrm{d}[W_t^i, M_t] = 0 \quad \forall i$$

and

$$\mathrm{d}[W_t^i, W_t^j] = \delta_{ij}\mathrm{d}t,$$

where $[.,.]$ represents the quadratic covariation. Here, δ_{ij} is the Kronecker delta. We assume that $\sigma > 0$ and $\rho \in [0,1)$ are constants and ρ is the correlation of the market factor of each asset.

By writing this in terms of distance to default process $X_t^i = (\ln A_t^i - \ln B^i)/\sigma$, we can reinterpret this as

$$\begin{aligned}
\mathrm{d}X_t^i &= \mu\mathrm{d}t + \sqrt{1-\rho}\mathrm{d}W_t^i + \sqrt{\rho}\mathrm{d}M_t,\ t < \tau_0^i, \\
X_t^i &= 0,\ t \geq \tau_0^i, \\
X_0^i &= x_\circ^i > 0, \\
\tau_0^i &= \inf\{t : X_t^i = 0\}
\end{aligned} \tag{3.29}$$

for $i = 1, 2, \ldots, N$, and where $\mu = (r - \frac{1}{2}\sigma^2)/\sigma$.

3.2.1 Large portfolio approximation

We introduce further notation to state the results in this setting. A more complete explanation of these definitions can be found in Bush et al. (2011). Let $(\Omega^M, \mathcal{F}^M, \mathbb{P}^M)$ be a probability space supporting a one-dimensional Brownian motion $(M_t, \mathcal{F}_t)$. Let $\mathcal{G}^M$ denote a σ-algebra of predictable sets $\Omega^M \times (0,\infty)$ associated with the filtration $\mathcal{F}_t^M$ and $H^1((0,\infty)) = \{f : f \in L^2((0,\infty)), f' \in L^2((0,\infty))\}$, where $L^2((0,\infty)) = \{\int_0^\infty f^2 \mathrm{d}x < \infty\}$. We write $L^2(\Omega^M \times (0,T), \mathcal{G}^M, H^1((0,\infty))) = \{f(\omega,t,\cdot) : f(\omega,t,\cdot) \in H^1((0,\infty)), f(\omega,t,\cdot)$ is $\mathcal{F}_t^M$-measurable$, \mathbb{E}^M[\int_0^T \parallel f(\omega,t) \parallel_{H^1}^2 \mathrm{d}t] < \infty\})$. See Appendix A for notation.

Let $\bar{\nu}_t^N$ denote the equally weighted empirical measure for the entire portfolio given by

$$\bar{\nu}_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^i} \tag{3.30}$$

where δ_x is the Dirac measure at the point x. The following is from Bush et al. (2011):

Theorem 3.2.1. *The limit empirical measure $\bar{\nu}_t = \lim_{N\to\infty} \bar{\nu}_t^N$ exists and is a probability measure with a natural decomposition into two components, $\bar{\nu}_t = L_t\delta_0 + \nu_t$. The measure ν_t is a measure on $(0,\infty)$ with density $v(t,x)$, which is the unique solution in $L^2(\Omega^M \times (0,T), \mathcal{G}^M, H^1((0,\infty)))$ of the SPDE*

$$\begin{cases} \mathrm{d}v = -\frac{1}{\sigma}(r - \frac{1}{2}\sigma^2)v_x \mathrm{d}t + \frac{1}{2}v_{xx}\mathrm{d}t - \sqrt{\rho}v_x \mathrm{d}M(t), \\ v(0,x) = v_0(x),\ v(t,0) = 0. \end{cases} \tag{3.31}$$

The weight of the Dirac mass at 0 is

$$L_t = 1 - \int_0^\infty v(t,x)\mathrm{d}x. \tag{3.32}$$

Therefore, prices of typical large portfolio credit products are functions of this proportionate loss function L_t. (Bush et al., 2011)

We show below how we can derive this SPDE from the structural model of distance to default. Let $\mathbb{R}_+ = [0,\infty)$. We write $\mathscr{P}(\mathbb{R}_+)$ for the set of probability measures on $\mathbb{R}_+$ and $\mathscr{P}(C_{\mathbb{R}_+}[0,\infty))$ for the set of probability measures on $C_{\mathbb{R}_+}[0,\infty)$ where the topology on spaces of measures is always that of weak convergence. We write $C_K^\infty(0,\infty)$ as the set of continuous functions on $(0,\infty)$ bounded by a constant K that are infinitely differentiable.

First we start with our model for X_t^n given by equation (3.29). By recalling how we defined the empirical measure in eq. (3.30), and by using the notation

$$\langle \bar{\nu}_t^N, \phi \rangle = \int \phi \, \mathrm{d}\nu_t^N = \frac{1}{N} \sum_{i=1}^{N} \phi(X_t^i),$$

we can derive the heuristics of the proof. The Itō formula tells us that

$$\begin{aligned}
\mathrm{d}\langle \bar{\nu}_t^N, \phi \rangle &= \frac{1}{N} \sum_{n=1}^{N} (\phi'(X_t^n)\mathrm{d}X_t^n + \frac{1}{2}\phi''(X_t^n)(\mathrm{d}X_t^n)^2) \\
&= \frac{1}{N} \sum_{n=1}^{N} \mu\phi'(X_t^n)\mathrm{d}t \\
&\quad + \sqrt{1-\rho}\frac{1}{N} \sum_{n=1}^{N} (\phi'(X_t^n)\mathrm{d}W_t^n \\
&\quad + \sqrt{\rho}\frac{1}{N} \sum_{n=1}^{N} (\phi'(X_t^n)\mathrm{d}M_t \\
&\quad + \frac{1}{2}\frac{1}{N} \sum_{n=1}^{N} (\phi''(X_t^n)(\sqrt{1-\rho})^2\mathrm{d}t \\
&\quad + \frac{1}{2}\frac{1}{N} \sum_{n=1}^{N} (\phi''(X_t^n)(\sqrt{\rho})^2\mathrm{d}t \\
&= \mu\langle \nu_t^N, \phi' \rangle \mathrm{d}t + \sqrt{1-\rho}\frac{1}{N} \sum_{n=1}^{N} (\phi'(X_t^n)\mathrm{d}W_t^n) \\
&\quad + \sqrt{\rho}\langle \nu_t^N, \phi' \rangle \mathrm{d}M_t + \frac{1}{2}\langle \nu_t^N, \phi'' \rangle \mathrm{d}t
\end{aligned}$$

Heuristically, as $N \to \infty$

$$\mathrm{d}\langle \nu_t, \phi \rangle = \mu\langle \nu_t, \phi' \rangle \mathrm{d}t + \sqrt{\rho}\langle \nu_t, \phi' \rangle \mathrm{d}M_t + \frac{1}{2}\langle \nu_t, \phi'' \rangle \mathrm{d}t. \tag{3.33}$$

For a full proof, see Bush et al. (2011).

Then, by setting

$$\mathcal{A} = \mu\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial^2}{\partial x^2} \tag{3.34}$$

and by writing $\mathcal{A}^\dagger$ for the adjoint operator of $\mathcal{A}$, and by taking the integrals over all of $\mathbb{R}$

unless specified otherwise, we get

$$\begin{aligned}\int \phi(x)\cdot v(t,x)\mathrm{d}x &= \int \phi(x)v(0,x)\mathrm{d}x + \int_0^t \int \mathcal{A}\phi(x)v(s,x)\mathrm{d}x\mathrm{d}s + \int_0^t \int \sqrt{\rho}\phi'(x)v(s,x)\mathrm{d}x\mathrm{d}M_s \\ &= \int \phi(x)\left\{v(0,x) + \int_0^t \mathcal{A}^\dagger v(s,x)\mathrm{d}s - \int_0^t \frac{\partial}{\partial x}(\sqrt{\rho}v(s,x))\mathrm{d}M_s\right\}\mathrm{d}x.\end{aligned}$$

As this holds, $\forall\, \phi \in C_K^\infty(0,\infty)$, we have shown that we have a weak solution to the SPDE given by

$$v(t,x) = v(0,x) + \int_0^t \mathcal{A}^\dagger v(s,x)\mathrm{d}s - \int_0^t \frac{\partial}{\partial x}(\sqrt{\rho}v(s,x))\mathrm{d}M_s$$

with $v(t,0) = 0\ \forall t \in [0,T]$. Alternatively, we can write this in a differential form as

$$\mathrm{d}v(t,x) = -\mu\frac{\partial}{\partial x}v(t,x)\mathrm{d}t + \frac{1}{2}\frac{\partial^2}{\partial x^2}v(t,x)\mathrm{d}t - \sqrt{\rho}\frac{\partial}{\partial x}v(t,x)\mathrm{d}M_s,$$

with $v(t,0) = 0\ \forall t \in [0,T]$ and $v(0,x) = v_0(x)$. In summary, as the size of the portfolio N grows to infinity, meaning when we look at the asymptotic behaviour of our credit portfolio, we have that

$$\bar{\nu}_t^N \to \bar{\nu}_t$$

where $\bar{\nu}_t$ follows the SPDE in equation (3.33). This is the empirical measure of our portfolio. The stochastic PDE derived above describes the evolution of the distance to default of an infinite portfolio of assets whose dynamics are given by (3.28).

We can now use the limiting empirical measure ν_t to approximate the loss distribution for a portfolio of fixed size N whose assets follow (3.28). We do this by matching the initial condition, thus setting

$$v(0,x) = \frac{1}{N}\sum_{i=1}^{N}\delta_{X_0^i}(x)$$

where the $X_0^i > 0,\ i = 1,\ldots,N$ are the initial values for the distance to default of the assets in our fixed portfolio of size N that can be inferred from the firms' CDS spreads as mentioned in Chapters 2 and 4.

3.2.2 Moments of the Loss distribution

In this subsection, we take the theory developed by Bush et al. (2011) and combine it with the characterization introduced by Lucas et al. (2000) and Schönbucher (2001) in order to derive a theorem for the moments of the loss distribution. Define

$$\tau = \inf_{s\leq t}(\sqrt{\rho}Z_t + \sqrt{1-\rho}\epsilon_i^{(t)} < K_t),$$

for K_t the default threshold. Z and ϵ_i are independent Brownian motions.

Following the introduction of the Lucas model in theorem 3.1.3 and its particular case of equation (3.13), we can define the loss variable at time t as

$$\begin{aligned} L_t &= \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{\{\tau_i=t\}} \\ &\sim \mathbb{E}\left[\mathbb{1}_{\{\inf_{s\leq t}(\sqrt{\rho}Z_t+\sqrt{1-\rho}\epsilon^{(t)}<K_t)\}}|Z_s,\, s\leq t\right] \end{aligned} \tag{3.35}$$

The expectation of this variable is

$$\mathbb{E}[L_t] = \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\{\inf_{s\leq t}(\sqrt{\rho}Z_t+\sqrt{1-\rho}\epsilon^{(t)}<K_t)\}}|Z_s,\, s\leq t\right]\right].$$

Taking the square of this variable, by the conditional independence of the ϵ's, we get

$$\begin{aligned} L_t^2 &= \mathbb{E}\left[\mathbb{1}_1|Z_s,\, s\leq t\right]\mathbb{E}\left[\mathbb{1}_2|Z_s,\, s\leq t\right] \\ &= \mathbb{E}\left[\mathbb{1}_{\{\inf_{s\leq t}(\sqrt{\rho}Z_t+\sqrt{1-\rho}\epsilon_1^{(t)}<K_t)\}}\mathbb{1}_{\{\inf_{s\leq t}(\sqrt{\rho}Z_t+\sqrt{1-\rho}\epsilon_2^{(t)}<K_t)\}}|Z_s,\, s\leq t\right] \end{aligned}$$

with default threshold K_t. $\mathbb{1}_1$ and $\mathbb{1}_2$ represent the two indicator functions as in 3.35 with independent idiosyncratic factors ϵ_1 and ϵ_2.

Then by the law of iterated expectations, we get the second moment of the loss distribution:

$$\begin{aligned} \mathbb{E}[L_t^2] &= \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\{\inf_{s\leq t}(\sqrt{\rho}Z_t+\sqrt{1-\rho}\epsilon_1^{(t)}<K_t)\}}\mathbb{1}_{\{\inf_{s\leq t}(\sqrt{\rho}Z_t+\sqrt{1-\rho}\epsilon_2^{(t)}<K_t)\}}|Z_s,\, s\leq t\right]\right] \\ &= \mathbb{P}\left(\inf_{s\leq t}(\sqrt{\rho}Z_t+\sqrt{1-\rho}\epsilon_1^{(t)}<0), \inf_{s\leq t}(\sqrt{\rho}Z_t+\sqrt{1-\rho}\epsilon_2^{(t)}<0)\right) \\ &= \mathbb{P}(\tau_1\leq t, \tau_2\leq t), \end{aligned}$$

where $\tau = \inf_{s\leq t}(\sqrt{\rho}Z_t + \sqrt{1-\rho}\epsilon^{(t)} < K_t)$, and we can set $K_t = 0$ without loss of of generality. τ then becomes the first passage time to the barrier $K_t = 0$ of a surplus variable influenced by both systemic and independent idiosyncratic risk factors. We use ϵ_1 and ϵ_2 to refer to individual independent idiosyncratic risk's path. $\mathbb{P}(\cdot,\cdot)$ is the two-dimensional joint distribution function. Therefore, the second moment of the loss function defined in (3.35) is equal to the probability that two individual surplus functions will cross the default barrier before time t. This computation can be repeated n times in order to get the n-th moment of our loss distribution. This gives way to our theorem for the moments of the loss distribution.

Theorem 3.2.2 (Loss distribution's moments)**.** *Assume a dynamic structural model as in* (3.29) *(Bush et al., 2011). Moreover, assume that the single factor Z and idiosyncratic shocks ϵ_n follow independent Brownian motions. Define*

$$\tau_i = \inf_{s \le t} (\sqrt{\rho} Z_t + \sqrt{1-\rho}\epsilon_i^{(t)} < K_t), \tag{3.36}$$

for K_t the default threshold. Without loss of generality, assume that $K_t = 0$.

We define the loss variable at time t as

$$L_N(t) \quad = \quad \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\{\tau_i = t\}} \tag{3.37}$$

Then, as $N \to \infty$, the n-th moment of the loss distribution from the model of Bush et al. (2011) is

$$\mathbb{E}[L_t^n] = \mathbb{P}\left(\bigcap_{i=1,\dots,n} \{\tau_i \le t\} \right), \tag{3.38}$$

where $\mathbb{P}(\cdot,\cdot)$ is the n-dimensional joint distribution function.

This way, we can get all of the moments of the loss distribution L. By observing that the n-th moment is the joint probability of n defaults by time s, we see how studying the moments might give us good insight on the whole df. This is an interesting characteristic of this particular dynamic structural model. By being able to observe the loss distribution at a specific time, we are able to determine the moments, which correspond to the joint distribution of defaults.

Unfortunately, this ease of implementation comes at the cost of very simplifying assumptions. Assuming independence of the idiosyncratic shocks might not be that restrictive, but assuming a single factor, distributed along a Brownian motion in the Bush et al. (2011) model, is certainly far from reality. On the other hand, Lucas et al. (2000) have shown that a combination of thin-tailed distributions may create a heavy-tailed one. Similarly, we have shown that thin-tailed distribution could also create other thin-tailed ones, warning us to be careful when choosing our model's assumptions.

In our case, we would want to be able to understand better the tail of the distribution in order to better price senior and super-senior tranches of a CDO. Indeed, such tranches are often mispriced due to the very low amount of data on extreme events. In the next chapter, we will contribute a section on calibrating this model to market data and analyse its behaviour during the last financial crisis.

3.3 Dynamic Reduced-form Model

When looking at financial data and at default rates among firms, we often observe clustering of time of defaults. Given this empirical evidence, researchers have tried to propose models for portfolio analysis that could capture these key aspects of reality. The commonly called reduced-form models are based on market data and simulate defaults using random variables. These are not based on economic variables but the objective is to replicate the empirical observations. The most popular reduced-form model is the intensity based model, where defaults are simulated by an intensity-based jump process, and correlation can be applied between firm specific processes.

The papers (Giesecke et al., 2012), (Giesecke et al., 2013), and (Giesecke et al., 2014), introduce a model where a default occurs at an intensity following a mean-reverting jump-diffusion process driven by several terms. First, they assume a firm specific source of risk, driven by a square-root diffusion, that is independent of other aspects of the economy. Second, another source of risk is added by a systematic risk factor, influencing all firms, reflecting the state of the general economy, and generating diffusive correlation between the intensities. Finally, a contagion term affects the intensity by adding correlation with default rate in the pool. Thus, a systematic risk factor and the history of past defaults both add some dependence to pooled firms. Also, the impact of the history of defaults on the surviving firms fades away with time due to a recovery effect.

With this model in hand, they try to analyse the limit as the number of firms in the pool grows, in order to retrieve some macroscopic organization observations. This law of large numbers will allow them to identify typical behaviours. We notice that this is something we have been doing for all three models analysed in this thesis. It is thus worth remembering why we do this. In fact, by taking the number of names in our portfolio to grow to infinity, we are able to derive LLN, and it becomes much easier to find analytical results than with finite portfolio. On the other hand, we do this while keeping in mind that all of our real-life portfolios will be finite. Thus, our results must be robust for such finite portfolios. In fact, since many instruments used will be based on many underlyings, this approximation is considered to be flexible.

Giesecke et al. (2012) construct a point process model of correlated default timing in a portfolio of firms. Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is an underlying probability triple on which all of the random variables are defined. Let $\{W^n\}_{n\in\mathbb{N}}$ be a countable collection of standard Brownian motions. Each will represent a source of risk which is idiosyncratic to the specific firm n. Let $\{\mathfrak{e}_n\}_{n\in\mathbb{N}}$ be an i.i.d. collection of standard exponential random variables. Each $\mathfrak{e}_n$ will represent a normalized default time for the specific firm n. Finally, let V be a standard Brownian motion independent of the W^n's and $\mathfrak{e}_n$'s. The process V will drive a systematic risk factor process to which all firms are exposed.

