ON 2-CROSSING-CRITICAL GRAPHS WITH A V_8-MINOR

by

Alan Marcelo Arroyo Guevara

A thesis
presented to the University of Waterloo
in fulfilment of the
thesis requirement for the degree of
Master of Mathematics
in
Combinatorics and Optimization

Waterloo, Ontario, Canada, 2014

AUTOR'S DECLARATION

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Alan Marcelo Arroyo Guevara

ABSTRACT

The crossing number of a graph is the minimum number of pairwise edge crossings in a drawing of a graph. A graph G is k-crossing-critical if it has crossing number at least k, and any subgraph of G has crossing number less than k. A consequence of Kuratowski's theorem is that 1-critical graphs are subdivisions of $K_{3,3}$ and K_5. The graph V_{2n} is a $2n$-cycle with n diameters. Bokal, Oporowski, Richter and Salazar found in [6] all the critical graphs except the ones that contain a V_8 minor and no V_{10} minor.

We show that a 4-connected graph G has crossing number at least 2 if and only if for each pair of disjoint edges there are two disjoint cycles containing them. Using a generalization of this result we found limitations for the 2-crossing-critical graphs remaining to classify. We showed that peripherally 4-connected 2-crossing-critical graphs have at most 4001 vertices. Furthermore, most 3-connected 2-crossing-critical graphs are obtainable by small modifications of the peripherally 4-connected ones.

ACKNOWLEDGEMENTS

I wish to acknowledge the support provided by Bruce Richter for his incredible guidance during this work. His enthusiastic assistance, patience, and encouragement were vital to make this thesis a joyful experience.

I would like to express thanks to the University of Waterloo for their financial support that let me pursue my research.

Thanks to Bertrand Guenin and Jim Geelen for showing a genuine interest in this work and for their valuable comments.

I would also like to express gratitude to my undergraduate supervisor Hortensia Galeana who introduced me to Graph Theory.

Finally, I would like to thank all my family members and my friends who have been constant support during these years. One day I hope to give back as much as you have given to me.

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1 INTRODUCTION

An enigmatic relation exists between the drawing of a graph, and its combinatorial properties. Kuratowski's Theorem is a prime example of this, characterizing planar graphs as those that do not contain either a $K_{3,3}$ or a K_5 subdivision. Many questions and important results in Graph Theory have been inspired from Kuratowski's Theorem.

The *crossing number* $cr(G)$ of a graph G, is the minimum number of pairwise crossings of edges in a drawing of G in the plane. For an integer k, a graph G is k-*crossing-critical* if $cr(G) \geqslant k$ and $cr(G') < k$ for any subgraph G' of G. Notice that a graph is planar if and only if it does not contain a 1-critical graph. Thus, Kuratowski's Theorem can be thought as a description of the set of 1-critical graphs.

The previous discussion lead us to an important question:

> *Given positive integer* k*, it possible to determine all* k*-crossing-critical graphs? And, more generally, it is possible to give a combinatorial characterization of graphs having crossing number* $k-1$*?*

A complete characterization of 2-crossing-critical is currently not known. Bloom, Kennedy and Quintas were the first to exhibit 21 such graphs [5]. Širáň discovered in [11] an infinite family of 2-crossing-critical graphs. More infinite families of these graphs were shown later by Kochol [7]. In [9], Richter proved that there are exactly 8 cubic 2-crossing-critical graphs.

A more recent attempt of characterizing 2-crossing-critical is close to completely solving this problem. The Möbius ladder V_{2n} is a graph consisting of a $2n$-cycle $(v_0, v_1, ..., v_{2n-1}, v_0)$ together with the n chords $v_i v_{i+n}$ (with the indices read modulo $2n$). In [6], Bokal, Oporowski, Richter and Salazar found all the 2-crossing-critical graphs, except the finite set of graphs that are 3-connected and contain a V_8 minor but no V_{10} minor.

Oporowski developed a list of 201 2-crossing-critical graphs having a V_8 minor and no V_{10} minor [8]. In her master's thesis, Urrutia-Shroeder [14] claimed to find 326 3-connected in this class; however, only 214 of those graphs were in fact 2-crossing critical. In [2], Austin found in total 312 3-connected 2-crossing-critical graphs having a V_8 minor and no V_{10} minor.

Austin [2] defined a *fully covered* graph, a combinatorial concept that forces a graph to have crossing number at least two. Austin found all fully covered, 3-connected 2-crossing critical graphs having a V_8 minor and no V_{10} minor. This list is not complete, since there are at least 8 examples of not fully covered graphs. Understanding why these examples are 2-crossing-critical led us to a series of unexpected results.

The graphs considered in this work are allowed to have multiple edges but no loops. An exception is made in Chapter 3, where all the graphs are simple and we explicitly specify it.

Two edges e,f in a graph G are *separated by cycles* if there is a pair of disjoint cycles C_1, C_2 in G, such that $e \in E(C_1)$ and $f \in E(C_2)$. This combinatorial property is related to the crossing number of a graph as follows: *Any non-planar graph such that every pair of disjoint edges is separated by cycles has crossing number at least* 2. The converse is not true in general. The graph $K_{3,4}$ has crossing number 2, but each pair of disjoint edges is not separated by cycles. However, we realize that 2-crossing-critical graphs having a V_8 minor and no V_{10} minor in Oporowski's list, including those 8 not fully-covered examples, satisfy that each pair of disjoint edges is separated by cycles.

What makes $K_{3,4}$ different from the other graphs? In this work we provide a complete answer to this question. Furthermore, we give a complete characterization of "almost 4-connected" graphs having crossing number at least 2, in terms of disjoint cycles. A remarkable consequence of our main result is a combinatorial characterization of graphs having crossing number 1: *a non-planar 4-connected graph* G *has* $cr(G) = 1$ *if and only if there exists a pair of edges that is not separated by cycles.*

In Chapter 2 we introduce the basic notions used in this thesis. In order to facilitate comprehension, non-standard definitions are given in the same section in which they are first used.

In Chapter 3 we study the 2-linkage problem. Given 4 distinct vertices x_1,y_1,x_2,y_2 in a simple graph G, the *2-linkage* problem consists of determining whether or not there are two disjoint paths P_1, P_2 connecting the pairs (x_1, y_1), (x_2, y_2), respectively. Seymour characterized all simple graphs having a 2-linkage [10]. Later, Thomassen introduced the concept of an (x_1, x_2, y_1, y_2)-web; using this concept, he gave a different characterization of simple graphs with no (x_1, y_1) (x_2, y_2)-linkage [12]. We adapted Thomassen's characterization, in order to obtain an equivalent statement, which is more useful in the context of studying disjoint cycles in a graph. Our characterization is slightly more general than one Mohar stated (without proof) in [4], since we do not require the graph to be 2-connected.

Chapter 4 contains the main results of the thesis. Here we characterize "almost 4-connected" graphs having crossing number at least 2 and state some immediate consequences.

Chapters 5, 6, and 7 are devoted to a detailed study of 3-connected, 2-crossing-critical graphs with V_8 but no V_{10} minor. These are technical sections, whose goal is to limit the possibilities more than in [6]. In [6], it is shown that if a 2-crossing-critical graph G has a V_8 minor but no V_{10} minor, then G has slightly fewer that three million vertices. We show that in fact a graph has at most 4,001 vertices.

More interestingly, in Chapter 5, we use the disjoint cycles characterization of Chapter 4 to obtain strong restrictions on the structure of a 2-crossing-critical graph with a V_8 minor but no V_{10} minor. In Chapter 7, these restrictions are applied to bound the size of such a graph. It now seems to be a much smaller step to using a computer to finally determine all of these 2-crossing-critical graphs.

Since the the first attempts to understand 2-crossing critical graphs, a particular structure has been noticed. Oporowski was the first person to point out that every large peripherally 4-connected 2-crossing critical is being composed of smaller graphs that later were called "tiles".

The set of *tiles* consists of 42 graphs. Each tile is obtained as a combinations of two *frames* (Figure 1.0.1), and 13 *pictures* (Figure 1.0.2), in such a way that a picture is inserted into a frame by identifying two squares. A given picture may be inserted with a 180° rotation. Observe that each picture produces either 4 or 2 tiles depending on whether or not the picture is invariant under 180° rotations.

Figure 1.0.1

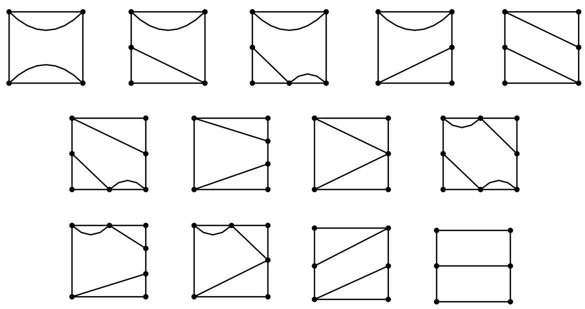

Figure 1.0.2

Given a positive integer m, a *composition of the tiles* T_0, T_1,..., T_{2m-1}, T_{2m} consists of an sequential identification of vertices of consecutive tiles in the sequence (identifying vertices on the right of T_{i-1} with vertices on the left of T_i), so that the tiles having an odd index in the sequence are flipped as in Figure 1.0.3. A *twisted cycling* of this composition of tiles consists of identifying the vertices on the left hand side of the composition (from top to bottom) with the vertices on the right hind side (from bottom to top), similarly as we construct a Möbius strip using a rectangular strip of paper (the obtained graph is similar to the example in Figure 1.0.3 where the vertices having the same label are identified).

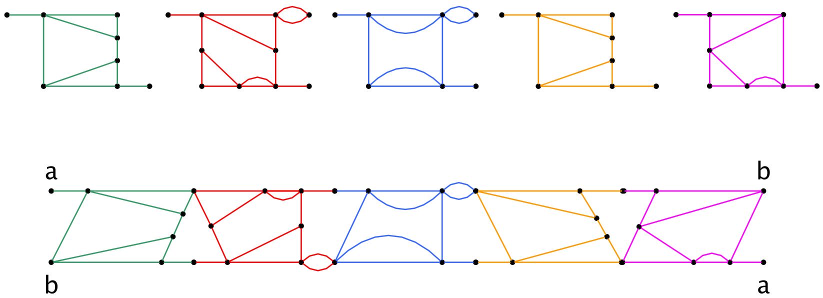

Figure 1.0.3

One of the main results in [6] is the characterization of the 3-connected 2-crossing-critical graphs having a V_{10} minor. Bokal, Oporowski, Richter and Salazar showed that these graphs are exactly those that are a twisted cycling of a composition of tiles.

On the other side, 3-connected 2-crossing critical graphs having no V_8 are also described in [6] using Robertson's characterization of V_8-free graphs. Thus, for completing the characterization of 2-crossing-critical graphs, it remains to classify those graphs that have a V_8 minor and no V_{10} minor.

2 PRELIMINARIES

2.1 DRAWINGS OF A GRAPH

The only two surfaces we will consider along this work are the plane $\mathbb{R}^2$ and the real projective plane $\mathbb{R}P^2$, so the reader might choose a definition of surface where these two surfaces fit in. Given a graph G and a surface Σ, a *drawing* of G in Σ is a mapping of G into Σ, where *drawing*

- vertices of G are represented as distinct points in Σ;
- any xy-edge is represented as an xy- open arc that does not contain the image of any vertex of G;
- there are only finitely many points of intersection between edge-arcs; and
- no three edge-arcs intersect in a single point.

An *embedding* is a drawing in which the set of edge-arcs is pairwise disjoint. From now, given a graph drawn in a surface, we make no distiction between a vertex and its image in the surface, or from an edge and the closure of its image. *embedding*

A *face* of a drawing of G in Σ is the closure of a component of $\Sigma \setminus G$. *face*

A *crossing* x between two edges e, f is a point in $e \cap f$ that is not a vertex. It is important to mention that we are only considering embeddings in the projective plane (drawings without crossings), and general drawings in the plane. An *optimal drawing* $D = D[G]$ of G, is a drawing in which the total number of crossings between edges in G is minimized. The crossing number $cr(G)$ of a graph G is the number of crossings in a optimal drawing of G in the plane. *crossing*

A graph G is *planar* if $cr(G) = 0$. Kuratowski's theorem is a well known result that characterizes planar graphs in terms minimal forbidden subgraphs. Indeed this set of minimal obstructions consists of subdivisions of K_5 and $K_{3,3}$. The study of minimal obstructions for graphs having crossing number 1 is discussed in Chapter 5. *planar*

Given a subgraph H of a graph G and a drawing D of G, we use $D[H]$ to denote the drawing of H induced by D. A *1-drawing* D of G is a drawing with at most 1 crossing. *1-drawing*

2.2 BRIDGES

A bridge is a concept that arises naturally when one study embeddings of graphs in surfaces.

Definition 2.2.1. *Let* G *be a graph and* H *be a proper subgraph of* G*. An* H-bridge B *is a subgraph of* G *such that either*

H-*bridge*

(a) B *is an edge in* $E(G) \setminus E(H)$ *between two vertices in* H*; or*

(b) B *obtained from a component* F *of* $G - V(H)$ *by adding all the edges with one end in* F *and one end in* H*, together with their ends in* H*.*

nucleus
trivial bridge
attachment
H-*avoiding*

In (b), F *is the* nucleus $\text{nuc}(B)$ *of* B*, while in (a),* B *is* trivial *and* $\text{nuc}(B)$ *is empty. An* attachment *of* B *is a vertex in* $\text{att}(B) = V(B) \setminus \text{nuc}(B)$*. For* $x, y \in V(G)$*, an* xy*-path* P *in* G *is* H-avoiding *if* $P \cap H \subseteq \{x, y\}$*.*

Definition 2.2.2. *Let* C *be a cycle in a graph* G*. Let* B *and* B' *be distinct* C*-bridges.*

residual arc

- *The* residual arcs *of* B *are the* B*-bridges in* $C \cup B$*; if* B *has at least two attachments, then these are subarcs of* C *having ends in* B *but internally disjoint to* B*.*

overlap

- *The* C*-bridges* B *and* B' *do not* overlap *if all the attachments of* B *are in the same residual arc of* B'*; otherwise, they* overlap.

overlap diagram

- *The* overlap diagram *is the graph whose vertices are* C*-bridges; with two vertices adjacent if they overlap.*

bipartite overlap diagram

- *A cycle* C *has* bipartite overlap diagram *if the overlap diagram of* C *is bipartite.*

Definition 2.2.3. *Let* C *be a cycle on a graph* G*, and let* B *be a* C*-bridge. Then* B *is a* planar C-bridge *if* $C \cup B$ *is planar. Otherwise,* B *is a non-planar* C*-bridge.*

planar C-*bridge*

In previous definition, notice that the fact of B being a planar graph does not guarantee B is a planar C-bridge. The following is a well known result that characterizes planar graphs [13].

Theorem 2.2.4. *Let* G *be a planar graph and let* C *be a cycle of* G*. Then* G *has bipartite overlap diagram and all* C*-bridges planar.*

Observe that 2.2.4 give a sufficient condition for a graph to have a 1-drawing.

Definition 2.2.5. *Let* H *be a subgraph of a graph* G *and let* D *be a drawing of* G *in the plane. Then,* H *is* clean *in* D *if no edge of* H *is crossed in* D*.*

clean

Next result from [6] is a direct consequence of 2.2.4.

Lemma 2.2.6. *Let* G *be a graph and let* C *be a cycle having bipartite overlap diagram. If there is a* C*-bridge* B *such that every other* C*-bridge is planar and* $C \cup B$ *has a 1-drawing in which* C *is clean, then,* $\text{cr}(G) \leqslant 1$*.*

Proof. Suppose e and f are crossed edges in a 1-drawing D of $C \cup B$. Let x denote the crossing of e and f. Consider the graph G' obtained from G by deleting e and f, and adding x to the set of vertices so that x is adjacent only to the four ends of e and f. Observe that C has bipartite overlap diagram in G' and every C-bridge in G' is planar. Theorem 2.2.4 implies that G' is planar. The embedding of G' yields a 1-drawing of G. Hence $cr(G) \leqslant 1$. □

3

2-LINKAGE AND DISJOINT CYCLES

3.1 INTRODUCTION

All the graphs considered in this chapter are simple and connected. We start by defining the concept of 2-linkage.

Definition 3.1.1. *Let* G *be a graph with* $|V(G)| \geqslant 4$*, and* x_1, x_2, y_1, y_2 *distinct vertices called the terminals. Then* G *has an* $(x_1, y_1)(x_2, y_2)$-linkage *if there is an* x_1y_1*-path and an* x_2y_2*-path disjoint paths in* G*. Any such pair of paths constitute an* $(x_1, y_1)(x_2, y_2)$*-linkage.*

terminal

$(x_1, y_1)(x_2, y_2)$-*linkage*

Our motivation to study 2-linkages arises from the problem of finding disjoint cycles through two particular edges. This problem is introduced and solved in Section 3.5; its connection to crossing numbers is the topic of Chapter 4.

Graphs having an $(x_1, y_1)(x_2, y_2)$-linkage were completely characterized by Seymour [10]. In [12], Thomassen introduced the concept of (x_1, x_2, y_1, y_2)-web (see Section 3.3); using this concept, he characterized a graph with no $(x_1, y_1)(x_2, y_2)$-linkage as a subgraph of an (x_1, x_2, y_1, y_2)-web. Mohar realized that graphs might have small cuts, in this work called "reducible cuts", that are irrelevant in order to determine whether a graph has a 2-linkage. In [4] he stated, without a proof, a characterization of 2-connected graphs having no 2-linkage, in terms of reduced graphs (graphs having no reducible cuts).

The following is the main result of this chapter; it is a slightly more general than Mohar's result.

Theorem 3.1.2. *Let* G *be a connected simple graph with set* $\{x_1, x_2, y_1, y_2\}$ *of four distinct terminals and suppose* G *is reduced. There is no* (x_1, y_1) (x_2, y_2)*-linkage in* G *if and only if* G *has a planar embedding with* $(x_1, x_2,$ $y_1, y_2)$ *occurring this cyclic order in the boundary walk of some face of* G.

In Section 3.2 we define reduced graphs. We will see that any simple graph has a reduced minor G' so that solving the 2-linkage problem in G is equivalent to solving this problem in G'. In Sections 3.4 we give two proofs of Theorem 3.1.2. One uses Thomassen's characterization, and the other adapts Thomassen's argument in the context of reduced graphs.

Finally, in Section 3.5 we show the relation between the 2-linkage problem and the problem of deciding whether or not there are two disjoint cycles passing through two predetermined disjoint edges.

In this section, we formally introduce *reduced graphs*, thereby making complete the statement of Theorem 3.1.2. Our two proofs will be given in the next two sections.

terminal Given a graph G, a set T of *terminals* is a set of distiguished vertices of G. We start defining a reduced graph.

Definition 3.2.1. *Let G be a graph with set T of four distinct terminals. Then*

strongly reduced

- G *is* strongly reduced *if, for every set X of three distinct vertices of G, every component of $G - X$ contains a vertex in T; and*

reduced

- G *is* reduced *if, for every set X of three distinct vertices of G, $G - X$ has at most one terminal-free component, and this component is trivial (contains exactly one vertex).*

It is immediate that any strongly reduced graph is reduced.

Definition 3.2.2. *Let G be a graph with set T of four distinct terminals. A* reduction *of G is a graph obtained from G and a set X of three vertices, by either:*

reduction

(r1) replacing any terminal-free component K of $G - X$ with an edge between each pair of attachments of K (for each such edge that is not already in G); or

r1-reduction

(r2) replacing all but one terminal-free components of $G - X$ by edges between each pair of attachments of these components, and contracting the remaining terminal-free component to a single vertex.

r2-reduction

Next observation describes the relation between Definitions 3.2.1, 3.2.2.

Observation 3.2.3. *Let G be a simple graph, and T a set of four terminals.*

- *A graph is strongly reduced if and only if has no r1-reductions.*
- *A graph is reduced if and only if has no r2-reductions.*

We now show the importance of reductions in the study of a 2-linkage in a graph.

Lemma 3.2.4. *Let G' be a reduction of a graph G with a set T of four distinct terminals. Then any 2-linkage problem with terminals in T has a solution in G if and only if it has a solution in G'.*

Proof. Let $T = \{x_1, x_2, y_1, y_2\}$. Suppose we want to determine whether or not there is an $(x_1, y_1)(x_2, y_2)$-linkage in G. Let X be a set of three vertices G, such that G' is obtained from G by applying a reduction on X.

Observe that, for $i \in \{1,2\}$, any $x_i y_i$ path P in G that uses vertices in a terminal-free component K of $G - X$ can be represented as an $x_i y_i$-path P' in G', either by using the new added edges in X (in case K is deleted from G) or by using the single vertex k that represents the component K in G' (in case K is contracted to a single vertex k). Conversely, any path P' in G' using either a new added edge or the contracted component can be identified to a path in G using some vertices in K.

Since two disjoint paths in G cannot both have vertices in K, G has an $(x_1, y_1)(x_2, y_2)$-linkage if and only if G' has $(x_1, y_2)(x_2, y_2)$-linkage. □

Although the notion of strong reducibility is sufficient to solve the 2-linkage in a specific graph, the importance of distinguishing between reduced and strongly reduced relies on the fact that the weaker notion of reducibility is used in the induction hypothesis for proving Theorem 3.1.2.

The following observation is a consequence of the previous result and reduces the 2-linkage problem to the problem restricted to reduced graphs.

Observation 3.2.5. *Let* H *be a simple graph and* T *be a set of four terminals.*

- *If a graph* G *is obtained from* H *by doing a sequence of r1-reductions and* G *has no r1-reduction, then* G *is strongly reduced and the linkage problems for* G *and* H *are equivalent.*

- *If a graph* G *is obtained from* H *by doing a sequence of r2-reductions and* G *has no r2-reduction, then* G *is reduced and the linkage problems for* G *and* H *are equivalent.*

In the following we prove that irreducibility is preserved under edge addition.

Lemma 3.2.6. *Let* G *and* G' *be graphs such that* G *is a spanning connected subgraph of* G'*. Suppose* T *is a set of four terminals in* G*. If* G *is reduced, then* G' *is reduced.*

Proof. Suppose G' is not reduced. Then G' has a set X of vertices so that either K is a terminal-free nontrivial component in $G' - X$, or H and J are two trivial components in $G' - X$.

Case 1. $G' - X$ *has a terminal-free nontrivial component* K.

Observe that either $G - X$ has a nontrivial component K' contained in K, or all the components of $G - X$ contained in K are trivial. In the former case K' is terminal-free, and therefore G is not reduced, a contradiction. In the latter case $G - X$ has at least two terminal-free components, again contradicting the fact that G is reduced.

Case 2. $G' - X$ *has two terminal-free trivial components* H, J.

In this case H and J are also terminal-free trivial components of $G - X$, contradicting the fact that G is reduced. □

disconnected terminals

separate

Two terminals x_i, y_i are *disconnected* by a cut set X if x_i and y_i are in distinct components in $G - X$. Letting R, S, T be vertex subsets, S *separates* R from T if every RT path has a vertex in S. Observe that the concept of disconnect and separate differ slightly. If x_i and y_i are disconnected by X, then $\{x_i\}$ and $\{y_i\}$ are separated by X. However the converse is not necessarily true. In Figure 3.2.1 we exhibit a graph where $\{x_i\}$ and $\{y_i\}$ are separated by X, but x_i and y_i are not disconnected by X.

minimal cut set

A cut set X in a graph G is *minimal* if there is no cut set X' properly contained in X. Observe that if X is a minimal cut set, then, for any $x \in X$ and any component H in $G - X$, there is an edge connecting x to a vertex in H. In what follows we will use *cut* and *cut set* interchangeably.

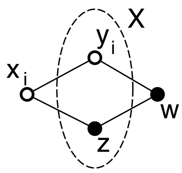

Figure 3.2.1

3.3 PROOF OF 3.1.2 USING THOMASSEN'S CHARACTERIZATION

In this section we give a short proof of Theorem 3.1.2, using Thomassen's result. But first, we need some terminology.

web

Definition 3.3.1. *Suppose G_0 is a simple plane graph such that the unbounded face is bounded by the four cycle $S_0 = x_1, x_2, y_1, y_2$, and such that every other face is bounded by a 3-cycle. Suppose in addition that any 3-cycle in G_0 is facial (hence G_0 has no separating 3-cycle). For each 3-cycle S of G_0 we add a complete graph K_S inside the face bounded by S, and join all vertices of K_S to all vertices of S. The resulting graph G is called an* (x_1, x_2, y_1, y_2)-web *with frame S_0 and rib G_0.*

Thomassen characterized simple graphs having no $(x_1, y_1)(x_2, y_2)$-linkage by describing maximal graphs (with respect to edge addition) having no $(x_1, y_1)(x_2, y_2)$-linkage.

Theorem 3.3.2 (Thomassen [12])**.** *Let G be a connected simple graph and x_1, x_2, y_1, y_2 distinct vertices. If G has no linkage $(x_1, y_1)(x_2, y_2)$-linkage and the addition of any edge to G results in a graph containing an $(x_1, y_1)(x_2, y_2)$-linkage, then G is an (x_1, x_2, y_1, y_2)-web.*

We adapted the proof of Theorem 3.3.2 to give a longer proof of Theorem 3.1.2 that is independent of Thomassen's result. This proof is presented in next section.

Our aim in this section is to prove Theorem 3.1.2 in order to characterize graphs having no $(x_1, y_1)(x_2, y_2)$-linkage in terms of reduced graphs.

Proof of Theorem 3.1.2. If G is a planar graph where the vertices x_1, x_2, y_1, y_2 are in this cyclic order in the boundary walk of a face (which we may assume is the exterior face) of a planar embedding of G, then G has no $(x_1, y_1)(x_2, y_2)$-linkage. Otherwise, we can add the edges x_1x_2, x_2y_1, y_1y_2, and y_2x_1 in the outside face to obtain an embedding of a K_4-subdivision. Now add a new vertex in the exterior face and join it to the four vertices. This give us a planar embedding of K_5, which is not possible.

Now suppose G is a reduced planar graph that has no (x_1, y_1) (x_2, y_2)-linkage. Add all possible edges to G to obtain a graph G' that has no $(x_1, y_1)(x_2, y_2)$-linkage, and any edge addition to G' creates a 2-linkage. Theorem 3.3.2 implies that G' is an (x_1, x_2, y_1, y_2)-web, and Lemma 3.2.6 implies that G' is reduced. Thus if G_0 is the rib of G', then each facial 3-cycle C of G_0 bounds a face containing at most one vertex of G'. Therefore G' is planar. Delete all vertices in $E(G') \setminus E(G)$ to G' in order to obtain a planar embedding of G so that x_1, x_2, y_1, y_2 occur in this cyclic order in the boundary walk of some face of G. □

3.4 ALTERNATIVE PROOF

In order to keep this chapter self-contained, we present an alternative proof of Theorem 3.1.2 that does not depend on Theorem 3.3.2. However, the ideas involved in the proof came from the proof of Thomassen's result.

Lemma 3.4.1. *Let G be a strongly reduced, connected, simple graph and x_1, x_2, y_1, y_2 terminals in G such that G has no $(x_1, y_1)(x_2, y_2)$-linkage. If X is a cut in G with $|X| \leqslant 3$, then:*

(i) every component of $G - X$ has a terminal;

(ii) at least one of the pairs (x_1, y_1) and (x_2, y_2) is disconnected by X; and

(iii) If $|X| \leqslant 2$ and K is a component of G such that $|K \cap \{x_1, x_2, y_1, y_2\}| = 1$, then $|V(K)| = 1$.

