Novel Approaches to Gravity Scattering Amplitudes

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Quantum Field Theory (QFT) provides the essential background for formulating the standard model of elementary particles and, moreover, practically all other theories attempting to explore the physical laws of nature at the sub-atomic level. One of the main observables in QFT are the scattering amplitudes, physical quantities which encode the information of the scattering process of particles. Accordingly, having authentic, well-defined and feasible prescriptions for the calculations of amplitudes is of huge importance to theoretical physicists. Actual calculations show that the text-book prescription, the Feynman method, besides in general being very cumbersome also hides some of the beautiful mathematical features of amplitudes. The last decade has seen tremendous efforts and achievements to improve such calculations, particularly in supersymmetric gauge theories, which have also led to better understanding of QFT itself. Among the known physically and mathematically interesting quantum field theories is perturbative gravity and its supersymmetric version, $\mathcal{N} = 8$ supergravity– much less understood than gauge theory. Following the developments in gauge theory, this dissertation mainly aims at exploring scattering amplitudes in gravity as a quantum field theory, using the modern approaches to QFT. The goal is not only to improve our understanding of gravity amplitudes by applying part of the known modern methods of calculations to it but also to introduce and develop new ones.

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Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

The main concepts needed to introduce the objectives and to establish the methods used in this thesis are overviewed in this section. We will mainly discuss what is now widely known as the modern methods in calculating scattering amplitudes for which also some preliminaries are briefly reviewed. Our focus will be on only the materials which are essential to know for the rest of this dissertation; hence, our literature review has tried to be concisely directed.

1.1 S-Matrix and Scattering Amplitudes

To "observe" the sub-atomic world and explore its underlying physics, *scattering* experiments are carried out. Transition rates and cross-sections are the main observables in a scattering process both of which can be calculated from the evolution operator, the Scattering matrix S, introduced to particle physicists by Heisenberg in the 1940s. When sandwiched between *in* and *out* states – experimentally prepared at $-\infty$ and $+\infty$ – the S-matrix gives its elements, the *scattering amplitudes*, which encode the information to calculate the probability amplitude for the transition from an initial i to a final f state, $S_{fi} = \langle \text{out}|\text{in}\rangle$. [1]

[1] Equivalently, S-matrix elements can be written in terms of *free particle* states Φ as $S_{fi} = \langle \Phi_f|S|\Phi_i\rangle$.

Now let us look at this procedure from a calculative point of view. Quantum field theory describes field interactions by a Lagrangian from which physical observables can be derived. The traditional mathematical toolkit for calculations in quantum field theories is the Feynman method. Once we have a Lagrangian for the quantum field theory under study, we can extract the Feynman rules from it to calculate the scattering amplitudes $\mathcal{A}$ of different processes allowed in our quantum field theory to occur. The amplitude is a gauge invariant and on-shell object which as Feynman method establishes, is the sum of possible Feynman diagrams at each level of perturbation. Given the scattering amplitudes, one can calculate the differential cross-section $\frac{d\sigma}{d\Omega}$. Usually we are interested in an unpolarized differential cross-section because it is difficult to prepare external states with definite spins. Therefore, we sum over spin of final states and average over spin of initial states before comparing $\frac{d\sigma}{d\Omega}$ with experimental measurements. By integrating over angles and considering suitable symmetry factors for identical final states, the total cross-section σ can be found. Both σ and $\frac{d\sigma}{d\Omega}$ are physical quantities which are of interest to experimentalists, however, for theorists, the most interesting physical quantity is the scattering amplitude itself. Feynman's rules have come to be the main and standard method in calculations of scattering amplitudes. Although clear and simple to utilize in principle, in several situations they are unfit in practice. The practical and also main issue with Feynman's method is the huge complexity of calculations when the number of external scattered particles increases.

The number of Feynman diagrams contributing to a scattering process of n particles, say 2 incoming and $n-2$ outgoing, increases factorially [1, 2]. As an example, for a pure gluon scattering process, table 1.1 shows how many diagrams are needed:

n	4	5	6	7	8
number of Feynman diagrams	4	25	220	2485	34300

Table 1.1: Number of Feynman diagrams in n-gluon scattering, [1].

Although implementing Feynman's method on a computer can partially solve the problem of the large number of diagrams, still it is not efficient for higher point amplitudes.

Moreover, on the analytic side, one has to suffer from dealing with all contributing Feynman diagrams, even though at the end of the calculation, many would cancel and the result becomes simple and compact.

Let us talk about the origin of this complexity. The Lagrangian is a local function in space-time. The Feynman rules, derived from the Lagrangian, hence make locality manifest. Also, as a physical quantum theory, QFT is unitary. In the Feynman method, both space-time locality and quantum unitarity are made manifest at the expense of introducing gauge redundancies. Consequently, S-matrix calculations with Feynman's method contain many unnecessary intermediate steps while the final result is unexpectedly simple. It turns out that the underlying simplicity of scattering amplitudes cannot be seen when computed by a method which makes both locality and unitarity manifest. This observation was a key to many of the last few years' advancements in the quest for a new (non-local) formulation of QFT where other physical principles are more manifest than locality. However, there have been attempts other than Feynman's approach to calculate the S-matrix of elementary particles since the '60s.

1.1.1 Unitarity and Analyticity

As a unitary operator, S-matrix satisfies $S^\dagger S = 1$. Also, it can be decomposed into a no-scattering part plus transition (interaction) matrix T: $S = 1 + iT$. Combining the two relations, unitarity requires that

$$-i(T - T^\dagger) = 2\text{Im}T = TT^\dagger, \tag{1.1.1}$$

also known as the optical theorem. This statement can be examined order by order in a perturbative quantum field theory. When a complete set of states (containing summation over all possible internal states and integration over the on-shell momenta of the internal states) is inserted between T and $T^\dagger$ on the right hand side, (1.1.1) describes the imaginary part of the T-matrix as a sum of terms each representing two amplitudes connected by different possible on-shell internal states. This sum resembles the usual Feynman's perturbative expansion but is in fact different from it because the internal states of this expansion are all on-shell. The on-shellness and momentum conservation require that in the integral of the right hand side, there exists $\delta(P^2 - m^2)$, where P is the sum of incoming momenta and m is the mass of the internal particle. For this to happen, the external momenta cannot be real and have to be extended to the complex plane.

Moreover, since the amplitude has a pole as $\frac{1}{P^2-m^2}$, the usual Feynman propagator, the left hand side of (1.1.1), happens to be the discontinuity of T which also satisfies

$$\mathrm{Disc}(\frac{1}{P^2 - m^2}) = \lim_{\epsilon \to 0} \left(\frac{1}{(P^2 - m^2) + i\epsilon} - \frac{1}{(P^2 - m^2) - i\epsilon} \right) = \delta(P^2 - m^2). \qquad (1.1.2)$$

This is in fact what is called the particle-pole correspondence. Every particle which is allowed to be produced by analytic continuation corresponds to a pole of the amplitude.

An analytic function can be made by knowing all its singularities. As an analytic function of the kinematical variables, S-matrix has different types of singularities at different orders of perturbation. The information about the singularities was hoped to be enough to reconstruct amplitudes in the '60s under the S-matrix program. The claim was that all singularities of the S-matrix correspond to physical processes.[2] The aim of the S-matrix program was to develop a theory to describe the strong interaction for massive particles like pions, using unitarity and analyticity properties of the S-matrix.

As Quantum Chromo Dynamics (QCD), a quantum field theory based on a Lagrangian, was established in the '70s correctly describing the strong interaction, the S-matrix viewpoint became less attractive. However, later on in the '90s, it was realized that the natural home to the approach is in fact massless theories. In particular, certain one-loop amplitudes were built from their branch-cut discontinuities using the optical theorem (1.1.1) [4, 5, 6]. This approach, called the unitarity cut method in which only on-shell tree amplitudes are used as input, revived the hope of bootstrapping higher order amplitudes in a

[2] For a review of the S-matrix program see [3].

perturbative expansion.

1.2 Preliminaries

In this thesis, the two theories that we will study are (massless) pure Yang-Mills and pure gravity and, in some cases, the supersymmetric versions of them, all at the first order of perturbation (tree level). This section introduces the concepts and conventions which we will use to study our favourite theories, thus we skip reviewing other interesting material. As was touched on, since the goal is to avoid complicated approaches in calculating scattering amplitudes, our formalism is built towards this goal.

A big step to simplify calculations of gauge theory tree amplitudes is to first exclude the group theory (color) factors from the amplitude and calculate the color-stripped amplitude which depends only on kinematical variables. This manipulation is called *color-ordering* or *color-decomposition* since in the color-stripped amplitude, also called partial amplitude, only a fixed order of gluons appear. Each gluon carries a colour label a which also labels the generators of $U(N)$. The gauge group of QCD is $SU(3)$ whose generators, T^a, are related to Gell-Mann matrices by $T^a = \frac{1}{2}\lambda^a$ for $a = 1 \cdots 8$. The full amplitude is of course obtained from summing over all the non-cyclic permutations of partial amplitudes each multiplied by its corresponding trace factor,

$$\mathcal{A}_n = \sum \mathrm{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} \cdots T^{a_{\sigma(n)}}) A_n(\sigma(1), \sigma(2), \cdots, \sigma(n)). \tag{1.2.1}$$

$\mathcal{A}_n$ is the full tree-level amplitude of n gluons preserving momentum conservation (extracting $\delta^4\left(\sum_{i=1}^n p^i\right)$) and $A_n(\sigma(1), \sigma(2), \cdots, \sigma(n))$ is a partial amplitude with a specific order of the n gluons. To show why (1.2.1) works and the full amplitude can be written in terms of single-trace terms, recall that the group generators obey

$$\mathrm{Tr}(T^a T^b) = \delta^{ab}, \tag{1.2.2}$$

$$[T^a, T^b] = i f^{abc} T^c, \tag{1.2.3}$$

from which the structure constant can be written as

$$f^{abc} = -i(\mathrm{Tr}(T^a T^b T^c) - \mathrm{Tr}(T^a T^c T^b)). \tag{1.2.4}$$

Further manipulation writes the contractions of two or more structure constants, which appear in the expressions for amplitudes, as the sum of single traces of T^a's with different orderings, and hence proves the color-decomposition of gauge theory amplitudes (1.2.1).

1.2.1 Spinor Helicity Formalism

In their most general form, scattering amplitudes are functions of momenta, spin and polarization vectors (for particles with spin), and also colour. We saw that it is possible to colour-strip Yang-Mills amplitudes and hence we do so from now on and only consider partial amplitudes unless specified. For massless particles, the information of spin is encoded in the helicity in 4-dimensional space-time. So, momenta, helicities and polarization vectors are the ingredients of amplitudes of massless particles in four dimensions. Moreover, in four dimensions, a null momentum vector can be written in terms of two Weyl spinors. These spinors together with the helicity can fix the state's polarization vector up to gauge redundancies. Let us see these all in detail.

To every four-momentum p_μ, one can associate a Hermitian matrix $P_{\alpha\dot{\alpha}}$ by contraction with Pauli matrices,

$$P_{\alpha\dot{\alpha}} \equiv p_\mu \sigma^\mu_{\alpha\dot{\alpha}}, \tag{1.2.5}$$

where each index α and $\dot{\alpha}$ runs over 1 and 2. This relation directly shows that $\det(P_{\alpha\dot{\alpha}}) = p_\mu p^\mu = -m^2$ (on-shell), and hence, massless particles have associated matrices with vanishing determinants. Any rank one matrix (with zero determinant) can be represented as the product of two spinors,

$$P_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}. \tag{1.2.6}$$

Spinors λ_α (holomorphic) and $\tilde{\lambda}_{\dot{\alpha}}$ (anti-holomorphic) are in the $(1/2, 0)$ and $(0, 1/2)$ representations of the Lorentz group respectively.[3] Moreover, direct computation shows that for real momenta in Lorentz signature, the two spinors are complex conjugate of each other, $\tilde{\lambda}_{\dot{\alpha}} = \pm\overline{\lambda}_\alpha$ where the sign is determined by the sign of the energy of the particle. However, when the momenta are complex-valued, the corresponding spinors are considered as independent variables.[4]

For given λ and $\tilde{\lambda}$, one can build a unique momentum p, while the reverse is not true. Notice that simultaneous rescaling,

$$\lambda \rightarrow\ t\lambda \quad \text{and} \quad \tilde{\lambda} \rightarrow t^{-1}\tilde{\lambda}, \tag{1.2.7}$$

for a non-zero complex parameter t, leaves the momentum invariant. For real momenta, t is unit modulus whereas when momenta are complex it can be any complex number. This is in fact the little group ($U(1) = SO(2)$) scaling of the spinors. As we will see, this redundancy plays an important role in constructing functions which transform similar to scattering amplitudes under the Lorentz group.

As is known, inner products of momenta are Lorentz invariant objects used in quantum field theories to construct amplitudes. It is obvious that the next step in the spinor helicity formalism is to find $SL(2,\mathbb{C})$-invariants. Contracting with the anti-symmetric Levi-Civitas

[3]In the literature, alternatives of the notations used here are $|\lambda\rangle$ and $|\tilde{\lambda}]$ for the holomorphic and anti-holomorphic spinors respectively.

[4]Our convention for complex conjugation is to use bar instead of asterisk, keeping asterisk for the purpose of showing a specific value of a quantity.

tensor ϵ, one can define the following products for light-like spinors labeled by a and b,

$$\langle\lambda^a\lambda^b\rangle \equiv \det(\lambda^a\lambda^b) = \epsilon^{\alpha\beta}\lambda^a_\alpha\lambda^b_\beta = \lambda^a_1\lambda^b_2 - \lambda^a_2\lambda^b_1, \tag{1.2.8}$$

$$[\tilde{\lambda}^a\tilde{\lambda}^b] \equiv \det(\tilde{\lambda}^a\tilde{\lambda}^b) = \epsilon^{\dot{\alpha}\dot{\beta}}\tilde{\lambda}^a_{\dot{\alpha}}\tilde{\lambda}^b_{\dot{\beta}} = \tilde{\lambda}^a_{\dot{1}}\tilde{\lambda}^b_{\dot{2}} - \tilde{\lambda}^a_{\dot{2}}\tilde{\lambda}^b_{\dot{1}}, \tag{1.2.9}$$

and moreover, using (1.2.5) and (1.2.6), the brackets can be related to the inner product of the corresponding momenta,

$$2p^a \cdot p^b = \langle\lambda^a\lambda^b\rangle[\tilde{\lambda}^a\tilde{\lambda}^b]. \tag{1.2.10}$$

For simplicity, the angular and square brackets defined above are usually written as $\langle ab\rangle$ and $[ab]$. We will see how calculations by means of the new variables are much simpler and more compact than with the usual momentum space variables. Another Lorentz invariant quantity which will be repeatedly used in calculations is $[a|P|b\rangle \equiv [a\tilde{\lambda}_P]\langle\lambda_P b\rangle$ when P is light-like.

It is also useful to mention that the momentum conservation condition for scattering of n massless particles in terms of spinors is

$$\sum_{i=1}^{n} p^i = 0 \quad\Longrightarrow\quad \sum_{i=1}^{n} \lambda^i\tilde{\lambda}^i = 0. \tag{1.2.11}$$

The other useful relation for the calculations is the Schouten identity between any four holomorphic and also anti-holomorphic spinors,

$$\langle ij\rangle\langle kl\rangle + \langle ik\rangle\langle lj\rangle + \langle il\rangle\langle jk\rangle = 0, \tag{1.2.12}$$

$$[ij][kl] + [ik][lj] + [il][jk] = 0 \tag{1.2.13}$$

which is in fact the consequence of the dependence of any three vectors on a plane (here, 2-component vectors j, k and l).

Having introduced λ and $\tilde{\lambda}$, and picking a helicity for a massless particle with spin 1, the polarization vectors can be determined as follows[5]:

$$\epsilon^-_{\alpha\dot\alpha} = \frac{\lambda_\alpha \tilde\mu_{\dot\alpha}}{[\tilde\lambda\tilde\mu]} \quad \text{for negative-helicity particle,} \tag{1.2.14}$$

$$\tilde\epsilon^+_{\alpha\dot\alpha} = \frac{\mu_\alpha \tilde\lambda_{\dot\alpha}}{\langle\mu\lambda\rangle} \quad \text{for positive-helicity particle.} \tag{1.2.15}$$

These definitions guarantee $\epsilon^i \cdot p^i = 0$ as well as the correct scaling of the polarization vector for every particle i when the spinors μ and $\tilde\mu$ are arbitrary.

It now becomes clear that the scattering amplitude of massless gauge bosons only depends on λ, $\tilde\lambda$ and the helicity of external states, hence $A(\{\lambda^i, \tilde\lambda^i, h_i\})$. Furthermore, in an amplitude, vertices and propagators do not scale under the little group transformation (1.2.7) but the external states do. From (1.2.14), one can see that $\epsilon^\pm(\lambda, \tilde\lambda)$ rescales with t^{-2h} when (1.2.7) is applied. These observations directly determine the scaling of the amplitude under little group transformation of each of the external states with helicity h_i,

$$A_n(\{\lambda^1, \tilde\lambda^1, h_1\}, \dots, \{t_i\lambda^i, t^{-1}\tilde\lambda^i, h_i\}, \dots, \{\lambda^n, \tilde\lambda^n, h_n\}) = t^{-2h_i} A_n(\dots, \{\lambda^i, \tilde\lambda^i, h_i\}, \dots). \tag{1.2.16}$$

1.2.2 Constructing 3-Particle Amplitudes

3-particle amplitudes are very special objects in massless theories. For three massless particles, momentum conservation implies that

$$0 = (p^3)^2 = (-p^1 - p^2)^2 = \langle 12\rangle[12], \tag{1.2.17}$$

so either $\langle 12\rangle = 0$ or $[12] = 0$. Suppose that the square brackets vanish; then one can write that $\langle 1|p^1 + p^3|3] = -\langle 12\rangle[23] = 0$ which indicates that $[23] = 0$. Similarly, one can find the other square brackets [13] vanish too. We conclude that the on-shell 3-particle amplitude is

[5] We hope that the context makes it clear when we refer to polarization vectors $\epsilon_{\alpha\dot\alpha}$, or the Levi-Civita tensor $\epsilon^{\alpha\beta}$ and $\epsilon^{\dot\alpha\dot\beta}$.

a function of either angular or square brackets of external massless particles. An immediate consequence is that when momenta are all real, since $\bar{\lambda}^i = \tilde{\lambda}^i$ for each particle i, on-shell 3-particle amplitude is identically zero because all its kinematical variables vanish. We will later on see that non-vanishing *complex* 3-particle amplitudes play an important role in constructing amplitudes with higher points.

To compute a non-vanishing 3-particle amplitude, suppose that it depends on holomorphic spinors,

$$A_3(\{\lambda^1, h_1\}, \{\lambda^2, h_2\}, \{\lambda^3, h_3\}) = c_H \langle 12\rangle^{m_1} \langle 23\rangle^{m_2} \langle 31\rangle^{m_3}. \tag{1.2.18}$$

(1.2.16) easily fixes the exponents of brackets in terms of the helicities; hence, up to a constant factor, the 3-particle amplitude of massless external states is given by

$$A_3^H(\{\lambda^1, h_1\}, \{\lambda^2, h_2\}, \{\lambda^3, h_3\}) = c_H \langle 12\rangle^{h_3 - h_1 - h_2} \langle 23\rangle^{h_1 - h_2 - h_3} \langle 31\rangle^{h_2 - h_3 - h_1}. \tag{1.2.19}$$

Similar arguments when we assume that the amplitude only depends on anti-holomorphic spinors yield

$$A_3^A(\{\tilde{\lambda}^1, h_1\}, \{\tilde{\lambda}^2, h_2\}, \{\tilde{\lambda}^3, h_3\}) = c_A [12]^{-h_3 + h_1 + h_2} [23]^{-h_1 + h_2 + h_3} [31]^{-h_2 + h_1 + h_3}. \tag{1.2.20}$$

We now need another condition to choose between (1.2.19) and (1.2.20) which is given by the real momenta limit. As we discussed earlier, all brackets vanish for three on-shell momenta. Therefore, it is clear from the relations above that $h_1 + h_2 + h_3$ determines whether the amplitude is holomorphic or anti-holomorphic. For a finite c_A, if the sum of helicities is negative, the anti-holomorphic amplitude A_3^A blows up while A_3^H vanishes. One can then conclude that c_A must be zero to avoid an unphysical result. Similarly, when $h_1 + h_2 + h_3 > 0$, only A_3^A vanishes and c_H must be set to zero.

For pure Yang-Mills, $h = \pm 1$, (1.2.19) and (1.2.20) together with the real momentum condition can completely fix 3-particle amplitudes with different helicities as follows

$$A_3(1^-, 2^-, 3^+) = g_{YM} \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle}, \tag{1.2.21}$$

$$A_3(1^+, 2^+, 3^-) = g_{YM} \frac{[12]^3}{[23][31]}. \tag{1.2.22}$$

Also note that the coupling constant of Yang-Mills is dimensionless and the mass dimension of an n-particle amplitude in $d = 4$ is $4 - n$ followed from dimensional analysis (the cross-section has mass dimension -2 or dimension of area). Our results above are also compatible with the dimensional constraints since both square and angular brackets have mass dimension 1 (recall that $\langle pq \rangle [pq] = 2p.q$). Moreover, one can easily check that if all three helicities are either plus or minus, the corresponding A_3's cannot have the correct mass dimension 1 and hence are ruled out. Therefore, $A_3(+, +, +) = A_3(-, -, -) = 0$.

Similar calculations can be done for 3-particle amplitudes of other massless theories including gravity. We will comment on that after introducing gravity as a quantum field theory in 1.2.4.

1.2.3 MHV Amplitudes in Yang-Mills

In the 1980s, some very important properties of n-gluon scattering amplitudes were discovered. Conjectured by Parke and Taylor [7] and proven by Berends and Giele [8],

$$A(1^+, \ldots, n^+) = 0, \tag{1.2.23}$$

$$A(1^+, \ldots, i^-, \ldots, n^+) = 0, \tag{1.2.24}$$

$$A(1^+, \ldots, i^-, \ldots, j^-, \ldots, n^+) = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}. \tag{1.2.25}$$

The first two results hold the same when $+$ and $-$ helicity signs are exchanged and for the last expression, under the same sign flip, one just needs to replace square brackets with angular brackets, $\langle\ \rangle \to [\ ]$. This amplitude with two negative helicities among other positive ones is called the Maximally Helicity Violating amplitude (MHV) and its conjugate amplitude is called $\overline{\text{MHV}}$:

$$A(1^-, \ldots, i^+, \ldots, j^+, \ldots, n^-) = \frac{[ij]^4}{[12][23]\cdots[n1]}. \tag{1.2.26}$$

The simple and compact form of the Parke-Taylor formula, especially when written in spinor helicity notation, brings a lot of simplicity to computations. The next-to-simplest amplitudes, with three negative helicities, are called Next-to-MHV or NMHV and similarly an amplitude with k+2 negative helicities is called N^kMHV amplitude. It is easy to see that five (and less) gluon amplitudes at tree level can be completely determined by the formulas above, MHV or $\overline{\text{MHV}}$.

In super Yang-Mills theory, the full tree amplitude can be decomposed as a sum over different sectors:

$$A_n = A_n^{\text{MHV}} + A_n^{\text{NMHV}} + A_n^{\text{N}^2\text{MHV}} + \cdots + A_n^{\overline{\text{MHV}}}, \tag{1.2.27}$$

where the N^kMHV sector is defined as all amplitudes which are connected to N^kMHV gluon amplitude by supersymmetry. The amplitude in the MHV sector has in fact the very form of its non-supersymmetric partner, using Grassmann variables $\bar{\eta}$ which are the eigenstates of supercharge $\bar{Q}$,

$$A_n^{\text{MHV}} = \frac{\delta^{2\mathcal{N}}(\sum_i \lambda_i \bar{\eta}_i)}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}. \tag{1.2.28}$$

1.2.4 Einstein's Gravity as a Quantum Field Theory

The main focus of this thesis will be on gravity at the first order of perturbation. To establish what we mean by perturbative gravity and also appreciate the modern methods for calculating gravity amplitudes, we return to the Lagrangian formulation of the theory for the moment[6]. Like other quantum field theories, in the next step one can read off the

[6] In this thesis we only study gravity at its *weak* coupling limit; meaning that those classical solutions of Einstein's equation like black holes or Friedmann-Robertson-Walker metric are not our areas of concern. In other words, we have a *perturbative* point of view whereas non-perturbative solutions (e.g. black holes in gravity and monopoles in Yang-Mills) go beyond the scope of this thesis.

Feynman rules from the Lagrangian and then calculate scattering amplitudes of particles associated with quantum fields.

General relativity describes gravity at the classical level through Einstein-Hilbert action,

$$S_{EH} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} R, \tag{1.2.29}$$

where R is the Ricci scalar and $2\kappa^2 = 16\pi G_N$. Working with pure gravity, we can exclude S_{matter} from the Einstein-Hilbert action. When expanded perturbatively around a flat metric, $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, the action contains vertices to all orders. This in fact happens because both $\sqrt{-g}$ (not the determinant itself) and the inverse metric, $g^{\mu\nu}$ which appears in the Christoffel symbols in R, get infinite number of terms in the series expansion. Each term in the expanded action contains two derivatives since the Ricci scalar does so. Hence, each vertex carries two powers of momentum.

Keeping the terms to the second order in $\kappa h_{\mu\nu}$, one obtains the Fierz-Pauli Lagrangian in terms of the graviton field:

$$L = \frac{1}{4}\partial^\mu h^{\nu\rho}\partial_\mu h_{\nu\rho} - \frac{1}{2}\partial^\mu h^{\nu\rho}\partial_\nu h_{\mu\rho} + \frac{1}{2}\partial^\mu h \partial^\lambda h_{\lambda\mu} - \frac{1}{4}\partial^\mu h \partial_\mu h, \tag{1.2.30}$$

where h is the trace of $h_{\mu\nu}$. This action has gauge redundancy $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \eta_\nu + \partial_\nu \eta_\mu$, and in order to extract any physical quantity, one must first fix the gauge. Our choice is the de Donder gauge, $\partial^\mu h_{\mu\nu} = \frac{1}{2}\partial_\nu h$. A bit of calculation yields the final form of the Lagrangian from which to extract the propagator:

$$L = -\frac{1}{2}\partial_\mu h_{\nu\rho} V^{\nu\rho\alpha\beta}\partial^\mu h_{\alpha\beta}, \qquad V^{\nu\rho\alpha\beta} = \frac{1}{2}\eta^{\nu\alpha}\eta^{\rho\beta} - \frac{1}{4}\eta^{\nu\rho}\eta^{\alpha\beta}. \tag{1.2.31}$$

With k being the momentum of the graviton, one can now write the propagator which is inversely quadratic in momentum,

$$P^{\nu\rho\alpha\beta} = \frac{\eta^{\nu\alpha}\eta^{\rho\beta} + \eta^{\alpha\rho}\eta^{\beta\nu} - \eta^{\nu\rho}\eta^{\alpha\beta}}{k^2 - i\epsilon}. \tag{1.2.32}$$

As is known, gravitons – associated with quantization of $h_{\mu\nu}$– are massless spin-2 particles with two (± 2) helicity states. Similar to massless gluons, along with helicities, gravitons can be labeled by holomorphic, λ, and anti-holomorphic, $\tilde{\lambda}$, spinors. Polarization tensors associated with each state can be written as products of spin-1 polarization vectors,

$$\epsilon^{\pm}_{\mu\nu}(k_i) = \epsilon^{\pm}_{\mu}(k_i)\epsilon^{\pm}_{\nu}(k_i). \tag{1.2.33}$$

Moreover, different reference momenta can be used in the definitions for $\epsilon^{\pm}_{\mu}$ and $\epsilon^{\pm}_{\nu}$.

We already said that there exist infinite interaction terms (quadratic in momentum) in the Lagrangian. It is therefore obvious how complicated calculations are with Feynman method. However, one lesson from new on-shell methods in Yang-Mills was that the gauge redundancies in a theory are responsible for complications in calculations. In fact, gravity has more redundancies: diffeomorphism invariance. The role of the infinite vertices in Einstein-Hilbert action is to maintain the diffeomorphism invariance of the action. While little group properties fix the 3-graviton amplitude, one's dream would be to build higher-point amplitudes recursively, and in fact there exists a recursion relation, Britto-Cachazo-Feng-Witten (BCFW) [9, 10], to do the job. As a summary, only a certain set of interaction terms is needed to construct graviton amplitudes using methods which we will review in the rest of this chapter.

To complete the discussion, let us calculate 3-particle amplitudes in gravity, another example of a massless theory. Similar to pure Yang-Mills, 3-graviton amplitudes can be fixed by little group properties and dimensional analysis. With $h = \pm 2$, (1.2.19) and (1.2.20) yield

$$M_3(1^-, 2^-, 3^+) = \left(\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\right)^2, \tag{1.2.34}$$

$$M_3(1^+, 2^+, 3^-) = \left(\frac{[12]^3}{[23][31]}\right)^2. \tag{1.2.35}$$

Notice "$A_3^2 = M_3$" in these relations (apart from the coupling constants).

Through this thesis, M_n is used to denote n-particle tree-level amplitudes in gravity.

1.2.5 MHV Amplitudes in Gravity

Like generic amplitudes in gravity, MHV amplitudes are much more complicated than their Yang-Mills counterparts. In fact, there have been different versions of graviton's MHV formula; and just very recently, a formula similar to and still more complicated than Parke-Taylor's was found.

The first version is Berends-Giele-Kuijf (BGK) formula [11]. Almost 20 years later, two other versions were found. One was the Mason-Skinner's [12] derived by background field calculations and integration in twistor space. Later on in chapter 3 we will use this version to study 12-graviton NMHV amplitudes. A year after, another MHV formula was introduced by Nguyen et al [13] with a different perspective on summing over the terms.

For the MHV amplitude $M_n\left(1^-, 2^+, \ldots, (n-1)^+, n^-\right)$ Mason-Skinner formula is

$$\begin{aligned} M_n^{\text{MHV}} &= \kappa^{n-2}\delta^4\left(\sum_{i=1}^{n} p^i\right)\frac{\langle 1\ n\rangle^8}{\langle 1\ n-1\rangle\langle n-1\ n\rangle\langle n\ 1\rangle}\left(\frac{1}{\langle 1\ 2\rangle\langle 2\ 3\rangle\cdots\langle n-1\ n\rangle\langle n\ 1\rangle}\right. \\ &\times\left.\prod_{k=2}^{n-2}\frac{\langle n|p_{n-1}+\ldots+p_{k+1}|k]}{\langle k\ n\rangle} + (\text{permutations of labels } \{2,\ldots,n-2\})\right) \\ &= \frac{\langle 1\ n\rangle^6}{\langle 1\ n-1\rangle\langle n-1\ n\rangle}\left(\frac{1}{\langle 1\ 2\rangle\langle 2\ 3\rangle\cdots\langle n-1\ n\rangle}\right. \\ &\times\left.\prod_{k=2}^{n-2}\frac{\langle n|-p_1-\ldots-p_{k-1}|k]}{\langle k\ n\rangle} + (\text{permutations of labels } \{2,\ldots,n-2\})\right) \end{aligned} \tag{1.2.36}$$

which is completely symmetric in the external states, except for the factor $\langle 1\ n\rangle$ representing the two negative helicities. Notice that this formula represents the full amplitude. It is seen from (1.2.36) that gravity's MHV amplitude is not holomorphic, opposed to Yang-Mills.