Fix an $N \in \mathbb{N}$, $n \in \{1, 2, \ldots, N\}$ and consider the following system:

$$\begin{aligned}
d\lambda_t^{N,n} &= -\alpha_{N,n}(\lambda_t^{N,n} - \bar{\lambda}_{N,n})dt + \sigma_{N,n}\sqrt{\lambda_t^{N,n}}dW_t^n \\
&\quad + \beta_{N,n}^C dL_t^N + \beta_{N,n}^S \lambda_t^{N,n} dX_t
\end{aligned} \tag{3.39}$$

$$\lambda_0^{N,n} = \lambda_{\circ,N,n}$$

$$dX_t = b_0(X_t)dt + \sigma_0(X_t)dV_t, \quad t > 0 \tag{3.40}$$

$$X_0 = x_\circ$$

$$L_t^N = \frac{1}{N}\sum_{n=1}^N \mathbb{1}_{[\mathfrak{e}_n,\infty)}\left(\int_{s=0}^t \lambda_s^{N,n} ds\right). \tag{3.41}$$

Here, we use $\mathbb{1}$ as the indicator function. The initial condition value $x_\circ$ of X is fixed. $\beta_{N,n}^C \in \mathbb{R}_+ = [0, \infty)$ and $\beta_{N,n}^S \in \mathbb{R}$ are constants which represent the exposure of the n-th firm in the pool to L^N and X, respectively. The $\alpha_{N,n}$, $\bar{\lambda}_{N,n}$ and $\sigma_{N,n}$ are in $\mathbb{R}_+$ and characterize the dynamics of the firms. Their meanings for each N and n will be discussed below. We can also define L^N in a more standard way using stopping times. In particular, define

$$\tau^{N,n} \overset{\text{def}}{=} \inf\{t \geq 0 : \int_{s=0}^t \lambda_s^{N,n} ds \geq \mathfrak{e}_n\}.$$

Then $\mathbb{1}_{[\mathfrak{e}_n,\infty)}\left(\int_{s=0}^t \lambda_s^{N,n} ds\right) = \mathbb{1}_{\{\tau^{N,n} \leq t\}}$ and thus

$$L_t^N = \frac{1}{N}\sum_{n=1}^N \mathbb{1}_{\{\tau^{N,n} \leq t\}}$$

The process L^N represents the loss rate in a portfolio of N names, assuming a loss given defaults of one unit. The process $\lambda^{N,n}$ represents the intensity, or conditional event rate, of the n-th name in the pool. More precisely, $\lambda^{N,n}$ is the density of the Doob-Meyer compensator to the default indicator $\mathbb{1}_{\{\tau^{N,n} \leq t\}}$ (see Appendix C. The results of Giesecke et al. (2013) imply that the system has a unique solution such that $\lambda^{N,n} \geq 0$, $\forall\ N \in \mathbb{N}$, $n \in \{1, 2, \ldots, N\}$ and $t \geq 0$. Thus, the model is well-posed. This jump-diffusion model is empirically motivated and addresses several channels of default clustering. First, an intensity is driven by an idiosyncratic source of risk represented by a Brownian motion W^n, and a source of systemic risk common to all firms in the diffusion process X. The sensitivity of $\lambda^{N,n}$ to changes in X is measured by the parameter $\beta_{N,n}^S \in \mathbb{R}$. The second channel for default clustering is modelled through the feedback term $\beta^C_{N,n}\mathrm{d}L^N_t$. It creates a contagion effect and this self-exciting effect is empirically found to be an important channel of clustering defaults.

The main reason why we introduced the $\alpha_{N,n}$, $\bar{\lambda}_{N,n}$ and $\sigma_{N,n}$ on top of the $\beta^C_{N,n}$ and $\beta^S_{N,n}$ is that we allow for a heterogeneous pool. This implies that the intensity dynamics of each name can be different. This is in fact an interesting aspect of the model which makes it model flexible. It is an important fact in practice, as homogeneity is rarely observed. On the other hand, too much heterogeneity implies greater difficulty in extracting large numbers results. We shall therefore be careful when using these types. Thus, the different constants defined what Giesecke et al. call "types"

$$\mathrm{p}^{N,n} \stackrel{\text{def}}{=} (\alpha_{N,n}, \bar{\lambda}_{N,n}, \sigma_{N,n}, \beta^C_{N,n}, \beta^S_{N,n}); \tag{3.42}$$

the $\mathrm{p}^{N,n}$ take values in parameter space $\mathcal{P} \stackrel{\text{def}}{=} \mathbb{R}^4 \times \mathbb{R}$. In order to expect a regular macroscopic behaviour of L^N as $N \to \infty$, the $\mathrm{p}^{N,n}$'s and the $\lambda_{\circ,N,n}$'s should have enough regularity as $N \to \infty$. For each $N \in \mathbb{N}$, define

$$\pi^N \stackrel{\text{def}}{=} \frac{1}{N}\sum_{n=1}^{N} \delta_{\mathrm{p}^{N,n}}$$

and

$$\Lambda^N_\circ \stackrel{\text{def}}{=} \frac{1}{N}\sum_{n=1}^{N} \delta_{\lambda_{\circ,N,n}}$$

as elements of $\mathscr{P}(\mathcal{P})$ and $\mathscr{P}(\mathbb{R}^+)$ respectively, for $\mathscr{P}(A)$ representing the power set of A. The π^N and Λ^N represent the empirical measures of the equally weighted types and initial intensities of the portfolio containing N names, respectively. The model studied in Giesecke et al. (2013) requires three main conditions for a regular behaviour. First, assume sufficient regularity for the types $\mathrm{p}^{N,n}$ and the initial distributions $\lambda_{\circ,N,n}$. Second, the π^N's and $\Lambda^N_\circ$'s must all have compact support. Finally, regarding the systemic risk process X, its corresponding SDE must have a unique solution and there must be a function $u(x)$ such that $\sigma_0(x)u(x) = -b_0(x)$ that satisfies the Novikov condition for every $T > 0$. This means

$$\mathbb{E}[e^{\frac{1}{2}\int_0^T |u(X_s)|^2 \mathrm{d}s}] < \infty.$$

With this definition, we assure the well posedness of the model and can develop a law of large numbers for the portfolio loss rate L_N. In order to develop the LLN, we must work in a system containing more information than the loss rate. For each $N \in \mathbb{N}$ and $n \in \{1, 2, \dots N\}$ define

$$\mathrm{M}_t^{N,n} \stackrel{\text{def}}{=} \mathbb{1}_{[0,\mathfrak{e}_n)}\left(\int_{s=0}^t \lambda_s^{N,n} \mathrm{d}s\right) = \mathbb{1}_{\{\tau^{N,n}>t\}}$$

for $\tau^{N,n}$ is as defined previously. So $\mathrm{M}_t^{N,n} = 1$ means that the n-th name is still alive by time t; otherwise $\mathrm{M}_t^{N,n} = 0$. Thus $\mathrm{M}^{N,n}$ is non-increasing and right-continuous and we easily see that

$$\mathrm{M}_t^{N,n} + \int_{s=0}^t \lambda_s^{N,n} \mathrm{M}_s^{N,n} \mathrm{d}s$$

is a martingale. Now, to define the empirical distribution of the names that are still alive, we need to define $\hat{\mathcal{P}} \stackrel{\text{def}}{=} \mathcal{P} \times \mathbb{R}_+$, and for each $N \in \mathbb{N}$, define $\hat{\mathrm{p}}_t^{N,n} \stackrel{\text{def}}{=} (\mathrm{p}^{N,n}, \lambda_t^{N,n})$ for all $n \in \{1, 2, \dots, N\}$ and $t \geq 0$. Thus, the empirical distribution is defined as

$$\mu_t^N \stackrel{\text{def}}{=} \frac{1}{N} \sum_{n=1}^N \delta_{\hat{\mathrm{p}}_t^{N,n}} \mathrm{M}_t^{N,n}.$$

This allows us to keep track of the type and intensity for those asset which are still "alive". This obviously implies that

$$L_t^N = 1 - \mu_t^N(\hat{\mathcal{P}}),\ t \geq 0.$$

As we are interested in tracking the portfolio loss, this representation for the loss distribution in terms of the empirical distribution of the active names is crucial.

We provide the heuristic derivation of the paper's main theorem. For a complete proof, we refer to Giesecke et al. (2013), but the following derivation is not inspired by their proof. First, for every $f \in \mathcal{C}^\infty(\hat{\mathcal{P}})$ and μ in the set of sub-probability measures E (i.e. defective probability measure such that $\mu(\Omega) \leq 1$, for Ω being the possible universe), define

$$\begin{aligned}
\langle f, \mu \rangle_E &\stackrel{\text{def}}{=} \int_{\hat{\mathrm{p}} \in \hat{\mathcal{P}}} f(\hat{\mathrm{p}}) \mu(\mathrm{d}\hat{\mathrm{p}}) \\
&\equiv \frac{1}{N} \sum_{n=1}^N f(\hat{\mathrm{p}}_t^{N,n}) \mathrm{M}_t^{N,n} \text{ as } N \to \infty.
\end{aligned}$$

Since $\mathrm{M}_t^{N,n}$ is a discontinuous function acting as a jump in our process, we have to rely on Itō's product rule for jump processes, given in Shreve (2004). Hence

$$f(\hat{\mathrm{p}}_t^{N,n})\mathrm{M}_t^{N,n} = f(\hat{\mathrm{p}}_0^{N,n})\mathrm{M}_0^{N,n} + \int_0^t f(\hat{\mathrm{p}}_s^{N,n,})\mathrm{d}\mathrm{M}_s^{N,n,C} + \int_0^t \mathrm{M}_s^{N,n}\mathrm{d}f(\hat{\mathrm{p}}_s^{N,n,C}) \\ + \left[f(\hat{\mathrm{p}}^{N,n,C}), \mathrm{M}^{N,n,C}\right](t) + \sum_{0\le s\le t}\left[f(\hat{\mathrm{p}}_s^{N,n})\mathrm{M}_s^{N,n} - f(\hat{\mathrm{p}}_{s-}^{N,n})\mathrm{M}_{s-}^{N,n}\right] \tag{3.43}$$

where $f(\hat{\mathrm{p}}^{N,n,C})$ and $\mathrm{M}_s^{N,n,C}$ represent the continuous part of $f(\hat{\mathrm{p}}^{N,n})$ and $\mathrm{M}_s^{N,n}$, respectively. Also, $f(\hat{\mathrm{p}}_{s-}^{N,n})\mathrm{M}_{s-}^{N,n}$ corresponds to the function's value at time $(s - \mathrm{d}s)$, as both are left-continuous by definition. This formulation allows us to investigate the rate of change of $\langle f, \mu\rangle_E$, namely

$$\mathrm{d}\langle f, \mu\rangle_E = \frac{1}{N}\sum_{n=1}^{N}\mathrm{d}\left(f(\hat{\mathrm{p}}_t^{N,n})\mathrm{M}_t^{N,n}\right).$$

From the particularities of Itō calculus with jumps presented in (3.43), we get that

$$\mathrm{d}\left(f(\hat{\mathrm{p}}_t^{N,n})\mathrm{M}_t^{N,n,C}\right) = f(\hat{\mathrm{p}}_t^{N,n})\mathrm{d}\mathrm{M}_s^{N,n,C} + \mathrm{M}_t^{N,n}\mathrm{d}f(\hat{\mathrm{p}}_t^{N,n}) + \mathrm{d}\mathrm{M}_t^{N,n,C}\mathrm{d}f(\hat{\mathrm{p}}_t^{N,n}) \\ + \mathrm{d}\left(\sum_{0\le s\le t}\left[f(\hat{\mathrm{p}}_s^{N,n})\mathrm{M}_s^{N,n} - f(\hat{\mathrm{p}}_{s-}^{N,n})\mathrm{M}_{s-}^{N,n}\right]\right).$$

Before going further in the derivation, it is worth noticing key aspects of our formula and proceed to some simplifications. First, note that $\mathrm{d}\mathrm{M}_t^{N,n,C}$ is a delta function, therefore $\mathrm{d}\mathrm{M}_t^{N,n,C} = 0$ almost everywhere, since $\mathrm{M}_t^{N,n}$ is an indicator function being either 1 or 0 in its continuous parts. Second, a similar reasoning applies to the $\sum_{0\le s\le t}\left[f(\hat{\mathrm{p}}_s^{N,n})\mathrm{M}_s^{N,n} - f(\hat{\mathrm{p}}_{s-}^{N,n})\mathrm{M}_{s-}^{N,n}\right]$. Indeed, we have

$$\sum_{0\le s\le t}\left[f(\hat{\mathrm{p}}_s^{N,n})\mathrm{M}_s^{N,n} - f(\hat{\mathrm{p}}_{s-}^{N,n})\mathrm{M}_{s-}^{N,n}\right] = \begin{cases} 0 & \text{if no jump occurs within } [0,t] \\ -f(\hat{\mathrm{p}}_{s-}^{N,n}) & \text{if a jump occurs at time } s \end{cases}$$

Furthermore, since $\mathrm{M}_t^{N,n}$ serves as an indicator for a hazard rate, we know that, at each time t, jumps occur with intensity λ_t. Therefore

$$\mathrm{d}\left(\sum_{0\le s\le t}\left[f(\hat{\mathrm{p}}_s^{N,n})\mathrm{M}_s^{N,n} - f(\hat{\mathrm{p}}_{s-}^{N,n})\mathrm{M}_{s-}^{N,n}\right]\right) = -f(\hat{\mathrm{p}}_t^{N,n})\mathrm{M}_t^{N,n}\lambda_t^{N,n}\mathrm{d}t$$

and this leaves us with

$$\mathrm{d}\left(f(\hat{\mathrm{p}}_t^{N,n})\mathrm{M}_t^{N,n,C}\right) \;=\; \mathrm{M}_t^{N,n}\mathrm{d}f(\hat{\mathrm{p}}_t^{N,n}) - \lambda_t f(\hat{\mathrm{p}}_t^{N,n})\mathrm{M}_t^{N,n}\mathrm{d}t.$$

Therefore, continuing our expansion of $\mathrm{d}\langle f,\mu\rangle_E$ according to Itō's lemma for the chain rule and to the definitions of $\hat{\mathrm{p}}_t^{N,n}$ and (3.39), we get

$$\begin{aligned}
\mathrm{d}\langle f,\mu\rangle_E \;&=\; \frac{1}{N}\sum_{n=1}^{N}\left[\mathrm{M}_t^{N,n}\mathrm{d}f(\hat{\mathrm{p}}_t^{N,n}) - \lambda_t f(\hat{\mathrm{p}}_t^{N,n})\mathrm{M}_t^{N,n}\mathrm{d}t\right] \\
&=\; \frac{1}{N}\sum_{n=1}^{N}\mathrm{M}_t^{N,n}\left[f'(\hat{\mathrm{p}}_t^{N,n})\mathrm{d}\hat{\mathrm{p}}_t^{N,n} + \frac{1}{2}f''(\hat{\mathrm{p}}_t^{N,n})\left(\mathrm{d}\hat{\mathrm{p}}_t^{N,n}\right)^2 - \lambda_t f(\hat{\mathrm{p}}_t^{N,n})\mathrm{d}t\right] \\
&=\; \frac{1}{N}\sum_{n=1}^{N}\mathrm{M}_t^{N,n}\Bigg[f'(\hat{\mathrm{p}}_t^{N,n})\Big(-\alpha_{N,n}(\lambda_t^{N,n} - \bar{\lambda}_{N,n})\mathrm{d}t + \sigma_{N,n}\sqrt{\lambda_t^{N,n}}\mathrm{d}W_t^n \\
&\quad + \beta_{N,n}^C\mathrm{d}L_t^N + \beta_{N,n}^S\lambda_t^{N,n}\mathrm{d}X_t\Big) + \frac{1}{2}f''(\hat{\mathrm{p}}_t^{N,n})\left(\sigma_{N,n}^2\lambda_t^{N,n}\mathrm{d}t + \left(\beta^S\lambda_t^{N,n}\right)^2(\mathrm{d}X_t)^2\right) \\
&\quad - \lambda_t f(\hat{\mathrm{p}}_t^{N,n})\mathrm{d}t\Bigg]
\end{aligned}$$

Then by using the definition of $\mathrm{d}X$ found in (3.40), we can find

$$(\mathrm{d}X_t)^2 = \sigma_0^2(X_t)\mathrm{d}t$$

and using (3.41) and the formula for jump processes , we can get

$$\mathrm{d}L_t = \frac{1}{N}\sum_{n=1}^{N}\lambda_t^{N,n}\mathrm{M}_t^{N,n}\mathrm{d}t$$

With these simplifications, and by reorganizing the terms to collect the $\mathrm{d}t$, $\mathrm{d}W_t$, and $\mathrm{d}V_t$ together, we get

$$
\begin{aligned}
\mathrm{d}\langle f,\mu\rangle_E \;=\;& \frac{1}{N}\sum_{n=1}^{N}\mathrm{M}_t^{N,n}\Bigg[\left(\frac{1}{2}\sigma_{N,n}^2\lambda_t^{N,n}f''(\hat{\mathrm{p}}_t^{N,n})-\alpha_{N,n}(\lambda_t^{N,n}-\bar{\lambda}_{N,n})f'(\hat{\mathrm{p}}_t^{N,n})-\lambda_t f(\hat{\mathrm{p}}_t^{N,n})\right)\mathrm{d}t\\
&+\beta_{N,n}^C f'(\hat{\mathrm{p}}_t^{N,n})\frac{1}{N}\sum_{n=1}^{N}\lambda_t^{N,n}\mathrm{M}_t^{N,n}\mathrm{d}t\\
&+\left(\beta_{N,n}^S\lambda_t^{N,n}b_0(X_t)f'(\hat{\mathrm{p}}_t^{N,n})+\frac{1}{2}\left(\beta^S\right)^2\left(\lambda_t^{N,n}\right)^2\sigma_0^2(X_t)f''(\hat{\mathrm{p}}_t^{N,n})\right)\mathrm{d}t\\
&+\beta_{N,n}^S\lambda_t^{N,n}\sigma_0(X_t)f'(\hat{\mathrm{p}}_t^{N,n})\mathrm{d}V_t\Bigg]\\
&+\frac{1}{N}\sum_{n=1}^{N}\mathrm{M}_t^{N,n}\sigma_{N,n}\sqrt{\lambda_t^{N,n}}\mathrm{d}W_t^n
\end{aligned}
$$

We can thus define some operators to simplify the notation. Let

$$
\begin{aligned}
(\mathcal{L}_1 f)(\hat{\mathrm{p}}) &= \tfrac{1}{2}\sigma^2\lambda\tfrac{\partial^2 f}{\partial\lambda^2}(\hat{\mathrm{p}})-\alpha(\lambda-\bar{\lambda})\tfrac{\partial f}{\partial\lambda}(\hat{\mathrm{p}})-\lambda f(\hat{\mathrm{p}})\\
(\mathcal{L}_2 f)(\hat{\mathrm{p}}) &= \beta^C\tfrac{\partial f}{\partial\lambda}(\hat{\mathrm{p}})\\
(\mathcal{L}_3^x f)(\hat{\mathrm{p}}) &= \beta^S\lambda b_0(x)\tfrac{\partial f}{\partial\lambda}(\hat{\mathrm{p}})+\tfrac{1}{2}(\beta^S)^2\lambda^2\sigma_0^2(x)\tfrac{\partial^2 f}{\partial\lambda^2}(\hat{\mathrm{p}})\\
(\mathcal{L}_4^x f)(\hat{\mathrm{p}}) &= \beta^S\lambda\sigma_0(x)\tfrac{\partial f}{\partial\lambda}(\hat{\mathrm{p}}).
\end{aligned}
\tag{3.44}
$$

We also define

$$
\mathcal{Q}(\hat{\mathrm{p}}) \overset{\text{def}}{=} \lambda. \tag{3.45}
$$

the generator $\mathcal{L}_1$ corresponds to the diffusive part of the intensity with killing rate λ, and $\mathcal{L}_2$ is the macroscopic effect of contagion on the surviving intensities at any given time. Operators $\mathcal{L}_3^x$ and $\mathcal{L}_4^x$ are related to the exogenous systematic risk X.

