A Structural Modelling Approach to Closed End Bond Funds

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

This thesis develops a model of closed end bond funds that helps us better understand a recent finding in the literature. In 2012 Elton et al. published an empirical study of closed end bond funds (CEBFs) and they suggested that the use of leverage in CEBFs could explain the fact that these funds had higher returns than those of comparable open end funds. This thesis provides a framework for estimating the impact of leverage on expected return and risk in this context. We use a Merton type approach to model both unlevered and levered CEBFs. The assets of a CEBF are primarily risky bonds. Each of these risky bonds can be analysed in terms of options under the Merton approach. We create an unlevered CEBF model by extending Merton's model [28] to a multi-firm framework to represent a CEBF composed of several risky bonds. We then add leverage by assuming the CEBF issues debt. This permits us to model the securities of a levered closed end bond fund as compound options. The equity and debt of the CEBF can be decomposed into options on a portfolio of options. This framework enables us to compute the expected rate of return and standard deviation of an unlevered and levered CEBF. We obtain results that are comparable to those observed in Elton et al. [17].

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Chapter 1

Thesis Roadmap

Closed end funds (CEFs) have long been an interesting topic to both investors and academics. Recent empirical findings in Elton, Gruber, Blake, and Shachar [17] (Elton et al.) have presented several interesting observations regarding the use of leverage in closed end bond funds (CEBFs). Elton et al. analyze CEBF data from 1996 to 2006. They find that closed end bond funds have higher returns than comparable open ended funds. They postulate that it is the leverage employed by CEBFs that causes the difference in returns of levered CEBFs over open ended bond funds (OEBF) and unlevered CEBFs. No quantitative analysis or intuitive mathematical justification is provided in [17] as to why there is a difference in returns. In this thesis we develop an unlevered and levered CEBF model in order to show that it is use of leverage in levered CEBFs that causes the difference in returns of levered CEBFs over unlevered CEBFs.

Chapter 2 begins by introducing CEFs as one of the different types of mutual funds and gives some estimates of their relative importance compared to the universe of mutual funds. We introduce some basic information regarding the different asset classes of CEFs (based on the types of assets that the fund holds) as well as where, when, and how CEFs trade. In section 2.4 we explain how CEFs differ from open end funds (OEFs) in where, when, and how they trade. In section 2.5 we choose a specific example of a CEBF, the *PIMCO Income Opportunity Fund*, and present the information that is typically contained in the fund prospectus. Next we discuss the so called CEF discount/premium and why the existence of the discount/premium is termed *The Closed End Fund Puzzle*.

Section 2.6 provides a short literature review of various models and frameworks that have contributed a better understanding of the CEF discount/premium. Although this thesis does not rely on any of the models presented in section 2.6 directly, they highlight which factors are important determinants of the CEF discount. In section 2.7 we outline the key findings of Elton et al. [17]. These findings serve as motivation for the models developed in this thesis. We describe the construction of our structural credit risk model in Chapter 3. Elton et al. [17] suggested that it is the use of leverage in CEBFs that allows them to achieve higher returns from OEBF and from unlevered CEBFs. We describe some of the costs, both to the fund and the investor, that are associated with the use of leverage in CEBFs. Both Cherkes [10] and Elton et al. [17] provide reviews of the CEBF literature.

The main contributions of this thesis are in Chapter 3 where we develop a structural credit risk model of an unlevered CEBF and a levered CEBF. The assets of a CEBF consist of risky bonds which can be modelled using the Merton framework. The unlevered CEBF is financed solely by equity and the levered CEBF is financed by debt and equity. First we obtain expressions for the market value of the equity and its standard deviation in an unlevered CEBF. Then we determine the market value and standard deviation of the debt and equity of a levered CEBF. The chapter begins with our construction of a multi-firm version of Merton's structural credit risk model [28]. In sections 3.2.1 and 3.2.2 we outline and construct the unlevered and levered CEBF structural models.

The second contribution is in section 3.3 where we quantify the impact of varying firm correlations and volatilities on the market value and standard deviation of an unlevered and levered CEBF. This provides a convenient framework for evaluating the impact of a financial crisis on the debt and equity of closed end bond fund. In a crisis both volatility of the underlying firms and the correlations across firms increases. The increase in volatility makes the firm's bonds more risky so the market value of the assets of a CEBF will fall. However, an increase in correlations among firms means that a call option written on a portfolio of risky bonds will increase in value other things equal. Hence an increase in correlation across firms without any associated change in each firm's volatility, will not affect the toal assets of a CEBF but it will transfer wealth from the debtholder of a levered CEBF to the equityholder of the CEBF.

The third contribution of this thesis is in Chapter 4 where we show that expected rate of return of the equityholder of the levered CEBF is higher than that of the unlevered CEBF. Furthermore we show that our levered CEBF model can achieve return results comparable to those observed by Elton et al. [17]. First we derive the expected rate of return (ERR) under the physical measure (the Real World $\mathbb{P}$-measure) of a risky bond (RB) that is held for a period but priced under the Risk-Neutral or $\mathbb{Q}$-measure. We use the model for the ERR on a European put option developed by in Rubinstein's [31]. Next we extend Rubinstein's framework to both a unlevered CEBF and a levered CEBF using the models introduced in sections 3.2.1 and 3.2.2. In section 4.3 we obtain the expected rate of return, variance of the rate of return, and Sharpe Ratio of the unlevered CEBF.

Subsequently in section 4.4 we obtain the expected rate of return, variance of the rate of return, and Sharpe Ratio for the debtholder and equityholder of the levered CEBF. Under the same model assumptions as in section 3.3 the levered CEBF equityholder achieves a higher ERR than the unlevered CEBF equityholder. Section 4.5 shows that we achieve similar expected return and variance of return to those of Elton et al. [17].

Chapter 5 summarizes our results and discusses some future areas of research in closed end bond funds.

Chapter 2

An Overview of Closed End Bond Funds

We begin this chapter with a gentle introduction to CEFs as a type of mutual fund. Some basic facts about CEFs are presented regarding the different asset classes of CEFs and how CEFs differ from open end funds (OEFs) in where, when, and how they trade. We present the information that is typically contained in the fund prospectus with a specific example of a CEBF, the *PIMCO Income Opportunity Fund*. Next we discuss the so called CEF discount/premium and provide a short literature review of various models and frameworks surrounding it. The most important part of this chapter is our outlining of the key findings of Elton et al. [17]. These findings serve as motivation for the models developed in Chapters 3 and 4. We would like to mention that in this chapter we do not contribute any new academic findings or models. Lastly we would like to mention that the results presented in this thesis are based on CEBF information collected and observed in the U.S. markets and not the Canadian markets.

2.1 An Introduction to Mutual Funds

As per Elton and Gruber [16], there are four distinct types of mutual funds[1],

- Open end funds (OEFs);

[1]From Elton and Gruber [16], we found that when people refer to mutual funds they typically mean open end funds.

- Exchange traded funds (ETFs);
- Closed end funds (CEFs);
- Unit investment trusts (UITs);

Note that exchange traded funds (ETFs) are either considered OEFs or UITs that trade on an exchange. Based on information collected from the 2012 Investment Company Fact Book [23], summarized in the table below, it is clear that OEFs are by far the most popular investment vehicle, followed by ETFs. At this point we are not differentiating funds based on which asset class of investments they purchase, such as stock funds, bond funds, etc.

Fund Type	**Total Net Assets at 12/31/2011 (USD)**
Open end funds	$11.6 Trillion
Exchange traded funds	1.0 Trillion
Closed end funds	239 Billion
Unit investment trusts	60 Billion

Table 2.1.1: Mutual Fund Investment Breakdown

This thesis is concerned with CEFs (specifically CEBFs) and we will not concern ourselves with details from the other structures. We will, however, mention some of the fund return characteristics and comparisons of OEF to CEF returns from Elton et al. [17].

2.2 Classes of Closed End Funds

CEFs are typically grouped into the four main categories based on the type of assets the fund holds[2].

- *Tax Free Income Strategy Funds* typically invest in municipal bonds. These are bonds that are issued by a municipality, city, or state requiring funding for various public or government projects. The interest that is paid to the municipal bondholder is not taxed at the federal income tax level. For individuals that purchase resident state specific bonds any income provided by these bonds is not taxed at the state income tax level. Without taking into account the tax savings at the state and federal level,

[2]This information was obtained from *Closed End Fund Connect* [18]

it is difficult to make a comparison of tax free investments to taxed investments. It is important to note that municipal bonds are one of the largest groups of investments made by CEFs.

- *Taxable Income Strategy Funds* include government, investment grade, corporate, convertible, high yield, and unrated bonds with differing term to maturities. These funds offer exposure to senior loans, preferred investments, or unsecured junior debt.

- *U.S. Equity Funds* earn their returns from U.S. stock returns, option premiums (European, American, and exotics), option profits, and dividends from common and preferred shares. This category includes fund investments in real estate investment trusts. Real estate investment trusts are publicly traded companies that own commercial real estate property such as appartments, shopping centers, hotels, etc. The fund generates income from dividends based on rent and income received from the underlying real estate rent and appreciation/depreciation of the property.

- *Non - U.S. Funds* are funds that invest in international stocks and bonds. Global stock funds have equity investments across different countries, regions, and credit ratings. Global bond funds can give investors exposure to sovereign debt which not only gives rise to foreign interest rate and credit rating but also to foreign currency exposure. With foreign country investment exposure comes exposure to foreign corporations, legal systems, political risk, legal, accounting, less liquid markets, and other risks.

From the 2012 Investment Company Fact Book [23], we observe a historical trend where more than half of the existing CEFs were bond funds. The following graph shows the number of closed end bond and stock funds from 2001 to 2011.

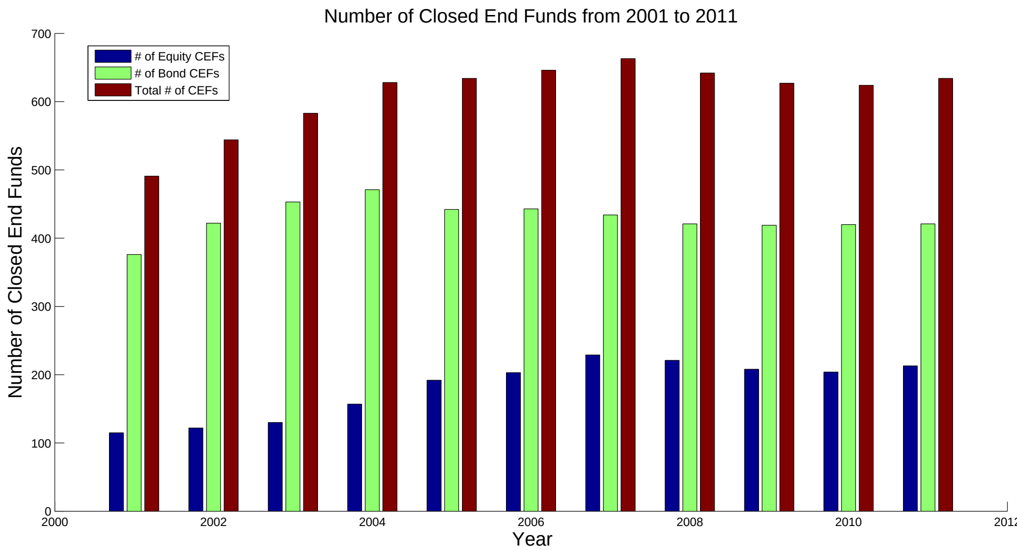

Figure 2.2.1: Annual number of Closed End Stock and Bonds Funds

Elton et al. [17] found that of those CEFs that are bond funds, most of the bonds being held by the fund are in fact corporate bonds. The table below is provided by Table 5 in Elton et al. [17].

Asset Type	**Closed End Fund**	**Open End Fund**
Government Bonds	8.50%	9.60%
Mortgages	7.60%	10.20%
Corporate Bonds	67.50%	69.10%
Foreign Bonds	11.00%	8.90%
Stocks	0.50%	0.60%
Preferred	0.60%	1.50%
Convertibles	0.30%	0.50%
Other	4.10%	0.00%

Table 2.2.1: Closed and Open End Fund Return Summary

Table 2.2.1 shows that the CEBFs contain a substantial amount of risky bonds in their fund portfolios. The fact that CEBFs are composed of risky bonds will be a key point in the construction of our structural model of CEBFs in Chapter 3.

2.3 An Introduction to Closed End Funds and the Fund Discount

A CEF is a type of mutual fund in which at fund initial public offering (IPO)[3] a fixed number of common shares are issued in order to raise capital for the fund manager to invest. Subsequent to the IPO no more common shares can be issued and the fund's common shares are then traded in a secondary market (an exchange). As per Elton and Gruber [16], CEF shares can be purchased or sold at the market price at any time in which the market is open. In addition to reporting the price per share (denoted P), closed and open end funds report the net asset value (denoted NAV) per share. The NAV is calculated as the total market value of the underlying assets of the fund less expenses divided by the number of shares outstanding. The net asset value and the price per share are typically not equal. The CEF common shares are said to be trading at a premium when the fund share price is above the NAV and trading at a discount when the fund share price is below the NAV. We define the discount at time t, to be,

$$DISC_t = NAV_t - P_t \tag{2.3.1}$$

where NAV_t and P_t are the net asset value and price per share at time t of the CEF respectively. Naturally a negative discount is a positive premium and vice versa. An investor is purchasing a fund at a discount at time t at a price P_t if the $NAV_t > P_t$ and at a premium otherwise.

2.4 Comparing Closed and Open End Funds

There are several important differences between open and closed end funds. For OEFs the number of outstanding common shares and how the shares are traded is different from CEFs. For OEFs the common share price is determined as total market value of the portfolio divided by the total number of common shares outstanding. Investors purchase and sell common shares at the current market price directly to the fund where additional shares are issued from the fund as necessary to satisfy investor demands. As per Elton and Gruber [16], OEFs can be bought and sold throughout the day, however, the price is set at the net asset value per share at the end of the trading day. Since the net asset

[3] A fund initial public offering is also be referred to as the point of fund inception.

value and price per share are set equal at the end of each trading day for open end funds, the net asset value and the price per share are usually equal. Whereas for CEFS they are typically not equal since the shares trade on an exchange and are not directly determined by the net asset value but rather by market forces.

2.5 Example of a Closed End Fund

In this section we take a detailed look at a specific CEF and the fund prospectus to get a broad picture of how a CEF is organized. Our choice of CEF is the *PIMCO Income Opportunity Fund* which is a CEBF. When a particular investor wishes to look at the fund objective they have or wish to purchase they look into a fund's prospectus. A fund prospectus is a legal document that must be filed with the Securities and Exchange Commission (SEC) and details certain required information regarding the fund management. One of the key pieces of information that one would see in a fund's prospectus is the fund's *Investment Objective*. From it's fund prospectus [1], we can see the investment objective is to maximize capital appreciation and fund income. The investment objective also lays out its investment strategy in how the fund will choose, allocate and manage fund assets. From [1] we can see that the *PIMCO Income Opportunity Fund* uses a dynamic allocation strategy, which is an active global macroeconomic approach to investment analysis. Its portfolio management strategies focus on quality of credit rating, time to maturity of the assets, geographic region, and industrial sector. The *PIMCO Income Opportunity Fund* seeks to maintain an average portfolio duration of roughly 2 to 8 years. The portfolio management strategies are used in order to earn a fund return in excess of the risk free rate while minimizing interest rate risk exposure.

The prospectus also outlines the different types of leverage that the fund can and plans to use. Bank loans, reverse purchase agreements, commercial paper, and issuing preferred shares are some of the methods of financing that the fund can use to generate leverage. The section on leverage also discusses the limitations on specific types of leverage that the fund intends to have as a maximum percentage of total assets. For example, according to [1], the *PIMCO Income Opportunity Fund* can add leverage in the form of reverse purchase agreements not to exceed 20% of the fund's total assets.

We note that the *PIMCO Income Opportunity Fund* invests in corporate, municipale, government, foreign, emerging market, and high yield debt securities. In addition to bonds, the fund can invest in mortgages, other asset backed securities, inflation protected securities, options, and complex derivatives. The fund can use reverse purchases, dollar rolls and leverage in the form of preferred shares. The *PIMCO Income Opportunity Fund* can also make short sales or take a short position. The assets that the fund purchases can be of U.S. or of any other global market.

The graph on the left below shows daily observations of the NAV and share price P of the fund from July 2, 2007 to December 31, 2009[4]. The graph to the right shows the fund discount/premium over the same time period.

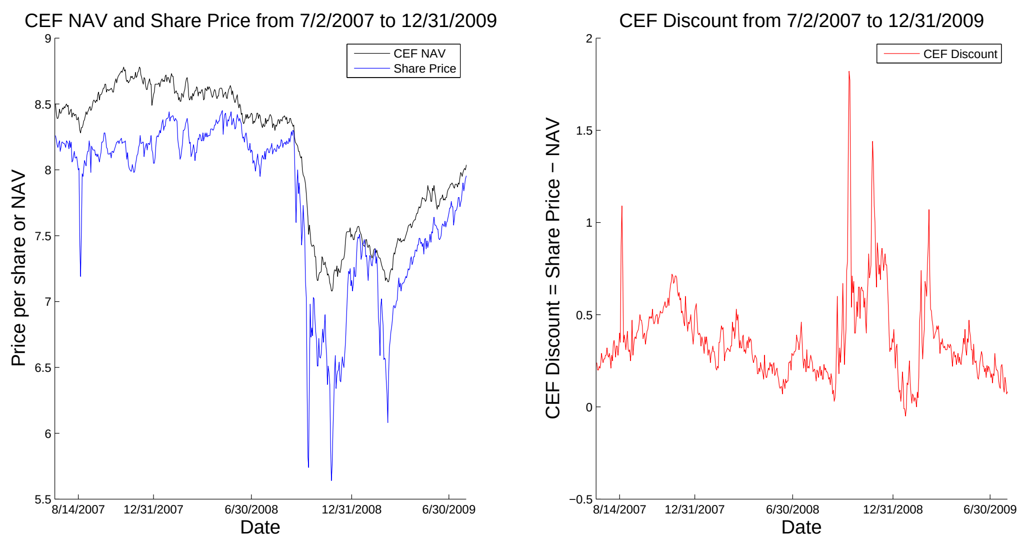

Figure 2.5.1: *PIMCO Income Opportunity Fund*

The propectus outlines the different *shareholder fees* that are paid by the shareholders from their investment in the fund. According to [1] there are sale charges imposed by purchases that the fund makes and annual expenses. The annual expenses are management fees, interest expenses on reverse purchase agreements, and other expenses.

Part of the fund prospectus also provides a general outline of the risks associated with an investment made in the fund. Some of the risks associated with the *PIMCO Income Opportunity Fund* are:

- *Market Risk*: changes in the fund asset value due to fluctuations in the equity and bond markets;

[4]The data for the graphs in figure 2.5.1 was obtained from Bloomberg using ticker code PTY US Equity.

- *Interest Rate Risk*: changes in the interest rate levels that affect the value of the bonds of the portfolio;
- *Credit Risk*: there are two types, credit risk default and downgrade. Credit risk default is the risk of a counterparty defaulting on its financial obligations and loss of coupon payments and principal in partial or in full value. Credit risk downgrade is the risk of decrease in value of a debt security upon which the external rating decreased;
- *Currency Risk*: risk of changes in the exchange rate could negatively impact the value of the fund portfolio;
- *Emerging Market Risk*: risks inherent in investing in emerging market countries such as less liquid markets, increased economic insecurity, political instability, and regulatory uncertainty;
- *Leverage Risk*: more leverage results in a larger net asset value which magnifies the impact of changes in interest rates (larger market and interest rate risk);

2.6 The Closed End Fund Puzzle

In the academic literature, two questions have typically been asked about closed end funds,

- Why do closed end funds exist over their more liquid counterpart, open end funds?
- What is a rational economic explanation for the observed share price to net asset value discount/premium at which closed end funds sell?

The unexplained existence of the share price to net asset value discount/premium for a CEF is termed *The Closed End Fund Puzzle*. *The Closed End Fund Puzzle* is a long-standing puzzle in modern finance. The existing literature that has attempted to provide possible explanations for the observed market discount/premium have been based on a variety of approaches. These include empirical regression models, investor sentiment models, and quantitative models that have tried to explain the CEF discount/premium. The consensus so far is that no single one of these explanations fully explains the existence of the discount/premium. Each one can explain only a small portion of the observed discount/premium. This section is a review of some of the models that have been developed and their contributions to a better understanding of the discount/premium and CEFs themselves. We would like to stress there are no new contributions to this problem in this thesis. This section serves as a literature review of existing models. For further information about any particular model the reader is referred to the original paper(s).

2.6.1 The First Contributor

One of the first papers in analyzing and explaining the *Closed End Fund Puzzle* was by Burton Malkiel [27]. He composed the list below detailing some of the observed behaviours of CEF share prices and portfolios which can serve to better understand and provide an explanation for the CEF discount/premium. The data set used at the time of publication of [27] consisted of 24 major CEFs observed over 1967 to 1974.

- Unrealized Capital Appreciation;
- Distribution Policy;
- Restricted, or Letter Stock;
- Holdings of Foreign Stock;
- Fund and Managerial Performance;
- Asset Turnover in the Portfolio;
- Management Fee;

It is widely believed that CEFs have a tax liability built into their share price that comes from their unrealized capital appreciation potential. In theory, the larger the amount of unrealized capital appreciation a fund has, the larger the fund discount. This comes from the fact that the fund investor must pay a capital gains tax on the distributions from the fund. Hence a larger fund will have a larger distribution and thus a larger tax. The tax amount that the investor will pay depends on both the size of the fund and the amount of time in which the investor has held the fund.

The fund capital distribution policy, with respect to how a CEF realizes and distributes capital gains (in the form of dividends), is hypothesized to be an influencial factor in the CEF discount/premium. It is presumed that they could have three possible effects:

- The fund's capital gains realization and distribution policy will decrease the fund's unrealized capital appreciation which in turn will lower the future tax liability;

- The fund capital distribution policy would be directly favourable to a specific investor tax bracket. The low tax bracket investors will prefer the fund to have regular set distribution policy. This is in order for them to avoid transaction costs involved in fund sales. However, high tax bracket investors would prefer the fund to realize and reinvest all their capital gains;

- The capital gains paid from the fund in the form of dividends are considered a distribution of part of the fund portfolio. As long as the sum of the amount distributed from the fund and the unrealized appreciation are less than the discount the investors benefit from the fund selling at a discount;

It is known that some CEFs invest a large portion of their assets in restricted and letter stock. Letter stocks require written consent of the buyer to confirm that they have been purchased for a considerable investment holding. Typically restricted and letter stocks are purchased at a discount to market price, however, they are considered very illiquid securities. Funds that hold these types of securities are expected to sell at a discount. Certain funds will invest solely in foreign assets benefiting from exchange and tax controls. Fund discounts may be driven by past fund performance and managerial ability or reputation. Fund portfolio management requires fairly active management and portfolio turnover which incurs frequent transaction costs and taxes to the shareholders. The larger the portfolio the higher the cost to the shareholders. Although active management of the portfolio has the best intentions of steering the portfolio in order to earn higher returns, it is believed that the costs can outweigh the potential gains. It is believed that a portfolio that incurs more transaction costs from buying and selling more assets, will yield a larger discount. At the time of [27], in 1977, no relationship was discovered between management fees and fund return indicating no correlation between the two. As a result a higher fee is a cost solely borne by the investor and hence it is believed that a higher fee implies a larger discount.

Cross sectional empirical estimates, using 24 CEFs from 1967 to 1974, show the relationship between the model factors and the observed fund discount/premium. The regression results of the empirical data shows that the factors listed below have a small but noticeable explanation of the CEF discount/premium.

- fund unrealized capital gains;

- distribution policy of capital gains;

- holding of foreign securities, and letter stocks;

One of their additional observations was that the fund discount decreases as the market decreases and increases as the market increases. This result suggests that CEFs are a particularly attractive investment opportunity for diversification purposes. They have a negative covariance with respect to the market. Aoun [2] provides recent evidence that this observation is not always true. During the credit crises of 2008-2010 the CEF discount/premium widened.

2.6.2 Individual Investor Sentiment Model

In addition to the *Closed End Fund Puzzle*, Lee, Schleifer, and Thaler [25] (Lee et al.) add the empirical observations of CEFs listed below to the list of unexplained behaviour of CEFs.

- CEF shares start selling, shortly after their IPO, at a premium (of at least 10%) to their NAV;
- Most CEF shares move to sell at a discount (of over 10%) to their NAV within the first 3 months of trading on secondary markets after selling at a premium to their NAV;
- Discounts are subject to wide fluctuations over time, moving from selling at a large discount to a large premium, however, they have been observed to be mean reverting;
- Share prices tend to increase resulting in a narrowing discount around the time in which a fund announces termination through either fund conversion (to an OEF) or asset liquidation;

The list of factors below are strongly believed to be possible determinants of the *Closed End Fund Puzzle*. They are also believed to help explain some of the other observed unexplained market anomalies of CEFs in the above list.

- *Agency Costs*, in terms of high management fees and the notion of below average future expected managerial performance lower the NAV and cause the fund discount/premium;
- *Tax Liabilities*, that must be paid by the fund if assets are sold, on unrealized appreciations that are not taken into account in the NAV thus causing the NAV to be overvalued;

- *Illiquidity of holding assets* are overvalued in the calculations of the NAV;

Lee et al. [25] show that none of the above factors can explain more than a small portion of the *Closed End Fund Puzzle*. Agency costs, management fees and future managerial experience, do not account for the range of fluctuations in the discount, from being large premiums to large discounts. As well, they also fail to account for why the funds are purchased by investors at a premium shortly after fund inception. Despite the fact that investors are aware that the funds have a history of then moving from selling at a premium to varying discount levels. When fund assets are sold, they are taxed at a capital gains tax rate which is not taken into account in the calculation of the NAV. This tax theory implies that the fund discount should increase when the market increases. It was noted in Malkiel [27] that the tax liability theory can create a fund discount of not more than 6%. With regards to the illiquid holding asset theory, Lee et al. [25] note that some of the largest CEFs observed during 1968 to 1986 hold only very liquid traded securities.

An obvious question comes to mind for a fund that trades at a discount. Why can't an individual buy up all the shares at a discount to the NAV thereby taking control of the fund and begin to liquidate the fund assets at the NAV thereby making a profit? Grossman and Hart [21] outline the reasons explaining that this is difficult to accomplish in real time. The common shareholders will not sell their shares to bidders without receiving full NAV. On the other hand bidders will not pay the full price of the NAV since they will not profit on the transaction. Furthermore there will be resistance from the fund management as well as the regulatory agency to such fund liquidations. There is yet another obvious question that comes to mind for a CEF selling at a discount. Why can't an individual create an arbitrage opportunity from the fund discount by short selling the CEF portfolio while purchasing the common shareholder stock? The reasons for this are listed below.

- Shortselling the CEF portfolio will involve shorting each component of the fund, and these funds are often composed of hundreds of assets for which rebalancing the portfolio in *real time* may be difficult and time consuming;

- Transaction costs on rebalancing the portfolio will become very costly given the number of assets involved;

- The hedge is not perfect in finite time (since the fund can be maintained indefinitely) and when the discount widens the investors takes a loss on the short position in the portfolio;

- Short sellers will not receive full proceeds from their short sale and may need to liquidate some of the portfolio on demand;

Lee et al. [25] construct an investor sentiment model for explaining the CEF discount/premium and related market anomalies of the funds. The authors postulate that the CEF discount/premium is caused by changes in individual investor sentiment on the future expectations of fund performance. Zweig [34] first suggested a model in which the individual investor's fund expectations is a key factor in the CEF discount/premium. The model conjectured by Lee et al. is based on that constructed by DeLong, Shleifer, Summers, and Waldman [15] (DeLong et al.). The model constructed in DeLong et al. [15] is an asset pricing model that assumes the existence of both *rational investors* and *noise traders*. The rational investors have short investment horizons with the primary concern being the value of their holding assets in secondary markets. This implies that their primary concern is not the present value of their dividends. The noise traders have investment sentiment that is random and is not completely predictable by the rational investors. The existence of noise traders imposes an extra risk to the rational traders. When the rational investor chooses to either buy or sell an asset, there is a positive probability that the noise trader will act in an adverse manner causing the price to shift against those expected by the rational investor.