The simplest and most compact form for MHV amplitudes in gravity was introduced by Hodges in 2012 [14]. The formula enjoys special determinants, closely related to soft factors, and has also appeared in different works since introduced, e.g. [15, 16]. Stripping momentum conserving δ-function and Grassmannian δ-function, Hodges's tree-level MHV amplitude in $\mathcal{N}=8$ supergravity is given by

$$M_n^{MHV} = (-1)^{n+1}\sigma(ijk,rst)\frac{|\Phi^H|^{rst}_{ijk}}{\langle ij\rangle\langle jk\rangle\langle ki\rangle\langle rs\rangle\langle st\rangle\langle tr\rangle}, \tag{1.2.37}$$

where

$$\sigma(ijk,rst) = \text{sgn}((ijk12\ldots\not{i}\not{j}\not{k}\ldots n) \to (rst12\ldots\not{r}\not{s}\not{t}\ldots n),$$

and $|\Phi^H|^{rst}_{ijk}$ is the $(n-3)\times(n-3)$ minor of the matrix

$$(\Phi^H)^i_j = \frac{[ij]}{\langle ij\rangle},\ i\neq j, \qquad (\Phi^H)^i_i = -\sum_{j\neq i}\frac{[ij]\langle jx\rangle\langle jy\rangle}{\langle ij\rangle\langle ix\rangle\langle iy\rangle}, \tag{1.2.38}$$

obtained by deleting the columns r,s,t and rows i,j,k. Here x and y are two arbitrary spinors.

We will again see Hodges's formula in chapter 4 where the tree-level n-particle amplitudes in $\mathcal{N}=8$ supergravity, a new formulation introduced by Cachazo and Geyer [15], is studied to prove correct soft limit and parity symmetry.

1.3 Modern Methods in Calculations of Scattering Amplitudes

Introducing alternatives to the Feynman method goes back to the '80s. Based on Feynman's rules, Berends and Giele developed a recursive method to calculate QCD amplitudes [8]. Their method uses colour-ordering, gauge invariance and some other conditions to simplify intermediate steps of calculations but also makes one of the states in the Feynman diagram off-shell (gauge-dependent). To do so, they replace one of the polarization vectors by an off-shell propagator. The off-shell n-point "amplitude" whose $n-1$ legs are on-shell is called the n-point gluon current J_n^μ for which they write a recursion relation. In the end, the off-shell momentum is set on-shell and the leg is multiplied by its polarization vector. Although this method is faster and numerically more efficient than Feynman's, it still produces long expressions and also needs off-shell information.

In this thesis, we are interested in the recent alternatives to the Feynman method, the modern *on-shell* methods of calculation which have proven enormously simpler than their ancestors. In on-shell methods, the lower-point amplitudes used in calculating higher ones are themselves gauge invariant and on-shell. These modern methods also elucidate many underlying mathematical structures of amplitudes which are obscured in the Lagrangian formulation of quantum field theories. Complex analysis plays a crucial role in formulating these methods. We will see, for instance, in the case of BCFW recursion relations 1.3.1 [9, 10], using complex analysis makes the calculations of amplitudes on-shell.

In our review in the rest of this section, two main developments in calculating tree-level amplitudes in gauge and gravity theories will be studied, however not in the historical order: BCFW recursion relations and Cachazo-Svrcek-Witten (CSW) expansion [17].

Since the introduction of the CSW expansion and BCFW recursion relations, the methods have been extensively used in calculations of tree and loop-level amplitudes in QCD, $\mathcal{N} = 4$ super Yang-Mills theory, general relativity and $\mathcal{N} = 8$ supergravity. For a review of on-shell methods, see for instance [18, 19] and the references listed there.

Although we will not review other new advancements in the amplitude area, it is worthwhile to briefly mention some of them here: generalized unitarity and maximal cuts for computing loop-level amplitudes in $\mathcal{N} = 4$ super Yang-Mills and QCD [5, 6, 4]; Kawai-Lewellen-Tye (KLT) relations [20] which at tree level write the amplitudes in gravity as the product of two gauge theory amplitudes times some kinematical factors; and Bern-Carrasco-Johansson (BCJ) relations [21, 22] for Yang-Mills and its supersymmetric extension which propose a duality between the kinematic and colour factors of tree-level amplitudes if written only in terms of cubic vertices; and much more.

One of the main roots of the increasing interest and developments in modern approaches to quantum field theory is Witten's twistor string theory, which appeared in 2003 [23]. In this work, Yang-Mills amplitudes are transformed from momentum space to Penrose's twistor space [24] and shown to live on certain holomorphic curves in twistor space. CSW expansion and BCFW recursion relations were found soon after the appearance of this work and later on extended in [25, 26] where the supersymmetric version of BCFW deformation was also introduced. Further works studied the amplitudes and in particular BCFW in twistor space [27, 28]. Inspired by the twistor space formulation of BCFW, a dual formulation for the S-matrix of $\mathcal{N} = 4$ super Yang-Mills was proposed in 2009 [29]. The duality connects the leading singularities of planar N^kMHV amplitudes to simple contour integrals over the Grassmannian manifold of k-planes in n-dimensions. This work has garnered huge interest among quantum field theorists and has already led to several advances. [7]

Another breakthrough was finding the all-loop integrand for the scattering amplitudes in the planar limit of $\mathcal{N} = 4$ super Yang-Mills [38]. In this work, we see two important generalizations: BCFW recursion relations from tree-level amplitudes to all loop orders and the Grassmannian duality from leading singularities to the full amplitude. Very recently, with a more abstract mathematical viewpoint, scattering amplitudes in planar theories were connected to the positive Grassmannian, a mathematical structure [39]. The significance of on-shell diagrams is clearly seen in this approach. Among the very recent developments in amplitude area are polytopes [40, 41] and scattering equations [42, 43, 44].

This quick review does not cover all of what many quantum field theorists have done to explore an important part of the fundamental physics, scattering amplitudes. The last ten years have seen many fundamental discoveries in planar gauge theories, and in particular $\mathcal{N} = 4$ super Yang-Mills. Much has been done, yet much remains to do, especially for gravity!

We first review the BCFW recursion relations and then the CSW expansion for gauge theory amplitudes and, closely related to it, Risager's method [45]. Extension to gravity is also discussed and is part of the focus of this thesis too, later on in chapter 5.

[7] The reader is referred to [30, 31, 32, 33, 34, 35, 36, 37] for a sample of more recent progress.

1.3.1 BCFW Recursion Relations

In 2004-2005, Britto, Cachazo, Feng and Witten introduced and proved recursion relations, BCFW [9, 10], for obtaining Yang-Mills tree-level amplitudes from *on-shell* lower point amplitudes. The process elegantly uses complex analysis to derive a physical on-shell amplitude from a complexified one by using the Cauchy's theorem. In the first step, BCFW shifts the (on-shell) momenta of two external particles to the complex plane in such a way that the total momenta sum to zero, as they did before the shift was applied. For particles labeled by i and j, the BCFW deformation is

$$p^i \to p^i(z) = p^i + zq, \qquad p^j \to p^j(z) = p^j - zq, \tag{1.3.1}$$

for an arbitrary vector q and the complex variable z. It is obvious from the shift that the momentum conservation is preserved. Moreover, demanding $q^2 = 0$ and $p^i \cdot q = p^j \cdot q = 0$, one can see that the deformed momenta stay on-shell. Equivalently, this deformation can be made on the holomorphic and anti-holomorphic spinors of the two particles i and j:

$$\lambda^i \to \lambda^i(z) = \lambda^i + z\lambda^j, \qquad \tilde{\lambda}^j \to \tilde{\lambda}^j(z) = \tilde{\lambda}^j - z\tilde{\lambda}^i, \tag{1.3.2}$$

keeping $\tilde{\lambda}^i$ and λ^j undeformed.

The amplitude, two of whose external momenta are deformed as above, is now an analytic function of z through some of its propagators (and vertices): $A(z)$. However, the question is to derive the non-deformed amplitude, $A(z = 0)$, as it is our physical quantity.

Notice that at tree level, $A(z)$ can only have simple poles. In fact, this observation is the key to making BCFW so simple. No other type of singularity, e.g. branch-cuts, exists in tree amplitudes. Simple poles in $A(z)$ belong to the shifted propagators $1/P^2(z)$. In the next step, assuming that $A(z)$ vanishes as z approaches infinity, BCFW applies the Cauchy's theorem to $A(z)$, with contour $\mathcal{C}$ enclosing all the poles of the integrand, in the following way:

$$\frac{1}{2\pi i}\oint_{\mathcal{C}} dz \frac{A(z)}{z} = 0, \tag{1.3.3}$$

which by residue theorem implies that

$$A(0) = -\sum_{\substack{\text{poles of}\\ A(z)}} \text{Res}(\frac{A(z)}{z}). \tag{1.3.4}$$

Close to a simple pole of the amplitude, where a propagator blows up, one can write the following factorization (to left (L) and right (R)) for the amplitude:

$$\lim_{P_{k,l}^2(z)\to 0}(P_{k,l}^2(z)A_n(z)) = \sum_h A_L(k,\cdots,l,-P_{k,l}^{-h}(z))A_R(P_{k,l}^h(z),l+1,\cdots,k-1). \tag{1.3.5}$$

On the right hand side, the sum is over different helicities of the internal particle connecting the two sub-amplitudes A_L and A_R. To be more precise, this helicity is of the two emergent on-shell particles which come to exist after the corresponding propagator $\frac{1}{P_{k,l}^2(z)}$ blows up. The physical meaning of (1.3.5) is that when a propagator blows up, the dominant contribution to the amplitude is from the Feynman diagrams which are split into two parts by that propagator. This relation also gives the residue of the amplitude for the pole

$$z_{k,l} = -\frac{P_{k,l}^2}{[j|P_{k,l}|i\rangle} \tag{1.3.6}$$

when particles i and j (1.3.2) do not belong to the same side, A_L or A_R. A few more steps shows that

$$A_n(0) = \sum_{\substack{\text{poles } z_{k,l}\\ h}} A_L(k,\cdots,l,-P_{k,l}^{-h}(z_{k,l}))\frac{1}{P_{k,l}^2}A_R(P_{k,l}^h(z_{k,l}),l+1,\cdots,k-1), \tag{1.3.7}$$

which states the full physical amplitude is written in terms of lower-point on-shell amplitudes and is called the BCFW recursion relations. The process above can be repeated for each lower-point amplitude down to 4-particle. The 3-particle amplitudes, as was seen, can be fixed by group theory information and play the role of the bricks of our amplitude construction.

As was said, the assumption we made here before using the Cauchy's theorem (1.3.3) is that the complex amplitude vanishes as $z \to \infty$. This needs to be proved separately for different quantum field theories under study. Our two favourite theories are Yang-Mills gauge theory and gravity.

For tree-level amplitude in Yang-Mills, the z dependence comes from vertices, polarization vectors and propagators. The two types of vertices are cubic, which is linear in momentum, and quadratic, which is momentum-independent. A BCFW-deformed propagator depends on the complex variable as $1/z$. The helicities of the polarization vectors for particles i and j can be chosen so that each polarization vector depends on z as $1/z$. Hence, if say m number of vertices, and therefore $m-1$ propagators, are affected by the shift, the amplitude at large z behaves as

$$\lim_{z\to\infty} A(z) \sim z^m \frac{1}{z^{m-1}} \frac{1}{z^2} = \frac{1}{z}, \tag{1.3.8}$$

which guarantees (1.3.3). This in fact happens when particles i and j have helicities $(+,+)$, $(-,-)$ and $(+,-)$, while if the helicities were chosen as $(-,+)$ the amplitude would blow as z^3. It is obvious that one can always choose two particles to deform so that the BCFW recursion relations work for Yang-Mills.

For BCFW to work for tree-level graviton amplitudes, a proof based on the background field method is presented in [46]. This gives us great power to calculate tree graviton amplitudes using recursion relations and based on 3-graviton amplitudes as the building blocks. As already mentioned, all other infinitely many interaction terms are irrelevant to the on-shell tree amplitudes in gravity.

1.3.2 CSW Expansion

In their seminal work [17], Cachazo, Svrcek and Witten (CSW) proposed a recursive expansion for Yang-Mills which writes tree-level amplitudes in terms of only MHV ones. It is also called MHV-vertex expansion. This work directly followed Witten's twistor string theory [23] and later on was extended to one-loop calculations [47]. The rules of the expansion are similar to Feynman rules except for the vertices which are off-shell continuations of MHV amplitudes. The off-shellness is provided by introducing an auxiliary spinor into the calculations; for an internal propagator with momentum P, we pick up an auxiliary spinor $\tilde{\eta}$ and set λ_P as

$$\lambda_P = P|\tilde{\eta}]. \tag{1.3.9}$$

It was also shown that, similar to Feynman rules, CSW rules can be derived from the Yang-Mills Lagrangian in a particular form of light-cone gauge [48, 49]. Possible interaction vertices are cubic $(++-)$ and $(+--)$ and quadratic $(++--)$. These works show that using a field redefinition, it is possible to remove all $\overline{\text{MHV}}$ vertices $(++-)$ and reproduce the amplitudes from only on-shell MHV ones.

In a closely related work, the equivalence of MHV diagrams and Feynman diagrams of certain twistor actions for gauge theories was shown in [50]. The usual Yang-Mill Lagrangian, which is manifestly Lorentz invariant, is derived from a particular gauge-fixing of twistor action while the MHV Lagrangian is obtained from choosing a different gauge, the light-cone gauge.

In twistor space, an MHV amplitude is represented by a line (degree 1 holomorphic curve). The terms in the CSW expansion are each product of MHV amplitudes, hence, some lines in twistor space. For the N^{k-2}MHV amplitude in momentum space, there will be $k-1$ lines in twistor space, each representing an MHV sub-diagram. Two lines in twistor space intersect at a point which in momentum space corresponds to the internal line between the sub-amplitudes. Hence, one can show a generic term in the CSW expansion of NMHV tree amplitudes, also localized in twistor space, as in figure 1.1, where negative

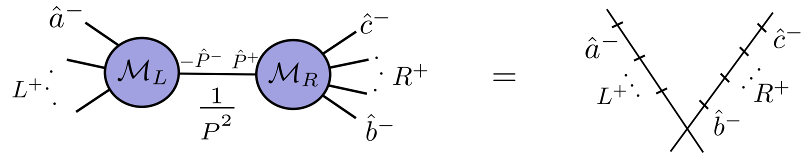

Figure 1.1: A CSW term in momentum space and twistor space.

helicities are a, b and c.

1.3.3 Risager's Method

In 2005, Risager [45] presented a BCFW-like proof of the CSW rules to compute gluon amplitudes. In his approach, an auxiliary spinor similar to CSW's is used to apply a certain complex deformation on external momenta. As in the BCFW method, the residue theorem is used to recover the non-deformed Yang-Mills amplitude.[8]

The simplicity of the CSW expressions in gauge theory motivated similar computations for gravity. However, when applied to gravity, Risager's method only produces graviton amplitudes up to eleven particles in the Next-to-MHV sector. The failure of Risager's method to yield an MHV-expansion for NMHV gravity was first discovered by Bianchi et al. [53] using numerical techniques and later analytically confirmed by Benincasa et al. [54]. Also, recently the difference of the pure 12-graviton NMHV amplitude with its Risager's expansion was analytically calculated by Conde and the author of this thesis [55]. The main problem with gravity is that no off-shell definition of MHV amplitudes has been found which could provide a CSW expansion for gravity, whereas in Yang-Mills the holomorphic nature of MHV amplitudes allows a simple off-shell extension.

Here we briefly review Risager's method. Throughout this section, we only concentrate on pure gluon and graviton amplitudes in the NMHV sector which contain three negative helicity and $n-3$ positive helicity particles. For the moment we denote both Yang-Mills and gravity tree amplitudes by A_n.

[8] Similar analysis for super Yang-Mills theory is presented in [51, 52]

Using the spinor-helicity notation for the on-shell momentum of particle i, $p^i = \lambda^i \tilde{\lambda}^i$, Risager's expansion is obtained from a complex deformation on the anti-holomorphic spinors, $\tilde{\lambda}^i$, of the external negative helicity particles. Since we restricted ourselves to the NMHV amplitudes, Risager's deformation with a complex variable z is on three spinors,

$$\begin{cases} \tilde{\lambda}^a(z) = \tilde{\lambda}^a + z\,\langle b\,c\rangle \tilde{\eta} \\ \tilde{\lambda}^b(z) = \tilde{\lambda}^b + z\,\langle c\,a\rangle \tilde{\eta} \\ \tilde{\lambda}^c(z) = \tilde{\lambda}^c + z\,\langle a\,b\rangle \tilde{\eta} \end{cases} \tag{1.3.10}$$

where the negative helicities are labeled by a, b and c, and $\tilde{\eta}$ is an arbitrary spinor [9]. The coefficients $\langle ab\rangle, \langle bc\rangle$ and $\langle ca\rangle$ have been chosen to maintain momentum conservation after the deformation.

Through this deformation, the amplitude becomes a rational function of z, $A_n(z)$. One can apply the residue theorem to construct the amplitude A_n from the poles of $A_n(z)$. Risager's expansion, A_n^{Ris}, is obtained by summing the residues of $A_n(z)$:

$$A_n^{\text{Ris}} = \sum_{a,L^+} A_L\left(\hat{a}^-, L^+, \left(-\hat{P}\right)^-\right) \frac{1}{(p_a + P_{L^+})^2}\, A_R\left(\hat{P}^+, \hat{b}^-, \hat{c}^-, R^+\right)\,, \tag{1.3.11}$$

where $\hat{P} = \hat{p}_a + P_{L^+}$ is Risager's deformed momentum flowing in the internal propagator, and L^+ (R^+) denotes the subset of external positive-helicity particles in the left (right) sub-amplitude in (1.3.11). The hatted momenta are evaluated at the position of the poles which makes the corresponding propagator on-shell. Also, note that each term in the expansion is the product of two MHV amplitudes and a propagator.

As a requirement of the residue theorem, $A_n(z)$ must vanish at infinity for the expansion to be valid, which is the case for Yang-Mills [45]. Using this deformation repeatedly, Risager reproduced CSW rules for Yang-Mills amplitudes [45]. On the other hand, for gravity, it was shown numerically [53] and analytically [54] that the behaviour of the amplitude under Risager's shift is $M_n(z) \sim z^{n-12}$ when $z \to \infty$. Risager's MHV-vertex expansion for

[9] Not to be confused with the Grassmann parameter $\tilde{\eta}$ in the supersymmetric BCFW deformation.

gravity, hence, is not valid when applied to amplitudes of more than eleven gravitons. We will see in chapter 3 how to modify Risager's expansion for gravity.

1.4 About This Thesis

In this thesis, we move towards the goal of probing scattering amplitudes from different angles and most of our focus is on tree-level graviton amplitudes. In the amplitudes community, gravity has shown to require much more intricate calculations and therefore has been much less developed compared to its cousin Yang-Mills theory. For gravity, there is no simplification similar to color-ordering in Yang-Mills and one needs to compute many more individual sub-amplitudes to write the full graviton amplitude even at tree-level. Not only is gravity the closest theory to Yang-Mills and, hence, the next natural path to take after gauge theory, but also it has more hidden and dark sides to examine, making it more alluring to us. That being said, we have more than enough motivation to explore gravity with our approach, the new on-shell techniques in scattering amplitudes. Along the road, the hope is that the newly developed techniques guide us to better understand the longest-known force in nature at short distances.

We will treat gravity perturbatively and focus on the first level, tree amplitudes. Risager's method is used in most parts of our calculations and in different problems. It is thus worth first better examining the method itself which we will do using Yang-Mills. The second chapter is then devoted to Yang-Mills and the rest are centred around gravity.

In chapter 2, we discuss how Risager's method is allowed to contain more than one complex variable in the shift and is then used to compute Yang-Mills tree amplitudes. This generalization to several complex variables generally creates more terms in the CSW expansion than the usual one-variable shift. This may sound unnecessary but the novelty of the method is that new types of terms appear in the expansion which do not show up in the usual Risager's method. We study next-to-MHV 5- and 6-gluon amplitudes and find that the new terms are exactly of the form of *soft terms* in which the momentum of one of the external gluons approaches zero and the amplitude factorizes to a soft factor times a lower-point amplitude. The interpretation is that now soft terms do contribute to building an amplitude. Indeed, subsequent works have shown that certain types of amplitudes can be made solely from soft terms.

Chapter 3 discusses the failure of Risager's method (tree-level) in computing 12- and higher graviton amplitudes at the NMHV sector. As was said in the last section, numeric and analytic computations show that Risager's method cannot produce correct amplitudes when the number of external gravitons exceeds eleven. We show where this number comes from and compute the missing piece for Risager's expansion to match the full physical amplitude. We call this term *the residue at infinity* since the failure of Risager's method happens at large values of the complex variable z. In summary, the result of this chapter is the first analytic expression for 12-graviton NMHV amplitude.

Our study of gravity amplitudes continues to investigate two important properties of a recently proposed formula for tree-level n-particle amplitude of $\mathcal{N} = 8$ supergravity by Cachazo and Geyer [15]. The formula was derived from the supersymmetric version of the Kawai-Lewellen-Tye (KLT) relations [20] which relates the amplitudes of $\mathcal{N} = 4$ super Yang-Mills and $\mathcal{N} = 8$ supergravity. Parity symmetry and soft limit of the formula, two important consistency checks, are proved in chapter 4 and strongly validate the proposal to be the complete tree-level amplitude of supergravity in all R-charge sectors. In our proof, we follow the steps used for checking the same properties in $\mathcal{N} = 4$ super Yang-Mills by Witten [23] and Roiban-Spradlin-Volovich [56]. Finally, we use the Cachazo-Geyer formula to explicitly compute MHV and $\overline{\text{MHV}}$ amplitudes.

As an attempt to find analytic expressions for gravity amplitudes, in chapter 5 we show that there exists an expansion for gravity amplitudes whose terms are exactly CSW-like. Explicit calculations are done for pure 6- and 7-graviton NMHV amplitudes and we find that the terms of our expansion agree with the Risager terms. The novel fact is that the method can be applied to any graviton amplitude with $n \geq 12$ where Risager's method does not work. In summary, the ingredients of our calculations are the global residue theorem and δ-function relaxation (originally introduced for Yang-Mills [33]) applied to a particular form of the tree amplitudes in $\mathcal{N} = 8$ supergravity, called the link representation, [57] and [58]. This work is in principle a candidate for expressing tree-level gravity amplitudes in analytic form.

In the end, we close with some concluding remarks and future directions in chapter 6.

This dissertation consists of four distinct research projects and publications, all directed by my supervisor whose contribution to the works is invaluable. I have very much employed the fruitful collaborations with Eduardo Conde, Brenda Penante and Grigory Sizov and greatly appreciate the insightful comments of Bo Feng, Song He, the journal referees of my papers and especially David Skinner.

Chapter 2 is based on the paper [59] by the author of this thesis:

S. Rajabi, "***Higher Codimension Singularities Constructing Yang-Mills Tree Amplitudes***," JHEP **1308**, 037 (2013) [arXiv:1101.5208 [hep-th]].

The material of chapter 3 presents the work of E. Conde and the author in [55]:

E. Conde and S. Rajabi, "***The Twelve-Graviton Next-to-MHV Amplitude from Risager's Construction***," JHEP **1209**, 120 (2012) [arXiv:1205.3500 [hep-th]].

Two other collaborative research and co-authored publications are captured in chapter 4 [60]:

B. Penante, S. Rajabi and G. Sizov, "***Parity Symmetry and Soft Limit for the Cachazo-Geyer Gravity Amplitude***," JHEP **1211**, 143 (2012) [arXiv:1207.4289 [hep-th]].

and chapter 5 [61]:

B. Penante, S. Rajabi and G. Sizov, "***CSW-like Expansion for Einstein Gravity***," JHEP **1305**, 004 (2013) [arXiv:1212.6257 [hep-th]].

Chapter 2

Higher Codimension Singularities Constructing Yang-Mills Tree Amplitudes

Yang-Mills tree-level amplitudes contain singularities of codimension one like collinear and multi-particle factorizations, codimension two such as soft limits, as well as higher codimension singularities. Traditionally, BCFW-like deformations with one complex variable were used to explore collinear and multi-particle channels. Higher codimension singularities need more complex variables to be reached. In this chapter, along with a discussion on higher singularities and the role of the global residue theorem in this analysis, we specifically consider soft singularities. This is done by extending Risager's deformation to a $\mathbb{C}^2$-plane, i.e., two complex variables. The two-complex-dimensional deformation is then used to recursively construct Yang-Mills tree amplitudes.

2.1 Introduction

As was mentioned in section 1.3.3, BCFW-like deformations were used to give a direct proof of the CSW-expansion of amplitudes in pure Yang-Mills by Risager [45], and in super Yang-Mills [51] and [52]. Risager's deformation, applied only to $\tilde{\lambda}$ of negative helicity particles, contains an auxiliary anti-holomorphic spinor, $\tilde{\eta}$, similar to CSW's reference spinor,

$$\tilde{\lambda}^i \longrightarrow \tilde{\lambda}^i + z\alpha^i\tilde{\eta}, \tag{2.1.1}$$

where α^i are constant. For an amplitude with k negative helicities, we can fix two of α^i's using momentum conservation,

$$\sum_i \alpha^i\lambda^i = 0. \tag{2.1.2}$$

For NMHV amplitudes, we saw that α^i's can be chosen as in (1.3.10).

On a $\mathbb{C}^2$-plane where $\tilde{\lambda}^i$ lives, Risager's deformation constrains the shifted spinor to a strip made by the original $\tilde{\lambda}^i$ and the shift, $z\alpha^i\tilde{\eta}$. The full $\mathbb{C}^2$-plane hence cannot be reached by the shift. This observation suggests the idea of generalization of Risager's deformation in order for $\tilde{\lambda}^i$'s to have access to entire space. In fact it is more natural to deform spinors in a two-complex-dimensional plane, $\mathbb{C}^2$, for which we need two complex variables. It is then natural to ask how Risager's method with two complex variables would provide the CSW expansion for Yang-Mills amplitudes. This is the question we address in this chapter.

We follow the steps to reconstruct the physical non-deformed amplitude. The novelty is that now the amplitude generically receives contributions from channels that were not accessible before. In our examples, these new contributions are soft limits (in each of which one of the external deformed momenta vanishes), and double-factorization channels. Therefore, the full amplitude can be reconstructed using two codimension one and a single codimension two singularities.

Using generalized deformations, we will have access to channels of interaction which are out of reach by one-variable BCFW or Risager's shifts. Although multi-variable deformations may make the calculations heavier, we found it interesting that amplitudes could have representations in terms of other types of physical singularity, especially soft singularities. This was our main motivation for using complex multi-variable analysis in calculations of scattering amplitudes. When the amplitude depends on several complex variables, the generalization of the residue theorem to several variables, the global residue theorem (reviewed in section 2.3), can be applied. In multi-variable analysis where all the shifts are linear, the Cauchy's theorem can be applied several times to build non-deformed amplitudes. However, with generic deformations (e.g. non-linear shifts), the only possible way to solve the multi-variable problem is applying the global residue theorem.

Here we restrict ourselves to color-ordered Yang-Mills tree-level amplitudes. We discuss general BCFW-like deformations and the necessity of applying the global residue theorem in section 2.2. Risager's two-variable deformation is introduced in particular. In section 2.3, residues in multi-dimensional complex analysis and the global residue theorem, our mathematical tool, are briefly reviewed. Calculations of NMHV 5- and 6-particle amplitudes with two complex variables and appearance of soft terms, as new contributions, are given in sections 2.4 and 2.6 respectively. Section 2.5 generalizes the argument to the n-particle N^{k-2}MHV amplitudes. We discuss that with the introduced deformation, there is no more singular term in the corresponding residue theorem, except the known collinear, multi-particle and soft singularities. We finally make some concluding remarks in section 2.7.

2.2 General Deformations and the Global Residue Theorem

Through general deformations on holomorphic and anti-holomorphic spinors, scattering amplitudes are generic functions of several complex variables. The simplest linear deformation with one variable is BCFW by which the Cauchy's theorem generates non-deformed amplitudes. In BCFW and also Risager's one-variable methods, not all but some of the singularities of amplitudes can be reached. These singularities are collinear, where two external momenta are orthogonal ($p^i.p^j = 0$), and multi-particle ($(\sum_i p^i)^2 = 0$ for a subset of external particles) [1]. We call them codimension one singularities where each of them can be determined by one condition on external momenta. It is clear that one complex variable in the deformation is enough for solving the condition and finding the pole.

Applying a linear two-variable Risager's deformation, amplitudes exhibit codimension two singularities: (codimension one)$\times$(codimension one), and soft singularities. The former corresponds to a two-factorization channel of interaction where each singularity can be of collinear or multi-particle type with codimension one. Therefore each diagram of this type has two different poles which can be completely determined by two variables. A soft singularity arises where an external momentum vanishes, and as a result the contribution of this process to the amplitude contains a singular factor. For a soft momentum of a massless particle, there are again two equations to determine the pole, since each index α or $\dot{\alpha}$ in $P_{\alpha\dot{\alpha}}$ runs over 1 and 2, hence we need exactly two variables to solve the equations.

Having linear deformations, one can apply Cauchy's theorem to the complexified amplitude which has now two linear polynomials in the denominator,

$$\oint \oint dz_1 dz_2 \frac{1}{(az_1 + bz_2 + c)(a' z_1 + b' z_2 + c')}. \tag{2.2.1}$$

These polynomials are denominators of propagators, which become on-shell, or of the soft factors. We first carry out, e.g., z_2-integral in which the corresponding pole is considered as a function of the other variable, $z_2^* = z_2^*(z_1)$. We will finally find $1/(ab' - a'b)$ after the second integration. The same result can be obtained from the global residue theorem which will be discussed in the next section.