Then, heuristically, as $N\to\infty$, the idiosyncratic noises average out, and we get the following theorem. For a rigorous proof, see Giesecke et al. (2013).

Theorem 3.3.1. *We have that $\mu_{(\cdot)}^N$ converges in distribution to $\bar{\mu}_{(\cdot)}$ in $D_E[0,T]$. The evolution of $\bar{\mu}_{(\cdot)}$ is given by the measure evolution equation*

$$
\begin{aligned}
\mathrm{d}\langle f,\bar{\mu}_t\rangle_E \;&=\left\{\langle\mathcal{L}_1 f,\bar{\mu}_t\rangle_E+\langle\mathcal{Q},\bar{\mu}_t\rangle_E\langle\mathcal{L}_2 f,\bar{\mu}_t\rangle_E+\langle\mathcal{L}_3^{X_t}f,\bar{\mu}_t\rangle_E\right\}\mathrm{d}t\\
&\quad+\langle\mathcal{L}_4^{X_t}f,\bar{\mu}_t\rangle_E\mathrm{d}V_t,\ \forall f\in\mathcal{C}^\infty(\hat{\mathcal{P}})\ a.s.
\end{aligned}
\tag{3.46}
$$

Suppose that there is a solution of the nonlinear SPDE

$$\begin{aligned}\mathrm{d}v(t,\hat{\mathrm{p}}) &= \left\{\mathcal{L}_1^\star v(t,\hat{\mathrm{p}}) + \mathcal{L}_3^{\star,X_t} v(t,\hat{\mathrm{p}}) + \left(\int_{\hat{\mathrm{p}}'\in\hat{\mathcal{P}}} \mathcal{Q}(\hat{\mathrm{p}}')v(t,\hat{\mathrm{p}}')\mathrm{d}\hat{\mathrm{p}}'\right)\mathcal{L}_2^\star v(t,\hat{\mathrm{p}})\right\}\mathrm{d}t \\ &\quad + \mathcal{L}_4^{X_t,\star} v(t,\hat{\mathrm{p}})\mathrm{d}V_t,\ t>0,\ \hat{\mathrm{p}}\in\hat{\mathcal{P}} \end{aligned} \tag{3.47}$$

where $\mathcal{L}_i^\star$ denote adjoint operators, with the initial condition

$$\lim_{t\searrow 0} v(t,\hat{\mathrm{p}})\mathrm{d}\hat{\mathrm{p}} = \pi \times \Lambda_\circ.$$

Then

$$\bar{\mu}_t = v(t,\hat{\mathrm{p}})\mathrm{d}\hat{\mathrm{p}}$$

Note that the SPDE (3.47) should be supplied with appropriate boundary conditions. In the work below we will assume the conditions

$$v(t,\lambda=0,\mathrm{p}) = v(t,\lambda=\infty,\mathrm{p}) = 0.$$

With this theorem in hand, we might think that we would be in a good position to model our credit portfolio containing a large number of names. In fact, many steps are left to be taken. Indeed, even though this reduced-form model does not give an economical meaning to its parameters, saving us from some rigid specification, the large amount of parameters in its types creates another kind of problem. We are now left with the problem of correctly identifying the type of each name in our portfolio, which might be quite complicated. This feature might push us away from the model in that the practicality of the heterogeneity of the types is very low and we will in fact refer to a homogeneous pool of assets in order to derive our more detailed approximation.

3.3.1 Homogeneous pool

In order to study the model more in depth, Giesecke et al. (2013) develop the case of a homogeneous portfolio. This gives further insight into the SPDE governing the limit density. Therefore, $\hat{\mathrm{p}}^{N,n} = \hat{\mathrm{p}}$ for all $N \in \mathbb{N}$ and $n \in \{1,2,\ldots,N\}$. The SPDE takes the form

$$\begin{aligned}\mathrm{d}v(t,\lambda) &= \left\{\mathcal{L}_1^\star v(t,\lambda) + \mathcal{L}_3^{\star,X_t} v(t,\lambda) + \beta^C\left(\int_0^\infty \lambda v(t,\lambda)\mathrm{d}\lambda\right)\mathcal{L}_2^\star v(t,\lambda)\right\}\mathrm{d}t \\ &\quad + \mathcal{L}_4^{\star,X_t} v(t,\lambda)\mathrm{d}V_t, \quad t,\ \lambda>0 \\ v(0,\lambda) &= \Lambda_\circ(\lambda), \\ v(t,0) &= \lim_{\lambda\to\infty} v(t,\lambda) = 0\end{aligned}$$

where the adjoint operators are given by

$$\begin{aligned}
\mathcal{L}_1^\star v(t,\lambda) &= \frac{\partial^2}{\partial\lambda^2}\left(\frac{1}{2}\sigma^2\lambda v(t,\lambda)\right) + \frac{\partial}{\partial\lambda}(\alpha(\lambda-\bar{\lambda})v(t,\lambda)) - \lambda v(t,\lambda) \\
\mathcal{L}_2^\star v(t,\lambda) &= -\frac{\partial v(t,\lambda)}{\partial\lambda} \\
\mathcal{L}_3^{\star,x} v(t,\lambda) &= \frac{\partial^2}{\partial\lambda^2}\left(\frac{1}{2}(\beta^S)^2\lambda^2\sigma_0^2(x)v(t,\lambda)\right) - \frac{\partial}{\partial\lambda}(\beta^S\lambda b_0(x)v(t,\lambda)) \\
\mathcal{L}_4^{\star,x} v(t,\lambda) &= -\beta^S\sigma_0(x)\frac{\partial}{\partial\lambda}(\lambda v(t,\lambda)).
\end{aligned}$$

We can then define the *limiting portfolio loss* L by

$$L_t \overset{\text{def}}{=} 1 - \int_0^\infty v(t,\lambda)\mathrm{d}\lambda, \quad t \geq 0.$$

For large N, Theorem 3.3.1 suggests the large portfolio approximation

$$L_t^N \approx L_t, \quad t \geq 0.$$

Of course, with this large portfolio approximation, we can proceed to some modelling and risk management, but the interesting part of the model is now reduced greatly. Indeed, the homogeneous pool is not very realistic. Plus, its numerical results is too simplistic and does not capture the dynamics of the market. Below we present a numerical implementation as well as an example of the homogeneous pool model. As we will see, it exhibits too much consistency and not enough clustering.

3.3.2 Numerical implementation

Approaches might differ according to the parameters and the set-up. In the case of an homogeneous pool, we will concentrate on the case where $\beta^C > 0$ and $\beta^S > 0$. We provide the finite difference scheme in the case of general diffusion $\mathrm{d}X_t = b_0(X_t)\mathrm{d}t + \sigma_0(X_t)\mathrm{d}V_t$. Since the term $\beta^S\sigma_0(X_t)(\lambda\nu)_\lambda\mathrm{d}V_t$ must be non-anticipating, the future time-steps being dependent on the path of V_t, an implicit method such as Crank-Nicholson cannot be used. Indeed, all of the other factors are deterministic, but this last term involving $\mathrm{d}V_t$ makes the problem dependent on past behaviour since the contagion factor transmits the risk through time as well as through the space of names. This is in contrast to the model developed in Bush et al. (2011), where we use the Crank-Nicholson method. In this latter method, the randomness at each time is independent of the randomness at any other time, each step receiving an independent shock through a Brownian motion and the general movement being passed through a PDE solving the heat equation with drift. Hence in this case, we can observe the effect of the systemic factor at certain points in time and use an implicit method between the observation dates. More details will be given on this technique in Chapter 4.

We therefore provide an explicit scheme. This has the disadvantages of only first-order accuracy in time and conditional stability. [1] The time-step is denoted by Δ and the mesh-size by δ. Let $v_{i,j} = v(i\Delta, j\delta)$, $\lambda_j = j\delta, X_i = X_{i\Delta}$, and $\Delta V_i = V_{i\Delta} - V_{(i-1)\Delta}$, for $i = 0, \dots, N$ and $j = 0, \dots, J$. We then have the explicit finite difference scheme as

$$
\begin{aligned}
v_{i,j} = \Delta[& \frac{\mathcal{I}_{i-1}}{2\delta} - \frac{\sigma^2}{2\delta} + \frac{\sigma^2 \lambda_j}{2\delta^2} + (\beta^S \sigma_0(X_{i-1})^2)\frac{\lambda_j^2}{2\delta^2} + \frac{\alpha(\bar{\lambda} - \lambda_j)}{2\delta} + \beta^S \frac{\lambda_j}{2\delta}(b_0(X_{i-1}) + \sigma_0(X_{i-1})\frac{\Delta V_i}{\Delta}) \\
& - (\beta^S \sigma_0(X_{i-1}))^2 \frac{\lambda_j}{\delta}] v_{i-1,j-1} \\
& + \Delta[\frac{1}{\Delta} + \alpha - \frac{\sigma^2 \lambda_j}{\delta^2} - (\beta^S \sigma_0(X_{i-1}))^2 \frac{\lambda_j}{\delta} - \lambda_j - \beta^S (b_0(X_{i-1}) + \sigma_0(X_{i-1})\frac{\Delta V_i}{\Delta}) \\
& + (\beta^S \sigma_0(X_{i-1}))^2] v_{i-1,j} \\
& + \Delta[-\frac{\mathcal{I}_{i-1}}{2\delta} + \frac{\sigma^2}{2\delta} + \frac{\sigma^2 \lambda_j}{2\delta^2} + (\beta^S \sigma_0(X_{i-1})^2)\frac{\lambda_j^2}{2\delta^2} - \frac{\alpha(\bar{\lambda} - \lambda_j)}{2\delta} - \beta^S \frac{\lambda_j}{2\delta}(b_0(X_{i-1}) + \sigma_0(X_{i-1})\frac{\Delta V_i}{\Delta}) \\
& + (\beta^S \sigma_0(X_{i-1}))^2 \frac{\lambda_j}{\delta}] v_{i-1,j+1}
\end{aligned}
$$

with boundary conditions $v_{i,0} = v_{i,J} = 0$ and $\mathcal{I}_{i-1} = \sum_{j=1}^{J} \delta \frac{v_{i-1,j} + v_{i-1,j-1}}{2}$. Here $\mathcal{I}_{i-1}$ and the functions $b_0(\cdot)$ and $\sigma_0(\cdot)$ relying on X_{i-1} are the reasons why we cannot rely on an implicit method. Giesecke et al. (2012) propose an approximate criterion for the time-step size, based on the criterion for a deterministic diffusion PDE where $\sigma_0 = 1$. They have numerically tested the approximation and found that $\Delta \lesssim \frac{\delta^2}{(\beta^S \lambda_{\max})^2}$ is sufficient to ensure stability.

With a homogeneous pool, we simulate the cumulative distribution function of the portfolio loss through Monte Carlo simulation and get figure 3.4. We notice a lot of small defaults, and no real cluster, as predicted by the model. This is caused by the fact that we can only simulate an homogeneous portfolio, which greatly diminishes the power of the model.

We thus recognize how the defaults seem to occur rather consistently. We do not observe clusters or extreme events. Even by varying the systematic and contagion risk

[1] Giesecke et al. (2013) mention a "method of moments" to proceed to the numerical implementation Giesecke et al. (2012). Due to lack of description about this method, we rely on a finite difference approximation.

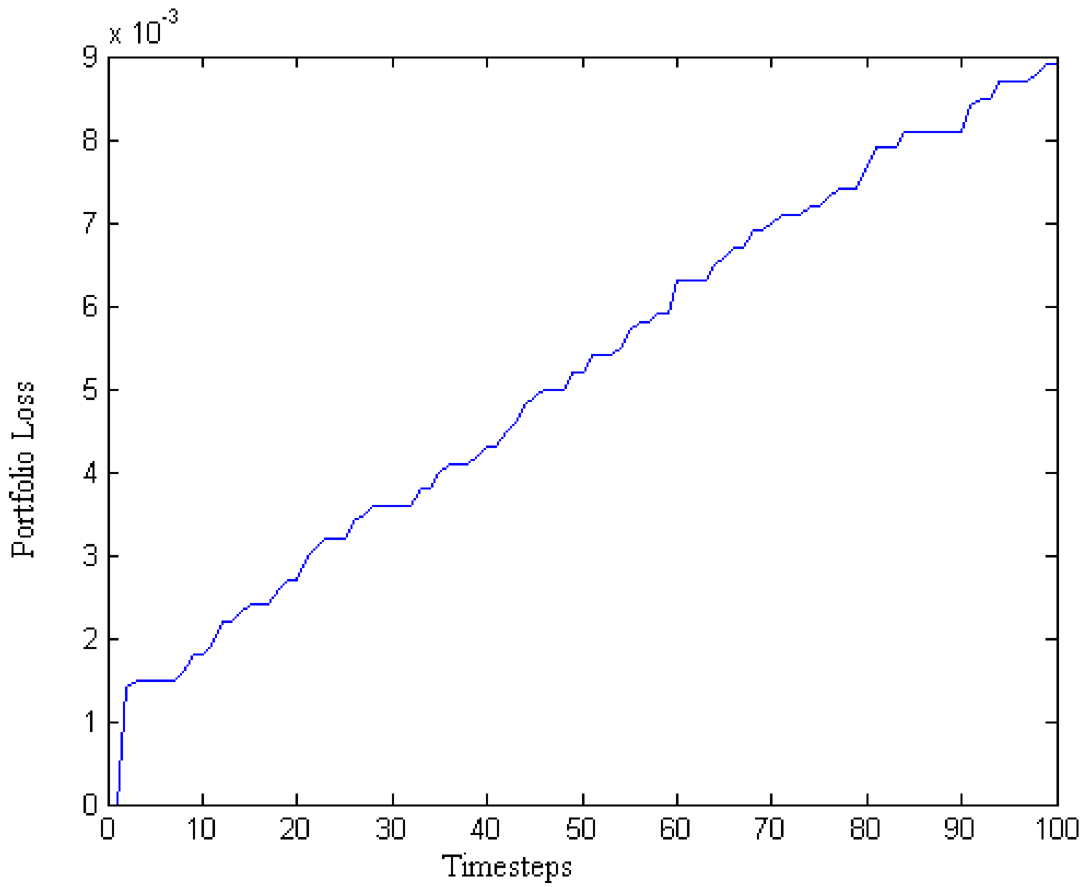


Figure 3.4: Portfolio loss progression through time in homogeneous pool

parameters, we do not vary greatly the results. Therefore, this model seems quite rigid in its homogeneous form, and complicated to implement in its heterogeneous form.

3.4 Comparison

Having presented these three models, we are now in a position to analyse their strengths and shortcomings. This is an important part of the analysis as it takes on a practical perspective of the latest financial modelling techniques. All of the three models start by tackling the cost of simulations problem. Indeed, by deriving a Law of Large Numbers for infinite portfolios and using it to approximate large portfolios, it lowers the amount of computations required, thereby reducing the costs of simulation. All of these three models approach the problem from a practical perspective. It is clearly stated in Bush et al. (2011), Bujok and Reisinger (2012), and in Giesecke et al. (2014) that they want to reproduce the empirical measure of the portfolio for ease of implementation and practical purposes. Furthermore, Lucas et al. (2000) use an already popular model and adds the convergence to a theoretical distribution theorem, combining the ease of implementation of the simple model to the theoretical results of tail behaviour needed in risk management.

Unfortunately, all three models have some serious drawbacks. First, Lucas et al. (2000) uses a simplistic, one-period factor model, and assumes normally distributed and independent random variables. This has been empirically proven to be counter factual. On the other hand, they prove that despite this simplification can lead to a heavy-tailed distribution, which is a very important result for the validity of this model. Otherwise, to derive their tail expansion and give it an economic interpretation, they assume homogeneity in the portfolio, which can be very restrictive. Also, we have shown that the way we define the factors' distribution function can lead to some completely different tail behaviours. This can be seen as both a strength and a weakness. It is a strength because it shows that the model is flexible enough to allow for thin-tailed distributions. It is a drawback because it requires us to be careful when choosing our factors' distributions functions. Indeed, if we were able to find a thin tailed distribution, that might indicate that using a normal distribution for the factors and shocks might not create enough heavy-tail characterization in our surplus variable.

Second, when looking at the model introduced by Bush et al. (2011), we observe the same simplistic characterization of Lucas et al. (2000), namely with one market factor and one idiosyncratic shock, with all being independent from each other. As mentioned, this paper builds on CID models, whose drawbacks have been underlined in §2.3. The strong assumptions required to derive the results might render this model less appealing, but the ease of representation of the loss variable and the intuitive economic meaning are working in the model's favour. Also, as the authors mention, and as will be pointed out in the next chapter, the fast convergence of the method is very practical. Indeed, even though the results are derived with $N \to \infty$, the number of names in our portfolio needed to observe the results within an acceptable range is only 120. See Bujok and Reisinger (2012) for details. Finally, the connection with the static models, and the Theorem 3.2.2 for the moments of the loss distribution are definitely add to the appeal of the model proposed by Bush et al. (2011).