The main thesis in Lee et al. [25] is that changes in the fund discount/premium are driven by the different individual investor sentiments of the holders and traders. The main implication is that the individual investor sentiment that affects CEFs also impacts other funds that are held by individual investors, particularly small firm investments (also termed *The Small Firm Effect*). Lee et al. [25] model the risk of changes in CEF share prices coming from changes in the noise trader sentiments about the fund price. As well as the actual market value changes in the assets held by the portfolio. It is also a requirement of the model to assume that noise traders would rather trade or hold the CEF itself as oppose to the assets that compose the CEF. Under the individual investor sentiment model investing in the CEF is riskier than investing in the portfolio of assets that it is composed of. This is so since the required rate of return on share assets held by the fund must yield an average higher rate of return than the assets held by the fund (since the assets held by the fund can easily be purchased on the open market).

The model of Lee et al. [25] is consistent with the other three main components of the *Closed End Fund Puzzle* listed at the beginning of this section. First of all, the theory suggests that irrational investors must be the investors that purchase CEFs shortly after their IPO. This follows since rational investors would not purchase funds that follow a regular pattern of selling at a premium shortly after their IPO then subsequently selling at a discount. Secondly the theory requires fluctuations in the fund discount over time which is driven by changes in investor sentiment about future returns on the fund. This in itself is a requirement of the model. Lastly, the increase in CEF share price after the announcement of fund liquidation (or conversion to an OEF) stems from the fact that as long as the fund is selling at a discount an investor knows that they can create a profit. The profit is generated by short selling the fund portfolio and purchasing the shares and holding them until the end of the existence of the fund. At liquidation, or fund conversion, the fund NAV and price per share will converge together (or nearly together) and the profit is realized.

They suggest that model has the additional implications listed below.

- Changes in a fund discount are believed to be correlated across various funds;
- New CEFs are created when long standing funds are selling at a premium (or small discount);
- The investor sentiment that affects the discount on CEFs affects other funds that are completely unrelated to CEFs;

The data collected is over the time period of 1960 to 1987 from Wiesenberger's Investment Company Series annual survey of mutual funds as well as from 1956 to 1985 of the Wall Street Journal (WSJ). Of the 87 original funds, 68 were used in the analysis and others were removed due to missing data. The empirical evidence presented in [25] shows that the discount/premium on funds is in fact correlated across funds and asset classes. The theory that new CEFs are started when long standing funds are being sold at a premium is tested. Lee, Schleifer, and Thaler [25] present some evidence that funds have been issued when the average CEFs are selling at a small discount or premium. The theory is difficult to test since it takes time to register a new fund that is being issued and the market may already obtain information about the fund through news prior to the fund actually beginning to trade on the market.

The theory that the discount/premium on CEFs affects other completely unrelated funds is tested by analyzing the funds' correlation with portfolios of other asset classes. In Table IV of [25], when the fund returns are grouped into deciles and the value weighted portfolio of CEF returns is compared to a value weighted portfolio returns of the New York stock Exchange firms, they note that the discount on CEFs narrows when the correlation between them and small capitalization stocks increases. The implication that discount on CEFs narrows when the correlation between them and small capitalization stocks increases, is that the *Closed End Fund Puzzle* is a phenomenon that is being caused by the *Small Firm Effect*.

One of the main concerns in the analysis conducted by [25] is the stability of the results over time. The total time period used in Table IV is divided equally and the value weighted portfolio of CEF returns is compared to a value weighted portfolio returns of the New York Stock Exchange firms. In each subsample, the results are considerably stronger in the first half and much weaker in the second half than when the entire sample is used. The main reason for this is believed to be the increase in ownership of small firms by larger institutions thereby diminishing the influence of the individual investor during the time period between 1975 to 1985. This theory is tested and a weaker correlation between changes in the fund discount and small stock firms is observed in the second half period confirming the theory.

Lee et al. [25] conclude that the changes in share prices and discounts of the CEFs are due to changes in individual investor sentiment about the funds and their future expected returns. A consequence of individual investor sentiment driving CEF share price and their asset portfolio selling at a discount, is that other funds that are held by individual investors are at risk to trade, on average, at a discount from their intrinsic value. Another consequence of individual investor sentiment driving CEF share prices and discounts is other market shares will exhibit changes in prices due to investor sentiment.

2.6.3 Arguments against the Individual Investor Sentiment Model

The conclusions drawn in Lee et al. [25] were strongly rejected by Chen, Kan, and Miller [8] (Chen et al.). Chen et al. [8] claim that the empirical results of Lee et al. [25] in respect to the co-movement of the CEF discounts and small firm returns are not strong or robust enough to claim that the *Closed End Fund Puzzle* and the *Small Firm Effect* are being driven by small investor sentiment. The main arguments in Chen et al. [8] are that the correlation between the CEF discount and small firm returns decreases substantially in the second data set from Lee et al. [25] when the primary sample is divided in two to the point that one cannot claim an impact of investor sentiment. In the test for correlation between the CEF returns and small firm returns, Chen et al. [8] claim that Lee et al. [25] have confounded the effect of institutional ownership with other extraneous factors.

Chen et al. [8] perform a revised test whereby the smallest decile of small firms is divided into two groups (subportfolios), one with small firms that have less than 10% institutional ownership and the other with all small firms that have greater than 10% institutional ownership. The percentage of institutional ownership is measured as of the end of the previous year. In order for the investor sentiment theory to be true the two subportfolios should behave very differently. The subportfolio of small firms that have less than 10% institutional ownership should exhibit strong correlation between the CEF discount and fund returns when the sample size data is divided in half and the whole data set is used. Neither of which is observed. The subportfolio with small firms that have greater than 10% institutional ownership should exhibit weak correlation between the CEF discount and fund returns in both the cases when the sample size data is divided in half and the whole data set is used, neither of which is observed. Both when the sample size data, from 1965 to 1985, is divided in half and the whole data set is used, there is virtually no distinction between the performance of the two subportfolios (in terms of R-Squared and regression coefficients).

2.6.4 Arguments in favour of the Investor Sentiment Model

In a subsequent paper, Chopra, Lee, Shleifer, and Thaler [12] (Chopra et al.) defend the work of Lee et al. [25] from the criticisms of Chen et al. [8]. They test the results and comments of Chen et al. [8] used to support their claim that the relationship between CEF discounts and small firm returns is the same regardless of significant or insignificant institutional ownership.

More evidence is provided to show the impact of institutional ownership of shares and the relationship of their changes in returns with those of changes in the CEF discounts proposed in Lee et al. [25]. For each year of data, the firms are divided into deciles by market capitalization and then ranked and grouped into one of three categories based on size of institutional holdings (high, medium, and low). From this the changes in returns and the fund discount within each group within each decile can be analyzed with respect to firm ownership. The data set used spans the entire Spectrum database of 13-F SEC filings of all NYSE and AMEX firms between 1981 and 1990 which is a larger data set and independent from Lee et al. [25] or Chen et al. [8]. The results show that in each of the deciles, except the first, there is stronger evidence of co-movement of low institutional firm ownership with changes in fund discount than co-movement of medium and high institutional ownership firms and the fund discount. Their tests, in a test using many more firms, confirm the original result of Lee et al. [25] that the shares that have low institutional ownership increase more in value when the discount on CEFs decreases than shares that have a large institutional ownership.

It is of interest to note that despite the fact that there are weak points in Lee et al. [25], Chen et al. [8] do not point out the strengths of Lee et al. [25] such as the observed phenomenon that the investor sentiment model does capture but rather the points that were admitted to be weak to begin with.

2.6.5 A Liquidity Premium Model

Lee et al. [25] first noticed that most CEF shares begin to sell at a premium (of over 10%) to their NAV after their IPO. Within the first 3 months of trading on secondary markets after selling at a premium to their NAV they start to trade at a discount (of over 10%) to their NAV within the first 3 months of trading on secondary markets. Empirical evidence provided in Cherkes, Sagi, and Stanton [11] (Cherkes et al.) shows that CEFs invest a substantial amount of their capital in illiquid assets. CEFs provide investors liquid access to the relatively illiquid assets that CEFs invest in since the fund shares are sold to other investors in secondary markets without the underlying assets begins sold themselves. It is clear that CEFs provide a liquidity benefit to their shareholders in the sense that they themselves do not have to hold illiquid assets.

From their observations, as well as those of Lee et al. [25], Cherkes et al. [11] derive a model that predicts CEF IPO and post-IPO behaviour. Their model is built on the relation between the proportion of assets that the fund holds that are illiquid and the fund liquidity premium. This relation is postulated to be negative since new CEF IPOs are believed to be more liquid than previously existing CEFs.

The market value of the fund, P_t, is determined as the sum of the present value of the fund's future cash flows discounted at interest rate r plus the sales proceeds from selling the fund assets at an optimal stopping time, a stochastic variable denoted by τ. The optimal stopping time τ is determined by the shareholders when they choose to liquidate the fund. Note that while the fraction of the cost of liquidating a fund (K) is larger than the fraction of the fund cash flows that management receives (k), it is never optimal to liquidate the fund. Under the condition that cost of liquidating a fund is larger than the fraction of the fund cash flows that management receives, P_t can be written in closed form.

$$\begin{aligned} P_t &= \mathbb{E}_t\left[\int_t^\tau (1-k)C_{t'}e^{-\int^{t'} trdt''}dt'\right] + (1-K)\mathbb{E}_t\left[e^{-r\tau}NAV_\tau\right] \\ &= C_t(1-k)\mathbb{E}_t\left[\int_t^\tau e^{-(r-g)(t'-t)}dt'\right] + (1-K)\mathbb{E}_t\left[e^{-r\tau}NAV_\tau\right] \end{aligned}$$

The net asset value of the fund at time t is determined as the expected present value of CEF future dividends. The interest rate used for discounting is the sum of the risk free rate r and the liquidity premium ρ_t.

$$NAV_t = \mathbb{E}\left[\int_t^\infty C_{t'}e^{-\int_t^{t'}(r+\rho_{t''})dt''}dt'\right]$$

Cherkes et al. [11] show that in equilibrium, when the liquidity premium ρ_t follows a Reflected Geometric Brownian Motion process, there exists distributions for NAV_t and P_t and corresponding parameters for an existing equilibrium. In this equilibrium there is an attainable liquidity premium level such that the closed end fund can enter the market competitively.

The CEF premium distribution greatly depends on the fact that the liquidity premium, in the absence of CEFs, follows a Geometric Brownian Motion process. As well as the attainable liquidity premium level in which the fund enters the market at a competitive price. The authors determine the liquidity premium stationary distribution. Using the stationary distribution, the CEF expected premium can be determined at a specific time after IPO of the fund. Under a set of model parameters, the model can show the CEF market price falls from selling at a premium to at a discount as is observed in actual CEFs that are traded in the market.

2.6.6 Further Papers on the Closed End Fund Puzzle

For more papers on the *Closed End Fund Puzzle* see Berk and Stanton [3], Brenan and Jain [6], Gemmill and Thomas [19], or Ross [30].

2.7 Leverage in Closed End Funds

A recent empirical study by Elton et al. [17] hypothesizes that the ability of CEFs to use leverage (in the form of debt and preferred shares) is the main reason for the existence of CEFs. First we begin with an introduction to the use of leverage in CEFs.

2.7.1 History of Leverage in Closed End Funds

Prior to 2003 debt[5] was the primary use of leverage financing (for non-municipal bonds). Then preferred shares were used up until 2008. It is important to note that preferred share dividend payments are tax exempt to the shareholder. Also note that the CEBF debt interest payments are not tax exempt to the debtholder. For the shareholder of a CEF it is more advantageous if the fund issues preferred shares as opposed to debt.

[5] This information was obtained from the 2012 Investment Company Fact Book [23] and Elton et al. [17].

Preferred shares have a structure that is similar to a debt instrument but ultimately it is considered an equity instrument. In the event of firm liquidation, preferred shareholders have preference over common shareholders, but, not over debtholders. The claim that preferred shareholders have on liquidation proceeds is equal to the par value of the shares, whereas common shareholders only have a residual claim on firm assets after debtholders and preferred shareholders. Typically preferred shareholders have no voting rights and do not participate in the actual returns of the fund value, however, their dividend payment is senior in priority to that of common shareholders. Naturally dividends from preferred shares are not guaranteed in the same way that interest payments are to the debtholders.

There are both fixed and auction market preferred shares (AMPS)[6]. For fixed rate shares the dividend rate on preferred shares is fixed whereas the AMPS have rates that can change when there is an auction. Before the credit crisis of 2008, dividend paying AMPS were by far the most commonly issued type of preferred share by CEFs. Dividend rates on AMPS are reset during the auction markets which were held either weekly or monthly. In these auction markets a broker submits bids to an auction agent. The bid orders are filled with the available shares and the sell orders are filled as long as there are bid orders. The bids that are filled receive dividends at the rates set by the dealers or at the market clearing rate.

When the credit crises of 2008 began, there was a large demand for liquidity in the equities market and in particular many investors in AMPS wanted to sell their shares. In the auction markets sellers outnumbered buyers, causing an imbalance in supply and demand. Although not required, dealers would occasionally enter the auction and buy up any shares to prevent the auctions from failing. With a large number of outstanding sell orders the auction markets began to fail. Once a few of the weekly or monthly auctions failed all the subsequent ones failed. Failure in the auction markets was not believed to have caused problems with respect to liquidity of the CEF common shareholder stock. The credit quality of the CEFs during the 2008 crisis was not believed to be a major cause of the failed auctions.

Since the failure of the auction markets, CEFs have been replacing their AMPS by redeeming their outstanding preferred share balances. Replacing and redeeming the AMPS was achieved by issuing loans and extended notes while maintaining leverage and re-issuing a new class of preferred shares. Boards of directors have duties to both the common and preferred shareholders when considering refinancing options[7]. The board of directors must consider the impact that the refinancing decisions will have on common shareholders'

[6]This information was obtained from the 2012 Investment Company Fact Book [23]
[7]Noted in BlackRock [22].

share value when deciding on how to refinance the preferred shareholder value. Most of the refinancing options available involve bank notes and tender offers which may not be the optimal refinancing decisions for common shareholders. The reason to redeem AMPS is to provide liquidity to the preferred shareholders affected by the frozen auction markets (essentially there is no liquidty in these markets) in which rates can no longer be set by auction mechanisms. Financial firms are designing alternative forms of leverage in order to benefit both the preferred and common shareholders.

One of the recent developments[8] in the CEF capital structure is the use of puttable preferred stock in order to redeem AMPS while keeping the firm leveraged as much as possible. The puttable preferred share pays dividends that vary in the rate paid. The main difference is that the dividend rate is not set in an auction process, but rather through remarketing runs. Remarketing runs occur when financial firms provide notice of a dividend rate and remarketing agents solicit existing holders and buyers to gauge interest in buying and selling. For interested buyers and sellers, agents make a match at the lowest dividend rate as long as there are sellers for the bidders. Liquidity providers are third party members that are obliged to purchase all excess sell orders over bids in the remarketing. Under SEC rules, money market funds, banks and insurance companies can purchase puttable preferred shares.

2.7.2 Leverage and the Expected Rate of Return of CEBFs

As noted in Elton et al. [17], most CEBFs use leverage to increase fund capital in order to further increase the return of their funds. The leverage can be in the form of preferred stock, reverse purchase agreements, dollar rolls, commercial paper, bank loans, and notes. As we pointed out in section 2.7.1 the most commonly used method to lever a CEBF was to issue preferred shares (specifically AMPS). Under Section 18 of the Investment Company Act of 1940, CEFs are only permitted to issue one class of preferred shares, meaning all preferred shares issued by a fund must be identical. One of the possible reasons for using preferred shares as the method of financing is that a CEF can have debt up to 50% of shareholder equity (common) and can issue preferred shares up to 100% of shareholder equity. Not only is it typical for CEBFs to use leverage they tend to use the maximum amount of leverage that is allowed by the SEC. In their empirical analysis Elton et al. [17] found that almost all CEBFs make extensive use of their ability to lever, when compared to their matched OEBF. As noted before, OEBF do not tend to use leverage.

[8]Obtained from *Investment Company Institute* [24].

It is believed[9] that CEBF managers would alter leverage strategically by investing in higher yielding longer term to maturity assets when current interest rates are low. This was done to produce higher expected returns and variance of returns for levered funds in the long term. The larger the difference between the long term and short term rate the greater the profit. As the yield curve flattens out the difference between long term and short term rate decreases. A flattening of the yield curve results in a lower return to the fund and to the common shareholders (since the preferred shareholders do not share in the fund gains and losses). Elton et al. [17] find that CEBFs tend to issue leverage strategically when current interest rates are low, however, they maintain a constant amount of leverage over time (that is to say they do not reduce their leverage strategically).

2.7.3 Cost of Borrowing and Leveraging

Elton et al. [17] note that the CEF borrowing cost to the common shareholders will depend on whether or not the CEF uses leverage and whether that leverage is in the form of debt and/or preferred shares. Note that when the fund uses debt as leverage the total cost of borrowing is the sum of the interest payments on the debt and the increased management fee collected. The increased management fee collected is in the form of the management fee on the increased total assets of the fund. Also note that when using preferred shares to lever the fund the total cost of borrowing is the sum of the preferred share dividend and the increased management fee collected. Elton et al. [17] estimated the increased annual management fee due to leverage to be 51 basis points (bps) by looking at the average change in administrative costs and management fees charged for a fund in the two years prior to borrowing. The list below are the fees charged to CEBFs for the use of leverage in the form of either preferred shares or debt.

- the interest payments on bonds used to finance the CEBF leverage;
- the dividend payments on preferred shares paid from the fund to the preferred shareholders;
- administrative costs and management fees charged to the common shareholders as a fee by the fund paid for assets under management (included in total expenses reported) paid by the common shareholders to the fund which was estimated by Elton et al. to be 65bps municipal and non municipal bond funds from 1996 to 2006;

[9]By Elton et al. [17].

- flotation costs paid from the fund to the broker handling the issuance of AMPS which are paid weekly or monthly (from Elton et al. this cost averaged 33 bps per year per dollar of preferred share issued which averaged 48bps for non-municipal bonds);

2.7.4 Comparison of Open and Closed End Funds Returns

Elton et al. [17] match CEBFs and OEBFs by issuing company and investment objectives to compare the portfolio composition and returns of the funds. When they are comparing differences in portfolio composition they look at differences in asset maturities, credit ratings of the assets being held, as well as the amount of cash being held by the funds.

No statistically significant results were found in differences between investment grade, non-investment grade bonds, unrated bonds, and even when comparing at the detailed credit rating level in the portfolio holdings of matched CEBFs and OEBFs. When comparing asset maturities of matched CEBFs and OEBFs, with short term assets being thought of as more liquid, the percentage of assets held in short time to maturity is actually higher for CEBFs than for OEBFs. Based on the data in Elton et al. [17], the authors conjecture that the CEBFs do not employ more illiquid asset portfolios than their matched OEBF counterparts. The median percentage of assets being held as cash in matched CEBFs and OEBFs were found to be 0.6% to 2.3% respectively in which this difference would impact returns by less than 8 basis points (bps)[10]. Hence the impact due to CEBFs holding a lower cash position than OEBFs is negligible.

The time series correlations of returns between matched CEBFs and OEBFs are analyzed for time patterns of returns between different portfolio managers. When analyzing pairwise correlations of returns for all CEBFs and OEBFs that had different portfolio managers, Elton et al. [17] find that OEBFs and CEBFs that have the same fund manager and are issued by the same company have higher correlated returns than CEBFs of the same type but with different fund managers.

When Elton et al. [17] are comparing returns between CEBFs and OEBFs they are comparing the mean and variance of returns on both the fund assets and share price. Naturally the return on fund assets and share price should be identical for OEFs whereas for CEFs they will not be equal. For comparison purposes the returns between CEBFs and OEBFs are presented before and after including the costs of leverage in the form of debt and preferred shares. Table 2.7.1 below, which is Table 5 from Elton et al. [17],

[10]The authors determine that the returns would be impacted by less than 8bps by taking the difference in cash between CEBFs and OEBFs and adding it to return on the fund assets before expenses.

displays the pre-expense return on assets for matched CEBF and OEBF with the same fund objectives, have the same fund family, and having the same fund manager.

Return on All Assets	**Open End Funds**	**Closed End Funds**	**Difference**
All Funds	6.40%	6.35%	−0.05%
Municipal Funds	6.05%	6.03%	−0.02%
Non-Municipal Funds	6.92%	6.83%	−0.09%

Table 2.7.1: CEBF and OEBF Annual Return on Assets

One can see from Table 2.7.1 that the returns on assets for OEBFs and CEBFs are very similar. This empirical evidence is not consistent with the theory that CEFs invest in riskier securities or more illiquid securities to earn higher returns since there was also no difference in the detailed credit rating level in the portfolio holdings of matched CEBFs and OEBFs as well.

Table 2.7.2 below, which is Table 9 from Elton et al. [17], shows both the annual return on net asset value NAV (including cost of levering) and return on share price P (return to the common shareholders).

Bond Fund Type	**Return on NAV**	**Return on Share Price P**
All Open End Bond Funds	5.27%	5.27%
All Closed End Bond Funds	6.72%	8.08%
Unlevered Open End Bond Funds	5.77%	5.77%
Unlevered Closed End Bond Funds	5.77%	7.01%
Levered Open End Bond Funds	5.07%	5.07%
Levered Closed End Bond Funds	6.82%	8.40%

Table 2.7.2: Closed and Open End Bond Fund Return Summary

From Table 2.7.2 we can see that the return on NAV and the return on share price P are equal for OEBFs but as expected they are not equal for CEBFs. For CEBFs we see that the return on share price P is higher than the return on NAV for both levered and unlevered CEBFs. Also both the return on NAV and the return on P are larger for levered CEBFs as opposed to unlevered CEBFs which is as expected. Table 2.7.3[11] below, which is information collected from Elton et al. [17], shows both the standard deviation of the return on net asset value NAV and the standard deviation of return on share price P.

[11]The entries labeled N/A indicates no information was provided in Elton et al. [17].

	Standard Deviation	
Bond Fund Type	NAV **Returns**	P **Returns**
All Open End Bond Funds	4.63%	1.24%
All Closed End Bond Funds	5.56%	3.40%
Unlevered Open End Bond Funds	N/A	1.19%
Unlevered Closed End Bond Funds	N/A	2.71%
Levered Open End Bond Funds	N/A	1.24%
Levered Closed End Bond Funds	N/A	3.51%

Table 2.7.3: Closed and Open End Bond Std. Dev. of Return Summary

From Table 2.7.3 we can see that both the standard deviation of the NAV returns and P returns are higher for CEBFs than for OEBFs. More important is the fact that the standard deviation of the levered returns are higher than the unlevered returns for CEBFs (as well as for OEBFs). From the table we can see that levered CEBF are riskier than unlevered CEBF. In the next chapter we develop a model for both an unlevered and levered CEBF. In Chapter 4 we use the model to generate returns from both the unlevered and levered CEBFs to compare against those from Elton et al. [17] presented in this section.

Chapter 3

Modelling Closed End Bond Funds using Structural Credit Risk Models

As we discussed in section 2.7, recent findings in Elton et al. [17] observed that almost all closed end bond funds lever their assets with debt (as well as preferred shares). They noted that leverage typically increases returns of a levered CEBF over an unlevered CEBF as well as variability in returns. This implies that the decision to lever the CEBF is a key component to the CEBF's future return performance. In this chapter, as the principle contribution of this thesis, we develop a structural model of an unlevered CEBF that is financed solely by equity and a levered CEBF which is financed by debt and equity. Both the unlevered and levered CEBF are multi-firm structural credit risk models based on Merton's risky bond model [28]. The choice for using Merton's risky bond model [28] was motivated by the empirical findings of Elton et al. [17] that we discussed in section 2.2 in that most of the bonds held by CEBFs are in fact risky bonds. In section 3.1 we extend Merton's risky bond model [28] to apply to several firms.

As a technical point, when we refer to the underlying firms of the risky bonds that the CEBF has purchased we will refer to them as *firms*. Note that this distinction is necessary as to distinguish between when we consider the debtholders of the CEBF and the bondholders of the firms in which the CEBF has invested in. Our goal is to analyze and compare the expected value and standard deviation of the equityholder market value of an unlevered CEBF with that of the debtholder and equityholder of a levered CEBF. Our analysis will focus on the impact of adding leverage, when the underlying firm correlations change, and when the firm volatilities change.

It is important to note that the market value and standard deviation of the equityholder market value of an unlevered CEBF are determined under the Risk Neutral $\mathbb{Q}$-measure. Deriving the unlevered and levered CEBF framework under the Risk Neutral $\mathbb{Q}$-measure serves as a introductory point for our results in Chapter 4. In Chapter 4 we will analyze both the unlevered and levered CEBF that are held for a period h under the Real World $\mathbb{P}$-measure and valued under the Risk Neutral $\mathbb{Q}$-measure for the remaining time $T-h$ until maturity at time T. The motivation for analyzing the standard deviation of equityholder and debtholder is driven from the standard deviation results presented in section 2.7 from Elton et al. [17].

In section 3.3 we provide numerical results to illustrate how the unlevered and levered CEBF market value and standard deviation change when the CEBF adds leverage, when firm correlations change, and when firm volatilities change. Using our model, we show that whether the correlations of the firms increase or decrease, for an unlevered CEBF, the market value remains the same. Whereas for a levered CEBF the market value of the debtholder and equityholder positions of the CEBF will change depending on whether the correlations increase or decrease. In a levered CEBF, regardless of whether the correlations increase or decrease, the sum of the market values of the debt and equityholder positions will remain the same. From seeing asset volatilities skyrocket to record highs during the *Credit Crises of 2008*, one cannot help but to consider the impact that increased volatilies will have on the unlevered and levered CEBF. We analyze the case when the firm volatilities of the risky bonds increase (in both the cases where the firm correlations are low and high), for both an unlevered and levered CEBF. We find that as the volatilities increase the market value to the equityholder of an unlevered CEBF as well as the debt and equityholder of a levered CEBF all decrease. This result implies that the total market value of the CEBF drops when firm volatilities increase, as was seen of many debt and equityholders of corporations during the *Credit Crises of 2008*.

3.1 Multi Firm Merton Model of Risky Bonds

We begin by extending the single firm debt valuation model from Merton's risky bond model [28] to a multi-firm model. Note that the notation and method of construction of a Multi Firm Merton model come from Sundaram [33]. Consider holding a portfolio of n firms each of which has risky debt (risky bond) outstanding. We denote the value of the ith firm's assets at time t as $S_i(t)$, for $i = 1, 2, ..., n$. As in Merton [28], we make the simple assumption regarding each firm's debt structure being that each firm has one issue of debt outstanding. Each debt is in the form of a zero coupon bond with face value (FV) K_i, $i = 1, 2, ..., n$, due at future time T_i, $i = 1, 2, ..., n$, respectively.