With generic deformations, propagators will have higher degree polynomials in denom-

[1] One can think of a particular auxiliary spinor in Risager's deformation by which some external momenta can be soft, but generically soft singularities are not visible in this way. Consider the deformation $\tilde{\lambda}^i \to \tilde{\lambda}^i + z\alpha^i \tilde{\lambda}^1$ on negative helicity particles which include particle 1. It can be immediately seen that $\tilde{\lambda}^1(z)$ vanishes at $z^* = -1/\alpha^1$. This is in fact the pole of all the diagrams in which particle 1 is collinear with any other particle.

inators corresponding to different types of singularities. In case these polynomials are irreducible, Cauchy's theorem does not work and the global residue theorem has to be applied. This theorem, the generalization of Cauchy's one variable residue theorem, is the only tool in calculations with more complex variables and higher degrees.

Toward having the goal of presenting amplitudes which makes different singularities manifest, in this chapter we extend Risager's deformation to $\mathbb{C}^2$ and will see amplitudes expose codimension two singularities. As was discussed in the introduction, Risager's shift naturally needs to be defined in a two-complex-dimensional plane. Therefore, our generalized deformation on negative helicities will be,

$$\tilde{\lambda}^i \longrightarrow \tilde{\lambda}^i + \alpha^i(z_1\tilde{\zeta}_1 + z_2\tilde{\zeta}_2), \tag{2.2.2}$$

with $\tilde{\zeta}_1$ and $\tilde{\zeta}_2$ being two reference spinors, and α^i are determined in such a way that momentum conservation is preserved. Although with this linear deformation it is possible to recover the non-deformed amplitude using Cauchy's theorem, we will apply the global residue theorem in our calculations.

2.3 Review of Residues in Multi-dimensional Complex Analysis

Starting by a linear deformation in two complex variables, z_1 and z_2, on $\tilde{\lambda}^i$'s of negative helicity particles, generalization of Risager's deformation, the amplitude will be a rational function of both variables. In analogy with one-variable analysis, we study the following contour integral from which the physical amplitude, $A(0,0)$, can be obtained

$$\oint dz_1 dz_2 \, \frac{A(z_1, z_2)}{z_1 z_2}, \tag{2.3.1}$$

where the denominator of $A(z_1, z_2)$ factorizes into pieces coming from deformed propagators. Therefore, the full integrand of (2.3.1) can be written as $\dfrac{g(z_1, z_2)}{f_1(z_1, z_2) f_2(z_1, z_2)}$, where z_1, z_2 and the factors of the denominator of A could arbitrarily belong to f_1 or f_2. The functions f_1, f_2 and g are polynomials, and g is regular at zeros of the denominator.

Now, let Γ be the set of all the zeros of f_1 and f_2,

$$\Gamma = \{P = (z_1^*, z_2^*) | f_1(z_1^*, z_2^*) = f_2(z_1^*, z_2^*) = 0\}. \tag{2.3.2}$$

The *Global Residue Theorem* for any f_1 and f_2, states that

$$\sum_{P \in \Gamma} \text{Res} \left(\frac{A(z_1, z_2)}{z_1 z_2} \right)_P = 0, \tag{2.3.3}$$

when the degree condition

$$\deg(g) < \deg(f_1) + \deg(f_2) - 2 \tag{2.3.4}$$

is satisfied.[2]

Since there are different ways to group factors of the denominator into f_1 and f_2, there exist different residue theorems for a given function $A(z_1, z_2)$. Each term in (2.3.3) is a contour integral for small ϵ as follows,

$$\begin{aligned} \text{Res} \left(\frac{g(z_1, z_2)}{f_1(z_1, z_2) f_2(z_1, z_2)} \right)_P &= \frac{1}{(2\pi i)^2} \oint_{|f_1|=\epsilon, |f_2|=\epsilon} dz_1 dz_2 \, \frac{g(z_1, z_2)}{f_1(z_1, z_2) f_2(z_1, z_2)} \\ &= \frac{g(z_1^*, z_2^*)}{(2\pi i)^2} \oint_{|u|=|v|=\epsilon} \frac{du}{u} \frac{dv}{v} \det \left(\frac{\partial(f_1, f_2)}{\partial(z_1, z_2)} \right)^{-1} \end{aligned} \tag{2.3.5}$$

where in the last line we performed a change of variables, $u = f_1$ and $v = f_2$, so the corresponding Jacobian, evaluated at $P = (z_1^*, z_2^*)$, appears inside the integral.

[2] With n complex variables and therefore n maps, $(f_1, \cdots, f_n) : \mathbb{C}^n \to \mathbb{C}^n$, the degree condition generalizes to $\deg(g) < \deg(f_1) + \cdots + \deg(f_n) - n$. This condition is analogous to having no pole at infinity in the usual BCFW or Risager's deformation with one complex variable.

As can be seen above, the integration factorizes into two pieces, each on a $\mathbb{C}^1$-plane similar to one variable analysis. The full contour is therefore $S^1 \times S^1 \subset \mathbb{C}^2$ which unlike the one variable case does not fully enclose the pole. This is in fact one important difference between one and several complex integrals. Each of these integrals around the defined contour equals 1, therefore the residue is given by

$$\text{Res}\left(\frac{g(z_1, z_2)}{f_1(z_1, z_2) f_2(z_1, z_2)}\right)_P = g(z_1^*, z_2^*) \det\left(\frac{\partial(f_1, f_2)}{\partial(z_1, z_2)}\right)^{-1} (z_1^*, z_2^*). \tag{2.3.6}$$

While having a determinant, the ordering of arguments is important. We fix the orientation of contours in such a way that f_1 always comes before f_2 in the Jacobian. With the order reversed there will be a minus sign for the residue.

Now in case f_1 contains z_1 and f_2 contains z_2, one possible solution for $f_1 = f_2 = 0$ would be $z_1^* = z_2^* = 0$. It is obvious that $A(z_1, z_2)$ is not singular at $(0, 0)$ since this corresponds to no deformation on the amplitude. Therefore, (2.3.6) gives the physical non-deformed amplitude,

$$\text{Res}\left(\frac{A(z_1, z_2)}{z_1 z_2}\right)(0, 0) = A(0, 0). \tag{2.3.7}$$

This simply fixes our convention for the definition of f_1 and f_2. In order for (2.3.3) to contain $A(0, 0)$ as one of the terms, z_1 and z_2 have to belong to different functions. Applying this convention, we will use (2.3.3) and (2.3.6) for calculations in the following sections provided the degree condition is satisfied.

In the same way that Risager [45] proved the $z^{-(k-1)}$ behavior of N^{k-2}MHV Yang-Mills amplitudes under k-line shift, k being the number of negative helicities, we show that the degree condition (2.3.4) is satisfied with our 2-variable deformation. The most dangerous Feynman diagrams are those with only cubic vertices. Performing this deformation on $\tilde{\lambda}$ of all negative helicity particles, one finds that $\deg(g) = m$ when there are m cubic vertices. The reason is that each cubic vertex depending on a deformed momentum is linear in z_1 and z_2.

In the denominator of amplitudes we have contributions from $m-1$ propagators and k polarization vectors. Each propagator linearly depends on complex variables. On the other hand, the $z_{1,2}$-dependence of polarization vectors depends on their helicities. For negative helicities we have

$$\epsilon^{\mu(-)}(p) = \frac{\lambda^{\alpha}(p)\sigma^{\mu}\tilde{\lambda}^{\dot{\alpha}}(q)}{\sqrt{2}[pq]}, \tag{2.3.8}$$

where q is an auxiliary spinor. The deformation is on $\tilde{\lambda}(p)$, so each polarization vector with negative helicity contributes a $+1$ to the degree of denominator. Since we are working with $A(z_1, z_2)/z_1 z_2$, the total degree of denominator will be $\deg(f_1)+\deg(f_2) = (m-1)+k+2$. The degree condition then says $m < m+k+1-2$ or $1 < k$ which is true. Therefore, the validity condition of the global residue theorem is satisfied for our 2-variable deformation on n-particle N^{k-2}MHV Yang-Mills tree amplitudes.

2.4 5-Particle NMHV Tree Amplitude with 2-variable Shifts

The aim is to calculate Yang-Mills tree amplitudes using the global residue theorem. As a case in point, we consider the split-helicity NMHV 5-particle amplitude, $A(---++)$, under the deformation

$$\hat{\tilde{\lambda}}^{i}(z_1, z_2) = \tilde{\lambda}^{i} + \alpha^{i}\tilde{\eta}, \tag{2.4.1}$$

where $i = 1, 2, 3$ and we choose $\tilde{\eta} = z_1\tilde{\lambda}^{4}+z_2\tilde{\lambda}^{5}$. Using momentum conservation, a nontrivial solution for α^i is $\alpha^i = \langle jk\rangle$ where i, j and k cyclically take values of 1, 2 and 3.

Since we are working with color-ordered amplitudes, the z-dependent propagators are those with $\hat{P}_{12}^2, \hat{P}_{23}^2, \hat{P}_{34}^2,$ and $\hat{P}_{51}^2$ which together with z_1 and z_2 are the factors in f_1 and f_2. The diagram with particles 4 and 5 being on one sub-diagram does not contribute, since $P_{4,5}^2$ has no $z_{1,2}$-dependence.

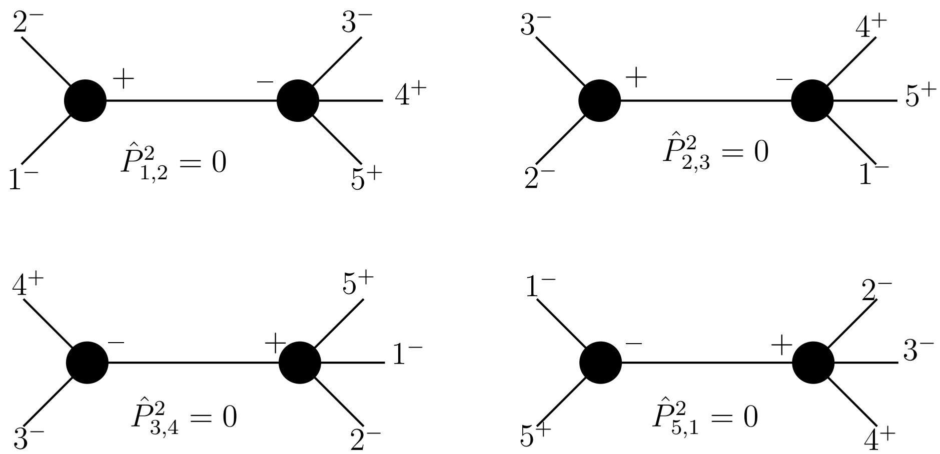

Figure 2.1: BCFW diagrams of 5-particle NMHV amplitude.

The simplest choice

$$f_1 = z_1, \quad f_2 = z_2 \hat{P}^2_{1,2} \hat{P}^2_{2,3} \hat{P}^2_{3,4} \hat{P}^2_{5,1}, \tag{2.4.2}$$

results in the 1-variable Risager's deformation since one of the complex variables, z_1, is zero throughout calculations.

Apart from $(0,0)$, $f_1 = f_2 = 0$ has 4 solutions. Hence, there are four terms, all with collinear/multi-particle singularities, in the sum of the residues,

$$A(0,0) = -\sum_{poles \neq (0,0)} \text{Res}(\frac{A(z_1, z_2)}{z_1 z_2}), \tag{2.4.3}$$

corresponding to the four diagrams in Fig. 2.1 [3].

In the next example we consider

$$f_1 = z_1 \hat{P}^2_{1,2}, \quad f_2 = z_2 \hat{P}^2_{2,3} \hat{P}^2_{3,4} \hat{P}^2_{5,1}. \tag{2.4.4}$$

[3] With this choice for $\tilde{\eta}$, $P^2_{1,5}$ is independent of both complex variables when $f_1 = z_1 = 0$. Therefore, the residue corresponding to this channel vanishes.

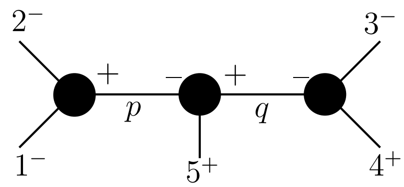

Figure 2.2: Double factorization channel in 5-particle NMHV amplitude.

This time, solutions to $f_1 = f_2 = 0$ are coming from

$$z_1 = 0, \begin{cases} \hat{P}^2_{2,3} = 0 \\ \hat{P}^2_{3,4} = 0 \\ \hat{P}^2_{5,1} = 0 \end{cases} \quad \text{or} \quad \hat{P}^2_{1,2} = 0, \begin{cases} z_2 = 0 \\ \hat{P}^2_{2,3} = 0 \\ \hat{P}^2_{3,4} = 0 \\ \hat{P}^2_{5,1} = 0 \end{cases}, \tag{2.4.5}$$

in addition to $z_1 = z_2 = 0$ which corresponds to the non-deformed amplitude.

Having $z_1 = 0$ or $z_2 = 0$ in any system of equations, the problem reduces to 1-variable Risager's deformation with a collinear singularity. The corresponding diagrams are exactly those in Fig. 2.1.

Let's now consider $\hat{P}^2_{1,2} = \hat{P}^2_{3,4} = 0$, Fig. 2.2, with solutions

$$z_1^* = \frac{s_{12} - s_{34}}{\langle 12 \rangle [54] \langle 35 \rangle}, \quad z_2^* = \frac{[34]}{\langle 12 \rangle [45]}. \tag{2.4.6}$$

In this double factorization channel we have three sub-amplitudes multiplying and forming the diagram,

$$\frac{\langle 12 \rangle^3}{\langle 2p \rangle \langle p1 \rangle} \frac{1}{\langle 12 \rangle [\hat{1}\hat{2}]} \frac{[q5]^3}{[5p][pq]} \frac{1}{\langle 34 \rangle [\hat{3}4]} \frac{\langle q3 \rangle^3}{\langle 34 \rangle \langle 4q \rangle} = \frac{[45]^3}{[1^*5][3^*2^*][\hat{1}\hat{2}][\hat{3}4]}, \tag{2.4.7}$$

where the starred spinors are evaluated at (2.4.6) and the hatted ones depend on z_1 and z_2. Using (2.3.6), the residue for this process can be obtained,

$$\frac{\langle 12 \rangle^3 \langle 35 \rangle^2 [45]}{\langle 34 \rangle \langle 25 \rangle \langle 15 \rangle \langle 45 \rangle [34] (s_{12} - s_{34})}. \tag{2.4.8}$$

The next system of equations is $\hat{P}^2_{1,2} = \hat{P}^2_{2,3} = 0$ with a shared deformed momentum $\hat{p}_2$. Similarly, in $\hat{P}^2_{1,2} = \hat{P}^2_{5,1} = 0$, the last equations, $\hat{p}_1$ is shared. One can easily see that there is no way to draw a diagram with correct factorizations for any of the cases at hand. For a detailed examination of these processes we write the former as

$$\begin{cases} [\hat{1}\hat{2}] = 0 \\ [\hat{2}\hat{3}] = 0 \end{cases}, \tag{2.4.9}$$

with some solutions z_1^* and z_2^*. Simple calculations show that $\tilde{\lambda}^2(z_1^*, z_2^*) = 0$. In fact from (2.4.9) one can see that evaluated at (z_1^*, z_2^*), $\tilde{\lambda}^1 \| \tilde{\lambda}^2$ and $\tilde{\lambda}^2 \| \tilde{\lambda}^3$ but $\tilde{\lambda}^1$ and $\tilde{\lambda}^3$ are not parallel. Therefore we can conclude that $\tilde{\lambda}^2(z_1^*, z_2^*) = 0$.

The shifted momentum of particle 2 is being soft. This means that the contribution of this channel comes from the soft limit $\hat{\tilde{\lambda}}^2 \to 0$ of the full amplitude.

In general, Yang-Mills tree amplitudes in the soft limit of one of the momenta factorize into two parts, an amplitude without the soft particle and a singular factor,

$$A_n(\ldots, i-1, i, i+1, \ldots) \xrightarrow{p_i \to 0} \text{Soft}(i-1, i, i+1) A_{n-1}(\ldots, i-1, i+1, \ldots), \tag{2.4.10}$$

where clearly A_{n-1} has no singularity at the limit $p_i \to 0$.

The soft factor, first computed by Weinberg [62], in spinor-helicity notation is

$$\text{Soft}(i-1, i, i+1) = \begin{cases} \dfrac{\langle i\!-\!1\, i\!+\!1\rangle}{\langle i\!-\!1\, i\rangle\langle i\, i\!+\!1\rangle}, & \text{if } \lambda^i \to 0 \\ \dfrac{[i\!-\!1\, i\!+\!1]}{[i\!-\!1\, i][i\, i\!+\!1]}, & \text{if } \tilde{\lambda}^i \to 0 \end{cases}. \tag{2.4.11}$$

Having this behavior, one can find the residue of $\dfrac{A(z_1, z_2)}{z_1 z_2}$ in this limit.

For the case where $\tilde{\lambda}^2(z_1^*, z_2^*)$ is soft we plug $\dfrac{[\hat{1}\hat{3}]}{[\hat{1}\hat{2}][\hat{2}\hat{3}]} A(\hat{1}^-, \hat{3}^-, 4^+, 5^+)$ into (2.3.6), and the residue will be

$$\frac{\langle 13\rangle^3 [45]}{\langle 34\rangle\langle 45\rangle\langle 51\rangle[42][52]}. \tag{2.4.12}$$

Similarly, the solutions of the system of equations $[5\hat{1}] = 0$ and $[\hat{1}\hat{2}] = 0$ satisfy $\tilde{\lambda}^1(z_1^*, z_2^*) = 0$, and the contribution of this channel to the amplitude comes from

$$A(\hat{1}^-, \hat{2}^-, \hat{3}^-, 4^+, 5^+) \xrightarrow{\hat{\tilde{\lambda}}^1 \to 0} \frac{[5\hat{2}]}{[5\hat{1}][\hat{1}\hat{2}]} A(\hat{2}^-, \hat{3}^-, 4^+, 5^+), \tag{2.4.13}$$

with the residue being

$$-\frac{\langle 23\rangle^3 [45]}{\langle 34\rangle\langle 45\rangle\langle 52\rangle [41][51]}. \tag{2.4.14}$$

Finally, we add up all the relevant terms and the known result of NMHV 5-particle amplitude can be obtained,

$$A(1^-, 2^-, 3^-, 4^+, 5^+) = \frac{[45]^3}{[12][23][34][51]}. \tag{2.4.15}$$

One can consider other combinations in f_1 and f_2 and apply the residue theorem. In 5-particle amplitude for any choice of these functions there are always collinear (via single or double factorizations) and soft singularities.

In amplitudes with more particles we will have multi-particle singularities as well (in the 5-particle example collinear and multi-particle singularities are the same). This may result in some difficulties in finding the residues, as there will be more shared particles between simultaneous equations. We will see that these cases often result in vanishing residues, and soft singularities are the only ones in addition to previously known collinear and multi-particle singularities. The double factorization channels, which also appear in the expansion, are in fact made out of collinear and/or multi-particle singularities.

2.5 N^{k-2}MHV Amplitudes

For a general discussion on the singularities of 2-variable deformed amplitudes, we consider the most general N^{k-2}MHV amplitude where k negative helicities are randomly distributed, $A_n(+, \cdots, i_1^-, \cdots, i_2^-, \cdots, i_k^-, \cdots, +)$. As before, the two variable deformations are only on negative helicities. Using $\tilde{\lambda}$'s of two particles, $\tilde{\eta}$ can be defined, and the deformation will be

$$\tilde{\lambda}^i \longrightarrow \tilde{\lambda}^i + \alpha^i \tilde{\eta}(z_1, z_2). \tag{2.5.1}$$

There are infinite families of α^i for $k > 3$ which can be turned into more complex variables. For $k = 3$, as was seen in our 5-particle example, we can fix these coefficients $\alpha^a = \langle bc \rangle$ up to an overall factor, where a, b, and c cyclically take the indices of negative helicities.

Similar to previous example, we first determine the $z_{1,2}$-dependent propagators on which the factorizations take place. Hence, at least one but not all of the deformed momenta are included between particles A and B in the set of denominators of propagators, $\mathcal{P} = \{\hat{P}^2_{A,B} = (p_A + \cdots + p_B)^2\}$. As stated before, $f_1(z_1, z_2)$ and $f_2(z_1, z_2)$ contain z_1 and z_2 respectively as well as an arbitrary grouping of elements of $\mathcal{P}$. We will explain what possible channels do contribute to the full amplitude by different ways of getting $f_1 = f_2 = 0$.

In case $z_1 = 0$ or $z_2 = 0$, the one-variable shift, the corresponding residue follows from a collinear or multi-particle channel depending on how many particles are forming $\hat{P}^2_{A,B}$.

For cases where two members of $\mathcal{P}$ simultaneously vanish,

$$\begin{cases} \hat{P}^2_{a,b} = 0 \\ \hat{P}^2_{c,d} = 0 \end{cases}, \tag{2.5.2}$$

depending on how indices overlap, different events may happen. One can imagine various orderings and coincidences of particles as follows: 1) $a < c < d < b$, 2) $a < c < b = d$, 3) $a < c < b < d$, 4) $a < b = c < d$, where any other distribution is equivalent to one of these cases. For instance, using momentum conservation one can see that the case where the two sets are completely separated, $a < b < c < d$, is exactly the first ordering which results in double factorization.

Therefore, 1 says that the diagram has three sub-amplitudes, one with particles $\{c, \cdots, d, p\}$, the other in the middle with $\{a, \cdots, c-1, -p, d+1, \cdots, b, q\}$, and rest of the particles are in the third sub-amplitude as in Fig.2.3(a). The two singularities here can either be collinear or multi-particle.

Next, we have 2 again with double factorization, $\{a, \cdots, c-1, -p, q\}$ in the middle, $\{c, \cdots, d, p\}$ on the left and the rest in the third sub-amplitude, Fig.2.3(b). Again as in 1, the process can have two collinear or multi-particle singularities.

In 3, the overlap is again non-empty but we cannot find any Feynman diagram associated with the given propagators. In fact the amplitude cannot factorize in this way. One can also check that there is no soft singularity at the solutions of (2.5.2), (z_1^*, z_2^*). Therefore, (z_1^*, z_2^*) does not correspond to any pole of the amplitude. We support our argument by explicit evaluation of the residue of $A_6(-+-+-+)$ at (z_1^*, z_2^*) and find that it vanishes.

In 4, the two sets share only a single particle. This may lead us to conclude that the shared particle, if deformed, is soft as was seen in the 5-particle example. It is true only if both singularities are collinear. To see this, assume that $\tilde{\lambda}^b(z_1^*, z_2^*) \to 0$ is a solution to (2.5.2) when $b = c$. Hence we will have $\hat{P}^2_{a,b-1} = 0$ and $\hat{P}^2_{b+1,d} = 0$ which are independent of $\tilde{\lambda}^b$ and therefore are not necessarily valid unless $a = b-1$ and $b+1 = d$. Having said that, (2.5.2) reduces to $[ab] = [bd] = 0$ which is equivalent to both singularities being collinear.

One may also imagine a case where the two sets of indices coincide, $a = c < b = d$. In fact this can never happen to Feynman diagrams since there is no double pole in propagators.

We conclude that using 2-variable Risager's deformation in n-point N^{k-2}MHV amplitudes, collinear, multi-particle and soft singularities of tree amplitudes can be probed. With a generic one-variable shift, soft channels do not contribute to amplitudes and it is the second complex variable which is necessary for probing these channels.

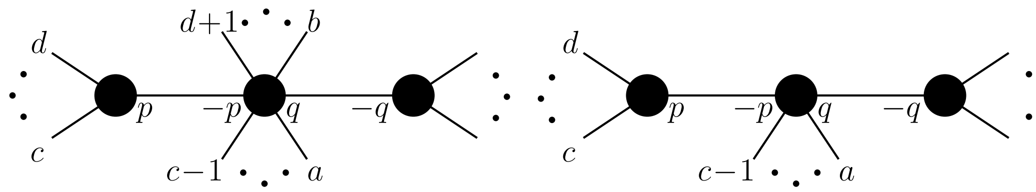

Figure 2.3: Double factorization channels in N^{k-2}MHV amplitude.

2.6 6-Particle NMHV Amplitude

In this section we compute the 6-particle amplitude with alternating helicities, $A(-+-+-+)$. We choose particles 2 and 4 in the definition of the reference spinor. Since each pair of adjacent momenta in this helicity configuration is deformed, the number of complex propagators with Risager's shift is maximum. Together with three multi-particle propagators, we can arbitrarily define f_1 and f_2, e.g.,

$$f_1(z_1,z_2) = z_1\hat{P}^2_{1,2}\hat{P}^2_{4,5}\hat{P}^2_{1,3}, \quad f_2(z_1,z_2) = z_2\hat{P}^2_{2,3}\hat{P}^2_{3,4}\hat{P}^2_{5,6}\hat{P}^2_{6,1}\hat{P}^2_{2,4}\hat{P}^2_{3,5}. \tag{2.6.1}$$

As was discussed before, the contributions from $z_1=0$ or $z_2=0$ are the usual Risager's terms. The corresponding residues are, Fig. 2.4,

$$\text{(a) } z_1 = \hat{P}^2_{2,3} = 0: \quad \frac{\langle 15\rangle^4[24]^3}{\langle 45\rangle\langle 56\rangle\langle 61\rangle[23][34]\langle 1|2+3|4]\langle 4|2+3|4]}, \tag{2.6.2}$$

$$\text{(b) } z_1 = \hat{P}^2_{5,6} = 0: \quad \frac{\langle 13\rangle^4[64]^3}{\langle 12\rangle\langle 23\rangle\langle 34\rangle[45][56]\langle 1|2+3|4]\langle 4|5+6|4]}, \tag{2.6.3}$$

$$\text{(c) } z_1 = \hat{P}^2_{6,1} = 0: \quad \frac{\langle 35\rangle^4[46]^3}{\langle 23\rangle\langle 34\rangle\langle 45\rangle[61][14]\langle 5|2+3|4]\langle 2|1+6|4]}, \tag{2.6.4}$$

$$\text{(d) } z_1 = \hat{P}^2_{2,4} = 0: \quad \frac{-\langle 15\rangle^4\langle 23\rangle^2[24]^4}{\langle 34\rangle\langle 56\rangle\langle 61\rangle[34]P^2_{2,4}\langle 1|2+3|4]\langle 4|2+3|4]\langle 5|2+3|4]}, \tag{2.6.5}$$

$$(\mathrm{e})\ z_1 = \hat{P}_{3,5}^2 = 0: \quad \frac{\langle 35\rangle^2 \langle 1|3+5|4]^4}{\langle 12\rangle\langle 34\rangle\langle 45\rangle\langle 61\rangle[34][45]P_{3,5}^2\langle 6|3+5|4]\langle 2|1+6|4]}, \tag{2.6.6}$$

$$(\mathrm{f})\ \hat{P}_{4,5}^2 = z_2 = 0: \quad \frac{-\langle 13\rangle^4[24]^3}{\langle 61\rangle\langle 12\rangle\langle 23\rangle[45][52]\langle 3|4+5|2]\langle 6|4+5|2]}, \tag{2.6.7}$$

$$(\mathrm{g})\ \hat{P}_{1,3}^2 = z_2 = 0: \quad \frac{\langle 13\rangle^2\langle 5|1+3|2]^4}{\langle 12\rangle\langle 23\rangle\langle 45\rangle\langle 56\rangle[12][23]P_{1,3}^2\langle 6|4+5|2]\langle 4|1+3|2]}. \tag{2.6.8}$$

There is no contribution from $z_1 = \hat{P}_{3,4}^2 = 0$ and $z_2 = \hat{P}_{1,2}^2 = 0$ since the particles we chose in the definition of $\tilde{\eta}$ make both $P_{3,4}^2$ and $P_{1,2}^2$ independent of z_1 and z_2. Therefore the corresponding residues vanish.