Proof. Let $T = \{x_1, x_2, y_1, y_2\}$. (i) Suppose $G - X$ has a component K with $K \cap T = \emptyset$. Let X' be a set of size 3 so that $X \subseteq X' \subset G - K$. Observe that K is a terminal-free component of $G - X'$, which contradicts the fact that G is strongly reduced.

(ii) Suppose this is not true. Then either all four terminals are in the same X-bridge or the pair (x_1, y_1) is in one X-bridge, while the pair (x_2, y_2) is in a different X-bridge. In both cases, we have a contradiction: in the former, there is a component of $G - X$ with no terminals; in the latter for $i \in \{1, 2\}$, there is an X-avoiding $x_i y_i$-path P_i in the X bridge containing x_i andy_i. Then (P_1, P_2) is an $(x_1, y_1)(x_2, y_2)$-linkage in G.

(iii) Let X' be a set with three vertices so that $X \cup (T \cap V(K)) \subseteq X' \subseteq V(G) - (V(K) \setminus T)$. Since G is strongly reduced, $G - X'$ does not contain a component contained in $K \setminus T$. Therefore $K \setminus T$ is empty, and $|V(K)| = 1$ as required. □

We are now able to prove Theorem 3.1.2.

Alternative proof of Theorem 3.1.2. The necessity follows as in the first proof.

Suppose that G is an reduced graph with no $(x_1, y_1)(x_2, y_2)$-linkage. We aim to show that G has a planar embedding in which x_1, x_2, y_1, y_2 occur in this cyclic order in the unbounded face.

Let $T = \{x_1, x_2, y_1, y_1\}$. We proceed by induction on the number of vertices. If $|V| = 4$ then the result holds because G is a subgraph of $K_4 - x_i y_i$ for some $i \in \{1, 2\}$.

Let G' be the graph obtained from G by adding any of the edges x_1x_2, x_1y_2, y_1x_2, and y_1y_2 not already present in G. Notice that G' is reduced (Lemma 3.2.6), and has no (x_1, y_1) (x_2, y_2)-linkage. Moreover, if G' has a planar embedding in which x_1, x_2, y_1, y_2 occur in this cyclic order in the unbounded face, then this induces a planar embedding of G with the desired properties. Thus, we may assume $G = G'$, and hence x_1x_2, x_2y_1, y_1y_2, y_2x_1 are in $E(G)$. We denote by C_T the 4-cycle $x_1x_2y_1y_2$.

Claim 3.4.2. *Either G is strongly reduced or has the required embedding.*

Proof. Suppose G is not strongly reduced. Because G is reduced, there is a set X of vertices of size 3 and a non-terminal vertex v so that v is a component of $G - X$. If we apply an r1-reduction to v, that is, if we delete v from G and add edges between each pair of neighbours of v, we get a reduced graph, because the degree of each vertex in the new graph G' is at least the degree in G. By induction hypothesis, G' has an embedding D in the plane in which x_1, x_2, y_1, y_2 occur in this cyclic order in the boundary walk of the exterior face.

If $\deg_G(v) \leqslant 2$, then we can easily extend D to the required planar embedding of G. Suppose $\deg_G(v) = 3$ and let $X = \{z_1, z_2, z_3\}$. We claim X is a facial 3-cycle in $D[G']$. Otherwise X is a 3-cut that disconnects vertices that lie inside and outside the regions bounded by the 3-cycle induced by X. All the terminals are in the boundary walk of the exterior face. Therefore $G' - X$ contains a terminal-free component K. This implies that X is a set such that $G - X$ has two terminal-free components (K and v), contradicting the fact that G is reduced.

Because X is a facial 3-cycle, we can extend D to a planar embedding of G (placing v inside the face bounded by X), and this embedding satisfies the required properties. □

From now on, we will assume G is strongly reduced. The fact that C_T is a cycle and Lemma 3.4.1 imply the following observation.

Observation 3.4.3. *For any cut X of G with $|X| \leqslant 3$, there are precisely two components K, J of $G - X$, and, for some $i \in \{1, 2\}$, $x_i \in V(K)$, $y_i \in V(J)$, and $\{x_j, y_j\} \subseteq X$*

Claim 3.4.4. *G is 3-connected.*

Proof of Claim. Let X be a cut in G with $|X| \leqslant 2$. In this case, by Observation 3.4.3, we may assume $X = \{x_2, y_2\}$, and $G - X$ has exactly two components K, J, so that $x_1 \in K$ and $y_1 \in J$. Lemma 3.4.1(iii) implies that K and J are trivial components, therefore G has exactly four vertices. This contradicts the assumption that G has at least 5 vertices. □

Claim 3.4.5. *Suppose G has a 4-cut X so that at least one component of $G - X$ has no teminal. Let K be the union of the components of $G - X$ having no terminal. Then either $|K| = 1$ or G has the desired embedding.*

Proof of Claim. Suppose first that the deletion of some set Y of at most three vertices from G leaves no TX-path. Then there is a component J of $G - Y$ so that $J \cup Y$ contains T and yet is different from G. This contadicts the fact that G is strongly reduced. We concluded from Menger's Theorem that there are four disjoint TX-paths. Label the vertices of X as x'_1, x'_2, y'_1, y'_2 so that there are disjoint paths P_1, P_2, P_3, and P_4 joining, respectively, $x_1 - x'_1$, $x_2 - x'_2$, $y_1 - y'_1$ and $y_2 - y'_2$.

Since G is strongly reduced, any component of $G - X$ not containing any terminal has four attachments. Therefore if there are two components in $G - X$ not containing any terminal, then there is an $(x_1, y_1)(x_2, y_2)$-linkage. Thus K consists of a single component of $G - X$.

Suppose that in $K \cup X$ there is a vertex $z \in K$ such that z disconnects $\{x'_1, x'_2\}$ from $\{y'_1, y'_2\}$. Let (R_1, R_2) be a partition of the components of $(K \cup X) - z$ such that x'_1, $x'_2 \in R_1$, and y'_1, $y'_2 \in R_2$. Either one of $R_1\{z, x'_1, x'_2\}$ and $R_2 - \{z, y'_1, y'_2\}$ is not empty and G is not strongly reduced, or K is just z, as required. Hence we may assume that there is no such z.

Since there is no $(x'_1, y'_1)(x'_2, y'_2)$-linkage in $K \cup X$ (otherwise there is an $(x_1, y_1)(x_2, y_2)$-linkage in G). Menger's Theorem implies that there are two disjoint paths P_5, P_6 in $K \cup X$ connecting $x'_1 - y'_2$ and $x'_2 - y'_1$ respectively.

Observe that we can add the edge $x_2'y_1'$ in G and we still have no $(x_1, y_1)(x_2, y_2)$-linkage in the new graph, otherwise we can use P_5 and P_6 to produce an $(x_1, y_1)(x_2, y_2)$-linkage in G. Likewise we can add the edge $x_1'y_2'$ and we have no $(x_1, y_1)(x_2, y_2)$-linkage.

If we repeat the preceding discussion starting with the sets $\{x_1', y_2'\}$ and $\{x_2', y_1'\}$ instead of $\{x_1', x_2'\}$ and $\{y_1', y_2'\}$, we see that we can add the edges $x_1'x_2'$ and $y_1'y_2'$ without creating an $(x_1, y_1)(x_2, y_2)$-linkage. Let G' be the graph obtained after adding all four of these edges. Lemma 3.2.6 implies G' is strongly reduced. It is evident that an embedding of G' witnessing there is no linkage implies that G has such embedding; so we may assume that $G = G'$.

Notice that $x_1'y_1'$ and $x_2'y_2' \notin E(G)$, otherwise G has an $(x_1, y_1)(x_2, y_2)$ -linkage. Therefore the graph induced by $X = \{x_1', x_2', y_1', y_2'\}$ is the 4 cycle $C' = x_1'x_2'y_1'y_2'$.

Let G/K be the graph obtained by contracting K into a vertex k. Observe that G/K has no $(x_1, x_2)(y_1, y_2)$-linkage. We claim that G/K is strongly reduced (with respect to T). Otherwise, there is a 3-cut Y in G/K with a terminal-free component K'. As G is strongly reduced, the vertex k of G/K obtained by contracting K is in Y. Since $|Y| \leqslant 3 < |X|$, we may choose the labelling so that $x_1' \notin Y$. Let H be the component of $G - Y$ containing x_1'.

Case 1. $C' \subseteq G[H \cup Y]$.

Let J be a terminal-free component of $(G/K) - Y$. Since k is adjacent only to vertices in X, J connects only to $Y \setminus \{k\}$, showing J to be a terminal-free component of $G - (Y \setminus \{k\})$, a contradiction to the fact that G is strongly reduced.

Case 2. C' *is not contained in* $G[H \cup Y]$.

Since x_2' and y_2' are both adjacent to x_1', we have that $x_2', y_2 \in H \cup Y$. Therefore, $y_1' \notin H \cup Y$; let H' be the component of $G/K - Y$ containing y_1'. It follows that $x_2', y_2' \in Y$, so $Y = \{x_2', y_2', k\}$.

Let H' be the component containing y_1'. H and H' are the only components of $G/K - Y$, since there is no $(x_1'y_1')(x_2', y_2')$-linkage. If, say H is nontrivial, then $\{x_2', y_2', x_1'\}$ separates K and H' from $H - x_1'$ in G, showing G is not strongly reduced. Therefore H, and similarly H', is trivial. But now G/K has exactly 5 vertices and we see that $G - X = K$, contradicting the assumption that X was a 4-cut in G. Therefore, G/K is strongly reduced.

The induction hypothesis implies that G/K has a planar embedding where x_1, x_2, y_1, y_2 occur in this cyclic order in the boundary walk of the exterior face.

Since $\{x_1', x_2'\}$ is not a 2-cut in G/K, the 3-cycle $x_1'x_2'k$ is a facial cycle of G/K. Likewise $x_2'y_1'k$, $y_1'y_2'k$ and $y_2'x_1'k$ are facial cycles of the embedding of G/K.

If $G[K \cup X]$ is not strongly reduced with respect to $T' = \{x_1', x_2', y_1', y_2'\}$, then there is a set Y of size 3 so that $G[K \cup X] - Y$ has a terminal-free component H. Since X is disjoint from H, the rest of G, which attaches at X, has no adjacency to H, so H is a terminal-free component of $G - Y$, contradicting the fact that G is strongly reduced. Therefore, $G[K \cup X]$ is strongly reduced.

Since $G[K \cup X]$ has no $(x_1' y_1')(x_2', y_2')$-linkage, the induction hypothesis implies that $G[K \cup X]$ has a planar embedding where x_1', x_2', y_1', y_2' occur in the exterior face. Moreover, the exterior face is the 4-cycle $x_1' x_2' y_1' y_2'$ since X is not a 4-cut in $K \cup X$. Now, gluing the embeddings of G/K and $G[K \cup X]$ along the cycle C' we obtain a planar embedding of G satisfying the required conditions. □

Claim 3.4.6. *Let* X *be a 3-cut in* G*. If* $G - X$ *has a component with exactly one terminal, then either there is a component of* $G - X$ *consisting of a single vertex and that vertex is a terminal, or* G *has the desired embedding.*

Proof of claim. Suppose that G has a 3-cut X such that $G - X$ has one component H that contains exactly one terminal (say x_1). By Lemma 3.4.1 and Observation 3.4.3, we may assume that there is exactly one component J of $G - X$ distinct from H containing the terminal y_1 and the presence of C_T shows that $\{x_2, y_2\} \in X$. Let b be the third vertex in X. If either $|J| = 1$ or $|H| = 1$, then the claim holds, hence we may assume $|J|, |H| \geqslant 2$.

Let G/H be the graph obtained by contracting H to a vertex h, and G/J be obtained by contracting J to a vertex j. Observe that there is no $(h, y_1)(x_2, y_2)$-linkage in G/H, as otherwise we can find an $(x_1, x_2)(y_1, y_2)$-linkage in G.

We can add edges $x_2 b$, and $y_2 b$, and G has no $(x_1, x_2)(y_1, y_2)$-linkage in G. Hence we may assume $x_2 b$ and $y_2 b$ are originally in G.

We claim that G/H is strongly reduced with respect to the terminals h, x_2, y_1, y_2. Otherwise, there is a set Y of three vertices so that $(G/H) - Y$ has a terminal-free component K. If $h \notin Y$, then K is a terminal-free component of $G - Y$, contradicting the fact that G is strongly reduced; thus, $h \in Y$. If $b \in V(K)$, then its neighbours x_2, $y_2 \in Y$, so all the neighbours of h are in $G[K \cup Y]$. Then $Y \setminus \{h\}$ is a 2-cut in G, contradicting that G is 3-connected. Therefore $b \notin V(K)$, so all neighbours of h are not in K, so K attaches only to $Y \setminus \{h\}$ in both G/H and in G, contradicting the 3-connection of G.

By the induction hypothesis, G/H has an embedding in the plane where h, x_2, y_1, y_2 occur in this cyclic order in the boundary of the exterior face. Observe that the 3-cycles h, b, x_2, and h, b, y_2 are facial cycles in G/H (otherwise G is not strongly reduced). Likewise we can prove that G/J has a planar embedding such that x_1, x_2, j, y_2 occur in this cyclic order in the boundary of the exterior face, and such that the 3-cycles j, b, x_2, and j, b, y_2 are facial.

Now, gluing together the embeddings of G/H and G/J we obtain the desired embedding. □

Let e be any edge with endpoints a, b, not both terminals. Trivially, there is no $(x_1, y_1)(x_2, y_2)$-linkage in G/e. Our aim is to show the following.

Claim 3.4.7. G/e *has a planar embedding where* x_1, x_2, y_1, y_2 *occur in this cyclic order in the exterior face.*

Proof of claim. If G/e is strongly reduced, then this follows from the induction hypothesis.

In the alternative, there is a set X of three vertices in G/e so that $G/e - X$ has a terminal-free component J. Observe that necessarily e is in X; otherwise X is reducible in G.

Let $X' = (X - e) \cup \{a, b\}$. Observe that X' is a cut set in G and $|X'| = 4$. Notice that J is a component in $G - X'$ with no terminals. By Claim 3.4.5, we may assume $|J| = 1$ and any component in $G - X'$ distinct from J contains a terminal.

We showed that for any reducible cut X in G/e, there exists at most one reducible terminal-free component, and this component is trivial. Then G/e is reduced and by induction hypothesis, G/e has the required embedding. □

Claim 3.4.8. G *is planar.*

Proof of claim. By the previous observations, if we contract any edge not joining two terminals, the result is a planar graph. Suppose G is not planar. Then by Kuratowski's theorem G has a subgraph F that is a subdivision of $K_{3,3}$ of K_5.

Observation 3.4.9. F *satisfies the following:*

(1) every terminal is in F*; and*

(2) any edge incident with a vertex of degree 2 in F *has both ends in* T*.*

Proof of Observation. Suppose some terminal, say x_1, is not in F. Since G is 3-connected and x_1 is not adjacent to y_1, there is an edge e incident with x_1 that is not incident with any other terminal. Then G/e is not planar, contradicting Claim 3.4.7.

Now observe that any edge incident to a vertex of degree 2 in F must be an edge between two terminals, because if not we can contract it, obtaining a non-planar graph, contradicting the initial observation in the paragraph. □

Therefore G is obtained from K_5 or $K_{3,3}$ by possibly inserting one or two vertices of degree 2 and then adding some edges such that the resulting graph is 3-connected. We claim that any of these possible graphs has an $(x_1, y_1)(x_2, y_2)$-linkage.

If there are two adjacent terminals $x_i y_i$, then, as G is 3-connected, $G - \{x_i, y_i\}$ is 1-connected, therefore G has a 2-linkage.

Case 1. F *is a subdivision of* $K_{3,3}$.

In this case, F is obtained from subdividing an edge e 0, 1, or 2 times. The vertices subdividing the edge and the endpoints of e are terminals.

Subcase 1.2. F *is obtained from subdividing* e *0 times.*

The vertices of one of the pairs (x_1, y_1) and (x_2, y_2) are in distinct independent sets of F. Then G has a 2-linkage.

Subcase 1.2. F *is obtained from subdividing* e *1 time.*

Hence we may assume that $e = x_2y_2$ and x_1 subdivides e. Let $\{x_2, a, b\}, \{y_2, c, d\}$ be the bipartition of the $K_{3,3}$. Because G is 3-connected and every neighbour of x_1 is either in T of is an F-node, x_1 is adjacent to at least one vertex in $\{a, b, c, d\}$. We may assume that x_1 is adjacent to a. If $y_1 \in \{a, c, d\}$, then G has a 2-linkage, a contradiction. Therefore, $y_1 = b$. As G is strongly reduced, setting $X = \{x_2, a, b\}$, the component of $G - X$ containing c must contain a terminal. The same holds for d. We may choose the labelling of c and d so that there is a path from c to either x_1 or y_2 that does not include either the other or d. Any such path completes a 2-linkage in G, a contradiction.

Subcase 1.3. e *is subdivided 2 times.*

We can assume that x_1, x_2, y_1, y_2 is the path that is obtained from subdividing e. Let $\{x_2, a, b\}, \{y_2, c, d\}$ be the bipartition of the $K_{3,3}$. Since G is 3-connected, x_2 and y_1 are adjacent to some vertices in $\{a, b, c, d\}$. If there are distinct z, y in $\{a, b, c, d\}$, so that z is adjacent to x_2,and y is adjacent to y_1, then using these edges we can complete a 2-linkage in G, a contradiction. Thus, we may assume that both x_2, y_1 are only adjacent to only one vertex in $\{a, b, c, d\}$, say a. Then $\{a, x_1, y_2\}$ is a 3-cut in G, such that its deletion contains a terminal-free component, a contradiction.

Case 2. F *is a subdivision of* K_5.

This follows from a similar analysis as in Case 1; in every case we obtain a 2-linkage. Therefore G is planar. □

Let z be a non-terminal vertex adjacent to x_1. Either G/x_1z is 3-connected, or there exists a vertex w such that $\{x_1, z, w\}$ is a 3-cut. In the latter case, there is a component K of $G - X$ such that contains at most one terminal t and, by Claim 3.4.6, we may assume K is a trivial component. Hence tz is such that G/tz is 3-connected. Thus, in both cases, we can find an edge e so that not all its endpoints are terminals and G/e is 3-connected.

Since G is 3-connected and planar, it has a unique embedding in the plane. The unique embedding of G/e is obtained from that of G by contracting e. By the induction, x_1, x_2, y_1, y_2 occur in this cyclic order on some face of G/e. A straightforward verification that this is also true for the embedding of G, as required.

□

3.5 DISJOINT CYCLES

In this section we are interested in studying the following problem.

Problem 3.5.1. *Given two disjoint edges* e, f *in a graph* G*, decide if there are disjoint cycles* C_e *and* C_f*, containing* e *and* f*, respectively.*

separated by cycles

If C_e and C_f exist, then we say that e and f are *separated by cycles*. Let $e = x_1y_1$ and $f = x_2y_2$; observe that the disjoint cycles C_e and C_f exist if and only if $G - e - f$ has an $(x_1, y_1)(x_2, y_2)$-linkage. Hence we reduced the problem of finding these disjoint cycles to a 2-linkage problem.

In Chapter 4 we will see how disjoint cycles play an important role in the study of graphs with crossing number at least two. Indeed, we will show that for 4-connected graphs the problem of deciding whether a graph has crossing number at least 2 reduces to Problem 3.5.1.

Lemma 3.5.2. *Let* $e = x_1y_1, f = x_2y_2$ *be disjoint edges of a connected, strongly reduced, simple graph* G*. Then either* e *and* f *are separated by cycles, or there is a 1-drawing of* G *in which* e *and* f *cross.*

Proof. Suppose that e and f are not in disjoint cycles. Let $G' = G - e - f$ and let $T = \{x_1, x_2, y_1, y_2\}$.

Case 1. G' *is disconnected.*

We may choose the labelling so that x_1 and y_1 are in different components of G'. Let K be the component of G' containing x_1 and let X be a minimal cut set in G containing x_1 and at most one other vertex, necessarily a terminal, in K.

Subcase 1.1. $|X| = 1$.

We may assume that $X = \{x_1\}$. Let K be the component of G' containing x_1.

Suppose $|K| = 1$. Observe that $\deg_G(x_1) = 1$, and hence $\{y_1\}$ is a cut disconnecting x_1 from $\{x_2, y_2\}$ in G. Since G is strongly reducible, $\{y_1, x_2, y_2\}$ is not a 3 cut in G'. Then $V(G) = T$. Therefore G' is contained in the graph in Figure 3.5.1, which clearly has the required embedding.

From now on, we may assume that $\deg(t) \geqslant 2$ for all $t \in T$, and consequently $|K| \geqslant 2$. If $K - T$ has a vertex v, then set $X = \{x_1, x_2, y_2\}$ and note that v and y_1 are in different components of $G - X$. As the component of $G - X$ containing v is terminal-free, G is not strongly reduced, a contradiction. Therefore, $K - T$ has no vertex. Since $\{x_1\}$ is a cut in G and $|K| \geqslant 2$, then $\{x_2, y_2\} \subset K$.

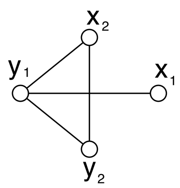

Figure 3.5.1

Let z be a vertex adjacent to y_1 and distinct to x. The cut $\{x_1, y_1, x_2\}$ in G disconnects z from y_2 has a terminal-free component (the one containing z), contradicting the fact that G is strongly reduced.

Subcase 1.2. $|X| = 2$.

We may choose the labelling of T so that $X = \{x_1, x_2\}$.

Lemma 3.4.1 (i) implies that $G - X$ has exactly two components H, H'; each containing y_1, y_2 respectively. By Lemma 3.4.1(iii) H and H' are trivial components. Therefore $|V(G)| = 4$, and e, f are disjoint edges and since K_4 has a 1-drawing in which e and f cross, G has a 1-drawing in which e and f cross.

Case 2. G' *is connected.*

By Theorem 3.1.2 it suffices to prove that G' is strongly reduced for obtaining an embedding of G' where x_1, x_2, y_1, y_2 occur in this cyclic order in the exterior face. This give us the required 1-drawing of G.

Suppose for the sake of contradiction that G' is not strongly reduced. Then there is a set X with three vertices so that $G' - X$ has a terminal free-component K. Observe that X is a cut in G since K has no terminals. But then G is not strongly reduced, a contradiction. □

We end this section by solving Problem 3.5.1 in case when G is a reduced graph.

Theorem 3.5.3. *Let* $e = x_1y_1, f = x_2, y_2$ *be disjoint edges of a connected, reduced, simple graph* G. *Then either* e *and* f *are separated by cycles, or there is a 1-drawing of* G *in which* e *and* f *cross.*

Proof. Suppose that e and f are not in disjoint cycles. We may assume edges x_1x_2, x_2y_1, y_1y_2, y_2x_1 are in G, since the new graph G' obtained by adding these edges to G is reduced (3.2.6) and, e and f are separated by cycles in G if and only are separated by cycles in G'. Moreover, a 1-drawing of G' in which e and f cross certifies that G has the required 1-drawing.

Claim 3.5.4. *If* X *is a set of 3 vertices in* G, *then at most one component of* $G - X$ *is terminal-free. If there is a terminal-free component, then it is trivial and* $G - X$ *has precisely two components.*

Proof of claim. Since the four terminals induce a K_4, there is a component K' of $G - X$ so that the terminals are all in $G[K' \cup X]$. Since G is reduced, $G - X$ has at most one terminal-free component K and, if it exists, K is trivial. Thus, the only possible components are K and K'. □

We proceed by induction on the number of vertices of G. If G has 4 vertices, since K_4 has an embedding in which two disjoint edges are crossed, then the statement holds for G. Suppose G has more than 4 vertices.

If G is strongly reduced, then the result follows from Lemma 3.5.2. Suppose G has a set X of 3 vertices such that $G - X$ contains a terminal-free component. Since G is reduced, $G - X$ contains exactly one terminal-free component K, and this component is trivial. Let z be the vertex in K.

Let G'' be the graph obtained by deleting z and adding the edges between the neighbours of z. This graph G'' is clearly reduced. Observe that e and f are not in disjoint cycles in G''. Then, the induction hypothesis implies that G'' has a 1-drawing in which e and f are crossed.

If $deg(z) \leqslant 2$, we can easily extend the embedding of G'' to an embedding of G by attaching z to its neighbours, and placing z inside an inner face containing its neighbours. This embedding of G satisfies the required conditions. Thus, we may assume $deg(z) = 3$. Claim 3.5.4 implies that the 3-cycle induced by X in G'' has exactly one X-bridge. Therefore X is a facial cycle of G'' and we can extend the embedding of G'' to an embedding of G satisfying the required properties. □

4

DISJOINT CYCLES AND CROSSING NUMBER

4.1 INTRODUCTION

In this chapter we exhibit a relation between the crossing number of a graph and each pair of disjoint cycles. In Section 4.3 we prove the main result of this thesis. It is almost a triviality that if each pair of disjoint edges in a non-planar graph G are cycle separated; then $cr(G) \geqslant 2$ (for more details see 4.2.3). The main point of this work is to prove the following converse.

Theorem. *Let G be a 4-connected non-planar graph. Then, each pair of disjoint edges is separated by cycles if and only if $cr(G) \geqslant 2$.*

In the next section we introduce some relevant concepts and stablish preliminary results. A slightly stronger and more useful version of the above theorem is proved in Section 4.3.

4.2 PRELIMINARIES

In this section we present some preliminary observations about pairs of edges that do not cross in a 1-drawing of a non-planar graph.