There are two claimholders being considered in this analysis, the equityholder and the bondholder. The ith bondholder receives the value K_i at maturity times T_i if $S_i(T_i) > K_i$ otherwise he/she receives the residual value of the firm's assets $S_i(T_i)$. The ith equityholder receives each of $S_i(T_i) - K_i$ if $S_i(T_i) > K_i$, otherwise they receive nothing since the bondholders are first claimholders of the firm's residual asset value in the case of default. To summarize, we have

$$\text{Bondholder payment on firm } i \text{ at time } T_i \text{ is } \begin{cases} K_i \text{ if } S_i(T_i) > K_i \\ S_i(T_i) \text{ otherwise} \end{cases}$$

$$\text{Equityholder payment on firm } i \text{ at time } T_i \text{ is } \begin{cases} S_i(T_i) - K_i \text{ if } S_i(T_i) > K_i \\ 0 \text{ otherwise} \end{cases}$$

A key result in Merton [28], was that each of the bondholder payments at time T_i can be rewritten as $K_i - (K_i - S_i(T_i))_+$ for firms $i = 1, 2, ..., n$. It is clear that the first term in the above expression, K_i, is the payoff at time T_i of a long position in a default free zero coupon bond with face value K_i that matures at time T_i. The second term, $-(K_i - S_i(T_i))_+$, is the payoff at time T_i of a short position in a put option on the assets of the firm $S_i(T_i)$ with strike price K_i at maturity date T_i.

The price of a risky bond at time t is denoted by $RB_i(t)$, for each of $i = 1, \ldots, n$, as defined in Merton [28] and can be determined as

$$RB_i(t) = K_i e^{-r(T_i - t)} - \underbrace{P_i(S_i(t), K_i, r, t, T_i, \sigma_i^2)}_{*} \tag{3.1.1}$$

where $*$ is a European put option on firm i. The option price is defined under the Risk Neutral $\mathbb{Q}$-measure as

$$P_i(S_i(t), K_i, r, t, T_i, \sigma_i^2) = e^{-r(T_i - t)} \mathbb{E}^{\mathbb{Q}} \left[(S_i(T) - K_i)_+ | \mathcal{F}_i(t) \right] \tag{3.1.2}$$

which can be evaluated using the Black and Scholes [5] European option pricing framework. Note that $\mathcal{F}_i(t)$ is defined as the sigma algebra (or available information) for $S_i(t)$ at time t. We determine the yield to maturity ($YTM_i(t)$) for each of the risky bonds $RB_i(t)$, for $i = 1, \ldots, n$ by

$$YTM_i(t) = \frac{1}{T_i - t} \log\left(K_i / RB_i(t)\right). \tag{3.1.3}$$

As is noted in Merton [28], formally (3.1.1) shows the value at time t of the decomposition at time T_i has two components, the long position in the bond and a short position in the put. Clearly only the bond is risk free whereas the put option will change in value depending on the firm value $S_i(\cdot)$. The risk inherent in the bondholder's payment is represented by the value of the short position in a put option on the assets of the firm.

3.2 Structural Modelling of Closed End Bond Funds

We assume that a closed end bond fund, denoted as $V(\cdot)$, owns n risky bonds. We will assume, for computational simplicity and for other reasons explained later, that each of the n risky bonds has time to maturity T^*. That is to say $T_i = T^*$ for all $i = 1, ..., n$ where the ith bond has time to maturity T_i. Under this definition, we have the value of the CEBF at time T^* as

$$V(T^*) = \sum_{i=1}^{n} \left[K_i - (K_i - S_i(T^*))_+\right]. \tag{3.2.1}$$

It is important to note that (3.2.1), where $S_i(t_i)$ follows a Geometric Brownian motion as in the Black Scholes framework, cannot be used to model an OEBF. In section 2.4 we noted that for OEFs the net asset value and price per share are typically set equal at the end of the trading day. A feature that is not taken into account in (3.2.1).

In section 3.2.1 we determine the market value and standard deviation of an unlevered CEBF. In section 3.2.2 we determine the market value and standard deviation of a levered CEBF for both the debtholder and the equityholder of the fund.

3.2.1 The Unlevered Closed End Bond Fund

First we consider a CEBF financed solely by equity, an unlevered CEBF, denoted by $V_{UL}(.)$ (where the subscript $_{UL}$ refers to unlevered CEBF). The CEBF owns n risky bonds, where the ith bond has time to maturity T_i, as outlined in section 3.1. At time T^*, (where $T^* < T_i$ for all $i = 1, ..., n$) the unlevered CEBF value can be written as the sum of its component risky bonds, hence

$$V_{UL}(T^*) = \sum_{i=1}^{n} \left[K_i - (K_i - S_i(T^*))_+\right]. \tag{3.2.2}$$

The market value of the unlevered CEBF $V_{UL}(T^*)$ at time t, with $t < T^*$, can be written as,

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T^*-t)}V_{UL}(T^*)|\mathcal{F}(t)\right] = \sum_{i=1}^{n} K_i e^{-r(T^*-t)} - \sum_{i=1}^{n} \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T^*-t)}(K_i - S_i(T^*))_+|\mathcal{F}(t)\right] \tag{3.2.3}$$

Note that $\mathcal{F}(t)$ is the sigma algebra (or available information) for all of $S_1(t), \ldots, S_n(t)$ at time t. The risk neutral variance of the unlevered CEBF $V_{UL}(T^*)$ has value at time t,

$$\begin{aligned}
&\mathbb{V}\mathrm{ar}^{\mathbb{Q}}\left[e^{-r(T^*-t)}V_{UL}(T^*)|\mathcal{F}(t)\right] \\
&= \mathbb{V}\mathrm{ar}^{\mathbb{Q}}\left[\sum_{i=1}^{n} (K_i e^{-r(T^*-t)} - e^{-r(T^*-t)}\,(K_i - S_i(T^*))_+)|\mathcal{F}(t)\right] \\
&= \mathbb{V}\mathrm{ar}^{\mathbb{Q}}\left[\sum_{i=1}^{n} e^{-r(T^*-t)}\,(K_i - S_i(T^*))_+|\mathcal{F}(t)\right] \\
&= \sum_{i=1}^{n} e^{-2r(T^*-t)}\,\underbrace{\mathbb{V}\mathrm{ar}^{\mathbb{Q}}\left[(K_i - S_i(T^*))_+|\mathcal{F}(t)\right]}_{*} \\
&+ \sum_{i\neq j}^{n} 2e^{-r(T^*-t)}e^{-r(T^*-t)}\,\underbrace{\mathbb{C}\mathrm{ov}^{\mathbb{Q}}\left[(K_i - S_i(T^*))_+, (K_j - S_j(T^*))_+|\mathcal{F}(t)\right]}_{**}
\end{aligned} \tag{3.2.4}$$

Note that analytical formulas for $*$ and $**$ in (3.2.4) are derived under the Black and Scholes framework in section A.1. It is also important to note that the market value and variance of the unlevered CEBF $V_{UL}(T^*)$ are conditional on observing each of $S_1(t), \ldots, S_n(t)$ at time t.

3.2.2 The Levered Closed End Bond Fund

Next we consider a CEBF financed by both debt and equity, a levered CEBF. Suppose the CEBF has one issue of debt outstanding in the form of a zero coupon bond with face value K^* and time to maturity T^*. The CEBF debtholder receives the value K^* at maturity times T^* if $V(T^*) > K^*$ otherwise he/she receives the residual value of the CEBF's assets $V(T^*)$. The CEBF debtholder payment at time T^* can rewritten as $K^* - (K^* - V(T^*))_+$ where K^* is the payoff at time T^* of a long position in a default free zero coupon bond with face value K^* that matures at time T^*. The second term,$-(K^* - V(T^*))_+$, is the payoff at time T^* of a short position in a put option on the assets of the CEBF $V(T^*)$ with strike price K^* at maturity date T^*. The equityholder receives $V(T^*) - K^*$ if $V(T^*) > K^*$ otherwise they receive nothing since the debtholder is the first claimholder of the CEBF's residual asset value in the case of default. To summarize, we have

$$\text{Debtholder payment at time } T^* \text{ is } \begin{cases} K^* \text{ if } V(T^*) > K^* \\ V(T^*) \text{ otherwise} \end{cases}$$

$$\text{Equityholder payment at time } T^* \text{ is } \begin{cases} V(T^*) - K^* \text{ if } V(T^*) > K^* \\ 0 \text{ otherwise} \end{cases}$$

We denote the value of the debtholder of the CEBF as $V_{L-dt}(\cdot)$ where the subscript $_{L-dt}$ refers to the debtholder of the levered CEBF. Similarly we denote the value of the equityholder of the CEBF as $V_{L-eq}(\cdot)$ where the subscript $_{L-eq}$ refers to the equityholder of the levered CEBF.

The value of the debtholder of the levered CEBF can be written as

$$V_{L-dt}(T^*) = K^* - (K^* - \sum_{i=1}^{n} [K_i - (K_i - S_i(T^*))_+])_+ \tag{3.2.5}$$

and similarly, the value of the equityholder of the levered CEBF can be written as

$$V_{L-eq}(T^*) = (\sum_{i=1}^{n} [K_i - (K_i - S_i(T^*))_+] - K^*)_+. \tag{3.2.6}$$

At time t, the risk neutral value of the debtholder of the levered CEBF has market value

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}} \left[e^{-r(T^*-t)} V_{L-dt}(T^*) | \mathcal{F}(t) \right] &= K^* e^{-r(T^*-t)} \\ &\quad - \mathbb{E}^{\mathbb{Q}} \left[e^{-r(T^*-t)} (K^* - \sum_{i=1}^{n} [K_i - (K_i - S_i(T^*))_+])_+ | \mathcal{F}(t) \right] \end{aligned} \tag{3.2.7}$$

and similarly the risk neutral value of the equityholder of the levered CEBF has market value

$$\mathbb{E}^{\mathbb{Q}} \left[e^{-r(T^*-t)} V_{L-eq}(T^*) | \mathcal{F}(t) \right] = \mathbb{E}^{\mathbb{Q}} \left[e^{-r(T^*-t)} (\sum_{i=1}^{n} [K_i - (K_i - S_i(T^*))_+] - K^*)_+ | \mathcal{F}(t) \right]. \tag{3.2.8}$$

The risk neutral variance of the debtholder of the levered CEBF has value at time t, that can be written as

$$\begin{aligned}&\mathbb{V}\mathrm{ar}^{\mathbb{Q}}\left[e^{-r(T^*-t)}V_{L-dt}(T^*)|\mathcal{F}(t)\right]\\&=\mathbb{V}\mathrm{ar}^{\mathbb{Q}}\left[K^*e^{-r(T^*-t)}-e^{-r(T^*-t)}(K^*-\sum_{i=1}^{n}[K_i-(K_i-S_i(T^*))_+])_+|\mathcal{F}(t)\right].\end{aligned}\qquad(3.2.9)$$

The risk neutral variance of the equityholder of the levered CEBF has value at time t, that can be written as

$$\mathbb{V}\mathrm{ar}^{\mathbb{Q}}\left[e^{-r(T^*-t)}V_{L-eq}(T^*)|\mathcal{F}(t)\right]=\mathbb{V}\mathrm{ar}^{\mathbb{Q}}\left[e^{-r(T^*-t)}(\sum_{i=1}^{n}[K_i-(K_i-S_i(T^*))_+]-K^*)_+|\mathcal{F}(t)\right].\qquad(3.2.10)$$

3.3 Numerical Results for an Unlevered and Levered Closed End Bond Fund

The market value of the unlevered CEBF given in (3.2.3) is no more than the sum of n risky bonds as defined in section 3.1. As a result we can determine the market value (using (3.2.3)) and standard deviation (using (3.2.4)) of an unlevered CEBF under the Black Scholes framework [5]. With regards to a levered CEBF since all of (3.2.7)–(3.2.10) are of the basket option form, none of the equations have closed form solutions. We use Monte Carlo simulation to determine the market value and standard deviation (Std. Dev.) for the equityholder and debtholder. We assume that the unlevered and levered CEBFs have purchased 5 risky bonds whose individual firm prices, volatility, and face value (FV) of debt are detailed in Table 3.3.1. We refer the reader to section A.2 for a more detailed outline of the economic and model assumptions used in this simulation analysis.

Firm	**Initial Firm Price**	**Firm Volatility**	**Firm FV of Debt**
1	$S_1(0)=10$	$\sigma_1=0.15$	$K_1=40$
2	$S_2(0)=20$	$\sigma_2=0.15$	$K_2=40$
3	$S_3(0)=30$	$\sigma_3=0.15$	$K_3=40$
4	$S_4(0)=40$	$\sigma_4=0.15$	$K_4=40$
5	$S_5(0)=50$	$\sigma_5=0.15$	$K_5=40$

Table 3.3.1: Closed End Fund Asset Data

Assuming the Black and Scholes framework [5], we can determine the discounted risk-free (Rf) bond price and European put option price. Using this information we can determine the risky bond (RB) price and yield to maturity (YTM) for each of the bonds in Table 3.3.1.

Firm	Rf Bond Price	Put Option Price	RB Price	YTM
1	31.15	21.153	9.995	27.73%
2	31.15	11.513	19.639	14.23%
3	31.15	4.6737	26.478	8.25%
4	31.15	1.5529	29.599	6.02%
5	31.15	0.46942	30.683	5.30%

Table 3.3.2: Risky Bond Prices

Remark 3.3.1. From Table 3.3.2 we can see that as $S_i(0)$ increases the RB price increases since the value of the put option price decreases as $S_i(0)$ increases. This causes the RB price to increase as can be seen from (3.1.1). It's evident that as the risky bond prices increase the yield to maturity also decreases.

By direct application of (3.2.3) and (3.2.4) we can determine the unlevered CEBF market value and the standard deviation. One important modelling assumption that we haven't discussed yet is the correlations of the firms in which the CEBF has purchased. We use two different firm correlation matrices for the 5 firms, one with a high correlation structure and one with a low correlation structure. This is done in order to compare the differences in resulting market values and standard deviations under two different economic settings. One correlation matrix has a high correlation structure with an average correlation of 0.79 and the other is a low correlation structure with an average correlation of 0.12. The specific details of the two correlation matrices used are outlined in section A.2.

Correlation Environment	Low Correlation	High Correlation
Unlevered CEBF Market Value	116.4	116.4
Unlevered CEBF Value Standard Deviation	10.55	16.21

Table 3.3.3: Unlevered CEBF Market Value and Standard Deviation

Using the economic assumptions of Table 3.3.2, we analyze the market value and standard deviation of the levered CEBF debt and equityholder for varying face values of CEBF debt K^*. Considering the market value of an unlevered CEBF from Table 3.3.3 of 116.4 we consider the range of FV of CEBF debt K^* to be $K^* = 90, 100, 110$, and 120. These FV of

CEBF debt correspond to a levered CEBF which employs a ratio of FV of CEBF to market value of an unlevered CEBF of roughly $77\%, 86\%, 95\%$, and 103% respectively. Note that we analyze the market value and standard deviation of the levered CEBF debtholder and equityholder under both the high and low correlation environments.

Low Correlation Environment				
CEBF Strike Price K^*	$K^* = 90$	$K^* = 100$	$K^* = 110$	$K^* = 120$
Debtholder Portion Market Value	70.09	77.88	85.66	93.36
Std. Dev. of Debtholder Value	0.024	0.119	0.312	0.883
Equityholder Portion Market Value	46.39	38.54	30.82	23.12
Std. Dev. of Equityholder Value	10.47	10.46	10.42	10.21
Total Levered CEBF Value	116.48	116.48	116.48	116.48
High Correlation Environment				
CEBF Strike Price K^*	$K^* = 90$	$K^* = 100$	$K^* = 110$	$K^* = 120$
Debtholder Portion Market Value	70.02	77.67	85.17	92.41
Std. Dev. of Debtholder Value	0.88	1.67	2.80	4.31
Equityholder Portion Market Value	46.45	38.80	31.30	24.06
Std. Dev. of Equityholder Value	15.98	15.61	14.96	13.93
Total Levered CEBF Value	116.47	116.47	116.47	116.47

Table 3.3.4: Levered CEBF Market Value and Standard Deviation

Remark 3.3.2. From Table 3.3.4 we see that changing from a low to a high correlation environment decreases the market value of the debtholder position. This increases the market value of the corresponding equityholder for a levered CEBF. We also see that changing from a low to a high correlation environment increases the standard deviation of CEBF value of both the debtholder and equityholder. This increase happens for each FV of CEBF debt K^*. Despite the change in the market values of the equity and the debt of the CEBF as the correlations change the total market value of the levered CEBF remains the same. The total market value is constant for each value of K^*.

One cannot help but think of the impact of the change in firm volatilities on the unlevered and levered CEBF market value and the standard deviation. We can determine the unlevered CEBF market value and the standard deviation assuming the firm volatilities $\sigma_i = 0.15$ and then $\sigma_i = 0.50$ for all $i = 1, \ldots, 5$.

Firm Volatilities $(i = 1, \ldots, 5)$	$\sigma_i = 0.15$	$\sigma_i = 0.50$
Unlevered CEBF Market Value	116.4	82.37
Unlevered CEBF Value Standard Deviation	10.55	26.77

Table 3.3.5: Unlevered CEBF Market Value and Standard Deviation

Similarly we analyze the market value and standard deviation of the levered CEBF for the debtholder and equityholder for varying FV of CEBF debt K^*. We assume the firm volatilities are $\sigma_i = 0.15$ and then $\sigma_i = 0.50$ for all $i = 1, \ldots, 5$. Note that we only analyze the market value and standard deviation of the levered CEBF for the debtholder and equityholder under the low correlation environment.

Firm Volatilities Equal to 0.15				
CEBF Strike Price K^*	$K^* = 90$	$K^* = 100$	$K^* = 110$	$K^* = 120$
Debtholder Portion Market Value	70.09	77.88	85.66	93.36
Std. Dev. of Debtholder Value	0.024	0.119	0.312	0.883
Equityholder Portion Market Value	46.39	38.54	30.82	23.12
Std. Dev. of Equityholder Value	10.47	10.46	10.42	10.21
Total Levered CEBF Value	116.42	116.42	116.42	116.42
Firm Volatilities Equal to 0.50				
CEBF Strike Price K^*	$K^* = 90$	$K^* = 100$	$K^* = 110$	$K^* = 120$
Debtholder Portion Market Value	64.30	69.12	73.07	76.17
Std. Dev. of Debtholder Value	11.02	13.80	16.55	19.11
Equityholder Portion Market Value	18.02	13.21	9.25	6.16
Std. Dev. of Equityholder Value	19.57	17.06	14.36	11.62
Total Levered CEBF Value	82.31	82.32	82.32	82.32

Table 3.3.6: Levered CEBF Market Value and Standard Deviation

Remark 3.3.3. From Tables 3.3.5 and 3.3.6 we see that increasing the firm volatilities from 0.15 to 0.5 decreases the market value of the debtholder and equityholder for a levered and unlevered CEBF. Since both the market values of the equity and the debt of the levered CEBF decrease, the total market value of the levered CEBF decreases as the firm volatilities increase. We also see that increasing the firm volatilities increases the standard deviation of CEBF value of both the debt and equityholder at each value of K^*. Increasing the firm volatilities also increases the standard deviation of an unlevered CEBF.

We further explore the impact of increasing firm volatilities in sections 3.3.2 and 3.3.3.

We continue next in section 3.3.1 to further analyze the impact of changing firm correlations on the market value of the unlevered and levered CEBF.

3.3.1 Impact of Varying Firm Correlations

As noticed in remark 3.3.2, the firm correlations will impact the market value and standard deviation (Std. Dev.) of the equityholder and debtholder value in a levered CEBF. The firm correlations will only impact the standard deviation of the equityholder value in an unlevered CEBF as can be seen from Table 3.3.3. To further understand the impact of changes in firm correlations on an unlevered CEBF market value and standard deviation, we vary the firm correlation assumption, holding all other model assumptions constant. We calculate (3.2.3) and (3.2.4) with the assumption of the low correlation environment, then high correlation environment. Then we increase the correlations to all firms having pair-wise correlations equal to 0.8 and then 0.9, while holding all other modelling assumptions constant. The results are presented in figure 3.3.1 below.

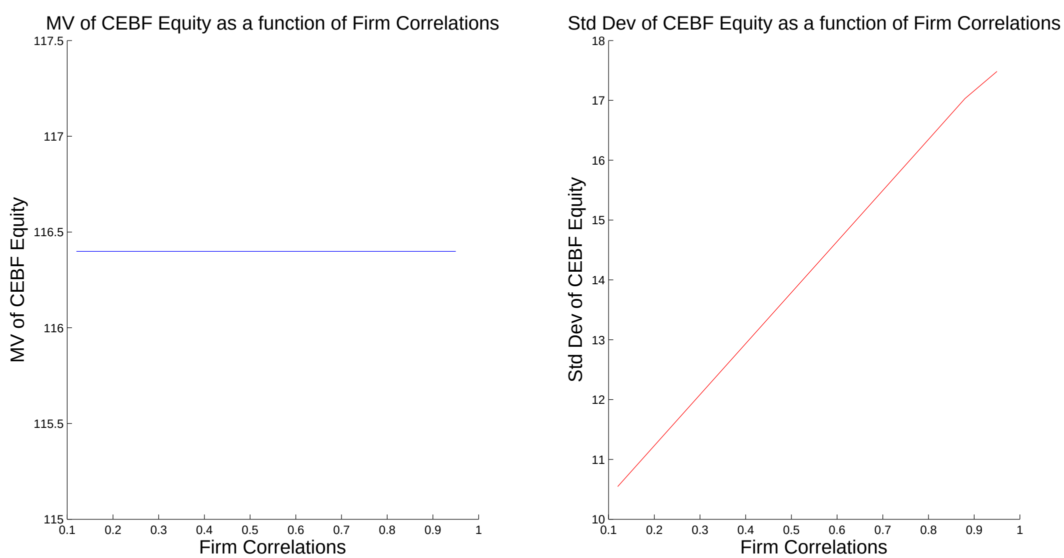

Figure 3.3.1: MV and Std. Dev. of Unlevered CEBF as a function of Firm Correlations

Remark 3.3.4. From figure 3.3.1 we see that the market value of the unlevered CEBF remains constant. This is not surprising given the result in Table 3.3.3 and the fact that

(3.2.3) is no more than the sum of n risky bonds as defined in section 3.1. The model does not depend on the firm correlations. On the other hand the standard deviation of the unlevered CEBF value linearly increases as the firm correlations increase from 0 towards 1.

Next we consider the impact of varying firm correlations on a levered CEBF, as outlined by (3.2.5) and (3.2.6), on the calculations of the market value and standard deviations of the debt and equityholder. We calculate the market values of the debt and equityholder using (3.2.7) and (3.2.8) as well as the standard deviations using (3.2.9) and (3.2.10). First we consider the low correlation environment, then high correlation environment, to all firms having pairwise correlations equal to 0.85 and then 0.95, while holding all other modelling assumptions constant. The debtholder market value and standard deviation are presented in figure 3.3.2 below and the equityholder market value and standard deviation are presented in figure 3.3.3.

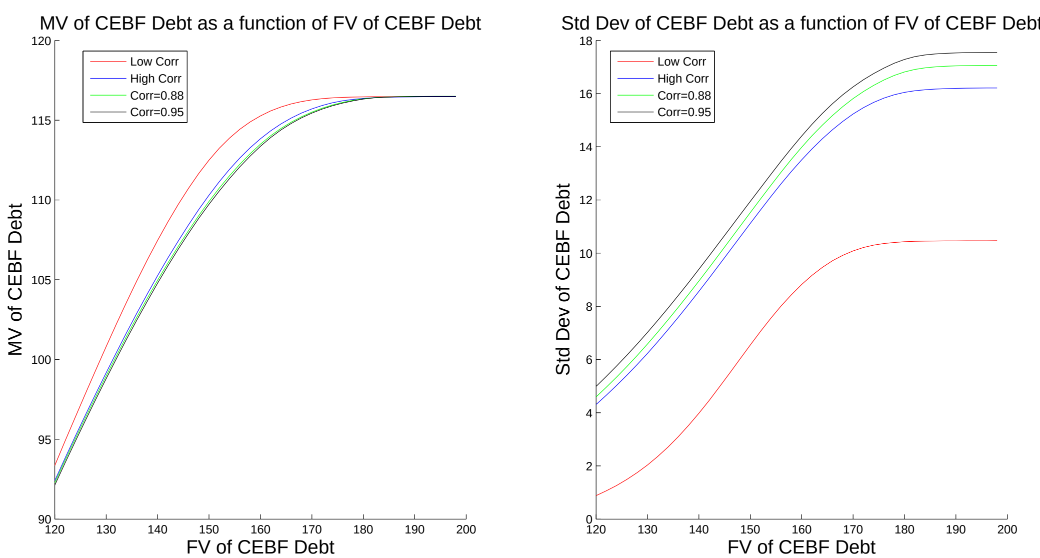

Figure 3.3.2: MV and Std. Dev. of Levered CEBF Debtholder Position as a function of Firm Correlations

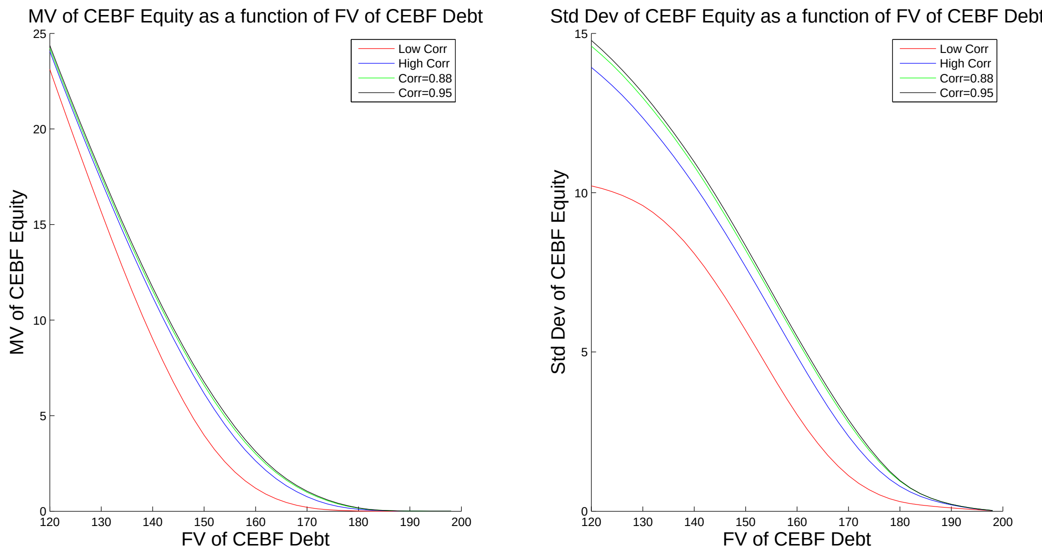

Figure 3.3.3: MV and Std. Dev. of Levered CEBF Equityholder Position as a function of Firm Correlations

Remark 3.3.5. In figure 3.3.2 as the firm correlations increase towards 1 the market value of the debtholder value decreases. On the contrary, as the firm correlations increase the standard deviation of the debtholder increases (becomes more risky). In figure 3.3.3 as the correlations increase the market value of the equityholder of the fund increases. The standard deviation of the CEBF value of the equityholder is lower when the asset correlations of the firms increase (as was the case for the standard deviation of the debtholder).

When considering a levered CEBF the market values of the equity and the debt of the CEBF will change as the correlations change. Despite the changing correlations the total market value of the CEBF remains the same (as was noted in remark 3.3.2). However, we noticed that the put option inherent in each of the credit risky bonds becomes more valuable as the correlations increase. This implies that the risky CEBF debt becomes more risky and hence less valuable which in turn means that the market value of the CEBF equity becomes more valuable.

