Notions of Dependence with Applications in Insurance and Finance

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Many insurance and finance activities involve multiple risks. Dependence structures between different risks play an important role in both theoretical models and practical applications. However, stochastic and actuarial models with dependence are very challenging research topics. In most literature, only special dependence structures have been considered. However, most existing special dependence structures can be integrated into more-general contexts. This thesis is motivated by the desire to develop more-general dependence structures and to consider their applications.

This thesis systematically studies different dependence notions and explores their applications in the fields of insurance and finance. It contributes to the current literature in the following three main respects. First, it introduces some dependence notions to actuarial science and initiates a new approach to studying optimal reinsurance problems. Second, it proposes new notions of dependence and provides a general context for the studies of optimal allocation problems in insurance and finance. Third, it builds the connections between copulas and the proposed dependence notions, thus enabling the constructions of the proposed dependence structures and enhancing their applicability in practice.

The results derived in the thesis not only unify and generalize the existing studies of optimization problems in insurance and finance, but also admit promising applications in other fields, such as operations research and risk management.

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Dedicated to My Dear Parents

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List of Symbols

Stochastic Orders

$\le_{a.s.}$	almost sure inequality	6
$\le_{cx}$	convex order	9
$\le_{cv}$	concave order	9
$\le_{dcx}$	directional convex order	45
$\le_{hr}$	hazard rate order	6
$\le_{hr:j}$	joint hazard rate order	74
$\le_{icx}$	increasing convex order	9
$\le_{icv}$	increasing concave order	9
$\le_{idcx}$	increasing directional convex order	45
$\le_{lo}$	lower orthant order	10
$\le_{lr}$	likelihood ratio order	6
$\le_{lr:j}$	joint likelihood ratio order	74
$\le_{mgf}$	moment generating function order	9
$\le_{mom}$	moments order	8
$\le_{rh}$	reverse hazard rate order	6
$\le_{sl}$	stop-loss order	10

$=_{st}$	equality in distribution	6
$\leq_{st}$	(multivariate) usual stochastic order	6, 11
$\leq_{uo}$	upper orthant stochastic order	10

Abbreviations

AI	arrangement increasing	76
AD	arrangement decreasing	76
CI	conditionally increasing	23
CIS	conditionally increasing in sequence	23
CTDAI	conditionally tail density arrangement increasing	101
CUOAI	conditionally upper orthant arrangement increasing	78
LOAI	lower orthant arrangement increasing	96
PDS	positive dependence through the stochastic ordering	23
PDUO	positive dependence through the upper orthant ordering	24
SAI	stochastic arrangement increasing	111
SI	stochastically increasing	22
TDAI	tail density arrangement increasing	101
UOAI	upper orthant arrangement increasing	101
WSI	weakly stochastically increasing	24

Functional Classes

Chapter 1

Introduction

1.1 Background

In both practical and theoretical studies, long-standing strategic problems widely exist, such as how to design a insurance policy, or how to make an investment. Essentially, these kinds of problems can be formulated as optimization problems. This thesis focuses on modeling and solving optimization problems in insurance and finance.

In the field of insurance, reinsurance is an important instrument for risk management. In order to reduce the exposure to risk, an insurer chooses to enter into a reinsurance arrangement to cede part of its risk to a reinsurer. One of the main concerns of the insurer is how to design an optimal reinsurance arrangement so as to minimize its retained risk. Such an optimal reinsurance problem is a complex topic in actuarial science and has been extensively studied during the past decades. In the classical setting of optimal reinsurance problems, an insurer's risk is modeled by a single risk and a lot of results have been derived.

The drawback of the single-risk model is that it is too simple to capture the complexity of insurance risk in the real world. In practice, an insurer is likely to run more than one line of business. In this case, the insurance risks have to be modeled by a random vector. The difficulties of the multivariate risk model come from two sources. First, in insurance practice, different lines of business are likely to adopt different reinsurance arrangements separately, in what is called individualized strategy. The individualized strategy is a vector of ceded functions other than a single ceded function, which means that the optimal reinsurance problem is an optimization over multiple functionals. Second, dependence structures between risks from different lines of business can be complicated. To cope with these difficulties, seeking solutions to optimal reinsurance problems is divided into two steps: identifying the optimal reinsurance form and determining parameters of the optimal reinsurance form. In the literature, optimal reinsurance problems with special dependence structures have been studied and the excess-of-loss strategy has proved to be the optimal form in most cases.

With an optimal reinsurance form known as an individualized excess-of-loss form, the optimal reinsurance problem reduces to identifying the optimal parameters, or specifically, the optimal retentions. Essentially, it is an extreme value problem of a multivariate function. However, without the joint distribution function of the risk random variables, the extreme value problem is still difficult to solve. In that case, many studies explore the qualitative properties of the optimal retentions. This approach actually puts the study of the optimal reinsurance problem into a different context, as an optimal allocation problem.

Optimal allocation problems are commonly seen in the fields of insurance, finance and operations research. For example, investors are concerned about how to allocate investment weights in different assets; a system designer wants to determine an optimal issuing order of system components so as to maximize the duration of the system. Again, with assumptions of special dependence structures, many optimal allocation problems have been studied and the solutions to optimal reinsurance problems have been qualitatively analyzed in the same context.

It is worthwhile to point out that, in the aforementioned optimization problems, the dependence structure between the multiple risks plays an essential role. The ways in which the risks depend on each other directly affect what optimal strategy will be adopted. It is also the complexity of the dependence structure that brings the main difficulty in solving those models. To simplify the models, many studies have focused only on special dependence structures such as independence, comonotonicity, exchangeability or the common shock model.

Motivated by the limitations of the dependence structures in the literature, this thesis aims to develop general dependence structures so as to unify as well as to extend the existing studies.

Roughly speaking, there are two types of dependence structures. Dependence structures of type I are fully characterized by copulas, regardless of marginal distributions. Typical examples of this type include independence and comonotonicity. Type I dependence is a property of copula and thus is preserved under (strictly) increasing transformations. For instance, for comonotonic random variables, their increasing transformations still result in comonotonic random variables. Dependence structures of type II involve properties of both copula and marginal distributions, for example, exchangeable random variables. It is easy to see that type II dependence is not preserved under increasing transformations, since increasing transformations of exchangeable random variables do not necessarily have the same marginal distributions.

Solving different optimization problems calls for different dependence structures. In order to identify the optimal reinsurance form, independence or comonotonicity between insurance risks has been assumed in most of the literature. In Chapter 3 of the thesis, we employ the concept of "positive dependence through sequence" (PDS) and develop a related concept to study the optimal reinsurance problem. The concept of PDS was initially proposed by Shaked (1977), and has been applied in the field of statistics. We introduce this concept to model dependence structures of insurance risks for two reasons. First, PDS is a positive dependence structure. In insurance practice, risks from different lines of business are often supposed to be positively correlated since they are more likely to be affected by common external factors in the same way. Second, PDS is a very general dependence structure, which includes independence and comonotonicity as special cases. In this sense, this thesis generalizes the studies of optimal reinsurance problems.

In order to study the optimal allocation problems, stronger assumptions of dependence structures are needed. In qualitative analyses of optimal allocation of policy limits/deductibles, Cheung (2007), Zhuang et al. (2009), and many others have made the assumptions of independence and comonotonicity with marginal distributions ordered in certain stochastic orders. These assumptions are special cases of type II dependence structures. In light of this observation, some general dependence structures of type II are developed in Chapter 4. With the introduction of these new dependence structures, we have unified the existing studies of optimal allocation problems and also explore more applications in Chapter 5.

Below is a brief summary of the rest of the chapters.

In Chapter 2, we examine the notion of PDS and propose a related concept PDUO. We derive invariant properties of these notions of positive dependence. More importantly, we build a relation between PDS/PDUO and copulas. This relation allows us to construct PDS/PDUO random vectors easily through copulas and thus greatly enhances applicability of these notions. In addition, we generalize the invariant property of copulas, which is widely used in the fields of not only finance and insurance but also statistics, economics and other areas.

In Chapter 3, we use the notions of PDS and PDUO to study the optimal reinsurance problem, thereby broadening approaches to address this optimization problem. We prove that the individualized excess-of-loss reinsurance strategy is optimal, not only in the traditional sense of minimizing risk measures of the total retained risk, but also in the sense of minimizing the ruin probability or stochastically maximizing the ruin time in the collective model, and maximizing the expected utility in the model with a random initial wealth. On the other hand, we also generalize some classical results about multivariate stochastic orders.

In Chapter 4, we work on the dependence structures of type II. We develop an existing notion and propose some new notions through the arrangement increasing property of joint distribution functions or joint density functions. We systematically analyze properties of these notions. We develop different characterizations of these dependence structures and illustrate how to construct them through copulas. Special dependence structures that have been considered in the literature are proved to belong to these new dependence structures.

In Chapter 5, we apply the dependence structures proposed in Chapter 4 to study optimal allocation problems in insurance, finance as well as operations research. We restudy the following problems: optimal allocation of policy limits/deductibles with discount factor, optimal portfolio selections and optimal stochastic scheduling problems. The results derived in Chapter 5 unify and extend previous studies in optimal allocation problems.

In Chapter 6, we provide concluding remarks.

In order to make the thesis self-contained, some standard contents about stochastic orders, risk measures and copulas are firstly introduced.

1.2 Stochastic Orders

Due to the randomness, it is difficult to compare two random variables directly. Many stochastic orders are proposed for the comparison of random variables. Below we state some commonly used results about stochastic orders, which can be found in Shaked and Shanthikumar (2007).

Through out this thesis, we assume that all random variables are defined on the common probability space $(\Omega, \mathcal{F}, \mathbb{P})$; expectations under $\mathbb{P}$ are assumed to be finite whenever we write them; the notation of '$\leq_{a.s.}$ ($\geq_{a.s.}$)' means the inequality '$\leq$ ($\geq$)' holds almost surely on the common probability space $(\Omega, \mathcal{F}, \mathbb{P})$; and the notation of '$=_{st}$' means the equality holds in distribution.

Definition 1.2.1 Let X and Y be two random variables with distribution functions $F_X(x) = 1 - \bar{F}_X(x)$ and $F_Y(x) = 1 - \bar{F}_Y(x)$.

X is said to be smaller than Y in the *usual stochastic order*, denoted as $X \leq_{st} Y$, if $\bar{F}_X(x) \leq \bar{F}_Y(x)$ for all $x \in \mathbb{R}$;

X is said to be smaller than Y in the *hazard rate order*, denoted as $X \leq_{hr} Y$, if $\bar{F}_Y(x)/\bar{F}_X(x)$ is increasing in $x \in \mathbb{R}$ such that $\bar{F}_X(x) > 0$;

X is said to be smaller than Y in the *reversed hazard rate order*, denoted as $X \leq_{rh} Y$, if $F_Y(x)/F_X(x)$ is increasing in $x \in \mathbb{R}$ such that $F_X(x) > 0$.

Assume that X and Y have density functions $f_X(x)$ and $f_Y(x)$. X is said to be smaller than Y in the *likelihood ratio order*, denoted as $X \leq_{lr} Y$, if $f_Y(x)/f_X(x)$ is increasing in $x \in \mathbb{R}$ such that $f_X(x) > 0$. □

For the likelihood ratio order, the assumption of the existence of the density functions is not essential. The following is an equivalent definition of the likelihood ratio order, which avoids the density function.

Definition 1.2.2 Random variable X is said to be smaller than random variable Y in the *likelihood ratio order*, denoted as $X \leq_{lr} Y$, if

$$\mathbb{P}\{X \in A\}\mathbb{P}\{Y \in B\} \geq \mathbb{P}\{X \in B\}\mathbb{P}\{Y \in A\}$$

for all $A, B \in \mathcal{B}(\mathbb{R})$ such that $\sup A \leq \inf B$. □

The usual stochastic order has a functional characterization: $X \leq_{st} Y$ if and only if $\mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)]$ for any increasing function $u(x)$ such that the expectations exist.

It is known that the stochastic orders defined above have the following implications:

$$X \leq_{lr} Y \Longrightarrow X \leq_{hr} Y \; (X \leq_{rh} Y) \Longrightarrow X \leq_{st} Y.$$

The following relation between the hazard rate order and the reverse hazard rate can be found in Theorem 1.B.41 in Shaked and Shanthikumar (2007).

Proposition 1.2.3 $X \leq_{hr} Y$ if and only if $-X \geq_{rh} -Y$. □

Definition 1.2.4 Random variables $X_1, ..., X_n$ are said to be *comonotonic*, if

$$\mathbb{P}\{X_1 \leq x_1, ..., X_n \leq x_n\} = \min\{\mathbb{P}\{X_1 \leq x_1\}, ..., \mathbb{P}\{X_n \leq x_n\}\}.$$

We also refer this case as that random vector $(X_1, ..., X_n)$ is comonotonic. □

For comonotonic random variables X and Y, there exists a random variable Z and nondecreasing functions f, g such that $X = f(Z)$ and $Y = g(Z)$.

Proposition 1.2.5 If random variables X and Y are comonotonic, then $X \leq_{st} Y$ if and only if $X \leq_{a.s.} Y$.

Proof. The proof for the "if" part is obvious.

For the "only if" part, noting that $\{X > Y\} = \cup_{r\in\mathbb{Q}}\{X > r > Y\}$, where $\mathbb{Q}$ is the collection of all the rational numbers, we have

$$\begin{aligned} \mathbb{P}\{X > Y\} &= \mathbb{P}\left\{\cup_{r\in\mathbb{Q}}\{X > r > Y\}\right\} \leq \sum_{r\in\mathbb{Q}} \mathbb{P}\{X > r \geq Y\} \\ &= \sum_{r\in\mathbb{Q}} \left(\mathbb{P}\{X > r\} - \mathbb{P}\{X > r, Y > r\}\right). \end{aligned} \tag{1.2.1}$$

Since X and Y are comonotonic and $X \leq_{st} Y$, we have

$$\mathbb{P}\{X > r, Y > r\} = \min\{\mathbb{P}\{X > r\}, \mathbb{P}\{Y > r\}\} = \mathbb{P}\{X > r\}.$$

Therefore, $\mathbb{P}\{X > Y\} = 0$ from (1.2.1), which means $X \leq_{a.s.} Y$. □

Definition 1.2.6 Let X and Y be two nonnegative random variables.

X is said to be smaller than Y in the *moments order*, denoted as $X \leq_{mom} Y$, if

$$\mathbb{E}[X^m] \leq \mathbb{E}[Y^m] \qquad \text{for all } m \in \mathbb{N}.$$

X is said to be smaller than Y in the *moment generating function order*, denoted as $X \leq_{mgf} Y$, if

$$\mathbb{E}[e^{sX}] \leq \mathbb{E}[e^{sY}],$$

for all $s > 0$ such that the above expectations exist. □

Noting that $e^{sx} = \sum_{n=0}^{\infty} \frac{1}{n!}(sx)^n$, we have $X \leq_{mom} Y$ implies $X \leq_{mgf} Y$ given that moment generating functions of X and Y exist.

Definition 1.2.7 Let X and Y be two random variables.

X is said to be smaller than random variable Y in the *convex order*, denoted as $X \leq_{cx} Y$, if $\mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)]$ for any convex function $u(x)$ such that the expectations exist.

X is said to be smaller than random variable Y in the *increasing convex order*, denoted as $X \leq_{icx} Y$, if $\mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)]$ for any increasing convex function $u(x)$ such that the expectations exist.

X is said to be smaller than random variable Y in the *concave order*, denoted as $X \leq_{cv} Y$, if $\mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)]$ for any concave function $u(x)$ such that the expectations exist.

X is said to be smaller than random variable Y in the *increasing concave order*, denoted as $X \leq_{icv} Y$, if $\mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)]$ for any increasing concave function $u(x)$ such that the expectations exist. □

Obviously, $X \leq_{cx} (\leq_{cv}) Y \Rightarrow X \leq_{icx} (\leq_{icv}) Y$, and $X \leq_{st} Y$ implies that $X \leq_{icx} (\leq_{icv})Y$. With the assumption of $\mathbb{E}[X] = \mathbb{E}[Y]$, $X \leq_{icx} (\leq_{icv})Y$ implies that $X \leq_{cx} (\leq_{cv})Y$.

Insurance practice often involves the comparison of the stop-loss premiums of two risks, which motivates the definition of the stop-loss order.

Definition 1.2.8 Random variable X is said to be smaller than random variable Y in the *stop-loss order*, denoted as $X \leq_{sl} Y$, if $\mathbb{E}[(X-t)_+] \leq \mathbb{E}[(Y-t)_+]$ for all $t \in \mathbb{R}$. □

It is easy to verify that, $X \leq_{sl} Y$ if and only if $X \leq_{icx} Y$; see for example Shaked and Shanthikumar (2007).

The following result, known as Ohlin's Lemma, provides a useful sufficient condition for the convex order, and the proof can be found in Lemma 3 of Ohlin (1969).

Lemma 1.2.9 Let X be a random variable, h_1 and h_2 be increasing functions such that $\mathbb{E}[h_1(X)] = \mathbb{E}[h_2(X)]$. If there exists $\alpha \in \mathbb{R} \cup \{+\infty\}$ such that $h_1(x) \geq h_2(x)$ for all $x < \alpha$ and $h_1(x) \leq h_2(x)$ for all $x > \alpha$, then $h_1(X) \leq_{cx} h_2(X)$. □

When it comes to comparison of random vectors, the univariate stochastic orders can be generalized into multivariate cases. In the following, we focus on the usual stochastic order and investigate its generalizations in three different forms.

Definition 1.2.10 Let $(X_1, ..., X_n)$ and $(Y_1, ..., Y_n)$ be two random vectors.

$(X_1, ..., X_n)$ is said to be smaller than $(Y_1, ..., Y_n)$ in the *upper orthant order*, denoted as $(X_1, ..., X_n) \leq_{uo} (Y_1, ..., Y_n)$, if $\mathbb{P}\{X_1 > z_1, ..., X_n > z_n\} \leq \mathbb{P}\{Y_1 > z_1, ..., Y_n > z_n\}$ for any $(z_1, ..., z_n) \in \mathbb{R}^n$.

$(X_1, ..., X_n)$ is said to be smaller than $(Y_1, ..., Y_n)$ in the *lower orthant order*, denoted as $(X_1, ..., X_n) \leq_{lo} (Y_1, ..., Y_n)$, if $\mathbb{P}\{X_1 \leq z_1, ..., X_n \leq z_n\} \geq \mathbb{P}\{Y_1 \leq z_1, ..., Y_n \leq z_n\}$ for any $(z_1, ..., z_n) \in \mathbb{R}^n$.

$(X_1, ..., X_n)$ is said to be smaller than $(Y_1, ..., Y_n)$ in the *usual stochastic order*, denoted as $(X_1, ..., X_n) \leq_{st} (Y_1, ..., Y_n)$, if $\mathbb{E}[u(X_1, ..., X_n)] \leq \mathbb{E}[u(Y_1, ..., Y_n)]$ for any increasing function $u : \mathbb{R}^n \to \mathbb{R}$ such that the expectations exist. □

Clearly, among the above three stochastic orders, $\leq_{st}$ is the strongest, and it implies the other two.

Shaked and Shanthikumar (2007) have proposed functional characterizations of upper orthant order and lower orthant order (Theorem 6.G.1), which is restated below.

Proposition 1.2.11 Let $\mathbf{X} = (X_1, ..., X_n)$ and $\mathbf{Y} = (Y_1, ..., Y_n)$ be two random vectors.

(i) $\mathbf{X} \leq_{uo} \mathbf{Y}$ if and only if $\mathbb{E}\left[\prod_{i=1}^n g_i(X_i)\right] \leq \mathbb{E}\left[\prod_{i=1}^n g_i(Y_i)\right]$ for all univariate nonnegative increasing functions $g_i, i = 1, ..., n$.

(ii) $\mathbf{X} \leq_{lo} \mathbf{Y}$ if and only if $\mathbb{E}\left[\prod_{i=1}^n h_i(X_i)\right] \geq \mathbb{E}\left[\prod_{i=1}^n h_i(Y_i)\right]$ for all univariate nonnegative decreasing functions $h_i, i = 1, ..., n$. □

From Proposition 1.2.11 we can build a relation between $\leq_{uo}$ and $\leq_{lo}$. Specifically, $(X_1, ..., X_n) \leq_{uo} (Y_1, ..., Y_n)$ if and only if $(-X_1, ..., -X_n) \geq_{lo} (-Y_1, ..., -Y_n)$.

It follows from Proposition 1.2.11 that the upper/lower orthant order is closed under certain increasing transformations. Specifically, $(X_1, ..., X_n) \leq_{uo} (\leq_{lo})(Y_1, ..., Y_n)$ implies $(g_1(X_1), ..., g_n(X_n)) \leq_{uo} (\leq_{lo})(g_1(Y_1), ..., g_n(Y_n))$ for all univariate increasing functions $g_1, ..., g_n$. Details can be seen in Theorem 6.G.3 in Shaked and Shanthikumar (2007).

Lemma 1.2.12 If $(X_1, ..., X_n) \leq_{uo} (Y_1, ..., Y_n)$, then

$$\sum_{k=1}^n X_i \leq_{mgf} \sum_{k=1}^n Y_i \qquad \text{and} \qquad \sum_{k=1}^n X_i \leq_{mom} \sum_{k=1}^n Y_i.$$

Proof. Since $\leq_{mom}$ implies $\leq_{mgf}$, we only need to show the second inequality.

For any given $m \in \mathbb{N}$, consider the following set

$$K = \left\{ (k_1, ..., k_n) \middle| \sum_{i=1}^{n} k_i = m, \text{ and } k_1, ..., k_n \in \mathbb{N} \right\}.$$

According to Proposition 1.2.11, we have $\mathbb{E}\left[\prod_{i=1}^{n} X_i^{k_i}\right] \leq \mathbb{E}\left[\prod_{i=1}^{n} Y_i^{k_i}\right]$, for any $k_1, ..., k_m \in \mathbb{N}$, then

$$\begin{aligned} \mathbb{E}\left[\left(\sum_{i=1}^{n} X_i\right)^m\right] &= \sum_{(k_1,...,k_n)\in K} \frac{m!}{\prod_{i=1}^{n} k_i!} \mathbb{E}\left[\prod_{i=1}^{n} X_i^{k_i}\right] \\ \leq \sum_{(k_1,...,k_n)\in K} \frac{m!}{\prod_{i=1}^{n} k_i!} \mathbb{E}\left[\prod_{i=1}^{n} Y_i^{k_i}\right] &= \mathbb{E}\left[\left(\sum_{i=1}^{n} Y_i\right)^m\right]. \end{aligned}$$

□

1.3 Risk Measures and Utility Functions

In order to compare different random variables, another approach is to use the concept of risk measure. A risk measure is a function defined on a set of random variables, and assigns a real number to a random variable in the set. A risk measure should satisfy certain conditions, and detailed discussions can be found in Delbaen (2000) and many others. A traditional way to define risk measure is through expectation. For a random variable X, its risk measure is defined to be $\rho(X) = \mathbb{E}[u(X)]$, where $u(x)$ is a function to be determined. An individual who is risk averse tends to take a risk with smaller risk measure. In insurance and finance, one important criterion is to minimize the risk measure of a risk.

A concept dual to the risk measure is the expected utility. The concept of utility function was initially proposed by Daniel Bernoulli in 1738 and then fully developed by von Neumann and Morgenstern (1947). A systematical discussion of the applications of utility functions in actuarial science can be found in Gerber and Pafumi (1998). Utility functions help make decisions on financial activities. For example, consider two risky assets W_1 and W_2, an investor with the utility function $u(x)$ will choose W_2 over W_1 if W_2 has a greater utility, that is, $\mathbb{E}[u(W_2)] \geq \mathbb{E}[u(W_1)]$.

There are many different choices of utility functions, and different utility functions reflect different attitudes toward the risky assets. A fundamental assumption of the utility function is the increasingness, in the sense that the utility is supposed to increase as the the wealth increases. For a risk-averse individual, an additional assumption of the utility function is the concavity. The increasingness and concavity of the utility function have the following interpretation: when the amount of the wealth is small, a small increase in the wealth results in a large increase in the utility. However, once this investor has accumulated a certain amount of wealth, there is little increase in the utility for additional dollars earned. In this case, the individual is risk-averse in the sense that she does not care much about the additional dollars earned. An example of risk-averse utility function is the exponential utility function $u(x) = \frac{1}{\gamma}(1 - e^{-\gamma x}), \gamma > 0$.

For notational convenience, we define

$$
\begin{aligned}
\mathcal{U}_{icx} &= \{u(x) : u(x) \text{ is increasing convex}\}, \\
\mathcal{U}_{icv} &= \{u(x) : u(x) \text{ is increasing concave}\}, \\
\mathcal{U}_{exp}^{+} &= \{u(x) : u(x) = e^{\gamma x}, \gamma > 0\}, \\
\mathcal{U}_{exp}^{-} &= \{u(x) : u(x) = 1 - e^{-\gamma x}, \gamma > 0\}.
\end{aligned}
$$

In the fields of insurance and finance, many optimization problems use the criterion of minimizing the risk measure or maximizing the expected utility. To some degree, these two criteria are equivalent. For example, let $\{W_\lambda, \lambda \in \Lambda\}$ be the total wealth under different strategies for some given index set Λ. Then the optimization problem aiming at maximizing the expected utility can be formulated as

$$\max_{\lambda \in \Lambda} \mathbb{E}[u(W_\lambda)], \qquad \forall\, u \in \mathcal{U}. \tag{1.3.1}$$

On the other hand, the negative of the wealth can be taken as the risk. Let $\mathcal{U}^* = \{-u(-x) : u(x) \in \mathcal{U}\}$, then the above optimization problem is equivalent to:

$$\min_{\lambda \in \Lambda} \mathbb{E}[u(-W_\lambda)], \qquad \forall\, u \in \mathcal{U}^*. \tag{1.3.2}$$

which can be interpreted as minimizing the risk measure of the risk.

Recall Section 1.1, the functional characterizations of most stochastic orders connect stochastic orders to optimization criteria: either minimizing risk measures or maximizing expected utilities. For example, in Problem (1.3.1) and Problem (1.3.2), we set $\mathcal{U} = \mathcal{U}_{icv}$, then $\mathcal{U}^* = \mathcal{U}_{icx}$. The two problems can be interpreted as minimizing the risk in the sense of increasing convex order.

We furthermore define the following functional classes:

$$\begin{aligned} \mathcal{U}_{st} &= \{u(x) : u(x) \text{ is increasing}\}, \\ \mathcal{U}_{cx} &= \{u(x) : u(x) \text{ is convex}\}, \\ \mathcal{U}_{mom} &= \{u(x) : u(x) = x^n, n = 1, 2, ...\}. \end{aligned}$$

Clearly,

$$\begin{aligned}
X \le_{st} Y &\iff \mathbb{E}[u(X)] \le \mathbb{E}[u(Y)], \qquad \forall\, u \in \mathcal{U}_{st}, \\
X \le_{icx} Y &\iff \mathbb{E}[u(X)] \le \mathbb{E}[u(Y)], \qquad \forall\, u \in \mathcal{U}_{icx}, \\
X \le_{mgf} Y &\iff \mathbb{E}[u(X)] \le \mathbb{E}[u(Y)], \qquad \forall\, u \in \mathcal{U}_{exp}^{+}, \\
X \le_{mom} Y &\iff \mathbb{E}[u(X)] \le \mathbb{E}[u(Y)], \qquad \forall\, u \in \mathcal{U}_{mom}.
\end{aligned}$$

Throughout this thesis, we focus on the traditional risk measures, defined by the form $\rho(X) = \mathbb{E}[u(X)]$. Recently, there are more risk measures introduced to the field of insurance, such as CTE and other law invariant coherent risk measures. Recent studies on the applications of CTE measure in actuarial science can be seen in Cai et al. (2008), Tan et al. (2011) and many others. Kusuoka (2001) and Bäuerle and Müller (2006) have shown that the law-invariant coherent risk measure has a relation to convex order, which allows potentials of extending our work to the general risk measure.

1.4 Copulas and Survival Copulas

Copula is a common tool to model dependence structure. A copula, denoted as $C(u_1, ..., u_n)$, is the joint distribution function of a random vector $(U_1, ..., U_n)$ with $U_i, i = 1, ..., n$, all uniformly distributed on $[0, 1]$. The random vector $(U_1, ..., U_n)$ is called the generator of the copula C. We say that random vector $(X_1, ..., X_n)$ is linked by (or has) the copula $C(u_1, ..., u_n)$, if

$$F(x_1, ..., x_n) = C(F_1(x_1), ..., F_n(x_n)),$$

where $F(x_1, ..., x_n)$ is the joint distribution function of $(X_1, ..., X_n)$ and $F_i(x_i)$ is the marginal distribution of $X_i, i = 1, ..., n$.

From Sklar's theorem, we know that for any given random vector $(X_1, ..., X_n)$, its copula always exists. It is worthwhile to point out, the uniqueness of copula only holds when the random vector has continuous marginal distributions. Assume that $(X_1, ..., X_n)$ has continuous marginal distributions. Then $F_i(X_i) \sim UNIF([0,1]), i = 1, ..., n$, and $(F_1(X_1), ..., F_n(X_n))$ is the generator of the unique copula of $(X_1, ..., X_n)$.

Two commonly used dependence structures are independence and comonotonicity. They correspond to the following copula:

$$C^{ind}(u_1, ..., u_n) = u_1 u_2 ... u_n \qquad \text{and} \qquad C^c(u_1, ..., u_n) = \min\{u_1, ..., u_n\}.$$

It is known that, with certain assumptions, copulas have the invariant property. Specifically, if $(X_1, ..., X_n)$ has the copula C, then $(f_1(X_1), ..., f_n(X_n))$ also has the copula C for any strictly increasing functions $f_i(x), i = 1, ..., n$. If $(X_1, ..., X_n)$ has the copula C and has continuous marginal distribution functions, then $(f_1(X_1), ..., f_n(X_n))$ also has the copula C for any increasing continuous functions $f_i(x), i = 1, ..., n$. See, for example, Proposition 5.6. of McNeil et al. (2005), Proposition 4.7.4 of Denuit et al. (2006), Theorem 3.4.3 of Nelsen (1999), and Theorem 2.8 of Cherubini et al. (2004). We generalize these invariant properties in Chapter 2.

A concept parallel to the copula is the survival copula. Let $\widehat{C}(u_1, ..., u_n)$ be a copula. We say that random vector $(X_1, ..., X_n)$ has the survival copula $\widehat{C}$, if

$$\bar{F}(x_1, ..., x_n) = \mathbb{P}\{X_1 > x_1, ..., X_n > x_n\} = \widehat{C}(\bar{F}_1(x_1), ..., \bar{F}_n(x_n)),$$

where $\bar{F}_i(x) = 1 - F_i(x)$ is the survival function of X_i, $i = 1, ..., n$.

A different concept to be distinguished with the survival copula is the survival function of a copula. The survival function of copula C, denoted by $\bar{C}$, is the survival function of the generator of C. If $(X_1, ..., X_n)$ has continuous marginal distribution functions, the unique copula of $(X_1, ..., X_n)$ is generated by $(U_1, ..., U_n) = (F_1(X_1), ..., F_n(X_n))$. In that case, it is easy to verify that the copula $\widehat{C}$ generated by $(1 - U_1, ..., 1 - U_n) = (\bar{F}_1(X_1), ..., \bar{F}_n(X_n))$ is the survival copula of $(X_1, ..., X_n)$. With continuous marginal distributions, it is easy to verify the following relation between the survival copula $\widehat{C}$ and the survival function of copula $\bar{C}$ (see McNeil et al. (2005)).

$$\bar{C}(u_1, ..., u_n) = \widehat{C}(1 - u_1, ..., 1 - u_n). \tag{1.4.1}$$

We also have the following relation between the copula and the survival copula:

Proposition 1.4.1 Assume that $C(u_1, ..., u_n)$ is a copula of the random vector $(X_1, ..., X_n)$. Then $C(u_1, ..., u_n)$ is a survival copula of $(-X_1, ..., -X_n)$.

Proof. We need to verify that, for any $x_1, ..., x_n$,

$$\begin{aligned} & \mathbb{P}\{-X_1 > -x_1, ..., -X_n > -x_n\} = C(\mathbb{P}\{-X_1 > -x_1\}, ..., \mathbb{P}\{-X_n > -x_n\}) \\ \iff & \mathbb{P}\{X_1 < x_1, ..., X_n < x_n\} = C(\mathbb{P}\{X_1 < x_1\}, ..., \mathbb{P}\{X_n < x_n\}). \end{aligned}$$

Noting that C is a copula of $(X_1, ..., X_n)$, we have

$$\mathbb{P}\{X_1 \le x_1 - \delta, ..., X_n \le x_n - \delta\} = C(\mathbb{P}\{X_1 \le x_1 - \delta\}, ..., \mathbb{P}\{X_n \le x_n - \delta\}),$$

for any δ. Since $C(u_1, ..., u_n)$ is continuous in each $u_k, k = 1, ..., n$, then

$$
\begin{aligned}
& \mathbb{P}\{X_1 < x_1, ..., X_n < x_n\} \\
= \ & \lim_{\delta \downarrow 0} \mathbb{P}\{X_1 \leq x_1 - \delta, ..., X_n \leq x_n - \delta\} \\
= \ & \lim_{\delta \downarrow 0} C(\mathbb{P}\{X_1 \leq x_1 - \delta\}, ..., \mathbb{P}\{X_n \leq x_n - \delta\}) \\
= \ & C(\lim_{\delta \downarrow 0} \mathbb{P}\{X_1 \leq x_1 - \delta\}, ..., \lim_{\delta \downarrow 0} \mathbb{P}\{X_n \leq x_n - \delta\}) \\
= \ & C(\mathbb{P}\{X_1 < x_1\}, ..., \mathbb{P}\{X_n < x_n\}). \qquad \square
\end{aligned}
$$

We are particularly interested in a special class of copulas: Archimedean copulas. Let $\Psi : (0, 1] \to [0, \infty)$ be invertible and satisfy: (i) $\Psi(1) = 0$, $\lim_{x \downarrow 0} \Psi(x) = \infty$, and (ii) $\Lambda(x)$ is completely monotonic, specifically, $(-1)^k \Lambda^{(k)}(x) = (-1)^k \frac{d^k}{dx^k} \Lambda(x) \geq 0$ for all $k = 0, 1, ...$, where $\Lambda(x) = \Psi^{-1}(x)$. Define

$$
C(u_1, ..., u_n) = \Lambda \left(\sum_{k=1}^{n} \Psi(u_k) \right), \qquad u_1, ..., u_n \in [0, 1]. \tag{1.4.2}
$$

Then $C(u_1, ..., u_n)$ is a copula, which is called Archimedean copula.

The complete monotonicity implies an important properties of the generator of an Archimedean copula, which will be used in later studies. We state the property below.

Remark 1.4.2 If $\Lambda(x)$ is completely monotonic, then $(-1)^k \Lambda^{(k)}(x)$ is decreasing for all $k = 1, 2, ...$

Chapter 2

Notions of Positive Dependence and Their Relations with Copulas

2.1 Introduction

Notions of positive dependence and copulas play important roles in modeling dependent risks. The invariant properties of notions of positive dependence and copulas under (strictly) increasing transformations are often used in the studies of economics, finance, insurance and many other fields. In the literature, some of these invariant properties have been proved, while some were stated without proofs and are assumed to hold.

In this chapter, we examine the notions of the conditionally increasing (CI), the conditionally increasing in sequence (CIS), the positive dependence through the stochastic ordering (PDS), and the positive dependence through the upper orthant ordering (PDUO). The definitions of these notions will be stated later.

We first use two counterexamples to show that the statements in Theorem 3.10.19 of

Müeller and Stoyan (2002) about the invariant properties of CI and CIS under increasing transformations are not true. The counterexamples motivate us to verify the statements of Theorem 3.10.19 of Müeller and Stoyan (2002) about the invariant properties of other notions of positive dependence under increasing transformations. Actually, it is easy to prove that most of the notions of positive dependence mentioned in Theorem 3.10.19 of Müeller and Stoyan (2002) are preserved under increasing transformations. However, it is not easy to verify if those notions defined through conditional expectations or conditional survival functions, such as CI, CIS, and PDS, are preserved under increasing transformations. It is straightforward to show that PDS is preserved under strictly increasing transformations. Indeed, Theorem 2.1 of Block et al. (1985) states that the negative dependence through the stochastic ordering (NDS), which is the counterpart of PDS, is preserved under strictly increasing transformations. More discussion about the notion of positive dependence and negative dependence can be found in Lehmann (1966), Block and Ting (1981) and Block et al. (1982). To the best of our knowledge, the proof for the invariant property of PDS is not available. Indeed, the proof is not trivial and needs further investigation on the concept of the conditional expectation.

Under certain conditions, some notions of positive dependence of a random vector are the properties of its copula in the sense that a random vector has a notion of positive dependence if and only if its copula has the same notion. In addition, some notions of positive dependence of a random vector can be characterized by its copula. Actually, we will show that a random vector $(X_1, ..., X_n)$ is PDS (PUDO) if and only if $(F_1(X_1), ..., F_n(X_n))$ is PDS (PDUO). Consequently, if $(X_1, ..., X_n)$ has the continuous marginal distribution functions, then $(X_1, ..., X_n)$ is PDS (PDUO) if and only if its copula is PDS (PDUO).

A very useful property of a copula is the invariance under strictly increasing transformations on the components of a continuous random vector or under increasing and continuous transformations on the components of any random vector. See, for example, Proposition 5.6. of McNeil et al. (2005), Proposition 4.7.4 of Denuit et al. (2006), Theorem 3.4.3 of Nelsen (1999), and Theorem 2.8 of Cherubini et al. (2004). Using the properties of generalized left-continuous and right-continuous inverse functions, we give rigorous proofs for the invariant properties of copulas under increasing transformations on the components of any random vector.

The invariant properties of notions of positive dependence and copulas under increasing transformations are often used in the studies of economics, finance, insurance and many other fields. It is necessary for one to give a detailed study of these invariant properties. In this chapter, '$=_{st}$' means 'equal in distribution'.