Soft channels appear where $\hat{P}_{1,2}^2 = \hat{P}_{6,1}^2 = 0$, and $\hat{P}_{4,5}^2 = \hat{P}_{5,6}^2 = 0$ with $\hat{p}_1$ and $\hat{p}_5$ being zero respectively.

$$(\mathrm{h})\ \hat{P}_{1,2}^2 = \hat{P}_{6,1}^2 = 0: \quad \frac{\langle 35\rangle^4[42]}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 62\rangle[12][14]}, \tag{2.6.9}$$

$$(\mathrm{i})\ \hat{P}_{4,5}^2 = \hat{P}_{5,6}^2 = 0: \quad \frac{\langle 13\rangle^4[42]}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 46\rangle\langle 61\rangle[45][52]}. \tag{2.6.10}$$

There are also double factorization channels,

$$(\mathrm{j})\ \hat{P}_{1,2}^2 = \hat{P}_{3,5}^2 = 0: \quad \frac{-\langle 35\rangle^4\langle 61\rangle^2[24]}{\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 63\rangle\langle 26\rangle\langle 12\rangle[12]\langle 6|3+5|4]}, \tag{2.6.11}$$

$$(\mathrm{k})\ \hat{P}_{1,3}^2 = \hat{P}_{5,6}^2 = 0: \quad \frac{\langle 13\rangle^4\langle 45\rangle^3[24]}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle\langle 56\rangle\langle 64\rangle\langle 4|5+6|4]\langle 4|1+3|2]}. \tag{2.6.12}$$

Considering the diagram of $\hat{P}_{2,3}^2 = \hat{P}_{1,3}^2 = 0$, one might naively guess that this channel vanishes. In fact it does if $\tilde{\lambda}^p$ is parallel to $\hat{\tilde{\lambda}}^1$, which is not true. More careful observation shows that λ^p evaluated at the corresponding poles of the diagram is parallel to λ^1, and therefore the process does contribute to the amplitude. The same happens to the channel $\hat{P}_{4,5}^2 = \hat{P}_{3,5}^2 = 0$ where here λ^p is proportional to λ^3. Hence, we have two more double

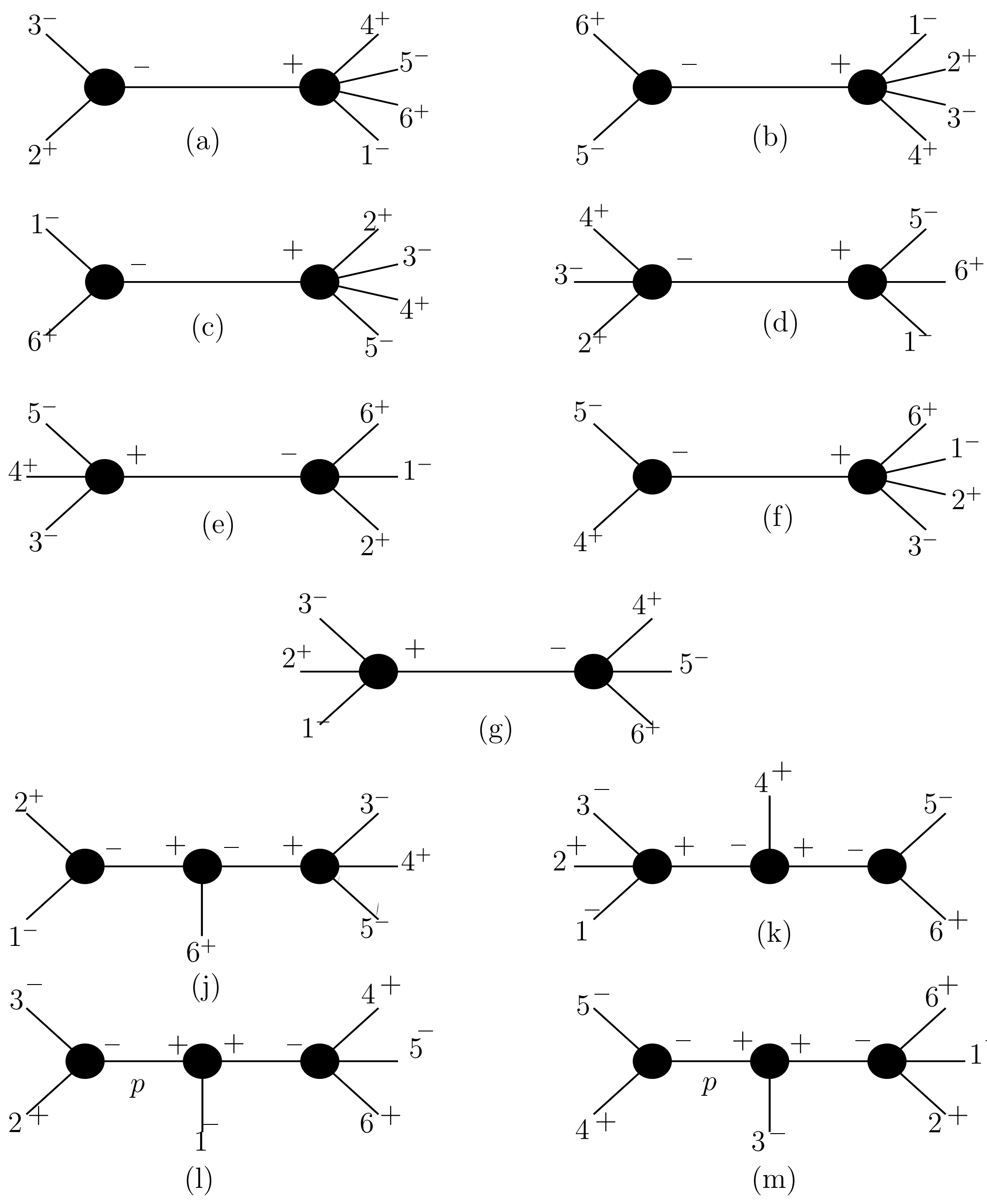

Figure 2.4: BCFW diagrams of 6-particle NMHV amplitude.

factorization channels contributing to the amplitude with residues,

$$\text{(l) } \hat{P}_{1,3}^2 = \hat{P}_{2,3}^2 = 0: \quad \frac{-\langle 15\rangle^4\langle 13\rangle^2[24]}{\langle 12\rangle\langle 23\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle\langle 14\rangle[23]\langle 1|2+3|4]}, \tag{2.6.13}$$

$$\text{(m) } \hat{P}_{4,5}^2 = \hat{P}_{3,5}^2 = 0: \quad \frac{\langle 13\rangle^4\langle 35\rangle^2[24]}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 36\rangle\langle 61\rangle[45]\langle 3|4+5|2]}. \tag{2.6.14}$$

Finally the full amplitude is given by (2.4.3) which agrees with the known result of 6-particle NMHV amplitude [9]:

$$\begin{aligned} A(1^-,2^+,3^-,4^+,5^-,6^+) &= \frac{\langle 13\rangle^4[46]^4}{\langle 12\rangle\langle 23\rangle[45][56]P_{1,3}^2\langle 3|1+2|6]\langle 1|2+3|4]} \\ &+ \frac{[26]^4\langle 35\rangle^4}{[61][12]\langle 34\rangle\langle 45\rangle P_{3,5}^2\langle 3|4+5|6]\langle 5|4+3|2]} \\ &+ \frac{\langle 15\rangle^4[24]^4}{\langle 23\rangle\langle 34\rangle[56][61]P_{2,4}^2\langle 5|4+3|2]\langle 1|2+3|4]}. \end{aligned} \tag{2.6.15}$$

2.7 Concluding Remarks

In this chapter, we studied the analytic structure of Yang-Mills tree level scattering amplitudes by a new deformation on external momenta. Using the power of multi-variable complex analysis, especially the global residue theorem, physical amplitudes can be written recursively in a way similar to BCFW method. The degree condition, under which the global residue theorem is valid, was proved for generic n-particle N^{k-2}MHV amplitudes where the two-complex-variable deformation is on $\tilde{\lambda}$ of $(-)$ helicities.

While with a generic one variable Risager's deformation, collinear and multi-particle singularities of tree amplitudes can be probed, the generalized 2-variable shift reveals soft channels as well as the two other types of singularity. This generalization is actually the natural way of defining all negative helicity deformation since it allows the deformed $\tilde{\lambda}$'s to live on the entire $\mathbb{C}^2$. We computed Yang-Mills 5- and 6-particle NMHV amplitudes at tree level with this generalization and discussed that in a general n-particle N^{k-2}MHV

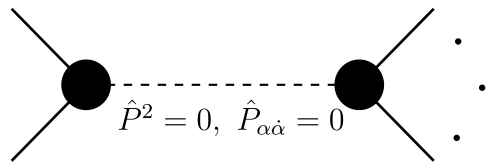

Figure 2.5: Collinear-soft singularity

amplitude the only singularities that can exist correspond to collinear, multi-particle and soft channels.

For each collinear or multi-particle singularity, there is one condition on the sum of external momenta. This means that one complex variable is enough to solve the condition and find the corresponding pole, as was the case in BCFW and Risager's deformations. On the other hand, softness of particles results in two conditions. Having two equations, we need two complex variables in the deformation as was shown in our work. This simply indicates the necessity of introducing more variables in the deformations when there are more types of singularity to be investigated.

As an example of higher codimension singularities, one can consider a BCFW-like shift by which an internal deformed momentum is not only on-shell ($P^2 = 0$), but also soft ($P_{\alpha\dot{\alpha}} = \lambda_\alpha\tilde{\lambda}_{\dot{\alpha}} = 0$). Here we have a codimension 3 singularity. Therefore, there are three conditions to define the poles for which we need three complex variables. Figure 2.5 shows a collinear-soft singularity of codimension 3.

There could be other deformations one can consider to investigate more interesting singularities. Under the experienced deformations, depending on which spinors are deformed, either $\langle ij\rangle$ or $[ij]$ is the singularity of an on-shell propagator with $P^2 = \langle ij\rangle[ij]$. As an example, in (2.6.15) we have both $\langle 12\rangle$ and $[12]$, but only the latter is singular under our two-dimensional Risager's deformation. A generalized deformation could present amplitudes in such a way that both of these brackets vanish.

Probing new singularities can be thought of as part of the motivation for using multi-dimensional complex analysis. We also saw that the global residue theorem has to be used in generic cases while having higher degree singularities with more than one complex variable.

There have also been other attempts to make use of soft limits in order to construct amplitudes. We would like to mention the link between our work and a more recent development for expanding amplitudes. It has been shown that MHV gauge theory and gravity amplitudes with equal and fewer than seven points and certain NMHV gauge theory amplitudes with any number of external legs can be built using *only* soft singularities [63]. The construction uses the inverse soft method introduced in [29] and BCFW recursion relations: by adding a single particle to the amplitude, a BCFW term whose one sub-amplitude is a three-particle will be produced. This work [63] shows that all BCFW terms of the amplitudes, described above, can be built using the inverse soft method. The difference between gauge theory and gravity constructions is that gravity's soft factor is a sum of many terms, hence is more complicated as there is no color-ordering in gravity.

Generalization of this method to all tree-level amplitudes of $\mathcal{N} = 4$ super Yang-Mills is given in [64] by a systematic inverse soft limit instruction. This method makes the Yangian symmetry [65] and the soft limit of the amplitudes manifest.

Chapter 3

The Twelve-Graviton Next-to-MHV Amplitude from Risager's Construction

The MHV or CSW expansion of tree-level Yang-Mills amplitudes provides an elegant and simple way of obtaining analytic formulas for S-matrix elements. Although Risager's method, a systematic approach to obtain the MHV expansion, works for Yang-Mills amplitudes, it fails to provide an MHV expansion already for Next-to-MHV gravity amplitudes with more than eleven particles, as shown by Bianchi, Elvang and Freedman in 2008 [53]. This fact implies that in this sector there is a contribution at infinity starting at $n = 12$. In this chapter, we determine the explicit analytic form of this residue at infinity for $n = 12$. Together with the terms of Risager's expansion, the residue at infinity completes the first full CSW-like analytic expression of the twelve-graviton NMHV amplitude. Our technique can also be used to compute the residue at infinity for higher points.

3.1 Introduction and Summary

Tree-level gravity amplitudes are objects of genuine theoretical interest. Although in practice they can be constructed with BCFW recursion relations [10, 54, 46], it is of great interest to have analytic formulas for them (see for instance [11, 66, 67, 12, 68, 69, 13]). In particular, the simpler these formulas are, the more insight they contain about tree-level gravity. A great step towards this goal has been recently taken by Hodges, who found an elegant formula for MHV gravity amplitudes [14]. His work renewed the interest in developing an MHV-vertex expansion for gravity amplitudes.

Applying Risager's technique to gravity amplitudes was the next natural goal after the method was successful in Yang-Mills theory [70]. However, as verified by numerical calculations in [53] and later by analytic means in [54], graviton amplitudes in the Next-to-MHV sector depend on the reference spinor of Risager's deformation, starting at twelve particles.

In this chapter, we address the question of how Risager's expansion disagrees with the physical amplitude in the NMHV sector of gravity, *i.e.* we study the tree-level amplitude $M_n(1^-, 2^-, 3^-, 4^+, \ldots, n^+)$ for $n \geq 12$, and develop a procedure to determine this discrepancy. As an illustration, we present the explicit result in the case of the twelve-graviton amplitude.

Let us specify our notation. We denote the NMHV amplitude of our interest simply by M_n. We use the spinor-helicity formalism and represent the momenta of the gravitons as $p^i = \lambda^i \tilde{\lambda}^i$, $(i = 1, \ldots, 12)$. The Risager shift deforms the anti-holomorphic spinors of the three negative-helicity particles as

$$\begin{cases} \tilde{\lambda}^1(w) = \tilde{\lambda}^1 + w\,\langle 2\ 3\rangle \tilde{\eta} \\ \tilde{\lambda}^2(w) = \tilde{\lambda}^2 + w\,\langle 3\ 1\rangle \tilde{\eta} \\ \tilde{\lambda}^3(w) = \tilde{\lambda}^3 + w\,\langle 1\ 2\rangle \tilde{\eta} \end{cases} ; \qquad M_n \to M_n(w)\,, \tag{3.1.1}$$

where w is the complex variable that we associate with the Risager shift (we later associate

z with BCFW shifts). We have then a one-parameter family of amplitudes $M_n(w)$. We call $M_n = M_n(0)$ the *physical amplitude*, for obvious reasons, while we denote by *Risager's expansion*, M_n^{Ris}, the sum of residues of $M_n(w)$ at its poles on the complex plane. The Risager expansion can be expressed as the following MHV-vertex decomposition:

$$M_n^{\text{Ris}} = \sum_{a,L^+} M_{n_L}\left(\hat{a}^-, L^+, (-I)^-\right) \frac{1}{P_L^2} M_{n_R}\left(I^+, \hat{b}^-, \hat{c}^-, R^+\right) , \tag{3.1.2}$$

where by P_L we mean $P_L = p_a + P_{L^+} = p_a + \sum_{l_i \in L^+} p_{l_i}$, and the labels a, b, c denote negative-helicity gravitons, whereas $l, l_1, l_2, \ldots, r_1, r_2, \ldots$ denote positive-helicity ones. L^+ (R^+) denotes the subset of external positive-helicity gravitons in the left (right) sub-amplitudes in (3.1.2). We use n_L (n_R) for the number of external legs in the left (right), so that $n + 2 = n_L + n_R$. The hats on particles a, b, c indicate that, in each of the sub-amplitudes of (3.1.2), their momenta must be evaluated with (3.1.1) at the appropriate value of $w = w^*$ (the one that makes $(p_{\hat{a}} + P_{L^+})^2 = 0$). The momentum of the graviton I (opposite to the momentum of graviton $-I$) is determined by momentum conservation.

It is known [53] (see also appendix B of [54]) that Risager's deformation fails to give a valid recursion relation for $n \geq 12$, since $M_n(w) \sim w^{n-12}$ as $w \to \infty$. In order to fix the expansion for $n \geq 12$, one needs to compute the residue at infinity, that we denote by $\mathcal{R}_n$, which can be defined as

$$\mathcal{R}_n = M_n - M_n^{\text{Ris}} . \tag{3.1.3}$$

Of course, we have that $\mathcal{R}_n = 0$ for $n < 12$.

Our method for computing $\mathcal{R}_n$ is as follows: we perform a BCFW complex deformation on two external legs, making $\mathcal{R}_n \to \mathcal{R}_n(z)$, that allows us to recover the original object $\mathcal{R}_n$ from the residues at its poles. This can be done since under certain BCFW deformations, $M_n^{\text{Ris}}(z) \to 0$ at large z, as we discuss in section 3.2.2. It happens that the z-dependent poles of $\mathcal{R}_n(z)$ can be split into physical and unphysical ones. The physical poles are, as usual, of the form $1/P^2(z)$ where $P(z)$ is the sum of external momenta in one sub-amplitude. The unphysical poles depend on the reference spinor $\tilde{\eta}$. The result for the

n-point residue at infinity is then

$$\mathcal{R}_n = -\sum_{\text{phys}} \text{Res}\left[\frac{\mathcal{R}_n(z)}{z}\right] - \sum_{\text{unphys}} \text{Res}\left[\frac{\mathcal{R}_n(z)}{z}\right] . \tag{3.1.4}$$

We explicitly calculate $\mathcal{R}_{12}$ (performing the BCFW deformation on particles 1 and 4), and get

$$\begin{aligned}\sum_{\text{unphys}} \text{Res}\left[\frac{\mathcal{R}_{12}}{z}\right] &= -\left(\langle 1\ 2\rangle\langle 2\ 3\rangle\langle 3\ 1\rangle\right)^6 \prod_{k=5}^{12} \frac{[k\ \tilde{\eta}]}{\langle 1\ k\rangle\langle 2\ k\rangle\langle 3\ k\rangle} \\ \times \sum_{l=5}^{12} \frac{[4\ l]^6}{[4\ \tilde{\eta}]^2\,[l\ \tilde{\eta}]^2} &\frac{\langle 4\ l\rangle\,[4\ l]}{\langle 1|p_4+p_l|\tilde{\eta}]\langle 2|p_4+p_l|\tilde{\eta}]\langle 3|p_4+p_l|\tilde{\eta}]} \frac{\langle 1\ l\rangle\langle 2\ l\rangle\langle 3\ l\rangle}{[l\ \tilde{\eta}]} ,\end{aligned} \tag{3.1.5}$$

for the sum of residues at the unphysical poles (we have used the standard notation $\langle i|\sum_j p_j|\tilde{\eta}] = \sum_j \langle i\ j\rangle\,[j\ \tilde{\eta}]$), and at the physical poles we have:

$$\begin{aligned}\sum_{\text{phys}} \text{Res}\left[\frac{\mathcal{R}_{12}}{z}\right] = &- \langle 3\ 1\rangle \frac{[4\ \tilde{\eta}]\,\langle 1\ 2\rangle^2}{\langle 2\ 4\rangle\langle 1\ 4\rangle^2} \text{Res}\left[M_{11}^A(w), \infty\right] \\ &- \langle 1\ 2\rangle \frac{[4\ \tilde{\eta}]\,\langle 1\ 3\rangle^2}{\langle 3\ 4\rangle\langle 1\ 4\rangle^2} \text{Res}\left[M_{11}^B(w), \infty\right] ,\end{aligned} \tag{3.1.6}$$

where M_{11}^A and M_{11}^B are eleven-point NMHV amplitudes that are obtained by "dissolving" particle 4 into particles $(1,2)$ and $(1,3)$ respectively, and one performs the Risager shift (3.1.1) on them to obtain $M_{11}^A(w)$ and $M_{11}^B(w)$. The precise meaning of "dissolving" is defined in section 3.2.3.

As usual with formulas obtained from BCFW, (3.1.5) and (3.1.6) are asymmetric in the set of positive helicity particles (note that the deformed particle 4 is special in the formulas), but the sum of them is indeed invariant under permutation of the positive labels.

3.2 The Residue at Infinity

The behavior of the deformed amplitude $M_n(w)$ at infinity, $M_n(w) \sim 1/w^{12-n}$, makes it clear that starting at $n = 12$ particles, there is a contribution at infinity, $\mathcal{R}_n$, that must be added to the Risager expansion in order to recover the physical amplitude. However, it is useful to have an alternative perspective on the existence of this residue at infinity, namely a more physical reason why it appears. Notice that it could be that this contribution at infinity vanished for $n > 12$. We show that this is not the case.

For BCFW two-particle deformations, the presence of a contribution at infinity is related to the BCFW amplitude missing some physical factorization channels [71]. For the Risager three-particle deformation, the situation is different. Risager's expansion does not miss any physical pole; rather, it contains extra residues as well as some unphysical poles. Let us be more specific.

3.2.1 Physical Meaning of the Residue at Infinity

One can ask about the factorization channels of the NMHV scattering amplitude of n gravitons that are correctly reproduced by Risager's expansion M_n^{Ris}. In view of the definition (3.1.3), one can also search for the poles of $\mathcal{R}_n$.

It turns out that all the poles of the physical amplitude M_n are already present in M_n^{Ris}. Moreover, most of the residues of M_n^{Ris} at these poles give the expected factorization of the physical amplitude M_n. It happens that the physical factorization fails in two classes of channels: the ones corresponding to the poles $\langle l_1\ l_2\rangle$ and $\langle a\ l\rangle$ (recall l, l_1, l_2 denote positive-helicity gravitons and a is a negative-helicity graviton).

We first study the limit $\langle l_1\ l_2\rangle \to 0$. In this limit, the only singular diagrams in the expansion (3.1.2) of M_n^{Ris} are the ones that have both gravitons l_1, l_2 either on the left or on the right sub-amplitude. It is then easy to see that we have the following factorization:

$$\lim_{\langle l_1\ l_2\rangle\to 0}\langle l_1\ l_2\rangle\,[l_1\ l_2]\,M_n^{\text{Ris}} = M_3\left(l_1^+, l_2^+, p_{l_1 l_2}^-\right)M_{n-1}^{\text{Ris}}\,, \tag{3.2.1}$$

where it is understood that in the $(n-1)$-point Risager expansion l_1 and l_2 are substituted by a positive-helicity graviton with on-shell momentum $p_{l_1 l_2} = p_{l_1} + p_{l_2}$. Equation (3.2.1) implies that the residue at this type of pole is the physical one as long as $M_{n-1}^{\text{Ris}} = M_{n-1}$, which holds for $n < 13$.

The limit $\langle a\ l\rangle \to 0$ is a little bit more subtle. The singular diagrams of M_n^{Ris} are now those where particles $\hat{a}$ and l are on the same sub-amplitude, and a three-particle amplitude $M_3\left(\hat{a}^-, l^+, -J^+\right)$ factorizes out since $p_J = p_{\hat{a}} + p_l$ becomes on-shell. The subtlety arises because $p_{\hat{a}} \neq p_a$, and such a three-point amplitude is not common to all the diagrams in the Risager expansion. Taking into account that

$$M_3\left(\hat{a}^-, l^+, -J^+\right) = \frac{[\hat{a}\ l]^2}{[a\ l]^2}\,M_3\left(a^-, l^+, -p_{al}^+\right)\,, \tag{3.2.2}$$

we can write (schematically)

$$\lim_{\langle a\ l\rangle\to 0}\langle a\ l\rangle\,[a\ l]\ M_n^{\text{Ris}} = M_3\left(a^-, l^+, -p_{al}^+\right)\sum\frac{[\hat{a}\ l]}{[a\ l]}\begin{pmatrix}\text{term in the Risager}\\ \text{expansion for } M_{n-1}\end{pmatrix}\,, \tag{3.2.3}$$

where the sum is over the terms of the Risager expansion of an $(n-1)$-point amplitude M_{n-1} with the same external states as M_n, but where a and l combine into a negative-helicity graviton with momentum $p_{al} = p_a + p_l$. Now, computing

$$\frac{[\hat{a}\ l]}{[a\ l]} - 1 = \hat{w}\,\frac{[\tilde{\eta}\ l]}{[a\ l]}\langle b\ c\rangle\,, \tag{3.2.4}$$

it is straightforward to see that

$$\lim_{\langle a\ l\rangle\to 0}\langle a\ l\rangle\,[a\ l]\ M_n^{\text{Ris}} = M_3\left(a^-, l^+, -p_{al}^+\right)\left(M_{n-1}^{\text{Ris}} + \langle b\ c\rangle\frac{[\tilde{\eta}\ l]}{[a\ l]}\text{Res}\left[M_{n-1}(w), \infty\right]\right)\,. \tag{3.2.5}$$

For our Risager's deformation (3.1.1), we recall to the reader that the residue for an n-point NMHV amplitude can be written as:

$$\text{Res}\left[M_n(w),\infty\right] = \sum_{a,L^+} M_{n_L}\left(\hat{a}^-, L^+, (-I)^-\right)\frac{1}{\langle b\ c\rangle\langle a|P_L|\tilde{\eta}]}M_{n_R}\left(I^+, \hat{b}^-, \hat{c}^-, R^+\right)\ , \quad (3.2.6)$$

where the notation is as in (3.1.2). The implication of equation (3.2.5) is that we have the proper physical factorization at the poles $\langle a\ l\rangle$ when, besides the previous condition $M_{n-1}^{\text{Ris}} = M_{n-1}$, it also happens that $\text{Res}\left[M_{n-1}(w),\infty\right] = 0$. For this last condition to hold, $w\, M_{n-1}(w)$ must vanish at infinity, or equivalently $M_{n-1}(w)$ must vanish faster than $1/w$, which happens only for $n < 12$.

Interestingly, the necessity of the residue at infinity of lower-point amplitudes to vanish also happens when reconstructing tree-level graviton amplitudes with the BCFW technique, as noted by Toro and Schuster in [72]. They saw that in order to prove that the BCFW expansion for an n-graviton amplitude has the correct factorization in this very same channel $\langle a\ l\rangle$, $(n-1)$-graviton amplitudes need to vanish faster than $1/z$ under BCFW deformations.

In addition to this failure to correctly reproduce physical poles, a careful analysis of the different terms in the Risager expansion (3.1.2) shows that for $n \geq 12$, unphysical poles of the form $\langle a|P_L|\tilde{\eta}]$ appear in M_n^{Ris}. More precisely, they appear in the denominator with the power $\langle a|P_L|\tilde{\eta}]^{n-7-n_L}$.

In order to have an intuition of why $n = 12$ is special, let us look at a given diagram of the Risager expansion (3.1.2) with $L^+ = \{l_1, ..., l_{n_L-2}\}$ and $R^+ = \{r_2, ..., r_{n_R-2}\}$. The contribution of this diagram to the expansion is

$$M_{n_L}\left(\hat{a}^-, L^+, (-I)^-\right)\frac{1}{P_L^2}M_{n_R}\left(I^+, \hat{b}^-, \hat{c}^-, R^+\right)\ , \quad (3.2.7)$$

where M_{n_L} and M_{n_R} are MHV amplitudes. These amplitudes could contain poles of the form $\langle a\ I\rangle = \langle a|P_L|\tilde{\eta}]$, where by convention we use $\lambda^I = P_L|\tilde{\eta}] = [a\ \tilde{\eta}]\,\lambda^a + \sum_{l_i\in L^+}[i\ \tilde{\eta}]\,\lambda^{l_i}$.

To check this possibility we can use any explicit analytic expression of MHV amplitudes. We do it using the Mason-Skinner formula [12], which reads for the MHV amplitude $M_n\left(1^-, 2^+, \ldots, (n-1)^+, n^-\right)$ as

$$
\begin{aligned}
M_n^{\text{MHV}} = & \frac{\langle 1\ n\rangle^8}{\langle 1\ n-1\rangle\langle n-1\ n\rangle\langle n\ 1\rangle}\Bigg(\frac{1}{\langle 1\ 2\rangle\langle 2\ 3\rangle\cdots\langle n-1\ n\rangle\langle n\ 1\rangle} \\
& \times \prod_{k=2}^{n-2}\frac{\langle n|p_{n-1}+\ldots+p_{k+1}|k]}{\langle k\ n\rangle} + (\text{permutations of labels } \{2,\ldots,n-2\})\Bigg) \\
= & \frac{\langle 1\ n\rangle^6}{\langle 1\ n-1\rangle\langle n-1\ n\rangle}\Bigg(\frac{1}{\langle 1\ 2\rangle\langle 2\ 3\rangle\cdots\langle n-1\ n\rangle} \\
& \times \prod_{k=2}^{n-2}\frac{\langle n|-p_1-\ldots-p_{k-1}|k]}{\langle k\ n\rangle} + (\text{permutations of labels } \{2,\ldots,n-2\})\Bigg)\,.
\end{aligned}
\tag{3.2.8}
$$

With our convention, $\lambda^I = P_L|\tilde{\eta}]$, the Mason-Skinner formula yields for M_{n_L} a factor $\langle a\ I\rangle^6 = \langle a|P_L|\tilde{\eta}]^6$ in the numerator, since a and I are the negative-helicity particles on the left sub-amplitude. One can notice that the power of this factor is the fingerprint of $\mathcal{N}=8$ SUSY (it was initially eight, before canceling two powers of the same factor in the denominator). In the expression for M_{n_R}, identifying $\hat{b}\equiv 1$, $\hat{c}\equiv n$ and $I\equiv n-1$ when using Mason-Skinner formula (3.2.8), $\langle a\ I\rangle$ appears only through the denominator of the complexified momentum $p_{\hat{b}}$, since with Risager deformation (3.1.1) we evaluate the sub-amplitude at

$$
\hat{w} = -\frac{P_L^2}{\langle b\ c\rangle\langle a|P_L|\tilde{\eta}]}\,. \tag{3.2.9}
$$

Therefore, from the product inside Mason-Skinner formula (3.2.8) we get n_R-3 powers of $\langle a|P_L|\tilde{\eta}]$ in the denominator of M_{n_R}. In total, for the whole Risager diagram we have the power

$$
\frac{\langle a\ I\rangle^6}{\langle a\ I\rangle^{n_R-3}} = \langle a|P_L|\tilde{\eta}]^{9-n_R}\,. \tag{3.2.10}
$$

So, in order to have a pole of this type, n_R needs to be at least ten. Considering that $n_L+n_R=n+2$ and $n_L\geq 3$, we simply see that n has to be at least eleven to produce this unphysical pole. Naively one would expect that M_{11}^{Ris} would contain the pole[1] $[l\ \tilde{\eta}]$.

[1] Although in this case $\langle a|P_L|\tilde{\eta}] = \langle a\ l\rangle\,[l\ \tilde{\eta}]$, notice that these diagrams with three-point amplitudes do

However, there are three Risager diagrams contributing to this pole (the ones with $a = 1, 2, 3$ and $L^+ = \{l\}$) and, only in the case of $n = 11$, a cancellation happens when summing over the three diagrams (see appendix 5.A for details). The pole $[l\ \tilde{\eta}]$ is then spurious for $n = 11$, as we knew beforehand since $M_{11}^{\text{Ris}} = M_{11}$. Hence, the twelve-particle amplitude is the place for the first appearance of the unphysical poles $\langle a|P_L|\tilde{\eta}]$.

Combining all the information spelled out in this subsection, we know what the poles of $\mathcal{R}_n$ are, and we can write the schematic expression[2]

$$\mathcal{R}_{12} = \frac{\mathcal{P}_{12}}{\prod\limits_{a,l,l_1,l_2} \langle a|p_{l_1} + p_{l_2}|\tilde{\eta}]\,[l\ \tilde{\eta}]^2\,\langle a\ l\rangle}\,, \tag{3.2.11}$$

$$\mathcal{R}_n = \frac{\mathcal{P}_n}{\prod\limits_{\substack{a,l,l_1,l_2\\ n_L<n-7}} \langle a|P_L|\tilde{\eta}]^{n-7-n_L}\langle a\ l\rangle\langle l_1\ l_2\rangle}\,, \qquad n > 12\,, \tag{3.2.12}$$

where $\mathcal{P}_n$ is some polynomial of the momenta of n scattering gravitons.

3.2.2 A BCFW Computation of the Residue at Infinity

In virtue of (3.2.11)-(3.2.12), we know the poles of the contribution at infinity $\mathcal{R}_n$. Moreover, we also know the residues of $\mathcal{R}_n$ at them. At the physical poles $\langle a\ l\rangle$ and $\langle l_1\ l_2\rangle$, the residues are determined by (3.2.1) and (3.2.5). And at the unphysical poles $\langle a|P_L|\tilde{\eta}]$, the residues come from just one diagram in the Risager expansion, namely the one with particles a, L^+ on the left blob (see Figure 3.1), which is the only one that has the factor $1/\langle a|P_L|\tilde{\eta}]^{n-7-n_L}$.

This information is enough to implement a one-parameter complex deformation on the momenta of some gravitons, turn the residue at infinity into a function $\mathcal{R}_n(z)$ of a complex variable z, and recover $\mathcal{R}_n$ from the residues at the poles of this function, as long as it

not contribute to the poles $\langle a\ l\rangle$, since this factor cancels in (3.2.9).

[2] In writing formula (3.2.12), when $L^+ = \{l\}$, by $\langle a|P_L|\tilde{\eta}]$ we understand just $[l\ \tilde{\eta}]$.

