Risk Measure Approaches to Partial Hedging and Reinsurance

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Hedging has been one of the most important topics in finance. How to effectively hedge the exposed risk draws significant interest from both academicians and practitioners.

In a complete financial market, every contingent claim can be hedged perfectly. In an incomplete market, the investor can eliminate his risk exposure by superhedging. However, both perfect hedging and superhedging usually call for a high cost. In some situations, the investor does not have enough capital or is not willing to spend that much to achieve a zero risk position. This brings us to the topic of partial hedging.

In this thesis, we establish the risk measure based partial hedging model and study the optimal partial hedging strategies under various criteria. First, we consider two of the most common risk measures known as Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR). We derive the analytical forms of optimal partial hedging strategies under the criterion of minimizing VaR of the investor's total risk exposure. The knock-out call hedging strategy and the bull call spread hedging strategy are shown to be optimal among two admissible sets of hedging strategies. Since VaR risk measure has some undesired properties, we consider the CVaR risk measure and show that bull call spread hedging strategy is optimal under the criterion of minimizing CVaR of the investor's total risk exposure. The comparison between our proposed partial hedging strategies and some other partial hedging strategies, including the well-known quantile hedging strategy, is provided and the advantages of our proposed partial hedging strategies are highlighted. Then we apply the similar approaches in the context of reinsurance. The VaR-based optimal reinsurance strategies are derived under various constraints. Then we study the optimal partial hedging strategies under general risk measures. We provide the necessary and sufficient optimality conditions and use these conditions to study some specific hedging

strategies. The robustness of our proposed CVaR-based optimal partial hedging strategy is also discussed in this part. Last but not least, we propose a new method, simulation-based approach, to formulate the optimal partial hedging models. By using the simulation-based approach, we can numerically obtain the optimal partial hedging strategy under various constraints and criteria. The numerical results in the examples in this part coincide with the theoretical results.

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Chapter 1

Introduction

1.1 Background

Hedging has been one of the most important topics in finance. How to effectively hedge the exposed risk draws significant interest from both academicians and practitioners. Under the classical option pricing theory, when the market is complete, the payout of any contingent claim can be duplicated perfectly by a self-financing portfolio and this gives rise to the so-called perfect hedging strategy. When the market is incomplete, the perfect hedging is typically not possible and the superhedging strategy has been proposed as an alternative. The superhedging strategy involves seeking the cheapest self-financing portfolio with payout no smaller than that of the contingent claim in all scenarios. While superhedging ensures that the hedger always has sufficient fund to cover his future obligation arising from the sale of the contingent claim, the strategy, however, is too costly to be of practical interest in most cases.

Perfect hedging strategy or superhedging strategy not only requires large initial amount of capital, but also erodes the chance of making higher profit. Therefore, instead of eliminating their risk exposure completely, some investors are inclined to or have to control it within an acceptable level, which is equivalent to minimizing the exposed risk subject to some constraints. It is to resort to the partial hedging which hedges the future obligation only partially. A natural question is what is the optimal partial hedging strategy with a given initial amount of capital.

1.2 Literature Review

In this section, we provide a brief literature review on optimal partial hedging models that are relevant to the thesis.

The pioneering work of optimal partial hedging is attributed to Föllmer and Leukert (1999) who propose a hedging strategy that maximizes the probability of meeting the future obligation under a given budget constraint. This strategy is commonly known as quantile hedging. More specifically, by using a very elegant idea, they translate the optimal hedging problem into the problem of finding the most powerful test. They model the price process as a semimartingle and use Neyman-Pearson Lemma to derive quantile hedging strategy which is able to meet the future obligation with maximal probability under the objective measure $\mathbb{P}$ when there is a hedging budget constraint. An idea that is closely related to quantile hedging also appears in the literature of portfolio management, see Browne (1999a, 1999b, 1999c, 2000) and the reference therein.

The classical quantile hedging has been generalized in a number of interesting directions. One extension is to study quantile hedging under more sophisticated market structures. For example, Spivak and Cvitanić (1999) study the problem of quantile hedging and rederive the complete market solution by using a duality method which is developed in utility maximization literature. They also demonstrate how to modify their approach to deal with the problem in a market with partial information. They define a market with partial information as a market where the hedger only knows a prior distribution of the vector of returns of the risky assets. Krutchenko and Melnikov (2001) study quantile hedging strategy under a special case of jump-diffusion market. They obtain the hedging strategy by deducing the corresponding stochastic differential equation. Bratyk and Mishura (2008) consider the incomplete market with several fractional Brownian motions and independent Brownian motions, which is a more complicated market structure. They estimate the successful probability for quantile hedging when the price process model defined by two Wiener processes and two fractional Brownian motions.

Generally, the objective of quantile hedging strategy is to maximize the probability of meeting the future obligation. Essentially, by maximizing that probability, the shortfall is evaluated in terms of a binary loss function. Therefore, another extension is to investigate the partial hedging strategies using some other optimization criteria, as opposed to maximizing the probability of meeting the future obligation as in quantile hedging. The optimal partial hedging in Föllmer and Leukert (2000), for example, takes into account the size of the shortfall instead of the probability of its occurrence. They use a loss function l to describe the investor's attitude towards the shortfall while deriving the optimal hedging strategy. In particular, the hedging strategies that minimize the investor's expected shortfall[1] are derived. Nakano (2004) attempts to minimize some coherent risk measures of the shortfall under the similar model setting as that in Föllmer and Leukert (1999). He represents the risk measure as the expected value of the loss under a certain probability measure, and then addresses the optimization problem by constructing the most powerful test in a way similar to Föllmer and Leukert (2000). Though the most powerful test is

[1] In Föllmer and Leukert (2000), expected shortfall means expected value of the shortfall risk.

expressed quite explicitly, the optimal hedging strategy can not be derived in most cases. Nakano (2004) also considers the optimal hedging strategy which minimizes the Conditional Value-at-Risk (CVaR) of the shortfall risk. The Conditional Value-at-Risk (CVaR) of the shortfall risk under confidence level $(1-\alpha)$ is the average value of the shortfall in the $\alpha\%$ worst cases. The optimal hedging strategy is derived in a very special case, in which CVaR of the shortfall risk is same as the expected value of the shortfall risk under the physical probability measure. In this special case, the hedging strategy which minimizes CVaR of the shortfall risk is the same as the hedging strategy which minimizes the expected value of the shortfall risk, which has been derived in Föllmer and Leukert (2000). Rudloff (2007) considers the similar hedging problem in the incomplete market by using convex risk measures. More recently, Melnikov and Smirnov (2012) study the optimal hedging strategies by minimizing the Conditional Value-at-Risk of the portfolio in a complete market. By exploiting the results from Föllmer and Leukert (2000) and Rockafellar and Uryasev (2002), they derive some semi-explicit solutions. Many other generalizations along this direction can be seen in Cvitanić (2000), Nakano (2007), Sekine (2004) and references therein.

Another important generalization of the idea of quantile hedging is to apply this idea to some specific financial and insurance contracts. Some interesting references are Sekine (2000), Melnikov and Skornyakova (2005), Wang (2009), Klusik and Palmowski (2011) and the references therein.

Clearly, the criterion of optimization plays a critical role in constructing the optimal partial hedging strategies. Different criteria usually induce different optimal partial hedging strategies. Different ways to characterize risk will produce different hedging strategies. However, sometimes, the investor may not have a specific risk measure in mind. Probably, the investor wants to adopt a class of risk measures instead of a specific risk measure. This will lead to hedging problem under the general risk measures, which is a generalization of the risk measure based optimal hedging. Furthermore, by considering the hedging strategies under the general risk measures, we will be able to have some insights into the robustness of the optimal hedging strategy with respect to the risk measures. It will be interesting to study whether there exists optimal partial hedging strategy which is robust with respect to the risk measures. Here, by saying that the optimal strategy is robust with respect to the risk measures, we mean that the optimal strategy would not change dramatically when the risk measure changes. However, few previous literature addressed the problem of optimal partial hedging under a general risk measure or the robustness of the optimal partial hedging strategies with respect to the risk measures. Though Nakano (2004) use a quite general way to express the coherent risk measures, he does not investigate the partial hedging strategies from the perspective of general risk measures. In the context of insurance, some works have been done on finding the optimal reinsurance strategy when the risk is measured by a general risk measure. Interested readers can refer to Gajek and Zagrodny (2004), Balbás et al. (2009) and the references therein.

1.3 Review of Quantile Hedging Strategy

Since quantile hedging strategy, which is proposed by Föllmer and Leukert (1999), is one of the most popular and important partial hedging strategies in academic, we will briefly describe the ideas and results of quantile hedging in this section.

Assume that the discounted price process of the underlying is given as a semimartingale $S = (S_t)_{t\in[0,T]}$ on a probability space with filtration $(\mathcal{F}_t)_{t\in[0,T]}$. $\mathbb{P}$ denotes the physical probability measure, while $\mathbb{Q}$ denotes a equivalent martingale measure. In the complete case, $\mathbb{Q}$ will be unique. A self-financing strategy, which is defined by an initial capital $V_0 \geqslant 0$ and by a predictable process ζ, will be called admissible if the resulting value process V defined by

$$V_t = V_0 + \int_0^t \zeta_s dS_s \quad \forall t \in [0,T], \quad \mathbb{P} - a.s.$$

satisfies

$$V_t \geqslant 0 \quad \forall t \in [0,T], \quad \mathbb{P} - a.s.$$

Consider a contingent claim given by a nonnegative measurable random variable H. In the complete case, there exists a perfect hedge. Equivalently, there exists a predictable process ζ^H such that

$$\mathbb{E}^{\mathbb{Q}}[H|\mathcal{F}_t] = H_0 + \int_0^t \zeta_s^H dS_s \quad \forall t \in [0,T], \quad \mathbb{P} - a.s.$$

where $\mathbb{E}^{\mathbb{Q}}$ denotes the expectation under the equivalent martingale measure $\mathbb{Q}$. It means that the contingent claim H can be duplicated by the self-financing trading strategy (H_0, ζ^H) if the investor is able and willing to spend the required initial capital $H_0 = \mathbb{E}^{\mathbb{Q}}[H]$ on hedging. But if the hedging budget $\tilde{V}_0$ is less than H_0, then it is impossible to perform the perfect hedge. In this case, we need to find an optimal partial hedging strategy under some criterion. Quantile hedging strategy is the optimal strategy under the criterion of maximizing the probability that the hedging is successful. The hedge is successful means that the payoff of the hedging portfolio is larger than or equal to the payoff of the contingent claim. Thus the problem of quantile hedging is equivalent to looking for an admissible strategy (V_0, ζ) in order to maximize the following probability

$$\mathbb{P}\left[V_0 + \int_0^T \zeta_s dS_s \geqslant H\right]$$

under the constraint $V_0 \leqslant \tilde{V}_0$.

In Föllmer and Leukert (1999), the set $V_T \geqslant H$ is called the "success set" corresponding to the admissible strategy (V_0, ζ), where V_T is the time T value of the strategy. The idea of solving the problem is to translate the problem to finding the most powerful test. This can be done in two steps. The first step is to reduce the problem to the construction of a success set of maximal probability, which is the following proposition from Föllmer and Leukert (1999).

Proposition 1.3.1. *Let $\tilde{A} \in \mathcal{F}_T$ be a solution of the problem*

$$\mathbb{P}[A] = max$$

under the constraint

$$\mathbb{E}^{\mathbb{Q}}[HI_A] \leqslant \tilde{V}_0,$$

where $\mathbb{Q}$ is the unique equivalent martingale measure. Let $\tilde{\zeta}$ denote the perfect hedge for the knockout option $\tilde{H} = HI_{\tilde{A}}$, i.e.,

$$\mathbb{E}^{\mathbb{Q}}[HI_{\tilde{A}}|\mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}[HI_{\tilde{A}}] + \int_0^t \tilde{\zeta}_s dS_s \quad \forall t \in [0,T], \quad \mathbb{P} - a.s.$$

Then $(V_0, \tilde{\zeta})$ solves the quantile hedging problem and the corresponding success set coincides almost surely with $\tilde{A}$.

The second step is addressing the problem that how to construct the maximal success set. This can be solved by applying the Neyman-Pearson lemma. In order to use the Neyman-Pearson lemma, Föllmer and Leukert (1999) introduce another measure $\mathbb{Q}^*$ given by

$$\frac{d\mathbb{Q}^*}{d\mathbb{Q}} = \frac{H}{\mathbb{E}^{\mathbb{Q}}[H]} = \frac{H}{H_0}.$$

With this, the price of the knockout option HI_A can be expressed as a constant times the probability of A under the probability measure $\mathbb{Q}^*$, i.e.

$$\mathbb{E}^{\mathbb{Q}}[HI_A] = \mathbb{E}^{\mathbb{Q}^*}\left[HI_A \cdot \frac{H_0}{H}\right] = H_0 \cdot \mathbb{Q}^*(A).$$

Therefore, the budget constraint can be rewritten as

$$\mathbb{Q}^*(A) \leqslant \frac{\tilde{V}_0}{H_0}.$$

With these notations, the maximal success set can be proved to be of the form

$$\tilde{A} = \left\{ \frac{d\mathbb{P}}{d\mathbb{Q}} > const \cdot \frac{d\mathbb{Q}^*}{d\mathbb{Q}} \right\}.$$

The following theorem from Föllmer and Leukert (1999) tells us how to construct quantile hedging strategy which maximizes the probability that the hedge is successful.

Theorem 1.3.1. *Assume that the set $\tilde{A}$ satisfies*

$$\mathbb{Q}^*(\tilde{A}) = \frac{\tilde{V}_0}{H_0}$$

where the set $\tilde{A}$ is of the form

$$\tilde{A} = \left\{ \frac{d\mathbb{P}}{d\mathbb{Q}} > const \cdot H \right\}.$$

Then quantile hedging strategy is given by $(\tilde{V}_0, \tilde{\zeta})$ where $\tilde{\zeta}$ is the perfect hedge for the knockout option $HI_{\tilde{A}}$.

Define the level

$$\tilde{a} = \inf \left\{ a : \mathbb{Q}^* \left[\frac{d\mathbb{P}}{d\mathbb{Q}} > a \cdot H \right] \leqslant \frac{\tilde{V}_0}{H_0} \right\}.$$

Then the set $\tilde{A}$ in the above theorem can be written as

$$\tilde{A} = \left\{ \frac{d\mathbb{P}}{d\mathbb{Q}} > \tilde{a} \cdot H \right\}$$

as long as $\mathbb{P}\left[\frac{d\mathbb{P}}{d\mathbb{Q}} = \tilde{a} \cdot H\right] = 0$.

In the case that $\mathbb{P}\left[\frac{d\mathbb{P}}{d\mathbb{Q}} = \tilde{a} \cdot H\right] > 0$, Föllmer and Leukert (1999) consider the partial hedging strategy which maximizes the “success ratio”, which is defined as follows

Definition 1.3.1. *For any admissible strategy* (V_0, ζ)*, the corresponding "success ratio" is defined as*

$$\varphi = I_{\{H \leqslant V_T\}} + \frac{V_T}{H} I_{\{V_T < H\}}.$$

By using the similar idea, Föllmer and Leukert (1999) obtain the optimal hedging strategy, which is stated in the following theorem:

Theorem 1.3.2. *Let* $\tilde{\zeta}$ *denote the perfect hedge for the contingent claim* $\tilde{H} = H\tilde{\varphi}$ *where* $\tilde{\varphi}$ *is defined as follows*

$$\tilde{\varphi} = I_{\{\frac{d\mathbb{P}}{d\mathbb{Q}} > \tilde{a}H\}} + \gamma I_{\{\frac{d\mathbb{P}}{d\mathbb{Q}} = \tilde{a}H\}}$$

where

$$\gamma = \frac{\frac{\tilde{V}_0}{H_0} - \mathbb{Q}^* \left[\frac{d\mathbb{P}}{d\mathbb{Q}} > \tilde{a}H \right]}{\mathbb{Q}^* \left[\frac{d\mathbb{P}}{d\mathbb{Q}} = \tilde{a}H \right]}.$$

Then $(\tilde{V}_0, \tilde{\zeta})$ *maximizes the expected success ratio under all admissible strategies* (V_0, ζ) *with* $V_0 \leqslant \tilde{V}_0$.

The problem of minimizing the hedging cost for a given probability of success hedge is also solved in Föllmer and Leukert (1999) by using the same idea.

In Föllmer and Leukert (2000), the optimal partial hedging strategy which minimizes the expected shortfall is considered. By saying expected shortfall, Föllmer and Leukert (2000) refer to the expected value of the investor's shortfall risk. Throughout the thesis, we will use the same definition of expected shortfall as that in Föllmer and Leukert (2000) and call this optimal partial hedging strategy as "expected shortfall hedging strategy".

Using the same notations, the problem can be formulated as to minimize the expected shortfall

$$\mathbb{E}^{\mathbb{P}} \left[(H - V_T)_+ \right]$$

under the constraint $V_0 \leqslant \tilde{V}_0$. By using the same idea, Föllmer and Leukert (2000) apply the Neyman-Pearson lemma to obtain the optimal strategy. The results are stated in the following theorem.

Theorem 1.3.3. *Let $\tilde{\zeta}$ denote the perfect hedge for the contingent claim $\tilde{H} = H\tilde{\varphi}$ where $\tilde{\varphi}$ is defined as follows*

1. *If* $\mathbb{Q}\left[\left\{\frac{d\mathbb{P}}{d\mathbb{Q}} = \tilde{a}\right\} \bigcap \{H > 0\}\right] > 0$,

$$\tilde{\varphi} = I_{\{\frac{d\mathbb{P}}{d\mathbb{Q}} > \tilde{a}\}} + \gamma I_{\{\frac{d\mathbb{P}}{d\mathbb{Q}} = \tilde{a}\}}$$

   where

$$\gamma = \frac{\tilde{V}_0 - \int_{\{\frac{d\mathbb{P}}{d\mathbb{Q}} > \tilde{a}\}} H d\mathbb{Q}}{\int_{\{\frac{d\mathbb{P}}{d\mathbb{Q}} = \tilde{a}\}} H d\mathbb{Q}}$$

$$\tilde{a} = \inf\{a| \int_{\{\frac{d\mathbb{P}}{d\mathbb{Q}} > \tilde{a}\}} H d\mathbb{Q} \leqslant \tilde{V}_0\}$$

2. *If* $\mathbb{Q}\left[\left\{\frac{d\mathbb{P}}{d\mathbb{Q}} = \tilde{a}\right\} \bigcap \{H > 0\}\right] = 0$,

$$\tilde{\varphi} = I_{\{\frac{d\mathbb{P}}{d\mathbb{Q}} > \tilde{a}\}}.$$

Then $(\tilde{V}_0, \tilde{\zeta})$ maximizes the expected success ratio under all admissible strategies (V_0, ζ) with $V_0 \leqslant \tilde{V}_0$. Equivalently, $(\tilde{V}_0, \tilde{\zeta})$ minimizes the expected shortfall.

1.4 The Objectives and Outline

The main objective of the thesis is to develop theoretically sound and practical solutions to the risk measures based optimal partial hedging problems. Such an objective is achieved by several steps in the thesis. First, since VaR and CVaR are popular and significant risk measures in both academia and industry, the VaR-based and CVaR-based optimal partial hedging strategies are studied in detail respectively. Explicit strategies are derived under these two criteria. Then we consider the optimal partial hedging strategies under general risk measures. We also study the robustness of the optimal partial hedging strategies with respect to risk measures. Lastly, we consider the simulation-based partial hedging models. In the simulation-based models, we can numerically obtain the optimal partial hedging strategies with flexible constraints under various objectives. Since there is some intrinsic similarities between hedging and reinsurance, we will use the similar idea and approach to address some interesting reinsurance models in the thesis as well.

There are several significant differences between our proposed partial hedging strategies and those in literature. First of all, most of the existing research on partial hedging share the same idea of formulating the optimal partial hedging problem as one of identifying the most powerful test. In the thesis, we will be using a totally different approach to investigate the partial hedging problem. We will solve the optimal partial hedging problem by first investigating an optimal partition between the hedged loss and the retained loss, and then analyzing the specific hedging strategy. Secondly, in most of the literature mentioned in Section 1.2, the optimal hedging strategy is closely related to the market structure, or more specifically, to the price process. Once the price process can not accurately describe the movement of the security's price accurately, the derived hedging strategy may be very different from the optimal strategy. In this thesis, we will consider the problem of optimal hedging strategy without imposing specific assumptions on the price process. Therefore, the hedging strategy we derived is independent of the market structure. This is one of the main differences between the hedging strategies we proposed and most hedging strategies in literature. Thirdly, another main difference between the hedging strategies we proposed and most hedging strategies in literature is the assumptions on the hedging strategy, which will be discussed in more detail in the following chapters. Last but not least, we consider some optimal hedging problems that is rare in literature. Few literature considers the optimal hedging problems under the general risk measures. In the thesis, we provide the sufficient and necessary optimality conditions of the partial hedging strategies under the general risk measures. By doing this, we are able to not only characterize the optimal strategies under different classes of risk measures, but also analyze the robustness of the optimal strategies with respect to the risk measures. In order to relaxing the assumptions imposed on the hedged loss functions, we reformulate the hedging problem by using the simulation-based method, which is rare in the context of hedging in literature. As we will see in the examples in Chapter 6, the numerical results in the our simulation-based hedging model coincide with our established theoretical results.

The rest of the thesis is organized as follows. In Chapter 2, we formulate the partial hedging model under the criterion of minimizing VaR of the investor's total exposed risk. We analytically derive the optimal form of the partial hedging strategy under two different admissible sets of the hedged loss functions. Some numerical examples and comparison between our proposed VaR-based hedging strategies and quantile hedging strategy are provided. Our result shows that the VaR-based hedging strategies we proposed are more robust than quantile hedging strategy in the sense that the structures of our proposed VaR-based hedging strategies are model independent while the structure of quantile hedging strategy is sensitive to the model specifications. In Chapter 3, we analytically derive the optimal form of the partial hedging strategy which minimizes CVaR of the investor's total exposed risk under two different market assumptions, namely no arbitrage pricing and stop-loss order preserving pricing. We further compare our proposed CVaR-based partial hedging strategies with some other partial hedging strategies, including quantile hedging strategy, expected shortfall hedging strategy proposed in Föllmer and Leukert (2000), and

VaR-based partial hedging strategies that are derived in Chapter 2. In Chapter 4, we will extend our previous results and ideas in the context of reinsurance. We study the optimal reinsurance problem under the criteria of minimizing VaR and a newly proposed monotonic piecewise premium principle. This class of premium principles is quite general in that it encompasses many of the commonly studied premium principles. Additionally, we will also investigate the optimal reinsurance in the context of multiple reinsurers as well as two new variants of the optimal reinsurance models. In Chapter 5, we consider the partial hedging problem under the general risk measures. The necessary and sufficient optimality conditions of the hedging strategy are provided. The robustness of the optimal hedging strategy derived in Chapter 3 is reinvestigated in this chapter. As an example of general risk measures, we also discuss the partial hedging strategies under general spectral risk measures. In Chapter 6, we use the simulation-based approach to reformulate the partial hedging models. A numerical example in Black-Scholes model is studied in detail in this chapter. The numerical results in this chapter support the theoretical results in the previous chapters. Some preliminary analyses on the convergence of the simulation-based solutions are also conducted in this part. At the end of the thesis, we state some potential research topics in Chapter 7.

The following flowchart (Figure 1.1) provides a road map on how the entire thesis is structured. Notation used in the flow chart will be explained in the respective chapter.

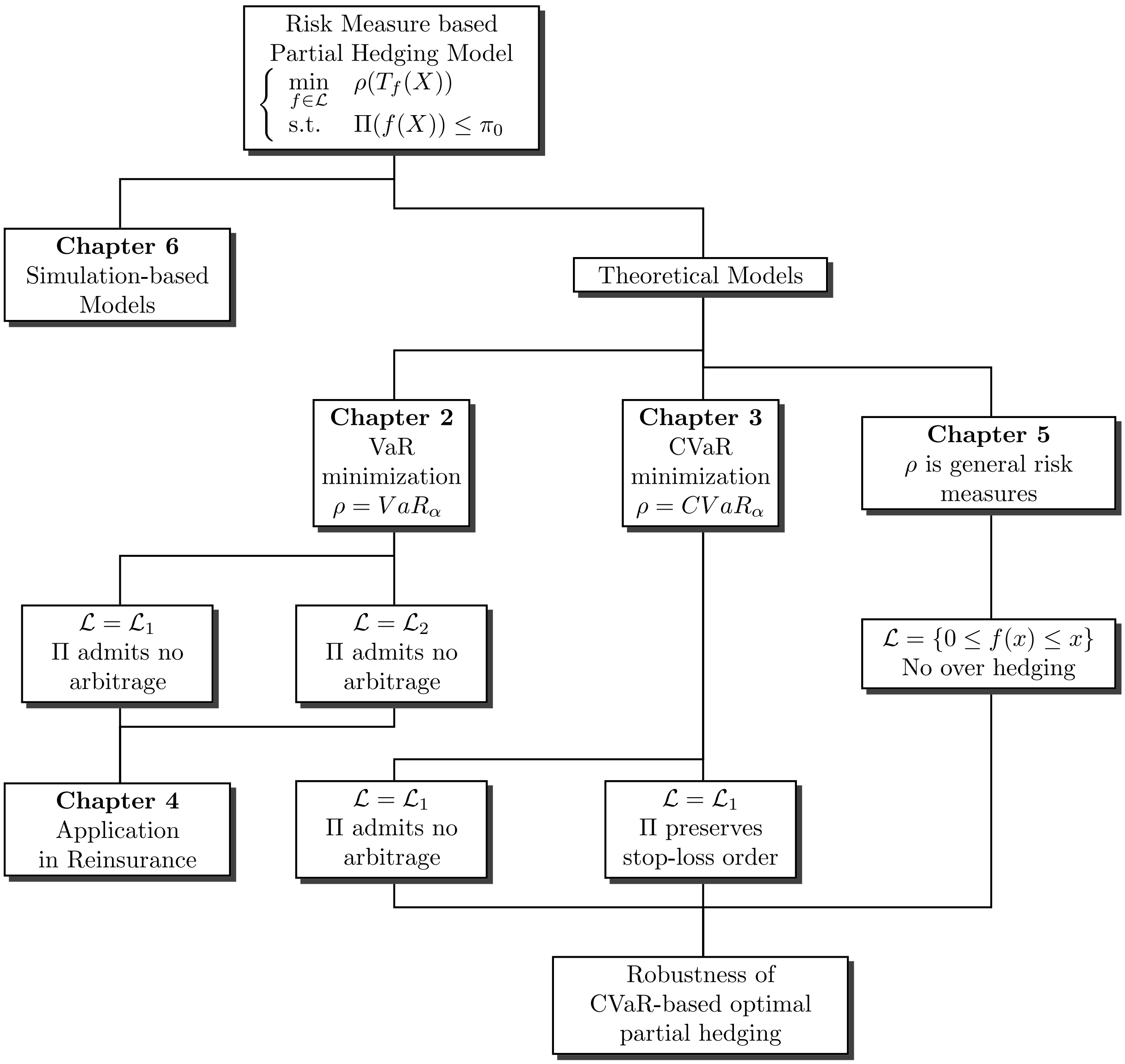

Figure 1.1: structure of thesis

Chapter 2

VaR Minimization Models

2.1 Preliminaries

Given the popularity of using Value-at-Risk (VaR) as a risk measure in literature and in practice, in this chapter we will study the optimal partial hedging strategies which minimize Value-at-Risk (VaR) of the investor's total exposed risk given some budget constraint.

In this chapter, a general risk measure based optimal partial hedging model is first proposed. Then by confining ourselves to a special case which involves minimizing Value-at-Risk (VaR) of the total exposed risk of a hedger for a given hedging budget constraint, we derive the analytic solutions under two admissible sets of hedging strategies (see Subsection 2.1.2 for their definitions and justifications).

2.1.1 Model Description and Notations

We suppose that a hedger is exposed to a future obligation X at time T and that his objective is to hedge X. We emphasize that X can be any function of the index or the price of a specific stock, i.e. $X = H(S_t, 0 \leqslant t \leqslant T)$, where S_t denotes the time t value of the index or price of a specific stock and H is a functional. Without loss of generality, we assume that X is a non-negative random variable with cumulative distribution function (c.d.f.) $F_X(x) = \mathbb{P}(X \leqslant x)$ and $\mathbb{E}(X) < \infty$ under the physical probability measure $\mathbb{P}$.

Our approach of addressing the optimal partial hedging problem is conducted in two steps. In the first step, we study the optimal partitioning of X into $f(X)$ and $R_f(X)$; i.e. $X = f(X) + R_f(X)$. Here $f(X)$ denotes the part of the payout to be hedged with a predetermined budget, and $R_f(X)$ represents the part of the payout to be retained. We use π_0 to denote the initial hedging budget. As functions of x, we call $f(x)$ and $R_f(x)$ the hedged loss function and the retained loss function respectively. In the second step, we investigate the possibility of replicating the time-T payout $f(X)$ in the market.

Let Π denote the risk pricing functional so that $\Pi(X)$ is the time-0 market price of the contingent claim with payout X at time T. Similarly, $\Pi(f(X))$ is the time-0 market price of $f(X)$ and this also corresponds to the time-0 cost of performing hedging strategy f. In this chapter, we do not need to specify the pricing functional $\Pi(\cdot)$, but we assume that it admits no arbitrage opportunity in the market.

Assuming the initial cost of performing hedging strategy f accumulates with interest at a risk-free rate r, then $T_f(X)$, which is defined as

$$T_f(X) = R_f(X) + e^{rT} \cdot \Pi(f(X)), \tag{2.1}$$

can be interpreted as the hedger's total time-T risk exposure from implementing the partial hedge strategy f since $R_f(X)$ denotes the time-T retained risk exposure. Note that $T_f(X)$

also succinctly captures the risk and reward tradeoff of the partial hedging strategy. On one hand, if the hedger is more conservative in that he is willing to spend more on hedging, then a greater portion of the initial risk will be hedged so that the retained risk $R_f(X)$ will be smaller. On the other hand, if the hedger is more aggressive in that he is willing to spend less on hedging, then this can be achieved at the expense of a higher retained risk exposure $R_f(X)$. Consequently, the problem of partial hedging boils down to the optimal partitioning of X into $f(X)$ and $R_f(X)$ for a given hedging budget constraint π_0, and one possible formulation of the optimal partial hedging problem can be described as follows:

$$\begin{cases} \min\limits_{f \in \mathcal{L}} & \rho(T_f(X)) \\ \text{s.t.} & \Pi(f(X)) \leqslant \pi_0, \end{cases} \tag{2.2}$$

where $\rho(\cdot)$ is an appropriately chosen risk measure for quantifying the total risk exposure $T_f(X)$ and $\mathcal{L}$ denotes an admissible set of hedged loss functions.

We emphasize that the risk measure based partial hedging model (2.2) is quite general in that it permits an arbitrary risk measure as long as it reflects and quantifies the hedger's attitude towards risk. Risk measures such as Value-at-Risk (VaR), Conditional Value-at-Risk (CVaR), Entropic risk measure, variance, among many others, are reasonable choices. In this chapter, we analyze the optimal partial hedging model by setting the risk measure ρ to Value-at-Risk (VaR). Despite its shortcomings such as lacking coherence property (see Artzner et al., 1999), VaR remains prominent among financial institutions and regulatory authorities for quantifying risk (see Jorion, 2006). Formally, VaR is defined as follows:

Definition 2.1.1. *The VaR of a non-negative variable X at the confidence level $(1-\alpha)$ with $0 < \alpha < 1$ is defined as*

$$VaR_\alpha(X) = \inf\{x \geqslant 0 : \mathbb{P}(X > x) \leqslant \alpha\}.$$

The constant α, which is typically a small value such as 1% or 5%, reflects the the desired confidence level of the investor.

Although we confine ourselves to VaR risk measure in the following analysis in this chapter, our derivations can also be applicable to tail conditional median (TCM), see Kou et al. (2012). TCM is defined as follows:

Definition 2.1.2. *The TCM of a non-negative variable X at the level α with $0 < \alpha < 1$ is defined as*

$$TCM_\alpha(X) = median\{X|X \geqslant VaR_\alpha(X)\}.$$

If neither $\mathrm{VaR}_{\frac{1+\alpha}{2}}(X)$ nor $TCM_\alpha(X)$ equals to the discontinuity in the distribution of X, then $TCM_\alpha(X) = \mathrm{VaR}_{\frac{1+\alpha}{2}}(X)$.

2.1.2 Desired Properties of Hedged Loss Functions

In addition to specifying the risk measure ρ in model (2.2), we also need to define the admissible set $\mathcal{L}$; otherwise, the formulation is ill-posed in that a position with an infinite number of certain assets (long or short) in the market is optimal. Similar issue has been observed in quantile hedging and CVaR hedging, and a standard technique of alleviating this issue is to impose some additional conditions or constraints in the optimization problem. For example, the hedged loss functions in both quantile hedging of Föllmer and Leukert (1999) and CVaR dynamic hedging of Melnikov and Smirnov (2012) are restricted to be nonnegative. Alexander, et al. (2004), on the other hand, introduce an additional term (which reflects the cost of holding an instrument) to the objective function in a CVaR-based hedging problem.

Before specifying the admissible sets of hedged loss functions, we now consider the following properties:

P1. Not globally over-hedged: $f(x) \leqslant x$ for all $x \geqslant 0$.

P2. Not locally over-hedged: $f(x_2) - f(x_1) \leqslant x_2 - x_1$ for all $0 \leqslant x_1 \leqslant x_2$.

P3. Nonnegativity of the hedged loss: $f(x) \geqslant 0$ for all $x \geqslant 0$.

P4. Monotonicity of the hedged loss function: $f(x_2) \geqslant f(x_1)$ $\forall$ $0 \leqslant x_1 \leqslant x_2$.

Note that property **P2** is equivalent to the following

P2'. Monotonicity of the retained loss function: $R_f(x_2) \geqslant R_f(x_1)$ $\forall$ $0 \leqslant x_1 \leqslant x_2$.

In this chapter, we analyze the optimal partial hedging strategy under two overlapping admissible sets of hedged loss functions. The first set assumes that the hedged loss functions satisfy properties **P1**-**P3** while the second set imposes property **P4** in addition to **P1**-**P3**. Without loss of too much generality we assume that the retained loss function $R_f(x)$ is left continuous with respect to x. These two admissible sets, with formal definitions given below, are labeled as $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively:

$$\begin{aligned}\mathcal{L}_1 =& \{0 \leqslant f(x) \leqslant x : R_f(x) \equiv x - f(x) \text{ is a nondecreasing and left} \\ & \text{continuous function}\},\end{aligned} \tag{2.3}$$

$$\begin{aligned}\mathcal{L}_2 =& \{0 \leqslant f(x) \leqslant x : \text{both } R_f(x) \text{ and } f(x) \text{ are nondecreasing functions,} \\ & R_f(x) \text{ is left continuous}\}.\end{aligned} \tag{2.4}$$

Note that $\mathcal{L}_2 \subset \mathcal{L}_1$.

We now provide some justifications on the above properties for the hedged loss functions. Property **P1** is reasonable as it ensures that the hedged loss should be uniformly bounded from above by the original risk to be hedged. Property **P2** indicates that the increment of the hedged part should not exceed the increment of the risk itself. If the hedger feels comfortable having a nondecreasing retained loss function, then **P2** will be necessary. While imposing **P2** makes the admissible set of the hedging functions more restrictive, it is reassuring from the numerical examples to be presented in Subsection 2.3.2 that the expected shortfalls of our proposed VaR-based optimal partial hedging strategies are still significantly smaller than that under quantile hedging strategy. Moreover, it will become clear shortly that with property **P2**, the resulting optimal partial hedging strategy will be model independent. This means that the structure of the optimal hedging strategy remains unchange irrespective of the assumptions on the dynamics of the underlying asset price.

We note that it is possible to relax property **P2** to a relatively weaker condition of the form

$$R_f(x_2) \geqslant R_f(VaR_\alpha(X)) \geqslant R_f(x_1) \; \forall 0 \leqslant x_1 \leqslant VaR_\alpha(X) \leqslant x_2$$

where $1-\alpha$ is the confidence level adopted by the hedger. This can be accomplished by a simple modification in the proof of our main results in Theorem 2.2.1 and Theorem 2.2.3.

Property **P3** is not only commonly imposed in the literature related to quantile hedging, its importance is further highlighted in the following example which shows that the partial hedging problem (2.2) is still ill-posed if we only impose properties **P1** and **P2**.

Example 2.1.1. *Suppose we wish to partially hedge a payout X, which is nondecreasing as a function of the stock price S so that S is nondecreasing in X as well. Take a constant K_0 large enough such that $K_0 > VaR_\alpha(S)$, and consider the hedged loss $f_n(X) = -n(S-K_0)_+$ indexed by positive integers n. Clearly, in this case both properties* **P1** *and* **P2** *are satisfied by the hedged loss $f_n(X)$. Since $K_0 > VaR_\alpha(S)$ and X is nondecreasing in S, $VaR_\alpha(X) = VaR_\alpha(X - f_n(X))$ for any $n > 0$, which implies that, if we do not consider the premium received by the hedger, the payout of selling the call option with strike price K_0 will not* *affect VaR of the hedger's risk exposure. Therefore, by selling one unit of the call option with strike price K_0, the hedger can decrease his VaR by the premium he receives, which is the price of the call option. It follows that the more units the hedger sells of the call options, the smaller VaR of his total exposed risk. In this case, the optimal hedging strategy is to sell an infinite units of the call options with strike price K_0. With this hedging strategy, VaR of the hedger's total exposed risk is negative infinity. However, such a hedging strategy is not a desirable hedging strategy as it is obviously a kind of gamble.* □

Remark 2.1.1. *(a) In Example 2.1.1, selling the call option on the stock S with strike price K_0 is not the only choice to decrease VaR of the hedger's total exposed risk. In fact, selling any contract whose payout is zero with probability larger than $1 - \alpha$ is able to decrease VaR of the hedger's total exposed risk.*

(b) Example 2.1.1 indicates that if we only impose properties **P1** *and* **P2***, the optimal hedging strategy is to sell as many "lotteries" as possible. Here, the term "lottery" refers to a financial contract whose payout is zero with very high probability (larger than $1 - \alpha$ in the above example).*

(c) The situation illustrated above is not unique to the VaR-based partial hedging model. It also occurs in the context of quantile hedging; see Section 1.3 for more details.

We assume that the hedger's primary objective is to hedge the payout X rather than gambling. Consequently, selling "lottery" is not an acceptable partial hedging strategy. This situation can be avoided by imposing some additional constraints on the admissible set $\mathcal{L}$, in addition to properties **P1** and **P2**. This leads to property **P3**; the same condition is also imposed in Föllmer and Leukert (1999) to eliminate the ill-posedness of the quantile hedging problem.

Apart from analyzing the optimal hedging strategy under properties **P1-P3**, we are also interested in the optimal solution of the partial hedging problem by imposing the monotonicity condition on the hedged loss function (i.e. property **P4**). By doing so, the admissible set $\mathcal{L}_2$ is even more restrictive than the admissible set $\mathcal{L}_1$. However, the monotonicity condition of the hedged loss function sometimes is crucial, especially when the hedger has a greater concern with the tail risk. Property **P4** ensures that the protection level will not decline as the risk exposure X gets larger. Without such a condition, it is possible for the hedger to have some or full protection for small losses and yet no protection against the extreme losses. This phenomenon seems counter-intuitive, particularly from the risk management point of view. We will further highlight this situation in the numerical examples in Subsection 2.3.2.

As will become clear later, we can see that by restricting the hedged loss functions in either $\mathcal{L}_1$ or $\mathcal{L}_2$, the optimal partial hedging strategy will not be some extreme gambling strategies.

2.2 VaR optimization

Recall that our proposed optimal partial hedging model corresponds to the optimization problem (2.2). By using VaR as the relevant risk measure ρ for a given confidence level $1-\alpha \in (0,1)$, optimization problem (2.2) can be rewritten as follows

$$\begin{cases} \min\limits_{f\in\mathcal{L}} & VaR_\alpha(T_f(X)) \\ \text{s.t.} & \Pi(f(X)) \leqslant \pi_0, \end{cases} \tag{2.5}$$

The objective of this section is to identify the solution to the optimization problem (2.5) under either the admissible set $\mathcal{L}_1$ as defined in (2.3) or $\mathcal{L}_2$ as defined in (2.4). These two cases are discussed in details in Subsections 2.2.1 and 2.2.2 respectively.

2.2.1 Optimality of the Knock-out Call Hedging

This subsection focuses on the VaR-based optimal partial hedging problem under the admissible set $\mathcal{L}_1$ as defined in (2.3). We will show that the so-called knock-out call hedging is optimal among all the hedging strategies in $\mathcal{L}_1$. We achieve this objective by demonstrating that given any partial hedging strategy f from the admissible set $\mathcal{L}_1$, the knock-out call hedging strategy g_f constructed from f leads to a smaller VaR of the total risk exposure of the hedger. More precisely, suppose g_f is constructed from $f \in \mathcal{L}_1$ as follows:

$$g_f(x) = \begin{cases} (x + f(v) - v)_+ \quad , & \text{if} \quad 0 \leqslant x \leqslant v, \\ \\ 0, & \text{if} \quad x > v, \end{cases} \tag{2.6}$$

where $v = \mathrm{VaR}_\alpha(X)$ and $(x)_+$ equals to x if $x > 0$ and zero otherwise. We first note that for any $f \in \mathcal{L}_1$, the function g_f constructed according to (2.6) is an element in $\mathcal{L}_1$. Second, for an arbitrary choice of f, $g_f(X)$ is the knock-out call option written on X with strike $v - f(v)$ and knock-out barrier v. For any given hedged loss function $f \in \mathcal{L}_1$, (2.6) provides a corresponding hedged loss function $g_f \in \mathcal{L}_1$ in the form of a knock-out call hedging strategy. If we can demonstrate that the hedged loss function g_f outperforms the hedged loss function f in the sense that former function results in a smaller VaR of the hedger's risk exposure, then we can conclude that the knock-out call hedging g_f is optimal among all the admissible strategies in $\mathcal{L}_1$. The following Theorem 2.2.1 confirms our assertion.

Theorem 2.2.1. *Assume that the market is complete and the pricing functional* Π *admits no arbitrage opportunity in the market. Then, the knock-out call hedged loss function* g_f *of the form* (2.6) *satisfies the following properties: for any* $f \in \mathcal{L}_1$*,*

(a) $\Pi(f(X)) \leqslant \pi_0$ *implies* $\Pi(g_f(X)) \leqslant \pi_0$*, and*

(b) $VaR_\alpha(T_{g_f}(X)) \leqslant VaR_\alpha(T_f(X))$.

Proof: (a) It follows from properties **P1-P3** that, for any $f \in \mathcal{L}_1$,

$$f(x) \geqslant (x + f(v) - v)_+ = g_f(x), \forall\ 0 \leqslant x \leqslant v$$

and

$$g_f(x) = 0 \leqslant f(x), \forall\ x > v.$$

Thus, $g_f(x) \leqslant f(x), \forall\, x \geqslant 0$. The assumption of no arbitrage implies $\Pi(g_f(X)) \leqslant \Pi(f(X))$, which in turn leads to the required result.

(b) The translation invariance property of the VaR risk measure leads to

$$\begin{aligned}
\mathrm{VaR}_\alpha(T_f(X)) &= \mathrm{VaR}_\alpha(R_f(X)) + e^{rT} \cdot \Pi(f(X)) \\
&= R_f(\mathrm{VaR}_\alpha(X)) + e^{rT} \cdot \Pi(f(X)) \\
&= \mathrm{VaR}_\alpha(X) - f(\mathrm{VaR}_\alpha(X)) + e^{rT} \cdot \Pi(f(X)) \\
&\geqslant \mathrm{VaR}_\alpha(X) - g_f(\mathrm{VaR}_\alpha(X)) + e^{rT} \cdot \Pi(g_f(X)) \\
&= \mathrm{VaR}_\alpha(T_{g_f}(X)),
\end{aligned}$$

where the second equality is due to the left continuity and nondecreasing properties of $R_f(x)$ and Theorem 1 in Dhaene et al. (2002). □

Remark 2.2.1. *(a) Theorem 2.2.1 indicates that the knock-out call hedging strategy is optimal among all the strategies in $\mathcal{L}_1$. We note that the optimal knock-out call hedging is a knock-out call written on the risk X itself, instead of the one written on the asset that underlies the risk X.*

(b) The optimality of the knock-out call hedging is model independent. It does not depend on the dynamic of the underlying or the specific pricing functional.

(c) If the knock-out call option on the risk X is available from the financial market, then the optimal partial hedging strategy can easily be implemented via a simple static hedging strategy. Otherwise, the optimal partial hedging strategy is to replicate a knock-out call option on the risk X. The examples in Subsection 2.3.1 will exemplify this point.

In incomplete market, not every contingent claim is attainable. So if the market is incomplete, it is possible that the optimal partial hedging strategy in Theorem 2.2.1 is not attainable. However, Theorem 2.2.1 can be generalized to incomplete market with the concept of the super replication. A super replication portfolio of the time-T payoff A is the portfolio with the value that is at least as great as that of A at time-T. The super replication price of the time-T payoff A is defined as the minimum cost of constructing a super replication portfolio of the time-T payoff A. Mathematically, the super replication price of the time-T payoff A, which is denoted as $\Pi_S(A)$, can be expressed as follows

$$\Pi_S(A) = \inf\{\Pi(B) \;\big|\; \text{time } T \text{ value of } B \geqslant \text{time } T \text{ value of } A\} \tag{2.7}$$

The following corollary extends the results in Theorem 2.2.1 to incomplete market. In incomplete market, when the partial hedging strategy is not attainable, the super replication strategy of the partial hedging strategy is adopted.

Corollary 2.2.1. *Assume that the market is incomplete and there is no arbitrage opportunity in the market. For any attainable $f \in \mathcal{L}_1$, we can construct the knock-out call hedged* *loss function g_f of the form* (2.6)*. Denote g_f^S as the least cost super replication strategy of g_f among $\mathcal{L}_1$, i.e. $g_f^S \in \mathcal{L}_1$, $g_f^S(X)$ is attainable and $\Pi(g_f^S(X)) = \inf\{\Pi(\tilde{g}_f(X)) \mid \tilde{g}_f(X) \geqslant g_f(X), \tilde{g}_f \in \mathcal{L}_1\}$. Then the strategy g_f^S satisfies the following properties:*

(a) $\Pi(f(X)) \leqslant \pi_0$ *implies* $\Pi(g_f^S(X)) \leqslant \pi_0$, *and*

(b) $VaR_\alpha(T_{g_f^S}(X)) \leqslant VaR_\alpha(T_f(X))$.

Proof: (a) According to the proof of Theorem 2.2.1, we know that $g_f(x) \leqslant f(x), \forall\, x \geqslant 0$, which means that f is a super replication portfolio of g_f. Note that $f \in \mathcal{L}_1$, the definition of g_f^S immediately implies that $\Pi(g_f^S(X)) \leqslant \Pi(f(X))$, which leads to the desired result.

(b) Similar to the proof of part (b) of Theorem 2.2.1, we have

$$\begin{aligned}
\mathrm{VaR}_\alpha(T_f(X)) &= \mathrm{VaR}_\alpha(R_f(X)) + \Pi(f(X)) \\
&= R_f(\mathrm{VaR}_\alpha(X)) + \Pi(f(X)) \\
&= \mathrm{VaR}_\alpha(X) - f(\mathrm{VaR}_\alpha(X)) + \Pi(f(X)) \\
&\geqslant \mathrm{VaR}_\alpha(X) - g_f^S(\mathrm{VaR}_\alpha(X)) + \Pi(g_f^S(X)) \\
&= \mathrm{VaR}_\alpha(T_{g_f^S}(X)),
\end{aligned}$$

which completes the proof. □

Remark 2.2.2. *(a) Corollary 2.2.1 can be considered as an extension of Theorem 2.2.1. When the optimal partial hedging strategy in Theorem 2.2.1 is not attainable, the optimal strategy is to perform the least cost super replication strategy of the knock-out call hedging strategy among $\mathcal{L}_1$. When the optimal partial hedging strategy in Theorem 2.2.1 is attainable, its least cost super replication strategy is itself.*

(b) Presumably, when the knock-out call hedging strategy is not attainable, the super replication strategy of the knock-out call hedging strategy is not a knock-out call option on the risk any more. However, in such case, we still say that the optimal partial hedging strategy is knock-out call hedging strategy, for the reason that the optimal strategy is induced by the knock-out call hedging strategy.

In the following, we mainly focus on the case of complete market. Unless stating particularly, we assume the market is complete.