Observation 4.2.1. *Let D be a 1-drawing of a non-planar graph G. Then*

- *there is no self-crossing edge in D; and*
- *adjacent edges of G do not cross in D.*

Lemma 4.2.2. *Disjoint cycles do not cross in a 1-drawing of a graph G.*

Proof. Let D be a 1-drawing of G. Suppose G has disjoint cycles C_1 and C_2 that are crossed in D. Observe that neither C_1 nor C_2 can intersect itself (otherwise D has a least 2 crossings). Therefore, C_1 and C_2 are simple closed curves in the plane. Notice that, any time C_2 crosses into C_1, it must also cross out (this is a consequence of Jordan's Curve Theorem). Then C_1 and C_2 intersect at least two times, which contradicts that D has one crossing. □

Theorem 4.2.3. *Let G be a non-planar graph. Suppose that each pair of disjoint edges e_1, e_2 is separated by cycles. Then $cr(G) \geqslant 2$.*

Proof. Consider an optimal drawing D of G. Since G is nonplanar, there is a crossing pair e_1 and e_2 of edges in D. By Observation 4.2.1, e_1 and e_2 are not adjacent. The hyphothesis implies that there exists a pair of cycles (C_1, C_2) that separates e_1 and e_2. By Lemma 4.2.2, the optimal drawing is not a 1-drawing. Therefore, $cr(G) \geqslant 2$. □

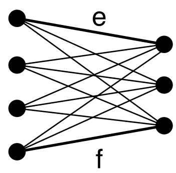

Figure 4.2.1

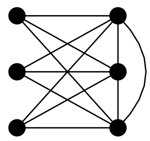

Figure 4.2.2

The graph $K_{3,4}$ (Figure 4.2.1) has crossing number 2. However, it is not true that every pair of disjoint edges are separated by cycles. For instance, e and f are disjoint and they are not separated by cycles ($K_{3,4}$ does not have disjoint cycles). This motivates the question: under what conditions is the converse of Theorem 4.2.3 true? In other words, under what circumstances, if a graph has crossing number at least 2, then each pair of disjoint edges are separated by cycles? We will show that it is enough to require high connectivity.

peripherally 4-connected

Definition 4.2.4. *A graph G is* peripherally 4-connected *if it is 3-connected and, for any 3-cut X of G and any partition of the components of $G - X$ into two non-null subgraphs H and K, at least one of H and K has just one vertex.*

Observe that if X is a 3-cut in a peripherally 4-connected graph G, then $G - X$ has 2 or 3 components. In the former case one of the two components is a trivial component. In case $G - X$ has 3 components, necessarily all of them are trivial, and therefore G is a subgraph of the graph shown in Figure 4.2.2.

linked

Definition 4.2.5. *Let G be a graph, and $e = x_1y_1$, $f = x_2y_2$ distinct edges in G. The edges e and f are* linked *if either e and f are incident to a common vertex or there is a 3-cut X in G such that $X \subset \{x_1, y_1, x_2, y_2\}$ and $\{x_1, y_1, x_2, y_2\} \setminus X$ is the vertex in a trivial component of $G - X$. Otherwise e and f are unlinked.*

Observe that if two edges are linked, then there are no disjoint cycles separating them. Thus, a pair of unlinked edges e, f is potentially separable by cycles. A sufficient condition that prevents e,f to be separated by cycles, is when G has $cr(G) = 1$ and e and f are crossed in a 1-drawing of G. This follows from the fact that there is no $(x_1, y_1)(x_2, y_2)$-linkage in $G - e - f$. The next result shows that the converse is true for peripherally 4-connected simple graphs.

Theorem 4.2.6. *Let G be a peripherally 4-connected simple graph. Let e and f be unlinked edges in G. Then either e and f are in disjoint cycles or there is a 1-drawing of G in which they cross.*

Proof. If G is 4-connected, then G is reduced for any pair of disjoint edges e, f. Then the result follows from Theorem 3.5.3.

Suppose that G is not 4-connected. Let $e = x_1y_1$ and $f = x_2y_2$. If G has a 3-cut X such that $G - X$ has at least three components, then G is isomorphic to a subgraph of the graph in Figure 4.2.2. In this case, e and f are disjoint edges of the $K_{3,3}$ since e and f are unlinked. Since G is simple, it is easy to see that there is a 1-drawing with e and f crossing.

Thus, we may assume that $G - X$ has two components for any 3-cut X. We claim G is reduced with respect to terminals x_1, y_1, x_2, y_2. Otherwise, there is a 3-cut X so that there is a union J of terminal-free components $G - X$ having at least two vertices. Since there are 4 terminals and $|X| = 3$, there is a component K of $G - X$ with at least one terminal. Since $G - X$ has only two components, J is a component of $G - X$. As G is peripherally 4-connected, K must have a single vertex. Therefore $X \subseteq \{x_1, y_1, x_2, y_2\}$ and e and f are linked, a contradiction. Thus G is reduced. Now theorem 3.5.3 yields the result. □

4.3 MAIN RESULTS

Our next theorem is the main result of the thesis and justifies the definition of linked edges. We emphasize that we are allowing graphs to have multiple edges.

Theorem 4.3.1. *Let G be a peripherally 4-connected non-planar graph. Then each pair of unlinked edges is separated by cycles if and only if $\mathrm{cr}(G) \geqslant 2$.*

Proof. Suppose that G is a graph such that each pair of unlinked edges is separated by cycles. For the sake of contradiction, suppose $e = x_1y_1$ and $f = x_2y_2$ is a crossed pair in a 1-drawing D of G. By Observation 4.2.1, e and f are disjoint, and by Lemma 4.2.2, e and f are linked. We may assume that $X = \{x_1, y_1, x_2\}$ is a 3-cut in G so that y_2 is a component K of $G - X$. Let $\mathrm{si}(G)$ be the simplification of G. Since G is non-planar, D induces a 1-drawing of $\mathrm{si}(G)$ in which e and f cross.

If $G - X$ has at least three components, then $\mathrm{si}(G)$ is contained in the graph in Figure 4.2.2. Since $\mathrm{si}(G)$ is nonplanar, the $K_{3,3}$ in Figure 4.2.2 is contained in G and e and f are disjoint edges in the $K_{3,3}$. Thus, they are unlinked, a contradiction. Therefore, we may assume $G - X$ has two components.

Evidently, $D - e$ is a planar embedding of $G - e$ and the vertex y_2 has only the neighbours x_1, y_1, x_2. There is some face F of $D - e$ incident with all three of x_1, y_1, and y_2. We may add e into this face to produce a planar embedding of G, contradicting the assumptionn that G is not planar. Therefore, $\mathrm{cr}(G) \geqslant 2$, as required.

For the converse, suppose now that G has $\mathrm{cr}(G) \geqslant 2$. Let $e = x_1y_1$ and $f = x_2y_2$ be unlinked edges in G, and let $\mathrm{si}(G)$ be the simplification of G. Observe that $\mathrm{si}(G)$ is peripherally 4-connected. If e and f are simple edges of G, there is no 1-drawing of G' so that e and f cross. Thus, Theorem 4.2.6 implies that there are disjoint cycles separating e and f in $\mathrm{si}(G)$, and therefore e and f are separated by disjoint cycles in G.

If e and f both have parallel edges, then e and f are separated by cycles. Hence, we may assume that the pair (x_1, y_1) is joined by multiple edges, and f is a simple edge. Let C_1 be a cycle of length 2 containing e. We claim that there exists a cycle C_2 containing f, and disjoint to C_1. Otherwise, f is a cut-edge in $G - \{x_1, y_1\}$. Since G is not planar, $|V(G)| \geqslant 5$ and either $\{x_1, y_1, x_2\}$ or $\{x_1, y_1, y_2\}$ is a 3-cut. We may assume that $X = \{x_1, y_1, x_2\}$ is a 3-cut.

If $G - X$ has three components, then $\mathrm{si}(G)$ is contained in the graph of Figure 4.2.2. Since G is non-planar, so is $\mathrm{si}(G)$, so the $K_{3,3}$ is contained in $\mathrm{si}(G)$. Because e and f are unlinked, both are in the $K_{3,3}$. Since e is in a 2-cycle C, the 4-cycle $K_{3,3} - \{x_1, y_1\}$ contains f and is disjoint from C, as required.

We may assume K, J are the components of $G - X$ so that $y_2 \in K$. Because e and f are unlinked, K is not trivial and hence J is trivial. Observe that if $|K| \geqslant 3$, then $X' = \{x_1, y_1, y_2\}$ is a 3-cut such that $J \cup \{x_2\}$ and $K \setminus \{y_2\}$ are nontrivial components of $G - X'$, contradicting that G is peripherally 4-connected. On the other hand, if $|K| = 2$, then $|V(G)| = 5$. Observe that $\mathrm{si}(G)$ is not a complete graph, otherwise X is not a 3-cut. Thus $\mathrm{si}(G)$ is a proper subgraph of K_5, and hence G is planar. In every case we obtained a contradiction, therefore e and f are separated by cycles.

□

Observe that, in a 4-connected graph, two edges are disjoint if and only if they are unlinked. This observation lead us to a converse of Theorem 4.2.3.

Theorem 4.3.2. *Let G be a 4-connected non-planar graph. Then, each pair of disjoint edges is separated by cycles if and only if $\mathrm{cr}(G) \geqslant 2$.*

The contrapositive of this statement is a somewhat surprisingly tidy, combinatorial characterization of 4-connected graphs having crossing number 1.

Theorem 4.3.3. *Let G be a 4-connected non-planar graph. Then, $\mathrm{cr}(G) = 1$ if and only if there exists a pair of disjoint edges that cannot be separated by cycles.*

5

2-CROSSING-CRITICAL GRAPHS

5.1 INTRODUCTION

Given a positive integer k, a graph is k-*crossing-critical* if the crossing number $cr(G)$ is at least k, but every proper subgraph H has $cr(H) < k$. A consequence of Kuratowski's theorem is that 1-crossing-critical graphs are the $K_{3,3}$ and K_5 subdivisions. In the same spirit one can ask a more general question.

k-*crossing-critical*

Problem 5.1.1. *Given a positive integer* k*, characterize the set of* k*-crossing-critical graphs.*

Insertion and suppression of degree 2 vertices do not affect the crossing number of a graph. This means that any subdivision of a k-crossing-critical graph is also k-crossing-critical. We simplify our study by restricting our graphs to have vertices of degree at least 3 (clearly no vertex of degree 1 nor a loop appears in a crossing-critical graph).

In this work, we focus on 2-crossing-critical graphs. Bloom, Kennedy, and Quintas first exhibit 21 2-crossing-critical graphs [5]. In [11], Širáň constructed infinitely many 2-crossing-critical graphs. The only 2-crossing-critical graph with crossing number greater than 2 is the Cartesian product $C_3 \square C_3$. Richter found the eight cubic 2-crossing-critical graphs [9] .

The Möbius ladder V_{2n}, is a graph consisting of a $2n$-cycle with n diameters (see Def. 5.2.1). In [6], Bokal, Oporowski, Richter and Salazar found all the 2-crossing-critical graphs, except the finite set of those graphs that are 3-connected and contain a V_8 minor but no V_{10}. A graph has a V_{2n} minor if and only if has a V_{2n} topological minor. We caution the reader that, in the following sections, we refer to a V_{2n} topological minor K in a graph G, by just saying K is a V_{2n} minor in G. They also show how to obtain all the not 3-connected 2-crossing-critical graphs from the 3-connected ones. So from now on we can assume all graphs are 3-connected, and we use $\mathcal{M}_2^3$ to denote the set of 3-connected 2-crossing-critical graphs.

Oporowski developed a list of 201 2-crossing-critical graphs with a V_8 minor and no V_{10} minor [8]. In her master's thesis, Urrutia-Shroeder [14] claimed to find 326 3-connected graphs in this class; however, only 214 of those graphs were in fact 2-crossing critical.

In [2], Austin found in total 312 3-connected 2-crossing-critical having a V_8 minor and no V_{10} minor. For finding these graphs, Austin defined the concept of fully covered graph, a combinatorial definition that forces a graph to have a crossing number at least 2. In her work, Austin showed that in Oporowski's list all but 8 graphs satisfy this definition. Our Theorem 4.3.1, and all theory behind, was inspired by this concept of fully covered.

The intention of what follows in this work is to present a new approach to the problem of describing all 2-crossing-critical graphs having a V_8 minor and no V_{10} minor. The idea is to reduce this problem to a computational problem, that is, enumerate all possible graphs having V_8 minor and no V_{10} minor that might be 2-crossing-critical. At the end, an algorithm can quickly check if the graphs are indeed 2-crossing-critical.

In order to achieve this goal, we need some bounds and restrictions on the possibilities for 2-crossing-critical graphs having a V_8 minor, and no V_{10}. In this chapter we give some properties that graphs in this family must satisfy.

As the reader might guess, we will show how the combinatorial characterization of graphs having crossing number at least two (Theorem 4.3.1) offers a new approach to understand 2-crossing-critical graphs (see Section 5.4). Another central idea, is to carefully select a "minimal" V_8 minor (with respect to some property defined latter), in order to obtain more restrictions on any 2-crossing-critical graph. This "minimal" V_8 is called unpolluted (Section 5.6).

5.2 GRAPHS HAVING A V_8 MINOR

In this section we define the basic concepts related to a V_{2n} minor.

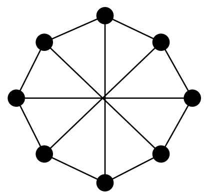

Figure 5.2.1

V_{2n} **Definition 5.2.1.** *The graph* V_{2n} *consists of*

rim R
- *a* $2n$*-cycle* $[v_0, v_1, ..., v_{2n-1}, v_0]$ *called the* rim R*; and*

spoke
- *the n* spokes $v_i v_{n+i}$*, for* $i = 0, ..., n-1$ *(the indices are read modulo* $2n$*).*

In our present work, we will focus on graphs having a V_8 minor (see Figure 5.2.1) and no V_{10} minor. We will denote $V_8 \cong H \subseteq G \not\supseteq V_{10}$ when a graph G satisfies these properties, and H is a subgraph of G topologically equivalent to V_8.

The class of graphs containing a V_8 minor and no V_{10} minor is not well-quasi-ordered with respect to the topological minor order. To see this, replace an edge e (say $e = v_0v_1$) by a path of length k, and double each edge of the path. This gives a famililly (over all positive integers k) of graphs, no one topollogically contained in another. To make this family 3-connected, we can add edges from the vertex v_5 to vertices in the path.

Definition 5.2.2. *Let* G *be a graph, and* H *be such that* $V_8 \cong H \subseteq G \not\supseteq V_{10}$.

- *The* H-nodes *are the vertices of* H *corresponding to* $v_0, v_1, ..., v_7$ *in* V_8. H-*node*
- *The* rim branch r_i *is the path in* H *connecting consecutive* H-*nodes* v_i *and* v_{i+1} *corresponding to edge* v_iv_{i+1} *of* V_8. *rim branch*
- *The* spoke s_i *is the path in* H *connecting* H-*nodes* v_iv_{i+4} *corresponding to edge* v_iv_{i+4} *of* V_8. *spoke*
- *For* $i = 0, 1, 2, 3, 4$, *the cycle* $r_i \cup s_i \cup r_{i+1} \cup s_{i+1}$ *is the* H-quad Q_i. H-*quad*

5.3 REPRESENTATIVITY IN THE PROJECTIVE PLANE

In the study of 2-critical graphs, the relationship between a graph and its embeddability in the projective plane has been useful for understanding 2-crossing-critical graphs. Richter was the first one to use this relation to determine all eight cubic 2-crossing-critical graphs [9]. The embedding of 2-critical graphs in the projective plane was a core idea for determining all 2-critical graphs with a V_{10} minor [6]. In this section we review these considerations.

It is well known result that there is a set 103 graphs that minimally do not embed in the projective plane $\mathbb{R}P^2$ [1]. All these graphs have crossing number at least 2, some of them are 2-crossing-critical (enumerated in [6]), the rest have subgraphs with crossing number 2 that embedd in $\mathbb{R}P^2$. Thus, we can restrict our problem of finding the 2-crossing-critical graphs by looking only to those graphs that have an embedding in the projective plane.

Definition 5.3.1. *Let* G *be a graph embedded in the projective plane* $\mathbb{R}P^2$.

1. *The* representativity $\text{rep}(G)$ *of* G *is the largest integer* n *so that every non-contractible, simple, closed curve in* $\mathbb{R}P^2$ *intersects* G *in at least* n *points (may intersect some vertices of* G*);* *representativity*
2. G *is* n-representative *if* $n \geqslant \text{rep}(G)$*;*
3. G *is* embedded with representativity n *if* $\text{rep}(G) = n$.

Barnette [3] and Vitray [15] proved that every 3-representative embedding of the projective plane topologically contains one graph of a list of 15. Vitray pointed out that each of these graphs has crossing number at least 2. This implies that the set of 3-representative 2-crossing-critical-graphs is contained in this finite set. All 1-representative graphs are planar; this means that we can restrict ourselves only to graphs with representativity 2.

Let G have a representativity 2 embedding in $\mathbb{R}P^2$ and suppose G has a V_8 minor H, but no V_{10} minor. Let γ be a non-contractible, simple, closed curve meeting the graph G in exactly two points a and b. Is not hard to see that we may choose γ so that γ only intersects G in $V(G)$. We claim the points a, b are in the rim R. Otherwise, suppose a or b is not in R. Deleting the vertex $v \in \{a, b\}$ not in R from H, we obtain a graph with representativity 1; therefore, this is a planar graph. Observe that $H - v$ contains a $V_6 = K_{3,3}$, so that $H - v$ is not planar, a contradiction.

One model of the projective plane is a closed disk where antipodal points in the boundary are identified. Without loss of generality we will assume γ is the curve defined by the points in the boundary. The points a and b are represented by pairs of antipodal points in γ, and the rim R is a contractible curve that intersects γ only in a and b.

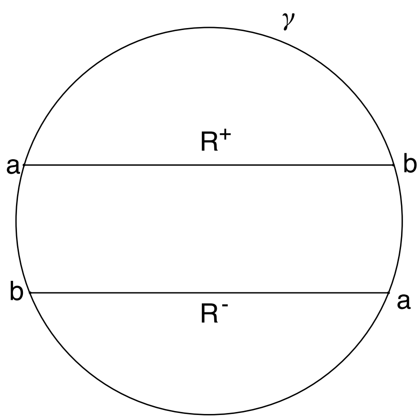

Figure 5.3.1

Observe that there are two faces of R: a closed Möbius strip $\mathfrak{M}$ and a closed disk $\mathfrak{D}$. The points a and b divide R into two arcs, denoted R^+ and R^-. Observe that as the end points of the spokes are interlaced, hence at most one spoke is contained in $\mathfrak{D}$; if there is one, call this spoke the *exposed spoke*. This analysis tell us that there are basically two types of embeddings of V_8 in $\mathbb{R}P^2$. *Type* A has an exposed spoke embedded in $\mathfrak{D}$. *Type* B has no exposed spokes. During the rest of this work, we will assume (by relabelling) that Type A and Type B embeddings are like in Figure 5.3.2, so s_0 is the exposed spoke in the type A embedding.

exposed spoke

Type A

Type B

Regardless the type of embedding, $H \cup \gamma$ divides the projective plane into seven regions. In the following sections, given a graph G with a V_8 minor H, in order to give a precise description of the 7 regions of the embedding of G in the two types of embeddings (from now on, these are called H-*regions*), we will use the labelling as in Figure 5.3.3. One important fact, of which we must be aware, is that curve γ might intersect H in H-nodes. More specifically, it is possible to have:

H-*regions*

- either $a = v_0$ or $a = v_7$;
- in Type A embeddings we can have $b = v_4$ or $b = v_5$; and
- in Type B embeddings we can have $b = v_3$ or $b = v_4$.

Thus there are nine different posibilities for each type of embedding.

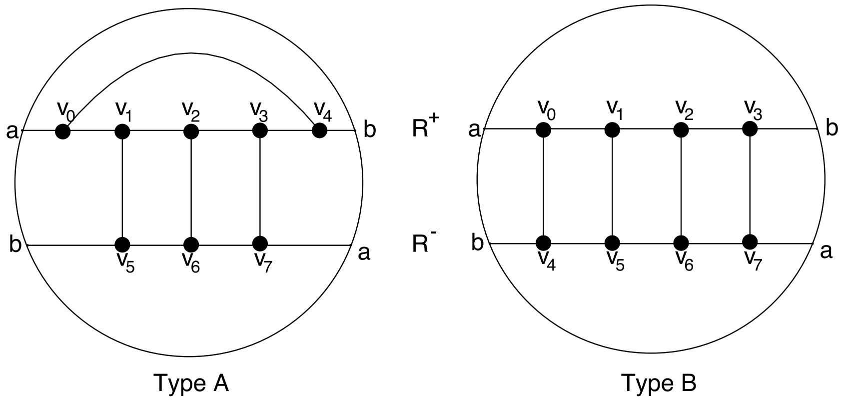

Figure 5.3.2

Observe that for any H-bridge B, the nucleus $Nuc(B)$ is contained in one of the H-regions. This means that the problem of describing all 2-crossing-critical graphs with representativity two having a V_8 minor and no V_{10} minor is reduced to studying the H-bridges in each of the H-regions for the two types of embeddings. In other words, if we know all possible ways in which we can attach bridges in all 7 regions, then considering all possible combinations and testing whether or not they are 2-critical, we can obtain all the unknown 2-crossing-critical graphs.

The skeptical reader might argue about the difficulties for the proposed approach in the previous paragraph. *A priori* the nucleus of an H-bridge could be any graph, and the number of attachments can be a huge number. None of the known examples of 2-crossing-critical graphs having a V_8 minor and no V_{10} have "big" H-bridges. In fact, in [6], a bound for the number of vertices of a 2-crossing-critical graph having no V_{10} was found (less than 3,000,000 vertices). Our aim now is to significantly improve this, by showing that there is a limited number of possibilities for each of the regions.

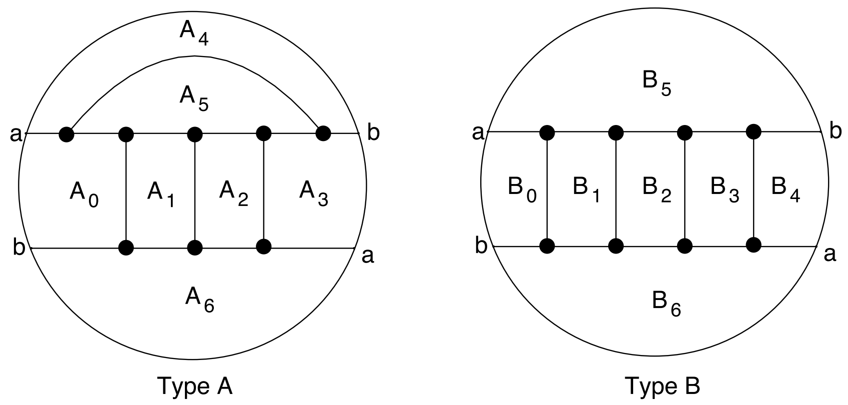

Figure 5.3.3

5.4 DISJOINT CYCLES IN CROSSING CRITICAL GRAPHS

In this chapter we use the properties of disjoint cycles and crossing number given in previous chapters to obtain some restrictions on the set of edges of a 2-crossing-critical graph G with V_8 minor and no V_{10}. At the end of this section, we prove Theorem 5.4.8. This result is a combinatorial characterization of 2-crossing-critical graphs having a V_8 minor and no V_{10} in terms of the edges in the V_8 minor.

Recall that disjoint edges e and f in a graph G are *cycle separated* if there is a pair (C_1, C_2) of disjoint cycles in G so that $e \in E(C_1)$ and $f \in E(C_2)$.

The following proposition states that crossings in a 1-drawing of a graph having a V_8 minor must be between rim branches that are not so close together.

Lemma 5.4.1. *Let G be a graph containing a subdivision H of V_8. If D is a 1-drawing of G with edges e and f crossed, then e and f are both in H and there is an i so that $e \in r_i$ and, for some $j \in \{i+3, i+4, i+5\}$, $f \in r_j$.*

Proof. We begin with a simple observation.

Claim 5.4.2. *e and f are edges in the rim R.*

Claim proof. If not, then we may assume $e \notin R$. Since $D[G-e]$ has no crossing, $G-e$ is planar. If $e \notin H$, then $V_8 \subseteq G-e$ shows $G-e$ is not planar. If $e \in H$, then e is in some spoke s of H. Since $H-s$ is a subdivision of $K_{3,3}$, $G-e$ is again not planar, a contradiction. □

If $j \in \{i-1, i, i+1\}$, then D yields a 1-drawing of V_8 with the rim edge r_i either self-crossing or crossing an adjacent rim branch. However, Observation 4.2.1 implies that this is not possible, a contradiction.

The remaining case is $j \in \{i-2, i+2\}$. In this instance, the disjoint H-quads Q_i and Q_{i+2} separate $e \in r_i$ and $f \in r_{i-2} \cup r_{i+2}$. □

Definition 5.4.3. *Let G be a crossing critical graph having $V_8 \cong H \subset G \not\supseteq V_{10}$. If e and f are edges of H and there is an i so that $e \in r_i$ and $f \in r_{i+3} \cup r_{i+4} \cup r_{i+5}$, then e and f are* crossable edges *and (e, f) is a* crossable pair.

crossable edges
crossable pair

We have the following result.

Lemma 5.4.4. *Let G be a peripherally 4-connected 2-crossing-critical graph with $V_8 \cong H \subseteq G$. Let h be an edge in G not in the rim R. Then there are no edges parallel to h.*

Proof. Suppose that h' is a parallel edge to h. Then $G - h'$ has a 1-drawing D, since G is 2-crossing-critical (and is not planar because $H - h'$ is not planar). Observe that h is not crossed since h does not belong a crossable pair. Then we can obtain a 1-drawing of G by adding h' close enough to h in D, a contradiction. □

Definition 5.4.5. *Let G be a crossing critical graph having $V_8 \cong H \subset G \not\supseteq V_{10}$. Let e, f, h be disjoint edges in G.*

- *h is (e, f)-inessential, if there exists a pair (C_1, C_2) that separates (e, f) and $h \notin E(C_1) \cup E(C_2)$.*
- *h is* inessential *in G, if it is (e, f)-inssential for every pair of crossable edges (e, f).*

inessential

Our next observation links the study of disjoint cycles and 2-crossing-critical graphs.

Observation 5.4.6. *Let G be a 2-crossing-critical graph so that $V_8 \cong H \subset G$. Then G has no inessential edges.*

Proof. Suppose for the sake of contradiction that h is an inessential edge in G. Because G is 2-crossing-critical, there is a 1-drawing D of $G - h$; let e, f be crossed edges in D. By Lemma 5.4.1, (e, f) is a crossable pair. Since h is (e, f)-inessential, there exists a pair of disjoint cycles (C_1, C_2) separating (e, f), and so that $h \neq E(C_1) \cup E(C_2)$. Then, C_1 and C_2 are two disjoint cycles crossing in a 1-drawing, contradicting Lemma 4.2.2. □

Observe that if G is 2-connected non-planar graph embedded in the projective plane, then every face is cycle The proof of this observation follows from the fact that the embedding of subdivisions of $K_{3,3}$ and K_5 in the projective plane satisfy this property. The proof is similar to the analog result in the plane: In a 2-connected plane graph, every face is bounded by a cycle. If G is a graph having a representativity 2 embedding, then we can make strong assumptions on cycles separating two disjoint edges, as our next result claims.

Lemma 5.4.7. *Let G be a graph having a representativity 2 embedding in $\mathbb{R}P^2$, and $V_8 \cong H \subset G \not\supseteq V_{10}$. If (C_1, C_2) are disjoint cycles separating a pair of edges (e, f), then there exists a pair of cycles (C_1', C_2') separating (e, f) in which either C_1' or C_2' bounds a face.*

Proof. Any two non-contractible curves in $\mathbb{R}P^2$ cross. Therefore, one of C_1 and C_2 is contractible. We may choose the labelling so that C_1 bounds a closed disc Δ_1 and $C_2 \cap \Delta_1 = \emptyset$. There is a unique face F of G contained in Δ_1 and incident with e. Since G is 3-connected and nonplanar, F is bounded by a cycle C_1', as required. □

Finally we end this section by proving that if G is a graph having crossing number 2 and such that $V_8 \cong H \subseteq G$, then any pair (e, f) of crossable edges is separated by cycles.

Theorem 5.4.8. *Let G be a peripherally 4-connected graph with $V_8 \cong H \subseteq G$. Then $\mathrm{cr}(G) \geqslant 2$ if and only if any pair e, f of crossable edges is cycle separated.*

Proof. Suppose first that $\mathrm{cr}(G) \geqslant 2$. By Theorem 4.3.1, it is enough to prove that e and f are unlinked in G. Edges e and f are disjoint by definition. Observe that V_8 is peripherally 4-connected and no three of x_1, y_1, x_2, y_2 are adjacent to a single vertex. Therefore, for any $X \subset \{x_1, y_1, x_2, y_2\}$ of size 3, $H - X$ is connected and contains the vertex in $\{x_1, y_1, x_2, y_2\} \setminus X$. Therefore e and f are unlinked.