The rest of the chapter is organized as follows. In Section 2.2, we revisit several notions of positive dependence including the stochastically increasing (SI), CI, CIS, PDS, and PDUO. We use two counterexamples to show that the statements in Theorem 3.10.19 of Müeller and Stoyan (2002) about the invariant properties of CI and CIS under increasing transformations are not true. We prove that CIS and CI are preserved under strictly increasing transformations. We give rigorous proofs for the invariant properties of SI, PDS, and PDUO under strictly increasing transformations. These invariant properties enable us to show that a continuous random vector is PDS (PDUO) if and only if its copula is PDS (PDUO). In Section 2.3, using the properties of the generalized inverse functions, we also give a rigorous proof for the invariant property of copulas under increasing transformations on any random vector. This result generalizes Proposition 5.6. of McNeil et al. (2005) and Proposition 4.7.4 of Denuit et al. (2006). In Section 2.4, we give the characterization of PDUO in terms of survival copulas.

2.2 The Invariant Properties of the Notions of Positive Dependence

In the literature, there are several notions of positive dependence, which describe positive dependence for two random variables or two random vectors. We refer to Müeller and Stoyan (2002) and Shaked and Shanthikumar (2007) for a detailed treatment of these topics and to Denuit et al. (2006) for their applications in actuarial science and insurance. In this section, we focus on the notions of SI, CI, CIS, PDS, PDUO. The notions of SI, CI, CIS, and PDS and their properties can be found in Block et al. (1985), Joe (1997), Shaked (1977), and references therein. The PDUO will be defined in this section. More notions of positive dependence can be found in Colangelo et al. (2005) and references therein. A new characterization of CIS is given in Fernández-Ponce et al. (2011).

We recall that for a random vector $\mathbf{Y} = (Y_1, ..., Y_n)$, a support of $\mathbf{Y}$, denoted by $S(\mathbf{Y})$ or $S(Y_1, ..., Y_n)$, is a Borel set of $\mathbb{R}^n$ such that $\mathbb{P}\{\mathbf{Y} \in S(\mathbf{Y})\} = 1$.

Definition 2.2.1 Let $(X_1, ..., X_n)$ be a random vector and Y be a random variable.

(1) Y is said to be stochastically increasing (SI) in random vector $(X_1, ..., X_n)$, denoted as $Y \uparrow_{SI} (X_1, ..., X_n)$, if $\mathbb{P}\{Y > y \,|\, X_1 = x_1, ..., X_n = x_n\}$ is increasing in $(x_1, ..., x_n) \in S(X_1, ..., X_n)$ for all $y \in \mathbb{R}$, or equivalently, $Y \uparrow_{SI} (X_1, ..., X_n)$ if and only if $\mathbb{E}[u(Y) \,|\, X_1 = x_1, ..., X_n = x_n]$ is increasing in $(x_1, ..., x_n) \in S(X_1, ..., X_n)$ for any increasing function u such that the conditional expectation exists.

(2) $(X_1, ..., X_n)$ is said to be stochastically increasing (SI) in random variable Y, denoted as $(X_1, ..., X_n) \uparrow_{SI} Y$, if $\mathbb{E}\left[u(X_1, ..., X_n) \,|\, Y = y\right]$ is increasing in $y \in S(Y)$ for any increasing function $u : \mathbb{R}^n \to \mathbb{R}$ such that the conditional expectation exists.

(3) $(X_1, ..., X_n)$ is said to be conditionally increasing in sequence (CIS) if $X_i \uparrow_{SI} (X_1, ..., X_{i-1})$ for all $i = 2, ..., n$.

(4) $(X_1, ..., X_n)$ is said to be positively dependent through the stochastic ordering (PDS) if $(X_1, ..., X_{i-1}, X_{i+1}, ..., X_n) \uparrow_{SI} X_i$ for all $i = 1, ..., n$.

(5) $(X_1, ..., X_n)$ is said to be conditionally increasing (CI) if $(X_{\pi(1)}, ..., X_{\pi(n)})$ is CIS for all permutations π of $(1, ..., n)$. □

The following is some preliminary properties of stochastic increasing.

Proposition 2.2.2 Let $(X_1, ..., X_n)$ be a random vector and Y be a random variable. If $(X_1, \cdots, X_n) \uparrow_{SI} Y$, then

(i) $(Y, X_1, \cdots, X_n) \uparrow_{SI} Y$.

(ii) $u(X_1, \cdots, X_n) \uparrow_{SI} Y$ for any increasing function $u : \mathbb{R}^n \to \mathbb{R}^k$ where $k \in \mathbb{N}$.

Proof. Let $u : \mathbb{R}^{n+1} \to \mathbb{R}$ be an increasing function. For any $y_1, y_2 \in S(Y)$ such that $y_1 \leq y_2$, we have

$$\begin{aligned} & \mathbb{E}\left[u(Y, X_1, ..., X_n)|Y = y_1\right] = \mathbb{E}\left[u(y_1, X_1, ..., X_n)|Y = y_1\right] \\ \leq \; & \mathbb{E}\left[u(y_2, X_1, ..., X_n)|Y = y_1\right] \leq \mathbb{E}\left[u(y_2, X_1, ..., X_n)|Y = y_2\right] \\ = \; & \mathbb{E}\left[u(Y, X_1, ..., X_n)|Y = y_2\right], \end{aligned}$$

which implies that $(Y, X_1, \cdots, X_n) \uparrow_{SI} Y$ according to Definition 2.2.1(2).

For any increasing function $\phi : \mathbb{R}^k \to \mathbb{R}$, $\phi \circ u : \mathbb{R}^n \to \mathbb{R}$ is also increasing. Then $\mathbb{E}\left[\phi(u(X_1, ..., X_n))|Y = y\right]$ is increasing in y since $(X_1, \cdots, X_n) \uparrow_{SI} Y$. Therefore, we have $u(X_1, \cdots, X_n) \uparrow_{SI} Y$ according to the definition of SI. □

The natural extensions of $(X_1, ..., X_n) \uparrow_{SI} Y$ and PDS are to define a notion of positive dependence by using the weaker condition of the conditional survival function $\mathbb{P}\{X_1 > x_1, ..., X_n > x_n | Y = y\}$ instead of using the stronger condition of the condition expectation $\mathbb{E}[u(X_1, ..., X_n)|Y = y]$. Thus, we can define two notions of positive dependence which are weaker than the dependence of SI and PDS.

Definition 2.2.3 Let $\mathbf{X} = (X_1, ..., X_n)$ be a random vector and Y be a random variable.

(1) $(X_1, ..., X_n)$ is said to be weakly stochastically increasing (WSI) in Y, denoted as $(X_1, ..., X_n) \uparrow_{WSI} Y$, if $\mathbb{P}\{X_1 > x_1, ..., X_n > x_n \,|\, Y = y\}$ is increasing in $y \in S(Y)$ for any $(x_1, ..., x_n) \in \mathbb{R}^n$.

(2) $(X_1, ..., X_n)$ is said to be positively dependent through the upper orthant ordering (PDUO) if $(X_1, ..., X_{i-1}, X_{i+1}, ..., X_n) \uparrow_{WSI} X_i$ for all $i = 1, 2, ..., n$. □

Note that $\mathbf{X} \uparrow_{WSI} Y$ actually means $\mathbf{X} \,|\, Y = y_1 \leq_{uo} \mathbf{X} \,|\, Y = y_2$ for any $y_1, y_2 \in S(Y)$ with $y_1 < y_2$. Combining this observation and Proposition 1.2.11, we immediately get the following property about weakly stochastically increasing.

Proposition 2.2.4 If $(X_1, ..., X_n) \uparrow_{WSI} Y$, then for any nonnegative increasing functions $f_i, i = 1, ..., n$, $\mathbb{E}\left[\prod_{i=1}^n f_i(X_i)|Y = y\right]$ is increasing in $y \in S(Y)$.

It is clear that for two random variables X and Y, $X \uparrow_{WSI} Y$ is equivalent to $X \uparrow_{SI} Y$, and for a bivariate random vector (X_1, X_2), (X_1, X_2) is PDUO if and only if (X_1, X_2) is PDS. In general, we have SI $\Longrightarrow$ WSI and PDS $\Longrightarrow$ PDUO. In addition, we will see that PDUO can be characterized by the survival copulas for continuous random vectors. Hence, it is easy to construct a continuous PDUO random vector by copulas.

From the definitions, we know that CI, CIS, PDS, and PDUO describe the notions of positive dependence for a random vector and are defined by using conditional expectations or conditional survival functions. We summarize their implications as follows:

$$\text{CI} \Longrightarrow \text{CIS}$$

and

$$\text{CI} \Longrightarrow \text{PDS} \Longrightarrow \text{PDUO}.$$

Theorem 3.10.19 of Müeller and Stoyan (2002) states (without proofs) that several common notions of positive dependence including CIS, CI and PDS are preserved under increasing transformations. We point out the "increasing" in their result should be understood as "strictly increasing". We shall first construct two counterexamples to demonstrate that invariant property of CIS/CI does not hold for nondecreasing transformations:

Example 2.2.5 (CIS is not preserved under general increasing transformations) Let X and Y be two independent random variables. Then it always holds that $X + Y \uparrow_{SI} X$. Now, assume that X and Y have the following probability mass functions: $\mathbb{P}(X = 0) = \mathbb{P}(X = 1) = 0.5$, $\mathbb{P}(Y = 0) = 0.4$, $\mathbb{P}(Y = 1) = 0.2$, and $\mathbb{P}(Y = 2) = 0.4$. Then it is easy to check that

$$\mathbb{P}(X > 0 \,|\, X + Y = 1) = \mathbb{P}(X = 1 \,|\, X + Y = 1) = 2/3,$$
$$\mathbb{P}(X > 0 \,|\, X + Y = 2) = \mathbb{P}(X = 1 \,|\, X + Y = 2) = 1/3.$$

Then $\mathbb{E}[X \,|\, X + Y = 1] = 2/3 > \mathbb{E}[X \,|\, X + Y = 2] = 1/3$, which means that $\mathbb{E}[X \,|\, X + Y]$

is not increasing in $X+Y$.

Let $X_1 = X$, $X_2 = X+Y$, and $X_3 = X$. Then (X_1, X_2, X_3) is CIS since $X_2 = X+Y \uparrow_{SI} X_1 = X$ and $\mathbb{E}[u(X_3) \,|\, X_1 = x_1, X_2 = x_2] = \mathbb{E}[u(X) \,|\, X = x_1, X+Y = x_2] = u(x_1)$ is increasing in x_1 and x_2 for any increasing function u. Now, consider the increasing transformations of $f_1(x) = 1$ and $f_2(x) = f_3(x) = x$, then $(f_1(X_1),\, f_2(X_2),\, f_3(X_3)) = (1,\, X+Y,\, X)$ is not CIS since $\mathbb{E}[f_3(X_3) \,|\, f_1(X_1),\, f_2(X_2)] = \mathbb{E}[X \,|\, 1,\, X+Y] = \mathbb{E}[X \,|\, X+Y]$ is not increasing in $X+Y$. Therefore, CIS is not preserved under increasing transformations. □

Example 2.2.6 (CI is not preserved under general increasing transformations) Assume that the conditional distribution of X, conditioning on Y, is given by the left table below. For instance, from the table, $\mathbb{P}\{X = 1 \,|\, Y = 2\} = 0.2$. Assume that the marginal distribution of Y is $\mathbb{P}\{Y = i\} = 1/3, i = 0, 1, 2$. Thus, the conditional distribution of Y, conditioning on X, is given by the right table below. For example, from the table, $\mathbb{P}\{Y = 2 \,|\, X = 1\} = 2/7$.

$X \,|\, Y$	0	1	2
0	0.4	0.2	0.4
1	0.2	0.3	0.5
2	0.2	0.2	0.6

$Y \,|\, X$	0	1	2
0	1/2	1/4	1/4
1	2/7	3/7	2/7
2	4/15	1/3	2/5

It is easy to verify that $X \uparrow_{SI} Y$ and $Y \uparrow_{SI} X$. Consider the random vector $\mathbf{V} = (V_1,\, V_2,\, V_3) = (X,\, Y,\, X)$. Obviously, $\mathbf{V} = (V_1,\, V_2,\, V_3)$ is CI. Consider the increasing transformations of $f_1(x) = (x-1)_+,\ f_2(x) = f_3(x) = x$. Now we examine the CI property of the random vector $(f_1(V_1),\, f_2(V_2),\, f_3(V_3)) = ((X-1)_+,\, Y,\, X)$.

Note that

$$\mathbb{E}\,[X\,|\,((X-1)_+,\,Y)=(0,1)]=\mathbb{E}\,[X\,|\,0\le X\le 1,\,Y=1]=0.6,$$
$$\mathbb{E}\,[X\,|\,((X-1)_+,\,Y)=(0,2)]=\mathbb{E}\,[X\,|\,0\le X\le 1,\,Y=2]=0.5<0.6,$$

which means that $\mathbb{E}\,[X|((X-1)_+,\,Y)=(x,\,y)]$ is not increasing in y. Thus X is not stochastically increasing in $((X-1)_+,\,Y)$. Therefore, $(f_1(V_1),\,f_2(V_2),\,f_3(V_3))=((X-1)_+,\,Y,\,X)$ is not CI. □

Motivated by the counterexamples, we feel it is necessary to reexamine the invariant property of CIS/CI. In the following, we shall rigorously prove that CIS/CI is invariant under strictly increasing transformations.

Definition 2.2.7 For an increasing function $g:\mathbb{R}\to\mathbb{R}$, we denote the generalized left-continuous inverse function of g by $g^{-1}:\mathbb{R}\to[-\infty,\infty]$ and the generalized right-continuous inverse function of g by $g^{-1+}:\mathbb{R}\to[-\infty,\infty]$, which are defined as $g^{-1}(y)=\inf\{x\,|\,g(x)\ge y\}$ and $g^{-1+}(y)=\sup\{x\,|\,g(x)\le y\}$ with the convention $\inf\{\emptyset\}=\infty$ and $\sup\{\emptyset\}=-\infty$. □

Proposition 2.2.8 Assume that random vector $(X_1,...,X_n)$ is CIS. Then for any strictly increasing functions f_i, $i=1,...,n$, random vector $(f_1(X_1),...,f_n(X_n))$ is also CIS.

Proof. For any $k\in\{2,3,...,n\}$, we have $\sigma(f_1(X_1),...,f_{k-1}(X_{k-1}))\subset\sigma(X_1,...,X_{k-1})$. Thus,

$$\begin{aligned}
&\mathbb{E}[f_k(X_k)\,|\,f_1(X_1),...,f_{k-1}(X_{k-1})]\\
=\;&\mathbb{E}[\mathbb{E}[f_k(X_k)\,|\,X_1,...,X_{k-1}]\,|\,f_1(X_1),...,f_{k-1}(X_{k-1})]\\
=\;&\mathbb{E}[h_k(X_1,...,X_{k-1})\,|\,f_1(X_1),...,f_{k-1}(X_{k-1})], \qquad (2.2.1)
\end{aligned}$$

where $h_k(x_1, ..., x_{k-1}) = \mathbb{E}\left[f_k(X_k) \,|\, X_1 = x_1, ..., X_{k-1} = x_{k-1}\right]$. Since $X_k \uparrow_{SI} (X_1, ..., X_{k-1})$, by the definition of $\uparrow_{SI}$, we know that h_k is increasing in each argument. Recalling that f_i is strictly increasing, we have f_i^{-1} is increasing and $f_i^{-1}(f_i(x)) = x$. Therefore, by (2.2.1), we have

$$
\begin{aligned}
& \mathbb{E}[f_k(X_k) \,|\, f_1(X_1), ..., f_{k-1}(X_{k-1})] \\
= \; & \mathbb{E}[h_k(f_1^{-1}(f_1(X_1)), ..., f_{k-1}^{-1}(f_{k-1}(X_{k-1}))) \,|\, f_1(X_1), ..., f_{k-1}(X_{k-1})] \qquad (2.2.2) \\
= \; & h_k(f_1^{-1}(f_1(X_1)), ..., f_{k-1}^{-1}(f_{k-1}(X_{k-1}))) = g(f_1(X_1), ..., f_{k-1}(X_{k-1})),
\end{aligned}
$$

where $g(x_1, ..., x_{k-1}) = h_k(f_1^{-1}(x_1), ..., f_{k-1}^{-1}(x_{k-1}))$ is increasing in each argument, which means $f_k(X_k) \uparrow_{SI} (f_1(X_1), ..., f_{k-1}(X_{k-1}))$. □

Proposition 2.2.8 means that CIS property is invariant under strictly increasing transformations. Following this result, we immediately get the invariant property of CI.

Corollary 2.2.9 Assume that random vector $(X_1, ..., X_n)$ is CI. Then for any strictly increasing functions f_i, $i = 1, ..., n$, random vector $(f_1(X_1), ..., f_n(X_n))$ is also CI.

In the following, we shall show that the notions of PDS and PDUO are invariant under general increasing transformations. In doing so, we need some properties of increasing functions and conditional expectations.

For a set $A \subseteq \mathbb{R}$, we denote the inverse image of the set A under function $g : \mathbb{R} \to \mathbb{R}$ by $g^{-1}(A) = \{x \in \mathbb{R} \,|\, g(x) \in A\}$. Thus, for any $y \in \mathbb{R}$, $g^{-1}(\{y\}) = \{x \in \mathbb{R} \,|\, g(x) = y\}$.

For an increasing function g, we define the following three sets:

$$\begin{aligned}
F_0 &= \{y \in \mathbb{R} \,|\, g^{-1}(\{y\}) = \emptyset\} \\
&= \{y \in \mathbb{R} \,|\, \text{there does not exist a point } x \in \mathbb{R} \text{ such that } g(x) = y\}, \\
F_1 &= \{y \in \mathbb{R} \,|\, g^{-1}(\{y\}) \text{ contains exactly one element}\} \\
&= \{y \in \mathbb{R} \,|\, \text{there exists exactly one point } x \in \mathbb{R} \text{ such that } g(x) = y\}, \\
F_2 &= \{y \in \mathbb{R} \,|\, g^{-1}(\{y\}) \text{ contains more than one element}\} \\
&= \{y \in \mathbb{R} \,|\, \text{there exist more than one point } x \in \mathbb{R} \text{ such that } g(x) = y\}.
\end{aligned}$$

Moreover, for an increasing function g, we recall that g has at most countably many points of discontinuities and that if g is discontinuous at x, then the left and right limits of g at x exist with $g(x-) < g(x+)$. Furthermore, since g is increasing, the sets F_0, F_1, F_2 are mutually disjoint and $F_0 \cup F_1 \cup F_2 = \mathbb{R}$. Note that if $g(x) \in F_1$, then $g^{-1}(g(x)) = x$.

Lemma 2.2.10 If g is an increasing function, then the set F_2 is countable.

Proof. For any $y \in F_2$, there exist two points $x_1(y) < x_2(y)$ in $\mathbb{R}$ such that $x_1(y), x_2(y) \in g^{-1}(\{y\})$, and then $g(x_1(y)) = g(x_2(y)) = y$. Thus $g(x) = y$ for any $x \in (x_1(y), x_2(y))$ since g is increasing. Note that for any $y_1 \neq y_2 \in F_2$, the open intervals $(x_1(y_1), x_2(y_1))$ and $(x_1(y_2), x_2(y_2))$ are disjoint. Therefore, there is a one-to-one mapping between F_2 and the set of the mutually disjoint open intervals of $\mathbb{R}$ and thus F_2 is countable. □

For increasing function g and random variables X and $Y = g(X)$, denote $F_3 = \{y \in F_2 \,|\, \mathbb{P}\{Y = y\} > 0\}$ and $F_4 = \{y \in F_2 \,|\, \mathbb{P}\{Y = y\} = 0\}$. Then F_3 and F_4 are disjoint and $F_3 \cup F_4 = F_2$. From Lemma 2.2.10, we know that F_4 is countable. Thus, $\mathbb{P}\{Y \in F_4\} = 0$.

Note that $\mathbb{P}\{Y \in F_0\} = \mathbb{P}\{X \in \emptyset\} = 0$ and $F_0 \cup F_1 \cup F_3 \cup F_4 = \mathbb{R}$. Hence, $\mathbb{P}\{Y \in F_1 \cup F_3\} = 1 - \mathbb{P}\{Y \in F_4\} - \mathbb{P}\{Y \in F_0\} = 1$.

Furthermore, for any function u such that $\mathbb{E}\big[|u(X)|\big] < \infty$ and for $y \in F_3$, we define

$$q_u(y) = \frac{\mathbb{E}[u(X)\,\mathbb{I}\{Y = y\}]}{\mathbb{P}\{Y = y\}}.$$

Proposition 2.2.11 Let $\mathbb{E}\big[|u(X)|\big] < \infty$ and g be an increasing function. Then

$$\mathbb{E}[u(X)\,|\,g(X)] = u(X)\,\mathbb{I}\{g(X) \in F_1\} + q_u(g(X))\,\mathbb{I}\{g(X) \in F_3\}. \tag{2.2.3}$$

Proof. Let X be defined on the probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. Note that if $g(X) \in F_1$, then $g^{-1}(g(X)) = X$. Denote $Y = g(X)$ and

$$m_u(Y) = u(g^{-1}(Y))\,\mathbb{I}\{Y \in F_1\} + q_u(Y)\,\mathbb{I}\{Y \in F_3\}. \tag{2.2.4}$$

Thus, according to the definition of the conditional expectation, to prove the expression (2.2.3), it is sufficient to show $\mathbb{E}[u(X)\,\mathbb{I}_A] = \mathbb{E}[m_u(Y)\,\mathbb{I}_A]$ for all $A \in \sigma(Y)$. For any $A \in \sigma(Y)$, there exists $B \in \mathcal{B}(\mathbb{R})$ such that $A = \{Y \in B\} = \{\omega \in \Omega \,|\, Y(\omega) \in B\}$. Recalling that $g^{-1}(Y) = g^{-1}(g(X)) = X$ if $Y = g(X) \in F_1$ and $\mathbb{P}\{Y \in F_1 \cup F_3\} = 1$, we have

$$\begin{aligned}
\mathbb{E}[u(X)\,\mathbb{I}_A] &= \mathbb{E}[u(X)\,\mathbb{I}\{Y \in B\}] = \mathbb{E}[u(X)\,\mathbb{I}\{Y \in B \cap (F_1 \cup F_3)\}] \\
&= \mathbb{E}[u(X)\,\mathbb{I}\{Y \in B \cap F_1\}] + \mathbb{E}[u(X)\,\mathbb{I}\{Y \in B \cap F_3\}] \\
&= \mathbb{E}[u \circ g^{-1}(Y)\,\mathbb{I}\{Y \in B \cap F_1\}] + \mathbb{E}\left[\sum_{y_0 \in B \cap F_3} u(X)\,\mathbb{I}\{Y = y_0\}\right] \\
&= \mathbb{E}[u \circ g^{-1}(Y)\,\mathbb{I}\{Y \in B \cap F_1\}] + \sum_{y_0 \in B \cap F_3} \mathbb{E}[u(X)\,\mathbb{I}\{Y = y_0\}] \qquad (2.2.5) \\
&= \mathbb{E}[u \circ g^{-1}(Y)\,\mathbb{I}\{Y \in B \cap F_1\}] + \sum_{y_0 \in B \cap F_3} q_u(y_0)\,\mathbb{P}\{Y = y_0\} \qquad (2.2.6) \\
&= \mathbb{E}[u \circ g^{-1}(Y)\,\mathbb{I}\{Y \in B \cap F_1\}] + \mathbb{E}[q_u(Y)\,\mathbb{I}\{Y \in B \cap F_3\}] \\
&= \mathbb{E}[m_u(Y)\,\mathbb{I}\{Y \in B\}] = \mathbb{E}[m_u(Y)\,\mathbb{I}_A],
\end{aligned}$$

where (2.2.5) holds by the Lebesgue convergence theorem and (2.2.6) holds by the definition of $q_u(y)$. □

Corollary 2.2.12 For increasing function g and random variable X, it holds that $X \uparrow_{SI} g(X)$.

Proof. By Proposition 2.2.11, we have $\mathbb{E}[u(X) \,|\, g(X)] = m_u(Y)$, where $m_u(Y)$ is defined by (2.2.4). In order to prove $X \uparrow_{SI} g(X)$, it is sufficient to show that, for any increasing function u, $m_u(y) = u \circ g^{-1}(y)\,\mathbb{I}\{y \in F_1\} + q_u(y)\,\mathbb{I}\{y \in F_3\}$ is increasing in $y \in F_1 \cup F_3$, which is a support of Y since $\mathbb{P}(Y \in F_1 \cup F_3) = 1$.

For any set $A \subseteq \mathbb{R}$ and the function u, we denote $u(A) = \{u(x) \,|\, x \in A\}$, $\sup\{A\} = \sup\{x \,|\, x \in A\}$ and $\inf\{A\} = \inf\{x \,|\, x \in A\}$. Let $B(y) = g^{-1}(\{y\})$, then $B(y) \neq \emptyset$ for any

$y \in F_1 \cup F_3$. For $y \in F_3$, we have

$$\begin{aligned} q_u(y) &= \frac{\mathbb{E}[u(X)\,\mathbb{I}\{Y = y\}]}{\mathbb{P}\{Y = y\}} = \frac{\mathbb{E}[u(X)\,\mathbb{I}\{X \in B(y)\}]}{\mathbb{P}\{Y = y\}} \\ &\leq \frac{\mathbb{E}\left[\sup\{u(B(y))\}\,\mathbb{I}\{X \in B(y)\}\right]}{\mathbb{P}\{Y = y\}} = \sup\{u(B(y))\}. \end{aligned}$$

Similarly, $q_u(y) \geq \inf\{u(B(y))\}$. If $y \in F_1$, then $\inf\{u(B(y))\} = u(g^{-1}(y)) = \sup\{u(B(y))\}$ since $B(y) = \{g^{-1}(y)\}$ is a single point set in this case. Since for any fixed $y \in F_1 \cup F_3$, $m_u(y)$ is of the form either $u(g^{-1}(y))$ or $q_u(y)$, we have $\inf\{u(B(y))\} \leq m_u(y) \leq \sup\{u(B(y))\}$.

Consider $y_1 < y_2 \in F_1 \cup F_3$. For any $x_1 \in B(y_1), x_2 \in B(y_2)$, we have $g(x_1) = y_1 < y_2 = g(x_2)$, and then $x_1 < x_2$ since g is increasing. Thus, $u(x_1) \leq u(x_2)$ and then $\sup\{u(B(y_1))\} \leq \inf\{u(B(y_2))\}$. Therefore $m_u(y_1) \leq \sup\{u(B(y_1))\} \leq \inf\{u(B(y_2))\} \leq m_u(y_2)$. □

Proposition 2.2.13 Let $\mathbf{X} = (X_1, ..., X_n)$ be an n-dimensional random vector and Y be a random variable. If $\mathbf{X} \uparrow_{SI} Y$, then $f(\mathbf{X}) \uparrow_{SI} g(Y)$ for any increasing functions $f : \mathbb{R}^n \to \mathbb{R}^k$ and $g : \mathbb{R} \to \mathbb{R}$, where $k \in \mathbb{N}$.

Proof. First, it is easy to show that $\mathbf{X} \uparrow_{SI} Y \Longrightarrow f(\mathbf{X}) \uparrow_{SI} Y$. Indeed, for any increasing function $h : \mathbb{R}^k \to \mathbb{R}$, $h \circ f : \mathbb{R}^n \to \mathbb{R}$ is also increasing. By the definition of $\uparrow_{SI}$, we know that $\mathbb{E}[h \circ f(X_1, ..., X_n) \,|\, Y = y]$ is increasing in $y \in S(Y)$, which means $f(X_1, ..., X_n) \uparrow_{SI} Y$.

Then, to complete the proof, it is sufficient to show that $\mathbf{X} \uparrow_{SI} Y \Longrightarrow \mathbf{X} \uparrow_{SI} g(Y)$. Denote $Z = g(Y)$, note that $\sigma(Z) \subset \sigma(Y)$. Thus, for any increasing function $u : \mathbb{R}^n \to \mathbb{R}$, we have

$$\mathbb{E}[u(\mathbf{X}) \,|\, Z] = \mathbb{E}[\mathbb{E}[u(\mathbf{X}) \,|\, Y] \,|\, Z] = \mathbb{E}[h_u(Y) \,|\, Z], \tag{2.2.7}$$

where $h_u(Y) = \mathbb{E}[u(\mathbf{X}) \,|\, Y]$ is increasing in Y since $\mathbf{X} \uparrow_{SI} Y$. By the properties of conditional expectations, we know that (2.2.7) implies $\mathbb{E}[u(\mathbf{X}) \,|\, Z = z] = \mathbb{E}[h_u(Y) \,|\, Z = z]$ for all $z \in S(Z)$, where $S(Z)$ is a support of Z. By Corollary 2.2.12, we have $Y \uparrow_{SI} Z$ and thus $h_u(Y) \uparrow_{SI} Z$. Therefore $\mathbb{E}[u(\mathbf{X}) \,|\, Z = z] = \mathbb{E}[h_u(Y) \,|\, Z = z]$ is increasing in $z \in S(Z)$, which implies that $\mathbf{X} \uparrow_{SI} Z$. □

From Proposition 2.2.13, we immediately get the following property.

Proposition 2.2.14 If random vector $(X_1, ..., X_n)$ is PDS, then $(f_1(X_1), ..., f_n(X_n))$ is PDS for any increasing functions f_i, $i = 1, ..., n$. □

Corollary 2.2.15 Random vector $(X_1, ..., X_n)$ is PDS if and only if $(F_1(X_1), ..., F_n(X_n))$ is PDS, where F_i is the distribution function of X_i, $i = 1, ..., n$.

Proof. Since $F_i(x), i = 1, ..., n$ are increasing, according to Proposition 2.2.14, we have $(X_1, ..., X_n)$ PDS implies $(F_1(X_1), ..., F_n(X_n))$ PDS. On the other hand, from Proposition A.4 of McNeil et al. (2005), we know that $X_i = F_i^{-1} \circ F_i(X_1)$ holds with probability 1 for all $i = 1, ..., n$. By Proposition A.3(i) of McNeil et al. (2005), $F_i^{-1}, i = 1, ..., n$ are increasing. Thus, if $(F_1(X_1), ..., F_n(X_n))$ is PDS, by Proposition 2.2.14, we have $(F_1^{-1} \circ F_1(X_1), ..., F_n^{-1} \circ F_n(X_n))$ is PDS, and hence $(X_1, ..., X_n)$ is PDS. □

Proposition 2.2.16 Assume that that $g(x)$ and $g_i(x)$, $i = 1, 2, ..., n$, are increasing functions. For random vector $\mathbf{X} = (X_1, ..., X_n)$ and random variable Y, if $\mathbf{X} = (X_1, ..., X_n) \uparrow_{WSI} Y$, then $(g_1(X_1), ..., g_n(X_n)) \uparrow_{WSI} g(Y)$.

Proof. Since $\mathbf{X} \uparrow_{WSI} Y$, we have $\mathbf{X} \,|\, Y = y_1 \leq_{uo} \mathbf{X} \,|\, Y = y_2$ for any $y_1, y_2 \in S(Y)$ with $y_1 < y_2$. Thus, by Theorem 6.G.3 of Shaked and Shanthikumar (2007), we know that the upper orthant order $\le_{uo}$ is preserved under componentwise increasing transformations. Thus we have $(g_1(X_1), ..., g_n(X_n)) \,|\, Y = y_1 \le_{uo} (g_1(X_1), ..., g_n(X_n)) \,|\, Y = y_2$ for any $y_1, y_2 \in S(Y)$ with $y_1 < y_2$, which means that $h(y) = \mathbb{P}\{g_1(X_1) > x_1, ..., g_n(X_n) > x_n \,|\, Y = y\}$ is increasing in $y \in S(Y)$ for any $(x_1, ..., x_n) \in \mathbb{R}^n$.

On the other hand, since $\sigma(g(Y)) \subseteq \sigma(Y)$, we have

$$\begin{aligned} & \mathbb{E}[\mathbb{I}\{g_1(X_1) > x_1, ..., g_n(X_n) > x_n\} \,|\, g(Y)] \\ = \;& \mathbb{E}[\mathbb{E}\,[\mathbb{I}\{g_1(X_1) > x_1, ..., g_n(X_n) > x_n\} \,|\, Y] \,|\, g(Y)] = \mathbb{E}[h(Y) \,|\, g(Y)]. \end{aligned}$$

According to Corollary 2.2.12, we have $Y \uparrow_{SI} g(Y)$, thus $\mathbb{E}[h(Y) \,|\, g(Y) = y]$ is increasing in $y \in S(g(Y))$, which implies $\mathbb{P}\{g_1(X_1) > x_1, ..., g_n(X_n) > x_n \,|\, g(Y) = y\}$ is increasing in $y \in S(g(Y))$ for any $(x_1, ..., x_n) \in \mathbb{R}^n$. □

Corollary 2.2.17 Assume that $g_i(x), i = 1, ..., n$, are increasing functions. If random vector $\mathbf{X} = (X_1, ..., X_n)$ is PDUO, then $(g_1(X_1), ..., g_n(X_n))$ is PDUO.

Proof. The proof follows immediately from the definition of PDUO and Proposition 2.2.16. □

Corollary 2.2.18 Let F_i be the distribution function of X_i for $i = 1, ..., n$. Then, $(X_1, ..., X_n)$ is PDUO if and only if $(F_1(X_1), ..., F_n(X_n))$ is PDUO.

Proof. The proof is similar to that for Corollary 2.2.15 and is omitted. □

Corollaries 2.2.15 and 2.2.18 imply that if $(X_1, ..., X_n)$ has continuous marginal distributions F_i, $i = 1, 2, ..., n$, then $(X_1, ..., X_n)$ is PDS (PDUO) if and only if its copula is PDS (PDUO).

2.3 Generalized Inverse Functions and the Copula Invariance

For the inverse functions g^{-1} and g^{-1+} defined in Definition 2.2.7, it is easy to check that g^{-1} is left-continuous while g^{-1+} is right-continuous. The generalized inverse functions of increasing functions appear in many studies. Below, we prove a property of the generalized inverse functions, which will be used to derive the invariant property of copulas under increasing transformations.

Proposition 2.3.1 Let $g : \mathbb{R} \to \mathbb{R}$ be an increasing function and $x, z \in \mathbb{R}$.

(i) If g is left continuous, then $g(x) \leq z$ if and only if $x \leq g^{-1+}(z)$.

(ii) If g is right continuous, then $g(x) \geq z$ if and only if $x \geq g^{-1}(z)$.

(iii) The following implications hold: $x < g^{-1+}(z) \Longrightarrow g(x) \leq z \Longrightarrow x \leq g^{-1+}(z)$.

Proof. (i) If $g(x) \leq z$, then $x \in \{y \,|\, g(y) \leq z\}$ and thus $x \leq \sup\{y \,|\, g(y) \leq z\} = g^{-1+}(z)$. Conversely, if $x \leq g^{-1+}(z)$, then $g(x) \leq g(g^{-1+}(z))$ since g is increasing. Because $g^{-1+}(z)$ is the supremum of the set $\{y \,|\, g(y) \leq z\}$, there exists a series $\{x_n\}_{n=1}^{\infty}$ in the set such that $g(x_n) \leq z$ and $x_n \uparrow g^{-1+}(z)$ as $n \to \infty$. Since g is left-continuous, then $g(g^{-1+}(z)) = \lim_{n\to\infty} g(x_n) \leq z$. Thus, $g(x) \leq g(g^{-1+}(z)) \leq z$.

(ii) The statement is from Proposition A.3(iv) in McNeil et al. (2005).

(iii) Assume that $x < g^{-1+}(z)$. If $g(x) > z$, then $g(x) > g(y)$ for all $y \in \{t : g(t) \leq z\}$. Hence, $x > y$ for all $y \in \{t \,|\, g(t) \leq z\}$ since g is increasing. Thus, $x \geq \sup\{t \,|\, g(t) \leq z\} = g^{-1+}(z)$, which contradicts the assumption of $x < g^{-1+}(z)$. Therefore,

$x < g^{-1+}(z) \implies g(x) \le z$. Furthermore, assume $g(x) \le z$, then $x \in \{y \,|\, g(y) \le z\}$ and thus $x \le \sup\{y \,|\, g(y) \le z\} = g^{-1+}(z)$. □

Lemma 2.3.2 Assume that random variable X_i has continuous marginal distribution function F_i for $i = 1, ..., n$ and $(X_1, ..., X_n)$ has copula C. If $f_1, ..., f_n$ are increasing functions, then $(f_1(X_1), ..., f_n(X_n))$ also has the copula C.

Proof. Note that $\mathbb{P}\{X_1 \le x_1, ..., X_n \le x_n\} = C(F_1(x_1), ..., F_n(x_n))$. For any $i = 1, ..., n$, we have $\mathbb{P}\{X_i = f_i^{-1+}(z_i)\} = 0$ since X_i has a continuous distribution function. According to Proposition 2.3.1 (iii), we have for any $i = 1, ..., n$, $\{X_i < f_i^{-1+}(z_i)\} \subseteq \{f_i(X_i) \le z_i\} \subseteq \{X_i \le f_i^{-1+}(z_i)\}$, which, together with $\mathbb{P}\{X_i = f_i^{-1+}(z_i)\} = 0$, implies that $F_{f_i(X_i)}(z_i) = \mathbb{P}\{f_i(X_i) \le z_i\} = \mathbb{P}\{X_i \le f_i^{-1+}(z_i)\} = F_i \circ f_i^{-1+}(z_i)$. Therefore,

$$\begin{aligned}\mathbb{P}\{f_1(X_1) \le z_1, ..., f_n(X_n) \le z_n\} &= \mathbb{P}\{X_1 \le f_1^{-1+}(z_1), ..., X_n \le f_n^{-1+}(z_n)\} \\ = C(F_1 \circ f_1^{-1+}(z_1), ..., F_n \circ f_n^{-1+}(z_n)) &= C(F_{f_1(X_1)}(z_1), ..., F_{f_n(X_n)}(z_n)),\end{aligned}$$

which means that C is also a copula of $(f_1(X_1), ..., f(X_n))$. □

Theorem 2.3.3 Assume that $f_1, ..., f_n$ are increasing functions. If random vector $(X_1, ..., X_n)$ has copula C, then $(f_1(X_1), ..., f_n(X_n))$ also has the copula C.

Proof. Since random vector $(X_1, ..., X_n)$ has copula C, by the last paragraph of the proof for Theorem 5.3 of McNeil et al. (2005), we know that there exists a uniform random vector $(U_1, ..., U_n)$ defined on $[0, 1]^n$ such that $(U_1, ..., U_n)$ has distribution function $C(u_1, ..., u_n)$ and $(F_1^{-1}(U_1), ..., F_n^{-1}(U_n)) =_{st} (X_1, ..., X_n)$.