If we denote $d = \mathrm{VaR}_\alpha(X) - f(\mathrm{VaR}_\alpha(X)) = v - f(v)$, then the knock-out call function g_f defined in (2.6) can be succinctly represented as

$$g_f(x) = (x-d)_+ \cdot \mathbb{1}(x \leqslant v),$$

where $\mathbb{1}(\cdot)$ is the indicator function. Furthermore, it follows from Theorem 2.2.1 that the VaR-based partial hedging problem (2.5) under admissible set $\mathcal{L}_1$ can equivalently be rewritten as

$$\begin{cases} \min\limits_{0 \leqslant d \leqslant v} & \mathrm{VaR}_\alpha\left(X - (X-d)_+ \cdot \mathbb{1}(X \leqslant v) + e^{rT} \cdot \Pi\left[g_f(X)\right]\right) \\ \\ \text{s.t.} & \Pi[g_f(X)] \equiv \Pi\left[(X-d)_+ \cdot \mathbb{1}(X \leqslant v)\right] \leqslant \pi_0. \end{cases} \tag{2.8}$$

This is simply an optimization problem of only one variable and technically it is easily solved as demonstrated in the following theorem.

Theorem 2.2.2. *Assume that the market is complete and the pricing functional* Π *admits no arbitrage opportunity in the market.*

(a) *If the hedging budget* $\pi_0 \geqslant \Pi\left[X \cdot \mathbb{1}(X \leqslant v)\right]$, *then the optimizer to problem* (2.8) *is* $d^* = 0$ *and the corresponding minimal VaR of the hedger's total risk exposure at time* T *is* $e^{rT} \cdot \Pi\left[X \cdot \mathbb{1}(X \leqslant v)\right]$.

(b) If the hedging budget $\pi_0 < \Pi\left[X \cdot \mathbb{1}(X \leqslant v)\right]$*, then the optimizer to problem* (2.8) *is given by the solution* d^* *to the following equation*

$$\Pi\left[(X-d)_+ \cdot \mathbb{1}(X \leqslant v)\right] = \pi_0, \tag{2.9}$$

and the corresponding minimal VaR of the hedger's total risk exposure at time T *is* $d^* + e^{rT} \cdot \pi_0$*.*

Proof: First note that the pricing formula $\Pi\left[(X-d)_+ \cdot \mathbb{1}(X \leqslant v)\right]$ in the constraint is clearly nonincreasing in d. Thus, it is sufficient to show that the objective in problem (2.8) is nondecreasing in d as well.

Let $B(x) = x - (x-d)_+ \cdot \mathbb{1}(x \leqslant v)$. Then, the objective in problem (2.8) can be expressed as $\mathrm{VaR}_\alpha(B(X) + e^{rT} \cdot \Pi(g_f(X)))$. Moreover, the function B is obviously left continuous and nondecreasing, and hence a direct application of Theorem 1 in Dhaene et at. (2002) implies that $\mathrm{VaR}_\alpha(B(X)) = B(\mathrm{VaR}_\alpha(X))$, which, together with the fact that $d < \mathrm{VaR}_\alpha(X) \equiv v$, leads to

$$\begin{aligned}
\mathrm{VaR}_\alpha(B(X)) &= \mathrm{VaR}_\alpha\left(X - (X-d)_+ \cdot \mathbb{1}(X \leqslant v)\right) \\
&= \mathrm{VaR}_\alpha(X) - \left[\mathrm{VaR}_\alpha(X) - d\right]_+ \\
&= d.
\end{aligned}$$

Consequently, the translation invariance property of VaR implies equivalence between the objective function in (2.8) and the following expression

$$\mathrm{VaR}_\alpha(B(X) + e^{rT} \cdot \Pi(g_f(X))) = d + e^{rT} \cdot \Pi\left[(X-d)_+ \cdot \mathbb{1}(X \leqslant v)\right]. \tag{2.10}$$

Hence, it remains to show that the right-hand-side of (2.10) is indeed nondecreasing in d. We verify this by contradiction.

Assume that (2.10) is not nondecreasing in d. Then, there must exist two constants d_1 and d_2 satisfying $d_1 < d_2$ and

$$d_1 + e^{rT} \cdot \Pi\left[(X - d_1)_+ \cdot \mathbb{1}(X \leqslant v)\right] > d_2 + e^{rT} \cdot \Pi\left[(X - d_2)_+ \cdot \mathbb{1}(X \leqslant v)\right]. \tag{2.11}$$

Indeed, this condition implies an arbitrage opportunity which can be exploited by constructing the following portfolio:

(i) selling the contract $(X - d_1)_+ \cdot \mathbb{1}(X \leqslant v)$,

(ii) buying the contract $(X - d_2)_+ \cdot \mathbb{1}(X \leqslant v)$,

(iii) putting the net premium $\Delta := \Pi\left[(X - d_1)_+ \cdot \mathbb{1}(X \leqslant v)\right] - \Pi\left[(X - d_2)_+ \cdot \mathbb{1}(X \leqslant v)\right]$ in the bank account to earn interest at a constant rate r.

Since Π is assumed to admit no arbitrage opportunity, we must have $\Delta \geqslant 0$, which means that there is no initial cost to create the above portfolio. Nevertheless, its payoff at the expiration date T is positive almost surely as shown below:

$$\begin{aligned}
& e^{rT} \cdot \Pi\left[(X - d_1)_+ \cdot \mathbb{1}(X \leqslant v)\right] - e^{rT} \cdot \Pi\left[(X - d_2)_+ \cdot \mathbb{1}(X \leqslant v)\right] \\
& + (X - d_2)_+ \cdot \mathbb{1}(X \leqslant v) - (X - d_1)_+ \cdot \mathbb{1}(X \leqslant v) \\
\geqslant & e^{rT} \cdot \Pi\left[(X - d_1)_+ \cdot \mathbb{1}(X \leqslant v)\right] - e^{rT} \cdot \Pi\left[(X - d_2)_+ \cdot \mathbb{1}(X \leqslant v)\right] + d_1 - d_2 \\
> & 0,
\end{aligned}$$

where the first step is due to the fact that

$$(X - d_1)_+ \cdot \mathbb{1}(X \leqslant v) - (X - d_2)_+ \cdot \mathbb{1}(X \leqslant v) \leqslant d_2 - d_1,$$

and the second step is because of (2.11). The existence of an arbitrage opportunity violates our assumption on the pricing functional Π and thus this completes the proof. □

Remark 2.2.3. *By Theorem 2.2.2, the optimal partial hedged loss is given by* $f(X) = X \cdot \mathbb{1}(X \leqslant v)$ *for sufficiently large hedging budget (no less than* $\Pi\left[X \cdot \mathbb{1}(X \leqslant v)\right]$*). This implies that the optimal strategy is to hedge the entire risk up to the threshold level* v*. If the risk* X *is so large that it exceeds* v*, then the optimal hedging strategy is not to hedge at all. On the other hand when the hedging budget is limited, it is then optimal to hedge* $f(X) = (X - d^*)_+ \cdot \mathbb{1}(X \leqslant v)$ *and exhaust the entire hedging budget by determining the positive retention* d^* *which satisfies* (2.9)*. Note that in either scenario, it is optimal not to hedge at all when the risk is so extreme that it exceeds* v*. Even though such a hedged loss function seems counterintuitive, it can still be optimal, since the hedged loss function does not need to be nondecreasing under the admissible set* $\mathcal{L}_1$*.*

2.2.2 Optimality of the Bull Call Spread Hedging

In this subsection, we investigate the optimal solution of the VaR-based partial hedging problem under the admissible set $\mathcal{L}_2$ as defined in (2.4). Recall that compared to the admissible set $\mathcal{L}_1$ analyzed in the preceding subsection, the admissible set $\mathcal{L}_2$ is more restrictive in that it imposes the additional monotonicity condition on the hedged loss functions. As a result, the undesirable characteristic of the optimal hedging solution observed in the last subsection (see Remark 2.2.3) is excluded.

We will see shortly that the same technique can be used to derive the optimal hedging strategy under the more restrictive admissible set $\mathcal{L}_2$, and the so-called bull call spread hedging is an optimal hedging strategy to the VaR-based partial hedging problem. To proceed, for any hedged loss function $f \in \mathcal{L}_2$, we construct h_f as follows:

$$\begin{aligned} h_f(x) &= \min\left\{(x + f(\mathrm{VaR}_\alpha(X)) - \mathrm{VaR}_\alpha(X))_+, f(\mathrm{VaR}_\alpha(X))\right\}, \\ &= (x-d)_+ - (x-v)_+. \end{aligned} \tag{2.12}$$

Recall that $v = \mathrm{VaR}_\alpha(X)$ and $d = \mathrm{VaR}_\alpha(X) - f(\mathrm{VaR}_\alpha(X))$. Clearly, for any $f \in \mathcal{L}_2$, h_f constructed according to (2.12) is also an element in $\mathcal{L}_2$. The function $h_f(X)$ is commonly known as the bull call spread written on X; i.e. it consists of a long and a short call option written on the same underlying risk X with respective strike prices d and v such that $0 \leqslant d \leqslant v$. The following Theorem 2.2.3 states that the bull call spread on the underlying risk X is an optimal partial hedging strategy among $\mathcal{L}_2$.

Theorem 2.2.3. *Assume that the market is complete and the pricing functional* Π *admits no arbitrage opportunity in the market. Then, the bull call spread hedged loss function* h_f *defined in* (2.12) *satisfies the following properties: for any* $f \in \mathcal{L}_2$,

(a) $\Pi(f(X)) \leqslant \pi_0$ *implies* $\Pi(h_f(X)) \leqslant \pi_0$, *and*

(b) $VaR_\alpha(T_{h_f}(X)) \leqslant VaR_\alpha(T_f(X))$.

Proof: The proof is similar to that of Theorem 2.2.1. (a). Due to the no-arbitrage assumption on Π, the result $\Pi(h_f(X)) \leqslant \Pi(f(X))$ follows if we can show that $h_f(x) \leqslant f(x)$ for all $x \geqslant 0$. Indeed, properties **P1-P3** imply $f(x) \geqslant [x + f(v) - v]_+ = h_f(x)$ for $0 \leqslant x \leqslant v$, and property **P4** implies $h_f(x) = f(v) \leqslant f(x), \forall\ x \geqslant v$. (b). The proof is in parallel with that of part (b) of Theorem 2.2.1 and hence is omitted. □

Remark 2.2.4. *The comments we made in Remark 2.2.1 for the solutions among* $\mathcal{L}_1$ *are similarly applicable to the solutions among* $\mathcal{L}_2$ *established in Theorem 2.2.3. In particular, we draw the following remarks.*

(a) Theorem 2.2.3 indicates that the bull call spread hedging strategy is optimal among all the strategies in $\mathcal{L}_2$. *We note again that the optimal hedging strategy is to construct a*

bull call spread on the risk X, instead of a bull call spread on the asset that underlies X.

(b) The optimality of bull call spread hedging is model independent. It does not depend on the dynamic of the underlying or the specific pricing functional.

(c) If the bull call spread written on the risk X is available from the financial market, then the optimal partial hedging can be achieved via a simple static hedging strategy.

Similar to the last subsection, we can extend the results in Theorem 2.2.3 to incomplete market.

Corollary 2.2.2. *Assume that the market is incomplete and there is no arbitrage opportunity in the market. For any attainable $f \in \mathcal{L}_2$, we can construct the bull call spread hedged loss function h_f of the form (2.12). Denote h_f^S as the least cost super replication strategy of h_f among $\mathcal{L}_2$, i.e. $h_f^S \in \mathcal{L}_2$, $h_f^S(X)$ is attainable and $\Pi(h_f^S(X)) = \inf\{\Pi(\tilde{h}_f(X)) \mid \tilde{h}_f(X) \geqslant h_f(X), \tilde{h}_f \in \mathcal{L}_2\}$. Then the strategy h_f^S satisfies the following properties:*

(a) $\Pi(f(X)) \leqslant \pi_0$ *implies* $\Pi(h_f^S(X)) \leqslant \pi_0$, *and*

(b) $VaR_\alpha(T_{h_f^S}(X)) \leqslant VaR_\alpha(T_f(X))$.

Proof: The proof is in parallel with that of Corollary 2.2.1 and hence is omitted. □

Remark 2.2.5. *The comments we made in Remark 2.2.2 are similarly applicable here.*

Based on the results from Theorem 2.2.3, it is easy to see that the VaR-based partial hedging problem (2.5) under admissible set $\mathcal{L}_2$ can be equivalently cast as

$$\begin{cases} \min\limits_{0 \leqslant d \leqslant v} & \mathrm{VaR}_\alpha \left\{X - (X-d)_+ + (X-v)_+ + e^{rT} \cdot \Pi\left[h_f(X)\right]\right\} \\ \text{s.t.} & \Pi\left[h_f(X)\right] = \Pi\left[(X-d)_+ - (X-v)_+\right] \leqslant \pi_0. \end{cases} \tag{2.13}$$

The optimal partial hedging problem is similarly reduced to an optimization problem of a single variable. Consequently, we have the following Theorem 2.2.4 as a counterpart of Theorem 2.2.2 that we have established in the previous subsection.

Theorem 2.2.4. *Assume that the market is complete and the pricing functional* Π *admits no arbitrage opportunity.*

(a) If the hedging budget $\pi_0 \geqslant \Pi\left[X-(X-v)_+\right]$*, then the optimizer of the problem (2.13) is* $d^*=0$ *and the corresponding minimal VaR of the hedger's total risk exposure at the expiration date* T *is* $e^{rT}\cdot\Pi\left[X-(X-v)_+\right]$.

(b) If the hedging budget $\pi_0 < \Pi\left[X-(X-v)_+\right]$*, then the optimizer of the problem (2.13) is given by the solution* d^* *to the following equation*

$$\Pi\left[(X-d^*)_+-(X-v)_+\right]=\pi_0, \tag{2.14}$$

and the corresponding minimal VaR of the hedger's total risk exposure at the expiration date T *is* $d^*+e^{rT}\cdot\pi_0$.

Proof: The proof is similar to that of Theorem 2.2.2 and hence is omitted. □

Remark 2.2.6. *Theorem 2.2.4 provides a very simple way of identifying the parameter values of the optimal hedged loss function. If the hedging budget is sufficiently large (i.e. greater than or equal to* $\Pi\left[X-(X-v)_+\right]$*), then the optimal strategy is to hedge all the risk except in the tail. In this case the optimal hedged loss is given by* $f(X)=\min(X,v)$*. However, if the hedging budget is limited, then the optimal strategy is to implement the bull call spread hedging and exhaust the entire hedging budget by determining the positive retention* d^* *with equation* (2.14).

2.3 Partial Hedging Examples: VaR vs. Quantile

In the previous section, we have analyzed the optimal hedged loss functions among the admissible sets $\mathcal{L}_1$ (see (2.3)) and $\mathcal{L}_2$ (see (2.4)) respectively. The optimal solution among $\mathcal{L}_1$ is the knock-out call hedging as formally established in Subsection 2.2.1 while the optimal solution among $\mathcal{L}_2$ is the bull call spread hedging as shown in Subsection 2.2.2. In Remarks 2.2.1 and 2.2.4, we respectively commented that the knock-out call hedging and the bull call spread hedging can usually be achieved by a static strategy in many situations. Subsection 2.3.1 provides some examples to further illustrate such a statement. The contingent claim X we will consider include a short position of stock, a European put option, an Asian call option and a barrier option. Subsection 2.3.2 gives some interesting comparison between our proposed partially hedge strategy and the quantile hedging strategy.

2.3.1 Partial Hedging Examples

Example 2.3.1. *Suppose that a hedger has short sold a stock and has to pay back the stock at time T. Now he intends to partially hedge the time-T payout $X = S_T$ of the short position. The optimal VaR-based partial hedging strategies under the respective admissible sets $\mathcal{L}_1$ and $\mathcal{L}_2$ are as follows:*

(a) Under admissible set $\mathcal{L}_1$: According to Theorem 2.2.1, the optimal hedging strategy among $\mathcal{L}_1$ is to hedge the part of loss given by

$$(X - d^*)_+ \cdot \mathbb{1}(X \leqslant v) = (S_T - d^*)_+ - (S_T - v)_+ - (v - d^*) \cdot \mathbb{1}(S_T \leqslant v),$$

where d^* is determined by the hedging budget π_0 as specified in Theorem 2.2.2. We assume that $v \equiv VaR_\alpha(X) = VaR_\alpha(S_T)$ is known. Given that knock-out call option on the under-

lying stock is available in the market, then from Theorem 2.2.2, the optimal partial hedging strategy can be constructed as follows, depending on the relative magnitude of π_0*:*

(i) *If the hedging budget* π_0 *is large enough such that* $\pi_0 \geqslant \Pi\left[S_T \cdot \mathbb{1}(S_T \leqslant v)\right]$*, then the optimal hedging strategy is to long the knock-out call option on the underlying stock with barrier level* v *and strike price* 0*. Under this strategy, the hedger perfectly hedges the risk except its tail. So he only retains the risk in the tail, but retains all the tail risk.*

(ii) *If the hedging budget* π_0 *is of small amount satisfying* $\pi_0 < \Pi\left[S_T \cdot \mathbb{1}(S_T \leqslant v)\right]$*, then the optimal hedging strategy is to long the knock-out call option on the underlying stock with barrier level* v *and strike price as low as possible so as to exhaust the entire hedging budget. Consequently, the budget constraint is binding in this case and the hedger again retains all the tail risk.*

(b) Under admissible set $\mathcal{L}_2$*: It follows from Theorem 2.2.3 that the optimal hedging strategy among* $\mathcal{L}_2$ *is to hedge the part of risk given by*

$$(X - d^*)_+ - (X - v)_+ = (S_T - d^*)_+ - (S_T - v)_+,$$

where d^* *depends on the hedging budget* π_0 *as specified in Theorem 2.2.4. Again, we assume that* $v \equiv VaR_\alpha(X) = VaR_\alpha(S_T)$ *is known. Given that the call option* $(S_T - v)_+$ *is available in the market, then from Theorem 2.2.4, the optimal partial hedging strategy can be constructed as follows, depending on the relative magnitude of* π_0*:*

(i) *If the hedging budget* π_0 *is large enough such that* $\pi_0 \geqslant S_0 - \Pi\left[(S_T - v)_+\right]$*, then the optimal hedging strategy is to long the stock and short a call option on the stock with strike price* v*. Under this strategy, the hedger only retains some risk in the tail.*

(ii) If the hedging budget π_0 is of small amount satisfying $\pi_0 < S_0 - \Pi[(S_T - v)_+]$, then the optimal hedging strategy is to first short a call option on the underlying stock with strike price v. The proceeds received from the short position, i.e. $\Pi[(S_T - v)_+]$, together with the initial hedging budget π_0, is used to invest in a call option on the same underlying stock with a strike price as low as possible so as to exhaust the entire amount of $\Pi[(S_T - v)_+] + \pi_0$. Consequently, the budget constraint is binding in this case and the hedging strategy mimics a bull call spread on the underlying stock.

For brevity, the remaining examples only discuss the optimal partial hedging strategies among $\mathcal{L}_2$. The optimal partial hedging strategies among $\mathcal{L}_1$ can be constructed in a similar fashion.

Example 2.3.2. *This example is concerned with partial hedging a European put option with its time-T payout given by $X = (K - S_T)_+$. Using Theorem 2.2.3, the optimal hedging strategy among $\mathcal{L}_2$ is to hedge the part of risk given by*

$$\begin{aligned}(X - d^*)_+ - (X - v)_+ &= ((K - S_T)_+ - d^*)_+ - ((K - S_T)_+ - v)_+ \\ &= (K - d^* - S_T)_+ - (K - v - S_T)_+,\end{aligned}$$

where d^* is determined by the hedging budget π_0 as specified in Theorem 2.2.4.

As in Example 2.3.1, we assume that $v \equiv VaR_\alpha(X)$ is known and the market price of the European put option with strike price $(K - v)_+$ (i.e. $\Pi[(K - v - S_T)_+]$) is observable from the market. Then, the optimal hedging strategy can be constructed according to Theorem 2.2.4 as follows:

(i) If the hedging budget π_0 is large enough such that $\pi_0 \geqslant \Pi[(K - S_T)_+] - \Pi[(K - v - S_T)_+]$, then the optimal hedging strategy is to long a European put option on the stock with

strike price K and at the same time short a European put option on the same underlying stock with strike price $(K-v)_+$.

(ii) If the hedging budget is of small amount satisfying $\pi_0 < \Pi\left[(K-S_T)_+\right] - \Pi\left[(K-v-S_T)_+\right]$, then the optimal hedging strategy consists of a short position in a European put option on the stock with strike price $(K-v)_+$ and a long position in a European put option on the same stock with a strike price as high as possible to exhaust the entire budget and the proceeds received from the short position.

Remark 2.3.1. *(a) In the previous example, if the options used to construct the hedging portfolio are not available in the market, we may directly replicate the payout of these options by a continuously rebalancing strategy on the stock.*

(b) The European call option can be partially hedged in a way similar to the European put option. We omit the specific procedure for brevity.

Example 2.3.3. *Suppose a hedger is to partially hedge an Asian call option with a time-T payout given by $X = \left(\frac{1}{T}\int_0^T S_t dt - K\right)_+$. According to Theorem 2.2.3, the optimal hedging strategy among $\mathcal{L}_2$ is to hedge the part of risk given by*

$$\begin{aligned}(X-d^*)_+ - (X-v)_+ &= \left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)_+ - d^*\right]_+ - \left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)_+ - v\right]_+ \\ &= \left(\frac{1}{T}\int_0^T S_t dt - K - d^*\right)_+ - \left(\frac{1}{T}\int_0^T S_t dt - K - v\right)_+,\end{aligned}$$

where d^* is determined by the hedging budget π_0 according to Theorem 2.2.4.

Again, we assume that $v \equiv VaR_\alpha(X)$ is known and the option price $\Pi\left[\left(\frac{1}{T}\int_0^T S_t dt - K - v\right)_+\right]$ can be observed from the market. Then, Theorem 2.2.4 implies the following optimal hedging strategies *(i) If the hedging budget is large enough such that*

$$\pi_0 \geqslant \Pi\left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)_+\right] - \Pi\left[\left(\frac{1}{T}\int_0^T S_t dt - K - v\right)_+\right],$$

then the optimal hedging strategy is to long an Asian call option on the stock with strike price K and at the same time short an Asian option on the same stock with strike price $(K+v)$.

(ii) If, however, the hedging budget is relatively small with

$$\pi_0 < \Pi\left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)_+\right] - \Pi\left[\left(\frac{1}{T}\int_0^T S_t dt - K - v\right)_+\right],$$

then the optimal hedging strategy is to first short an Asian call option on the stock with strike price $(K+v)$, and then use the proceeds, together with the hedging budget

$$\Pi\left[\left(\frac{1}{T}\int_0^T S_t dt - K - v\right)_+\right] + \pi_0,$$

to invest in an Asian call option on the same stock with a strike price as low as possible so as to exhaust the entire amount.

Remark 2.3.2. *While Example 2.3.3 illustrates how to apply Theorem 2.2.4 to construct the partial hedging strategy for the Asian call option, we can similarly construct the optimal hedging strategy for the Asian put option. However, when the strike price is floating, instead of being fixed, Theorem 2.2.4 cannot be applied directly for an effective hedging strategy, as in this case, the optimal hedged loss obtained from Theorem 2.2.4 may not be attainable in the market.*

Example 2.3.4. *Suppose a hedger is to partially hedge an up-and-in call option with a time-T payout $X = (S_T - K)_+ \cdot \mathbb{1}\left(\max_{0\leqslant t\leqslant T} S_t \geqslant H\right)$, where S_t is the time-t price of the stock.*

According to Theorem 2.2.3, the optimal hedging strategy among $\mathcal{L}_2$ is to hedge the part of risk given by

$$\begin{aligned}
& (X-d^*)_+ - (X-v)_+ \\
= & \left[(S_T-K)_+ \cdot \mathbb{1}\left(\max_{0\leqslant t\leqslant T} S_t \geqslant H\right) - d^*\right]_+ - \left[(S_T-K)_+ \cdot \mathbb{1}\left(\max_{0\leqslant t\leqslant T} S_t \geqslant H\right) - v\right]_+ \\
= & (S_T-K-d^*)_+ \cdot \mathbb{1}\left(\max_{0\leqslant t\leqslant T} S_t \geqslant H\right) - (S_T-K-v)_+ \cdot \mathbb{1}\left(\max_{0\leqslant t\leqslant T} S_t \geqslant H\right),
\end{aligned}$$

where d^* is determined by the hedging budget π_0 according to Theorem 2.2.4.

As in the previous examples, we assume we can accurately determine the value of $v \equiv VaR_\alpha(X)$ and can observe from the market the corresponding price

$$\Pi\left[(S_T-K-v)_+ \cdot \mathbb{1}\left(\max_{0\leqslant t\leqslant T} S_t \geqslant H\right)\right]$$

of the barrier option. As a result, Theorem 2.2.4 implies the following optimal partial hedging strategies.

(i) If the hedging budget is large enough such that

$$\pi_0 \geqslant \Pi\left[(S_T-K)_+ \cdot \mathbb{1}\left(\max_{0\leqslant t\leqslant T} S_t \geqslant H\right)\right] - \Pi\left[(S_T-K-v)_+ \cdot \mathbb{1}\left(\max_{0\leqslant t\leqslant T} S_t \geqslant H\right)\right],$$

then the optimal hedging strategy is to long an up-and-in call option on the stock with strike price K and barrier H and at the same time short an up-and-in call option on the same stock with strike price $(K+v)$ and barrier H.

(ii) If, however, the hedging budget is relatively small such that

$$\pi_0 < \Pi\left[(S_T-K)_+ \cdot \mathbb{1}\left(\max_{0\leqslant t\leqslant T} S_t \geqslant H\right)\right] - \Pi\left[(S_T-K-v)_+ \cdot \mathbb{1}\left(\max_{0\leqslant t\leqslant T} S_t \geqslant H\right)\right],$$

then the optimal hedging strategy is to first short an up-and-in call option on the stock with strike price $(K+v)$ and barrier H, and then the proceeds together with the *hedging budget, i.e.* $\Pi\left[(S_T-K-v)_+\cdot\mathbb{1}\left(\max\limits_{0\leqslant t\leqslant T}S_t\geqslant H\right)\right]+\pi_0$*, are used to invest in an up-and-in call option on the same stock with barrier* H *and a strike price as low as possible to exhaust the entire amount.*

Remark 2.3.3. *While Example 2.3.4 illustrates how to apply Theorem 2.2.4 to construct the partial hedging strategy for the up-and-in call option, we can similarly construct the optimal hedging strategy for other types of barrier options.*

2.3.2 A Comparison between VaR-based Partial Hedging and Quantile Hedging

In this subsection, we will conduct an example to highlight the difference between our proposed VaR-based partial hedging strategy and the well-known quantile hedging strategy proposed by Föllmer and Leukert (1999), which has been described in detail in Section 1.3.

Example 2.3.5. *We assume that the standard Black-Scholes market applies so that the dynamic of the stock price process is governed by the following stochastic differential equation:*

$$dS_t = S_t m dt + S_t \sigma dW_t, \quad t \geqslant 0,$$

where W *is a Wiener process under the physical measure* $\mathbb{P}$*, and* σ *and* m *are respectively the constant volatility and return rate of the underlying stock. The contingent claim that we are interested in hedging is a European call option with payout* $X_T = (S_T - K)_+$*. We use the same set of parameter values as that of one numerical example in Föllmer and Leukert (1999):*

$$S_0 = 100, \quad r = 0, \quad m = 0.08, \quad T = 0.25 \ \text{ and } \ K = 110.$$

To establish the optimal partial hedging strategy, we need to further specify the values of the volatility σ and the hedging budget π_0. We consider the following three scenarios:

(i) $\sigma = 0.3$, $\pi_0 = 1.5$,

(ii) $\sigma = 0.3$, $\pi_0 = 0.5$,

(iii) $\sigma = 0.2$, $\pi_0 = 0.5$.

Using the Black-Scholes formula, the prices of the corresponding European call options are

$$P_C = \begin{cases} 2.50, & \text{for } \sigma = 0.3; \\ 0.95, & \text{for } \sigma = 0.2. \end{cases}$$

By comparing the budget of the respective hypothetical volatility scenario to the above option prices, it is clear that the European call options can not be hedged perfectly. Given the limited hedging budget, it is therefore instructive and useful to develop alternate partial hedging strategies involving quantile hedging, knock-out call hedging, and bull call spread hedging. These three hedging strategies are discussed in detail below.

(a) Quantile hedging strategy:

For our assumed Black-Scholes model, Föllmer and Leukert (1999) show that quantile hedging strategy admits different form depending on the relative magnitude of m and σ. In particular we need to consider the following two cases:

(1) When $m \leqslant \sigma^2$, the hedged loss function of quantile hedging strategy is given by $f(X_T) = X_T \mathbb{1}_{\{X_T < c\}}$ and in our European call option case, this becomes

$$(S_T - K)_+ - (S_T - c)_+ - (c - K) \cdot \mathbb{1}(S_T > c), \tag{2.15}$$

where the constant c is determined by the following two equations through an auxiliary variable b:

$$\begin{cases} c = S_0 \exp\left(\sigma b - \frac{1}{2}\sigma^2 T\right) \\ \\ \pi_0 = P_C - S_0 \Phi\left(\dfrac{-b + \sigma T}{\sqrt{T}}\right) + K\Phi\left(\dfrac{-b}{\sqrt{T}}\right). \end{cases} \tag{2.16}$$

In the above, Φ denotes the standard normal cumulative distribution function.

(2) When $m > \sigma^2$, the hedged loss function of quantile hedging strategy is given by $f(X_T) = X_T \mathbb{1}_{\{X_T < c_1, \text{ or } X_T > c_2\}}$ and in our European call option case, this becomes

$$(S_T - K)_+ - (S_T - c_1)_+ - (c_1 - K) \cdot \mathbb{1}_{(S_T > c_1)} + (S_T - c_2)_+ + (c_2 - c_1)\mathbb{1}_{S_T > c_2}, \tag{2.17}$$

where c_1 and c_2 are two distinct constants satisfying the following system of equations with auxiliary variables b_1, b_2 and λ:

$$\begin{cases} c_1^{\frac{m}{\sigma^2}} = \lambda(c_1 - K)_+ \\ c_2^{\frac{m}{\sigma^2}} = \lambda(c_2 - K)_+ \\ c_1 = S_0 \exp\left(\sigma b_1 - \frac{1}{2}\sigma^2 T\right) \\ c_2 = S_0 \exp\left(\sigma b_2 - \frac{1}{2}\sigma^2 T\right) \\ \pi_0 = P_C - S_0\Phi\left(\frac{-b_1 + \sigma T}{\sqrt{T}}\right) + K\Phi\left(\frac{-b_1}{\sqrt{T}}\right) + S_0\Phi\left(\frac{-b_2 + \sigma T}{\sqrt{T}}\right) + K\Phi\left(\frac{-b_2}{\sqrt{T}}\right). \end{cases} \tag{2.18}$$

With the above setup, we are now ready to obtain quantile hedging strategies under each of the three scenarios (i)-(iii) specified above. For scenarios (i) and (ii), we have $m < \sigma^2$ so that quantile hedging strategy is of the form (2.15). The required constant c can be deduced by substituting the corresponding parameter values into the set of equations (2.16). To summarize, quantile hedging strategy is of the form

$$(S_T - 110)_+ - (S_T - 129.47)_+ - 19.47 \cdot \mathbb{1}(S_T > 129.47)$$

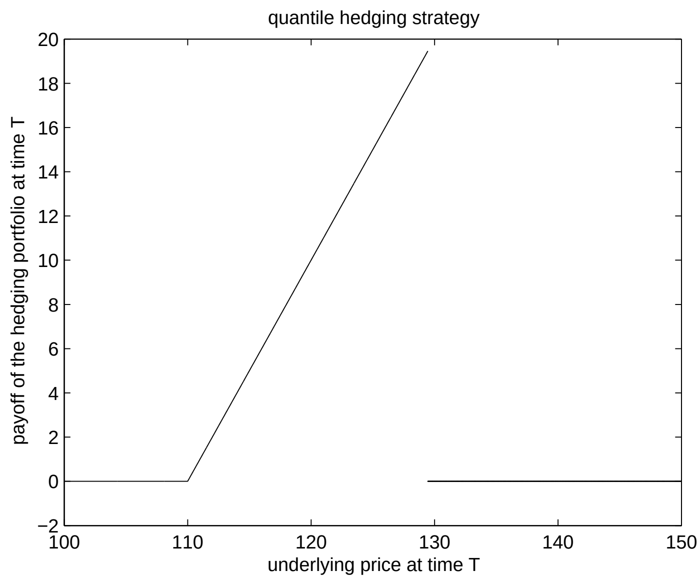

Figure 2.1: Optimal quantile hedging strategy in scenario (i)

for scenario (i), and

$$(S_T - 110)_+ - (S_T - 118.69)_+ - 8.69 \cdot \mathbb{1}(S_T > 118.69)$$

for scenario (ii). For scenario (iii), we have $m > \sigma^2$ so that quantile hedging strategy is of the form (2.17), and the respective constants c_1 and c_2 can be obtained by solving the system of equations (2.18) based on the assumed parameter values. The resulting optimal quantile hedging strategy becomes

$$(S_T-110)_+-(S_T-119.98)_+-9.98\cdot\mathbb{1}(S_T > 119.98)+(S_T-1323)_++1203.02\cdot\mathbb{1}(S_T > 1323).$$

The optimal hedged loss functions for the three scenarios are demonstrated in Figure 2.1, Figure 2.2 and Figure 2.3 respectively.

(b) Knock-out call hedging strategy:

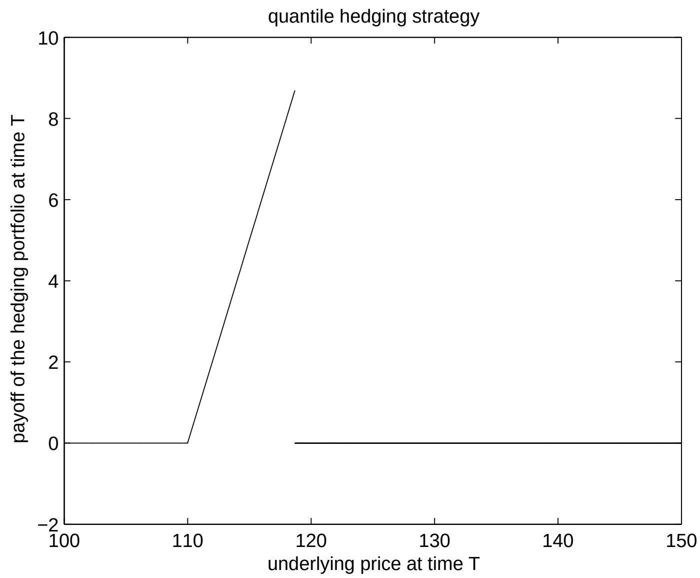

Figure 2.2: Optimal quantile hedging strategy in scenario (ii)

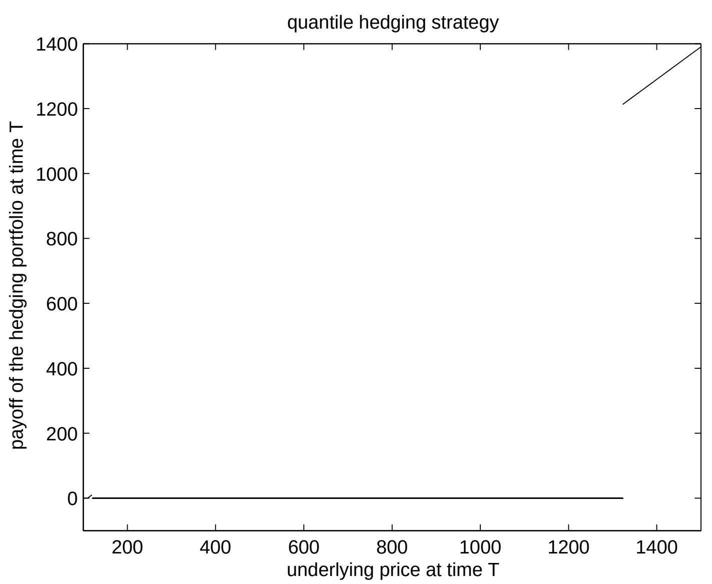

Figure 2.3: Optimal quantile hedging strategy in scenario (iii)

We consider the optimal partial hedging strategies by minimizing $\mathrm{VaR}_{0.95}$ of the hedger's total risk exposure among the admissible set $\mathcal{L}_1$ defined in (2.3). Using Theorems 2.2.1 and 2.2.2, the optimal choice of the hedger is to adopt the following knock-out call hedging strategy

$$[S_T - (K + d^*)]_+ - [S_T - (K + v)]_+ - (v - d^*)\mathbb{1}_{(S_T \geqslant K+v)},$$

where $v = \mathrm{VaR}_{0.95}((S_T - K)_+)$ and d^* is again implied by the budget π_0 as asserted in Theorem 2.2.2. These values are readily determined and are summarized as follows for the three scenarios:

$$\begin{cases} v = 19.11, \quad d^* = 0, & \text{for } \sigma = 0.3 \text{ and } \pi_0 = 1.5, \\ v = 19.11, \quad d^* = 6.67, & \text{for } \sigma = 0.3 \text{ and } \pi_0 = 0.5, \\ v = 9.66, \quad d^* = 0, & \text{for } \sigma = 0.2 \text{ and } \pi_0 = 0.5. \end{cases}$$

Accordingly, the corresponding optimal knock-out call hedging strategies are

$$\begin{cases} (S_T - 110)_+ - (S_T - 129.11)_+ - 19.11\mathbb{1}_{(S_T \geqslant 129.11)}, & \text{for } \sigma = 0.3 \text{ and } \pi_0 = 1.5, \\ (S_T - 116.67)_+ - (S_T - 129.11)_+ - 12.44\mathbb{1}_{(S_T \geqslant 129.11)}, & \text{for } \sigma = 0.3 \text{ and } \pi_0 = 0.5, \\ (S_T - 110)_+ - (S_T - 119.66)_+ - 9.66\mathbb{1}_{(S_T \geqslant 119.66)}, & \text{for } \sigma = 0.2 \text{ and } \pi_0 = 0.5. \end{cases}$$

The optimal hedged loss functions are illustrated in Figure 2.4, Figure 2.5 and Figure 2.6 respectively.

(c) Bull call spread hedging strategy:

We consider the optimal partial hedging strategies by minimizing $\mathrm{VaR}_{0.95}$ of the hedger's total risk exposure among the admissible set $\mathcal{L}_2$ defined in (2.4). It follows from Theorems 2.2.3 and 2.2.4 that the optimal bull call spread hedging strategy is of the form

$$[S_T - (K + d^*)]_+ - [S_T - (K + v)]_+,$$

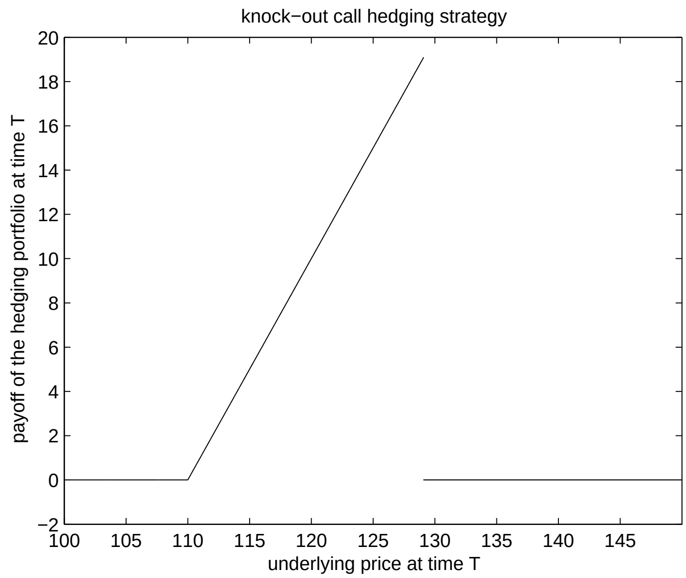

Figure 2.4: Optimal knock out call hedging strategy in scenario (i)

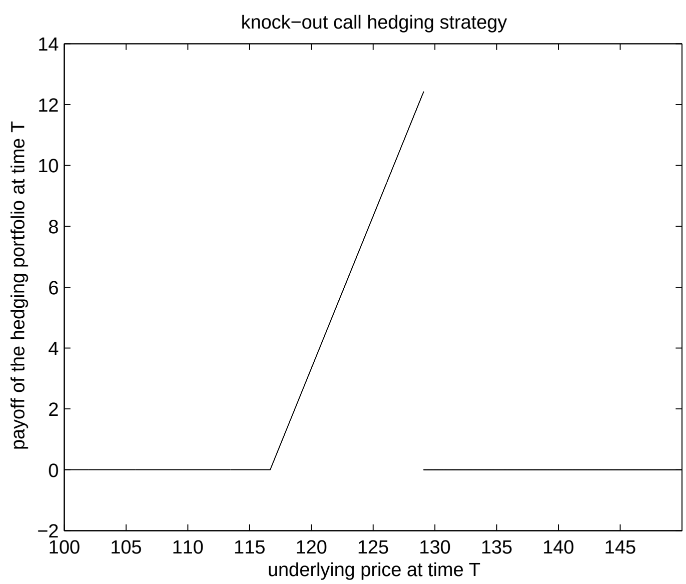

Figure 2.5: Optimal knock out call hedging strategy in scenario (ii)

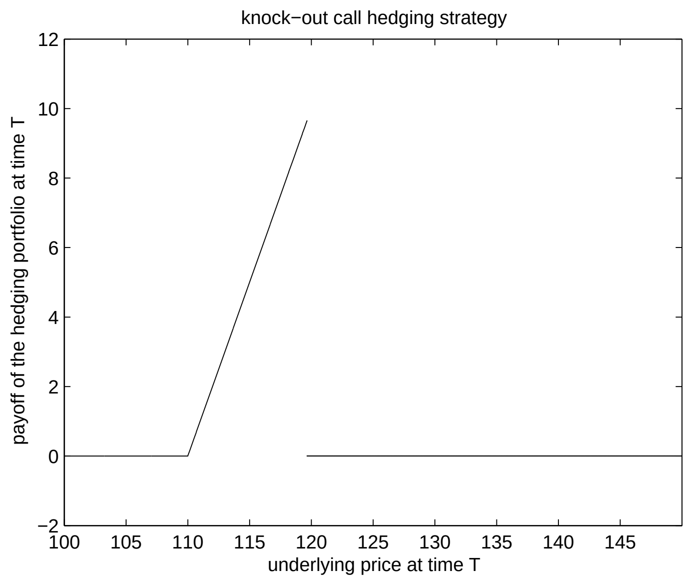

Figure 2.6: Optimal knock out call hedging strategy in scenario (iii)

where $v = \mathrm{VaR}_{0.95}((S_T - K)_+)$ and d^* is determined by the budget π_0 as asserted in Theorem 2.2.4. These values are easily determined and are summarized as follows for the three specified scenarios:

$$\begin{cases} v = 19.11, \quad d^* = 3.30, & \text{for } \sigma = 0.3 \text{ and } \pi_0 = 1.5, \\ v = 19.11, \quad d^* = 10.88, & \text{for } \sigma = 0.3 \text{ and } \pi_0 = 0.5, \\ v = 9.66, \quad d^* = 2.18, & \text{for } \sigma = 0.2 \text{ and } \pi_0 = 0.5. \end{cases}$$

Accordingly, the corresponding optimal bull call spread hedging strategies are

$$\begin{cases} (S_T - 113.30)_+ - (S_T - 129.11)_+, & \text{for } \sigma = 0.3 \text{ and } \pi_0 = 1.5, \\ (S_T - 120.88)_+ - (S_T - 129.11)_+, & \text{for } \sigma = 0.3 \text{ and } \pi_0 = 0.5, \\ (S_T - 112.18)_+ - (S_T - 119.66)_+, & \text{for } \sigma = 0.2 \text{ and } \pi_0 = 0.5. \end{cases}$$

The optimal hedged loss functions are illustrated in Figure 2.7, Figure 2.8 and Figure 2.9 respectively.

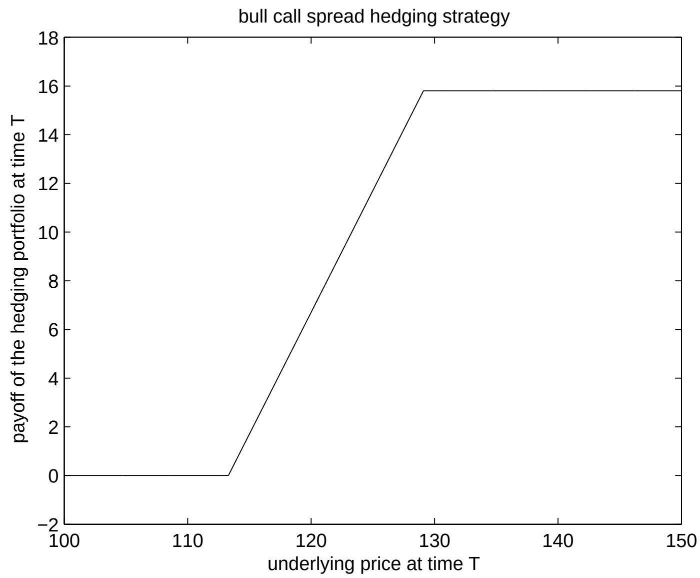

Figure 2.7: Optimal bull call spread hedging strategy in scenario (i)

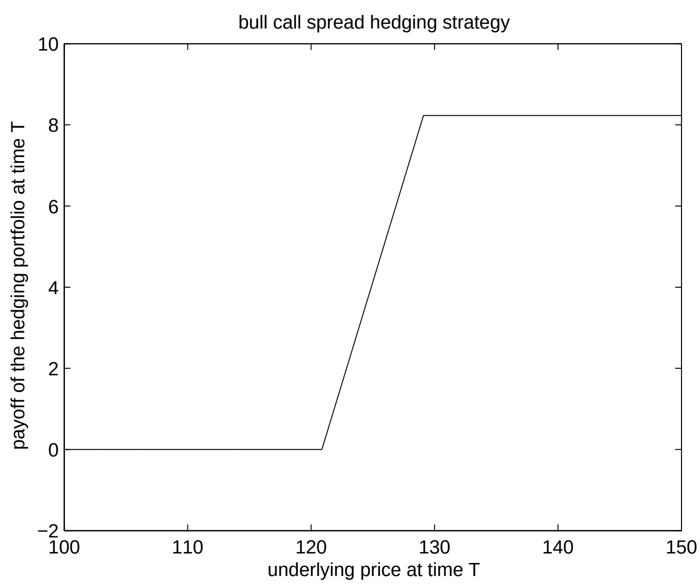

Figure 2.8: Optimal bull call spread hedging strategy in scenario (ii)

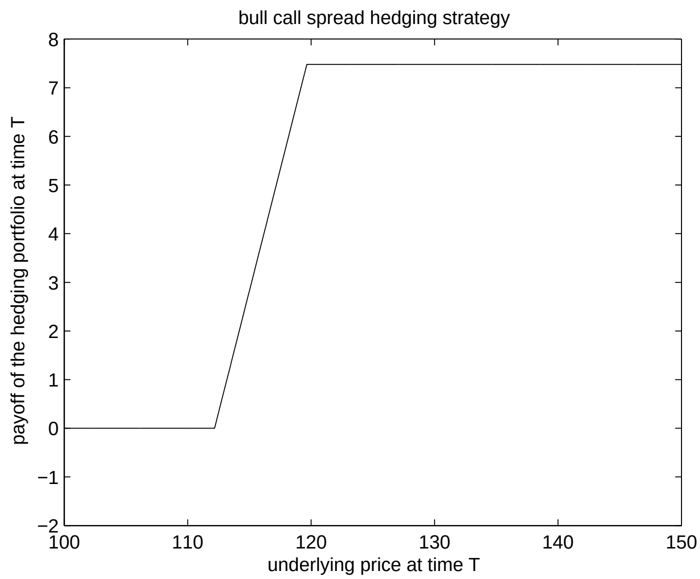

Figure 2.9: Optimal bull call spread hedging strategy in scenario (iii)

Based on these numerical results, we draw the following observations with respect to the optimal hedging strategies.

a). Let us first consider quantile hedging. Recall that scenario (i) has a higher hedging budget than scenario (ii) and this is their only difference. As a result, the shapes of both optimal quantile hedging are the same for both scenarios; see Figure 2.1 and Figure 2.2. The European call option is fully hedged for $S_T \leqslant 118.69$. For $S_T > 118.69$, the quantile hedging under scenario (ii) changes drastically from the fully hedged position to the naked position, as induced by the limited hedging budget, and the hedger is exposed to the entire potential obligation of $S_T - 110$. Moreover, because the first scenario has a higher budget, the option remains to be hedged until S_T increases to 129.47, beyond which the hedger is again exposed to the naked position, as in the second scenario. From the risk management viewpoint, the above optimal hedging strategies seem to be counterintuitive, since generally a hedger should be more concerned with larger losses. Yet the strategy dictated by quantile hedging only produces perfect hedging for small losses and completely no hedging for large losses. This phenomenon is attributed to the criterion stipulated by quantile hedging that it only focuses on the likelihood of a successful hedge while ignores completely the tail risk.