3.3.2 Impact of Varying Firm Volatilities - Low Correlation Environment

Of particular interest is the impact of changes in the firm volatilities on the market value and standard deviation of the unlevered and levered CEBF. We calculate (3.2.3) and (3.2.4) when $\sigma_i, i = 1, 2, ..., 5$ are increased from 0.15 to $0.25, 0.5$, and 0.85 holding all other model assumptions constant. Note that we assume the low correlation environment when determining (3.2.3) and (3.2.4). The results are presented in figure 3.3.4 below.

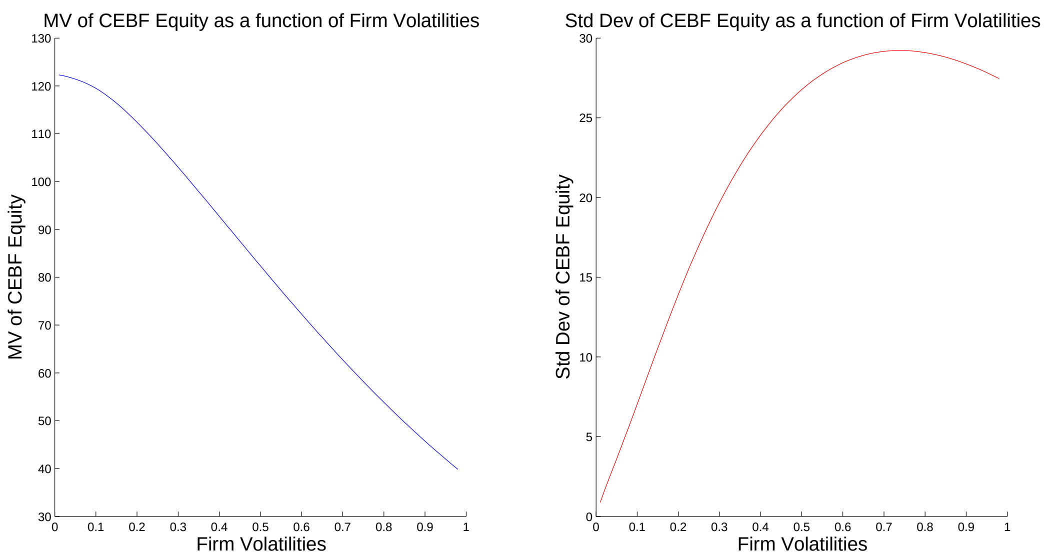

Figure 3.3.4: MV and Std. Dev. of Unlevered CEBF as a function of Firm Volatilities - Low Correlation Environment

Remark 3.3.6. From figure 3.3.4 we see that the market value of the unlevered CEBF decreases and the standard deviation of the unlevered CEBF increases while the firm volatilities increase. The market value of the unlevered CEBF decreases as the firm volatilities increase since the put options inherent in each of the risky bonds increase in value as the firm volatilities increase. This decreases the value of the risky bonds hence decreasing the market value of the unlevered CEBF.

The standard deviation of the unlevered CEBF increases then peaks and begins to decrease at high values of firm volatility. The unlevered CEBF becomes relatively riskier as the firm volatilities increase since the market value of the unlevered CEBF also decreases as the firm volatilities increase. For firm volatilities greater than 0.8 the unlevered CEBF becomes less risky as the rate of decrease of the market value of the unlevered CEBF decreases for high values of firm volatilities.

Next we consider the impact of varying firm volatilities on a levered CEBF on the calculations of the market value and standard deviations of the debt and equityholder. We calculate the market values of the debt and equityholder using (3.2.7) and (3.2.8) as well as the standard deviations using (3.2.9) and (3.2.10). As was the case for the unlevered CEBF, we assume the low correlation environment in this section while holding all other modelling assumptions constant. The debtholder market value and standard deviation are presented in figure 3.3.5 below. The equityholder market value and standard deviation are presented in figure 3.3.6.

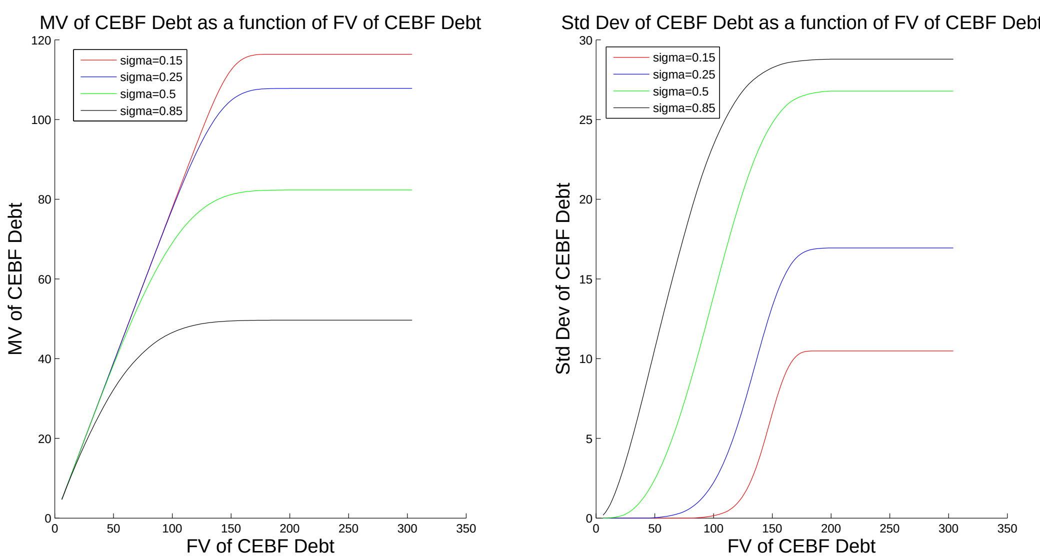

Figure 3.3.5: MV and Std. Dev. of Levered CEBF Debtholder Position as a function of Firm Volatilities - Low Correlation Environment

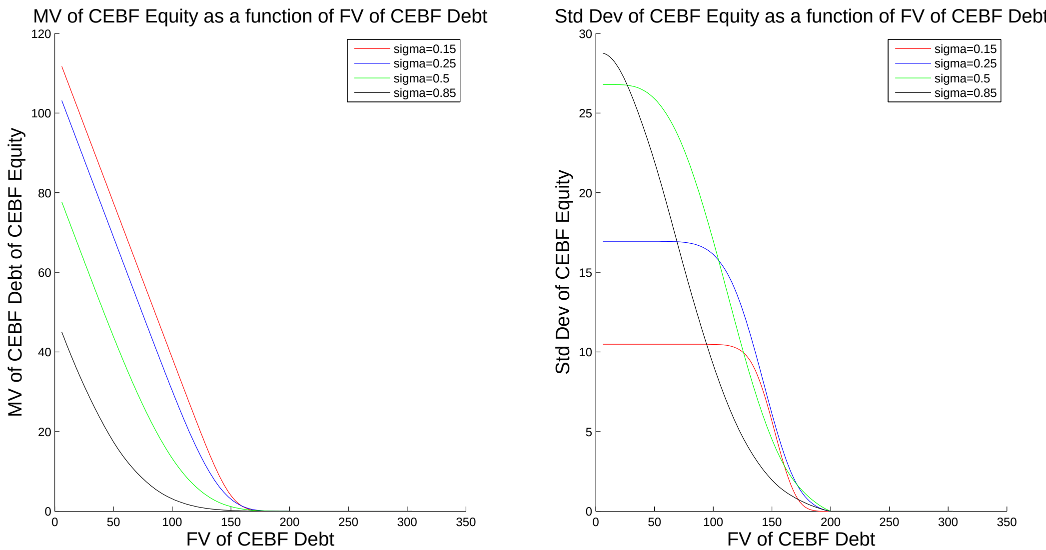

Figure 3.3.6: MV and Std. Dev. of Levered CEBF Debtholder Position as a function of Firm Volatilities - Low Correlation Environment

Remark 3.3.7. From figure 3.3.5 we see that as the firm volatilies increase the debtholder market value decreases. As the face value of CEBF debt increases the larger the decrease in market value. The levered CEBF debtholder market value decreases as the firm volatilities increase since the put options inherent in each of the risky bonds that the CEBF holds increase in value as the firm volatilities increase. As the put options increase in value this decreases the value of the risky bonds that the CEBF holds which increases the value of the put option inherent to the debtholder of the CEBF which decreases the market value of debtholder position of the CEBF. When the firm volatilies increase the standard deviation of the CEBF debtholder value increases since an increase in the firm volatilities increases the value of the put option inherent in the risky bonds. Increasing the value of the inherent put options of the risky bonds decreases the value of the risky bonds. Decreasing the value of the risky bonds increases the value of the put option inherent on the CEBF debt and hence increases the standard deviation of the CEBF debtholder value. As the face value of CEBF debt K^* inceases the standard deviation of the CEBF debtholder value increases. This is due to the fact that an increase in the face value of the CEBF increases the value of the put option inherent to the CEBF debtholder which increases the standard deviation of the CEBF debtholder value.

With respect to the market value of the CEBF equityholder, from figure 3.3.6 we see that increasing the firm volatilities decreases the market value of the equityholder position. As the FV of CEBF debt increases the market value of the equityholder position decreases. The market value of the levered CEBF equityholder position decreases as the firm volatilities increase since the put options inherent in each of the risky bonds that the CEBF holds increase in value as the firm volatilities increase. As the put options increase in value this decreases the value of the risky bonds that the CEBF holds which decreases the value of the inherent call option of the equityholder on the CEBF. Increasing the firm volatilies decreases the standard deviation of the equityholder value. As the firm volatilities increase the value of the put option inherent in the risky bonds increases. This decreases the value of the risky bonds and decreases the value of the call option inherent to the CEBF equityholder. Since the call option inherent to the CEBF equityholder has decreased the standard deviation of the CEBF equityholder also decreases. As the face value of CEBF debt K^* increases the CEBF equityholder standard deviation decreases. This is due to the fact that an increase in the face value of the CEBF debt K^* decreases the value of the call option inherent to the CEBF equityholder which decreases the CEBF equityholder standard deviation.

Remark 3.3.8. Recall that in remark 3.3.6 we noted that increasing the firm volatilities decreases market value of the unlevered CEBF and increases the standard deviation of the unlevered CEBF. Since we noted in remark 3.3.7 that increasing the firm volatilities decreases both the levered CEBF debt and equityholder market values (and hence the total CEBF market value) we can conclude that increasing the firm volatilities decreases the market value of the CEBF regardless of whether or not the CEBF is unlevered or levered.

3.3.3 Impact of Varying Firm Volatilities - High Correlation Environment

In section 3.3.2 we analyzed the impact of changes in firm volatilities on an unlevered and levered CEBF market value and standard deviation assuming the low correlation environment. In this section we determine the impact of changes in firm volatilities on an unlevered and levered CEBF market value and standard deviation assuming the high correlation environment. For an unlevered CEBF market value and standard deviation, we calculate (3.2.3) and (3.2.4) when $\sigma_i, i = 1, 2, ..., 5$ are increased from 0.15 to $0.25, 0.5$, and 0.85 under a high correlation environment and present the results in figure 3.3.7 below.

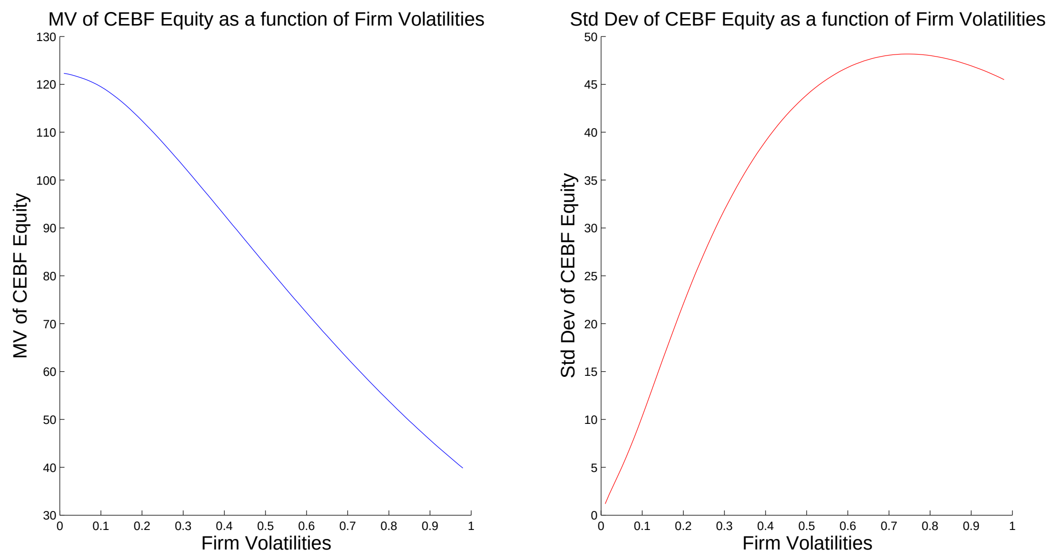

Figure 3.3.7: MV and Std. Dev. of Unlevered CEBF as a function of Firm Volatilities - High Correlation Environment

Remark 3.3.9. From figure 3.3.7 we see that the market value of the unlevered CEBF decreases and the standard deviation unlevered CEBF increases while the firm volatilities increase. When comparing figure 3.3.7 to figure 3.3.4, when the low correlation environment was used, we note that there is no difference in the market value of the unlevered CEBF curves (whether the low or high correlation environments were used). This is due to the fact that changes in the firm correlations do not impact the market value of the unlevered CEBF, as was noted in remark 3.3.4.

Note, however, that there is a difference in the standard deviation of the unlevered CEBF from a low to high correlation environment in that the higher correlation environment leads to a higher standard deviation of the unlevered CEBF for all levels of firm volatilities (as can been seen in the difference in standard deviation of the unlevered CEBF from figures 3.3.4 and 3.3.7). As in figure 3.3.4 the standard deviation of the unlevered CEBF increases then peaks and begins to decrease at high values of firm volatility (the explanation for this is discussed in remark 3.3.6).

Next we consider the impact of varying firm volatilities on a levered CEBF, as outlined by (3.2.5) and (3.2.6), and the impact on the market value and standard deviations of the debt and equityholder. We calculate the market values of the debt and equityholder positions using (3.2.7) and (3.2.8) as well as the standard deviations using (3.2.9) and (3.2.10) assuming the high correlation environment while holding all other modelling assumptions constant. The debtholder market value and standard deviation are presented in figure 3.3.8 below. The equityholder market value and standard deviation are presented in figure 3.3.9.

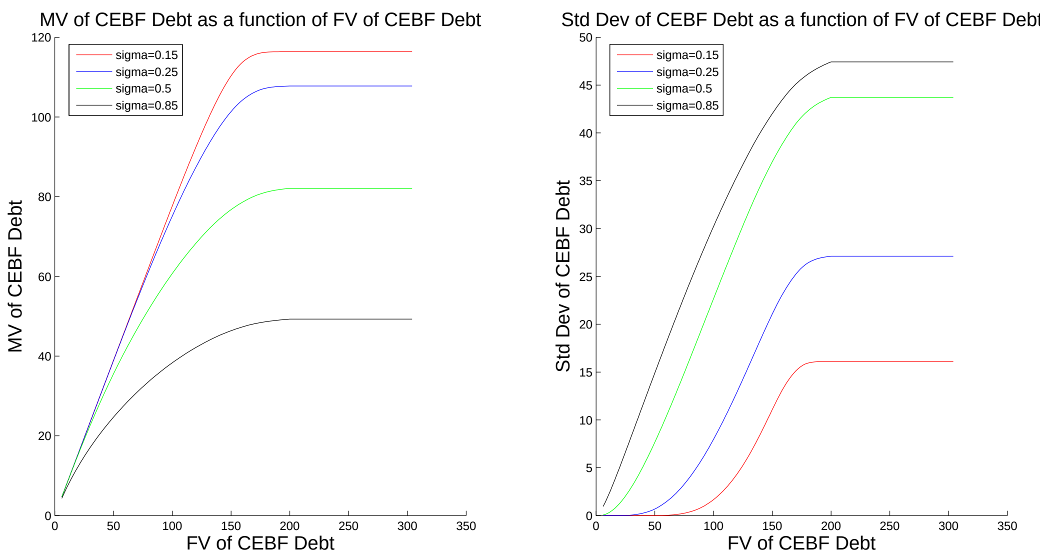

Figure 3.3.8: MV and Std. Dev. of Levered CEBF Debtholder Position as a function of Firm Volatilities - High Correlation Environment

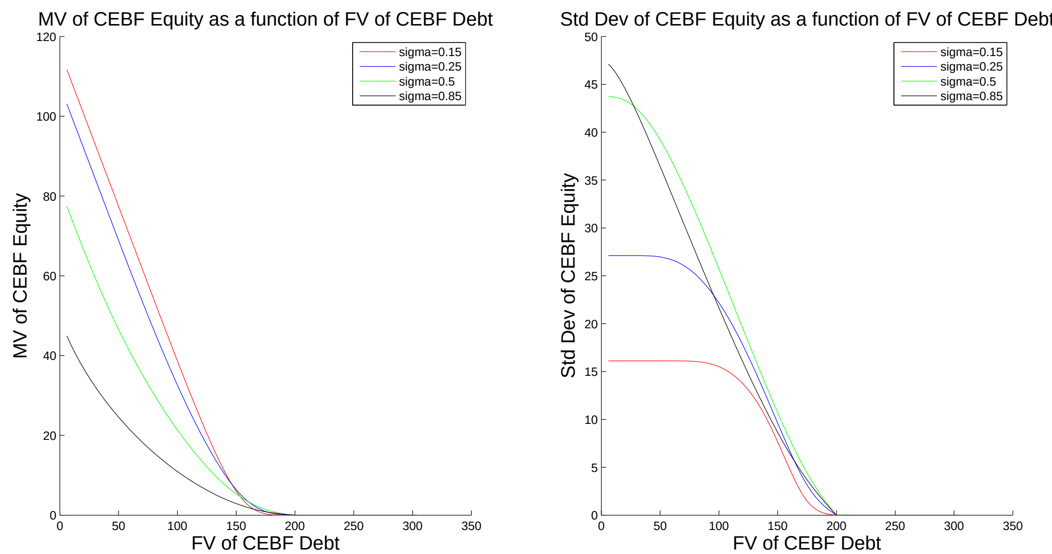

Figure 3.3.9: MV and Std. Dev. of Levered CEBF Equityholder Position as a function of Firm Volatilities - High Correlation Environment

Remark 3.3.10. When comparing the market values and standard deviations of the CEBF debtholder under a low correlation, figure 3.3.5, and high correlation environment, figure 3.3.8, there are minimal differences in the market values but the high correlation environment yields a higher standard deviation for all firm volatilities. When comparing the market values and standard deviations of the CEBF equityholder under a low correlation, figure 3.3.6, and high correlation environment, figure 3.3.9, again there are minimal differences in the market values but the high correlation environment yields a higher standard deviation for all firm volatilities.

Chapter 4

Portfolio Analysis of Closed End Bond Funds

In Chapter 3 we developed a model to determine the market value and standard deviation of an unlevered CEBF and a levered CEBF (for both the debt and equityholders). As mentioned in section 2.7.4, Elton et al. [17] noted that leverage typically increases expected returns and the variance of returns of a levered CEBF over an unlevered CEBF. In this chapter we move to provide a more mathematical and intuitive explanation of why a levered CEBF has higher expected returns and variance over an unlevered CEBF. To determine the expected return of both an unlevered and levered CEBF we first begin by introducting and extending the work of Rubinstein [31]. Rubinstein [31] determined the expected return of a European vanilla option held for a period h under the Real World $\mathbb{P}$-measure and valued under the Risk Neutral $\mathbb{Q}$-measure for the remaining time $T-h$ until option maturity at time T, where $T > h$. Next we proceed to determine the return on a credit risky bond and then to an unlevered and levered CEBF (as defined in sections 3.2.1 and 3.2.2). Our goal is to compare the expected return and Sharpe Ratio (as defined in Sharpe [32]) of an unlevered CEBF to the equityholder and debtholder Sharpe ratios of a levered CEBF. We also compare our expected returns and variance of returns against those from Elton et al. [17] presented in section 2.7.4.

4.1 Expected rate of return of a European Put Option

What is the rate of return on holding a European call or put option? This interesting question was answered by Rubinstein's [31] option pricing framework[1], who defines the rate of return on an European put option held for a period of h as,

$$\mathbb{E}^{\mathbb{P}}\left[R_P(h)\right] = \left[\frac{\mathbb{E}^{\mathbb{P}}\left[P(S(h), K, r, h, T, \sigma^2)\right]}{P(S(0), K, r, 0, T, \sigma^2)}\right]^{1/h} - 1 \tag{4.1.1}$$

where the denominator is the price of a European put option using the classical Black and Scholes European option pricing framework. From (4.1.1) we can see that the option is held for a period h under the Real World $\mathbb{P}$-measure and valued under the Risk Neutral $\mathbb{Q}$-measure for the remaining time $T - h$ until option maturity at time T, where $T > h$. From Rubinstein's [31] option pricing framework, the numerator is the expected value of a European put option evaluated assuming the following asset price process $S(t)$ (pointed out by Cheng [9]) on the time interval $0 \leq t \leq T$,

$$\begin{aligned} dS(t) &= \mu S(t)dt + \tilde{\sigma} S(t) dW_t \text{ for } 0 \leq t \leq h \\ dS(t) &= rS(t)dt + \sigma S(t) dW_t \text{ for } h \leq t \leq T. \end{aligned} \tag{4.1.2}$$

where μ and $\tilde{\sigma}$ are the Geometric Brownian Motion (GBM) model parameters on the interval $0 \leq t \leq h$ and r and σ are the GBM model parameters on the interval $h \leq t \leq T$. In summary, using (3.1.2), this requires us to determine

$$\mathbb{E}^{\mathbb{P}}\left[P(S(h), K, r, h, T, \sigma^2)\right] = \mathbb{E}^{\mathbb{P}}\left[e^{-r(T-h)}\mathbb{E}^{\mathbb{Q}}\left[(S(T) - K)_+ | \mathcal{F}(h)\right]\right] \tag{4.1.3}$$

which results in,

$$\mathbb{E}^{\mathbb{P}}\left[P(S(h), K, r, h, T, \sigma^2)\right] = Ke^{-r(T-h)}\Phi(-\tilde{x} + 0.5\sigma^*\sqrt{T}) - S(0)e^{\mu h}\Phi(-\tilde{x} - 0.5\sigma^*\sqrt{T}) \tag{4.1.4}$$

where

$$\begin{aligned} \sigma^* &= \sqrt{\tilde{\sigma}^2 \tfrac{h}{T} + \sigma^2 \tfrac{(T-h)}{T}} \qquad & m &= \mu - \tfrac{1}{2}\tilde{\sigma}^2 \\ \tilde{x} &= \tfrac{\log\left(S(0)\mu'/Ke^{-r(T-h)}\right)}{\sigma^*\sqrt{T}} \qquad & \mu' &= e^{\mu h} \end{aligned} \tag{4.1.5}$$

For a proof see section B.2.1.

Although Rubinstein [31] assumes different volatilities, σ and $\tilde{\sigma}$, in (4.1.2) there is no reason for them not to be equal. For the remainder of this chapter we will assume that σ and $\tilde{\sigma}$ are equal for all asset price processes that follow (4.1.2).

[1] The notation used in this chapter with regards to Rubinstein's [31] framework follows his original work, however, we also make extensive use of the measure theoretic notation of his framework that was outlined by Cheng [9].

4.2 Expected rate of return of a risky bond

We define the annualized expected rate of return of a risky bond over a holding period h, denoted as $\mathbb{E}^{\mathbb{P}}[R_{RB}(h)]$ (where the subscript $_{RB}$ denotes risky bond), to be

$$\mathbb{E}^{\mathbb{P}}[R_{RB}(h)] = \left[\frac{\mathbb{E}^{\mathbb{P}}[RB(S(h), K, r, h, T, \sigma^2)]}{RB(S(0), K, r, 0, T, \sigma^2)}\right]^{1/h} - 1 \tag{4.2.1}$$

The denominator of (4.2.1) can be determined from (3.1.1) and the numerator can be obtained from (4.1.4). The numerator results in

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[RB(S(h), K, r, h, T, \sigma^2)\right] &= Ke^{-r(T-h)} - Ke^{-r(T-h)}\Phi(-\tilde{x} + 0.5\sigma^*\sqrt{T}) \\ &\quad + S(0)e^{\mu h}\Phi(-\tilde{x} - 0.5\sigma^*\sqrt{T}).\end{aligned} \tag{4.2.2}$$

For a proof of (4.2.2) see section B.2.1.

4.3 Portfolio Analysis for an Unlevered Closed End Bond Fund

Now we extend the result of one risky bond presented in section 4.2 to a portfolio of risky bonds to form a CEBF as defined in (3.2.1). Note that for notational convenience, we have for each of $i = 1, \ldots, n$ risky bonds, as defined in (3.1.1),

$$RB_i(h) = RB(S_i(h), K_i, r, h, T_i, \sigma_i^2). \tag{4.3.1}$$

We define the annualized expected rate of return of an unlevered CEBF over a holding period h, denoted as $\mathbb{E}^{\mathbb{P}}[R_{UL}(h)]$ (where the subscript $_{UL}$ denotes unlevered CEBF), to be

$$\mathbb{E}^{\mathbb{P}}[R_{UL}(h)] = \left[\sum_{i=1}^{n} \frac{\mathbb{E}^{\mathbb{P}}(RB_i(h))}{\sum_{i=1}^{n} RB_i(0)}\right]^{1/h} - 1. \tag{4.3.2}$$

We can determine the annualized variance[2] of the rate of return over a holding period h of an unlevered CEBF under Rubinstein's [31] option pricing framework as

$$\begin{aligned}\mathbb{V}\mathrm{ar}^{\mathbb{P}}\left[R_{UL}(h)\right] &= \mathbb{V}\mathrm{ar}^{\mathbb{P}}\left[\sum_{i=1}^{n}\frac{RB_i(h)}{\sqrt{h}\sum_{i=1}^{n}RB_i(0)}\right] \\ &= \frac{1}{h(\sum_{i=1}^{n}RB_i(0))^2}\mathbb{V}\mathrm{ar}^{\mathbb{P}}\left[\sum_{i=1}^{n}RB_i(h)\right] \\ &= \sum_{i=1}^{n}\frac{1}{h(\sum_{i=1}^{n}RB_i(0))^2}\underbrace{\mathbb{V}\mathrm{ar}^{\mathbb{P}}\left[RB_i(h)\right]}_{*} \\ &+ 2\sum_{i\neq j}^{n}\frac{1}{h(\sum_{i=1}^{n}RB_i(0))^2}\underbrace{\mathbb{C}\mathrm{ov}^{\mathbb{P}}\left[RB_i(h), RB_j(h)\right]}_{**} \end{aligned} \tag{4.3.3}$$

For a portfolio of risky bonds, as defined in (3.2.1), under Rubinstein's [31] option pricing framework, h is limited to the life of the shortest option time to expiration (or less). Note that analytical formulas for $*$ and $**$ in (4.3.3) are derived under Rubinstein's [31] framework in section B.2.2 and section B.2.3.