Thus, $(f_1(X_1), ..., f_n(X_n)) =_{st} (f_1(F_1^{-1}(U_1)), ..., f_n(F_n^{-1}(U_n)))$ or $(f_1(X_1), ..., f_n(X_n))$ and $(f_1(F_1^{-1}(U_1)), ..., f_n(F_n^{-1}(U_n)))$ have the same joint distribution function and hence they have the same copula. On the other hand, since $f_i \circ F_i^{-1}$ is increasing and U_i has the continuous marginal distribution function, by Lemma 2.3.2, we know that $(U_1, ..., U_n)$ and $(f_1(F_1^{-1}(U_1)), ..., f_n(F_n^{-1}(U_n)))$ have the same copula C. Therefore, $(f_1(X_1), ..., f_n(X_n))$ has the copula C as well. □

Theorem 2.3.3 generalizes Proposition 4.7.4 of Denuit et al. (2006) and Proposition 5.6. of McNeil et al. (2005).

Combining Theorem 2.3.3 and Proposition 1.4.1, we get the following invariant property about survival copulas.

Corollary 2.3.4 Assume that $f_1, ..., f_n$ are increasing functions. If random vector $(X_1, ..., X_n)$ has survival copula C, then $(f_1(X_1), ..., f_n(X_n))$ also has survival copula C.

Proof. According to Proposition 1.4.1, we know that $(-X_1, ..., -X_n)$ has copula C. Define $g_i(x) = -f_i(-x), i = 1, ..., n$. Then $g_i(x)$ is increasing for $i = 1, ..., n$. Combining with Theorem 2.3.3, we have $(g_1(X_1), ..., g_n(X_n))$ has copula C, i.e., $(-f_1(X_1), ..., -f_n(X_n))$ has copula C. Therefore, $(f_1(X_1), ..., f_n(X_n))$ also has survival copula C. □

Following Corollary 2.3.4, we immediately get

Corollary 2.3.5 If $C(u_1, ..., u_n)$ is a copula (survival copula) of random vector $(X_1, ..., X_n)$, then $C(u_1, ..., u_n)$ is a survival copula (copula) of $(f_1(X_1), ..., f_n(X_n))$ for any decreasing functions $f_1(x), ..., f_n(x)$. □

2.4 The Characterization of PDUO in Terms of Survival Copulas

If the distribution function F_i of X_i is continuous for $i = 1, ..., n$, then $F_i(X_i)$ has the uniform distribution over $[0, 1]$ and thus the joint distribution function of $(F_1(X_1), ..., F_n(X_n))$ is the unique copula of $(X_1, ..., X_n)$, which links the marginal distributions of $X_1, ..., X_n$. This means that the PDS and PDUO properties of a continuous random vector can be characterized by their copulas. Actually, if the joint distribution function of a continuous random vector $(X_1, ..., X_n)$ is linked by a Gaussian copula, then $(X_1, ..., X_n)$ is PDS if and only if all off-diagonal elements of the covariance matrix in the Gaussian copula are non-negative. See, for example, Joe (1997). Now, using Corollary 2.2.18, we could develop a sufficient and necessary condition for PDUO in terms of survival copulas.

Let $(X_1, ..., X_n)$ be a random vector with marginal distribution functions $F_1, ..., F_n$ and marginal survival functions $\bar{F}_i(x) = 1 - F_i(x), i = 1, ..., n$. The joint survival function of the random vector is denoted by $\bar{F}(x_1, ..., x_n) = \mathbb{P}\{X_1 > x_1, ..., X_n > x_n\}$, which is linked by a copula $\widehat{C}$ as

$$\bar{F}(x_1, ..., x_n) = \widehat{C}(\bar{F}_1(x_1), ..., \bar{F}_n(x_n)).$$

Such a copula $\widehat{C}$ is called a survival copula of the random vector $(X_1, ..., X_n)$. See, for example, McNeil et al. (2005).

Proposition 2.4.1 Assume that $(X_1, ..., X_n)$ has continuous marginal distribution functions $F_1, ..., F_n$ with survival copula $\widehat{C}$. Then $(X_1, ..., X_n)$ is PDUO if and only if $\widehat{C}$ is concave in each argument.

Proof. Denote $U_i = F_i(X_i), i = 1, 2, ..., n$, then U_i has a uniform distribution over $[0, 1]$. According to Corollary 2.2.18, it is sufficient to show that $(U_1, ..., U_n)$ is PDUO if and only if $\widehat{C}$ is concave in each argument.

Let $\bar{C}$ be the survival function of $(U_1, ..., U_n)$. Note that the survival function $\bar{C}$ is decreasing in each argument and thus differentiable with respect with each argument almost everywhere. Thus, for any $u_k \in [0, 1]$ and $k \in \{1, 2, ..., n\}$,

$$
\begin{aligned}
& \mathbb{P}\{U_1 > u_1, ..., U_{k-1} > u_{k-1}, U_{k+1} > u_{k+1}, ..., U_n > u_n \,|\, U_k = u_k\} \\
= & \lim_{\Delta u \downarrow 0} \left\{ \frac{\mathbb{P}\{U_1 > u_1, ..., U_k > u_k, ..., U_n > u_n\}}{\mathbb{P}\{U_k \in (u_k, u_k + \Delta u]\}} \right. \\
& \qquad \left. - \frac{\mathbb{P}\{U_1 > u_1, ..., U_k > u_k + \Delta u, ..., U_n > u_n\}}{\mathbb{P}\{U_k \in (u_k, u_k + \Delta u]\}} \right\} \\
= & \lim_{\Delta u \downarrow 0} \frac{\bar{C}(u_1, ..., u_k, ..., u_n) - \bar{C}(u_1, ..., u_k + \Delta u ..., u_n)}{\Delta u} \\
= & -\frac{\partial}{\partial u_k} \bar{C}(u_1, ..., u_k, ..., u_n). \qquad (2.4.1)
\end{aligned}
$$

Hence, $(U_1, ..., U_{k-1}, U_{k+1}, ..., U_n) \uparrow_{WSI} U_k$ if and only if $-\frac{\partial}{\partial u_k} \bar{C}(u_1, u_2, ..., u_n)$ is increasing in $u_k \in [0, 1]$ (almost everywhere) for any fixed $u_1, ..., u_{k-1}, u_{k+1}, ..., u_n$ if and only if $\bar{C}$ is concave in each argument.

Recall (1.4.1), we know that $\bar{C}(u_1, u_2, ..., u_n) = \widehat{C}(1 - u_1, 1 - u_2, ..., 1 - u_n)$. Hence, $\bar{C}(u_1, ..., u_k, ..., u_n)$ is concave in u_k if and only if $\widehat{C}(u_1, ..., u_k, ..., u_n)$ is concave in u_k. Therefore, we conclude that continuous random vector $(X_1, ..., X_n)$ is PDUO if and only if their survival copula $\widehat{C}$ is concave in each argument. □

Proposition 2.4.1 enables one to easily construct a PDUO random vector by choosing a copula such that the copula is concave in each argument. We give such an example below.

Example 2.4.2 Assume that that a continuous random vector $(X_1, ..., X_n)$ is linked by a survival Archimedean copula

$$\widehat{C}(u_1, u_2, ..., u_n) = \Psi^{-1}\left(\sum_{k=1}^{n} \Psi(u_k)\right), \qquad u_k \in [0, 1],\ k = 1, 2, ..., n. \tag{2.4.2}$$

Rewrite (2.4.2), we get $\Psi(\widehat{C}) = \sum_{k=1}^{n} \Psi(u_k)$. Then, differentiating with respect to u_k on both sides of the equation, we have $\Psi'(\widehat{C}) \times \frac{\partial \widehat{C}}{\partial u_k} = \Psi'(u_k)$. Therefore, $\frac{\partial \widehat{C}}{\partial u_k} = \frac{\Psi'(u_k)}{\Psi'(\widehat{C})}$ and

$$\frac{\partial^2 \widehat{C}}{\partial u_k^2} = \frac{\Psi'(\widehat{C})\Psi''(u_k) - \Psi''(\widehat{C})\frac{\partial \widehat{C}}{\partial u_k}\Psi'(u_k)}{[\Psi'(\widehat{C})]^2} = \frac{[\Psi'(u_k)]^2}{\Psi'(\widehat{C})} \times \left[\frac{\Psi''(u_k)}{[\Psi'(u_k)]^2} - \frac{\Psi''(\widehat{C})}{[\Psi'(\widehat{C})]^2}\right].$$

Recall that the survival copula $\widehat{C} = \widehat{C}(u_1, u_2, ..., u_n)$ is a copula. By the Fréchet bounds for copulas, we have $\widehat{C} = \widehat{C}(u_1, u_2, ..., u_n) \le \min\{u_1, ..., u_n\} \le u_k$. Thus, $\widehat{C}(u_1, ..., u_n)$ is concave in each argument, or $\frac{\partial^2 \widehat{C}}{\partial u_k^2} \le 0$ for each $k \in \{1, ..., n\}$, if

$$\frac{\Psi''(x)}{[\Psi'(x)]^2} \text{ is increasing in } x \in [0, 1]. \tag{2.4.3}$$

Hence, if the joint survival function of a continuous random vector $(X_1, ..., X_n)$ is linked by the Archimedean copula (1.4.2), then $(X_1, ..., X_n)$ is PDUO if (2.4.3) holds.

Two examples of the Archimedean copula satisfying (2.4.3) are the multivariate Gumbel copula with $\Psi(x) = (-\ln x)^\theta$, $\theta \ge 1$ and the multivariate Clayton copula with $\Psi(x) = x^{-\theta} - 1$, $\theta > 0$. In both cases, the condition (2.4.3) holds. We refer to Müller and Scarsini (2005) for a detailed study of the relationships between Archimedean copulas and other notions of positive dependence. □

Chapter 3

Optimal Reinsurance with Positive Dependence

3.1 Introduction

Let $\{X_i, i \geq 1\}$ be random variables. Assume that an insurer has n lines of business or the insurance portfolio of an insurer has n policyholders. The loss or claim in line i or for policy holder i is X_i, $i = 1, ..., n$. Without reinsurance, the total loss/claim of the insurer is $S_n = \sum_{i=1}^{n} X_i$, which is called the individual risk model. However, each line of business or each policyholder may produce a large claim. To protect from a potential huge loss, the insurer applies reinsurance strategy I_i to the loss in line i. With the reinsurance strategy I_i, the insurer retains the part of the loss in line i, which is $I_i(X_i)$, and a reinsurer covers the rest of the loss, which is $X_i - I_i(X_i)$, where the function $I_i(x)$ is increasing in $x \geq 0$ and satisfies $0 \leq I_i(x) \leq x$ for $i = 1, 2, ..., n$. Thus, the total retained loss of the insurer is $S_n^I = I_1(X_1) + I_2(X_2) + \cdots + I_n(X_n)$ and the total loss covered by the reinsurer is $S_n - S_n^I$, where we use $I = (I_1, ..., I_n)$ to denote the n-dimensional reinsurance policy. Such a policy I is called an individualized reinsurance strategy.

In the reinsurance contract I, the insurer needs to pay a reinsurance premium to the reinsurer. As in Denuit and Vermandele (1998) and Vanheerwaarden et al. (1989), we assume that the reinsurance premium is charged by the expected value principle and is fixed to a constant \$$P$, which means that the reinsurance premium is equal to $(1+\theta_R)\mathbb{E}[S_n - S_n^I] = P$, where $\theta_R > 0$ is called the security loading of the reinsurer. In this way, the insurer can control her cost or budget for the reinsurance contract at the amount of P. Note that $(1+\theta_R)\mathbb{E}[S_n - S_n^I] = P$ is equivalent to assuming that $\mathbb{E}[S_n^I]$ is fixed and equal to a constant $p = \mathbb{E}[S_n] - P/(1+\theta_R)$, which means that the expected retained loss of the insurer is fixed. We are interested in the following class of admissible reinsurance strategies:

$$\mathcal{D}_n^p = \left\{ I = (I_1, ..., I_n) \;\middle|\; \begin{array}{l} I_i(x) \text{ is increasing in } x \geq 0 \text{ with} \\ 0 \leq I_i(x) \leq x \text{ for } i = 1, ..., n \text{ and } \mathbb{E}[S_n^I] = p > 0 \end{array} \right\}. \quad (3.1.1)$$

In particular, when $I_i(x) = x \wedge d_i$ for $i = 1, ..., n$, the reinsurance $I = (I_1, ..., I_n)$ is called the individualized excess-of-loss strategy and $(d_1, ..., d_n)$ is called the retention vector of the individualized excess-of-loss strategy.

In this chapter, we study what the optimal reinsurance strategy $I^* = (I_1^*, ..., I_n^*) \in \mathcal{D}_n^p$ is for the insurer under certain optimization criteria. We use a unified criterion and study the following optimization problem:

$$\inf_{I \in \mathcal{D}_n^p} \mathbb{E}[u(S_n^I)] \quad (3.1.2)$$

for a convex function u.

This optimization criterion (3.1.2) includes the criteria of minimizing the variance of the total retained loss of the insurer; maximizing the expected exponential utility for the insurer; maximizing the expected concave utility function for the insurer; and so on.

When $X_1, ..., X_n$ are exchangeable random variables, Denuit and Vermandele (1998) showed that the optimal reinsurance strategy for problem (3.1.2) is the excess-of-loss strategy with the equal retentions for each line of business. A further study of Denuit and Vermandele (1998) about optimal reinsurance with exchangeable risks can be found in Denuit and Vermandele (1999).

However, in individualized reinsurance treaties, one is often concerned about dependent risks, and in particular positively dependent risks. For example, in a two-line insurance business with life insurance and non-life insurance, the property losses and the numbers of death in earthquakes, tornadoes, and hurricanes are usually positively dependent. Roughly speaking, two risks are positively dependent if a large value of one risk results in a potential large value of the other. Several notions of positive dependence were proposed to describe such dependent risks in the literature.

In this chapter, we assume that the risks in the individual risk model are positively dependent through the stochastic ordering (PDS), which is defined in Chapter 2. We show that when $X_1, ..., X_n$ are PDS dependent risks, the optimal reinsurance strategy for problem (3.1.2) is the excess-of-loss strategy. To do so, we denote $\mathcal{D}_n^{p*}$ by all individualized excess-of-loss strategies in $\mathcal{D}_n^p$, namely

$$\mathcal{D}_n^{p*} = \left\{ I^d = (I^{d_1}, ..., I^{d_n}) \,\middle|\, I^d \in \mathcal{D}_n^p,\ I^{d_i}(x) = x \wedge d_i,\ d_i \geq 0,\ i = 1, ..., n \right\}.$$

This subclass $\mathcal{D}_n^{p*}$ is uniquely determined by the retention vector $(d_1, ..., d_n)$ and there is a one-to-one mapping between D_n^{p*} and L_n^p that is defined as

$$L_n^p = \Big\{(d_1, ..., d_n) \,\Big|\, d_i \geq 0,\ i = 1, ..., n \text{ and } \mathbb{E}\Big[\sum_{i=1}^{n}(X_i \wedge d_i)\Big] = p > 0\Big\}. \tag{3.1.3}$$

We will show that for the PDS dependent risks $X_1, ..., X_n$ and a convex function u,

$$\inf_{I \in \mathcal{D}_n^p} \mathbb{E}[u(S_n^I)] = \inf_{I \in \mathcal{D}_n^{p*}} \mathbb{E}[u(S_n^I)], \tag{3.1.4}$$

which means that the optimal strategies for problem (3.1.2) are the individualized excess-of-loss strategies and that the infinite-dimensional optimization problem (3.1.2) is reduced to the feasible finite-dimensional optimization problem:

$$\inf_{(d_1,...,d_n) \in L_n^p} \mathbb{E}\Big[u\Big(\sum_{i=1}^{n}(X_i \wedge d_i)\Big)\Big]. \tag{3.1.5}$$

The rest of the chapter is organized as follows. In Section 3.2, we recall the notions of several positive dependence including the stochastically increasing (SI) and the positive dependence through the stochastic ordering (PDS). In Section 3.3, we first improve the results about convex order of PDS random vectors in Müller and Scarsini (2005). We then prove that the individualized excess-of-loss strategy is the optimal reinsurance form for the insurer with PDS dependent risks. This extends the study in Denuit and Vermandele (1998) on individualized reinsurance strategies. In Section 3.4, we use a two-line insurance business model to illustrate how to derive the explicit expressions for the retention vector $(d_1^*, d_2^*) \in L_2^p$ in the optimal individualized excess-of-loss strategy such that $\mathbb{E}\Big[u\Big(X_1 \wedge d_1^* + X_2 \wedge d_2^*\Big)\Big] = \inf_{(d_1,d_2) \in L_2^p} \mathbb{E}\Big[u\Big(X_1 \wedge d_1 + X_2 \wedge d_2\Big)\Big]$. In Section 3.5, we apply the results in previous sections to study the collective risk model. The individualized excess-of-loss strategy is proved optimal in the sense of minimizing the ruin probability or stochastically maximizing the ruin time. In Section 3.6, we study the optimal reinsurance problem with a random initial wealth. The excess-of-loss strategy is proved to preserve its optimality under certain assumptions on the dependent structure between the initial wealth and the risk.

3.2 Optimal Reinsurance Form with Dependent Risks

In this section, we first prove that two convolution preservation results of the convex order for SI and PDS random vectors in Lemma 3.2.5 and Theorem 3.2.6. Then, we can determine the optimal reinsurance forms with the PDS dependent risks in Propositions 3.2.10 and 3.2.12 for the individual risk model and the collective risk model, respectively.

Müeller and Stoyan (2002) introduced the concept of directional convex order for random vectors.

Definition 3.2.1 A multivariate function $\phi : \mathbb{R}^n \to \mathbb{R}$ is said to be directional convex, if

$$\phi(\mathbf{x_1} + \mathbf{y}) - \phi(\mathbf{x_1}) \leq \phi(\mathbf{x_2} + \mathbf{y}) - \phi(\mathbf{x_2}),$$

for all $\mathbf{x_1} \leq \mathbf{x_2}$ and $\mathbf{y} \geq 0$.

Definition 3.2.2 Random vector $\mathbf{X} = (X_1, ..., X_n)$ is said to be smaller than $\mathbf{Y} = (Y_1, ..., Y_n)$ in the *directional convex order*, denoted as $\mathbf{X} \leq_{dcx} \mathbf{Y}$, if $\mathbb{E}\left[\phi(\mathbf{X})\right] \leq \mathbb{E}\left[\phi(\mathbf{Y})\right]$ for all directional convex function ϕ such that the above expectations exist.

Random vector $\mathbf{X} = (X_1, ..., X_n)$ is said to be smaller than $\mathbf{Y} = (Y_1, ..., Y_n)$ in the *increasing directional convex order*, denoted as $\mathbf{X} \leq_{idcx} \mathbf{Y}$, if $\mathbb{E}\left[\phi(\mathbf{X})\right] \leq \mathbb{E}\left[\phi(\mathbf{Y})\right]$ for all increasing directional convex function ϕ such that the above expectations exist.

Lemma 3.2.3 Assume that random vectors $\mathbf{X} = (X_1, ..., X_n)$ and $\mathbf{Y} = (Y_1, ..., Y_n)$ are both comonotonic.

(i) If $X_i \leq_{cx} Y_i$ for all $i = 1, ..., n$, then $\mathbf{X} \leq_{dcx} \mathbf{Y}$.

(ii) If $X_i \leq_{icx} Y_i$ for all $i = 1, ..., n$, then $\mathbf{X} \leq_{idcx} \mathbf{Y}$.

Proof. See Lemma 2.12.13 in Müeller and Stoyan (2002) . □

Corollary 3.2.4 Let X be random variable and f, g, h be increasing functions.

(i) If $g(X) \leq_{cx} h(X)$, then $\mathbb{E}[f(X)g(X)] \leq \mathbb{E}[f(X)h(X)]$;

(ii) If $g(X) \leq_{icx} h(X)$ and $f(x), g(x) \geq 0$, then $\mathbb{E}[f(X)g(X)] \leq \mathbb{E}[f(X)h(X)]$.

Proof. (i). Apparently, random vector $(f(X), g(X))$ is comonotonic, so is $(f(X), h(X))$. Since $f(X) \leq_{cx} f(X)$ and $g(x) \leq_{cx} h(X)$, then $(f(X), g(X)) \leq_{dcx} (f(X), g(X))$ according to Lemma 3.2.3.

Consider the function $\phi_1(x, y) = xy$, it is easy to verify that $\phi_1(x, y)$ is directional convex. According to Definition 3.2.2, we have $\mathbb{E}[\phi_1(f(X), g(X))] \leq \mathbb{E}[\phi_1(f(X), h(X))]$, i.e., $\mathbb{E}[f(X)g(X)] \leq \mathbb{E}[f(X)h(X)]$.

(ii) Consider the function $\phi_2(x, y) = xy\mathbb{I}\{x \geq 0, y \geq 0\}$. Note that $\phi_2(x, y)$ is increasing directional convex. Following the same arguments as in (i), we have $\mathbb{E}[f(X)g(X)] = \mathbb{E}[\phi_2(f(X), g(X))] \leq \mathbb{E}[\phi_2(f(X), h(X))] = \mathbb{E}[f(X)h(X)]$. □

Lemma 3.2.5 Let X and Y be random variables. If $Y \uparrow_{SI} X$, then $h_1(X) + Y \leq_{cx} h_2(X) + Y$ for any increasing functions h_1 and h_2 such that $h_1(X) \leq_{cx} h_2(X)$.

Proof. It is sufficient to show that $h_1(X)+Y \leq_{sl} h_2(X)+Y$, or equivalently, to show that $\mathbb{E}[(h_1(X)+Y-t)_+] \leq \mathbb{E}[(h_2(X)+Y-t)_+]$ for all $t \in \mathbb{R}$.

It is easy to verify that $(x-t)_+ - (y-t)_+ \leq \mathbb{I}\{x>t\} \times (x-y)$ for any $x, y, t \in \mathbb{R}$, then

$$\begin{aligned} & \mathbb{E}\left[(h_1(X)+Y-t)_+ - (h_2(X)+Y-t)_+\right] \\ \leq\ & \mathbb{E}\left[\mathbb{I}\{(h_1(X)+Y)>t\} \times (h_1(X)-h_2(X))\right] \\ =\ & \mathbb{E}\left[\mathbb{E}\left[\mathbb{I}\{(h_1(X)+Y)>t\}|X\right] \times (h_1(X)-h_2(X))\right] \\ =\ & \mathbb{E}\left[p_t(X)(h_1(X)-h_2(X))\right], \end{aligned} \tag{3.2.1}$$

where the function $p_t(x) = \mathbb{E}\left[\mathbb{I}\{(h_1(x)+Y)>t\}\,|\,X=x\right] \geq 0$ is well defined since $0 \leq \mathbb{I}\{x>t\} \leq 1$.

From Proposition 2.2.2, we know that $(Y, X) \uparrow_{SI} X$ and $h_1(X)+Y \uparrow_{SI} X$. Therefore, the function $p_t(x) = \mathbb{E}\left[\mathbb{I}\{h_1(x)+Y>t\}\,|\,X=x\right]$ is increasing in x since $\mathbb{I}\{x>t\}$ is increasing in x. Thus, both $(p_t(X), h_1(X))$ and $(p_t(X), h_2(X))$ are comonotonic vectors. According to Corollary 3.2.4, we have $\mathbb{E}\left[p_t(X)h_1(X)\right] \leq \mathbb{E}\left[p_t(X)h_2(X)\right]$, which completes the proof by (3.2.1). $\square$

Lemma 3.2.5 is an interesting result and will be used to prove the following Theorem 3.2.6. Also, Lemma 3.2.5 generalizes Theorems 1 and 2 of Aboudi and Thon (1995), in which they presented the optimal insurance policies when the insurance risk has positively dependent relationships with the random initial wealth.

Theorem 3.2.6 Let $(X_1, ..., X_n)$ be a PDS random vector, and f_i, g_i be increasing functions such that $f_i(X_i) \leq_{cx} g_i(X_i)$ for $i = 1, \cdots, n$. Then $\sum_{k=1}^n f_k(X_k) \leq_{cx} \sum_{k=1}^n g_k(X_k)$.

Proof. According to Proposition 2.2.2, we have $\sum_{i=1}^{k-1} f_i(X_i) + \sum_{i=k+1}^{n} g_i(X_i) \uparrow_{SI} X_k$ for any $k = 1, \cdots, n$, where $\sum_{k=i}^{j} a_k$ is defined to be 0 for $i > j$. Applying Lemma 3.2.5, we have for any $k = 1, ..., n$,

$$\sum_{i=1}^{k-1} f_i(X_i) + \sum_{i=k+1}^{n} g_i(X_i) + f_k(X_k) \leq_{cx} \sum_{i=1}^{k-1} f_i(X_i) + \sum_{i=k+1}^{n} g_i(X_i) + g_k(X_k),$$

or equivalently,

$$\sum_{i=1}^{k} f_i(X_i) + \sum_{i=k+1}^{n} g_i(X_i) \leq_{cx} \sum_{i=1}^{k-1} f_i(X_i) + \sum_{i=k}^{n} g_i(X_i). \tag{3.2.2}$$

By applying the relationship (3.2.2) repeatedly from $k = n$ to $k = 1$ and using the transitive property of the convex order, we have

$$\begin{aligned} \sum_{i=1}^{n} f_i(X_i) &\leq_{cx} \sum_{i=1}^{n-1} f_i(X_i) + \sum_{i=n}^{n} g_i(X_i) \leq_{cx} \sum_{i=1}^{n-2} f_i(X_i) + \sum_{i=n-1}^{n} g_i(X_i) \\ &\leq_{cx} \cdots \leq_{cx} \sum_{i=1}^{1} f_i(X_i) + \sum_{i=2}^{n} g_i(X_i) \leq_{cx} \sum_{i=1}^{n} g_i(X_i). \end{aligned}$$

It completes the proof. □

Using Theorem 3.2.6, we can prove the convolution preservation of the convex order for two random vectors with the same PDS copula in the following corollary.

Corollary 3.2.7 Assume that random vectors $(Y_1, ..., Y_n)$ and $(Z_1, ..., Z_n)$ have the same PDS copula. If $Y_k \leq_{cx} Z_k$ for $k = 1, ..., n$, then $\sum_{k=1}^{n} Y_k \leq_{cx} \sum_{k=1}^{n} Z_k$.

Proof. Let F_i and G_i be the distributions of Y_i and Z_i, respectively. Let the common PDS copula be $C(u_1, ..., u_n) = \mathbb{P}\{U_1 \leq u_1, ..., U_n \leq u_n\}$ for some uniform random vector $(U_1, ..., U_n)$ defined on $[0,1]^n$. Then, $(U_1, ..., U_n)$ is a PDS random vector. From the last paragraph of the proof for Theorem 5.3 of McNeil et al. (2005), we know that $(Y_1, ..., Y_n) =_{st} (F_1^{-1}(U_1), ..., F_n^{-1}(U_n))$ and $(Z_1, ..., Z_n) =_{st} (G_1^{-1}(U_1), ..., G_n^{-1}(U_n))$, where F_i^{-1} and G_i^{-1} are the left-continuous generalized inverses of F_i and G_i and they are increasing. Thus, $\sum_{k=1}^n Y_k \leq_{cx} \sum_{k=1}^n Z_k$ by Theorem 3.2.6. □

Remark 3.2.8 We point out that for all non-negative constants $\alpha_1, ..., \alpha_n$, $(Y_1, ..., Y_n)$ and $(\alpha_1 Y_1, ..., \alpha_n Y_n)$ have the same copula, and $(\alpha_1 Z_1, ..., \alpha_n Z_n)$ and $(Z_1, ..., Z_n)$ have the same copula. Thus, if $(Y_1, ..., Y_n)$ and $(Z_1, ..., Z_n)$ have the same PDS copula, and $Y_k \leq_{cx} Z_k$ for $k = 1, ..., n$, then by Corollary 3.2.7, we have $\sum_{k=1}^n \alpha_k Y_k \leq_{cx} \sum_{k=1}^n \alpha_k Z_k$ since $Y_k \leq_{cx} Z_k \Longrightarrow \alpha_k Y_k \leq_{cx} \alpha_k Z_k$ for $k = 1, ..., n$. Hence, Corollary 3.2.7 extends Corollary 3.12.15 of Müeller and Stoyan (2002) about the preservation of the convex order under non-negative linear combinations of CI random variables since CI $\Longrightarrow$ PDS. □

Similar as in Theorem 3.2.6, we have an analogous result for PDUO random vector below.

Theorem 3.2.9 Let $(X_1, ..., X_n)$ be a PDUO random vector, and $f_i, g_i : \mathbb{R}^+ \to \mathbb{R}^+$ be increasing functions such that $f_i(X_i) \leq_{icx} g_i(X_i)$ for $i = 1, ..., n$. Then $\mathbb{E}\left[\prod_{i=1}^n f_k(X_k)\right] \leq \mathbb{E}\left[\prod_{i=1}^n g_k(X_k)\right]$.

Proof. It is sufficient to show that $\mathbb{E}\left[\prod_{i=1}^n f_i(X_i)\right] \leq \mathbb{E}\left[\prod_{i=1}^{n-1} f_i(X_i) \times g_n(X_n)\right]$. Denote $W = \prod_{i=1}^{n-1} f_i(X_i)$, then $\mathbb{E}[W|X_n] = h(X_n)$ is increasing in X_n by Proposition 2.2.4. Therefore, applying Corollary 3.2.4, we have

$$\begin{aligned} & \mathbb{E}\left[W \times f_n(X_n)\right] = \mathbb{E}\left[\mathbb{E}\left[W \times f_n(X_n)|X_n\right]\right] = \mathbb{E}\left[h(X_n) f_n(X_n)\right] \\ \leq \ & \mathbb{E}\left[h(X_n) g_n(X_n)\right] = \mathbb{E}\left[\mathbb{E}\left[W \times g_n(X_n)|X_n\right]\right] = \mathbb{E}\left[W \times g_n(X_n)\right]. \end{aligned}$$ □

Using Theorem 3.2.6 and 3.2.9, we can show in the following propositions that, in the presence of PDS or PDUO, the optimal reinsurance for the optimization problem 3.1.2 is the individualized excess-of-loss strategy.

Proposition 3.2.10 Assume that random vector $(X_1, \cdots, X_n)$ is PDS, then for any reinsurance policy $I = (I_1, ..., I_n) \in \mathcal{D}_n^p$, there exists retention vector $(d_1, ..., d_n) \in L_n^p$ such that

$$\sum_{i=1}^{n}(X_i \wedge d_i) \leq_{cx} \sum_{i=1}^{n} I_i(X_i),$$

where d_i is determined by $\mathbb{E}[X_i \wedge d_i] = \mathbb{E}[I_i(X_i)]$, $i = 1, ..., n$.

Proof. Since $0 \leq \mathbb{E}[I_k(X_k)] \leq \mathbb{E}[X_k]$ and the function $g(x) = \mathbb{E}[X_k \wedge x]$ is continuous and increasing in $x \in [0, \infty)$ with $g(0) = 0$ and $g(\infty) = \mathbb{E}[X_k]$, there exists $d_k \in [0, \infty]$ such that $g(d_k) = \mathbb{E}[X_k \wedge d_k] = \mathbb{E}[I_k(X_k)]$. Note that $0 \leq I_k(x) \leq x$ for all $x \geq 0$. Thus, according to Lemma 1.2.9, we have $X_k \wedge d_k \leq_{cx} I_k(X_k)$ for $k = 1, \cdots, n$. Therefore, $\sum_{i=1}^{n}(X_i \wedge d_i) \leq_{cx} \sum_{i=1}^{n} I_i(X_i)$ from Theorem 3.2.6. □

Proposition 3.2.11 Assume that random vector $(X_1, \cdots, X_n)$ is PDUO, then for any reinsurance policy $I = (I_1, ..., I_n) \in \mathcal{D}_n^p$, there exists retention vector $(d_1, ..., d_n) \in L_n^p$ such that

$$\mathbb{E}\left[u\left(\sum_{i=1}^{n}(X_i \wedge d_i)\right)\right] \leq \mathbb{E}\left[u\left(\sum_{i=1}^{n} I_i(X_i)\right)\right], \text{ for any } u \in \mathcal{U}_{exp}^{+} \cup \mathcal{U}_{mom},$$

where d_i is determined by $\mathbb{E}[X_i \wedge d_i] = \mathbb{E}[I_i(X_i)]$, $i = 1, ..., n$.

Proof. The conclusion follows immediately from Theorem 3.2.9 if $u \in \mathcal{U}^{+}_{exp}$, i.e., $u(x) = e^{\gamma x}$ with $\gamma > 0$.

In the case that $u \in \mathcal{U}_{mom}$, i.e., $u(x) = x^m$ with $m \in \mathbb{N}$, consider the following set

$$K = \left\{ (k_1, ..., k_n) \middle| \sum_{i=1}^{n} k_i = m, \text{ and } k_1, ..., k_n \in \mathbb{N} \right\}.$$

Since $X_i \wedge d_i \leq_{cx} I_i(X_i)$, then $(X_i \wedge d_i)^{k_i} \leq_{icx} (I_i(X_i))^{k_i}$ for any $k_i \in \mathbb{N}$. From Theorem 3.2.9, we have $\mathbb{E}\left[\prod_{i=1}^{n}(X_i \wedge d_i)^{k_i}\right] \leq \mathbb{E}\left[\prod_{i=1}^{n}(I_i(X_i))^{k_i}\right]$. Therefore,

$$\begin{aligned} \mathbb{E}\left[\left(\sum_{i=1}^{n}(X_i \wedge d_i)\right)^m\right] &= \sum_{(k_1,...,k_n)\in K} \frac{m!}{\prod_{i=1}^{n} k_i!}\mathbb{E}\left[\prod_{i=1}^{n}(X_i \wedge d_i)^{k_i}\right] \\ \leq \quad \sum_{(k_1,...,k_n)\in K} \frac{m!}{\prod_{i=1}^{n} k_i!}\mathbb{E}\left[\prod_{i=1}^{n}(I_i(X_i))^{k_i}\right] &= \mathbb{E}\left[\left(\sum_{i=1}^{n} I_i(X_i)\right)^m\right] \end{aligned}$$ □

Now, we apply the above result to consider the optimal reinsurance in a collective risk model. In this model, we assume that the number of claims in the insurance portfolio of an insurer is a counting random variable N and the amount of claim i is X_i, $i = 1, 2, ...$ and that the reinsurance strategy I_i is applied to claim i for $i = 1, 2, ...$, where $I_i(x)$ satisfies the same conditions assumed in the individual risk model, namely $I_i(x)$ is increasing in $x \geq 0$ and $0 \leq I_i(x) \leq x$ for $i = 1, 2, ...$ In this case, the total retained loss for the insurer is $\sum_{i=1}^{N} I_i(X_i)$.

Proposition 3.2.12 Let $\{X_1, X_2, \cdots\}$ be a sequence of random variables and N be a counting random variable independent of $\{X_1, X_2, \cdots\}$. If for any $n = 2, 3, ...$, the random vector $(X_1, \cdots, X_n)$ is PDS, then for any $I_i(x)$, $i = 1, 2, ...$, there exist $d_i \in [0, \infty]$, $i =$

$1, 2, ...$, such that

$$\sum_{i=1}^{N}(X_i \wedge d_i) \leq_{cx} \sum_{i=1}^{N} I_i(X_i),$$

where d_i is determined by $\mathbb{E}[X_i \wedge d_i] = \mathbb{E}[I_i(X_i)]$, $i = 1, 2, ...$

Proof. According to Proposition 3.2.10, $\sum_{i=1}^{n}(X_i \wedge d_i) \leq_{cx} \sum_{i=1}^{n} I_i(X_i)$ for any fixed n. Thus for any convex function u, we have

$$\mathbb{E}\left[u\left(\sum_{i=1}^{n}(X_i \wedge d_i)\right)\right] \leq \mathbb{E}\left[u\left(\sum_{i=1}^{n} I_i(X_i)\right)\right].$$

Therefore,

$$\begin{aligned} \mathbb{E}\left[u\left(\sum_{i=1}^{N}(X_i \wedge d_i)\right)\right] &= \sum_{n=0}^{\infty} \mathbb{P}\{N = n\}\, \mathbb{E}\left[u\left(\sum_{i=1}^{n}(X_i \wedge d_i)\right)\right] \\ \leq \sum_{n=0}^{\infty} \mathbb{P}\{N = n\}\, \mathbb{E}\left[u\left(\sum_{i=1}^{n} I_i(X_i)\right)\right] &= \mathbb{E}\left[u\left(\sum_{i=1}^{N} I_i(X_i)\right)\right], \end{aligned}$$

which means $\sum_{i=1}^{N}(X_i \wedge d_i) \leq_{cx} \sum_{i=1}^{N} I_i(X_i)$. □

If $X_1, X_2, ...$ are a sequence of independent random variables, then for any $n = 2, 3, ...$, the random vector $(X_1, X_2, \cdots, X_n)$ is PDS. Furthermore, if $X_1, X_2, ...$ are a sequence of comonotonic random variables or there exist a random variable Z and a sequence of increasing functions $\{f_i, i = 1, 2, ...\}$ such that $X_i = f_i(Z), i = 1, 2, ...$, then for any $n = 2, 3, ...$, the random vector $(X_1, X_2, \cdots, X_n)$ is PDS. Propositions 3.2.10 and 3.2.12 mean that the excess-of-loss reinsurance is the optimal strategy for an insurer to minimize certain risk measures of the retained loss.

3.3 Explicit Expressions for the Retentions in the Optimal Individualized Excess-of-loss Strategy

In this section, we illustrate how to derive the explicit expressions for the retentions in the optimal individualized excess-of-loss strategy. In general, it is difficult to derive such expressions due to the complexity of dependent risks. Here, we consider the bivariate case and assume that the company has two lines of business or $n = 2$ in the individual risk model. We assume that X_1 and X_2 are nonnegative random variables with distribution functions F_1 and F_2, respectively.