We now compare the quantile hedging strategies between scenario (ii) and scenario (iii); see Figure 2.2 and Figure 2.3. The only difference between these two scenarios is the volatility parameter σ. By merely decreasing σ from 30% to 20%, it is striking to learn that the shape of the hedging strategy changes quite substantially. In particular, in scenario (iii) the call option is perfectly hedged up to $S_T = 119.98$ and then completely unhedged, just like the first two scenarios. More interestingly, when S_T becomes really large such as exceeding 1323, the option is completely hedged again. The optimal hedged loss function displayed in Figure 2.3 seems to indicate that it is flat at zero for most of S_T. However, it should be pointed out that this is just an optical illusion due to the scale of the plot. Figure 2.10 magnifies the portion of the optimal hedged loss function for $100 \leqslant S_T \leqslant 150$ and confirms that for low values of S_T, the optimal hedged loss function from scenario (iii) resembles the first two scenarios.

b). Unlike quantile hedging, the optimal knock-out call partial hedging strategy has the same consistent shape in all three scenarios (see Figure 2.4 - 2.6). Moreover, their shapes resemble those of the quantile hedging strategies in the first two scenarios. Just to elaborate, the knock-out call hedging strategy for scenario (i) provides a perfect hedge for S_T up to 129.11 and then switches to a naked position for $S_T > 129.11$. On the other hand, the optimal partial hedging under the lower hedging

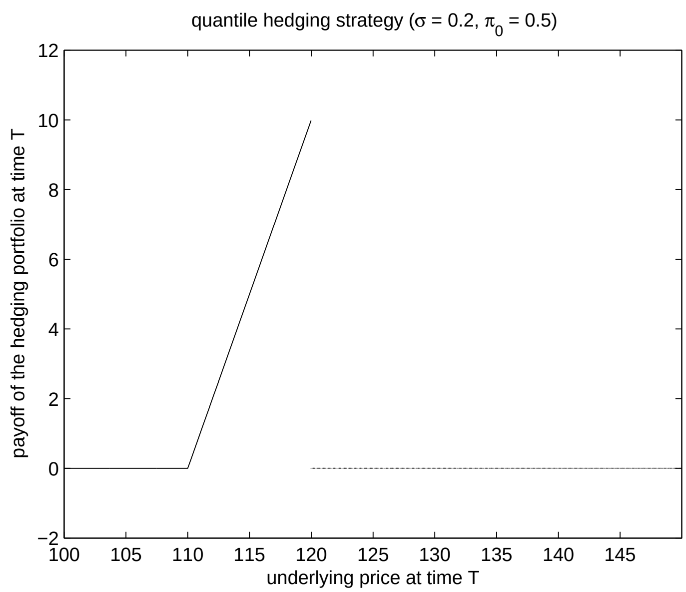

Figure 2.10: The optimal quantile hedging strategy for scenario (iii) over the range $100 \leqslant S_T \leqslant 150$

budget of scenario (ii) is accomplished at the expense of not perfectly hedging the call option. In particular, the hedger absorbs the loss of amount $S_T - 110$ for $110 < S_T < 116.67$ and up to a fixed amount of 6.67 for $S_T \in [116.67, 129.11]$. For $S_T > 129.11$ the hedger does not hedge anything at all as in the first scenario.

c). The optimal partial hedging under the bull call spread strategy generates a very different but more desirable solution (see Figure 2.7 - 2.9). First, we emphasize that the shapes of the optimal hedged loss functions are again consistently the same among the three scenarios; they are all bull call spread strategies. Second, the bull call spread hedging provides some partial hedging, even for large losses. This contradicts the preceding two methods (except the quantile hedging under scenario (iii)) which do not provide any protection on the right tail. This is a consequence of imposing the nondecreasing property **P4** on the hedged loss functions. Third, because of enforcing some partial hedging on large losses, the bull call spread strategies sacrifice the chance of perfect hedging for small losses. To see this, let us recall that for scenario (i) the knock-out call strategy perfectly hedges the call option for $S_T \in [110, 129.11]$. For the bull call spread hedging, the optimal strategy only begins partial hedging from $S_T = 113.30$ using an option that pays $S_T - 113.30$ for $S_T \in [113.30, 129.11]$. This implies that over the same range of stock prices, the hedger is exposed to a constant loss of 3.30 for the bull call spread hedging while zero loss for the knock-out call strategy. Fourth, while the bull call spread hedging provides some partial hedging on the tail, it is still not satisfactory in view that the amount being hedged remains constant after a threshold level. For instances, when $S_T > 129.11$ the optimal bull call spread hedging yields a constant hedged amount of 15.81 for scenario (i). This implies that the hedger is still subject to a potential loss of $S_T - 110 - 15.81 = S_T - 125.81$ for $S_T > 129.11$.

d). The plots of the optimal strategies in Figures 2.1-2.3 again highlight the sensitivity of the shapes of the hedged loss functions for quantile hedging with respect to the parameter values of the assumed model. The shape of the optimal hedged loss function depends on the ratio m/σ^2, whether it being greater or smaller than 1. In contrast, Figure 2.4 - 2.6 and Figure 2.7 - 2.9 re-assure that the optimal hedging strategies are always the knock-out call strategy and the bull call spread strategy respectively, despite of changes in parameter values. These results demonstrate the stability or the robustness of our proposed VaR-based hedging strategy in that the optimal hedging strategy always admits the same structure and it is independent of the assumed market model.

e). Additional insight on these hedging strategies can be gained by comparing the expected shortfall of the hedger under each of these three strategies. The results, which are depicted in Table 2.1, indicate that the expected shortfall of the hedger's total risk under the bull call spread hedging strategy is always the smallest among the three hedging strategies and in all three scenarios. This is consistent with our intuition as the bull call hedging strategy is derived as an optimal solution under the additional assumption of **P4**, which reflects the hedger's concern on the right tail risk. In other words, bull call spread hedging provides some partial hedging on the tail risk.

2.4 Concluding Remark

In this chapter, we propose a general framework for determining an optimal partial hedging strategy. The proposed model involves minimizing an arbitrary risk measure of a hedger's risk exposure. We derive the analytic solutions by specializing to the Value-at-Risk measure and under two admissible classes of hedging strategies. We analytically obtain the optimal

	Quantile hedging	Knock-out call hedging	Bull call spread hedging
Scenario (i)	1.35	1.36	1.25
Scenario (ii)	2.56	2.52	2.48
Scenario (iii)	0.72	0.72	0.66

Table 2.1: Expected shortfall of the hedger under each of the optimal partial hedging strategies

hedging solution as either the knock-out call hedging strategy, which involves constructing a knock-out call on the payout, or the bull call spread hedging strategy, which involves constructing a bull call spread on the payout. Through many examples, we show that, in implementing our optimal hedging strategies, we often only need to hedge an instrument which has the same structure (with different parameter values though) as the risk we aim to partially hedge. Therefore, if such an instrument exists in the market, we are then able to achieve our objective by a static hedging strategy. Even if such an instrument is not available in the market, our results provide some important insights on which part of the risk should be hedged as an optimal partial hedging strategy.

In comparison to the well-known quantile hedging, our proposed VaR-based partial hedging has a number of advantages. Notably, the structure of the optimal hedging strategy is independent of the assumed market model, the optimal solution is relatively easy to determine, and it is also better at capturing the tail risk when we impose the monotonicity on the hedging strategy.

Although the proposed VaR-based partial hedging model and the resulting optimal strategies have the above appealing features, it is also important to point out their potential limitations. In particular, VaR suffers from the typical criticisms that it is not a coherent risk measure and that it is a quantile risk measure. The latter property implies that as long as the probability of loss is within the prescribed tolerance of the hedger, the optimal VaR-based partial hedging strategy is to leave the risk unhedged. For example, if the probability of a loss on a particular risk exposure is less than 5%, then the optimal VaR-based partial hedging strategy under 95% confidence level is not to hedge any part of the risk. Besides, the property P2 we imposed on the hedged loss functions would exclude some hedging strategies.

While we have confined our analysis to VaR, it should be emphasized that our proposed partial hedging model is quite general in that it can be applied to other risk measures including Conditional Value-at-Risk (CVaR). It would be of great interest to investigate the optimal hedging strategy under CVaR since CVaR is known to have some desirable properties. These include the coherence property, spectral property, and the capability of capturing the tail risk. Melnikov and Smirnov (2012) investigate the problem of partial hedging by minimizing CVaR of the portfolio in the complete market. Their solution exploits the properties of CVaR risk measure and also relies on the Neyman-Pearson lemma approach, a method which is used extensively in the literature related to quantile hedging. On the other hand, the proposed model and the approach used in this chapter provide a possible different perspective on studying the optimal partial hedging problem involving CVaR. We report this in detail in the next chapter.

Chapter 3

CVaR Minimization Models

In the previous chapter, we deal with the optimal partial hedging strategies in the Value-at-Risk (VaR) minimization models. VaR is relatively easy to calculate and very intuitive. But VaR has its limitations, such as being insensitive to the tail risk, not belonging to the class of coherent risk measures due to violating the sub-additivity property, and so forth. As an alternative, Conditional Value-at-Risk (CVaR), which can be derived from VaR, is of the class of coherent risk measures. In literature, CVaR is sometimes called conditional tail expectation (CTE), expected shortfall, average Value-at-Risk, worst case expectation. Since its coherence and tail-sensitivity, CVaR is becoming a more and more popular risk measure. Therefore, in this chapter, we will investigate the CVaR-based optimal partial hedging.

The main objective in this chapter is to identify the optimal hedged loss function f to the optimization problem (2.2) by assuming the risk measure ρ being CVaR_α with a fixed constant α in $(0, 1)$ reflecting the associated confidence level. We will consider the CVaR-based optimal partial hedging under two different market structures. The technical method we will use in this chapter is similar to that used in Chapter 2.

3.1 Preliminaries

3.1.1 Model Description

We use the same notations as those in Chapter 2. We mathematically restate the optimal partial hedging problem (2.2) here

$$\begin{cases} \min\limits_{f\in\mathcal{L}} & \rho(T_f(X)) \\ \text{s.t.} & \Pi(f(X)) \leqslant \pi_0 \end{cases}$$

where, $\rho(\cdot)$ is a risk measure employed by the investor, Π represents the risk pricing functional so that $\Pi(f(X))$ is the market price of hedging the risk $f(X)$, $T_f(X)$ denotes $R_f(X) + e^{rT} \cdot \Pi(f(X))$ the total risk exposure of the investor under such a partial hedging strategy at time T (maturity of the contract X), r is the risk free return rate, π_0 is the hedging budget (an amount up to which the investor is willing to spend on hedging), $\mathcal{L}$ is the admissible set of hedged loss functions. For presentation convenience, we assume that the risk free interest rate is constant hereafter, though our results can easily be extended to the case with a stochastic interest rate.

Since we mainly consider the risk measure $\rho(\cdot)$ as CVaR risk measures in this chapter, we rewrite the above optimization problem as

$$\begin{cases} \min\limits_{f\in\mathcal{L}} & CVaR_\alpha(T_f(X)) \\ \text{s.t.} & \Pi(f(X)) \leqslant \pi_0 \end{cases} \tag{3.1}$$

Here, we provide the definitions of CVaR risk measures as follows

Definition 3.1.1. *The CVaR of X at the confidence level* $(1-\alpha)$*, where* $0<\alpha<1$*, is defined as*

$$CVaR_\alpha(X)=\frac{1}{\alpha}\int_0^\alpha VaR_s(X)ds.$$

where $VaR_\alpha(X)$ *has the same definition as that in Chapter 2*

$$VaR_\alpha(X)=\inf\{x\geqslant 0:\mathbb{P}(X>x)\leqslant\alpha\}.$$

In Chapter 2, we mainly consider the VaR-based optimal partial hedging strategies by restricting the risk measure $\rho(\cdot)$ as VaR risk measure in the above optimization problem. Because CVaR is a more desirable risk measure than VaR, as it is more adequate in capturing the magnitude of the risk on the tail (see Artzner, et al., 1999), the optimal partial hedging strategy derived under this risk measure will be of more relevant and more important. Hence, in this section, we will study the CVaR-based optimal partial hedging model by restricting the risk measure $\rho(\cdot)$ as CVaR risk measure. While the same basic framework is used, it is important to emphasize that the construction technique employed in Chapter 2 cannot be directly applied in the present chapter. Instead, we need to impose some different assumptions on the feasible hedging strategies and utilize a different construction method in order to derive analytically the optimal partial hedging strategy. Furthermore, the optimal solution is analyzed under two different classes of pricing functionals, see Subsection 3.1.3 for the detailed illustration.

There are some similarities between our proposed optimal partial hedging strategies and the optimal hedging strategies investigated in Melnikov and Smirnov (2012), for the reason that CVaR is the optimization criterion in both cases. However, our proposed CVaR-based partial hedging is significantly distinct from that in Melnikov and Smirnov (2012) in several aspects. The contrast between our proposed risk measure based partial hedging with the existing hedging strategies in literature provided in Section 1.2 also applies here. The repetitious details are not given here. Another significant distinct between our proposed CVaR-based partial hedging in this chapter and the hedging strategies in Melnikov and Smirnov (2012) is the admissible set of the hedging strategies. Our admissible set of the hedging strategies is less general in the sense that we impose two additional assumptions on the hedged loss function, namely not globally over-hedged assumption (Property P1) and not locally over-hedged assumption (Property P2), in addition to the usual nonnegativity assumption. As justified in Subsection 2.1.2, these additional assumptions are reasonable. More importantly, there are several important advantages to analyzing our proposed optimal partial hedging model, albeit it is more restrictive. First, the analysis of our CVaR-based optimal partial hedging is considerable simpler and more transparent because we use a two-step procedure which is similar to that in Chapter 2. In contrast, Melnikov and Smirnov (2012) follow the same procedure as Föllmer and Leukert (1999) and study the optimal partial hedging strategies by searching for the most powerful test for a corresponding hypothesis testing problem. Second, in the general case the solution to the optimal partial hedging strategy of Melnikov and Smirnov (2012) is only semi-explicit. Even if we were to confine ourselves to one of the simplest models such as the Black-Scholes model, the procedure for obtaining the optimal hedging strategy can be quite involved as we will demonstrate in Example 3.4.3. In other words, it in general, can be challenging to numerically solve for the partial hedging strategy of Melnikov and Smirnov (2012). On the other hand, the optimal partial hedging strategies of our proposed model in this chapter are explicit and are relatively easy to deduce, not just for the Black-Scholes model but also for other more sophisticated and more complex market models. Third, the optimal partial hedging strategies of our proposed model have the same functional form, irrespective of the specification of the market model assumptions. The functional form of the optimal partial hedging strategy of Melnikov and Smirnov (2012), on the other hand, could be sensitive to the specification of the market models as well as the corresponding parameter values. For instance, for the Black-Scholes model considered in Example 3.4.3, the optimal partial hedging strategy of Melnikov and Smirnov (2012) for hedging a European call option could either be a knock-in call or a knock-out call, depending on the relative magnitude of interest rate and the drift coefficient. This implies that the optimal partial hedging strategy needs to be evaluated case by case depending on the values of these parameters. In contrast, our proposed optimal partial hedging strategy is consistently a bull call spread. Because of this property, the optimal partial hedging of our proposed model is said to be more robust than that of Melnikov and Smirnov (2012) in the sense that the structure of our proposed CVaR-based hedging strategies is model independent while the structure of hedging strategy proposed in Melnikov and Smirnov (2012) is sensitive to the model specifications.

3.1.2 Desired Class of Partial Hedging Strategies

In Chapter 2, we have shown the necessity of imposing some constraints on the hedged loss functions. Therefore, in this subsection, we will consider the first three desirable properties of the hedged loss functions we discussed in Chapter 2. These desired properties would specify the admissible set $\mathcal{L}$ in formulation (3.1). The desirable properties are restated as follows

P1. Not globally over-hedged: $f(x) \leqslant x$ for all $x \geqslant 0$;

P2. Not locally over-hedged: $f(x_2) - f(x_1) \leqslant x_2 - x_1$ for all $0 \leqslant x_1 \leqslant x_2$;

P3. Nonnegativity of the hedged loss: $f(x) \geqslant 0$ for all $x \geqslant 0$.

Note that property **P2** is equivalent to the following;

P2'. Monotonicity of the retained loss function: $R_f(x_2) \geqslant R_f(x_1) \; \forall \; 0 \leqslant x_1 \leqslant x_2$.

Unlike VaR risk measure, CVaR risk measure takes into account the magnitude of the risk on the tail. Hence, it is not necessary for us to have the nondecreasing property on the hedged loss function, as property **P4** in Chapter 2. However, as will become clear later, the optimal hedged loss function is still nondecreasing.

Similar to Chapter 2, without lose of generality, we assume that the retained loss function $R_f(x)$ is a left continuous function with respect to x. Therefore, the admissible set of the hedged loss functions we consider in this chapter is the same as $\mathcal{L}_1$, which is the more general admissible set of the hedged loss functions in Chapter 2. We restate the admissible set $\mathcal{L}_1$ as follows

$$\begin{aligned} \mathcal{L}_1 = &\{0 \leqslant f(x) \leqslant x : R_f(x) \equiv x - f(x) \text{ is a nondecreasing and left} \\ &\text{continuous function}\} \end{aligned} \tag{3.2}$$

3.1.3 Assumptions on the Pricing Method

In Chapter 2, we only assume that the market pricing function excludes arbitrage in the market. In order to deal with the CVaR optimization problem, we need some stronger assumptions on the market pricing function $\Pi(\cdot)$. In this chapter, we will study the CVaR-based optimal partial hedging under two different market pricing functions.

No arbitrage pricing

Arbitrage free condition is one of the most common and basic assumptions imposed in the financial market. Therefore, the first market pricing function we consider in this chapter is similar to the one in Chapter 2, which admits no arbitrage opportunity in the market. In Chapter 2, we only assume that there is no arbitrage in the market when studying the VaR-based partial hedging. However, in this chapter, in order to obtain the explicit optimal partial hedging strategy under CVaR risk measure, we need to impose an extra assumption on the market pricing function.

Other than no arbitrage, we further assume that $\Pi(X) \leqslant \frac{1}{\alpha} \cdot e^{-rT} \cdot \mathbb{E}^{\mathbb{P}}[X]$ holds for any contract X, where r is the risk-free rate, $\Pi(\cdot)$ is the market pricing function and $(1-\alpha)$ is the confidence level of CVaR risk measure. This assumption can be justified as follows: if there exists a contract X_0 that has nonnegative payout such that $e^{rT} \cdot \Pi(X_0) > \frac{1}{\alpha} \cdot \mathbb{E}^{\mathbb{P}}[X_0]$, then we can claim that $e^{rT} \cdot \Pi(X_0) > CVaR_\alpha(X_0)$ due to the fact that $\frac{1}{\alpha} \cdot \mathbb{E}^{\mathbb{P}}[X_0] > CVaR_\alpha(X_0)$. In this case, it is obvious that investor can decrease his/her CVaR risk exposure by short selling the contract X_0, because the price of contract X_0 is larger than the risk of short selling contract X_0 measured by CVaR. And investor can make his/her CVaR risk exposure to be negative infinity by short selling infinite units of contract X_0. Therefore, it is reasonable to have this assumption when we consider CVaR as the risk measure.

Besides of the reasonability of this assumption, we also want to emphasize that this assumption is very mild. Usually the confidence level of CVaR risk measure $(1-\alpha)$ is very close to 1, so $\frac{1}{\alpha}$ is a very large constant, which implies that the aforementioned assumption imposed on the pricing function is very easy to be satisfied.

Stop-loss order preserving pricing

In a complete market, the price of a contingent claim is uniquely determined as the cost needed to replicate the claim. In reality, however, the financial market is far from being complete. Under market frictions like illiquidity or transaction costs, contingent claims can incorporate some inevitable intrinsic risk that cannot be completely hedged. In this case, those contingent claims could not be replicated and hence could not be priced solely with no arbitrage pricing argument. In an incomplete market, the arbitrage free price is no longer unique, and there are quite a few prevalent pricing approaches, among which is the utility based indifference pricing (UBIP) method. As we can see in Section 3.3, UBIP method is among the class of stop-loss order preserving pricing. Therefore, as one of the complements of the no arbitrage pricing, the stop-loss order preserving pricing method is the second market pricing function we consider in this chapter.

To conclude this subsection, we recall the definition of stop-loss ordering between two risks and provide the definition of the stop-loss order preserving pricing as below.

Definition 3.1.2. *Suppose X and Y are two random variables with finite means under a probability measure $\mathbb{P}$. If*

$$\mathbb{E}^{\mathbb{P}}\left[(X-d)_+\right] \leqslant \mathbb{E}^{\mathbb{P}}\left[(Y-d)_+\right], \quad \forall\, d \in \mathbb{R},$$

then we say that X is smaller than Y in stop-loss order under $\mathbb{P}$ and is denoted by $X \leqslant_{sl}^{\mathbb{P}} Y$.

We remark that the stop-loss ordering may be defined in some other equivalent ways (see Hurlimann (1998)).

Definition 3.1.3. *For a given pricing functional $\Pi(\cdot)$, if $\Pi(X) \leqslant \Pi(Y)$ for any $X \leqslant_{sl}^{\mathbb{P}} Y$, then we say that the pricing functional $\Pi(\cdot)$ preserves stop-loss order.*

3.2 CVaR Optimization under No Arbitrage Pricing

In this section, we will study the optimal CVaR-based partial hedging with the assumption of sufficient hedging budget under the no arbitrage pricing functional. First we rewrite the optimization problem (3.1) by assuming there is sufficient hedging budget

$$\min_{f \in \mathcal{L}} CVaR_{\alpha}(T_f(X)) \tag{3.3}$$

We restate the assumptions imposed on the pricing function $\Pi(\cdot)$ in this section as follows

Assumption 3.2.1. *(a)* $\Pi(\cdot)$ *admits no arbitrage opportunities.*

(b) $\Pi(X) \leqslant \frac{1}{\alpha} \cdot e^{-rT} \cdot \mathbb{E}^{\mathbb{P}}[X]$ *holds for any contract X, where r is the risk-free rate and T is the time to maturity of contract X.*

With the above assumptions on the market pricing function and the admissible set of hedged loss functions which is stated in (3.2), we are ready to construct the optimal hedging strategy. The first step of the construction is done by the following lemma.

Lemma 3.2.1. *For a given random variable X and any function $f \in \mathcal{L}_1$, we can construct a function $\hat{g}_f$ as follows:*

$$\hat{g}_f(x) = (x + f(v) - v)_+. \tag{3.4}$$

where $v = VaR_{\alpha}(X)$ and $(x)_+$ equals to x if $x > 0$ and zero otherwise. Then $\hat{g}_f \in \mathcal{L}_1$.

Proof: Obviously, the above function $\hat{g}_f$ is well defined. Therefore, it is sufficient to show that the function $\hat{g}_f \in \mathcal{L}_1$ holds for any function $f \in \mathcal{L}_1$.

Note that $f \in \mathcal{L}_1$, we have $f(x) \leqslant x$ holds for any x. In particular, For a given random variable X, $f(v) \leqslant v$ where $v = \mathrm{VaR}_{\alpha}(X)$. Thus, $\hat{g}_f(x) = (x + f(v) - v)_+ \leqslant x$ holds for any x. It is clear that $\hat{g}_f(x) \geqslant 0$ for any x. Then we can conclude that $0 \leqslant \hat{g}_f(x) \leqslant x$ for any x.

According to the definitions of retained loss function, for any x, we have

$$R_{\hat{g}_f}(x) = \begin{cases} x, & \text{if} \quad 0 \leqslant x \leqslant v - f(v), \\ \\ v - f(v), & \text{if} \quad x > v - f(v), \end{cases}$$

From the above expression, it is straightforward that $R_{\hat{g}_f}(x)$ is nondecreasing and left continuous with respect to x. Therefore, we can conclude that $\hat{g}_f \in \mathcal{L}_1$ holds for any function $f \in \mathcal{L}_1$, according to definition 3.2. □

Remark 3.2.1. *For a given random variable X and any function $f \in \mathcal{L}_1$, if we denote $d = v - f(v)$, then the constructed hedged loss function can be rewritten as*

$$\hat{g}_f(X) = (X - d)_+ \tag{3.5}$$

Therefore, the partial hedging strategy corresponding to the hedged loss function $\hat{g}_f$ is to construct a call on the contingent claim.

The following lemma provides a comparison between the constructed hedging function and the original hedging function.

Lemma 3.2.2. *For a given contract X and any hedging function $f \in \mathcal{L}_1$, if the hedging function $\hat{g}_f$ is constructed as in Lemma 3.2.1, then $\hat{g}_f(x) \leqslant f(x)$ when $x \leqslant v$, and $\hat{g}_f(x) \geqslant f(x)$ when $x > v$.*

Proof: When $x \leqslant v - f(v)$, according to the definition of function $\hat{g}_f$ in Lemma 3.2.1, we can claim that $\hat{g}_f(x) = 0$. From the definition of $\mathcal{L}_1$ which is stated in (3.2), we know that $f(x) \geqslant 0$ for any x. Therefore, we can conclude that $\hat{g}_f(x) \leqslant f(x)$ when $x \leqslant v - f(v)$.

When $x \geqslant v - f(v)$, according to the definition of function $\hat{g}_f$ in Lemma 3.2.1, we have $\hat{g}_f(x) = x + f(v) - v$. In this case,

$$\begin{aligned} f(x) - \hat{g}_f(x) &= (v - f(v)) - (x - f(x)) \\ &= R_f(v) - R_f(x) \end{aligned}$$

From the definition of $\mathcal{L}_1$ which is stated in (3.2), we know that R_f is a nondecreasing function. Therefore, we have $R_f(v) - R_f(x) \geqslant 0$, which implies that $\hat{g}_f(x) \leqslant f(x)$, when $v - f(v) \leqslant x \leqslant v$, and $R_f(v) - R_f(x) \leqslant 0$, which implies that $\hat{g}_f(x) \geqslant f(x)$, when $x > v$

From the above analysis, we can conclude that $\hat{g}_f(x) \leqslant f(x)$ when $x \leqslant v$, and $\hat{g}_f(x) \geqslant f(x)$ when $x > v$. □

Now we can state the main result of this subsection, which says that for a given payoff X with maturity T and any partial hedging strategy $f \in \mathcal{L}_1$, the partial hedging strategy $\hat{g}_f(X)$ will outperform the partial hedging strategy $f(X)$ under Assumption 3.2.1.

Theorem 3.2.1. *Assume that the market is complete and the market pricing function satisfies Assumption 3.2.1. For any contract X with maturity T and any hedging function $f \in \mathcal{L}_1$, we can construct a call hedging strategy $\hat{g}_f \in \mathcal{L}_1$ as* (3.4)*, or equivalently* (3.5)*, such that*

$$CVaR_\alpha(T_{\hat{g}_f}(X)) \leqslant CVaR_\alpha(T_f(X))$$

Therefore, the call hedging strategy is the optimal form of hedging in the sense of minimizing CVaR of the total exposed risk at maturity of the investor.

Proof: For brevity of expressions, we assume that $\mathbb{P}(X = v) = 0$ in the following proof. According to the definition of $T_f(X)$ given in (2.1) and the translation invariance property of CVaR risk measure, we have

$$
\begin{aligned}
&CVaR_\alpha(T_f(X))\\
=&CVaR_\alpha(R_f(X)+e^{rT}\cdot\Pi(f(X)))\\
=&CVaR_\alpha(R_f(X))+e^{rT}\cdot\Pi(f(X))\\
=&CVaR_\alpha(R_f(X))+e^{rT}\cdot\Pi(f(X)\cdot\mathbb{1}(X\leqslant v))+e^{rT}\cdot\Pi(f(X)\cdot\mathbb{1}(X>v))\\
=&\frac{1}{\alpha}\mathbb{E}^{\mathbb{P}}\left[R_f(X)\cdot\mathbb{1}(X>v)\right]+e^{rT}\cdot\Pi(f(X)\cdot\mathbb{1}(X\leqslant v))+e^{rT}\cdot\Pi(f(X)\cdot\mathbb{1}(X>v))
\end{aligned}
$$

where r is the risk-free rate. The third equality is because of the linearity of the pricing functional, which is implied by the no arbitrage property in Assumption 3.2.1. The fourth equality is due to the definition of CVaR risk measure.

According to Lemma 3.2.2, we know that $\hat{g}_f(x)\geqslant f(x)$ when $x>v$. Now we construct a contract Y which matures at T and has payoff as follows

$$
Y=\begin{cases} 0, & \text{if} \quad X\leqslant v,\\ \\ \hat{g}_f(X)-f(X), & \text{if} \quad X>v, \end{cases}
$$

It is clear that Y is nonnegative, thus Y is a well-defined contract. From Assumption 3.2.1, we know that

$$
\Pi(Y)\leqslant\frac{1}{\alpha}\cdot e^{-rT}\cdot\mathbb{E}^{\mathbb{P}}\left[Y\right]
$$

According to the construction of Y, the above inequality is equivalent to

$$
\Pi\left[(\hat{g}_f(X)-f(X))\cdot\mathbb{1}(X>v)\right]\leqslant\frac{1}{\alpha}\cdot e^{-rT}\cdot\mathbb{E}^{\mathbb{P}}\left[(\hat{g}_f(X)-f(X))\cdot\mathbb{1}(X>v)\right]
$$

Due to the linearity of the expectation and the pricing functional, we can rewrite the above inequality as follows

$$
\begin{aligned}
&e^{rT}\cdot\Pi\left[\hat{g}_f(X)\cdot\mathbb{1}(X>v)\right]+\frac{1}{\alpha}\cdot\mathbb{E}^{\mathbb{P}}\left[(X-\hat{g}_f(X))\cdot\mathbb{1}(X>v)\right]\\
\leqslant&e^{rT}\cdot\Pi\left[f(X)\cdot\mathbb{1}(X>v)\right]+\frac{1}{\alpha}\cdot\mathbb{E}^{\mathbb{P}}\left[(X-f(X))\cdot\mathbb{1}(X>v)\right]
\end{aligned}
$$

From Lemma 3.2.2, we know that $\hat{g}_f(x) \leqslant f(x)$ when $x \leqslant v$, which implies that $\hat{g}_f(X \cdot \mathbb{1}(X \leqslant v)) \leqslant f(X \cdot \mathbb{1}(X \leqslant v))$. Since the pricing functional $\Pi(\cdot)$ admits no arbitrage, we can claim that

$$\Pi(\hat{g}_f(X) \cdot \mathbb{1}(X \leqslant v)) \leqslant \Pi(f(X) \cdot \mathbb{1}(X \leqslant v))$$

From the above two inequalities, we have

$$\begin{aligned}
&e^{rT} \cdot \Pi(\hat{g}_f(X) \cdot \mathbb{1}(X \leqslant v)) + e^{rT} \cdot \Pi\left[\hat{g}_f(X) \cdot \mathbb{1}(X > v)\right] + \frac{1}{\alpha} \cdot \mathbb{E}^{\mathbb{P}}\left[(X - \hat{g}_f(X)) \cdot \mathbb{1}(X > v)\right] \\
\leqslant & e^{rT} \cdot \Pi(f(X) \cdot \mathbb{1}(X \leqslant v)) + e^{rT} \cdot \Pi\left[f(X) \cdot \mathbb{1}(X > v)\right] + \frac{1}{\alpha} \cdot \mathbb{E}^{\mathbb{P}}\left[(X - f(X)) \cdot \mathbb{1}(X > v)\right]
\end{aligned}$$

which is equivalent to

$$CVaR_\alpha(T_{\hat{g}_f}(X)) \leqslant CVaR_\alpha(T_f(X))$$

□

Remark 3.2.2. *(a) The comments we made in Remark 2.2.1 and Remark 2.2.4 for the solutions to the VaR-based optimal partial hedging problem in Chapter 2 are similarly applicable here for the CVaR case. In particular, we draw the following conclusions.*

(i) Theorem 3.2.1 indicates that the call hedging strategy is optimal among all the strategies in $\mathcal{L}_1$ under Assumption 3.2.1. We note again that the optimal call hedging strategy is to construct a call on the risk X and not on the asset that underlies X.

(ii) The optimality of call hedging is model independent. It does not depend on the dynamic of the underlying or the specific pricing functional.

(iii) If the call option written on the risk X is available in the financial market, then the optimal partial hedging can be achieved via a simple static hedging strategy.

(b) In this section, we study the CVaR-based optimal partial hedging problem by assuming there is sufficient budget constraint. The reason why we removed the budget constraint is because we have imposed a relatively mild assumption on the pricing functional. As we can see in the next section, when we assume the pricing functional preserves stop-loss order, we will be able to solve the CVaR-based optimal partial hedging problem with any given budget constraint.

(c) Although we assume the hedging budget is sufficient in this section, the cost of constructing the partial hedging strategy is included in the total risk exposure. Therefore, the tradeoff between the cost and effect of the hedging strategy is still being considered in the objective function.

3.3 CVaR Optimization under Stop-loss Order Preserving Pricing

In this section, we will study the optimal CVaR-based partial hedging with budget constraint under the stop-loss order preserving pricing functional, i.e. we assume that the pricing functional Π in our CVaR based formulation (3.1) preserves the stop-loss order. One typical example in finance satisfying this property is the utility based indifference pricing (UBIP). In this section, we first summarize some fundamental facts about UBIP and make some relevant remarks. We then introduce one special but important UBIP called the marginal utility-based pricing (MUBP).

3.3.1 Utility Based Indifference Pricing

In a complete market, the price of a contingent claim is uniquely determined as the cost needed to replicate the claim. In reality, however, the financial market is far from being complete, particularly when we realize the existence of those factors such as transactions costs, non-traded securities, portfolio constraints and so forth. When the market is incomplete, the arbitrage free price is no longer unique, and there are quite a few prevalent pricing approaches, among which is the UBIP method. One of the pioneering works is attributed to Hodges and Neuberger (1989) where the authors discussed how to apply the UIBP method for options in the presence of transaction cost. Some other interesting literature include Henderson and Hobson (2004), Musiela and Zariphopoulou (2004), Mania and Schweizer (2005), Klöppel and Schweizer (2007), Monoyios (2008) and the references therein.

While the UBIP method can further be categorized into many different groups depending on the choices of utility functions and some other relevant factors, the method is based on the same basic idea, where the attitude of every investor in the market towards risks is assumed to be fully described by a utility function and all investors are assumed to maximize their expected utility of wealth. The utility indifference price of a given contract is then defined as the contract price which makes no difference to the investor's expected utility whether an investor chooses to add the contract into or exclude it from the portfolio. As such, the utility indifference price does not rely on the completeness of the market and does not distinguish between the presence or the absence of the market frictions.

Below are some fundamental properties of the UBIP.

(1) Recovery of complete market price.

If the contract can be replicated in the market, then the utility indifference price coincides with the cost of replicating that contract. A brief justification of this property can be found in Henderson and Hobson (2004). It is a very desirable and important property of the utility indifference pricing, since it makes the utility indifference price compatible with the complete market price.

(2) Monotonicity.

If the payoff of contract A is always larger than or equal to that of contract B, then the utility indifference price of contract A is also larger than or equal to that of contract B. This property guarantees that the utility indifference price of a contingent claim lies between the super-replicating price and the sub-replicating price.

(3) Concavity.

The utility indifference price of the convex combination of contract A and contract B is larger than or equal to the convex combination of the utility indifference prices of contract A and contract B. This is due to the concavity of the utility function.

(4) Preserving stop-loss order.

As will be shown in Proposition 3.3.1 below, as long as the preference of the investor can be described by an increasing concave utility function and the investor is aiming at maximizing the expected utility of wealth, the market price must preserve the stop-loss order, ie, $\Pi(X) \leqslant \Pi(Y)$ if $X \leqslant_{sl}^{\mathbb{P}} Y$. So as a special case, UBIP preserves stop-loss order.

The following proposition shows the intrinsic connections between stop-loss ordering preserving pricing and the utility theory.

Proposition 3.3.1. *If the preference of the investors can be fully described by an increasing concave utility function and the investors are aiming at maximizing his expected utility of wealth, then the market price must preserve the stop-loss order, ie,* $\Pi(X) \leqslant \Pi(Y)$ *whenever* $X \leqslant_{sl}^{\mathbb{P}} Y$.

Proof: We begin by assuming that an investor, with an initial cash endowment w, has an increasing concave utility function $U(\cdot)$. When the investor's trading strategy is $\xi \in \Xi$, where Ξ denotes the admissible set of trading strategies, the cash value of his dynamic portfolio at time t is denoted as $W_w^{\xi}(t)$. The objective of the investor is to maximize his expected utility of wealth at the terminal time T as given below:

$$V(w) = \sup_{\xi \in \Xi} \mathbb{E}\left[U(W_w^{\xi}(T))\right].$$

Next, we will then complete the proof by contradiction. We assume that there exists two random variables X and Y satisfying $\Pi(X) > \Pi(Y)$ and $X \leqslant_{sl}^{\mathbb{P}} Y$ simultaneously. Indeed, if we let ξ^* be the optimal strategy of the investor, we have the following contradiction: for small enough $\delta > 0$,

$$\begin{aligned}
V(w) &= \mathbb{E}\left[U(W_w^{\xi^*}(T))\right] \\
&\geqslant \mathbb{E}\left[U\left(W^{\xi^*}_{w+\delta\frac{\Pi(X)}{\Pi(Y)}}(T) - \frac{\delta}{\Pi(Y)}X\right)\right] \\
&\geqslant \mathbb{E}\left[U\left(W^{\xi^*}_{w+\delta\frac{\Pi(X)}{\Pi(Y)}}(T) - \frac{\delta}{\Pi(Y)}Y\right)\right] \\
&\geqslant \mathbb{E}\left[U\left(W_w^{\xi^*}(T) + \delta\left(\frac{\Pi(X)}{\Pi(Y)} - 1\right)\right)\right] \\
&> \mathbb{E}\left[U(W_w^{\xi^*}(T))\right] \\
&= V(w).
\end{aligned}$$

The first and the third inequalities are due to the optimality of ξ^* while the second inequality follows from

$$-\mathbb{E}\left[U\left(W^{\xi^*}_{w+\delta\frac{\Pi(X)}{\Pi(Y)}}(T)-\frac{\delta}{\Pi(Y)}X\right)\right]\leqslant-\mathbb{E}\left[U\left(W^{\xi^*}_{w+\delta\frac{\Pi(X)}{\Pi(Y)}}(T)-\frac{\delta}{\Pi(Y)}Y\right)\right],$$

which in turn follows from the increasing property and convexity of

$$-\left[U\left(W^{\xi^*}_{w+\delta\frac{\Pi(X)}{\Pi(Y)}}(T)-\frac{\delta}{\Pi(Y)}X\right)\right],$$

as a function of X and the assumption $X \leqslant^{\mathbb{P}}_{sl} Y$.

□

As discussed above, in contrast to the complete market price and many alternative pricing methods, the UBIP is generally non-linear (concave), a property inherited from the concavity assumption on the utility functions of the investors. At the first glance, it seems that the non-linear property may cause UBIP to be an inappropriate pricing method as it contradicts to the principle of no arbitrage. Nevertheless, on one hand, the utility indifference pricing still can eliminate the arbitrage opportunity in the market if we take into account those practical factors such as the existence of market friction. On the other hand, a marginal version of the UBIP developed by Davis (1997) is indeed a linear pricing functional. This marginal utility-based pricing (MUBP) gives the utility indifference price for an infinitesimal position in claims, which is unique, lies in the no-arbitrage interval and also lies in the bid and ask utility-based price for a finite position in claims. Consequently, the MUBP functional is endowed with those desirable properties such as recovery of complete market price, monotonicity, stop-loss order preserving. More interestingly, the MUBP pricing functional has a very elegant representation as the risk neutral pricing. By using the Markov process theory, Davis (1997) proved that the marginal utility-based pricing (MUBP) can be expressed as a discounted expectation under a unique probability measure and a unique discount factor. For the rigorous definition of the MUBP, we refer to Davis (1997); see also Monoyios (2008) for a different but equivalent definition.

In this section, we will first investigate the CVaR-based optimal partial hedging strategies under the general stop-loss order preserving pricing functional, then we will further study the optimal strategies under the marginal utility-based pricing functional.

3.3.2 Under Stop-loss Order Preserving Pricing

Recall that our proposed CVaR-based optimal partial hedging model corresponds to the optimization problem (3.1). Therefore, our objective here is to identify the solution to the optimization problem (3.1) under the admissible set of $\mathcal{L}_1$ specified in (3.2). In this subsection, we assume the pricing functional preserves stop-loss order so that all the UBIP functionals apply.

To proceed, it is useful to present the following two lemmas.

Lemma 3.3.1. *For a given random variable X and any function $f \in \mathcal{L}_1$, let*

$$g_f(x) = \min\left\{(x + f(VaR_\alpha(X)) - VaR_\alpha(X))_+, \bar{u}\right\}, \tag{3.6}$$

where $\bar{u}$ is determined by letting $CVaR_\alpha(R_f(X)) = CVaR_\alpha(R_{g_f}(X))$, and $R_f(X) := X - f(X)$ so that $R_{g_f}(X) = X - g_f(X)$. Then, g_f is well defined and $g_f \in \mathcal{L}_1$ for any $g \in \mathcal{L}_1$.

Proof: To show that function g_f is well-defined, it is sufficient to verify the equation $\mathrm{CVaR}_\alpha(R_f(X)) = \mathrm{CVaR}_\alpha(R_{g_f}(X))$. To this end, we first note that

$$R_{g_f}(x) = x - g_f(x) = x - \min\left\{(x-d)_+, \bar{u}\right\} \tag{3.7}$$

where $d := \mathrm{VaR}(X) - f(\mathrm{VaR}(X)) \geqslant 0$. From (3.7), $\mathrm{CVaR}_\alpha(R_{g_f}(X))$ is continuous as a function of $\bar{u}$. Moreover, when $\bar{u} = 0$, $R_{g_f}(x) = x \geqslant R_f(x)$, and hence $\mathrm{CVaR}_\alpha(R_{g_f}(X)) \geqslant \mathrm{CVaR}_\alpha(R_f(X))$. Thus, to show that g_f is well defined, it is sufficient to establish

$$\lim_{\bar{u}\to\infty} \mathrm{CVaR}_\alpha(R_{g_f}(X)) \leqslant \mathrm{CVaR}_\alpha(R_f(X)). \tag{3.8}$$

Indeed, since $R_f(x) = x - f(x)$ is nondecreasing and left continuous, and using Theorem 1 in Dhaene et al. (2002), we have $R_f(\mathrm{VaR}_\alpha(X)) = \mathrm{VaR}_\alpha(R_f(X))$ and hence

$$\begin{aligned}
\mathrm{CVaR}_\alpha(R_f(X)) &\geqslant \mathrm{VaR}_\alpha\left(R_f X\right) = R_f(\mathrm{VaR}_\alpha(X)) \\
&= \mathrm{VaR}_\alpha(X) - f(\mathrm{VaR}_\alpha(X)) \\
&= d. \qquad (3.9)
\end{aligned}$$

Moreover,

$$\begin{aligned}
\lim_{\bar{u}\to\infty} \mathrm{CVaR}_\alpha(R_{g_f}(X)) &= \mathrm{CVaR}_\alpha\left(\lim_{\bar{u}\to\infty} R_{g_f}(X)\right) \\
&= \mathrm{CVaR}_\alpha\left(X - (X-d)_+\right) \\
&= \mathrm{CVaR}_\alpha\left(\min\{X, d\}\right) \\
&\leqslant d. \qquad (3.10)
\end{aligned}$$

Combining (3.9) and (3.10) yields (3.8).

To show $g_f \in \mathcal{L}_1$, we need to demonstrate that $R_{g_f}(x)$ is nondecreasing and left continuous as a function of x. It is clear if we notice the expression (3.7). Thus, the proof is complete. □

For a given payoff X, we are going to prove that the cost of the hedging strategy $g_f(X)$ will not be higher than that of the hedging strategy $f(X)$ under the assumption that the market pricing function $\Pi(\cdot)$ preserves the stop-loss order.

Lemma 3.3.2. *For a given payout X and any function $f \in \mathcal{L}_1$, the function $g_f \in \mathcal{L}_1$ as constructed in Lemma 3.3.1 is smaller than $f(X)$ in stop-loss order under the physical probability measure $\mathbb{P}$:*

$$g_f(X) \leqslant_{sl}^{\mathbb{P}} f(X). \qquad (3.11)$$

Hence, if the market pricing functional Π *preserves the stop-loss ordering, we have*

$$\Pi(g_f(X)) \leqslant \Pi(f(X)).$$

Proof: We only need to establish (3.11). Let u_α denote a random variable uniformly distributed on $[0, \alpha]$, and assume it is independent of all other random variables involved in the chapter. If

$$g_f(VaR_{u_\alpha}(X)) \leqslant_{sl}^{\mathbb{P}} f(VaR_{u_\alpha}(X)) \tag{3.12}$$

holds, then, for any $d \in \mathbb{R}$, we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[(g_f(X)-d)_+\right] &= \int_0^1 (g_f(\mathrm{VaR}_s(X))-d)_+\,ds \\
&= \int_0^\alpha (g_f(\mathrm{VaR}_s(X))-d)_+\,ds + \int_\alpha^1 (g_f(\mathrm{VaR}_s(X))-d)_+\,ds \\
&= \alpha\mathbb{E}^{\mathbb{P}}\left[(g_f(\mathrm{VaR}_{u_\alpha}(X))-d)_+\right] + \int_\alpha^1 (g_f(\mathrm{VaR}_s(X))-d)_+\,ds \\
&\leqslant \alpha\mathbb{E}^{\mathbb{P}}\left[(f(\mathrm{VaR}_{u_\alpha}(X))-d)_+\right] + \int_\alpha^1 (f(\mathrm{VaR}_s(X))-d)_+\,ds \\
&= \int_0^1 (f(\mathrm{VaR}_s(X))-d)_+\,ds \\
&= \mathbb{E}^{\mathbb{P}}\left[(f(X)-d)_+\right],
\end{aligned}$$

which leads to the desired result (3.11). Thus, it remains to show (3.12).

To demonstrate (3.12), we shall use a well known sufficient condition for the stop-loss order (see, for example, Rolski et al. (1999)). For two random variables Z_1 and Z_2 with finite means, a sufficient condition for $Z_1 \leqslant_{sl}^{\mathbb{P}} Z_2$ is as follows:

(i) $\mathbb{E}^{\mathbb{P}}[Z_1] \leqslant \mathbb{E}^{\mathbb{P}}[Z_2]$, and

(ii) There exists $t_0 \in \mathbb{R}$ such that $\mathbb{P}(Z_1 \leqslant t) \leqslant \mathbb{P}(Z_2 \leqslant t)$ for $t < t_0$ while $\mathbb{P}(Z_1 \leqslant t) \geqslant \mathbb{P}(Z_2 \leqslant t)$ for $t > t_0$.

Consequently, we only need to verify the above two conditions with $Z_1 = g_f(VaR_{u_\alpha}(X))$ and $Z_2 = f(VaR_{u_\alpha}(X))$. In fact, using Theorem 1 in Dhaene et al. (2002) and using the nonincreasing and left continuous property of $R_f(x) = x - f(x)$ as a function of x, we have

$$\begin{aligned}\mathrm{CVaR}_\alpha(R_f(X)) &= \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_s(R_f(X))ds \\ &= \frac{1}{\alpha}\int_0^\alpha R_f(\mathrm{VaR}_s(X))ds \\ &= \frac{1}{\alpha}\int_0^\alpha (\mathrm{VaR}_s(X) - f(\mathrm{VaR}_s(X)))ds \\ &= \mathrm{CVaR}_\alpha(X) - \mathbb{E}^{\mathbb{P}}\left[f(\mathrm{VaR}_{u_\alpha}(X))\right]\end{aligned}$$

and similarly

$$\mathrm{CVaR}_\alpha(R_{g_f}(X)) = \mathrm{CVaR}_\alpha(X) - \mathbb{E}^{\mathbb{P}}\left[g_f(\mathrm{VaR}_{u_\alpha}(X))\right].$$

The above results, combining with the fact that g_f is constructed such that $\mathrm{CVaR}_\alpha(R_f(X)) = \mathrm{CVaR}_\alpha(R_{g_f}(X))$, imply

$$\mathbb{E}^{\mathbb{P}}\left[f(\mathrm{VaR}_{u_\alpha}(X))\right] = \mathbb{E}^{\mathbb{P}}\left[g_f(\mathrm{VaR}_{u_\alpha}(X))\right].$$

This means that condition (i) in the above is met by $Z_1 = g_f(VaR_{u_\alpha}(X))$ and $Z_2 = f(VaR_{u_\alpha}(X))$.