As per Sharpe [32], we define the Sharpe Ratio for an unlevered CEBF over a holding period h to be

$$\lambda_{UL}(h) = \frac{\mathbb{E}^{\mathbb{P}}\left[R_{UL}(h)\right] - r}{\sqrt{\mathbb{V}\mathrm{ar}^{\mathbb{P}}(R_{UL}(h))}}. \tag{4.3.4}$$

4.4 Portfolio Analysis for Levered Closed End Bond Fund

Next we extend the results presented in section 4.3 from an unlevered CEBF to a levered CEBF that includes a debtholder and equityholder as defined in section 3.2.2. We define the annualized expected rate of return to the equityholder of levered CEBF over a holding period h, denoted as $\mathbb{E}\left[R_{L-eq}(h)\right]$ (where the subscript $_{L-eq}$ denotes equityholder portion

[2]As noted on page 331 of Cox and Rubinstein [14] there is no clear way to annualize volatility, hence we proceed as Cox and Rubinstein did to annualize by dividing by $\sqrt{h}$.

of the levered CEBF), under Rubinstein's [31] option pricing framework, as

$$\mathbb{E}^{\mathbb{P}}\left[R_{L-eq}(h)\right] = \left[\frac{\mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T^*-h)}\tilde{V}_{L-eq}(T^*)|\mathcal{F}(h)\right]\right]}{\mathbb{E}^{\mathbb{Q}}\left[e^{-rT^*}V_{L-eq}(T^*)\right]}\right]^{1/h} - 1 \tag{4.4.1}$$

We define the annualized expected rate of return to the debtholder of levered CEBF over a holding period h, denoted as $\mathbb{E}\left[R_{L-dt}(h)\right]$ (where the subscript $_{L-dt}$ denotes debtholder portion of the levered CEBF), under Rubinstein's [31] option pricing framework, as

$$\mathbb{E}^{\mathbb{P}}\left[R_{L-dt}(h)\right] = \left[\frac{\mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T^*-h)}\tilde{V}_{L-dt}(T^*)|\mathcal{F}(h)\right]\right]}{\mathbb{E}^{\mathbb{Q}}\left[e^{-rT^*}V_{L-dt}(T^*)\right]}\right]^{1/h} - 1 \tag{4.4.2}$$

The variance of the rate of return to the equityholder of the levered CEBF over a holding period h is

$$\mathbb{V}\mathrm{ar}^{\mathbb{P}}(R_{L-eq}(h)) = \frac{1}{h\mathbb{E}^{\mathbb{Q}}\left[e^{-rT^*}V_{L-eq}(T^*)\right]^2}\mathbb{V}\mathrm{ar}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T^*-h)}\tilde{V}_{L-eq}(T^*)|\mathcal{F}(h)\right]\right]. \tag{4.4.3}$$

The variance of the rate of return to the debtholder of the levered CEBF over a holding period h is

$$\mathbb{V}\mathrm{ar}^{\mathbb{P}}(R_{L-dt}(h)) = \frac{1}{h\mathbb{E}^{\mathbb{Q}}\left[e^{-rT^*}V_{L-dt}(T^*)\right]^2}\mathbb{V}\mathrm{ar}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T^*-h)}\tilde{V}_{L-dt}(T^*)|\mathcal{F}(h)\right]\right]. \tag{4.4.4}$$

We define the Sharpe Ratio, over a holding period h, for the equityholder of a levered CEBF as

$$\lambda_{L-eq}(h) = \frac{\mathbb{E}^{\mathbb{P}}\left[R_{L-eq}(h)\right] - r}{\sqrt{\mathbb{V}\mathrm{ar}^{\mathbb{P}}(R_{L-eq}(h))}} \tag{4.4.5}$$

and similarly the debtholder Sharpe Ratio for levered CEBF is defined as

$$\lambda_{L-dt}(h) = \frac{\mathbb{E}^{\mathbb{P}}\left[R_{L-dt}(h)\right] - r}{\sqrt{\mathbb{V}\mathrm{ar}^{\mathbb{P}}(R_{L-dt}(h))}}. \tag{4.4.6}$$

4.5 Numerical Examples for Portfolios of Levered and Unlevered Closed End Bond Funds

In this section we present numerical results of expected rate of return and standard deviation of the rate return of the unlevered and levered CEBF using the framework in sections 4.3 and 4.4. Note that we will use the same economic and modeling assumptions as in section 3.3. Table 4.5.1 lists the additional model assumptions required for our analysis. Note that in this section we only present the results using the low correlation matrix assumption introduced in section 3.3.

Asset	**Price t=0**	**RW Asset ERR**	**RW Asset Volatility**
1	$S_1(0) = 10$	$\mu_1 = 0.078$	$\tilde{\sigma}_1 = 0.15$
2	$S_2(0) = 20$	$\mu_2 = 0.078$	$\tilde{\sigma}_2 = 0.15$
3	$S_3(0) = 30$	$\mu_3 = 0.078$	$\tilde{\sigma}_3 = 0.15$
4	$S_4(0) = 40$	$\mu_4 = 0.078$	$\tilde{\sigma}_4 = 0.15$
5	$S_5(0) = 50$	$\mu_5 = 0.078$	$\tilde{\sigma}_5 = 0.15$

Table 4.5.1: Closed End Bond Fund Asset Data

First we determine the expected rate of return of each risky bond using (4.2.2). We also determine the rate of return on the inherent put option in the risky bond using (4.1.1) for each of the firms listed in Table 4.5.1 for holding periods of $h = 1$ and $h = 2$.

Firm i	h=1		h=2	
	Put Option Return	**RB Return**	**Put Option Return**	**RB Return**
1	3.72%	8.11%	3.69%	8.11%
2	0.63%	7.76%	0.52%	7.74%
3	−3.71%	6.69%	−3.90%	6.64%
4	−7.93%	5.81%	−8.16%	5.78%
5	−11.61%	5.38%	−11.85%	5.37%

Table 4.5.2: Risky Bond Returns

Remark 4.5.1. From Table 4.5.2 we can see that as the initial firm price $S_i(0)$ increases the put option return decreases and quickly becomes negative. Negative returns on European put options in reality are certainly possible and well discussed in Coval and Shumway [13]. The reason that the returns become negative as $S_i(0)$ increases is that the numerator in

(4.1.1) decreases in value faster than the denominator. This is due to the fact that $S_i(0)$ grows exponentially in terms of μ_i over the period h. The larger the value of $S_i(0)$ the faster the numerator will decrease in value over the denominator in (4.1.1).

As $S_i(0)$ increases the rates of return on the risky bonds decrease since, per (4.2.2), as $S_i(0)$ increases the value of a risky bond will increase. This is due to the fact that the value of the inherent put option on the risky bond decreases as $S_i(0)$ increases. The reason that the rate of return on a risky bond decreases even though the value of the risky bond increases is that in the ratio (4.2.1) both the numerator and the denominator are increasing at a decreasing rate. This is due to the fact that $S_i(0)$ grows exponentially in terms of μ_i over the period h and the inherent put option become less in-the-money faster.

We also determine the standard deviation of return of each risky bond for each of the firms listed in Table 4.5.1 for holding periods of $h = 1$ and $h = 2$.

Firm i	**RB Std. Dev.** $h = 1$	**RB Std. Dev.** $h = 2$
1	16.29%	17.07%
2	14.17%	15.00%
3	8.30%	8.50%
4	3.74%	3.83%
5	1.47%	1.56%

Table 4.5.3: Risky Bond Returns

Remark 4.5.2. From Table 4.5.3 we can see that as the initial firm price $S_{0(i)}$ increases the risky bond return standard deviation decreases. The reason that the standard deviation decreases as $S_i(0)$ increases is that the inherent put option becomes less in-the-money as $S_i(0)$ increases. As a put option becomes less in-the-money its standard deviation will decrease.

Next we determine the expected rate of return and standard deviation of return, using Rubinstein's framework, for an unlevered CEBF using (4.3.2) and (4.3.3) respectively. Using this information we can determine the Sharpe Ratio for an unlevered CEBF as per (4.3.4) for holding periods (h) of one and two years.

Holding Period	$\mathbb{E}^{\mathbb{P}}[R_{UL}(h)]$	$\sqrt{\mathbb{V}\mathbf{ar}^{\mathbb{P}}[R_{UL}(h)]}$	$\lambda_{UL}(h)$
$h = 1$	6.42%	3.96%	0.3597
$h = 2$	6.40%	4.13%	0.3392

Table 4.5.4: Unlevered CEBF Expected Returns as a function of holding period (h)

Remark 4.5.3. From Table 4.5.4 we see that for an unlevered CEBF the Sharpe Ratio decreases as the holding period of return h increases. The decrease in Sharpe Ratio is driven by a decreasing unlevered CEBF expected return and an increasing standard deviation of return as h increases.

Since the expected return and standard deviation of return ((4.4.1) and (4.4.3) respectively) for the equityholder of the levered CEBF are of the basket option form, neither of the equations have closed form solutions. We use nested Monte Carlo simulation to evaluate (4.4.1) and (4.4.3). The outer set of nested simulations are performed across the Real World $\mathbb{P}$-measure and inner nested simulations are valued under the Risk Neutral $\mathbb{Q}$-measure. Note that we value the expected return and standard deviation of return for different FV of CEBF debt K^* as was done in section 3.3.

Holding Period	**CEBF debt** K^*	$\mathbb{E}^{\mathbb{P}}\left[R_{L-eq}(h)\right]$	$\sqrt{\mathbb{V}\mathbf{ar}^{\mathbb{P}}\left[R_{L-eq}(h)\right]}$	$\lambda_{L-eq}(h)$
$h=1$	10	6.54%	4.25%	0.3626
	20	6.65%	4.58%	0.3604
	30	6.78%	4.97%	0.3583
$h=2$	10	6.48%	4.43%	0.3336
	20	6.58%	4.77%	0.3314
	30	6.70%	5.13%	0.3292

Table 4.5.5: Levered Closed End Fund Equityholder Performance

Remark 4.5.4. From Table 4.5.5, for a fixed FV of CEBF debt K^*, the Sharpe Ratio for the equityholder of the levered CEBF decreases as h increases. The decrease in the Sharpe Ratio for the equityholder of a levered CEBF is driven by a decrease in the expected rate of return and an increase in the standard deviation of return for the levered CEBF as h increases. As the FV of CEBF debt K^* increases the expected rate of return and standard deviation of the rate of return both increase which also causes the Sharpe Ratio to increase.

Next we determine the expected rate of return and standard deviation of the rate return as well as Sharpe Ratio for the debtholder of a levered CEBF using (4.4.2), (4.4.4), and (4.4.6). Similarly for the debtholder of the levered CEBF, we use nested Monte Carlo simulation to evaluate (4.4.2) and (4.4.4). Again the outer set of nested simulations are performed under the Real World $\mathbb{P}$-measure and inner nested simulations are valued under the Risk Neutral $\mathbb{Q}$-measure.

Remark 4.5.5. Debtholder return is 5.13% for both $h=1$ and $h=2$, for the same face values of CEBF debt K^* in Table 4.5.5, with a standard deviation of the rate return of virtually zero. The near zero standard deviation does not come as a surprise since for low face values of CEBF debt K^* the inherent put option to the debtholder will be out of the money. The debtholder will thus receive a virtually riskless expected rate of return for low face values of CEBF debt K^*. As the face values of CEBF debt K^* increases (as the CEBF takes on more debt) the debtholder standard deviation of the rate return increases (the risk increases). The debtholder standard deviation of the rate return reaches 1% at roughly $K^* = 140$ in our example.

4.6 Comparing Portfolio Returns

We reduce the firm volatilities from $\sigma_i = 0.15$ to $\tilde{\sigma}_i = 0.1$ for $i = 1, \ldots, 5$ in Table 3.3.1 (and subsequently in Table 4.5.1). We regenerate the expected rate of return and standard deviation of return for an unlevered CEBF in Table 4.5.4 with the firm volatilties of $\tilde{\sigma}_i = 0.1$ for $i = 1, \ldots, 5$.

Holding Period	$\mathbb{E}^{\mathbb{P}}\left[R_{UL}(h)\right]$	$\sqrt{\mathbb{V}\mathbf{ar}^{\mathbb{P}}\left[R_{UL}(h)\right]}$	$\lambda_{UL}(h)$
$h = 1$	6.34%	2.61%	0.5151
$h = 2$	6.32%	2.72%	0.4860

Table 4.6.1: Unlevered CEBF Expected Returns as a function of holding period (h)

Remark 4.6.1. We find our expected unlevered CEBF return of 6.34% below the overall average of 6.35% and lower than the non-municipal fund average 6.83% in Table 2.7.1. Our expected unlevered CEBF return is slightly lower than the return on share price P and higher than the return on NAV from Table 2.7.2. Note, however, that our expected unlevered CEBF return has not been adjusted for expenses in order to be compared to the return on NAV. When reducing the unlevered return by 65bps (the estimated management fee discussed in section 2.7.3), our unlevered return is comparable to the return on NAV for an unlevered CEBF.

When comparing the standard deviation of returns from Table 4.6.1 to that of Elton et al. [17] in Table 2.7.3 for unlevered CEBFs we find comparable results of 2.61% to 2.71%.

Next we determine the expected return and standard deviation of return for different FV of CEBF debt K^* for the equityholder with the firm volatilties of $\tilde{\sigma}_i = 0.1$ for $i = 1, \ldots, 5$.

Holding Period	**CEBF debt** K^*	$\mathbb{E}^{\mathbb{P}}\left[R_{L-eq}(h)\right]$	$\sqrt{\mathbb{V}\mathbf{ar}^{\mathbb{P}}\left[R_{L-eq}(h)\right]}$	$\lambda_{L-eq}(h)$
$h=1$	10	6.39%	2.83%	0.4892
	20	6.48%	3.05%	0.4860
	30	6.59%	3.29%	0.4829
	40	6.72%	3.58%	0.4798
	50	6.87%	3.93%	0.4767

Table 4.6.2: Levered Closed End Fund Equityholder Performance

Remark 4.6.2. Comparing the expected levered CEBF equityholder returns in Table 4.6.2 to the overall average CEBF return in Table 2.7.1 our returns are comparable to the overall average of 6.35%. When comparing our results against those of Elton et al. in table 2.7.2 for levered funds our equityholder returns are lower than the return on share price of 8.08%. We can also see that our expected levered CEBF equityholder returns are higher than the return on NAV for higher face values of CEBF debt K^*. One can clearly see from Table 4.6.2 that increasing the face value of CEBF debt K^* increases both equityholder expected return and standard deviation. Upon deducting the average leverage cost (the leverage cost assumption is outlined further in section A.3) at high FV of CEBF debt K^* our expected levered CEBF equityholder returns are comparable to those of levered CEBF return on NAV. When comparing the standard deviation of returns from Table 4.5.5 to those in Table 2.7.3, for levered CEBFs we find 3.58% at $K^* = 40$ comparable to 3.51% from Elton et al. [17].

Unfortunately there are no debtholder CEBF returns reported in Elton et al. [17] in Table 2.7.2. Using $\tilde{\sigma}_i = 0.1$ for $i = 1, \ldots, 5$ yields a return to the debtholder of 5.13% with near zero standard deviation (for the same reasons outlined in remark 4.5.5). The debtholder return is higher than the risk free rate outlined in the pricing assumptions in section A.2. This implies that the debtholder is earning a return that is slightly higher than the risk free rate which is as expected.

Chapter 5

Conclusions and Future Research

In summary, one of our contributions of this thesis was determining the impact of changes in firm volatilies and correlations on an unlevered and levered CEBF market value and standard deviation. In remark 3.3.5 we noted that for a levered CEBF the market values of the equity and the debt of the CEBF will change as the correlations change even though the total market value of the CEBF remains the same. If all the correlations increase the total value of the CEBFs assets does not change. With regards to firm volatilities, we noted in remark 3.3.7 that increasing the firm volatilities decreases the market value of equityholder. As the FV of CEBF debt increases, the market value of levered CEBF equityholder decreases. The market value of the levered CEBF to the equityholder decreases as the firm volatilities increase since the put options inherent in each of the risky bonds increase in value as the firm volatilities increase. As the put options increase in value this decreases the value of the risky bonds which decreases the value of the call option of equityholder position of the CEBF. In remark 3.3.8 we concluded that increasing the firm volatilities decreases the market value of the CEBF regardless of whether or not the CEBF is unlevered or levered.

Our simulated results in section 4.5 show that the equityholder of levered CEBFs has a higher expected rate of return and standard deviation of the rate of return than an unlevered CEBF, as was noted in Elton et al. [17]. The levered CEBF expected rate of return and standard deviation of the rate of return (for different holding periods h and CEBF debt K^*) of Table 4.5.5 are clearly larger than those of the unlevered CEBF in Table 4.5.4. The resulting returns we simulated in Table 4.6.1 and 4.6.2 are comparable to those in Tables 2.7.2 and 2.7.3 from Elton et al. [17].

Despite the fact that our model of an unlevered and levered CEBF can achieve expected rates of return and standard deviation of returns similar to those observed by Elton et al. [17] there is certainly room for improvement. Our model relies on several simplifying assumptions which, given time, can be improved. First of all, for computational simplicity we only considered a CEBF composed of five risky bonds when in fact most CEBF are composed of hundreds of bonds. Our definition of a CEBF, (3.2.1), heavily relies on the Merton risky bond model [28] definition which does not consider the possibility of the bond defaulting before maturity. Some models that would incorporate the possibility of the bond defaulting before maturity would be the famous Black and Cox [4] *First Passage Time* model or using the approximate simulation techniques for correlated *First Passage Time* models in Metzler[29]. A Copula model approach (such as from Li [26]) would be interesting to explore as opposed to using a basket-option pricing model approach as we did. Bush et al. [7] develop a limiting distribution of surviving bonds from a portfolio of bonds and then use it to price options. This density can be used in the framework that we have developed instead of assuming the Black and Scholes framework. In the levered CEBF model outlined in section 3.2.2 we only consider the debtholder and equityholder and that all the fund leverage is supplied by the debtholder. Our model does not consider preferred shareholders as a separate class of financing for the CEBF leverage. As we discused in section 2.7, a CEF can lever with both debt and preferred shares. An extension of our levered CEBF model would be to include preferred shares as a separate class of leverage financing and having them as another senior claimant over equityholders (but less senior than debtholders). Furthermore the maximum amount of debt allowed for both preferred share and debtholder financing should also be included in the model (as discussed in section 2.7.2).

Even though this entire thesis is concerned with CEBFs, we observed in figure 2.2.1 some changes in the assets being held by CEFs. The figure shows the percentage of closed end bond and stock funds changing from nearly 75% and 25%, respectively in 2001 to nearly a 65% and 35%, respectively in 2011. It would be interesting to spend some time conjecturing and exploring reasons for the change in asset holdings of CEFs. We can add this question to list of unexplained behaviours of CEFs!

APPENDICES

Appendix A

Chapter 3 Formulas and Economic Assumptions

This appendix contains the derivations of several formulas from Chapter 3.

A.1 Unlevered CEBF Company Formulas

A.1.1 Variance of Two European Put Options

Our goal is to determine the variance at maturity T of a European put option payoff written on an underlying asset $S(\cdot)$, under the classical market model assumptions of Black and Scholes [5]

$$\begin{aligned}\mathbb{V}\mathrm{ar}^{\mathbb{Q}}\left[e^{-r(T-t)}(K-S(T))_+|\mathcal{F}(t)\right] &= e^{-2r(T-t)}\underbrace{\mathbb{E}^{\mathbb{Q}}\left[(K-S(T))_+^2|\mathcal{F}(t)\right]}_{*} \\ &\quad - (\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(K-S(T))_+|\mathcal{F}(t)\right])^2 \qquad \text{(A.1)}\end{aligned}$$

From [5] we have the following model assumptions,

- r is the constant risk free interest rate;
- δ is the constant dividend yield on the underlying asset $S(\cdot)$;
- σ is the constant volatility of the underlying asset $S(\cdot)$;

- $\mathcal{F}(t)$ is defined as the sigma algebra (or available information) for $S(t)$ at time t;
- at time T, conditional on $S(t)$, the underlying asset process $S(T)$ follows the distribution,
  $\ln(S(T)/S(t))|\mathcal{F}(t) \sim \mathcal{N}(\mu', \sigma')$ where $\mu' = (r - \delta - \frac{\sigma^2}{2})(T-t)$ and $\sigma' = \sigma\sqrt{T-t}$;

We can derive $*$ under the Black and Scholes framework as follows

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[(K - S(T))_+^2|\mathcal{F}(t)\right] &= \int_{-\infty}^{\infty}\left[(K - S(0)e^x)_+^2\right]\frac{1}{\sigma'}\phi(\frac{x-\mu'}{\sigma'})dx \\
&= \int_{-\infty}^{\ln(K/S(0))}\left[K^2 - 2KS(0)e^x + (S(0))^2e^{2x}\right]\frac{e^{-\frac{1}{2}(\frac{x-\mu'}{\sigma'})^2}}{\sigma'\sqrt{2\pi}}dx \\
&= K^2\int_{-\infty}^{\ln(K/S(0))}\frac{e^{-\frac{1}{2}(\frac{x-\mu'}{\sigma'})^2}}{\sigma'\sqrt{2\pi}}dx - 2KS(0)\int_{-\infty}^{\ln(K/S(0))}e^x\frac{e^{-\frac{1}{2}(\frac{x-\mu'}{\sigma'})^2}}{\sigma'\sqrt{2\pi}}dx \\
&+ (S(0))^2\int_{-\infty}^{\ln(K/S(0))}e^{2x}\frac{e^{-\frac{1}{2}(\frac{x-\mu'}{\sigma})^2}}{\sigma'\sqrt{2\pi}}dx \qquad \text{(A.2)}
\end{aligned}$$

where

$$\frac{1}{\sigma'}\phi(\frac{x-\mu'}{\sigma'}) = \frac{e^{-\frac{1}{2}(\frac{x-\mu'}{\sigma'})^2}}{\sigma'\sqrt{2\pi}}. \qquad \text{(A.3)}$$

The first integral in (A.2) can be determined as follows.

$$\begin{aligned}
K^2\int_{-\infty}^{\ln(K/S(0))}\frac{e^{-\frac{1}{2}(\frac{x-\mu'}{\sigma'})^2}}{\sigma'\sqrt{2\pi}}dx &= K^2\int_{-\infty}^{\frac{\ln(K/S(0))-\mu'}{\sigma'}}\frac{e^{-\frac{1}{2}u^2}}{\sqrt{2\pi}}du \\
&= K^2\Phi(\frac{\ln(K/S(0)) - \mu'}{\sigma'}) \\
&= K^2\Phi(\frac{-(\ln(S(0)/K) + (r - \delta - \frac{\sigma^2}{2})(T-t))}{\sigma\sqrt{T-t}}) \\
&= K^2\Phi(-d_2) \qquad \text{(A.4)}
\end{aligned}$$

The second integral in (A.2) can be determined as follows.

$$-2KS(0)\int_{-\infty}^{\ln(K/S(0))}\frac{e^xe^{-\frac{1}{2}(\frac{x-\mu'}{\sigma'})^2}}{\sqrt{2\pi}\sigma'}dx = -2KS(0)\int_{-\infty}^{\frac{\ln(K/S(0))-\mu'}{\sigma'}}\frac{e^{\frac{-(u^2+2u\sigma'+2\mu')}{2}}}{\sqrt{2\pi}}du$$

$$
\begin{aligned}
&= -2KS(0)e^{\mu'} \int_{-\infty}^{\frac{\ln(K/S(0))-\mu'}{\sigma'}} \frac{e^{\frac{-(u^2-2u\sigma'+\sigma'^2-\sigma'^2)}{2}}}{\sqrt{2\pi}} du \\
&= -2KS(0)e^{\mu'+\frac{\sigma'^2}{2}} \int_{-\infty}^{\frac{\ln(K/S(0))-\mu'}{\sigma'}} \frac{e^{\frac{-(u-\sigma')^2}{2}}}{\sqrt{2\pi}} du \\
&= -2KS(0)e^{\mu'+\frac{\sigma'^2}{2}} \int_{-\infty}^{\frac{\ln(K/S(0))-\mu'-\sigma'^2}{\sigma'}} \frac{e^{\frac{-v^2}{2}}}{\sqrt{2\pi}} dv \\
&= -2KS(0)e^{(r-\delta-\frac{\sigma^2}{2})(T-t)+\frac{\sigma^2(T-t)}{2}} \Phi(-d_1) \\
&= -2KS(0)e^{(r-\delta)(T-t)} \Phi(-d_1) \qquad \text{(A.5)}
\end{aligned}
$$

The third integral in (A.2) can be determined as follows.

$$
\begin{aligned}
(S(0))^2 \int_{-\infty}^{\ln(K/S(0))} \frac{e^{2x} e^{-\frac{1}{2}(\frac{x-\mu'}{\sigma'})^2}}{\sigma'\sqrt{2\pi}} dx &= (S(0)^2) \int_{-\infty}^{\frac{\ln(K/S(0))-\mu'}{\sigma'}} \frac{e^{\frac{-(u^2-4u\sigma'-4\mu')}{2}}}{\sqrt{2\pi}} du \\
&= (S(0)^2)e^{2\mu'} \int_{-\infty}^{\frac{\ln(K/S(0))-\mu'}{\sigma'}} \frac{e^{\frac{-(u^2-4u\sigma'+4\sigma'^2-4\sigma'^2)}{2}}}{\sqrt{2\pi}} du \\
&= (S(0)^2)e^{2\mu'+2\sigma'^2} \int_{-\infty}^{\frac{\ln(K/S(0))-\mu'}{\sigma'}} \frac{e^{\frac{-(u-2\sigma')^2}{2}}}{\sqrt{2\pi}} du \\
&= (S(0)^2)e^{2\mu'+2\sigma'^2} \int_{-\infty}^{\frac{\ln(K/S(0))-\mu'-2\sigma'^2}{\sigma'}} \frac{e^{\frac{-v^2}{2}}}{\sqrt{2\pi}} dv \\
&= (S(0)^2)e^{2\mu'+2\sigma'^2} \Phi(\frac{\ln(K/S(0))-\mu'-2\sigma'^2}{\sigma'}) \\
&= (S(0)^2)e^{2((r-\delta)(T-t))+\sigma^2(T-t)} \Phi(-d_1-\sigma\sqrt{T-t}) \qquad \text{(A.6)}
\end{aligned}
$$

Combining the results of (A.4), (A.5), and (A.6) yields the following result for (A.2)

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[e^{-2r(T-t)}(K-S(T))_+^2|\mathcal{F}(t)\right] &= e^{-2r(T-t)}K^2\Phi(-d_2) - 2e^{-2r(T-t)}KS(0)e^{(r-\delta)(T-t)}\Phi(-d_1) \\
&\quad + e^{-2r(T-t)}(S(0)^2)e^{2((r-\delta)(T-t))+\sigma^2(T-t)}\Phi(-d_1-\sigma\sqrt{T-t}). \qquad \text{(A.7)}
\end{aligned}
$$

Combining (A.7) with the use of the Black Scholes European put option pricing formula yields the following result for (A.1)

$$
\mathbb{V}\mathrm{ar}^{\mathbb{Q}}\left[e^{-r(T-t)}\,(K-S(T))_+\,|\mathcal{F}(t)\right] = e^{-2r(T-t)}K^2\Phi(-d_2) - 2e^{-2r(T-t)}KS(0)e^{(r-\delta)(T-t)}\Phi(-d_1)
$$

$$+ e^{-2r(T-t)}(S(0)^2)e^{2((r-\delta)(T-t))+\sigma^2(T-t)}\Phi(-d_1 - \sigma\sqrt{T-t})$$
$$- (Ke^{-r(T-t)}\Phi(-d_2) - S(0)e^{-\delta(T-t)}\Phi(-d_1))^2. \qquad \text{(A.8)}$$

A.1.2 Covariance of Two European Put Options

Consider two asset price processes $S_1(t)$ and $S_2(t)$, with correlation ρ, that are both Geometric Brownian Motion (GBM) processes for $0 \leq t \leq T$

$$dS_1(t) = (r - \delta_1)S_1(t)dt + \sigma_1 S_1(t)dZ_1(t)$$
$$dS_2(t) = (r - \delta_2)S_2(t)dt + \sigma_2 S_2(t)dZ_2(t)$$

where we have the following model assumptions

- r is the constant risk free interest rate;
- δ_i is the constant dividend yield on the underlying asset $S_i(\cdot)$ for $i = 1, 2$;
- σ_i is the constant volatility of the underlying asset $S_i(\cdot)$ for $i = 1, 2$;
- $\mathcal{F}(t)$ is defined as the sigma algebra (or available information) for $S_1(t)$ and $S_2(t)$ at time t;
- $Z_1(t)$ and $Z_2(t)$ are Brownian Motion processes under the risk neutral measure such that $dZ_1(t)dZ_2(t) = \rho dt$, where ρ is the correlation between the two processes;

Using a result from Ghasem[20], we have that the stock price processes $S_1(t)$ and $S_2(t)$ have joint distribution, conditional on $\mathcal{F}(t)$,

$$(X, Y) = (\ln(S_1(T_1)/S_1(t)), \ln(S_2(T_2)/S_2(t)))|\mathcal{F}(t) \sim \mathcal{BVN}(\mu'_x, \mu'_y, \sigma'_x, \sigma'_y, \rho)$$

where

- ρ is the correlation between the two processes X and Y;
- $\mu'_i = (r - \delta_i - \frac{\sigma_i^2}{2})(T_j - t)$ and $\sigma'_i = \sigma_i\sqrt{T_j - t}$ is used for $i = x, y$ and $j = 1, 2$;

- (X, Y) have joint density function,

$$\frac{1}{\sigma'_x}\frac{1}{\sigma'_y}\phi_2(\frac{x-\mu'_x}{\sigma'_x}, \frac{y-\mu'_y}{\sigma'_y}; \rho) = \frac{e^{\frac{-1}{2(1-\rho^2)}\left[(\frac{x-\mu'_x}{\sigma'_x})^2 - 2\rho(\frac{x-\mu'_x}{\sigma'_x})(\frac{y-\mu'_y}{\sigma'_y}) + (\frac{y-\mu'_y}{\sigma'_y})^2\right]}}{2\pi\sigma'_x\sigma'_y\sqrt{1-\rho^2}}. \tag{A.9}$$

Recall that for a pair of Bivariate Normal random variables each marginal random variable is Normally distributed, hence we have that,

$X = \ln(S_1(T_1)/S_1(t))\mathcal{F}(t) \sim \mathcal{N}(\mu'_x, \sigma'_x)$ and $Y = \ln(S_2(T_2)/S_2(t))\mathcal{F}(t) \sim \mathcal{N}(\mu'_y, \sigma'_y)$ with correlation ρ.