To avoid tedious arguments, throughout this section, we assume $\overline{F}_1(d_1) = 1 - F_1(d_1) > 0$ and $\overline{F}_2(d_2) = 1 - F_2(d_2) > 0$ for any $d_1, d_2 \in \mathbb{R}$. We will derive the explicit expressions for $(d_1^*, d_2^*) \in L$ such that

$$\mathbb{E}[u\big(X_1 \wedge d_1^* + X_2 \wedge d_2^*\big)] = \inf_{(d_1,d_2)\in L} \mathbb{E}\Big[u\big(X_1 \wedge d_1 + X_2 \wedge d_2\big)\Big], \tag{3.3.1}$$

where

$$L = L_2^p = \left\{(d_1, d_2) \,\middle|\, \int_0^{d_1} \overline{F}_1(x)dx + \int_0^{d_2} \overline{F}_2(x)dx = p > 0,\ \ d_1, d_2 \geq 0\right\}.$$

Moreover, we assume $p < \mathbb{E}[X_1] + \mathbb{E}[X_2]$. Otherwise, if $p \geq \mathbb{E}[X_1] + \mathbb{E}[X_2]$, then $L = \{(\infty, \infty)\}$ or $L = \emptyset$.

To derive the explicit solutions given in Theorems 3.3.4 and 3.3.5, we need the following Lemmas 3.3.1-3.3.3.

Lemma 3.3.1 On the set L, the mapping from d_1 to d_2 is one-to-one. Denote the mapping as $d_2 = L(d_1)$. Then, $L(d_1)$ is continuous, differentiable and strictly decreasing in d_1, with $\frac{\partial d_2}{\partial d_1} = -\frac{\overline{F}_1(d_1)}{\overline{F}_2(d_2)}$. □

Proof. To show that the mapping is one-to-one, it suffices to show that for any (d_1, d_2) and $(d_1', d_2') \in L$, $d_1 = d_1'$ if and only if $d_2 = d_2'$. First assume $d_1 = d_1'$. Recalling that

$$\int_0^{d_1'} \overline{F}_1(x)dx + \int_0^{d_2'} \overline{F}_2(x)dx = \int_0^{d_1} \overline{F}_1(x)dx + \int_0^{d_2} \overline{F}_2(x)dx = p, \tag{3.3.2}$$

we have $\int_0^{d_2'} \overline{F}_2(x)dx = \int_0^{d_2} \overline{F}_2(x)dx$, or $\int_{d_2}^{d_2'} \overline{F}_2(x)dx = 0$, which implies $d_2 = d_2'$ since $\overline{F}_2(s) > 0, \forall x \in \mathbb{R}$. Similarly, $d_2 = d_2'$ implies $d_1 = d_1'$. Therefore, on L, there is a one-to-one mapping from d_1 to d_2.

Differentiating the second equation in (3.3.2) with respect to d_1 on both sides, we have $\overline{F}_1(d_1) + \overline{F}_2(d_2)\frac{\partial d_2}{\partial d_1} = 0$, which implies that $\frac{\partial d_2}{\partial d_1} = -\frac{\overline{F}_1(d_1)}{\overline{F}_2(d_2)} < 0$. Thus $L(d_1)$ is strictly decreasing. □

Lemma 3.3.1 means that the set L is a continuous and strictly decreasing curve in the first quadrant and the inverse function L^{-1} of L is also continuous, differentiable and strictly decreasing.

To avoid tedious discussions, in the following, we further assume $\mathbb{E}[X_1] < p$ and $\mathbb{E}[X_2] < p$. Thus, both limits of $\lim_{d_2 \to \infty} L^{-1}(d_2)$ and $\lim_{d_1 \to \infty} L(d_1)$ exist on the set L. We denote by $\underline{d}_1 = \lim_{d_2 \to \infty} L^{-1}(d_2)$ and $\underline{d}_2 = \lim_{d_1 \to \infty} L(d_1)$. Therefore, $(\underline{d}_1, \infty)$ is the domain of the function $L(d_1)$ with $\lim_{d_1 \downarrow \underline{d}_1} L(d_1) = \infty$ and $\underline{d}_2 = \lim_{d_1 \to \infty} L(d_1)$. Furthermore, on the set L, $d_1 \downarrow \underline{d}_1 \Longleftrightarrow d_2 \to \infty$.

In the following, we denote

$$M(d_1, d_2) = \mathbb{E}\,[u(X_1 \wedge d_1 + X_2 \wedge d_2)], \quad (d_1, d_2) \in L.$$

Note that $M(d_1, d_2) = M(d_1, L(d_1))$ is a univariate function of d_1 on the set L.

Lemma 3.3.2 Let function u be continuous and monotonic such that $\mathbb{E}\big[|u(X_1 + X_2)|\big] < \infty$. Then $M(d_1, d_2) = M(d_1, L(d_1))$ is continuous in $d_1 \in (\underline{d}_1,\, \infty)$ with

$$\lim_{d_1 \to \infty} M(d_1, L(d_1)) = M(\infty,\, \underline{d}_2) = \mathbb{E}\,[u(X_1 + X_2 \wedge \underline{d}_2)].$$

and

$$\lim_{d_1 \downarrow \underline{d}_1} M(d_1, L(d_1)) = \lim_{d_2 \to \infty} M(L^{-1}(d_2), d_2) = M(\underline{d}_1,\, \infty) = \mathbb{E}\,[u(X_1 \wedge \underline{d}_1 + X_2)].$$

Proof. Since $u(x)$ is monotonic, and $X_1, X_2 \geq 0$, then $|u(X_1 \wedge d_1 + X_2 \wedge d_2)|$ is bounded from above by either $|u(0)|$ or $|u(X_1 + X_2)|$, both of which are integrable. Therefore, according to Lebesgue dominated convergence theorem, for any $d_1 \in (\underline{d}_1,\, \infty)$, we have

$$\begin{aligned}\lim_{s \to d_1} M(s, L(s)) &= \mathbb{E}[\lim_{s \to d_1} u(X_1 \wedge s + X_2 \wedge L(s))] \\ &= \mathbb{E}[u(X_1 \wedge d_1 + X_2 \wedge L(d_1))] = M(d_1, L(d_1)),\end{aligned}$$

which means that $M(s, L(s))$ is continuous at d_1.

Similarly,

$$
\begin{aligned}
\lim_{d_1\to\infty} M(d_1, L(d_1)) &= \lim_{d_1\to\infty} \mathbb{E}[u(X_1\wedge d_1 + X_2 \wedge L(d_1))] \\
&= \mathbb{E}[\lim_{d_1\to\infty} u(X_1\wedge d_1 + X_2 \wedge L(d_1))] \\
&= \mathbb{E}[u(X_1 + X_2\wedge L(\infty))] = \mathbb{E}[u(X_1 + X_2 \wedge \underline{d}_2)],
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{d_1\downarrow \underline{d}_1} M(d_1, L(d_1)) &= \lim_{d_2\to\infty} M(L^{-1}(d_2), d_2) \\
&= \lim_{d_2\to\infty} \mathbb{E}[u(X_1\wedge L^{-1}(d_2) + X_2\wedge d_2)] \\
&= \mathbb{E}[\lim_{d_2\to\infty} u(X_1\wedge L^{-1}(d_2) + X_2\wedge d_2)] \\
&= \mathbb{E}[u(X_1\wedge L^{-1}(\infty) + X_2)] = \mathbb{E}[u(X_1\wedge \underline{d}_1 + X_2)].
\end{aligned}
$$

□

Lemma 3.3.3 Assume that $u(x) \in C^1(\mathbb{R})$, i.e., $u'(x)$ is continuous on $\mathbb{R}$. Then the right derivative $\frac{\partial^+}{\partial d_1} M(d_1, d_2)$ is right continuous in $d_1 \in (\underline{d}_1,\ \infty)$ and

$$
\begin{aligned}
\frac{\partial^+}{\partial d_1} M(d_1, d_2) = \overline{F}_1(d_1)\Big(&\mathbb{E}[u'(d_1 + X_2\wedge d_2)\,|\,X_1 > d_1] \\
&-\mathbb{E}[u'(d_2 + X_1\wedge d_1)\,|\,X_2 > d_2]\Big).
\end{aligned}
\tag{3.3.3}
$$

□

Proof. Denote $f(\omega, s) = u(X_1(\omega)\wedge s + X_2(\omega)\wedge L(s))$, then $M(d_1, d_2) = \mathbb{E}\left[f(\omega, d_1)\right] = \int_\Omega f(\omega, s)\mathbb{P}(d\omega)$. Note that for any fixed $\omega \in \Omega$, the right derivative of $f(\omega, s)$ with respect to s exists for any $s \in (\underline{d}_1, \infty)$ and

$$\frac{\partial^+}{\partial s} f(\omega, s) = u'(X_1 \wedge s + X_2 \wedge L(s)) \left(\mathbb{I}\{X_1 > s\} + \mathbb{I}\{X_2 > L(s)\} \frac{\partial L(s)}{\partial s} \right).$$

Let $[a, d_1] \subset (\underline{d}_1, \infty)$, then for any $(\omega, s) \in \Omega \times [a, d_1]$, we have $0 \leq X_1 \wedge s + X_2 \wedge L(s) \leq s + L(s) \leq d_1 + L(a) < \infty$, since $L(s)$ is decreasing. Therefore $u'(X_1 \wedge s + X_2 \wedge L(s))$ is bounded on $\Omega \times [a, d_1]$ since $u'(x)$ is continuous and thus bounded on the closed interval $[0,\ d_1 + L(a)]$. Also, by Lemma 3.3.1, we have

$$\left| \mathbb{I}\{X_1 > s\} + \mathbb{I}\{X_2 > L(s)\} \frac{\partial L(s)}{\partial s} \right| \leq 1 + \left| \frac{\partial L(s)}{\partial s} \right| = 1 + \frac{\overline{F}_1(s)}{\overline{F}_2(L(s))} \leq 1 + \frac{\overline{F}_1(a)}{\overline{F}_2(L(a))} < \infty.$$

Therefore, $\frac{\partial^+}{\partial s} f(\omega, s)$ is bounded on $\Omega \times [a, d_1]$. Denote the bound as A, then

$$\int_a^{d_1} \mathbb{E}\left[\left| \frac{\partial^+}{\partial s} f(\omega, s) \right| \right] ds \leq A\,(d_1 - a) < \infty.$$

According to Fubini's theorem, we can exchange the order of integration and expectation:

$$\int_a^{d_1} \mathbb{E}\left[\frac{\partial^+}{\partial s} f(\omega, s) \right] ds = \mathbb{E}\left[\int_a^{d_1} \frac{\partial^+}{\partial s} f(\omega, s) ds \right].$$

For any fixed $\omega \in \Omega$, it is easy to verify that $u(x)$ and $g(s) = X_1(\omega) \wedge s + X_2(\omega) \wedge L(s)$ satisfies Lipschitz condition on $[0,\ d_1 + L(a)]$ and on $[a,\ d_1]$ respectively. Therefore $f(\omega, s) = u \circ g(s)$ also satisfies Lipschitz condition on $[a, d_1]$, and thus is absolute continuous on $[a, d_1]$. Then $f(\omega, s)$ is differentiable with respect to s almost everywhere on $[a, d_1]$, and the derivative is equal to the right derivative. By Fundamental Theorem II of Lebesgue integral, we have

$$\int_a^{d_1} \frac{\partial^+}{\partial s} f(\omega, s) ds = \int_a^{d_1} \frac{\partial}{\partial s} f(\omega, s) ds = f(\omega, d_1) - f(\omega, a).$$

Therefore,

$$\begin{aligned} & \int_a^{d_1} \mathbb{E}\left[\frac{\partial^+}{\partial s} f(\omega, s)\right] ds = \mathbb{E}\left[\int_a^{d_1} \frac{\partial^+}{\partial s} f(\omega, s) ds\right] \\ = \ & \mathbb{E}[f(\omega, d_1) - f(\omega, a)] = M(d_1, d_2) - \mathbb{E}\left[f(\omega, a)\right]. \end{aligned} \tag{3.3.4}$$

Since $\frac{\partial^+}{\partial s} f(\omega, s)$ is right continuous in s and is bounded on $[a, d_1]$, according to Lebesgue dominated convergence theorem, we have $\mathbb{E}\left[\frac{\partial^+}{\partial s} f(\omega, s)\right]$ is right continuous in s.

It is easy to show that if $g(x)$ is right continuous and integrable on closed interval I and $G(x) = \int_a^x g(t)dt$, where $a \in I$, then $\frac{\partial^+}{\partial x} G(x) = g(x), \forall x \in I$. Thus, taking right derivative on both sides of (5.2.5), we get

$$\begin{aligned} \frac{\partial^+}{\partial d_1} M(d_1, d_2) &= \frac{\partial^+}{\partial d_1} \int_a^{d_1} \mathbb{E}\left[\frac{\partial^+}{\partial s} f(\omega, s)\right] ds = \mathbb{E}\left[\frac{\partial^+}{\partial d_1} f(X, d_1)\right] \qquad (3.3.5) \\ &= \mathbb{E}\left[u'(X_1 \wedge d_1 + X_2 \wedge d_2)\left(\mathbb{I}\{X_1 > d_1\} + \mathbb{I}\{X_2 > d_2\} \frac{\partial d_2}{\partial d_1}\right)\right] \\ &= \mathbb{E}[u'(d_1 + X_2 \wedge d_2)\, \mathbb{I}\{X_1 > d_1\}] - \frac{\overline{F}_1(d_1)}{\overline{F}_2(d_2)} \mathbb{E}[u'(X_1 \wedge d_1 + d_2)\, \mathbb{I}\{X_2 > d_2\}] \\ &= \overline{F}_1(d_1)\Big(\mathbb{E}\left[u'(d_1 + X_2 \wedge d_2) \,|\, X_1 > d_1\right] - \mathbb{E}\left[u'(d_2 + X_1 \wedge d_1) \,|\, X_2 > d_2\right]\Big). \end{aligned}$$

The last equality follows from the fact that $\mathbb{E}[X\, \mathbb{I}\{Y \in B\}] = \mathbb{E}[X \,|\, Y \in B]\, \mathbb{P}\{Y \in B\}$ if $\mathbb{P}\{Y \in B\} > 0$. The right continuity of $\frac{\partial^+}{\partial d_1} M(d_1, d_2)$ is from (5.2.6). □

Now, applying the above preliminarily results, we can determine the optimal solutions to the following problems.

$$\mathbb{E}\big[\big(X_1 \wedge d_1^* + X_2 \wedge d_2^*\big)^2\big] = \inf_{(d_1,d_2)\in L} \mathbb{E}\,[(I^{d_1}(X_1) + I^{d_2}(X_2))^2], \tag{3.3.6}$$

$$\mathbb{E}\big[\exp\{s(X_1 \wedge d_1^* + X_2 \wedge d_2^*)\}\big] = \inf_{(d_1,d_2)\in L} \mathbb{E}\,[\exp\{s\,(I^{d_1}(X_1) + I^{d_2}(X_2))\}], \tag{3.3.7}$$

where $s > 0$ is fixed.

Theorem 3.3.4 Assume that (X_1, X_2) is PDS and $\mathbb{E}\big[(X_1+X_2)^2\big] < \infty$. For $d_1 \in (\underline{d}_1, \infty)$, define

$$C_1(d_1) = \mathbb{E}[(X_2 - L(d_1)) \wedge 0 \,|\, X_1 > d_1] - \mathbb{E}[(X_1 - d_1) \wedge 0 \,|\, X_2 > L(d_1)].$$

Denote $r_1 = \sup\{d_1 \,|\, C_1(d_1) < 0\}$ and $r_2 = \inf\{d_1 \,|\, C_1(d_1) > 0\}$. Then $\underline{d}_1 < r_1 \le r_2 < \infty$ and for any $d_1^* \in [r_1,\, r_2]$, the retention vector $(d_1^*,\, L(d_1^*))$ is a solution to (3.3.6).

Proof. By setting $u(x) = x^2$ in (5.2.4) and noticing $d_2 = L(d_1)$, we have

$$\begin{aligned}
&\frac{\partial^+ M(d_1, d_2)}{\partial d_1} \\
= {}& 2\,\overline{F}_1(d_1)\Big(\mathbb{E}[(d_1 + X_2 \wedge d_2) \,|\, X_1 > d_1]\,\mathbb{E}[(d_2 + X_1 \wedge d_1) \,|\, X_2 > d_2]\Big) \\
= {}& 2\,\overline{F}_1(d_1)\Big(\mathbb{E}[(d_1 + X_2 \wedge d_2) - (d_1 + d_2) \,|\, X_1 > d_1] \\
&\qquad - \mathbb{E}[(d_2 + X_1 \wedge d_1) - (d_1 + d_2) \,|\, X_2 > d_2]\Big) \\
= {}& 2\,\overline{F}_1(d_1)\Big(\mathbb{E}[(X_2 - d_2) \wedge 0 \,|\, X_1 > d_1] - \mathbb{E}[(X_1 - d_1) \wedge 0 \,|\, X_2 > d_2]\Big) \\
= {}& 2\,\overline{F}_1(d_1)\, C_1(d_1).
\end{aligned} \tag{3.3.8}$$

Now we show that $C_1(d_1)$ is an increasing function of d_1 in $(\underline{d}_1, \infty)$. In doing so, let $d_1, d_1' \in (\underline{d}_1,\, \infty)$ and $d_1 < d_1'$. Since $X_2 \uparrow_{SI} X_1$, we have $X_2 | (X_1 > d_1) \le_{st} X_2 | (X_1 > d_1')$, see, for example, Barlow and Proschan (1981). Therefore, since $(x-L(d_1))\wedge 0$ is increasing in x and $L(d_1) > L(d_1')$, by the definition of $\leq_{st}$, we have

$$\begin{aligned} \mathbb{E}[(X_2 - L(d_1)) \wedge 0 \,|\, X_1 > d_1] &\leq \mathbb{E}[(X_2 - L(d_1)) \wedge 0 \,|\, X_1 > d_1'] \\ &\leq \mathbb{E}[(X_2 - L(d_1')) \wedge 0 \,|\, X_1 > d_1'], \end{aligned} \tag{3.3.9}$$

which means that $\mathbb{E}[(X_2-L(d_1))\wedge 0 \,|\, X_1 > d_1]$ is increasing in d_1. Similarly, since $(x-d_1)\wedge 0$ is increasing in x and $d_1' > d_1$, we have

$$\begin{aligned} \mathbb{E}[(X_1 - d_1) \wedge 0 \,|\, X_2 > L(d_1)] &\geq \mathbb{E}[(X_1 - d_1) \wedge 0 \,|\, X_2 > L(d_1')] \\ &\geq \mathbb{E}[(X_1 - d_1') \wedge 0 \,|\, X_2 > L(d_1')]. \end{aligned}$$

Thus $\mathbb{E}[(X_1 - d_1) \wedge 0 \,|\, X_2 > L(d_1)]$ is decreasing in d_1. Therefore $C_1(d_1)$ is increasing in $d_1 \in (\underline{d}_1, \infty)$.

In the following, we examine the limits of $C_1(d_1)$ at two endpoints $\underline{d}_1$ and ∞ of the interval $(\underline{d}_1, \infty)$. For a fixed $d > d_1 > \underline{d}_1$, by (3.3.9), we have

$$\mathbb{E}[(X_2 - L(d_1)) \wedge 0 \,|\, X_1 > d_1] \leq \mathbb{E}[(X_2 - L(d_1)) \wedge 0 \,|\, X_1 > d].$$

Then by the monotone convergence theorem, we have

$$\begin{aligned} &\lim_{d_1 \downarrow \underline{d}_1} \mathbb{E}[(X_2 - L(d_1)) \wedge 0 \,|\, X_1 > d_1] \\ \leq\ &\lim_{d_1 \downarrow \underline{d}_1} \mathbb{E}[(X_2 - L(d_1)) \wedge 0 \,|\, X_1 > d] \\ =\ &\mathbb{E}[\lim_{d_1 \downarrow \underline{d}_1} (X_2 - L(d_1)) \wedge 0 \,|\, X_1 > d] = -\infty, \end{aligned} \tag{3.3.10}$$

where, the first limit exists because $\mathbb{E}\left[(X_2 - L(d_1)) \wedge 0 \,|\, X_1 > d_1\right]$ is an increasing function of d_1 and the last equality follows from the fact that $\lim_{d_1 \downarrow \underline{d}_1} L(d_1) = \infty$.

Since $X_1 \geq 0$, we have $\mathbb{E}[(X_1 - d_1) \wedge 0 \,|\, X_2 > L(d_1)] \geq \mathbb{E}[(-d_1) \wedge 0 \,|\, X_2 > L(d_1)] = -d_1$. Then, $\lim_{d_1 \downarrow \underline{d}_1} \mathbb{E}[(X_1 - d_1) \wedge 0 \,|\, X_2 > L(d_1)] \geq \lim_{d_1 \downarrow \underline{d}_1}(-d_1) = -\underline{d}_1$, which, together with (3.3.10) and the definition of $C_1(d_1)$, implies $\lim_{d_1 \downarrow \underline{d}_1} C_1(d_1) = -\infty$. Thus, there exists $d_1 > \underline{d}_1$ such that $C(d_1) < 0$, which implies $\{d_1 \,|\, C_1(d_1) < 0\} \neq \emptyset$ and $r_1 = \sup\{d_1 \,|\, C_1(d_1) < 0\} > \underline{d}_1$.

Similarly, we have $\lim_{d_1 \uparrow \infty} \mathbb{E}[(X_2 - L(d_1)) \wedge 0 \,|\, X_1 > d_1] \geq -\underline{d}_2$ and $\lim_{d_1 \uparrow \infty} \mathbb{E}[(X_1 - d_1) \wedge 0 \,|\, X_2 > L(d_1)] \leq -\infty$. Therefore, $\lim_{d_1 \uparrow \infty} C_1(d_1) = \infty$ and thus $\{d_1 \,|\, C_1(d_1) > 0\} \neq \emptyset$ and $r_2 = \inf\{d_1 \,|\, C_1(d_1) > 0\} < \infty$.

Since $C_1(d_1)$ is increasing in d_1, for any $x \in \{d_1 \,|\, C_1(d_1) < 0\}$, $y \in \{d_1 \,|\, C_1(d_1) > 0\}$, we have $x < y$, thus $r_1 = \sup\{d_1 \,|\, C_1(d_1) < 0\} \leq \inf\{d_1 \,|\, C_1(d_1) > 0\} = r_2$. According to the definitions of r_1 and r_2, we have $C_1(d_1) < 0$ for all $d_1 \in (\underline{d}_1, r_1)$ and $C_1(d_1) > 0$ for all $d_1 \in (r_2, \infty)$. Moreover, if $d_1 > r_1$, then $C_1(d_1) \geq 0$; and if $d_1 < r_2$, then $C_1(d_1) \leq 0$. Therefore, $C_1(d_1) = 0$ for all $d_1 \in (r_1, r_2)$.

By (3.3.8), we know that $\frac{\partial^+}{\partial d_1} M(d_1, L(d_1)) = 2\,\overline{F}_1(d_1)\,C(d_1)$ has the same sign as $C_1(d_1)$ on $(\underline{d}_1, \infty)$. Hence, $M(d_1, L(d_1))$ is strictly decreasing on $(\underline{d}_1, r_1)$, strictly increasing on (r_2, ∞), and a constant on (r_1, r_2) and thus a constant on $[r_1, r_2]$ since $M(d_1, L(d_1))$ is continuous. Therefore, $\inf_{d_1 \in (\underline{d}_1, \infty)} M(d_1, L(d_1)) = M(d_1^*, L(d_1^*))$ for any $d_1^* \in [r_1, r_2]$. Notice that $M(d_1, L(d_1))$ is continuous in $d_1 \in (\underline{d}_1, \infty)$, strictly decreasing on $(\underline{d}_1, r_1)$, and strictly increasing on (r_2, ∞). Thus, according to Lemma 3.3.2, for any $d_1^* \in [r_1, r_2]$, $M(d_1^*, L(d_1^*)) < \lim_{d_1 \to \infty} M(d_1, L(d_1)) = M(\infty, \underline{d}_2)$ and $M(d_1^*, L(d_1^*)) < \lim_{d_1 \downarrow \underline{d}_1} M(d_1, L(d_1)) = M(\underline{d}_1, \infty)$. Hence, $\inf_{d_1 \in [\underline{d}_1, \infty]} M(d_1, L(d_1)) = M(d_1^*, L(d_1^*))$ for any $d_1^* \in [r_1, r_2]$. It completes the proof of the theorem. $\square$

Theorem 3.3.5 Let $s > 0$ and assume (X_1, X_2) is PDS and $\mathbb{E}\left[\exp\{s\,(X_1 + X_2)\}\right] < \infty$.

For $d_1 \in (\underline{d}_1, \infty)$, let

$$C_2(d_1) = \mathbb{E}[\exp\{s\,(X_2 - L(d_1)) \wedge 0\} \,|\, X_1 > d_1\,] - \mathbb{E}[\exp\{s\,(X_1 - d_1) \wedge 0\} \,|\, X_2 > L(d_1)\,].$$

Denote $r_1 = \sup\{d_1 \,|\, C_2(d_1) < 0\}$ and $r_2 = \inf\{d_1 \,|\, C_2(d_1) > 0\}$. Then we have $\underline{d}_1 < r_1 \leq r_2 < \infty$, the retention vector $(d_1^*,\, L(d_1^*))$ is a solution to (3.3.7)and for any $d_1^* \in [r_1,\, r_2]$.

Proof. By setting $u(x) = e^{sx}$ in (5.2.4) and noticing $d_2 = L(d_1)$, we have

$$\begin{aligned}
\frac{\partial^+ M(d_1, d_2)}{\partial d_1} &= \overline{F}_1(d_1)\Big(\mathbb{E}[s\,\exp\{s\,(X_2 \wedge d_2)\} \,|\, X_1 > d_1\,] \\
&\qquad - \mathbb{E}[s\,\exp\{s\,(X_1 \wedge d_1)\} \,|\, X_2 > d_2]\Big) \\
&= s\,e^{s\,(d_1+d_2)}\,\overline{F}_1(d_1)\Big(\,\mathbb{E}[\exp\{s\,(X_2 - d_2) \wedge 0\} \,|\, X_1 > d_1] \\
&\qquad - \mathbb{E}\,[\exp\{s\,(X_1 - d_1) \wedge 0\} \,|\, X_2 > d_2]\,\Big) \\
&= s\,e^{s\,(d_1+d_2)}\,\overline{F}_1(d_1)\,C_2(d_1).
\end{aligned}$$

Then, using the same arguments as in Theorem 3.3.4, we complete the proof. The details are omitted. □

3.4 Applications in the Collective Risk Model

In the collective risk model, the surplus process of an insurer is modeled as following.

$$U(t) = u + c\,t - \sum_{i=1}^{N(t)} X_i, \tag{3.4.1}$$

where u is the initial surplus, c is the premium rate, $\{X_1, X_2, ...\}$ are claim sizes which are mutually independent and identically distributed as the generic random variable X, and $N(t)$ is a Poisson process independent of $\{X_1, X_2, ...\}$ with the arrival intensity $\lambda > 0$. In addition, we assume that $c > \lambda \mathbb{E}[X]$ to avoid ruin with probability 1.

The collective risk model (3.4.1) has been extensively studied. For more details, see, for example, Gerber and Shiu (1998) and many others. One essential quantity of the model is the ruin probability, which measures the potential sustainability of the insurance company in a long term. In optimization problems, one of our concerns is to minimize the ruin probability. However, it is usually difficult to derive an explicit expression for the ruin probability in general cases. As an alternative, we turn to study the adjustment coefficient γ, which is defined as the the smallest positive root of the following Lundberg equation.

$$\lambda + c\,s - \lambda\,\mathbb{E}\left[e^{s\,X}\right] = 0. \tag{3.4.2}$$

The adjustment coefficient plays an important role in ruin analysis. In the absence of the explicit expression for the ruin probability, the adjustment coefficient provides an upper bound for the ruin probability, which is given by the following Lundberg inequality.

$$\psi(u) = \mathbb{P}\left\{\inf_{t\geq 0} U(t) < 0\right\} \leq e^{-\gamma u}, \qquad u \geq 0.$$

The classical proof for Lundberg inequality involves the technique of induction; details can be found in Klugman et al. (2012). Willmot and Yang (1996) provided a different proof using a martingale approach, which reveals the probabilistic essence of the adjustment coefficient. Their proof uses the fact that $\{e^{-\gamma\,U(t)}, t \geq 0\}$ is a martingale, which actually motivates the definition of the adjustment coefficient. It is worth pointing out that, the Lundberg inequality still holds in the Sparre-Anderson model, only except that the definition of the adjustment coefficient needs to be slightly modified.

In this section, we consider the case that each policy claim has n different sources of risks, modeled by the random vector $\mathbf{X}_i = (X_{i,1}, ..., X_{i,n})$. Assume that random vectors $\{\mathbf{X}_i, i = 1, 2, ...\}$ are mutually independent and identically distributed as the generic random vector $\mathbf{X} = (X_1, ..., X_n)$. The insurer applies the individualized reinsurance strategy $(I_1, ..., I_n)$ to each claim. Then the total retained risk for the k^{th} claim is $S^I_{k,n} = I_1(X_{k,1}) + ... + I_n(X_{k,n})$. Noting that $\{S^I_{k,n}, k = 1, 2, ...\}$ are independent and identically distributed random variables, we use the notation $S^I_n = I_1(X_1) + ... + I_n(X_n)$ to represent their generic random variable. The surplus process after reinsurance is represented by

$$U^I(t) = u + (c - P)\,t - \sum_{k=1}^{N(t)} S^I_{k,n}, \tag{3.4.3}$$

where c is the insurance premium rate and P is the reinsurance premium rate.

Again, we adopt the expectation premium principle to calculate the reinsurance premium, which implies that $\mathbb{E}[S^I_n] = \mathbb{E}[\sum_{i=1}^n I_i(X_i)]$ is a constant, denoted as p. We also assume $c - P > \lambda \mathbb{E}[S^I_n] = \lambda p$ to avoid the trivial case, namely, ruin with probability 1.

For model (3.4.3), our target is to find the optimal reinsurance strategy $I = (I_1, ..., I_n)$ so as to

1. Minimize the ruin probability of the insurer's surplus process, or
2. Maximize the adjustment coefficient of the insurer's surplus process, thereby minimizing the upper bound of the ruin probability.

We use the notations γ^I and $\psi^I(u)$ to denote the adjustment coefficient and the ruin probability in the model (3.4.3). We still adopt the previously defined notations $\mathcal{D}_n^p$ and $\mathcal{D}_n^{p*}$ to represent different admissible strategy classes. With these notations, the optimization problems are formulated as:

$$\min_{I\in\mathcal{D}_n^p} \psi^I(u), \tag{3.4.4}$$

$$\max_{I\in\mathcal{D}_n^p} \gamma^I. \tag{3.4.5}$$

Recalling Pollaczek-Khinchine formula, the ruin probability $\psi^I(u)$ is expressed as

$$\psi^I(u) = \frac{\lambda\,\mathbb{E}\left[S_n^I\right]}{c-P} - \left(1 - \frac{\lambda\,\mathbb{E}\left[S_n^I\right]}{c-P}\right) K^I(u), u \geq 0,$$

where

$$K^I(x) = \sum_{k=1}^{\infty} \left(\frac{\lambda\,\mathbb{E}\left[S_n^I\right]}{c-P}\right)^k \times G_I^{*k}(x), \text{ and } G_I(x) = \frac{\int_0^x \bar{F}_{S_n^I}(y)dy}{\mathbb{E}\left[S_n^I\right]},$$

and $G_I^{*k}(x)$ is the $n-$fold convolution of the function $G(x)$.

Proposition 3.4.1 Consider model (3.4.3), if $(X_1, ..., X_n)$ is PDS, then

$$\min_{I\in D_n^p} \psi^I(u) = \min_{I\in D_n^{p*}} \psi^I(u), \text{ for any } u \geq 0.$$

Proof. According to Proposition 3.2.10, we know that, for any reinsurance strategy $I \in \mathcal{D}_n^p$, there exists $I^* \in \mathcal{D}_n^{p*}$ such that $S_n^{I^*} \leq_{cx} S_n^I$. Noting that $\mathbb{E}\left[S_n^{I^*}\right] = \mathbb{E}\left[S_n^I\right]$, we have $G_{I^*}(x) \geq G_I(x)$ for all $x \geq 0$, and thus $G_{I^*}^{*k}(x) \geq G_I^{*k}(x)$ for all $k \in \mathbb{N}$ and all $x \geq 0$. Recall that $c - P > \lambda\,\mathbb{E}\left[S_n^I\right]$, we have $\psi^{I^*}(u) \leq \psi^I(u)$ for any $u \geq 0$. □

Proposition 3.4.2 Consider model (3.4.3). If $(X_1, ..., X_n)$ is PDUO, then

$$\max_{I \in D_n^p} \gamma^I = \max_{I \in D_n^{p*}} \gamma^I.$$

Proof. For any reinsurance strategy $I = (I_1, ..., I_n) \in \mathcal{D}_n^p$, there exists $I^* = (x \wedge d_1, ..., x \wedge d_n) \in \mathcal{D}_n^{p*}$ such that $\mathbb{E}[X_i \wedge d_i] = \mathbb{E}[I_i(X_i)]$. Then $X_i \wedge d_i \leq_{cx} I_i(X_i)$ for all $i = 1, ..., n$, according to Ohlin's Lemma. For any $s > 0$, define functions $f_i(x) = e^{s\,(x \wedge d_i)}$ and $g_i(x) = e^{s\,I_i(x)}$ for $i = 1, ..., n$. Obviously, $f_i(x), g_i(x), i = 1, ..., n$, are all nonnegative and increasing. Note that $f_i(X_i) = e^{s(X_i \wedge d_i)} \leq_{icx} e^{sI(X_i)} = g_i(X_i)$. Since $(X_1, ..., X_n)$ is PDUO, according to Proposition 3.2.9, we have

$$\mathbb{E}\left[e^{s\,S_n^{I^*}}\right] = \mathbb{E}\left[\prod_{i=1}^n f_i(X_i)\right] \leq \mathbb{E}\left[\prod_{i=1}^n g_i(X_i)\right] = \mathbb{E}\left[e^{s\,S_n^I}\right].$$

Recall that γ^{I^*} and γ^I satisfy Lundberg equation (3.4.2), i.e.,

$$\begin{aligned} \lambda + (c - P)\,\gamma^{I^*} - \lambda\,\mathbb{E}\left[e^{\gamma^{I^*}\,S_n^{I^*}}\right] &= 0, \\ \lambda + (c - P)\,\gamma^I - \lambda\,\mathbb{E}\left[e^{\gamma^I\,S_n^I}\right] &= 0. \end{aligned}$$

Then γ^{I^*} and γ^I can be viewed as smallest (also unique) positive solutions to the following two systems respectively.

$$\begin{cases} y = \lambda + s(c - P) \\ y = \lambda \mathbb{E}\left[e^{s\,S_n^{I^*}}\right] \end{cases}, \qquad \begin{cases} y = \lambda + s(c - P) \\ y = \lambda \mathbb{E}\left[e^{s\,S_n^I}\right] \end{cases}.$$

The relation $\mathbb{E}\left[e^{s\,S_n^{I^*}}\right] \leq \mathbb{E}\left[e^{s\,S_n^I}\right]$ indicates that the curve $\left\{(y, s)\,\middle|\,y = \mathbb{E}\left[e^{s\,S_n^{I^*}}\right]\right\}$ is below the curve $\left\{(y, s)\,\middle|\,y = \mathbb{E}\left[e^{s\,S_n^I}\right]\right\}$. Therefore, the latter curve upward crosses the line

$\{(y,s)\,|y=\lambda+s(c-P)\}$ first, which means $\gamma^{I^*}\geq\gamma^I$. □

Proposition 3.4.1 and Proposition 3.4.2 indicate that, under certain assumptions of positive dependence, the individualized excess-of-loss is the optimal reinsurance form in the sense of minimizing ruin probability or upper bound of ruin probability. Note that ruin probability is merely one characteristic of the ruin time. In order to do more accurate studies, we shall approach the distribution function of the ruin time random variable directly in the following.

Consider the model (3.4.1), the ruin time is defined as

$$\tau(u)=\inf\{t>0:U(t)<0|U(0)=u\}.$$

Note that $\tau(u)\geq 0$ is an improper random variable, and its distribution function

$$\psi(u,t)=\mathbb{P}\{\tau(u)\leq t\}$$

is also referred to as the finite-time ruin probability.

We use the notation $\tau^I(u)$ to denote the ruin time in the model (3.4.3), i.e.,

$$\tau^I(u)=\inf\{t>0:U^I(t)<0|U^I(0)=u\},$$

and use $\psi^I(u,t)$ to denote its distribution function or the finite-time ruin probability. We shall focus on the model with zero initial capital, namely, $u=0$.

In model (3.4.1), in the case that $u=0$, Seal (1978) provided an expression for the survival function of the ruin time $\tau(0)$.

$$\mathbb{P}\{\tau(0) > t\} = \frac{1}{(c-P)t} \int_0^{(c-P)t} \mathbb{P}\left\{\sum_{k=1}^{N(t)} X_k \leq y\right\} d\,y, \qquad t > 0. \tag{3.4.6}$$

In model (3.4.3), assume $(X_1, ..., X_n)$ is PDS. According to Proposition 3.2.12 we have, for any $I \in \mathcal{D}_n^p$, there exists $I^* \in \mathcal{D}_n^{p^*}$ such that $S_n^{I^*} \leq_{cx} S_n^I$. Then $\sum_{k=1}^{N(t)} S_{k,n}^{I^*} \leq \sum_{k=1}^{N(t)} S_{k,n}^I$ according to the closure property of $\leq_{cx}$ under independent compounding. Therefore,

$$\begin{aligned} \int_0^{(c-P)t} \mathbb{P}\left\{\sum_{k=1}^{N(t)} S_{k,n}^{I^*} \leq y\right\} d\,y &= \mathbb{E}\left[\left((c-P)t - \sum_{k=1}^{N(t)} S_{k,n}^{I^*}\right)_+\right] \\ \geq \ \mathbb{E}\left[\left((c-P)t - \sum_{k=1}^{N(t)} S_{k,n}^{I}\right)_+\right] &= \int_0^{(c-P)t} \mathbb{P}\left\{\sum_{k=1}^{N(t)} S_{k,n}^{I} \leq y\right\} d\,y, \end{aligned}$$

which, according to (3.4.6), implies that $\mathbb{P}\{\tau^{I^*}(0) > t\} \geq \mathbb{P}\{\tau^I(0) > t\}$ for all $t > 0$, or $\mathbb{P}\{\tau^{I^*}(0) \leq t\} \leq \mathbb{P}\{\tau^I(0) \leq t\}$ for all $t > 0$. Based on the above discussions, we may conclude that the individualized excess-of-loss reinsurance strategy uniformly minimizes the finite-time ruin probability for any time horizontal. Formally, we have

Proposition 3.4.3 Consider model (3.4.3). If $(X_1, ..., X_n)$ is PDS, then

$$\min_{I \in D_n^p} \psi^I(0,t) = \max_{I \in D_n^{p*}} \psi^I(0,t), \text{ for all } t > 0. \qquad \square$$

Essentially, Proposition 3.4.3 indicates that the individualized excess-of-loss strategy maximizes the ruin time $\tau^I(0)$ in the usual stochastic order.