To verify condition (ii), we first note that $\mathrm{VaR}_{u_\alpha}(X) \geqslant \mathrm{VaR}_\alpha(X)$ and $\mathrm{VaR}_{u_\alpha}(X) - f(\mathrm{VaR}_{u_\alpha}(X)) \geqslant \mathrm{VaR}_\alpha(X) - f(\mathrm{VaR}_\alpha(X))$ due to the nondecreasing property of $R_f(x)$, and consequently, for any $t < \bar{u}$,

$$\begin{aligned}\mathbb{P}\left(g_f(\mathrm{VaR}_{u_\alpha}(X)) \leqslant t\right) &= \mathbb{P}(\mathrm{VaR}_{u_\alpha}(X) - \mathrm{VaR}_\alpha(X) + f(\mathrm{VaR}_\alpha(X)) \leqslant t) \\ &\leqslant \mathbb{P}(f(\mathrm{VaR}_{u_\alpha}(X)) \leqslant t),\end{aligned} \tag{3.13}$$

where the equality is due to the construction (3.6) for g_f. Moreover, for any $t > \bar{u}$, the construction (3.6) of g_f implies that

$$\mathbb{P}(g_f(\mathrm{VaR}_{u_\alpha}(X)) \leqslant t) = 1 \geqslant \mathbb{P}(f(\mathrm{VaR}_{u_\alpha}(X)) \leqslant t. \tag{3.14}$$

By (3.13) and (3.14), condition (ii) is also satisfied by $Z_1 = g_f(\mathrm{VaR}_{u_\alpha}(X))$ and $Z_2 = f(\mathrm{VaR}_{u_\alpha}(X))$, and thus the proof is complete. $\square$

Remark 3.3.1. *Let $v = VaR_\alpha(X)$, $d = VaR_\alpha(X) - f(VaR_\alpha(X))$ (which is obviously nonnegative) and $u = \bar{u} + d$. Then, the function g_f given in* (3.6) *can be rewritten as $g_f(x) = \min\{(x-d)_+, u-d\}$, where $0 \leqslant d \leqslant v$. Since $\bar{u}$ is determined by the equation $CVaR_\alpha(R_f(X)) = CVaR_\alpha(R_{g_f}(X))$, $\bar{u} \geqslant 0$ and hence $u \geqslant 0$; consequently, g_f can be further rewritten as follows*

$$g_f(x) = (x-d)_+ - (x-u)_+ \tag{3.15}$$

where $0 \leqslant d \leqslant v$, $u \geqslant d$. This implies that $g_f(X)$ is the payout of a bull call spread written on X.

Now we can state the main result of this subsection. The following theorem shows that the optimal hedged loss function has a form as given in (3.6) (or equivalently (3.15)).

Theorem 3.3.1. *Assume that the market pricing functional preserves the stop-loss order. For any attainable hedging strategy $f \in \mathcal{L}_1$, we assume that the hedging strategy g_f as constructed in* (3.6) *is attainable. Then, hedging strategy g_f belongs to $\mathcal{L}_1$ and satisfies*

$$CVaR_\alpha(T_{g_f}(X)) \leqslant CVaR_\alpha(T_f(X)) \tag{3.16}$$

Moreover, $\Pi(f(X)) \leqslant \pi_0$ implies $\Pi(g_f(X)) \leqslant \pi_0$. Therefore, bull call spread hedging is the optimal form of hedging under any budget constraint among $\mathcal{L}_1$ in the sense of minimizing CVaR of the total exposed risk at maturity of the investor.

Proof: For any function $f \in \mathcal{L}_1$, it follows from Lemma 3.3.1 that $g_f \in \mathcal{L}_1$, and from Lemma 3.3.2 that $\Pi(g_f(X)) \leqslant \Pi(f(X))$; thus we must have $\Pi(g_f(X)) \leqslant \pi_0$ as long as

$\Pi(f(X)) \leqslant \pi_0$. It remains to verify inequality (3.16). This can be justified as follows:

$$\begin{aligned} \mathrm{CVaR}_\alpha(T_{g_f}(X)) &= \mathrm{CVaR}_\alpha(R_{g_f}(X)) + e^{rT}\Pi(g_f(X)) \\ &= \mathrm{CVaR}_\alpha(R_f(X)) + e^{rT}\Pi(g_f(X)) \\ &\leqslant \mathrm{CVaR}_\alpha(R_f(X)) + e^{rT}\Pi(f(X)) \\ &= \mathrm{CVaR}_\alpha(T_f(X)), \end{aligned}$$

where the first and the last equalities are due to the translation invariance property of the risk measure CVaR, the second equality is because of the construction of g_f, and the inequality results from the fact that $\Pi(g_f(X)) \leqslant \Pi(f(X))$. □

For brevity, it is convenient to introduce the following function:

$$G(x; d, u) = (x-d)_+ - (x-u)_+ \text{ for } u \geqslant d, \text{ and } x, d, u \in \mathbb{R}. \tag{3.17}$$

Remark 3.3.2. *(a) With the help of Theorem 3.3.1, for the solution to the CVaR-optimization problem, it is sufficient for us to concentrate on the set of hedged loss functions of the form* (3.6) *or equivalently* (3.15). *This means that it suffices to focus on the following 2-dimensional optimization problem with decision variables d and u:*

$$\begin{cases} \min\limits_{0 \leqslant d \leqslant v, u \geqslant d} & CVaR_\alpha \left\{X - (X-d)_+ + (X-u)_+ + e^{rT} \cdot \Pi\left[G(X; d, u)\right]\right\} \\ s.t. & \Pi\left[G(X; d, u)\right] \equiv \Pi\left[(X-d)_+ - (X-u)_+\right] \leqslant \pi_0 \end{cases} \tag{3.18}$$

which can obviously be simplified, using the translation invariance property of CVaR risk measure, to

$$\begin{cases} \min\limits_{0 \leqslant d \leqslant v, u \geqslant d} & d + CVaR_\alpha\left[(X-u)_+\right] + e^{rT} \cdot \Pi\left[G(X; d, u)\right] \\ s.t. & \Pi\left[G(X; d, u)\right] \equiv \Pi\left[(X-d)_+ - (X-u)_+\right] \leqslant \pi_0. \end{cases} \tag{3.19}$$

Once we obtain the optimal solution (d^*, u^*) *to the above 2-dimensional optimization problem* (3.19), *the optimal hedged loss function to our CVaR-minimization problem is*

$$f^*(x) = (x - d^*)_+ - (x - u^*)_+ .$$

(b) The comments we made in Remark 2.2.1 and Remark 2.2.4 for the solutions to the VaR-based optimal partial hedging problem in Chapter 2 are similarly applicable here for the CVaR case. In particular, we draw the following conclusions.

(i) The above theorem shows that the optimal hedging strategy, if possible, is to buy a call option on the payout X *with strike price* d^**, while selling a call option on the payout* X *with strike price* u^**, where* d^* *and* u^* *are the optimizers of the optimization problem (3.19).*

(ii) Since we did not specify the dynamics of the underlying assets and the specific pricing functional, the result in Theorem 3.3.1 is model independent.

(iii) As long as the call options on the payout X *exist in the market, our proposed partial hedging strategy, which is the bull call spread hedging strategy, can be replicated by the portfolio of options without rebalancing. Therefore, in many cases, our proposed partial hedging strategy is a static hedging strategy.*

(c) In Theorem 3.2.1, we derive the optimal partial hedging strategies with the assumption of sufficient hedging budget. As we discussed in Remark 3.2.2, when we consider the stop-loss order preserving pricing, Theorem 3.3.1 provides us the optimal partial hedging strategies with any given hedging budget.

(d) Theorem 3.3.1 is consistent with Theorem 3.2.1. First of all, the optimal partial hedging strategies in Theorem 3.2.1 is of a special form of the optimal partial hedging s-trategies in Theorem 3.3.1. Secondly, if we further assume the pricing functional preserves

stop-loss order in Theorem 3.2.1, we can obtain the optimal partial hedging strategies under any given hedging budget. Thirdly, if we further assume the pricing functional satisfies Assumption 3.2.1 in Theorem 3.3.1, we can conclude that the optimal partial hedging strategy is to construct a call on the contingent claim.

3.3.3 Under the Marginal Utility-based Pricing Method

According to the previous subsection (see Theorem 3.3.1 and Remark 3.3.2), for the specific optimal partial hedging strategy, we need to solve the optimization problem (3.19), which is not trivial in the general case. In this subsection, we will analyze the optimal partial hedging strategies under the marginal utility-based pricing (MUBP) method. As introduced in Subsection 3.3.1, the MUBP functional can be expressed as a discounted expectation under a uniquely determined probability measure $\mathbb{Q}$, and hence it is linear. Without lose of generality, hereafter we assume that it admits the following representation

$$\Pi(Z) = e^{-rt}\mathbb{E}^{\mathbb{Q}}[Z] \text{ for any time-}t \text{ contingent claim } Z \tag{3.20}$$

where $\mathbb{E}^{\mathbb{Q}}$ denotes the expectation under $\mathbb{Q}$, e^{-rt} is the discount factor. If the risk free rate is constant in the market, then the discount rate is the same as the risk free rate (see Davis (1997)). For the details on how to obtain the probability measure $\mathbb{Q}$ and the discount rate r in more general cases, we refer to Davis (1997).

Under the MUBP functional with the presentation (3.20), the optimization problem (3.19) can be simplified as shown in the following proposition.

Proposition 3.3.2. *Assume the pricing functional* Π *is such that* $\Pi(Z) = e^{-rt}\mathbb{E}^{\mathbb{Q}}[Z]$ *for any time-t contingent payout* Z*, where* $\mathbb{E}^{\mathbb{Q}}$ *denotes the expectation under a fixed probability measure* $\mathbb{Q}$*, and* r *is the discount rate. If* $\mathbb{P}(X = v) = 0$*, the solutions to the following* *optimization problem* (3.21) *solve problem (3.19) and they share the same optimal value.*

$$\begin{cases} \min\limits_{0\leqslant d\leqslant v, u\geqslant v} & \left\{d + \frac{1}{\alpha}\mathbb{E}^{\mathbb{P}}(X-u)_+ + e^{rT}\cdot \Pi\left[G(X;d,u)\right]\right\} \\ s.t. & \Pi\left[(X-d)_+\right] - \Pi\left[(X-u)_+\right] \leqslant \pi_0. \end{cases} \tag{3.21}$$

Proof: We first note that $\mathrm{CVaR}_\alpha\left([X-u]_+\right) = \dfrac{1}{\alpha}\mathbb{E}\left[(X-u)_+\right]$ for $u \geqslant v$. Thus, problem (3.19) reduces to (3.21) if we confine to $u \geqslant v$. Consequently, it suffices to show that that the optimal value of problem (3.19) for $u < v$ cannot be smaller than that for $u = v$.

We now consider a generic point (d_1, u_1) from the feasible set of problem (3.19) with $u_1 < v$. Let d_2 be a number satisfying the following equation

$$\Pi(G(X; d_2, v)) = \Pi(G(X; d_1, u_1)),$$

where G is defined in (3.17). The last equation is obviously equivalent to

$$\int_{d_1}^{u_1} \mathbb{Q}(X > x)dx = \int_{d_2}^{v} \mathbb{Q}(X > x)dx,$$

which, along with the fact that $\mathbb{Q}(X > x)$ is decreasing as function of x, further implies $v - d_2 \geqslant u_1 - d_1$. Consequently,

$$\begin{aligned} d_2 + CVaR_\alpha\left[(X-v)_+\right] &= d_2 + CVaR_\alpha(X) - v \\ &\leqslant d_1 + CVaR_\alpha(X) - u_1 \\ &= d_1 + CVaR_\alpha\left[(X-u_1)_+\right]. \end{aligned}$$

This means that the optimal value of the problem (3.19) for $u < v$ cannot be smaller than that for $u = v$. Thus, the proof is complete. □

The following lemma gives us a necessary condition under which the investor will utilize all the hedging budget to construct the optimal partial hedging strategy.

Lemma 3.3.3. *Assume that the conditions in Proposition 3.3.2 are satisfied with a hedging budget* $\pi_0 \leqslant e^{-rT}\mathbb{E}^{\mathbb{Q}}[(X-\tilde{d})_+ - (X-v)_+]$, *where*

$$v = VaR_\alpha(X) \ \ and \ \ \tilde{d} = \sup\{d \in \mathbb{R} : Q(X \leqslant d) = 0\}. \tag{3.22}$$

Then the budget constraint in problem (3.21) *is binding under the optimal hedging strategy.*

Proof: Let (d^*, u^*) be one optimal solution to problem (3.21). We will complete the proof by contradiction. We first note that the objective function in problem (3.19) is nondecreasing as a function of d as indicated below:

$$\frac{\partial}{\partial d}\left(d + \frac{1}{\alpha}\mathbb{E}^{\mathbb{P}}(X-u)_+ + \int_d^u \mathbb{Q}(X>x)dx\right) = 1 - \mathbb{Q}(X>d) \geqslant 0.$$

Thus, if the constraint in problem (3.21) is loose at (d^*, u^*), we would have $d^* \in [0, \tilde{d})$, and consequently

$$e^{-rT}\mathbb{E}\left[(X-d^*)_+ - (X-u^*)_+\right] < \pi_0 \leqslant e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\left(X-\tilde{d}\right)_+ - (X-v)_+\right].$$

It follows from the last inequality that

$$\mathbb{E}^{\mathbb{Q}}\left[(X-u^*)_+\right] - \mathbb{E}^{\mathbb{Q}}\left[(X-v)_+\right] > \mathbb{E}^{\mathbb{Q}}\left[(X-d^*)_+\right] - \mathbb{E}^{\mathbb{Q}}\left[\left(X-\tilde{d}\right)_+\right] \geqslant 0$$

and hence $u^* < v$, which contradicts the assumption that (d^*, u^*) is an optimal solution to problem (3.21). Hence the proof is complete. □

Remark 3.3.3. *(a) When the hedging budget* $\pi_0 \leqslant e^{-rT}\mathbb{E}^{\mathbb{Q}}[(X-\tilde{d})_+ - (X-v)_+]$, *by using Lemma 3.3.3, we can rewrite the optimization problem (3.21) as follows*

$$\begin{cases} \min\limits_{0\leqslant d\leqslant v, u\geqslant v} & \left\{d + \dfrac{1}{\alpha}\mathbb{E}^{\mathbb{P}}(X-u)_+ \equiv d + \dfrac{1}{\alpha}\displaystyle\int_u^\infty \mathbb{P}(X>x)dx\right\} \\ s.t. & \Pi\left[(X-d)_+\right] - \Pi\left[(X-u)_+\right] = \pi_0. \end{cases} \tag{3.23}$$

(b) In many interesting situations, $\tilde{d}$ defined in (3.22) *is obviously 0, and hence for a sufficient small α, $e^{-rT}\mathbb{E}^{\mathbb{Q}}[(X-\tilde{d})_+ - (X-v)_+]$ is very close to the market price of the full payout X. Therefore, an investigation in such a case is of interest.*

Combining Lemma 3.3.3 and Remark 3.3.3 provides an explicit way of identifying the optimal hedged loss function as stated in the following theorem:

Theorem 3.3.2. *Assume that the conditions in Lemma 3.3.3 are satisfied. Then, the optimal hedged loss function g_f^* is given by*

$$g_f^*(x) = (x-d^*)_+ - (x-u^*)_+,$$

where (d^*, u^*) satisfies the following equations

$$\begin{cases} e^{-rT} \displaystyle\int_{d^*}^{u^*} \mathbb{Q}(X > x)dx = \pi_0 \\ \mathbb{P}(X > u^*) = \alpha \cdot \dfrac{\mathbb{Q}(X > u^*)}{\mathbb{Q}(X > d^*)}. \end{cases} \tag{3.24}$$

Proof: Consider the following Lagrangian function of problem (3.23):

$$L(d, u, \lambda) = d + \frac{1}{\alpha}\int_u^{+\infty} \mathbb{P}(X > x)dx + \lambda\left(e^{-rT} \cdot \int_d^u \mathbb{Q}(X > x)dx - \pi_0\right).$$

By letting

$$\frac{\partial L}{\partial d} = \frac{\partial L}{\partial u} = \frac{\partial L}{\partial \lambda} = 0,$$

we immediately obtain the desired results. □

The following corollary provides one of the possible candidates of the optimal hedged loss functions.

Corollary 3.3.1. *Assume the conditions in Lemma 3.3.3 are satisfied, then one of the possible optimal hedging functions is given by*

$$g_f^*(x) = (x - d^*)_+,$$

where d^* is the solution to the following equation

$$e^{-rT} \int_{d^*}^{+\infty} \mathbb{Q}(X > x) dx = \pi_0.$$

This means that constructing a call option on the contract X is one of the possible optimal hedging strategies.

Proof: The result follows trivially from Theorem 3.3.2 by taking $u^* = \infty$. □

3.4 Comparison with Other Partial Hedging Strategies

In this section, we will present two numerical examples to compare and contrast our proposed CVaR-based hedging to other strategies that have appeared in the literatures, namely the well-known quantile hedging and the expected shortfall hedging respectively developed by Föllmer and Leukert (1999, 2000), and the two hedging strategies proposed in Chapter 2 from a perspective of minimizing VaR of the hedger's resulting risk exposure. We use the same numerical setting as that in Chapter 2, which is also same as that in Föllmer and Leukert (1999). In Example 3.4.1, we analyze and evaluate all five hedging strategies in term of the shapes of optimal hedged loss functions, the expected shortfall (the expected value of the hedger's shortfall) and CVaR of the hedger's total risk exposure. By confining ourselves to our proposed CVaR-based hedging strategy and the expected shortfall hedging strategy, Example 3.4.2 is then used to highlight the relative effectiveness of these strategies. Finally, we present the third example to compare our proposed CVaR-based partial hedging to the hedging stategy in Melnikov and Smirnov (2012). In Example 3.4.3, we can see the advantages of our proposed CVaR-based partial hedging.

Since the CVaR-based optimal partial hedging strategies in Section 3.2 are consistent with the CVaR-based optimal partial hedging strategies in Section 3.3, and the CVaR-based optimal partial hedging strategies in Section 3.3 is given under any hedging budget, we will only consider the CVaR-based optimal partial hedging strategies derived in Section 3.3 in the following numerical examples.

The confidence levels for both CVaR and VaR are set to be 95% throughout this section.

Example 3.4.1. *We consider the Black-Scholes model with the dynamics of the stock price* S_t *at time* t *given by*

$$dS_t = S_t(\sigma dW_t + mdt)$$

where W *is a Wiener process under the physical probability measure* $\mathbb{P}$*,* σ *and* m *are, respectively, the constant volatility and the return rate. The objective is to hedge a European call option with payoff* $X_T = (S_T - K)_+$*, where* T *is the expiration date and* K *is the strike price. We further assume*

$$T = 0.25, \quad K = 110, \quad r = 0, \quad S_0 = 100, \quad m = 0.08,$$

where r *is the risk free rate,* S_0 *is the initial stock price, and we have the following three distinct sets of values for the volatility* σ *and the hedging budget* π_0*:*

(i) $\sigma = 0.3$*,* $\pi_0 = 1.5$*,*

(ii) $\sigma = 0.3$, $\pi_0 = 0.5$,

(iii) $\sigma = 0.2$, $\pi_0 = 0.5$.

Using the Black-Scholes formula, the prices of the corresponding call options are

$$P_C = \begin{cases} 2.50 & \textit{for } \sigma = 0.3 \\ 0.95 & \textit{for } \sigma = 0.2. \end{cases}$$

Since the hedging budget for $\sigma = 0.3$ *is either 0.5 or 1.5 and for* $\sigma = 0.2$ *is 0.5, this implies that we do not have sufficient fund to perfectly hedge the option. If the hedger is still interested in some sort of hedging subject to the limited budget, partial hedging is one alternative. We now investigate the following four optimal partial hedging strategies:*

(a) Quantile hedging strategy. *Using the results derived in Föllmer and Leukert (1999), the quantile hedging strategies for the three cases are*

(i): $(S_T - 110)_+ - (S_T - 129.47)_+ - 19.47 \cdot \mathbb{1}(S_T > 129.47)$

(ii): $(S_T - 110)_+ - (S_T - 118.69)_+ - 8.69 \cdot \mathbb{1}(S_T > 118.69)$

(iii): $(S_T - 110)_+ - (S_T - 119.98)_+ - 9.98 \cdot \mathbb{1}(S_T > 119.98) + (S_T - 1323)_+ + 1213 \cdot \mathbb{1}(S_T > 1323)$.

(b) VaR-based hedging strategy. *As shown in Chapter 2 that when VaR is used as the criterion, the knock-out call hedging strategy and the bull call spread hedging strategy are respectively the optimal hedging strategies for the two different admissible sets. At 95% confidence level, the optimal knock-out call hedging and the optimal bull call spread hedging are*

(b.1) Knock-out call hedging strategy:

(i): $(S_T - 110)_+ - (S_T - 129.11)_+ - 19.11\mathbb{1}_{(S_T \geqslant 129.11)}$

(ii): $(S_T - 116.67)_+ - (S_T - 129.11)_+ - 12.44\mathbb{1}_{(S_T \geqslant 129.11)}$

(iii): $(S_T - 110)_+ - (S_T - 119.66)_+ - 9.66\mathbb{1}_{(S_T \geqslant 119.66)}$.

(b.2) Bull call spread hedging strategy:

(i): $(S_T - 113.30)_+ - (S_T - 129.11)_+$

(ii): $(S_T - 120.88)_+ - (S_T - 129.11)_+$

(iii): $(S_T - 112.18)_+ - (S_T - 119.66)_+$.

(c) Expected shortfall hedging strategy. *Using the results from Föllmer and Leukert (2000), the optimal hedging strategies under the expected shortfall hedging are*

(i): $(S_T - 110)_+ \cdot \mathbb{1}_{(S_T \geqslant 123.85)}$

(ii): $(S_T - 110)_+ \cdot \mathbb{1}_{(S_T \geqslant 137.31)}$

(iii): $(S_T - 110)_+ \cdot \mathbb{1}_{(S_T \geqslant 119.19)}$.

(d) CVaR-based partial hedging. *It has been shown earlier that under our proposed CVaR-based hedging strategy, the optimal solution is to follow the bull call spread hedging strategy of the form*

$$[S_T - (K + d^*)]_+ - [S_T - (K + u^*)]_+,$$

where d^* and u^* can be determined numerically from (3.24) *of Theorem 3.3.2. The corresponding values for all three cases at 95% confidence level are*

(i): $d = 5.13$ *and* $u = \infty$*;*

(ii): $d = 15.08$ *and* $u = \infty$*;*

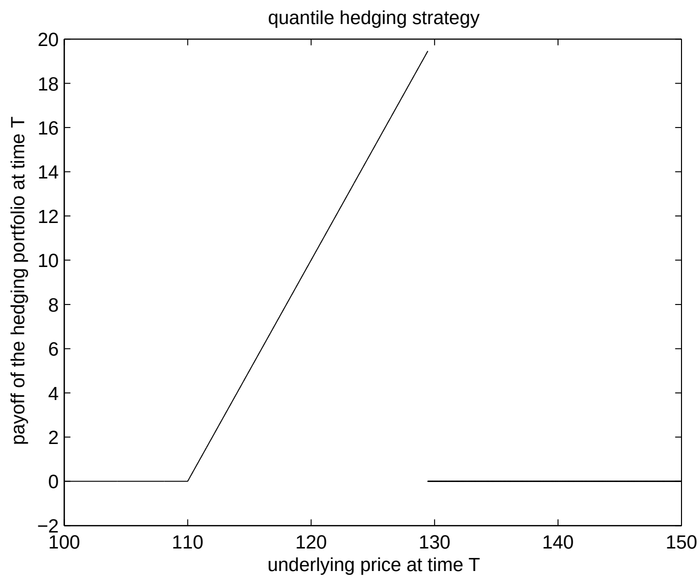

Figure 3.1: Optimal Quantile hedging strategy in scenario (i)

(iii): $d = 3.72$ *and* $u = \infty$.

This leads to the following optimal hedged loss functions:

(i): $(S_T - 115.13)_+$;

(ii): $(S_T - 125.08)_+$;

(iii): $(S_T - 113.72)_+$.

Figures 3.1-3.15 provide a graphical comparison of all the optimal hedged loss functions for all three cases of parameter values and all the five aforementioned hedging strategies. While all these strategies are optimal depending on their adopted objectives, there are some notable differences among the optimal hedged loss functions, as depicted in Figures 3.1-3.15. In particular, one key distinction among them is that strategies such as quantile hedging and the expected shortfall hedging are of the type "all-or-nothing" while the strategy such as

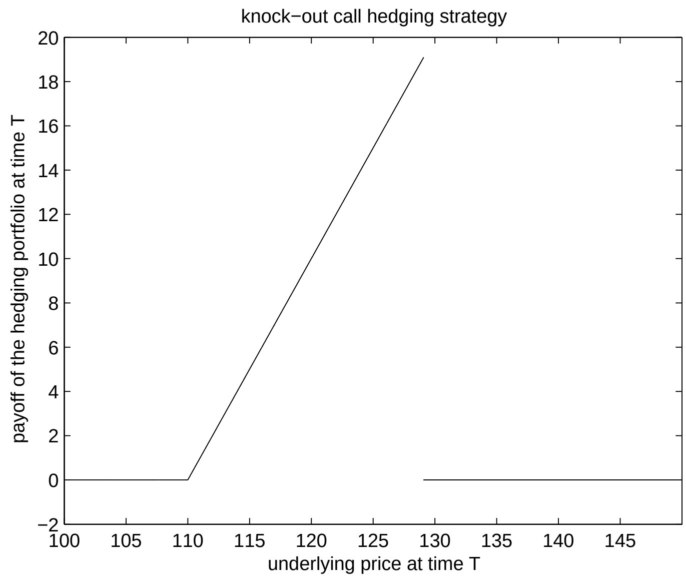

Figure 3.2: Optimal VaR-based Knock-out call hedging strategy in scenario (i)

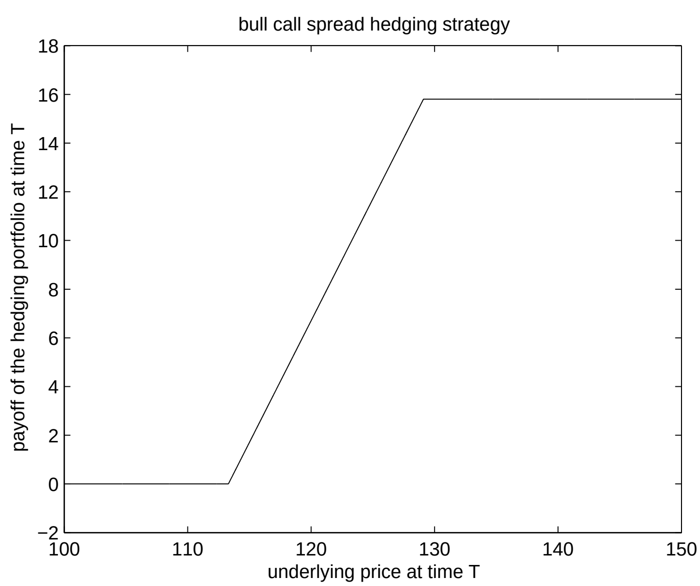

Figure 3.3: Optimal VaR-based Bull call spread hedging strategy in scenario (i)

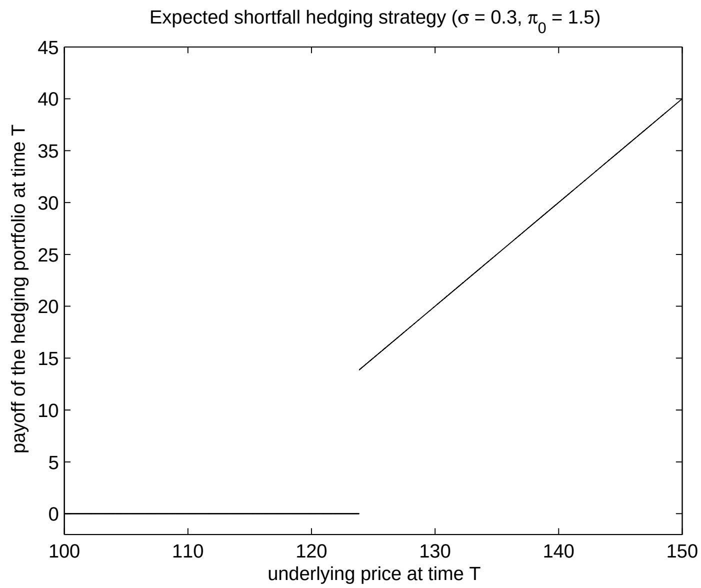

Figure 3.4: Optimal Expected shortfall hedging strategy in scenario (i)

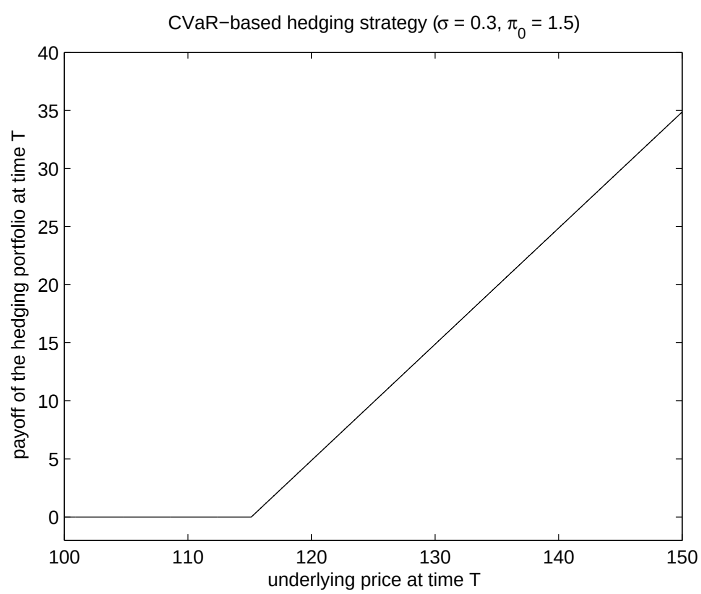

Figure 3.5: Optimal CVaR-based hedging strategy in scenario (i)

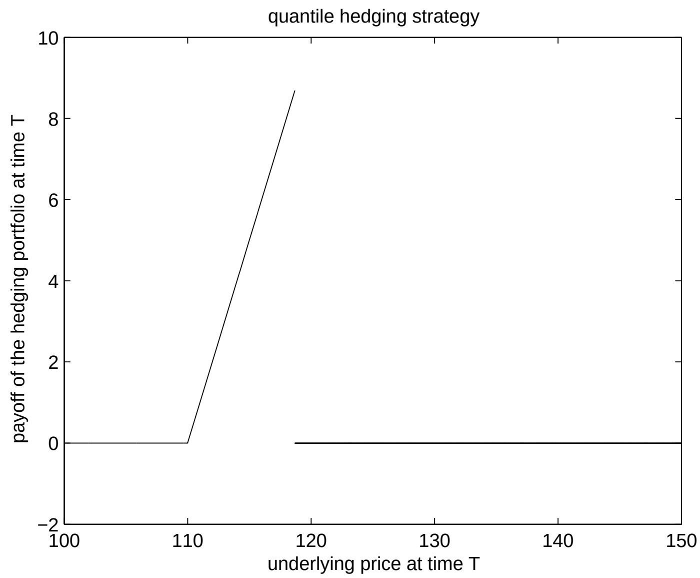

Figure 3.6: Optimal Quantile hedging strategy in scenario (ii)

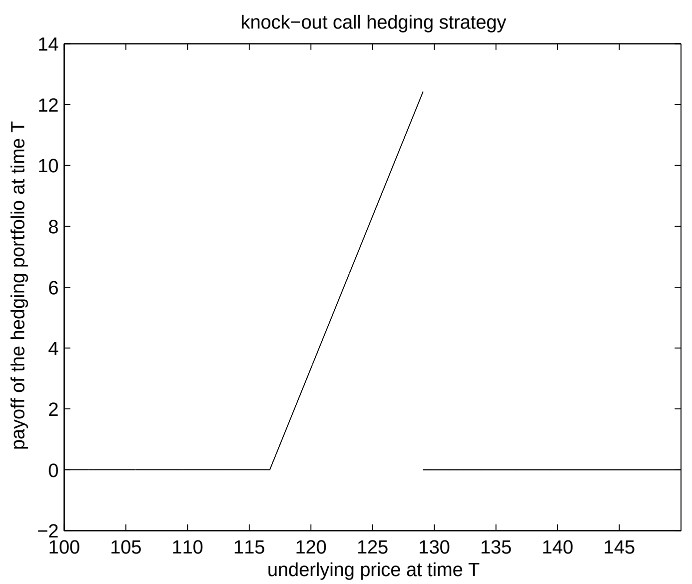

Figure 3.7: Optimal VaR-based Knock-out call hedging strategy in scenario (ii)

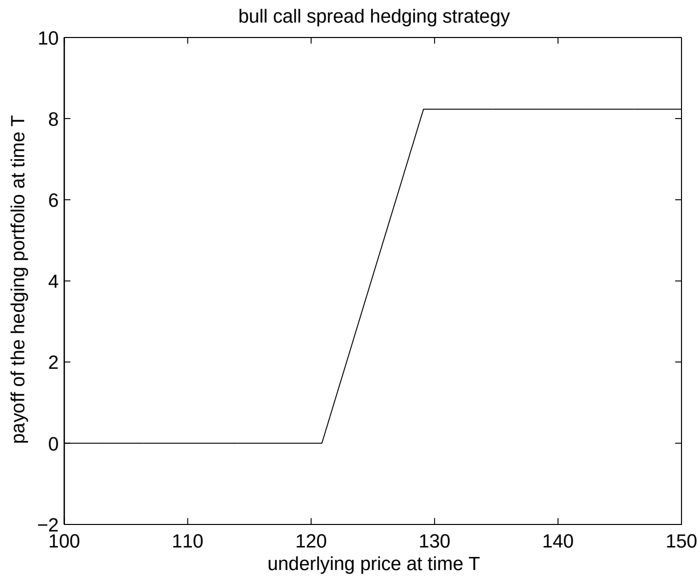

Figure 3.8: Optimal VaR-based Bull call spread hedging strategy in scenario (ii)

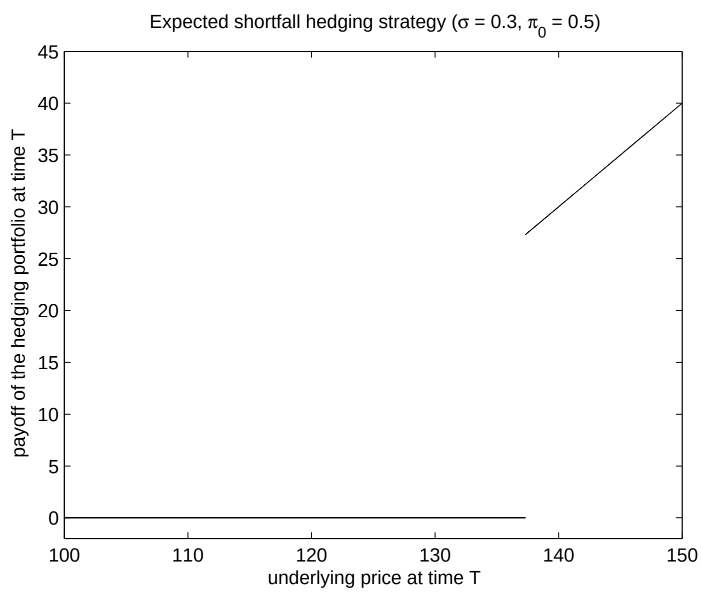

Figure 3.9: Optimal Expected shortfall hedging strategy in scenario (ii)

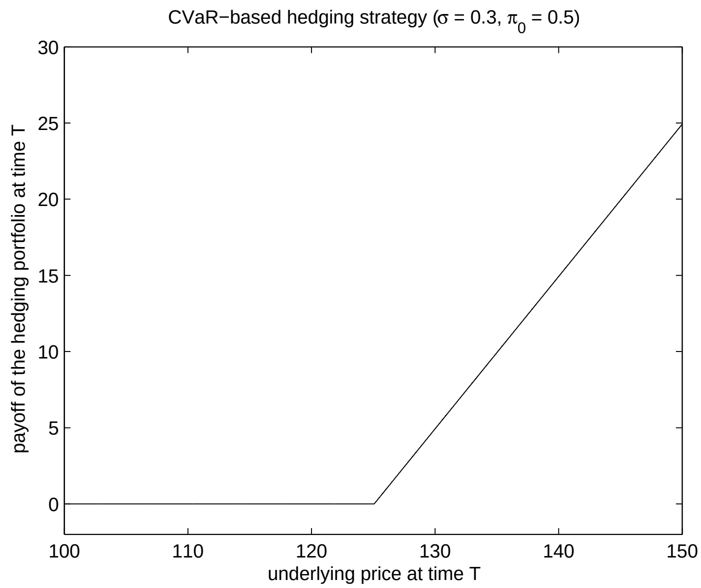

Figure 3.10: Optimal CVaR-based hedging strategy in scenario (ii)

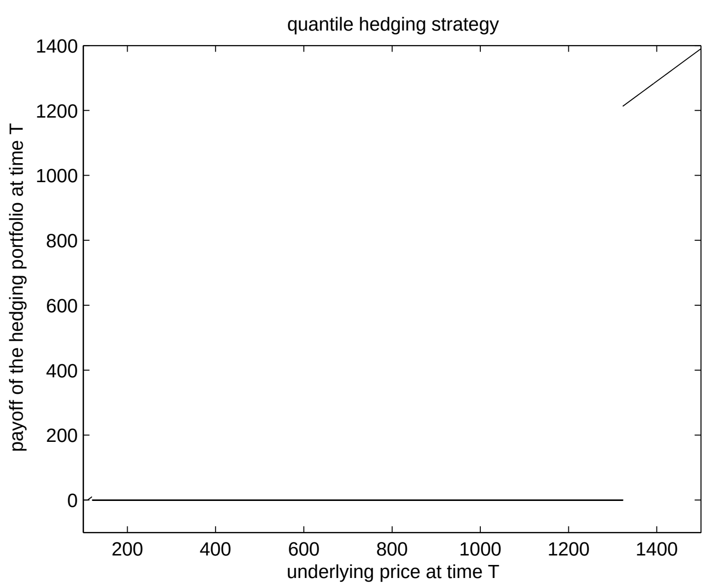

Figure 3.11: Optimal Quantile hedging strategy in scenario (iii)

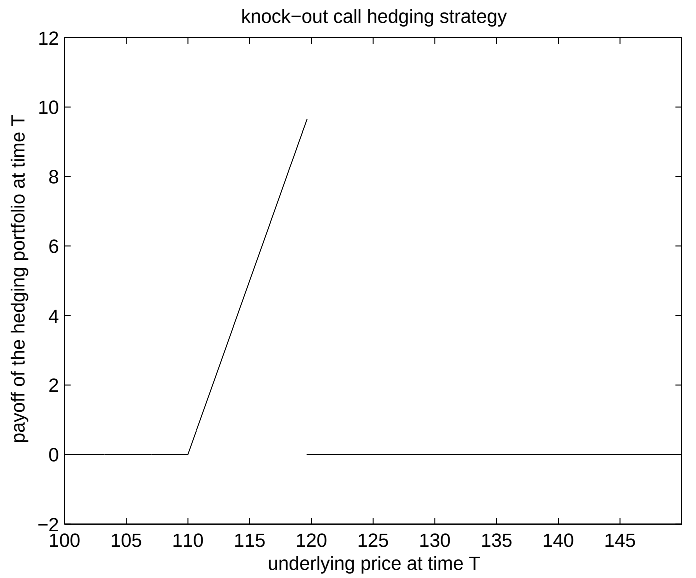

Figure 3.12: Optimal VaR-based Knock-out call hedging strategy in scenario (iii)

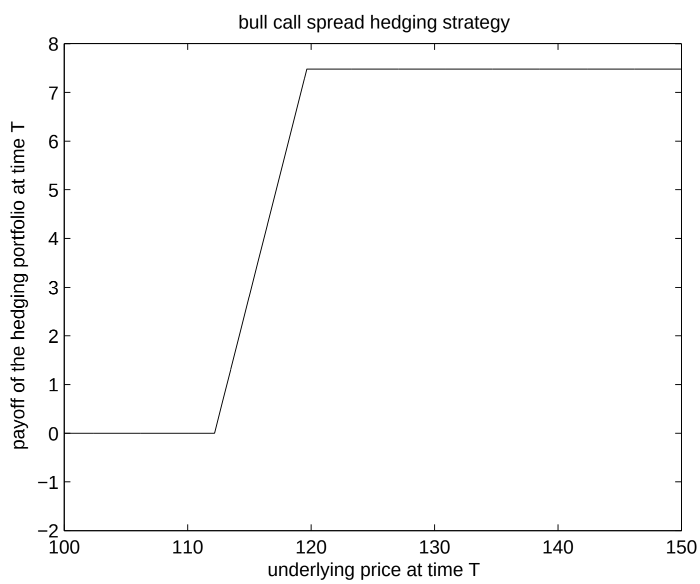

Figure 3.13: Optimal VaR-based Bull call spread hedging strategy in scenario (iii)

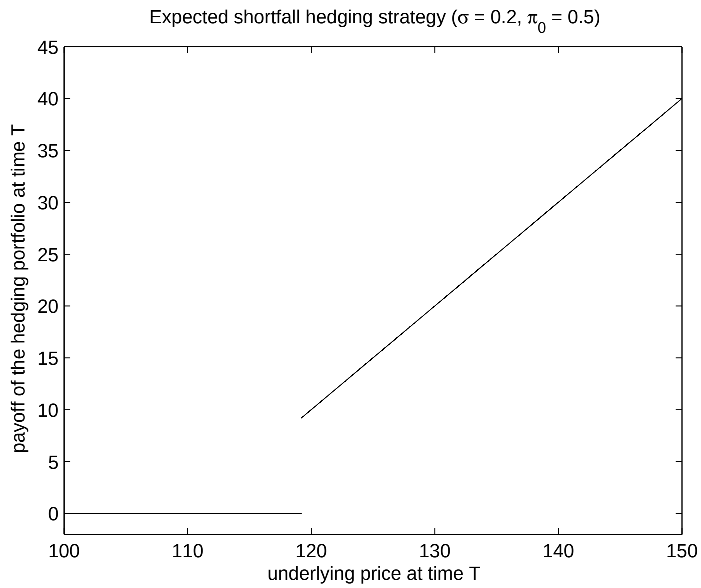

Figure 3.14: Optimal Expected shortfall hedging strategy in scenario (iii)

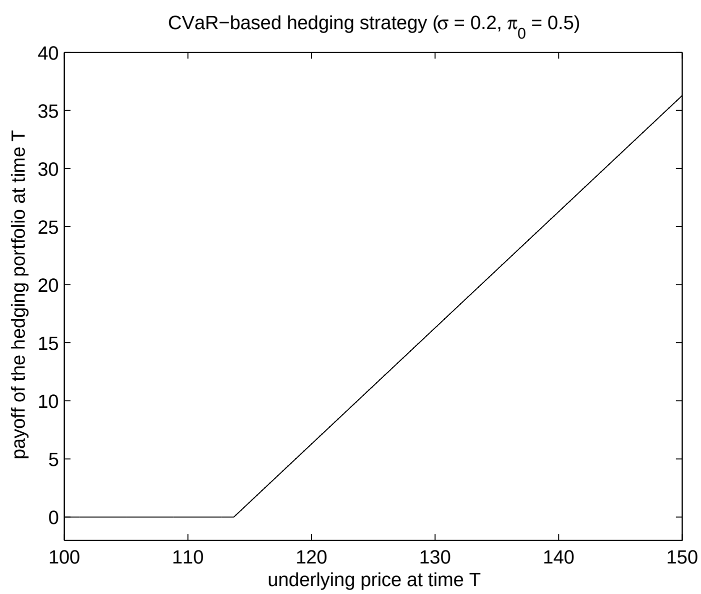

Figure 3.15: Optimal CVaR-based hedging strategy in scenario (iii)

the CVaR-based hedging is not. By "all-or-nothing", we mean that the hedger is perfectly hedged for some part of the losses but unhedge for other parts of losses. To elaborate this point, let us consider quantile hedging the risk exposure in case (i). In this case, as long as $S_T \leqslant 129.47$ *the hedge is perfect. The perfect hedging, however, is accomplished at the expense of having a naked exposure for* $S_T > 129.47$*; ie, no protection is provided whenever* $S_T > 129.47$ *and as a result the hedger is exposed to an unlimited loss due to the infinite payout of the European call option. The optimal hedged loss function of hedging the call option for the expected shortfall hedging is also another "all-or-nothing" strategy. The main difference here is that the expected shortfall hedging perfectly hedges large losses while leaving the small losses unhedged. For instance in case (i) of our example the expected shortfall hedging yields a strategy with perfect hedge for* $S_T > 123.85$ *but for* $110 < S_T < 123.85$*, the strategy does not provide any protection. Hence the maximum risk exposure is capped at 13.85, which is an improvement over quantile hedging with an infinite risk exposure. For this reason, the expected shortfall hedging is considered more desirable than quantile hedging in this example.*

In contrast, the CVaR-based hedging is not a "all-or-nothing" strategy. In fact it is truly a partial hedging strategy in the sense that whenever there is a loss, the hedging position is never perfect unless the hedging budget is large enough for implementing a perfect hedge. In other words, the hedger typically needs to absorb a certain level of loss whenever there is a payout from the option. The potential loss, however, is normally kept to a manageable level so that the tail risk is managed effectively. For instance, under the CVaR-based hedging the maximum loss exposure of the hedger is never more than 5.13 in case (i). This compares favorably to quantile hedging with an infinite loss and the expected shortfall hedging with a maximum loss of 13.85.

The differences among the optimal loss functions also have important implication on *the effectiveness of the hedge. By construction, each strategy is optimal to the designated criterion. It is therefore of great interest to evaluate the performance of these "optimal" strategies if other criteria have been used instead. Table 3.1 provides some insights. The values in the table are the resulting expected shortfall of the hedging strategy and CVaR of the hedger's total risk exposure for each of the five "optimal" hedging strategies. It is not surprising that the expected shortfall hedging and our proposed CVaR-based hedging, respectively, lead to the smallest expected shortfall of the hedging strategy and the smallest CVaR of the hedger's total risk exposure. It is, however, important to note that while the expected shortfall hedging has the smallest expected shortfall of the hedging strategy (by design), other hedging strategies also have expected shortfalls that are close to the optimal value. This is desirable for other hedging strategies even though they are not specifically design to achieve this. In contrast, if CVaR of the hedger's total risk exposure were used to assess these strategies, the situation is quite different. In this case, other hedging strategies have a much larger CVaR than the optimal CVaR-based hedging strategy. Let us exemplify this by considering case (i). The CVaR of the hedger's total risk exposure for other strategies ranges from 1.9 times to 4.4 times relative to the CVaR-based hedging strategy. This cautions the hedger that while other strategies may be optimal in their designated criteria, but when the optimal strategy is used to measure CVaR of the total risk exposure, the resulting CVaR can be unreasonably large and hence leave the hedger open to unexpectedly large loss exposure. This also suggests the sensitivity of these hedging strategies and their ineffectiveness in managing the tail risk as measured by CVaR. This phenomenon is even more pronounced when we consider the relative effectiveness of the CVaR-based hedging and the expected shortfall hedging on hedging a put option. We will demonstrate this in the next example.*

Example 3.4.2. *The set up of this example is similar to Case (i) of Example 3.4.1 (ie,*

Table 3.1: The resulting total risk exposure of hedging a call option

	Expected shortfall of the hedging strategy			CVaR of hedger's total risk exposure		
Case	(i)	(ii)	(iii)	(i)	(ii)	(iii)
CVaR hedging	1.20	2.45	0.62	6.63	15.58	4.22
Quantile hedging	1.35	2.56	0.72	28.46	28.17	14.83
Knock-out call hedging	1.36	2.52	0.72	29.15	28.17	15.34
Bull call spread hedging	1.25	2.48	0.66	13.36	19.94	7.87
Expected shortfall hedging	1.16	2.43	0.60	12.76	21.07	7.48

$\sigma = 0.03$ *and* $\pi_0 = 1.5$*) except that we are interested in hedging a European put option with payoff* $X_T = (95 - S_T)_+$*. Based on the assumed parameter values, the Black-Scholes put option price is* $P_P = 3.6659$ *and thus is is not possible to perfectly hedge the put option from the given budget* $\pi_0 = 1.5$*. The purpose of this example is to compare our proposed CVaR-based hedging to the expected shortfall hedging.*

For this example, the optimal hedged loss function becomes

$$(95 - S_T)_+ \cdot \mathbb{1}_{(S_T \geqslant 83.0297)} \tag{3.25}$$

for the expected shortfall hedging and

$$(87.8215 - S_T)_+ \tag{3.26}$$

for our proposed CVaR-based hedging.