The motivation to determine the covariance of two European put options, using risk neutral pricing framework, was given by the desire to determine the variance of a portfolio of two European put options (with different times to maturity, strike prices, and underlying (correlated) assets).

$$\begin{aligned}
&\mathbb{V}\mathrm{ar}^{\mathbb{Q}}\left[e^{-r(T_1-t)}\left(K_1 - S_1(T_1)\right)_+ + e^{-r(T_2-t)}\left(K_2 - S_2(T_2)\right)_+ |\mathcal{F}(t)\right] \\
&= e^{-2r(T_1-t)}\mathbb{V}\mathrm{ar}^{\mathbb{Q}}\left[(K_1 - S_1(T_1))_+|\mathcal{F}(t)\right] + e^{-2r(T_2-t)}\mathbb{V}\mathrm{ar}^{\mathbb{Q}}\left[(K_2 - S_2(T_2))_+|\mathcal{F}(t)\right] \\
&+ 2e^{-r(T_1-t)}e^{-r(T_2-t)}\mathbb{C}\mathrm{ov}^{\mathbb{Q}}\left[(K_1 - S_1(T_1))_+, (K_2 - S_2(T_2))_+|\mathcal{F}(t)\right]
\end{aligned} \tag{A.10}$$

We can determine the variance of a put option payoff at maturity for $i = 1, 2$ from the result in A.1.1. The more challenging result is to determine

$$\begin{aligned}
&\mathbb{C}\mathrm{ov}^{\mathbb{Q}}\left[(K_1 - S_1(T_1))_+, (K_2 - S_2(T_2))_+|\mathcal{F}(t)\right] \\
&= \underbrace{\mathbb{E}^{\mathbb{Q}}\left[(K_1 - S_1(T_1))_+(K_2 - S_2(T_2))_+|\mathcal{F}(t)\right]}_{**} - (\mathbb{E}^{\mathbb{Q}}\left[(K_1 - S_1(T_1))_+|\mathcal{F}(t)\right])\times \\
&(\mathbb{E}^{\mathbb{Q}}\left[(K_2 - S_2(T_2))_+|\mathcal{F}(t)\right]).
\end{aligned} \tag{A.11}$$

Of course the challenging part is determining $**$ which can be calculated as follows.

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[(K_1 - S_1(T_1))_+(K_2 - S_2(T_2))_+|\mathcal{F}(t)\right] \\
&= \int_{-\infty}^{\ln(K_1/S_1(0))}\int_{-\infty}^{\ln(K_2/S_2(0))}(K_1 - S_1(T_1))(K_2 - S_2(T_2))\frac{1}{\sigma'_x}\frac{1}{\sigma'_y}\phi_2(\frac{x-\mu'_x}{\sigma'_x}, \frac{y-\mu'_y}{\sigma'_y}; \rho)dydx \\
&= \int_{-\infty}^{\ln(K_1/S_1(0))}\int_{-\infty}^{\ln(K_2/S_2(0))}(K_1 - S_1(0)e^x)(K_2 - S_2(0)e^y)\frac{1}{\sigma'_x}\frac{1}{\sigma'_y}\phi_2(\frac{x-\mu'_x}{\sigma'_x}, \frac{y-\mu'_y}{\sigma'_y}; \rho)dydx
\end{aligned}$$

$$
\begin{aligned}
&= K_1K_2\int_{-\infty}^{\ln(K_1/S_1(0))}\int_{-\infty}^{\ln(K_2/S_2(0))}\frac{1}{\sigma'_x}\frac{1}{\sigma'_y}\phi_2(\frac{x-\mu'_x}{\sigma'_x},\frac{y-\mu'_y}{\sigma'_y};\rho)dydx \\
&\quad -K_1S_2(0)\int_{-\infty}^{\ln(K_1/S_1(0))}\int_{-\infty}^{\ln(K_2/S_2(0))}e^y\frac{1}{\sigma'_x}\frac{1}{\sigma'_y}\phi_2(\frac{x-\mu'_x}{\sigma'_x},\frac{y-\mu'_y}{\sigma'_y};\rho)dydx \\
&\quad -K_2S_1(0)\int_{-\infty}^{\ln(K_1/S_1(0))}\int_{-\infty}^{\ln(K_2/S_2(0))}e^x\frac{1}{\sigma'_x}\frac{1}{\sigma'_y}\phi_2(\frac{x-\mu'_x}{\sigma'_x},\frac{y-\mu'_y}{\sigma'_y};\rho)dydx \\
&\quad +S_1(0)S_2(0)\int_{-\infty}^{\ln(K_1/S_1(0))}\int_{-\infty}^{\ln(K_2/S_2(0))}e^{x+y}\frac{1}{\sigma'_x}\frac{1}{\sigma'_y}\phi_2(\frac{x-\mu'_x}{\sigma'_x},\frac{y-\mu'_y}{\sigma'_y};\rho)dydx
\end{aligned}
$$

The first integral can be determined as follows.

$$
\begin{aligned}
&K_1K_2\int_{-\infty}^{\ln(K_1/S_1(0))}\int_{-\infty}^{\ln(K_2/S_2(0))}\frac{1}{\sigma'_x}\frac{1}{\sigma'_y}\phi_2(\frac{x-\mu'_x}{\sigma'_x},\frac{y-\mu'_y}{\sigma'_y};\rho)dydx \\
&= K_1K_2\int_{-\infty}^{\ln(K_1/S_1(0))}\int_{-\infty}^{\ln(K_2/S_2(0))}\frac{e^{\frac{-1}{2(1-\rho^2)}\left[(\frac{x-\mu'_x}{\sigma'_x})^2-2\rho(\frac{x-\mu'_x}{\sigma'_x})(\frac{y-\mu'_y}{\sigma'_y})+(\frac{y-\mu'_y}{\sigma'_y})^2\right]}}{2\pi\sigma'_x\sigma'_y\sqrt{1-\rho^2}}dxdy \\
&= K_1K_2\int_{-\infty}^{\frac{\ln(K_1/S_1(0))-\mu'_x}{\sigma'_x}}\int_{-\infty}^{\frac{\ln(K_2/S_2(0))-\mu'_y}{\sigma'_y}}\frac{e^{\frac{-1}{2(1-\rho^2)}\left[u^2-2\rho uv+v^2\right]}}{2\pi\sqrt{1-\rho^2}}dvdu \qquad \text{(A.12)}\\
&= K_1K_2\Phi_2(\frac{\ln(K_1/S_1(0))-\mu'_x}{\sigma'_x},\frac{\ln(K_2/S_2(0))-\mu'_y}{\sigma'_y};\rho) \\
&= K_1K_2\Phi_2(\frac{\ln(K_1/S_1(0))-(r-\delta_1-\frac{\sigma_1^2}{2})(T_1-t)}{\sigma_1\sqrt{(T_1-t)}},\frac{\ln(K_2/S_2(0))-(r-\delta_2-\frac{\sigma_2^2}{2})(T_2-t)}{\sigma_2\sqrt{(T_2-t)}};\rho) \\
&= K_1K_2\Phi_2(-d_{2(1)},-d_{2(2)};\rho) \qquad \text{(A.13)}
\end{aligned}
$$

In order to obtain (A.12) we use the substitution $u=\frac{x-\mu'_x}{\sigma'_x}$ and $v=\frac{y-\mu'_y}{\sigma'_y}$. The second integral can be determined as follows.

$$
\begin{aligned}
&-K_1S_2(0)\int_{-\infty}^{\ln(K_1/S_1(0))}\int_{-\infty}^{\ln(K_2/S_2(0))}e^y\frac{1}{\sigma'_x}\frac{1}{\sigma'_y}\phi_2(\frac{x-\mu'_x}{\sigma'_x},\frac{y-\mu'_y}{\sigma'_y};\rho)dydx \\
&= -K_1S_2(0)\int_{-\infty}^{\ln(K_1/S_1(0))}\int_{-\infty}^{\ln(K_2/S_2(0))}e^y\frac{e^{\frac{-1}{2(1-\rho^2)}\left[(\frac{x-\mu'_x}{\sigma'_x})^2-2\rho(\frac{x-\mu'_x}{\sigma'_x})(\frac{y-\mu'_y}{\sigma'_y})+(\frac{y-\mu'_y}{\sigma'_y})^2\right]}}{2\pi\sigma'_x\sigma'_y\sqrt{1-\rho^2}}dydx \\
&= -K_1S_2(0)e^{\mu'_y}\int_{-\infty}^{\frac{\ln(K_1/S_1(0))-\mu'_x}{\sigma'_x}}\int_{-\infty}^{\frac{\ln(K_2/S_2(0))-\mu'_y}{\sigma'_y}}\frac{e^{\frac{-1}{2(1-\rho^2)}\left[u^2-2\rho uv+v^2\right]-2(1-\rho^2)\sigma_y v}}{2\pi\sqrt{1-\rho^2}}dvdu
\end{aligned}
$$

$$
\begin{aligned}
&= -K_1 S_2(0) e^{\mu'_y} \int_{-\infty}^{\frac{\ln(K_1/S_1(0))-\mu'_x}{\sigma'_x}} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\ln(K_2/S_2(0))-\mu'_y}{\sigma'_y}} \frac{e^{\frac{-1}{2(1-\rho^2)}(v-\rho u)^2 - 2(1-\rho^2)\sigma_y v}}{\sqrt{2\pi(1-\rho^2)}} dv du \\
&= -K_1 S_2(0) e^{\mu'_y + \frac{\sigma_y^2(1-\rho^2)}{2}} \times \\
&\quad \int_{-\infty}^{\frac{\ln(K_1/S_1(0))-\mu'_x}{\sigma'_x}} \frac{e^{-\frac{(u^2-2\rho u \sigma'_y)}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\ln(K_2/S_2(0))-\mu'_y}{\sigma'_y}} \frac{e^{\frac{-1}{2(1-\rho^2)}(v-(\rho u+\sigma'_y(1-\rho^2)))^2}}{\sqrt{2\pi(1-\rho^2)}} dv du \qquad \text{(A.14)} \\
&= -K_1 S_2(0) e^{\mu'_y + \frac{\sigma'^2_y}{2}} \int_{-\infty}^{\frac{\ln(K_1/S_1(0))-\mu'_x}{\sigma'_x}} \frac{e^{-\frac{(u-\rho\sigma'_y)^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\ln(K_2/S_2(0))-\mu'_y}{\sigma'_y}} \frac{e^{\frac{-1}{2(1-\rho^2)}(v-(\rho u+\sigma'_y(1-\rho^2)))^2}}{\sqrt{2\pi(1-\rho^2)}} dv du \\
&= -K_1 S_2(0) e^{\mu'_y + \frac{\sigma'^2_y}{2}} \int_{-\infty}^{\frac{\ln(K_1/S_1(0))-\mu'_x}{\sigma'_x}} \frac{e^{-\frac{(u-\rho\sigma'_y)^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\ln(K_2/S_2(0))-\mu'_y}{\sigma'_y}} \frac{e^{\frac{-1}{2(1-\rho^2)}(v-(\rho u+\sigma'_y(1-\rho^2)))^2}}{\sqrt{2\pi(1-\rho^2)}} dv du \\
&= -K_1 S_2(0) e^{\mu'_y + \frac{\sigma'^2_y}{2}} \int_{-\infty}^{\frac{\ln(K_1/S_1(0))-\mu'_x}{\sigma'_x}-\rho\sigma'_y} \frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\ln(K_2/S_2(0))-\mu'_y}{\sigma'_y}-\sigma'_y} \frac{e^{\frac{-1}{2(1-\rho^2)}(z-\rho k)^2}}{\sqrt{2\pi(1-\rho^2)}} dz dk \\
&= -K_1 S_2(0) e^{\mu'_y + \frac{\sigma'^2_y}{2}} \int_{-\infty}^{\frac{\ln(K_1/S_1(0))-\mu'_x}{\sigma'_x}-\rho\sigma'_y} \int_{-\infty}^{\frac{\ln(K_2/S_2(0))-\mu'_y}{\sigma'_y}-\sigma'_y} \frac{e^{\frac{-1}{2(1-\rho^2)}\left[z^2-2\rho z k+k^2\right]}}{2\pi\sqrt{1-\rho^2}} dz dk \\
&= -K_1 S_2(0) e^{(r-\delta_2)(T_2-t)} \Phi_2(-d_{2(1)} - \rho\sigma_2\sqrt{(T_2-t)}, -d_{2(2)} - \sigma_2\sqrt{(T_2-t)}; \rho) \\
&= -K_1 S_2(0) e^{(r-\delta_2)(T_2-t)} \Phi_2(-d_{2(1)} - \rho\sigma_2\sqrt{(T_2-t)}, -d_{1(2)}; \rho) \qquad \text{(A.15)}
\end{aligned}
$$

(A.14) comes from applying transformation $k = u - \rho\sigma_y$ and $z - \rho k = v - (\rho u + \sigma_y(1-\rho^2))$ (hence $z = \rho k + v - (\rho u + \sigma_y(1-\rho^2))$). The third integral can be determined similarly to the second integral above.

$$
\begin{aligned}
&-K_2 S_1(0) \int_{-\infty}^{\ln(K_1/S_1(0))} \int_{-\infty}^{\ln(K_2/S_2(0))} e^x \frac{1}{\sigma'_x}\frac{1}{\sigma'_y}\phi_2(\frac{x-\mu'_x}{\sigma'_x}, \frac{y-\mu'_y}{\sigma'_y}; \rho) dy dx \\
&\quad = -K_2 S_1(0) \int_{-\infty}^{\ln(K_2/S_2(0))} \int_{-\infty}^{\ln(K_1/S_1(0))} e^x \frac{e^{\frac{-1}{2(1-\rho^2)}\left[(\frac{x-\mu'_x}{\sigma'_x})^2 - 2\rho(\frac{x-\mu'_x}{\sigma'_x})(\frac{y-\mu'_y}{\sigma'_y}) + (\frac{y-\mu'_y}{\sigma'_y})^2\right]}}{2\pi\sigma'_x\sigma'_y\sqrt{1-\rho^2}} dx dy \\
&\quad = -K_2 S_1(0) e^{(r-\delta_1)(T_1-t)} \Phi_2(-d_{2(1)} - \sigma_1\sqrt{(T_1-t)}, -d_{2(2)} - \rho\sigma_1\sqrt{(T_1-t)}; \rho) \\
&\quad = -K_2 S_1(0) e^{(r-\delta_1)(T_1-t)} \Phi_2(-d_{1(1)}, -d_{2(2)} - \rho\sigma_1\sqrt{(T_1-t)}; \rho) \qquad \text{(A.16)}
\end{aligned}
$$

The fourth integral can be determined as follows.

$$
S_1(0) S_2(0) \int_{-\infty}^{\ln(K_1/S_1(0))} \int_{-\infty}^{\ln(K_2/S_2(0))} e^{x+y} \frac{1}{\sigma'_x}\frac{1}{\sigma'_y}\phi_2(\frac{x-\mu'_x}{\sigma'_x}, \frac{y-\mu'_y}{\sigma'_y}; \rho) dy dx
$$

$$
\begin{aligned}
&= S_1(0)S_2(0)\int_{-\infty}^{\ln(K_1/S_1(0))}\int_{-\infty}^{\ln(K_2/S_2(0))} e^{x+y}\frac{e^{\frac{-1}{2(1-\rho^2)}\left[(\frac{x-\mu'_x}{\sigma'_x})^2-2\rho(\frac{x-\mu'_x}{\sigma'_x})(\frac{y-\mu'_y}{\sigma'_y})+(\frac{y-\mu'_y}{\sigma'_y})^2\right]}}{2\pi\sigma'_x\sigma'_y\sqrt{1-\rho^2}}dydx \\
&= S_1(0)S_2(0)e^{\mu'_x+\mu'_y}\int_{-\infty}^{u^*}\int_{-\infty}^{v^*}\frac{e^{\frac{-1}{2(1-\rho^2)}\left[u^2-2\rho uv+v^2-2(1-\rho^2)\sigma'_y v-2(1-\rho^2)\sigma'_x u\right]}}{2\pi\sqrt{1-\rho^2}}dvdu \\
&= S_1(0)S_2(0)e^{\frac{2\mu'_x+2\mu'_y+\left[(1-\rho^2){\sigma'_y}^2+(\sigma'_x+\rho\sigma'_y)^2\right]}{2}}\times \\
&\quad \int_{-\infty}^{u^*}\frac{e^{\frac{-(u-(\sigma'_x+\rho\sigma'_y))^2}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{v^*}\frac{e^{\frac{-1}{2(1-\rho^2)}\left[v-(\rho u+(1-\rho^2)\sigma'_y)\right]^2}}{\sqrt{2\pi(1-\rho^2)}}dvdu && \text{(A.17)} \\
&= S_1(0)S_2(0)e^{\frac{2\mu'_x+2\mu'_y+\left[(1-\rho^2){\sigma'_y}^2+(\sigma'_x+\rho\sigma'_y)^2\right]}{2}}\int_{-\infty}^{u^*-\sigma'_x-\rho\sigma'_y}\frac{e^{\frac{-k^2}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{v^*-\rho\sigma_x-\sigma'_y}\frac{e^{\frac{-1}{2(1-\rho^2)}[z-\rho k]^2}}{\sqrt{2\pi(1-\rho^2)}}dzdk \\
&= S_1(0)S_2(0)e^{\frac{2\mu'_x+2\mu'_y+\left[(1-\rho^2){\sigma'_y}^2+(\sigma'_x+\rho\sigma'_y)^2\right]}{2}}\times \\
&\quad \int_{-\infty}^{u^*-\sigma'_x-\rho\sigma'_y}\int_{-\infty}^{v^*-\rho\sigma'_x-\sigma'_y}\frac{e^{\frac{-1}{2(1-\rho^2)}\left[z^2-2\rho zk+k^2\right]}}{2\pi\sqrt{(1-\rho^2)}}dzdk && \text{(A.18)} \\
&= S_1(0)S_2(0)e^{\frac{2\mu'_x+2\mu'_y+\left[(1-\rho^2){\sigma'_y}^2+(\sigma'_x+\rho\sigma'_y)^2\right]}{2}}N_2(-d_{2(1)}-\sigma'_x-\rho\sigma'_y,-d_{2(2)}-\sigma'_y-\rho\sigma'_x;\rho) \\
&= S_1(0)S_2(0)A(r,\hat{\delta},\hat{T},\hat{\sigma})\Phi_2(-d_{1(1)}-\rho\sigma_2\sqrt{T_2-t},-d_{1(2)}-\rho\sigma_1\sqrt{T_1-t};\rho) && \text{(A.19)}
\end{aligned}
$$

Where in (A.18) we use the notation $u^* = \frac{\ln(K_1/S_1(0))-\mu'_x}{\sigma'_x}$ and $v^* = \frac{\ln(K_2/S_2(0))-\mu'_y}{\sigma'_y}$. As well as in (A.19) we have

$$
A(r,\hat{\delta},\hat{T},\hat{\sigma}) = e^{\frac{\left[2(r-\delta_1-\frac{\sigma_1^2}{2})(T_1-t)+2(r-\delta_2-\frac{\sigma_2^2}{2})(T_2-t)+(1-\rho^2)\sigma_2^2(T_2-t)+(\sigma_1\sqrt{T_1-t}+\rho\sigma_2\sqrt{T_2-t})^2\right]}{2}}.
$$

Note (A.18) comes from applying the substitutions $k = u-(\sigma'_x+\rho\sigma'_y)$ and $z-\rho k = v-(\rho u+\sigma_y(1-\rho^2))$ (hence $z = v-\rho'_x-\sigma'_y$).

Combining the resulting integrals of (A.13)–(A.19) with the Black and Scholes European option pricing model for a European put option for each of $S_1(t)$ and $S_2(t)$ we have a resulting closed form solution for (A.12). With a closed form solution for (A.12) yields a closed form solution for (A.11). By applying (A.8) to each of $S_1(t)$ and $S_2(t)$, this results in a closed form solution for (A.10) as desired.

A.2 Simulation Assumptions

In section 3.3, we make the following economic assumptions to value the unlevered and levered CEBF of n risky bonds at time T^*:

- The number of simulations $M = 10,000$;
- Both the unlevered and levered CEBF companies have purchased $n = 5$ risky bonds;
- Each firm asset price process, $S_i(t)$, underlying the risky bond follows a Geometric Brownian Motion (GBM) process

$$\frac{dS_i(t)}{S_i(t)} = \mu_i dt + \sigma_i dZ_i(t)$$

  where $Z_i(t)$ is a Brownian motion process for each $i = 1, \ldots, n$;
- The time to maturity of the firm debt is $T^* = 5$ years;
- The CEBF face value of debt is varied from $K^* = 5$ to $K^* = 150$;
- The risk free rate is $r = 0.05$[1];

In order to determine the covariance of two credit risky bonds, shown earlier to be $**$ in (3.2.4), we must consider the correlations between the different firms (i, j) where $i, j = 1, \ldots, 5$. Note that these are instantaneous correlations that we assume are held constant for the entire period of observation. We consider two different correlation matrices between the firms, one where the firms are weakly correlated (average correlation of 0.12) and one where they are strongly correlated (average correlation of 0.79).

For evaluating (3.2.4), and (3.2.7)–(3.2.10) we assume two different correlation matrices, a low and a high correlation environment, between the underlying 5 firms. Below we define the low correlation environment (ρ_{Low})

$$\rho_{Low} = \begin{bmatrix} 1 & 0.05 & 0.15 & 0.10 & 0.07 \\ 0.05 & 1 & 0.08 & 0.05 & 0.25 \\ 0.15 & 0.08 & 1 & 0.10 & 0.15 \\ 0.10 & 0.05 & 0.10 & 1 & 0.20 \\ 0.07 & 0.25 & 0.15 & 0.20 & 1 \end{bmatrix}$$

[1] $r = 5\%$ is the average return observed on the 5-year US treasury rate over 1996 to 2006 rounded to the nearest integer.

Below we define the high correlation environment (ρ_{High}).

$$\rho_{High} = \begin{bmatrix} 1 & 0.60 & 0.80 & 0.75 & 0.65 \\ 0.60 & 1 & 0.70 & 0.90 & 0.80 \\ 0.80 & 0.70 & 1 & 0.85 & 0.90 \\ 0.75 & 0.90 & 0.85 & 1 & 0.95 \\ 0.65 & 0.80 & 0.90 & 0.95 & 1 \end{bmatrix}$$

A.3 Expense Assumptions

Table A.3.1 below is from Table 6 in Elton et al. [17] which shows the cost of borrowing for non-municipal CEBFs which include the preferred dividend over the amount of the preferred shares plus the management fee. For debt the cost is the interest over the amount of the loan. We only include the estimated total cost of borrowing based on the management fee from the original table since there is only a small difference from assuming a flat fee of 51bps.

Year	**# with Preferred Shares**	**Total Cost**	**# with Debt**	**Total Cost**
1996	1	3.47%	5	5.45%
1997	1	5.20%	6	6.06%
1998	1	5.65%	7	6.93%
1999	1	6.00%	7	6.60%
2000	1	8.22%	7	7.83%
2001	3	8.30%	7	6.75%
2002	5	4.13%	7	3.83%
2003	7	2.36%	7	2.42%
2004	11	2.34%	7	2.53%
2005	11	3.87%	7	4.09%
2006	11	5.49%	7	6.01%
Mean	4.82	5.19%	6.64	5.32%

Table A.3.1: Non-Municipal CEBFs Cost of Levering

Appendix B

Chapter 4 Formulas and Economic Assumptions

This appendix contains the derivations of several formulas from Chapter 4. We first begin by proving a few lemmas that are used later on.