3.5 Optimal Reinsurance with Random Initial Wealth

In this section, we consider the classical optimal reinsurance problem. Let X be the insurer's original risk. If applying the reinsurance strategy I, the insurer's retained risk would be $I(X)$. The insurer's objective is to find the optimal reinsurance strategy I so as to minimize the risk measure of the retained risk. Traditionally, we employ the form $\rho(X) = \mathbb{E}[u(X)]$ to measure the risk X, where $u(x)$ is (increasing) convex. Then the optimization problem is stated as $\min_I \mathbb{E}[u(I(x))]$.

There is a different way to interpret the optimization problem. Assume that insurer's initial wealth is w. Then insurer's total wealth after reinsurance arrangement is $w-I(X)-P$, where P is the reinsurance premium which is assumed to be a constant. Another optimization criterion is to maximize the expected utility of the insurer's total wealth. The optimization problem is expressed as $\max_I \mathbb{E}[v(w-p-I(X))]$, where $v(x)$ is a utility function, which is usually assumed to be increasing and concave.

Define $u^*(x) = -v(w-P-x)$. Obviously, $u^*(x)$ is increasing and convex. We have

$$\max_I \mathbb{E}[v(w-P-I(X))] \Longleftrightarrow \min_I \mathbb{E}[u^*(I(X))].$$

In this sense, minimizing the insurer's retained risk and maximizing expected utility of the insurer's total wealth are equivalent. However, it has to be pointed out that, the equivalence is based on the fact that the initial wealth w is a constant, which is not usually the case in practice. In practice, the initial wealth is usually related to the potential risk. In this section, we shall study the optimal reinsurance model with a random initial wealth. We denote the initial wealth by a random variable W and we only focus on single risk model.

The optimal reinsurance problem has been studied by Hong et al. (2011) and references therein. Most of the studies work on certain optimal reinsurance form (for example, excess-of-loss or quota share reinsurance) and try to identify the optimal parameters. In this section, under the dependence structure of PDS, we identify the optimal reinsurance form in the first place.

We formulate the optimization problem as follows.

$$\max_{I \in \mathcal{D}_1^p} \mathbb{E}[v(W - P - I(X))], \qquad \text{for all increasing concave function } v,$$

or

$$\min_{I \in \mathcal{D}_1^p} \mathbb{E}[u(I(X) - W)], \qquad \text{for all } u \in \mathcal{U}_{icx}, \tag{3.5.1}$$

where $\mathcal{D}_1^p$ is the admissible strategy class defined by

$$\mathcal{D}_1^p = \left\{ I(x) \;\middle|\; \begin{array}{l} I(x) \text{ is increasing in } x \geq 0; \\ 0 \leq I(x) \leq x \text{ and } \mathbb{E}[I(X)] = p > 0; \\ |I(x_1) - I(x_2)| \leq |x_1 - x_2| \text{ for any } x_1, x_2 \geq 0. \end{array} \right\}. \tag{3.5.2}$$

Note that, compared to the admissible strategy class $\mathcal{D}_n^p$ defined by (3.1.1), $\mathcal{D}_1^p$ adds the assumption of $|I(x_1) - I(x_2)| \leq |x_1 - x_2|$. This assumption is called slow growth assumption, which means that the retained risk is not supposed to increase faster than the original risk. This assumption actually implies that the ceded function $\bar{I}(x) = x - I(x)$ is also increasing.

Proposition 3.5.1 If $-W \uparrow_{SI} X$, then the optimal solution to (3.5.1) is $I^* = x \wedge d^*$, where d^* is determined by $\mathbb{E}[X \wedge d^*] = p$.

Proof. For any $I \in \mathcal{D}_1^p$, we know that $I^*(X) \leq_{cx} I(X)$. Since $-W \uparrow_{SI} X$, according to Theorem 3.2.6, we have $E[u(I^*(X) - W)] \leq_{cx} E[u(I(X) - W)]$ for all $u \in \mathcal{U}_{cx}$. □

Proposition 3.5.2 If $(W - X) \uparrow_{SI} X$, then the optimal solution to (3.5.1) is $I^* = (x - d^*)_+$, where d^* is determined by $\mathbb{E}[(X - d^*)_+] = p$.

Proof. Denote $W' = W - X$, $\bar{I}(x) = x - I(x)$, then

$$\mathbb{E}\left[u(I(X) - W)\right] = \mathbb{E}\left[u(-\bar{I}(X) - W')\right] = \mathbb{E}\left[v(\bar{I}(X) + W')\right],$$

where $v(x) = u(-x)$ is also a convex function.

Noting that $\bar{I}^*(X) = X \wedge d^* \leq \bar{I}(X)$, and $\bar{I}(x), \bar{I}^*(x)$ are increasing, according to Theorem 3.2.6, we have $\mathbb{E}\left[v(\bar{I}^*(X) + W')\right] \leq \mathbb{E}\left[v(\bar{I}(X) + W')\right]$. □

Proposition 3.5.1 and Proposition 3.5.2 respectively give the optimal reinsurance form to Problem (3.5.1) when the initial wealth W and the risk X are negative dependent or "strongly" positive dependent. The case that W and X are "regularly" positive dependent still needs to be studied.

Chapter 4

Dependence Notions through Arrangement Increasing Functions

Chapter 3 has studied the optimal reinsurance problem with multiple risks. With certain assumptions of positive dependence between the risks, the individualized excess-of-loss form was proved to be optimal. Therefore, the infinite dimensional optimization problem $\min_{I\in\mathcal{D}_n^p}\mathbb{E}[u(\sum_{k=1}^n I_k(X_k))]$ is reduced to a finite dimensional optimization problem $\min_{I\in\mathcal{D}_n^{p*}}\mathbb{E}[u(\sum_{k=1}^n I_k(X_k))]$. The conversion of the problem only completes the first step: identifying the optimal reinsurance form. To completely solve the optimal reinsurance problem, we need to determine the parameters of the reinsurance form.

Section 3.3 investigates two optimization problems with bivariate risks and specific risk measures, and derives explicit solutions. However, when multivariate risks and general risk measures are involved, it is difficult to derive explicit solutions. Alternatively, many studies turn to analyze the quantitative properties of the optimal solutions; see, for example Cheung (2007), Zhuang et al. (2009), and Hu and Wang (2010). In the literature, only a few special dependence structures have been considered. This limitation motivates us to develop more general dependence structures. In this chapter, we first revisit the notion of joint likelihood ratio order and its multivariate version proposed by Shanthikumar and Yao (1991) and derive new characterizations and properties for these two notions. Furthermore, we propose new dependence notions of UOAI, LOAI, TDAI and CTDAI, and systematically develop their properties.

4.1 Preliminaries

In the literature, the following three dependence structures are commonly used in the studies of optimal allocation problems.

(A1) $X_1, ..., X_n$ are comonotonic and $X_1 \leq_{st} ... \leq_{st} X_n$;

(A2) $X_1, ..., X_n$ are mutually independent and $X_1 \leq_{hr} ... \leq_{hr} X_n$;

(A3) $X_1, ..., X_n$ are mutually independent and $X_1 \leq_{lr} ... \leq_{lr} X_n$.

Note that these dependence structures can be rephrased as $(X_1, ..., X_n)$ has a comonotonic/independent copula with specially ordered marginal distributions. In light of this observation, we want to develop new dependence structures which involve properties of both copulas and marginal distributions.

To get some inspirations, we first recall some concepts of bivariate stochastic orders proposed by Shanthikumar and Yao (1991).

Definition 4.1.1 Let X and Y be two random variables. X is said to be smaller than Y in the *joint hazard rate order*, denoted as $X \leq_{hr:j} Y$, if

$$\frac{\partial}{\partial x}\bar{F}(x,y) \leq \frac{\partial}{\partial x}\bar{F}(y,x), \qquad \text{for all } x \leq y,$$

where $\bar{F}(x,y)$ is the joint survival function of the random vector (X, Y).

X is said to be smaller than Y in the *joint likelihood ratio order*, denoted as $X \leq_{lr:j} Y$, if

$$f(x,y) \leq f(y,x), \qquad \text{for all } x \leq y,$$

where $f(x,y)$ is the joint density function or joint probability function of (X, Y).

In Definition 4.1.1, the existence of the joint density function and the existence of the partial derivatives of the joint survival function are assumed. As a matter of fact, these assumptions are not essential. Shanthikumar and Yao (1991), Righter and Shanthikumar (1992) and Aly and Kochar (1993) have developed some functional characterizations to avoid these assumptions. We restate some of the functional characterizations later in Section 4.3 and Section 4.4.

In Shanthikumar and Yao (1991), the joint hazard rate order and the joint likelihood ratio order are referred as bivariate stochastic orders. Accordingly, we refer to the stochastic orders defined in Section 1.2 ($\leq_{st}, \leq_{hr}, \leq_{lr}$) as univariate stochastic orders. Our main concern is how to generalize these bivariate stochastic orders into multivariate cases. The difficulty lies in the fact that, unlike univariate stochastic orders, the bivariate stochastic orders are not transitive. For instance, $X_1 \leq_{lr} X_2$ and $X_2 \leq_{lr} X_3$ imply $X_1 \leq_{lr} X_3$. However, it is not the case for bivariate stochastic orders, namely, $X_1 \leq_{lr:j} X_2$ and $X_2 \leq_{lr:j}$

X_3 do not necessarily imply $X_1 \leq_{lr:j} X_3$, as shown by the following example.

Example 4.1.2 *The joint likelihood ratio order $\leq_{lr:j}$ is not transitive.*

Let (X_1, X_3) have the following joint distribution: $p_{00} = p_{11} = 0.1, p_{10} = 0.15, p_{01} = 0.1, p_{02} = 0.5, p_{12} = 0.05$, where $p_{ij} = \mathbb{P}\{X_1 = i, X_3 = j\}$ is the joint mass function.

The marginal probability mass functions of X_1 and X_3 are $p_1(0) = 0.7, p_1(1) = 0.3$ and $p_3(0) = 0.25, p_3(0) = 0.2, p_3(2) = 0.55$. On the other hand, let X_2 be independent of X_1, X_3 and have the mass function $p_2(0) = 0.6, p_2(1) = 0.4$. Then it is easy to verify that $X_1 \leq_{lr:j} X_2$ and $X_2 \leq_{lr:j} X_3$. But $X_1 \not\leq_{lr:j} X_3$ since $p_{01} > p_{10}$. □

Note that, if $X \leq_{hr:j} Y$, then $\int_x^y \frac{\partial}{\partial s}\bar{F}(s, y)ds \leq \int_x^y \frac{\partial}{\partial s}\bar{F}(y, s)ds$ for $x \leq y$, which implies that $\bar{F}(x, y) \geq \bar{F}(y, x)$ for all $x \leq y$. Shaked and Shanthikumar (2007) has shown that $X \leq_{hr} Y$ if and only if $\bar{F}_X(x)\bar{F}_Y(y) \geq \bar{F}_X(y)\bar{F}_Y(x)$ for all $x \leq y$, and $X \leq_{lr} Y$ if and only if $f_X(x)f_Y(y) \geq f_X(y)f_Y(x)$ for all $x \leq y$ given that the density functions exist. Essentially, those stochastic orders are characterized by a common property of the joint survival function or the joint density function, which is arrangement increasing. This observation motivates us to develop general stochastic dependence structures based on arrangement increasing functions.

Before we state the definition of arrangement increasing, we first introduce some notations for convenience. Throughout Chapter 4 and Chapter 5, we refer to the real-valued vector $(x_1, ..., x_n)$ as $\mathbf{x}$, and refer to the random vector $(X_1, ..., X_n)$ as $\mathbf{X}$. Accordingly, $\mathbf{X} > (<)\, \mathbf{x}$ means $X_i > (<)\, x_i$ for all $i = 1, ..., n$. Let $\mathbf{f} = (f_1, ..., f_n)$ be a vector-valued function, then $\mathbf{f}(\mathbf{x})$ represents $(f_1(x_1), ..., f_n(x_n))$. In particular, $\mathbf{X} \wedge \mathbf{x} = (X_1 \wedge x_1, ..., X_n \wedge x_n)$ and $(\mathbf{X} - \mathbf{d})_+ = ((X_1 - d_1)_+, ..., (X_n - d_n)_+)$. We use the notation $\mathbf{a} \cdot \mathbf{b}$ to denote the inner product of two vectors, namely, $\mathbf{a} \cdot \mathbf{b} = \sum_{k=1}^n a_k b_k$ where $\mathbf{a} = (a_1, ..., a_n)$ and $\mathbf{b} = (b_1, ..., b_n)$.

In particular, we use the notation $\mathbf{x} \cdot \mathbf{e}$ to denote the sum of the components of $\mathbf{x}$, i.e., $\sum_{i=1}^{n} x_i$, where $\mathbf{e} = (1, ..., 1)$ has the same dimension of $\mathbf{x}$.

For any set $K = \{i_1, ..., i_k\} \subset \{1, ..., n\}$ where $i_1 < ... < i_k$, we denote by $|K|$ the cardinality of K, i.e., $|K| = k$. We also $\mathbf{x}_K = (x_k, k \in K) = (x_{i_1}, ..., x_{i_k})$. For convenience, the vector $\mathbf{x}$ is also referred to as $(\mathbf{x}_K, \mathbf{x}_{\bar{K}})$, where $\bar{K} = \{1, ..., n\} \setminus K$; in particular, we write $\overline{ij} = \{1, ..., i-1, i+1, ..., j-1, j+1, ..., n\}$. Let $(\pi(1), ..., \pi(n))$ be any permutation of $\{1, ..., n\}$, and we define $\pi(\mathbf{x}) = (x_{\pi(1)}, ..., x_{\pi(n)})$. We are particularly interested in a special class of permutation: transposition π_{ij}, where $\pi_{ij}(k) = k$ for $k \neq i, j$ and $\pi_{ij}(i) = j, \pi(j) = i$. Let A be a subset of $\mathbb{R}^n$, and denote $\pi_{ij}(A) = \{\mathbf{x} : \pi_{ij}(\mathbf{x}) \in A\}$.

Definition 4.1.3 A multivariate function $f(\mathbf{x})$ is said to be arrangement increasing (AI), if $f(\mathbf{x}) \geq f(\pi_{ij}(\mathbf{x}))$ for any $\mathbf{x} \in \mathbb{R}^n$ and any $i < j$ such that $x_i \leq x_j$.

A multivariate function $f(\mathbf{x})$ is said to be arrangement decreasing (DI), if $f(\mathbf{x}) \leq f(\pi_{ij}(\mathbf{x}))$ for any $\mathbf{x} \in \mathbb{R}^n$ and any $i < j$ such that $x_i \leq x_j$. □

Obviously, $f(x_1, ..., x_n)$ is arrangement decreasing if and only if $-f(x_1, ..., x_n)$ is arrangement increasing.

More discussions about arrangement increasing functions can be found in Marshall and Olkin (1979). It is easy to verify the following properties of arrangement increasing functions.

Proposition 4.1.4 Arrangement increasing functions have the following properties.

(i) If $f(\mathbf{x}) = f(x_1, ..., x_n) : \mathbb{R}^n \mapsto \mathbb{R}$ is arrangement increasing, then $g(\mathbf{x}_K) := f(\mathbf{x}_K, \mathbf{x}_{\bar{K}})$ is arrangement increasing for any fixed $\mathbf{x}_{\bar{K}} \in \mathbb{R}^{n-|K|}$.

(ii) If $h_1(x_1, ..., x_n)$ and $h_2(x_1, ..., x_n)$ are nonnegative and arrangement increasing, then $h_1(x_1, ..., x_n) \times h_2(x_1, ..., x_n)$ is also arrangement increasing.

(iii) If $h(x_1, ..., x_n)$ is arrangement increasing, then $g(h(x_1, ..., x_n))$ is arrangement increasing for any increasing function $g(x) : \mathbb{R} \to \mathbb{R}$. □

Proposition 4.1.4 (ii) and (iii) are obvious. As an example of Proposition 4.1.4 (i), consider the linear function $f(x_1, x_2, x_3) = x_1 + 2x_2 + 3x_3$. $f(x_1, x_2, x_3)$ is arrangement increasing. If x_3 is fixed, $g(x_1, x_2) = x_1 + 2x_2 + 3x_3$ is also arrangement increasing.

In order to characterize the bivariate stochastic orders, we first introduce some notations. For any bivariate function $g(x, y)$, denote $\Delta g(x, y) = g(x, y) - g(y, x)$. Define two classes of functions as follows:

$$\begin{aligned} \mathcal{G}_{hr} &= \{g(x, y) : \Delta g(x, y) \text{ is increasing in } y, \forall y \geq x\}, \\ \mathcal{G}_{lr} &= \{g(x, y) : \Delta g(x, y) \geq 0, \forall y \geq x\}. \end{aligned}$$

Shanthikumar and Yao (1991) derived the following results.

(i) $X \leq_{hr:j} Y$ if and only if $\mathbb{E}[g(X, Y)] \geq \mathbb{E}[g(Y, X)]$ for all $g \in \mathcal{G}_{hr}$;

(ii) $X \leq_{lr:j} Y$ if and only if $\mathbb{E}[g(X, Y)] \geq \mathbb{E}[g(Y, X)]$ for all $g \in \mathcal{G}_{lr}$;

(iii) $X \leq_{hr} Y$ if and only if $\mathbb{E}[g(X^*, Y^*)] \geq \mathbb{E}[g(Y^*, X^*)]$ for all $g \in \mathcal{G}_{hr}$, where $X^* =_{st} X, Y^* =_{st} Y$ and X^* is independent of Y^*;

(iv) $X \leq_{lr} Y$ if and only if $\mathbb{E}[g(X^*, Y^*)] \geq \mathbb{E}[g(Y^*, X^*)]$ for all $g \in \mathcal{G}_{lr}$, where $X^* =_{st} X, Y^* =_{st} Y$ and X^* is independent of Y^*.

To generalize the bivariate stochastic orders, we also introduce some classes of multivariate functions. For a multivariate function $g(x_1, ..., x_n)$, for any $1 \le i < j \le n$, denote $\Delta_{ij} g(\mathbf{x}) = g(\mathbf{x}) - g(\pi_{ij}(\mathbf{x}))$. Define

$$\begin{aligned} \mathcal{G}^{ij}_{ctdai} &= \{g(x_1, ..., x_n) : \Delta_{ij} g(x_1, ..., x_n) \text{ is increasing in } x_j \text{ for any } x_j \ge x_i\}, \\ \mathcal{G}^{ij}_{sai} &= \{g(x_1, ..., x_n) : \Delta_{ij} g(x_1, ..., x_n) \ge 0 \text{ for any } x_j \ge x_i\}. \end{aligned}$$

Obviously, $\mathcal{G}^{ij}_{ctdai}$ and $\mathcal{G}^{ij}_{sai}$ are natural generalizations of $\mathcal{G}_{hr}$ and $\mathcal{G}_{lr}$. It is easy to verify that $\mathcal{G}_{lr} \subset \mathcal{G}_{hr}$ and $\mathcal{G}^{ij}_{sai} \subset \mathcal{G}^{ij}_{ctdai}$.

As a matter of fact, a function in $\mathcal{G}^{ij}_{sai}$ can be considered "partially" arrangement increasing, which is a weaker concept than arrangement increasing. For instance, $f(x_1, x_2, x_3) = x_1 + 2x_2 + x_3 \in \mathcal{G}^{12}_{sai}$ is not arrangement increasing. But for a fixed x_3, the bivariate function $g(x_1, x_2) \equiv f(x_1, x_2, x_3)$ is arrangement increasing.

4.2 (Conditionally) Upper Orthant Arrangement Increasing

Definition 4.2.1 Random vector $\mathbf{X} = (X_1, ..., X_n)$ is said to be upper orthant arrangement increasing (UOAI) if its joint survival function $\bar{F}(x_1, ..., x_n)$ is arrangement increasing.

Definition 4.2.2 Random vector $\mathbf{X} = (X_1, ..., X_n)$ is said to be conditionally upper orthant arrangement increasing (CUOAI), if $(X_i, X_j)|\mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}$ is UOAI for any $1 \le i < j \le n$ and any fixed $\mathbf{x}_{\overline{ij}} \in S(\mathbf{X}_{\overline{ij}})$, where $\overline{ij} = \{1, ..., n\} \setminus \{i, j\}$ and $S(\mathbf{X}_{\overline{ij}})$ is the support of the random vector $\mathbf{X}_{\overline{ij}}$.

Obviously, CUOAI and UOAI are equivalent in the bivariate case. In general cases, CUOAI implies UOAI, as shown by the following proposition.

Proposition 4.2.3 If random vector $\mathbf{X} = (X_1, ..., X_n)$ is CUOAI, then $(X_1, ..., X_n)$ is UOAI.

Proof. From Definition 4.2.2, for any $1 \leq i < j \leq n$ and $x_i \leq x_j$, we have

$$\mathbb{P}\{X_i > x_i, X_j > x_j | \mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}\} \geq \mathbb{P}\{X_i > x_j, X_j > x_i | \mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}\},$$

for any $\mathbf{x}_{\overline{ij}} \in S(\mathbf{X}_{\overline{ij}})$, which means

$$\mathbb{E}\left[\mathbb{I}\{X_i > x_i, X_j > x_j\} | \mathbf{X}_{\overline{ij}}\right] \geq_{a.s.} \mathbb{E}\left[\mathbb{I}\{X_i > x_j, X_j > x_i\} | \mathbf{X}_{\overline{ij}}\right].$$

Therefore,

$$\begin{aligned}
& \mathbb{P}\{X_i > x_i, X_j > x_j, \mathbf{X}_{\overline{ij}} > \mathbf{x}_{\overline{ij}}\} \\
= \; & \mathbb{E}\left[\mathbb{I}\{X_i > x_i, X_j > x_j\} \times \mathbb{I}\{\mathbf{X}_{\overline{ij}} > \mathbf{x}_{\overline{ij}}\}\right] \\
= \; & \mathbb{E}\left[\mathbb{E}\left[\mathbb{I}\{X_i > x_i, X_j > x_j\} | \mathbf{X}_{\overline{ij}}\right] \times \mathbb{I}\{\mathbf{X}_{\overline{ij}} > \mathbf{x}_{\overline{ij}}\}\right] \\
\geq \; & \mathbb{E}\left[\mathbb{E}\left[\mathbb{I}\{X_i > x_j, X_j > x_i\} | \mathbf{X}_{\overline{ij}}\right] \times \mathbb{I}\{\mathbf{X}_{\overline{ij}} > \mathbf{x}_{\overline{ij}}\}\right] \\
= \; & \mathbb{P}\{X_i > x_j, X_j > x_i, \mathbf{X}_{\overline{ij}} > \mathbf{x}_{\overline{ij}}\}. \qquad \square
\end{aligned}$$

Below we derive some preliminary properties of the notion of UOAI.

Lemma 4.2.4 If $\mathbf{X} = (X_1, ..., X_n)$ is UOAI, then

$$\begin{aligned} & \mathbb{P}\{X_i > x_i, X_j > x_j, \mathbf{X}_{K_1} \geq \mathbf{x}_{K_1}, \mathbf{X}_{K_2} > \mathbf{x}_{K_2}\} \\ \geq\ & \mathbb{P}\{X_i > x_j, X_j > x_i, \mathbf{X}_{K_1} \geq \mathbf{x}_{K_1}, \mathbf{X}_{K_2} > \mathbf{x}_{K_2}\}, \end{aligned} \tag{4.2.1}$$

$$\begin{aligned} & \mathbb{P}\{X_i \geq x_i, X_j > x_j, \mathbf{X}_{K_1} \geq \mathbf{x}_{K_1}, \mathbf{X}_{K_2} > \mathbf{x}_{K_2}\} \\ \geq\ & \mathbb{P}\{X_i > x_j, X_j \geq x_i, \mathbf{X}_{K_1} \geq \mathbf{x}_{K_1}, \mathbf{X}_{K_2} > \mathbf{x}_{K_2}\}, \end{aligned} \tag{4.2.2}$$

$$\begin{aligned} & \mathbb{P}\{X_i > x_i, X_j \geq x_j, \mathbf{X}_{K_1} \geq \mathbf{x}_{K_1}, \mathbf{X}_{K_2} > \mathbf{x}_{K_2}\} \\ \geq\ & \mathbb{P}\{X_i \geq x_j, X_j > x_i, \mathbf{X}_{K_1} \geq \mathbf{x}_{K_1}, \mathbf{X}_{K_2} > \mathbf{x}_{K_2}\}, \end{aligned} \tag{4.2.3}$$

$$\begin{aligned} & \mathbb{P}\{X_i \geq x_i, X_j \geq x_j, \mathbf{X}_{K_1} \geq \mathbf{x}_{K_1}, \mathbf{X}_{K_2} > \mathbf{x}_{K_2}\} \\ \geq\ & \mathbb{P}\{X_i \geq x_j, X_j \geq x_i, \mathbf{X}_{K_1} \geq \mathbf{x}_{K_1}, \mathbf{X}_{K_2} > \mathbf{x}_{K_2}\}, \end{aligned} \tag{4.2.4}$$

for any $1 \leq i < j \leq n$ and $x_i < x_j$ and any $(\mathbf{x}_{K_1}, \mathbf{x}_{K_2}) \in \mathbb{R}^{n-2}$, where $K_1 \cup K_2 = \{1, ..., n\} \setminus \{i, j\}$ and $K_1 \cap K_2 = \emptyset$.

Proof. It is sufficient to show the above inequalities hold for the case $n = 2$. The first inequality holds obviously. For any $x_1 < x_2$, there exists an increasing series $\{x_1^n\} \subset \mathbb{R}$ such that $\lim_{n\to\infty} x_1^n = x_1$ and thus $x_1^n < x_2$ for all n. Since (X_1, X_2) is UOAI, we have

$$\mathbb{P}\{X_1 > x_1^n, X_2 > x_2\} \geq \mathbb{P}\{X_1 > x_2, X_2 > x_1^n\}. \tag{4.2.5}$$

Denote $A_n = \{X_1 > x_1^n, X_2 > x_2\}, B_n = \{X_1 > x_2, X_2 > x_1^n\}$, then $A_n \supset A_{n+1}, B_n \supset B_{n+1}$ and thus

$$\lim_{n\to\infty} A_n = \cap_{n=1}^{\infty} A_n = \{X_1 \geq x_1, X_2 > x_2\}, \lim_{n\to\infty} B_n = \cap_{n=1}^{\infty} B_n = \{X_1 > x_2, X_2 \geq x_1\}.$$

Let $n \to \infty$ in (4.2.5), according to the continuous property of probability measure, we have

$$\mathbb{P}\{X_1 \geq x_1, X_2 > x_2\} \geq \mathbb{P}\{X_1 > x_2, X_2 \geq x_1\}.$$

Using similar approximation argument, we can prove $\mathbb{P}\{X_1 \geq x_1, X_2 > x_2\} \geq \mathbb{P}\{X_1 > x_2, X_2 \geq x_1\}$ and $\mathbb{P}\{X_1 \geq x_1, X_2 \geq x_2\} \geq \mathbb{P}\{X_1 \geq x_2, X_2 \geq x_1\}$ for any $x_1 < x_2$. $\square$

Proposition 4.2.5 If $(X_1, ..., X_n)$ is UOAI, then $(f(X_1), ..., f(X_n))$ is UOAI for any increasing function $f(x)$.

Proof. The proof is straightforward by combining Lemma 4.2.4 and the fact that $f^{-1}((x, \infty))$ has the form of either $[a_x, \infty)$ or (a_x, ∞) for any increasing function $f(x)$, where $a_x \in \mathbb{R}$ depends on x. $\square$

Proposition 4.2.6 If random vector $\mathbf{X} = (X_1, ..., X_n)$ is UOAI, then

(i) $X_i \leq_{st} X_{i+1}$ for all $1 \leq i \leq n-1$.

(ii) $\mathbf{X}_K = (X_{i_1}, ..., X_{i_k})$ is UOAI for any $K = \{i_1, ..., i_k\}$, where $1 \leq i_1 < ... < i_k \leq n$.

(iii) Let K_1, K_2, K_3 be mutually disjoint subsets of $\{1, ..., n\}$ such that $|K_1| = |K_2|$ and $\max K_1 < \min K_2$, then

$$(X_k, k \in K_1 \cup K_3) \leq_{uo} (X_k, k \in K_2 \cup K_3).$$

Proof. (i) It suffices to show that $X_1 \leq_{st} X_2$.

Since $\bar{F}(x_1, ..., x_n)$ is arrangement increasing, we have

$$\begin{aligned}\mathbb{P}\{X_1 > x\} &= \mathbb{P}\{X_1 > x, X_2 > -\infty, ..., X_n > -\infty\} = \bar{F}(x, -\infty, -\infty, ..., -\infty)\\ &\leq \bar{F}(-\infty, x, -\infty, ..., -\infty) = \mathbb{P}\{X_2 > x\},\end{aligned}$$

which means $X_1 \leq_{st} X_2$.

(ii) Note that

$$\bar{F}_{\mathbf{X}_K}(\mathbf{x}_K) = \mathbb{P}\{\mathbf{X}_K > \mathbf{x}_K\} = \mathbb{P}\{\mathbf{X}_K > \mathbf{x}_K, \mathbf{X}_{\bar{K}} > -\infty\} = \lim_{\mathbf{x}_{\bar{K}} \to -\infty} \bar{F}_{\mathbf{X}}(\mathbf{x}_K, \mathbf{x}_{\bar{K}}).$$

According to Proposition 4.1.4(i), we know that $\bar{F}_{\mathbf{X}}(\mathbf{x}_K, \mathbf{x}_{\bar{K}})$ is arrangement increasing for any fixed $\mathbf{x}_{\bar{K}}$. Therefore, $\bar{F}_{\mathbf{X}_K}(\mathbf{x}_K)$ is arrangement increasing, which means $\mathbf{X}_K$ is UOAI.

(iii) For simplicity, assume $K_1 = \{1\}, K_2 = \{2\}$ and $K_3 = \{3, ..., n\}$, then we need to show that

$$(X_1, X_3, ..., X_n) \leq_{uo} (X_2, X_3, ..., X_n).$$

Since $(X_1, ..., X_n)$ is UOAI, we have for any fixed $x, x_3, ..., x_n \in \mathbb{R}$,

$$\begin{aligned}&\mathbb{P}\{X_1 > x, X_3 > x_3, ..., X_n > x_n\} = \mathbb{P}\{X_1 > x, X_2 > -\infty, X_3 > x_3, ..., X_n > x_n\}\\ \leq\ &\mathbb{P}\{X_1 > -\infty, X_2 > x, X_3 > x_3, ..., X_n > x_n\} = \mathbb{P}\{X_2 > x, X_3 > x_3, ..., X_n > x_n\},\end{aligned}$$

which implies $(X_1, X_3, ..., X_n) \leq_{uo} (X_2, X_3, ..., X_n)$. □

Proposition 4.2.7 If random variables $X_1, ..., X_n$ are mutually independent, then the random vector $(X_1, ..., X_n)$ is UOAI if and only if $X_i \leq_{hr} X_{i+1}$ for all $1 \leq i \leq n-1$. If

$X_1, ..., X_n$ are comonotonic, then $(X_1, ..., X_n)$ is UOAI if and only if $X_i \leq_{st} X_{i+1}$ for all $1 \leq i \leq n-1$.

Proof. Denote by $\bar{F}(\mathbf{x}) = \bar{F}(x_1, ..., x_n)$ the joint survival function of $(X_1, ..., X_n)$, by $\bar{F}_i(x)$ the survival function of $X_i, i = 1, ..., n$. Define $\sup X = \sup\{x : \mathbb{P}\{X > x\} > 0\}$.

(i) Independent case.

The "only if" part, assume $(X_1, ..., X_n)$ is UOAI, thus (X_i, X_{i+1}) is UOAI from Proposition 4.2.6(i), then $\mathbb{P}\{X_i > x_i, X_{i+1} > x_{i+1}\} \geq \mathbb{P}\{X_i > x_{i+1}, X_{i+1} > x_i\}$ for any $x_i \leq x_{i+1} < \sup X_{i+1}$. Noting that X_i and X_{i+1} are independent, we have $\bar{F}_i(x_i)\bar{F}_{i+1}(x_{i+1}) \geq \bar{F}_i(x_{i+1})\bar{F}_{i+1}(x_i)$ for any $x_i \leq x_{i+1} < \sup X_{i+1}$, which means $X_i \leq_{hr} X_{i+1}$.

The "if" part, assume $X_i \leq_{hr} X_{i+1}, i = 1, ..., n-1$. According to Definition 4.2.1, we need to show that $\bar{F}(\mathbf{x})$ is arrangement increasing, i.e.,

$$\bar{F}(\mathbf{x}) \leq \bar{F}(\pi_{ij}(\mathbf{x})) \text{ for any } \mathbf{x} \in \mathbb{R}^n \text{ and } i < j \text{ such that } x_i \geq x_j.$$

We only need to show the inequality for the case $i = 1, j = 2$. Since $X_1 \leq X_2$, we have $\bar{F}_1(x_1)\bar{F}_2(x_2) \leq \bar{F}_1(x_2)\bar{F}_2(x_1)$ for any $x_1 \geq x_2$. Therefore,

$$\bar{F}_1(x_1)\bar{F}_2(x_2) \times \prod_{k=3}^{n} \bar{F}_k(x_k) \leq \bar{F}_1(x_2)\bar{F}_2(x_1) \times \prod_{k=3}^{n} \bar{F}_k(x_k),$$

which implies that $\bar{F}(\mathbf{x}) \leq \bar{F}(\pi_{12}(\mathbf{x}))$ since $X_1, ..., X_n$ are mutually independent.

(ii) Comonotonic case.

The "only if" part holds from Proposition 4.2.6(i).

The "if" part, assume $X_i \leq_{st} X_{i+1}$ for all $1 \leq i \leq n-1$, thus $\bar{F}_i(x_j) \leq \bar{F}_j(x_j)$ for any $1 \leq i < j \leq n$. Then for any $x_i \leq x_j$, we have $\bar{F}_j(x_i) \geq \bar{F}_j(x_j) \geq \bar{F}_i(x_j)$ and

$\bar{F}_i(x_i) \geq \bar{F}_i(x_j)$. Therefore, $\min\{\bar{F}_i(x_i), \bar{F}_j(x_j)\} \geq \bar{F}_i(x_j) = \min\{\bar{F}_i(x_j), \bar{F}_j(x_i)\}$. Noting that $\bar{F}(\mathbf{x}) = \min\{\bar{F}_k(x_k), k = 1, ..., n\}$ due to the comonotonicity of $\{X_1, ..., X_n\}$, we have

$$\begin{aligned}\bar{F}(\mathbf{x}) &= \min\{\bar{F}_k(x_k), k = 1, ..., n\} = \min\{\min\{\bar{F}_i(x_i), \bar{F}_j(x_j)\}, \min\{\bar{F}_k(x_k), k \in \overline{ij}\}\} \\ &\geq \min\{\min\{\bar{F}_i(x_j), \bar{F}_j(x_i)\}, \min\{\bar{F}_k(x_k), k \in \overline{ij}\}\} = \bar{F}(\pi_{ij}(\mathbf{x})).\end{aligned}$$

□

Proposition 4.2.7 constructs two special cases of UOAI random vectors, which demonstrates that UOAI dependence involves not only properties of copulas but also properties of marginal distributions. In the following, we show how to construct more general UOAI/CUOAI random vectors through copulas and specially ordered marginal distributions.

First, we recall an important property of convex (concave) functions. Consider the line determined by two points located on the graph of a convex function. If we fix one point and move another point along the graph to the right, then the slope of the line between the two point increases. This observation is summarized as the following lemma.

Lemma 4.2.8 If $f(t)$ is convex (concave), then the slope function:

$$L_f(t_1, t_2) = \frac{f(t_1) - f(t_2)}{t_1 - t_2}, \qquad t_1 < t_2,$$

is increasing (decreasing) in both t_1 and t_2. □

Definition 4.2.9 A multivariate function $f(\mathbf{x})$ is said to be super-modular, if for any

$\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, it satisfies

$$f(\mathbf{x} \vee \mathbf{y}) + f(\mathbf{x} \wedge \mathbf{y}) \geq f(\mathbf{x}) + f(\mathbf{y}),$$

where $\mathbf{x} \vee \mathbf{y} = \max\{\mathbf{x}, \mathbf{y}\}$ and $\mathbf{x} \wedge \mathbf{y} = \min\{\mathbf{x}, \mathbf{y}\}$. □

Consider a bivariate function $g(x, y)$. A different version of the definition of super-modularity is, $g(x, y)$ is super-modular if and only if $g(x_2, y_2) - g(x_2, y_1) \geq g(x_1, y_2) - g(x_1, y_1)$ for all $x_1 \leq x_2$ and $y_1 \leq y_2$. Detailed discussions about the super-modular function and its applications can be found in Shaked and Shanthikumar (2007) and references therein. Given the existence of cross partial derivatives, $f(x_1, ..., x_n)$ is super-modular if and only if $\frac{\partial^2}{\partial x_i \partial x_j} f(x_1, ..., x_n) \geq 0$ for all $1 \leq i \neq j \leq n$. Obviously, a joint distribution function or a copula is super-modular. Furthermore, we have the following invariant property.

Lemma 4.2.10 Let multivariate function $f(x_1, ..., x_n)$ be increasing in x_i for all $i = 1, ..., n$. If $f(x_1, ..., x_n)$ is super-modular, then $u(f(x_1, ..., x_n))$ is super-modular for any increasing convex function u.

Proof. If suffices to prove the bivariate case.