Conversely, suppose that any pair e, f of crossable edges is cycle separated. Observe that $\mathrm{cr}(G) \geqslant 1$ since $H \subseteq G$. Suppose G has a 1-drawing D in which e and f are crossed edges. By Lemma 5.4.1, e and f is a crossable pair. However, this is not possible since e and f are cycle separated, contradicting Lemma 4.2.2. Hence $\mathrm{cr}(G) \geqslant 2$. □

5.5 H-BRIDGES

Given a graph G, and $V_8 \cong H \subset G$, there are different kinds of H-bridges in G. In this section we define some concepts related to these H-bridges in G.

The following result from [6, Lemma 4.8.2] constraints the H-bridges of G to be acyclic. Recall that $\mathcal{M}_2^3$ denotes the set of 3-connected 2-crossing-critical graphs.

Theorem 5.5.1. *Let $G \in \mathcal{M}_2^3$, then any H-bridge has no cycles.*

Theorem 5.5.1 implies each H-bridge in G is a tree, a fact we will use without further reference.

We make a distinction between trivial H-bridges in G. A *jump* is a trivial H-bridge that has both attachments in the rim R. A *diagonal* is a trivial H-bridge that has attachments in the H-nodes v_i, v_{i+5} for some i. A *semidiagonal* is a trivial H-bridge with attachments v_i, z, where z is a vertex in the interior of either r_{i+3} or r_{i+4}.

jump

diagonal

semidiagonal

Definition 5.5.2. *Let* G *be a graph with a representativity 2 embedding in* $\mathbb{R}P^2$ *and* $V_8 \cong H \subset G$. *Consider two points* x, y *in* H.

- $\mathrm{span}(x, y)$ *is the* xy*-subpath in* H *with the fewest* H*-nodes (this definition applies only to points in* H *where this* xy*-path is unique, otherwise it is not defined).*
- *Let* B *be an* H*-bridge, and suppose* x, y *are* B*-attachments such that* $\mathrm{att}(B) \subseteq \mathrm{span}(x, y)$. *We define* $\mathrm{span}(B) = \mathrm{span}(x, y)$.

span

Definition 5.5.3. *Let* G *be a graph with a representativity 2 embedding in* $\mathbb{R}P^2$ *and* $V_8 \cong H \subset G$. *Suppose* P *and* P' *are* R*-avoiding* $xy-$, $x'y'-$*paths respectively, contained in an* H*-region* $E \subseteq \mathfrak{D}$ *and such that* $x, y, x', y' \in R$. *Then:*

- P wraps P' *if* P' *is contained in the closed disc bounded by* P *and the* xRy*-arc contained in* E.
- *Let* B, B' *be two* H*-bridges contained in an* H*-region* $E \subseteq \mathfrak{D}$. *Suppose* x', y' *are attachments in* B' *such that all the attachments of* B' *are contained in the* $x'Ry'$*-arc contained in* E. *Then* B wraps B' *if there exists an* R*-avoiding* xy*-path in* B *that wraps any* $x'y'$*-path in* B'.

wrap

5.6 UNPOLLUTED V_8

A graph that has a V_8 minor might have multiple subgraphs that are V_8 subdivisions. One key idea for reducing the size of the H-bridges is to define an order on the V_8 subdivisions in any graph. At the end we will work with a V_8 that is minimal with respect to this order.

Definition 5.6.1. *Let* G *be a graph embedded in* $\mathbb{R}P^2$ *with representativity 2. Then* G *is* Type B-free *if, for every subdivision* H *of* V_8 *in* G*, the subembedding of* H *is type* A. *Let* H *be any subdivision of* V_8 *in* G *and let* $\mathfrak{M}$ *be the Möbius strip of* H *(with respect to the embedding).*

Then H *is* unpolluted *if all the following statements are true.*

unpolluted

(I) There is no subdivision H' *of* V_8 *in* G *such that the Möbius strip* $\mathfrak{M}'$ *of* H' *with respect to the embedding is properly contained in the region* $\mathfrak{M}$.
(II) If H *is Type* A*, then:*

a. G *is type* B*-free; and*

b. if $i \in \{0, 3, 5\}$*, then there is no* $H' \cong V_8$ *having the same Möbius strip* $\mathfrak{M}' = \mathfrak{M}$ *and corresponding region* A_i' *so that* $A_i' \subsetneq A_i$.

(III) If H *is Type* B*, then:*

a. there is no $H' \cong V_8$ *having the same Möbius strip* $\mathfrak{M}' = \mathfrak{M}$ *and corresponding region* B'_0 *so that* $B'_0 \subsetneq B_0$*;*

b. there is no $H' \cong V_8$ *having the same Möbius strip* $\mathfrak{M}' = \mathfrak{M}$ *and corresponding region* B'_4 *so that* $B'_4 \subsetneq B_4$ *subject to III.a;*

a. there is no $H' \cong V_8$ *having the same Möbius strip* $\mathfrak{M}' = \mathfrak{M}$*, and corresponding regions* B'_i*, for* $i = 1, ..., 7$*, so that* $B'_0 = B_0$*,* $B'_4 = B_4$ *and* $B'_2 \subsetneq B_2$*.*

Observe that any graph G that has a V_8 minor and that it is embedded in the projective plane with representativity 2, has an unpolluted subgraph $H \cong V_8$. We can find an unpolluted V_8 from a fixed a embedding of G and a V_8 minor H as follows. If there is a V_8 minor H' whose Möbius strip is contained in the Möbius strip of H, replace H by H'. We can do this until no V_8 has a Möbius strip properly contained in the Möbius strip of H. In case H has a type A embedding, for this fixed Möbius strip choose a V_8 so that for $i = 0, 3, 5$, it minimizes the number of vertices and edges inside A_i. In case H has a type B embedding, choose a V_8 that minimizes the number of vertices and edges inside B_0. We can clearly choose H so that minimizes the number of edges and vertices inside B_4 and B_2, subject to the previous restrictions.

We begin with some elementary properties of an unpolluted V_8.

Lemma 5.6.2. *Let* G *be a graph embedded in* $\mathbb{RP}^2$ *with representativity 2. Consider an embedding* D *of* G *in* $\mathbb{RP}^2$*, and let* $V_8 \cong H \subseteq G \not\supseteq V_{10}$ *such that* H *is unpolluted. Let* x *and* y *be distinct vertices in* H*. Suppose* P *is an* H*-avoiding* xy*-path contained in* $\mathfrak{M}$*. The following situations are not possible:*

(i) x *is in the interior of the rim branch* r_i *and* y *is in the interior of the rim branch* r_{i+4}*;*

(ii) x *and* y *both lie in a rim branch* r_i*;*

(iii) x *and* y *are in the interior of* s_i*,* s_{i+1} *respectively;*

(iv) x *is in* r_{i-1} *or in* r_i *and* y *is in the interior of the spoke* s_i*;*

If H *has a Type* A *embedding, none of the following is possible.*

(v) P *is contained in* A_0*,* x *and* y *lie in* $\mathrm{span}(a, v_1)$*, and* $\{x, y\} \neq \{v_7, v_1\}$*;*

(vi) P *is contained in* A_3*,* x *and* y *lie in* $\mathrm{span}(v_3, b)$*, and* $\{x, y\} \neq \{v_3, v_5\}$*;*

(vii) P *is contained in* A_0*,* x *is in* $\mathrm{span}(a, v_1) \setminus \{v_7\}$ *and* y *is in the interior of* s_1*;*

(viii) P *is contained in* A_3, x *is in* $span(v_3, b) \setminus \{v_5\}$*, and* y *is in the interior of* s_3*;*

(ix) P *is contained in* A_0, $x \in R^+$, $y \in R^-$*, except when* $x \in span(a, v_0)$ *and* $y = v_5$*, or* $x = v_7$ *and* $y \in span(b, v_5)$*, or* $x = v_1$ *and* $y = v_4$*; and*

(x) P *is contained in* A_3, $x \in R^+$, $y \in R^-$*, except when* $x \in span(v_4, b)$ *and* $y = v_7$*, or* $x = v_5$ *and* $y \in span(v_7, a)$*, or* $x = v_3$ *and* $y = v_0$*.*

Proof. If G has a path as in (i), then $H \cup P$ is a subdivision of V_{10} in G, a contradiction.

Suppose G has a path as in (ii)-(viii). The rim and the spokes can be redefined in order to obtain $H' \cong V_8$, such that its corresponding Möbius strip $\mathfrak{M}'$ is properly contained in $\mathfrak{M}$, contradicting the fact that H is unpolluted (in Figure 5.6.1 we ilustrate some of the Type A cases, showing how the rim and the spokes are defined in order to obtain a V_8 subdivision having a Möbius strip contained in $\mathfrak{M}$).

Suppose P is a path satisfying (ix) or (x). By symmetry, we may assume we are in case (ix). If $y \neq v_5$ and $y \neq v_4$, then by (i), x must be in $span(a, v_0)$. As $v_7 \neq x$, we can redefine P to be the spoke s_0 in order to obtain a Type B drawing, contradicting the fact that H is unpolluted (see Figure 5.6.2).

Now suppose $y = v_5$. Then $x \in span(v_0, v_1) \setminus \{v_0\}$. Observe that we can use P to redefine spoke s_1 in order to obtain a subgraph $H' \cong V_8$ such that the region A_0' is properly contained in A_0 (see Figure 5.6.2). This contradicts that H is unpolluted.

Suppose $y = v_4$. Since P is a path not satisfying (ix), $x \neq v_5$ and $x \neq v_1$. Observe that since $x \in span(v_7, v_0) \cup span(v_0, v_1)$, the rim R together with the spokes xv_4, s_1, s_2 and s_3 form a Type B V_8 minor, which is not possible since H is unpolluted (see Def. 5.6.1-I.a) . □

We now study, in Type A embeddings, restrictions on the paths of H-bridges contained in regions A_4, A_5 and A_6.

Lemma 5.6.3. *Let* D *be a 2-representative embedding of* G *and let* $V_8 \cong H \subseteq G \not\supseteq V_{10}$ *such that* H *is unpolluted and* $D[H]$ *is Type A. Let* x *and* y *be distinct vertices in* H*. Suppose* P *is an* xy*-path in* G*. The following situations are not possible:*

(i) P *is an* R*-avoiding path contained in* A_5*, distinct from* s_0*, and, for some* $i \in \{1, 2\}$*, the rim branch* r_i *is contained in the* xRy*-arc contained in* A_5*, and* v_i, $v_{i+1} \notin \{x, y\}$*, except when* $x = v_0$ *and* $y \in r_2 \setminus v_2$ *or when* $x = v_4$ *and* $y \in r_1 \setminus v_2$*;*

(ii) P *is an* H*-avoiding path contained in* A_6, $v_0, v_4 \notin \{x, y\}$*, and, for some* $i \in \{1, ..., 8\}$, r_i *is contained in the* xRy*-arc contained in* A_6*, and* v_i, $v_{i+1} \notin \{x, y\}$*; and*

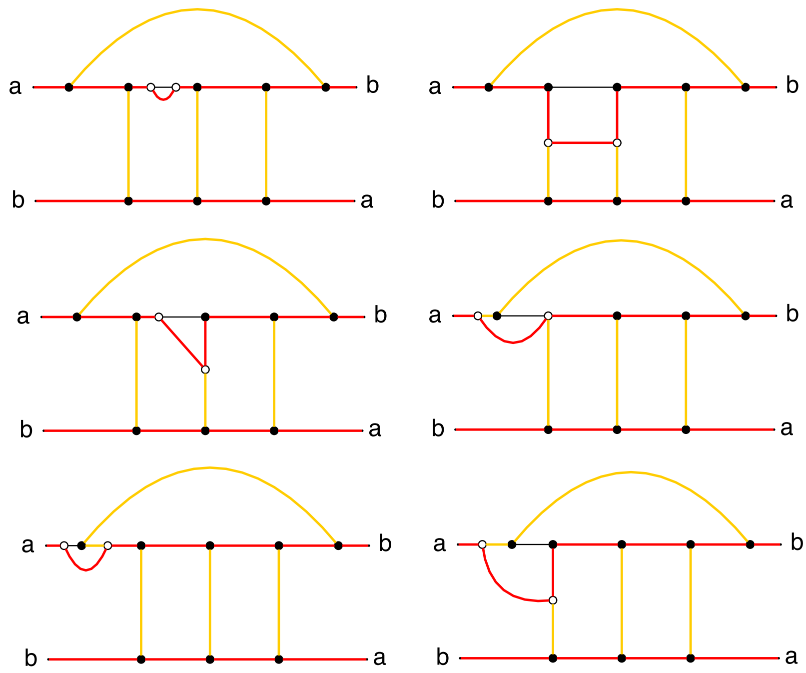

Figure 5.6.1

(iii) P *is an* H*-avoiding path contained in* A_4*,* $x \in \mathrm{span}(a, v_0) \setminus \{v_0\}$*,* $y \in \mathrm{span}(v_4, b) \setminus \{v_4\}$*, and* $\{x, y\} \cap \{v_5, v_7\} = \emptyset$*.*

Proof. Suppose P is as in (i), if either $x = v_0$ and $y \in \mathrm{span}(v_3, v_4) \setminus \{v_3\}$, or $x = v_4$ and $y \in \mathrm{span}(v_0, v_1) \setminus \{v_1\}$, then we can use the xy-path to redefine the spoke s_0 in order that $H' \cong V_8$ has the same Möbius strip, and its correspondent region A'_5 is properly contained in A_5 (see Figure 5.6.3) , contradicting that H is unpolluted.

Hence we may assume x, y belong to the open $v_0 R^+ v_4$-arc contained in A_5. In this case, G has a V_8 subdivision having a Type B embedding (above graph in Figure 5.6.4 shows how to get this V_8 subdivision), a contradiction.

If P is as in (ii)-(iii), we can always obtain a V_8 subdivision having a Type B embedding (see graphs below in Figure 5.6.4). □

5.7 H-bridges inside Möbius strip are trivial

Using the results in the preceding section, we are able to prove an important observation about H-bridges contained in the Möbius strip of an unpolluted V_8.

For simplicity we only consider peripherally 4-connected 2-crossing-critical graphs.

P_2^4 **Definition 5.7.1.** *Let* P_2^4 *denote the set of pairs* (G, H)*, in which* G *is a*

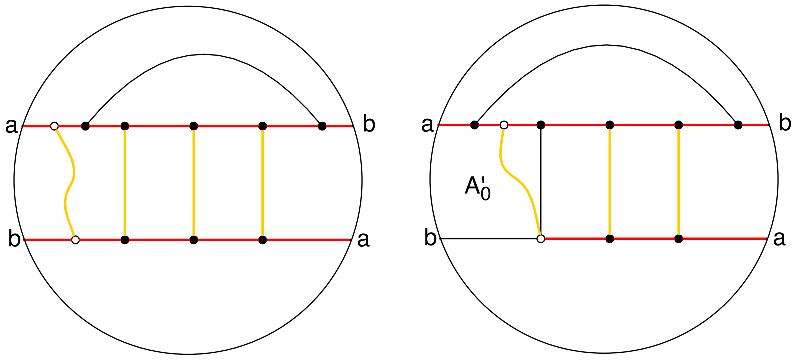

Figure 5.6.2

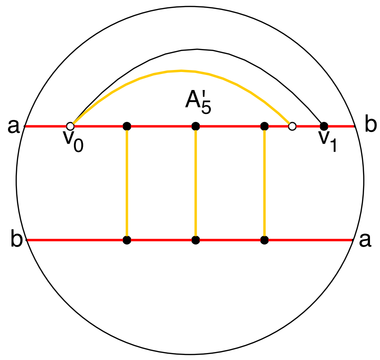

Figure 5.6.3

peripherally 4-connected 2-crossing-critical graph, containing a subdivision H *of* V_8 *and no* V_{10} *subdivision, so that* G *has a 2-representative embedding in* $\mathbb{R}P^2$*, and* H *is unpolluted.*

Theorem 5.7.2. *Let* $(G, H) \in \mathbf{P}_2^4$*. If* B *is an* H*-bridge contained in the Möbius strip of* H*, then* B *is an edge.*

Before we proceed to prove the theorem, we will show the following.

Lemma 5.7.3. *Let* $(G, H) \in \mathbf{P}_2^4$*. Suppose* B *is an* H*-bridge having one attachment* w *not in the spoke* s *and other attachment in the interior of* s*. Then:*

(i) w *is the only attachment not in* s*;*

(ii) H *has a type* A *embedding, and either* $w = a = v_7$*,* $B \subseteq A_0$*, or* $w = b = v_5$*,* $B \subseteq A_3$*; and*

(iii) B *is the edge* zw*, and there is no other* H*-bridge* B'*, distinct from* B*, that has attachments in both the interior of* s *and in* $H - s$*.*

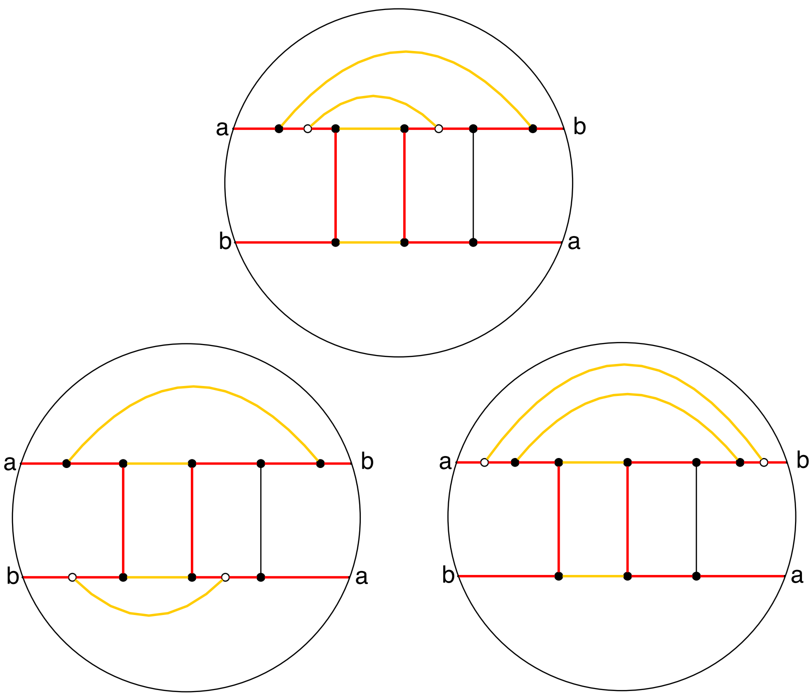

Figure 5.6.4

Proof. Lemma 5.6.2(iii) shows that w is not in the interior of a different spoke. Parts (iv), (vii) and (viii) of the same lemma show w is not in $R - s$, unless G has a type A embedding and either $B \subset A_0$, $w = v_7$, or $B \subset A_3$, $w = v_5$. Hence (i) and (ii) hold.

By symmetry, we may assume $B \subseteq A_0$, $w = a = v_7$. Suppose B has an attachment $y \in \mathrm{att}(B) \setminus \{z, w\}$. Uniqueness of w implies that $\mathrm{att}(B) \setminus \{w\} \subset s$, and hence $y \in s$. Using an H-avoiding zy-path P in B, we can replace s_0 with P plus a bit of s_0 to obtain a new subdivision H' of V_8 so that the H-region A_0' corresponding to H' is properly contained in A_0. However, H is unpolluted, so the existence of H' violates condition (II.b) in Definition 5.6.1, a contradicition. Hence $B = wz$.

□

Proof of Theorem 5.7.2. Suppose for the sake of contradiction that G has a nontrivial H-bridge B contained in $\mathfrak{M}$. Since G is 3-connected, B has at least three attachments.

By Lemma 5.7.3, the proof may be split into two cases: $\mathrm{att}(B) \subset R$ and $\mathrm{att}(B) \subseteq s$.

Case 1. $\mathrm{att}(B)$ *is contained in* R.

Condition (ii) in 5.6.2 shows that no two attachments of B are in the same rim branch. Since $|\mathrm{att}(B)| \geqslant 3$, H is Type A and B is contained in either A_0 or A_3; we may assume B is contained in A_0 and B has an attachment in each of ar_7v_0, r_0, and br_4v_5.

Condition (v) of 5.6.2 shows the first two must be $a = v_7$ and v_1, respectively. Now Condition (ix) of 5.6.2 shows there is no H-avoiding path from v_1 to the attachment in r_4, unless the attachment in r_4 is $b = v_4$. Hence we have that $\text{att}(B) = \{a = v_7, b = v_4, v_1\}$. However, in this case H is not unpolluted since we can define a V_8 minor H' having a Möbius strip contained in $\mathfrak{M}$ as shown in Figure 5.7.1. This contradiction shows $|\text{att}(B)| = 2$.

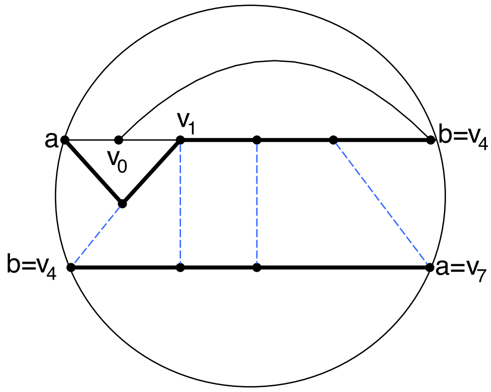

Figure 5.7.1

Case 2. $\text{att}(B)$ *is contained in a spoke* s.

Since B is not trivial, B has an attachment z in the interior of s. We divide the proof in cases depending on whether the H-regions incident to s are both bounded by quads or not.

Subcase 2.1. H *is type* A *and* $s = s_1$ *or* $s = s_3$*;*

Without loss of generality suppose $s = s_1$. Lemma 5.7.3 implies that every H-bridge B with an attachment in the interior of s_1 either has all its attachments in s_1 or at v_7. Setting $X = \{v_7, v_1, v_5\}$, we see that z is in a component K of $G - X$ that does not contain v_2 and v_6. Morever, $\text{nuc}(B)$ is also in K. But then G is not peripherally 4-connected, a contradiction that settles this case.

Subcase 2.2. H *is type* A *and* $s = s_2$*; or* H *is type* B.

In this case, let v_i, v_{i+4} be the ends of s. Lemma 5.7.3 implies that $\{v_i, v_{i+4}\}$ is a 2-cut disconnecting z from R, contradicting the fact that G is 3-connected. □

Using Lemma 5.7.3, and the fact that G is peripherally 4-connected, we obtain the following.

Corollary 5.7.4. *Let* $(G, H) \in \mathbf{P}_2^4$*. Let* s *be a spoke contained in* $\mathfrak{M}$*. Then either:*

- *s is an edge;*
- *$s = s_1$ has one internal vertex that is adjacent only to v_1, v_5, and v_7; or*
- *$s = s_3$ has one internal vertex adjacent only to v_3, v_7 and v_5.*

Now we can prove that there are no H-bridges having all its attachments in a spoke contained in the Möbius strip.

Theorem 5.7.5. *Let $(G, H) \in \mathbf{P}^4_2$. Let s be a spoke contained in $\mathfrak{M}$. Then there is no H-bridge whose attachments are contained in s.*

Proof. Let v_i and v_{i+4} be the ends of s. Suppose B is a H-bridge so that $\mathrm{att}(B) \subset s$. Theorem 5.7.2 implies that B is an edge attached to S. According to Corollary 5.7.4, we have three possible cases.

Case 1. *s is an edge.*

In this case B is an edge parallel to s, which is not possible by Corollary 5.4.4, since s is not a crossable edge.

Case 2. *s is the subdivided edge v_1, z, v_5.*

If B is parallel to either v_1z or zv_5, then G is not 2-crossing critical because of Corollary 5.4.4 (since v_1z and zv_5 are not crossable edges). Hence $B = v_1v_5$. Moreover, B is the only H-bridge having all its attachments in s, since there are no edges parallel to B.

Since G is 2-crossing-critical, $G - B$ has a 1-drawing D'. Observe that since s is not crossed, there exists a face F of $D'[G - B]$, incident with all of v_1, z, and v_5 in which we can draw edge B inside F, and B is not crossed. This yields a 1-drawing of G, a contradiction.

Case 3. *s is the subdivided edge v_3, z, v_7.*

The proof is symmetric to Case 2. □

Previous theorems in this section and the fact that H is an unpoluted V_8, completely determine the possible subgraphs induced by any H-region inside the Möbius strip $\mathfrak{M}$. For instance, our next result is an easy consequence of Theorems 5.7.2, 5.7.5 and Lemma 5.6.2. But first we need a definition.

$\mathbf{A}^4_2$ $\mathbf{B}^4_2$ **Definition 5.7.6.** *We partition $\mathbf{P}^4_2$ into two sets $\mathbf{A}^4_2$, $\mathbf{B}^4_2$, the former consisting of the pairs (G, H) with H being Type A and the latter having H of Type B.*

We urged the reader to remember this notation since it will be used in most of the theorems in the next two chapters.

Corollary 5.7.7. *Let $(G, H) \in \mathbf{P}^4_2$. Then:*

(i) if $(G, H) \in \mathbf{A}^4_2$, then the H-bridges inside A_1 and A_2 are diagonals or semidiagonals; and

(ii) if $(G, H) \in \mathbf{B}_2^4$*, then the* H*-bridges inside* B_0, B_1, B_2, B_3, B_4 *are diagonals and semidiagonals.*

After this result, the reader might wonder if a 2-crossing-critical graph has "many" semidiagonals inside an H-region E from the previous theorem. The answer is that we can bound the number of semidiagonals inside E; for this purpose, in next chapter we introduce the concept of a horn subgraph of G, and we will see how this concept is intimately related to the description of some substructures that G might contain.

For now, we can make an observation when G has a type B embedding. This observation fully describes the regions B_0, B_2, B_4 and follows from the fact that H is unpolluted.

Observation 5.7.8. *Let* $(G, H) \in \mathbf{B}_2^4$*. Then* B_0 *and* B_2 *have no semidiagonals in their interiors, while* B_0, B_2 *and* B_4 *have at most one diagonal each in their interiors.*

□

5.8 FORBIDDEN CYCLES IN 2-CROSSING-CRITICAL GRAPHS

In this section we recall some results used in the classification of graphs having a V_{10} minor [6]. A box in a graph (a concept that we will define later) is a forbidden cycle in a 2-crossing-critical graph. This enable us to add some restrictions on cycles in a 2-crossing-critical graph.

The main result of this section (Lemma 5.8.11) has the following nice consequence.

> *"If* $(G, H) \in \mathbf{P}_2^4$ *and* e *is a jump whose ends are contained in the union of two consecutive* H*-rim branches, then the cycle* $C = \mathrm{span}(e) + e$ *is a face of* G*."*

This implies that no H-bridge is contained inside the closed disk bounded by C. Informally speaking, this limits the possibility of many nested "small" jumps.

Definition 5.8.1. *A cycle* C *in a graph* K *is a* K-prebox *if, for each edge* e *of* C, $K - e$ *is not planar.* K-*prebox*

Definition 5.8.2. *Let* H *be a subgraph of a graph* G*. Then* $H^{\#}$ *is the graph induced by* $E(G) \setminus E(H)$. $H^{\#}$

Observe that if H is a subgraph of G and B is an H-bridge, then $B^{\#}$ is the union of C and all C-bridges other than B.