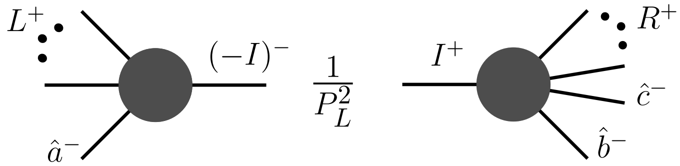

Figure 3.1: Contribution to the residue of $\mathcal{R}_n$ at the unphysical pole $\langle a|P_L|\tilde{\eta}]$.

vanishes at $z \to \infty$. We can actually use a BCFW shift:

$$\tilde{\lambda}^{(i)}(z) = \tilde{\lambda}^i - z\,\tilde{\lambda}^j\,, \qquad \lambda^j(z) = \lambda^j + z\,\lambda^i\,. \tag{3.2.13}$$

When the helicities of particles (i,j) are respectively $(-,+),(-,-),(+,+)$, we know that M_n vanishes as $1/z^2$. It is easy to check that under the last two shifts, the worst diagrams in the Risager expansion go as[3] $1/z$. From the definition (3.1.3), it is obvious that $\mathcal{R}_n(z)$ will vanish at infinity. We can then write the usual BCFW integral:

$$\oint \mathrm{d}z\,\frac{\mathcal{R}_n(z)}{z} = 0 \qquad \Longrightarrow \qquad \mathcal{R}_n = -\sum_{\text{poles } z^*} \text{Res}\left[\frac{\mathcal{R}_n(z)}{z}; z^*\right]\,, \tag{3.2.14}$$

where z^* are the points at which some factor in the denominator of $\mathcal{R}_n(z)$ becomes zero. Notice that there is a difference with respect to the usual BCFW reconstruction of an amplitude. Not all the poles of $\mathcal{R}_n(z)$ are simple poles, since in the denominator of $\mathcal{R}_n$ some factors come with a higher power than one (see (3.2.12)).

Having given a procedure to compute the residue at infinity $\mathcal{R}_n$ for any n, let us illustrate it explicitly by computing the first non-zero contribution, that happens for $n = 12$.

[3]Actually, it seems that the Risager expansion M_n^{Ris} vanishes as $1/z^2$ for the shifts $(-,-),(+,+)$, exactly as the physical gravity amplitude does. We checked this numerically for $n \leq 16$. The shift $(-,+)$ is not so nicely behaved, as the worst Risager diagrams go as $1/z^{13-n}$. Indeed, the Risager expansion does not vanish under the shift $(-,+)$ for $n \geq 13$. For the sake of completeness, we mention that under the $(+,-)$ shift the worst Risager diagrams behave as z^{n-5}, and Risager's expansion displays this large-z behavior for $n \geq 12$.

3.2.3 Residue at Infinity of the Twelve-Point Amplitude

Following the steps outlined above, we show how to compute $\mathcal{R}_{12}$. Its poles were written explicitly in (3.2.11). The BCFW shifts (3.2.13) of the type $(-,+)$ (which is valid for $n=12$) involve the fewest number of them. We use for instance the (1,4) shift:

$$\tilde{\lambda}^1(z) = \tilde{\lambda}^1 - z\,\tilde{\lambda}^4\,, \qquad \lambda^4(z) = \lambda^4 + z\,\lambda^1\,. \tag{3.2.15}$$

Indeed, it is possible to recover $\mathcal{R}_{12}$ from only the residues at the following (simple) poles:

$$\langle 2\;4\rangle\,,\;\langle 3\;4\rangle\,,\;\;\langle 2|p_4+p_l|\tilde{\eta}]\,,\;\langle 3|p_4+p_l|\tilde{\eta}] \quad \text{with } l=5,\ldots,12\,, \tag{3.2.16}$$

which happen, respectively, at the following values of z:

$$-\frac{\langle 2\;4\rangle}{\langle 2\;1\rangle}\,,\;\; -\frac{\langle 3\;4\rangle}{\langle 3\;1\rangle}\,,\;\; -\frac{\langle 2|p_4+p_l|\tilde{\eta}]}{\langle 2\;1\rangle\,[4\;\tilde{\eta}]}\,,\;\; -\frac{\langle 3|p_4+p_l|\tilde{\eta}]}{\langle 3\;1\rangle\,[4\;\tilde{\eta}]}\,. \tag{3.2.17}$$

On the one hand, the first two poles coincide with physical ones, and the residue can be computed directly from equation (3.2.5). We get

$$\begin{aligned}\sum_{\text{phys}} \text{Res}\left[\frac{\mathcal{R}_{12}(z)}{z}\right] = &- \langle 3\;1\rangle \frac{[4\;\tilde{\eta}]\,\langle 1\;2\rangle^2}{\langle 2\;4\rangle\langle 1\;4\rangle^2}\text{Res}\left[M_{11}^A(w),\infty\right] \\ &- \langle 1\;2\rangle \frac{[4\;\tilde{\eta}]\,\langle 1\;3\rangle^2}{\langle 3\;4\rangle\langle 1\;4\rangle^2}\text{Res}\left[M_{11}^B(w),\infty\right]\,,\end{aligned} \tag{3.2.18}$$

with M_{11}^A and M_{11}^B being the following eleven-point amplitudes:

$$M_{11}^A = M_{11}\left((1^A)^-,(2^A)^-,3^-,5^+,\ldots,12^+\right)\,, \tag{3.2.19}$$

$$M_{11}^B = M_{11}\left((1^B)^-,2^-,(3^B)^-,5^+,\ldots,12^+\right)\,, \tag{3.2.20}$$

where notice that particle 4 has disappeared, and particles 1 and 2, and 1 and 3 respectively, have been deformed as

$$P_{1^A} = \lambda^{(1)}\left(\tilde{\lambda}^{(1)} + \frac{\langle 2\ 4\rangle}{\langle 2\ 1\rangle}\tilde{\lambda}^{(4)}\right), \qquad P_{2^A} = \lambda^{(2)}\left(\tilde{\lambda}^{(2)} + \frac{\langle 1\ 4\rangle}{\langle 1\ 2\rangle}\tilde{\lambda}^{(4)}\right), \tag{3.2.21}$$

$$P_{1^B} = \lambda^{(1)}\left(\tilde{\lambda}^{(1)} + \frac{\langle 3\ 4\rangle}{\langle 3\ 1\rangle}\tilde{\lambda}^{(4)}\right), \qquad P_{3^B} = \lambda^{(3)}\left(\tilde{\lambda}^{(3)} + \frac{\langle 1\ 4\rangle}{\langle 1\ 3\rangle}\tilde{\lambda}^{(4)}\right). \tag{3.2.22}$$

One can say that particle 4 has been "dissolved" into particles 1 and 2 in M_{11}^A, and into particles 1 and 3 in M_{11}^B.

On the other hand, the residues at the second two types of (unphysical) poles in (3.2.16) can be extracted from just the diagrams of M_{12}^{Ris} where $L^+ = \{4, l\}$ and $a = 2, 3$, which we draw in Figure 3.2.

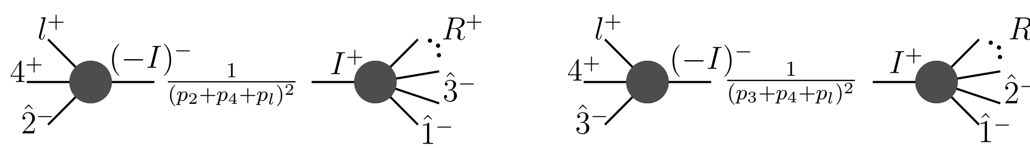

Figure 3.2: The only diagrams contributing to the residues at the unphysical poles $\langle 2|p_4 + p_l|\tilde{\eta}]$ (left) and $\langle 3|p_4 + p_l|\tilde{\eta}]$ (right).

One just needs to compute the piece proportional to $1/\langle a|P_L|\tilde{\eta}]$ of these diagrams. After some simplifications, the result can be compactly written in formula (3.1.5):

$$\begin{aligned}\sum_{\text{unphys}} \text{Res}\left[\frac{\mathcal{R}_{12}}{z}\right] &= -\left(\langle 1\ 2\rangle\langle 2\ 3\rangle\langle 3\ 1\rangle\right)^6 \prod_{k=5}^{12} \frac{[k\ \tilde{\eta}]}{\langle 1\ k\rangle\langle 2\ k\rangle\langle 3\ k\rangle} \\ &\times \sum_{l=5}^{12} \frac{[4\ l]^6}{[4\ \tilde{\eta}]^2\,[l\ \tilde{\eta}]^2} \frac{\langle 4\ l\rangle\,[4\ l]}{\langle 1|p_4 + p_l|\tilde{\eta}]\langle 2|p_4 + p_l|\tilde{\eta}]\langle 3|p_4 + p_l|\tilde{\eta}]} \frac{\langle 1\ l\rangle\langle 2\ l\rangle\langle 3\ l\rangle}{[l\ \tilde{\eta}]}\,.\end{aligned} \tag{3.2.23}$$

Finally, the sum of the two contributions (3.2.18) and (3.2.23) gives the residue at infinity of the twelve-point amplitude (3.1.4) we were looking for. It is quite remarkable that such a complex object admits such a simple expression. Even though (3.2.18) and (3.2.23) separately are not invariant under permutation of positive labels, their sum is.

Therefore the result for the residue at infinity is invariant under permutation of positive (and negative) labels. Both $M_{12}^{\rm Ris}$ and $\mathcal{R}_{12}$ depend on the reference spinor $\tilde{\eta}$, but this dependence disappears when we add both terms[4], and we obtain a compact expression for the physical amplitude M_{12}.

3.3 Discussion on Soft Limits

The method we proposed to determine the residue at infinity of the twelve-point amplitude from Risager's construction was based on the knowledge of its poles, or equivalently, on the failure of Risager's expansion to provide the correct factorization channels of the physical amplitude. One could think of other possibilities that could also lead to the determination of this residue at infinity.

One such possibility is to look at other singular kinematic limits of the physical amplitude, *e.g.*, soft limits. In gravity, the soft factors (which we denote by $S^{\pm}$), defined by

$$\lim_{p_{n+1}\to 0} \frac{M_{n+1}\left(1,\ldots,n,(n+1)^{\pm}\right)}{S_{n+1}^{\pm}\, M_n\left(1,\ldots,n\right)} = 1\,, \tag{3.3.1}$$

are universal[5]:

$$S_{n+1}^{-} = \sum_{i=1}^{n} \frac{[i\ \mu_1]\,[i\ \mu_2]}{[n+1\ \mu_1]\,[n+1\ \mu_2]} \frac{\langle n+1\ i\rangle}{[n+1\ i]}\,, \tag{3.3.2}$$

$$S_{n+1}^{+} = \sum_{i=1}^{n} \frac{\langle i\ \mu_1\rangle\langle i\ \mu_2\rangle}{\langle n+1\ \mu_1\rangle\langle n+1\ \mu_2\rangle} \frac{[n+1\ i]}{\langle n+1\ i\rangle}\,, \tag{3.3.3}$$

where μ_1, μ_2 are arbitrary reference spinors, and the sums above are independent of their choice [62, 11, 13]. We can check if these soft limits (of both negative- and positive-helicity gravitons) are correctly reproduced by Risager's expansion $M_n^{\rm Ris}$.

[4] Although this is not obvious from the corresponding analytic expressions, we checked numerically that the final result does not depend on the reference spinor $\tilde{\eta}$, as well as checking that it agrees with the twelve-graviton amplitude computed via a (more time-consuming) BCFW deformation.

[5] This means that they depend just on the helicity of the soft graviton, and not on the helicities of the others, *i.e.* they are the same in all sectors.

What we find is that soft limits where the momentum of a negative-helicity graviton vanishes produce the correct behavior (3.3.2) on Risager's expansion. However, when the soft graviton has a positive helicity, we obtain the non-trivial behavior:

$$\lim_{p_l\to 0} M_{n+1}^{\rm Ris} = \lim_{p_l\to 0}\left[S_{n+1}^{+} M_n^{\rm Ris} + \frac{\langle 1\ 2\rangle\langle 2\ 3\rangle\langle 3\ 1\rangle}{\langle l\ 1\rangle\langle l\ 2\rangle\langle l\ 3\rangle}\,[\tilde\eta\ l]\,{\rm Res}\,[M_n(w),\infty]\right], \tag{3.3.4}$$

where it should be understood that what coincides is the leading order of both sides of the equality, and the graviton l of M_{n+1} is not present in M_n. Formula (3.3.4) tells us that soft limits are correctly reproduced by $M_n^{\rm Ris}$ as long as $M_{n-1}^{\rm Ris} = M_{n-1}$ and ${\rm Res}\,[M_{n-1}(w),\infty]$ vanishes, which happens only for $n < 12$. The reason for the failure of the twelve-point Risager's expansion to have the right soft limits, namely the non-vanishing of the residue at infinity of the eleven-point amplitude, is exactly the same as the reason why it fails to account for the right residues at the physical poles. Notice the similarities between expressions (3.3.4) and (3.2.5).

3.4 Concluding Remarks

The main question addressed in this chapter was to find the discrepancy between what Risager's expansion produces for the 12-graviton tree-level amplitude in the NMHV sector and the physical amplitude which could be found by any guaranteed method *e.g.* BCFW recursion relations. Our method for computing this discrepancy, the residue at infinity, enjoyed a BCFW technique, namely a 2-particle complex deformation and using the residue theorem to recover the non-deformed $\mathcal{R}_{12}$ from the complexified $\mathcal{R}_n(z)$.

We also proposed another way of computing the residue at infinity based on the knowledge of other singular kinematic limits of the physical amplitude, *e.g.* soft limits. Given the recent interest in soft factors [63, 64], and the fact that they play a crucial role in Hodges formula for MHV amplitudes [14], it seems of obvious interest to further explore the possibility of recovering $\mathcal{R}_n$ from soft limits. Trying to use the formalism of [71] for computing contributions at infinity offers another interesting possibility to recover $\mathcal{R}_n$. The advantage of this direction would be to have the residue at infinity expressed as a sum of products of MHV amplitudes.

Appendix 3.A Spurious Poles of the Eleven-Point Risager's Expansion

In this Appendix we come back to a technical subtlety of our analysis of Section 3.2.1. There we showed analytically how Risager's expansion contains unphysical poles for $n \geq 12$. Actually, our analysis naively predicts the presence of an unphysical pole $[l\ \tilde{\eta}]$ already for the eleven-point Risager's expansion, which we know it is not the case as $M_{11}^{\text{Ris}} = M_{11}$. Let us see this fact explicitly.

There are three Risager diagrams, that we call $M^{(1,l)}$, $M^{(2,l)}$, $M^{(3,l)}$, contributing with a $1/\left[l\ \tilde{\eta}\right]$ factor to the eleven-point Risager's expansion M_{11}^{Ris}. In the expansion (3.1.2) they are the ones that have a three-particle amplitude with $(1, l, -I)$, $(2, l, -I)$ and $(3, l, -I)$ respectively on the left sub-amplitude. What we have to see is that

$$\lim_{[l\ \tilde{\eta}]\to 0} [l\ \tilde{\eta}]\left(M^{(1,l)} + M^{(2,l)} + M^{(3,l)}\right) = 0\,. \tag{3.A.1}$$

Let us compute the piece proportional to $1/\left[l\ \tilde{\eta}\right]$ of these diagrams. The left sub-amplitude (leaving $a = 1, 2, 3$ generic) is

$$M_{n_L} = M_3\left(\hat{a}^-, l^+, (-I)^-\right) = \frac{\langle a\ l\rangle^2}{[a\ \tilde{\eta}]^2}\,[l\ \tilde{\eta}]^6\ , \tag{3.A.2}$$

where we are taking into account that

$$\hat{\tilde{\lambda}}^{(a)} = \frac{[a\ \tilde{\eta}]}{[l\ \tilde{\eta}]}\tilde{\lambda}^{(l)}\,, \quad \lambda^{(I)} = (P_a + P_l)|\tilde{\eta}]\,, \quad \tilde{\lambda}^{(I)} = \frac{\tilde{\lambda}^{(l)}}{[l\ \tilde{\eta}]}\,. \tag{3.A.3}$$

The right sub-amplitude is an MHV amplitude, and it looks complicated if we use Mason-

Skinner formula (3.2.8). But we just need to keep the leading order in $1/[l\ \tilde{\eta}]$. Identifying $\hat{b} \equiv 1$, $\hat{c} \equiv n$, $l \equiv n-1$, and using the following result[6] (which is just an elaborated consequence of Schouten identity):

$$\sum_{S_n} \frac{1}{\langle \alpha\ a_{i_1}\rangle\langle a_{i_1}\ a_{i_2}\rangle \cdots \langle a_{i_{n-1}}\ a_{i_n}\rangle\langle a_{i_n}\ \beta\rangle\langle \beta\ \alpha\rangle} = \frac{-\langle \alpha\ \beta\rangle^{n-2}}{\langle \alpha\ a_1\rangle \cdots \langle \alpha\ a_n\rangle\langle a_1\ \beta\rangle \cdots \langle a_n\ \beta\rangle}\,, \tag{3.A.4}$$

where S_n stands for the permutation group of n elements; we get

$$M_{n_R} \sim -\frac{\langle b\ c\rangle^6 \langle b|P_a+P_l|\tilde{\eta}]^5 \langle c|P_a+P_l|\tilde{\eta}]^5}{[l\ \tilde{\eta}]^7} \prod_{\substack{k=4\\k\neq l}}^{11} \frac{[k\ l]}{\langle k|P_a+P_l|\tilde{\eta}]\langle b\ k\rangle\langle c\ k\rangle}\,. \tag{3.A.5}$$

Finally, to leading order in $1/[l\ \tilde{\eta}]$, we obtain

$$M^{(a,l)} \sim -\frac{1}{[l\ \tilde{\eta}]} \frac{\langle a\ l\rangle\langle b\ c\rangle^6 \langle b|P_a+P_l|\tilde{\eta}]^5 \langle c|P_a+P_l|\tilde{\eta}]^5}{[a\ \tilde{\eta}]^2\,[a\ l]} \prod_{\substack{k=4\\k\neq l}}^{11} \frac{[k\ l]}{\langle k|P_a+P_l|\tilde{\eta}]\langle b\ k\rangle\langle c\ k\rangle}\,. \tag{3.A.6}$$

With this we can check if (3.A.2) is satisfied. In the limit $[l\ \tilde{\eta}] \to 0$ we can put, up to unimportant scaling factors, $\tilde{\lambda}^{(l)} = \tilde{\eta}$. Then we have

$$\begin{aligned} \lim_{[l\ \tilde{\eta}]\to 0} [l\ \tilde{\eta}] \left(M^{(1,l)} + M^{(2,l)} + M^{(3,l)}\right) &= \left(\langle 1\ 2\rangle\langle 2\ 3\rangle\langle 3\ 1\rangle\right)^5 \\ &\times \prod_{\substack{k=4\\k\neq l}}^{11} \frac{[k\ l]}{\langle 1\ k\rangle\langle 2\ k\rangle\langle 3\ k\rangle} \left(\langle 1\ l\rangle\langle 2\ 3\rangle + \langle 2\ l\rangle\langle 3\ 1\rangle + \langle 3\ l\rangle\langle 1\ 2\rangle\right) = 0\,, \end{aligned} \tag{3.A.7}$$

where we just used Schouten identity in the last line. This shows that the pole $[l\ \tilde{\eta}]$, appearing in three of the Risager diagrams, is spurious as it cancels when summing over them. Notice also that this cancellation is only possible when $n = 11$, where $\langle b|P_a+P_l|\tilde{\eta}]^5$ and $\langle c|P_a+P_l|\tilde{\eta}]^5$ in (3.A.5) come exactly with the power five.

[6] Identity (3.A.4) was first presented in the context of QED amplitudes (see Section 8.2 of [1]).

Chapter 4

Parity Symmetry and Soft Limit for the Cachazo-Geyer Gravity Amplitude

In this chapter, we continue our study of gravity amplitudes in the light of a recent proposed formula for the tree-level n-particle $\mathcal{N} = 8$ supergravity amplitudes by Cachazo and Geyer [15]. We prove that the formula satisfies parity symmetry and soft limit behavior expected for graviton scattering amplitudes. After a review of the Cachazo-Geyer formula, we will move on to the details of the proofs and at the end present a sample calculation, MHV and $\overline{\text{MHV}}$ amplitudes, using the formula.

4.1 Introduction

The Lagrangian approach to gravity suggests that even the tree-level gravity is indeed a much more complex problem than Yang-Mills, in which a naive perturbative approach becomes computationally unfeasible already for a very small number of particles. Surprisingly, a huge development in gravity amplitudes has been recently made by Hodges [14] to write tree-level MHV amplitudes in a much simpler way than before.

Amazingly, triggered by Hodges MHV formula, two distinct novel formulas for tree-level supergravity amplitudes of all R-charge sectors were recently proposed by Cachazo-Geyer [15] and Cachazo-Skinner [16].

Both proposals, which are analogous to the Witten-Roiban-Spradlin-Volovich's (Witten-RSV) twistor string formulation of $\mathcal{N} = 4$ super Yang-Mills [23, 56], can be understood as huge steps toward finding a twistor-string formulation of gravity. Hodges-like determinants, reviewed in section 1.2.5, are important ingredients in the two formulas, which make them so simple and elegant.

On the one hand, the Cachazo-Geyer formula is derived from the supersymmetric version of the Kawai-Lewellen-Tye (KLT) relations [20] which relate the maximally supersymmetric amplitudes in Yang-Mills and gravity. The Cachazo-Skinner formula, on the other hand, emerges from studying the BCFW relations for gravity in super-twistor space.

In this chapter, we consider the Cachazo-Geyer proposal and study two consistency checks, namely the parity invariance and soft limit behavior of the formula. Our proofs use the results given by RSV [56] and Witten [73] for the parity and soft limit checks in $\mathcal{N} = 4$ super Yang-Mills, and present the validity of these properties in the gravity formula. These two checks are strong evidences that the proposal is the complete tree-level S-matrix of supergravity. Finally, we explore the possibility of using the known results in super Yang-Mills to compute amplitudes in gravity from the proposed formula, and we show the explicit computations for MHV and $\overline{\text{MHV}}$ amplitudes.

This chapter is organized as follows: In section 4.2 we review the Cachazo-Geyer and

Witten-RSV formulas. The proofs for the parity invariance and soft limit of the formula are presented in sections 4.3 and 4.4 respectively. Finally, in section 4.5 we calculate the MHV and $\overline{\text{MHV}}$ amplitudes from the formula.

4.2 The Cachazo-Geyer and Witten-RSV Formulas

Analogous to the Witten-RSV formulation, the Cachazo-Geyer formula for n particle tree-level supergravity amplitudes in the k^{th} sector is

$$\begin{aligned}\mathcal{M}_{n,k} =& \frac{1}{\text{Vol}(\text{GL}(2))}\int d^{2n}\sigma d^{2k}\rho \frac{H_n(\sigma)}{J_n(\sigma,\rho)}\prod_{\alpha=1}^{k}\delta^2\left(\sum_{a=1}^{n} C^V_{\alpha a}(\sigma)\tilde{\lambda}_a\right) \\ &\times \delta^{0|8}\left(\sum_{a=1}^{n} C^V_{\alpha a}(\sigma)\tilde{\eta}_a\right)\prod_{a=1}^{n}\delta^2\left(\sum_{\alpha=1}^{k}\rho_\alpha C^V_{\alpha a}(\sigma)-\lambda_a\right),\end{aligned} \tag{4.2.1}$$

whose ingredients we explain below:

- $C^V(\sigma)$ is a $k\times n$ matrix obtained from the world-sheet variables $(\sigma_{1a},\sigma_{2a}),\ a=1\ldots n$, via the Veronese map

$$\begin{aligned} V: G(2,n) &\to G(k,n) \\ \Sigma_{2\times n} &\mapsto C^V(\sigma)_{k\times n} \end{aligned}. \tag{4.2.2}$$

  Each column of Σ transforms as

$$\begin{pmatrix}\sigma_{1a}\\ \sigma_{2a}\end{pmatrix} \overset{V}{\mapsto} \begin{pmatrix}(\sigma_{1a})^{k-1}\\ (\sigma_{1a})^{k-2}\sigma_{2a}\\ \vdots \\ (\sigma_{2a})^{k-1}\end{pmatrix}, \qquad C^V_{\alpha a}(\sigma) = (\sigma_{1a})^{k-\alpha}(\sigma_{2a})^{\alpha-1}. \tag{4.2.3}$$

- In order to write H_n let us first define the Hodges-like $n\times n$ matrix Φ_n with elements

$$
\begin{aligned}
(\Phi_n)_{ab} &= \frac{s_{ab}}{(a\,b)^2}, \qquad \text{for } a\neq b \\
(\Phi_n)_{aa} &= -\sum_{\substack{b=1\\ b\neq a}}^{n} \frac{s_{ab}}{(a\,b)^2}\frac{(b\,l)(b\,r)}{(a\,l)(a\,r)}.
\end{aligned}
\tag{4.2.4}
$$

We recall to the reader that $(a\,b)$ are 2×2 minors of Σ, defined as

$$
(a\,b) = \sigma_{1a}\sigma_{2b} - \sigma_{1b}\sigma_{2a}. \tag{4.2.5}
$$

The matrix Φ_n has rank $n-3$. A non-degenerate matrix can be obtained by deleting three rows and three columns of Φ_n. This matrix is denoted by $\Phi^{(abc)}_{n(def)}$. Finally, H_n is defined as

$$
H_n = (-1)^{n+1}\frac{1}{(a\,b)(b\,c)(c\,a)} \times \frac{1}{(d\,e)(e\,f)(f\,d)}|\Phi^{(abc)}_{n(def)}|. \tag{4.2.6}
$$

- The last ingredient J_n is obtained in a similar fashion as H_n. First define the $2n+2k$ vectors grouping the sets of variables $\mathcal{V}$ and equations $\mathcal{E}$:

$$
\begin{aligned}
\mathcal{V} &= \{\rho_{11},\rho_{12},\ldots,\rho_{k1},\rho_{k2},\sigma_{11},\sigma_{21},\ldots,\sigma_{1n},\sigma_{2n}\} \\
\mathcal{E} &= \{E_{11},E_{12},\ldots,E_{k1},E_{k2},F_{11},F_{21},\ldots,F_{1n},F_{2n}\}
\end{aligned}
\tag{4.2.7}
$$

where

$$
E_{\alpha\underline{\dot\alpha}} = \sum_{a=1}^{n} C^V_{\alpha a}(\sigma)\tilde\lambda_{a\underline{\dot\alpha}}, \qquad F_{\underline{\alpha}a} = \sum_{\alpha=1}^{k}\rho_{\alpha\underline{\alpha}}C^V_{\alpha a}(\sigma), \tag{4.2.8}
$$

here $\underline{\alpha},\underline{\dot\alpha} = 1,2$ are holomorphic and anti-holomorphic spinor indices, respectively. We then construct the matrix

$$
(K_n)_{IJ} = \frac{\partial\mathcal{E}_J}{\partial\mathcal{V}_I}, \qquad K_n = \begin{pmatrix} \left(\frac{\partial E}{\partial\rho}\right)_{2k\times 2k} & \left(\frac{\partial F}{\partial\rho}\right)_{2k\times 2n} \\ \left(\frac{\partial E}{\partial\sigma}\right)_{2n\times 2k} & \left(\frac{\partial F}{\partial\sigma}\right)_{2n\times 2n} \end{pmatrix}. \tag{4.2.9}
$$

Again, we know that since the system of equations contains momentum conservation, its rank is $2n+2k-4$. Therefore, a non-degenerate matrix can be obtained by deleting four rows and four columns of K_n. Choosing the rows to be the ones corresponding to $\{\sigma_{1a}, \sigma_{2a}, \sigma_{1b}, \sigma_{2b}\}$ and the columns to $\{F_{1c}, F_{2c}, F_{1d}, F_{2d}\}$, and denoting the remaining matrix by $K_{n(cd)}^{\;(ab)}$, J_n is finally given by

$$J_n = \frac{1}{(a\,b)^2[c\,d]^2}|K_{n(cd)}^{\;(ab)}|. \tag{4.2.10}$$

The number of integration variables in (4.2.1) after gauge fixing the GL(2) redundancy is $2n+2k-4$, the same number of δ-functions under the integral after pulling out the momentum conserving $\delta^4\left(\sum_{a=1}^n p_a\right)$. Hence, the integral is completely localized. The resulting amplitude is therefore computed from the solutions $(\sigma_{1a}^*, \sigma_{2a}^*, \rho_{\alpha\underline{\alpha}}^*)$ of the system of equations

$$\sum_{\alpha=1}^{k} \rho_\alpha C_{\alpha a}^V(\sigma) = \lambda_a, \qquad \sum_{a=1}^{n} C_{\alpha a}^V(\sigma)\tilde{\lambda}_a = 0, \tag{4.2.11}$$

by evaluating the integrand with the corresponding Jacobian factor at each solution and summing over all contributions. The Cachazo-Geyer formula can be finally written as

$$\begin{aligned}\mathcal{M}_{n,k} &= \delta^4\left(\sum_{a=1}^n p_a\right) \sum_{\substack{\text{Solutions}\\(\rho^*,\sigma_1^*,\sigma_2^*)}} \frac{H_n}{J_n^2} \prod_{\alpha=1}^{k} \delta^{0|8}\left(\sum_{a=1}^n C_{\alpha a}^V(\sigma)\tilde{\eta}_a\right)\Bigg|_{(\rho^*,\sigma_1^*,\sigma_2^*)} \\ &= \delta^4\left(\sum_{a=1}^n p_a\right) M_{n,k}.\end{aligned} \tag{4.2.12}$$

For the proofs presented in this work, it turns out to be more convenient to write (4.2.1) in terms of other set of variables (ρ, σ, ξ) where σ and ξ are related to the world-sheet variables σ_1 and σ_2 as

$$\xi_a = (\sigma_{1a})^{k-1}, \qquad \sigma_a = \frac{\sigma_{2a}}{\sigma_{1a}}. \tag{4.2.13}$$

In these variables, (4.2.1) reads

$$\begin{aligned}\mathcal{M}_{n,k} =& \frac{1}{\mathrm{Vol(GL(2))}}\int \frac{d^n\xi d^n\sigma d^{2k}\rho}{(k-1)^{2n}}\prod_{a=1}^{n}\xi_a^{\frac{k-3}{k-1}}\frac{H_n(\sigma,\xi)}{J_n'(\sigma,\xi,\rho)}\prod_{\alpha=1}^{k}\delta^2\left(\sum_{a=1}^{n}C^V_{\alpha a}(\xi,\sigma)\tilde{\lambda}_a\right)\\ &\times\delta^{0|8}\left(\sum_{a=1}^{n}C^V_{\alpha a}(\xi,\sigma)\tilde{\eta}_a\right)\prod_{a=1}^{n}\delta^2\left(\sum_{\alpha=1}^{k}\rho_\alpha C^V_{\alpha a}(\xi,\sigma)-\lambda_a\right),\end{aligned} \tag{4.2.14}$$

where $C^V_{\alpha a}(\xi,\sigma)=\xi_a\sigma_a^{\alpha-1}$ and J_n' is the Jacobian obtained by solving the δ-functions with respect to the variables (ρ,σ,ξ), related to J_n as

$$J_n' = \frac{J_n}{(k-1)^n}\prod_{a=1}^{n}\xi_a^{\frac{3-k}{k-1}}. \tag{4.2.15}$$

In the following, we will use known results in $\mathcal{N}=4$ super Yang-Mills theory by Witten-RSV to verify that (4.2.12) indeed obeys parity invariance and reproduces the correct soft factor [74, 62].