Assume that $f(x, y)$ is increasing and super-modular. For any $x_1 \leq x_2$ and $y_1 \leq y_2$, denote $f(x_2, y_2) = a$, $f(x_2, y_1) = b$, $f(x_1, y_2) = c$ and $f(x_1, y_1) = d$. Without loss of generality, assume $b \geq c$. Since $f(x, y)$ is increasing and super-modular, we have $a \geq b \geq c \geq d$ and $a - b \geq c - d$. Noting that $u(x)$ is increasing and convex, we have

$$u(a) - u(b) \geq u'^{+}(b)(a - b) \geq u'^{+}(c)(a - b) \geq u'^{+}(c)(a - b) \geq u(c) - u(d),$$

which implies that $u(f(x, y))$ is 2−increasing. □

Lemma 4.2.11 Assume that random variables $X_1, ..., X_n$ with marginal survival functions $\bar{F}_1(x), ..., \bar{F}_n(x)$ satisfy $X_1 \leq_{hr} ... \leq_{hr} X_n$. For a multivariate function $H(u_1, ..., u_n)$, if

(i) $H(u_1, ..., u_n)$ is arrangement decreasing and is increasing in $u_k, k = 1, ..., n$; and

(ii) there exists a strictly increasing function $g(x)$ such that

$$g(H(u_1, ..., u_{k-1}, e^t, u_{k+1}, ..., u_n)) \text{ is concave in } t \in (-\infty, 0],$$

for all $k = 1, ..., n$, and $g(H(u_1, ..., u_n))$ is super-modular,

then $H(\bar{F}_1(x_1), ..., \bar{F}_n(x_n))$ is arrangement increasing.

Proof. Without loss of generality, it is sufficient to show that

$$H(\bar{F}_1(x_1), \bar{F}_2(x_2), ..., \bar{F}_n(x_n)) \geq H(\bar{F}_1(x_2), \bar{F}_2(x_1), ..., \bar{F}_n(x_n)) \text{ for all } x_1 \leq x_2.$$

For strictly increasing function $g(x)$, it is equivalent to

$$g(H(\bar{F}_1(x_1), \bar{F}_2(x_2), ..., \bar{F}_n(x_n))) \geq g(H(\bar{F}_1(x_2), \bar{F}_2(x_1), ..., \bar{F}_n(x_n))), \tag{4.2.6}$$

for all $x_1 \leq x_2$.

For any fixed $x_1, x_3, ..., x_n$, define $q(u) = g(H(\bar{F}_1(x_1), u, ..., \bar{F}_n(x_n)))$, then $q \circ e(t) = q(e^t)$ is increasing and concave in $t \in (-\infty, 0]$. Therefore, $L_{q \circ e}(t_1, t_2)$ is nonnegative and decreasing in t_1, t_2. Note that

$$\begin{aligned}
& g(H(\bar{F}_1(x_1), \bar{F}_2(x_2), ..., \bar{F}_n(x_n))) - g(H(\bar{F}_1(x_1), \bar{F}_1(x_2), ..., \bar{F}_n(x_n))) \\
= \; & q(\bar{F}_2(x_2)) - q(\bar{F}_1(x_2)) = q \circ e(\log(\bar{F}_2(x_2))) - q \circ e(\log(\bar{F}_1(x_2))) \\
= \; & L_{q\circ e}(\log(\bar{F}_2(x_2)), \log(\bar{F}_1(x_2))) \times (\log(\bar{F}_2(x_2)) - \log(\bar{F}_1(x_2))) \\
= \; & L_{q\circ e}(\log(\bar{F}_2(x_2)), \log(\bar{F}_1(x_2))) \times \log\left(\frac{\bar{F}_2(x_2)}{\bar{F}_1(x_2)}\right) \\
\geq \; & L_{q\circ e}(\log(\bar{F}_2(x_1)), \log(\bar{F}_1(x_1))) \times \log\left(\frac{\bar{F}_2(x_1)}{\bar{F}_1(x_1)}\right) \qquad (4.2.7) \\
= \; & g(H(\bar{F}_1(x_1), \bar{F}_2(x_1), ..., \bar{F}_n(x_n))) - g(H(\bar{F}_1(x_1), \bar{F}_1(x_1), ..., \bar{F}_n(x_n))) \\
\geq \; & g(H(\bar{F}_1(x_2), \bar{F}_2(x_1), ..., \bar{F}_n(x_n))) - g(H(\bar{F}_1(x_2), \bar{F}_1(x_1), ..., \bar{F}_n(x_n))). \qquad (4.2.8)
\end{aligned}$$

Inequality (4.2.7) holds because $\bar{F}_k(x_2) \leq \bar{F}_k(x_1)$ for $k = 1, 2$ and $\frac{\bar{F}_2(x_2)}{\bar{F}_1(x_2)} \geq \frac{\bar{F}_2(x_1)}{\bar{F}_1(x_1)}$ since $X_1 \leq_{hr} X_2$. Inequality (4.2.8) holds because $g(H(u_1, u_2, ..., u_n))$ is super-modular.

Recalling that $H(u_1, ..., u_n)$ is arrangement decreasing and $\bar{F}_1(x_2) \leq \bar{F}_1(x_1)$, we have

$$g(H(\bar{F}_1(x_1), \bar{F}_1(x_2), ..., \bar{F}_n(x_n))) \geq g(H(\bar{F}_1(x_2), \bar{F}_1(x_1), ..., \bar{F}_n(x_n))),$$

which implies (4.2.6) according to (4.2.8). □

Proposition 4.2.12 Assume that random variables $X_1, ..., X_n$ are linked by a survival copula $C(u_1, ..., u_n)$. If

(i) $X_k \leq_{hr} X_{k+1}$ for all $k = 1, ..., n-1$,

(ii) $C(u_1, ..., u_n)$ is arrangement decreasing, and

(iii) there exists a strictly increasing convex function $g(x)$ such that

$$g(C(u_1, ..., u_{k-1}, e^t, u_{k+1}, ..., u_n)) \text{ is concave in } t \leq 0 \text{ for any } k = 1, ..., n, \tag{4.2.9}$$

then $(X_1, ..., X_n)$ is UOAI.

Proof. It is easy to verify that the survival copula $C(u_1, ..., u_n)$ satisfies the conditions (ii) and (iii) in Lemma 4.2.11. Therefore, the joint survival function $C(\bar{F}_1(x_1), ..., \bar{F}_n(x_n))$ is arrangement increasing, which implies that $(X_1, ..., X_n)$ is UOAI. □

The following corollary shows that Archimedean copulas with certain restrictions satisfy conditions (ii) and (iii) in Proposition 4.2.12.

Corollary 4.2.13 Assume that random variables $X_1, ..., X_n$ are linked by an Archimedean survival copula $C(u_1, ..., u_n)$ given in (1.4.2). If

(i) $X_k \leq_{hr} X_{k+1}$ for all $k = 1, ..., n-1$, and

(ii) $\Psi(e^t)$ is convex in $t \in (-\infty, 0]$,

then $(X_1, ..., X_n)$ is UOAI.

Proof. First, $C(u_1, ..., u_n)$ is symmetric and thus arrangement decreasing. Second, let $g(x) = -\Psi(x) = -\Lambda^{-1}(x)$, then $g'(x) = -\frac{1}{\Lambda'(\Lambda^{-1}(x))}$. Noting that $\Lambda'(x)$ is negative and increasing, and $\Lambda^{-1}(x) = \Psi(x)$ is decreasing, we have $g'(x)$ is positive and increasing, which means that $g(x)$ is increasing convex. On the other hand, $g(C(u_1, ..., u_n)) = -\sum_{k=1}^{n} \Psi(u_k)$ satisfies condition (4.2.9) given that $\Psi(e^t)$ is convex. Therefore, $(X_1, ..., X_n)$ is UOAI according to Proposition 4.2.12. □

Proposition 4.2.14 Assume that random vector $\mathbf{X} = (X_1, ..., X_n)$ with positive joint density function $f(x_1, ..., x_n)$ is linked by an Archimedean survival copula given in (1.4.2). If $X_k \le_{hr} X_{k+1}$ for $k = 1, ..., n-1$ and $\Psi(e^t)$ is convex in $t \in (-\infty, 0]$, then $\mathbf{X}$ is CUOAI.

Proof. Without loss of generality, it is sufficient to show that $(X_1, X_2)|(X_3, ..., X_n) = (x_3, ..., x_n)$ is UOAI for any fixed $x_3, ..., x_n$, or equivalently

$$\mathbb{P}\{X_1 > x_1, X_2 > x_2|(X_3, ..., X_n) = (x_3, ..., x_n)\}$$

is arrangement increasing.

Note that

$$\begin{aligned}
& \mathbb{P}\{X_1 > x_1, X_2 > x_2|(X_3, ..., X_n) = (x_3, ..., x_n)\} \\
= \; & (-1)^{n-2} \times \frac{\partial^{n-2}}{\partial x_3 \cdots \partial x_n} \mathbb{P}\{X_1 > x_1, ..., X_n > x_n\} \times \frac{1}{f_{(X_3, ..., X_n)}(x_3, ..., x_n)} \\
= \; & (-1)^{n-2} \times \frac{\partial^{n-2}}{\partial x_3 \cdots \partial x_n} C(\bar{F}_1(x_1), ..., \bar{F}_n(x_n)) \times \frac{1}{f_{(X_3, ..., X_n)}(x_3, ..., x_n)} \\
= \; & (-1)^{n-2} \Lambda^{(n-2)} \left(\sum_{k=1}^{n} \Psi(\bar{F}_k(x_k)) \right) \times \prod_{k=3}^{n} (-\Psi'(\bar{F}_k(x_k))) \times \frac{\prod_{k=3}^{n} f_k(x_k)}{f_{(X_3, ..., X_n)}(x_3, ..., x_n)}.
\end{aligned}$$

It is sufficient to show that $(-1)^{n-2}\Lambda^{(n-2)}\left(\sum_{k=1}^{n} \Psi(\bar{F}_k(x_k))\right)$ is arrangement increasing for any fixed $x_3, ..., x_n$.

Consider the function $H(u_1, u_2) = -\Psi(u_1) - \Psi(u_2)$, then $H(u_1, u_2)$ satisfies condition (i) and condition (ii) in Lemma 4.2.11 with $g(x) \equiv x$. Recalling that $X_1 \le_{hr} X_2$, according to Lemma 4.2.11 we have $H(\bar{F}_1(x_1), \bar{F}_2(x_2)) = -\Psi(\bar{F}_1(x_1)) - \Psi(\bar{F}_2(x_2))$ is arrangement increasing. Therefore $(-1)^{n}\Lambda^{(n-2)}\left(\sum_{k=1}^{n} \Psi(\bar{F}_k(x_k))\right)$ is arrangement increasing since $(-1)^{n-2}\Lambda^{(n-2)}(-x)$ is decreasing from Remark 1.4.2. $\square$

There are many well known copulas satisfying the condition that $\Psi(e^t)$ is convex. For instance, Gumbel copula $\Psi(x) = (-\log x)^\alpha$ with $\alpha \geq 1$ and Clayton copula $\Psi(x) = x^{-\theta} - 1$ with $\theta > 0$. Noting that independent copula and comonotonic copula are special Gumbel copulas with $\alpha = 1$ and $\alpha = \infty$, respectively, Proposition 4.2.13 actually includes Proposition 4.2.7 as a special case.

Lemma 4.2.15 Let $g_1(x), ..., g_n(x)$ be differentiable functions such that $g_1(0) = ... = g_n(0) = 0$. Denote by $F(x_1, ..., x_n)$ and $\bar{F}(x_1, ..., x_n)$ the joint distribution function and joint survival function of random vector $(X_1, ..., X_n)$.

(i) If random variables $X_1, ..., X_n$ are nonnegative, then

$$\mathbb{E}\left[\prod_{i=1}^n g_i(X_i)\right] = \int_0^\infty \cdots \int_0^\infty \bar{F}(x_1, ..., x_n) \prod_{i=1}^n g_i'(x_i)\, dx_1 \cdots dx_n,$$

(ii) and if random variables $X_1, ..., X_n$ are non-positive, then

$$\mathbb{E}\left[(-1)^n \prod_{i=1}^n g_i(X_i)\right] = \int_{-\infty}^0 \cdots \int_{-\infty}^0 F(x_1, ..., x_n) \prod_{i=1}^n g_i'(x_i)\, dx_1 \cdots dx_n,$$

provided that the expectations exist.

Proof. (i) Noting that $\bar{F}(x_1, ..., x_n) = \mathbb{E}[\prod_{i=1}^n \mathbb{I}\{x_i < X_i\}]$, according to Fubini's theorem, we have

$$\begin{aligned}
& \int_0^\infty \cdots \int_0^\infty \bar{F}(x_1, ..., x_n) \prod_{i=1}^n g_i'(x_i)\, dx_1 \cdots dx_n \\
= \; & \int_0^\infty \cdots \int_0^\infty \mathbb{E}\left[\prod_{i=1}^n \mathbb{I}\{x_i < X_i\}\right] \prod_{i=1}^n g_i'(x_i)\, dx_1 \cdots dx_n \\
= \; & \mathbb{E}\left[\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^n g_i'(x_i)\mathbb{I}\{x_i < X_i\}\, dx_1 \cdots dx_n\right] \\
= \; & \mathbb{E}\left[\prod_{i=1}^n \int_0^\infty g_i'(x_i)\mathbb{I}\{x_i < X_i\}\, dx_i\right] = \mathbb{E}\left[\prod_{i=1}^n \int_0^{X_i} g_i'(x_i)\, dx_i\right] = \mathbb{E}\left[\prod_{i=1}^n g(X_i)\right].
\end{aligned}$$

The inequality of (ii) can be proved by the same argument. □

Proposition 4.2.16 Let $g_i(x), i = 1, ..., n$, be continuous differentiable increasing functions such that $g_1(0) = ... = g_n(0) \geq 0$. Assume that there exists $1 \leq i < j \leq n$ such that $g_i'(x) \leq g_j'(x)$ and $\prod_{k=1}^n g_k'(x_k) \in \mathcal{G}_{sai}^{ij}$. If $(X_1, ..., X_n)$ is nonnegative and UOAI, then

$$\mathbb{E}[g(X_1, ..., X_n)] \geq \mathbb{E}[g(\pi_{ij}(X_1, ..., X_n))],$$

where $g(x_1, ..., x_n) = \prod_{k=1}^n g_k(x_k)$.

Proof. We only give the proof for the case $i = 1, j = 2$.

Case 1: $g_k(0) = 0$ for all $k = 1, ..., n$. According to Lemma 4.2.15, we have

$$
\begin{aligned}
& \mathbb{E}\left[g_1(X_1)g_2(X_2)\prod_{k=3}^{n} g_k(X_k)\right] = \int_{[0,\infty)^n} \bar{F}(x_1,...,x_n)g_1'(x_1)g_2'(x_2)\prod_{k=3}^{n} g_k'(x_k)\,d\,x_1...d\,x_n \\
= \;& \int_{[0,\infty)^{n-2}}\int_0^\infty\int_0^\infty \bar{F}(x_1,x_2,...,x_n)g_1'(x_1)g_2'(x_2)\prod_{k=3}^{n} g_k'(x_k)\,d\,x_1\,d\,x_2\,d\,x_3...d\,x_n \\
= \;& \int_{[0,\infty)^{n-2}}\int_0^\infty \left(\int_0^{x_2} + \int_{x_2}^\infty\right) \bar{F}(x_1,x_2,...,x_n)g_1'(x_1)g_2'(x_2)\prod_{k=3}^{n} g_k'(x_k)\,d\,x_1\,d\,x_2\,d\,x_3...d\,x_n \\
= \;& \int_{[0,\infty)^{n-2}}\int_0^\infty\int_0^{x_2} \big(\bar{F}(x_1,x_2,...,x_n)g_1'(x_1)g_2'(x_2) + \bar{F}(x_2,x_1,...,x_n)g_1'(x_2)g_2'(x_1)\big) \\
& \qquad \times \prod_{k=3}^{n} g_k'(x_k)\,d\,x_1\,d\,x_2\,d\,x_3...d\,x_n \\
\geq \;& \int_{[0,\infty)^{n-2}}\int_0^\infty\int_0^{x_2} \big(\bar{F}(x_1,x_2,...,x_n)g_1'(x_2)g_2'(x_1) + \bar{F}(x_2,x_1,...,x_n)g_1'(x_1)g_2'(x_2)\big) \\
& \qquad \times \prod_{k=3}^{n} g_k'(x_k)\,d\,x_1\,d\,x_2\,d\,x_3...d\,x_n \qquad (4.2.10) \\
= \;& \int_{[0,\infty)^n} \bar{F}(x_1,...,x_n)g_1'(x_2)g_2'(x_1)\prod_{k=3}^{n} g_k'(x_k)\,d\,x_1...d\,x_n = \mathbb{E}\left[g_1(X_2)g_2(X_1)\prod_{k=3}^{n} g_k(X_k)\right],
\end{aligned}
$$

where Inequality (4.2.10) holds because $\bar{F}(x_1, ..., x_n) \in \mathcal{G}_{sai}^{12}$ and $\prod_{k=1}^{n} g_k'(x_k) \in \mathcal{G}_{sai}^{12}$.

Case 2: $g_k(0) \neq 0$ for all $k = 1, ..., n$. Without loss of generality, assume $g_k(0) = 1$. Define $h_i(x) = g_i(x) - 1$ for $i = 1, ..., n$. Then we have $h_k(0) = 0$ and $g(x_1, ..., x_n) =$

$\prod_{k=1}^{n}(h_k(x_k)+1)$. Note that

$$g(X_1,...,X_n) = (h_1(X_1)+1)(h_2(X_2)+1) \times \prod_{k=3}^{n}(h_k(X_k)+1)$$

$$= (h_1(X_1)h_2(X_2) + h_1(X_1) + h_2(X_2) + 1) \times \sum_{K\subset\{3,...,n\}} \prod_{k\in K} h_k(X_k), \qquad (4.2.11)$$

$$\mathbf{g}(\pi_{12}(X_1,...,X_n)) = (h_1(X_2)+1)(h_2(X_1)+1) \times \prod_{i=3}^{n}(h_k(X_k)+1)$$

$$= (h_1(X_2)h_2(X_1) + h_1(X_2) + h_2(X_1) + 1) \times \sum_{K\subset\{3,...,n\}} \prod_{k\in K} h_k(X_k). \qquad (4.2.12)$$

Following the proof for case 1, it is easy to verify that

$$\mathbb{E}\left[h_1(X_1)h_2(X_2) \times \prod_{k\in K} h_k(X_k)\right] \geq \mathbb{E}\left[h_1(X_2)h_2(X_1) \times \prod_{k\in K} h_k(X_k)\right], \qquad (4.2.13)$$

for any $K \subset \{3,...,n\}$.

On the other hand, according to Proposition 4.2.6 (iii), we have $(X_1, \mathbf{X}_K) \leq_{uo} (X_2, \mathbf{X}_K)$. Noting that $h_2 - h_1$ is nonnegative increasing, from Lemma 1.2.12, we have

$$\mathbb{E}\left[(h_2(X_1) - h_1(X_1)) \times \prod_{i\in K} h_i(X_i)\right] \leq \mathbb{E}\left[(h_2(X_2) - h_1(X_2)) \times \prod_{i\in K} h_i(X_i)\right],$$

which implies

$$\mathbb{E}\left[(h_1(X_1) + h_2(X_2)) \times \prod_{i\in K} h_i(X_i)\right]$$

$$\geq \mathbb{E}\left[(h_1(X_2) + h_2(X_1)) \times \prod_{i\in K} h_i(X_i)\right], \qquad (4.2.14)$$

for any $K \subset \{3, ..., n\}$.

Combining (4.2.11),(4.2.12),(4.2.13) and (4.2.14), the proof is completed. $\square$

Proposition 4.2.16 can be considered as a "quasi" functional characterization of the notion of UOAI. It should be pointed out that, the assumption of differentiability of $g_k(x)$ can be weakened to single sided differentiability, which is to be applied in Chapter 5.

From Proposition 4.2.16, we can draw some conclusion on the joint moment generating function of UOAI random vectors:

Corollary 4.2.17 If $\mathbf{X} = (X_1, ..., X_n)$ is UOAI, then the moment generating function $M_{\mathbf{X}}(t_1, ..., t_n) = \mathbb{E}\left[\prod_{k=1}^n e^{t_k X_k}\right]$ is arrangement increasing in $t_k \geq 0, k = 1, ..., n$.

Proof. It is sufficient to show that

$$\mathbb{E}\left[e^{t_1 X_1} e^{t_2 X_2} \prod_{k=3}^{n} e^{t_k X_k}\right] \geq \mathbb{E}\left[e^{t_1 X_2} e^{t_2 X_1} \prod_{k=3}^{n} e^{t_k X_k}\right], \tag{4.2.15}$$

for any $0 \leq t_1 \leq t_2$ and $t_k \geq 0, k = 3, ..., n$. Define $g_k(x) = e^{t_k x}, k = 1, ..., n$, then $g_k(x), k = 1, ..., n$, satisfies the conditions in Proposition 4.2.16, which implies (4.2.15). $\square$

Corollary 4.2.18 Let $\mathbf{X} = (X_1, ..., X_n)$ be nonnegative UOAI random vector. For any vector $\mathbf{a} = (a_1, ..., a_n) \geq \mathbf{0}$, if there exists $1 \leq i < j \leq n$ such that $a_i \leq a_j$, then $\mathbf{a} \cdot \mathbf{X} \geq_{mgf} \mathbf{a} \cdot \pi_{ij}(\mathbf{X})$. $\square$

From Shaked and Shanthikumar (2007), moment generating function order is implied by moments order. As a matter of fact, the moment generating function order in Corollary 4.2.18 could be strengthened to moments order.

Corollary 4.2.19 Let $\mathbf{X} = (X_1, ..., X_n)$ be nonnegative UOAI random vector. For any $\mathbf{a} = (a_1, ..., a_n) \geq \mathbf{0}$, if there exists $1 \leq i < j \leq n$ such that $a_i \leq a_j$, then $\mathbf{a} \cdot \mathbf{X} \geq_{mom} \mathbf{a} \cdot \pi_{ij}(\mathbf{X})$.

Proof. It is sufficient to show that

$$\mathbb{E}\left[\left(a_1X_1 + a_2X_2 + \sum_{i=3}^{n} a_iX_i\right)^m\right] \geq \mathbb{E}\left[\left(a_1X_2 + a_2X_1 + \sum_{i=3}^{n} a_iX_i\right)^m\right].$$

Note that

$$\left(a_1X_1 + a_2X_2 + \sum_{i=3}^{n} a_iX_i\right)^m = \sum_{j=0}^{m} \binom{m}{j} (a_1X_1 + a_2X_2)^j \left(\sum_{i=3}^{n} a_iX_i\right)^{m-j}.$$

For any j that is odd,

$$(a_1X_1 + a_2X_2)^j = \sum_{l=0}^{[\frac{j}{2}]} \binom{j}{l} \left(a_1^l X_1^l a_2^{j-l} X_2^{j-l} + a_1^{j-l} X_1^{j-l} a_2^l X_2^l\right),$$

for any j that is even

$$(a_1X_1 + a_2X_2)^j = \sum_{l=0}^{[\frac{j}{2}]-1} \binom{j}{l} \left(a_1^l X_1^l a_2^{j-l} X_2^{j-l} + a_1^{j-l} X_1^{j-l} a_2^l X_2^l\right) + \binom{j}{\frac{j}{2}} (a_1X_1a_2X_2)^{j/2}.$$

On the other hand,

$$\left(\sum_{i=3}^{n} a_iX_i\right)^{m-j} = \sum_{(l_3,...,l_n)\in K} \prod_{i=3}^{n} c_i X_i^{l_i},$$

where $K = \{(l_3, ..., l_n) | \sum_{i=3}^{n} l_i \text{ and } l_3, ..., l_n \in \mathbb{N}\}$, and $c_3, ..., c_n$ are nonnegative constants.

Then it is sufficient to show that

$$\begin{aligned} & \mathbb{E}\left[\left(a_1^l X_1^l a_2^{j-l} X_2^{j-l} + a_1^{j-l} X_1^{j-l} a_2^l X_2^l\right) \times \prod_{i=3}^{n} c_i X_i^{l_i}\right] \\ \geq \ & \mathbb{E}\left[\left(a_1^l X_2^l a_2^{j-l} X_1^{j-l} + a_1^{j-l} X_2^{j-l} a_2^l X_1^l\right) \times \prod_{i=3}^{n} c_i X_i^{l_i}\right] \end{aligned}$$

for any $j = 1, ..., n$ and $l \leq j/2$, which is equivalent to,

$$\begin{aligned} & \mathbb{E}\left[(a_1^l a_2^{j-l} - a_1^{j-l} a_2^l) X_1^l X_2^{j-l} \prod_{i=3}^{n} X_i^{l_i}\right] \\ \geq \ & \mathbb{E}\left[(a_1^l a_2^{j-l} - a_1^{j-l} a_2^l) X_1^{j-l} X_2^l \prod_{i=3}^{n} X_i^{l_i}\right]. \end{aligned} \tag{4.2.16}$$

Note that $a_1 \leq a_2$ and $l \leq j - l$. If $l \neq 0$, inequality (4.2.16) holds directly from Proposition 4.2.16; if $l = 0$, inequality (4.2.16) holds from the fact that $(X_2, X_3, .., X_n) \geq_{uo} (X_1, X_3, ..., X_n)$ (according to Proposition 4.2.6 (iii)) and Lemma 1.2.12. □

A parallel concept to UOAI is LOAI, which is defined as follows:

Definition 4.2.20 Random vector $(X_1, ..., X_n)$ is said to be lower orthant arrangement increasing (LOAI), if its joint distribution function $F(x_1, ..., x_n) = \mathbb{P}\{X_1 \leq x_1, ..., X_n \leq x_n\}$ is arrangement increasing.

Lemma 4.2.21 Random vector $(X_1, ..., X_n)$ is LOAI if and only if $(-X_n, ..., -X_1)$ is UOAI.

Proof. We give the proof for the bivariate case for simplicity.

“$\Rightarrow$” Assume that (X_1, X_2) is LOAI. For any $x_1 \le x_2$, we have $-x_2 \le -x_1$, and then

$$\begin{aligned} & \mathbb{P}\{-X_2 \ge x_1, -X_1 \ge x_2\} = \mathbb{P}\{X_1 \le -x_2, X_2 \le -x_1\} \\ \ge \; & \mathbb{P}\{X_1 \le -x_1, X_2 \le -x_2\} = \mathbb{P}\{-X_2 \ge x_2, -X_1 \ge x_1\}. \end{aligned} \tag{4.2.17}$$

Since inequality (4.2.17) holds for any $x_1 \le x_2$, it also holds that

$$\mathbb{P}\{-X_2 \ge x_1 + \delta, -X_1 \ge x_2 + \delta\} \ge \mathbb{P}\{-X_2 \ge x_2 + \delta, -X_1 \ge x_1 + \delta\}$$

for any $\delta > 0$. Taking the limit of $\delta \downarrow 0$, we have

$$\mathbb{P}\{-X_2 > x_1, -X_1 > x_2\} \ge \mathbb{P}\{-X_2 > x_2, -X_1 > x_1\},$$

which means $(-X_2, -X_1)$ is UOAI.

“$\Leftarrow$” Assume that $(-X_2, -X_1)$ is UOAI. For any $x_1 \le x_2$, we have

$$\begin{aligned} & \mathbb{P}\{X_1 < x_1 + \delta, X_2 < x_2 + \delta\} = \mathbb{P}\{-X_2 > -x_2 - \delta, -X_1 > -x_1 - \delta\} \\ \ge \; & \mathbb{P}\{-X_2 > -x_1 - \delta, -X_1 > -x_2 - \delta\} = \mathbb{P}\{X_1 < x_2 + \delta, X_2 < x_1 + \delta\}, \end{aligned}$$

for any $\delta > 0$. Let $\delta \downarrow 0$, we get

$$\mathbb{P}\{X_1 \le x_1, X_2 \le x_2\} \ge \mathbb{P}\{X_1 \le x_2, X_2 \le x_1\},$$

for any $x_1 \le x_2$, i.e., (X_1, X_2) is LOAI. □

Lemma 4.2.21 builds a relation between UOAI and LOAI. Based on this relation, we can easily derive similar properties of LOAI random vectors based on the properties of

UOAI random vectors.

Proposition 4.2.22 If random vector $(X_1, ..., X_n)$ is LOAI, then $(f(X_1), ..., f(X_n))$ is LOAI for any increasing function $f(x)$.

Proof. Noting that $-f(-x)$ is increasing, according to Lemma 4.2.21 and Proposition 4.2.5, we have

$$\begin{aligned} &(X_1, ..., X_n) \text{ is LOAI } \Rightarrow (-X_n, ..., -X_1) \text{ is UOAI} \\ \Rightarrow\ &(-f(X_n), ..., -f(X_1)) \text{ is UOAI } \Rightarrow (f(X_1), ..., f(X_n)) \text{ is LOAI.} \end{aligned} \qquad \square$$

Corollary 4.2.23 Let $f(x)$ be a decreasing function.

(i) If random vector $(X_1, ..., X_n)$ is UOAI, then $(f(X_n), ..., f(X_1))$ is LOAI.

(ii) If random vector $(X_1, ..., X_n)$ is LOAI, then $(f(X_n), ..., f(X_1))$ is UOAI.

Similar as Proposition 4.2.6, the LOAI dependence also implies lower orthant order between random vectors formulated by marginalizations.

Proposition 4.2.24 Let K_1, K_2, K_3 be mutually disjoint subsets of $\{1, ..., n\}$ such that $|K_1| = |K_2|$ and $\max K_1 < \min K_2$. If $(X_1, ..., X_n)$ is LOAI, then

$$(X_k, k \in K_1 \cup K_3) \leq_{lo} (X_k, k \in K_2 \cup K_3).$$

Proof. For simplicity, assume $K_1 = \{1\}, K_2 = \{2\}$ and $K_3 = \{3, ..., n\}$, then we need to show

$$(X_1, X_3, ..., X_n) \leq_{lo} (X_2, X_3, ..., X_n).$$

Since $(X_1, ..., X_n)$ is LOAI, we have for any fixed $x, x_3, ..., x_n \in \mathbb{R}$,

$$\begin{aligned} & \mathbb{P}\{X_1 \le x, X_3 \le x_3, ..., X_n \le x_n\} = \mathbb{P}\{X_1 \le x, X_2 \le \infty, X_3 \le x_3, ..., X_n \le x_n\} \\ \ge \; & \mathbb{P}\{X_1 \le \infty, X_2 \le x, X_3 \le x_3, ..., X_n \le x_n\} = \mathbb{P}\{X_2 \le x, X_3 \le x_3, ..., X_n \le x_n\}, \end{aligned}$$

which implies $(X_1, X_3, ..., X_n) \le_{lo} (X_2, X_3, ..., X_n)$. □

Similar as UOAI, independent or comonotonic random variables with certain marginal distributions can formulate a LOAI random vector.

Corollary 4.2.25 Let random variables $X_1, ..., X_n$ be mutually independent. Then random vector $(X_1, ..., X_n)$ is LOAI if and only if $X_1 \le_{rh} ... \le_{rh} X_n$.

Proof. Combine Proposition 4.2.7 and Lemma 4.2.21, it is sufficient to show that

$$X_1 \le_{rh} ... \le_{rh} X_n \Leftrightarrow -X_n \le_{hr} ... \le_{hr} -X_1.$$

The equivalence holds according to Proposition 1.2.3. □

Following the same argument, we have

Corollary 4.2.26 Let random variables $X_1, ..., X_n$ be comonotonic. Then random vector $(X_1, ..., X_n)$ is LOAI if and only if $X_1 \le_{st} ... \le_{st} X_n$. □

Furthermore, we can construct LOAI random vectors through copulas, as Proposition 4.2.12.

Proposition 4.2.27 Assume that random variables $X_1, ..., X_n$ are linked by Archimedean copula $C(u_1, ..., u_n)$. If

(i) $X_1 \leq_{rh} ... \leq_{rh} X_n$,

(ii) $C(u_1, ..., u_n)$ is arrangement increasing, and

(iii) there exists a strictly increasing convex function $g(x)$ such that

$$g(C(u_1, ..., u_{k-1}, e^t, u_{k+1}, ..., u_n)) \text{ is concave in } t \in (-\infty, 0] \text{ for any } k = 1, ..., n,$$

then $(X_1, ..., X_n)$ is LOAI.

Proof. This result can be proved following the same argument as the proof for Proposition 4.2.12. However, the relation between LOAI and UOAI given by Lemma 4.2.21 provides a shortcut for its proof, which is given below.

First, $X_1 \leq_{rh} ... \leq_{rh} X_n$ implies $-X_n \leq_{hr} ... \leq_{hr} -X_1$. According to Proposition 1.4.1, we know that as the survival copula of $(X_1, ..., X_n)$, $C(u_1, ..., u_n)$ is also a copula of $(-X_1, ..., -X_n)$. Then, $C_\downarrow(u_1, ..., u_n) \equiv C(u_n, ..., u_1)$ is a copula of $(-X_n, ..., -X_1)$. Recall that $C(u_1, ..., u_n)$ is arrangement increasing, then $C_\downarrow(u_1, ..., u_n)$ is arrangement decreasing. Obviously, $C_\downarrow(u_1, ..., u_n)$ satisfies condition (iii) in Proposition 4.2.12. Therefore, $(-X_n, ..., -X_1)$ is UOAI according to Proposition 4.2.12, which means that $(X_1, ..., X_n)$ is LOAI. $\square$

4.3 (Conditionally) Tail Density Arrangement Increasing

In this section, we focus on the generalization of the hazard rate order. There are two ways to generalize this concept into multivariate cases.

Definition 4.3.1 Random vector $(X_1, ..., X_n)$ with a joint density function is said to be tail density arrangement increasing (TDAI), if

$$\frac{\partial}{\partial x_i}\bar{F}(x_1, .., x_i, ..., x_j, ..., x_n) \le \frac{\partial}{\partial x_i}\bar{F}(x_1, .., x_j, ..., x_i, ..., x_n),$$

for any $1 \le i < j \le n$ and $x_i \le x_j$.

Definition 4.3.2 Random vector $(X_1, ..., X_n)$ is said to be conditionally tail density arrangement increasing (CTDAI), if

$$\frac{\partial}{\partial x_i}\mathbb{P}\{X_i > x_i, X_j > x_j | \mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}\} \le \frac{\partial}{\partial x_i}\mathbb{P}\{X_i > x_j, X_j > x_i | \mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}\},$$

for any $1 \le i < j \le n$ and $x_i \le x_j$.

It is easy to verify that TDAI implies UOAI, and CTDAI implies CUOAI. Obviously, "$(X_1, ..., X_n)$ is CTDAI" is equivalent to "$(X_i, X_j)|\mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}$ is TDAI for all $1 \le i < j \le n$ and all $\mathbf{x}_{\overline{ij}} \in S(\mathbf{X}_{\overline{ij}})$". It is clear that, in the case $n = 2$, (X, Y) is CTDAI if and only if (X, Y) is TDAI if and only if $X \le_{hr:j} Y$. In the case that $n \ge 3$, CTDAI implies TDAI.

Proposition 4.3.3 If random vector $(X_1, ..., X_n)$ is CTDAI, then $(X_1, ..., X_n)$ is TDAI.

Proof. The conclusion is straightforward by noting that

$$\begin{aligned} & \frac{\partial}{\partial x_i}\mathbb{P}\{X_i > x_i, X_j > x_j | \mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}\} \\ = \; & (-1)^{n-2}\frac{\partial^{n-2}}{\partial \mathbf{x}_{\overline{ij}}}\frac{\partial}{\partial x_i}\mathbb{P}\{X_1 > x_1, ..., X_n > x_n\} \times (f_{\mathbf{X}_{\overline{ij}}}(\mathbf{x}_{\overline{ij}}))^{-1}, \end{aligned}$$

where $f_{\mathbf{X}_{\overline{ij}}}(\mathbf{x}_{\overline{ij}})$ is the joint density function of $\mathbf{X}_{\overline{ij}}$, and by noting that $\frac{\partial}{\partial x_i}\bar{F}(x_1, ..., x_n)$ is a multiple integration of $(-1)^{n-2}\frac{\partial^{n-2}}{\partial \mathbf{x}_{\overline{ij}}}\frac{\partial}{\partial x_i}\mathbb{P}\{X_1 > x_1, ..., X_n > x_n\}$. □

In the case that $X_1, ..., X_n$ are mutually independent, "$(X_1, ..., X_n)$ is CTDAI" is equivalent to "$(X_1, ..., X_n)$ is TDAI". Indeed, we have the following result.

Proposition 4.3.4 Assume that random variables $X_1, ..., X_n$ are mutually independent. $(X_1, ..., X_n)$ is CTDAI if and only if $(X_1, ..., X_n)$ is TDAI if and only if $X_1 \le_{hr} ... \le_{hr} X_n$.

Proof. Due to the independence between $X_1, ..., X_n$, it suffices to show that (X_1, X_2) is TDAI if and only if $X_1 \le_{hr} X_2$, which follows from Shanthikumar and Yao (1991). $\square$

Recall that the definition of the joint hazard rate order given in Definition 4.1.1 assumes the existence of partial derivatives of the joint survival function. As mentioned in Section 4.1, this assumption is not essential. In the following, we state several equivalent characterizations of the joint hazard rate order from the probabilistic and functional aspects, which avoid the existence of the partial derivatives of the joint survival function and can serve as the generalized versions of the definition of joint hazard rate order.

Definition 4.3.5 Let A be a subset of $\mathbb{R}^n$. A is said to be an *upper set*, if the indicator function $\mathbb{I}\{(x_1, ..., x_n) \in A\}$ is increasing in $x_1, ..., x_n$.

A is said to be a *partial upper set* with respect to a subset $K \subset \{1, ..., n\}$, or a $K-$upper set, if $\mathbb{I}\{(x_1, ..., x_n) \in A\}$ is increasing in x_k for all $k \in K$.

Proposition 4.3.6 The following statements are equivalent.

(i) $X \le_{hr:j} Y$

(ii) $\mathbb{E}[g(X, Y)] \ge \mathbb{E}[g(Y, X)]$ for all $g(x, y) \in \mathcal{G}_{hr}$.

(iii) $\mathbb{P}\{(X, Y) \in A\} \ge \mathbb{P}\{(Y, X) \in A\}$ for all $\{2\}-$upper set $A \subset \{(x, y) | \, x \le y\}$.

Proof. The equivalence between (i) and (ii) has been proved by Shanthikumar and Yao (1991) in Theorem 3.17. In the following, we only prove the equivalence between (ii) and (iii).