Recall definition of a cycle C to have a bipartite overlap diagram (Def. 2.2.2).

box

Definition 5.8.3. *A cycle* C *in a graph* G *is a* box *in* G, *if* C *has bipartite overlap diagram in* G *and, there is a planar* C*-bridge* B *so that* C *is a* $B^\#$*-prebox.*

Theorem 5.8.4. [6, Lemma 5.12] *If* G *is a 2-crossing-critical graph, then* G *has no box.*

Before we prove this, first we need to show a basic result.

Lemma 5.8.5. [6, Lemma 5.4] *If* K *is a graph and* C *is a* K*-prebox, then, for any 1-drawing* D *of* K, C *is clean in* D. *Recall that* C *is clean in* D *means no edge of* C *is crossed in* D *(Def. 2.2.5)*

Proof. Suppose e is an edge in C such that e is crossed. Then $D[K - e]$ has no crossings, contradicting the assumption that $K - e$ is not planar. □

Proof of Theorem 5.8.4. Suppose C is a box in G. Let B be a planar C-bridge B so that C is a $B^\#$-prebox. As $B^\#$ is a proper subgraph of G, $B^\#$ has a 1-drawing D. Lemma 5.8.5 implies C is clean in D. Since C has bipartite overlap diagram and G is not planar, G has a non-planar C-bridge B' (Theorem 2.2.4). Observe that any crossing in $D[B' \cup C]$ involves edges in B', otherwise C is not clean in D. Therefore the crossing in $B^\#$ is between edges in B'. Applying Lemma 2.2.6 to C and B', we obtain that $cr(G) \leqslant 1$, a contradiction. Thus G has no box. □

Definition 5.8.6. *Let* G *be a graph,* $V_{2n} \cong K \subseteq G$, $n \geqslant 3$, *and let* F *be a subgraph of* G. *Then:*

claw

1. *a* claw *is a subdivision of* $K_{1,3}$ *with* centre *the vertex of degree 3 and* talons *the vertices of degree 1;*
2. *an* $\{x, y, z\}$-claw *is a claw with talons* x, y *and* z*;*
3. *an* open K-claw *is the subgraph of* K *obtained from a claw in* K *consisting of the three* K*-branches incident with an* K*-node, which is the centre of the open* K*-claw, but with the three talons deleted;*

K-close

4. F *is* K-close *if* $F \cap K$ *is contained either in a closed* K*-branch or in an open* K*-claw.*

Informally speaking, a K-close graph F is a graph such that the points in $F \cap K$ are close to each other relative to K. In the following sections, we will focus on K-close graphs such that $K \cong V_6$ is obtained by deleting a spoke from $H \cong V_8 \subseteq G$. The following observation states that when two vertices are in a K-close subgraph for $K \cong V_6 \subset H \cong V_8$, $\text{span}(x, y)$ is defined.

Observation 5.8.7. *Let* x, y *be vertices in* H, *a subgraph of* G *topologically equivalent to* V_8. *Suppose* $\{x, y\}$ *is* K*-close for some* $K \cong V_6 \subset H$. *Let* J *be a subgraph of* K *containing* x, y *and so that* J *is either a closed rim* K*-branch or an open* $\{z_1, z_2, z_3\}$*-claw in* K. *Then* $\mathrm{span}(x, y)$ *is defined and is equal to the* xy*-path in* J.

From now on, we refer to $\mathrm{span}(x, y)$ only when $\{x, y\}$ is K-close for $K \cong V_6 \subset H \cong V_8$.

The following two results help us to find circumstances in which cycles are preboxes and have bipartite overlap diagram.

Lemma 5.8.8. [6, Lemma 5.3] *Let* C *be a* K*-close cycle, for some* $K \cong V_6$. *Then* C *is a* $(C \cup K)$*-prebox.*

Proof. Let e be in C. If e is not in K, then $K \subseteq (C \cup K) - e$. Thus $(C \cup K) - e$ is not planar.

Suppose e is in K. Since C is K-close, $C \cap K$ is contained in either a closed K-branch b, or is contained in an open K-claw Y. There is a K-avoiding path P in $C - e$ having ends in both components of either $b - e$ or $Y - e$. In the former case, $(K - e) \cup P \subseteq (C \cup K) - e$ contains a V_6. In the latter case, $(Y - e) \cup P$ contains a different claw that has the same talons as Y, so again $(K - e) \cup P$ and, $(C \cup K) - e$ contains a V_6. □

Next result from [6] is useful for determining whether a cycle of a graph embedded in the projective plane has bipartite overlap diagram.

Lemma 5.8.9. [6, Corollary 5.17] *Let* G *be a graph embedded in* $\mathbb{RP}^2$ *and let* C *be a cycle of* G. *Let* B *be a* C*-bridge that* $\mathrm{nuc}(B)$ *contains a non-contractible cycle. Then* C *is contractible,* C *has bipartite overlap diagram, and every* C*-bridge other than* B *is planar.*

Corollary 5.8.10. *Let* G *be a graph having a representativity 2 embedding* D *in* $\mathbb{RP}^2$, *and* $V_8 \cong H \subseteq G$. *Let* x, y *be vertices in* H, *and* P *be an* H*-avoiding* xy*-path so that* P *is* K*-close for some* $V_6 \cong K \subset H$. *Let* C *be the cycle* $P \cup \mathrm{span}(x, y)$ *and let* M_C *be the* C*-bridge such that* $H \subseteq M_c \cup C$. *Then* C *is contractible,* C *has bipartite overlap diagram, and for every* C*-bridge* B', *distinct from* M_C, $C \cup B'$ *is planar.*

Proof. Since $\mathrm{span}(x, y)$ is K-close for some $K \cong V_6$, there exists a non-contractible cycle in $H \setminus \mathrm{span}(x, y) = H - C \subseteq M_C$. Apply Lemma 5.8.9 to C and $B = M_C$ to obtaining the result. □

Now we are able to prove the main result of this section.

Lemma 5.8.11. *Let* $G \in \mathcal{M}_2^3$, $V_8 \cong H \subseteq G$ *so that* G *has a representativity 2 embedding in* $\mathbb{RP}^2$. *Let* x, y *be vertices in* H, *and* P *be an* H*-avoiding* xy*-path so that* P *is* K*-close for some* $V_6 \cong K \subset H$. *Let* C *be the cycle* $P \cup \mathrm{span}(x, y)$ *and let* M_C *be the* C*-bridge such that* $H \subseteq M_c \cup C$. *Then* C *bounds a face in the embedding of* G.

Proof. C is contracible by Corollary 5.8.10. Suppose C is not a face. Then, there is a C-bridge B contained in the closed disk Δ bounded by C. Observe that M_C has no vertices inside Δ, so $B \neq M_C$. Corollary 5.8.10 implies that C has bipartite overlap diagram and B is a planar C-bridge.

Recall C is K-close, Lemma 5.8.8 implies that C is a $(C \cup K)$-prebox, and since $C \cup K \subseteq C \cup H \subseteq B^\#$, consequently C is a $B^\#$-prebox. Thus C is a box in G, contradicting Theorem 5.8.4. □

5.9 BRIDGES HAVING CLOSE ATTACHMENTS

Let $G \in \mathcal{M}_2^3$, $V_8 \cong H \subseteq G$. In this section, we prove several technical lemmas restricting the location of attachments of H-bridges.

Theorem 5.9.1. [6, Corollary 7.4] *Let $G \in \mathcal{M}_2^3$, $V_8 \cong H \subseteq G$ so that G has a representativity 2 embedding in $\mathbb{R}P^2$. Let B be an H-bridge. Let K be a subgraph of H, such that $K \cong V_6$. Let Q be a K-closed branch, contained in H. Then B has at most two attachments in Q.*

Proof. Suppose G has three attachments x, z and y. For any distinct $r, s \in \{x, y, x\}$, the cycle $C_{r,s} = rQs \cup rYs$ satisfies 5.8.11 and so is contractible. Pairwise, they intersect in paths, so the union of some two of the discs they bound is the third one. That largest one does not bound a face, contradicting 5.8.11. □

Next corollary is an immediate consequence of Theorem 5.9.1.

Corollary 5.9.2. *Let $G \in \mathcal{M}_2^3$, $V_8 \cong H \subseteq G$ so that G has a representativity 2 embedding in $\mathbb{R}P^2$. Let B be an H-bridge. Then, for any $i \in \{0, 1, ..., 7\}$, B has at most two attachments in $r_i \cup r_{i+1}$.* □

Our next corollary is a little less obvious.

Corollary 5.9.3. *Let $G \in \mathcal{M}_2^3$, $V_8 \cong H \subseteq G$ so that G has a representativity 2 embedding in $\mathbb{R}P^2$. Let B be an H-bridge, and suppose that B is K-close for some $V_6 \cong K \subset H$. If $x, y \in \mathrm{att}(B)$ are such that $\mathrm{span}(x, y) = \mathrm{span}(B)$, then B is the edge xy and wraps no other H-bridge.*

Proof. Theorem 5.9.1 implies that B does not have attachments distinct from x and y. Therefore B is an xy-path. As G is 3-connected, this path has length 1, therefore $B = xy$. Finally, $C = B \cup \mathrm{span}(B)$ bounds a face by Lemma 5.8.11, therefore B wraps no H-bridge. □

Corollary 5.9.4. *Let $G \in \mathcal{M}_2^3$, $V_8 \cong H \subseteq G$ so that G has a representativity 2 embedding in $\mathbb{R}P^2$. Suppose B is an H-bridge that has K-close attachments x, y for some $V_6 \cong K \subset H$. Let P be an H-avoiding xy-path contained in B. Then P wraps no H-bridge.*

Proof. As in previous theorems, cycle $C = P \cup \mathrm{span}(P)$ is a face in G. Therefore P wraps no bridge. □

Definition 5.9.5. *Let* G *be a graph with* $V_8 \cong H \subseteq G$*. Let* P_1, P_2 *be* H*-avoiding* x_1y_1*-,*x_2y_2*-paths respectively, so that* x_1, y_1, x_2, y_2 *are in the rim* R*. Let* B *an* H*-bridge. Suppose* P_1, P_2 *and* B *are all contained in an* H*-region* E *not inside the Möbius strip.*

- P_1 *is* small *if* $\{x, y\}$ *is* K*-close for some* $V_6 \cong K \subseteq H$*. Otherwise,* P_1 *is* large*;* *small path*
- P_1 *and* P_2 *are* independent *if* $\{x_1, x_2\} \cap \{x_2, y_2\} = \emptyset$*.* *independent paths*
- P_1 *and* P_2 *are* parallel *if they are independent and one of them wraps the other.* *parallel paths*
- B *is* small *if every* H*-avoiding path* P *contained in* B *is small; otherwise* B *is* large*.* *small* H*-bridge*

Observation 5.9.6. *Let* G *be a graph with* $V_8 \cong H \subseteq G$*. Let* P *be a large* xy*-path contained in the* H*-region* E*, and let* Q *be the minimum subarc in* $E \cap R$ *containing* x *and* y*. Then* Q *contains at least two consecutive rim branches* r_i, r_{i+1}*, and at least one of them is in the interior of* Q*.* □

The following observation is a consequence of Corollary 5.9.2 and the fact that a non-trivial H-bridge has at least three attachments.

Observation 5.9.7. *Let* G *be a graph with* $V_8 \cong H \subseteq G$*. Then any non-trivial* H*-bridge* B *that has all its attachments in the rim* R *is large.* □

Finally we prove that the number of parallel large paths contained in an H-region is bounded. This may be viewed as the first of our results limiting the H-bridges.

Lemma 5.9.8. *Let* $G \in \mathcal{M}_2^3$*,* $V_8 \cong H \subseteq G$ *so that* G *has a representativity 2 embedding in* $\mathbb{R}P^2$*. Then* G *does not have three large paths that are pairwise parallel.*

Proof. Suppose for the sake of contradiction that P_1, P_2, P_3 are paths that are pairwise parallel. We may assume P_2 wraps P_1 and P_3 wraps P_2. Let x_1, y_1 be the ends of P_1.

Since P_1 is large, Observation 5.9.6 implies that $Q_1 = x_1Ry_1$ must contain a rim branch r_j in its interior. Now we can find a V_{10} having rim $R' = (R - (r_j \cup r_{j+4})) \cup (s_j \cup s_{j+1})$ and spokes P_3, P_2, P_1, r_j and r_{j+4}, a contradiction. □

6

CLASSIFYING 3-CONNECTED, 2-CROSSING-CRITICAL GRAPHS HAVING V_8 MINOR AND NO V_{10}

In this Chapter we describe 2-crossing-critical-graphs having a V_8 minor and no V_{10} that have a 2-representative embedding in $\mathbb{R}P^2$, by giving conditions on the H-bridges contained inside each of the H-regions.

Horn subgraphs, a concept constantly used in this Chapter, is defined in Section 6.2 and more carefully explored in Section 6.3. The horns help us to find bounds on the number of H-bridges inside some H-regions. It is remarkable that Lemma 6.3.16, which is the key result for finding these bounds, ultimately depends on the results that relate disjoint cycles and the crossing number of a graph.

In [6, Section 15], it is shown that, with 4 exceptions, a 3-connected 2-crossing-critical graph can be reduced to a 3-connected 2-crossing-critical graph in which every 3-cut has two components, one of which is either a single vertex incident with 0, 1, 2 or 3 parallel pairs of edges or is precisely the configuration in Figure 6.0.1. In order to verify the utility of our approach, we simplify matters by considering only those 2-crossing-critical graphs that reduce to a peripherally 4-connected graph; in other words, we ignore the configuration in the figure. These are obviously simpler than the most geneneral case, yet still retain sufficient complication.

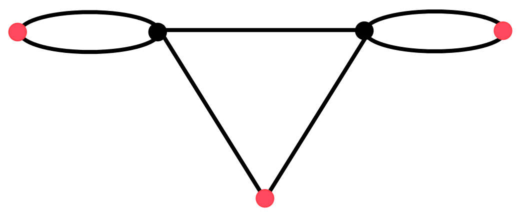

Figure 6.0.1

The objective in this chapter is to give a description of the seven different H-regions (see Figure 5.3.3) for type A and type B embeddings. This description is summarized in the following result.

Theorem 6.0.9. *Let $(G, H) \in \mathbf{P}_2^4$. Then any H-bridge has at most 5 attachments and has at most 3 vertices in its nucleus.*

(a) If the embedding of H is Type A, then:

- *the contents of the H-regions A_0 and A_3 are determined in Theorem 6.6.3;*

- *the contents of the* H*-regions* A_1 *and* A_2 *are determined in Theorem* 6.4.2;
- *the contents of the* H*-region* A_4 *is determined in Theorem* 6.2.9;
- *the contents of the* H*-region* A_5 *is determined in Theorem* 6.7.9.
- *the contents of the* H*-region* A_6 *is determined in Theorem* 6.1.5.

(b) *If the embedding of* H *is Type* B*, then*

- *the contents of the* H*-regions* B_0, B_2, B_4 *are determined in Corollary* 5.7.7 *and Observation* 5.7.8;
- *the contents of the* H*-regions* B_1, B_3 *are determined in Theorem* 6.4.3;
- *the contents of the* H*-regions* B_5, B_6 *are determined in Theorems* 6.8.2, 6.8.3.

In [6] it was proved that in any 3-connected 2-crossing-critical graph G, such that $V_8 \cong H \subset G$ (H not necessarily unpolluted), each H-bridge B has $|att(B)| \leqslant 45$. Our new approach allow us to improve this upper bound to 5 when we considerate H to be unpolluted. Furthermore, we can enumerate all possible H-bridges inside of almost every H-region, including all those contained inside the Möbius strip.

6.1 BRIDGES INSIDE A_6

In this section, we assume the induced embedding in $\mathbb{R}P^2$ of the unpolluted subdivision H of V_8 is Type A. With this hypothesis, we can find some restrictions on the H-bridges embedded in H-region A_6 (see Figure 5.3.3 or 6.1.1). All the graphs we are considering in this Chapter are peripherally 4-connected.

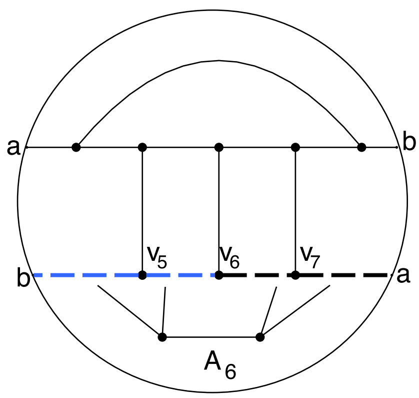

Figure 6.1.1

Theorem 6.1.1. *Let* $(G, H) \in \mathbf{A}_2^4$*. If* B *be an* H*-bridge contained in* A_6*. Then* B *has at most 4 attachments and* $|nuc(B)| \leqslant 2$.

Proof. Observe that by Theorem 5.9.2, B has at most two attachments in each of $Q_1 = \mathrm{span}(b, v_6)$ and in $Q_2 = \mathrm{span}(v_6, a)$. Since $R^- = Q_1 \cup Q_2$, then B has at most 4 attachments in R^-. Recall that G is peripherally 4-connected, so each vertex in $\mathrm{nuc}(B)$ has degree at least 3. We know by Theorem 5.5.1 that B is a tree, so necessarily $|\mathrm{nuc}(B)| \leqslant 2$. □

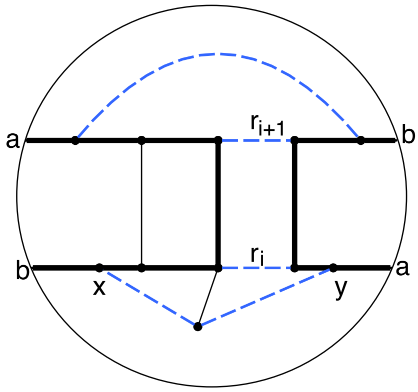

Figure 6.1.2

Theorem 6.1.2. *Let* $(G, H) \in \mathbf{A}_2^4$. *Let* B *is an* H*-bridge contained in* A_6 *such that neither* v_0 *nor* v_4 *are attachments of* B, *then* B *is an edge, and the minimal subpath in* $A_6 \cap R$ *containing all attachments of* B *does not contain a rim branch in its interior.*

Proof. Let Q be the minimal subpath in $A_6 \cap R$ containing all attachments of B. We begin by proving the following.

Claim 6.1.3. *No rim branch is contained in the interior of* Q.

Proof of Claim. Suppose Q contains a rim branch r_i in its interior. Let x and y be the attachments of B so that $Q = x(R \cap A_6)y$. Let P be an H-avoiding xy-path contained in B. Since neither v_0 nor v_4 are attachments of B, $x, y \notin \{v_0, v_4\}$. Thus, we can define $H' \cong V_8 \subseteq G$, having rim $R' = (R - \{r_i, r_{i+4}\}) \cup (\{s_i, s_{i+1}\})$, and spokes s_0, r_i, r_{i+4}, P (example in Figure 6.1.2). Notice that $D[H']$ is a type B embedding, contradicting that H is unpolluted. □

Lemma 5.9.7 implies that B is not a non-trivial bridge, otherwise B is large, and Observation 5.9.6 implies that Q contains a rim branch in its interior, which is not possible by the previous Claim. Therefore B is an edge satisfying the required conditions. □

In a Type A embedding of H, if $a = v_0$ and $b = v_4$, then we can redraw spoke s_0 inside $\mathfrak{M}$. In this case we can obtain a Type B embedding of H. This is not possible when H is unpolluted. Hence we have the following.

Observation 6.1.4. *Let* $(G, H) \in \mathbf{A}_2^4$. *Then either;*

(i) $a, b \notin \{v_0, v_4\}$*; or*

(ii) $a = v_0$ *and* $b \neq v_4$*; or*

(iii) $b = v_4$ *and* $a \neq v_0$. □

Our next result follows directly from Lemma 6.1.2 and the preceding observation.

Theorem 6.1.5. *Let* $(G, H) \in \mathbf{A}_2^4$. *Let* B *be an* H*-bridge contained in* A_6. *Then* B *is either an edge, or is a non-trivial* H*-bridge so that* $|\mathrm{nuc}(B)| \leqslant 2$, $|\mathrm{att}(B)| \leqslant 4$*, and either*

- G *is in case* 6.1.4*(ii), and* $v_0 \in \mathrm{att}(B)$*; or*
- G *is in case* 6.1.4*(iii), and* $v_4 \in \mathrm{att}(B)$.

6.2 BRIDGES INSIDE A_4

In this section, we describe the bridges contained in the H-region A_4 (see Figure 5.3.3 or 6.2.1). For this, we introduce the concept of horn subgraph. We begin limiting the size of the H-bridges.

Lemma 6.2.1. *Let* $(G, H) \in \mathbf{A}_2^4$. *Let* B *be an* H*-bridge contained in* A_4. *Suppose* B *has attachments*

$$x \in \mathrm{span}(a, v_0) \setminus \{v_0\} \text{ and } y \in \mathrm{span}(v_4, b) \setminus \{v_4\}.$$

Then $x = a = v_7$ *and* $y = b = v_5$.

Proof. There is an H-avoiding xy-path P in B. By Lemma 5.6.3(iii), either $x = a = v_7$ or $y = b = v_5$. We may assume $x = a = v_7$. If $y \neq v_5$, then G has a V_8 minor having a type B embedding, having rim $R' = (R - (r_1 \cup r_5)) \cup (s_1 \cup s_2)$ and spokes P, s_0, r_1, r_5. Thus G is not unpolluted, a contradiction. Therefore $y = b = v_5$. □

Our next result is an immediate consequence of Lemma 5.8.11.

Observation 6.2.2. *Let* $(G, H) \in \mathbf{A}_2^4$. *Let* P *be an* H*-avoiding* xy*-path contained in* A_4*, with* $\{x, y\} \subset (\mathrm{span}(a, v_0) \cup s_0) \setminus \{v_4\}$ *or* $\{x, y\} \subset (s_0 \cup \mathrm{span}(v_4, b)) \setminus \{v_0\}$. *Then,* $P \cup \mathrm{span}(x, y)$ *bounds a face contained in* A_4.

Theorem 6.2.3. *Let* $(G, H) \in \mathbf{A}_2^4$. *Let* B *be an* H*-bridge contained in* A_4. *Suppose* B *has attachments* $x \in \mathrm{span}(a, v_0) \setminus \{v_0\}$, $y \in \mathrm{span}(v_4, b) \setminus \{v_4\}$. *Then either*

- B *is the edge* v_7v_5*, and can be embedded in* A_6 *instead of* A_4*;*
- B *is* $K_{1,3}$*, having attachments* v_7, v_5*, and a vertex in* s_0*;*

- B *has the four attachments* v_7*,* v_5*,* v_0*,* v_4 *and* $|\mathrm{nuc}(B)| = 1$ *or* 2*.*

Proof. By Lemma 6.2.1, the only attachments of B in $(\mathrm{span}(a, v_0) \setminus \{v_0\}) \cup (\mathrm{span}(v_4, b) \setminus \{v_4\})$ are v_5 and v_7. Observe this implies $a = v_7$ and $b = v_5$. If B has no attachment in s_0, then B is the edge v_7v_5. Therefore we can embed B in A_6, and B does not overlap any H-bridge in A_6.

Suppose B has an attachment y in s_0. If y is in the interior of s_0, apply Observation 6.2.2 to H-avoiding v_7y- and v_5y- paths. In this case B has only these attachments. Since G is 3-connected and B is a tree, we have $B = K_{1,3}$.

Thus, we may assume B has no attachments in the interior of s_0. In particular either $y = v_0$ or $y = v_4$. If $\mathrm{att}(B) = \{v_7, y, v_5\}$, then we are done. Otherwise B has another attachment y', and $\{y, y'\} = \{v_0, v_4\}$. Then, applying Observation 6.2.2 to the v_7v_0-, v_0v_4-, and v_4v_5-paths in B, we obtain that $\mathrm{att}(B) = \{v_7, v_5, v_0, v_4\}$, and, since G is 3-connected and B is a tree, $|\mathrm{nuc}(B)| \leqslant 2$. □

Definition 6.2.4. *Let* $(G, H) \in \mathbf{P}_2^4$*.*

(i) *A bridge* B *is* A_4-degenerate*, if* B *is one of the* H*-bridges described in Theorem 6.2.3.* *A_4-degenerate*

(ii) *The spoke* s_0 *is* completely wrapped *if there is an* H*-avoiding path* $P \subseteq A_4$ *that wraps* s_0*, so that* P *and* s_0 *are internally disjoint, and* P *is not contained in an* A_4*-degenerate bridge.* *completely wrapped*

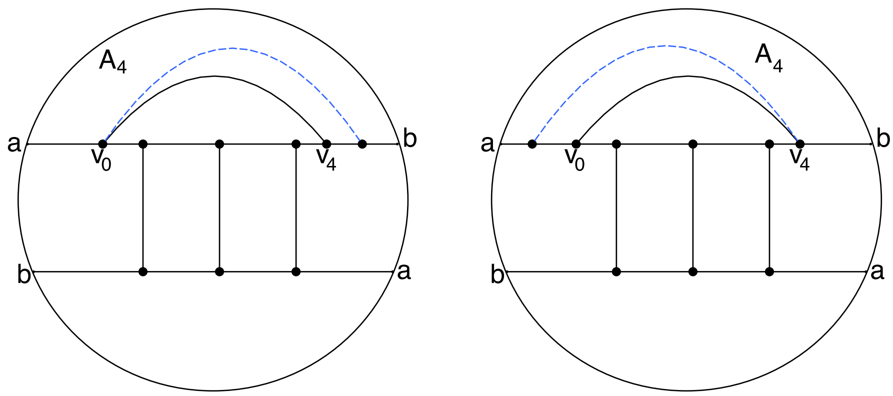

Figure 6.2.1

Observe that if s_0 is completely wrapped, by Lemma 6.2.1, there exists an H-avoiding xy-path P such that either $x = v_0$ and $y \in \mathrm{span}(v_4, b)$, or $x \in \mathrm{span}(a, v_0)$ and $y = v_4$ (see Figure 6.2.1). In case the spoke s_0 is not completely wrapped, the bridges in A_4 are easy to describe, as the following observation asserts.

Observation 6.2.5. *Let* $(G, H) \in \mathbf{P}_2^4$*. If* s_0 *is not completely wrapped, then any bridge* B *contained in* A_4 *is either*

- *A_4-degenerate; or*
- *an edge whose ends are contained* $\mathrm{span}(a, v_0) \cup s_0 \cup \mathrm{span}(v_4, b)$.

Proof. If s_0 is not completely wrapped, then any H-bridge B is either A_4-degenerate, or $\mathrm{att}(B) \subseteq (\mathrm{span}(a, v_0) \cup s_0) \setminus \{v_4\}$ or $\mathrm{att}(B) \subseteq (s_0 \cup \mathrm{span}(v_4, b)) \setminus \{v_0\}$. In the latter case B is an edge because of Observation 6.2.2. □

We will focus our discussion on graphs having the spoke s_0 completely wrapped (Figure 6.2.1). Let P be an H avoiding xy-path that wraps s_0, and is not contained in an A_4-degenerate H-bridge. Lemma 6.2.1 implies that P must contain either v_0 or v_4 as an endpoint. Notice that the embedding in $\mathbb{R}P^2$ shows that it is not possible to have two H-avoiding P and P', such that both wrap s_0, $v_0 \notin P$, and $v_4 \notin P'$. Thus, all H-avoiding paths contained in A_4 are edges incident with a common vertex, either v_0 or v_4. From now on, we assume that all these paths contain v_0.