B.1 Some Useful Lemmas

Lemma B.1.1. *Let A and B are real valued constants and $Z \sim \mathcal{N}(0,1)$ then*

$$\int_{-\infty}^{\infty} \Phi(A+Bz)\phi(z)dz = \Phi(\frac{A}{\sqrt{1+B^2}}) \tag{B.1}$$

Proof.

$$\begin{aligned}
\int_{-\infty}^{\infty} \Phi(A+Bz)\phi(z)dz &= \mathbb{E}_z\left[\Phi(A+Bz)\right] \\
&= \mathbb{E}_z\left[P_{U|z}(U \leq A+Bz|z)\right] \\
&= \mathbb{E}_z\left[\mathbb{E}_{U|z}\left[\mathbb{I}_{(U\leq A+Bz|z)}\right]\right] \\
&= \mathbb{E}_z\left[\mathbb{E}_{U|z}\left[\mathbb{I}_{(U\leq A+Bz|z)}\right]\right] \\
&= \mathbb{E}_U\left[\mathbb{I}_{(U\leq A+BZ)}\right] \\
&= \Phi(\frac{A}{\sqrt{1+B^2}})
\end{aligned}$$

□

Lemma B.1.2. *Let A_1, A_2, B_1, and B_2 be real valued constants and $Z \sim \mathcal{N}(0,1)$ then we have*

$$\int_{-\infty}^{\infty} \Phi(A_1 + B_1 z)\Phi(A_2 + B_2 z)\phi(z)dz = \Phi_2(\frac{A_1}{\sqrt{1+B_1^2}}, \frac{A_2}{\sqrt{1+B_2^2}}; \frac{B_1 B_2}{\sqrt{1+B_1^2}\sqrt{1+B_2^2}}) \tag{B.2}$$

Proof.

$$\begin{aligned}
&\int_{-\infty}^{\infty} \Phi(A_1 + B_1 z)\Phi(A_2 + B_2 z)\phi(z)dz \\
&= \mathbb{E}_z \left[\Phi(A_1 + B_1 z)\Phi(A_2 + B_2 z)\right] \\
&= \mathbb{E}_z \left[P_{U_1|z}(U_1 \leq A_1 + B_1 z|z) P_{U_2}(U_2 \leq A_2 + B_2 z|z)\right] \\
&= \mathbb{E}_z \left[\mathbb{E}_{U_1|z}\left[\mathbb{I}_{(U_1 \leq A_1 + B_1 z|z)}\right] \mathbb{E}_{U_2|z}\left[\mathbb{I}_{(U_2 \leq A_2 + B_2 z|z)}\right]\right] \\
&= \mathbb{E}_z \left[\mathbb{E}_{U_1|z}\left[\mathbb{I}_{(U_1 \leq A_1 + B_1 z|z)}\right] \mathbb{E}_{U_2|z}\left[\mathbb{I}_{(U_2 \leq A_2 + B_2 z|z)}\right]\right] \\
&= \mathbb{E}_z \left[\mathbb{E}_{(U_1,U_2|z}\left[\mathbb{I}_{(U_1 \leq A_1 + B_1 z|z)}\mathbb{I}_{(U_2 \leq A_2 + B_2 z|z)}\right]\right] \\
&= \mathbb{E}_{(U_1,U_2,Z)}\left[\mathbb{I}_{(U_1 \leq A_1 + B_1 Z)}\mathbb{I}_{(U_2 \leq A_2 + B_2 Z)}\right] \\
&= \mathbb{E}_{(U_1,U_2,Z)}\left[\mathbb{I}_{(U_1 - B_1 Z \leq A_1)}\mathbb{I}_{(U_2 - B_2 Z \leq A_2)}\right] \\
&= \mathbb{E}_{(U_1^*,U_2^*)}\left[\mathbb{I}_{(U_1^* \leq \frac{A_1}{\sqrt{1+B_1^2}})}\mathbb{I}_{(U_2^* \leq \frac{A_2}{\sqrt{1+B_2^2}})}\right] \\
&= P_{(U_1^*,U_2^*)}(U_1^* \leq \frac{A_1}{\sqrt{1+B_1^2}}, U_2^* \leq \frac{A_2}{\sqrt{1+B_2^2}}; \frac{B_1 B_2}{\sqrt{1+B_1^2}\sqrt{1+B_2^2}}) \\
&= \Phi_2(\frac{A_1}{\sqrt{1+B_1^2}}, \frac{A_2}{\sqrt{1+B_2^2}}; \frac{B_1 B_2}{\sqrt{1+B_1^2}\sqrt{1+B_2^2}})
\end{aligned}$$

□

Corollary B.1.3. *Let A and B are real valued constants and $Z \sim \mathcal{N}(0,1)$ then*

$$\int_{-\infty}^{\infty} \Phi(A + Bz)^2 \phi(z)dz = \Phi_2(\frac{A}{\sqrt{1+B^2}}, \frac{A}{\sqrt{1+B^2}}; \frac{B^2}{1+B^2}) \tag{B.3}$$

Proof.

$$\int_{-\infty}^{\infty} \Phi(A + Bz)^2 f(z)dz = \mathbb{E}_z\left[\Phi(A + Bz)^2\right]$$

$$
\begin{aligned}
&= \mathbb{E}_z \left[P_{U_1|z}(U_1 \le A + Bz|z) P_{U_2|z}(U_2 \le A + Bz|z) \right] \\
&= \mathbb{E}_z \left[\mathbb{E}_{U_1|z} \left[\mathbb{I}_{(U_1 \le A+Bz|z)} \right] \mathbb{E}_{U_2|z} \left[\mathbb{I}_{(U_2 \le A+Bz|z)} \right] \right] \\
&= \mathbb{E}_z \left[\mathbb{E}_{(U_1,U_2)|z} \left[\mathbb{I}_{(U_1 \le A+Bz|z)} \mathbb{I}_{(U_2 \le A+Bz|z)} \right] \right] \\
&= \mathbb{E}_{(U_1,U_2,Z)} \left[\mathbb{I}_{(U_1 \le A+BZ)} \mathbb{I}_{(U_2 \le A+BZ)} \right] \\
&= \mathbb{E}_{(U_1,U_2,Z)} \left[\mathbb{I}_{(U_1 - BZ \le A)} \mathbb{I}_{(U_2 - BZ \le A)} \right] \\
&= \mathbb{E}_{(U_1^*,U_2^*)} \left[\mathbb{I}_{(U_1^* \le \frac{A}{\sqrt{1+B^2}})} \mathbb{I}_{(U_2^* \le \frac{A}{\sqrt{1+B^2}})} \right] \\
&= \Phi_2\left(\frac{A}{\sqrt{1+B^2}}, \frac{A}{\sqrt{1+B^2}}; \frac{B^2}{1+B^2}\right)
\end{aligned}
$$

□

Lemma B.1.4. *Let A_1, A_2, B_1, and B_2 be real valued constants and*

$$(Z_1, Z_2) \sim \mathcal{BVN}(0, 0, 1, 1, \rho)$$

then

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(A_1 + B_1 z_1)\Phi(A_2 + B_2 z_2)\phi_2(z_1, z_2; \rho) dz_1 dz_2 = \Phi_2\left(\frac{A_1}{\sqrt{1+B_1^2}}, \frac{A_2}{\sqrt{1+B_2^2}}; \rho^*\right) \tag{B.4}$$

where $\rho^* = \frac{B_1 B_2 \rho}{\sqrt{1+B_1^2}\sqrt{1+B_2^2}}$.

Proof.

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(A_1 + B_1 z_1)\Phi(A_2 + B_2 z_2)\phi_2(z_1, z_2; \rho) dz_1 dz_2 \\
&= \mathbb{E}_{(Z_1,Z_2)} \left[\Phi(A_1 + B_1 Z_1)\Phi(A_2 + B_2 Z_2) \right] \\
&= \mathbb{E}_{(Z_1,Z_2)} \left[P_{U_1|Z_1}(U_1 \le A_1 + B_1 Z_1 | Z_1) P_{U_2|Z_2}(U_2 \le A_2 + B_2 Z_2 | Z_2) \right] \\
&= \mathbb{E}_{(Z_1,Z_2)} \left[\mathbb{E}_{U_1|Z_1} \left[\mathbb{I}_{(U_1 \le A_1 + B_1 Z_1 | Z_1)} \right] \mathbb{E}_{U_2|Z_2} \left[\mathbb{I}_{(U_2 \le A_2 + B_2 Z_2 | Z_2)} \right] \right] \\
&= \mathbb{E}_{(Z_1,Z_2)} \left[\mathbb{E}_{U_1|(Z_1,Z_2)} \left[\mathbb{I}_{(U_1 \le A_1 + B_1 Z_1 | (Z_1,Z_2))} \right] \mathbb{E}_{U_2|(Z_1,Z_2)} \left[\mathbb{I}_{(U_2 \le A_2 + B_2 Z_2 | (Z_1,Z_2))} \right] \right] \\
&= \mathbb{E}_{(Z_1,Z_2)} \left[\mathbb{E}_{(U_1,U_2)|(Z_1,Z_2)} \left[\mathbb{I}_{(U_1 \le A_1 + B_1 Z_1 | (Z_1,Z_2))} \mathbb{I}_{(U_2 \le A_2 + B_2 Z_2 | (Z_1,Z_2))} \right] \right] \\
&= \mathbb{E}_{(U_1,U_2)} \left[\mathbb{I}_{(U_1 \le A_1 + B_1 Z_1)} \mathbb{I}_{(U_2 \le A_2 + B_2 Z_2)} \right] \\
&= \mathbb{E}_{(U_1,U_2,Z_1,Z_2)} \left[\mathbb{I}_{(U_1 \le A_1 + B_1 Z_1, U_2 \le A_2 + B_2 Z_2)} \right]
\end{aligned}
$$

$$= P_{(U_1^*,U_2^*)}(U_1^* \leq \frac{A_1}{\sqrt{1+B_1^2}}, U_2^* \leq \frac{A_2}{\sqrt{1+B_2^2}}; \frac{B_1 B_2 \rho}{\sqrt{1+B_1^2}\sqrt{1+B_2^2}})$$
$$= \Phi_2(\frac{A_1}{\sqrt{1+B_1^2}}, \frac{A_2}{\sqrt{1+B_2^2}}; \frac{B_1 B_2 \rho}{\sqrt{1+B_1^2}\sqrt{1+B_2^2}})$$

□

B.2 Formula Derivations - Portfolio Analysis of Unleverd CEBFs

Our goal is to determine the expected return of a European vanilla option held for a period h under the Real World $\mathbb{P}$-measure and valued under the Risk Neutral $\mathbb{Q}$-measure for the remaining time $T-h$ until option maturity at time T, where $T > h$. The method to do this is outlined in Rubinstein [31], where an asset process $S(t)$ on the time interval $0 \leq t \leq T$ follows (as was pointed out by Cheng [9])

$$\begin{aligned} dS(t) &= \mu S(t)dt + \tilde{\sigma} S(t) dW_t \text{ for } 0 \leq t \leq h \\ dS(t) &= rS(t)dt + \sigma S(t) dW_t \text{ for } h \leq t \leq T. \end{aligned} \tag{B.1}$$

B.2.1 Return on a European Put Option and a Risky Bond

This section contains the derivation of (4.1.4) from (4.1.3) in section 4.1 on page 49. First let

$$-x_1 = -\frac{y}{\sigma\sqrt{T-h}} - \frac{\log\left(S(0)/Ke^{-r(T-h)}\right) - 0.5\sigma^2(T-h)}{\sigma\sqrt{T-h}} \tag{B.2}$$

and

$$-x_2 = -\frac{y}{\sigma\sqrt{T-h}} - \frac{\log\left(S(0)/Ke^{-r(T-h)}\right) + 0.5\sigma^2(T-h)}{\sigma\sqrt{T-h}} \tag{B.3}$$

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}(P(S(h), K, r, h, T, \sigma^2)) &= \mathbb{E}^{\mathbb{P}}\left[e^{-r(T-h)}\mathbb{E}^{\mathbb{Q}}\left[(S(T)-K)_+ | \mathcal{F}(h)\right]\right] \\ &= \int_{-\infty}^{\infty} \left[Ke^{-r(T-h)}\Phi(-x_2) - S(0)e^y\Phi(-x_1)\right] \frac{1}{\tilde{\sigma}\sqrt{h}}\phi(\frac{y-mh}{\tilde{\sigma}\sqrt{h}})dy \end{aligned}$$

$$
\begin{aligned}
&= Ke^{-r(T-h)} \int_{-\infty}^{\infty} \Phi(-x_2) \frac{e^{\frac{-(y-mh)^2}{2\tilde{\sigma}^2 h}}}{\tilde{\sigma}\sqrt{2\pi h}} dy \\
&- \int_{-\infty}^{\infty} S(0)e^{y}\Phi(-x_1) \frac{e^{\frac{-(y-mh)^2}{2\tilde{\sigma}^2 h}}}{\tilde{\sigma}\sqrt{2\pi h}} dy \\
&= Ke^{-r(T-h)}\Phi(\frac{-\frac{mh+\log(S(0)/Ke^{-r(T-h)})-0.5\sigma^2(T-h)}{\sigma\sqrt{T-h}}}{\sqrt{1+(\frac{\sqrt{\tilde{\sigma}^2 h}}{\sigma\sqrt{T-h}})^2}}) \\
&- S(0)e^{\mu h}\Phi(-\frac{\frac{mh+\log(S(0)/Ke^{-r(T-h)})+0.5\sigma^2(T-h)+\tilde{\sigma}^2 h}{\sigma\sqrt{T-h}}}{\sqrt{1+(\frac{\sqrt{\tilde{\sigma}^2 h}}{\sigma\sqrt{T-h}})^2}}) \qquad \text{(B.4)} \\
&= Ke^{-r(T-h)}\Phi(-\tilde{x}+0.5\sigma^*\sqrt{T}) - S(0)e^{\mu h}\Phi(-\tilde{x}-0.5\sigma^*\sqrt{T}) \qquad \text{(B.5)}
\end{aligned}
$$

where

$$
\begin{array}{ll}
\sigma^* = \sqrt{\tilde{\sigma}^2 \frac{h}{T} + \sigma^2 \frac{(T-h)}{T}} & m = \mu - \frac{1}{2}\tilde{\sigma}^2 \\
\tilde{x} = \frac{\log(S(0)\mu'/Ke^{-r(T-h)})}{\sigma^*\sqrt{T}} & \mu' = e^{\mu h}
\end{array} \qquad \text{(B.6)}
$$

Note that (B.4) results from a direct application of (B.1).

Using (B.5), we can determine (4.1.3) under Rubinstein's [31] framework for a holding period of return, h, shown below.

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}(RB(h)) &= Ke^{-r(T-h)} - Ke^{-r(T-h)}\Phi(-\frac{\log(S(0)\mu'/Ke^{-r(T-h)}) - 0.5\sigma^{*2}T}{\sigma^*\sqrt{T}}) \\
&+ S(0)e^{\mu h}\Phi(-\frac{\log(S(0)\mu'/Ke^{-r(T-h)}) + 0.5\sigma^{*2}T}{\sigma^*\sqrt{T}}) \qquad \text{(B.7)}
\end{aligned}
$$

B.2.2 Variance of a Risky Bond

For notation simplicity we denote a European put option at time h, $P_i(h)$ on an underlying asset $S_i(h)$ for $i = 1, \ldots, n$.

$$
P_i(h) = P_i(S_i(h), K_i, r, h, T_i, \sigma_i^2) \qquad \text{(B.8)}
$$

and

$$
RB_i(h) = K_i e^{-r(T_i-h)} - P_i(S_i(h), K_i, r, h, T_i, \sigma_i^2) \qquad \text{(B.9)}
$$

From properties of variances of random variables, the variance of a risky bond reduces to,

$$\mathbb{V}\mathrm{ar}^{\mathbb{P}}(RB_i(h)) = \mathbb{E}^{\mathbb{P}}(P_i(h)^2) - (\mathbb{E}^{\mathbb{P}}(P_i(h)))^2. \tag{B.10}$$

The challenging exercise is to determine

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}(P_i(h)^2) &= \int_{-\infty}^{\infty} \left[Ke^{-r(T-h)}\Phi(-x_2) - S(0)e^y\Phi(-x_1)\right]^2 \frac{1}{\tilde{\sigma}\sqrt{h}}\phi(\frac{y-mh}{\tilde{\sigma}\sqrt{h}})dy \\
&= K^2e^{-2r(T-h)} \int_{-\infty}^{\infty} \Phi(-x_2)^2 \frac{1}{\tilde{\sigma}\sqrt{h}}\phi(\frac{y-mh}{\tilde{\sigma}\sqrt{h}})dy \\
&- 2Ke^{-r(T-h)}S(0) \int_{-\infty}^{\infty} \Phi(-x_1)\Phi(-x_2)e^y \frac{1}{\tilde{\sigma}\sqrt{h}}\phi(\frac{y-mh}{\tilde{\sigma}\sqrt{h}})dy \\
&+ (S(0))^2 \int_{-\infty}^{\infty} \Phi(-x_1)^2 e^{2y} \frac{1}{\tilde{\sigma}\sqrt{h}}\phi(\frac{y-mh}{\tilde{\sigma}\sqrt{h}})dy
\end{aligned} \tag{B.11}$$

The first integral in (B.11) can be evaluated as

$$K^2e^{-2r(T-h)} \int_{-\infty}^{\infty} \Phi(-x_2)^2 \frac{1}{\tilde{\sigma}\sqrt{h}}\phi(\frac{y-mh}{\tilde{\sigma}\sqrt{h}})dy \tag{B.12}$$

$$= K^2e^{-2r(T-h)}\Phi_2(-\frac{\frac{mh+\log(S(0)/Ke^{-r(T-h)})-0.5\sigma^2(T-h)}{\sigma\sqrt{T-h}}}{\sqrt{1+(\frac{\sqrt{\tilde{\sigma}^2h}}{\sigma\sqrt{T-h}})^2}}, -\frac{\frac{mh+\log(S(0)/Ke^{-r(T-h)})-0.5\sigma^2(T-h)}{\sigma\sqrt{T-h}}}{\sqrt{1+(\frac{\sqrt{\tilde{\sigma}^2h}}{\sigma\sqrt{T-h}})^2}}; \rho')$$

$$= K^2e^{-2r(T-h)}\Phi_2(-\tilde{x}+0.5\sigma^*\sqrt{T}, -\tilde{x}+0.5\sigma^*\sqrt{T}; \rho') \tag{B.13}$$

where $\rho' = \frac{\sqrt{\tilde{\sigma}^2h}\sqrt{\tilde{\sigma}^2h}}{\sqrt{\tilde{\sigma}^2h+\sigma^2(T-h)}\sqrt{\tilde{\sigma}^2h+\sigma^2(T-h)}}$. Note that (B.12) comes from a direct application of (B.3). The second integral can be evaluated as

$$-2Ke^{-r(T-h)}S(0) \int_{-\infty}^{\infty} \Phi(-x_1)\Phi(-x_2)e^y \frac{1}{\tilde{\sigma}\sqrt{h}}\phi(\frac{y-mh}{\tilde{\sigma}\sqrt{h}})dy \tag{B.14}$$

$$= -2Ke^{-r(T-h)}S(0)e^{\mu h}\times$$

$$\Phi_2(-\frac{\frac{mh+\log(S(0)/Ke^{-r(T-h)})+0.5\sigma^2(T-h)+\tilde{\sigma}^2h}{\sigma\sqrt{T-h}}}{\sqrt{1+(\frac{\sqrt{\tilde{\sigma}^2h}}{\sigma\sqrt{T-h}})^2}}, -\frac{\frac{mh+\log(S(0)/Ke^{-r(T-h)})-0.5\sigma^2(T-h)+\tilde{\sigma}^2h}{\sigma\sqrt{T-h}}}{\sqrt{1+(\frac{\sqrt{\tilde{\sigma}^2h}}{\sigma\sqrt{T-h}})^2}}; \rho')$$

$$= -2Ke^{-r(T-h)}S(0)e^{\mu h}\Phi_2(-\tilde{x}-0.5\sigma^*\sqrt{T}, -\tilde{x}+0.5\sigma^*\sqrt{T} - \frac{\tilde{\sigma}^2h}{\sigma^*\sqrt{T}}; \rho') \tag{B.15}$$

Note that (B.14) comes from a direct application of (B.2). The third integral can be evaluated as

$$
\begin{aligned}
&(S(0))^2 \int_{-\infty}^{\infty} \Phi(-x_1)^2 e^{2y} \frac{1}{\tilde{\sigma}\sqrt{h}} \phi(\frac{y - mh}{\tilde{\sigma}\sqrt{h}}) dy \\
&= (S(0))^2 e^{2\mu h + \tilde{\sigma}^2 h} \times \\
&\Phi_2(-\frac{\frac{mh + \log{(S(0)/Ke^{-r(T-h)})} + 0.5\sigma^2(T-h) + 2\tilde{\sigma}^2 h}{\sigma\sqrt{T-h}}}{\sqrt{1 + (\frac{\sqrt{\tilde{\sigma}^2 h}}{\sigma\sqrt{T-h}})^2}}, -\frac{\frac{mh + \log{(S(0)/Ke^{-r(T-h)})} + 0.5\sigma^2(T-h) + 2\tilde{\sigma}^2 h}{\sigma\sqrt{T-h}}}{\sqrt{1 + (\frac{\sqrt{\tilde{\sigma}^2 h}}{\sigma\sqrt{T-h}})^2}}) \\
&= (S(0))^2 e^{2\mu h + \tilde{\sigma}^2 h} \Phi_2(-\tilde{x} - 0.5\sigma^* \sqrt{T} - \frac{\tilde{\sigma}^2 h}{\sigma^* \sqrt{T}}, -\tilde{x} - 0.5\sigma^* \sqrt{T} - \frac{\tilde{\sigma}^2 h}{\sigma^* \sqrt{T}}; \rho') \qquad \text{(B.16)}
\end{aligned}
$$

Combining (B.13)–(B.16) with use of (B.5) yields a closed form solution for (B.10) as desired.

B.2.3 Covariance of a Risky Bond

Our next step is to determine the covariance between two risky bonds at maturity similarly to that of two European call options at maturity from Cox and Rubinstein [14]. We first show that for the covariance of two risky bonds $RB_1(h)$ and $RB_2(h)$ we have

$$\mathbb{C}\text{ov}^{\mathbb{P}}(RB_1(h), RB_2(h)) = \mathbb{E}^{\mathbb{P}}(P_1(h)P_2(h)) - \mathbb{E}^{\mathbb{P}}(P_1(h))\mathbb{E}^{\mathbb{P}}(P_2(h)). \tag{B.17}$$

The challenging exercise is determining

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\left[P_1(S_1(h), K_1, r, h, T_1, \sigma_1^2)P_2(S_2(h), K_2, r, h, T_2, \sigma_2^2)\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Pi_{j=1}^{j=2}\left[K_j e^{-r(T_j-h)}\Phi(-x_{2(j)} + \sigma_j\sqrt{T_j - h}) - S_j(0)e^{y_j}N(-x_{1(j)})\right]\frac{1}{\tilde{\sigma}_1\sqrt{h}}\frac{1}{\tilde{\sigma}_2\sqrt{h}}\phi_2(\frac{y_1 - m_1 h}{\tilde{\sigma}_1\sqrt{h}}, \frac{y_2 - m_2 h}{\tilde{\sigma}_2\sqrt{h}}; \rho)dy_1 dy_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_1 K_2 e^{-r(T_1-h)}e^{-r(T_2-h)}\Phi(-x_{2(1)})\Phi(-x_{2(2)})\frac{1}{\tilde{\sigma}_1\sqrt{h}}\frac{1}{\tilde{\sigma}_2\sqrt{h}}\phi_2(\frac{y_1 - m_1 h}{\tilde{\sigma}_1\sqrt{h}}, \frac{y_2 - m_2 h}{\tilde{\sigma}_2\sqrt{h}}; \rho)dy_1 dy_2 \\
&- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[K_1 e^{-r(T_1-h)}S_2(0)e^{y_2}\Phi(-x_{2(1)})\Phi(-x_{1(2)})\right]\frac{1}{\tilde{\sigma}_1\sqrt{h}}\frac{1}{\tilde{\sigma}_2\sqrt{h}}\phi_2(\frac{y_1 - m_1 h}{\tilde{\sigma}_1\sqrt{h}}, \frac{y_2 - m_2 h}{\tilde{\sigma}_2\sqrt{h}}; \rho)dy_1 dy_2 \\
&- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[S_1(0)e^{y_1}K_2 e^{-r(T_2-h)}\Phi(-x_{1(1)})\Phi(-x_{2(2)})\right]\frac{1}{\tilde{\sigma}_1\sqrt{h}}\frac{1}{\tilde{\sigma}_2\sqrt{h}}\phi_2(\frac{y_1 - m_1 h}{\tilde{\sigma}_1\sqrt{h}}, \frac{y_2 - m_2 h}{\tilde{\sigma}_2\sqrt{h}}; \rho)dy_1 dy_2 \\
&+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[S_1(0)S_2(0)\Phi(-x_{1(1)})\Phi(-x_{1(2)})e^{y_1}e^{y_2}\right]\frac{1}{\tilde{\sigma}_1\sqrt{h}}\frac{1}{\tilde{\sigma}_2\sqrt{h}}\phi_2(\frac{y_1 - m_1 h}{\tilde{\sigma}_1\sqrt{h}}, \frac{y_2 - m_2 h}{\tilde{\sigma}_2\sqrt{h}}; \rho)dy_1 dy_2
\end{aligned} \tag{B.18}$$

where, for the jth asset, we have

$$-x_{1(j)} = \frac{-y_j}{\sigma_j\sqrt{T_j - h}} - \frac{\log\left(S_j(0)/K_j e^{-r(T_j-h)}\right) + 0.5\sigma_j^2(T_j - h)}{\sigma_j\sqrt{T_j - h}} \tag{B.19}$$

and

$$-x_{2(j)} = \frac{-y_j}{\sigma_j\sqrt{T_j - h}} - \frac{\log\left(S_j(0)/K_j e^{-r(T_j-h)}\right) - 0.5\sigma_j^2(T_j - h)}{\sigma_j\sqrt{T_j - h}} \tag{B.20}$$

as well as

- the correlation between the two firms is ρ;
- the covariance of the two firms is $\sigma_{12} = \rho\tilde{\sigma}_1\tilde{\sigma}_2$;
- $m_j h = (\mu_j - \frac{\tilde{\sigma}_j^2}{2})h$ for $j = 1, 2$;
- $\sigma_j^* = \sqrt{\tilde{\sigma}_j^2 \frac{h}{T_j} + \sigma_j^2(1 - \frac{h}{T_j})}$ for $j = 1, 2$;
- $\rho^* = \frac{\sigma_{12}}{\sigma_1^*\sigma_2^*\sqrt{T_1T_2}}$;
- $\mu_j' = e^{\mu_j h}$ for $j = 1, 2$;

The first integral in (B.18) is determined below.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_1K_2e^{-r(T_1-h)}e^{-r(T_2-h)}\Phi(-x_{2(1)})\Phi(-x_{2(2)})\frac{1}{\tilde{\sigma}_1\sqrt{h}}\frac{1}{\tilde{\sigma}_2\sqrt{h}}\phi_2(\frac{y_1 - m_1h}{\tilde{\sigma}_1\sqrt{h}}, \frac{y_2 - m_2h}{\tilde{\sigma}_2\sqrt{h}}; \rho)dy_1dy_2$$

$$= K_1K_2e^{-r(T_1-h)}e^{-r(T_2-h)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi(-x_{2(1)})\Phi(-x_{2(2)})\phi_2(z_1, z_2; \rho)dz_1dz_2$$

$$= K_1K_2e^{-r(T_1-h)}e^{-r(T_2-h)}\times$$

$$\Phi_2(\frac{-\frac{\mu_1h+\log(S_1(0)/K_1e^{-r(T_1-h)})-0.5\sigma_1^2(T_1-h)}{\sigma_1\sqrt{T_1-h}}}{\sqrt{1+(\frac{\sqrt{\tilde{\sigma}_1^2h}}{\sigma_1\sqrt{T_1-h}})^2}}, \frac{-\frac{\mu_2h+\log(S_2(0)/K_2e^{-r(T_2-h)})-0.5\sigma_2^2(T_2-h)}{\sigma_2\sqrt{T_2-h}}}{\sqrt{1+(\frac{\sqrt{\tilde{\sigma}_2^2h}}{\sigma_2\sqrt{T_2-h}})^2}}; \rho^*) \tag{B.21}$$

$$= K_1K_2e^{-r(T_1-h)}e^{-r(T_2-h)}\Phi_2(-\tilde{x}_{2(1)}, -\tilde{x}_{2(2)}; \rho^*) \tag{B.22}$$

Note that (B.21) comes from a direct application of (B.4). Also note that

$$\rho^* = \frac{\rho\frac{\sqrt{\tilde{\sigma}_1^2h}}{\sigma_1\sqrt{T_1-h}}\frac{\sqrt{\tilde{\sigma}_2^2h}}{\sigma_2\sqrt{T_2-h}}}{\sqrt{1+(\frac{\sqrt{\tilde{\sigma}_1^2h}}{\sigma_1\sqrt{T_1-h}})^2}\sqrt{1+(\frac{\sqrt{\tilde{\sigma}_2^2h}}{\sigma_2\sqrt{T_2-h}})^2}}. \tag{B.23}$$

The second integral is determined below.