In order to do so, it is instructive to review the Witten-RSV formulation of gauge theory amplitudes in terms of Witten's twistor string [23]. The super Yang-Mills n-point partial amplitudes in the k^{th} sector are given by

$$\begin{aligned}\mathcal{A}_{n,k}(1,\ldots,n) =& \frac{1}{\mathrm{Vol(GL(2))}}\frac{1}{(k-1)^n}\int\frac{d^n\sigma d^n\xi d^{2k}\rho}{\prod_{a=1}^{n}\xi_a(\sigma_a-\sigma_{a+1})}\prod_{\alpha=1}^{k}\delta^2\left(\sum_{a=1}^{n}C^V_{\alpha a}(\xi,\sigma)\tilde{\lambda}_a\right)\\ &\times\delta^{0|8}\left(\sum_{a=1}^{n}C^V_{\alpha a}(\xi,\sigma)\tilde{\eta}_a\right)\prod_{a=1}^{n}\delta^2\left(\sum_{\alpha=1}^{k}\rho_\alpha C^V_{\alpha a}(\xi,\sigma)-\lambda_a\right).\end{aligned} \tag{4.2.16}$$

Once again, this integral is completely localized and the amplitude is given by

$$\mathcal{A}_{n,k}(1,\ldots,n) = \frac{\delta^4\left(\sum_{a=1}^{n} p_a\right)}{(k-1)^n} \sum_{\substack{\text{Solutions}\\(\rho^*,\sigma^*,\xi^*)}} \frac{1}{\prod_{a=1}^{n}\xi_a(\sigma_a-\sigma_{a+1})} \frac{1}{J'_n} \prod_{\alpha=1}^{k} \delta^{0|4}\left(\sum_{a=1}^{n} C^V_{\alpha a}(\xi,\sigma)\tilde{\eta}_a\right)\Bigg|_{(\rho^*,\sigma^*,\xi^*)}$$
$$= \delta^4\left(\sum_{a=1}^{n} p_a\right) A_{n,k}(1,\ldots,n), \tag{4.2.17}$$

with J' being the Jacobian obtained by solving equations (5.4.8) with respect to the variables (ρ,σ,ξ).

4.3 Parity Invariance

In this section we check the parity symmetry of the Cachazo-Geyer formula. Given a scattering amplitude, the parity conjugated one is obtained by swapping $k \leftrightarrow n-k$, $\lambda \leftrightarrow \tilde{\lambda}$ and Fourier transforming the Grassmann supersymmetric parameters $\tilde{\eta} \leftrightarrow \eta$. Being more explicit, this statement for gravity reads

$$\mathcal{M}_{n,k}(\lambda,\tilde{\lambda},\tilde{\eta}) = \int d^{8n}\eta \exp\left(i \sum_{\substack{a=1\\A=1,\ldots,8}}^{n} \tilde{\eta}^A_a \eta^A_a\right) \widetilde{\mathcal{M}}_{k,n-k}(\tilde{\lambda},\lambda,\eta). \tag{4.3.1}$$

We start from the result stated by RSV that for each solution (ρ^*,σ^*,ξ^*) of the (n,k) system of equations (5.4.8) there corresponds a solution $(\tilde{\rho},\tilde{\sigma},\tilde{\xi})$ of the $(n,n-k)$ system of equations of the parity conjugated amplitude

$$\sum_{\beta=1}^{n-k} \tilde{\rho}_\beta \tilde{C}^V_{\beta a}(\tilde{\sigma}) = \tilde{\lambda}_a, \qquad \sum_{a=1}^{n} \tilde{C}^V_{\beta a}(\tilde{\sigma})\lambda_a = 0, \tag{4.3.2}$$

where $\tilde{C}^V(\tilde{\sigma})$ is an $(n-k)\times n$ matrix also obtained via the Veronese map (4.2.3) from some $2\times n$ matrix $\tilde{\Sigma}$. The conjugated solutions are obtained by performing the change of variables

$$\tilde{\sigma}_a = \sigma_a, \qquad \tilde{\xi}_a = \frac{1}{\xi_a \prod_{b\neq a}^{n}(\sigma_a - \sigma_b)}. \tag{4.3.3}$$

From now on we denote $\sigma_{ab} \equiv \sigma_a - \sigma_b$. We want to show that each individual solution is parity invariant, that is

$$\frac{H_n}{J_n^2}\prod_{\alpha=1}^{k}\delta^{0|8}\left(\sum_{a=1}^{n} C^V_{\alpha a}(\sigma)\tilde{\eta}_a\right)\Bigg|_{(\rho^*,\sigma_1^*,\sigma_2^*)} = \int d^{8n}\eta e^{i\eta_a\tilde{\eta}^a}\frac{\tilde{H}_n}{\tilde{J}_n^2}\prod_{\beta=1}^{n-k}\delta^{0|8}\left(\sum_{a=1}^{n}\tilde{C}^V_{\beta a}\eta_a(\tilde{\sigma})\right)\Bigg|_{(\tilde{\sigma}_1^*,\tilde{\sigma}_2^*,\tilde{\rho}^*)}, \tag{4.3.4}$$

where $(\tilde{\sigma}_1^*, \tilde{\sigma}_2^*, \tilde{\rho}^*)$ are the solutions of (4.3.2).

In super Yang-Mills, the correspondence between individual solutions leads to the identity

$$\frac{\prod_{\alpha=1}^{k}\delta^{0|4}\left(\sum_{a=1}^{n} C^V_{\alpha a}(\sigma)\tilde{\eta}_a\right)}{(k-1)^n J' \prod_{a=1}^{n}\xi_a\sigma_{a+1\,a}}\Bigg|_{(\rho^*,\sigma^*,\xi^*)} = \int d^{4n}\eta e^{i\eta_a\tilde{\eta}^a}\frac{\prod_{\beta=1}^{n-k}\delta^{0|4}\left(\sum_{a=1}^{n}\tilde{C}^V_{\beta a}(\tilde{\sigma})\eta_a\right)}{(n-k-1)^n\tilde{J}'\prod_{a=1}^{n}\tilde{\xi}_a\tilde{\sigma}_{a+1\,a}}\Bigg|_{(\tilde{\rho}^*,\tilde{\sigma}^*,\tilde{\xi}^*)}. \tag{4.3.5}$$

We see immediately that the factor $\prod_{a=1}^n \sigma_{a+1\,a} = \prod_{a=1}^n \tilde{\sigma}_{a+1\,a}$ cancels in both sides as a consequence of (4.3.3) and the equality between each solution. This identity when written in terms of the variables $(\rho, \sigma_1, \sigma_2)$ as in (4.2.1) reads

$$\begin{gathered}\prod_{\alpha=1}^{k}\delta^{0|4}\left(\sum_{a=1}^{n} C^V_{\alpha a}(\sigma)\tilde{\eta}_a\right)\prod_{a=1}^{n}\xi_a^{-\frac{2}{k-1}}J^{-1}\Bigg|_{(\rho^*,\sigma_1^*,\sigma_2^*)} = \\ \int d^{4n}\eta e^{i\eta_a\tilde{\eta}^a}\prod_{\beta=1}^{n-k}\delta^{0|4}\left(\sum_{a=1}^{n}\tilde{C}^V_{\beta a}(\tilde{\sigma})\eta_a\right)\prod_{a=1}^{n}\tilde{\xi}_a^{-\frac{2}{n-k-1}}\tilde{J}^{-1}\Bigg|_{(\tilde{\rho}^*,\tilde{\sigma}_1^*,\tilde{\sigma}_2^*)}.\end{gathered} \tag{4.3.6}$$

The idea is to split the fermionic $\delta^{0|8}$ from (4.3.4) into two copies of $\delta^{0|4}$, and then apply (4.3.6) twice. Now it is left to work out what H_n is in terms of the coordinates (ξ, σ). Each 2×2 minor is given by $(i\, j) = (\xi_i \xi_j)^{\frac{1}{k-1}} \sigma_{ji}$, thus we can factorize the ξ dependence of the matrix $\Phi_n(\xi, \sigma)$ and write instead $\Phi'_n(\sigma)$:

$$(\Phi_n)_{ab} = (\xi_a \xi_b)^{-\frac{2}{k-1}} \frac{s_{ab}}{(\sigma_{ab})^2} = (\xi_a \xi_b)^{-\frac{2}{k-1}} (\Phi'_n)_{ab}(\sigma), \qquad \text{for } a \neq b \tag{4.3.7}$$

$$(\Phi_n)_{aa} = -\xi_a^{-\frac{4}{k-1}} \sum_{b \neq a} \frac{s_{ab}}{(\sigma_{ab})^2} \frac{\sigma_{lb}\sigma_{rb}}{\sigma_{la}\sigma_{ra}} = \xi_a^{-\frac{4}{k-1}} (\Phi'_n)_{aa}(\sigma). \tag{4.3.8}$$

With this, H_n is given by

$$\begin{aligned} H_n =& (-1)^{n+1} \prod_{i=1}^{n} \frac{\xi_i^{-\frac{4}{k-1}}}{(\xi_a \xi_b \ldots \xi_f)^{\frac{2}{k-1}}} \frac{1}{(\xi_a \xi_b \ldots \xi_f)^{-\frac{2}{k-1}} (\sigma_{ba} \ldots \sigma_{df})} |\Phi'_n(\sigma)^{(abc)}_{(def)}| \\ =& (-1)^{n+1} \prod_{i=1}^{n} \xi_i^{-\frac{4}{k-1}} \frac{|\Phi'_n(\sigma)^{(abc)}_{(def)}|}{(\sigma_{ba} \ldots \sigma_{df})}. \end{aligned} \tag{4.3.9}$$

Splitting $\tilde{\eta}^A,\ A = 1, \ldots, 8$, into two Grassmann parameters $\tilde{\eta}^A_{\mathrm{L}}, \tilde{\eta}^A_{\mathrm{R}},\ A = 1, \ldots, 4$, we can write

$$\begin{aligned} \frac{H_n}{J_n^2} \prod_{\alpha=1}^{k} \delta^{0|8} \left(\sum_{a=1}^{n} C^V_{\alpha a}(\sigma) \tilde{\eta}_a \right) \Bigg|_{(\rho^*, \sigma_1^*, \sigma_2^*)} = H_n \prod_{\alpha=1}^{k} \left(\frac{\delta^{0|4} \left(\sum_{a=1}^{n} C^V_{\alpha a}(\sigma) \tilde{\eta}^a_{\mathrm{L}} \right)}{J_n} \right) \times \\ \left(\frac{\delta^{0|4} \left(\sum_{a=1}^{n} C^V_{\alpha a}(\sigma) \tilde{\eta}^a_{\mathrm{R}} \right)}{J_n} \right) \Bigg|_{(\rho^*, \sigma_1^*, \sigma_2^*)} . \end{aligned} \tag{4.3.10}$$

Then, using (4.3.6) twice and merging the two Grassmann $\eta^A_{\mathrm{L,R}}$, $A = 1, \ldots, 4$, conjugated to $\tilde{\eta}^A_{L,R}$, into an 8-component η^A, we get

$$\frac{H_n}{J_n^2} \prod_{\alpha=1}^{k} \delta^{0|8} \left(\sum_{a=1}^{n} C^V_{\alpha a}(\sigma) \tilde{\eta}_a \right) \Bigg|_{(\rho^*, \sigma_1^*, \sigma_2^*)} = \int d^{8n} \eta e^{i\eta_a \tilde{\eta}^a} \frac{\tilde{H}_n}{\tilde{J}_n^2} \prod_{\beta=1}^{n-k} \delta^{0|8} \left(\sum_{a=1}^{n} \tilde{C}^V_{\beta a}(\tilde{\sigma}) \eta_a \right) \Bigg|_{(\tilde{\rho}^*, \tilde{\sigma}_1^*, \tilde{\sigma}_2^*)}, \tag{4.3.11}$$

with

$$\tilde{H}_n = - \prod_{a=1}^{n} \tilde{\xi}_a^{- \frac{2}{n-k-1}} \frac{|\Phi'_n(\sigma)^{(abc)}_{(def)}|}{(\sigma_{ba} \ldots \sigma_{df})}. \tag{4.3.12}$$

We conclude that each solution is invariant under parity transformation. This implies that the whole amplitude satisfies (4.3.1).

4.4 Soft Graviton Limit

In this section, we show that the Cachazo-Geyer formula reproduces the correct soft factor for gravity amplitudes. In order to do so, we first recall to the reader the soft limit for super Yang-Mills amplitudes: if particle 1 has positive helicity and we take its momentum to zero, the amplitude factorizes in the following way:

$$A_n \xrightarrow{p_1 \to 0} \frac{\langle 2\, n \rangle}{\langle n\, 1 \rangle \langle 1\, 2 \rangle} A_{n-1}. \tag{4.4.1}$$

If particle 1 has negative helicity, we simply conjugate the soft factor.

We start from the RSV formula (4.2.17) in terms of the variables $(\rho, \sigma_1, \sigma_2)$:

$$A_{n,k} = \sum_{\substack{\text{Solutions} \\ (\rho^*, \sigma_1^*, \sigma_2^*)}} \prod_{a=1}^{n} \frac{\xi_a^{-\frac{2}{k-1}}}{\sigma_{a+1\, a}} \frac{1}{J_n} \prod_{\alpha=1}^{k} \delta^{0|4} \left(\sum_{a=1}^{n} C^V_{\alpha a}(\sigma) \tilde{\eta}_a \right) \Bigg|_{(\rho^*, \sigma_1^*, \sigma_2^*)}. \tag{4.4.2}$$

Under the soft limit $(\lambda_1, \tilde{\lambda}_1) \to (0,0)$ and $\tilde{\eta}_1 = 0$, J_n factorizes into

$$J_n = J_{n-1} D, \tag{4.4.3}$$

with D being a 2×2 matrix that carries all dependence on the soft particle. To see this, let us recall from (4.2.10) the definition $J_n = \frac{1}{(a\,b)^2[c\,d]^2}|K_{n(cd)}^{(ab)}|$ and look at the matrix K_n under such limit. If we arrange the columns and rows in order to put the dependence on the soft particle on the last two ones, then

$$K_n = \begin{pmatrix} K_{n-1} & \vdots \\ A & D \end{pmatrix}, \tag{4.4.4}$$

where A is a $2 \times 2(k+n-1)$ matrix in which all non-zero entries are of the kind $\frac{\partial E_{\alpha\underline{\dot{\alpha}}}}{\partial(\sigma_{11},\sigma_{21})} \propto \tilde{\lambda}_1^{\dot{\alpha}}$. Therefore, in the soft limit all its entries become zero and

$$\det K_n = \det K_{n-1} \det D. \tag{4.4.5}$$

Choosing the deleted rows and columns $\{a, b, c, d\} \neq 1$, the factorization of $|K_{n(cd)}^{(ab)}|$ translates into the factorization (4.4.3) of J_n.

We extract the factors depending on the soft particle, and use the soft limit of the RSV formula (4.2.17)

$$A_{n,k} = \sum_{\substack{\text{Solutions} \\ (\rho^*,\sigma^*,\xi^*)}} \xi_1^{-\frac{2}{k-1}} \frac{\sigma_{2n}}{\sigma_{1n}\sigma_{21}} \prod_{a=2}^{n} \frac{1}{\xi_a(\sigma_{a+1a})} \frac{\prod_{\alpha=1}^{k} \delta^{0|4}\left(\sum_{a=1}^{n} C_{\alpha a}(\sigma,\xi)\tilde{\eta}_a\right)}{J_{n-1}} \frac{1}{D}\Bigg|_{(\rho^*,\sigma^*,\xi^*)}, \tag{4.4.6}$$

where we multiplied and divided by σ_{2n}[1] in order to obtain the correct measure for A_{n-1}.

[1]Not to be confused with the elements of the $2 \times n$ matrix Σ of (4.2.2).

Comparing (4.4.6) with (4.4.1), we can find the factor D at each solution

$$\frac{1}{D} = \frac{\langle 2\, n\rangle}{\langle 1\, 2\rangle\langle n\, 1\rangle} \times \frac{\xi_1^{\frac{2}{k-1}}\sigma_{n1}\sigma_{12}}{\sigma_{n2}}\Bigg|_{(\rho^*,\sigma^*,\xi^*)} = \frac{\langle 2\, n\rangle}{(2\, n)}\frac{(1\, 2)}{\langle 1\, 2\rangle}\frac{(n\, 1)}{\langle n\, 1\rangle}\Bigg|_{(\rho^*,\sigma_1^*,\sigma_2^*)}, \tag{4.4.7}$$

which will be useful below in the calculation of the soft limit. It is crucial to notice that D does not know anything about the ordering of the particles, so in (4.4.7) we can replace 2 and n by any other two particles. We will now make use of these facts to calculate the soft limit in gravity. Recall the Cachazo-Geyer formula (4.2.12)

$$M_{n,k} = \sum_{\substack{\text{Solutions}\\(\rho^*,\sigma_1^*,\sigma_2^*)}} \frac{H_n}{J_n^2}\prod_{\alpha=1}^{k}\delta^{0|8}\left(\sum_{a=1}^{n} C_{\alpha a}^{V}(\sigma)\tilde{\eta}_a\right)\Bigg|_{(\rho^*,\sigma_1^*,\sigma_2^*)}. \tag{4.4.8}$$

In the soft limit H_n also factorizes as

$$H_n = H_{n-1}\sum_{i=2}^{n}\frac{s_{1i}(i\, l)(i\, r)}{(a\, i)^2(a\, l)(a\, r)}. \tag{4.4.9}$$

To see this, let us recall from (4.3.9)

$$H_n = (-1)^{n+1}\prod_{i=1}^{n}\xi_i^{-\frac{4}{k-1}}\frac{|\Phi_n'(\sigma)_{(def)}^{(abc)}|}{(\sigma_{ba}\ldots\sigma_{df})}.$$

The determinant $|\Phi_n'(\sigma)_{(def)}^{(abc)}|$ in the limit $(\lambda_1,\tilde{\lambda}_1)\to(0,0)$ can be approximated as

$$|\Phi_n'(\sigma)_{(def)}^{(abc)}| = \left|\begin{array}{c|ccc} -\sum_{i=2}^{n}\frac{s_{1i}}{(\sigma_{1\,i})^2}\frac{\sigma_{l\,i}\sigma_{r\,i}}{\sigma_{l\,1}\sigma_{r\,1}} & \frac{s_{12}}{(12)^2} & \cdots & \frac{s_{1n}}{(1\,n)^2} \\ \hline \frac{s_{21}}{(21)^2} & & & \\ \vdots & & \Phi_{n-1}'(\sigma)_{(def)}^{(abc)} & \\ \frac{s_{n1}}{(n\,1)^2} & & & \end{array}\right| \approx -\sum_{i=2}^{n}\frac{s_{1i}}{(\sigma_{1\,i})^2}\frac{\sigma_{l\,i}\sigma_{r\,i}}{\sigma_{l\,1}\sigma_{r\,1}}|\Phi_{n-1}'(\sigma)_{(def)}^{(abc)}| \tag{4.4.10}$$

by expanding in the first row and keeping only the first order in s_{1i}, which is small in this limit. Here we assumed that the set of three rows and three columns that are deleted from Φ_n do not contain the first row or first column. The σ_{ij} of (4.4.10) combine with the ξ dependence of (4.3.9) to give (4.4.9). Thus, in the soft limit, the gravity amplitude takes the form

$$M_{n,k} = \sum_{\substack{\text{Solutions}\\(\rho^*,\sigma_1^*,\sigma_2^*)}} \sum_{c=2}^{n} \frac{s_{1c}(c\,l)(c\,r)}{(1\,c)^2(1\,l)(1\,r)} \frac{H_{n-1}}{J_{n-1}^2} \frac{1}{D^2}\Bigg|_{(\rho^*,\sigma_1^*,\sigma_2^*)}, \tag{4.4.11}$$

where the factor H_{n-1}/J_{n-1}^2 gives the lower-point amplitude $M_{n-1,k}$.

Now we can use the expression (4.4.7) for D with convenient replacements for the labels 2 and n. There are two copies of D in the formula, so choosing for the first one $\{2,n\} \to \{c,l\}$ and for the second one $\{2,n\} \to \{c,r\}$, we obtain

$$\begin{aligned} M_{n,k} &= \sum_{\substack{\text{Solutions}\\(\rho^*,\sigma_1^*,\sigma_2^*)}} \sum_{c=2}^{n} \frac{s_{1c}(c\,l)(c\,r)}{(1\,c)^2(1\,l)(1\,r)} \frac{H_{n-1}}{J_{n-1}^2} \left(\frac{\langle c\,l\rangle}{(c\,l)}\frac{(1\,c)}{\langle 1\,c\rangle}\frac{(l\,1)}{\langle l\,1\rangle}\right)\left(\frac{\langle c\,r\rangle}{(c\,r)}\frac{(1\,c)}{\langle 1\,c\rangle}\frac{(r\,1)}{\langle r\,1\rangle}\right)\Bigg|_{(\rho^*,\sigma_1^*,\sigma_2^*)} \\ &= \sum_{\substack{\text{Solutions}\\(\rho^*,\sigma_1^*,\sigma_2^*)}} \sum_{c=2}^{n} \frac{[1c]\langle cl\rangle\langle cr\rangle}{\langle 1c\rangle\langle 1l\rangle\langle 1r\rangle} \times M_{n-1,k}(2,\ldots,n)\Bigg|_{(\rho^*,\sigma_1^*,\sigma_2^*)}. \end{aligned} \tag{4.4.12}$$

Since for *every* solution we obtain a factor that depends only on the external data, the full amplitude obtains the same factor. In other words, we have obtained the well-known expression for the gravitational soft limit

$$\lim_{p_{1^+}\to 0} \mathcal{M}_{n,k}(1^+,2,\ldots,n) = \sum_{c=2}^{n} \frac{[1c]\langle cl\rangle\langle cr\rangle}{\langle 1c\rangle\langle 1l\rangle\langle 1r\rangle} \times \mathcal{M}_{n-1,k}(2,\ldots,n). \tag{4.4.13}$$

4.5 Calculating Gravity Amplitudes from super Yang-Mills Results

It can be noticed that in both proofs in the previous sections we avoided calculating the clumsiest part of the formula (4.2.1) — the Jacobian J_n. This was achieved by making use of the knowledge of the corresponding super Yang-Mills result. One can ask if it is possible to use the same trick also for computing amplitudes. It is trivially true for MHV, because the Jacobian is 1 in this case. In this section, we will show that it also works for $\overline{\text{MHV}}$ whose Jacobian is not trivial. Obviously, $\overline{\text{MHV}}$ amplitudes can be obtained from the Hodges' MHV formula [14] by parity conjugation. However, in order to illustrate using super Yang-Mills results for calculating gravity amplitudes, we will show explicitly how the formula (4.2.1) reproduces the $\overline{\text{MHV}}$ amplitudes.

According to Hodges, a reduced[2] tree-level MHV amplitude in $\mathcal{N} = 8$ supergravity is given by

$$\bar{M}(1,2,\ldots,n) = (-1)^{n+1}\sigma(ijk,rst)\frac{|\Phi^H|^{rst}_{ijk}}{\langle ij\rangle\langle jk\rangle\langle ki\rangle\langle rs\rangle\langle st\rangle\langle tr\rangle}, \qquad (4.5.1)$$

where

$$\sigma(ijk,rst) = \text{sgn}((ijk12\ldots \not{i}\not{j}\not{k}\ldots n) \to (rst12\ldots \not{r}\not{s}\not{t}\ldots n),$$

and $|\Phi^H|^{rst}_{ijk}$ is the $(n-3)\times(n-3)$ minor of the matrix

$$(\Phi^H)^i_j = \frac{[ij]}{\langle ij\rangle},\ i \neq j, \qquad (\Phi^H)^i_i = -\sum_{j\neq i}\frac{[ij]\langle jx\rangle\langle jy\rangle}{\langle ij\rangle\langle ix\rangle\langle iy\rangle}, \qquad (4.5.2)$$

obtained by deleting the columns r, s, t and rows i, j, k. Here x and y are two arbitrary spinors.

[2] Following Hodges, we call reduced amplitude an amplitude with stripped momentum conserving δ-function and Grassmannian δ-functions.

First let us recall that each of the integral formulas for super Yang-Mills (4.2.16) and gravity (4.2.1) can be written as a sum over solutions of the δ-functions in the integrand (4.2.12,4.2.17). In the MHV case the integral receives contribution only from one solution, which by using the GL(2) "gauge freedom" can be written as

$$\begin{pmatrix} \sigma_{1a} & \dots & \sigma_{1n} \\ \sigma_{2a} & \dots & \sigma_{2n} \end{pmatrix} = \begin{pmatrix} \lambda_1^a & \dots & \lambda_1^n \\ \lambda_2^a & \dots & \lambda_2^n \end{pmatrix}, \qquad \rho^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \rho^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{4.5.3}$$

or equivalently, in terms of (ρ, σ, ξ)

$$\begin{cases} \xi^a & = (\lambda_1^a)^{k-1} \\ \sigma^a & = \lambda_2^a / \lambda_1^a \\ \rho_\alpha^\beta & = \delta_\alpha^\beta \end{cases} . \tag{4.5.4}$$

Therefore, on this solution the minors $(\sigma_a \sigma_b)$ become inner products $\langle a\, b \rangle$ and the matrix Φ_n of (4.2.4) reduces to the Hodges' matrix Φ^H (4.5.2).

On the MHV solution (4.5.3), $J_n = 1$. Indeed, J_n appears as a determinant of resolving the δ-functions in (4.2.1), which in the MHV case takes the form

$$\int d^{2n}\sigma d^2\rho_1 d^2\rho_2\, \delta^2\left(\sum_{a=1}^{n} \sigma_{1a}\tilde{\lambda}_a\right) \delta^2\left(\sum_{a=1}^{n} \sigma_{2a}\tilde{\lambda}_a\right) \prod_{a=1}^{n} \delta^2\left(\rho_1\sigma_{1a} + \rho_2\sigma_{2a} - \lambda^a\right). \tag{4.5.5}$$

On the solution, $\lambda^a = \sigma^a$, so the first two δ-functions combine to the momentum conserving $\delta^4\left(\sum_{a=1}^{n} \lambda_{\underline{\alpha}}^a \tilde{\lambda}_{\underline{\dot{\alpha}}}^a\right)$. The last $2n$ δ-functions in (4.5.5) integrated over σ can be written in a form

$$(\det R)^{-n} \int d^{2n}\sigma \prod_{a=1}^{n} \delta^2(\sigma^a - R^{-1}\lambda^a) = (\det R)^{-n}, \qquad R = \begin{pmatrix} \rho_{\underline{1}}^1 & \rho_{\underline{1}}^2 \\ \rho_{\underline{2}}^1 & \rho_{\underline{2}}^2 \end{pmatrix}. \tag{4.5.6}$$

But on the MHV solution, R is equal to the identity matrix, so no factor is produced in (4.5.5) and $J_n = 1$.

Substituting $J_n = 1$ into the Cachazo-Geyer formula (4.2.12) and pulling out the momentum conserving and Grassmannian δ-functions, we see that the reduced amplitude is equal to H_n. As (4.2.4) reduces to the Hodges' matrix (4.5.2), we conclude that the Cachazo-Geyer formula (4.2.1) reproduces the Hodges' formula (4.5.1) in the MHV case.

Now we consider $\overline{\text{MHV}}$ amplitudes. First let us understand how (4.2.16) reproduces the well-known Parke-Taylor formula for MHV amplitudes in super Yang-Mills

$$A_{\text{MHV}} = \frac{\prod_{\alpha=1}^{2} \delta^{0|4}\left(\sum_{a=1}^{n} C_{\alpha a}(\sigma)\tilde{\eta}_a\right)}{J_n \prod_{a=1}^{n}(\sigma_a \sigma_{a+1})} = \frac{\delta^{0|4}\left(\sum_{a=1}^{n} \lambda_a^1 \tilde{\eta}^a\right)\delta^{0|4}\left(\sum_{a=1}^{n} \lambda_a^2 \tilde{\eta}^a\right)}{\langle 12\rangle \ldots \langle n1\rangle}. \tag{4.5.7}$$

The corresponding formula for $\overline{\text{MHV}}$ can be obtained from MHV by parity conjugation. The only solution contributing to the amplitude is

$$A_{\overline{\text{MHV}}} = \frac{\prod_{\beta=1}^{n-2} \delta^{0|4}\left(\sum_{a=1}^{n} \tilde{C}_{\beta a}(\tilde{\sigma})\tilde{\eta}_a\right)}{\tilde{J}_n \prod_{a=1}^{n}(\tilde{\sigma}_a \tilde{\sigma}_{a+1})} = \int d^{4n}\eta \, e^{i\eta_a \tilde{\eta}^a} \frac{\delta^{0|4}\left(\sum_{a=1}^{n} \tilde{\lambda}_a^1 \eta^a\right)\delta^{0|4}\left(\sum_{a=1}^{n} \tilde{\lambda}_a^2 \eta^a\right)}{[12]\ldots[n1]}. \tag{4.5.8}$$

In order to make it clear that this is the conjugate of an MHV amplitude, we wrote the LHS in terms of the $(n-2)\times n$ matrix $\tilde{C}_{\beta a} = \tilde{\xi}_a \tilde{\sigma}_a^{\beta-1}$ where $\begin{pmatrix} \tilde{\sigma}_1 \\ \tilde{\sigma}_2 \end{pmatrix} = \tilde{\xi}^{\frac{1}{n-3}} \begin{pmatrix} 1 \\ \tilde{\sigma} \end{pmatrix}$ is a solution of the parity conjugated system (4.3.2).