Third, and finally, the model suggested by Giesecke et al. (2012) has pros and cons. The *types* it introduces allow more flexibility for the user of the model. Unfortunately, it is the kind of flexibility that can be harmful, as it is very difficult to define such types, and using them becomes more of a hassle than an empowerment. Even the authors themselves do not spend much time using them, preferring to study homogeneous pools of assets in order to derive their results and numerical tests. So even if the very convenient aspect of taking into account systematic and contagion risk into the model, the difficulty of defining the model reduces its practicality and hence its advantages. Also, the numerical implementation of this model is hampered as we are required to use a less precise, less stable forward Euler scheme because of the forward looking nature of the contagion effect. Finally, as was mentioned in §2.4, the model does not have a clear economic interpretation. So even if it has a greater capacity at reproducing the empirical facts, the model is not easy to interpret and the possible conclusions we could draw from it are reduced.

The above analysis has shown a number of strengths and weaknesses of the existing models in the literature and identified various problems and difficulties when modelling credit risks. We thus want to try to understand the intricacies of the numerical implementation of one of these models, namely the one developed by Bush et al. (2011). Indeed, as it has been presented as a novel and supposedly better approach than the traditional ones, it would be interesting to know if it performs well in reproducing the market and if it would be practical to price instruments such as CDOs. As we have mentioned, a good model should be dynamic, so we will not use the model developed by Lucas et al. (2000). Furthermore, we have mentioned the restrictions and difficulties in implementing the model developed by Giesecke et al. (2012). Therefore, in the next chapter, we will be analysing numerically the performance of Bush et al. (2011), using the method developed in Bujok and Reisinger (2012) to calibrate our results.

Chapter 4

Numerical Analysis of a Dynamic Structural Model

It will be important to be able to numerically prove the model's capabilities with market data. We first start by explaining the numerical implementation of the model and the novel approach we used to circumvent potential problems. Then, a replication of the results in Bush et al. (2011) is done in order to validate our method vis-a-vis theirs. Finally, we test the calibration of the model through the different phases of the financial crisis of 2007. We start with data from mid-2007 and show the performance of the model through the crisis until the end of 2008, at the midst of the crisis.

4.1 Numerical implementation

In order to study the particularities of this model, we proceeded to reproduce their numerical results. To do so, we replicate the simulation results reported in Bush et al. (2011) according to the methods adopted in this thesis. When expressing the SPDE (3.33) in an integral form, we get:

$$\langle \phi, \nu_t \rangle = \langle \phi, \nu_0 \rangle + \int_0^t \langle \mathcal{A}\phi, \nu_s \rangle \mathrm{d}s + \int_0^t \langle \sqrt{\rho}\phi', \nu_s \rangle \mathrm{d}M_s, \ \forall \phi \in C_K^\infty(0, \infty) \tag{4.1}$$

where $\mathcal{A}$ defines the second order linear operator

$$\mathcal{A} = \mu \frac{\partial}{\partial x} + \frac{1}{2} \frac{\partial^2}{\partial x^2}.$$

From Theorem 3.2.1, it follows that the density v describes the non-absorbed elements. In particular, v satisfies

$$(\phi, v(t,\cdot)) = (\phi, v(0,\cdot)) + \int_0^t (\mathcal{A}\phi, v(s,\cdot))\mathrm{d}s + \sqrt{\rho}\int_0^t (\phi', v(s,\cdot))\mathrm{d}M_s,$$

where we write $(\cdot,\cdot)$ for the L^2 inner product. Integrating by parts, noting $v \in H_0^1$ with dense subspace $C_K^\infty(0,\infty)$ (Bush et al., 2011), we get

$$(\phi, v(t,\cdot)) + \int_0^t a(\phi, v(s,\cdot))\mathrm{d}s = (\phi, v(0,\cdot)) + \sqrt{\rho}\int_0^t (\phi', v(s,\cdot))\mathrm{d}M_s$$

for all $\phi \in H_0^1$, where

$$a(\phi, v) = \frac{1}{2}(\phi', v') - \sqrt{\rho}(\phi', v).$$

The model implicitly uses continuous monitoring for the defaults. In practice, the closest we can get to continuous monitoring is the daily announcement of defaults. In order to make the model computationally more tractable, Bush et al. (2011) assume that default is detected on spread payment dates, i.e. quarterly.

Of course, such an assumption must be tested in order to confirm that it is not counter factual. By calculating survival probabilities for $m = 2, \ldots, 252$ monitoring dates i.e. from half-year to daily monitoring, at the CDS maturity time of $t = 5$, Bujok and Reisinger (2012) observe that the change in precision is small. Survival probabilities decrease as the number of monitoring dates increase, but the difference between their quarterly monitoring and the daily monitoring, which are the maximum realistic monitoring times per year, is only 0.59 percentage point, dropping from 0.8761 for quarterly monitoring to 0.8702 for daily monitoring.

As is pointed out in Broadie et al. (1997) Broadie et al. (1997), the error for discrete monitoring a barrier option is of the order $o\left(\frac{1}{\sqrt{n}}\right)$, where n is the number of monitoring dates. Thus, the probability of a jump in an interval of length ΔT is $O(\Delta T)$ and that of an undetected down-up combination event is $O(\Delta T^2)$. Therefore, discretely monitoring the defaults quarterly is reasonable and makes the model more practical.

By making the assumption that defaults can only be observed at a discrete set of times, we can set up a modified SPDE problem to solve for v. We assume that if a firm's value is below the default barrier on one of the observation dates T_i, it is considered defaulted and therefore removed from the basket. Thus, we have to solve the problem

$$
\begin{aligned}
\mathrm{d}v = -\frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right) v_x \mathrm{d}t + \frac{1}{2} v_{xx} \mathrm{d}t - \sqrt{\rho} v_x \mathrm{d}M(t) \qquad & t \in (T_k, T_{k+1}),\, 0 \le k \le n \\
v(0,x) = v_0(x) \qquad & \\
v(T_k, x) = 0 \qquad & \forall\, x \le 0,\, 0 < k \le n
\end{aligned} \tag{4.2}
$$

This way, we can use the solution to the SPDE without the boundary condition. This is possible since, by making the boundary condition inactive within the intervals (T_k, T_{k+1}), it does not matter whether default has occurred or not. Since Brownian driver produces normally distributed increments with a variance proportional to the time elapsed, we proceed to apply the market factor only at the end of the interval, simulating the effect of the market factor as accumulated over (T_k, T_{k+1}). Therefore, by observing the behaviour of equation (4.2) without the market factor, we simply need to solve the PDE

$$
u_t = \frac{1}{2}(1-\rho) u_{xx} - \frac{1}{\sigma}(r - \frac{1}{2}\sigma^2) u_x
$$

with $u(0,x) = v_0(x)$, then apply the market factor as a shift to the solution $u(t,x)$ to obtain

$$
v(t,x) = u(t, x - \sqrt{\rho} M_t), \quad \forall\, x \in \mathbb{R},\, t > 0.
$$

Therefore, in our numerical set up, we have

$$
v(t,x) = \begin{cases} 0 & \text{if } x \le 0 \wedge t \in \{T_k, 1 \le k \le n\} \\ v^{(k)}(t - T_k, x - \sqrt{\rho}(M(t) - M(T_k))) & \text{else if } t \in (T_k, T_{k+1}],\, 0 \le k \le n \end{cases} \tag{4.3}
$$

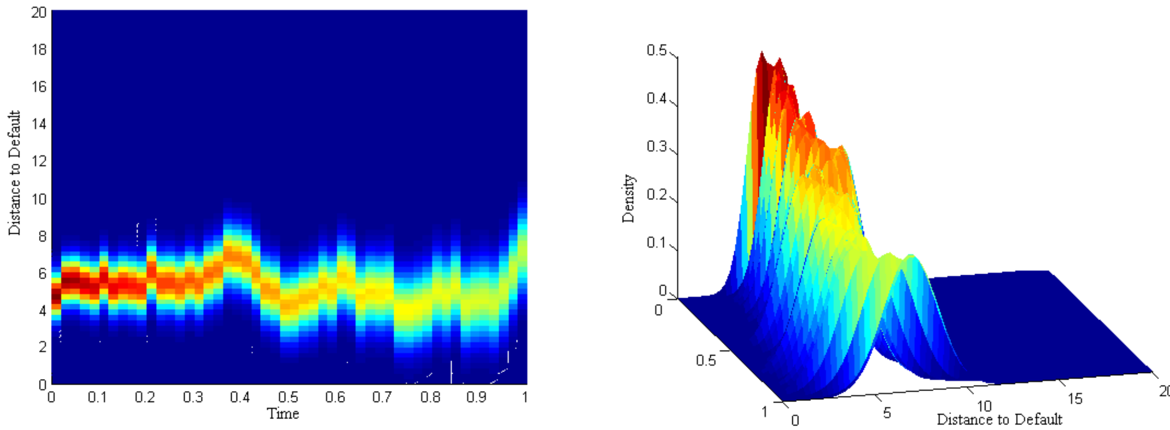
where $v^{(k)}$ is the solution to the deterministic problem

$$
\begin{aligned}
v_t^{(k)} &= \frac{1}{2}(1-\rho) v_{xx}^{(k)} - \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right) v_x^{(k)}, \quad t \in (0,\tau) = (0, T_{k+1} - T_k) \\
v^{(k)}(0,x) &= v(T_k, x)
\end{aligned} \tag{4.4}
$$

assuming payment dates are equally spaced with intervals $\tau = T_{k+1} - T_k$.

This translates into the inductive steps for $k = 0, \ldots, n-1$:

1. Start with $v^{(0)} = v_0(x)$.
2. Solve the PDE (4.4) numerically in the interval $(0, T_1)$, to obtain $v^{(0)}(T_1, x)$



(a) Ordinates: Distance-to-default
Abscissa: Time

(b) Perspective of the density's SPDE

Figure 4.1: Sample path of portfolio's density through time

3. Simulate $M(T_1)$, evaluate $v(T_1, x)$ according to (4.3).

4. For $k > 0$, having computed $v(T_k, x)$ in the previous step, use this as the initial condition for $v^{(k)}$, and repeat until $k = n$.

The finite difference method used is approximating the PDE (4.4) on a support $[x_{min}, x_{max}]$, with asymptotic boundary conditions $v(t, x_{min}) = v(t, x_{max}) = 0$. A suitable choice of $x_{min} < 0$ and $x_{max} > 0$ can reduce the localization error for a given path M as small as needed, assuming the initial distribution is localized. To lay out the approximation, we introduce a grid $x_0 = x_{min}, x_1 = x_{min} + \Delta x, \ldots, x_{min} + j\Delta x, \ldots, x_J = x_{min} + J\Delta x = x_{max}$, where $\Delta x = (x_{max} - x_{min})/J$, time steps $t_0 = 0, t_1 = \Delta t, \ldots, t_I = I\Delta t = \tau$, where $\Delta t = \tau/I$. Figure 4.1 shows an example of one sample path calculated by the method mentioned above. We can notice the shifts at every quarter of a year.

Finally, an approximation v_j^i to $v(t_i, x_j)$ can be made by finding the solution to the scheme

$$\begin{aligned}\frac{v_j^i - v_j^{i-1}}{\Delta t} &= \theta\left\{\frac{1}{2}(1-\rho)\frac{v_{j+1}^i - 2v_j^i + v_{j-1}^i}{\Delta x^2} - \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right)\frac{v_{j+1}^i - v_{j-1}^i}{2\Delta x}\right\} \\ &\quad +(1-\theta)\left\{\frac{1}{2}(1-\rho)\frac{v_{j+1}^{i-1} - 2v_j^{i-1} + v_{j-1}^{i-1}}{\Delta x^2} - \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right)\frac{v_{j+1}^{i-1} - v_{j-1}^{i-1}}{2\Delta x}\right\}\end{aligned} \tag{4.5}$$

When expressed in matrix terms, this scheme becomes

$$B_\theta v_i = F_\theta v_{i-1}$$

with B_θ a tridiagonal matrix representing the backward portion of the θ-method and the F_θ representing its forward computations. In this case, as we are simply solving a particular case of the heat equation with drift within the time period between two default observation dates, we can use an implicit method. Indeed, these calculations are independent of the past or future path of the market factor represented by the Brownian motion M. In comparison with the previous model of Giesecke et al. (2012), where the dependence on the past value of the loss function was needed in order to transmit the contagion effect, no such thing is needed here, and, hence, each step is self contained. Therefore, for the matrix B_θ, the elements on the the *main diagonal* are

$$1 + \theta \frac{\Delta t}{\Delta x^2}(1-\rho),$$

while those on the *upper* diagonal are

$$-\frac{\theta}{2}\frac{\Delta t}{\Delta x}\left(\frac{(1-\rho)}{\Delta x} - \frac{(r-\frac{1}{2}\sigma^2)}{\sigma}\right),$$

and those on the *lower* diagonal are

$$-\frac{\theta}{2}\frac{\Delta t}{\Delta x}\left(\frac{(1-\rho)}{\Delta x} + \frac{(r-\frac{1}{2}\sigma^2)}{\sigma}\right).$$

All of the other elements are set to 0. Similarly, the matrix F_θ is defined as a tridiagonal matrix such that the elements on the *main* diagonal are

$$1 + (1-\theta) \frac{\Delta t}{\Delta x^2}(1-\rho),$$

those on the *upper* diagonal are

$$-\frac{(1-\theta)}{2}\frac{\Delta t}{\Delta x}\left(\frac{(1-\rho)}{\Delta x} - \frac{(r-\frac{1}{2}\sigma^2)}{\sigma}\right),$$

and those on the *lower* diagonal are

$$-\frac{(1-\theta)}{2}\frac{\Delta t}{\Delta x}\left(\frac{(1-\rho)}{\Delta x} + \frac{(r-\frac{1}{2}\sigma^2)}{\sigma}\right).$$

All of the other elements are set to 0.

This scheme is second order accurate in Δx. The method used in our computations is the Crank-Nicholson (C-N) scheme with $\theta = \frac{1}{2}$, which is second order accurate in Δt and unconditionally stable in the l_2-norm for sufficiently smooth solutions.

4.1.1 Initial Distance to Default

Unfortunately, we are not facing a smooth solution *per se*. Indeed, since the problem starts with a collection of point masses in the form of δ-distribution for distances to default for each name in our portfolio, our initial distribution is not smooth. The C-N scheme becomes rather erratic and unstable. Fortunately, the backward Euler scheme $\theta = 1$ is strongly $\mathcal{A}$-stable (see Appendix C), even if only first order accurate in Δt, and Bujok and Reisinger (2012) suggest to use it in what is called a Rannacher start-up (Rannacher, 1984) in order to address the instability of the C-N scheme.

In our case, we circumvent the problem by using a kernel-smoothing density estimation for the initial distribution, provided by the MATLAB function `ksdensity()`. A *kernel* is defined in Wasserman (2004) to be any smooth function C such that $C(x) \geq 0$, $\int C(x)\mathrm{d}x = 1$, $\int xC(x)\mathrm{d}x = 0$ and $\sigma_C^2 \equiv \int x^2 C(x)\mathrm{d}x > 0$. An example of a kernel is the Gaussian (Normal) kernel $C(x) = (2\pi)^{-1/2}e^{-x^2/2}$, which will be used in our modelling. Thus, the kernel density estimation is defined as

$$\widehat{f}(x) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{h}C\left(\frac{x - X_i}{h}\right).$$

where h is a positive number called the *bandwidth* which controls the amount of smoothing. For h close to 0, $\widehat{f}_n$ is simply a set of spikes, one at each data point X_i. This allows us to have a smooth initial condition when our input data is discontinuous. Therefore, the solution of the initial distribution is smooth, and the C-N scheme works properly.

When implying the initial distance to default for our model, we rely on market data for CDS spreads for the names forming the iTraxx Europe series 6. By assuming that the distance to default $X_t = \frac{1}{\sigma}\ln\left(\frac{A_t}{B_t}\right)$ as in equation (3.29), we base X_0 on 5 years CDS spreads. Since a CDS spread implies a default probability computed as in (2.3), it is possible to infer the distance to default of a firm by using the distribution of the first hitting time problem. As is explained in Björk (2009), let

$$\iota = \inf\{t \geq 0 | X_t = 0\}$$

and define the absorbing process

$$X_{\text{Observed}} = X(t \wedge \iota).$$

The minimum process is defined as

$$m_X(t) = \inf_{0 \leq s \leq t} X(s).$$

Since our model defined in (3.29) assumes independence of the two Brownian motions W_t and M_t, we have that

$$X \sim N(\mu t + x_\circ, t).$$

Given that $x_\circ \geq 0$, which we know because all the companies we are starting with in our portfolio are active and have not defaulted, we get from Björk (2009) the distribution function for the running minimum $F_{m(t)}(x)$ to be

$$F_{m(t)}(x) = N\left(\frac{x - x_\circ - \mu t}{\sigma\sqrt{t}}\right) - \exp\left\{2 \cdot \frac{\mu(x - x_\circ)}{\sigma^2}\right\} N\left(\frac{x - x_\circ + \mu t}{\sigma\sqrt{t}}\right),$$

for $\mu = (r - \frac{1}{2}\sigma^2)/\sigma$, and σ being the common unique volatility of the assets and of the market, by assumption.

Therefore, our initial distance to default is the X_0 that solves the equation

$$F_{m(T)}(0, X_0) = P^{\text{Def}}(0, 0, T, X_0). \tag{4.6}$$

In our case, we chose T to be our earliest maturity date, i.e., $T = 5$ years. As mentioned in Section 2.1, the earliest and most liquid maturity will be used as reference to derive the first hitting time problem's initial distribution. We solved this equation numerically using `fsolve()` function from MATLAB for each firm contained in the iTraxx Europe Main series 6 for our dates, then proceeded to the kernel smoothing estimation to get the initial density of our SPDE.

Different market volatilities, represented by σ, will produce different results, and σ mainly influences the initial distance to default. For illustration, we show in figure 4.2 the initial density of our portfolio's distance to default for different values of the volatility parameter σ. We notice that, as the volatility σ increases, so does the distance to default. This is explained by the fact that the CDS spread we extracted implies a particular default probability P^{def}. When we increase the volatility, we increase the movements amplitude of our random variable. So for a fixed distance to default, a higher volatility implies a higher probability of default. Therefore, to match the market's implied probability of default, we must move our initial distance to default away from the origin.