(ii)⇒(iii). For any $\{2\}-$upper set $A \subset \{(x,y)|\, x \le y\}$, consider the indicator function $h(x,y) = \mathbb{I}\{(x,y) \in A\}$. Obviously, $h(x,y)$ in increasing in y since A is a $\{2\}-$upper set. Noting that $\Delta h(x,y) = h(x,y)$ if $y > x$, and $\Delta h(x,y) = 0$ if $y = x$, we have $\Delta h(x,y) = h(x,y)$ is increasing in $y \ge x$. Therefore $h(x,y) \in \mathcal{G}_{hr}$, which implies that

$$\mathbb{P}\{(X,Y) \in A\} = \mathbb{E}[h(X,Y)] \ge \mathbb{E}[h(Y,X)] = \mathbb{P}\{(Y,X) \in A\}.$$

(iii)⇒(ii). Consider any function $g(x,y) \in \mathcal{G}_{hr}$. Noting that $\Delta g(y,x) = -\Delta g(x,y)$ and $\Delta g(x,y) = 0$ if $x = y$, we have

$$\begin{aligned} \mathbb{E}[\Delta g(X,Y)] &= \mathbb{E}[\Delta g(X,Y)\mathbb{I}\{Y \ge X\}] + \mathbb{E}[\Delta g(X,Y)\mathbb{I}\{Y \le X\}] \\ &= \mathbb{E}[\Delta g(X,Y)\mathbb{I}\{Y \ge X\}] - \mathbb{E}[\Delta g(Y,X)\mathbb{I}\{X \ge Y\}]. \end{aligned} \tag{4.3.1}$$

Since $\Delta g(X,Y)\mathbb{I}\{Y \ge X\} \ge 0$, we have

$$\begin{aligned} &\mathbb{E}[\Delta g(X,Y)\mathbb{I}\{Y \ge X\}] = \int_0^\infty \mathbb{P}\{\Delta g(X,Y)\mathbb{I}\{Y \ge X\} > z\}d\,z \\ &= \int_0^\infty \mathbb{P}\{\Delta g(X,Y) > z, Y \ge X\}dz = \int_0^\infty \mathbb{P}\{(X,Y) \in A_z\}d\,z, \end{aligned} \tag{4.3.2}$$

where $A_z = \{(x,y)|\Delta g(x,y) > z, y \ge x\}$. Similarly,

$$\mathbb{E}[\Delta g(Y,X)\mathbb{I}\{X \ge Y\}] = \int_0^\infty \mathbb{P}\{(Y,X) \in A_z\}dz. \tag{4.3.3}$$

Recalling that $g(x,y) \in \mathcal{G}_{hr}$, it is easy to verify that $A_z \subset \{(x,y)|\, x \le y\}$ and A_z is a $\{2\}-$upper set for any fixed $z \ge 0$, then $\mathbb{P}\{(X,Y) \in A_z\} \ge \mathbb{P}\{(Y,X) \in A_z\}$ for any $z \ge 0$. Combining with (4.3.2) and (4.3.3), we have

$$\mathbb{E}[\Delta g(X,Y)\mathbb{I}\{Y \ge X\}] \ge \mathbb{E}[\Delta g(Y,X)\mathbb{I}\{X \ge Y\}],$$

which implies $\mathbb{E}[g(X,Y)] \ge \mathbb{E}[g(Y,X)]$ according to (4.3.1). $\square$

Proposition 4.3.6 (ii) is actually a functional characterization of the joint hazard rate order, which has been proposed by Shanthikumar and Yao (1991). The contribution of this proposition is to develop a probabilistic characterization (iii) and to build connection with the functional characterization. The proof of (iii)⇒(ii) reveals the essence of the functional characterization and gives some inspirations on how to generalize the joint hazard rate order into multivariate cases.

Below we propose a functional characterization for the notion of CTDAI. First, we derive a property of the class $\mathcal{G}^{ij}_{ctdai}$.

Lemma 4.3.7 Let $u(x)$ be an increasing and convex function. If $g(x_1,...,x_n) \in \mathcal{G}^{ij}_{ctdai}$ and $g(x_1,...,x_n)$ is increasing in each x_i, then $u(g(x_1,...,x_n)) \in \mathcal{G}^{ij}_{ctdai}$.

Proof. For simplicity, assume that $u(x)$ and $g(x_1,...,x_n)$ are differentiable, and denote by $g'_i(x_1,...,x_n)$ the partial derivative of g with respect to the i^{th} argument. Then $g(x_1,...,x_n) \in \mathcal{G}^{ij}_{ctdai}$ implies $g'_j(x_1,...,x_n) \ge g'_i(\pi_{ij}(x_1,...,x_n))$ for all $x_j \ge x_i$.

Note that $u'(x)$ is nonnegative and increasing, then

$$\begin{aligned} & \frac{\partial}{\partial x_j} u(g(x_1, ..., x_n)) = u'(g(x_1, ..., x_n)) \times g'_j(x_1, ..., x_n) \\ \geq \; & u'(g(\pi_{ij}(x_1, ..., x_n))) \times g'_i(\pi_{ij}(x_1, ..., x_n)) = \frac{\partial}{\partial x_j} u(g(\pi_{ij}(x_1, ..., x_n))). \end{aligned}$$

□

Proposition 4.3.8 Assume that random vector $(X_1, ..., X_n)$ has a joint density function. Then the following statements are equivalent.

(i) $(X_1, ..., X_n)$ is CTDAI.

(ii) $\mathbb{E}[g(\mathbf{X})] \geq \mathbb{E}[g(\pi_{ij}(\mathbf{X}))]$ for any $1 \leq i < j \leq n$ and any $g(x_1, ..., x_n) \in \mathcal{G}^{ij}_{ctdai}$.

(iii) $\mathbb{P}\{\mathbf{X} \in A\} \geq \mathbb{P}\{\pi_{ij}(\mathbf{X}) \in A\}$ for any $1 \leq i < j \leq n$ and any $\{j\}-$upper set $A \subset \{(x_1, ..., x_n) \in \mathbb{R}^n | \, x_i \leq x_j\}$.

Proof. (i)⇒(ii). For any $1 \leq i < j \leq n$ and any fixed $\mathbf{x}_{\overline{ij}} \in S(\mathbf{X}_{\overline{ij}})$, according to Definition 4.3.2, we know that $(X_i, X_j)|\mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}$ is TDAI. For any $g(\mathbf{x}) \in \mathcal{G}^{ij}_{ctdai}$, denote $h(x_i, x_j) \equiv g(x_1, ..., x_i, ..., x_j, ..., x_n)$, then $h(x_i, x_j) \in \mathcal{G}^0_{hr}$. Then according to Proposition 4.3.6, we have $\mathbb{E}\left[h(X_i, X_j)\right] \geq \mathbb{E}\left[h(X_j, X_i)\right]$, or

$$\mathbb{E}\left[g(X_1, ..., X_i, ..., X_j, ..., X_n)|\mathbf{X}_{\overline{ij}}\right] \geq \mathbb{E}\left[g(X_1, ..., X_j, ..., X_i, ..., X_n)|\mathbf{X}_{\overline{ij}}\right].$$

Taking expectation on both sides of the above inequality, we have $\mathbb{E}[g(\mathbf{X})] \geq \mathbb{E}[g(\pi_{ij}(\mathbf{X}))]$.

(ii)⇒(iii). For any $1 \leq i < j \leq n$ and any $\{j\}-$upper set $A \subset \{(x_1, ..., x_n) \in \mathbb{R}^n | \, x_i \leq x_j\}$, define $h(x_1, ..., x_n) = \mathbb{I}\{(x_1, ..., x_n) \in A\}$. It is easy to verify that $h(x_1, ..., x_n) \in \mathcal{G}^{ij}_{ctdai}$. Then $\mathbb{P}\{\mathbf{X} \in A\} = \mathbb{E}[h(\mathbf{X})] \geq \mathbb{E}[h(\pi_{ij}(\mathbf{X}))] = \mathbb{P}\{\pi_{ij}(\mathbf{X}) \in A\}$.

(iii)⇒(i). Without loss of generality, it is sufficient to show that

$$\frac{\partial}{\partial x_1}\mathbb{P}\{X_1 > x_1, X_2 > x_2|\mathbf{X}_{\overline{12}} = \mathbf{x}_{\overline{12}}\} \leq \frac{\partial}{\partial x_1}\mathbb{P}\{X_1 > x_2, X_2 > x_1|\mathbf{X}_{\overline{12}} = \mathbf{x}_{\overline{12}}\}, \quad (4.3.4)$$

for any $x_1 \leq x_2$ and any fixed $\mathbf{x}_{\overline{12}} \in S(\mathbf{X}_{\overline{12}})$.

For any $x_1 \leq x_2$, define $A_t = (x_1 - t, x_1] \times (x_2, \infty) \times A_{\overline{12}}$, where $t > 0$ and $A_{\overline{12}} \in \sigma(\mathbf{X}_{\overline{12}})$. Then A_t is a $\{2\}-$upper set and $A_t \subset \{(x_1, ..., x_n) \in \mathbb{R}^n | x_1 \leq x_2\}$. Therefore, we have $\mathbb{P}\{(X_1, X_2, ..., X_n) \in A_t\} \geq \mathbb{P}\{(X_2, X_1, ..., X_n) \in A_t\}$, or

$$\mathbb{P}\{x_1 - t < X_1 \leq x_1, X_2 > x_2, \mathbf{X}_{\overline{12}} \in A_{\overline{12}}\} \geq \mathbb{P}\{x_1 - t < X_2 \leq x_1, X_1 > x_2, \mathbf{X}_{\overline{12}} \in A_{\overline{12}}\}.$$

which is equivalent to

$$\begin{aligned} & \mathbb{E}[\mathbb{E}[\mathbb{I}\{x_1 - t < X_1 \leq x_1, X_2 > x_2\}|\mathbf{X}_{\overline{12}}] \times \mathbb{I}\{\mathbf{X}_{\overline{12}} \in A_{\overline{12}}\}] \\ \geq \; & \mathbb{E}[\mathbb{E}[\mathbb{I}\{x_1 - t < X_2 \leq x_1, X_1 > x_2\}|\mathbf{X}_{\overline{12}}] \times \mathbb{I}\{\mathbf{X}_{\overline{12}} \in A_{\overline{12}}\}] \end{aligned}$$

According to Lemma 4.4.4, we have

$$\mathbb{E}[\mathbb{I}\{x_1 - t < X_1 \leq x_1, X_2 > x_2\}|\mathbf{X}_{\overline{12}}] \geq_{a.s.} \mathbb{E}[\mathbb{I}\{x_1 - t < X_2 \leq x_1, X_1 > x_2\}|\mathbf{X}_{\overline{12}}]$$

or

$$\mathbb{P}\{x_1 - t < X_1 \leq x_1, X_2 > x_2|\mathbf{X}_{\overline{12}} = \mathbf{x}_{\overline{12}}\} \geq \mathbb{P}\{x_1 - t < X_2 \leq x_1, X_1 > x_2|\mathbf{X}_{\overline{12}} = \mathbf{x}_{\overline{12}}\},$$

for any $\mathbf{x}_{\overline{12}} \in S(\mathbf{X}_{\overline{12}})$.

By dividing $t > 0$ on both sides of the above inequality and letting $t \searrow 0$, we have

$$\frac{\partial}{\partial x_1}\mathbb{P}\{X_1 > x_1, X_2 > x_2 | \mathbf{X}_{\overline{12}}\} \le \frac{\partial}{\partial x_1}\mathbb{P}\{X_1 > x_2, X_2 > x_1 | \mathbf{X}_{\overline{12}}\},$$

which implies (4.3.4). □

Proposition 4.3.8 provides a functional characterization and a probabilistic characterization of the notion of CTDAI, which also serve as two generalized versions of the definition of CTDAI. The advantage of these generalized definitions is that they do not require the existence of joint density function. It has to be pointed out that, although being assumed in Proposition 4.3.8, the existence of the joint density function is not essential for the proof of the implication of (iii)⇒(ii). Alternatively, we can prove (iii)⇒(ii) using the similar arguments as in the proof of Proposition 4.3.6.

Following the functional characterization of CTDAI, it is easy to derive the invariant property of CTDAI.

Proposition 4.3.9 If random vector $(X_1, ..., X_n)$ is CTDAI, then $(f(X_1), ..., f(X_n))$ is CTDAI for any increasing function $f(x)$.

Proof. The proof is straightforward by noting the fact that

$$g(x_1, ..., x_n) \in \mathcal{G}^{ij}_{ctdai} \Longrightarrow g(f(x_1), ..., f(x_n)) \in \mathcal{G}^{ij}_{ctdai},$$

for any increasing function $f(x)$. □

Proposition 4.3.10 Assume that random vector $(X_1, ..., X_n)$ has a joint density function and the marginal distributions satisfy $X_1 \le_{hr} ... \le_{hr} X_n$. Let $(X_1, ..., X_n)$ be linked by an Archimedean survival copula $C(u_1,...,u_n) = \Psi^{-1}(\sum_{k=1}^n \Psi(u_k))$. If $\Psi(e^t)$ is convex in $t \in (-\infty, 0]$, then $(X_1,...,X_n)$ is CTDAI.

Proof. It suffices to show that

$$\frac{\partial}{\partial x_1}\mathbb{P}\{X_1 > x_1, X_2 > x_2 | \mathbf{X}_{\overline{12}} = \mathbf{x}_{\overline{12}}\} \leq \frac{\partial}{\partial x_1}\mathbb{P}\{X_1 > x_2, X_2 > x_1 | \mathbf{X}_{\overline{12}} = \mathbf{x}_{\overline{12}}\}, \quad (4.3.5)$$

for any $x_1 \leq x_2$.

Note that $\bar{F}(x_1,...,x_n) = \Lambda\left(\sum_{k=1}^n \Psi(\bar{F}_k(x_k))\right)$ and $\sum_{k=1}^n \Psi(\bar{F}_k(x_k)) = \Psi(\bar{F}(x_1,...,x_n))$. Then,

$$\begin{aligned}
& \mathbb{P}\{X_1 > x_1, X_2 > x_2 | \mathbf{X}_{\overline{12}} = \mathbf{x}_{\overline{12}}\} \\
= & (-1)^{n-2}\frac{\partial^{n-2}}{\partial x_3 ... \partial x_n}\bar{F}(x_1,...,x_n) \times \frac{1}{f_{(X_3,...,X_n)}(x_3,...,x_n)} \\
= & (-1)^{n-2}\Lambda^{(n-2)}\left(\sum_{k=1}^n \Psi(\bar{F}_k(x_k))\right) \times \prod_{k=3}^n(-\Psi'(\bar{F}_k(x_k))) \times \frac{\prod_{k=3}^n f_k(x_k)}{f_{(X_3,...,X_n)}(x_3,...,x_n)} \\
= & (-1)^{n-2}\Lambda^{(n-2)}\left(\Psi(\bar{F}(x_1,...,x_n))\right) \times h(x_3,...,x_n),
\end{aligned}$$

where $h(x_3,...,x_n) = \prod_{k=3}^n(-\Psi'(\bar{F}_k(x_k))) \times \frac{\prod_{k=3}^n f_k(x_k)}{f_{(X_3,...,X_n)}(x_3,...,x_n)} \geq 0$.

Therefore,

$$\begin{aligned}
& -\frac{\partial}{\partial x_1}\mathbb{P}\{X_1 > x_1, X_2 > x_2 | \mathbf{X}_{\overline{12}} = \mathbf{x}_{\overline{12}}\} \\
= & (-1)^{n-1}\Lambda^{(n-1)}\left(\Psi(\bar{F}(x_1,x_2,...,x_n))\right) \\
& \times \Psi'(\bar{F}_1(x_1))(-f_1(x_1)) \times h(x_3,...,x_n), \quad (4.3.6)
\end{aligned}$$

and

$$
\begin{aligned}
& -\frac{\partial}{\partial x_1}\mathbb{P}\{X_1 > x_2, X_2 > x_1 | \mathbf{X}_{\overline{12}} = \mathbf{x}_{\overline{12}}\} \\
= \quad & (-1)^{n-1}\Lambda^{(n-1)}\left(\Psi(\bar{F}(x_2, x_1, ..., x_n))\right) \\
& \quad \times\Psi'(\bar{F}_2(x_1))(-f_2(x_1)) \times h(x_3, ..., x_n). \qquad (4.3.7)
\end{aligned}
$$

According to Corollary 4.2.13, we know that $\bar{F}(x_1, x_2, ..., x_n) \geq \bar{F}(x_2, x_1, ..., x_n)$ for any $x_1 \leq x_2$. On the other hand, $\Psi(x)$ is decreasing and $(-1)^{n-1}\Lambda^{(n-1)}(x)$ is decreasing from Remark 1.4.2, then

$$
(-1)^{n-1}\Lambda^{(n-1)}\left(\Psi(\bar{F}(x_1, x_2, ..., x_n))\right) \geq (-1)^{n-1}\Lambda^{(n-1)}\left(\Psi(\bar{F}(x_2, x_1, ..., x_n))\right). \qquad (4.3.8)
$$

Since $\Psi(e^t)$ is convex, we have $\frac{d}{dt}\Psi(e^t) = \Psi'(e^t)e^t$ is increasing and thus $\Psi'(x)x$ is increasing. Since $\bar{F}_1(x_1) \leq \bar{F}_2(x_1)$, we have

$$
-\bar{F}_1(x_1) \times \Psi'(\bar{F}_1(x_1)) \geq -\bar{F}_2(x_1) \times \Psi'(\bar{F}_2(x_1)). \qquad (4.3.9)
$$

Recalling that $X_1 \leq_{hr} X_2$, we have

$$
\frac{f_1(x_1)}{\bar{F}_1(x_1)} \geq \frac{f_2(x_1)}{\bar{F}_2(x_1)}. \qquad (4.3.10)
$$

A combination of (4.3.9) and (4.3.10) implies that

$$
\Psi'(\bar{F}_1(x_1))(-f_1(x_1)) \geq \Psi'(\bar{F}_2(x_1))(-f_2(x_1)). \qquad (4.3.11)
$$

Combing (4.3.6),(4.3.7),(4.3.8) and (4.3.11), we get (4.3.5). □

Similarly as Corollary 4.2.19, we can compare different linear combinations of components of a CTDAI random vector in certain stochastic orders.

Proposition 4.3.11 If random vector $\mathbf{X} = (X_1, ..., X_n)$ is CTDAI, then

$$\mathbf{a} \cdot \mathbf{X} \geq_{icx} \mathbf{a} \cdot \pi_{ij}(\mathbf{X}),$$

for any vector $\mathbf{a} = (a_1, ..., a_n) \geq \mathbf{0}$ such that $1 \leq i < j \leq n$ and $a_i \leq a_j$.

Proof. It is sufficient to show that

$$\mathbb{E}\left[u\left(a_1X_1 + a_2X_2 + \sum_{k=3}^{n} a_kX_k\right)\right] \geq \mathbb{E}\left[u\left(a_2X_1 + a_1X_2 + \sum_{k=3}^{n} a_kX_k\right)\right]$$

for all $u \in \mathcal{U}_{icx}$ and any $a_1 \leq a_2$.

Consider the function $g(x_1, ..., x_n) = u\left(\sum_{k=1}^{n} a_kx_k\right)$. Noting that

$$\Delta_{12}g(x_1, ..., x_n) = u\left(a_1x_1 + a_2x_2 + \sum_{k=3}^{n} a_kx_k\right) - u\left(a_2x_1 + a_1x_2 + \sum_{k=3}^{n} a_kx_k\right),$$

we have

$$\frac{\partial}{\partial x_2}\Delta_{12}g(x_1, ..., x_n) = a_2u'(\mathbf{a} \cdot \mathbf{x}) - a_1u'(\mathbf{a} \cdot \pi_{12}(\mathbf{x})).$$

Since $u \in \mathcal{U}_{icx}$, we have $u'(x) \geq 0$ and $u'(x)$ is increasing. Noting that $a_2 \geq a_1$ and $\mathbf{a} \cdot \mathbf{x} \geq \mathbf{a} \cdot \pi_{12}(\mathbf{x})$, we have $\frac{\partial}{\partial x_2}\Delta_{12}g(x_1, ..., x_n) \geq 0$, which means $g(x_1, ..., x_n) \in \mathcal{D}_{ctdai}^{12}$.

According to the functional characterization of CTDAI, we know that

$$\mathbb{E}\left[u\left(a_1X_1+a_2X_2+\sum_{k=3}^{n}a_kX_k\right)\right]=\mathbb{E}\left[g(X_1,X_2,...,X_n)\right]$$
$$\geq\ \mathbb{E}\left[g(X_2,X_1,...,X_n)\right]=\mathbb{E}\left[u\left(a_2X_1+a_1X_2+\sum_{k=3}^{n}a_kX_k\right)\right]. \qquad \square$$

4.4 Stochastic Arrangement Increasing

Shanthikumar and Yao (1991) extended the definition of the joint likelihood ratio order into multivariate cases. In this section, we redefine the notion using a multivariate functional characterization and derive many important properties.

Definition 4.4.1 Random vector $\mathbf{X}=(X_1,...,X_n)$ is said to be stochastically arrangement increasing (SAI), if for any $1\leq i<j\leq n$, $\mathbb{E}\left[g(\mathbf{X})\right]\geq\mathbb{E}\left[g(\pi_{ij}(\mathbf{X}))\right]$ for any $g(x_1,...,x_n)\in\mathcal{G}_{sai}^{ij}$ such that the expectations exist.

Obviously, SAI implies CTDAI since $\mathcal{G}_{ctdai}^{ij}\subset\mathcal{G}_{sai}^{ij}$. Recall that the notions of UOAI (CUOAI) and TDAI (CTDAI) are characterized by joint survival functions. Similarly, SAI can also be characterized by the joint density function, given that the joint density function exists. Shanthikumar and Yao (1991) have stated this result without proof. In the following, we provide a detailed proof.

Proposition 4.4.2 Assume that $\mathbf{X}$ has joint density function $f(\mathbf{x})$. Then $\mathbf{X}$ is SAI if and only if $f(\mathbf{x})$ is arrangement increasing.

Proof. First assume $f(\mathbf{x})$ is arrangement increasing. Without loss of generality, it is sufficient to show that

$$\mathbb{E}[g(X_1, X_2, ..., X_n)] \geq \mathbb{E}[g(X_2, X_1, ..., X_n)] \text{ for any } g \in \mathcal{G}_{sai}^{12}.$$

Noting that $(g(x_1, x_2, \mathbf{x}_{\overline{12}}) - g(x_2, x_1, \mathbf{x}_{\overline{12}}))(f(x_1, x_2, \mathbf{x}_{\overline{12}}) - f(x_2, x_1, \mathbf{x}_{\overline{12}})) \geq 0$, we have

$$\begin{aligned}
&g(x_1, x_2, \mathbf{x}_{\overline{12}})f(x_1, x_2, \mathbf{x}_{\overline{12}}) + g(x_2, x_1, \mathbf{x}_{\overline{12}})f(x_2, x_1, \mathbf{x}_{\overline{12}}) \\
&\geq g(x_2, x_1, \mathbf{x}_{\overline{12}})f(x_1, x_2, \mathbf{x}_{\overline{12}}) + g(x_1, x_2, \mathbf{x}_{\overline{12}})f(x_2, x_1, \mathbf{x}_{\overline{12}}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\left[g(\mathbf{X})\right] = \int_{\mathbb{R}^n} g(x_1, x_2, \mathbf{x}_{\overline{12}})f(x_1, x_2, \mathbf{x}_{\overline{12}})\mathbf{dx} \\
= &\int_{x_1 > x_2} g(x_1, x_2, \mathbf{x}_{\overline{12}})f(x_1, x_2, \mathbf{x}_{\overline{12}})\mathbf{dx} + \int_{x_1 < x_2} g(x_1, x_2, \mathbf{x}_{\overline{12}})f(x_1, x_2, \mathbf{x}_{\overline{12}})\mathbf{dx} \\
&\qquad + \mathbb{E}\left[g(\mathbf{X})\mathbb{I}\{X_1 = X_2\}\right] \\
= &\int_{x_1 > x_2} g(x_1, x_2, \mathbf{x}_{\overline{12}})f(x_1, x_2, \mathbf{x}_{\overline{12}})\mathbf{dx} + \int_{x_1 > x_2} g(x_2, x_1, \mathbf{x}_{\overline{12}})f(x_2, x_1, \mathbf{x}_{\overline{12}})\mathbf{dx} \\
&\qquad + \mathbb{E}\left[g(\mathbf{X})\mathbb{I}\{X_1 = X_2\}\right] \\
= &\int_{x_1 > x_2} (g(x_1, x_2, \mathbf{x}_{\overline{12}})f(x_1, x_2, \mathbf{x}_{\overline{12}}) + g(x_2, x_1, \mathbf{x}_{\overline{12}})f(x_2, x_1, \mathbf{x}_{\overline{12}}))\mathbf{dx} \\
&\qquad + \mathbb{E}\left[g(\mathbf{X})\mathbb{I}\{X_1 = X_2\}\right] \\
\geq &\int_{x_1 > x_2} (g(x_2, x_1, \mathbf{x}_{\overline{12}})f(x_1, x_2, \mathbf{x}_{\overline{12}}) + g(x_1, x_2, \mathbf{x}_{\overline{12}})f(x_2, x_1, \mathbf{x}_{\overline{12}}))\mathbf{dx} \\
&\qquad + \mathbb{E}\left[g(\mathbf{X})\mathbb{I}\{X_1 = X_2\}\right] \\
= &\int_{\mathbb{R}^n} g(x_2, x_1, \mathbf{x}_{\overline{12}})f(x_1, x_2, \mathbf{x}_{\overline{12}})\mathbf{dx} = \mathbb{E}\left[g(\pi_{\overline{12}}(\mathbf{X}))\right].
\end{aligned}$$

Now assume that $\mathbf{X}$ is SAI, without loss of generality, it is sufficient to show that $f(x_1, x_2, \mathbf{x}_{\overline{12}}) \geq f(x_2, x_1, \mathbf{x}_{\overline{12}})$ for any $x_1 < x_2$ and any $\mathbf{x}_{\overline{12}} \in \mathbb{R}^{n-2}$. Since the density function exists, we have

$$f(x_1, x_2, \mathbf{x}_{\overline{12}}) = \lim_{t_1 \to 0} \lim_{t_2 \to 0} \frac{1}{4t_1 t_2} \lim_{\mathbf{t}_{\overline{12}} \to \mathbf{0}} \frac{2^{2-n}}{\prod_{i=3}^{n} t_i} \mathbb{P}\{|X_i - x_i| \leq t_i, i = 1, \cdots, n\}. \quad (4.4.1)$$

Noting that for any given $\mathbf{x} \in \mathbb{R}^n$ such that $x_1 < x_2$, there exist $t_1, t_2 \geq 0$ such that $x_1 + t_1 \leq x_2 - t_2$. It is easy to verify that the function $h(\mathbf{y}) = \prod_{i=1}^{n} \mathbb{I}\{y_i \in [x_i - t_i, x_i + t_i]\} \in \mathcal{G}_{sai}^{12}$, thus

$$\begin{aligned}
& \mathbb{P}\{|X_1 - t_1| \leq t_1, |X_2 - t_2| \leq t_2, |X_i - t_i| \leq t_i, i = 3, \cdots, n\} \\
= \ & \mathbb{E}\left[\mathbb{I}\{|X_1 - t_1| \leq t_1, |X_2 - t_2| \leq t_2, |X_i - t_i| \leq t_i, i = 3, \cdots, n\}\right] \\
\geq \ & \mathbb{E}\left[\mathbb{I}\{|X_2 - t_1| \leq t_1, |X_1 - t_2| \leq t_2, |X_i - t_i| \leq t_i, i = 3, \cdots, n\}\right] \\
= \ & \mathbb{P}\{|X_2 - t_1| \leq t_1, |X_1 - t_2| \leq t_2, |X_i - t_i| \leq t_i, i = 3, \cdots, n\}.
\end{aligned}$$

Combining with (4.4.1), we have $f(x_1, x_2, \mathbf{x}_{\overline{12}}) \geq f(x_2, x_1, \mathbf{x}_{\overline{12}})$ for any $x_1 < x_2$. □

The following two propositions show that the SAI property is preserved under marginalization and conditioning:

Proposition 4.4.3 If $\mathbf{X} = (X_1, ..., X_n)$ is SAI, then $\mathbf{X}_K = (X_k, k \in K)$ is SAI for any non-empty $K \subseteq \{1, ..., n\}$.

Proof. Without loss of generality, assume $K = \{1, ..., |K|\}$. For any $1 \leq i < j \leq |K|$, consider any function $g(x_1, ..., x_{|K|}) : \mathbb{R}^{|K|} \to \mathbb{R}$ such that $g \in \mathcal{G}_{sai}^{ij}$. Define $h(\mathbf{x}) = h(x_1, ..., x_n) \equiv g(x_1, ..., x_{|K|})$. It is easy to verify that $h(\mathbf{x}) \in \mathcal{G}_{sai}^{ij}$ for any $1 \leq i < j \leq |K|$.

Since $(X_1, ..., X_n)$ is SAI, we have

$$\mathbb{E}[g(\mathbf{X}_K)] = \mathbb{E}[h(\mathbf{X})] \geq \mathbb{E}[h(\pi_{ij}(\mathbf{X}))] = \mathbb{E}[g(\pi_{ij}(\mathbf{X}_K))],$$

which implies $\mathbf{X}_K$ is SAI according to Definition 4.4.1. $\square$

Lemma 4.4.4 Let $\mathbf{X}$ be a random vector defined on the space $(\Omega, \mathcal{F}, \mathbb{P})$ and $f(\mathbf{x})$ be a multivariate function. If $\mathbb{E}[f(\mathbf{X})\mathbb{I}(A)] \leq 0$ for all $A \in \mathcal{F}$, then $f(\mathbf{X}) \leq_{a.s.} 0$.

Proof. Define $A = \{\omega \in \Omega : f(\mathbf{X}(\omega)) > 0\}$, we want to show that $\mathbb{P}(A) = 0$. Otherwise, assume $\mathbb{P}(A) > 0$. Denote $A_n = \{\omega \in \Omega : f(\mathbf{X}(\omega)) \geq 1/n\}$. Then the sets sequence $\{A_n, n = 1, 2, ...\}$ is increasing and converges to A, therefore $\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(A) > 0$. Then there exists $N \in \mathbb{N}^+$ such that $\mathbb{P}(A_N) > 0$, and thus

$$\mathbb{E}[f(\mathbf{X}(\omega))\mathbb{I}\{\omega \in A_N\}] \geq \mathbb{E}\left[\frac{1}{N}\mathbb{I}\{\omega \in A_N\}\right] = \frac{1}{N}\mathbb{P}(A_N) > 0,$$

which conflicts with the fact that $\mathbb{E}[f(\mathbf{X})\mathbb{I}(A)] \leq 0$ for all $A \in \mathcal{F}$. $\square$

Proposition 4.4.5 If $\mathbf{X}$ is SAI, then $\mathbf{X}_K|\mathbf{X}_{\bar{K}} = \mathbf{x}_{\bar{K}}$ is SAI for any $K \subset \{1, ..., n\}$ and any $\mathbf{x}_{\bar{K}} \in S(\mathbf{X}_{\bar{K}})$.

Proof. Without loss of generality, assume $K = \{1, ..., |K|\}$. For any $i < j \leq |K|$, consider any function $g(x_1, ..., x_{|K|}) : \mathbb{R}^{|K|} \to \mathbb{R}$ such that $g \in \mathcal{G}_{sai}^{ij}$. Define $h(\mathbf{x}) = g(x_1, ..., x_{|K|})\mathbb{I}\{(x_{|K|+1}, ..., x_n) \in A\}$, where $A \in \sigma(X_{|K|+1}, ..., X_n)$. It is easy to verify that $h(\mathbf{x}) \in \mathcal{G}_{sai}^{ij}$. Following the definition of SAI, we have $\mathbb{E}\left[h(\mathbf{X})\right] \geq \mathbb{E}\left[h(\pi_{ij}(\mathbf{X}))\right]$, i.e., $\mathbb{E}\left[g(\mathbf{X}_K)\mathbb{I}\{\mathbf{X}_{\bar{K}} \in A\}\right] \geq \mathbb{E}\left[g(\pi_{ij}(X_1, ..., X_{|K|}))\mathbb{I}\{(X_{|K|+1}, ..., X_n) \in A\}\right]$. According to the property of conditional expectation, it is equivalent to

$$\mathbb{E}\left[\mathbb{E}\left[g(\mathbf{X}_K)|\mathbf{X}_{\bar{K}}\right]\mathbb{I}\{\mathbf{X}_{\bar{K}} \in A\}\right] \geq \mathbb{E}\left[\mathbb{E}\left[g(\pi_{ij}(\mathbf{X}_K))|\mathbf{X}_{\bar{K}}\right]\mathbb{I}\{\mathbf{X}_{\bar{K}} \in A\}\right].$$

Therefore, $\mathbb{E}\left[g(\mathbf{X}_K)|\mathbf{X}_{\bar{K}}\right] \geq_{a.s.} \mathbb{E}\left[g(\pi_{ij}(\mathbf{X}_K))|\mathbf{X}_{\bar{K}}\right]$ by Lemma 4.4.4. □

Proposition 4.4.6 $\mathbf{X}$ is SAI if and only if $(X_i, X_j)|\mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}$ is SAI for any $1 \leq i < j \leq n$ and any $\mathbf{x}_{\overline{ij}} \in S(\mathbf{X}_{\overline{ij}})$.

Proof. The "only if" part follows directly from Proposition 4.4.5.

As for the "if" part, consider any $1 \leq i < j \leq n$ and any function $g \in \mathcal{G}_{sai}^{ij}$. We want to show

$$\mathbb{E}\left[g(\mathbf{X})\right] \geq \mathbb{E}\left[g(\pi_{ij}(\mathbf{X}))\right].$$

Note that for any fixed $\mathbf{x}_{\overline{ij}} \in S(\mathbf{X}_{\overline{ij}})$, $g(x_i, x_j; \mathbf{x}_{\overline{ij}})$ is arrangement increasing as a bivariate function. Since $(X_i, X_j)|\mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}$ is SAI, we have $\mathbb{E}\left[g(\mathbf{X})|\mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}\right] \geq \mathbb{E}\left[g(\pi_{ij}(\mathbf{X}))|\mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}\right]$, which implies $\mathbb{E}\left[g(\mathbf{X})\right] \geq \mathbb{E}\left[g(\pi_{ij}(\mathbf{X}))\right]$ by taking expectation on $\mathbf{X}_{\overline{ij}}$. □

Recall that the likelihood ratio order could be characterized by distribution, i.e., $X \leq_{lr} Y$ if and only if $\mathbb{P}\{X \in A\}\mathbb{P}\{Y \in B\} \geq \mathbb{P}\{X \in B\}\mathbb{P}\{Y \in A\}$ for all measurable sets A, B such that $\sup A \leq \inf B$, see Shaked and Shanthikumar (2007). This fact motivates us to derive a similar characterization for SAI. In doing so, we first derive some equivalent characterizations for the bivariate SAI dependence (i.e., the joint likelihood ratio order).

Proposition 4.4.7 The following statements are equivalent.

(i) (X, Y) is SAI.

(ii) $\mathbb{P}\{(X,Y) \in I \times J\} \geq \mathbb{P}\{(X,Y) \in J \times I\}$ for all measurable sets $I, J \subset \mathbb{R}$ such that $\sup I \leq \inf J$.

(iii) $\mathbb{P}\{(X,Y) \in A\} \geq \mathbb{P}\{(Y,X) \in A\}$ for any measurable $A \subseteq \{(x,y)|x \leq y\}$.

Proof. (i)$\Longrightarrow$(iii) is obvious since $h(x,y) = \mathbb{I}\{(x,y) \in A\}$ is arrangement increasing for any measurable $A \subset \{(x,y) \in \mathbb{R}^2 : x \leq y\}$.

(iii)$\Longrightarrow$(ii) is straightforward.

(ii)$\Longrightarrow$(i). Consider any arrangement increasing function $g(x,y)$. We first assume $g(x,y) \geq 0$. For positive integer n, define

$$\mathcal{A}_n = \left\{ \left[\frac{i}{2^n}, \frac{i+1}{2^n}\right) \times \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right), i,j \in \mathbb{Z} \text{ and } -n2^n \leq j < i \leq n2^n - 1 \right\}$$

and

$$g_n(x,y) = \sum_{A \in \mathcal{A}_n} \inf_{(s,t) \in A} g(s,t) \times \mathbb{I}\{(x,y) \in A\},$$

where the minimum $\inf_{(x,y)\in A} g(x,y)$ always exists since $g(x,y) \geq 0$. It is easy to see that $\{g_n(x,y)\}$ is an increasing series and converges to $g(x,y) \times \mathbb{I}\{x > y\}$. Therefore, by monotone convergence theorem, we have

$$\mathbb{E}\left[g(X,Y)\mathbb{I}\{X > Y\}\right] = \lim_{n\to\infty} \mathbb{E}\left[g_n(X,Y)\right].$$

For any set $A \subset \mathbb{R}^2$, define its symmetric set as $A^s = \{(y,x) : (x,y) \in A\}$.