Through the transformation (4.3.3), $\tilde{\sigma}$ and $\tilde{\xi}$ are related to ξ and σ, which in turn can be expressed through the antiholomorphic part of kinematical data

$$\begin{aligned} \xi^a &= (\tilde{\lambda}_1^a)^{n-3}, \\ \sigma^a &= \frac{\tilde{\lambda}_2^a}{\tilde{\lambda}_1^a}. \end{aligned} \tag{4.5.9}$$

We will use our knowledge of the $\overline{\text{MHV}}$ amplitudes in super Yang-Mills to calculate them in gravity. As in the previous sections, we represent the formula (4.2.1) for gravity amplitudes as a sum over solutions. Similar to the MHV case, there is only one solution

$$M_{n,n-2} = \frac{H_n}{J_n^2} \prod_{\beta=1}^{n-2} \delta^{0|8} \left(\sum_{a=1}^{n} C_{\beta a}(\sigma) \tilde{\eta}^a \right) \Bigg|_{(\rho^*,\sigma_1^*,\sigma_2^*)} . \tag{4.5.10}$$

We split the 8-component fermionic δ-function into two 4-component ones, group each of them with one copy of $1/J_n$ and use the super Yang-Mills result (4.5.8). The two fermionic integrals over $d^{4n}\eta$ can be merged into one over $d^{8n}\eta$

$$M_{n,n-2} = H_n \left(\frac{(\tilde{\sigma}_1\tilde{\sigma}_2)\dots(\tilde{\sigma}_n\tilde{\sigma}_1)}{[12]\dots[n1]} \right)^2 \int d^{8n}\eta e^{i\eta_a\tilde{\eta}^a} \prod_{\underline{\dot{\alpha}}=1}^{2} \delta^{0|8} \left(\sum_{a=1}^{n} \tilde{\lambda}^a_{\underline{\dot{\alpha}}} \eta_a \right) . \tag{4.5.11}$$

Under the transformation (4.3.3), σ does not change while ξ does, so it makes sense to extract the ξ-dependence from H_n, as in (4.3.9):

$$H_n = \prod_{a=1}^{n} \tilde{\xi}_a^{-\frac{4}{n-k-1}} H_n', \tag{4.5.12}$$

where H_n' depends only on σ and thus does not change under the transformation (4.3.3). We can also extract the ξ-dependent factors from the minors

$$\frac{(\tilde{\sigma}_1\tilde{\sigma}_2)\dots(\tilde{\sigma}_n\tilde{\sigma}_1)}{[12]\dots[n1]} = \frac{\prod_{a=1}^{n} \tilde{\xi}_a^{\frac{2}{n-k-1}} \tilde{\sigma}_{a+1\,1}}{\prod_{a=1}^{n} \xi_a^{\frac{2}{k-1}} \sigma_{i+1\,1}} = \prod_{a=1}^{n} \frac{\tilde{\xi}_a^{\frac{4}{n-k-1}}}{\xi_a^{\frac{4}{k-1}}}. \tag{4.5.13}$$

Here we used the fact that

$$(\xi_a\xi_b)^{\frac{1}{k-1}} (\sigma_a - \sigma_b) = [a\, b]. \tag{4.5.14}$$

Substituting (4.5.12) and (4.5.13) into (4.5.11), we see that the $\tilde{\xi}$'s cancel

$$M_{n,n-2} = H'_n \prod_{a=1}^{n} \xi_a^{-\frac{4}{k-1}} \int d^{8n}\eta e^{i\eta_a \tilde{\eta}^a} \prod_{\underline{\dot{\alpha}}=1}^{2} \delta^{0|8}\left(\sum_{a=1}^{n} \tilde{\lambda}^a_{\underline{\dot{\alpha}}}\eta_a\right). \quad (4.5.15)$$

The ξ factor can be now absorbed into H'_n by defining

$$H_n^{\text{conj}} = H'_n \prod_{a=1}^{n} \xi_a^{-\frac{4}{k-1}}.$$

Then, H_n^{conj} can be calculated as a determinant of the matrix Φ_{conj}

$$\Phi_{ij}^{\text{conj}} = \frac{\langle i\, j\rangle}{[i\, j]}, i \neq j \qquad\qquad \Phi_{ii}^{\text{conj}} = -\sum_{j} \frac{\langle i\, j\rangle [j\, l][j\, r]}{[i\, j][i\, l][i\, r]}, \quad (4.5.16)$$

with three rows and three columns eliminated. Here we used again (4.5.14).

Finally, we obtain the following formula for $\overline{\text{MHV}}$ gravity amplitudes:

$$M_{n,n-2} = H_n^{\text{conj}} \int d^{8n}\eta e^{i\eta_a\tilde{\eta}^a} \delta^{0|4}\left(\sum_{a=1}^{n} \tilde{\lambda}^1_a \eta_a\right) \delta^{0|4}\left(\sum_{a=1}^{n} \tilde{\lambda}^2_a \eta_a\right). \quad (4.5.17)$$

Taking the fermionic integrations explicitly, we can represent the answer in the final form

$$M_{n,n-2} = H_n^{\text{conj}} \delta^{0|8}\left(\sum_{a\neq b}^{n} [a\, b] \prod_{c\neq a,b}^{n} \tilde{\eta}_c\right), \quad (4.5.18)$$

which one can check that is the parity conjugate of Hodges' formula for MHV amplitudes [14].

We can conclude that in the MHV and $\overline{\text{MHV}}$ cases there is no need to explicitly calculate the Jacobian J_n in the Cachazo-Geyer formula (4.2.1), because in the gravity $\overline{\text{MHV}}$ amplitude, J_n can be extracted from the super Yang-Mills counterpart. This is a surprising fact, since unlike the MHV case, in which $J_n = 1$, in the $\overline{\text{MHV}}$ case J_n is nontrivial, nevertheless this trick allows us to avoid computing it explicitly. This simplification hints that there may be a possibility that the calculation of amplitudes with arbitrary k does not require the computation of J_n explicitly, taking instead advantage of the corresponding super Yang-Mills result. Therefore, this could be a path for further simplifications of the formula (4.2.1).

4.6 Conclusion

We have proved in this chapter that the recently proposed Cachazo-Geyer formula (4.2.1) for all tree-level amplitudes in $\mathcal{N} = 8$ supergravity satisfies parity symmetry and behaves correctly in a soft-graviton limit. These properties provide evidence for the validity of the formula (4.2.1) in all k-sectors. Indeed, a k-preserving soft limit produces a lower-point amplitude with the same k. So, iteratively performing the k-preserving soft limit, each amplitude can be reduced to $\overline{\text{MHV}}$, which in turn can be related to MHV by parity conjugation. Thus, the consistency checks which we performed support the validity of the Cachazo-Geyer proposal.

Chapter 5

CSW-like Expansion for Einstein Gravity

This chapter continues the discussion of the CSW expansion for gravity amplitudes from a different perspective. Using the recent formula presented in [57, 58] for the link representation of tree-level $\mathcal{N} = 8$ supergravity amplitudes, we derive a CSW-like expansion for the Next-to-MHV 6- and 7-graviton amplitudes by using the global residue theorem; a technique introduced originally for Yang-Mills amplitudes [33]. We analytically check the equivalence of one of the CSW terms and its corresponding Risager's diagram. For the remaining 6-graviton and all 7-graviton terms, we numerically checked the agreement with Risager's expansion. We show that the conditions for the absence of contributions at infinity of the global residue theorem are satisfied for any number of particles. This means that our technique and Risager's should disagree starting at twelve particles where Risager's method is known to fail.

5.1 Introduction

As was shown in [23], Yang-Mills MHV amplitudes are localized on lines in supertwistor space. The CSW expansion in Yang-Mills tells us that NMHV amplitudes are localized on pairs of lines in twistor space. Gravity MHV amplitudes, on the other hand, are not localized on lines because they have derivative of a δ-function support on lines. However, in the NMHV sector, one can write amplitudes as integrals over the Grassmannian $G(3,n)$ which is the space of 3-planes in $\mathbb{C}^n$ [58]. A convenient way to represent a point in $G(3,n)$ is as a $3 \times n$ matrix modulo a $GL(3)$ action. This matrix can be thought of as n 3-vectors which are the columns of the matrix. The columns are denoted by $c_a \in \mathbb{C}^3$, or equivalently $[c_a] \sim \mathbb{CP}^2$ as they are non-zero. We call the space of 3-vectors the C-space and will discuss localizations in this space throughout this chapter.

In Yang-Mills, NMHV amplitudes are localized on pairs of lines in the C-space. Hence, localization in twistor space and $\mathbb{CP}^2$ are equivalent for Yang-Mills. In general we define CSW-localization to mean that NMHV amplitudes are localized only on pairs of lines in the C-space.

The main result of this chapter is to show that gravity has an expansion in which each term is CSW-like localized. The key ingredient in our technique is the link representation [57, 58] of the recently found formula for gravity amplitudes [16].

In [16], Cachazo and Skinner proposed a manifestly permutation invariant formula to compute all tree-level $\mathcal{N} = 8$ supergravity amplitudes in terms of higher degree rational maps to twistor space. The parity invariance and soft limits of the formula, two strong evidences for the proposal to be correct, were checked in [75]. This conjecture was later on proved in [58], showing that the formula admits the BCFW recursion relations and also produces 3-particle MHV and $\overline{\text{MHV}}$ amplitudes. Moreover, the formula behaves at large momenta (large BCFW's complex variable, z) as tree-level gravity amplitudes do, $1/z^2$ [46]. In [57] and independently in the same work of Cachazo-Mason-Skinner [58] the formula was presented in the *link representation*. This formulation was indeed another progress to further explore CSW in gravity. The link representation is reviewed in section 5.2.

In this chapter, we are looking for a formula for gravity analogous to the CSW. The analogy is in the C-space, not directly in twistor space. The procedure of our computations is explained in section 5.3 where we applied a global residue theorem (GRT)[76] to the link representation of the Cachazo-Skinner formula. The residue theorem writes the amplitude as a sum over terms which coincide with different localizations of particles on a pair of lines in the C-space. These are in fact the CSW terms for NMHV gravity amplitudes.

We computed the CSW terms for the NMHV 6- and 7-graviton scattering amplitudes in sections 5.4 and 5.5, respectively. The results obtained are compared to the known corresponding Risager's diagrams. We show that the two sides are in complete agreement. Both Yang-Mills and gravity have an expansion in terms of two lines in the C-space. Throughout this chapter, we graphically represent each term in Risager's expansion for gravity

$$\mathcal{M}_n^{\mathrm{Ris}} = \sum_{a,L^+} \mathcal{M}_L\left(\hat{a}^-, L^+, \left(-\hat{P}\right)^-\right) \frac{1}{(p_a + P_{L^+})^2} \mathcal{M}_R\left(\hat{P}^+, \hat{b}^-, \hat{c}^-, R^+\right), \tag{5.1.1}$$

as two intersecting lines shown in figure 5.1.

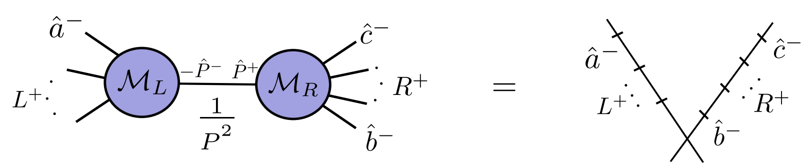

Figure 5.1: The CSW-like localization of NMHV amplitudes on lines in C-space.

The notations and conventions are as introduced and used before.

For 6- and 7-graviton amplitudes, there are 21 and 45 Risager diagrams respectively, which are the same as the localizations of the Cachazo-Skinner formula. This formula however, does not hold us back from exceeding 12-graviton amplitudes. Indeed, a power counting in section 5.6 shows that the global residue theorem is valid for any number of particles. In section 5.7 we discuss a possible reason for the discrepancy between Risager's and our technique.

5.2 Review of the Link Representation of Gravity in Momentum Space

Recently, a new formula for the tree-level S-matrix of $\mathcal{N} = 8$ supergravity for all R-charge sectors (labeled by $d = k - 1$) was proposed by Cachazo and Skinner [16]. The formula is written as an integral over degree d holomorphic maps from a Riemann sphere to $\mathcal{N} = 8$ supertwistor space $\mathbb{CP}^{3|8}$. The integral, however, is known to be completely localized by its bosonic δ-functions; it is therefore the analog of the twistor string formulation for $\mathcal{N} = 4$ super Yang-Mills tree amplitudes proposed by Witten [23] and later studied by Roiban, Spradlin and Volovich [56].

In twistor string formulation, world-sheet and external variables are connected through high degree polynomials inside δ-functions. To extract the result for the amplitude, one has to solve these high degree polynomials and sum the contribution of all solutions. We found that it is more convenient for our purpose to use the so-called *link representation* of gravity, written by He [57], Cachazo, Mason and Skinner [58], which is a "gauge fixed" Grassmannian $G(k, n)$ formulation. In the link representation, internal and external variables are linearly coupled in the δ-function constraints which makes our computations simpler to perform, but instead, the amplitude is now a contour integral in $(d-1)(\tilde{d}-1)$ variables, where $\tilde{d} = n - d - 2$.

For amplitudes having only gravitons as incoming particles, we choose the gauge such that the k columns associated with the negative helicity gravitons form an identity matrix. We label the gauge fixed columns by r and the remaining $n - k$ columns by a. Since the row index runs from 1 to k, we label them using r as well. The unfixed variables c_{ra} are called *link variables.*

In this chapter we concentrate on the NMHV sector, $d = 2$. We assign the negative helicity indices r, s and t to take values 1, 2, 3 and the positive ones a and b to run over $4, \ldots, n$. Therefore, the gauge will be

$$\begin{pmatrix} 1 & 0 & 0 & c_{14} & c_{15} & \ldots & c_{1n} \\ 0 & 1 & 0 & c_{24} & c_{25} & \ldots & c_{2n} \\ 0 & 0 & 1 & c_{34} & c_{35} & \ldots & c_{3n} \end{pmatrix}. \tag{5.2.1}$$

Minors of this matrix are linear in each of the link variables. Using this gauge and keeping track of corresponding signs (cyclicity),

$$\begin{aligned} c_{ra} &= (sta), \\ (rab) &= c_{sa}c_{tb} - c_{sb}c_{ta}, \end{aligned} \tag{5.2.2}$$

where $s \neq t \neq r$. For instance, $c_{14} = (234)$ and $(356) = c_{15}c_{26} - c_{16}c_{25}$.

Our computations are done using the link representation and therefore, instead of presenting the twistor string formula [16], let us directly write the n-particle N^{d-1}MHV link representation in momentum space as introduced in [58]. Using the gauge above and setting $d = 2$ and $\tilde{d} = n - 4$, the link representation for $\mathcal{M}(1^-, 2^-, 3^-, 4^+, \cdots, n^+)$ is[1]

$$\begin{aligned} \mathcal{M}_{n,2} = \int \prod_{r,a} \frac{dc_{ra}}{c_{ra}} (c_{3n-1}c_{3n})^{n-6} \frac{\left(D_{12}^{n-1n}\right)^{n-4}}{(c_{1n-1}c_{1n}c_{2n-1}c_{2n})^2} \phi^{(2)} \left(\frac{\langle r\, s\rangle}{H_{rs}^{n-1n}}\right) \tilde{\phi}^{(n-4)} \left(\frac{[a\, b]}{H_{12}^{ab}}\right) \\ \times \prod_{a=4}^{n-2} \frac{1}{\mathcal{V}_{an-1n}^{123}} \prod_{r=1}^{3} \delta^2(\tilde{\lambda}_r + c_{ra}\tilde{\lambda}_a) \prod_{a=4}^{n} \delta^2\left(\lambda_a - \lambda_r c_{ra}\right), \end{aligned} \tag{5.2.3}$$

where we sum over the repeated indices in the δ-functions. The ingredients of this formula are defined as follows:

- $$D_{rs}^{ab} = \begin{vmatrix} c_{ra} & c_{rb} \\ c_{sa} & c_{sb} \end{vmatrix}, \qquad H_{rs}^{ab} = \frac{D_{rs}^{ab}}{c_{ra}c_{rb}c_{sa}c_{sb}},$$

[1] Notice that our matrices $\phi^{(d)}$ and $\tilde{\phi}^{(\tilde{d})}$ are exactly $\phi^{(d)}$ and $\phi^{(\tilde{d})}$ in [58]. Also notice that the two matrices, defined in (5.2.4) and (5.2.5), have the same structure.

- $$\mathcal{V}^{12r}_{an-1n} = (1\,2\,r)(r\,a\,n-1)(n-1\,n\,1)(2\,a\,n) - (2\,r\,a)(a\,n-1\,n)(n\,1\,2)(r\,n-1\,1),$$

  We call $\mathcal{V}^{12r}_{an-1n}$ the Veronese polynomials and the reason for the name will be explained in the next section. The integration contour is defined around the zeroes of the $(d-1)(\tilde{d}-1) = n-5$ Veronese polynomials.

- $\phi^{(d)}$ is the determinant of a $d \times d$ matrix with elements

$$\begin{aligned} \phi_{rs} &= \frac{\langle r\,s\rangle}{H^{n-1n}_{rs}} \qquad \text{for } r \neq s \quad \text{and} \quad r,s \neq 1, \\ \phi_{rr} &= -\sum_{s\neq r} \frac{\langle r\,s\rangle}{H^{n-1n}_{rs}}, \end{aligned} \tag{5.2.4}$$

  with the convention for r and s as was said, leaving out 1,

- $\tilde{\phi}^{(\tilde{d})}$ is the determinant of a $\tilde{d} \times \tilde{d}$ matrix with elements

$$\begin{aligned} \tilde{\phi}_{ab} &= \frac{[a\,b]}{H^{ab}_{12}} \qquad \text{for } a \neq b \quad \text{and} \quad a,b \neq n, \\ \tilde{\phi}_{aa} &= -\sum_{b\neq a} \frac{[a\,b]}{H^{ab}_{12}}, \end{aligned} \tag{5.2.5}$$

  where indices a and b run over the rest of the particles, except for n.

$\phi^{(d)}$ and $\tilde{\phi}^{(\tilde{d})}$ are the pseudo-determinants, $\det'$, of the matrices Φ and $\tilde{\Phi}$ defined in (17) and (12) of [16] respectively, with the following particular choice of removed rows and columns: using cyclicity, the labels are chosen such that particle 1 has negative helicity and particle n has positive helicity. Then,

- from (17) all rows and columns a *plus* $r = 1$ are removed,

- from (12), all rows and columns r *plus* $a = n$ are removed.

5.3 A CSW-like Expansion for Gravity

In [33], a derivation of the CSW expansion for $\mathcal{N} = 4$ super Yang-Mills NMHV amplitudes from the Grassmannian is presented using a procedure called "relaxing δ-functions" (section 3 of [33]). Here we apply the same procedure to NMHV gravity amplitudes, using the link representation formula (5.2.3) written in momentum space.

We repeat formula (5.2.3) for the reader's convenience:

$$\mathcal{M}_{n,2} = \int \prod_{r,a} \frac{dc_{ra}}{c_{ra}} (c_{3n-1}c_{3n})^{n-6} \frac{\left(D_{12}^{n-1n}\right)^{n-4}}{(c_{1n-1}c_{1n}c_{2n-1}c_{2n})^2} \phi^{(2)} \left(\frac{\langle r\, s\rangle}{H_{rs}^{n-1n}}\right) \tilde{\phi}^{(n-4)} \left(\frac{[a\, b]}{H_{12}^{ab}}\right) \\ \times \prod_{a=4}^{n-2} \frac{1}{\mathcal{V}_{an-1n}^{123}} \prod_{r=1}^{3} \delta^2(\tilde{\lambda}_r + c_{ra}\tilde{\lambda}_a) \prod_{a=4}^{n} \delta^2\left(\lambda_a - \lambda_r c_{ra}\right). \tag{5.3.1}$$

Now we perform the following steps:

1. Pull out momentum conservation

$$\prod_{r=2}^{3} \delta^2(\tilde{\lambda}_r + c_{ra}\tilde{\lambda}_a) = \langle 23\rangle^2 \delta^4\left(\sum_{i=1}^{n} \lambda_i \tilde{\lambda}_i\right); \tag{5.3.2}$$

2. Split the remaining $\tilde{\lambda}_1$ "two component" δ-functions into two "one-component" δ-functions by projecting it on two arbitrary linearly independent spinors $[X|$ and $[Y|$:

$$\delta^2(\tilde{\lambda}_1 + c_{1a}\tilde{\lambda}_a) = [XY]\delta([1X] + c_{1a}[aX])\delta([1Y] + c_{1a}[aY]); \tag{5.3.3}$$

3. Solve the system with all λ δ-functions together with $\delta([1Y] + c_{1a}[aY])$, and relax $\delta([1X]+c_{1a}[aX])$, that is to replace the δ-function by one over its argument and treat the integral as a contour integral around the point where this argument is zero:

$$\delta([1X] + c_{1a}[aX]) \longrightarrow \frac{1}{[1X] + c_{1a}[aX]}; \tag{5.3.4}$$

4. Since we relaxed one δ-function, we increase by one the number of integration variables t_i, $i = 1, 2, \ldots, n-4$. The amplitude $\mathcal{M}_{n,2}$ is obtained by carrying the integrations on a contour defined around the zeros of the $n-5$ Veronese polynomials plus the zero of $[1X]+c_{1a}[aX]$. However, relaxing the δ-function means treating this point as a pole, so we can use the global residue theorem to "blow up the residue" and write the amplitude as minus the sum of residues of the other poles of the integrand. Being more precise, suppose we perform $n-5$ integrations to solve all Veronese constraints. Then, we are left with one integration, say in the variable t_1, and the integrand is a function of t_1. To obtain the amplitude, we should compute the residue due to the only remaining constraint $[1X]+c_{1a}(t_1)[aX] = 0$. However, if the integrand contains m other poles for $t^*_{1,j}$, $j = 1, \ldots, m$, then

$$\mathcal{M}_{n,2} = -\sum_{j=1}^{m} \text{Res}[\text{Integrand}]\Bigg|_{t^*_{1,j}}, \tag{5.3.5}$$

where these poles correspond to 3×3 minors in the denominator.

These are the steps for obtaining a CSW expansion for super Yang-Mills amplitudes from the Grassmannian formulation, and in this work we apply the same procedure for the gravity formula (5.2.3) to calculate an analogous expansion for 6- and 7-graviton NMHV amplitudes.

Let us now explain how this expansion (5.3.5) coincides with CSW. We discuss the localization in the space of 3-vectors, C-space, in which the columns of the $3 \times n$ matrix (5.2.1) live. The following discussion will be equivalent for both Yang-Mills and gravity amplitudes in the NMHV sector, as they both localize on pairs of lines in the C-space. For Yang-Mills, this localization is equivalent to the localization in twistor space, while for gravity the situation is different, as we know even the MHV amplitudes in gravity are not represented by lines in twistor space.

The reason $\mathcal{V}^{12r}_{an-1n}$ are called Veronese polynomials is that their simultaneous zeros imply that each column of C is obtained from applying a Veronese map from $\mathbb{CP}^1$ to $\mathbb{CP}^{k-1}$. In the computations of residues of (5.2.3), the C-matrix is evaluated on each of the poles of the integrand. This $3 \times n$ matrix, after a rescaling of one of its rows, can be depicted on a 2-plane. Since we have $n-5$ Veronese polynomials in (5.2.3), not all but some of the solutions make the points of the rescaled C-matrix lie on a conic. There also exist some non-conic configurations. The numerator, however, vanishes on the solutions leading to these configurations, so projects them out. In addition to setting the Veronese polynomials to zero, at any time there is a minor from the denominator set to zero, hence, three of the points must be collinear. In general, any five points define a unique conic. When three of these points are collinear, the conic is reducible and the configuration will be a degenerate conic (two intersecting lines) which has three points on one line and the others spreading on any of the two lines.

In summary, on the support of $n-5$ Veronese polynomials and the numerator we only see conic configurations. Each of the minors in the denominator then tells us which three points must be collinear. The outcomes are the terms in (5.3.5) which we define to be CSW terms.

Here we refer to our expansion not as a "CSW expansion for gravity", but as a CSW-like expansion. The reason is that "the canonical" way of generating the MHV expansion for gravity amplitudes is through Risager's procedure. It is not obvious a priori, however, that the procedure shown here will correspond to Risager's as happens for super Yang-Mills. That point being clear, we will refer to the channels as CSW-terms rather than CSW-like-terms for convenience of the notation.

5.4 6-Graviton Computation

For the 6-graviton NMHV amplitude, we have $n = 6$, $d = \tilde{d} = 2$. As already said, we choose a gauge fixing such that the indices r, a take values $r = 1, 2, 3$ and $a = 4, 5, 6$. Then (5.2.3) is written as

$$\mathcal{M}(1^-,2^-,3^-,4^+,5^+,6^+) = \int \prod_{\substack{r=1,2,3\\a=4,5,6}} \frac{dc_{ra}}{c_{ra}} (H_{12}^{56})^2 \phi^{(2)}\left(\frac{\langle r\, s\rangle}{H_{rs}^{56}}\right) \tilde{\phi}^{(2)}\left(\frac{[a\, b]}{H_{12}^{ab}}\right) \\ \times \frac{1}{\mathcal{V}} \prod_{r=1}^{3} \delta^2(\tilde{\lambda}_r + c_{ra}\tilde{\lambda}_a) \prod_{a=4}^{6} \delta^2(\lambda_a - c_{ra}\lambda_r), \tag{5.4.1}$$

where

$$H_{12}^{56} = \frac{c_{15}c_{26} - c_{16}c_{25}}{c_{15}c_{26}c_{16}c_{25}} \quad \text{and} \quad \mathcal{V} \equiv \mathcal{V}_{123}^{456}. \tag{5.4.2}$$

The functions $\phi^{(2)}$ and $\tilde{\phi}^{(2)}$ entering the formula (5.4.1) are determinants of arbitrary 2×2 minors of the following rank-2 matrices:

$$\Phi = \begin{pmatrix} -\frac{\langle 12\rangle}{H_{12}^{56}} - \frac{\langle 13\rangle}{H_{13}^{56}} & \frac{\langle 12\rangle}{H_{12}^{56}} & \frac{\langle 13\rangle}{H_{13}^{56}} \\ \frac{\langle 12\rangle}{H_{12}^{56}} & -\frac{\langle 12\rangle}{H_{12}^{56}} - \frac{\langle 23\rangle}{H_{23}^{56}} & \frac{\langle 23\rangle}{H_{23}^{56}} \\ \frac{\langle 13\rangle}{H_{13}^{56}} & \frac{\langle 23\rangle}{H_{23}^{56}} & -\frac{\langle 13\rangle}{H_{13}^{56}} - \frac{\langle 23\rangle}{H_{23}^{56}} \end{pmatrix}, \tag{5.4.3}$$

$$\tilde{\Phi} = \begin{pmatrix} -\frac{[45]}{H_{12}^{45}} - \frac{[46]}{H_{12}^{46}} & \frac{[45]}{H_{12}^{45}} & \frac{[46]}{H_{12}^{46}} \\ \frac{[45]}{H_{12}^{45}} & -\frac{[45]}{H_{12}^{45}} - \frac{[56]}{H_{12}^{56}} & \frac{[56]}{H_{12}^{56}} \\ \frac{[46]}{H_{12}^{46}} & \frac{[56]}{H_{12}^{56}} & -\frac{[46]}{H_{12}^{46}} - \frac{[56]}{H_{12}^{56}} \end{pmatrix}. \tag{5.4.4}$$

There are 9 integration variables and 12 bosonic δ-functions, four of which provide momentum-conservation (5.3.2). After pulling out the momentum conservation the integral (5.4.1) contains 8 constraints and 9 integration variables, so it should be complemented by the choice of an integration contour. The amplitude is given by integration along the contour encircling $\mathcal{V} = 0$.

Now let us start the transformations leading from the integral representation (5.4.1) to the CSW-like expansion of the gravity amplitude. First, according to the procedure explained in section 5.3, we split

$$\delta^2(\tilde{\lambda}_1 + c_{14}\tilde{\lambda}_4 + c_{15}\tilde{\lambda}_5 + c_{16}\tilde{\lambda}_6) = [XY]\delta\left([1X] + c_{14}[4X] + c_{15}[5X] + c_{16}[6X]\right) \times\delta\left([1Y] + c_{14}[4Y] + c_{15}[5Y] + c_{16}[6Y]\right), \tag{5.4.5}$$

and relax $\delta\left([1X] + c_{1a}[aX]\right)$. After expanding $(H_{12}^{56})^2$, $\phi^{(2)}$ and $\tilde{\phi}^{(2)}$ in (5.4.1) using the definition (5.4.2), the denominator of the integrand will contain polynomials of c_{ra} as well as $\mathcal{V}$ and $[1X]+c_{1a}[aX]$. Using (5.2.2), one can simplify the expression and write it in terms of entries and 3×3 minors of the C-matrix (5.2.1). All the factors in the denominator can then be split into two functions:

$$\begin{aligned} f_1 &= c_{15}c_{16}c_{25}c_{26}c_{34}(156)(256)(345)(364)\left([1X] + c_{1a}[aX]\right), \\ f_2 &= \mathcal{V}. \end{aligned} \tag{5.4.6}$$

The integral is now in two complex variables and the amplitude is then given by a multi-dimensional residue at the poles where both $[1X] + c_{1a}[aX]$ and $\mathcal{V}$ are zero.

According to the global residue theorem, we can also write the amplitude as minus the sum of all other residues in which $\mathcal{V}$ and one of the factors in f_1 other than $[1X] + c_{1a}[aX]$ are set to zero. Explicitly speaking, denoting a residue by the factors set to zero to compute it, one will have

$$\begin{aligned} \mathcal{M}_{6,2} = \{\mathcal{V}, [1X] + c_{1a}[aX]\} = &-\{\mathcal{V}, c_{15}\} - \{\mathcal{V}, c_{16}\} - \{\mathcal{V}, c_{25}\} - \{\mathcal{V}, c_{26}\} - \{\mathcal{V}, c_{34}\} \\ &-\{\mathcal{V}, (156)\} - \{\mathcal{V}, (256)\} - \{\mathcal{V}, (345)\} - \{\mathcal{V}, (364)\}. \end{aligned} \tag{5.4.7}$$

Notice that, since $\mathcal{V}$ is a polynomial of degree 4, each term splits yet into 4 terms. Notice also that many residues contribute to the same configuration, e.g., $\{(156), (126)\}$ and $\{(256), (126)\}$ shown in figure 5.2. The way to carefully deal with this and take care of all contributions for a given diagram will be explained in section 5.4.2.

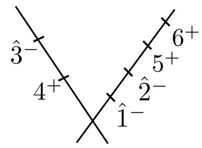

Figure 5.2: A configuration associated with $\{(156),(126)\}$ and $\{(256),(126)\}$.

Carrying on the computation, the δ-functions which we keep, impose the following set of seven equations

$$\begin{cases} [1Y] + c_{14}[4Y] + c_{15}[5Y] + c_{16}[6Y] = 0, \\ \lambda_4 = c_{14}\lambda_1 + c_{24}\lambda_2 + c_{34}\lambda_3, \\ \lambda_5 = c_{15}\lambda_1 + c_{25}\lambda_2 + c_{35}\lambda_3, \\ \lambda_6 = c_{16}\lambda_1 + c_{26}\lambda_2 + c_{36}\lambda_3, \end{cases} \tag{5.4.8}$$

or equivalently

$$\begin{cases} [1Y] + c_{14}[4Y] + c_{15}[5Y] + c_{16}[6Y] = 0, \\ c_{14}\langle 12\rangle + c_{34}\langle 32\rangle - \langle 42\rangle = 0, \\ c_{24}\langle 21\rangle + c_{34}\langle 31\rangle - \langle 41\rangle = 0, \\ c_{15}\langle 15\rangle + c_{25}\langle 25\rangle + c_{35}\langle 35\rangle = 0, \\ c_{25}\langle 21\rangle + c_{35}\langle 31\rangle - \langle 51\rangle = 0, \\ c_{16}\langle 16\rangle + c_{26}\langle 26\rangle + c_{36}\langle 36\rangle = 0, \\ c_{26}\langle 21\rangle + c_{36}\langle 31\rangle - \langle 61\rangle = 0. \end{cases} \tag{5.4.9}$$

The last transformation is performed for computational convenience and multiplies the Jacobian by a factor of $\langle 21\rangle\langle 51\rangle\langle 61\rangle$.