4.1.2 Monte Carlo Simulation

We now want to approximate the loss functional (3.32) at time T_k by

$$\widehat{L}_{T_k} = (1 - R)\left(1 - \int_0^{x_{max}} \widehat{v}(T_k, x)\mathrm{d}x\right) = (1 - R)\left(1 - \Delta x \sum_{j=1}^{J-1} v_j^{k+1,0}\right). \tag{4.7}$$

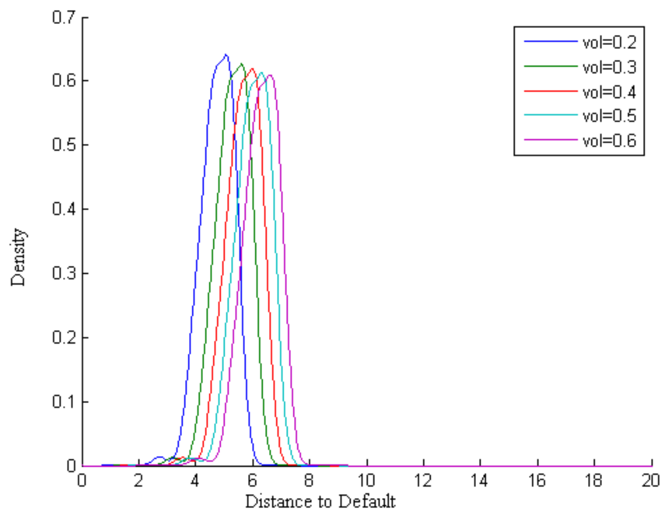


Figure 4.2: Different initial densities for different values of volatility σ

To compute the expected tranche losses and outstanding tranche notionals, we explicitly include the dependence on the Monte Carlo samples ϕ_i of $\sqrt{\rho}(M_t - M_{T_k})$ in (4.7) by writing $\widehat{L}_{T_k}(\phi)$, $\phi = (\phi_i)_{1\leq i\leq n}$, for ϕ_i independent outcomes. Then for N_{sims} simulations with samples $\phi^l = (\phi_i^l)_{1\leq i\leq n}$, $1 \leq l \leq N_{sims}$, we get

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[Z_{T_k}] &\approx \mathbb{E}^{\mathbb{Q}}[\max(d - \widehat{L}_{T_k}, 0) - \max(a - \widehat{L}_{T_k}, 0)] \\ &\approx \frac{1}{N_{sims}} \sum_{l=1}^{N_{sims}} \Big(\max(d - \widehat{L}_{T_k}(\phi^l), 0) - \max(a - \widehat{L}_{T_k}(\phi^l), 0)\Big). \qquad (4.8)\end{aligned}$$

When doing Monte Carlo, we must simulate multiple sample paths and average them. Figure 4.3 show some of those sample paths. We will be taking around 65,000 of them to compute expected losses and expected outstanding tranche notionals.

4.1.3 Convergence

We have tested the convergence of this method. This simulation took as initial condition the estimated Normal distribution $N(4.6, 0.16)$ as is proposed by the authors to characterize the distribution of the data found in the iTraxx Europe Series 6 on 22 February 2007, in order to speed the simulation and to concentrate on the observation of the convergence.

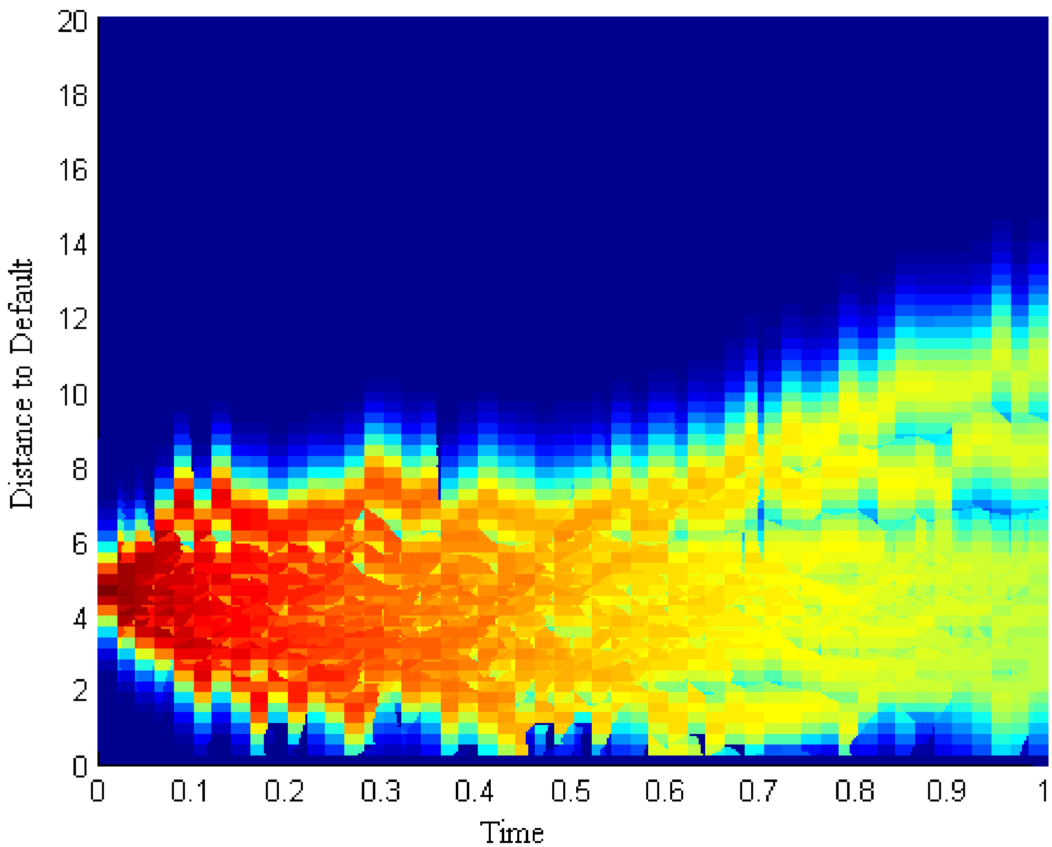


Figure 4.3: Simulation of 64 sample paths to illustrate the behaviour of our model. We observe that we get scenarios with multiple defaults as well as scenarios with very few.

The results obtained clearly show the fast convergence of the expected tranche loss. This simulation shown in Figure 4.4 was done for the tranche $0\% - 3\%$ of the CDO, with the ordinate corresponding to the expected tranche loss in % and the abscissa being the number k of Monte Carlo samples, such that $N_{sims} = 4^{k+1}$. Therefore, the expected tranche loss for the junior tranche is 1.575% with a Confidence Interval of $[1.558\%, 1.592\%]$. We notice the confidence interval starting rather wide, but narrowing down rather quickly, being less than 5 basis points after simply $4,000$ simulations.

For this estimator, we expect (weak and strong) convergence of order $\Delta x^2 + \Delta t^2$. To perform these simulations, direct guidance from Bush et al. (2011) has been obtained in

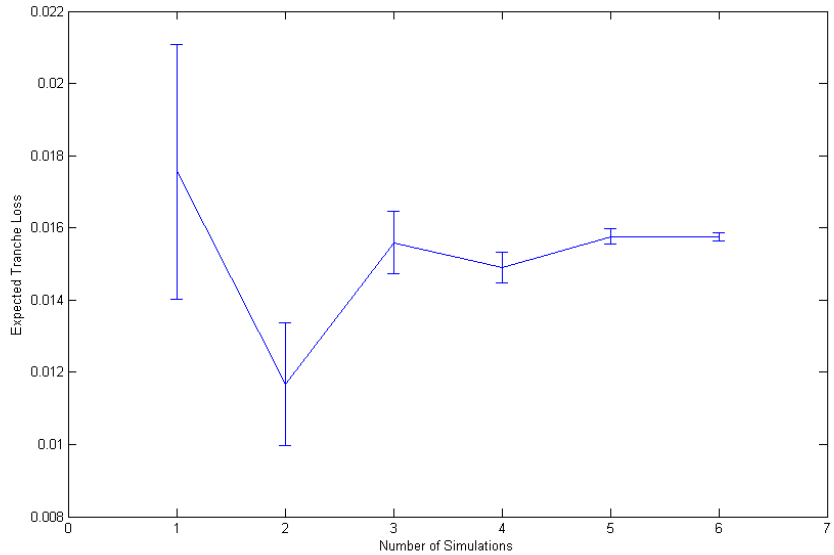


Figure 4.4: Monte Carlo estimators with 95%-confidence interval for expected losses in tranche $[0\% - 3\%]$ for $N = 4^{k+1}$ simulations. Ordinates are Expected Tranche Loss; Abscissa are k's

order to have values for Δx and Δt which have been empirically proven to give negligible discretisation error. We thus take $\Delta x = 0.28$ for $x \in [0, 20]$ and $\Delta t = 0.01$ for $\Delta t \in (T_k, T_{k+1}]$ and $\Delta T = 0.1$ for $T_0 = 0$ and $T_{end} = 5$, 7, 10 for the maturities.

4.2 Results Reproduction

In this section, we aim at testing the performance of the model, and the reproducibility of the results of Bush et al. (2011). To be an efficient model, we should be able to reproduce their pricing method, as well at get results close to market data, since that would imply appropriate calibration capacity. We will analyse the influence of the parameters on the loss function, as well as compare our results with the ones obtained by Bush et al. (2011). This will allow us to really judge whether the model performs well or if it needs improvement.

The results we present are the CDO tranche and index spreads computed as in (2.7), using (4.8) to approximate the expected outstanding tranche notional Z_{T_k} for V^{fee} and similarly for V^{prot}. The loss distribution is estimated by Monte Carlo simulation as in (4.7), using $65,000$ simulations of the empirical distribution v as in (4.3) and the methodology described in section 4.1. The approximated loss distribution for the Index is not scaled by the recovery rate $(1 - R)$, as is common practice.

We present in Table 4.1 the results obtained by replicating Bush et al. (2011)'s Table 2 with the finite difference method. For reference, Table 4.2 corresponds to the results from Table 2 in Bush et al. (2011). We notice that the results obtained by our simulations are fairly close to their results. This difference is caused by a different initial distribution, pulled from a different data set than their data set. On the other hand, we notice a similar difference from the market data as the authors, even while using what they considered their optimal parameters. This is not very encouraging for the usage of the model in practice.

Indeed, if both our results and their original results differ from the markets behaviour, this could mean that our model is not flexible enough. By testing for different values of ρ, we were able to see how much the prices change with the parameters. This allows us to assess that our model was indeed flexible. Unfortunately, varying the parameter ρ has not proven enough to efficiently calibrate all of the tranches spread. Indeed, when we fix a set of parameters that make our computations for one tranche closely match the market's tranche, we notice that often the rest of the CDO will start diverging from it rather quickly. We notice this especially for senior tranches.

This could mean that even if the model is flexible, it does not induce enough risk from extreme events. These types of events would be the ones affecting the spreads of senior tranches. This is a reason why the study of tail behaviour using the moments of the loss distribution that we introduced in Theorem 3.2.2 from Section 3.2.2 could help to assess how these events might affect our results. Another way of tackling this problem would be to add some random jumps to our diffusion process, in order to obtain a jump-diffusion process, as is done in Bujok and Reisinger (2012).

We will analyse into more details this ability of the model to calibrate in the next section, as we test the model's behaviour in the ramp-up that led to the crisis. We will show how the model is responsive to the economic environment, all the while missing out on some key aspects of credit risk management.

4.3 Statistical Significance

All of the spreads presented in the previous section have been computed using $4^8 = 65,536$ Monte Carlo simulations. As we extend their work and conduct our analysis based on our datasets, we set forth to provide the statistical significance of our results. In the next

	5 Years							
Tranche	Market	$\rho = 0.1$	$\rho = 0.2$	$\rho = 0.3$	$\rho = 0.4$	$\rho = 0.5$	$\rho = 0.6$	$\rho = 0.7$
0%-3%	7.25%	8.05%	5.05%	1.75%	-1.50%	-4.67%	-7.73%	-10.70%
3%-6%	40.29	19.88	60.62	88.60	104.33	110.89	110.70	105.17
6%-9%	10.38	0.88	11.78	28.30	43.11	54.29	61.57	64.88
9%-12%	4.75	0.04	2.80	10.72	20.98	30.83	38.70	44.31
12%-22%	1.68	0	0.25	2.11	6.14	11.62	17.80	23.75
22%-100%	N/A	0	0	0.01	0.10	0.38	0.90	1.77
	7 Years							
Tranche	Market	$\rho = 0.1$	$\rho = 0.2$	$\rho = 0.3$	$\rho = 0.4$	$\rho = 0.5$	$\rho = 0.6$	$\rho = 0.7$
0%-3%	22.06%	27.98%	19.88%	13.15%	7.16%	1.64%	-3.56%	-8.55%
3%-6%	109.31	145.85	193.08	206.59	205.77	197.09	183.29	165.39
6%-9%	31.83	19.52	59.08	86.30	102.52	110.88	113.33	110.87
9%-12%	14.89	2.65	19.97	40.48	57.24	69.31	76.93	80.65
12%-22%	4.65	0.11	3.21	10.84	20.65	30.59	39.67	47.07
22%-100%	N/A	0.00	0.01	0.15	0.58	1.41	2.66	4.40
	10 Years							
Tranche	Market	$\rho = 0.1$	$\rho = 0.2$	$\rho = 0.3$	$\rho = 0.4$	$\rho = 0.5$	$\rho = 0.6$	$\rho = 0.7$
0%-3%	38%	42.32%	31.95%	23.06%	15.01%	7.50%	0.31%	-6.71%
3%-6%	303.01	388.05	358.53	330.10	301.94	273.59	244.66	214.62
6%-9%	83.12	110.27	152.60	168.14	172.27	169.90	163.12	152.67
9%-12%	36.68	28.54	68.55	93.15	107.58	115.26	118.01	116.72
12%-22%	12.36	2.61	15.65	31.62	46.09	57.94	67.03	73.42
22%-100%	N/A	0.00	0.13	0.69	1.80	3.43	5.55	8.17

Table 4.1: Computed tranche spreads (bp) for varying values of the correlation parameter from our implementation of the model. The equity tranches are quoted as an upfront assuming a 500bp running spread. The model uses data from the iTraxx Main Series 6 index for Feb 22, 2007. Market levels shown are for this date. Model parameters used are the one from the Bush et al. (2011) calibration, i.e., $r = 0.042$, $\sigma = 0.22$, $R = 0.4$

	5 Year							
Tranche	Market	$\rho = 0.1$	$\rho = 0.2$	$\rho = 0.3$	$\rho = 0.4$	$\rho = 0.5$	$\rho = 0.6$	$\rho = 0.7$
0%-3%	7.19 %	7.55 %	4.99 %	2.14 %	-0.71 %	-3.48 %	-6.17 %	-8.78 %
3%-6%	41	15.6	55.6	86.4	106.1	116.2	119.5	117.4
6%-9%	10.8	0.7	9.1	25	40.3	54.5	65.2	71.7
9%-12%	5	0	2.2	8.2	18.8	28.6	37.2	45.4
12%-22%	1.8	0	0.2	1.7	4.9	9.8	16.1	22.5
22%-100%	0.9	0	0	0	0.1	0.3	0.7	1.5
	7 Year							
Tranche	Market	$\rho = 0.1$	$\rho = 0.2$	$\rho = 0.3$	$\rho = 0.4$	$\rho = 0.5$	$\rho = 0.6$	$\rho = 0.7$
0%-3%	22.1 %	27.45 %	19.97 %	13.79 %	8.31 %	3.27 %	-1.47 %	-6.04 %
3%-6%	110	130.6	183.3	202.2	206	201.5	191.6	177.8
6%-9%	32.5	15.3	52.4	80.5	99.1	110.6	116.1	116.9
9%-12%	15	1.8	17.4	37.1	54.3	67.1	76.5	82.7
12%-22%	4.9	0.1	2.3	8.9	19	29.9	39.5	47.9
22%-100%	2	0	0	0.1	0.4	1.1	2.3	4.1
	10 Year							
Tranche	Market	$\rho = 0.1$	$\rho = 0.2$	$\rho = 0.3$	$\rho = 0.4$	$\rho = 0.5$	$\rho = 0.6$	$\rho = 0.7$
0%-3%	38 %	42.51 %	32.51 %	24.13 %	16.65 %	9.71 %	3.11 %	-3.33 %
3%-6%	302.5	375.8	354.9	331.9	308.1	283.5	258.3	231.9
6%-9%	83	101.4	147.3	166.2	173.6	174.4	170.8	163.8
9%-12%	37	24.3	64.1	90.9	107.7	117.8	122.9	124.1
12%-22%	12.5	2	13.5	29.1	44.4	57.5	68.2	76.5
22%-100%	3.6	0	0.1	0.6	1.5	3	5.1	7.7

Table 4.2: Taken as Table 2 from Bush et al. (2011). Model tranche spreads (bp) for varying values of the correlation parameter. The equity tranches are quoted as an upfront assuming a 500bp running spread. The model is calibrated to the iTraxx Main Series 6 index for Feb 22, 2007. Market levels shown are for this date; model parameters are $r = 0.042$, $\sigma = 0.22$, $R = 0.4$

section, we want to test the model in a way that would take a reasonable amount of time for a practitioner to compute the spreads. We therefore performed $4^7 = 16,384$ simulations.

As we are dealing with an estimator which is a complex function of our simulated data, we rely on the bootstrap method (Wasserman, 2004) in order to pull out a standard error estimate from the model. The standard error we provide here therefore required us to perform more than $16,384$ simulations, but one could use these results for reference when using this model. The statistic we compute is the estimate of the tranche spread s, i.e., $\widehat{s}$. This is computed as the ratio of the estimated protection leg over the estimated fee leg, such that

$$\widehat{s} = \frac{\widehat{V}^{\text{prot}}}{\widehat{V}^{\text{fee}}}.$$

These two estimates are both composed of the sum of discounted averages through time. One, $\widehat{V}^{\text{prot}}$, takes the discounted sum of the average change in tranche notional, whereas the other, $\widehat{V}^{\text{fee}}$, takes the discounted sum of the outstanding tranche notional multiplied by the observation step size. We thus have

$$\frac{\widehat{V}^{\text{prot}}}{\widehat{V}^{\text{fee}}} = \frac{\sum_{t=1}^{T} \frac{1}{b(t)} (\bar{Z}_{t-1} - \bar{Z}_t)}{\sum_{t=1}^{T} \frac{\delta_t}{b(t)} \bar{Z}_t}, \tag{4.9}$$

where t is each observation date, which we assume to be 4 times per year to correspond to the dividend paying dates. Therefore, $\delta_t = 1/4$. Here, $\bar{Z}_t$ is computed as

$$\bar{Z}_t = \frac{1}{N} \sum_{n=1}^{N} Z_t^n,$$

where Z_t^n is the simulated outstanding tranche notional as in (2.4), using the loss approximation $\widehat{L}_t$ of equation (4.7). As we simulate the loss variable and perform some calibration to find the parameters of its distribution, we can replicate its empirical distribution with the sample distribution $\widehat{F}_N$. Therefore, as our estimated spread $\widehat{s}$ is a function of the loss variable $\widehat{L}$, we will perform the bootstrap variance estimation of $\widehat{s}$ by following the steps described in Wasserman (2004).