Note that with the put option, the hedger is concerned with declining stock prices. Hence to control the risk of large loss exposure, the hedger should pay special attention when the stock prices have depreciated substantially. We saw in the last example that the expected shortfall hedging strategy was effective at managing the tail risk of the call option by perfectly hedging large losses. What is striking (and counter-intuitive) about the expected shortfall hedging strategy in hedging the put option is that the optimal hedging function (3.25) suggests that it is optimal not to hedge large losses. More specifically, the optimal strategy dictated by the expected shortfall hedging is to only perfectly hedge the put option for $83.0297 \leqslant S_T \leqslant 95$ and is unhedged for $S_T < 83.0297$. Clearly with this strategy, the hedger exposed to undesirable potentially large losses. On the other hand, the optimal hedging strategy from the CVaR-based hedging is consistent with what we observed earlier. Whenever there is a payout from the put option, the hedger needs to incur some losses. However, the loss is never more than 7.1785, regardless of the level of the stock prices. In this aspect, the CVaR-based hedging can be considered as more effective in managing tail risk than the expected shortfall hedging.

The above phenomenon is further highlighted by comparing the resulting CVaR of the hedger's risk exposure and the expected shortfall of the hedged position under both hedging strategies. The results, which provide additional support in favor of the CVaR-based hedging, are shown in Table 3.2. While our proposed CVaR hedging is only designed to optimally minimize CVaR of the hedger's risk exposure, its expected shortfall of the hedging strategy is only slightly larger than that of the expected shortfall hedging. In contrast, even though the expected shortfall hedging optimally minimizes the expected shortfall of the hedged position, the resulting optimal strategy gives rise to CVaR of the hedger's total risk exposure that is a few times larger than the corresponding CVaR-based hedging. This calls for a concern with the expected shortfall hedging strategy in that while the expected shortfall of the hedging strategy is optimally minimized, the resulting CVaR of the hedger's risk exposure can be unexpectedly large. In fact, the CVaR of the hedger's total risk exposure resulting from the expected shortfall hedging strategy is even larger than that without any hedging. The reason is that the expected shortfall hedging strategy does not hedge the risk of the option beyond its VaR level and thus the additional hedging budget leads to an increase in CVaR.

Table 3.2: The effectiveness of hedging a put option using the CVaR-based hedging and the expected shortfall hedging

	Hedging Strategy	
	CVaR	Expected Shortfall
Expected shortfall of the hedging strategy	1.8479	1.6984
CVaR of hedger's total risk exposure	8.6785	22.3548

The following example is designed to compare and contrast our proposed CVaR-based optimal partial hedging to the CVaR hedging strategy of Melnikov and Smirnov (2012). As pointed out earlier that it is a non-trivial exercise to obtain the optimal hedging strategy of Melnikov and Smirnov (2012) in the general case. For this reason, we only focus on the Black-Scholes model below where the optimal partial hedging is derived in Melnikov and Smirnov (2012).

Example 3.4.3. *As in Example 3.4.1 and Example 3.4.2, we assume that the dynamics of the stock price S_t at time t are governed by*

$$dS_t = S_t(\sigma dW_t + mdt)$$

and that we are interested in partial hedging a European call option with parameter values

$$T = 0.25, \;\; K = 110, \;\; r = 0.05, \;\; S_0 = 100, \;\; \sigma = 0.3, \;\; \pi_0 = 1$$

and the following two scenarios of drift coefficient m:

(i) $m = 0.06$,

(ii) $m = 0.04$.

Under the the Black-Scholes model, the price of the call option is independent of the drift of the stock price. Hence the price of the European call option is calculated to be $P_C = 2.84$, irrespective of the drift coefficients.

Let us now investigate the optimal partial hedging of Melnikov and Smirnov (2012). The optimal form of their partial hedging depends on the relative magnitude of the risk-free rate r and the drift coefficient m. This entails analyzing the optimal hedging strategy of Melnikov and Smirnov (2012) separately depending on the hypothetical scenarios, as shown below:

(i) For $m = 0.06$ so that this corresponds to the case $r < m$. In this particular case, Melnikov and Smirnov (2012) demonstrates that the optimal hedging strategy is given by

$$(H - \hat{z}_1)_+ \cdot \mathbb{1}(S_T > e^{rT} \cdot \tilde{b}_1(\hat{z}_1)) \tag{3.27}$$

where $\hat{z}_1$ is the minimizer of function $c_1(z)$ defined by

$$c_1(z) = z + \frac{1}{1-\alpha} \cdot (S_0 \cdot e^{(m-r)T} \cdot \tilde{\Lambda}_+(K(z), \tilde{b}_1(z)) - K(z) \cdot \tilde{\Lambda}_-(K(z), \tilde{b}_1(z))),$$

and $\tilde{b}_1(z)$ is the solution for the following system

$$\begin{cases} S_0\Phi_+(b) - K(z)\Phi_-(b) = \pi_0 \\ b \geqslant K(z). \end{cases}$$

In the above,

$$K(z) = K \cdot e^{-rT} + z,$$

$$\tilde{\Lambda}_+(x,y) = \Phi_+(x \cdot e^{-(m-r)T}) - \Phi_+(y \cdot e^{-(m-r)T}),$$

$$\tilde{\Lambda}_-(x,y) = \Phi_-(x \cdot e^{-(m-r)T}) - \Phi_-(y \cdot e^{-(m-r)T}),$$

$$\Phi_+(x) = \Phi\left(\frac{\ln S_0 - \ln K}{\sigma\sqrt{T}} + 0.5\sigma\sqrt{T}\right),$$

$$\Phi_-(x) = \Phi\left(\frac{\ln S_0 - \ln K}{\sigma\sqrt{T}} - 0.5\sigma\sqrt{T}\right),$$

and $\Phi(\cdot)$ is the standard normal distribution. Solving the above system numerically, the optimal partial hedging strategy is in the form of $(S_T - 120.45)_+$.

(ii) For $m = 0.04$ so that this corresponds to the case $r > m$. In this scenario, the optimal partial hedging is given by

$$(H - \hat{z}_2)_+ \cdot \mathbb{1}(S_T < e^{rT} \cdot \tilde{b}_2(\hat{z}_2)) \tag{3.28}$$

where $\hat{z}_2$ is the minimizer of $c_2(z)$ defined by

$$c_2(z) = z + \frac{1}{1-\alpha} \cdot (S_0 \cdot e^{(m-r)T} \cdot \Phi_+(\tilde{b}_2(z) \cdot e^{-(m-r)T}) - K(z) \cdot \Phi_-(\tilde{b}_2(z) \cdot e^{-(m-r)T}),$$

and $\tilde{b}_2(z)$ is the solution for the following system

$$\begin{cases} S_0\Lambda_+(K(z), b) - K(z)\Lambda_-(K(z), b) = \pi_0 \\ b \geqslant K(z). \end{cases}$$

In the above,

$$\Lambda_+(x,y) = \Phi_+(x) - \Phi_+(y),$$

$$\Lambda_-(x,y) = \Phi_-(x) - \Phi_-(y),$$

where $K(z), \Phi_+(\cdot), \Phi_-(\cdot)$ and $\Phi(\cdot)$ are defined in the previous scenario.

By solving the above system numerically, the optimal partial hedging strategy is found to be $(S_T - 120.45)_+$.

It is interesting to note that in both cases, the optimal partial hedging strategies are identical though this is not true in general. More importantly, the above optimal form of the partial hedging satisfies all the assumptions of our proposed optimal partial hedging. This implies that our model should similarly deduce the same optimal partial hedging strategy. Indeed, it follows easily from Theorem 3.3.2 that for both cases $m = 0.06$ and $m = 0.04$, our proposed CVaR-based partial hedging is the same with the optimal hedged loss function given by $(S_T - K - d^*)_+ - (S_T - K - u^*)_+$, where d^* and u^* are determined by (3.24). Numerically solving (3.24) yields $d^* = 10.45$ and $u^* = \infty$ and the optimal hedged loss function is identical to that of Melnikov and Smirnov (2012).

Remark 3.4.1. *We now draw the following remarks based on the above Example 3.4.3.*

(a) The above example clearly demonstrates that the optimal partial hedging strategy of our proposed model is numerically much easier to obtain as it boils down to solving a two dimensional optimization problem given by (3.24). *While we have assumed the Black-Scholes model in the above example, it should be emphasized that if we were to consider a more sophisticated (and complex) model, the basic procedure of obtaining the optimal partial hedging strategy applies with the same level of complexity. In contrast, it can be very challenging to numerically obtain the optimal partial hedging strategy of Melnikov and Smirnov (2012), particularly if the model assumption deviates from the Black-Scholes model.*

(b) The above example again ascertains that the functional form of our proposed CVaR-based partial hedging is robust with respect to the specification of the underlying asset price process. It is always a bull call spread hedging on the risk itself. On the contrary,

the general functional form of the hedging strategy of Melnikov and Smirnov (2012) depends on the specific model for the underlying asset price process. In particular, as highlighted in (3.27) *and* (3.28) *that the optimal hedging strategy of Melnikov and Smirnov (2012) is either a knock-in call or a knock-out call, depending on the relative magnitude of the interest rate r and the drift coefficient m.*

(c) It is easy to show that for the Black-Scholes model with $m < r$, the optimal partial hedging strategy of Melnikov and Smirnov (2012) satisfies all the assumptions we imposed on the hedged loss function of our proposed CVaR-based hedging problem. Consequently, the optimal hedging strategy from both models are exactly the same, as confirmed in the numerical example above. Our approach, as opposed to Melnikov and Smirnov (2012), is easier and more flexible.

3.5 Concluding Remark

In this chapter, we discuss how to optimally hedge a contingent claim that minimizes CVaR of the hedger's total risk exposure. We analyze the CVaR-based partial hedging strategies under two different classes of pricing functionals, namely no arbitrage pricing and stop-loss order preserving pricing.

In Section 3.2, we prove that the optimal partial hedging strategy under no arbitrage pricing is to construct a call on the contingent claim when there is sufficient hedging budget. In this case, the optimal partial hedging problem boils down to a one-dimensional optimization problem and thus is very tractable.

In Section 3.3, our results show that a bull call spread on the claim itself is optimal provided that the pricing functional preserves the stop-loss order. The optimal partial hedging problem consequently boils down to solving a two-dimensional optimization problem and thus is very tractable. We discuss the optimal partial hedging strategies in more depth under the utility based indifference pricing methods.

In Section 3.4, many numerical examples are provided to demonstrate how to partially hedge the European call and put options using our proposed CVaR-based hedging. The effectiveness of our proposed partial hedging strategy is compared with the other hedging strategies in the literature such as quantile hedging, expected shortfall hedging and the CVaR-based partial hedging strategies proposed by Melnikov and Smirnov (2012). The comparison also includes the VaR-based hedging strategies proposed in Chapter 2. The results indicate that our proposed CVaR-based hedging has some competitive advantages in the sense of managing better the tail risk when compared to quantile hedging, VaR-based hedging and expected shortfall hedging strategies. Relative to the CVaR-based hedging strategies proposed by Melnikov and Smirnov (2012), our proposed CVaR-based partial hedging has the advantage of robustness, explicitness, tractability and transparency.

Although our proposed CVaR-based partial hedging strategies have some attractive properties, it is also important to point out that there are some potential limitations as well. Same as our VaR-based partial hedging strategies, property P2 we imposed on the hedged loss functions would exclude some hedging strategies. Besides, once the market is incomplete and the call option on the payout is not attainable, investor may not be able to perform our CVaR-based partial hedging strategies effectively.

Chapter 4

Optimal Reinsurance under VaR Criteria

Due to the intrinsic similarities between our proposed optimal partial hedging model and the optimal reinsurance model, the approach we utilized in Chapter 2 and Chapter 3 is applied in the context of reinsurance in this chapter.

4.1 Preliminaries

4.1.1 Introduction

Reinsurance is one of the most traditional and long standing risk management solutions, particularly from an insurer's point of view. Its strategic use not only leads to an effective risk mitigation, but also enhances an insurer's stability and profitability. Examples of the reinsurance contracts (or treaties) for which an insurer can transfer its risk to a reinsurer include quota-share reinsurance, stop-loss reinsurance, excess-of-loss reinsurance, surplus reinsurance, and so on. Because of the variety of these reinsurance treaties that exist in the marketplace, the insurers are therefore constantly seeking for better and more effective reinsurance strategies.

The quest for optimal risk management solution using reinsurance is an active area of research among academics, actuaries, and risk managers. In a typical formulation of an optimal reinsurance model, it involves at least the following three components. First is the criterion (i.e. objective) that determines the optimality of the reinsurance contracts. Second is the premium principle that specifies how the reinsurance premium is calculated. Third is the constraints, if any, that are imposed on the model. Examples of some typical constraints include the restriction on the structure of the reinsurance contracts and the reinsurance premium budget that an insurer could spend on reinsuring his risk via reinsurance. In this chapter, we will also demonstrate that by an ingenious specification of constraints could lead to an optimal reinsurance model with some desirable features including controlling the credit risk of the reinsurer and the counterparty risk of the insurer. The pioneering work on optimal reinsurance is attributed to Borch (1960), Kahn (1961) and Arrow (1963). In particular, Borch (1960) showed that the stop-loss reinsurance is optimal in the sense of minimizing the variance of the insurer's retained loss under the assumption of expected reinsurance premium principle. Confining to the expected reinsurance premium principle and the criterion of maximizing the expected utility of a risk-averse insurer's terminal wealth, Arrow (1963) also established that stop-loss reinsurance is optimal.

The classical optimal reinsurance models have been generalized in a number of interesting directions, with particular emphasis on the three aspects of the optimal reinsurance models discussed above, i.e. more sophistical criterion, more complex premium principles, and more involved constraints. Just to name a few, Young (1999), Kaluszka (2001, 2005), Kaluszka and Okolewski (2008) addressed the optimal reinsurance strategy by considering other premium principles such as Wang's premium principle, mean-variance premium principles, maximal possible claims principle, convex premium principles, etc. Cai and Tan (2007), and Cai et al. (2008), Balbas et al. (2009), Chi and Tan (2011), Tan et al. (2011) demonstrated that modern risk measures such as Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR) can be exploited in a reinsurance model for a viable risk management solution. More recently, Chi and Tan (2013) broadened the optimal reinsurance model by investigating the VaR and CVaR reinsurance models under a more general premium principle. They imposed some constraints on the ceded loss functions and assumed that the premium principle satisfies three basic axioms, namely distribution invariance, risk loading and stop-loss order preserving.

While the existing results have studied the optimal reinsurance solutions under a standard premium premium or a particular class of premium principles, in this chapter we propose a new class of premium principles which we call *monotonic piecewise premium principle* and show that the resulting optimal reinsurance model involving this new class of premium principles is still tractable. By piecewise premium principle we mean a premium principle that can be constructed by either concatenating a series of different premium principles or using the same premium principle but with different parameter values.

There are many advantages to investigating the optimal reinsurance model under this new class of premium principles. First and foremost is that the proposed monotonic piecewise premium principle is able to capture the risk attitude of the reinsurer easily and intuitively. If risk were segmented into different layers so that a higher layer of risk refers to a greater risk exposure with a larger potential catastrophic loss, then a reinsurer typically has a different level of risk attitude on each of these layers. This implies that different layers of risk may be priced differently. More specifically, a reinsurer, in general, demands a higher risk premium (i.e. higher risk loading) on a risk in higher layers than a risk in lower layers. The proposed monotonic piecewise premium principle provides an elegant way of addressing the differentiate in risk attitude. For example, if a reinsurer prefers to consistently using an expected premium principle to price all layers of risk, then the differentiate in risk attitude can be reflected by attaching a higher risk loading parameter of the expected premium principle when pricing a higher layer of risk. The piecewise nature of the premium principle also provides a greater flexibility in modeling a reinsurer's risk attitude by allowing the reinsurer to adopt different premium principles depending on the layers. For instance, the reinsurer may use the expected value premium principle when the claim is less than a certain threshold level, and Wang's premium principle when the claim exceeds that threshold. Similarly, if the reinsurer uses principle of equivalent utility to price the contracts, the reinsurer may choose different parameters or even different utility functions on different layers and this again leads to premium principle that is piecewise.

A second advantage to investigating the optimal reinsurance under the proposed monotonic piecewise premium principle is that it can be used to analyze the optimal reinsurance in the context of multiple reinsurers. This is facilitated by the fact that the piecewise nature of pricing layers of risk can be viewed as being reinsured by different reinsurers. Each reinsurer is reinsuring one or more layers of risk using its premium principle.

A third advantage is that it is a much wider class of premium principles in that it encompasses the stop-loss preserving class of premium principles considered in Chi and Tan (2013). The stop-loss preserving includes the following eight classical premium principles: net, expected value, exponential, proportional hazard, principle of equivalent utility, Wang's, Swiss, and Dutch. Moreover, the class we consider here also includes the premium principles which are monotonic and constructed by concatenating some combination of the above eight premium principle.

Another contribution of this chapter is to demonstrate that by meticulously imposing an appropriate constraint on the optimal reinsurance model, optimal reinsurance strategy with a certain desirable property can be obtained analytically. More specifically, we propose two variants of the optimal reinsurance models. The first model takes into consideration the reinsurers' willingness to reinsure when designing the reinsurance contract. Many of the studies on optimal reinsurance implicitly assume that the reinsurers will accept any reinsurance contracts proposed by the insurance companies. This, however, may not be the case in practice. It is possible that the reinsurers are not willing to, or not allowed to due to concern with credit risk or constraint on risk capital requirement. This issue can be addressed by imposing a limit on the losses that can be ceded to the reinsurer. The second model is motivated by the presence of the counterparty risk that the insurer is concerned with. In an ideal arrangement, losses that are ceded to the reinsurer become the obligation of the reinsurer and will be reimbursed to the insurer. However there are situations where the reinsurer might be facing cash flow strained or financial distress that impact its ability to meet its obligation. When this arises, the insurer is responsible for the losses that are supposedly to have been transferred to the reinsurer and hence ultimately bearing the counterparty risk. This suggests that when designing an optimal reinsurance strategy, insurer needs to take into consideration the presence of the counterparty risk. In this chapter, we propose a new optimal reinsurance model that reflects counterparty risk.

The basic setup of our optimal reinsurance model is to study the optimal reinsurance strategies that minimize Value-at-Risk (VaR) of the total exposed risk of an insurer given some budget constraint and under the monotonic piecewise reinsurance premium principles. The model description, including the definition of monotonic piecewise premium principle and constraints on the ceded loss functions are described in the remaining of this section. The optimal reinsurance strategies for the basic reinsurance model as well as its variants are derived in Sections 4.2 and 4.3. In particular, by only requiring the retained loss function to be nondecreasing, a truncated stop-loss reinsurance treaty can be optimal. On the other hand, if both retained loss function and ceded loss function are nondecreasing, then a limited stop-loss reinsurance treaty can be optimal. Numerical examples to compare and contrast our proposed models to the existing models are presented in Section 4.4. Section 4.5 provides some concluding remarks.

4.1.2 Model Description

Let X be the claim random variable that an insurer is obligated to pay. Without any loss of generality, we assume that X is a non-negative random variable with cumulative distribution function (c.d.f.) $F_X(x) = \mathbb{P}(X \leqslant x)$ and $\mathbb{E}(X) < \infty$. In the absence of reinsurance, the insurer's risk exposure is X. Let us now assume that the insurer is using reinsurance to cede part of his risk to a reinsurer. In this case, the claim X is divided into two parts, i.e. the ceded loss part, $f(X)$, and the retained loss part, $R_f(X)$. This means that $X = f(X) + R_f(X)$ and that a reinsurance contract (or treaty) is uniquely determined by either the ceded loss function $f(\cdot)$ or the retained risk function $R_f(\cdot)$. Here we focus on the ceded loss function $f(\cdot)$ to identify the reinsurance treaty. Under the reinsurance treaty f, the reinsurer is obligated to pay $f(X)$ to the insurer when a claim X arises. By transferring part of the risk to a reinsurer, the insurer incurs an additional cost in the form of reinsurance premium $\tilde{\Pi}(f(X))$ that is payable to the reinsurer. Note that the reinsurance premium is a function of the ceded loss function $f(\cdot)$ and the adopted premium principle. In the presence of reinsurance, the total risk exposure of the insurer is transformed from X to $T_f(X)$ where $T_f(X) = R_f(X) + \tilde{\Pi}(f(X))$.

A plausible risk measure based optimal reinsurance model (see for example Cai and

Tan (2007) and Chi and Tan (2013)) can be formulated as

$$\begin{cases} \min\limits_{f \in \mathcal{L}} & \rho(T_f(X)) \\ \text{s.t.} & \tilde{\Pi}(f(X)) \leqslant \pi_0, \end{cases} \tag{4.1}$$

where $\rho(\cdot)$ represents the risk measure that is adopted by the insurer, π_0 is the maximum budget an insurer could spend on reinsurance premium, and $\mathcal{L}$ is the admissible set of ceded loss functions.

In this chapter, we analyze the optimal reinsurance model by setting the risk measure ρ to Value-at-Risk (VaR). As we can see, the formulation of the optimal reinsurance model is very similar to that in Chapter 2. Formal definition of VaR is provided in Definition 2.1.1. Given a fixed constant α in $(0, 1)$ which reflects the desired confidence level of the insurer, the optimal reinsurance model (4.1) under VaR criterion simplifies to

$$\begin{cases} \min\limits_{f \in \mathcal{L}} & \mathrm{VaR}_\alpha(T_f(X)) \\ \text{s.t.} & \tilde{\Pi}(f(X)) \leqslant \pi_0. \end{cases} \tag{4.2}$$

As to the admissible set of the ceded loss functions, we adopt the ones we consider in Chapter 2 for the VaR-based optimal partial hedging model, which are specified in (2.3) and (2.4). We restate these two admissible sets as follows

$$\begin{aligned} \mathcal{L}_1 = \{ & 0 \leqslant f(x) \leqslant x : R_f(x) \equiv x - f(x) \ \text{ is a nondecreasing and left} \\ & \text{continuous function}\}, \end{aligned} \tag{4.3}$$

$$\begin{aligned} \mathcal{L}_2 = \{ & 0 \leqslant f(x) \leqslant x : \text{both } R_f(x) \text{ and } f(x) \text{ are nondecreasing functions,} \\ & R_f(x) \text{ is left continuous}\}. \end{aligned} \tag{4.4}$$

The above two admissible sets have been discussed in detail in Chapter 2. Here we only emphasize their interpretations in the context of reinsurance. First, the loss that is ceded to a reinsurer is nonnegative and uniformly bounded by the risk itself. The latter restriction ensures that the claim amount paid by the reinsurer does not exceed the original claim. Second, the retained loss function is at least a nondecreasing function so that the insurer needs to bear a correspondingly higher claim for a larger claim. Third, some argue that the ceded loss function should be nondecreasing, similar to the retained loss function. Ensuring both ceded loss function and the retained loss function to be nondecreasing has the advantage of reducing the insurer's moral hazard. It is for this reason that we also investigate the optimal reinsurance under the admissible class $\mathcal{L}_2$. Chi and Tan (2013) similarly analyzed the optimal reinsurance under $\mathcal{L}_2$ and stop-loss preserving class of premium principles.

The above two admissible sets $\mathcal{L}_1$ and $\mathcal{L}_2$ represent the two basic constraints we impose on the ceded loss functions and the retained loss functions. We will analyze the optimal reinsurance under these admissible sets. Additionally, we will study the optimal strategies under some other admissible sets of the ceded loss functions in order to reflect the willingness of the reinsurance companies to accept risk or the desire of the insurer to control counterparty risk.

4.1.3 Piecewise Premium Principle

Recall that one of the main contributions of this chapter is to derive analytically the optimal reinsurance strategy under a newly proposed class of premium principles known as the monotonic piecewise premium principle. This subsection begins by first describing the well-known stop-loss order preserving premium principle. Then we formally define the class of premium principles that is monotonic and piecewise. We conclude the subsection by presenting an example in order to contrast the difference between the new class of premium principles and the class of stop-loss order preserving premium principles.

According to Definition 3.1.3, the insurance premium principle is said to be preserving stop-loss order if it preserves stop-loss order. We can restate the definition in the context of insurance premium principle as follows:

Definition 4.1.1. *Suppose* $\tilde{\Pi}(\cdot)$ *is an insurance premium principle. If* $\tilde{\Pi}(X_1) \leqslant \tilde{\Pi}(X_2)$ *for any random variables* X_1 *and* X_2 *as long as they satisfy* $X_1 \leqslant_{sl} X_2$*, then we say that the insurance premium principle* $\tilde{\Pi}(\cdot)$ *is stop-loss order preserving.*

Now we introduce what we meant by a class of premium principles that is monotonic.

Definition 4.1.2. *Given any two risks* X *and* Y *such that* $X(\omega) \leqslant Y(\omega)$ *for all possible outcomes* ω*, then* $\tilde{\Pi}(\cdot)$ *is said to be a monotonic premium principle if* $\tilde{\Pi}(X) \leqslant \tilde{\Pi}(Y)$*.*

It should be emphasized that monotonicity is a mild condition on the premium principle. In particular, the class of monotonic premium principles includes the premium principles which preserve stop-loss ordering. The class of premium principles which preserves stop-loss ordering includes the following eight classical premium principles: net, expected value, exponential, proportional hazard, principle of equivalent utility, Wang's, Swiss, and Dutch. It is also worth mentioning that monotonicity allows a premium principle to have a very flexible piecewise structure. The piecewise premium principle is defined as follows:

Definition 4.1.3. *If there exist* $0 = a_0 < a_1 < \cdots < a_{n-1} < a_n = \infty$*,* $a_i \in \mathbb{R}$*,* $i = 0, 1, \ldots, n$ *such that for any random variable* X*,* $\tilde{\Pi}(X) = \sum_{i=1}^{n} \tilde{\Pi}_i(X \cdot \mathbb{1}_{X \in [a_{i-1}, a_i)})$*, where* $\mathbb{1}$ *denotes the indicator function and each* $\tilde{\Pi}_i(\cdot)$ *is a specific premium principle, then we say that the premium principle* $\tilde{\Pi}(\cdot)$ *is a piecewise premium principle. If additionally the* *piecewise premium principle satisfies the monotonicity property, then the resulting premium principle is both monotonic and piecewise.*

Note that any arbitrary classical premium principle is a special case of the above piecewise premium principle. This follows by setting $n = 1$ in Definition 4.1.3. For this reason we will mainly focus our analysis on the piecewise premium principle instead of the ordinary premium principle. Furthermore, the monotonic piecewise premium principle encompasses the stop-loss order preserving premium principle so that the former class of premium principles is more general than the latter class of premium principles. In fact, the following example confirms that a premium principle can be monotonic and piecewise and yet does not preserve the stop-loss ordering.

Example 4.1.1. *Using the notation in Definition 4.1.3, this example considers a monotonic piecewise premium principle with $n = 2$, $a_1 = 10$, and $\tilde{\Pi}_i, i = 1, 2$ are expectation premium principles with risk loading factors $\rho_1 = 0.1$ and $\rho_2 = 0.5$, respectively. This implies that $\tilde{\Pi}_1$ applies to the first layer of risk with loss less amount than 10 while $\tilde{\Pi}_2$ applies to the remaining layer with loss amount exceeds or equals to 10. Hence the piecewise premium principle is constructed by concatenating two expectation premium principles with the following representation:*

$$\tilde{\Pi}(X) = 1.1 \cdot \mathbb{E}\left[X \cdot \mathbb{1}_{X\in[0,10)}\right] + 1.5 \cdot \mathbb{E}\left[X \cdot \mathbb{1}_{X\in[10,\infty)}\right]. \tag{4.5}$$

Note that the expectation premium principle is monotonic and preserves stop-loss order and that the premium principle (4.5) *is a monotonic piecewise premium principle.*

Let us now consider the following two loss random variables X_1 and X_2 such that X_1 represents a deterministic loss of 10 in any scenario while X_2 equals to 5 with probability of 80% *and uniformly distributed between 5 and 55 with probability of* 20%. *It is easy to* *verify that both risks have the same expectations; i.e.* $\mathbb{E}[X_1] = \mathbb{E}[X_2] = 10$*. Furthermore, the following analysis confirms that* $X_1 \leqslant_{sl} X_2$*.*

(i). If $d \leqslant 5$*, we have*

$$\mathbb{E}[(X_1 - d)_+] = \mathbb{E}[X_1] - d = \mathbb{E}[X_2] - d = \mathbb{E}[(X_2 - d)_+].$$

(ii). If $5 < d \leqslant 10$*, we have*

$$\begin{aligned}\mathbb{E}[(X_2 - d)_+] - \mathbb{E}[(X_1 - d)_+] =& \frac{d + 55}{2} * \frac{55 - d}{50} * 0.2 - (10 - d) \\ =& -\frac{d^2}{500} + d - 3.95,\end{aligned}$$

which is increasing with respect to d *when* $5 < d \leqslant 10$*. Therefore, it achieves its minimum when* $d = 5$*, i.e.,*

$$\mathbb{E}[(X_2 - d)_+] - \mathbb{E}[(X_1 - d)_+] \geqslant 1 > 0.$$

(iii). If $d \geqslant 10$*, it is clear that*

$$\mathbb{E}[(X_1 - d)_+] = 0 \leqslant \mathbb{E}[(X_2 - d)_+].$$

Hence according to Definition 3.1.2, we have $X_1 \leqslant_{sl} X_2$*. It is easy to verify that* $\tilde{\Pi}(X_1) > \tilde{\Pi}(X_2)$*, which concludes that the premium principle* (4.5) *is not stop-loss order preserving premium principle though it is a monotonic piecewise premium principle.*

4.2 Optimality of Truncated Stop-loss Reinsurance Treaties

By assuming the premium principle is monotonic (see Definition 4.1.3) and the ceded loss functions need not be non-decreasing (i.e. the admissible set of ceded loss functions is given by $\mathcal{L}_1$ as defined in (4.3)), Subsection 4.2.1 shows that the truncated stop-loss reinsurance treaty is optimal to the reinsurance model (4.2). The same subsection also demonstrates that the basic reinsurance model can be extended to analyzing the optimal reinsurance treaties under the multiple reinsurers when the premium principle is of the form piecewise as defined in Definition 4.1.3. Two interesting extensions of the optimal reinsurance models are discussed in Subsections 4.2.2 and 4.2.3. In particular, Subsection 4.2.2 investigates the reinsurance model (4.2) under the additional constraint that a limit is imposed on the reinsurance treaty while Subsection 4.2.3 examines a generalization of the reinsurance model (4.2) that incorporates counterparty risk. Interestingly, both variants of the optimal reinsurance models still confirm the optimality of the truncated stop-loss reinsurance treaties.

4.2.1 Without Nondecreasing Assumption on the Ceded Loss Functions

In this subsection, we show that for the reinsurance model (4.2), the truncated stop-loss reinsurance strategy is optimal among all the strategies in $\mathcal{L}_1$. To proceed, for any ceded loss function f from the set $\mathcal{L}_1$, it is useful to consider the following function:

$$g_f(x) = \begin{cases} [x + f(v) - v]_+ \quad , & \text{if} \quad 0 \leqslant x \leqslant v, \\ 0, & \text{if} \quad x > v. \end{cases} \tag{4.6}$$

where $v = \mathrm{VaR}_\alpha(X)$. Note that by construction, g_f is also an element in $\mathcal{L}_1$. Clearly, if the ceded loss function of a reinsurance treaty takes the form g_f, then the reinsurance treaty is a truncated stop-loss reinsurance treaty. The following theorem shows that if the reinsurance premium is monotonic, the truncated stop-loss reinsurance treaty is the optimal form among all the admissible treaties in $\mathcal{L}_1$.

Theorem 4.2.1. *Assume the reinsurance premium principle* $\tilde{\Pi}(\cdot)$ *is a monotonic piecewise premium principle. Then, for any ceded loss function* $f \in \mathcal{L}_1$*, we can construct the ceded loss function* $g_f \in \mathcal{L}_1$ *according to* (4.6)*, and* g_f *satisfies the following properties:*

(a) $\tilde{\Pi}(f(X)) \leqslant \pi_0$ *implies* $\tilde{\Pi}(g_f(X)) \leqslant \pi_0$*. Equivalently, if the ceded loss function* f *satisfies the budget constraint, then the ceded loss function* g_f *will satisfy the budget constraint as well;*

(b) $VaR_\alpha(T_{g_f}(X)) \leqslant VaR_\alpha(T_f(X))$*.*

Proof: The proof is similar to that of Theorem 2.2.1, hence is omitted here. □

Remark 4.2.1. *(a) Theorem 4.2.1 indicates that the optimality of the truncated stop-loss reinsurance strategy is independent of the reinsurance premium principle. The truncated stop-loss reinsurance strategy is optimal among all the strategies in* $\mathcal{L}_1$ *as long as the premium principle is monotonic. The actual specification of the parameter values of the optimal ceded loss function then depends on the premium principle.*

(b) If we denote $d = v - f(v)$*, then the truncated stop-loss function* g_f *defined in* (4.6) *can be succinctly represented as*

$$g_f(x) = (x - d)_+ \cdot \mathbb{1}(x \leqslant v),$$

where $\mathbb{1}(\cdot)$ *denotes the indicator function. Furthermore, it follows from Theorem 4.2.1 that the VaR-based optimal reinsurance problem* (4.2)*, when the admissible set of ceded loss functions is* $\mathcal{L}_1$*, can equivalently be rewritten as*

$$\begin{cases} \min\limits_{0 \leqslant d \leqslant v} & VaR_\alpha\Big(X - (X-d)_+ \cdot \mathbb{1}(X \leqslant v) + \tilde{\Pi}\left[g_f(X)\right]\Big) \\ \\ s.t. & \tilde{\Pi}[g_f(X)] \equiv \tilde{\Pi}\left[(X-d)_+ \cdot \mathbb{1}(X \leqslant v)\right] \leqslant \pi_0. \end{cases}$$

The above optimization problem can be further simplified as follows

$$\begin{cases} \min\limits_{0 \leqslant d \leqslant v} & d + \tilde{\Pi}\left[g_f(X)\right] \\ \\ s.t. & \tilde{\Pi}[g_f(X)] \equiv \tilde{\Pi}\left[(X-d)_+ \cdot \mathbb{1}(X \leqslant v)\right] \leqslant \pi_0. \end{cases} \tag{4.7}$$

which is simply an optimization problem involving only one variable. Hence once the reinsurance premium principle is given, it is technically much easier to solve, as shown in the numerical examples in Section 4.4.

If there exist several reinsurers which adopt different premium principle in the market, then the insurance company will naturally take advantage of this when ceding its risk to the reinsurers. When determining the optimal reinsurance strategy, the insurance company will consider the existence of multiple reinsurers, and the premium principle is not so explicit as that in case of single reinsurer. The following theorem deals with the case of multiple reinsurers.

Theorem 4.2.2. *Assume that there are n reinsurance companies in the market and they adopt different premium principles, $\{\tilde{\Pi}_i(\cdot)\}_{i=1}^n$. Every premium principle $\tilde{\Pi}_i(\cdot)$ is a monotonic piecewise premium principle. We further assume that the insurance company will always seek the optimal way to cede his risk to the reinsurance companies in order to minimize the cost of reinsurance.*

Then, for any ceded loss function $f \in \mathcal{L}_1$, we can construct the ceded loss function g_f according to (4.6), and g_f satisfies the following properties:

(a) $\tilde{\Pi}(f(X)) \leqslant \pi_0$ *implies* $\tilde{\Pi}(g_f(X)) \leqslant \pi_0$. *Equivalently, if the ceded loss function f satisfies the budget constraint, then the ceded loss function g_f will satisfy the budget constraint as well;*

(b) $VaR_\alpha(T_{g_f}(X)) \leqslant VaR_\alpha(T_f(X))$.

Proof: Under the above assumptions, it is clear that the premium that the insurance company pays associated with the ceded loss function f is given by

$$\begin{aligned}\tilde{\Pi}(f(X)) &= \min_{\{A_i\}_{i=1}^n} \sum_{i=1}^n \tilde{\Pi}_i \left(f(X) \cdot \mathbb{1}_{f(X)\in A_i}\right) \\ &= \sum_{i=1}^n \tilde{\Pi}_i \left(f(X) \cdot \mathbb{1}_{f(X)\in A_i^f}\right)\end{aligned} \tag{4.8}$$

where $\bigcup_{i=1}^n A_i = \bigcup_{i=1}^n A_i^f = [0, +\infty)$. And $\{A_i^f\}_{i=1}^n$ is the optimal partition associated with the ceded loss function f in the sense that it minimizes the premium paid by the insurance company.

(a) According to the proof of Theorem 2.2.1, we know that $g_f(x) \leqslant f(x), \forall\ x \geqslant 0$. Therefore, for any set B, we have

$$g_f(X \cdot \mathbb{1}_{X\in B}) \leqslant f(X \cdot \mathbb{1}_{X\in B}).$$

The monotonicity of the premium principle $\tilde{\Pi}_i(\cdot)$ immediately implies that

$$\tilde{\Pi}_i(g_f(X \cdot \mathbb{1}_{X\in B})) \leqslant \tilde{\Pi}_i(f(X \cdot \mathbb{1}_{X\in B})),\ i = 0, 1, \dots, n.$$

We assume that $\{A_i^f\}_{i=1}^n$ is the optimal partition associated with the ceded loss function f in the sense that it minimizes the premium paid by the insurance company given a ceded loss f, which means that

$$\tilde{\Pi}(f(X)) = \sum_{i=1}^n \tilde{\Pi}_i \left(f(X) \cdot \mathbb{1}_{f(X)\in A_i^f}\right).$$

We further denote that $B_i^f = f^{-1}(A_i^f)$ as the inverse image of A_i^f under f, then the premium associated with the ceded loss function f expressed in the above expression can be rewritten as

$$\begin{aligned}\tilde{\Pi}(f(X)) =& \sum_{i=1}^{n} \tilde{\Pi}_i \left(f(X) \cdot \mathbb{1}_{f(X)\in A_i^f} \right) \\ =& \sum_{i=1}^{n} \tilde{\Pi}_i \left(f(X \cdot \mathbb{1}_{X\in B_i^f}) \right).\end{aligned}$$

With the similar notations, $B_i^{g_f} = g_f^{-1}(A_i^{g_f})$, where $\{A_i^{g_f}\}_{i=1}^n$ is the optimal partition associated with the ceded loss function g_f and $B_i^{g_f}$ is the inverse image of $A_i^{g_f}$ under g_f, we have

$$\begin{aligned}\tilde{\Pi}(g_f(X)) =& \sum_{i=1}^{n} \tilde{\Pi}_i \left(g_f(X) \cdot \mathbb{1}_{g_f(X)\in A_i^{g_f}} \right) \\ =& \sum_{i=1}^{n} \tilde{\Pi}_i \left(g_f(X \cdot \mathbb{1}_{X\in B_i^{g_f}}) \right) \\ \leqslant& \sum_{i=1}^{n} \tilde{\Pi}_i \left(g_f(X \cdot \mathbb{1}_{X\in B_i^{f}}) \right) \\ \leqslant& \sum_{i=1}^{n} \tilde{\Pi}_i \left(f(X \cdot \mathbb{1}_{X\in B_i^{f}}) \right) \\ =& \tilde{\Pi}(f(X))\end{aligned}$$

which is the desired result.

The proof of part (b) is same as that of Theorem 2.2.1, hence is omitted here. □

Remark 4.2.2. *(a) The above theorem identifies the optimal form of the ceded loss function when there are multiple reinsurance companies in the market. The optimal form of the reinsurance strategies is also a truncated stop-loss type contract.*

(b) In this theorem, we even do not need to assume that the premium principle $\tilde{\Pi}(\cdot)$ to be monotonic, though it is a weak and reasonable assumption on the premium principle. We just need to impose the monotonic assumption on each $\tilde{\Pi}_i(\cdot)$, which is the premium principle adopted by the i-th reinsurance company.

(c) *It is worth mentioning that the overall retained loss function is nondecreasing. Therefore, the reinsurance companies will accept this treaty if they only requires the retained loss functions to be nondecreasing for the concern of moral hazard.*

(d) *It is imperative to distinguish the works of Asimit and Badescu (2011) and Chi and Meng (2012) from ours as they have similarly investigated the optimal reinsurance in the context of multiple reinsurers. The key difference lies in how the ceded losses are distributed to the reinsurers. Their formulations assume that the ceded losses can be splitted up in that any loss is shared among the reinsurers while in our setup, the ceded losses are divided into layers and each reinsurer is responsible (entirely) for each layer of risk. Because the potential claim is assumed to be segmentable, their optimal reinsurance strategies and the corresponding minimal exposed risk may depend on the number of reinsurance companies in the market. Even if all the reinsurers are using the same premium principle, the number of reinsurers in the market may still affect insurer's optimal strategy and the corresponding optimal exposed risk level. In contrast, our proposed optimal strategy and the corresponding minimal exposed risk only depend on the premium principles adopted by the reinsurers.*

(e) *Remark 4.2.1 for Theorem 4.2.1 is similarly applied to Theorem 4.2.2.*

4.2.2 Exerting Limit on the Reinsurance Treaties

In general, reinsurance companies do not wish to reinsure catastrophic claims unless they are appropriately compensated. Some reinsurance companies may raise the risk loading factor on higher layers of coverage, which has been dealt with by considering the monotonic piecewise premium principles in the last subsection. Some reinsurance companies may choose to impose a limit on the reinsurance treaties. Another reason for reinsurance company to impose a limit on the reinsurance treaties may be due to regulatory constraint. In this subsection we will investigate an optimal reinsurance strategy in this case.

We suppose that the reinsurers are only willing to accept the reinsurance treaties subject to a limit. This implies that the maximal values of the ceded loss functions are bounded by a specified constant c_1. Hence the admissible set of the ceded loss functions is revised to

$$\begin{aligned}\mathcal{L}_1' = \{0 \leqslant f(x) \leqslant \min\{x, c_1\} : & R_f(x) \equiv x - f(x) \text{ is a nondecreasing} \\ & \text{and left continuous function}\}.\end{aligned} \tag{4.9}$$

Though the admissible set is different, we can still use the method similar to the last subsection to derive the optimal reinsurance strategies. This is summarized in the following corollary.

Corollary 4.2.1. *Assume the reinsurance premium principle $\tilde{\Pi}(\cdot)$ is a monotonic piecewise premium principle. Then, for any ceded loss function $f \in \mathcal{L}_1'$, we can construct the ceded loss function g_f according to* (4.6)*, and g_f satisfies the following properties:*

(a) $g_f \in \mathcal{L}_1'$.

(b) $\tilde{\Pi}(f(X)) \leqslant \pi_0$ *implies* $\tilde{\Pi}(g_f(X)) \leqslant \pi_0$. *Equivalently, if the ceded loss function f satisfies the budget constraint, then the ceded loss function g_f will satisfy the budget constraint as well;*

(c) $VaR_\alpha(T_{g_f}(X)) \leqslant VaR_\alpha(T_f(X))$.

Proof: According to the proof of Theorem 2.2.1, we know that $g_f(x) \leqslant f(x), \forall\, x \geqslant 0$. Therefore, $f(x) \leqslant \min\{x, c_1\}$ implies that $g_f(x) \leqslant \min\{x, c_1\}$. It is easy to verify that the retained loss function R_{g_f}, which is associated with the ceded loss function g_f, is nondecreasing and left continuous. Accordingly, we can conclude that $g_f \in \mathcal{L}'_1$.

Since the construction of g_f is the same as that in Theorem 4.2.1, part (b) and (c) follow immediately from Theorem 4.2.1. $\square$

Remark 4.2.3. *(a) Corollary 4.2.1 identifies an optimal form of the ceded loss function when there is a limit imposed on the reinsurance treaties. The optimal form of the reinsurance strategies is also a truncated stop-loss type contract.*

(b) Using the notations used in Remark 4.2.1, we can express the ceded loss function g_f as follows

$$g_f(x) = (x-d)_+ \cdot \mathbb{1}(x \leqslant v).$$

It is clear that $\max_{x \geqslant 0}\{g_f(x)\} = v - d$. Therefore, to make sure $g_f(x)$ is bounded by the constant c_1, we need $d \geqslant v - c_1$. By using these notations, we can simplify the optimization problem as follows

$$\begin{cases} \min\limits_{\max\{0, v-c_1\} \leqslant d \leqslant v} & d + \tilde{\Pi}\left[g_f(X)\right] \\ \\ s.t. & \tilde{\Pi}[g_f(X)] \equiv \tilde{\Pi}\left[(X-d)_+ \cdot \mathbb{1}(X \leqslant v)\right] \leqslant \pi_0. \end{cases} \tag{4.10}$$

4.2.3 In the Presence of Counterparty Risk

In an ideal reinsurance arrangement, the reinsurer is liable for any claim as stipulated in the reinsurance treaty and hence any claim that is ceded will be reimbursed by the reinsurer. The insurer is only concerned with the residual part of the risk. While this is true in theory, in practice the use of reinsurance exposes the insurer to another type of risk known as the counterparty risk. The counterparty risk arises when the reinsurer is not able to meet its obligation for reasons such as the company is having cash flow strained or facing insolvency/bankruptcy. When this occurs, the insurer is ultimately responsible for the part of the risk that is supposedly ceded to the reinsurer. This suggests that in the design of optimal reinsurance strategy, the creditworthiness of the reinsurer is one of the critical factors that cannot be ignored. Yet the counterparty risk is often neglected in most formulations of the optimal reinsurance models. The objective of this subsection is to demonstrate that by artfully modifying some of the constraints of the reinsurance models, the counterparty risk could be integrated to the optimal reinsurance models that we have discussed so far.

We first assume that the actual claim that is ceded to the reinsurer is so large that it exceeds a certain threshold, then the reinsurer is in financial stress and might not be able to meet its contractual obligation. In this case, the payment that is supposedly reimbursed to the insurer will be defaulted. We propose to reduce the counterparty risk by ensuring that the probability of the reinsurer not meeting its obligation does not exceed a certain acceptable tolerance level of the insurer. If $\bar{c}_1$ represents the threshold of the above reinsurer and $0 \leqslant \beta \leqslant 1$ denotes the desired tolerance level of the insurer, then the above condition is translated to the probabilistic constraint $\mathbb{P}(f(X) > \bar{c}_1) \leqslant \beta$. The parameter β is predetermined by the insurer and reflects the insurer's risk tolerance towards to counterparty risk. Clearly, the smaller the β, the less exposure the insurer is to counterparty risk. In the extreme case where $\beta = 0$, the counterparty risk is completely eliminated since the ceded claim can never exceed the threshold $\bar{c}_1$ and hence the counterparty risk will never be triggered.

The optimal reinsurance model (4.2) can easily be modified to reflect the above approach of controlling the counterparty risk. This is achieved by seeking an optimal reinsurance to the reinsurance model (4.2) with the admissible set of the ceded loss function revised to

$$\begin{aligned}\mathcal{L}_1'' =&\{0 \leqslant f(x) \leqslant x : \mathbb{P}(f(X) > \bar{c}_1) \leqslant \beta, R_f(x) \equiv x - f(x) \text{ is a nondecreasing} \\ &\text{and left continuous function}\}\end{aligned} \tag{4.11}$$

As in the last subsection, we can still use the same technique to derive the optimal reinsurance strategies even though the admissible set is different. The results are summarized in the following corollary.

Corollary 4.2.2. *Assume the reinsurance premium principle* $\tilde{\Pi}(\cdot)$ *is a monotonic piecewise premium principle. Then, for any ceded loss function* $f \in \mathcal{L}_1''$*, we can construct the ceded loss function* g_f *according to* (4.6)*, and* g_f *satisfies the following properties:*

(a) $g_f \in \mathcal{L}_1''$.

(b) $\tilde{\Pi}(f(X)) \leqslant \pi_0$ *implies* $\tilde{\Pi}(g_f(X)) \leqslant \pi_0$. *Equivalently, if the ceded loss function* f *satisfies the budget constraint, then the ceded loss function* g_f *will satisfy the budget constraint as well;*

(c) $VaR_\alpha(T_{g_f}(X)) \leqslant VaR_\alpha(T_f(X))$.

Proof: According to the proof of Theorem 2.2.1, we know that $g_f(x) \leqslant f(x), \forall\ x \geqslant 0$. Therefore, $\mathbb{P}(f(X) > \bar{c}_1) \leqslant \beta$ implies that $\mathbb{P}(g_f(X) > \bar{c}_1) \leqslant \beta$. It is easy to verify that the retained loss function R_{g_f}, which is associated with the ceded loss function g_f, is nondecreasing and left continuous. Accordingly, we can conclude that $g_f \in \mathcal{L}_1''$.