Our next definition is important in order to describe some subgraphs in G; it is independent to the study of H-bridges embedded in A_4.

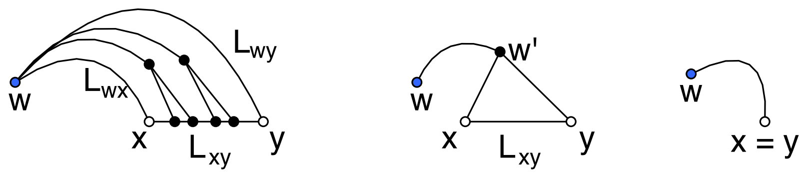

Figure 6.2.2

horn **Definition 6.2.6.** *Let* $(G, H) \in \mathbf{P}_2^4$. *A* (w, x, y, L)-horn M, *is a 4-tuple* (w, x, y, L) *consisting of* $w, x, y \in V(G)$, *and* L *a subgraph of* G *contained in an* H*-region* E_M *so that:*

- *x and y are vertices in the rim R of H;*
- *base of a horn* — *the* M-base L_{xy} *is the* xRy*-arc contained in* E_M*;*
- *apex of a horn* — *the vertex* w *is the* apex, *is in* H, *and is not in* L_{xy}*;*
- $L \cap H$ *is the union of* w *and* $L_{x,y}$ *and either:*

  (a) L *is contained in a disk* Δ_M *bounded by a cycle* C_M *containing* L_{xy} *and* w, *in which case* L *is a* standard horn*; or*

  (b) L *consists of a* $\{w, x, y\}$*-claw* K *together with* $L_{x,y}$, *in which case* Δ_M *is the union of* K *and the disc bounded by the unique cycle in* $K \cup L_{x,y}$ *and* M *is a* trihorn*; or* — *trihorn*

(c) $x = y$ *and* L *is just an* H*-avoiding* wx*-path, in which case* $\Delta_M = L$ *and* M *is* trivial. *trivial horn*

In every case, $L = G \cap \Delta_M$.

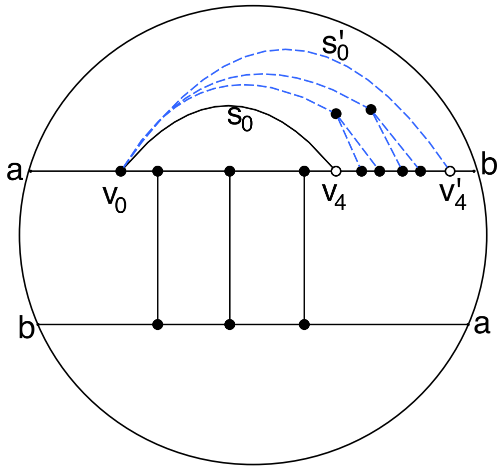

Figure 6.2.3

Horns might appear in many H-regions of G. For instance, suppose G has a type A embedding. Let s_0' be an H avoiding v_0v_4'-path contained in $A_4 \setminus \{v_7, v_5\}$ that wraps s_0, and wraps every other not A_4-degenerate H-bridge that also wraps s_0. Define M_{s_0} to be the $(v_0, v_4, v_4', L_{s_0})$-horn containing all the H-bridges wrapping s_0 and wrapped by s_0' (see Figure 6.2.3). For ease of notation set $C_{s_0} = C_{M_{s_0}}$, $\Delta_{s_0} = \Delta_{M_{s_0}}$.

Any not A_4-degenerate H-bridge B not contained in Δ_{s_0} has all attachments contained in either $\mathrm{span}(a, v_0)$ or $\mathrm{span}(v_4', b)$; by Corollary 5.9.3, B is an edge. Hence we have completely described all H-bridges in A_4 not contained in Δ_{s_0}.

The following result restricts the size of H-bridges inside a horn when the base is contained in a rim branch r_i. This is the case for M_{s_0}.

Lemma 6.2.7. *Let* $(G, H) \in \mathbf{P}_2^4$*, and let* M *be standard* (w, x, y, L)*-horn. Suppose* $L_{x,y}$ *is contained in two consecutive rim branches. Then:*

1. *any* H*-bridge* B *contained in the interior* Δ_M *such that* $w \in \mathrm{att}(B)$ *is either an edge* wz*, where* $z \in L_{xy}$*, or is a* $\{w, z_1, z_2\}$*-claw, with* z_1*,* $z_2 \in L_{xy}$
2. *for* $\{z, z'\} = \{x, y\}$*, if* B_{wz} *is the* H*-bridge containing the* wz*-path in* $C_M - z'$*, then* B_{wz} *is either an induced path or a claw.*

Proof. By Corollary 5.9.2, B has at most 2 attachments in L_{xy}. Thus, B has at most 3 attachments. Therefore, B is either K_2 or $K_{1,3}$. □

Definition 6.2.8. *Let* $(G, H) \in \mathbf{P}_2^4$. *Let* M *be a not a tri-horn, and suppose that* L_{wx} *and* L_{wy} *are* H*-avoiding, then:*

arm

- *An* arm *of* M *is either an* H*-bridge having attachments at* w *and in* L_{xy}*, or* $B_{wx} \cap \Delta_M$ *or* $B_{wy} \cap \Delta_M$ *(where* B_{wx} *and* B_{wy} *are as in* 6.2.7*). The latter two arms are the* boundary arms *of* M.

boundary arm

- *If* M *is degenerate (and no empty), we define* $L_{wx} (= L_{wy})$ *to be the only arm of* M.

finger

- *A* finger *of an arm* B *is any attachment of* B *in* L_{xy}.

Lemma 6.2.7 implies that any non-trivial H-bridge contained in L_{s_0} is an arm of M_{s_0}. The next Theorem is a summary of what we currently know about the H-bridges inside A_4.

Theorem 6.2.9. *Let* $(G, H) \in \mathbf{A}_2^4$. *Let* B *be an* H*-bridge inside* A_4. *Then either,*

(i) s_0 *is not completely wrapped, and* B *is described in Observation* 6.2.5*; or*

(ii) s_0 *is completely wraped and either:*

a. B *is an edge whose attachments are contained in* $\mathrm{span}(a, v_0)$ *or in* $\mathrm{span}(v_4, b)$*; or*

b. B *is* A_4*-degenerate; or*

c. B *is contained in the horn* M_{s_0}.

We end this section giving an observation that will be useful in Chapter 7 for bounding the number of vertices inside the spoke s_0.

Observation 6.2.10. *Let* $(G, H) \in \mathbf{A}_2^4$. *If* B *is a non-trivial* H*-bridge that has an attachment in the interior of* s_0*, then* B *is the only* H*-bridge in* A_4 *having an attachment in the interior of* s_0.

Proof. We split the proof into two cases, depending on whether B is an A_4-degenerate bridge.

Case 1. B *is an* A_4*-degenerate bridge*

In this case B is a $K_{1,3}$, having attachments v_7, v_5, and a vertex x in the interior of s_0, and $\mathrm{nuc}(B) = \{z\}$. Let $P_1 = (v_7, z, x)$, $P_2 = (x, z, v_5)$ be H-avoiding paths. Observation 6.2.2 shows that each of the cycles $\mathrm{span}(v_7, x) \cup P_1$ and $\mathrm{span}(x, v_5) \cup P_2$ bounds a face contained in A_4 (see Figure 6.2.4(i)). This implies that no other H-bridge in A_4 has an attachment in the interior of s_0.

Case 2. B *is not an* A_4*-degenerate bridge.*

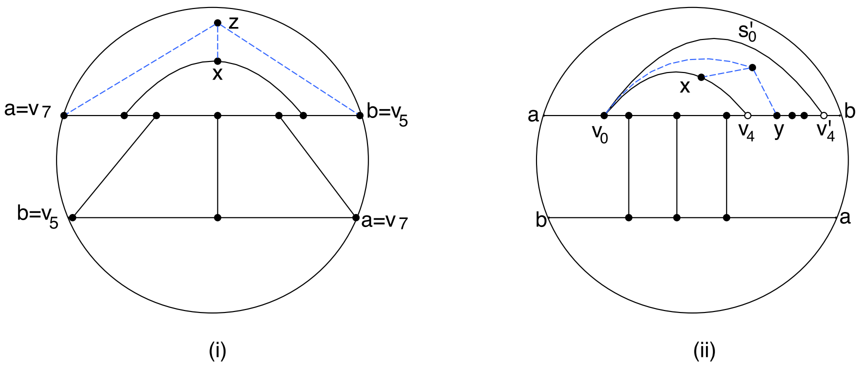

Figure 6.2.4

In this case B is a non-trivial H-bridge contained in the horn M_{s_0} having an attachment in the interior of s_0.

Since s_0 and $(s_0 \cup \mathrm{span}(v_4, b)) \setminus v_0$ are K-close for $K = H - s_0 \cong V_6$, (Theorem 5.9.1 shows that) neither $\mathrm{att}(B) \subseteq s_0$ nor $\mathrm{att}(B) \subset (s_0 \cup \mathrm{span}(v_4, b)) \setminus v_0$. Therefore, $\mathrm{att}(B) = \{v_0, x, y\}$, where x is in the interior of s_0, and $y \in \mathrm{span}(v_4, v_4')$.

Let P_1, P_2 be H-avoiding v_0x-, xy-paths, respectively, contained in B. Consider the cycles $C_1 = P_1 \cup \mathrm{span}(v_0, x)$, $C_2 = P_2 \cup \mathrm{span}(x, y)$. By Observation 6.2.2, each of C_1 and C_2 bounds a face contained in A_4 (see Figure 6.2.4(ii)). This prevents the existence of a distinct H-bridge contained in A_4 having attachments in the interior of s_0. □

6.3 HORNS

In this section we study properties of horns contained in 2-crossing-critical graphs. Some of these properties enable us to bound the number of H-bridges contained in a specific H-region.

Definition 6.3.1. *Let $G \in \mathbf{P}_2^4$, and let M be a horn contained in an H-region E.*

- *M is an* inside *horn if :* *inside horn*
  (a) the region E is contained in the Möbius strip $\mathfrak{M}$; and
  (b) if b is the boundary of E, then the basis and the apex of M are contained in distinct components of $b \cap R$ (we remark that for any $E \subseteq \mathfrak{M}$, $b \cap R$ consists of two components).
- *M is an* outside *horn if E is contained in $\mathfrak{D}$.* *outside horn*
- *If M and M' are two horns having the same apex, then M* contains *M' if $L_{M'} \subseteq L_M$.*

Notice that in the definition of an inside horn we are discarding the cases in which a horn has its basis and apex contained in the same component of $b \cap R$. These horns are easy to describe as follows.

Observation 6.3.2. *Let* $G \in \mathbf{P}_2^4$*, and let* M *be a horn contained in an* H*-region* $E \subseteq \mathfrak{M}$*. Let* b *be the boundary of* E*. If the basis and the apex of* M *are contained in the same components of* $b \cap R$*, then* H *has type* A *and either* M *is the edge* v_7v_1 *or* v_1v_4*.*

Proof. This is immediate from (ii), (v) and (vi) of 5.6.2. □

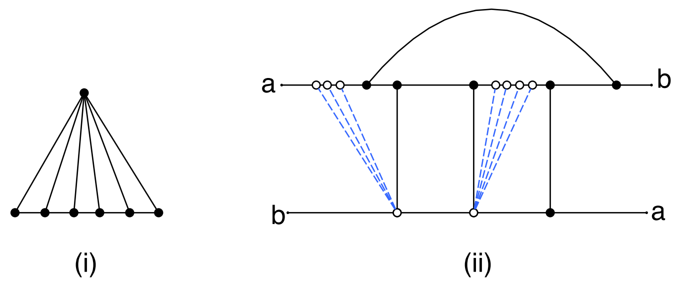

Figure 6.3.1

Definition 6.3.3. *Let* $(G, H) \in \mathbf{P}_2^4$*.*

elementary horn

- *A* (w, x, y, L)*-horn* M *is* elementary *if either it is trivial, it is a trihorn, or it has the property that each arm is either a path or topologically equivalent to* $K_{1,3}$*.*

k-pyramid

- *For a nonnegative integer* k*, the* k-pyramid *is the graph consisting of a length* k *path* P *and vertex* v *not in* P *but adjacent with every vertex of* P *(Figure 6.3.1(i) shows a 5-pyramid).*

basic horn

- *A* (w, x, y, L)*-horn* M *is* basic*, if* L *is isomorphic to a* k*-pyramid, for some* $k \geqslant 1$*.*

Observation 6.3.4. *Let* $(G, H) \in \mathbf{P}_2^4$*. Let* M *be an inside* (w, x, y, L)*-horn such that* L_{xy} *is contained in the interior of a rim branch* r_i*. Then* M *is basic.*

Proof. The base L_{xy} is contained in r_i, so either M is trivial or a trihorn or (by Lemma 6.2.7) the arms of M are topologically equivalent to K_2 or $K_{1,3}$. Therefore M is elementary. Since all the bridges contained inside the Möbius strip $\mathfrak{M}$ are edges (Theorem 5.7.2), M is not a trihorn and all the arms of M are edges. Therefore M is basic. □

induced horns

Definition 6.3.5. *Let* $(G, H) \in \mathbf{P}_2^4$*. Let* E *be an* H*-region,* r *an* R*-arc contained in the intersection of* R *with the boundary of* E*. The horn* $M_E(w, r)$ *contained in* E*, and* induced *by* r *and* w *(or simply* $M(w, r)$ *when* w *and* r *belong to a unique* H*-region) is the* (w, x, y, L)*-horn, such that*

- $L_{xy} \subseteq r$*;*
- *any* (w, x', y', L)*-horn with* $L'_{x'y'} \subseteq r$ *is contained in* $M(w, r)$*.*

In case there is no horn having apex w and base r, we say that the horn $M_E(w,r)$ is empty.

We are interested in studying inside horns. For instance, in a type A embedding of G, if $w = v_6$ and $r = r_2 \setminus \{v_2, v_3\}$, then the induced horn $M_{A_2}(w,r)$ is a k-basic horn containing all the semidiagonals from v_6 to r_2 (see Figure 6.3.1(ii)).

Definition 6.3.6. *Let $(G,H) \in \mathbf{P}_2^4$. Let r be a subpath of the rim R contained in a rim branch. Suppose r is contained in a single H-region E.*

- *If $E \subseteq \mathfrak{D}$, for each vertex z of $V(G) \setminus r$ incident with E, the z-*out-horn *of r is the induced horn $M_E(z,r)$;* *out-horn*
- *If $E \subseteq \mathfrak{D}$, a vertex z of $V(G) \setminus r$ incident with E is an* out-apex *of r if the z-out-horn is non-empty.* *out-apex*
- *For an inside horn M with base r, an M-out-horn is, for some out-apex z, a z-out-horn of r.*

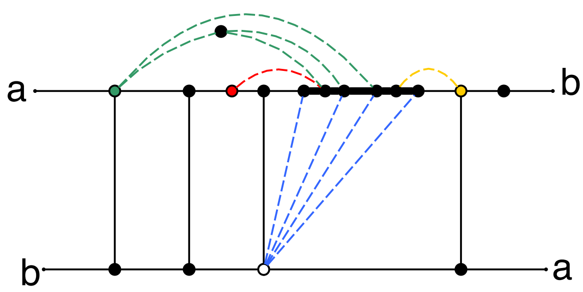

Figure 6.3.2

We use M_z to denote the z-out-horn $M_E(z,r)$. A *non-trivial* out-apex z of M, is an out-apex such that M_z is non-trivial.

In Figure 6.3.2 we illustrate the out-horns and the out-apices with respect to a path r contained in the interior of r_2.

Definition 6.3.7. *Let $(G,H) \in \mathbf{P}_2^4$, and let x, y be two distinct vertices in the rim R of H.*

- *An H-bridge B is* skew *to $\{x,y\}$ if B is contained in $\mathfrak{D}$ and has at least one attachment in each component of $R - \{x,y\}$.*
- *Let r be one of the xy-subpaths of R. An H-bridge B is* r-askew *if B is contained in $\mathfrak{D}$ and has at least one attachment in each of r and $R - r$.*

Observe that if $M = M(w,r)$ is an inside horn in G, then an H-bridge is r-askew, if and only if there exists an out-apex of M.

Observation 6.3.8. *Let* $(G, H) \in \mathbf{P}_2^4$. *Let* $M = M(w, r)$ *be an inside horn in* G, *having base* r *contained in the interior of a rim branch* r_i. *Then:*

(i) for any pair of consecutive fingers d_1, d_2 *in* M, *there exists an* H-*bridge* B r-*askew to* $\mathrm{span}(d_1, d_2)$; *and*

(ii) for any consecutive fingers d_1, d_2, d_3, d_4, *if there is no* H-*bridge skew to* $\{d_1, d_4\}$ *and having an attachment in* $r \setminus (\mathrm{span}(d_1, d_4))$, *then there exists an out-horn* M *that has a finger in* $\mathrm{span}(d_1, d_4)$.

Proof. (i) Otherwise let x, y be the vertices in $R \setminus \mathrm{span}(x, y)$ so that x, y is adjacent in R to d_1, d_2, respectively. Observe that $\{w, x, y\}$ is a 3-cut in G disconnecting $\{d_1, d_2\}$ from the rest of the graph. (ii) If this statement is false, then $\{w, d_1, d_4\}$ is 3-cut in G disconnecting $\{d_2, d_3\}$ from the rest of the vertices in the graph, a contradiction since G is peripherally-4-connected. □

Definition 6.3.9. *Let* $(G, H) \in \mathbf{P}_2^4$. *Let* r *be a path contained in the interior of a rim branch, and contained inside an* H-*region. An out-apex* z *of* r *is* close *if, for some subdivision* K *of* V_6 *contained in* H, *there is an arm of* M_z *that is* K-*close.*

We remark that the existence of one K-close arm implies all arms are K-close, because r is contained in the interior of a rim branch.

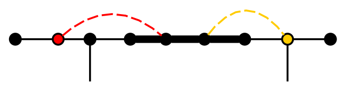

Figure 6.3.3

Lemma 6.3.10. *Let* $(G, H) \in \mathbf{P}_2^4$. *Let* r *be a path contained in the interior of a rim branch, and contained inside an* H-*region. Then* r *has at most* 2 *close out-apices and these are trivial.*

Proof. We first observe that, if M_z is an out-horn whose out-apex z is close, then M_z is trivial by Theorem 5.9.4. Furthermore M_z is the edge zx, for some $x \in r$, and cycle $\mathrm{span}(z, x) \cup zx$ bounds a face in the embedding of G (Lemma 5.8.11).

Likewise, if z' is another close apex of r, then the corresponding horn $M_{z'}$ is just an edge $z'x'$, also in the boundary of a face. Observe that neither zx nor $z'x'$ wraps the other. Then r has at most two close out-apices (see Figure 6.3.3). □

Definition 6.3.11. *Let* $(G, H) \in \mathbf{P}_2^4$. *Let* r *be a path contained in the interior of a rim branch* r_i, *and contained inside an* H-*region.*

- *A set* $S=\{M_{z_1}, M_{z_2},...,M_{z_d}\}$ *of non-close* M *out-horns is* independent *if, for every two out horns* $M_i, M_j \in S$*, the underlying graphs* L_{M_i} *and* L_{M_j} *are disjoint.*

independent horns

- *The maximum number of elements in an independent set* S *of* r *out-horns is the* maximum independence *of* r*, and* S *a maximum independent set. If* M *is an inside horn having base* r*, the maximum independence of* M *is the maximum independence of* r*.*

maximum independence

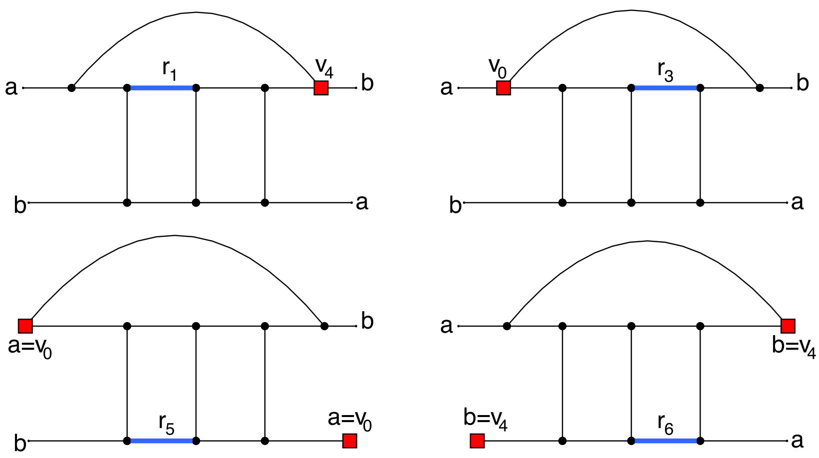

Figure 6.3.4

The following lemma bounds the number of fingers of an inside horn in terms of the outhorns.

Lemma 6.3.12. *Let* $(G, H) \in \mathbf{P}_2^4$*. Let* r *be a path contained in the interior of a rim branch* r_i *that is contained in the boundary of an* H*-region. Then:*

(a) r *has a maximum independence at most 2. If* H *is Type* A *and* $i \in \{1, 2, 5, 6\}$*, then* M *has maximum independence at most* 1*.*

(b) Let $M = M(w, r)$ *be an inside, non-trivial horn having basis* r*. Let* k *be the maximum independence of* r *and suppose each out-horn in a maximum independent set has at most* p M*-fingers in the interior of its base. If* $k = 0$*, then* M *has at most* 5 *fingers; if* $k = 1$*, then* M *has at most* $2p + 9$ *fingers; while if* $k = 2$*, then* M *has at most* $2p + 11$ *fingers.*

Proof. (a) Let k be the maximum independence of M. Observe that for any H-region $E \subset \mathfrak{M}$, and any $i \in \{1, ..., 7\}$, the existence of a set of three independent non-close M-out-horns M_1, M_2, M_3 imply the existence of three large paths pairwise parallel. Hence, by Lemma 5.9.8 we have $k \leqslant 2$.

Suppose H has a type A embedding. If $i = 1, 2, 5$ or 6 the only possible z-out-apex of a non-close out-horn is in each case v_4, v_0, a, b, respectively (for each of the considered rim branches, Figure 6.3.4 shows the places in which an out-apex can be).

(b) Let x, y be the fingers of M such that $\mathrm{span}(x, y) = r$.

Case 1. $k = 0$.

In this case the out-horns of M are close.

Lemma 6.3.10 implies that there are at most two close out horns of M; the out horns of M are trivial; and any close z-out-apex contains its finger g in $\mathrm{span}(x, y)$. Suppose $\mathrm{span}(z, g)$ contains x. Observation 6.3.8(i) implies that at most one M-finger lies in $\mathrm{span}(z, g) \setminus \{g\}$ (otherwise there are two consecutive fingers d_1, d_2 in the interior of $\mathrm{span}(z, g)$, and there is no H-bridge $\mathrm{span}(d_1, d_2)$-askew since $\mathrm{span}(z, g) \cup zg$ is a face).

Subcase 1.1. M *has no out-horns.*

In this case there is no H-bridge skew to $\mathrm{span}(x, y)$. Let u and v be vertices in $R \setminus \mathrm{span}(x, y)$ so that u is adjacent in R to x and v is adjacent in R to y, and $u \neq y$, $v \neq x$. Observe that $\{u, v, w\}$ is 3-cut that separates the fingers, which include x and y, from $G - \mathrm{span}(x, y)$, which is not possible since G is peripherally 4-connected. Therefore M has at least one out-horn.

Subcase 1.2. M *has exactly one out-horn.*

Suppose zg is the out-horn, where z is the out-apex, and g the finger. We may assume $x \in \mathrm{span}(z, g)$ and $y \notin \mathrm{span}(z, g)$. Recall that M has at most one M-finger in the interior of $\mathrm{span}(z, g)$. Observation 6.3.8(ii) implies that $\mathrm{span}(g, y)$ contains at most three M-fingers. Hence r has at most four M-fingers.

Subcase 1.3. M *has two out-horns.*

Suppose zg and $z'g'$ are the out-horns, where z and z' are the out-apices. We may assume $x \in \mathrm{span}(z, g)$ and $y \in \mathrm{span}(z', g')$. Observation 6.3.8(ii) implies that $\mathrm{span}(g, g')$ has at most three M-fingers, therefore r has at most five M-fingers.

Case 2. $k \geqslant 1$.

Let x' be the closest vertex to x in $\mathrm{span}(x, y)$, so that x' is in a base of some out-horn having a non-close out-apex. Likewise define y'.

Claim 6.3.13. *Let* $(G, H) \in \mathbf{P}_2^4$. *There are at most 8* M*-fingers in* $(\mathrm{span}(x, x') \cup \mathrm{span}(y', y))$.

Proof of Claim. Since M has non-close apices, at most one close z-out-horn contains its finger g in $\mathrm{span}(x, x')$. Observe that at most one M-finger is in the interior of $\mathrm{span}(z, g)$. Observation 6.3.8 shows that at most 3 M-fingers belong to $\mathrm{span}(g, x')$. Hence $\mathrm{span}(x, x')$ has at most 4 fingers, and similarly $\mathrm{span}(y', y)$ has at most 4 fingers. □

Subcase 2.2. $k = 1$.

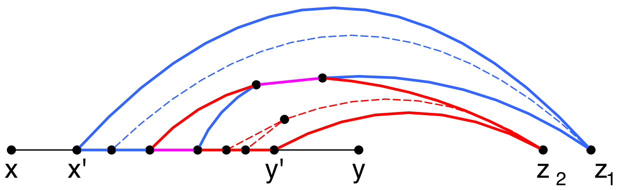

Figure 6.3.5

Claim 6.3.14. *There are at most* $2p+1$ *M-fingers in* $\mathrm{span}(x', y') \setminus \{x', y'\}$.

Proof of Claim. Let M_{z_1} and M_{z_2} be out horns containing x' and y' respectively. If $M_{z_1} = M_{z_2}$, then by assumption the base of M_{z_1} contains at most p M-fingers in its interior. If $M_{z_1} \neq M_{z_2}$, since $k = 1$, the bases of these out horns have non-empty intersection (see Figure 6.3.5) . Therefore the union of these bases is $\mathrm{span}(x', y')$, and hence $\mathrm{span}(x', y') \setminus \{x', y'\}$ contains at most $2p+1$ M-fingers. □

Subcase 2.1. $k = 2$.

Claim 6.3.15. *There are at most* $2p+3$ *M-fingers in* $\mathrm{span}(x', y') \setminus \{x', y'\}$

Proof of Claim. Observe that x' and y' are fingers for out-horns M_{z_1}, M_{z_2} respectively, with $\{M_{z_1}, M_{z_2}\} = S$. Suppose M_{z_1} has base $L_{x'x''}$ and M_{z_2} has base $L_{y''y'}$.

Inside each of the bases $L_{x'x''}$ and $L_{y'y''}$ there are at most p M-fingers by assumption. Observe that all H-bridges not contained in $\mathfrak{M}$ that have an attachment in $\mathrm{span}(x'', y'') \setminus \{x'', y''\}$ have all its attachments in $\mathrm{span}(x'', y'')$. Since G is peripherally 4-connected, there is at most one vertex in $\mathrm{span}(x'', y'') \setminus \{x'', y''\}$. Then $\mathrm{span}(x'', y'')$ contains at most three M-fingers, therefore there are at most $2p+3$ fingers in $\mathrm{span}(x', y')$. □

Using Claims 6.3.13, 6.3.14, 6.3.15; we obtain M has at most $2p+9$ fingers in case $k = 1$ and at most $2p+11$ fingers in case $k = 2$. □

The next result gives a good bound for p in Lemma 6.3.12; we remark that the proof depends on the study of disjoint cycles in 2-crossing-critical graphs.