1. Draw $\widehat{L}^1, \ldots, \widehat{L}^N \sim \widehat{F}_N$ for each $\widehat{L}^i = \left(\widehat{L}^i_{t_1}, \widehat{L}^i_{t_2}, \ldots, \widehat{L}^i_{t_T}\right)$.

2. Compute $\widehat{s}_N = g(\widehat{L}^1, \ldots, \widehat{L}^N)$.

3. Repeat steps 1 and 2, B times, to get $\widehat{s}_{N,1}, \dots, \widehat{s}_{N,B}$.

4. Let
$$v_{\text{boot}} = \frac{1}{B} \sum_{b=1}^{B} \left(\widehat{s}_{N,b} - \frac{1}{B} \sum_{r}^{B} \widehat{s}_{N,r} \right)^2. \tag{4.10}$$

where $g(\cdot)$ is the function (4.9).

For our purposes, we have used $B = 25$ and $N = 16,384$. This should give us a relatively good standard error.

4.4 Time Series Analysis

The time series analysis takes the values of CDS from before the crisis until the middle of the crisis in order to test the flexibility of the model and its ability to calibrate and to respond to rapid changes in the market. We took the risk-free rate as the value of a 5 years Euro-bond on AAA countries. This data was found on Bloomberg terminals. We thus try to calibrate the parameters σ and ρ of our model in order to match as closely as possible the market values for the iTraxx Europe Main series 6 CDO. The calibration problem we considered is given in Bujok and Reisinger (2012) and consists of a weighted least-squares problem. It states that

Problem 1. *Given market spreads at time t=0 of CDO tranches $C_0^j(T_i)$, and the CDO index $CI_0(T_i)$, for maturities T_i, $i = 1, \dots, \mathrm{M}$, tranches $j = 1, \dots, G$, and given spreads $s_0 = (s_0^1, \dots, s_0^N)$ for CDSs written on N underlying companies, solve the minimisation problem*

$$\sum_{i=1}^{M} \sum_{j=1}^{G} \alpha_i^j \left(C_0^j(T_i) - C_0^{j,\theta,x_0}(T_i) \right)^2 + \sum_{i=1}^{M} \alpha_i \left(CI_0(T_i) - CI_0^{\theta,x_0}(T_i) \right)^2 \to \min_{\theta}, \tag{4.11}$$

where $\theta = (\sigma, \rho)$, subject to

1. $x_0 = (x_0^1, \dots, x_0^N)$ *is a solution to*
$$s_0 = s_0^{\theta}(x_0) \tag{4.12}$$

2. $\rho \in [0, 1)$, $\sigma > 0$

where s^θ, C_0^{j,θ,x_0}, CI_0^{θ,x_0} *denote CDS, CDO tranche and index spreads calculated in the model with parameter vector* θ*,* x_0 *is a vector of initial distance-to-default,* $\alpha = (\alpha_i^j, \alpha_i)$ *is a scaling vector.*

The scaling factor α comes in handy because CDO tranches and index spreads have different orders of magnitude. We use α in order to make each observation roughly equally important. Bujok and Reisinger (2012) states that by choosing α so that the scaled market prices lie between 0.1 and 1, we get a good balance for each calibration. We have already discussed, in section 4.1 equation 4.6, the condition for equation (4.12) to be satisfied, therefore, we only need to implement the least squares minimisation function.

The calibration was done by testing all the parameter values for σ and ρ in the vectors

$$\begin{aligned}\sigma &= (0.1, 0.15, 0.2, 0.25, 0.3, 0.35, 0.4, 0.45, 0.5, 0.55, 0.6)\\ \rho &= (0.1, 0.2, 0.3, 0.40.5, 0.6, 0.7, 0.8, 0.9),\end{aligned}$$

and doing a grid-search because the optimization tool provided by MATLAB has not proven efficient in performing the optimization of our implementation. The available algorithms did not converge to an appropriate minimal solution. Therefore, our optimized values will be within those values. We performed the optimization problem stated by Bujok and Reisinger (2012) and computed the sum in (4.11) for all of the parameter values stated above. This gave us a surface as in Figure 4.5 for the data set of July 2007, and as in Figure 4.6 for the data set of December 2008. We clearly see a shift in the calibrated parameters. The correlation of our portfolio to the market factor has greatly increased, while our calibrated volatility has stayed relatively low all through the crisis. This low volatility is also observed throughout the calibrated time series presented hereafter.

Higher market correlation is to be expected. Indeed, during the crisis, most of the actors were deeply affected, and the credit market suffered severe shock all at once. It is therefore intuitive to observe a change in market correlation. On the other hand, this constantly low calibrated volatility is quite concerning. In economic terms, this would mean that the market does not fluctuate a lot around the mean return, even during the crisis of 2008. This could be explained by how the σ is set up in the model, as can be seen in equation (4.5). Indeed, as we observe high correlation and low volatility, this means that there is less influence given to the second derivative and more to the first derivative of the approximation of the heat equation with drift. This implies that, through time, in order to replicate the loss function that would generate the market spreads, the model gives more weight to the trend than the curvature of the density. The effect of the curvature of the empirical measure function would be buried by the large dependence on the market factor M_t inducing randomness in the model. This is a downside of the model as we know that

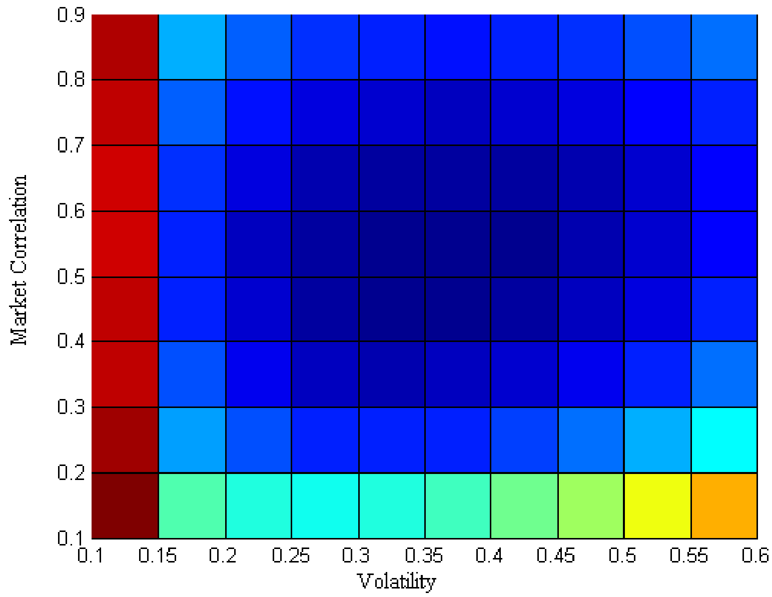


Figure 4.5: Surface produced by the minimization problem for each parameter tested on July 2007. Abscissa: σ, Ordinates: ρ.

during the crisis, the volatility went up by a factor of 3 from its level in October 2007 to March 2009 (Manda, 2010).

To explain this reaction of the model, we turn to our definition of the probability of default. As the CDS spreads were getting higher during the unravelling of the crisis, the probabilities of default as calculated in (2.3) were getting higher. This is required in order to have a distance to default $x_\circ$ as in (4.6) in accordance. Therefore it puts less emphasis on the volatility in the PDE and more on the initial distance to default definition. Since Bush et al. (2011) assume a single and constant volatility for both the individual underlying CDS and for the CDO, this introduces a bias in favour of low calibrated volatility. One way to remedy the situation would be to allow individual volatility for each name to be treated as exogenous, then compute our initial distance to default for each name, take the large number approximation and calibrate for the market volatility.

We now turn our attention on the time series analysis, and therefore refer to the tables contained in Appendix D. They show the time series of the calibrated tranche spreads for the 5, 7, and 10 years maturity CDO of iTraxx Europe Main series 6 from June 2007 until December 2008. We present also the market data for these spreads, which we used

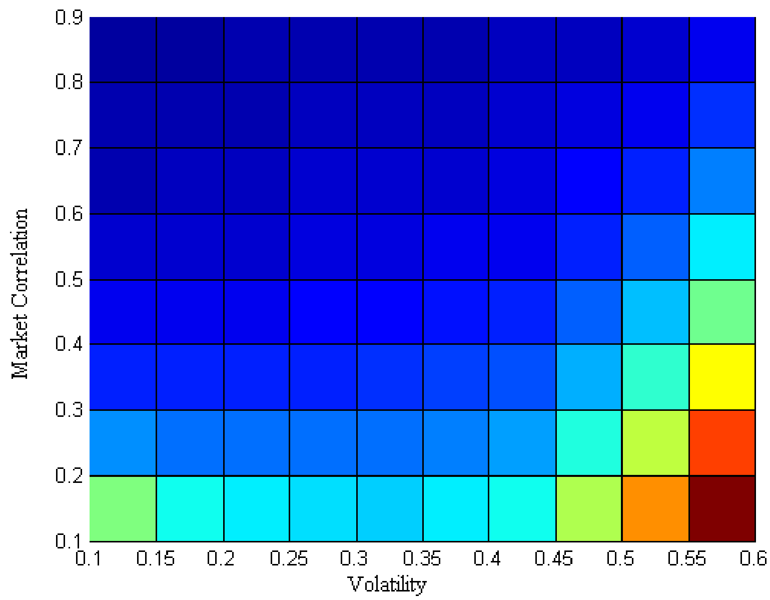


Figure 4.6: Surface produced by the minimization problem for each parameter tested on December 2008. Abscissa: σ, Ordinates: ρ.

to calibrate our parameters according to the grid-search method described above. Finally, at the bottom of each table, we define a relative error comparison for each tranche, in percentage points as

$$\text{Relative Error} = \frac{\text{Computed spread} - \text{Market spread}}{\text{Market spread}} \times 100\%.$$

We point out a missing data point, namely October 2007, as the data was too erratic at this date to be able to perform the calibration properly. This does not affect our analysis as we can still clearly observe the trend in the parameters, and the relative inefficiency of the calibrated tranche spreads.

We first notice that the magnitudes of the computed spreads are comparable to the ones of the observed market spreads. This is possibly the only encouraging observation. Indeed, most of the relative errors are quite large, some being as big as 200%, and on average the computed spreads being around 20% bigger than the market spreads. When observing the relative errors per tranche, we notice that they are quite big, but on average, the lower tranches, i.e., $0\% - 3\%$, $3\% - 6\%$, 6%-9%, are closer to the market spreads than the more senior tranches 9%-12%, 12%-22%, 22%-100%. This could be explained by a higher number of events happening in these tranches during the simulations, as opposed to the senior tranches, which could influence their pricing. This could be solved by an importance sampling method.

On the other hand, when we compare the relative errors through time, we observe the average relative error to be increasing as we get closer to the height of the crisis. This would mean that our model does not adapt well enough to changes in the market, and could be explained by the way it treats volatility as a unique constant for all assets. Indeed, during the crisis, volatility increased a lot and changed rapidly. The inability of the model to capture such behaviour would translate into the poor calibration results through time.

Moreover, when comparing the relative errors for different maturities, we notice that the longer term 10 years maturities are better calibrated than the shorter 7 years, and themselves better calibrated than the 5 years. Indeed, the average relative percentage error for 5 years maturity was 38.00%, while it was 20.55% for the 7 years maturity and -0.49% for 10 years. Of course, this takes into account both positive and negative error, and the standard deviations of these averages are great, but they are less important in longer maturities than shorter ones, being 52.95, 39.05, and 30.10, respectively. The model not incorporating enough volatility would again be an explanation for such a result. Longer term allow for more events to happen, whereas short term would rely mostly on the diffusion process, which is not representative of the market, therefore hampering the calibration process.

Finally, we could also change the calibration problem in order to have a better fit. As is pointed out in Bujok and Reisinger (2012), fitting all of the tranches will be very difficult, but for hedging purposes, one might prefer to put more weight on the index spreads. Therefore changing its α parameter in the calibration problem to allocate more importance to these spreads will affect the quality of the calibration. The senior tranche will be the ones that will be the worst replicating market data, but that is a trade-off one must be willing to make.

We might also add that performing the spreads calculations with more Monte Carlo simulations did not prove effective in getting more accurate results. As was mentioned in Section 4.1.3, the convergence is pretty fast. The spreads' calculations in Table D.1 to D.18 were performed with $N^{\text{sims}} = 4^7 = 16,384$ simulations, while we performed $N^{\text{sims}} = 4^9 = 262,144$ simulations for the spreads in the third column of D.19. We notice that the calculations do not change significantly when we increased the number of simulations. This shows that convergence is not an issue, but model definition might be.

In conclusion, this time series analysis of the model developed by Bush et al. (2011) has shown that the model needs improvement in order to be applicable to real-world problems. Indeed, it does not capture the volatility of the market, as well as being too stiff when being tested through the crisis. Its heavy reliance on the market factor and on the drift term to produce default events greatly hampers its capacity to explain the market's behaviour. A suggestion has been made by Bujok and Reisinger (2012) to add a jump term to its definition in order to add more influential shocks to the model. It has indeed proven efficient in recreating market spreads, but has not been tested through evolution of the crisis. This would be an important test to do in order to assess the ability of the model to react to economic turbulences. As we have shown here, the model dynamic diffusion model as developed in Bush et al. (2011) does not perform well in high stress periods.

We also suggest to add a contagion factor to the model as in Giesecke et al. (2012). This would reflect what is called in financial econometrics the *leverage effect*, that past negative returns of the market imply larger future volatility, as has been demonstrated in Bélisle and Egbewole (2012). Adding contagion and allowing for this leverage effect would put less strain on the volatility parameter σ, as this parameter clearly makes the model too stiff. Therefore, allowing more flexibility and better responsiveness to the market environment would hopefully allow better calibration results.

Chapter 5

Conclusion

In this work, we have presented the main concepts used to analyse credit risk and model its behaviour. New models that exhibit better market replication capability need to be developed. We have presented three such models, each following a key technique of reproduction.

The Lucas et al. (2000) model exhibiting a static single-period simplification of the market was used to study the tail behaviour of the loss distribution modelled. This is helpful when we are concerned with extreme events and large shocks. It is a key concept to take into account in risk management and hedging strategies. The authors suggested a form for the tail distribution, and we have noted that one ought to be careful when using this model. We demonstrated the limits of the tail index theorem.

The Bush et al. (2011) model extends the structural modelling theory to a dynamic setting. As Schönbucher (2003) mentioned, a good model should be dynamic, in order to take into account the distribution and time of defaults. We have presented the derivation of the main theorem, and have combined the theory developed by Lucas et al. (2000) with the dynamic model by developing a theorem about the moments of the loss distribution. This theorem enables us to study the tail behaviour of a structural diffusion dynamic model. This is of great importance as it places within reach the theory of extreme values in a dynamic setting.

The Giesecke et al. (2012) model was presented because of its reduced-form nature. Reduced-form models are very popular in practice, and the authors of this paper try to improve such model by adding some key component such as contagion and market systematic risk to the modelling of portfolio credit risk. Unfortunately, their implementation has not proven as promising as first presented, since the numerical implementation of their model is tedious due to the large number of parameters to calibrate. As Schönbucher (2003) mentioned, a good model must have a small number of parameters in order to produce better calibration results. As was shown, the simple homogeneous pool case of the model does not exhibit empirical evidence of clustering and contagion. However, the idea presented here can be used as a basis for improving the models in the future.

Finally, in Chapter 4, we have tested the numerical capabilities of the model developed by Bush et al. (2011). It has proven to be not as efficient as indicated by the authors, performing poorly in replicating the market data before and during the recent global financial crisis. Even if we have been able to reproduce the authors results, our numerical results have not been in favour of the model. When we tested their model through a time series of market data through the financial crisis, the model has not been able to reproduce consistently the market spreads, resulting in poor calibration, and counter intuitive results for the estimated parameter of σ. Indeed, the assumption that the volatility is unique and constant has been too restrictive for the model to survive the episodes of the financial crisis.

We have suggested an importance sampling method in order to get better calibration results for senior tranches, but in general, the modifications required to the model are more important. Indeed, we should add a jump factor to the diffusion in order to make the market more volatile. Also, adding a contagion factor that fades with time, as in Giesecke et al. (2012), would allow the model to exhibit what is called in financial econometrics the *leverage effect*. This could be of great value to increase the flexibility of the model without increasing the number of parameters in the model excessively and keeping the calibration relatively simple.

APPENDICES

Appendix A

Notation used for each model

Section 3.1 for Lucas et al. (2000)

$\mathbb{1}$: Indicator funtion

π : firm's credit loss

l : state of the firm, either active or defaulted

f : market risk factor

ϵ : idiosyncratic risk factor

Ω : Variance-covariance matrix of the market risk factor

ω : variance of the idiosyncratic risk factor

Φ : Standard Normal cumulative distribution function

ϕ : Standard Normal density function

Φ_α : Fréchet distribution with tail index α

Ψ_α : Weibull distribution with tail index α

Λ : Gumbel distribution

α : tail index

ρ : correlation of the surplus variable to the market risk factor

Section 3.2 for Bush et al. (2011)

Ω : sample space

M : Brownian motion

$\mathcal{F}$: filtration created by the Brownian motion M

$\mathcal{G}$: σ-algebra

$\mathscr{P}(A)$: power set of A

$\bar{\nu}$: empirical measure for the entire portfolio

ν: empirical measure for non-defaulted members of the portfolio

δ_x : Dirac delta function

σ : volatility of the portfolio

ρ : correlation to the market factor

ϕ : test function

$\mathcal{A}$: operator defined in (3.34)

$\mathcal{A}^\dagger$: adjoint operator of $\mathcal{A}$

ϵ : idiosyncratic risk factor, independent of market and other idiosyncratic factors.