Let us denote by S the system of equations formed by (5.4.9) and two additional equations: $\mathcal{V}$=0 and $(abc) = 0$, where (abc) is one of the minors in the denominator. Then, an individual residue is given by the following expression evaluated on the solution of the system S:

$$\frac{1}{J}\frac{\left([45][56]c_{15}c_{25}(364)+\text{cyclic}(4,5,6)\right)\left(\langle 12\rangle\langle 23\rangle c_{25}c_{26}(256)+\text{cyclic}(1,2,3)\right)(abc)}{(135)(136)(235)(236)(124)(156)(256)(345)(364)}\frac{[XY]}{([1X]+c_{1a}[aX])}, \tag{5.4.10}$$

where J is the Jacobian of solving the system S. The precise way of calculating the residues will be explained later in section 5.4.2.

5.4.1 Analytical Computation of $\{(135),(246)\}$

We calculate the residue of the integrand at the point $\{(135),(246)\}$. To see that this is indeed a simple multi-dimensional pole we represent the Veronese polynomials in the form $\mathcal{V}=(123)(345)(561)(246)-(234)(456)(612)(351)$. Setting (135) to zero makes f_1 in (5.4.6) vanish, since $(135)=c_{25}$. Also, the second term in $\mathcal{V}$ vanishes by this condition. Setting additionally (246) to zero sets $f_2=\mathcal{V}$ to zero and, as one can check, does not add any zeros among the factors in the denominator.

Using Mathematica to find the solution of the system (5.4.8), substituting it into (5.4.10) and simplifying the result, we obtain

$$\frac{[46]\langle 13\rangle^6\langle 2|4+6|5]\langle 2|4+6|Y]^5}{\langle 15\rangle\langle 24\rangle\langle 26\rangle\langle 35\rangle\langle 46\rangle\langle 1|3+5|Y]\langle 4|2+6|Y]\langle 6|2+4|Y]\langle 5|1+3|Y]\langle 3|1+5|Y](p_1+p_3+p_5)^2}. \tag{5.4.11}$$

One can see that this result coincides with a Risager's term which has the form

$$\mathcal{M}_L(\hat{1}^-,\hat{3}^-,5^+,\hat{P}^+)\frac{1}{P^2}\mathcal{M}_R(\hat{2}^-,4^+,6^+,-\hat{P}^-), \tag{5.4.12}$$

where $P=p_2+p_4+p_6$ is the momentum flowing through the link which is set on-shell and $\hat{P}$ is the Risager-deformed momentum $\hat{P}=\hat{p}_2+p_4+p_6$ as in (3.1.2).

5.4.2 Numerical Check of All Residues for 6 Gravitons: Taming Singular Configurations in Multi-Dimensional Residues

In order to numerically calculate the other channels, an immediate application of the global residue theorem leads to a computational problem: the part of the integrand besides the two zero factors in the denominator, which is supposed to be finite, can become ambiguous. This happens because the numerator and the denominator go to zero simultaneously and do not cancel with each other.

Consider for example the 2-4 channel shown in figure 5.3 in which particles 1 and 4 belong to $\mathcal{M}_L$ and particles 2, 3, 5 and 6 belong to $\mathcal{M}_R$.

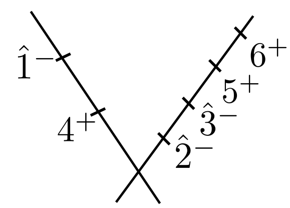

Figure 5.3: An example of a 2-4 channel.

One way to generate this configuration is by setting $\mathcal{V}$ and (256) to zero. Looking at the denominator of (5.4.10), however, we see that (235) and (236) are "accidentally" set to zero as well. For the residue to be finite, the numerator of (5.4.10) should also go to zero at the solution. Restricting ourselves to the submanifold of $G(3,6)$ defined by $\mathcal{V}=0$ equation, we can write all minors that go to zero as a multiple of say (235). More precisely, in this example we can use $\mathcal{V}=(462)(235)(514)(631)-(623)(351)(146)(254)=0$[2] to write $(236)\propto(235)$. Playing this game with all the minors that go to zero in the integrand, we note that both the numerator and denominator vanish as $(235)^5$, so they cancel and we can finally obtain a finite residue.

Although this method gives the correct results in this example, it requires an individual treatment of each residue, making the computations rather laborious. More importantly,

[2] Keep in mind that the condition $\mathcal{V}=0$ is permutation invariant, so we can make a choice of ordering of particles which is the most convenient for our purposes.

it cannot necessarily be applied to higher-point computations. For this reason, we use a procedure which can be directly applied in all cases which consists of the use of a *regulator*.

The singular residues are points in the Grassmannian on which the denominator of the integrand has a zero of second order or higher. The idea is to add to each minor in the denominator a small constant ϵ, the regulator,

$$\frac{1}{(abc)} \rightarrow \frac{1}{(abc)+\epsilon}, \tag{5.4.13}$$

which separates the higher order zeros in many zeros of first order. Also to make sure that the regulated minors do not accidentally vanish simultaneously, we may add different values of ϵ to different minors. After this, all poles are simple and the global residue theorem can be applied without restriction. Moreover, there is no need to regulate the Veronese polynomials, since they do not factorize when minors are regulated ($\epsilon \neq 0$).

The Veronese polynomials and each of the regulated minors in the denominator make a system of equations when set to zero. On each solution of each system of equations, there might be some minors which nearly vanish. We compute all possible minors on each solution and collect the almost-zero ones. These minors now define the localization of particles on the two crossing lines which makes a CSW channel. As an example, in figure 5.3, the only minors which are close to zero are (235), (236), (256) and (356). Hence, particles 2, 3, 5 and 6 lie on a line. And, there is also a line passing through any two free points; 1 and 4 here. As it is also clear from this example, many solutions may contribute to a given localization. Here, residues of the integrand on the solutions of the following systems of equations contribute to this channel: $\{\mathcal{V}=0 \text{ and } (235)+\epsilon=0\}$, $\{\mathcal{V}=0 \text{ and } (236)+\epsilon=0\}$ and $\{\mathcal{V}=0 \text{ and } (256)+\epsilon=0\}$. In the end, all contributions must be summed to give the corresponding CSW term.

This way, we computed the numerical values of the 21 configurations of 6 gravitons, depicted in figure 5.4.

The numerical values obtained are compared to the known corresponding Risager terms.

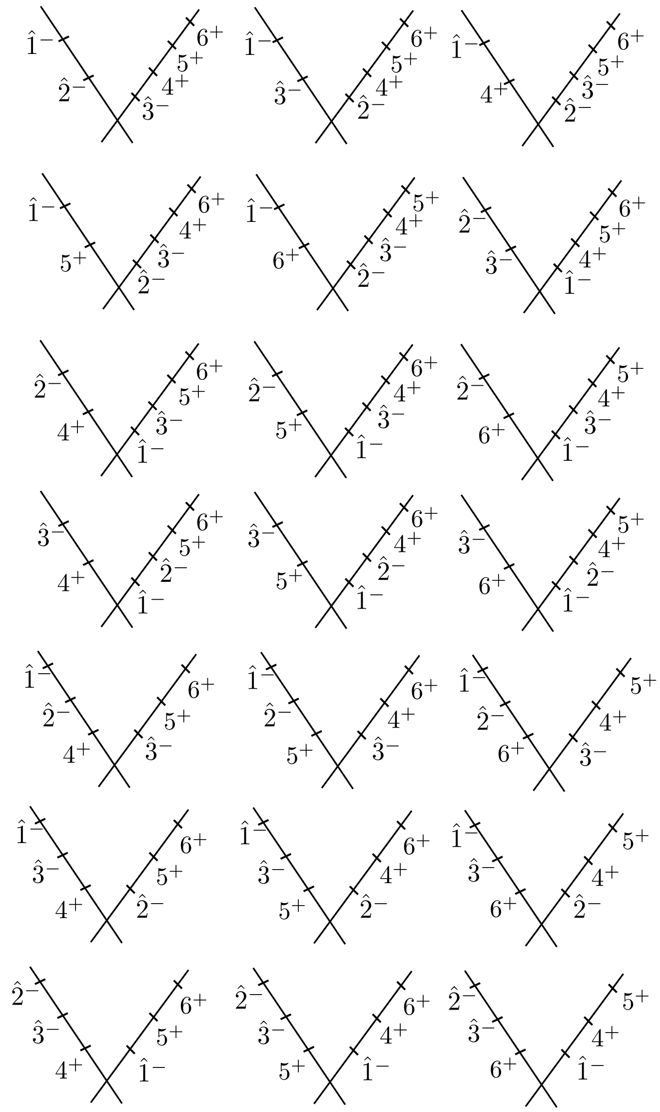

Figure 5.4: 6-graviton channels

As a result of this comparison we found, as happens to Yang-Mills, a complete agreement between our procedure and Risager's. Numerical results are given in the appendix for both 6- and 7-graviton cases.

5.5 7-Graviton Computation

For 7-graviton NMHV amplitudes, $n = 7$, $d = 2$ and $\tilde{d} = 3$, so the link representation (5.2.3) is

$$\mathcal{M}(1^-,2^-,3^-,4^+,5^+,6^+,7^+) = \int \prod_{\substack{r=1,2,3\\a=4,..,7}} \frac{dc_{ra}}{c_{ra}} c_{36}c_{37} \frac{(367)^3}{(c_{16}c_{17}c_{26}c_{27})^2}$$
$$\times \phi^{(2)}\left(\frac{\langle r\, s\rangle}{H_{rs}^{67}}\right) \tilde{\phi}^{(3)}\left(\frac{[a\, b]}{H_{12}^{ab}}\right) \frac{1}{\mathcal{V}_{467}^{123}\mathcal{V}_{567}^{123}} \prod_r \delta^2(\tilde{\lambda}_r + c_{ra}\tilde{\lambda}_a) \prod_a \delta^2(\lambda_a - c_{ra}\lambda_r). \tag{5.5.1}$$

Using the definitions of ϕ, one can rewrite the integrand as a rational expression whose denominator is

$$(136)(137)(236)(237)(124)(125)(167)(267)(345)(346)(347)(356)(357)\mathcal{V}_{467}^{123}\mathcal{V}_{567}^{123}. \tag{5.5.2}$$

According to the general procedure of section 5.3, we relax one component of the $\tilde{\lambda}_1$ δ-functions. After this, we get an additional factor of $[1X] + \sum_a c_{1a}[aX]$ in the denominator. To apply a global residue theorem these factors should be split into three functions:

$$f_1 = (136)(137)(236)(237)(124)(125)(167)(267)(345)(346)(347)(356)(357)\left([1X] + \sum_a c_{1a}[aX]\right),$$
$$f_2 = \mathcal{V}_{467}^{123},$$
$$f_3 = \mathcal{V}_{567}^{123}. \tag{5.5.3}$$

The amplitude is the residue of the integrand at the point where f_2, f_3 and the last factor in f_1 go to zero. Like in the 6-particle case, we get different CSW terms by computing residues at points where f_2, f_3 and one of the minors in f_1 go to zero. The GRT then tells us that all the CSW terms sum to the full amplitude, as it naturally should:

$$\mathcal{M}_{7,2} = \left\{ \left([1X] + \sum_a c_{1a}[aX] \right), f_2, f_3 \right\} = -\{(136), f_2, f_3\} - \{(137), f_2, f_3\} - \ldots - \{(357), f_2, f_3\}. \tag{5.5.4}$$

Notice that every term in the right-hand side can in principle contain more than one CSW term, because the system

$$\begin{cases} \text{minor} = 0, \\ \mathcal{V}^{123}_{467} = 0, \\ \mathcal{V}^{123}_{567} = 0. \end{cases} \tag{5.5.5}$$

can have more than one solution.

To compute the residues we perform a procedure similar to the one described in section 5.4.2. First we introduce the regulator to every minor in (5.5.2) and write a system of linear equations composed of the δ-functions and three equations setting a minor and the two $\mathcal{V}$'s to zero. After this, all residues are free from any accidental ambiguity and out of the $13 \times 4 \times 4$ solutions, many of them sum to give the 45 possible configurations.

If the configuration is such that these two Veronese polynomials are equal to zero, but some other Veronese polynomials which can be made of particles $1, \ldots, 7$ are not zero, then this configuration does not contribute, since the corresponding residue is zero (such configurations are called *spurious solutions*).

5.6 The General n NMHV Case: Power Counting

All the reasoning presented in section 5.3 relies on the assumption that the global residue theorem is applicable and our aim now is to verify when this is the case for gravity. To do so, we analyse how the integrand behaves for large values of the integration variables, that is, we look for n such that there is no pole at infinity.

In order to perform the power counting, consider formula (5.2.3) which we repeat here for convenience:

$$\mathcal{M}_{n,2} = \int \prod_{r,a} \frac{dc_{ra}}{c_{ra}} \prod_{a=4}^{n-2} (c_{3n-1}c_{3n})^{n-6} \frac{\left(D_{12}^{n-1n}\right)^{n-4}}{(c_{1n-1}c_{1n}c_{2n-1}c_{2n})^2} \phi^{(2)} \left(\frac{\langle r\, s\rangle}{H_{rs}^{n-1n}}\right) \tilde{\phi}^{(n-4)} \left(\frac{[a\, b]}{H_{12}^{ab}}\right)$$
$$\times \prod_{a=4}^{n-2} \frac{1}{\mathcal{V}_{an-1n}^{123}} \prod_{r=1}^{3} \delta^2(\tilde{\lambda}_r + c_{ra}\tilde{\lambda}_a) \prod_{a=4}^{n} \delta^2\left(\lambda_a - \lambda_r c_{ra}\right).$$

We know from section 2.4 of [29], that each minor is at most of degree one. We also know that in the link representation we can write each link variable and the factors D_{12}^{n-1n} as minors, so they are also linear in t_i, and keep in mind that each Veronese polynomial is of degree 4 in t_i.

Then, let us consider one variable t and look at how each factor in the integrand of (5.2.3) scales with it:

$$\prod_{r,a} \frac{1}{c_{ra}} \sim t^{9-3n},$$
$$(c_{3n-1}c_{3n})^{n-6} \sim t^{2n-12},$$
$$\frac{\left(D_{12}^{n-1n}\right)^{n-4}}{(c_{1n-1}c_{1n}c_{2n-1}c_{2n})^2} \sim \frac{t^{n-4}}{t^8} = t^{n-12},$$
$$\tilde{\phi}^{(n-4)} \left(\frac{[ab]}{H_{12}^{ab}}\right) \sim t^{3(n-4)},$$
$$\phi^{(2)} \left(\frac{\langle rs\rangle}{H_{rs}^{n-1n}}\right) \sim t^{6},$$
$$\prod_{a\neq n,n-1} \delta\left(\mathcal{V}_{n-1na}^{123}\right) \sim t^{-4(n-5)},$$
$$\text{Relaxed } \delta\text{-function} \sim t^{-1}.$$

Summing up, we get that the integrand scales as t^{-n-2}. The condition for the global residue theorem to apply is that the integrand should decay faster than $t^{-(n-4)}$ for each integration variable, that is $-n-2 \leqslant -n+4$. This condition is satisfied $\forall\, n$, so we conclude that the CSW-like expansion will be valid for all NMHV graviton amplitudes.

5.7 Conclusion

Building up on the recent results for $\mathcal{N}=8$ supergravity amplitudes — Hodges's MHV formula [14], the Cachazo-Skinner formula and its link representation — we carried on the application of the δ-function relaxation technique used in [33] for super Yang-Mills amplitudes to explore the new gravity territory.

From the link representation, we derived a CSW-like expansion for pure 6- and 7-graviton NMHV amplitudes. It was not obvious, however, that this expansion would coincide with Risager's, the known way up to now to produce an MHV-vertex expansion. We observed a complete agreement between the two expansions in the cases we worked on.

The most exciting feature of the new expansion is that in principle it works for any n for $d=2$, while Risager's method fails for $n \geq 12$. Further studies are necessary to verify if and when the two expansions stop agreeing. One possible reason for the disagreement of the two expansions is CSW-like configurations which are not the product of two MHV amplitudes but are still non-zero residues of the link formula. In other words, we may have a pair of lines in the C-space with all negative helicity particles on one line. This is obviously not a Risager's term. For small n, as was said, the numerator does not allow such configurations in the expansion; in fact, the residue will be zero. But, for higher n one can still have a valid expansion of residues, as the power counting shows, but there exist more terms which are not MHV $\times$ MHV. These extra terms must be the residues at infinity of the Risager method. It would be fascinating to explore these extra terms and find their physical meaning in momentum space.

Appendix 5.A Numerical Values for the 6- and 7-Graviton Residues and Risager Terms

Here we present the numerical values for different channels given by the Risager diagrams and our CSW-like terms for a given set of external data for the 6- and 7-graviton cases. As described in section 5.4.2, the residue computations are based on the use of a regulator ϵ for the minors of the integrand. We chose ϵ to take the value 0.0000001. Also, note that the amplitude is the sum of Risager terms but is minus the sum of residues as is known from the residue theorem. So, here, signs of the values of the two columns are opposite. The biggest errors in the 6- and 7-graviton computations are $1.6 \times 10^{-3}\%$ and $1.4 \times 10^{-2}\%$ respectively.

Our randomly generated data for the 6-graviton computation are as follows:

$$\lambda_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 8 \\ 5 \end{pmatrix}, \lambda_4 = \begin{pmatrix} 10 \\ 7 \end{pmatrix}, \lambda_5 = \begin{pmatrix} -7 \\ 9 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 9 \\ 5 \end{pmatrix},$$

$$\tilde{\lambda}_1 = \begin{pmatrix} -50 \\ 35 \end{pmatrix}, \tilde{\lambda}_2 = \begin{pmatrix} 71 \\ -25 \end{pmatrix}, \tilde{\lambda}_3 = \begin{pmatrix} -5 \\ 8 \end{pmatrix}, \tilde{\lambda}_4 = \begin{pmatrix} -4 \\ -8 \end{pmatrix}, \tilde{\lambda}_5 = \begin{pmatrix} -7 \\ 4 \end{pmatrix}, \tilde{\lambda}_6 = \begin{pmatrix} 9 \\ 1 \end{pmatrix},$$

and for the 7-graviton case we used:

$$\lambda_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \lambda_3 = \begin{pmatrix} -9 \\ -5 \end{pmatrix}, \lambda_4 = \begin{pmatrix} -3 \\ -3 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 9 \\ -6 \end{pmatrix}, \lambda_6 = \begin{pmatrix} -9 \\ -4 \end{pmatrix}, \lambda_7 = \begin{pmatrix} -10 \\ -8 \end{pmatrix},$$

$$\tilde{\lambda}_1 = \begin{pmatrix} 127 \\ 67 \end{pmatrix}, \tilde{\lambda}_2 = \begin{pmatrix} 56 \\ 57 \end{pmatrix}, \tilde{\lambda}_3 = \begin{pmatrix} -2 \\ 5 \end{pmatrix}, \tilde{\lambda}_4 = \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \tilde{\lambda}_5 = \begin{pmatrix} -2 \\ -1 \end{pmatrix}, \tilde{\lambda}_6 = \begin{pmatrix} 7 \\ -9 \end{pmatrix}, \tilde{\lambda}_7 = \begin{pmatrix} 7 \\ 10 \end{pmatrix}.$$

Channel	Risager's Term	Residue	Error
{1,2}{3,4,5,6}	-3.8035	3.8048	$3.4 \times 10^{-4}\%$
{1,3}{2,4,5,6}	0.027552142	-0.027552138	$1.5 \times 10^{-7}\%$
{1,4}{2,3,5,6}	-0.00024281392	0.00024281389	$1.2 \times 10^{-7}\%$
{1,5}{2,3,4,6}	-12.778	12.757	$1.6 \times 10^{-3}\%$
{1,6}{2,3,4,5}	0.0099509309	-0.0099509312	$3.0 \times 10^{-8}\%$
{2,3}{1,4,5,6}	-0.0014393998	0.0014393997	$6.9 \times 10^{-8}\%$
{2,4}{1,3,5,6}	0.0001347570	-0.0001347586	$1.2 \times 10^{-5}\%$
{2,5}{1,3,4,6}	0.2376382	-0.2376353	$1.2 \times 10^{-5}\%$
{2,6}{1,3,4,5}	-0.0000115706946	0.0000115706940	$5.2 \times 10^{-8}\%$
{3,4}{1,2,5,6}	-0.000011926600	0.000011926618	$1.5 \times 10^{-6}\%$
{3,5}{1,2,4,6}	-0.0135260863	0.0135260869	$4.4 \times 10^{-8}\%$
{3,6}{1,2,4,5}	-0.0010910475	0.0010910470	$4.6 \times 10^{-7}\%$
{1,2,4}{3,5,6}	-0.00160980	0.00160979	$6.2 \times 10^{-6}\%$
{1,2,5}{3,4,6}	15.50	-15.48	$1.3 \times 10^{-3}\%$
{1,2,6}{3,4,5}	-0.0011519435	0.0011519433	$1.7 \times 10^{-7}\%$
{1,3,4}{2,5,6}	-0.023460139	0.023460140	$4.3 \times 10^{-8}\%$
{1,3,5}{2,4,6}	$8.31380501 \times 10^{-7}$	$-8.31380545 \times 10^{-7}$	$5.3 \times 10^{-8}\%$
{1,3,6}{2,4,5}	-0.0000245346	0.0000245368	$9.0 \times 10^{-5}\%$
{1,4,5}{2,3,6}	-0.0005562082	0.0005562084	$3.6 \times 10^{-7}\%$
{1,4,6}{2,3,5}	0.0069317909	-0.0069317930	$3.0 \times 10^{-7}\%$
{1,5,6}{2,3,4}	6.4118974×10^{-7}	$-6.4118965 \times 10^{-7}$	$1.4 \times 10^{-7}\%$

Table 5.1: Numerical Values for 6-Graviton Computation

Channel	Risager's Term	Residue	Error
{1,2}{3,4,5,6,7}	-1.4302×10^{-7}	1.4327×10^{-7}	$1.7 \times 10^{-3}\%$
{1,3}{2,4,5,6,7}	-0.0012650646	0.0012650621	$2.0 \times 10^{-6}\%$
{1,4}{2,3,5,6,7}	4.075×10^{-8}	-4.099×10^{-8}	$6.1 \times 10^{-3}\%$
{1,5}{2,3,4,6,7}	0.000054264493	-0.000054264483	$1.8 \times 10^{-7}\%$
{1,6}{2,3,4,5,7}	-0.0003451595	0.0003451593	$5.8 \times 10^{-7}\%$
{1,7}{2,3,4,5,6}	-0.0011835542	0.0011835558	$1.4 \times 10^{-6}\%$
{2,3}{1,4,5,6,7}	-0.0024191261	0.0024191232	$1.2 \times 10^{-6}\%$
{2,4}{1,3,5,6,7}	2.869570×10^{-7}	-2.869555×10^{-7}	$5.2 \times 10^{-6}\%$
{2,5}{1,3,4,6,7}	2.553×10^{-7}	-2.589×10^{-7}	$1.4 \times 10^{-2}\%$
{2,6}{1,3,4,5,7}	-0.0001381087	0.0001381085	$1.4 \times 10^{-6}\%$
{2,7}{1,3,4,5,6}	0.021812	-0.021809	$1.3 \times 10^{-4}\%$
{3,4}{1,2,5,6,7}	-0.0000389804	0.0000389807	$7.7 \times 10^{-6}\%$
{3,5}{1,2,4,6,7}	0.000045447	-0.000045450	$6.6 \times 10^{-5}\%$
{3,6}{1,2,4,5,7}	-0.002108237	0.002108243	$2.8 \times 10^{-6}\%$
{3,7}{1,2,4,5,6}	-0.00326858	0.00326854	$1.2 \times 10^{-5}\%$
{1,2,4}{3,5,6,7}	-9.51196×10^{-8}	9.51141×10^{-8}	$5.8 \times 10^{-5}\%$
{1,2,5}{3,4,6,7}	-3.365311×10^{-8}	3.365362×10^{-8}	$1.5 \times 10^{-5}\%$
{1,2,6}{3,4,5,7}	5.972950×10^{-8}	-5.972944×10^{-8}	$1.0 \times 10^{-6}\%$
{1,2,7}{3,4,5,6}	-0.0204886	0.0204857	$1.4 \times 10^{-4}\%$
{1,3,4}{2,5,6,7}	-0.00033009914133	0.00033009914132	$3.0 \times 10^{-11}\%$
{1,3,5}{2,4,6,7}	-0.0010859253	0.0010859242	$1.0 \times 10^{-6}\%$
{1,3,6}{2,4,5,7}	-0.0058003020	0.0058003065	$7.8 \times 10^{-7}\%$
{1,3,7}{2,4,5,6}	0.000767141	-0.000767139	$2.6 \times 10^{-6}\%$
{1,4,5}{2,3,6,7}	$5.82339379 \times 10^{-6}$	$-5.82339358 \times 10^{-6}$	$3.6 \times 10^{-8}\%$
{1,4,6}{2,3,5,7}	-0.000048608115	0.000048608121	$1.2 \times 10^{-7}\%$
{1,4,7}{2,3,5,6}	-0.0016051200	0.0016051227	$1.7 \times 10^{-6}\%$
{1,5,6}{2,3,4,7}	0.0001486804	-0.0001486801	$2.0 \times 10^{-6}\%$
{1,5,7}{2,3,4,6}	0.0023658846	-0.0023658860	$5.9 \times 10^{-7}\%$
{1,6,7}{2,3,4,5}	0.0001890601	-0.0001890600	$5.3 \times 10^{-7}\%$
{2,3,4}{1,5,6,7}	0.0045052257	-0.0045052235	$4.9 \times 10^{-7}\%$
{2,3,5}{1,4,6,7}	-0.0000926669	0.0000926696	$2.9 \times 10^{-5}\%$
{2,3,6}{1,4,5,7}	0.002235755	-0.002235760	$2.2 \times 10^{-6}\%$
{2,3,7}{1,4,5,6}	$4.24184246 \times 10^{-6}$	$-4.24184210 \times 10^{-6}$	$8.5 \times 10^{-8}\%$
{2,4,5}{1,3,6,7}	$-4.0081097 \times 10^{-9}$	4.0081081×10^{-9}	$4.0 \times 10^{-7}\%$
{2,4,6}{1,3,5,7}	-0.0007386578	0.0007386558	$2.7 \times 10^{-6}\%$
{2,4,7}{1,3,5,6}	0.008541409	-0.008541422	$1.5 \times 10^{-6}\%$
{2,5,6}{1,3,4,7}	0.0001888818	-0.0001888815	$1.6 \times 10^{-6}\%$
{2,5,7}{1,3,4,6}	-0.0031819569	0.0031819561	$2.5 \times 10^{-7}\%$
{2,6,7}{1,3,4,5}	-0.0005354038	0.0005354037	$1.9 \times 10^{-7}\%$
{3,4,5}{1,2,6,7}	0.000044443578	-0.000044443556	$5.0 \times 10^{-7}\%$
{3,4,6}{1,2,5,7}	0.002633	-0.002671	$1.4 \times 10^{-2}\%$
{3,4,7}{1,2,5,6}	0.00395962	-0.00395958	$1.0 \times 10^{-5}\%$
{3,5,6}{1,2,4,7}	-0.006673719	0.00667372	$1.2 \times 10^{-6}\%$
{3,5,7}{1,2,4,6}	2.363790×10^{-7}	-2.363757×10^{-7}	$1.4 \times 10^{-5}\%$
{3,6,7}{1,2,4,6}	-0.000026712880	0.000026712871	$3.4 \times 10^{-7}\%$

Table 5.2: Numerical Values for 7-Graviton Computation

Chapter 6

Concluding Remarks

It is now time to look backward and summarize this work, although more detailed *concluding remarks* have been presented at the end of each chapter.

What have made the new developments in amplitude calculations so interesting and noteworthy are both effectiveness in calculations and novelty in attitude. While the former has led to producing compact algebraic expressions for amplitudes in a considerably shorter time, the latter has deepened our understanding of quantum field theory in many different ways.

As Risager's method was one of our favourites throughout this thesis, we examined itself earlier in chapter 2 by a generalization of the deformation to contain two complex variables rather than one. Unexpectedly, we saw how new physical terms, soft terms, appeared in the expansion for the case of Yang-Mills NMHV amplitudes. It would be very interesting if one would take this approach and apply it to gravity to see what could happen there. It is worth mentioning that here we are trying to better understand the analytic structure of amplitudes rather than to only compute them for which one may use shorter prescriptions.

We saw that one of the powerful approaches to construct tree-level amplitudes is a simple on-shell expansion which uses MHV amplitudes as input, the CSW expansion, systematically produced by Risager's method. When applied to gravity, Risager's method needs to be modified to produce correct amplitudes with higher than eleven gravitons. For the case of 12-graviton amplitude in the NMHV sector, chapter 3 presented the missing piece of Risager's method to make the physical amplitude. Our instruction can be also applied to higher point amplitudes for a complete modification of Risager's method which we leave to be done by interested readers. The main problem with gravity is that no off-shell definition of MHV amplitudes has been found which could provide a CSW expansion for gravity, whereas in Yang-Mills the holomorphic nature of MHV amplitudes allows a simple off-shell extension.

With a different approach, originally used for Yang-Mills amplitudes, we found an expansion for gravity tree amplitudes whose terms are CSW-like. In chapter 5, we explicitly showed our calculations and numerical checks for 6- and 7-graviton amplitudes and verified the validity of our method for any number of external gravitons. Some hints for further interesting works were also mentioned in 5.7.

As was seen, very recently, two novel formulas were proposed for tree-level amplitudes of $\mathcal{N} = 8$ supergravity in all R-charge sectors, Cachazo-Geyer and Cachazo-Skinner formulas. While the latter has been proved to be the correct amplitude, the former still needs a justification, providing that the parity symmetry and soft limit behaviour were acknowledged for the formula, Chapter 4. The next goal is hence obvious: to analytically prove the Cachazo-Geyer formula.

As a final comment, we would like to mention that it would be striking to explore loop-level gravity given the recent improvements at the tree level. Although we did not review the loop calculation techniques, as they were irrelevant to this work, there have been remarkable developments probing higher orders of perturbation, mainly, unitarity cut methods and leading singularities which also benefit from information about tree level. How new insights into tree-level amplitudes can be used for higher loops in gravity is worth revealing.

In the end and in a broader sense, these modern methods developed for calculating amplitudes can potentially be utilized in other areas of physics subject to suitable extensions and modifications.

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