Since the construction of g_f is the same as that in Theorem 4.2.1, part (b) and (c) follow immediately from Theorem 4.2.1. □

Remark 4.2.4. *(a) Corollary 4.2.2 identifies the optimal form of the ceded loss function which takes into consideration the presence of the counterparty risk. The optimal form of the reinsurance strategies is also a truncated stop-loss type contract.*

(b) Using the notations used in Remark 4.2.1, we can express the ceded loss function g_f *as follows*

$$g_f(x) = (x-d)_+ \cdot \mathbb{1}(x \leqslant v).$$

Therefore, $\mathbb{P}(g_f(X) > \bar{c}_1) \leqslant \beta$ *is equivalent to* $d > Q_1 - \bar{c}_1$ *where* $Q_1 = \inf\{q \geqslant 0 : \mathbb{P}(q < X \leqslant v) \leqslant \beta\}$*. By using these notations, we can simplify the optimization problem as follows*

$$\begin{cases} \min\limits_{\max\{0, Q_1 - \bar{c}_1\} \leqslant d \leqslant v} & d + \tilde{\Pi}\left[g_f(X)\right] \\ \\ s.t. & \tilde{\Pi}[g_f(X)] \equiv \tilde{\Pi}\left[(X-d)_+ \cdot \mathbb{1}(X \leqslant v)\right] \leqslant \pi_0. \end{cases} \tag{4.12}$$

(c) The optimal reinsurance model proposed in this subsection is more general than that in the last subsection in the sense that the model allows the insurance company to have additional flexibility in specifying its attitude towards the courterparty risk. The insurer's attitude towards the counterparty risk is reflected by $\bar{c}_1$ *and* β*. If we let* $\bar{c}_1 = c_1$ *and* $\beta = 0$*, the model in this subsection collapses to that in the last subsection. If we let* $\bar{c}_1 = +\infty$ *or* $\beta = 1$*, then the model in this subsection recovers the one in Subsection 4.2.1.*

4.3 Optimality of Limited Stop-loss Reinsurance Treaties

In the last section, we study a few variations of the optimal reinsurance model (4.2). All these variants share the same constraint that the ceded loss functions do not need to be nondecreasing and the same conclusion that the truncated stop-loss reinsurance treaties are optimal. These results imply that the losses that are ceded to the reinsurer do not need to increase with losses. In fact when the losses increase to a critical level, the losses ceded will reduce drastically to zero and remain at zero thereafter. This raises a concern to the reinsurer as reinsurance treaty of this type potentially triggers insurer's moral hazard. For this reason, reinsurers often prefer reinsurance treaties with the property that the ceded losses are at least non-decreasing with losses. As a result, the objective of this section is to investigate the optimal ceded loss function f to the optimization problem (4.2) when the premium principle is monotonic and there is a monotonic assumption imposed on the ceded loss functions. In this case, the admissible set of the ceded loss function corresponds to $\mathcal{L}_2$. Similarly, we will extend our results to the case of multiple reinsurers and investigate the optimal strategies if there is a limit on the reinsurance treaties or there exists counterparty risk.

4.3.1 With Nondecreasing Assumption on the Ceded Loss Functions

In this subsection, we assume the admissible set is $\mathcal{L}_2$ as defined in (4.4). We will show that the so-called limited stop-loss reinsurance strategy is optimal among all the strategies in $\mathcal{L}_2$. We will employ the same technique used in the previous section to derive the optimal solutions over $\mathcal{L}_2$. To proceed, for any ceded loss function f from set $\mathcal{L}_2$, we construct the following function h_f which is also an element in $\mathcal{L}_2$:

$$h_f(x) = \min\left\{[x - (v - f(v))]_+, f(v)\right\}, \tag{4.13}$$

where as defined previously $v = \mathrm{VaR}_\alpha(X)$.

It follows from the above representation that the reinsurance treaty with the ceded loss function $h_f(X)$ is commonly known as a limited stop-loss reinsurance treaty. The following theorem shows that if the reinsurance premium principle is monotonic, the limited stop-loss reinsurance treaty is the optimal form among all the admissible treaties in $\mathcal{L}_2$.

Theorem 4.3.1. *Assume the reinsurance premium principle $\tilde{\Pi}(\cdot)$ is a monotonic piecewise premium principle. Then, for any ceded loss function $f \in \mathcal{L}_2$, we can construct the ceded loss function $h_f \in \mathcal{L}_2$ according to* (4.13)*, and h_f satisfies the following properties:*

(a) $\tilde{\Pi}(f(X)) \leqslant \pi_0$ implies $\tilde{\Pi}(h_f(X)) \leqslant \pi_0$. Equivalently, if the ceded loss function f satisfies the budget constraint, then the ceded loss function h_f will satisfy the budget constraint as well;

(b) $VaR_\alpha(T_{h_f}(X)) \leqslant VaR_\alpha(T_f(X))$.

Proof: The proof is similar to that of Theorem 2.2.3, hence is omitted here. □

Remark 4.3.1. *All the comments in Remark 4.2.1 for Theorem 4.2.1 are analogously applicable to the present case. In particular, we make the following remarks:*

(a) Theorem 4.3.1 indicates that the optimality of the limited stop-loss reinsurance s-trategy is independent of the reinsurance premium principle. The limited stop-loss reinsurance strategy is optimal among all the strategies in $\mathcal{L}_2$ as long as the premium principle is monotonic.

(b) By denoting $d = v - f(v)$, the function h_f defined above can be rewritten as

$$h_f(x) = (x - d)_+ - (x - v)_+.$$

Based on the results from Theorem 4.3.1, it is easy to see that the VaR-based partial hedging problem (4.2) *can be equivalently cast as*

$$\begin{cases} \min\limits_{0\leqslant d\leqslant v} & VaR_\alpha \left\{X-(X-d)_+ + (X-v)_+ + \tilde{\Pi}\left[h_f(X)\right]\right\} \\ s.t. & \tilde{\Pi}\left[h_f(X)\right] \equiv \tilde{\Pi}\left[(X-d)_+ - (X-v)_+\right] \leqslant \pi_0. \end{cases}$$

The above optimization problem can be further simplified as follows

$$\begin{cases} \min\limits_{0\leqslant d\leqslant v} & d + \tilde{\Pi}\left[h_f(X)\right] \\ s.t. & \tilde{\Pi}\left[h_f(X)\right] \equiv \tilde{\Pi}\left[(X-d)_+ - (X-v)_+\right] \leqslant \pi_0. \end{cases} \tag{4.14}$$

The optimal reinsurance problem again reduces to an optimization problem of just a single variable.

Similar to the discussion in the last section, if there exist several reinsurance companies which adopt different premium principle in the market, then the insurance company will naturally take advantage of this when ceding his risk to the reinsurance companies. The following theorem, as a counterpart of Theorem 4.2.2, deals with the case of multiple reinsurance companies.

Theorem 4.3.2. *Assume that there are n reinsurance companies in the market and they adopt different premium principles, $\tilde{\Pi}_i(\cdot)\}_{i=1}^n$. Every premium principle $\tilde{\Pi}_i(\cdot)$ is a monotonic piecewise premium principle. We further assume that the insurance company will always seek the optimal way to cede his risk to the reinsurance companies in order to minimize the cost of reinsurance.*

Then, for any ceded loss function $f \in \mathcal{L}_2$, we can construct the ceded loss function h_f according to (4.13)*, and h_f satisfies the following properties:*

(a) $\tilde{\Pi}(f(X)) \leqslant \pi_0$ *implies* $\tilde{\Pi}(h_f(X)) \leqslant \pi_0$*. Equivalently, if the ceded loss function* f *satisfies the budget constraint, then the ceded loss function* g_f *will satisfy the budget constraint as well;*

(b) $VaR_\alpha(T_{h_f}(X)) \leqslant VaR_\alpha(T_f(X))$*.*

Proof: The proof is completely in parallel to that of Theorem 4.2.2 and hence is omitted. □

Remark 4.3.2. *All the comments in Remark 4.2.2 for Theorem 4.2.2 are analogously applicable here. We emphasize that the overall ceded loss function and retained loss function are both nondecreasing, though the ceded loss function with respect to the i-th reinsurer might not be. Therefore, the reinsurers will accept this treaty since there it reduces moral hazard.*

4.3.2 Exerting Limit on the Reinsurance Treaties

Similar to Subsection 4.2.2, here we study the optimal reinsurance strategies if there is a limit imposed on the reinsurance treaties. We suppose that the maximal values of the ceded loss functions are bounded by a specified constant c_2 so that the admissible set of the ceded loss functions changes to

$$\begin{aligned} \mathcal{L}'_2 = & \{0 \leqslant f(x) \leqslant \min\{x, c_2\} : \text{both } R_f(x) \text{ and } f(x) \text{ are nondecreasing functions,} \\ & R_f(x) \text{ is left continuous}\} \end{aligned} \tag{4.15}$$

Using the technique similar to the last section, we obtain the following corollary. The proof is also similar and hence is omitted.

Corollary 4.3.1. *Assume the reinsurance premium principle $\tilde{\Pi}(\cdot)$ is a monotonic piecewise premium principle. Then, for any ceded loss function $f \in \mathcal{L}_2'$, we can construct the ceded loss function h_f according to* (4.13)*, and h_f satisfies the following properties:*

(a) $h_f \in \mathcal{L}_2'$.

(b) $\tilde{\Pi}(f(X)) \leqslant \pi_0$ *implies* $\tilde{\Pi}(h_f(X)) \leqslant \pi_0$. *Equivalently, if the ceded loss function f satisfies the budget constraint, then the ceded loss function h_f will satisfy the budget constraint as well;*

(c) $VaR_\alpha(T_{h_f}(X)) \leqslant VaR_\alpha(T_f(X))$.

Remark 4.3.3. *All the comments in Remark 4.2.3 for Corollary 4.2.1 are analogously applicable here. In particular, using the notations used in Remark 4.3.1, the ceded loss function h_f can be expressed as*

$$h_f(x) = (x-d)_+ - (x-v)_+.$$

Since $\max\limits_{x\geqslant 0}\{h_f(x)\} = v-d$, $h_f(x)$ *is bounded by the constant* c_2 *is equivalent to* $d \geqslant v - c_2$. *Therefore, the optimization problem can be reformulated as follows*

$$\begin{cases} \min\limits_{\max\{0,v-c_2\}\leqslant d\leqslant v} & d + \tilde{\Pi}\left[h_f(X)\right] \\ s.t. & \tilde{\Pi}\left[h_f(X)\right] \equiv \tilde{\Pi}\left[(X-d)_+ - (X-v)_+\right] \leqslant \pi_0. \end{cases} \tag{4.16}$$

4.3.3 In the Presence of Courterparty Risk

As in Subsection 4.2.3, we model the counterpary risk by seeking an optimal ceded loss function such that the probability the ceded part exceeds the threshold $\bar{c}_2$, which is

$\mathbb{P}(f(X) > \bar{c}_2)$, is bounded by a predetermined parameter β. In this case, the admissible set of the ceded loss function is given by

$$\begin{aligned}\mathcal{L}_2'' =&\{0 \leqslant f(x) \leqslant x : \mathbb{P}(f(X) > \bar{c}_2) \leqslant \beta, \text{both } R_f(x) \text{ and } f(x) \text{ are nondecreasing} \\ &\quad \text{functions, } R_f(x) \text{ is left continuous}\}\end{aligned} \tag{4.17}$$

where $0 \leqslant \beta \leqslant 1$ is a predetermined parameter chosen by the insurance company.

Using the same technique, the following corollary gives optimal reinsurance strategy that reflects the counterparty risk. The proof is again omitted due to the similarity.

Corollary 4.3.2. *Assume the reinsurance premium principle* $\tilde{\Pi}(\cdot)$ *is a monotonic piecewise premium principle. Then, for any ceded loss function* $f \in \mathcal{L}_2''$*, we can construct the ceded loss function* g_f *according to* (4.13)*, and* g_f *satisfies the following properties:*

(a) $g_f \in \mathcal{L}_2''$.

(b) $\tilde{\Pi}(f(X)) \leqslant \pi_0$ *implies* $\tilde{\Pi}(g_f(X)) \leqslant \pi_0$*. Equivalently, if the ceded loss function* f *satisfies the budget constraint, then the ceded loss function* g_f *will satisfy the budget constraint as well;*

(c) $VaR_\alpha(T_{g_f}(X)) \leqslant VaR_\alpha(T_f(X))$.

Remark 4.3.4. *All the comments in Remark 4.2.4 for Corollary 4.2.2 are analogously applicable here. In particular, with the notations used in Remark 4.3.1, the ceded loss function* h_f *can be expressed as*

$$h_f(x) = (x - d)_+ - (x - v)_+.$$

Therefore, $\mathbb{P}(h_f(X) > \bar{c}_2) \leqslant \beta$ *is equivalent to* $d > Q_2 - \bar{c}_1$ *where* $Q_2 = VaR_{\max\{\alpha,\beta\}}(X)$.

By using these notations, we can simplify the optimization problem as follows

$$\begin{cases} \min\limits_{\max\{0, Q_2 - \bar{c}_2\} \leqslant d \leqslant v} & d + \tilde{\Pi}\left[h_f(X)\right] \\ s.t. & \tilde{\Pi}\left[h_f(X)\right] \equiv \tilde{\Pi}\left[(X-d)_+ - (X-v)_+\right] \leqslant \pi_0. \end{cases} \tag{4.18}$$

4.4 Examples

The objective of this section is to illustrate how the results obtained in the last two sections can be used to determine the optimal ceded loss functions by assuming the monotonic piecewise expected value premium principle with the following representation:

$$\tilde{\Pi}(X) = (1+\rho_1) \cdot \mathbb{E}(X \cdot \mathbb{1}_{X \in [0,a)}) + (1+\rho_2) \cdot \mathbb{E}(X \cdot \mathbb{1}_{X \in [a,+\infty)}) \tag{4.19}$$

where X is any random variable, a, ρ_1 and ρ_2 are fixed constants with $\rho_2 \geqslant \rho_1$. We note that the expected value premium principle is the simplest premium principle and it has been widely studied due to its tractability. The drawback of this premium principle is that the risk attitude of the reinsurer is assumed to be invariant to risk. This is inconsistent with practice since reinsurer often demands a higher level of compensation for larger risk. This issue is alleviated by using an expected value premium principle that is monotonic and piecewise since in this case, the higher layer of risk is penalized with a larger loading factor.

Using the monotonic piecewise expected value premium principle (4.19), Subsection 4.4.1 first derives the general expressions of the optimal ceded loss functions in term of parameters a, ρ_1 and ρ_2. By considering a specified set of numerical values, Subsection 4.4.2 then calculates explicitly the optimal ceded loss function. The optimal ceded loss functions are compared and contrast to some existing results.

We emphasize that while we have consistently used the piecewise expected value premium principle in our illustrations, the optimal reinsurance strategies under other piecewise premium principles, such as principle of equivalent utility but with piecewise parameter values, piecewise with expected premium principle and Wang's premium principle, piecewise with Dutch premium principle and Wang's premium principle, and so forth, can be calculated in a similar fashion.

4.4.1 Piecewise Expected Value Premium Principle

The general optimal ceded loss functions, in term of parameters a, ρ_1 and ρ_2, are derived in the following for the optimal reinsurance models that we have analyzed in the last two sections. The first part assumes that the ceded loss functions need not be nondecreasing while the second part imposes the monotonic constraint on the ceded loss functions.

VaR-minimization among $\mathcal{L}_1$

According to Theorem 4.2.1, the optimal ceded loss function is of the following form

$$f_1(x) = (x - d_1)_+ \cdot \mathbb{1}(x \leqslant v),$$

where $0 \leqslant d_1 < v$ and d_1 is yet to be determined. Recall that $v = \mathrm{VaR}_\alpha(X)$. It follows from (4.7) that VaR of the insurer's total exposed risk corresponding to the ceded loss function f_1 can be expressed as

$$\mathrm{VaR}_\alpha\left(T_{f_1}(X)\right) = d_1 + \tilde{\Pi}\left[(X - d_1)_+ \cdot \mathbb{1}(X \leqslant v)\right].$$

Now we will determine the optimal retention level d_1 under the assumed premium principle (4.19). Since the calculation of the reinsurance premium depends on the relationship between the ceded loss function and the constant a, we need to consider the following two different cases:

Case (i): $d_1 > v - a$

In this case, the ceded loss function $f_1(x) \leqslant v - d_1 < a$. Accordingly, the premium paid by the insurer can be calculated as

$$\begin{aligned}\tilde{\Pi}\left[(X-d_1)_+ \cdot \mathbb{1}(X \leqslant v)\right] &= (1+\rho_1)\int_{d_1}^{v}(X-d_1)dF_X(x)\\ &= (1+\rho_1)\int_{d_1}^{v}\bar{F}_X(x)dx - (1+\rho_1)(v-d_1)\bar{F}_X(v)\end{aligned}$$

where $\bar{F}_X(x) = 1 - F_X(x)$ is the complementary cumulative distribution function, which is also called survival function, of the random variable X. Therefore, the corresponding VaR of the insurer's total exposed risk in this case can be expressed as

$$\mathrm{VaR}_\alpha\left(T_{f_1}(X)\right) = d_1 + (1+\rho_1)\int_{d_1}^{v}\bar{F}_X(x)dx - (1+\rho_1)(v-d_1)\bar{F}_X(v).$$

Taking derivatives of the above expression of VaR with respect to d_1 yields

$$\frac{\partial \mathrm{VaR}_\alpha\left(T_{f_1}(X)\right)}{\partial d_1} = 1 + (1+\rho_1)\bar{F}_X(v) - (1+\rho_1)\bar{F}_X(d_1).$$

If $\frac{1}{1+\rho_1} + \bar{F}_X(v) < 1$, we denote $\gamma_1 = \bar{F}_X^{-1}\left(\frac{1}{1+\rho_1} + \bar{F}_X(v)\right)$, then it is obvious that $\gamma_1 < v$ and it is easy to verify that

$$\frac{\partial \mathrm{VaR}_\alpha\left(T_{f_1}(X)\right)}{\partial d_1}\begin{cases} < 0 & \text{if } d_1 < \gamma_1 \\ = 0 & \text{if } d_1 = \gamma_1 \\ > 0 & \text{if } d_1 > \gamma_1 \end{cases}.$$

If $\frac{1}{1+\rho_1} + \bar{F}_X(v) \geqslant 1$, we let $\gamma_1 = 0$.

Clearly, the reinsurance premium $\tilde{\Pi}\left[(X-d_1)_+ \cdot \mathbb{1}(X \leqslant v)\right]$ is decreasing with respect to the retention level d_1. We introduce a constant $\bar{\gamma}_1$ to reflect the minimum admissible retention level due to the budget constraint. If $\tilde{\Pi}\left[X \cdot \mathbb{1}(X \leqslant v)\right] \leqslant \pi_0$, then we denote $\bar{\gamma}_1 = 0$; otherwise we assume that the constant $\bar{\gamma}_1$ satisfies the following equation $\tilde{\Pi}\left[(X-\bar{\gamma}_1)_+ \cdot \mathbb{1}(X \leqslant v)\right] = \pi_0$. Therefore, $\tilde{\Pi}\left[(X-d_1)_+ \cdot \mathbb{1}(X \leqslant v)\right] \leqslant \pi_0$ if and only if $d_1 \geqslant \bar{\gamma}_1$. Now we can express the optimal ceded loss function $f_1(x)$ explicitly as follows

$$f_1(x) = \begin{cases} (x-\gamma_1)_+ \cdot \mathbb{1}(x \leqslant v), & \text{if } \gamma_1 > v-a \text{ and } \gamma_1 \geqslant \bar{\gamma}_1; \\ (x-\bar{\gamma}_1)_+ \cdot \mathbb{1}(x \leqslant v), & \text{if } \gamma_1 > v-a \text{ and } \gamma_1 < \bar{\gamma}_1; \\ (x-(v-a)^+)_+ \cdot \mathbb{1}(x \leqslant v), & \text{if } \gamma_1 \leqslant v-a \text{ and } (v-a) \geqslant \bar{\gamma}_1; \\ (x-\bar{\gamma}_1)_+ \cdot \mathbb{1}(x \leqslant v), & \text{if } \gamma_1 \leqslant v-a \text{ and } (v-a) < \bar{\gamma}_1; \end{cases}$$

where $(v-a)^+$ means the number which is larger than $(v-a)$ but infinitely close to $(v-a)$. The corresponding minimum VaR of the insurer's total exposed risk is

$$\begin{aligned} &\mathrm{VaR}_\alpha(T_{f_1}(X)) \\ &= \begin{cases} \gamma_1 + (1+\rho_1)\left[\int_{\gamma_1}^{v} \bar{F}_X(x)dx - (v-\gamma_1)\bar{F}_X(v)\right], & \text{if } \gamma_1 > v-a, \gamma_1 \geqslant \bar{\gamma}_1; \\ \bar{\gamma}_1 + \pi_0, & \text{if } \gamma_1 > v-a, \gamma_1 < \bar{\gamma}_1; \\ (v-a) + (1+\rho_1)\left[\int_{(v-a)^+}^{v} \bar{F}_X(x)dx - a\bar{F}_X(v)\right], & \text{if } \gamma_1 \leqslant v-a, (v-a) \geqslant \bar{\gamma}_1; \\ \bar{\gamma}_1 + \pi_0, & \text{if } \gamma_1 \leqslant v-a, (v-a) < \bar{\gamma}_1; \end{cases} \end{aligned}$$

Case (ii): $d_1 \leqslant v-a$

In this case, the premium paid by the insurer can be calculated as

$$
\begin{aligned}
&\tilde{\Pi}\left[(X-d_1)_+\cdot \mathbb{1}(X\leqslant v)\right]\\
=&(1+\rho_1)\int_{d_1}^{(a+d_1)^-}(X-d_1)dF_X(x)+(1+\rho_2)\int_{a+d_1}^{v}(X-d_1)dF_X(x)\\
=&(1+\rho_1)\left[\int_{d_1}^{(a+d_1)^-}\bar{F}_X(x)dx-a\bar{F}_X((a+d_1)^-)\right]\\
&+(1+\rho_2)\left[\int_{a+d_1}^{v}\bar{F}_X(x)dx-(v-d_1)\bar{F}_X(v)+a\bar{F}_X(a+d_1)\right]
\end{aligned}
$$

where $(v-a)^-$ means the number which is less than $(v-a)$ but infinitely close to $(v-a)$. Obviously, if the function $\bar{F}_X(x)$ is continuous at $x=a+d_1$, the above expression can be simplified a little bit. However, we allow the function $\bar{F}_X(x)$ to be not continuous. The corresponding VaR of the insurer's total exposed risk in this case can be expressed as

$$
\begin{aligned}
\mathrm{VaR}_\alpha\left(T_{f_1}(X)\right)=&d_1+(1+\rho_1)\left[\int_{d_1}^{(a+d_1)^-}\bar{F}_X(x)dx-a\bar{F}_X((a+d_1)^-)\right]\\
&+(1+\rho_2)\left[\int_{a+d_1}^{v}\bar{F}_X(x)dx-(v-d_1)\bar{F}_X(v)+a\bar{F}_X(a+d_1)\right].
\end{aligned}
$$

Therefore, the optimization problem in this case can be written as

$$
\begin{cases}
\min\limits_{0\leqslant d_1\leqslant v-a} & \Big\{d_1+(1+\rho_1)\left[\int_{d_1}^{(a+d_1)^-}\bar{F}_X(x)dx-a\bar{F}_X((a+d_1)^-)\right]\\
& +(1+\rho_2)\left[\int_{a+d_1}^{v}\bar{F}_X(x)dx-(v-d_1)\bar{F}_X(v)+a\bar{F}_X(a+d_1)\right]\Big\}\\
\text{s.t.} & (1+\rho_1)\left[\int_{d_1}^{(a+d_1)^-}\bar{F}_X(x)dx-a\bar{F}_X((a+d_1)^-)\right]\\
& +(1+\rho_2)\left[\int_{a+d_1}^{v}\bar{F}_X(x)dx-(v-d_1)\bar{F}_X(v)+a\bar{F}_X(a+d_1)\right]\leqslant\pi_0.
\end{cases}
$$

Once the constants a,ρ_1,ρ_2 and the distribution of the claim X are given, it is relatively easy to solve the above optimization problem. After solving the above optimization problem, we just need to compare the corresponding minimal VaR of the insurer's total exposed risk from cases (i) and (ii) to finalize the optimal reinsurance strategy.

VaR-minimization among $\mathcal{L}_2$

According to Theorem 4.3.1, the optimal ceded loss function is of the following form

$$f_2(x) = (x - d_2)_+ - (x - v)_+,$$

where $0 \leqslant d_2 < v$ and d_2 is still need to be determined. It follows from (4.14) that VaR of the insurer's total exposed risk corresponding to the ceded loss function f_2 can be expressed as

$$\mathrm{VaR}_\alpha\left(T_{f_2}(X)\right) = d_1 + \tilde{\Pi}\left[\min\{(X - d_2)_+, v - d_2\}\right].$$

Now we will determine the optimal retention level d_2 under the assumed premium principle (4.19). As before, we need to consider the following two cases:

Case (i): $d_2 > v - a$

In this case, the ceded loss function $f_2(x) \leqslant v - d_2 < a$. Accordingly, the premium paid by the insurer can be calculated as

$$\begin{aligned}\tilde{\Pi}\left[\min\{(X - d_2)_+, v - d_2\}\right] &= (1+\rho_1)\int_{d_2}^{v}(X - d_2)dF_X(x) + (1+\rho_1)(v - d_2)\bar{F}_X(v)\\ &= (1+\rho_1)\int_{d_2}^{v}\bar{F}_X(x)dx\end{aligned}$$

Therefore, the corresponding VaR of the insurer's total exposed risk in this case can be expressed as

$$\mathrm{VaR}_\alpha\left(T_{f_1}(X)\right) = d_2 + (1+\rho_1)\int_{d_2}^{v}\bar{F}_X(x)dx.$$

Taking derivatives of the above expression of VaR with respect to d_2 yields

$$\frac{\partial \mathrm{VaR}_\alpha\left(T_{f_2}(X)\right)}{\partial d_2} = 1 - (1+\rho_1)\bar{F}_X(d_2).$$

Let $\gamma_2 = \bar{F}_X^{-1}\left(\frac{1}{1+\rho_1}\right)$, then it is easy to verify that

$$\frac{\partial \mathrm{VaR}_\alpha\left(T_{f_2}(X)\right)}{\partial d_2}\begin{cases} < 0 & \text{if } d_2 < \gamma_2 \\ = 0 & \text{if } d_2 = \gamma_2 \\ > 0 & \text{if } d_2 > \gamma_2 \end{cases}$$

Clearly, the reinsurance premium $\tilde{\Pi}\left[\min\{(X-d_2)_+, v-d_2\}\right]$ is decreasing with respect to the retention level d_2. We introduce a constant $\bar{\gamma}_2$ to reflect the minimum admissible retention level due to the budget constraint. If $\tilde{\Pi}\left[\min\{X, v\}\right] \leqslant \pi_0$, then we denote $\bar{\gamma}_2 = 0$; otherwise we assume that the constant $\bar{\gamma}_2$ satisfies the following equation $\tilde{\Pi}\left[\min\{(X-\gamma_2)_+, v-\gamma_2\}\right] = \pi_0$. Therefore, $\tilde{\Pi}\left[\min\{(X-d_2)_+, v-d_2\}\right] \leqslant \pi_0$ if and only if $d_2 \geqslant \bar{\gamma}_2$. Now we can express the optimal ceded loss function $f_2(x)$ explicitly as follows

$$f_2(x) = \begin{cases} 0, & \text{if } \gamma_2 \geqslant v; \\ (x-\gamma_2)_+ - (x-v)_+, & \text{if } v-a < \gamma_2 < v \text{ and } \gamma_2 \geqslant \bar{\gamma}_2; \\ (x-\bar{\gamma}_2)_+ - (x-v)_+, & \text{if } v-a < \gamma_2 < v \text{ and } \gamma_2 < \bar{\gamma}_2; \\ (x-(v-a)^+)_+ - (x-v)_+, & \text{if } \gamma_2 \leqslant v-a \text{ and } (v-a) \geqslant \bar{\gamma}_2; \\ (x-\bar{\gamma}_2)_+ - (x-v)_+, & \text{if } \gamma_2 \leqslant v-a \text{ and } (v-a) < \bar{\gamma}_2; \end{cases}$$

where $(v-a)^+$ means the number which is larger than $(v-a)$ but infinitely close to $(v-a)$. The corresponding minimum VaR of the insurer's total exposed risk is

$$\mathrm{VaR}_\alpha(T_{f_2}(X)) = \begin{cases} v, & \text{if } \gamma_2 \geqslant v; \\ \gamma_2 + (1+\rho_1)\int_{\gamma_2}^{v} \bar{F}_X(x)dx, & \text{if } \gamma_2 > v-a, \gamma_2 \geqslant \bar{\gamma}_2; \\ \bar{\gamma}_2 + \pi_0, & \text{if } \gamma_2 > v-a, \gamma_1 < \bar{\gamma}_2; \\ (v-a) + (1+\rho_1)\int_{(v-a)^+}^{v} \bar{F}_X(x)dx, & \text{if } \gamma_2 \leqslant v-a, (v-a) \geqslant \bar{\gamma}_2; \\ \bar{\gamma}_2 + \pi_0, & \text{if } \gamma_2 \leqslant v-a, (v-a) < \bar{\gamma}_2; \end{cases}$$

Case (ii): $d_2 \leqslant v-a$

In this case, the premium paid by the insurer can be calculated as

$$
\begin{aligned}
&\tilde{\Pi}\left[\min\{(X-d_2)_+, v-d_2\}\right] \\
=&(1+\rho_1)\int_{d_2}^{(a+d_2)^-}(X-d_2)dF_X(x)+(1+\rho_2)\int_{a+d_2}^{v}(X-d_2)dF_X(x)+(1+\rho_2)(v-d_2)\bar{F}_X(v) \\
=&(1+\rho_1)\left[\int_{d_2}^{(a+d_2)^-}\bar{F}_X(x)dx-a\bar{F}_X((a+d_2)^-)\right]+(1+\rho_2)\left[\int_{a+d_2}^{v}\bar{F}_X(x)dx+a\bar{F}_X(a+d_2)\right]
\end{aligned}
$$

Similarly, we allow the function $\bar{F}_X(x)$ to be not continuous. The corresponding VaR of the insurer's total exposed risk in this case can be expressed as

$$
\begin{aligned}
&\mathrm{VaR}_\alpha\left(T_{f_2}(X)\right) \\
=&d_2+(1+\rho_1)\left[\int_{d_2}^{(a+d_2)^-}\bar{F}_X(x)dx-a\bar{F}_X((a+d_2)^-)\right]+(1+\rho_2)\left[\int_{a+d_2}^{v}\bar{F}_X(x)dx+a\bar{F}_X(a+d_2)\right]
\end{aligned}
$$

Therefore, the optimization problem in this case can be written as

$$
\begin{cases}
\min\limits_{0\leqslant d_2\leqslant v-a} & \Big\{d_2+(1+\rho_1)\left[\int_{d_2}^{(a+d_2)^-}\bar{F}_X(x)dx-a\bar{F}_X((a+d_2)^-)\right] \\
 & +(1+\rho_2)\left[\int_{a+d_2}^{v}\bar{F}_X(x)dx+a\bar{F}_X(a+d_2)\right]\Big\} \\
\text{s.t.} & (1+\rho_1)\left[\int_{d_2}^{(a+d_2)^-}\bar{F}_X(x)dx-a\bar{F}_X((a+d_2)^-)\right] \\
 & +(1+\rho_2)\left[\int_{a+d_2}^{v}\bar{F}_X(x)dx+a\bar{F}_X(a+d_2)\right]\leqslant\pi_0
\end{cases}
$$

As in the previous subsection, once the constants a, ρ_1, ρ_2 and the distribution of the claim X are given, the above optimization problems can be solved explicitly. The optimal reinsurance strategy is then given by the one that has the lower minimal VaR of the insurer's total exposed risk.

Remark 4.4.1. *(a) In the above example, we have derived the optimal ceded loss functions under the premium principle that is constructed by concatenating two expected* *value premium principles. If the piecewise premium principle is constructed from n expected value premium principles, similar steps apply although the derivation is more tedious and lengthy.*

(b) *In Chi and Tan (2011), they obtained the optimal reinsurance strategies among the admissible set $\mathcal{L}_1$ and $\mathcal{L}_2$ with the assumptions that there is no budget constraint and the premium principle is expected value premium principle. In this subsection, we demonstrate how to obtain the optimal reinsurance strategies among $\mathcal{L}_1$ and $\mathcal{L}_2$ when the premium principle is piecewise expected value premium principle, which is more general than the expected value premium principle. By setting $a = \infty$ (or equivalently $\rho_1 = \rho_2$) and removing the budget constraint, then our results collapse to those in Chi and Tan (2011).*

(c) *For the cases of exerting limit on the ceded loss functions and imposing the courter-party risk constraint, the optimal reinsurance strategies can be calculated similarly. Therefore, we will not discuss these cases in this example in detail. However, we will present the numerical results for these cases in the following subsection.*

4.4.2 Numerical Examples

In this subsection, we calculate numerically the optimal ceded loss functions by considering a concrete example. In particular, we assume the insurer faces a potential risk that follows the exponential distribution with mean 10 and the insurer is seeking reinsurance to reinsure its risk. We further assume the reinsurance budget is 5, the confidence levels of VaR is 95%, and the reinsurance premium principle is the piecewise premium principle (4.19) with parameter values $a = 10$, $\rho_1 = 0.1$ and $\rho_2 = 0.5$. This implies that when the claim is less than 10, the risk loading factor is 10%. When the claim is larger than 10, the risk loading increases to 50%.

With the above setup, we now utilize the analysis in the last subsection to calculate the optimal reinsurance strategies for the VaR minimization model when the admissible sets of the ceded loss functions are given by $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. We will also calculate the optimal reinsurance strategies when there is a limit imposed on the ceded loss and when there exists counterparty risk. These results are compared and contrasted to the existing results when there is no budget constraint.

(a) VaR minimization among $\mathcal{L}_1$:

In order to solve obtain the optimal strategy, we need to consider the two cases that are discussed in detail in the last subsection. Using the given parameter values, the optimal ceded loss function in case (i) is found to be $(X - 19.9573)_+ \cdot \mathbb{1}(X \leqslant 29.9573)$ with a non-binding budget constraint. Similarly, the optimal ceded loss function in case (ii) is $(X - 6.1539)_+ \cdot \mathbb{1}(X \leqslant 29.9573)$ and the budget constraint is binding. By comparing the VaR of these two cases, we conclude that the optimal ceded loss function is given by $(X - 6.1539)_+ \cdot \mathbb{1}(X \leqslant 29.9573)$ with the corresponding VaR value 11.1539.

When there is no budget constraint, the optimal ceded loss function is determined to be $(X - 2.6007)_+ \cdot \mathbb{1}(X \leqslant 29.9573)$ with the corresponding VaR value 10.5490.

Following the same procedure, we also calculate the optimal ceded loss function when there is a limit on the ceded loss functions or there is a counterparty risk constraint.

- When there is a limit constraint, say 20, on the ceded loss functions, the optimal ceded loss function is $(X - 9.9573)_+ \cdot \mathbb{1}(X \leqslant 29.9573)$, and the corresponding VaR value is 12.8586.

- When there is counterparty risk constraint, i.e. $\mathbb{P}(f(X) > 10) \leqslant 10\%$, the optimal ceded loss function is $(X - 8.9712)_+ \cdot \mathbb{1}(X \leqslant 29.9573)$ and the corresponding VaR value is 12.3324.

If the premium principle is the classical expected premium principle with risk loading factor $\rho = 0.1$ and there is no budget constraint, the optimal ceded loss function is easily obtained to be $(X - 0.4177)_+ \cdot \mathbb{1}(X \leqslant 29.9573)$. If the risk loading factor changes $\rho = 0.5$, then the optimal ceded loss function is $(X - 3.3314)_+ \cdot \mathbb{1}(X \leqslant 29.9573)$.

Table 4.1 summarizes the optimal reinsurance strategies in $\mathcal{L}_1$ for the various variants of the optimal reinsurance models.

Constraint	Premium principle	Optimal reinsurance strategy	VaR
Budget constraint	piecewise expected	$(X - 6.1539)_+ \cdot \mathbb{1}(X \leqslant 29.9573)$	11.1539
No budget constraint	piecewise expected	$(X - 2.6007)_+ \cdot \mathbb{1}(X \leqslant 29.9573)$	10.5490
Limit on reinsurance	piecewise expected	$(X - 9.9573)_+ \cdot \mathbb{1}(X \leqslant 29.9573)$	12.8586
Counterparty risk constraint	piecewise expected	$(X - 8.9712)_+ \cdot \mathbb{1}(X \leqslant 29.9573)$	12.3324
No budget constraint	expected ($\rho = 0.1$)	$(X - 0.4177)_+ \cdot \mathbb{1}(X \leqslant 29.9573)$	
No budget constraint	expected ($\rho = 0.5$)	$(X - 3.3314)_+ \cdot \mathbb{1}(X \leqslant 29.9573)$	

Table 4.1: Optimal Reinsurance Strategies Among $\mathcal{L}_1$

(b) VaR minimization among $\mathcal{L}_2$:

Similar calculations can be repeated to confirm that the optimal reinsurance strategies among $\mathcal{L}_2$ are given by Table 4.2.

Constraint	Premium principle	Optimal reinsurance strategy	VaR
Budget constraint	piecewise expected	$(X - 8.8578)_+ - (X - 29.9573)_+$	13.8578
No budget constraint	piecewise expected	$(X - 3.3240)_+ - (X - 29.9573)_+$	12.5740
Limit on reinsurance	piecewise expected	$(X - 9.9573)_+ - (X - 29.9573)_+$	14.3586
Counterparty risk constraint	piecewise expected	$(X - 13.0259)_+ - (X - 29.9573)_+$	16.0660
No budget constraint	expected ($\rho = 0.1$)	$(X - 0.9531)_+ - (X - 29.9573)_+$	
No budget constraint	expected ($\rho = 0.5$)	$(X - 4.0547)_+ - (X - 29.9573)_+$	

Table 4.2: Optimal Reinsurance Strategies Among $\mathcal{L}_2$

4.5 Concluding Remark

In this chapter, we investigate the VaR-based optimal reinsurance strategies under the monotonic piecewise premium principle. We consider several different admissible sets of the ceded loss functions. In the general model, we show that the truncated stop-loss or the limited stop-loss reinsurance strategy is optimal depending on whether the ceded loss functions are required to be nondecreasing. In both cases, we extend our results to the case of multiple reinsurers. Moreover, we also consider the cases of exerting a limit on the reinsurance treaties or existing courterparty risk constraint. We also use the piecewise expected value premium principle as an example to demonstrate how to apply our results to solve the optimal reinsurance problem. A numerical example is provided to highlight our results.

Chapter 5

General Risk Measures Minimization Models

In Chapter 2 and Chapter 3, we considered the optimal partial hedging strategies which minimize VaR and CVaR of the investor's total exposed risk respectively. However, sometimes, the investor may not have a specific risk measure in mind. Probably, he wants to adopt a class of risk measures instead of a specific risk measure. This will lead to the optimal hedging problem under the general risk measures. Furthermore, by considering hedging strategies under general risk measures, we will be able to gain some insight into the robustness of the optimal hedging strategy with respect to the risk measures. Therefore, in this chapter, we will study partial hedging strategies under the general risk measures.

5.1 Preliminaries

5.1.1 General Risk Measures

Following the ideas in Rockafellar et al. (2006) and Balbás et al. (2009), we can use a set of scenarios to represent a given risk measure.

Assume that the probability space $(\mathbf{\Omega}, \mathcal{F}, \mathbb{P})$ is composed of the set $\mathbf{\Omega}$, the σ-algebra $\mathcal{F}$ and the physical probability measure $\mathbb{P}$. Consider a couple of conjugate numbers $p \in [1, +\infty)$ and $q \in (1, +\infty]$, where $\frac{1}{p} + \frac{1}{q} = 1$. L^p denotes the Banach space of $\mathbb{R}$-valued random variables x on $\mathbf{\Omega}$ such that $\mathbb{E}(|x|^p)$ is finite, where $\mathbb{E}(\cdot)$ represents the expectation under the probability measure $\mathbb{P}$. L^q is defined similarly. Clearly, L^q is the dual space of L^p.

Let ρ, which is defined on the L^p space, be the general risk function that the investor uses to measure his risk. Then we can define the set induced by the risk measure ρ as follows

$$\Delta_\rho = \{z \in L^q : \mathbb{E}(xz) \leqslant \rho(x), \forall x \in L^p\} \tag{5.1}$$

Here, x is interpreted as the loss of the investor. If x represents the wealth of the investor, the $\mathbb{E}(xz)$ in the above expression should be replaced by $-\mathbb{E}(xz)$. According to the property of expectation, it is clear that the set Δ_ρ is convex. We further impose some assumptions on the set Δ_ρ.

Assumption 5.1.1. *Δ_ρ is compact. All the elements in Δ_ρ have the same nonnegative expectation, i.e.* $\mathbb{E}(z) = \tilde{E} \geqslant 0, \ \forall z \in \Delta_\rho$.

Assumption 5.1.1 is same as the assumption imposed on set induced by risk measures in Balbás et al. (2009). Even if with Assumption 5.1.1 imposed on Δ_ρ, Δ_ρ still can be quite general. According to the definition of Δ_ρ, we can represent the risk measure ρ as follows

$$\rho(x) = \max\{\mathbb{E}(xz) : z \in \Delta_\rho\} \quad \forall x \in L^p \tag{5.2}$$

According to an interpretation mentioned in Artzner et al. (1999), every element $z \in \Delta_\rho$ can be understood as a particular scenario. Mathematically, every element $z \in \Delta_\rho$, which can be viewed as the density of an alternative measure with respect to the physical probability measure $\mathbb{P}$, plays the role as distorting the physical probability measure $\mathbb{P}$. Therefore, every $\mathbb{E}(xz)$ can be interpreted as a distorted expectation of the investor's loss under the scenario that can be represented by z.

According to Balbás et al. (2009) and Rockafellar et al. (2006), the above representation of risk measures at least includes two important classes of risk measures, which are

1. Strictly expectation bounded risk measures defined in Rockafellar et al. (2006).

   It is easy to verify that if $\Delta_\rho = \{z \in L^q; \mathbb{E}(z) = 1\}$, which is a sufficient but not necessary condition, the corresponding risk measure ρ is of the class of strictly expectation bounded risk measures defined in Rockafellar et al. (2006).

2. General deviation measures defined in Rockafellar et al. (2006).

   It is easy to verify that if $\Delta_\rho = \{z \in L^q; \mathbb{E}(z) = 0\}$, which is a sufficient but not necessary condition, the corresponding risk measure ρ is of the class of general deviation measures defined in Rockafellar et al. (2006).

These two classes of risk measures include numerous risk measures, such as Conditional Value at Risk (CVaR), the Weighted Conditional Value at Risk mentioned in Cherny (2006), Wang measure, standard deviation, range-based deviation, CVaR-deviation, and so forth. For more related examples of risk measures, readers can refer to Rockafellar et al. (2006).

By using the above representation of risk measures, we can easily represent convex combinations. According to the Separation Theorems, if ρ is a convex combination of the risk measures satisfying Assumption 5.1.1, i.e.

$$\rho = \sum_{i=1}^{m} w_i \rho_i,$$

then the set induced by ρ is

$$\Delta_\rho = \sum_{i=1}^{m} w_i \Delta_{\rho_i}$$

where $\sum_{i=1}^{m} w_i \Delta_{\rho_i} = \{\sum_{i=1}^{m} w_i z_i : z_i \in \Delta_{\rho_i}\}$. Therefore, this representation enables us to study the optimal strategies under a convex combination of some risk measures. It is worth noting that if we choose

$$\rho = w\rho_0 + (1 - w)\mathbb{E},$$

it corresponds to the case when the investor wants to minimize the risk measured by ρ_0 and the expected value of his random loss. In this case, the investor aims at minimizing the risk measured by ρ_0 and maximizing his expected return while his tolerance of risk is measured by w.

5.1.2 Model Description

After characterizing the general risk measures, now we are ready to describe the problem of partial hedging. We assume the market is complete throughout this chapter. Let X be the time-T payoff that the investor wants to hedge. Similar to Chapter 2 and Chapter 3, here X can be any function of the index or the price of a specific stock, i.e. $X = H(S_t, 0 \leqslant t \leqslant T)$, where S_t denotes the time t value of the index or price of a specific stock and H is a functional. We assume that the contract that the investor wants to hedge is not a risk-free asset. Without loss of generality, we assume that X is a non-negative nonconstant random variable with cumulative distribution function $F_X(x) = \mathbb{P}(X \leqslant x)$ and there exists $p \in [1, +\infty)$ such that $X \in L^p$, which is equivalent to $\mathbb{E}(|X|^p) < \infty$.

The problem of optimal hedging can be solved in two steps. First, under the budget constraint and some given criteria, we try to find the optimal partitioning of X into $f(X)$ and $R_f(X)$ where $X = f(X) + R_f(X)$. Here, $f(X)$ denotes the part of the payoff which the investor wants to hedge, and $R_f(X)$ denotes the exposed part of the payoff. Then, the second step is to investigate the possibility of replicating the time-T payoff $f(X)$ in the market. In this chapter, we focus mainly on the first step. The first step can be formulated similarly as in (2.2), which is restated as follows

$$\begin{cases} \min\limits_{f \in \mathcal{L}} & \rho(T_f(X)) \\ \text{s.t.} & \Pi(f(X)) \leqslant \pi_0 \end{cases} \tag{5.3}$$

where, $\rho(\cdot)$ represents the risk measure adopted by the investor, $\Pi(f(X)))$ is the market price of $f(X)$, which is the cost of performing hedging strategy f, $T_f(X)$ denotes $R_f(X) + e^{rT} \cdot \Pi(f(X))$ the total risk exposure of the investor under such a partial hedging strategy at time T (maturity of the contract X), r is the risk free return rate, $\pi_0 > 0$ is the hedging budget (an amount up to which the investor is willing to spend on hedging), $\mathcal{L}$ is the admissible set of hedged part. Here, we do not need to specify $\Pi(\cdot)$, but we assume that $\Pi(\cdot)$ admits no arbitrage and there is no "overpricing" in the market. By saying that there is no "overpricing" in the market, we mean that there does not exist any financial contract whose price is large enough to completely cover the unitized risk of selling that contract. These assumptions are summarized as follows.

Assumption 5.1.2. *(a)* $\Pi(\cdot)$ *admits no arbitrage.*

(b) Let ρ *be a given risk measure that is not of the class of general deviation measures,* Δ_ρ *is the set induced by* ρ*. For any contract in the market* X_0*, there exists a* $z_0 \in \Delta_\rho$ *satisfies that*

$$e^{rT} \cdot \Pi(X_0) \leqslant \frac{1}{\tilde{E}} \mathbb{E}(X_0 z_0)$$

where $\tilde{E} > 0$ *is the expectation of the element in the set* Δ_ρ *under the physical probability measure. The above inequality strictly holds if* X_0 *is not a risk-free asset.*

Remark 5.1.1. *Intuitively, Assumption 5.1.2 (b) means that investor can not decrease his risk by merely selling any attainable contract in the market. The premium the investor gets by selling the contract is not sufficient to cover the unitized risk of the short position when the risk is not measured by deviation measure.*

Since in this chapter we are considering the partial hedging strategy, we assume that the hedging budget is less than the market price of the original contract, i.e. $\pi_0 < \Pi(X)$. Since the hedging budget is not sufficient, we exclude over-hedging in any scenario. We further assume that the payoff of the hedging portfolio remains nonnegative, because the hedging portfolio should not increase the debt of the investor in any scenario. This brings us to the third assumption:

Assumption 5.1.3. *The admissible set of the hedged part is* $\mathcal{L} = \{f : 0 \leqslant f(x) \leqslant x\}$

5.2 Optimal Partial Hedging Strategy

5.2.1 Optimality Conditions

Using the representation of the risk measure (5.2) and Assumption 5.1.3, we can rewrite the optimization problem (5.3) as follows

$$\begin{cases} \min\limits_{\theta\in\mathbb{R}, f\in\mathcal{L}} & \theta \\ \text{s.t.} & \theta \geqslant \mathbb{E}((R_f(X) + e^{rT}\cdot\Pi(f(X)))z), \quad \forall z\in\Delta_\rho \\ & \Pi(f(X)) \leqslant \pi_0. \end{cases} \tag{5.4}$$

The above optimization problem (5.4) is equivalent to the optimization problem (5.3) in the sense that f^* solves (5.3) if and only if (θ^*, f^*) solve (5.4). Furthermore, under the optimal case, $\theta^* = \rho(-R_{f^*}(X) - e^{rT}\cdot\Pi(f^*(X)))$, where (θ^*, f^*) is the optimizer of (5.4), is the risk of the investor at time T when using the optimal partial hedging strategy.