Lemma 6.3.16. *Let* $(G, H) \in \mathbf{P}_2^4$*. Let* $M = M(w, r)$ *be an inside, non-trivial induced horn in* G*, having basis* r *contained in the interior of a rim branch* r_i*. Let z be a non-trivial out-apex of* M*, and* M_z *its corresponding z-out-horn with base* $r' = L_{xy}$*. Then the interior of* L_{xy} *has at most 2 fingers of* M.

We need an elementary graph theoretic lemma.

Lemma 6.3.17. *Let C_1, C_2 be cycles of a graph G, and let $e \in E(C_1)$. Then there exists a cycle C in G such that $C \subseteq C_1 \cup C_2$ with $e \in C$ and $C \cap C_2$ is either empty, a path or a cycle.* □

Proof of Lemma 6.3.16. Let M_z be a (z, x, y, L)-horn. Suppose for the sake of contradiction that L_{xy} contains in its interior consecutive M-fingers $d_1, \ldots, d_k$, with $k \geqslant 3$, and so that d_i is the finger closest to x in L_{xy}.

We will show that wd_2 is inessential in G (see Definition 5.4.5). To this end, let (e, f) be a crossable pair.

Since G is peripherally-4-connected, Theorem 5.4.8 implies there exists a pair of disjoint cycles C_1, C_2 that separate (e, f). If $wd_2 \notin E(C_1) \cup E(C_2)$, then wd_2 is (e, f)-inessential, and we are done. So we may assume that wd_2 is one of the cycles, say C_1.

Define $\Delta_{d_1 d_k}$ to be the closed disk bounded by $span(d_1, d_k) \cup (d_k, w, d_1)$. Let $\Delta = \Delta_{d_1 d_k} \cup \Delta_{M_z}$. Observe that Δ is a closed disk because $\Delta_{d_1 d_k}$ and $\Delta_{M'_z}$ are closed disks intersecting in $span(d_1, d_k)$. Define C_Δ to be the boundary cycle of Δ and let P_{xy} be the xy-C_Δ-arc not containing w. Let N be the subgraph of G induced by the disk Δ. For $i = 1, 2$; define $T_i = (C_i - V(N)) \cup C_\Delta$.

Let C_{d_1} be the facial cycle containing $d_1 w$ and wd_2, and let C_{d_3} be the facial cycle containing wd_2 and wd_3.

Observe that by Lemma 5.4.7, we can suppose that either C_1 or C_2 bounds a face in the embedding of G.

Case 1. C_1 *bounds a face.*

In this case, the face F_1 bounded by C_1 is incident to wd_2 and e. Because $wd_2 \in C_1$, C_1 is either C_{d_1} or C_{d_3}. Notice that in any case e is contained in r_i. This implies that f is not in r_i, and therefore C_2 is not contained in Δ (hence T_2 is non-empty).

Let B_f be the C_Δ-bridge in T_2 containing f. If $C_2 \cap V(N) = \emptyset$, then define $C'_2 = C_2$. Otherwise, $B_f \cap C_2$ is a C_Δ-avoiding st-path in T_2. Moreover, s and t belong to P_{xy}. Let C'_2 be the cycle obtained by joining B_f and $sP_{x,y}t$, and let C'_1 be the cycle obtained by joining $d_1 w$, wd_k and $span(d_1 d_k)$. Then, C'_1 and C'_2 are disjoint cycles that separate e and f such that $wd_2 \notin E(C'_1) \cup E(C'_2)$. Hence wd_2 is (e, f)-inessential.

Case 2. C_2 *bounds a face of* G.

Suppose F_2 is the face bounded by C_2. Let $P_{F_2} = F_2 \cap C_\Delta$. Either F_2 is contained in Δ or not.

Subcase 2.1. F_2 *is contained in* Δ.

In this case, f is an edge contained in r_i, and F is a face contained in L (because C_2 does not contain w). Since (e, f) is a crossable pair, e is not an edge contained in r_i, and hence $T_1 \neq \emptyset$.

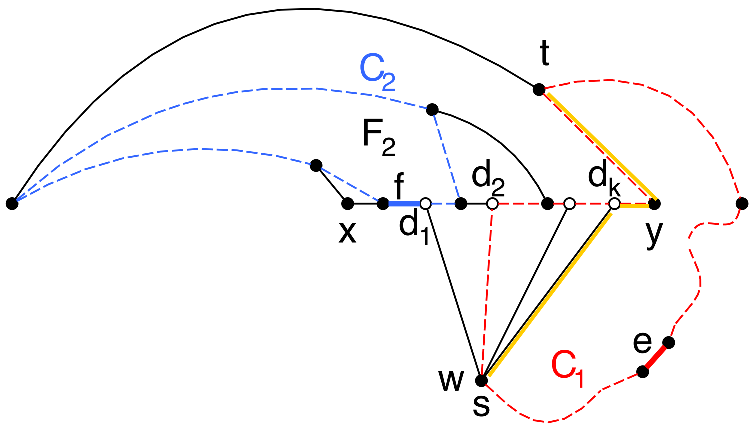

Figure 6.3.6

The horn M_z is an elementary horn because its base is contained in the -rim branch r_i. Therefore P_{F_2} is either a path contained in C_Δ or $P_{F_2} = \emptyset$. Let B_e be the C_Δ-bridge in T_1 containing e. B_e is a C_Δ-avoiding st-path, such that $s, t \in C_\Delta - C_2$. Let C_1' be the union of B_e and the st-C_Δ-subpath not containing P_{F_2} (in case $P_{F_2} = \emptyset$ we choose any st-C_Δ-arc). Observe that C_1' and C_2 are disjoint cycles separating e and f, and $wd_2 \notin E(C_1') \cup E(C_2)$ (see Figure 6.3.6).

Subcase 2.2. *F_2 is not contained in Δ.*

Suppose first that e is contained in $\mathrm{span}(d_1, d_k)$. Define C_1' to be the cycle obtained from $\mathrm{span}(d_1, d_2)$ by adding wd_1 and wd_k. Then (e, f) are cycle separated by (C_1', C_2), and $wd_2 \notin E(C_1') \cup E(C_2)$. Observe that, since M is a basic horn, $P_{F_2} \subseteq P_{xy}$ and F_2 is not Δ_M. Therefore C_1' and C_2 are disjoint, and hence wd_2 is (e, f)-inessential in this case.

Now suppose that $e \notin \mathrm{span}(d_1, d_k)$. Since $wd_2 \in C_1$, C_1 does not contain both wd_1 and wd_3. If $wd_1 \notin C_1$, we choose C_3 to be C_{d_1}, while if $wd_3 \notin C_1$, we choose C_3 to be C_{d_3} (see Figure 6.3.7). Notice that C_2 and C_3 are disjoint because F_2 is not contained in Δ and C_2 does not contain w. Thus, Lemma 6.3.17 implies we may assume $C_1 \cap C_3$ is an path P. Let P' be the st-arc in C_3 distinct from P. Let C_1' be the cycle obtained from C_1 by replacing P by P'. Because $e \notin \mathrm{span}(d_1, d_k)$, C_1' contains e since P do not contain e. Finally C_1' and C_2 are disjoint, separate e and f, and $wd_2 \notin E(C_1') \cup E(C_2)$.

In every case wd_2 is (e, f)-inessential, contradicting 5.4.6. □

Now we can use Lemmas 6.3.12 and 6.3.16 to obtain the following result.

Lemma 6.3.18. *Let $(G, H) \in \mathbf{P}_2^4$. Let $M = M(w, r)$ be an inside, non-trivial induced horn in G, having basis r contained the interior of a rim*

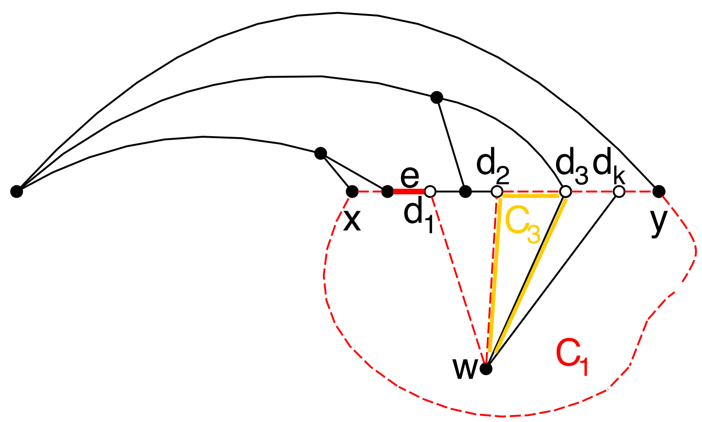

Figure 6.3.7

branch r_i. *If* M *has maximum independence* k*, then* M *has at most* 15 *fingers when* $k = 2$*, at most* 13 *fingers when* $k = 1$*, and at most* 5 *fingers when* $k = 0$.

6.4 BRIDGES INSIDE A_1, A_2, B_1, B_3

In Corollary 5.7.7 we stated that diagonals and semidiagonals are the only possible H-bridges inside H-regions A_1, A_2 in a type A embedding, and in regions B_1, B_3. In this section we use Lemma 6.3.18 to bound the number of semidiagonals inside A_1, A_2, B_1, B_3.

Lemma 6.4.1. *Let* $(G, H) \in \mathbf{P}^4_2$*. Let* v *be an* H*-node incident with the quad* Q_i *for some* $i \in \{0, 1, 2\}$*.*

(i) If $(G, H) \in \mathbf{A}^4_2$ *and* $i \in \{1, 2\}$*, then there are at most 13 semidiagonals incident with* v *and contained in* Q_i*.*

(ii) If $(G, H) \in \mathbf{B}^4_2$ *and* $i \in \{0, 2\}$*, then there are at most 15 semidiagonals incident with* v *and contained in* Q_i*.*

Proof. Let r be the minimal subpath of the H-rim branch in Q_i that contains all ends distinct from v of the semidiagonals in Q_i incident with v.

Let $M = M(v, r)$ be the inside horn induced by v and r. Clearly M contains all the semidiagonals incident with v as arms. If M is empty or degenerate, then there is at most 1 semidiagonal incident to v.

Otherwise, there are at least two semidiagonals in M. In case (i), Lemma 6.3.12(a) implies that the maximum independence of M is $k \leqslant 1$. Applying Lemma 6.3.18 (with $k \leqslant 1$), we obtain M has at most 13 fingers.

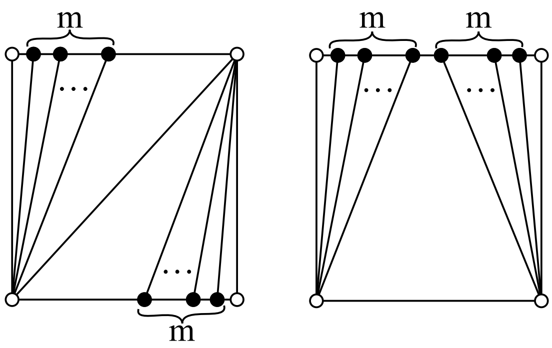

Figure 6.4.1

In case (ii), again by Lemma 6.3.12(a) we have that the maximum independence of M is $k \leqslant 2$. Applying Lemma 6.3.18 we obtain that M has at most 15 fingers. □

We finally remark that the graphs induced by the H regions A_1, A_2, B_1, B_3 are fully determined.

Theorem 6.4.2. *Let $(G, H) \in \mathbf{A}_2^4$. There are at most two pyramids in each of A_1 and A_2, each pyramid consisting of at most 13 semidiagonals (The posibilities are illustrated in Figure 6.4.1; m is 13). There may also be at most one diagonal.*

Theorem 6.4.3. *Let $(G, H) \in \mathbf{B}_2^4$. There are at most two pyramids in each of B_1 and B_3, each pyramid consisting of at most 15 semidiagonals (The posibilities are illustrated in Figure 6.4.1; m is 15). There may also be at most one diagonal.*

6.5 BRIDGES INSIDE B_4

In this section we fully describe the bridges inside B_4. Corollary 5.7.7 state that all the bridges inside B_4 are semidiagonals and diagonals. The definition of an unpolluted V_8-minor H implies the non-existence of semidiagonals inside B_0, since B_0 is minimal under considering all the V_8 minors having the same Möbius strip $\mathfrak{M}$. However, Figure 6.5.1 shows that this is not the case for B_4, since the minimality of the H-region is B_4 is restricted to the minimality of B_0. Although, the minimality of B_4 does restrict the existence of some semidiagonals as the following observation asserts.

Observation 6.5.1. *Let $(G, H) \in \mathbf{B}_2^4$. Then exactly one of the following holds:*

(i) there are no semidiagonals inside B_4;

(ii) $b = v_4$ and all the semidiagonals inside B_4 are of the form v_4x, where $x \in \mathrm{span}(v_7, a)$; or

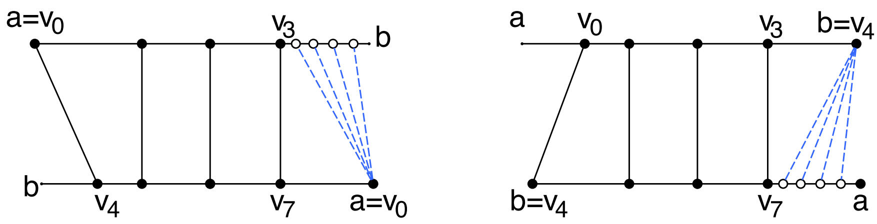

Figure 6.5.1

(iii) $a = v_0$ *and all the semidiagonals inside* B_4 *are of the form* v_0x*, where* $x \in \mathrm{span}(v_3, b)$.

Similar to the results obtained in Section 6.4, we have the following.

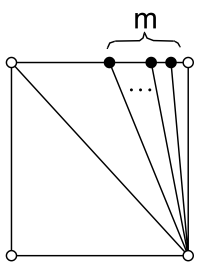

Figure 6.5.2

Theorem 6.5.2. *Let* $(G, H) \in \mathbf{B}_2^4$*. There are at most one pyramids in* B_4 *consisting of at most 15 semidiagonals (This is illustrated in Figure 6.5.2;* m *is 15). There may also be at most one diagonal.*

6.6 BRIDGES INSIDE A_0, A_3

Among the H-regions inside the Möbius strip $\mathfrak{M}$, A_0 and A_3 are the last that we classify. We must be careful, because the existence of some H-bridges inside these two regions depends on the existence of H-bridges inside the region A_4 and in A_6.

We summarize the possible H-bridges inside A_0 and A_3 in the following statement. By Theorem 5.7.2, we know all these H-bridges are edges.

Observation 6.6.1. *Let* $(G, H) \in \mathbf{A}_2^4$*. Then any* H*-bridge contained in* A_0 *is just an edge* xy*. Furthermore:*

(i) $y = v_5$ *and* $x \in \mathrm{span}(a, v_0)$*;*

(ii) $x = a = v_7$ *and* $y \in \mathrm{span}(b, v_5)$*;*

(iii) $x = a = v_7$ *and* $y = v_1$;

(iv) $x = a = v_7$ *and* y *in the interior of the spoke* s_1*, and there is exactly one edge of this kind; or*

(v) $x = v_1$ *and* $y = v_4 = b$.

The symmetric statement will apply if xy *is contained in* A_3*. That is:*

(i) $y = v_7$ *and* $x \in \mathrm{span}(v_4, b)$*; or*

(ii) $x = b = v_5$ *and* $y = v_3$*; or*

(iii) $x = b = v_5$ *and* $y \in \mathrm{span}(v_7, a)$.

(iv) $x = a = v_5$ *and* y *in the interior of the spoke* s_3*, and there is exactly one edge of this kind.*

(v) $x = v_3$ *and* $y = v_0 = a$

Next we determine bounds on the number of H-bridges in A_0 and A_3.

Theorem 6.6.2. *Suppose* $(G, H) \in \mathbf{A}_2^4$.

(a) In A_0*, there are at most 5 edges* xy *with* $x = a = v_7$ *and* $y \in \mathrm{span}(b, v_5) \setminus \{v_5, b\}$.

(b) In A_0*, there are at most 15 edges* xy *such that* $y = v_5$ *and* $x \in \mathrm{span}(a, v_0) \setminus \{a, v_0\}$.

Symmetric statements apply to A_3.

Proof. (a) Let r be the minimal subpath in $\mathrm{span}(b, v_5)$ containing one endpoint of all such edges. Let $M = M(v_7, r)$ be the induced horn whose arms are the edges we are considering.

Observe that M does not have any non-close horn, otherwise there exists a long H-avoiding xy path P contained in A_6, contradicting Theorem 6.1.2.

The observation in the previous paragraph implies that M has maximum independence 0. Using Lemma 6.3.18 we obtain the result.

(b) Let r be the minimal subpath in $\mathrm{span}(a, v_0)$ containing one endpoint of all such edges. Let $M = M(v_5, r)$ be the induced horn whose arms are the edges we are considering.

Lemma 6.3.12(a) implies M has maximum independence $k \leqslant 2$. Using Theorem 6.3.18 we obtain that M has at most 15 arms. □

Finally, we remark that the graphs induced by the H-regions A_0 and A_3 are completely determined.

Theorem 6.6.3. *Suppose* $(G, H) \in \mathbf{A}_2^4$*. The graph induced by* A_0 *is topologically equivalent to a subgraph of a graph in Figure 6.6.1. A symmetric statement apply to* A_3.

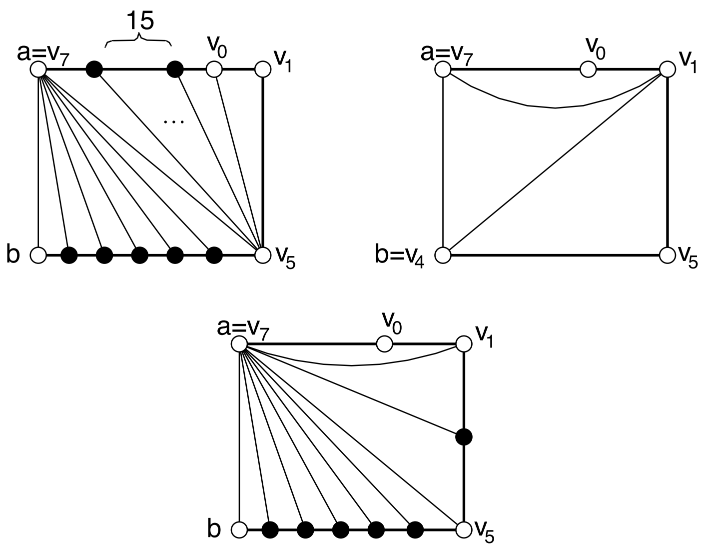

Figure 6.6.1

6.7 BRIDGES INSIDE A_5

In this section we classify the H-bridges inside A_5. It is possible that some H-bridges inside this H-region have attachments in the exposed spoke s_0. We will begin determining the non-trivial H-bridges that have no attachments in the interior of s_0.

Lemma 6.7.1. *Let* $(G, H) \in \mathbf{A}_2^4$. *Suppose* B *is a non-trivial* H*-bridge contained in* A_5 *and having no attachments in the interior of the spoke* s_0. *Let* x, y *be attachments of* B *such that all the attachments of* B *are contained in the* xRy*-arc contained in* A_5*, and suppose* x *is closest to* v_0 *with respect to the* v_0Rv_4*-arc contained in* A_5. *Then* $|\text{nuc}(B)| = 1$, $|\text{att}(B)| = 3$ *and either*

(i) $x = v_0$ *and* $y \in r_2 \setminus v_2$*; or*

(ii) $x \in r_1 \setminus v_2$ *and* $y = v_4$.

Proof. Observation 5.9.7 asserts that B is a large bridge. Hence, if P is an H-avoiding xy-path contained in B, P is large. Since the xRy-arc contained in A_5 contains a rim branch in its interior (Obs. 5.9.6), Lemma 5.6.3(i) shows that either $x = v_0$ and $y \in r_2 \setminus v_2$, or $x \in r_1 \setminus v_2$ and $y = v_4$. We may assume the former condition holds, since the second is symmetric.

If some attachment z of B lies in $r_0 \setminus \{v_0, v_1\}$, then let P be a zy-path in B. Then P satisfies Condition (i) of Lemma 5.6.3, a contradiction. By Corollary 5.9.2, there are at most two attachments of B in $r_1 \cup r_2$. This implies that $|\text{att}(B)| = 3$; since G is 3-connected and B is a tree, $|\text{nuc}(B)| = 1$. □

Now we describe H-bridges having an attachment in the interior of s_0.

Lemma 6.7.2. *Let* $(G, H) \in \mathbf{A}_2^4$. *If* B *be a non-trivial* H*-bridge contained in* A_5 *that has an attachment in the interior of* s_0*, then it has exactly one attachment in the interior of* s_0.

Proof. Suppose B has at least two distinct attachments x, y contained in the interior of s_0. Then there exists an H-avoiding xy-path P contained in A_5. Using s_0 and P we can construct an R-avoiding v_0v_4-path P' contained in A_5, and distinct from s_0. This contradicts Lemma 5.6.3(i). □

Lemma 6.7.3. *Suppose* $(G, H) \in \mathbf{A}_2^4$*, and let* B *be a non-trivial* H*-bridge* B *contained in* A_5 *that has an attachment in the interior of* s_0. *Then at most two attachments of* B *are in the* v_0Rv_4*-arc contained in* A_5.

Proof. Suppose B has at least three attachments in R. Let P be the minimal subpath of R in the boundary of A_5 and containing all vertices in $\mathrm{att}(B) \cap R$ and let x and y be the ends of P. Using an argument similar to that in the first paragraph of the proof of 6.7.1, we can show that either $x = v_0$ and $y \in r_2 \setminus v_2$, or $x \in r_1 \setminus v_2$ and $y = v_4$. We may assume the former case holds.

Let z be the attachment of B in the interior of s_0. Let P be an H-avoiding v_0z-path contained in B. Then there is a v_0v_4-path P' containing P and contained in $s_0 \cup P$. USing P' in place of s_0 produces a V_8 that contradicts Condition (II)(b) of the definition that H is unpolluted. Therefore, B has at most 2 attachments in the rim R.

□

Lemma 6.7.4. *Let* $(G, H) \in \mathbf{A}_2^4$. *Let* B *be a non-trivial* H*-bridge contained in* A_5 *that has an attachment in the interior of the spoke* s_0. *Then* B *is a* $K_{1,3}$ *with one talon in the interior of* s_0. *Furthermore, exactly one of the following holds:*

- *an attachment in each of* $r_1 \setminus \{v_2\}$ *and* $r_2 \setminus \{v_2\}$ *and* B *is the only* H*-bridge inside* A_5;
- B *has one attachment at* v_2 *and another in* $r_2 \setminus \{v_2\}$ *and* B *is the only* H*-bridge inside* a_5 *with attachments in* $\mathrm{span}(v_2, v_4)$.
- B *has one attachment at* v_2 *and another in* $r_1 \setminus \{v_2\}$ *and* B *is the only* H*-bridge inside* A_5 *with attachments in* $\mathrm{span}(v_0, v_2)$.

Proof. Because of Lemmas 6.7.2, 6.7.3, we know B has exactly one attachment z in the interior of s_0, and two attachments x, y in R. Since G is 3-connected and B is a tree, $\mathrm{nuc}(B)$ consists of exactly one vertex w, hence B is isomorphic to $K_{1,3}$.

We may assume x is the attachment closest to v_0 in $v_0R^+v_4$. The attachment x is not contained in $r_0 \setminus \{v_1\}$, as otherwise we violate minimality of A_5. Likewise y is not in $r_3 \setminus \{v_3\}$. What is left is: $x, y \in r_1 \cup r_2$.

The cycles $C_1 = (x, w, z) \cup \mathrm{span}(z, x)$, $C_2 = (x, w, y) \cup \mathrm{span}(x, y)$, and $C_3 = (y, w, z) \cup \mathrm{span}(z, y)$ play a prominent role in the proof.

Case 1. $v_2 \notin \{x, y\}$.

Suppose x, y are in the interior of the same rim branch. By symmetry we may assume $x, y \in r_1$. Because $y \neq v_2$, the path $P = yw \cup wz$ is K-close for $K = H - s_1 \cong V_6$. Cycle $C = P \cup \mathrm{span}(z, y)$ bounds a closed disk Δ contained in A_5 and containing edge wx. However, Lemma 5.8.11 implies that C bounds a face of G since $\mathrm{span}(z, y)$ is K-close, a contradiction. Hence $x \in r_1 \setminus \{v_2\}$, $y \in r_2 \setminus \{v_2\}$.

In this case, B is the only H-bridge inside A_5 since cycles C_1, C_2, C_3, are facial by Lemma 5.8.11.

Case 2. $x = v_2$.

In this case necessarily $y \in r_2 \setminus \{v_2\}$. Cycles C_2 and C_3 are facial because of Lemma 5.8.11. Hence B is the only H-bridge inside A_5 having attachments in $\mathrm{span}(v_2, v_4)$.

Case 3. $y = v_2$.

In this case we have that $x \in r_1 \setminus \{v_2\}$ and cycles C_1, C_2 are faces by Lemma 5.8.11. Hence B is the only H-bridge inside A_5 having attachments in $\mathrm{span}(v_0, v_2)$. □

Our next observation follows easily from Lemmas 6.7.1, 6.7.4.

Observation 6.7.5. *Let $(G, H) \in \mathbf{A}_2^4$. Then A_5 does not contain two non-trivial H-bridges B, B′ so that B has an attachment in the interior of s_0 and B′ does not.*

We studied the non-trivial H-bridges contained in A_5. We have already seen that some of them might have an attachment in the interior of s_0. We will describe the set of trivial H-bridges that have one attachment in the interior of s_0. The following observation follows from Lemma 5.6.3(i).

Observation 6.7.6. *Let $(G, H) \in \mathbf{A}_2^4$. Let B be a trivial H-bridge contained in A_5, suppose B has an attachment y in the interior of s_0. Then the attachment $x \in \mathrm{att}(B) \setminus \{y\}$ lies in $\mathrm{span}(v_1, v_3)$.* □

Definition 6.7.7. *Let $(G, H) \in \mathbf{A}_2^4$. Let T_{A_5} be the set of trivial bridges that have an attachment attachment in the interior of s_0. Partition T_{A_5} into three sets $T^1_{A_5} \cup T^2_{A_5} \cup T^3_{A_5}$ as follows:*

$T^1_{A_5}$
- *$T^1_{A_5}$ consists of the H-bridges xy for which y is the interior of s_0 and $x \in r_1 \setminus v_2$.*

$T^2_{A_5}$
- *$T^2_{A_5}$ consists of the H-bridges xy for which y is the interior of s_0 and $x = v_2$.*

- $T^3_{A_5}$ *consists of the* H*-bridges* xy *for which* y *is the interior of* s_0 *and* $x \in r_2 \setminus v_2$. $T^3_{A_5}$

The following result bounds the number of trivial H-bridges in A_5 having one attachment in the interior of s_0.