Section 3.3 for Giesecke et al. (2012)

$\mathscr{P}(A)$: power set of A

$\mathcal{P}$: space of types, defined in (3.42)

π : empirical measure of the portoflio's types

$\Lambda_\circ$: empirical measure of the portfolio's initial distribution of intensities

$\mathcal{L}$: operators defined in (3.3)

$\mathcal{Q}$: operator defined in (3.45)

Appendix B

Regular Variation

How do we define a distribution as having "heavy tails"? As Mikosch (1999) mentions, the notion of heaviness of a distribution's tail is not unique and only makes sense as a qualitative characterization proper to the context of the model studied. Nonetheless, we can still classify the tails of the distributions and will attempt in the following section to give a brief introduction to understand what we will be working with for the remainder of this chapter. This section is based on the report of Mikosch (1999), as well as on the very comprehensive textbook from Embrechts et al. (1997), and occasionally on Bingham et al. (1987).

We study regularly varying functions before introducing the concept of heavy or thin tails because it gives a characterization of the function. Having an idea of a "regularly varying" function, we can thereafter emphasize the key aspects of faster or slower varying function. This will prove useful when studying heavy tail distributions. We thus start with this introduction with some definitions relating to regular variation.

Definition B.0.1 (Karamata)**.** A positive measurable function f is called *regularly varying (at infinity)* with index $\alpha \in \mathbb{R}$ if

- It is defined on some neighbourhood $[x_0, \infty)$ of infinity.
- $$\lim_{x \to \infty} \frac{f(tx)}{f(x)} = t^{\alpha} \quad \forall t > 0.$$

If $\alpha = 0$, f is said to be *slowly varying (at infinity)*.

Remark B.0.2 (Mikosch, 1999). We can easily observe that every regularly varying function f with index α can be represented as

$$f(x) = x^{\alpha} L(x),$$

for L a slowly varying function.

A useful theorem concerning slowly varying function that gives some intuitive meaning to the notion is the following representation theorem:

Theorem B.0.3 (Mikosch, 1999). *A positive measurable function L on $[x_0, \infty)$ is slowly varying* if and only if *it can be written in the form*

$$L(x) = c(x) \exp\left\{ \int_{x_0}^{x} \frac{\epsilon(y)}{y} \mathrm{d}y \right\} \tag{B.1}$$

where $c(\cdot)$ is a measurable non-negative function such that $\lim_{x\to\infty} c(x) = c_0 \in (0, \infty)$ and $\epsilon(x) \to 0$ as $x \to \infty$.

We can therefore see that if L is slowly varying, for every $\epsilon > 0$

$$x^{-\epsilon} L(x) \to 0 \quad \text{and} \quad x^{\epsilon} L(x) \to \infty \quad \text{as } x \to \infty.$$

It is now useful to translate this notion of regularly varying function to probability theory by defining a regularly varying random variable.

Definition B.0.4 (Mikosch, 1999). A non-negative random variable X and its distribution are said to be *regularly varying with index* $\alpha \geq 0$ if the right distribution tail $\overline{F}_X(x) = 1 - F_X(x) = \mathbb{P}(X > x)$ is regularly varying with index $-\alpha$.

Once the concept of regularly varying r.v. is defined, we want to study the extreme value distributions of these r.v.'s. We will therefore consider a sequence of independent random variables X, X_1, X_2, X_3, ... with common distribution F, and their partial maxima sequence

$$M_1 = X_1, \quad M_n = \max_{i=1,\dots,n} X_i, \quad n \geq 2. \tag{B.2}$$

We can now define the only possible limit laws of the standardized maxima $c_n^{-1}(M_n - d_n), \quad n \geq 1$ as *max-stable distributions.*

Definition B.0.5. A non-degenerate random variable X and its distribution are called *max-stable* if they satisfy the relation

$$M_n \stackrel{d}{=} c_n X + d_n, \quad n \geq 2, \tag{B.3}$$

for i.i.d. X, X_1, X_2, X_3, ..., and appropriate constants $c_n > 0$, $d_n \in \mathbb{R}$.

Mikosch (1999) then concludes by stating an important relationship between max-stable distributions and limit distribution for maxima. He says that every max-stable distribution is a limit distribution for maxima of i.i.d. random variable, and that max-stable distributions are the *only* limit laws for normalized maxima.

Appendix C

Mathematical setting of the models

C.1 Structural model

This definition is due to Dahlquist (1963).

Definition C.1.1 ($\mathcal{A}$-stable)**.** A k-step method is called $\mathcal{A}$-stable if all solutions of the formula

$$\alpha_k x_{n+k} + \alpha_{k-1} x_{n+k-1} + \ldots + \alpha_0 x_n = h(\beta_k f_{n+k} + \ldots + \beta_0 f_n)$$

for the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t, x),\ x(0) = x_0,\ (x \in \mathbb{R}^s,\ t \geq 0)$$

tend to zero as $n \to \infty$ when the method is applied with fixed positiev h to any differential equation of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = qx$$

where q is a complex constant with negative real part.

C.2 Reduced-form model

The following definition is due to Doob (1953).

Definition C.2.1. A Doob-Meyer compensator A_n of an adapted stochastic process X_n with $\mathbb{E}[|X_n|] < \infty$ is an integrable process that allows the decomposition of a sub- (super-) martingale into a martingale M_n and the compensator A_n. It is such that

$$X_n = M_n + A_n.$$

Also, a sub- (super-)martingale can be compensated by A_n in order to make the result a martingale, such as

$$X_n - A_n = M_n.$$

Appendix D

Tables for calibrated tranche spreads and relative errors

The following tables present the results of our modelling of the Bush et al. (2011) model presented in Chapter 3, Section 3.2. We present the calibrated parameters through time for a time series from June 2007 to December 2008. The dates were chosen to capture the drastic change in the financial environment, mid-2007 being relatively calm and the end of 2008 being the height of the financial crisis. We present our computed spreads as well as the standard error of the simulations, noted by SE. For comparison, we add the market data for the studied dates, as well as the relative error between our computations and the market data. The relative error is represented by RE and is in percentage points. Note that all the tranches are quoted in basis points for the spread except the first 0%-3%, quoted in percentage of upfront.

5 years	2007-06-29		2007-07-31		2007-08-31	
Risk-free rate	$r = 4.43\%$		$r = 4.28\%$		$r = 4.12\%$	
Calibrated σ	0.15		0.35		0.15	
Calibrated ρ	0.20		0.50		0.30	
Tranches	Computed	SE		SE		SE
0%-3%	9.89	0.46	8.92	0.47	19.50	0.48
3%-6%	78.82	5.02	249.82	7.02	239.77	6.80
6%-9%	15.93	2.03	137.82	6.77	93.90	5.00
9%-12%	4.01	1.16	84.70	4.96	42.35	3.19
12%-22%	0.53	0.34	36.69	2.61	11.12	1.87
22%-100%	0.00	0.01	1.65	0.35	0.14	0.06
Index	37.92	0.72	63.75	1.39	62.59	1.26
Tranches	Market					
0%-3%	8.46		19.28		21.18	
3%-6%	47.92		139.42		110.66	
6%-9%	13.55		385.60		46.16	
9%-12%	5.92		166.12		28.65	
12%-22%	2.45		15.23		17.15	
22%-100%	N/A		1.44		10.25	
Index	22.74		47.20		41.53	
Tranches	RE in %					
0%-3%	16.868		-53.727		-7.929	
3%-6%	64.501		79.192		116.665	
6%-9%	17.517		-64.257		103.415	
9%-12%	-32.258		-49.011		47.789	
12%-22%	-78.178		140.926		-35.166	
22%-100%			14.547		-98.662	
Index	66.778		35.063		50.716	
Average RE	7.890		14.676		25.261	

Table D.1: Time Series of spreads computed and compared to actual market spreads for a CDO iTraxx Europe Main series 6 with maturity of 5 years

5 years	2007-09-28		2007-11-30		2007-12-31	
Risk-free rate	$r = 4.15\%$		$r = 3.91\%$		$r = 4.11\%$	
Calibrated σ	0.15		0.1		0.15	
Calibrated ρ	0.20		0.4		0.80	
Tranches	Computed	SE		SE		SE
0%-3%	20.63	0.62	20.25	0.57	-3.50	0.45
3%-6%	162.36	5.50	322.36	12.33	205.88	7.75
6%-9%	42.02	2.68	162.72	8.31	148.19	6.50
9%-12%	12.44	2.18	91.58	6.06	114.38	5.30
12%-22%	1.70	0.66	34.02	2.60	74.49	5.20
22%-100%	0.01	0.02	1.02	0.21	9.72	1.21
Index	54.31	0.91	77.28	1.97	67.78	3.10
Tranches	Market					
0%-3%	15.73		22.74		18.62	
3%-6%	60.02		126.06		117.45	
6%-9%	24.01		65.58		71.28	
9%-12%	14.30		51.67		45.31	
12%-22%	8.89		30.12		24.13	
22%-100%	5.11		13.63		10.69	
Index	34.46		49.63		48.38	
Tranches	RE in %					
0%-3%	31.147		-10.937		-118.776	
3%-6%	170.530		155.728		75.289	
6%-9%	75.058		148.111		107.914	
9%-12%	-13.006		77.256		152.429	
12%-22%	-80.859		12.948		208.774	
22%-100%	-99.801		-92.512		-9.057	
Index	57.614		55.719		40.119	
Average RE	20.098		49.473		65.242	

Table D.2: Time Series of spreads computed and compared to actual market spreads for a CDO iTraxx Europe Main series 6 with maturity of 5 years

5 years	2008-01-31		2008-02-29		2008-03-31	
Risk-free rate	$r = 3.59\%$		$r = 3.43\%$		$r = 3.70\%$	
Calibrated σ	0.10		0.20		0.15	
Calibrated ρ	0.60		0.80		0.70	
Tranches	Computed	SE		SE		SE
0%-3%	18.16	0.60	13.48	0.49	23.28	0.71
3%-6%	418.67	9.84	456.00	11.21	552.93	17.92
6%-9%	266.44	9.56	341.31	11.92	385.73	15.73
9%-12%	185.66	7.77	273.53	10.57	291.33	12.52
12%-22%	99.09	6.39	187.93	7.40	181.36	9.25
22%-100%	7.57	0.86	28.28	1.48	20.13	1.58
Index	108.38	2.74	156.37	4.34	160.68	5.70
Tranches	Market					
0%-3%	28.22		33.88		35.71	
3%-6%	273.91		437.03		409.98	
6%-9%	184.92		299.20		256.80	
9%-12%	123.79		209.05		176.73	
12%-22%	60.44		107.76		86.80	
22%-100%	18.00		46.00		40.00	
Index	77.08		120.50		112.00	
Tranches	RE in %					
0%-3%	-35.664		-60.201		-34.797	
3%-6%	52.850		4.339		34.869	
6%-9%	44.084		14.072		50.205	
9%-12%	49.985		30.846		64.852	
12%-22%	63.946		74.393		108.937	
22%-100%	-57.929		-38.523		-49.670	
Index	40.607		29.769		43.460	
Average RE	22.554		7.814		31.122	

Table D.3: Time Series of spreads computed and compared to actual market spreads for a CDO iTraxx Europe Main series 6 with maturity of 5 years

5 years	2008-04-30		2008-05-30		2008-06-30	
Risk-free rate	$r = 3.95\%$		$r = 4.27\%$		$r = 4.63\%$	
Calibrated σ	0.10		0.15		0.15	
Calibrated ρ	0.50		0.60		0.60	
Tranches	Computed	SE		SE		SE
0%-3%	27.53	0.63	20.15	0.55	27.45	0.78
3%-6%	464.26	13.77	426.81	11.12	554.24	17.21
6%-9%	262.97	9.86	266.05	9.65	352.91	13.02
9%-12%	166.59	6.48	183.09	9.90	246.80	9.48
12%-22%	74.90	5.01	96.89	6.12	133.75	6.37
22%-100%	3.74	0.57	6.95	0.82	10.16	0.91
Index	109.02	2.65	109.86	2.80	140.15	3.84
Tranches	Market					
0%-3%	27.04		28.52		28.63	
3%-6%	248.85		238.91		354.75	
6%-9%	152.93		147.12		218.22	
9%-12%	91.28		89.09		133.14	
12%-22%	30.43		39.18		60.40	
22%-100%	24.80		17.67		37.20	
Index	64.72		70.12		94.00	
Tranches	RE in %					
0%-3%	1.826		-29.371		-4.109	
3%-6%	86.563		78.653		56.233	
6%-9%	71.958		80.835		61.724	
9%-12%	82.519		105.519		85.375	
12%-22%	146.185		147.318		121.433	
22%-100%	-84.911		-60.668		-72.692	
Index	68.450		56.686		49.095	
Average RE	53.227		54.139		42.437	

Table D.4: Time Series of spreads computed and compared to actual market spreads for a CDO iTraxx Europe Main series 6 with maturity of 5 years

5 years	2008-07-31		2008-08-29		2008-09-30	
Risk-free rate	$r = 4.31\%$		$r = 4.11\%$		$r = 3.88\%$	
Calibrated σ	0.10		0.15		0.15	
Calibrated ρ	0.70		0.60		0.60	
Tranches	Computed	SE		SE		SE
0%-3%	20.21	0.63	28.25	0.67	33.95	0.68
3%-6%	466.37	14.23	568.64	15.36	697.51	15.12
6%-9%	305.84	9.73	360.31	11.76	457.55	11.25
9%-12%	220.39	8.91	250.75	9.31	325.45	9.42
12%-22%	129.21	6.52	134.43	6.87	179.59	7.15
22%-100%	12.10	1.04	9.92	0.94	14.15	1.19
Index	126.65	3.99	141.66	3.95	173.64	3.66
Tranches	Market					
0%-3%	25.24		32.69		37.46	
3%-6%	302.03		383.85		563.67	
6%-9%	172.27		224.35		295.00	
9%-12%	111.66		133.95		157.33	
12%-22%	55.64		55.85		67.25	
22%-100%	26.85		31.30		31.70	
Index	84.71		94.83		110.25	
Tranches	RE in %					
0%-3%	-19.940		-13.564		-9.357	
3%-6%	54.413		48.142		23.745	
6%-9%	77.537		60.601		55.103	
9%-12%	97.372		87.200		106.857	
12%-22%	132.217		140.701		167.052	
22%-100%	-54.953		-68.310		-55.377	
Index	49.519		49.379		57.497	
Average RE	48.024		43.450		49.360	

Table D.5: Time Series of spreads computed and compared to actual market spreads for a CDO iTraxx Europe Main series 6 with maturity of 5 years

5 years	2008-10-31		2008-11-28		2008-12-31	
Risk-free rate	$r = 3.58\%$		$r = 3.16\%$		$r = 2.95\%$	
Calibrated σ	0.10		0.10		0.1	
Calibrated ρ	0.40		0.80		0.9	
Tranches	Computed	SE		SE		SE
0%-3%	66.41	0.31	44.76	0.56	39.23	0.71
3%-6%	1354.65	21.44	980.66	20.84	936.97	28.96
6%-9%	772.13	18.11	675.08	16.92	679.61	22.96
9%-12%	488.49	14.69	510.60	14.67	533.15	19.29
12%-22%	213.94	8.55	322.08	10.77	357.10	15.48
22%-100%	8.56	0.86	40.65	2.54	56.79	3.29
Index	245.70	3.80	270.30	6.37	292.77	9.77
Tranches	Market					
0%-3%	55.96		55.96		37.70	
3%-6%	1083.00		1083.00		850.15	
6%-9%	567.00		567.00		397.55	
9%-12%	257.83		257.83		229.14	
12%-22%	83.70		83.70		75.46	
22%-100%	42.60		42.60		42.60	
Index	180.17		228.67		246.87	
Tranches	RE in %					
0%-3%	18.683		-20.008		4.053	
3%-6%	25.083		-9.450		10.213	
6%-9%	36.178		19.062		70.950	
9%-12%	89.458		98.036		132.677	
12%-22%	155.598		284.798		373.223	
22%-100%	-79.910		-4.575		33.299	
Index	36.376		18.205		18.592	
Average RE	40.210		55.152		91.858	

Table D.6: Time Series of spreads computed and compared to actual market spreads for a CDO iTraxx Europe Main series 6 with maturity of 5 years

7 years	2007-06-29		2007-07-31		2007-08-31	
Risk-free rate	$r = 4.43\%$		$r = 4.28\%$		$r = 4.12\%$	
Calibrated σ	0.15		0.35		0.15	
Calibrated ρ	0.20		0.50		0.30	
Tranches	Computed	SE		SE		SE
0%-3%	19.91	0.53	20.30	0.51	28.45	0.48
3%-6%	170.53	6.44	414.66	7.90	355.71	7.68
6%-9%	48.50	3.92	262.86	7.03	164.82	5.18
9%-12%	15.69	1.82	179.44	5.83	84.90	4.20
12%-22%	2.42	0.59	91.85	4.80	26.78	2.59
22%-100%	0.01	0.02	5.72	0.63	0.50	0.11
Index	48.45	0.86	101.34	2.23	76.42	1.31
Tranches	Market					
0%-3%	22.93		27.06		33.09	
3%-6%	115.63		199.81		197.17	
6%-9%	33.02		176.21		95.40	
9%-12%	14.12		90.10		56.78	
12%-22%	6.51		177.18		31.10	
22%-100%	N/A		2.39		17.38	
Index	32.36		59.43		51.50	
Tranches	RE in %					
0%-3%	-13.133		-25.007		-14.031	
3%-6%	47.484		107.527		80.415	
6%-9%	46.859		49.175		72.766	
9%-12%	11.123		99.156		49.523	
12%-22%	-62.860		-48.162		-13.892	
22%-100%			139.755		-97.139	
Index	49.732		70.529		48.384	
Average RE	11.315		56.139		18.004	

Table D.7: Time Series of spreads computed and compared to actual market spreads for a CDO iTraxx Europe Main series 6 with maturity of 7 years

7 years	2007-09-28		2007-11-30		2007-12-31	
Risk-free rate	$r = 4.15\%$		$r = 3.91\%$		$r = 4.11\%$	
Calibrated σ	0.15		0.1		0.15	
Calibrated ρ	0.20		0.4		0.80	
Tranches	Computed	SE		SE		SE
0%-3%	31.95	0.58	24.15	0.62	-3.15	0.52
3%-6%	299.56	9.11	358.95	9.56	239.81	7.70
6%-9%	103.73	3.74	193.47	7.75	179.22	6.27
9%-12%	39.39	2.26	115.10	5.92	142.28	4.83
12%-22%	7.19	1.07	46.35	2.67	97.93	4.72
22%-100%	0.04	0.03	1.65	0.25	14.43	1.22
Index	67.48	0.96	80.11	1.76	82.49	3.07
Tranches	Market					
0%-3%	28.69		31.50		28.82	
3%-6%	127.72		204.11		198.17	
6%-9%	62.08		105.66		107.43	
9%-12%	34.91		74.49		70.46	
12%-22%	21.42		42.88		39.05	
22%-100%	9.23		18.04		15.18	
Index	44.68		57.10		54.88	
Tranches	RE in %					
0%-3%	11.368		-23.327		-110.934	
3%-6%	134.543		75.862		21.017	
6%-9%	67.082		83.101		66.824	
9%-12%	12.838		54.523		101.933	
12%-22%	-66.428		8.076		150.823	
22%-100%	-99.521		-90.838		-4.954	
Index	51.030		40.302		50.318	
Average RE	15.844		21.100		39.290	