According to Assumption 5.1.1 and Assumption 5.1.2, the above optimization problem can be rewritten as follows

$$\begin{cases} \min\limits_{\theta\in\mathbb{R}, f\in\mathcal{L}} & \theta \\ \text{s.t.} & \theta \geqslant \mathbb{E}(R_f(X)\cdot z) + e^{rT}\cdot\Pi(f(X))\tilde{E}, \quad \forall z\in\Delta_\rho \\ & \Pi(f(X)) \leqslant \pi_0 \end{cases} \tag{5.5}$$

The above optimization problem is similar to the optimization problem in Balbás et al. (2009). Using Theorem 3 in Balbás et al. (2009) and the results in Luenberger (1969), we obtain the Karush-Kuhn-Tucker conditions of (5.5)

$$
\begin{cases}
\theta^* - \mathbb{E}(R_{f^*}(X) \cdot z^*) - e^{rT} \cdot \Pi(f^*(X))\tilde{E} = 0 \\
\theta^* - \mathbb{E}(R_{f^*}(X) \cdot z) - e^{rT} \cdot \Pi(f^*(X))\tilde{E} \geqslant 0, \quad \forall z \in \Delta_\rho \\
\tau^*(\Pi(f^*(X)) - \pi_0) = 0 \\
\Pi(f^*(X)) \leqslant \pi_0 \\
\mathbb{E}(R_{f^*}(X) \cdot z^*) + (\tilde{E} + \tau^*)e^{rT} \cdot \Pi(f^*(X)) \\
\quad \leqslant \mathbb{E}(R_f(X) \cdot z^*) + (\tilde{E} + \tau^*)e^{rT} \cdot \Pi(f(X)), \quad \forall\ 0 \leqslant f(X) \leqslant X, \Pi(f(X)) \leqslant \pi_0 \\
\theta^* \in \mathbb{R}, f^* \in \mathcal{L}, \tau^* \geqslant 0, z^* \in \Delta_\rho.
\end{cases}
\tag{5.6}
$$

Conditions (5.6) are necessary and sufficient conditions to the optimization problem (5.5) in the sense that (θ^*, f^*) solves the optimization problem (5.5) and (τ^*, z^*) solves its dual problem. The results in Balbás et al. (2009) and Luenberger (1969) guarantee that the Slater qualification holds and the pathological situation called "duality gap" does not happen in this optimization problem. It may be of interest to note that the optimal solutions to the above Karush-Kuhn-Tucker conditions, z^* and τ^*, describe the sensitivity of the optimal risk measured by ρ with respect to X and π_0 respectively.

We can give another set of alternative optimality conditions which is equivalent to (5.6)

$$
\begin{cases}
\mathbb{E}(R_{f^*}(X) \cdot z^*) \geqslant \mathbb{E}(R_{f^*}(X) \cdot z), \quad \forall z \in \Delta_\rho \\
\tau^*(\Pi(f^*(X)) - \pi_0) = 0 \\
\Pi(f^*(X)) \leqslant \pi_0 \\
\mathbb{E}(R_{f^*}(X) \cdot z^*) + (\tilde{E} + \tau^*)e^{rT} \cdot \Pi(f^*(X)) \\
\quad \leqslant \mathbb{E}(R_f(X) \cdot z^*) + (\tilde{E} + \tau^*)e^{rT} \cdot \Pi(f(X)), \quad \forall\ 0 \leqslant f(X) \leqslant X, \Pi(f(X)) \leqslant \pi_0 \\
f^* \in \mathcal{L}, \tau^* \geqslant 0, z^* \in \Delta_\rho
\end{cases}
\tag{5.7}
$$

where $\theta^* = \mathbb{E}(R_{f^*}(X) \cdot z^*) + e^{rT} \cdot \Pi(f^*(X))\tilde{E}$ is the optimal risk level.

The proof of the equivalence between these two set of conditions is similar to the proof of Theorem 4 in Balbás et al. (2009), hence is omitted here.

5.2.2 Pricing Kernel

In the optimization problem (5.5), both the objective function and the constraints involve the market price of hedging the risk $f(X)$. In order to study the optimization problem (5.5) in more detail, it is necessary for us to discuss the pricing method adopted in this subsection.

There are many different methods used to price financial contracts, especially in the incomplete market. In this chapter, we want to deal with all the problems under the physical probability measure. Therefore, we will use the method of pricing kernel to price the contracts. Pricing kernel is also called stochastic discount factor. In this chapter, we assume that the pricing kernel admits no arbitrage opportunity in the market.

Mathematically, if we use the method of pricing kernel, the price of a specific contract can be expressed as $\Pi(X) = \mathbb{E}(X \cdot Z_\Pi)$, where X is the time-T payoff of the contract, Z_Π is the time-T pricing kernel. Here, we do not need to specify the expression of Z_Π. However, there are some conditions which the pricing kernel Z_Π needs to satisfy. First of all, the pricing kernel is nonnegative in any scenario, which means that $\mathbb{P}(Z_\Pi \geqslant 0) = 1$. Secondly, when we use the pricing kernel to price the risk-free asset which pays \$1 at time T, we have $\Pi(1) = \mathbb{E}(1 \cdot Z_\Pi)$. It is obvious that the price of the risk-free asset which pays \$1 at time T is e^{-rT}. Therefore, the expectation of the pricing kernels which admits no arbitrage opportunity in the market should be given by the aforementioned constant, i.e. $\mathbb{E}(Z_\Pi) = e^{-rT}$. Lastly, when we use the pricing kernel notation, Assumption 5.1.2 in the last subsection can be rewritten as: for any contract in the market X_0, there exists a

$z_0 \in \Delta_\rho$ that satisfies

$$e^{rT} \cdot \mathbb{E}(X_0 \cdot Z_\Pi) < \frac{1}{\tilde{E}} \mathbb{E}(X_0 z_0)$$

where Δ_ρ is the set induced by ρ, $\tilde{E}$ is the expectation of the element in the set Δ_ρ under the physical probability measure.

The method of pricing kernel is quite general, and there are quite a few different economic meanings of pricing kernel in the literature. In the following, we will illustrate some of the well known economic interpretations of pricing kernel.

1. Interpreted as intertemporal marginal rate of substitution

   According to Rosenberg and Engle (2002), if the price of the contracts linked to the consumption of the investors, the pricing kernel can be determined based on the intertemporal marginal rate of substitution. The first-order conditions of the optimal consumption problem delineate the prices of the contracts and the pricing kernel. If the optimal consumption problem equipped with the time-separable utility function, the pricing kernel will be equal to the intertemporal marginal rate of substitution. If the investor's utility depends on some factors other than consumption, the pricing kernel will depend on additional state variables. Interested readers can refer to Startz (1989), Constantinides (1990), Campbell and Cochrane (1999).

2. Interpreted as state-price-per-unit-probability

   The pricing kernel can also be determined based on the state price density, which defines the prices of the contingent claims that pay one dollar at time T in one state of the world and nothing otherwise. The number of states do not need to be finite. If the pricing kernel is defined to be the quotient of the state price and the state probability density, then the method of pricing kernel is equivalent to the method of state pricing, which is also known as State Preference Theory.

3. Interpreted as the Radon-Nikodym derivative

   One of the most important pricing methods is risk-neutral pricing. Under the theory of risk-neutral pricing, the price of the financial contract equals to the discounted expected value of the payoff of the contract under the risk-neutral measure. Therefore, if the pricing kernel is defined to be the product of the discount factor and the Radon-Nikodym derivative of the risk-neutral measure with respect to the physical probability measure, then the method of pricing kernel is equivalent to the method of risk-neutral pricing.

5.2.3 Some Particular Partial Hedging Strategies

Using the pricing kernel notation $\Pi(X) = \mathbb{E}(X \cdot Z_\Pi)$, the optimality conditions (5.7) can be rewritten as follows

$$\begin{cases} \mathbb{E}(R_{f^*}(X) \cdot z^*) \geqslant \mathbb{E}(R_{f^*}(X) \cdot z), \quad \forall z \in \Delta_\rho \\ \tau^*(\mathbb{E}(f^*(X) \cdot Z_\Pi) - \pi_0) = 0 \\ \mathbb{E}(f^*(X) \cdot Z_\Pi) \leqslant \pi_0 \\ \mathbb{E}(R_{f^*}(X) \cdot z^*) + (\tilde{E} + \tau^*)e^{rT} \cdot \mathbb{E}(f^*(X) \cdot Z_\Pi) \\ \quad \leqslant \mathbb{E}(R_f(X) \cdot z^*) + (\tilde{E} + \tau^*)e^{rT} \cdot \mathbb{E}(f(X) \cdot Z_\Pi), \quad \forall\ f \in \mathcal{L}, \Pi(f(X)) \leqslant \pi_0 \\ f^* \in \mathcal{L}, \tau^* \geqslant 0, z^* \in \Delta_\rho \end{cases} \tag{5.8}$$

where $\theta^* = \mathbb{E}(R_{f^*}(X) \cdot z^*) + e^{rT} \cdot \mathbb{E}(f^*(X) \cdot Z_\Pi)\tilde{E}$ is the optimal risk level.

The following theorem delineates the optimal hedging strategy based on the conditions (5.8).

Theorem 5.2.1. *Assume that conditions (5.8) hold and $R_{f^*}(X)$ is not constant. Then the set $\{\omega : f^*(X)(\omega) = 0\}$ is not an empty set.*

Proof: We will prove this theorem by contradiction. Assume that $\{\omega : f^*(X)(\omega) = 0\}$ is an empty set.

First, we prove that $\{\omega : z^*(\omega) < (\tilde{E}+\tau^*)e^{rT} \cdot Z_\Pi\}$ is not an empty set. If $\{\omega : z^*(\omega) < (\tilde{E}+\tau^*)e^{rT} \cdot Z_\Pi\}$ is an empty set, then $z^* \geqslant (\tilde{E}+\tau^*)e^{rT} \cdot Z_\Pi$. Taking expectation on both sides, we have $\tilde{E} \geqslant (\tilde{E}+\tau^*)$. Therefore, we have $\tau^* = 0$, $z^* = \tilde{E} \cdot e^{rT} \cdot Z_\Pi$.

If $\tilde{E} > 0$, $z^* = \tilde{E} \cdot e^{rT} \cdot Z_\Pi$ contradicts with Assumption 5.1.2 because of the first condition in (5.8) and the assumption that $R_{f^*}(X)$ is not constant.

If $\tilde{E} = 0$, which corresponds to the case that ρ is of the class of general deviation measures, then $z^* = 0$. It is obvious that $\theta^* = 0$ which is the optimal risk level, since $\theta^* = \mathbb{E}(R_{f^*}(X) \cdot z^*) + e^{rT} \cdot \mathbb{E}(f^*(X) \cdot Z_\Pi)\tilde{E}$. Therefore, we have $\rho(R_{f^*}(X) + \Pi(f^*(X))) = 0$, which contradicts with the assumption that $R_{f^*}(X)$ is not constant.

Hence, $\{\omega : z^*(\omega) < (\tilde{E}+\tau^*)e^{rT} \cdot Z_\Pi\}$ is not an empty set.

Note that

$$
\begin{aligned}
&\mathbb{E}(R_{f^*}(X) \cdot z^*) + (\tilde{E}+\tau^*)e^{rT} \cdot \mathbb{E}(f^*(X) \cdot Z_\Pi) \\
=&\mathbb{E}(R_{f^*}(X) \cdot z^*) + (\tilde{E}+\tau^*)e^{rT} \cdot \mathbb{E}(X \cdot Z_\Pi) - (\tilde{E}+\tau^*)e^{rT} \cdot \mathbb{E}(R_{f^*}(X) \cdot Z_\Pi) \\
=&\mathbb{E}\left[R_{f^*}(X) \cdot (z^* - (\tilde{E}+\tau^*)e^{rT} \cdot Z_\Pi)\right] + (\tilde{E}+\tau^*)e^{rT} \cdot \mathbb{E}(X \cdot Z_\Pi)
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}(R_f(X) \cdot z^*) + (\tilde{E}+\tau^*)e^{rT} \cdot \mathbb{E}(f(X) \cdot Z_\Pi) \\
=&\mathbb{E}(R_f(X) \cdot z^*) + (\tilde{E}+\tau^*)e^{rT} \cdot \mathbb{E}(X \cdot Z_\Pi) - (\tilde{E}+\tau^*)e^{rT} \cdot \mathbb{E}(R_f(X) \cdot Z_\Pi) \\
=&\mathbb{E}\left[R_f(X) \cdot (z^* - (\tilde{E}+\tau^*)e^{rT} \cdot Z_\Pi)\right] + (\tilde{E}+\tau^*)e^{rT} \cdot \mathbb{E}(X \cdot Z_\Pi)
\end{aligned}
$$

therefore, if we let

$$
\bar{f} = \begin{cases} 0, & \text{when } z^* < (\tilde{E}+\tau^*)e^{rT} \cdot Z_\Pi \\ f^*, & \text{otherwise} \end{cases}
$$

then it is clear that $0 \leqslant \bar{f}(X) \leqslant X$ because $0 \leqslant f^*(X) \leqslant X$. Furthermore, we have $\bar{f} \leqslant f^*$, so $\Pi(\bar{f}(X)) \leqslant \Pi(f^*(X)) \leqslant \pi_0$.

According to the construction of $\bar{f}$ and the assumption that $\{\omega : f^*(X)(\omega) = 0\}$ is an empty set, we have

$$\mathbb{E}\left[R_{f^*}(X) \cdot (z^* - (\tilde{E} + \tau^*)e^{rT} \cdot Z_\Pi)\right] > \mathbb{E}\left[R_{\bar{f}}(X) \cdot (z^* - (\tilde{E} + \tau^*)e^{rT} \cdot Z_\Pi)\right]$$

which implies that

$$\begin{aligned} &\mathbb{E}(R_{f^*}(X) \cdot z^*) + (\tilde{E} + \tau^*)e^{rT} \cdot \mathbb{E}(f^*(X) \cdot Z_\Pi) \\ &> \mathbb{E}(R_{\bar{f}}(X) \cdot z^*) + (\tilde{E} + \tau^*)e^{rT} \cdot \mathbb{E}(\bar{f}(X) \cdot Z_\Pi) \end{aligned}$$

which contradicts with the fourth condition in (5.8).

□

Remark 5.2.1. *Theorem 5.2.1 shows that under the optimal partial hedging strategy, the investor will retain all the risk in some scenario, unless the risk retained by the investor is constant.*

Next, we show that using the same financial contract to perform the partially hedging can not be the optimal strategy, which means that there does not exist an optimal scale of writing contracts. This means that the scale of the investor's business does not affect the investor's risk measured by the general risk measures. As a special case, the proportional hedging strategy can not be optimal under the strictly expectation bounded risk measures or the general deviation measures. Furthermore, retaining all the risk without hedging can not be an optimal hedging strategy.

Theorem 5.2.2. *Assume that X is a risky asset and $\tilde{E} > 0$, then the proportional hedging strategy $f(X) = \lambda \cdot X$ for some $\lambda \in [0, 1]$ can not be an optimal partial hedging strategy among $\mathcal{L}$.*

Proof: We will prove this theorem by contradiction. Assume that the hedging strategy $f^*(X) = \lambda^* \cdot X$ is an optimal partial hedging strategy for some $\lambda^* \in [0, 1]$.

If $\lambda^* = 1$, it means that the investor hedges all his risk. Since the hedging budget $\pi_0 < \Pi(X)$, $\lambda^* = 1$ will violate the budget constraint, hence infeasible. Therefore, we have $\lambda^* < 1$, which means that $R_{f^*}(X) = (1 - \lambda^*)X$ is not constant because X is not constant.

According to Theorem 5.2.1, we can conclude that the set $\{\omega : f^*(X)(\omega) = 0\}$ is not a empty set, which yields $\lambda^* = 0$. This implies that the investor retains all his risk without hedging. In this case, the budget constraint is not binding. According to the second condition in (5.8), $\tau^* = 0$.

On the other hand, $z^* \leqslant (\tilde{E} + \tau^*)e^{rT} \cdot Z_\Pi$ must hold, otherwise $f^*(X) = 0$ will not satisfy the fourth condition in (5.8). Therefore, we have $z^* \leqslant \tilde{E} \cdot e^{rT} \cdot Z_\Pi$. If $z^* \neq \tilde{E} \cdot e^{rT} \cdot Z_\Pi$, taking expectation on both sides we have $\tilde{E} < \tilde{E}$, which is a contradiction. Therefore, we have $z^* = \tilde{E} \cdot e^{rT} \cdot Z_\Pi$. According to the first condition in (5.8), we have

$$\mathbb{E}(R_{f^*}(X) \cdot z^*) \geqslant \mathbb{E}(R_{f^*}(X) \cdot z), \quad \forall z \in \Delta_\rho$$

which can be rewritten as

$$\tilde{E} \cdot e^{rT} \cdot \mathbb{E}(X \cdot Z_\Pi) \geqslant \mathbb{E}(X \cdot z), \quad \forall z \in \Delta_\rho.$$

This contradicts with Assumption 5.1.2 because X is not a risk-free asset.

Hence, Theorem 5.2.2 is proved.

□

Theorem 5.2.3. *If X is a risky asset and ρ belongs to the class of general deviation measures, then the proportional hedging strategy $f(X) = \lambda \cdot X$ for some $\lambda \in [0, 1]$ can not be an optimal partial hedging strategy among $\mathcal{L}$.*

Proof: We will prove this theorem by contradiction. Assume that the proportional hedging strategy $f^*(X) = \lambda^* \cdot X$ is an optimal partial hedging strategy for some $\lambda^* \in [0, 1]$.

Since ρ is of the class of general deviation measures, we know that $\tilde{E} = 0$. Similar to the proof of Theorem 5.2.2, we can conclude that $\tau^* = 0$ and $z^* = \tilde{E} \cdot e^{rT} \cdot Z_\Pi = 0$.

In this case, it is obvious that $\theta^* = 0$ which is the optimal risk level, since $\theta^* = \mathbb{E}(R_{f^*}(X) \cdot z^*) + e^{rT} \cdot \mathbb{E}(f^*(X) \cdot Z_\Pi)\tilde{E}$. Therefore, we have $\rho((1 - \lambda^*)X + \lambda^* \cdot \Pi(X)) = (1 - \lambda^*)\rho(X) = 0$, which is equivalent to $\rho(X) = 0$. However, note that ρ is a deviation measure, $\rho(X) = 0$ implies that X is constant, which contradicts with the assumption that contract X is not a risk-free asset.

□

By using the above theorems, we can show that another specific class of hedging strategies can not be optimal under the general risk measures.

Theorem 5.2.4. *Assume that X is a risky asset and $\tilde{E} \geqslant 0$, then the hedging strategy of the form that $f(X) = X - (X - \beta)_+ = \min\{X, \beta\}$ for some constant $\beta \in [0, +\infty]$ can not be an optimal partial hedging strategy among $\mathcal{L}$. Here, X_+ is defined as $X_+ = \max\{X, 0\}$, which is the positive part of X.*

Proof: Assume that $f^*(X) = \min\{X, \beta^*\}$ for some $\beta^* \in [0, +\infty]$ is an optimal partial hedging strategy. In this case, the optimal retained part is $R_{f^*}(X) = (X - \beta)_+$.

Firstly, $\beta^* \neq +\infty$ due to the budget constraint. Therefore, $R_{f^*}(X)$ will not be constant. According to Theorem 5.2.1, the set $\{\omega : f^*(X)(\omega) = 0\}$ is not an empty set, which makes $\beta^* = 0$. However, Theorem 5.2.2 and Theorem 5.2.3 show that retaining all the risk without hedging can not be an optimal hedging strategy when $\tilde{E} \geqslant 0$, which means that $\beta^* \neq 0$.

Hence, Theorem 5.2.4 is proved.

□

In Chapter 3, the bull-call-spread hedging strategy is shown to be an optimal partial hedging strategy which minimizes the CVaR, which belongs to the class of strictly expectation bounded risk measures, of the investor's total exposed risk. For completeness, under the framework of this chapter, we show that the bull-call-spread hedging strategy can be optimal under general risk measures. The next theorem provides a sufficient condition under which the bull-call-spread hedging strategy is optimal.

Theorem 5.2.5. *Assume that $\tilde{E} \geqslant 0$. Then the bull-call-spread hedging strategy $f^*(X) = (X - d^*)_+ - (X - U^*)_+$ for some constant $0 < d^* < U^* \leqslant +\infty$ is an optimal partial hedging strategy among $\mathcal{L}$ if*

(a) $\mathbb{E}(f^*(X) \cdot Z_\Pi) = \pi_0$.

(b) there exists $\tau^* > 0$ and $z^* \in \Delta_\rho$ such that

- *(i)* $z^* \leqslant (\tilde{E} + \tau^*)e^{rT} \cdot Z_\Pi$.
- *(ii)* $z^*(\omega) = (\tilde{E} + \tau^*)e^{rT} \cdot Z_\Pi(\omega), \ \forall \ \omega \in \Omega_d = \{\omega \in \Omega : X(\omega) > d^*\}$.
- *(iii)* $\mathbb{E}(R_{f^*}(X) \cdot z^*) \geqslant \mathbb{E}(R_{f^*}(X) \cdot z), \ \forall \ z \in \Delta_\rho$, *where* $R_{f^*}(X) = X - f^*(X)$.

Proof: It is easy to verify that the optimality conditions (5.8) hold if the above conditions hold.

□

Remark 5.2.2. *Theorem 5.2.5 shows that it is possible, but not guaranteed, that the bull-call-spread hedging strategy is an optimal hedging strategy among $\mathcal{L}$ under general risk measures.*

5.3 Robustness with respect to Confidence Level

Under some assumptions, in Chapter 3, we show that the bull-call-spread hedging strategy is an optimal partial hedging strategy in the sense of minimizing CVaR of the investor's total exposed risk. Applying Theorem 5.2.5 to the results in Chapter 3, we can show that the optimal partial hedging strategy in Chapter 3 is strongly robust with respect to the confidence level $1-\alpha$. By saying that the optimal strategy is strongly robust with respect to the confidence level, we mean that the optimal strategy, not only the structure but also the parameters, remains unchanged even if the confidence level change. This interesting result is stated as the following theorem.

Theorem 5.3.1. *Assume that for a given confidence level $1-\alpha$, there exists an optimal partial hedging strategy which minimizes $CVaR_\alpha$ of the investor's total exposed risk. Then the optimal strategy also minimizes $CVaR_{\alpha_1}$ of the investor's total exposed risk for any α_1 which is small enough and $\alpha_1 \geqslant \alpha$.*

Proof: For a given confidence level $1-\alpha$, the existence of the optimal partial hedging strategy, denoted as f^*, guarantees the existence of $\tau^* > 0$ and $z^* \in \Delta_\rho$ which satisfy the optimality conditions (5.8) with the risk measure is $\rho = CVaR_\alpha$ and the set induced by ρ is

$$\Delta_\rho = \left\{ z \in L^2 : 0 \leqslant z \leqslant \frac{1}{\alpha}, \mathbb{E}(z) = 1 \right\}.$$

If the risk measure is given by $\rho_1 = CVaR_{\alpha_1}$ where α_1 is small enough and $\alpha_1 \geqslant \alpha$, then the set induced by ρ_1 is

$$\Delta_{\rho_1} = \left\{ z \in L^2 : 0 \leqslant z \leqslant \frac{1}{\alpha_1}, \mathbb{E}(z) = 1 \right\}.$$

For all small enough α_1, we have $z^* \in \Delta_{\rho_1}$. It is clear that $\Delta_{\rho_1} \subseteq \Delta_\rho$ because $\alpha_1 \geqslant \alpha$. Therefore, the optimal partial hedging strategy f^* also guarantees the existence of $\tau^* > 0$ and $z^* \in \Delta_\rho$ which satisfy the optimality conditions (5.8) when the risk measure change to be $\rho_1 = CVaR_{\alpha_1}$ where α_1 is small enough and $\alpha_1 \geqslant \alpha$. Theorem 5.3.1 is proved.

□

Remark 5.3.1. *For a given confidence level $1-\alpha$, in Chapter 3 we show that the optimal partial hedging strategy which minimizes $CVaR_\alpha$ of the investor's total exposed risk among $\mathcal{L}_1$ is the bull call spread hedging strategy. Theorem 5.3.1 shows that the bull call spread hedging strategy is strongly robust with respect to the confidence level $1-\alpha$ when $1-\alpha$ is large enough. The optimal partial hedging strategy will remain unchange even if the confidence level $1-\alpha$ changes, as long as $1-\alpha$ still lies in some interval.*

Corollary 5.3.1. *Assume that for a given confidence level $1-\alpha$, there exists an optimal partial hedging strategy, denoted as f^*, which minimizes $CVaR_\alpha$ of the investor's total exposed risk. If the optimal strategy f^* is optimal for two different confidence levels $1-\alpha_1$ and $1-\alpha_2$, where $1-\alpha_1 > 1-\alpha_2$. Then the strategy f^* is optimal under $CVaR_\alpha$ risk measure, as long as $\alpha \in [\alpha_1, \alpha_2]$.*

Proof: Assume that $\alpha \in [\alpha_1, \alpha_2]$.

Since f^* is the optimal partial hedging strategy under $CVaR_{\alpha_2}$, it guarantees the existence of $\tau^* > 0$ and $z^* \in \Delta_{\rho_2}$ that satisfy the optimality conditions (5.8) with the risk measure $\rho_2 = CVaR_{\alpha_2}$ and the set induced by ρ_2 is

$$\Delta_{\rho_2} = \left\{ z \in L^2 : 0 \leqslant z \leqslant \frac{1}{\alpha_2}, \mathbb{E}(z) = 1 \right\}.$$

Note that for $\alpha \leqslant \alpha_2$, we have $\Delta_{\rho_2} \subseteq \Delta_\rho$ where Δ_ρ is the set induced by $\rho = CVaR_\alpha$

$$\Delta_{\rho} = \left\{ z \in L^2 : 0 \leqslant z \leqslant \frac{1}{\alpha}, \mathbb{E}(z) = 1 \right\}.$$

Therefore, $z^* \in \Delta_\rho$.

Similarly, for $\alpha \geqslant \alpha_1$, we have $\Delta_\rho \subseteq \Delta_{\rho_1}$ where Δ_{ρ_1} is the set induced by $\rho_1 = CVaR_{\alpha_1}$

$$\Delta_{\rho_1} = \left\{ z \in L^2 : 0 \leqslant z \leqslant \frac{1}{\alpha_1}, \mathbb{E}(z) = 1 \right\}.$$

This guarantees that $\tau^* > 0$ and $z^* \in \Delta_\rho$ satisfy the optimality conditions (5.8) with the risk measure $\rho = CVaR_\alpha$. Therefore, the strategy f^* is also optimal for CVaR risk measure at the confidence level $1 - \alpha$.

□

5.4 Optimal Partial Hedging under Spectral Risk Measures

In Chapter 3, we show that bull-call-spread hedging strategy is optimal under CVaR risk measure. In Section 5.2, we conclude that bull-call-spread hedging strategy can be optimal under general risk measures. However, bull-call-spread hedging strategy might not be optimal for all the risk measures in the class of general risk measures. It is interesting to investigate if there is a class of risk measures under which bull-call-spread hedging strategy remains optimal.

In this section, we will show that the optimality of bull-call-spread hedging strategy in Chapter 3 can be generalized to a special class of risk measures, which is a set of spectral risk measures. Before giving the formal definition of spectral risk measure, we first introduce the definition of comonotonicity, which is discussed in detail in Dhaene et al. (2002).

Definition 5.4.1. *Two random variables X_1 and X_2 are said to be comonotonic if*

$$(X_1(\omega_2) - X_1(\omega_1)) \cdot (X_2(\omega_2) - X_2(\omega_1)) \geqslant 0, \forall \omega_1, \omega_2 \in \Omega.$$

We provide the definition of spectral risk measure and admissible risk spectrum as follows. For detailed discussion on the definition of spectral risk measure and admissible risk spectrum, see Acerbi et al. (2002), Adam et al. (2008) and the references therein.

Definition 5.4.2. *A risk measure, which is a functional over the set of random loss $X \to \rho(X) \in \mathbb{R}$, is called spectral risk measure if it satisfies the following axioms*

(i) Positive Homogeneity: for every random variable X and real number $\lambda > 0$, $\rho(\lambda X) = \lambda \rho(X)$.

(ii) Translation Invariance: for every random variable X and real number c, $\rho(X + c) = \rho(X) + c$.

(iii) Monotonicity: for any random variables X_1 and X_2 such that $X_1(\omega) \geqslant X_2(\omega), \forall \omega \in \Omega$, $\rho(X_1) \geqslant \rho(X_2)$.

(iv) Subadditivity: for any random variables X_1 and X_2, $\rho(X_1 + X_2) \leqslant \rho(X_1) + \rho(X_2)$.

(v) Comonotonic Additivity: for any comonotonic random variables X_1 and X_2, $\rho(X_1 + X_2) = \rho(X_1) + \rho(X_2)$.

Remark 5.4.1. *(a) It is easy to verify that CVaR risk measure belongs to the class of spectral risk measures.*

(b) Coherent risk measures as discussed in Artzner et al. (1999) conform to the above axioms (i) - (iv). Distortion risk measures as defined in Panjer et al. (1997) fulfil the above axioms (i) - (iii) and (v).

Definition 5.4.3. *A function $\phi : [0,1] \to \mathbb{R}$ is called an admissible risk spectrum if it satisfies the following properties*

(i) Nonnegativity: $\phi(s) \geqslant 0, \forall s \in [0,1]$.

(ii) Monotonicity: $\phi(s_1) \geqslant \phi(s_2), \forall 0 \leqslant s_1 \leqslant s_2 \leqslant 1$.

(iii) Unitization: $\int_0^1 \phi(s)ds = 1$.

Based on Definition 5.4.2 and Definition 5.4.3, one can obtain the following representation of spectral risk measures. For a thorough discussion and proof of this representation, we refer to Acerbi et al. (2002).

Proposition 5.4.1. *Any spectral risk measure ρ_ϕ has the following representation*

$$\rho_\phi(X) = \int_0^1 VaR_s(X) \cdot \phi(s)ds \tag{5.9}$$

where X is the random loss, and ϕ is the admissible risk spectrum associated with the spectral risk measure ρ_ϕ.

Remark 5.4.2. *(a) This representation provides revealing insight on the structure of the spectral risk measures. Any spectral risk measure is characterized by its risk spectrum, which distorts the loss distribution. The spectral risk measure in fact is a weighted average of the quantiles of the loss distribution. The weights in some sense reflect the investor's attitude towards risk.*

(b) *This representation enables us to construct spectral risk measures by purely specifying the risk spectrums. Two important types of spectral risk measures are exponential spectral risk measures and power spectral risk measures, which are associated with exponential utility and power utility respectively. See Dowd et al. (2008) for details about these two types of spectral risk measures.*

(c) *From this representation, we can see that spectral risk measures are closed under convex combinations. This result gives us more flexibility on constructing spectral risk measures. In particular, since expectation is a spectral risk measure, a convex combination of a given spectral risk measure and expectation is still a spectral risk measure.*

Spectral risk measures form a general class of risk measures. Below are some interesting examples of spectral risk measures and their corresponding risk spectrums.

Example 5.4.1. *(i) CVaR risk measure:*

$$\phi(s) = \begin{cases} \frac{1}{\alpha}, & \text{if} \quad 0 \leqslant s \leqslant \alpha, \\ \\ 0, & \text{if} \quad \alpha < s < 1. \end{cases}$$

(ii) Exponential spectral risk measures:

$$\phi(s) = \frac{k \cdot e^{-ks}}{1 - e^{-k}}, s \in [0, 1]$$

where k is a constant parameter.

(iii) Power spectral risk measures:

$$\phi(s) = \begin{cases} \gamma \cdot s^{\gamma - 1}, & \text{if} \quad 0 < \gamma < 1, \\ \\ \gamma \cdot (1 - s)^{\gamma - 1}, & \text{if} \quad \gamma > 1, \end{cases}$$

where γ is a constant parameter.

(iv) Spectral-Mean risk measures: for any given spectral risk measure ρ_ϕ, where ϕ is its risk spectrum, define a new risk spectrum as follows

$$\phi_\lambda(s) = \lambda + (1-\lambda)\cdot\phi(s), \lambda \in [0,1]\,.$$

Then ρ_{ϕ_λ} is a new spectral risk measure, which is a convex combination of the original spectral risk measure ρ_ϕ and expectation.

We proceed with the main result of this section. The following theorem shows that the optimality of our proposed CVaR-based optimal partial hedging strategy can be generalized to a set of spectral risk measures.

Theorem 5.4.1. *Assume that for a given confidence level $1-\alpha$, there exists an optimal partial hedging strategy, denoted as f^*, which minimizes $CVaR_\alpha$ of the investor's total exposed risk. If the optimal strategy f^* is optimal for two different confidence levels $1-\alpha_1$ and $1-\alpha_2$, where $1-\alpha_1 > 1-\alpha_2$. Then the strategy f^* is optimal under the spectral risk measure $\rho_{\tilde{\phi}}$, as long as the corresponding risk spectrum $\tilde{\phi}$ is admissible and satisfies*

(i) $\tilde{\phi}(s) = \phi(\alpha_1), \forall s \in [0,\alpha_1]$.

(ii) $\tilde{\phi}(s) = 0, \forall s > \alpha_2$.

Proof: For the spectral risk measure $\rho_{\tilde{\phi}}$, according to Proposition 5.4.1, we can have the following representation

$$\rho_{\tilde{\phi}}(X) = \int_0^1 VaR_s(X)\cdot\tilde{\phi}(s)ds$$

where X is the random loss.

In the following, we will prove that the spectral risk measure $\rho_{\tilde{\phi}}$ can be expressed in terms of CVaR risk measures.

$$\begin{aligned}
\rho_{\tilde{\phi}}(X) &= \int_0^1 VaR_s(X) \cdot \tilde{\phi}(s) ds \\
&= \tilde{\phi}(\alpha_1) \int_0^{\alpha_1} VaR_s(X) ds + \int_{\alpha_1}^{\alpha_2} VaR_s(X) \cdot \tilde{\phi}(s) ds \\
&= \alpha_1 \tilde{\phi}(\alpha_1) \cdot CVaR_{\alpha_1}(X) + \int_{\alpha_1}^{\alpha_2} VaR_s(X) \cdot \tilde{\phi}(s) ds \\
&= \alpha_1 \tilde{\phi}(\alpha_1) \cdot CVaR_{\alpha_1}(X) + \int_{\alpha_1}^{\alpha_2} \tilde{\phi}(s) d \left(\int_0^s VaR_\varepsilon(X) d\varepsilon \right) \\
&= \left\{ -s\tilde{\phi}(s) \cdot CVaR_s(X) \right\} \Big|_{\alpha_1}^{\alpha_2^+} + \int_{\alpha_1}^{\alpha_2} \tilde{\phi}(s) d \left(s \cdot CVaR_s(X) \right) \\
&= \int_{\alpha_1}^{\alpha_2} CVaR_s(X) \varphi(s) ds
\end{aligned}$$

where $\varphi(s) = s(\tilde{\phi}(s) - \tilde{\phi}(s^+))$.

Since $\tilde{\phi}$ is an admissible risk spectrum, we can conclude that $\varphi(s) \geqslant 0, \forall s \in [0,1]$. From Corollary 5.3.1, we know that the strategy f^* is optimal under $CVaR_\alpha$ risk measure, as long as $\alpha \in [\alpha_1, \alpha_2]$. Therefore, it is clear that the strategy f^* is optimal under the spectral risk measure $\rho_{\tilde{\phi}}$. □

Remark 5.4.3. *For a given confidence level $1-\alpha$, in Chapter 3 we show that the optimal partial hedging strategy which minimizes $CVaR_\alpha$ of the investor's total exposed risk among $\mathcal{L}_1$ is the bull call spread hedging strategy. Theorem 5.4.1 shows that our proposed CVaR-based optimal partial hedging strategies are robust with respect to risk measures, in the sense that its optimality can be generalized to a set of spectral risk measures.*

5.5 Concluding Remark

In this chapter, we discuss the optimal partial hedging strategies that minimize investor's total exposed risk measured by the general risk measures. We provide the necessary and sufficient conditions to the optimization problem. By utilizing the necessary and sufficient conditions, we prove that some classes of partial hedging strategies can not be optimal, while the bull-call-spread hedging strategy can be. Then we study the robustness of our proposed CVaR-based optimal partial hedging strategies. We demonstrate that our proposed CVaR-based optimal partial hedging strategy is strongly robust with respect to the confidence level. Furthermore, we consider a special class of risk measures, known as the spectral risk measures. We show that the optimality of our proposed CVaR-based partial hedging strategy can be generalized to a set of spectral risk measures.

Chapter 6

Simulation-based Hedging Models

Motivated by the empirical method of Weng (2009), the focus of this chapter is to use the similar idea to establish the simulation-based model and study the optimal partial hedging strategy. Because of the flexibility of the underlying method, we can relax the assumptions on the admissible set of the hedged loss functions. We can also consider some other optimization criteria while imposing more complicated constraints on the hedging strategies. Although we will mainly focus on the CVaR minimization model in the numerical examples and the preliminary analysis, this simulation-based model can be used to study the optimal hedging strategies under other criteria and with more complicated constraints.

6.1 Simulation-based Hedging Models

6.1.1 CVaR Minimization Model

We restate the theoretical CVaR minimization model as follows

$$\begin{cases} \min\limits_{f \in \mathcal{L}} & CVaR_\alpha(T_f(X)) \\ \text{s.t.} & \Pi(f(X)) \leqslant \pi_0 \end{cases} \tag{6.1}$$

where $\mathcal{L}$ is the admissible set of the hedged part, $T_f(X) = X - f(X) + e^{rT} \cdot \Pi(f(X))$ is the total exposed risk of the investor, π_0 is the budget of hedging.

In order to make the above optimization problem numerically tractable, we use the representation of CVaR which is discussed in detail in Rockafellar and Uryasev (2000). First, we define the following auxiliary function

$$F_\alpha(x, \xi) = \xi + \frac{1}{\alpha} \cdot \mathbb{E}^P \left[(x - \xi)_+\right]. \tag{6.2}$$

The following theorem, which is quoted from Theorem 10 in Rockafellar and Uryasev (2002), shows that CVaR can be calculated via calculating the minimum value of the above auxiliary function.

Theorem 6.1.1. *As a function of $\xi \in \mathbb{R}$, function $F_\alpha(x, \xi)$ defined in* (6.2) *is finite and convex (hence continuous), and*

$$CVaR_\alpha(X) = \min_{\xi \in \mathbb{R}} F_\alpha(X, \xi)$$

Moreover, $VaR_\alpha(X)$ is the lower endpoint of $\operatorname*{argmin}\limits_{\xi \in \mathbb{R}} F_\alpha(X, \xi)$.

The above theorem not only provides an alternative way to deal with the CVaR minimization problem, but also sheds some light on fundamental reason why CVaR has more appealing properties than VaR. The reason is that the optimal value in an optimization problem is usually much better behaved than the optimizer.

It follows from Theorem 6.1.1 that the optimization problem (6.1) can be solved by minimizing the following auxiliary optimization problem

$$\begin{aligned} \min_{\{f\in\mathcal{L},\xi\in\mathbb{R}\}} & \xi + \frac{1}{\alpha}\cdot\mathbb{E}^P\left[(T_f(X)-\xi)_+\right] \\ s.t. \qquad & \Pi(f(X)) \leqslant \pi_0. \end{aligned} \tag{6.3}$$

Based on the above auxiliary optimization problem, we can now use the simulation-based approach to study the CVaR minimization model. First, we assume a market model, then simulate a series of the payoffs of the contract, namely $\tilde{\mathbf{X}} = \{X_i\}_{i=1}^N$, according to the assumed values of the parameters.[1] The simulated payoffs $\{X_i\}_{i=1}^N$ can be viewed as the possible risk the investor is exposed to. Now the problem is to determine a series $\tilde{\mathbf{f}} = \{f_i\}_{i=1}^N$, which represents the hedged part and corresponds to the risk $\{X_i\}_{i=1}^N$, to minimize CVaR of the investor's total exposed risk. More specifically, the simulation-based optimization problem can be formulated as follows

$$\begin{aligned} \min_{\{\tilde{\mathbf{f}}\in\bar{\mathcal{L}},\xi\in\mathbb{R}\}} & \xi + \frac{1}{\alpha N}\sum_{i=1}^N\left[\left(X_i - f_i + \hat{\Pi}(\tilde{\mathbf{f}}) - \xi\right)_+\right] \\ s.t. \qquad & \hat{\Pi}(\tilde{\mathbf{f}}) \leqslant \pi_0 \end{aligned} \tag{6.4}$$

where $\bar{\mathcal{L}}$, which is the admissible set of $\tilde{\mathbf{f}}$, is determined according to admissible set of the hedging strategies. And $\hat{\Pi}(\cdot)$, which is the simulation-based pricing function, is determined according to $\Pi(\cdot)$.

In order to make the above optimization problem more tractable, we still need to calculate the simulation-based price of the hedged part, $\hat{\Pi}(\tilde{\mathbf{f}})$. It is well known that one

[1]The calibration of the parameters is not in the scope of this chapter, though the calibration is not trivial.

of the most common pricing methods is to discount the expected value of the payoff of the financial contract under the risk-neutral measure. However, if we use the method of risk-neutral pricing, two probability measures will be involved in the above optimization problem, which will make the problem much more complicated. Therefore, we will use the method of pricing kernel, which enables us to deal with the optimization problem merely in the physical probability measure, in the calculation of the simulation-based price of the hedged part, $\hat{\Pi}(\tilde{\mathbf{f}})$. Detailed discussion on pricing kernel has been provided in subsection 5.2.2. Under the simulation-based model, we denote the pricing kernel as $\tilde{\phi} = \{\phi_i\}_{i=1}^N$, which is one to one correspondence to the simulated scenarios. By introducing the pricing kernel, the optimization problem (6.4) can be formulated as the following optimization problem

$$\begin{aligned} &\min_{\{\tilde{\mathbf{f}}\in\tilde{\mathcal{L}},\xi\in\mathbb{R}\}} \xi + \frac{1}{\alpha N}\sum_{i=1}^{N}\left[\left(X_i - f_i + \frac{1}{N}\sum_{i=1}^{N}\phi_i f_i - \xi\right)_+\right] \\ &s.t. \qquad \frac{1}{N}\sum_{i=1}^{N}\phi_i f_i \leqslant \pi_0. \end{aligned} \tag{6.5}$$

6.1.2 Expected Shortfall Minimization Model

Expected shortfall hedging strategy has been discussed in detail in Föllmer and Leukert (2000). Here, we show that the expected shortfall hedging strategy can be studied with our simulation-based model as well. First, we restate the theoretical expected shortfall minimization model as follows

$$\begin{aligned} &\min_{f\in\mathcal{L}} \mathbb{E}^P\left[(X - f(X) + \Pi(f(X)))_+\right] \\ &s.t. \qquad \Pi(f(X)) \leqslant \pi_0 \end{aligned} \tag{6.6}$$

where $\mathcal{L}$ is the admissible set of the hedged part, f is the hedged loss function, π_0 is the budget of hedging. Here, the optimization problem (6.6) is similar to but different from that in Föllmer and Leukert (2000). We consider the cost of hedging as part of the risk in the objective function.

Similar to the CVaR minimization model in Subsection 6.1.1, we first simulate a series of the payoffs of the contract, namely $\tilde{\mathbf{X}} = \{X_i\}_{i=1}^N$, according to the assumed model with assumed values of the parameters. Then with the simulated payoffs $\{X_i\}_{i=1}^N$, we can obtain the optimal hedged part $\tilde{\mathbf{f}} = \{f_i\}_{i=1}^N$, which corresponds to the payoffs $\{X_i\}_{i=1}^N$, to minimize the investor's expected shortfall. More specifically, the simulation-based optimization problem can be formulated as follows

$$\begin{aligned} &\min_{\{\tilde{\mathbf{f}} \in \bar{\mathcal{L}}\}} \frac{1}{N} \sum_{i=1}^N u_i \\ s.t. \qquad & \hat{\Pi}(\tilde{\mathbf{f}}) \leqslant \pi_0 \\ & u_i \geqslant X_i - f_i + \hat{\Pi}(\tilde{\mathbf{f}}) \\ & u_i \geqslant 0 \end{aligned} \tag{6.7}$$

where $\bar{\mathcal{L}}$, which is the admissible set of $\tilde{\mathbf{f}}$, is determined according to admissible set of the hedging strategies. $\hat{\Pi}(\cdot)$, which is the simulation-based pricing function, is determined according to $\Pi(\cdot)$.

Similar to Subsection 6.1.1, by using the pricing kernel $\tilde{\phi} = \{\phi_i\}_{i=1}^N$, the optimization problem (6.7) can be formulated as the following optimization problem

$$\begin{aligned} &\min_{\{\tilde{\mathbf{f}} \in \bar{\mathcal{L}}\}} \frac{1}{N} \sum_{i=1}^N u_i \\ s.t. \qquad & \frac{1}{N} \sum_{i=1}^N \phi_i f_i \leqslant \pi_0 \\ & u_i \geqslant X_i - f_i + \frac{1}{N} \sum_{i=1}^N \phi_i f_i \\ & u_i \geqslant 0 \end{aligned} \tag{6.8}$$

6.1.3 CVaR Minimization under Expected Shortfall Constraint

In Subsection 6.1.1 and Subsection 6.1.2, we discussed the CVaR minimization model and expected shortfall minimization model with our simulation-based approach. Although both models can be solved analytically under some mild assumptions, it might not be the case for the CVaR minimization model that is with expected shortfall constraint. On the contrary, our simulation-based model has much more flexibility on extra constraints. As to be shown in the following, adding the expected shortfall constraint in the CVaR minimization model would not increase the complexity of our simulation-based model.

First, we state the CVaR minimization model with expected shortfall constraint as follows

$$
\begin{aligned}
& \min_{f \in \mathcal{L}} CVaR_{\alpha}(T_f(X)) \\
s.t. \quad & \Pi(f(X)) \leqslant \pi_0 \\
& \mathbb{E}^P\left[(X - f(X) + \Pi(f(X)))_+\right] \leqslant S
\end{aligned}
\tag{6.9}
$$

where $\mathcal{L}$ is the admissible set of the hedged part, f is the hedged loss function, π_0 is the budget of hedging, S is a predetermined constant reflecting the tolerance of the investor with respect to the expected shortfall.

Similar to the CVaR minimization model in Subsection 6.1.1, we first simulate a series of the payoffs of the contract,$\tilde{\mathbf{X}} = \{X_i\}_{i=1}^N$. Then we can try to find the optimal hedged part $\tilde{\mathbf{f}} = \{f_i\}_{i=1}^N$, which corresponds to the payoffs $\{X_i\}_{i=1}^N$, to minimize CVaR of the investor's total exposed risk under the constraint that the expected shortfall does not exceed a predetermined level. More specifically, the simulation-based optimization problem can be formulated as follows

$$
\begin{aligned}
&\min_{\{\tilde{\mathbf{f}}\in\bar{\mathcal{L}},\xi\in\mathbb{R}\}} \xi + \frac{1}{\alpha N}\sum_{i=1}^{N}\left[\left(X_i - f_i + \hat{\Pi}(\tilde{\mathbf{f}}) - \xi\right)_+\right] \\
s.t.\quad & \hat{\Pi}(\tilde{\mathbf{f}}) \leqslant \pi_0 \\
& u_i \geqslant X_i - f_i + \hat{\Pi}(\tilde{\mathbf{f}}) \\
& u_i \geqslant 0 \\
& \frac{1}{N}\sum_{i=1}^{N} u_i \leqslant S
\end{aligned}
\tag{6.10}
$$

where $\bar{\mathcal{L}}$, which is the admissible set of $\tilde{\mathbf{f}}$, is determined according to admissible set of the hedging strategies. $\hat{\Pi}(\cdot)$, which is the simulation-based pricing function, is determined according to $\Pi(\cdot)$.

Similar to Subsection 6.1.1, by using the pricing kernel $\tilde{\phi} = \{\phi_i\}_{i=1}^{N}$, the optimization problem (6.10) can be formulated as the following optimization problem

$$
\begin{aligned}
&\min_{\{\tilde{\mathbf{f}}\in\bar{\mathcal{L}},\xi\in\mathbb{R}\}} \xi + \frac{1}{\alpha N}\sum_{i=1}^{N}\left[\left(X_i - f_i + \frac{1}{N}\sum_{i=1}^{N}\phi_i f_i - \xi\right)_+\right] \\
s.t.\quad & \frac{1}{N}\sum_{i=1}^{N}\phi_i f_i \leqslant \pi_0 \\
& u_i \geqslant X_i - f_i + \frac{1}{N}\sum_{i=1}^{N}\phi_i f_i \\
& u_i \geqslant 0 \\
& \frac{1}{N}\sum_{i=1}^{N} u_i \leqslant S
\end{aligned}
\tag{6.11}
$$

Remark 6.1.1. *This subsection gives an example of incorporating other optimality criteria into a optimization problem as constraints. Besides of expected shortfall, we can consider* *other constraints, such as expected return, maximum loss, specific hedging constraints, and so forth.*

6.2 Numerical Examples

In order to solve the optimization problem (6.5), we still need to specify the market model and the admissible set of the hedged part. In this section, we assume that the market model is Black-Scholes model, and we will consider two different admissible sets of the hedged part.

In the standard Black-Scholes model, the underlying price process is described by a geometric Brownian Motion

$$dS_t = S_t(\sigma dW_t + \mu dt) \tag{6.12}$$

with initial value $S_0 = s_0$, where W is a Wiener process under the physical measure P, σ and μ are constants.