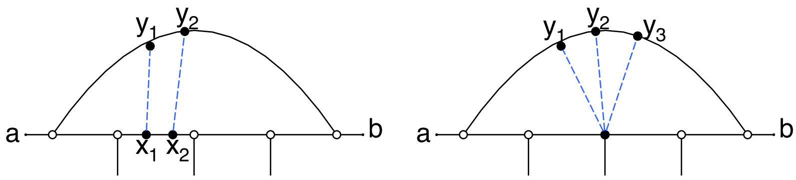

Figure 6.7.1

Lemma 6.7.8. *Let* $(G, H) \in \mathbf{A}^4_2$. *Then*

(i) the sets $T^1_{A_5}$, $T^3_{A_5}$ *have at most one edge; and*

(ii) $T^2_{A_5}$ *has at most two edges.*

Proof. (i) Suppose $T^1_{A_5}$ has two distinct edges x_1y_1, x_2, y_2. We may assume y_1 is closer to v_0 than y_2 in s_0, and x_1 is closer to x than x_2 in $r_0 \cup r_1$ (Figure 6.7.1) . Let C be the cycle obtained from the union of x_2y_2 and $\mathrm{span}(x_2, y_2)$. Since $\mathrm{span}(x_2, y_2)$ is K-close for $K = H - s_1 \cong V_6 \subseteq H$, the closed disk Δ bounded by C contained in A_5 is a face in the embedding. But this is not possible since x_1y_2 is an edge inside Δ, a contradiction. Therefore $T^1_{A_5}$ has at most one edge, and similarly $|T^3_{A_5}| \leqslant 1$.

(ii) Suppose $T^2_{A_5}$ has three distinct edges v_2y_1, v_2, y_2 and v_2y_3. We may assume these edges are consecutive and that y_i is the i-closest vertex to v_0 in s_0 (Figure 6.7.1).

Let $C = (y_1, v_2, y_3) \cup \mathrm{span}(y_1, y_3)$. Since $\mathrm{span}(y_1, y_3) \subseteq s_0$, $y_2 \in C$. Observe that C bounds a closed disk contained in A_5, so that v_2y_2 is inside Δ; moreover, $B = v_2y_2$ is the only C-bridge contained in Δ.

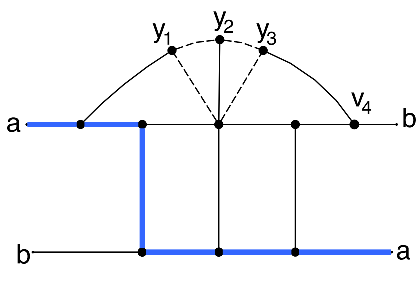

Figure 6.7.2

If we remove from G all the edges of the edges in C, the graph is not planar because it contains a V_6 minor. Hence C is a $B^{\#}$-prebox (Def. 5.8.1). Let M_c be the C-bridge containing $H - C$, since M_C contains a non-contractible cycle, for instance: $r_0 \cup s_0 \cup r_5 \cup r_6 \cup r_7$ (Figure 6.7.2). Lemma 5.8.9 implies that C has bipartite overlap diagram. Therefore C is a box in G, a contradicting Theorem 5.8.4. □

We summarize this section in the following Theorem.

Theorem 6.7.9. *Let $(G, H) \in \mathbf{A}_2^4$. Then A_5 contains at most 2 non-trivial H-bridges.*

(i) If B is a trivial H-bridge contained in A_5, then either:

a. the attachments of B are contained in the rim R; or

b. B has one attachment in the interior of s_0 and the rest in $\mathrm{span}(v_1, v_3)$. There are at most four edges having an attachment in the interior of s_0.

(ii) If A_5 contains exactly one non-trivial H-bridge B. Then either:

a. B has no attachments in the interior of s_0 and B is either as in 6.7.1(i) or as in 6.7.1(ii).

b. B has an attachment in the interior of s_0. In this case B is a bridge described in 6.7.4. If B is as in 6.7.4(i), then B is the only H-bridge inside A_5.

(iii) If A_5 contains exactly two non-trivial H-bridges, then one is as in 6.7.4(ii) and the other as in 6.7.4(iii).

6.8 BRIDGES INSIDE B_5, B_6

In this section we show that H-bridges inside B_5 and B_6 have at most 5 attachments. Clearly the two regions are symmetric, so we will consider only B_5.

Lemma 6.8.1. *Let $(G, H) \in \mathbf{B}_2^4$. Let B be an H-bridge in B_5 such that either $v_7 \notin \mathrm{att}(B)$ or $v_4 \notin \mathrm{att}(B)$. Then B has at most 4 attachments.*

Proof. By symmetry, we may assume $v_7 \notin \mathrm{att}(B)$. Let

$$P = \mathrm{span}(a, v_2) \setminus \{v_7, v_2\}, Q = \mathrm{span}(v_2, b).$$

Observe that $\mathrm{att}(B) \subseteq P \cup Q$ and $P \cap Q = \emptyset$. Since P and Q are $H - s_1$-, $H - s_4$-close respectively, Theorem 5.9.1 implies that B has at most two attachments in P and at most two attachments in Q. Hence B has at most 4 attachments. □

Theorem 6.8.2. *Let* $(G, H) \in \mathbf{B}_2^4$*. Let* B *be a* H*-bridge contained in* B_5*. Then* B *has at most 5 attachments, and if* $|\mathrm{att}(B)| = 5$*, then* $v_7, v_4 \in \mathrm{att}(B)$ *and* B *is the only* H*-bridge of* G*.*

Proof. Suppose B is a non-trivial H-bridge in B_5. If either v_7 or v_4 is not in $\mathrm{att}(B)$, then Lemma 6.8.1 shows that B has at most 4 attachments.

Suppose B has at least 5 attachments. We know v_7, $v_4 \in \mathrm{att}(B)$. Note that this implies $a = v_7$ and $b = v_4$. Note that $P = \mathrm{span}(v_7, v_2) \setminus \{v_7, v_2\}$ and $Q = \mathrm{span}(v_2, v_4)$ are disjoint, and $\mathrm{att}(B) \subseteq P \cup Q \cup \{v_7\}$. Because P and Q are $H - s_0$-, $H - s_3$-close, Theorem 5.9.1 implies that B has at most two attachments in P and at most two attachments in Q (one of them is v_4). Hence B has at most 5 attachments.

Now suppose B has exactly 5 attachments x_1, x_2, x_3, x_4, x_5 so that x_i is the i-closest to a. By 6.8.1, we may assume $x_1 = v_7 = a$ and $x_5 = v_4 = b$. Observe that each of $\mathrm{span}(v_7, v_1)$, $\mathrm{span}(v_2, v_4)$ contains at most 2 attachments of B. Hence $x_3 \in r_1 \setminus \{v_1, v_2\}$.

For $i = 1, 2, 3, 4$, $\{x_i, x_{i+1}\}$ is, for some $V_6 \cong K_i \subseteq H$, K_i-close. Therefore, letting C_i be the union of $\mathrm{span}(x_i, x_{i+1})$ and an H-avoiding $x_i x_{i+1}$-path in B, C_i is K_i-close.

For any crossable pair e, f of edges, let e be the one in R^+ (in this context, R^+ is the v_7v_4-subpath of R containing v_0). For some $i \in \{1, 2, 3, 4\}$, e is in C_i. Since C_i is close, we can find a non-contractible cycle C_i' in H disjoint to C_i:

- For $i = 1, 2$, if $e \in r_7 \cup r_0$, then $C_i' = r_4 \cup r_5 \cup s_2 \cup r_2 \cup r_3$, and if $e \in \mathrm{span}(v_1, x_3)$ then $C_i' = r_4 \cup r_5 \cup r_6 \cup s_3 \cup r_3$.
- For $i = 2, 3$, if $e \in \mathrm{span}(x_3, v_2)$ then $C_i' = r_4 \cup r_5 \cup r_6 \cup s_3 \cup r_3$, and if $e \in r_2 \cup r_3$ then $C_i' = r_7 \cup r_0 \cup s_0 \cup r_5 \cup r_6$.

Then e and f are cycle separated. Using Lemma 5.4.8, we obtain that $\mathrm{cr}(H \cup B) \geqslant 2$, and hence $G = H \cup B$. Therefore B is the only H-bridge of G. □

A symmetric statement holds for B_6.

Theorem 6.8.3. *Let* $(G, H) \in \mathbf{B}_2^4$*. Let* B *be a* H*-bridge either contained in* B_6*. Then* B *has at most 5 attachments, and if* $|\mathrm{att}(B)| = 5$*, then* $v_3, v_0 \in \mathrm{att}(B)$ *and* B *is the only* H*-bridge of* G*.*

7 UPPER BOUND ON THE NUMBER OF VERTICES

In [6] it was proved that 3-connected 2-crossing-critical graphs having a V_8 minor and no V_{10} have less than 3,000,000 vertices. Our aim in this chapter is to improve this bound by showing that if $(G, H) \in \mathbf{P}_2^4$, then $|V(G)| \leqslant 4,001$.

7.1 R-CLAWS

As we saw in the previous chapter, possibly many H-bridges of a graph $(G, H) \in \mathbf{P}_2^4$ have all their attachments on the rim R. In this section we bound the number of these H-bridges in terms of the number of vertices in R.

We begin by introducing some notation that will be used throughout this chapter.

Definition 7.1.1. *Let* $(G, H) \in \mathbf{P}_2^4$. *We define subsets of* $V = V(G)$ *as follows:*

- $V_{\mathfrak{M}} = V \cap \mathfrak{M}$ *is the set of vertices of* G *in the Möbius strip; and* $V_{\mathfrak{M}}$
- $V_R = V \cap R$ *is the set of vertices of* G *in the rim* R. V_R

Next we define R-claws.

Definition 7.1.2. *Let* $(G, H) \in \mathbf{P}_2^4$.

- *An* R-claw *is a non-trivial* H*-bridge* B *such that* $\mathrm{att}(B) \subseteq R$. R-*claw*
- *If a vertex* $z \in \mathrm{nuc}(B)$, *then* z *is a* head *of* B. *The set of* R*-claw heads is denoted by* V^{ch} *(the notation "ch" stands for "claw heads").* *head*
- $\mathrm{base}(B)$ *is the minimal subpath* P *of* R *contained in either* R^+ *of* R^-, *and so that* $\mathrm{att}(B) \subset P$. *base*

Observe that since the Möbius strip $\mathfrak{M}$ does not contain non-trivial bridges, all the R-claws are contained in $\mathfrak{D}$.

Lemma 7.1.3. *Let* $(G, H) \in \mathbf{P}_2^4$. *Then the set* V^{ch} *of* R*-claw-heads has at most* $2|V_R|$ *vertices.*

Proof. If there is an H-bridge B with at least 5 attachments, then (6.0.9) $\mathrm{nuc}(B)$ has at most 3 vertices and $G = H \cup B$. In this case, $|V_R| \geqslant 8$ and the result holds. Thus, we may assume every R-claw has at most 4 attachments and, therefore at most 2 R-claw heads.

We begin by proving the following claim.

Claim 7.1.4. *Let* $\mathfrak{B} = \{\mathtt{base}(B) \mid B$ *is an* R*-claw*$\}$.

(i) The set $\mathfrak{B}$ *consists of subpaths of* R *such that any pair of paths in this set are either internally disjoint, or one is contained in the other.*

(ii) Each R*-claw is associated to a unique base in* $\mathfrak{B}$.

Proof. (i) This is just a restatement of the well-known fact: any pair of H-bridges contained in $\mathfrak{D}$ that have all their attachments in R do not overlap. (ii) Since each R-claw has at least 3 attachments, two distinct R-claws have distinct bases, otherwise they overlap. □

Given $S \subset \mathfrak{B}$, a base $L \in S$ is minimal in S if L does not properly contain a base in S. We can order the elements $L_1, L_2, ..., L_k$ of $\mathfrak{B}$, so that L_i is minimal in the set $\{L_i, L_{i+1}, ..., L_k\}$ for all $i = 1, ..., k$. Let $B_1, ..., B_k$ be the H-bridges associated to the basis $L_1, ..., L_k$ respectively. Since L_1 does not ovelap with another base in $\mathfrak{B}$ (Claim 7.1.4), there is an attachment $x_1 \in \mathtt{att}(B_1)$ in the interior of L_1 , that is not in $L_2, ..., L_k$.

Remove $\mathtt{nuc}(B_1)$ from G. Observe that L_2 contains an attachment $x_2 \in \mathtt{att}(B_2)$ in the interior of L_2, that is not in $L_3, ..., L_k$; moreover, $x_2 \neq x_1$, since $x_1 \notin L_2$. Hence, we can inductively construct a sequence $x_1, ..., x_k$ of distinct vertices in R so that each one belong to $B_1, ..., B_k$ respectively. Therefore $|\mathfrak{B}| \leqslant |V_R|$. Finally, since each R-claw has at most 2 vertices in the nucleus (Theorem 6.0.9), we obtain $|V^{ch}| \leqslant 2|V_R|$. □

Lemma 7.1.5. *Let* $(G, H) \in \mathbf{P}^4_2$*. Let* r *be the number of vertices not in the Möbius strip* $\mathfrak{M}$ *that are not* R*-claw-heads. Then* G *has at most* $3|V_R| + r + 2$ *vertices.*

Proof. We begin counting number of vertices inside the Möbius strip.

Claim 7.1.6. *The Möbius strip* $\mathfrak{M}$ *contains at most* $|V_R| + 2$ *vertices.*

Proof of claim. We proved in Theorem 5.7.2 that there are no non-trivial H-bridges contained in $\mathfrak{M}$, hence all the vertices in $\mathfrak{M}$ are vertices in H. In Lemma 5.7.3 and Corollary 5.7.4 we showed that there are at most two spokes that contain at most one vertex each in their interiors. Hence, there are at most two vertices in the Möbius strip $\mathfrak{M}$ and not in R. □

Using Lemma 7.1.3, we bound the number of vertices in G not in $\mathfrak{M}$ as follows

$$|V \setminus V_{\mathfrak{M}}| = |V^{ch}| + |V \setminus (V_{\mathfrak{M}} \cup V^{ch})| \leqslant 2|V_R| + r.$$

Finally using Claim 7.1.6 and the previous inequality we obtain $|V| = |V_{\mathfrak{M}}| + |V \setminus V_{\mathfrak{M}}| \leqslant 3|V_R| + r + 2$. □

In this section we give a bound for the number of vertices in the rim R that depends on the number of vertices that are incident to an edge embedded inside the Möbius strip $\mathfrak{M}$.

Definition 7.2.1. *Let* $(G, H) \in \mathbf{P}_2^4$. *Define* V_R^{in} *to be the following union of vertex sets:*

$$V_R^{in} = \{a, b\} \cup \{v \in V \mid v \text{ is an } H\text{-node}\} \cup \{v \in V \mid v \text{ is incident to an edge embedded in the interior of } \mathfrak{M}\}.$$

V_R^{in}

We use V_R^{out} *to denote the complementary set* $V_R \setminus V_R^{in}$.

V_R^{out}

Our next observation follows from the fact that any vertex in G has degree at least 3, since G is 3-connected.

Observation 7.2.2. *Let* $(G, H) \in \mathbf{P}_2^4$. *If* $x \in V_R^{out}$, *then* x *is incident with an edge embedded in the interior of* $\mathfrak{D}$. □

Theorem 7.2.3. *Let* $(G, H) \in \mathbf{P}_2^4$. *Let* x *and* y *be distinct vertices in* V_R^{in} *such that* $\mathrm{span}(x, y)$ *contains no vertex of* V_R^{in} *in its interior. Then* $\mathrm{span}(x, y)$ *contains at most 12 vertices of* V_R^{out} *in its interior.*

Proof. Let $r = \mathrm{span}(x, y)$ and let k be the maximum independence of r (see Definition 6.3.11). Because $k \leqslant 2$, we split the proof into two cases, depending if $k = 0$ or if $k \geqslant 1$.

Case 1. $k = 0$.

In this case r has no non-close out-horns. By Lemma 6.3.10, there are at most two close out-horns of r, and these out-horns are trivial.

Subcase 1.1. *r has no out-horns.*

In this case $\mathrm{span}(x, y)$ has no vertex in V_R^{out}, otherwise $\{x, y\}$ is a 2-cut, which is not possible since G is 3-connected.

Subcase 1.2. *r has exactly one out-horn.*

Suppose zg is the out-horn, where z is the out-apex, and g the finger. Since zg is the only out-horn, either $\{z, x, y\}$ is a 3-cut in G and g is the only vertex in r, or $g \in \{x, y\}$ and r is an edge. In any case r contains at most one V_R^{out}-node.

Subcase 1.3. *r has two out-horns.*

Suppose zg and $z'g'$ are the out-horns, where z and z' are the out-apices. We may assume $x \in \mathrm{span}(z, g)$ and $y \in \mathrm{span}(z', g)$. Since the cycle C obtained from the union of $\mathrm{span}(z, g)$ and zg is facial, there is no vertex in V_R^{out} in the interior of $\mathrm{span}(x, g)$ (otherwise $\{x, g\}$ is a 2-cut). Similarly there is no vertex in V_R^{out} inside $\mathrm{span}(g', y)$. Finally there are no vertex in V_R^{out} in the interior of $\mathrm{span}(g, g')$, otherwise $\{g, g'\}$ is a 2-cut disconnecting them from the rest of the graph. Possibly g and g' are V_R^{out}-nodes, therefore $\mathrm{span}(x, y)$ has at most two V_R^{out}-nodes.

Case 2. $k \geqslant 1$

Let x' be the closest vertex to x in $span(x, y)$, so that x is in a base of some out-horn having a non-close out-apex of r. Likewise define y'.

Claim 7.2.4. *There are at most 2 vertices in* V_R^{out} *in*

$$(span(x, x') \cup span(y, y')) \setminus \{x', y'\}.$$

Proof of Claim. Since r has non-close apices, at most one close z-out-horn contains its finger g in $span(x, x')$. Since $span(z, g) \cup zg$ is a facial cycle, there is no vertex in V_R^{out} in the interior of $span(x, g)$. Observe that since $\{g, x'\}$ is not a 2-cut, there is no vertex in the interior of $span(g, x')$. Hence $span(x, x') \setminus \{x'\}$ has at most one vertex in V_R^{out}, and similarly $span(y', y) \setminus \{y'\}$ has at most one vertex in V_R^{out}. □

Claim 7.2.5. *If* M_z *is a non-close out-horn of* r*, then* M_z *has at most 5 vertices in its base.*

Proof of Claim. Suppose M_z has at least 6 vertices in its base. At most 4 M_z-fingers belong to boundary arms. Let w, w' be fingers in boundary arms of M_z such that $span(w, w')$ contains in its interior all the vertices of the base of M_z, except the M_z-fingers in the boundary arms. Since $span(w, w')$ contains at least two vertices in its interior, $\{z, w, w'\}$ is a 3-cut that disconnects the vertices in the interior of $span(w, w')$ from the vertices in $H - r$. This contradicts the fact that G is peripherally 4-connected. Thus M_z has at most 5 vertices in its base. □

Subcase 2.1. $k = 1$.

Let M_{z_1} and M_{z_2} be out-horns containing x' and y', respectively. If $M_{z_1} = M_{z_2}$, then by Claim 7.2.5 $span(x', y')$ contains at most 5 vertices in V_R^{out}. In case $M_{z_1} \neq M_{z_2}$, since $k = 1$, the bases of these out-horns have a non-empty intersection. Therefore $span(x', y')$ contains at most 9 vertices in V_R^{out}. Using 7.2.4 we have that $span(x, y)$ contains at most 9+2=11 vertices in V_R^{out}.

Subcase 2.2. $k = 2$.

Observe that x' and y' are fingers of two disjoint out-horns M_{z_1}, M_{z_2} respectively with $S = \{M_{z_1}, M_{z_2}\}$ a maximum independent set of r. Suppose M_{z_1} has base $L_{x'x''}$ and M_{z_2} has base $L_{y''y'}$.

By Claim 7.2.5, there are at most 5 vertices in V_R^{out} inside the bases $L_{x'x''}$, $L_{y''y'}$. Observe that there are no vertices in $span(x'', y'')$, as otherwise $\{x'', y''\}$ is a 2-cut in G. Hence $span(x', y')$ has at most 10 vertices in V_R^{out}. Using 7.2.4 and the preceding conclusion, we deduce that r has at most 12 vertices in V_R^{out}. □

Corollary 7.2.6. *Let* $(G, H) \in \mathbf{P}_2^4$. *Then* $|V(R)| \leqslant 13|V_R^{in}|$ □

This observation shows that, to bound the number of vertices in R, it suffices to bound $|V_R^{in}|$. This is our next step.

Theorem 7.2.7. *Let* $(G, H) \in \mathbf{P}_2^4$.

- *If* $(G, H) \in \mathbf{A}_2^4$, *then* V_R^{in} *has at most 102 vertices.*
- *If* $(G, H) \in \mathbf{B}_2^4$, *then* V_R^{in} *has at most 85 vertices.*

Proof. The set V_R^{in} contains 8 H-nodes and vertices a, b. Let $\overline{V_R^{in}}$ be the vertices in V_R^{in} that are neither H-nodes nor a nor b. We count the vertices in $\overline{V_R^{in}}$ in each of H-regions inside the Möbius strip:

Case 1. H *has a type* A *embedding.*

- Theorem 6.6.3 implies that each of A_0 and A_3 have at most $15 + 5 = 20$ vertices in $\overline{V_R^{in}}$.
- Theorem 6.4.2 implies that each of A_1, A_2 have at most 26 vertices in $\overline{V_R^{in}}$.

Hence $|\overline{V_R^{in}}|$ is at most 92. Therefore $|V_R^{in}| \leqslant 92 + 10 = 102$.

Case 2. H *has a type* B *embedding.*

- Theorem 6.4.3 implies that each of B_1 and B_3 has at most 30 vertices in $\overline{V_R^{in}}$.
- Theorem 6.5.2 implies that B_4 has at most 15 vertices in $\overline{V_R^{in}}$.
- Since neither B_0 nor B_2 has any semidiagonals, these regions have no vertices in $\overline{V_R^{in}}$.

Hence in total $|\overline{V_R^{in}}|$ is at most 75. Therefore $|V_R^{in}| \leqslant 75 + 10 = 85$. □

Corollary 7.2.8. *Let* $(G, H) \in \mathbf{P}_2^4$.

- *If* $(G, H) \in \mathbf{A}_2^4$, *then* V_R *has at most 1326 vertices.*
- *If* $(G, H) \in \mathbf{B}_2^4$, *then* V_R *has at most 975 vertices.*

7.3 UPPER BOUND

In this section we explicitly compute a bound for the number of vertices in G. Lemma 7.1.5 shows that it suffices to bound $|V_R|$ and the number of vertices in $\mathfrak{D}$ that are not claw-heads. We did the first in the preceding section and do the second in this section.

We begin this section giving an upper bound for the number of vertices in the interior of the exposed spoke s_0. Observe that these vertices are indeed not claw-heads.

Lemma 7.3.1. *Let* $(G, H) \in \mathbf{A}_2^4$. *Then the exposed spoke* s_0 *has at most 20 vertices in its interior.*

Proof. Let V_{s_0} be the set of vertices in the interior of s_0. Let $V_{s_0}^{A_5}$ be the set of vertices in V_{s_0} that are incident to an edge embedded in A_5. Thus, each vertex in $V_{s_0}^{A_5}$ is an attachment of an H-bridge contained inside A_5. In Theorem 6.7.9 we showed that there are at most 4 trivial H-bridges in A_5 having an attachment in the interior of s_0; furthermore, at most 2 non-trivial H-bridges have attachments in the interior of s_0. Since each of these H-bridges in A_5 have exactly one attachment in the interior of s_0, $V_{s_0}^{A_5}$ has at most 6 vertices.

Let $V_{s_0}^{A_4} = V_{s_0} \setminus V_{s_0}^{A_5}$. Each vertex in $V_{s_0}^{A_4}$ is in the interior of s_0, is incident to an edge embedded in A_4, and is not in $V_{s_0}^{A_5}$.

Claim 7.3.2. $V_{s_0}^{A_4}$ *has at most 14 vertices.*

Proof of Claim. We split the proof into two cases, depending on whether there exists a non-trivial H-bridge contained in A_4 having an attachment inside s_0.

Case 1. *A non-trivial* H*-bridge* B *contained in* A_4 *has one attachment in the interior of* s_0*.*

Observation 6.2.10 shows that in this case, exactly one H-bridge contained in A_4 has one attachment in the interior of s_0. Therefore $|V_{s_0}^{A_4}| = 1$.

Case 2. *There are no non-trivial* H*-bridges in* A_4 *having attachments in the interior of* s_0*.*

Let $S = \{v_0, v_4\} \cup V_{s_0}^{A_5}$. Since $V_{s_0}^{A_5}$ has at most 6 vertices, S has at most 8 vertices. Suppose s_0 $V_{s_0}^{A_4}$ has more than 14 vertices in its interior. Then there are two elements x and y in S so that $\mathrm{span}(x, y)$ contains at least 3 vertices in $V_{s_0}^{A_4}$, and $\mathrm{span}(x, y)$ does not contains a vertex from S in its interior. Let x' and y' be the neighbours of x and y, respectively, in $\mathrm{span}(x, y)$. If an H-bridge B has one attachment in the interior of $\mathrm{span}(x', y')$, then B is trivial. If it has an attachment outside $\mathrm{span}(x', y')$, then 6.2.2 implies it, together with its span, bounds a face. However, either x' or y' is incident with an edge inside that face, a contradiction. Therefore s_0 contains at most 14 vertices in $V_{s_0}^{A_4}$. □

Finally we have that $|V_s| = |V_{s_0}^{A_5}| + |V_{s_0}^{A_4}| \leqslant 6 + 14 = 20$. □

Lemma 7.3.3. *Let* $(G, H) \in \mathbf{P}_2^4$. *Then* G *has at most 23 vertices in* $V \setminus V_{\mathfrak{M}}$ *that are not claw-heads.*

Proof. If H has a type B embedding, since no spoke of H is embedded in $\mathfrak{D}$, all the H-bridges contained in $\mathfrak{D}$ are R-claws. Hence $|(V \setminus V_{\mathfrak{M}}) \setminus V_R^{ch}| = 0$. So we may assume H has a type A embedding.

There are two types of vertices in $V \setminus V_{\mathfrak{M}}$ that are not claw-heads:

- vertices in the interior of s_0; or
- vertices in the nucleus of a non-trivial H-bridge having an attachment in the interior of s_0.

In Lemma 7.3.1 we showed that there are at most 20 vertices of the first type in G. Let S be the set of vertices of the second type.

Vertices in S are either in A_4 or in A_5. Observation 6.2.10, and the fact that any non-trivial H-bridge inside A_4 that has one attachment in the interior of s_0 is isomorphic to a $K_{1,3}$, imply that there is at most 1 S-node in A_4. In Theorem 6.7.9 we obtained that there are at most two nontrivial H-bridges inside A_5 each having a nucleus of size 1. Then A_5 contains at most two vertices in S. Therefore $|S| \leqslant 3$.

Finally, we have that $|V \setminus (V_{\mathfrak{M}} \cup V^{ch})| \leqslant 20 + 3 = 23$. □

The main result of this Chapter is obtained from combining 7.1.5, 7.2.7, and 7.3.3.

Theorem 7.3.4. *Let* $(G, H) \in \mathbf{P}_2^4$. *Then* G *has at most 4001 vertices.* □

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