$$
\begin{aligned}
&-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[K_1 e^{-r(T_1-h)}S_2(0)e^{y_2}\Phi(-x_{2(1)})\Phi(-x_{1(2)})\right]\frac{1}{\tilde{\sigma}_1\sqrt{h}}\frac{1}{\tilde{\sigma}_2\sqrt{h}}\phi_2(\frac{y_1-m_1h}{\tilde{\sigma}_1\sqrt{h}},\frac{y_2-m_2h}{\tilde{\sigma}_2\sqrt{h}};\rho)dy_1dy_2 \\
&= -K_1e^{-r(T_1-h)}S_2(0)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[e^{z_2\sqrt{\tilde{\sigma}_2^2h}+m_2h}\Phi(-x_{2(1)})\Phi(-x_{1(2)})\right]\frac{e^{\frac{-1}{2(1-\rho^2)}\left[z_1^2-2\rho z_1z_2+z_2^2\right]}}{2\pi\sqrt{1-\rho^2}}dz_1dz_2 \\
&= -K_1e^{-r(T_1-h)}S_2(0)e^{m_2h}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi(-x_{2(1)})\Phi(-x_{1(2)})\frac{e^{\frac{-z_1^2}{2}}}{\sqrt{2\pi}}\frac{e^{\frac{-\left[(z_2-\rho z_1)^2-2(1-\rho^2)z_2\sqrt{\sigma_2^2h}\right]}{2(1-\rho^2)}}}{\sqrt{2\pi(1-\rho^2)}}dz_1dz_2 \\
&= -K_1e^{-r(T_1-h)}S_2(0)e^{m_2h+\frac{\tilde{\sigma}_2^2h}{2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi(-x_{2(1)})\Phi(-x_{1(2)})\frac{e^{\frac{-(z_1-\rho\sqrt{\tilde{\sigma}_2^2h})^2}{2}}}{\sqrt{2\pi}}\frac{e^{\frac{-(z_2-(\rho z_1+(1-\rho^2)\sqrt{\tilde{\sigma}_2^2h}))^2}{2(1-\rho^2)}}}{\sqrt{2\pi(1-\rho^2)}}dz_1dz_2 \\
&= -K_1e^{-r(T_1-h)}S_2(0)e^{\mu_2h}\times \\
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi(-x_{2(1)}-\frac{\rho\sqrt{\tilde{\sigma}_1^2h\tilde{\sigma}_2^2h}}{\sigma_1\sqrt{T_1-h}})\Phi(-x_{1(2)}-\frac{\tilde{\sigma}_2^2h}{\sigma_2\sqrt{T_2-h}})\frac{e^{\frac{-(u_1)^2}{2}}}{\sqrt{2\pi}}\frac{e^{\frac{-(u_2-\rho u_1)^2}{2(1-\rho^2)}}}{\sqrt{2\pi(1-\rho^2)}}du_1du_2 \qquad \text{(B.24)} \\
&= -K_1e^{-r(T_1-h)}S_2(0)e^{\mu_2h}\times \\
&\Phi_2(\frac{-\frac{m_1h+\log(S_1(0)/K_1e^{-r(T_1-h)})-0.5\sigma_1^2(T_1-h)}{\sigma_1\sqrt{T_1-h}}-\frac{\rho\sqrt{\tilde{\sigma}_1^2h\tilde{\sigma}_2^2h}}{\sigma_1\sqrt{T_1-h}}}{\sqrt{1+(\frac{\sqrt{\tilde{\sigma}_1^2h}}{\sigma_1\sqrt{T_1-h}})^2}},\frac{-\frac{m_2h+\log(S_2(0)/K_2e^{-r(T_2-h)})+0.5\sigma_2^2(T_2-h)}{\sigma_2\sqrt{T_2-h}}-\frac{\tilde{\sigma}_2^2h}{\sigma_2\sqrt{T_2-h}}}{\sqrt{1+(\frac{\tilde{\sigma}_2^2h}{\sigma_2\sqrt{T_2-h}})^2}};\rho^*) \\
&= -K_1e^{-r(T_1-h)}S_2(0)e^{\mu_2h}\times \\
&\Phi_2(-\frac{\log(S_1(0)\mu_1'/K_1e^{-r(T_1-h)})-0.5\sigma_1^{*2}T_1+\rho\sqrt{\tilde{\sigma}_1^2h\tilde{\sigma}_2^2h}}{\sigma_1^*\sqrt{T_1}},-\frac{\log(S_2(0)\mu_2'/K_2e^{-r(T_2-h)})+0.5\sigma_2^{*2}T_2}{\sigma_2^*\sqrt{T_2}};\rho_{12}^*) \\
&= -K_1e^{-r(T_1-h)}S_2(0)e^{\mu_2h}\Phi_2(-\tilde{x}_{2(1)}-\frac{\rho\sqrt{\tilde{\sigma}_1^2h\tilde{\sigma}_2^2h}}{\sigma_1^*\sqrt{T_1}},-\tilde{x}_{1(2)};\rho^*) \qquad \text{(B.25)}
\end{aligned}
$$

Note that (B.24) comes from applying the joint transformation $u_1 = z_1 - \rho\sqrt{\tilde{\sigma}_2^2h}$ and $u_2 - \rho u_1 = z_2 - (\rho z_1 +$

$(1-\rho^2)\sqrt{\tilde{\sigma}_2^2 h})$. The third integral is determined below.

$$
\begin{aligned}
&-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[S_1(0)e^{y_1}K_2e^{-r(T_2-h)}\Phi(-x_{1(1)})\Phi(-x_{2(2)})\right]\frac{1}{\tilde{\sigma}_1\sqrt{h}}\frac{1}{\tilde{\sigma}_2\sqrt{h}}\phi_2(\frac{y_1-m_1h}{\tilde{\sigma}_1\sqrt{h}},\frac{y_2-m_2h}{\tilde{\sigma}_2\sqrt{h}};\rho)dy_1dy_2\\
&=-S_1(0)K_2e^{-r(T_2-h)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[e^{z_1\sqrt{\tilde{\sigma}_1^2h}+m_1h}\Phi(-x_{1(1)})\Phi(-x_{2(2)})\right]\frac{e^{\frac{-1}{2(1-\rho^2)}\left[z_1^2-2\rho z_1z_2+z_2^2\right]}}{2\pi\sqrt{1-\rho^2}}dz_1dz_2\\
&=-S_1(0)K_2e^{-r(T_2-h)}e^{m_1h}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi(-x_{1(1)})\Phi(-x_{2(2)})\frac{e^{\frac{-z_2^2}{2}}}{\sqrt{2\pi}}\frac{e^{\frac{-\left[(z_1-\rho z_2)^2-2(1-\rho^2)z_1\sqrt{\tilde{\sigma}_1^2h}\right]}{2(1-\rho^2)}}}{\sqrt{2\pi(1-\rho^2)}}dz_1dz_2\\
&=-S_1(0)K_2e^{-r(T_2-h)}e^{m_1h+\frac{\tilde{\sigma}_1^2h}{2}}\times\\
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi(-x_{1(1)})\Phi(-x_{2(2)})\frac{e^{\frac{-(z_2-\rho\sqrt{\tilde{\sigma}_1^2h})^2}{2}}}{\sqrt{2\pi}}\frac{e^{\frac{-(z_1-(\rho z_2+(1-\rho^2)\sqrt{\tilde{\sigma}_1^2h}))^2}{2(1-\rho^2)}}}{\sqrt{2\pi(1-\rho^2)}}dz_1dz_2\\
&=-S_1(0)K_2e^{-r(T_2-h)}e^{\mu_1h}\times\\
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi(-x_{1(1)}-\frac{\tilde{\sigma}_1^2h}{\sigma_1\sqrt{T_1-h}})\Phi(-x_{2(2)}-\frac{\rho\sqrt{\tilde{\sigma}_1^2h\tilde{\sigma}_2^2h}}{\sigma_2\sqrt{T_2-h}})\frac{e^{\frac{-u_2^2}{2}}}{\sqrt{2\pi}}\frac{e^{\frac{-(u_1-\rho u_2)^2}{2(1-\rho^2)}}}{\sqrt{2\pi(1-\rho^2)}}du_1du_2\\
&=-S_1(0)K_2e^{-r(T_2-h)}e^{\mu_1h}\times\\
&\Phi_2(\frac{-\frac{m_1h+\log(S_1(0)/K_1e^{-r(T_1-h)})+0.5\sigma_1^2(T_1-h)}{\sigma_1\sqrt{T_1-h}}-\frac{\tilde{\sigma}_1^2h}{\sigma_1\sqrt{T_1-h}}}{\sqrt{1+(\frac{\sqrt{\tilde{\sigma}_1^2h}}{\sigma_1\sqrt{T_1-h}})^2}},\frac{-\frac{m_2h+\log(S_2(0)/K_2e^{-r(T_2-h)})-0.5\sigma_2^2(T_2-h)}{\sigma_2\sqrt{T_2-h}}-\frac{\rho\sqrt{\tilde{\sigma}_1^2h\tilde{\sigma}_2^2h}}{\sigma_2\sqrt{T_2-h}}}{\sqrt{1+(\frac{\sqrt{\tilde{\sigma}_2^2h}}{\sigma_2\sqrt{T_2-h}})^2}};\rho^*)\\
&=-S_1(0)K_2e^{-r(T_2-h)}e^{\mu_1h}\Phi_2(-\tilde{x}_{1(1)},-\tilde{x}_{2(2)}-\frac{\rho\sqrt{\tilde{\sigma}_1^2h\tilde{\sigma}_2^2h}}{\sigma_2^*\sqrt{T_2}};\rho^*) \qquad \text{(B.26)}
\end{aligned}
$$

The fourth integral is determined below.

$$
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[S_1(0)S_2(0)\Phi(-x_{1(1)})\Phi(-x_{1(2)})e^{y_1}e^{y_2}\right]\frac{1}{\tilde{\sigma}_1\sqrt{h}}\frac{1}{\tilde{\sigma}_2\sqrt{h}}\phi_2(\frac{y_1-m_1h}{\tilde{\sigma}_1\sqrt{h}},\frac{y_2-m_2h}{\tilde{\sigma}_2\sqrt{h}};\rho)dy_1dy_2
$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[S_1(0)S_2(0)\Phi(-x_{1(1)})\Phi(-x_{1(2)})e^{y_1}e^{y_2}\right]\frac{e^{\frac{-1}{2(1-\rho^2)}\left[(\frac{y_1-m_1h}{\sigma_1\sqrt{h}})^2-2\rho(\frac{y_1-m_1h}{\tilde{\sigma}_1\sqrt{h}})(\frac{y_2-m_2h}{\tilde{\sigma}_2\sqrt{h}})+(\frac{y_2-m_2h}{\tilde{\sigma}_2\sqrt{h}})^2\right]}}{2\pi\tilde{\sigma}_1\sqrt{h}\tilde{\sigma}_2\sqrt{h}\sqrt{1-\rho^2}}dy_1dy_2$$

$$= S_1(0)S_2(0)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\Phi(-x_{1(1)})\Phi(-x_{1(2)})e^{z_1\sqrt{\tilde{\sigma}_1^2h}+m_1h+z_2\sqrt{\tilde{\sigma}_2^2h}+m_2h}\right]\frac{e^{\frac{-1}{2(1-\rho^2)}\left[z_1^2-2\rho z_1z_2+z_2^2\right]}}{2\pi\sqrt{1-\rho^2}}dz_1dz_2$$

$$= S_1(0)S_2(0)e^{m_1h+\frac{\tilde{\sigma}_1^2h}{2}+m_2h+\frac{\tilde{\sigma}_2^2h}{2}+\rho\sqrt{\tilde{\sigma}_1h\tilde{\sigma}_2h}}\times$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi(-x_{1(1)})\Phi(-x_{1(2)})\frac{e^{\frac{-(z_1-(\sqrt{\tilde{\sigma}_1^2h}+\rho\sqrt{\tilde{\sigma}_2^2h}))^2}{2}}}{\sqrt{2\pi}}\frac{e^{\frac{-(z_2-(\rho z_1+(1-\rho^2)\sqrt{\tilde{\sigma}_2^2h}))^2}{2(1-\rho^2)}}}{\sqrt{2\pi(1-\rho^2)}}dz_1dz_2$$

$$= S_1(0)S_2(0)e^{\mu_1h+\mu_2h+\rho\sqrt{\tilde{\sigma}_1h\tilde{\sigma}_2h}}\times$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi(-x_{1(1)}-\frac{\tilde{\sigma}_1^2h+\rho\sqrt{\tilde{\sigma}_1^2h\tilde{\sigma}_2^2h}}{\sigma_1\sqrt{T_1-h}})\Phi(-x_{1(2)}-\frac{\tilde{\sigma}_2^2h+\rho\sqrt{\tilde{\sigma}_1^2h\tilde{\sigma}_2^2h}}{\sigma_2\sqrt{T_2-h}})\frac{e^{\frac{-u_2^2}{2}}}{\sqrt{2\pi}}\frac{e^{\frac{-(u_1-\rho u_2)^2}{2(1-\rho^2)}}}{\sqrt{2\pi(1-\rho^2)}}du_1du_2$$

$$= S_1(0)S_2(0)e^{\mu_1h+\mu_2h+\rho\sqrt{\tilde{\sigma}_1h\tilde{\sigma}_2h}}\times$$

$$\Phi_2\left(\frac{-\frac{m_1h+\log(S_1(0)/K_1e^{-r(T_1-h)})+0.5\sigma_1^2(T_1-h)+\tilde{\sigma}_1^2h+\rho\sqrt{\tilde{\sigma}_1^2h\tilde{\sigma}_2^2h}}{\sigma_1\sqrt{T_1-h}}}{\sqrt{1+(\frac{\sqrt{\tilde{\sigma}_1^2h}}{\sigma_1\sqrt{T_1-h}})^2}},\frac{-\frac{m_2h+\log(S_2(0)/K_2e^{-r(T_2-h)})+0.5\sigma_2^2(T_2-h)+\tilde{\sigma}_2^2h+\rho\sqrt{\tilde{\sigma}_1^2h\tilde{\sigma}_2^2h}}{\sigma_2\sqrt{T_2-h}}}{\sqrt{1+(\frac{\tilde{\sigma}_2^2h}{\sigma_2\sqrt{T_2-h}})^2}};\rho^*\right)$$

$$= S_1(0)S_2(0)e^{\mu_1h+\mu_2h+\rho\sqrt{\tilde{\sigma}_1^2h\tilde{\sigma}_2^2h}}\Phi_2(-\tilde{x}_{1(1)}-\frac{\rho\sqrt{\tilde{\sigma}_1^2h\tilde{\sigma}_2^2h}}{\sigma_1^*\sqrt{T_1}},-\tilde{x}_{1(2)}-\frac{\rho\sqrt{\tilde{\sigma}_1^2h\tilde{\sigma}_2^2h}}{\sigma_2^*\sqrt{T_2}};\rho^*) \tag{B.27}$$

where, for the jth asset, we have

$$-\tilde{x}_{1(j)} = -\frac{\log(S_j(0)\mu_j'/K_je^{-r(T_j-h)})+0.5\sigma_j^{*2}T_j}{\sigma_j^*\sqrt{T_j}} \tag{B.28}$$

and

$$-\tilde{x}_{2(j)} = -\frac{\log(S_j(0)\mu_j'/K_je^{-r(T_j-h)})-0.5\sigma_j^{*2}T_j}{\sigma_j^*\sqrt{T_j}} \tag{B.29}$$

Combining the results of (B.22)–(B.27) yields a closed form solution for (B.18).

Appendix C

Glossary of Notation

In this appendix we outline some important notation that is used in each chapter of this thesis.

C.1 Chapter 1

1. CEF: Closed end fund [Chapter 1 on page 1];
2. CEFs: Closed end funds [Chapter 1 on page 1];
3. CEBF: Closed end bond fund [Chapter 1 on page 1];
4. CEBFs: Closed end bond funds [Chapter 1 on page 1];
5. OEBF: Open end bond fund [Chapter 1 on page 1];
6. OEBFs: Open end bond funds [Chapter 1 on page 1];
7. RB: Risky bond [Chapter 1 on page 1];
8. ERR: Expected rate of return [Chapter 1 on page 1];

C.2 Chapter 2

1. OEF: Open end fund [Section 2.1 on page 4];
2. UIT: Unit investment trust [Section 2.1 on page 4];
3. CEF: Closed end fund [Section 2.1 on page 4];
4. USD: U.S. dollars [Table 2.1.1];
5. IPO: Initial Public Offering [Section 2.3 on page 8];
6. NAV_t: the net asset value of the closed end fund at time t [Section 2.3 on page 8];
7. P_t: the price per share of the closed end fund at time t [Section 2.3 on page 8];
8. $DISC_t$: the discount per share of the closed end fund at time t [Section 2.3 on page 8];
9. SEC: Securities and Exchange Commission [Section 2.5 on page 9];
10. AMPS: Auction market preferred shares [Subsection 2.7.1 on page 21];
11. bps: Basis Points [Subsection 2.7.4 on page 25];

C.3 Chapter 3

1. FV: Face value of debt [Section 3.1 on page 29];
2. $S_i(t)$: Asset i price at time t, for $i = 1, \ldots, n$ [Section 3.1 on page 29];
3. T_i: Asset i time to maturity, for all $i = 1, \ldots, n$ [Section 3.1 on page 29];
4. K_i: Face value of asset i, for all $i = 1, \ldots, n$ [Section 3.1 on page 29];
5. $RB_i(t)$: risky bond (RB) price at time t of firm i with firm price $S_i(t)$ [as defined (3.1.1)];
6. $P_i(S_i(t), K_i, r, t, T_i, \sigma_i^2)$: price of a European put option at time t written on firm i with firm price $S_i(t)$, asset volatility σ_i, with option contract maturity of T_i and risk free rate r [Section 3.1 on page 29];

7. $\mathcal{F}_i(t)$: is defined as the sigma algebra (or information) for $S_i(t)$ at time t [as defined in section 3.1 on page 29];

8. $\mathcal{F}(t)$ is the sigma algebra (or available information) for all of $S_1(t), \ldots, S_n(t)$ at time t [as defined in section 3.2.1 on page 31];

9. $YTM_i(t)$: is the yield to maturity on the credit risky bond $RB_i(t)$ priced on firm i at time t [as defined in (3.1.3)];

10. $V(T^*)$: closed end bond fund value at time T^* [as defined in (3.2.1)];

11. K^*: Face value of the debt taken by the CEBF [as defined in Section 3.2.2 on page 32];

12. T^*: Maturity of the debt K^* taken by the CEBF $V(\cdot)$ [Section 3.2 on page 31];

13. $V_{UL}(T^*)$: the unlevered CEBF value [as defined in (3.2.2)];

14. $V_{L-dt}(T^*)$: the value of the debtholder portion of the levered CEBF [as defined in (3.2.5)];

15. $V_{L-eq}(T^*)$: the value of the equityholder portion of the levered CEBF [as defined in (3.2.6)];

16. r: risk free rate [Section 3.3 on page 34];

17. σ_i: Asset i volatility according to the Black and Scholes European option pricing model [Section 3.3 on page 34];

18. Std. Dev.: Standard Deviation [Section 3.3.1 on page 38]

19. ρ_{Low}: 5×5 correlation matrix in a low correlation environment [Introduced in Section 3.3 on page 34 and defined in Section A.2 on page 69];

20. ρ_{High}: 5×5 correlation matrix in a high correlation environment [Introduced in Section 3.3 on page 34 and defined in Section A.2 on page 69];

C.4 Chapter 4

1. $\mathbb{E}^{\mathbb{P}}[\cdot]$: is the expected value evaluated under the Real World $\mathbb{P}$-measure [Section 4.1 on page 49];

2. $\mathbb{E}^{\mathbb{Q}}[\cdot]$: is the expected value evaluated under the Risk Neutral $\mathbb{Q}$-measure;

3. $\mathbb{E}^{\mathbb{P}}[R_P(h)]$: is the expected rate of return on a European put option priced using Rubinstein's option pricing model over a holding period h [as defined in (4.1.1)];

4. $\mathbb{E}^{\mathbb{P}}[R_{RB}(h)]$: is the expected rate of return on a European put option priced using Rubinstein's option pricing model over a holding period h [as defined in (4.2.1)];

5. $\mathbb{E}^{\mathbb{P}}[R_{L-eq}(h)]$: the expected equityholder rate of return of the levered CEBF over a holding period h [as defined in (4.4.1)]

6. $\mathbb{E}^{\mathbb{P}}[R_{L-dt}(h)]$: the exptected debtholder rate of return of the levered CEBF over a holding period h [as defined in (4.4.2)];

7. $\lambda_{UL}(h)$: equityholder Sharpe Ratio for an unlevered closed end bond fund [as defined in (4.3.4)];

8. $\lambda_{L-eq}(h)$: equityholder Sharpe Ratio for levered closed end bond fund [as defined in (4.4.5)];

9. $\lambda_{L-dt}(h)$: debtholder Sharpe Ratio for levered closed end bond fund [as defined in (4.4.6)];

10. μ_i: Asset i $\mathbb{P}$-measure expected rate of return for all $i = 1, \ldots, n$ under Rubinstein's option pricing model [Section 4.5 on page 53];

11. $\tilde{\sigma}_i$: firm i for all $i = 1, \ldots, n$, $\mathbb{P}$-measure volatility under Rubinstein's option pricing model [Section 4.5 on page 53];

C.5 Appendix A

1. r: is the constant risk free interest rate [as defined in section A.1.1 on page 61];

2. δ: is the constant dividend yield on the underlying asset $S(\cdot)$ [as defined in section A.1.1 on page 61];

3. σ: is the constant volatility of the underlying asset $S(\cdot)$ [as defined in section A.1.1 on page 61];

4. $\mu' = (r - \delta - \frac{\sigma^2}{2})(T - t)$: simplified notation [as defined in section A.1.1 on page 61];

5. $\sigma' = \sigma\sqrt{T-t}$: simplified notation [as defined in section A.1.1 on page 61];

6. $\mathcal{N}(\mu, \sigma)$: denotes a Normal distribution with mean μ and variance σ^2 [as defined in section A.1.1 on page 61];

7. $\phi(x)$: denotes the probability density function of a standard Normal density with mean zero and unit variance evaluated at x [as defined in section A.1.1 on page 61];

8. $\Phi(x)$: denotes the cumulative standard Normal distribution with mean zero and unit variance evaluated at x [as defined in section A.1.1 on page 61];

9. $d_1 = \frac{\log(S(0)/K)+(r-\delta+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$: as defined in the Black Scholes framework [as defined in section A.1.1 on page 61];

10. $d_2 = \frac{\log(S(0)/K)+(r-\delta-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$: as defined in the Black Scholes framework [as defined in section A.1.1 on page 61];

11. GBM: Geometric Brownian Motion process [as defined in section A.1.2 on page 64];

12. ρ: Correlation between two asset price processes $S_1(t)$ and $S_2(t)$ [as defined in section A.1.2 on page 64];

13. $\mathcal{BVN}(\mu_x, \mu_y, \sigma_x, \sigma_y, \rho)$: denotes a Bivariate Normal Distribution with means μ_x and μ_y with standard deviations σ_x and σ_y with correlation ρ [as defined in section A.1.2 on page 64];

14. $\mu_i' = (r - \delta_i - \frac{\sigma_i^2}{2})(T_j - t)$: for $i = x, y$ and $j = 1, 2$ [as defined in section A.1.2 on page 64];

15. $\sigma_i' = \sigma_i\sqrt{T_j - t}$: for $i = x, y$ and $j = 1, 2$ [as defined in section A.1.2 on page 64];

16. $\phi_2(u, v; \rho)$: denotes a probability density function of a standard Bivariate Normal distribution evaluated at u and v with correlation parameter ρ [as defined in section A.1.2 on page 64];

17. $\Phi_2(u, v; \rho)$: denotes a standard Bivariate Normal cumulative density function evaluated at u and v with correlation parameter ρ [as defined in section A.1.2 on page 64];

18. $d_{i(j)}$: denotes the value of d_i for $i = 1, 2$ as denoted in the standard Black and Scholes framework for the jth asset, $j = 1, 2$ [as defined in section A.1.2 on page 64];

C.6 Appendix B

1. $\sigma^* = \sqrt{\tilde{\sigma}^2 \frac{h}{T} + \sigma^2 \frac{(T-h)}{T}}$: weighted geometric average of $\tilde{\sigma}$ over a period h and σ over the period $T-h$ [see section B.2.1 on page 74];

2. $m = \mu - \frac{1}{2}\tilde{\sigma}^2$: Real-World $\mathbb{P}$-measure Lognormal model parameter [see section B.2.1 on page 74];

3. $\tilde{x} = \frac{\log(S(0)\mu'/Ke^{-r(T-h)})}{\sigma^* \sqrt{T}}$: Black-Scholes style analogue shorthand of $d_1 - \frac{1}{2}\sigma^2$ [see section B.2.1 on page 74];

4. $\mu' = e^{\mu h}$: expected rate of return under the Real-World $\mathbb{P}$-measure [see section B.2.1 on page 74]:

5. $P_i(h) = P_i(S_i(h), K_i, r, h, T_i, \sigma_i^2)$: is a European put option at time h written on firm i with firm price $S_i(h)$, asset volatility σ_i, with option contract maturity of T_i and risk free rate r [as defined in (B.8) in B.2.2 on page 75];

6. $RB_i(h) = K_i e^{-r(T_i-h)} - P_i(S_i(h), K_i, r, h, T_i, \sigma_i^2)$: is a risky bond at time h written on firm i with firm price $S_i(h)$, asset volatility σ_i, with option contract maturity of T_i and risk free rate r [as defined in (B.9) in B.2.2 on page 75];

7. $\rho' = \frac{\sqrt{\tilde{\sigma}^2 h}\sqrt{\tilde{\sigma}^2 h}}{\sqrt{\tilde{\sigma}^2 h + \sigma^2(T-h)}\sqrt{\tilde{\sigma}^2 h + \sigma^2(T-h)}}$: resulting implied correlation from the variance of European put option price [see section B.2.2 on page 75];

8. ρ: the correlation between the two firms [See section B.2.3 on page 78];

9. $\sigma_{12} = \rho\tilde{\sigma}_1\tilde{\sigma}_2$: the covariance between the two firms [See section B.2.3 on page 78];

10. $m_j h = (\mu_j - \frac{\tilde{\sigma}_j^2}{2})h$: Real-World $\mathbb{P}$-measure Lognormal model parameter for $j = 1, 2$: [See section B.2.3 on page 78];

11. $\sigma_j^* = \sqrt{\tilde{\sigma}_j^2 \frac{h}{T_j} + \sigma_j^2 (1 - \frac{h}{T_j})}$: weighted geometric average of $\tilde{\sigma}_j$ over a period h and σ_j over the period $T_j - h$ for assets $j = 1, 2$ [See section B.2.3 on page 78];

12. $\rho^* = \frac{\sigma_{12}}{\sigma_1^* \sigma_2^* \sqrt{T_1 T_2}}$: resulting firm correlation with volatilities σ_1^* and σ_2^* [See section B.2.3 on page 78];

13. $\mu_j' = e^{\mu_j h}$: expected rate of return of stock j under the Real-World $\mathbb{P}$-measure for $j = 1, 2$ [See section B.2.3 on page 78];

14. $-\tilde{x}_{1(j)} = -\frac{\log(S_j(0)\mu'_j/K_j e^{-r(T_j-h)})+0.5\sigma_j^{*2}T_j}{\sigma_j^*\sqrt{T_j}}$: Black-Scholes style analogue shorthand of d_1 for assets $j = 1, 2$ [See section B.2.3 on page 78];

15. $-\tilde{x}_{2(j)} = -\frac{\log(S_j(0)\mu'_j/K_j e^{-r(T_j-h)})-0.5\sigma_j^{*2}T_j}{\sigma_j^*\sqrt{T_j}}$: Black-Scholes style analogue shorthand of d_2 for assets $j = 1, 2$ [See section B.2.3 on page 78];

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