It can be easily shown that the Radon-Nykodym derivative at time t in the above standard Black-Scholes model is

$$D_t = \exp\left\{-\frac{1}{2}\lambda^2 t - \lambda W_t\right\},$$

where r is the risk-free rate and $\lambda = \frac{\mu - r}{\sigma}$ is the Sharp ratio of the underlying asset.

If we define the pricing kernel at time t as follows

$$\phi^{(t)} = \exp\left\{-\left(r + \frac{1}{2}\lambda^2\right)t - \lambda W_t\right\}, \tag{6.13}$$

then pricing path-independent financial contracts with the above pricing kernel is equivalent to using the method of risk-neutral pricing.

6.2.1 Nonnegative Constraint on the Hedged Part

In this part, we impose a nonnegative constraint on the hedged part, which means that the payoff of the hedging portfolio is nonnegative. In this case, the admissible set of the hedged part in the simulation-based model will be $\bar{\mathcal{L}}_1 = \{\tilde{\mathbf{f}} = \{f_i\}_{i=1}^N | f_i \geqslant 0, i = 1, 2, \ldots, N\}$. Thus, for $\mathbf{z} = \{z_i\}_{i=1}^N$, the optimization problem (6.5) can be rewritten as follows

$$\begin{aligned} \min_{\{\tilde{\mathbf{f}} \in \bar{\mathcal{L}}_1, \mathbf{z} \in \mathbb{R}^N, \xi \in \mathbb{R}\}} & \xi + \frac{1}{\alpha N} \sum_{i=1}^N z_i \\ s.t. \qquad & \frac{1}{N} \sum_{i=1}^N \phi_i f_i \leqslant \pi_0 \\ & z_i \geqslant X_i - f_i + \frac{1}{N} \sum_{i=1}^N \phi_i f_i - \xi, \quad i = 1, 2, \ldots, N. \\ & z_i \geqslant 0, \quad i = 1, 2, \ldots, N. \\ & f_i \geqslant 0, \quad i = 1, 2, \ldots, N. \end{aligned} \tag{6.14}$$

It is clear that the above optimization problem is just a linear programming problem. Now we will discuss some concrete examples.

Example 6.2.1. *We assume that the market model is Black-Scholes model, with parameter values* $r = 0$*,* $S_0 = 100$*,* $\sigma = 0.3$ *and* $\mu = 0.08$*. We wish to partially hedge the European call option with strike price* $K = 100$ *and maturity* $T = 0.25$*. Therefore, the payoff we will be hedging is* $X = (S_T - K)_+$*. We are aiming at minimizing CVaR of the investor's total exposed risk at the confidence level* 99%*, which means that* $\alpha = 0.01$*.*

According to the Black-Scholes formula, the price of this European call option P_C is 5.9785. Therefore, the investor has to spend 5.9785 if he wants to hedge this European call option perfectly. Now we assume that the investor is only willing to spend 1 on the hedging portfolio (i.e. $\pi_0 = 1$), which means that perfectly hedging is not possible.

In order to apply the simulation-based approach to find the optimal hedging strategy, we first simulate a series of stock prices according to the assumed underlying price process (6.12). Then we can obtain the series of the payoffs of the contract, i.e. the European call option. Once we get the simulated payoffs of the contract, we can compute the optimal hedged part by solving the linear programming problem (6.14).

Figure 6.1 shows the result of this numerical example. The number of simulations is $N = 3000$.

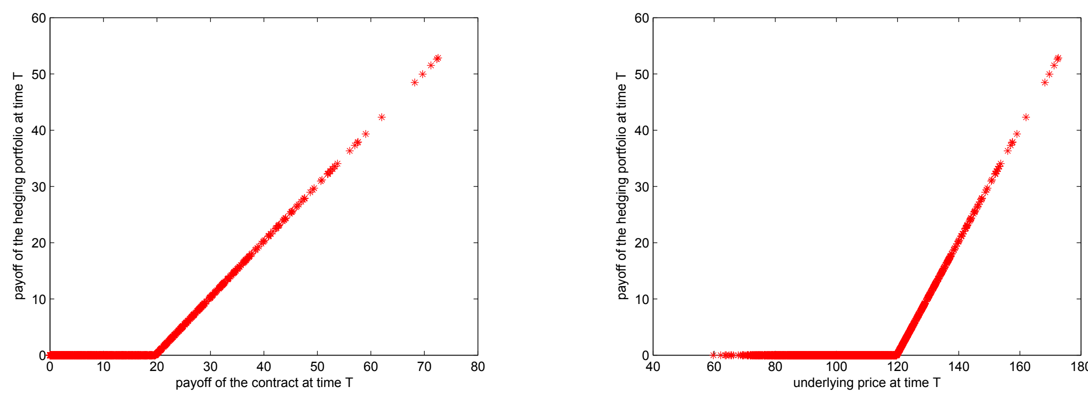

Figure 6.1: Left panel is the payoff of the hedging portfolio and the contract, right panel is the payoff of the hedging portfolio and underlying price. (With nonnegative constraint and $\pi_0 = 1$.)

We can change the hedging budget π_0 to see how the optimal hedging strategy change. Figures 6.2 - 6.6 show the optimal hedging strategies under various hedging budgets.

According to these figures, the optimal hedging strategy is to buy a call option on the contract that you are partially hedging, which is consistent with the theoretical result in Chapter 3 though we needed to impose more assumptions in the theoretical derivation. In

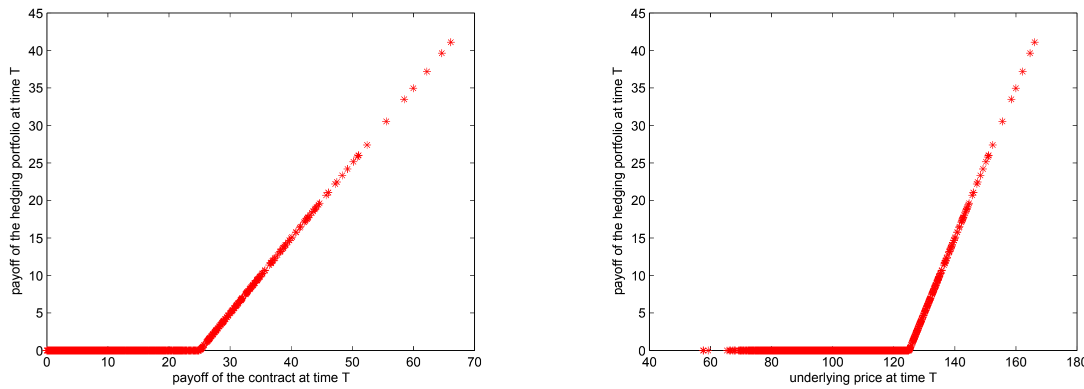

Figure 6.2: Left panel is the payoff of the hedging portfolio and the contract, right panel is the payoff of the hedging portfolio and underlying price. (With nonnegative constraint and $\pi_0 = 0.5$.)

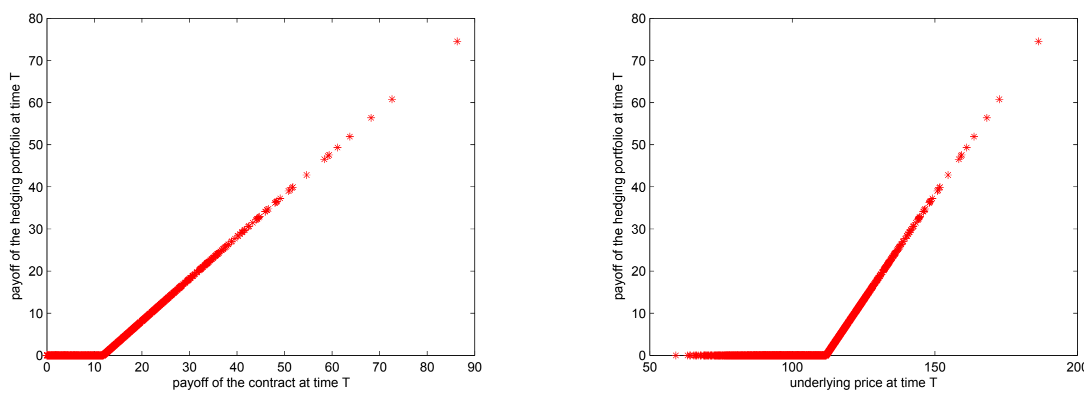

Figure 6.3: left panel is the payoff of the hedging portfolio and the contract, right panel is the payoff of the hedging portfolio and underlying price. (With nonnegative constraint and $\pi_0 = 2$.)

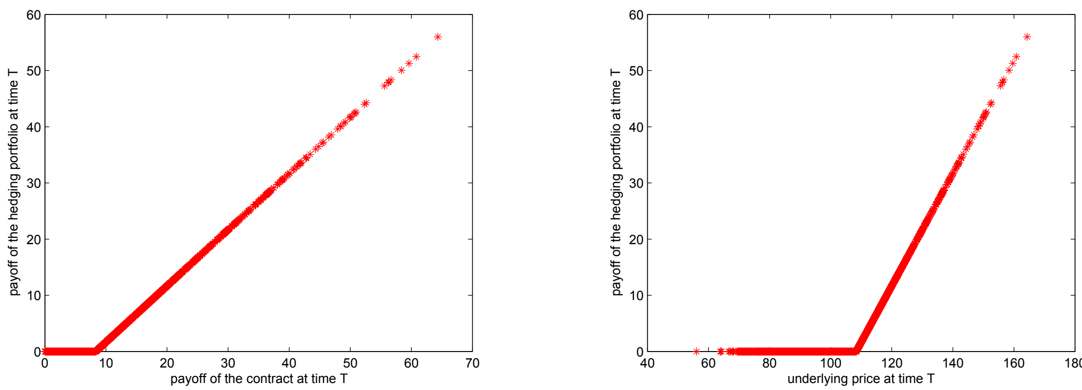

Figure 6.4: left panel is the payoff of the hedging portfolio and the contract, right panel is the payoff of the hedging portfolio and underlying price. (With nonnegative constraint and $\pi_0 = 3$.)

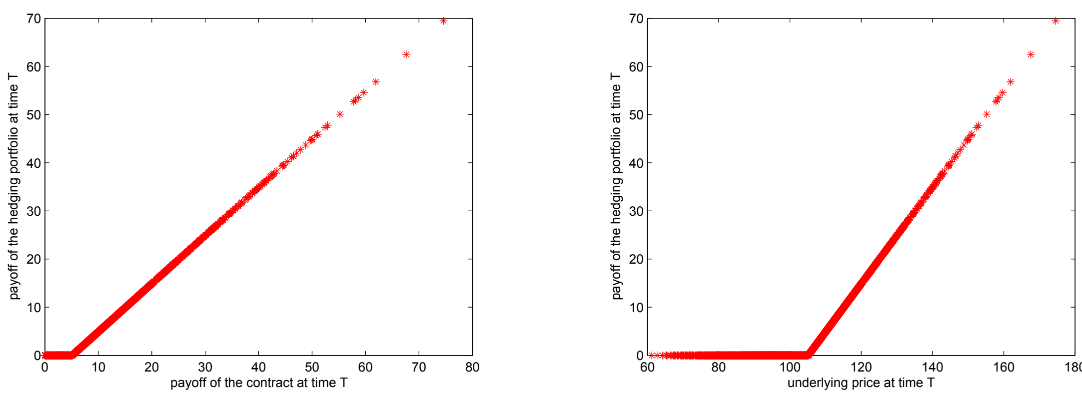

Figure 6.5: left panel is the payoff of the hedging portfolio and the contract, right panel is the payoff of the hedging portfolio and underlying price. (With nonnegative constraint and $\pi_0 = 4$.)

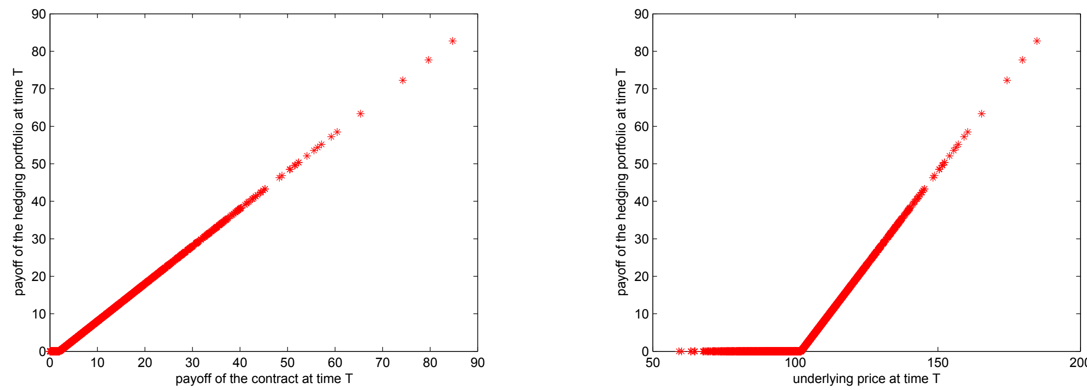

Figure 6.6: left panel is the payoff of the hedging portfolio and the contract, right panel is the payoff of the hedging portfolio and underlying price. (With nonnegative constraint and $\pi_0 = 5$.)

this example, the optimal hedging strategy is to buy a European call option with the same maturity and higher strike price. Furthermore, we can see that the shape of the optimal hedging strategy remains the same as the hedging budget changes. The larger the hedging budget is, the less risk the investor will retain.

In addition to looking at the impact of the hedging budget on the hedging portfolio, we now examine the effect of the confidence level. In particular, we change the confidence level from 99% to 95% so that α becomes 0.05. With the new confidence level, we recalculate the above numerical examples. The optimal partial hedging strategies remain unchanged in all scenarios. Even if the confidence level changes to 90%, the optimal partial hedging strategies still remain unchange in all scenarios. These phenomenons in fact are not surprising as they coincide with the results we obtained in Chapter 5 (see Theorem 5.3.1 and Corollary 5.3.1).

6.2.2 No Nonnegative Constraint on the Hedged Part

In this subsection, we remove the nonnegative constraint on the hedged part, which means that the admissible set of the hedged part in the simulation-based model will be on universal set. The hedging strategy is only constrained by the hedging budget. Once the nonnegative constraint is removed, the hedging strategy is allowed to be short-selling some contracts. In order to preclude the extreme gambling strategy, which is undesirable, we assume that the cost of hedging strategy is nonnegative. In this case, if we denote $\mathbf{z} = \{z_i\}_{i=1}^N$, then the optimization problem (6.5) can be rewritten as follows

$$\begin{aligned}
&\min_{\{\tilde{\mathbf{f}}\in\mathbb{R}^N,\mathbf{z}\in\mathbb{R}^N,\xi\in\mathbb{R}\}} \xi + \frac{1}{\alpha N}\sum_{i=1}^N z_i \\
&s.t. \qquad \frac{1}{N}\sum_{i=1}^N \phi_i f_i = \pi_0 \\
&\qquad\qquad \frac{1}{N}\sum_{i=1}^N \phi_i f_i \geqslant 0 \\
&\qquad\qquad z_i \geqslant X_i - f_i + \frac{1}{N}\sum_{i=1}^N \phi_i f_i - \xi, \quad i = 1, 2, \ldots, N. \\
&\qquad\qquad z_i \geqslant 0, \quad i = 1, 2, \ldots, N.
\end{aligned} \tag{6.15}$$

It is clear that the above optimization problem is just a linear programming problem. Now we will discuss some concrete examples.

Example 6.2.2. *We assume that the market model is the same as that in Example 6.2.1. Specifically, the market model is Black-Scholes model, with parameter values* $r = 0$, $S_0 = 100$, $\sigma = 0.3$ *and* $\mu = 0.08$. *We wish to partially hedge the same European call option as that in Example 6.2.1, which is with strike price* $K = 100$ *and maturity* $T = 0.25$.

According to the Black-Scholes formula, the price of this European call option P_C is 5.9785. Following the same steps as those in Example 6.2.1, we can solve the optimal hedging strategies under various hedging budgets. Figures 6.7 - 6.12 show the results in detail.

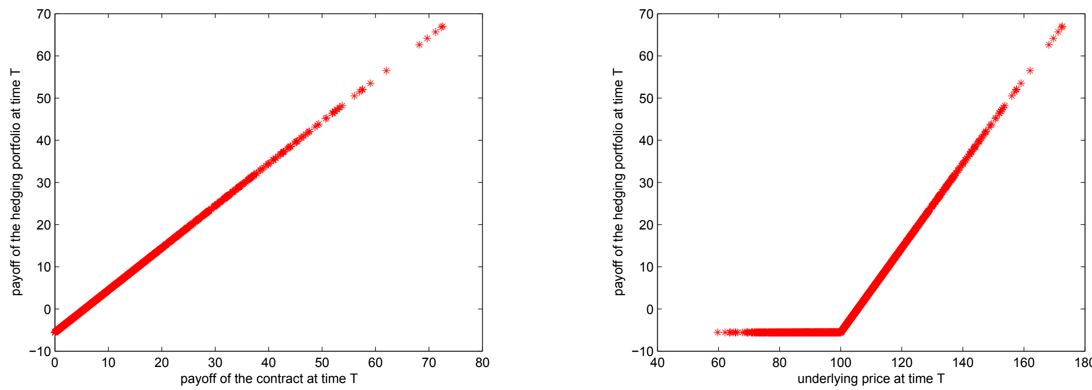

Figure 6.7: Left panel is the payoff of the hedging portfolio and the contract, right panel is the payoff of the hedging portfolio and underlying price. (Without nonnegative constraint and $\pi_0 = 0.5$.)

From these figures, we can see that the optimal partial hedging strategy, if there is no nonnegative constraint on the hedging strategy, is to replicate the exact shape of the original contract. Furthermore, the higher the hedging budget, the smaller the gap between the payoff of the hedging portfolio and the original contract.

Without the nonnegative constraint on the hedging strategy, the resulting optimal hedging strategy ensures that the investor attains a minimal CVaR of the total exposed risk. The downside of using such a hedging strategy is that the investor will suffer a constant loss in all scenarios.

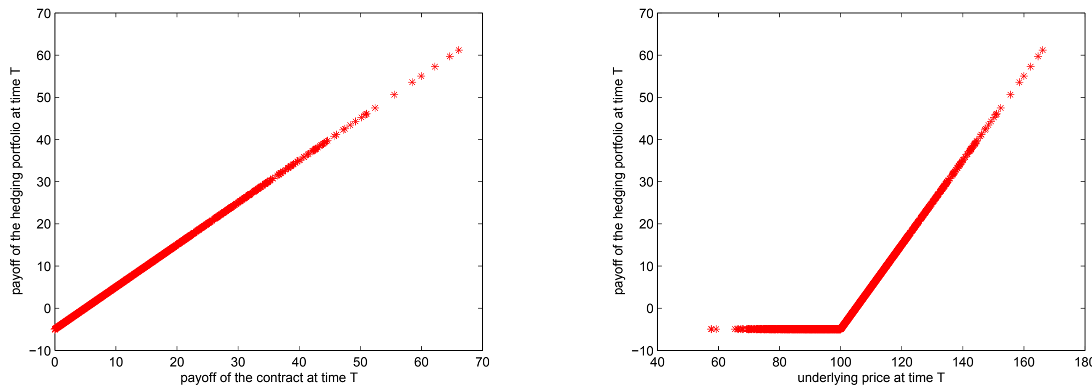

Figure 6.8: Left panel is the payoff of the hedging portfolio and the contract, right panel is the payoff of the hedging portfolio and underlying price. (Without nonnegative constraint and $\pi_0 = 1$.)

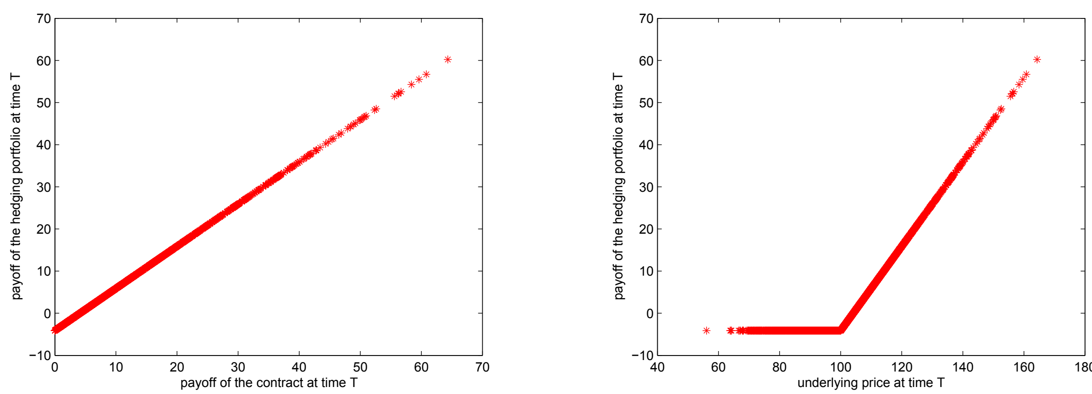

Figure 6.9: Left panel is the payoff of the hedging portfolio and the contract, right panel is the payoff of the hedging portfolio and underlying price. (Without nonnegative constraint and $\pi_0 = 2$.)

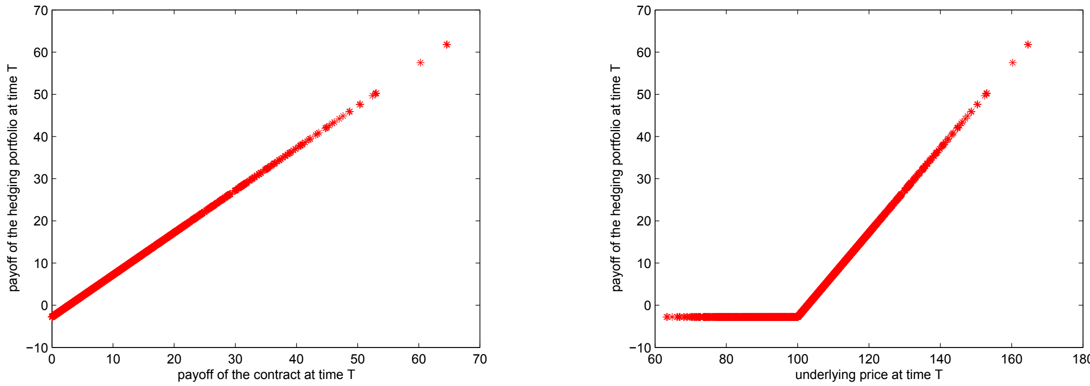

Figure 6.10: Left panel is the payoff of the hedging portfolio and the contract, right panel is the payoff of the hedging portfolio and underlying price. (Without nonnegative constraint and $\pi_0 = 3$.)

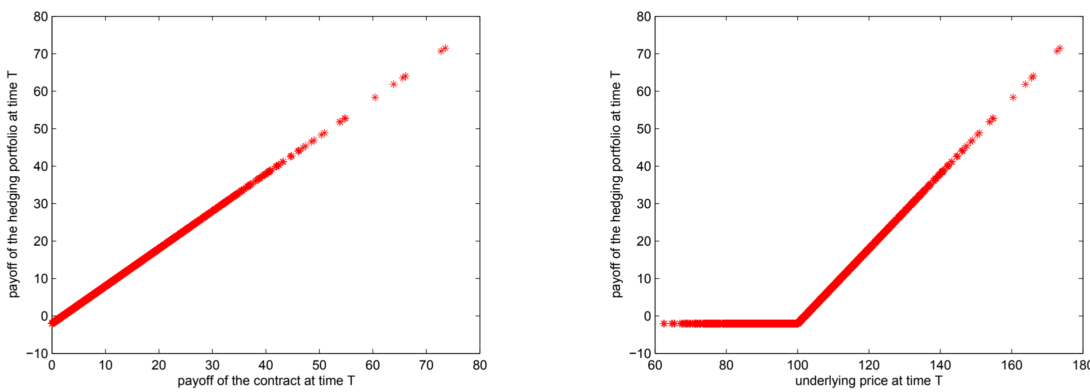

Figure 6.11: Left panel is the payoff of the hedging portfolio and the contract, right panel is the payoff of the hedging portfolio and underlying price. (Without nonnegative constraint and $\pi_0 = 4$.)

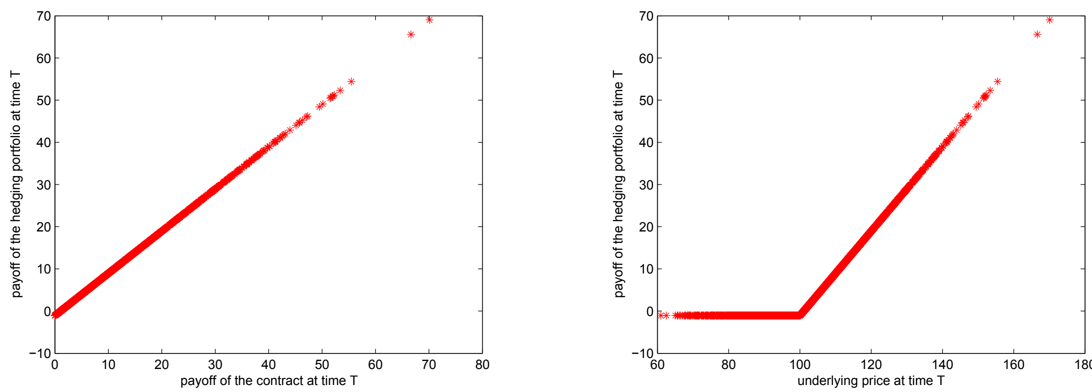

Figure 6.12: Left panel is the payoff of the hedging portfolio and the contract, right panel is the payoff of the hedging portfolio and underlying price. (Without nonnegative constraint and $\pi_0 = 5$.)

6.3 Convergence Analysis

In Section 6.2, we calculated the optimal partial hedging strategies with the simulation-based approach under some numerical examples. In this section, we will conduct some preliminary analysis on the solutions of the simulation-based model. We will use the theoretical results as benchmark to investigate the convergence of the solution from two perspectives. First of all, we are interested in the stability of the functional form of the partial hedging strategies. We will test if the solution to the simulation-based model will always generate the same functional form of partial hedging strategies. Then we will examine the convergence of the parameters. Even with the stability of the functional form, we can not conclude that solutions to the simulation-based model are reliable, since the parameters might not converge well. Therefore, it is important to examine the parameters as well.

As it is difficult to analyze the convergence of the simulation-based model theoretically, here we propose to address it numerically. In particular, we will apply the simulation-based model to the example we previously considered in Example 3.4.3 of Chapter 3. Recall that in this example, both our proposed CVaR-based partial hedging strategy and the CVaR hedging strategy of Melnikov and Smirnov (2012) were calculated numerically. Hence these results can be used as benchmark against the results from the simulation-based model. For convenience, we restate the specifications of Example 3.4.3 as follows

Example 6.3.1. *As in Example 3.4.3, we consider the Black-Scholes model with the dynamics of the stock price S_t at time t given by*

$$dS_t = S_t(\sigma dW_t + mdt)$$

where W is a Wiener process under the physical probability measure $\mathbb{P}$, σ and m are, respectively, the constant volatility and the return rate. We are interested in partial hedging a European call option with parameter values

$$T = 0.25, \;\; K = 110, \;\; r = 0.05, \;\; S_0 = 100, \;\; m = 0.06, \;\; \sigma = 0.3, \;\; \pi_0 = 1.$$

In Example 3.4.3, we show that both our proposed CVaR-based partial hedging strategy and the CVaR hedging strategy of Melnikov and Smirnov (2012) are to construct an European call option with strike 120.45, i.e. with the hedged loss function $(S_T - 120.45)_+$.

As we observed in the numerical examples presented in Section 6.2, if there is nonnegative constraint imposed on the hedged loss functions, the scatter plot of the simulation-based optimal partial hedging strategy is very close to its theoretical counterpart. So we consider to fit the simulation-based solutions to the following form of hedged loss function: $f(x) = k(x - d)_+$. Parameters k and d are determined by fitting to the optimal solutions of our simulation-based model. If our simulation-based optimal partial hedging strategy converges to the theoretical results, in this example we will see that the fitted values of k and d converge to 1 and 10.45 respectively.

Our analysis is done through the following five steps.

- Step 1: We simulate a series of stock prices according to the assumed underlying price process with the given parameters. We then obtain a series of payoffs of the contract, i.e. the payoffs of the European call option, denoted as $\{X_i\}_{i=1}^N$, where N is the number of simulations.

- Step 2: With the simulated payoffs of the contract, $\{X_i\}_{i=1}^N$, we compute the simulation-based optimal hedged part by solving the linear programming problem (6.14). Denote the optimal solution as $\{X_i, f_i^*\}_{i=1}^N$, where N is the number of simulations.

- Step 3: We fit the hedged loss function $f(x) = k(x-d)_+$ to the simulation-based solutions we got in Step 2, $\{X_i, f_i^*\}_{i=1}^N$, by using the ordinary least squares method. Denote the fitted values of k and d as $\hat{k}$ and $\hat{d}$ respectively.

- Step 4: Based on the fitted $\hat{k}$ and $\hat{d}$ in Step 3, we calculate the estimated hedged loss function $\hat{f}(x) = \hat{k}(x-\hat{d})_+$. We define the maximal discrepancy between our simulation-based solution and its implied hedged loss function as $\max_{1\leqslant i\leqslant N} |\hat{f}(X_i) - f_i^*|$.

- Step 5: We repeat Step 1 - Step 4 1000 times independently to estimate the mean and standard deviation of $\hat{k}$, $\hat{d}$ and the maximal discrepancy defined in Step 4.

Table 6.1 provides some convergence results on our simulation-based solutions. Column 1 shows the number of simulations we use for the simulation-based model. Gradually increasing the number of simulations from $N = 300$ to $N = 2000$ provides some indications of the convergence of the results from the simulation-based model. Column 2 and Column

3 are the average values of the fitted parameters $\hat{k}$ and $\hat{d}$ respectively. Column 4 provides the average values of maximal discrepancies between our simulation-based solution and its implied hedged loss function. The respective standard deviations are given in the parentheses.

No. of simulations	k	d	max discrepancy
300	1.00 (≈ 0)	10.40 (2.11)	$7.40 \times 10^{-10}(\approx 0)$
500	1.00 (≈ 0)	10.44 (1.62)	$3.36 \times 10^{-10}(\approx 0)$
700	1.00 (≈ 0)	10.47 (1.42)	$2.97 \times 10^{-10}(\approx 0)$
1000	1.00 (≈ 0)	10.46 (1.17)	$9.28 \times 10^{-11}(\approx 0)$
2000	1.00 (≈ 0)	10.44 (0.81)	$9.13 \times 10^{-11}(\approx 0)$

Table 6.1: Convergence results on simulation-based solutions, based on 1000 independent repetitions. (theoretical values: $k = 1$, $d = 10.45$.)

Remark 6.3.1. *(a) Table 6.1 shows that our simulation-based partial hedging strategies are very stable in this example. The maximal discrepancy between our simulation-based solution and its implied hedged loss function is very insignificant, even when the number of simulations is small. Moreover, the fitted value $\hat{k}$ is estimated as 1 with a almost zero standard deviation. This indicates that the simulation-based solution consistently yield the same structure of hedging strategy.*

(b) The fitted value $\hat{d}$ highly agrees with the analytical results, even when the number of simulations is small. However, it is relatively more sensitive to the number of simulations. As the number of simulations increases, although the average value of the fitted value $\hat{d}$ does not change a lot, the standard deviation drops substantially.

(c) *From Table 6.1, we can see that our simulation-based partial hedging strategy is consistent with the analytical results. This can be concluded by the observation that our simulation-based solution consistently yields the same structure of hedging strategy as that inferred by the analytical result, and the fitted value $\hat{d}$ agrees with its analytical counterpart.*

6.4 Concluding Remark

In this chapter, we propose a new numerical approach, simulation-based approach, to address the problem of optimal partial hedging. Under the example of CVaR minimization, expected shortfall minimization and CVaR minimization with expected shortfall constraint, we demonstrate how to formulate the corresponding simulation-based models. Then we calculate the simulation-based optimal CVaR hedging strategies under some numerical examples. Some preliminary analyses on the convergence of the simulation-based solutions are also conducted.

There are several advantages of our simulation-based model. Firstly, the simulation-based model is very intuitive and easy to understand. Secondly, the simulation-based model is very flexible. It allows much more flexibility in the optimality objective as well as the constraints. If the objective is complicated or there are some complicated constraints, many partial hedging problems would become too mathematically challenging to be solved analytically. However, they may still be handled in simulation-based model. Thirdly, the simulation-based model can be very efficient. In many cases, the optimal partial hedging problem can be formulated as a linear programming problem in the simulation-based model.

Albeit the aforementioned advantages, it is important to point out the limitations. One limitation lies in determining the the simulation-based pricing function, $\hat{\Pi}(\cdot)$. In Black-

Scholes model, with given parameters we can derive the simulation-based pricing function. However, in other models, it is not clear that how to specify $\hat{\Pi}(\cdot)$ and it demands further research. Another limitation is that large number of simulations has to be run in order to get an accurate solution. That would cause a lot of computational burden.

Chapter 7

Concluding Remarks and Future Research

In this thesis, we establish a risk measure based optimal partial hedging model. First, we confine our analysis to VaR risk measure. Under some mild technical assumptions, we explicitly derive the optimal partial hedging strategies which minimize VaR of the investor's total exposed risk. The knock-out call hedging and the bull call spread hedging strategy are shown to be optimal among two admissible sets of hedging strategies. Considering that VaR risk measure has some undesirable properties (such as lack of coherence), we study the CVaR minimization model and show that bull call spread hedging strategy is optimal among a given admissible set of hedging strategies. As a generalization, we utilize the similar methods to address the optimal reinsurance problem under the monotonic piecewise premium principles. We explicitly derive the VaR-based optimal reinsurance strategies under various constraints. The truncated stop loss reinsurance and limited stop loss reinsurance are proved to be optimal among two admissible sets of reinsurance treaties. Then, we consider the optimal partial hedging strategies under general risk measures. Necessary and sufficient conditions of the optimal strategies under general risk measures have been derived. With the results in the optimal partial hedging strategies under general risk measure, we investigate the robustness of our proposed CVaR-based optimal partial hedging strategies. We show that our proposed CVaR-based optimal partial hedging strategies are robust with respect to the market structure and risk measures. We also generalize the optimality of our proposed CVaR-based partial hedging strategy to a set of spectral risk measures. Furthermore, the optimal strategies are considered by using the simulation-based approach. The simulation-based approach is very intuitive and enables us to numerically obtain the optimal partial hedging strategies under various constraints. In some examples, the optimal strategies under the simulation-based model is calculated. The simulation-based solutions is consistent with the analytical results. Some preliminary analyses on the convergence of the simulation-based solutions are also conducted.

The potential research topics that can be explored in future are listed as follows

- Some researchers have generalized the idea of quantile hedging to some specified financial and insurance contracts, such as equity-linked insurance contracts, guaranteed minimum death benefits contracts, defaultable securities, etc. See Sekine (2000), Melnikov and Skornyakova (2005), Wang (2009), Klusik and Palmowski (2011) as examples. However, few literature considers to partially hedge these contracts in order to minimize the investor's total exposed risk measured by some risk measure. The optimal partial hedging strategies we derive in this thesis probably are more applicable for the ordinary financial contracts, such as options and futures. It is interesting to generalize our proposed partial hedging strategies to some other specified financial and insurance contracts.

- Reconsider our proposed partial hedging models with additional constraints. In reality, investors may not be able to construct his desired hedging strategy, due to the lack of some certain financial instruments. It can be reflected by imposing additional constraints in the optimal hedging models. For example, if the investor can only use European options to construct his hedging portfolio, then it is equivalent to restricting the hedged loss functions to be of the form as $\sum_{i=1}^{n_1} a_i(S_T - \hat{K}_i)_+ + \sum_{j=1}^{n_2} b_i(\tilde{K}_j - S_T)_+$. It will be interesting to study the optimal partial hedging strategies in such cases.

- Extend the results for VaR-based optimal reinsurance strategy to other optimality criteria, such as minimizing CVaR risk measure. In Chapter 4, the ideas and approaches in Chapter 2 are applied to study the optimal reinsurance strategies. Therefore, we may apply the ideas and approaches in Chapter 3 in the context of reinsurance to generalize the results in Chapter 4 to CVaR risk measure.

- The risk measures can be divided into several classes, such as expectation bounded risk measures, deviation risk measures and so on. In Chapter 5, necessary and sufficient conditions of the optimal partial hedging strategies under general risk measures have been derived. Robust optimal partial hedging strategies under a set of spectral risk measures have been obtained. It will be of interest to study the robust optimal partial hedging strategies under some other classes of risk measures.

- As pointed out in Section 6.4, one of the limitations of our simulation-based model lies in the determination of the simulation-based pricing function, $\hat{\Pi}(\cdot)$. This problem relates to the determination of pricing kernel. It will be of interest to study the pricing kernel in some other models. And it is also interesting to investigate the feasibility of specifying a market model via specifying a pricing kernel.

- Much work has been done in the field of robust portfolio optimization. Some interesting references are El Ghaoui et al. (2003), Zhu and Fukushima (2009), Huang et at. (2010) and the references therein. They assume that the investor does not know the exact values of the parameters, and then calculate the robust optimal portfolio selections under various criteria, such as minimizing the worse-case VaR, minimizing the worse-case CVaR, minimizing the relative CVaR, etc. In the simulation-based hedging model, similar ideas can be applied to study the robust optimal strategies which minimize the worst-case risk or the relative risk when the investor does not have complete information. This will lead to another type of robustness.

Bibliography

[1] Acerbi, C., 2002. Spectral Meausres of Risk: A Coherent Representation of Subjective Risk Aversion. *Journal of Banking and Finance*, 26(7):1505-1518.

[2] Adam, A., Houkari, M. and Laurent, J.-P., 2008. Spectral Risk Meausres and Portfolio Selection. *Journal of Banking and Finance*, 32(9):1870-1882.

[3] Alexander, S., Coleman, T.F. and Li, Y., 2004a. Minimizing CVaR and VaR for a portfolio of derivatives. *Journal of Banking and Finance*, 30(2):583-605.

[4] Alexander, S., Coleman, T.F. and Li, Y., 2004b. Derivative portfolio hedging based on CVaR. *Risk Measures for the 21st Century*, John Wiley & Sons, 2004, 339-363.

[5] Arrow, K.J., 1963. Uncertainty and the welfare economics of medical care. *American Economic Review*, 53(5): 941-973.

[6] Artzner, P., Delbaen, F., Eber, J.M. and Heath, D., 1999. Coherent measures of risk. *Mathematical finance*, 9(3):203-228.

[7] Asmit, A.V., Badescu, A.M. and Verdonck, T., 2013. Optimal risk transfer under quantile-based risk measures. *Insurance: Mathematics and Economics*, 53(1):252-265.

[8] Balbás, A., Balbás, B. and Heras, A., 2009. Optimal reinsurance with general risk measures. *Insurance: Mathematics and Economics*, 44(3):374-384.

[9] Borch, K., 1963. An attempt to determine the optimum amount of stop loss reinsurance. *In: Transactions of the 16th International Congress of Actuaries*, vol. I, 597 - 610.

[10] Bratyk, M. and Mishura, Y., 2008. The generalization of the quantile hedging problem for price process model involving finite number of brownian and fractional brownian motions. *Theory of Stachastic Processes*, 14(30):27-38.

[11] Browne, S., 1999a. Beating a moving target: Optimal portfolio strategies for outperforming a stochastic benchmark. *Finance and Stochastics*, 3(3):275-294.

[12] Browne, S., 1999b. Reaching goals by a deadline: Digital options and continuous-time active portfolio management. *Advance in Applied Probability*, 31(2):551-577.

[13] Browne, S., 1999c. The risk and reward of minimizing shortfall probability. *Journal of Portfolio Management*, 25(4):76-85.

[14] Browne, S., 2000. Risk-constrained dynamic active portfolio management. *Management Science*, 46(9):1188-1199.

[15] Cai, J. and Tan, K.S., 2007. Optimal retention for a Stop-Loss Reinsurance under the VaR and CTE Risk Measures. *ASTIN Bulletin*, 37(1):93-112.

[16] Cai, J., Tan, K.S., Weng C. and Zhang Y., 2008. Optimal reinsurance under VaR and CTE risk measures. *Insurance: Mathematics and Economics*, 43(1), 185-196.

[17] Campbell, J. and Cochrane, J., 1999. By force of habit: A consumption based explanation of aggregate stock market behavior. *Journal of Political Economy*, 107(2):205-251.

[18] Cherny, A.S., 2006. Weighted VaR and its properties. *Finance and Stochastics*, 10(3):367-393.

[19] Chi, Y. and Meng, H., 2013. Optimal reinsurance arrangements in the presence of two reinsurers. *Scandinavian Actuarial Journal* , forthcoming.

[20] Chi, Y. and Tan, K.S., 2011. Optimal reinsurance under VaR and CVaR risk measures: A simplied approach. *ASTIN Bulletin*, 41(2):487-509.

[21] Chi, Y. and Tan, K.S., 2013. Optimal reinsurance with general premium principles. *Insurance: Mathematics and Economics*, 52(2), 180-189

[22] Cong, J., Tan, K.S. and Weng, C., 2013. VaR-based optimal partial hedging. *ASTIN Bulletin* , 43(3):271-299.

[23] Cong, J., Tan, K.S. and Weng, C., 2014. CVaR-based optimal partial hedging. *Journal of Risk*, forthcoming.

[24] Constantinides, G.M., 1990. Habit formation: A resolution of the equity premium puzzle. *Journal of Political Economy*, 98(3):519-543.

[25] Cvitanić, J. and Karatzas, I., 1999. On dynamic measures of risk. *Finance and Stochastics*, 3(4):904-950.

[26] Cvitanić, J., 2000. Minimizing expected loss of hedging in incomplete and constrained markets. *SIAM Journal of Control and Optimization*, 38(4):1050-1066.

[27] Davis, M.H.A., 1997. Option pricing in incomplete markets. In *Mathematics of Derivative Securities* edited by M.A.H. Dempster and S.R. Pliska, 216-226. Cambridge: Cambridge University Press.

[28] Dhaene, J., Denuit, M., Goovaerts, M.J., Kaas, R. and Vyncke, D., 2002. The concept of comonotonicity in actuarial science and finance: theory. *Insurance: Mathematics and Economics*, 31(1):3-33.

[29] Dowd, K., Cotter, J. and Sorwar, G., 2008. Spectral Risk Measures: Properties and Limitations. *Journal of Financial Services Research*, 34(1): 61-75.

[30] Düring, B. and Lüders, E., 2005. Option price under generalized pricing kernel. *Review of Derivative Research*, 8(2):97-123.

[31] El Ghaoui, L., Oks, M. and Oustry, F., 2003. Worst-case Value-at-Risk and robust portfolio optimization: A conic programming approach. *Operations Research*, 51(4):543-556.

[32] Föllmer, H. and Leukert, P., 1999. Quantile hedging. *Finance and Stochastics*, 3:251-273.

[33] Föllmer, H. and Leukert, P., 2000. Efficient hedging: Cost verse shortfall risk. *Finance and Stochastics*, 4(2):117-146.

[34] Gajek, L. and Zagrodny, D., 2004. Optimal reinsurance under general risk measures. *Insurance: Mathematics and Economics*, 34(2):227-240.

[35] Huang, D., Zhu, S., Fabozzi, F.J. and Fukushima, M., 2010. Portfolio selection under distributional uncertainty: A relative robust CVaR approach. *European Journal of Operational Research*, 203(1):185-194.

[36] Hurlimann, W., 1998. On stop-loss order and the distortion pricing principle. *ASTIN Bulletin*, 28(1), 119-134.

[37] Henderson, V. and Honson, D., 2009. Utility indifference pricing: an overview. *Indifference Pricing: Theory and Applications, Camona, R. (ed), Princeton University Press.* 44 - 73.

[38] Hodges, S. and Neuberger, A., 1989. Optimal replication of contingent claims under transactions costs. *Review of Futures Markets*, 8(2):222-239.

[39] Jorion, P. Value at Risk: The New Benchmark for Managing Financial Risk. Third Edition. *McGraw-Hill*, 2006.

[40] Kahn, P.M., 1961. Some remarks on a recent paper by Borch. *ASTIN Bulletin*, 1(5):265-272.

[41] Kaluszka, M., 2001. Optimal reinsurance under mean-variance premium principles. *Insurance: Mathematics and Economics*, 28(1), 61-67.

[42] Kaluszka, M., 2005. Optimal reinsurance under convex principles of premium calculation. *Insurance: Mathematics and Economics*, 36(3), 375-398.

[43] Kaluszka, M. and Okolewski, A., 2008. An extension of Arrow's result on optimal reinsurance contract. *Journal of Risk and Insurance*, 75(2), 275-288.

[44] Klöppel, S. and Schweizer, M., 2007. Dynamic indiffernce valuation via convex risk measures. *Mathematical Finance*, 17(4):599-627.

[45] Klusik, P. and Palmowski, Z., 2011. Quantile hedging for equity-linked contracts. *Insurance: Mathematics and Economics*, 48(2):280-286.

[46] Kou, S., Peng, X. and Heyde, C.C., 2012. External risk measures and Basel accords. *Mathematics of Operations Research*, forthcoming. Available at SSRN: http://ssrn.com/abstract=2055634.

[47] Krutchenko, R. N. and Melnikov, A. V., 2001. Quantile hedging for a jump-diffusion financial market model. *Trends in Mathematics*, 2001, 215-229.

[48] Luenberger, D.G. Optimization by Vector Spaces Methods. *John Wiley and Sons*, 1969. New York.

[49] Mania, M. and Schweizer, M., 2005. Dynamic exponential utility indifference valuation. *Annuals of Applied Probability*, 15(3):2113-2143.

[50] Melnikov, A. and Skornyakova, V., 2005. Quantile hedging and its application to life insurance. *Statistics and Decisions*, 23(4):301-316.

[51] Melnikov, A. and Smirnov, I., 2012. Dynamic hedging of conditional value-at-risk. *Insurance: Mathematics and Economics*, 51(1):182-190.

[52] Monoyios, M., 2008. Utility indifference pricing with market incompleteness. *Nonlinear Models in Mathematical Finance: New Research Trends in Option Pricing, Ehrhardt, M. (ed), Nova Science Publishers, New York.* 67-100.

[53] Musiela, M. and Zariphopoulou, T., 2004. An example of indifference prices under exponential preferences. *Finance and Stochastics*, 8(2):229-239.

[54] Nakano, Y., 2004. Efficient hedging with coherent risk measure. *Journal of Mathematical Analysis and Applications*, 293(1):345-354.

[55] Nakano, Y., 2007. Minimizing coherent risk measures of shortfall in discrete-time models with cone constraints. *Applied Mathematical Finance*, 10(2):163-181.

[56] Panjer, H.H., Wang, S.S., Young, V.R., 1997. Axiomatic characterization of insurance prices. *Insurance: Mathematics and Economics*, 21(2), 173-183.

[57] Rockafellar, R.T. and Uryasev, S., 2000. Optimization of conditional Value-at-Risk. *Journal of Risk*, 2(3):21-41.

[58] Rockafellar, R.T. and Uryasev, S., 2002. Conditional Value-at-Risk for General Loss Distributions. *Journal of Banking and Finance*, 26(7):1443-1471.

[59] Rolski, T., Schmidli, H., Schmidt, V. and Teugels, J. Stochastic Processes for Insurance and Finance. *John Wiley and Sons*, 1999.

[60] Rosenberg, J.V. and Engle, R.F., 2002. Empirical pricing kernels. *Journal of Financial Economics*, 64(3):341-372.

[61] Rudloff, B., 2007. Convex hedging in incomplete markets. *Applied Mathematical Finance*, 14(5):437-452.

[62] Sekine, J., 2000. Quantile hedging for defaultable securities in an incomplete market. *Mathematical Economics (Japanese)*, 1165:215-231.

[63] Sekine, J., 2004. Dynamic Minimization of Worst Conditional Expectation of Shortfall. *Mathematical Finance*, 14(4):605-618.

[64] Spivak, G. and Cvitanić, J., 1999. Maximizing the probability of a perfect hedge. *The Annuals of Applied Probability*, 9(4):1303-1328.

[65] Startz, R., 1989. The stochastic behaviour of durable and non-durable consumption. *Review of Economics and Statistics*, 71(2):356-363.

[66] Tan K.S., Weng, C., and Zhang Y., 2011. Optimality of general reinsurance contracts under CTE risk measure. *Insurance: Mathematics and Economics*, 49(2), 175-187.

[67] Wang, Y., 2009. Quantile hedging for guaranteed minimum death benefits. *Insurance: Mathematics and Economics*, 45(3):449-458.

[68] Weng, C., 2009. Optimal reinsurance design: from an insurer's perspective. *Ph.D. thesis*

[69] Young, V.R., 1999. Optimal reinsurance under Wang's premium principles. *Insurance: Mathematics and Economics*, 25(2), 109-122.

[70] Zhu, S. and Fukushima, M., 2009. Worst-case conditional Value-at-Risk with application to robust portfolio management. *Operations Research*, 57(5):1155-1168.