Table D.8: Time Series of spreads computed and compared to actual market spreads for a CDO iTraxx Europe Main series 6 with maturity of 7 years

7 years	2008-01-31		2008-02-29		2008-03-31	
Risk-free rate	$r = 3.59\%$		$r = 3.43\%$		$r = 3.70\%$	
Calibrated σ	0.10		0.20		0.15	
Calibrated ρ	0.60		0.80		0.70	
Tranches	Computed	SE		SE		SE
0%-3%	20.40	0.71	17.54	0.50	26.84	0.70
3%-6%	428.43	8.45	507.17	8.44	576.12	14.04
6%-9%	282.32	8.27	394.74	9.80	415.33	13.29
9%-12%	201.95	7.61	324.93	8.99	323.64	10.65
12%-22%	113.82	5.24	232.67	6.80	210.62	7.77
22%-100%	9.44	0.78	38.82	1.52	25.98	1.42
Index	110.08	2.55	182.97	4.03	171.36	4.83
Tranches	Market					
0%-3%	36.92		40.91		44.36	
3%-6%	383.05		498.83		531.83	
6%-9%	249.79		326.96		319.38	
9%-12%	164.82		246.07		218.93	
12%-22%	73.95		125.37		102.64	
22%-100%	21.70		51.20		47.20	
Index	84.76		133.50		121.78	
Tranches	RE in %					
0%-3%	-44.739		-57.124		-39.507	
3%-6%	11.845		1.673		8.329	
6%-9%	13.025		20.731		30.044	
9%-12%	22.531		32.048		47.834	
12%-22%	53.916		85.591		105.195	
22%-100%	-56.482		-24.177		-44.947	
Index	29.880		37.053		40.709	
Average RE	4.282		13.685		21.094	

Table D.9: Time Series of spreads computed and compared to actual market spreads for a CDO iTraxx Europe Main series 6 with maturity of 7 years

7 years	2008-04-30		2008-05-30		2008-06-30	
Risk-free rate	$r = 3.95\%$		$r = 4.27\%$		$r = 4.63\%$	
Calibrated σ	0.10		0.15		0.15	
Calibrated ρ	0.50		0.60		0.60	
Tranches	Computed	SE		SE		SE
0%-3%	30.07	0.70	23.88	0.62	30.72	0.84
3%-6%	468.97	11.42	466.43	11.71	572.24	15.91
6%-9%	277.15	9.47	304.76	7.63	379.14	11.17
9%-12%	182.00	6.87	217.86	7.36	273.01	8.26
12%-22%	87.63	4.22	123.08	5.87	156.07	6.15
22%-100%	4.83	0.58	10.41	0.77	13.50	0.90
Index	106.50	2.53	119.51	2.61	144.18	3.51
Tranches	Market					
0%-3%	38.83		40.81		40.04	
3%-6%	385.08		381.39		491.61	
6%-9%	220.85		221.98		297.12	
9%-12%	144.45		142.99		193.28	
12%-22%	60.07		68.00		91.78	
22%-100%	30.60		24.82		46.20	
Index	81.02		86.28		110.33	
Tranches	RE in %					
0%-3%	-22.559		-41.474		-23.277	
3%-6%	21.788		22.297		16.403	
6%-9%	25.492		37.290		27.606	
9%-12%	25.996		52.358		41.254	
12%-22%	45.883		80.997		70.052	
22%-100%	-84.222		-58.040		-70.782	
Index	31.450		38.509		30.681	
Average RE	6.261		18.848		13.134	

Table D.10: Time Series of spreads computed and compared to actual market spreads for a CDO iTraxx Europe Main series 6 with maturity of 7 years

7 years	2008-07-31		2008-08-29		2008-09-30	
Risk-free rate	$r = 4.31\%$		$r = 4.11\%$		$r = 3.88\%$	
Calibrated σ	0.10		0.15		0.15	
Calibrated ρ	0.70		0.60		0.60	
Tranches	Computed	SE		SE		SE
0%-3%	20.07	0.70	32.13	0.68	38.15	0.80
3%-6%	433.99	12.26	595.96	12.78	718.28	16.24
6%-9%	291.87	9.03	396.60	9.09	491.22	13.71
9%-12%	215.10	7.88	287.19	8.40	362.66	10.67
12%-22%	129.63	6.15	163.95	5.52	212.04	8.41
22%-100%	13.11	0.99	14.04	0.86	19.10	1.35
Index	118.32	3.58	149.35	3.20	180.45	4.42
Tranches	Market					
0%-3%	36.50		44.37		47.67	
3%-6%	416.56		522.35		683.00	
6%-9%	235.92		306.05		373.00	
9%-12%	154.55		189.85		214.67	
12%-22%	86.48		88.35		101.00	
22%-100%	34.60		42.20		44.70	
Index	97.08		110.25		124.83	
Tranches	RE in %					
0%-3%	-45.013		-27.575		-19.976	
3%-6%	4.184		14.091		5.166	
6%-9%	23.718		29.587		31.693	
9%-12%	39.176		51.272		68.940	
12%-22%	49.884		85.574		109.937	
22%-100%	-62.115		-66.735		-57.278	
Index	21.874		35.463		44.552	
Average RE	4.530		17.382		26.148	

Table D.11: Time Series of spreads computed and compared to actual market spreads for a CDO iTraxx Europe Main series 6 with maturity of 7 years

7 years	2008-10-31		2008-11-28		2008-12-31	
Risk-free rate	$r = 3.58\%$		$r = 3.16\%$		$r = 2.95\%$	
Calibrated σ	0.10		0.10		0.1	
Calibrated ρ	0.40		0.80		0.9	
Tranches	Computed	SE		SE		SE
0%-3%	68.83	0.29	45.48	0.59	39.21	0.71
3%-6%	1296.62	20.05	880.59	19.02	827.96	23.12
6%-9%	758.09	14.76	618.24	13.45	610.10	19.74
9%-12%	491.30	12.36	475.26	10.80	486.96	16.81
12%-22%	228.12	7.32	309.12	8.25	338.24	13.87
22%-100%	10.72	0.90	43.00	2.09	58.78	2.82
Index	224.20	3.40	248.25	5.18	269.76	8.65
Tranches	Market					
0%-3%	64.21		64.21		47.27	
3%-6%	1140.00		1140.00		840.01	
6%-9%	627.33		627.33		427.64	
9%-12%	324.50		324.50		257.90	
12%-22%	115.35		115.35		91.47	
22%-100%	53.20		53.20		53.20	
Index	173.33		202.00		215.92	
Tranches	RE in %					
0%-3%	7.199		-29.163		-17.051	
3%-6%	13.739		-22.755		-1.434	
6%-9%	20.843		-1.449		42.666	
9%-12%	51.403		46.459		88.821	
12%-22%	97.761		167.986		269.775	
22%-100%	-79.846		-19.174		10.494	
Index	29.349		22.898		24.939	
Average RE	20.064		23.543		59.744	

Table D.12: Time Series of spreads computed and compared to actual market spreads for a CDO iTraxx Europe Main series 6 with maturity of 7 years

10 years	2007-06-29		2007-07-31		2007-08-31	
Risk-free rate	$r = 4.43\%$		$r = 4.28\%$		$r = 4.12\%$	
Calibrated σ	0.15		0.35		0.15	
Calibrated ρ	0.20		0.50		0.30	
Tranches	Computed	SE		SE		SE
0%-3%	27.34	0.52	29.59	0.50	34.32	0.43
3%-6%	257.38	6.70	545.63	9.40	426.88	7.79
6%-9%	93.19	3.89	379.85	7.61	222.46	6.01
9%-12%	36.91	2.83	281.06	6.41	125.17	4.15
12%-22%	7.20	0.94	164.11	4.80	45.23	2.69
22%-100%	0.05	0.02	13.63	0.77	1.15	0.17
Index	54.50	0.89	137.67	2.58	82.73	1.25
Tranches	Market					
0%-3%	39.56		39.00		45.20	
3%-6%	337.43		504.28		427.83	
6%-9%	97.82		2564.53		187.61	
9%-12%	45.42		656.38		97.42	
12%-22%	14.97		145.67		54.14	
22%-100%	N/A		4.40		21.38	
Index	45.91		73.23		66.33	
Tranches	RE in %					
0%-3%	-30.883		-24.143		-24.065	
3%-6%	-23.724		8.199		-0.221	
6%-9%	-4.734		-85.188		18.578	
9%-12%	-18.737		-57.180		28.490	
12%-22%	-51.883		12.662		-16.449	
22%-100%			210.150		-94.627	
Index	18.720		87.998		24.720	
Average RE	-15.892		21.785		-9.082	

Table D.13: Time Series of spreads computed and compared to actual market spreads for a CDO iTraxx Europe Main series 6 with maturity of 10 years

10 years	2007-09-28		2007-11-30		2007-12-31	
Risk-free rate	$r = 4.15\%$		$r = 3.91\%$		$r = 4.11\%$	
Calibrated σ	0.15		0.1		0.15	
Calibrated ρ	0.20		0.4		0.80	
Tranches	Computed	SE		SE		SE
0%-3%	39.37	0.53	25.50	0.65	-4.50	0.56
3%-6%	407.16	9.47	357.09	7.84	248.78	6.26
6%-9%	172.49	5.63	199.14	5.94	190.78	4.93
9%-12%	77.16	3.22	122.73	5.28	154.47	3.85
12%-22%	18.05	1.11	51.99	2.77	109.09	3.64
22%-100%	0.16	0.06	2.10	0.25	17.46	0.99
Index	74.20	0.99	75.48	1.49	88.35	2.53
Tranches						
0%-3%	40.79		40.42		38.17	
3%-6%	357.04		380.54		370.25	
6%-9%	136.13		163.91		163.92	
9%-12%	73.37		100.56		91.79	
12%-22%	39.73		58.24		50.46	
22%-100%	13.28		22.28		19.20	
Index	60.33		66.92		64.75	
Tranches	RE in %					
0%-3%	-3.479		-36.901		-111.795	
3%-6%	14.037		-6.163		-32.808	
6%-9%	26.713		21.493		16.388	
9%-12%	5.164		22.045		68.289	
12%-22%	-54.558		-10.726		116.207	
22%-100%	-98.760		-90.569		-9.071	
Index	22.992		12.804		36.450	
Average RE	-12.556		-12.574		11.952	

Table D.14: Time Series of spreads computed and compared to actual market spreads for a CDO iTraxx Europe Main series 6 with maturity of 10 years

10 years	2008-01-31		2008-02-29		2008-03-31	
Risk-free rate	$r = 3.59\%$		$r = 3.43\%$		$r = 3.70\%$	
Calibrated σ	0.10		0.20		0.15	
Calibrated ρ	0.60		0.80		0.70	
Tranches	Computed	SE		SE		SE
0%-3%	20.40	0.71	20.02	0.52	28.53	0.79
3%-6%	404.24	8.29	518.09	7.87	561.70	12.16
6%-9%	271.36	7.51	412.12	8.48	413.81	11.05
9%-12%	197.32	6.58	346.60	8.31	327.54	9.17
12%-22%	114.75	5.53	256.59	7.32	219.52	6.33
22%-100%	10.29	0.72	46.33	1.68	29.11	1.11
Index	102.83	2.53	195.05	4.44	168.89	3.70
Tranches	Market					
0%-3%	43.10		47.09		52.01	
3%-6%	478.17		591.31		676.83	
6%-9%	301.50		366.35		383.03	
9%-12%	204.04		282.46		249.90	
12%-22%	89.17		150.09		133.30	
22%-100%	26.18		56.30		52.80	
Index	91.33		140.75		127.59	
Tranches	RE in %					
0%-3%	-52.677		-57.492		-45.141	
3%-6%	-15.460		-12.382		-17.009	
6%-9%	-9.996		12.493		8.038	
9%-12%	-3.290		22.706		31.067	
12%-22%	28.687		70.961		64.683	
22%-100%	-60.673		-17.715		-44.876	
Index	12.601		38.577		32.374	
Average RE	-14.401		8.164		4.162	

Table D.15: Time Series of spreads computed and compared to actual market spreads for a CDO iTraxx Europe Main series 6 with maturity of 10 years

10 years	2008-04-30		2008-05-30		2008-06-30	
Risk-free rate	$r = 3.95\%$		$r = 4.27\%$		$r = 4.63\%$	
Calibrated σ	0.10		0.15		0.15	
Calibrated ρ	0.50		0.60		0.60	
Tranches	Computed	SE		SE		SE
0%-3%	30.34	0.79	25.56	0.60	32.06	0.78
3%-6%	438.34	10.02	464.60	10.73	550.83	13.30
6%-9%	265.65	8.17	314.09	7.50	373.74	10.54
9%-12%	178.12	6.50	231.44	6.05	274.89	7.76
12%-22%	88.70	3.47	136.46	4.94	162.36	5.19
22%-100%	5.30	0.55	12.82	0.64	15.38	0.78
Index	96.47	2.24	119.17	2.24	137.90	2.98
Tranches	Market					
0%-3%	46.58		48.91		47.49	
3%-6%	515.48		518.08		634.72	
6%-9%	288.93		288.33		369.25	
9%-12%	179.88		180.66		229.00	
12%-22%	82.85		87.27		124.75	
22%-100%	38.20		30.21		48.30	
Index	85.02		89.95		114.25	
Tranches	RE in %					
0%-3%	-34.854		-47.744		-32.506	
3%-6%	-14.963		-10.321		-13.216	
6%-9%	-8.056		8.932		1.217	
9%-12%	-0.975		28.108		20.040	
12%-22%	7.062		56.376		30.149	
22%-100%	-86.122		-57.573		-68.150	
Index	13.474		32.484		20.701	
Average RE	-17.776		1.466		-5.966	

Table D.16: Time Series of spreads computed and compared to actual market spreads for a CDO iTraxx Europe Main series 6 with maturity of 10 years

10 years	2008-07-31		2008-08-29		2008-09-30	
Risk-free rate	$r = 4.31\%$		$r = 4.11\%$		$r = 3.88\%$	
Calibrated σ	0.10		0.15		0.15	
Calibrated ρ	0.70		0.60		0.60	
Tranches	Computed	SE		SE		SE
0%-3%	17.92	0.75	34.11	0.74	40.34	0.89
3%-6%	382.69	10.41	581.60	10.61	694.66	15.35
6%-9%	260.20	7.82	398.60	7.83	485.75	11.51
9%-12%	193.36	7.17	296.25	7.00	367.12	10.41
12%-22%	118.44	5.52	177.35	5.29	223.70	7.28
22%-100%	12.62	0.87	16.91	0.68	22.33	1.27
Index	103.64	3.15	146.30	2.63	174.73	4.02
Tranches	Market					
0%-3%	43.29		52.80		55.17	
3%-6%	521.82		654.30		820.00	
6%-9%	291.80		376.00		444.00	
9%-12%	176.45		216.45		251.33	
12%-22%	86.83		109.95		119.83	
22%-100%	40.30		46.25		48.20	
Index	103.09		112.33		128.08	
Tranches	RE in %					
0%-3%	-58.616		-35.404		-26.872	
3%-6%	-26.661		-11.112		-15.285	
6%-9%	-10.828		6.010		9.402	
9%-12%	9.584		36.865		46.069	
12%-22%	36.411		61.300		86.681	
22%-100%	-68.688		-63.429		-53.679	
Index	0.527		30.237		36.420	
Average RE	-16.896		3.495		11.819	

Table D.17: Time Series of spreads computed and compared to actual market spreads for a CDO iTraxx Europe Main series 6 with maturity of 10 years

10 years	2008-10-31		2008-11-28		2008-12-31	
Risk-free rate	$r = 3.58\%$		$r = 3.16\%$		$r = 2.95\%$	
Calibrated σ	0.10		0.10		0.1	
Calibrated ρ	0.40		0.80		0.9	
Tranches	Computed	SE		SE		SE
0%-3%	69.93	0.31	44.76	0.60	37.61	0.75
3%-6%	1194.84	18.65	766.83	14.99	707.61	18.60
6%-9%	702.56	13.95	544.05	11.11	528.17	15.10
9%-12%	459.59	10.68	421.70	9.12	426.09	12.71
12%-22%	220.20	6.41	280.18	6.79	302.69	11.05
22%-100%	11.39	0.75	40.87	1.80	55.26	2.30
Index	194.38	2.91	217.25	4.52	236.68	6.94
Tranches	Market					
0%-3%	69.13		69.13		56.97	
3%-6%	1181.00		1181.00		882.56	
6%-9%	673.67		673.67		472.73	
9%-12%	362.00		362.00		293.03	
12%-22%	145.97		145.97		109.47	
22%-100%	58.40		58.40		58.40	
Index	163.83		180.50		190.25	
Tranches	RE in %					
0%-3%	1.162		-35.254		-33.984	
3%-6%	1.172		-35.069		-19.823	
6%-9%	4.289		-19.240		11.727	
9%-12%	26.957		16.492		45.407	
12%-22%	50.853		91.949		176.494	
22%-100%	-80.502		-30.012		-5.378	
Index	18.644		20.362		24.404	
Average RE	3.225		1.318		28.407	

Table D.18: Time Series of spreads computed and compared to actual market spreads for a CDO iTraxx Europe Main series 6 with maturity of 10 years

Tranches	Market data	$16,384$ simulations	$262,144$ simulations
0%-3%	37.70%	39.23%	39.39%
3%-6%	850.15	936.97	937.42
6%-9%	397.55	679.61	679.47
9%-12%	229.14	533.15	533.55
12%-22%	75.46	357.10	356.11
22%-100%	42.60	56.79	56.83
Index	246.87	292.77	292.89

Table D.19: Comparison of the calibrated spreads for 2008-12-31 calculated with 4^7 or 4^9 simulations. We observe no significant difference between the calculated spreads, but both are quite far from the market data.

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