Define $\mathcal{B}_n = \{A^s, A \in \mathcal{A}_n\}$, and

$$h_n(x,y) = \sum_{B \in \mathcal{B}_n} \inf_{(s,t)\in B} g(s,t) \times \mathbb{I}\{(x,y) \in B\},$$

then

$$\mathbb{E}\left[g(X,Y)\mathbb{I}\{X < Y\}\right] = \lim_{n\to\infty} \mathbb{E}\left[h_n(X,Y)\right].$$

Therefore,

$$\mathbb{E}\left[g(X,Y)\right] = \lim_{n\to\infty} \mathbb{E}\left[g_n(X,Y) + h_n(X,Y)\right] + \mathbb{E}\left[g(X,X)\mathbb{I}\{X = Y\}\right]. \quad (4.4.2)$$

$$\mathbb{E}\left[g(Y,X)\right] = \lim_{n\to\infty} \mathbb{E}\left[g_n(Y,X) + h_n(Y,X)\right] + \mathbb{E}\left[g(X,X)\mathbb{I}\{X = Y\}\right]. \quad (4.4.3)$$

In order to show $\mathbb{E}\left[g(X,Y)\right] \geq \mathbb{E}\left[g(Y,X)\right]$, it is sufficient to show that

$$\mathbb{E}\left[g_n(X,Y) + h_n(X,Y)\right] \geq \mathbb{E}\left[g_n(Y,X) + h_n(Y,X)\right] \text{ for all } n. \quad (4.4.4)$$

On the other hand, we have

$$\mathbb{E}\left[g_n(X,Y)\right] = \sum_{A\in\mathcal{A}_n} \inf_{(x,y)\in A} g(x,y) \times \mathbb{P}\{(X,Y) \in A\};$$

$$\mathbb{E}\left[h_n(X,Y)\right] = \sum_{A\in\mathcal{A}_n} \inf_{(x,y)\in A^s} g(x,y) \times \mathbb{P}\{(X,Y) \in A^s\};$$

$$\mathbb{E}\left[g_n(Y,X)\right] = \sum_{A\in\mathcal{A}_n} \inf_{(x,y)\in A} g(x,y) \times \mathbb{P}\{(X,Y) \in A^s\};$$

$$\mathbb{E}\left[h_n(Y,X)\right] = \sum_{A\in\mathcal{A}_n} \inf_{(x,y)\in A^s} g(x,y) \times \mathbb{P}\{(X,Y) \in A\}.$$

Note that for any $A \in \mathcal{A}_n$, A^s has the form of $I \times J$ with $I, J \subset \mathbb{R}$ and $\sup I \leq \inf J$. Based on the assumption of (ii), we have $\mathbb{P}\{(X,Y) \in A^s\} \geq \mathbb{P}\{(X,Y) \in A\}$ for any $A \in \mathcal{A}_n$. Noting that $g(x,y) \leq g(y,x)$ since g is arrangement increasing, we have

$$\inf_{(x,y)\in A} g(x,y) \leq \inf_{(x,y)\in A} g(y,x) = \inf_{(x,y)\in A^s} g(x,y).$$

Recall the arrangement inequality, i.e., $ab+cd \geq ad+bc$ for any constants a, b, c, d such that $a \leq c$ and $b \leq d$. Therefore, we have

$$\begin{aligned} &\inf_{(x,y)\in A} g(x,y) \times \mathbb{P}\{(X,Y) \in A\} + \inf_{(x,y)\in A^s} g(x,y) \times \mathbb{P}\{(X,Y) \in A^s\} \\ \geq &\inf_{(x,y)\in A} g(x,y) \times \mathbb{P}\{(X,Y) \in A^s\} + \inf_{(x,y)\in A^s} g(x,y) \times \mathbb{P}\{(X,Y) \in A\}, \end{aligned}$$

which implies (4.4.4) immediately.

Similarly, it can be proved that $\mathbb{E}\left[g(X,Y)\right] \geq \mathbb{E}\left[g(X,Y)\right]$ for any negative arrangement increasing function $g(x,y)$ such that the expectations exist.

For general arrangement increasing function $g(x,y)$, denote $g_+(x,y) = \max\{g(x,y), 0\}$ and $g_-(x,y) = \min\{g(x,y), 0\}$. Then $g_+(x,y)$ and $g_-(x,y)$ are both arrangement increasing, and $g(x,y) = g_+(x,y) + g_-(x,y)$. According to the above result, we have

$$\mathbb{E}\left[g_+(X,Y)\right] \geq \mathbb{E}\left[g_+(Y,X)\right] \text{ and } \mathbb{E}\left[g_-(X,Y)\right] \geq \mathbb{E}\left[g_-(Y,X)\right],$$

which implies $\mathbb{E}\left[g(X,Y)\right] \geq \mathbb{E}\left[g(Y,X)\right]$. □

Based on Propositions 4.4.6 and 4.4.7, we derive an equivalent characterization of SAI through distribution for random vector with high dimension below.

Proposition 4.4.8 $\mathbf{X} = (X_1, ..., X_n)$ is SAI if and only if

$$\mathbb{P}\{(X_i, X_j) \in A_{ij}, \mathbf{X}_{\overline{ij}} \in \mathbf{A}_{\overline{ij}}\} \geq \mathbb{P}\{(X_j, X_i) \in A_{ij}, \mathbf{X}_{\overline{ij}} \in \mathbf{A}_{\overline{ij}}\}, \tag{4.4.5}$$

for any $1 \leq i < j \leq n$ and any measurable sets $A_{ij} \subset \{(x, y)|\, x \leq y\}$ and $\mathbf{A}_{\overline{ij}} \subseteq \mathbb{R}^{n-2}$.

Proof. The only if part is straightforward by noting that the function $h(\mathbf{x}) = \mathbb{I}\{(x_i, x_j) \in A_{ij}, \mathbf{x}_{\overline{ij}} \in \mathbf{A}_{\overline{ij}}\} \in \mathcal{G}_{sai}^{ij}$ for any $1 \leq i < j \leq n$ and $A_{ij} \subseteq \{(x, y)|x \leq y\}$ and $\mathbf{A}_{\overline{ij}} \subseteq \mathbb{R}^{n-2}$.

For the if part, assume that (4.4.5) holds. It could be rewritten as

$$\begin{aligned} & \mathbb{E}\left[\mathbb{E}\left[\mathbb{I}\{(X_i, X_j) \in A_{ij}\}|\mathbf{X}_{\overline{ij}}\right] \times \mathbb{I}\{\mathbf{X}_{\overline{ij}} \in \mathbf{A}_{\overline{ij}}\}\right] \\ \geq \; & \mathbb{E}\left[\mathbb{E}\left[\mathbb{I}\{(X_j, X_i) \in A_{ij}\}|\mathbf{X}_{\overline{ij}}\right] \times \mathbb{I}\{\mathbf{X}_{\overline{ij}} \in \mathbf{A}_{\overline{ij}}\}\right]. \end{aligned}$$

According to Lemma 4.4.4 we have

$$\mathbb{E}\left[\mathbb{I}\{(X_i, X_j) \in A_{ij}\}|\mathbf{X}_{\overline{ij}}\right] \leq_{a.s.} \mathbb{E}\left[\mathbb{I}\{(X_j, X_i) \in A_{ij}\}|\mathbf{X}_{\overline{ij}}\right],$$

which means that $(X_i, X_j)|\mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}$ is SAI according to Proposition 4.4.7 (iii). Thus $\mathbf{X}$ is SAI according to Proposition 4.4.6. $\square$

Note that SAI is characterized by the density function, while UOAI/CUOAI/LOAI are characterized by the survival function. We should expect the implication from SAI to UOAI/CUOAI/LOAI.

Proposition 4.4.9 If $\mathbf{X}$ is SAI, then $\mathbf{X}$ is UOAI/LOAI/CUOAI.

Proof. Note that for any $x_1 < x_2$, $h(y_1, ..., y_n) = \mathbb{I}\{y_1 > x_1, ..., y_n > x_n\} \in \mathcal{G}_{sai}^{12}$. Since $\mathbf{X}$ is SAI, we have

$$\begin{aligned}\bar{F}(x_1, x_2, ..., x_n) &= \mathbb{E}\left[\mathbb{I}\{X_1 > x_1, ..., X_n > x_n\}\right] = \mathbb{E}\left[h(\mathbf{X})\right] \\ &\geq \mathbb{E}\left[h(\pi_{12}(\mathbf{X}))\right] = \bar{F}(x_2, x_1, ..., x_n).\end{aligned}$$

Following the same argument, it could be proved that $\bar{F}(\mathbf{x}) \geq \bar{F}(\pi_{ij}(\mathbf{x}))$ for any $1 \leq i < j \leq n$ such that $x_i < x_j$, which means that $\mathbf{X}$ is UOAI.

Similarly, note that $h(y_1, ..., y_n) = \mathbb{I}\{y_1 \leq x_1, ..., y_n \leq x_n\} \in \mathcal{G}_{sai}^{ij}$ for any $1 \leq i < j \leq n$ and $x_i < x_j$. Then $F(x_1, ..., x_n) = \mathbb{E}\left[h(\mathbf{X})\right] \geq \mathbb{E}\left[h(\pi_{ij}(\mathbf{X}))\right] = F(\pi_{ij}(x_1, ..., x_n))$, which implies that $\mathbf{X}$ is LOAI.

From Proposition 4.4.5, we know that $\mathbf{X}_K | \mathbf{X}_{\bar{K}} = \mathbf{x}_{\bar{K}}$ is SAI for any set $K \subset \{1, ..., n\}$, and thus $\mathbf{X}$ is CUOAI. □

Proposition 4.4.10 Assume that $\mathbf{X} = (X_1, ..., X_n)$ is mutually independent. Then $\mathbf{X}$ is SAI if and only if $X_i \leq_{lr} X_{i+1}$ for all $i = 1, ..., n-1$.

Proof. "$\Longrightarrow$" From Proposition 4.4.3, we know that (X_i, X_{i+1}) is SAI. According to Proposition 4.4.7, we have $\mathbb{P}\{X_i \in A, X_{i+1} \in B\} \geq \mathbb{P}\{X_i \in B, X_{i+1} \in A\}$ for any measurable sets $A, B \subset \mathbb{R}$ such that $\sup A \leq \inf B$. Since X_i, X_{i+1} are independent, we have $\mathbb{P}\{X_i \in A\}\mathbb{P}\{X_{i+1} \in B\} \geq \mathbb{P}\{X_i \in B\}\mathbb{P}\{X_{i+1} \in A\}$, which means $X_i \leq_{lr} X_{i+1}$.

"$\Longleftarrow$" By the transitivity property of $\leq_{lr}$, we know that $X_i \leq_{lr} X_j$ for any $1 \leq i < j \leq n$. Noting that if X and Y are independent, then $X \leq_{lr} Y \Rightarrow (X, Y)$ is SAI, we have (X_i, X_j) is SAI and thus $(X_i, X_j) | \mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}$ is SAI for any $\mathbf{x}_{\overline{ij}} \in S(\mathbf{X}_{\overline{ij}})$. Therefore, $\mathbf{X}$ is SAI by Proposition 4.4.6. □

Proposition 4.4.11 If $\mathbf{X}$ is SAI, then $X_i \leq_{st} X_{i+1}$ for $i = 1, ..., n-1$; if $X_i \leq_{a.s.} X_{i+1}$ for $i = 1, ..., n-1$, then $\mathbf{X}$ is SAI.

Proof. The first part is obvious. The second part is straightforward by noting that if $X_i \leq_{as} X_{i+1}$ for $i = 1, ..., n-1$, then we have $g(\mathbf{X}) \geq_{a.s.} g(\pi_{ij}(\mathbf{X}))$ for any $g \in \mathcal{G}_{sai}^{ij}$. □

Combining the above proposition with Proposition 1.2.5, we have

Corollary 4.4.12 Assume that $\mathbf{X} = (X_1, ..., X_n)$ is comonotonic. Then $\mathbf{X}$ is SAI if and only if $X_i \leq_{st} X_{i+1}$ for all $i = 1, ..., n-1$.

Remark 4.4.13 Proposition 4.4.10 and Corollary 4.4.12 imply that two special dependence structures studied in Cheung (2007) are incorporated in this section.

Proposition 4.4.14 If random vector $(X_1, ..., X_n)$ is SAI, then $(f(X_1), ..., f(X_n))$ is also SAI for any increasing function $f(x)$.

Proof. For any $1 \leq i < j \leq n$, consider any function $g(x_1, ..., x_n) \in \mathcal{G}_{sai}^{ij}$. Since $f(x)$ is increasing, it is easy to verify that $h(x_1, ..., x_n) = g(f(x_1), ..., f(x_n)) \in \mathcal{G}_{sai}^{ij}$, therefore

$$\begin{aligned} & \mathbb{E}\left[h(X_1, ..., X_n)\right] \geq \mathbb{E}\left[h(\pi_{ij}(X_1, ..., X_n))\right] \\ \iff \; & \mathbb{E}\left[g(f(X_1), ..., f(X_n))\right] \geq \mathbb{E}\left[g(\pi_{ij}(f(X_1), ..., f(X_n)))\right]. \end{aligned}$$ □

Corollary 4.4.15 Let random vector Z be independent of random vector $(X_1, ..., X_n)$. If $(X_1, ..., X_n)$ is SAI, then $(f(X_1, Z), ..., f(X_n, Z))$ is SAI for any bivariate function $f(x, z)$ increasing in x.

Proof. Since $f(x,z)$ is increasing in x, we have $(f(X_1,z),...,f(X_n,z))$ is SAI for any fixed z according to Proposition 4.4.14. For any $1 \le i < j \le n$, consider any function $g(x_1,...,x_n) \in \mathcal{G}_{sai}^{ij}$. We have

$$\begin{aligned} & \mathbb{E}\left[g(f(X_1,Z),...,f(X_n,Z))\right] = \mathbb{E}\left[\mathbb{E}\left[g(f(X_1,Z),...,f(X_n,Z))|Z\right]\right] \\ \ge \; & \mathbb{E}\left[\mathbb{E}\left[g(\pi_{ij}(f(X_1,Z),...,f(X_n,Z)))|Z\right]\right] = \mathbb{E}\left[g(\pi_{ij}(f(X_1,Z),...,f(X_n,Z)))\right], \end{aligned}$$

which implies that $(f(X_1,Z),...,f(X_n,Z))$ is SAI. □

Example 4.4.16 *Common shock model and inflation model.*

Let random vector $(X_1,...,X_n)$ be SAI and the common shock random variable Z be independent of $(X_1,...,X_n)$, then random vector $(X_1 \wedge Z,...,X_n \wedge Z)$ is also SAI.

If $Z \ge 0$, then random vector $(X_1\, Z,...,X_n\, Z)$ is also SAI.

Recalling Proposition 4.2.6 and Proposition 4.2.24, we know that, for a UOAI/LOAI random vector, some of its sub-random vectors can be compared in the upper/lower orthant order. Now with the SAI dependence structure, we can strengthen the upper/lower orthant order to the multivariate usual stochastic order.

Proposition 4.4.17 Let K_1, K_2, K_3 be mutually disjoint subsets of $\{1,...,n\}$ such that $|K_1| = |K_2|$ and $\max K_1 < \min K_2$. If $(X_1,...,X_n)$ is SAI, then

$$(X_k, k \in K_1 \cup K_3) \le_{st} (X_k, k \in K_2 \cup K_3).$$

Proof. For simplicity, assume $K_1 = \{1\}, K_2 = \{2\}$ and $K_3 = \{3, ..., n\}$. We need to show

$$(X_1, X_3, ..., X_n) \leq_{st} (X_2, X_3, ..., X_n).$$

Consider any increasing function $u(x_1, ..., x_{n-1}) : \mathbb{R}^{n-1} \to \mathbb{R}$. Define $g(x_1, ..., x_n) := u(x_2, x_3, ..., x_n)$. Then

$$g(x_1, x_2, ..., x_n) = u(x_2, x_3, ..., x_n) \geq u(x_1, x_3, ..., x_n) \geq g(x_2, x_1, ..., x_n),$$

for any $x_1 \leq x_2$. Since $(X_1, ..., X_n)$ is SAI, according to Definition 4.4.1, we have

$$\mathbb{E}[g(X_1, X_2..., X_n)] \geq \mathbb{E}[g(X_2, X_1..., X_n)],$$

which means that $\mathbb{E}[u(X_2, X_3..., X_n)] \geq \mathbb{E}[u(X_1, X_3..., X_n)]$. □

4.5 Summary

In this chapter, we have developed the new properties of the dependence notion of SAI and proposed new notions of dependence structures of UOAI, CUOAI, TDAI and CTDAI. As have been proved in the previous sections, these notions have the following implications.

$$\text{SAI} \Longrightarrow \text{CTDAI} \Longrightarrow \text{TDAI/CUOAI} \Longrightarrow \text{UOAI}.$$

Note that TDAI and CUOAI do not imply each other. We need to point out that all the above implications are strict. In other words, the reverses of those implications do not hold. We take the implication CTDAI $\Rightarrow$ CUOAI and CUOAI $\Rightarrow$ UOAI for example.

Example 4.5.1 *UOAI $\not\Rightarrow$ CUOAI, CUOAI $\not\Rightarrow$ CTDAI*

Let (X_1, X_2, X_3) be a discrete random vector with the following joint probability mass function: $\mathbb{P}\{(X_1, X_2, X_3) = (1, 2, 3)\} = p_1$, $\mathbb{P}\{(X_1, X_2, X_3) = (2, 1, 4)\} = p_2$, $\mathbb{P}\{(X_1, X_2, X_3) = (2, 3, 5)\} = p_3$ with $p_1+p_2+p_3 = 1$ and $p_1 > p_2$. Then it is easy to verify that (X_1, X_2, X_3) is UOAI but not CUOAI.

Furthermore, let (X, Y) be a random vector with the following joint probability mass function: $p_{00} = p_{11} = p_{22} = p_{01} = p_{10} = 0$, $p_{02} = 0.1$, $p_{12} = 0.4$, $p_{20} = 0.2$, and $p_{21} = 0.3$, where $p_{ij} = \mathbb{P}\{X = i, Y = j\}$ for $i, j = 0, 1, 2$. Then, it is easy to verify that (X, Y) is CUOAI but not CTDAI.

One important common property of these dependence structures is the invariant property under uniform increasing transformations. Specifically, if $(X_1, ..., X_n)$ is UOAI (or LOAI, TDAI, SAI), then $(f(X_1), ..., f(X_n))$ is also UOAI (or LOAI, TDAI, SAI) respectively for any increasing function $f(x)$. To make a comparison, we recall the invariant property of the positive dependence structures in Chapter 2: $(X_1, ..., X_n)$ is PDS/PDUO implies $(f_1(X_1), ..., f_n(X_n))$ is PDS/PDUO for any increasing function $f_1, ..., f_n$. We find that the assumption of the invariant property for PDS/PDUO is much weaker, in the sense that it does not require the increasing transformations to be uniform.

As a matter of fact, this difference originates from the different natures of Type I dependence and Type II dependence. As we know, positive dependence structures introduced in Chapter 2 belong to Type I dependence, which is a property of copulas and does not involve marginal distributions. However, on the other hand, UOAI/LOAI/TDAI/SAI is a Type II dependence structure, which involves the nature of marginal distributions. When applying different transformations on different components of a random vector, the nature of the marginal distributions is not necessarily preserved.

Proposition 4.2.12, Corollary 4.2.13, Proposition 4.2.14, Proposition 4.2.27, and Proposition 4.3.10 demonstrate that random vectors with dependence structures of UOAI, LOAI, CUOAI, and CTDAI can be constructed through copulas, especially Archimedean copulas. It turns out that a random vector $(X_1, ..., X_n)$ linked by an Archimedean survival copula $C(u_1, ..., u_n) = \Psi^{-1}(\sum_{k=1}^{n} \Psi(u_k))$ with $\Psi(e^t)$ convex and $X_1 \leq_{hr} ... \leq_{hr} X_n$ is CTDAI, and thus is also CUOAI, TDAI and UOAI.

Archimedean copulas which satisfy the condition that $\Psi(e^t)$ is convex are not difficult to find. For example, Gumbel copula $\Psi(x) = (-\log x)^\alpha$ with $\alpha \geq 1$ and Clayton copula $\Psi(x) = x^{-\theta} - 1$ with $\theta > 0$. Recall that Example 2.4.2 in Chapter 2, the above two classes of copulas are actually PDUO. In this sense, we build a connection between dependence notions of UOAI/CUOAI/TDAI/CTDAI and positive dependence notion of PDUO.

Chapter 5

Optimal Allocation Problems with Dependent Risks

5.1 Introduction

In this chapter, we apply the dependence structures proposed in Chapter 4 to study optimal allocation problems in the fields of insurance, finance as well as operations research.

In the field of insurance, we model by $\{X_i, i = 1, ..., n\}$ the losses/risks faced by a policyholder and by S_i the occurrence time of the loss for $i = 1, ..., n$. Typically, there are two types of insurance strategies: policy limit and deductible.

Under the policy limit strategy, the policyholder is granted an amount of $\$l > 0$ as total policy limits, which can be allocated arbitrarily among the n risks. Let $d_i \geq 0$ be the policy limit allocated to the risk X_i for $i = 1, ..., n$; then $d_1 + ... + d_n = l$. For each limit d_i on the risk X_i, the insurer covers $X_i \wedge d_i$ and the policyholder retains $X_i - X_i \wedge d_i = (X_i - d_i)_+$.

The total discounted retained loss of the policyholder is

$$\sum_{i=1}^{n} e^{-\delta S_i}(X_i - d_i)_+,$$

where $\delta \geq 0$ is the force of interest.

The policyholder may choose the optimal limits $d_1^*, ..., d_n^*$ to maximize the expected utility of the total discounted wealth:

$$\max_{\sum_{i=1}^{n} d_i = l} \mathbb{E}\left[u\left(\omega - \sum_{i=1}^{n} e^{-\delta S_i}(X_i - d_i)_+\right)\right], \tag{5.1.1}$$

where ω is the initial wealth of the policyholder after premium is paid, and u is an increasing and/or concave utility function.

The policyholder may also choose the optimal limits $d_1^*, ..., d_n^*$ to minimize the expected discounted total retained loss:

$$\min_{\sum_{i=1}^{n} d_i = l} \mathbb{E}\left[\sum_{i=1}^{n} e^{-\delta S_i}(X_i - d_i)_+\right]. \tag{5.1.2}$$

Under the strategy with deductible, the policyholder is granted an amount of $\$l > 0$ as total deductibles, which can be allocated arbitrarily among the n risks. Let $d_i \geq 0$ be the deductible allocated to the risk X_i for $i = 1, ..., n$; then $d_1 + ... + d_n = l$. For each deductible d_i on the risk X_i, the insurer covers $(X_i - d_i)_+$ and the policyholder retains $X_i - (X_i - d_i)_+ = X_i \wedge d_i$. The total discounted retained loss of the policyholder is

$$\sum_{i=1}^{n} e^{-\delta S_i}(X_i \wedge d_i).$$

The policyholder may choose the optimal limits $d_1^*, ..., d_n^*$ to maximize the expected utility of the discounted wealth, or minimize the total expected discounted retained loss:

$$\max_{\sum_{i=1}^n d_i = l} \mathbb{E}\left[u\left(\omega - \sum_{i=1}^n e^{-\delta S_i}(X_i \wedge d_i)\right)\right], \tag{5.1.3}$$

$$\min_{\sum_{i=1}^n d_i = l} \mathbb{E}\left[\sum_{i=1}^n e^{-\delta S_i}(X_i \wedge d_i)\right]. \tag{5.1.4}$$

Note that if $u(x)$ is increasing and/or concave, then $u^*(x) = -u(\omega - x)$ is increasing and/or convex. Therefore, the optimal allocation problems (5.1.1)-(5.1.4) are reduced to the following two types of optimal allocation problems:

$$\min_{\sum_{i=1}^n d_i = l} \mathbb{E}\left[u\left(\sum_{i=1}^n e^{-\delta S_i}(X_i \wedge d_i)\right)\right] \quad , \tag{5.1.5}$$

$$\min_{\sum_{i=1}^n d_i = l} \mathbb{E}\left[u\left(\sum_{i=1}^n e^{-\delta S_i}(X_i - d_i)_+\right)\right] \quad , \tag{5.1.6}$$

where u is an increasing and/or convex function.

The optimization problems (5.1.5) and (5.1.6) have been studied with certain assumptions. For example, Cheung (2007) has studied the case without interest rate, that is $\delta = 0$. In this case, the impacts of the occurrence times $S_1, ..., S_n$ disappear and the problems (5.1.5) and (5.1.6) are reduced to

$$\min_{\sum_{i=1}^n d_i = l} \mathbb{E}\left[u\left(\sum_{i=1}^n X_i \wedge d_i\right)\right], \tag{5.1.7}$$

$$\min_{\sum_{i=1}^n d_i = l} \mathbb{E}\left[u\left(\sum_{i=1}^n (X_i - d_i)_+\right)\right]. \tag{5.1.8}$$

Cheung (2007) has studied the optimization model (5.1.7) and obtained some qualitative properties of the optimal solutions. Hua and Cheung (2008b) generalize the studies in Cheung (2007). Zhuang et al. (2009) has studied the problems in the case of $\delta > 0$ and derived similar results as Cheung (2007). Li and You (2012) extended the studies in Zhuang et al. (2009) by modeling the dependence between the occurrence times $\{S_1, ..., S_n\}$ with specific Archimedean copulas. In all the above researches, special assumptions are made about the dependence structure of $(X_1, ..., X_n)$.

By the introduction of the dependence structures of UOAI/CUOAI, CTDAI and SAI, the optimal allocation problems can be studied in a more general context. In this chapter, Sections 5.2 and 5.3 focus on solving the optimal allocations of policy limits/deductibles, and Sections 5.4 and 5.5 explore interesting applications of the newly defined dependence structures in optimal portfolio selections and optimal scheduling problems.

5.2 Optimal Allocation of Policy Deductibles/Limits

Lemma 5.2.1 Assume that random vector $\mathbf{X} = (X_1, ..., X_n)$ is UOAI . Let $f, g_1, \cdots, g_{n-2}$ be nonnegative strictly increasing functions, and denote $\mathbf{g} = (g_1, \cdots, g_{n-2})$. Then,

$$(f(X_j \wedge x_i), \mathbf{g}(\mathbf{X}_{\overline{ij}})) \times \mathbb{I}\{X_i > x_j\} \leq_{uo} (f(X_i \wedge x_i), \mathbf{g}(\mathbf{X}_{\overline{ij}})) \times \mathbb{I}\{X_j > x_j\},$$

for any $1 \leq i < j \leq n$ such that $x_i \leq x_j$.

Proof. Denote $\mathbf{V} = (V_1, \mathbf{V}_{\overline{ij}}) = (f(X_i \wedge x_i), \mathbf{g}(\mathbf{X}_{\overline{ij}})) \times \mathbb{I}\{X_j > x_j\}$ and $\mathbf{W} = (W_1, \mathbf{W}_{\overline{ij}}) = (f(X_j \wedge x_i), \mathbf{g}(\mathbf{X}_{\overline{ij}})) \times \mathbb{I}\{X_i > x_j\}$.

Note that for any $\mathbf{z} < \mathbf{0}$, $\mathbb{P}\{\mathbf{V} > \mathbf{z}\} = \mathbb{P}\{\mathbf{W} > \mathbf{z}\} = 1$; and for any $\mathbf{z} \not< \mathbf{0}$ (i.e., there exists $z_i \geq 0$), $\mathbf{V} > \mathbf{z}$ implies $X_j > x_j$. Then for any $z_1 < 0$ and $\mathbf{z}_{\overline{ij}} < \mathbf{0}$, we have $\mathbb{P}\{\mathbf{V} > \mathbf{z}\} = \mathbb{P}\{\mathbf{W} > \mathbf{z}\} = 1$.

For any $z_1 < 0$ and $\mathbf{z}_{\overline{ij}} \not< \mathbf{0}$, we have

$$\begin{aligned}
\mathbb{P}\{\mathbf{V} > (z_1, \mathbf{z}_{\overline{ij}})\} &= \mathbb{P}\{\mathbf{V}_{\overline{ij}} > \mathbf{z}_{\overline{ij}}\} = \mathbb{P}\{X_i > -\infty, X_j > x_j, \mathbf{X}_{\overline{ij}} > \mathbf{g}^{-1}(\mathbf{z}_{\overline{ij}})\} \\
&\geq \mathbb{P}\{X_i > x_j, X_j > -\infty, \mathbf{X}_{\overline{ij}} > \mathbf{g}^{-1}(\mathbf{z}_{\overline{ij}})\} \qquad (5.2.1) \\
&= \mathbb{P}\{\mathbf{W}_{\overline{ij}} > \mathbf{z}_{\overline{ij}}\} = \mathbb{P}\{\mathbf{W} > (z_1, \mathbf{z}_{\overline{ij}})\},
\end{aligned}$$

where $\mathbf{g}^{-1} = (g_1^{-1}, \cdots, g_{n-2}^{-1})$ and g_k^{-1} is the generalized left-continuous inverse of the function g_k.

For any $z_1 \geq f(x_i)$, $\mathbb{P}\{\mathbf{V} > (z_1, \mathbf{z}_{\overline{ij}})\} = \mathbb{P}\{\mathbf{W} > (z_1, \mathbf{z}_{\overline{ij}})\} = 0$.

For any $0 \leq z_1 < f(x_i)$,

$$\begin{aligned}
\mathbb{P}\{\mathbf{V} > (z_1, \mathbf{z}_{\overline{ij}})\} = \mathbb{P}\{X_i > f^{-1}(z_1), X_j > x_j, \mathbf{X}_{\overline{ij}} > \mathbf{g}^{-1}(\mathbf{z}_{\overline{ij}})\} \\
\geq \mathbb{P}\{X_i > x_j, X_j > f^{-1}(z_1), \mathbf{X}_{\overline{ij}} > \mathbf{g}^{-1}(\mathbf{z}_{\overline{ij}})\} = \mathbb{P}\{\mathbf{W} > (z_1, \mathbf{z}_{\overline{ij}})\}. \qquad (5.2.2)
\end{aligned}$$

Inequalities (5.2.1) and (5.2.2) hold because $\mathbf{X}$ is UOAI and $f^{-1}(z_1) < x_i \leq x_j$.

Combining all all the cases (i.e., $z_1 < 0$, $0 \leq z_1 < f(x_i)$ and $z_1 \geq f(x_i)$), we have $\mathbb{P}\{\mathbf{V} > \mathbf{z}\} \geq \mathbb{P}\{\mathbf{W} > \mathbf{z}\}$ for any $\mathbf{z} \in \mathbb{R}^{n-1}$, which means $\mathbf{V} \geq_{uo} \mathbf{W}$. □

Corollary 5.2.2 Let $f_1(x), f_2(x)$ be two strictly increasing univariate functions valued on $[0, \infty)$. Assume that there exists $x_0 \in \mathbb{R}$ such that $f_1(x) \leq f_2(x)$ for all $x \leq x_0$. If (X_1, X_2) is UOAI, then

$$f_1(X_2 \wedge x_1) \times \mathbb{I}\{X_1 > x_2\} \leq_{st} f_2(X_1 \wedge x_1) \times \mathbb{I}\{X_2 > x_2\},$$

for any x_1, x_2 such that $x_1 \le x_0$ and $x_2 \ge x_1$.

Proof. According to Lemma 5.2.1, we have

$$f_1(X_2 \wedge x_1) \times \mathbb{I}\{X_1 > x_2\} \le_{st} f_1(X_1 \wedge x_1) \times \mathbb{I}\{X_2 > x_2\}$$

for any $x_1 \le x_2$.

Since $x_1 \le x_0$, we have $X_i \wedge x_1 \le x_0$ for $i = 1, 2$, and thus $f_1(X_1 \wedge x_1) \le_{a.s.} f_2(X_1 \wedge x_1)$. Therefore,

$$f_1(X_2 \wedge x_1) \times \mathbb{I}\{X_1 > x_2\} \le_{st} f_2(X_1 \wedge x_1) \times \mathbb{I}\{X_2 > x_2\}. \qquad \square$$

In order to study problem (5.1.7), we define the following objective function:

$$M(\mathbf{d}) = M(d_1, ..., d_n) = \mathbb{E}\left[u\left(\sum_{i=1}^{n} X_i \wedge d_i\right)\right], \mathbf{d} = (d_1, \cdots, d_n) \ge 0. \tag{5.2.3}$$

For notational convenience, the vector $(d_1, ..., d_{i-1}, x, d_{i+1}, ..., d_n)$ is also referred as $(x, \mathbf{d}_{\bar{i}})$. In particular, $M(x, \mathbf{d}_{\bar{i}}) = M(d_1, ..., d_{i-1}, x, d_{i+1}, ..., d_n)$.

Proposition 5.2.3 Assume that $u(x)$ is an increasing convex function defined on $\mathbb{R}$ such that $\mathbb{E}\left[\left|u\left(\sum_{i=1}^{n} X_i\right)\right|\right] < \infty$. Then $M_i'^{+}(\mathbf{d}) = \frac{\partial^+}{\partial d_i} M(\mathbf{d})$ is right continuous in d_i and

$$M_i'^{+}(\mathbf{d}) = \frac{\partial^+}{\partial d_i} M(\mathbf{d}) = \mathbb{E}\left[u'^{+}\left(\sum_{i=1}^{n} X_i \wedge d_i\right) \mathbb{I}\{X_i > d_i\}\right]. \tag{5.2.4}$$

Proof. Denote $f(\omega, \mathbf{d}) = u\left(\sum \mathbf{X}(\omega) \wedge \mathbf{d}\right)$, then $M(\mathbf{d}) = \mathbb{E}\left[f(\omega, \mathbf{d})\right] = \int_\Omega f(\omega, \mathbf{d}) \mathbb{P}(d\omega)$. Noting that for any fixed $\omega \in \Omega$, the right partial derivative of $f(\omega, \mathbf{d})$ with respect to each

d_i exists and from the chain rule of one-sided derivatives, we have

$$\frac{\partial^+}{\partial d_i} f(\omega, \mathbf{d}) = u'^+ \left(\sum_{i=1}^{n} X_i(\omega) \wedge d_i \right) \times \mathbb{I}\{X_i > d_i\}.$$

Let $d_i \in [a, s] \subseteq (0, \infty)$. For any fixed $d_1, ..., d_{i-1}, d_{i+1}, ..., d_n \geq 0$, we have $0 \leq \sum_{i=1}^{n} X_i(\omega) \wedge d_i \leq s + \sum_{j \neq i} d_j$, and thus $u'^+ \left(\sum_{i=1}^{n} X_i(\omega) \wedge d_i \right)$, as a univariate function of d_i, is bounded on $[a, s]$ since $u'^+(x)$ is increasing on $\mathbb{R}$. Therefore, $\frac{\partial^+}{\partial d_i} f(\omega, \mathbf{d})$ is bounded on $[a, s]$ for any fixed $d_1, ..., d_{i-1}, d_{i+1}, ..., d_n \geq 0$. Denote the bound as A, and we have

$$\int_a^s \mathbb{E} \left[\left| \frac{\partial^+}{\partial d_i} f(\omega, \mathbf{d}) \right| \right] \mathrm{d}d_i \leq A\,(s - a) < \infty.$$

According to Fubini's theorem, we can exchange the order of the integration and the expectation:

$$\int_a^s \mathbb{E} \left[\frac{\partial^+}{\partial d_i} f(\omega, \mathbf{d}) \right] \mathrm{d}d_i = \mathbb{E} \left[\int_a^s \frac{\partial^+}{\partial d_i} f(\omega, \mathbf{d}) \mathrm{d}d_i \right].$$

For any fixed $\omega \in \Omega$, it is easy to verify that $u(x)$ satisfies Lipschitz condition on $[0,\ s + \sum_{j \neq i} d_j]$ and $g(\omega, \mathbf{d}) = \sum_{i=1}^{n} X_i(\omega) \wedge d_i$ satisfies Lipschitz condition on $[a,\ s]$ as a function of d_i. Therefore $f(\omega, \mathbf{d}) = u \circ g(\mathbf{d})$ also satisfies Lipschitz condition on $[a, s]$, and thus is absolute continuous on $[a, s]$. Then $f(\omega, \mathbf{d})$ is differentiable with respect to d_i almost everywhere on $[a, s]$, and the derivative is equal to the right derivative. By Fundamental Theorem II of Lebesgue integral, we have

$$\int_a^s \frac{\partial^+}{\partial d_i} f(\omega, \mathbf{d}) \mathrm{d}d_i = \int_a^s \frac{\partial}{\partial d_i} f(\omega, \mathbf{d}) \mathrm{d}d_i = f(\omega, (s, \mathbf{d}_{\bar{i}})) - f(\omega, (a, \mathbf{d}_{\bar{i}})).$$

Therefore,

$$\begin{aligned}
&\int_a^s \mathbb{E}\left[\frac{\partial^+}{\partial d_i} f(\omega, \mathbf{d})\right] \mathrm{d}d_i = \mathbb{E}\left[\int_a^s \frac{\partial^+}{\partial d_i} f(\omega, \mathbf{d}) \mathrm{d}d_i\right] \\
= \ & \mathbb{E}[f(\omega, (s, \mathbf{d}_{\bar{i}})) - f(\omega, (a, \mathbf{d}_{\bar{i}}))] = M(s, \mathbf{d}_{\bar{i}}) - \mathbb{E}\left[f(\omega, a, \mathbf{d}_{\bar{i}})\right]. \qquad (5.2.5)
\end{aligned}$$

Since $\frac{\partial^+}{\partial d_i} f(\omega, \mathbf{d})$ is right continuous in d_i and is bounded on $[a, s]$, according to Lebesgue dominated convergence theorem, we have $\mathbb{E}\left[\frac{\partial^+}{\partial d_i} f(\omega, \mathbf{d})\right]$ is right continuous in d_i.

It is easy to show that if $g(x)$ is right continuous and integrable on closed interval I and $G(x) = \int_a^x g(t)dt$, where $a \in I$, then $\frac{\partial^+}{\partial x} G(x) = g(x), \forall x \in I$. Thus, taking right derivative on both sides of (5.2.5), we get

$$\begin{aligned}
\frac{\partial^+}{\partial s} M(s, \mathbf{d}_{\bar{i}}) &= \frac{\partial^+}{\partial s} \int_a^s \mathbb{E}\left[\frac{\partial^+}{\partial d_i} f(\omega, \mathbf{d})\right] \mathrm{d}d_i = \mathbb{E}\left[\frac{\partial^+}{\partial s} f(\omega, s, \mathbf{d}_{\bar{i}})\right] \qquad (5.2.6) \\
&= \mathbb{E}\left[u'^+\left((\mathbf{X}(\omega) \wedge (s, \mathbf{d}_{\bar{i}})) \cdot \mathbf{e}\right) \times \mathbb{I}\{X_i > s\}\right].
\end{aligned}$$

Replace s with d_i in (5.2.6), we get (5.2.4). The right continuity of $\frac{\partial^+}{\partial d_i} M(\mathbf{d})$ is from (5.2.6).
□

Theorem 5.2.4 Assume that $\mathbf{X}$ is UOAI and nonnegative. Then for any $i \leq j$ such that $d_i \geq d_j$, it holds that

$$\mathbb{E}\left[u\left((\mathbf{X} \wedge \mathbf{d}) \cdot \mathbf{e}\right)\right] \leq \mathbb{E}\left[u\left((\mathbf{X} \wedge \pi_{ij}(\mathbf{d})) \cdot \mathbf{e}\right)\right], \text{ for any } u \in \mathcal{U}_{exp}^+ \cup \mathcal{U}_{mom}. \qquad (5.2.7)$$

Proof. Without loss of generality, assume $i = 1, j = 2$, then it is sufficient to show that $M(d_1, d_2, \mathbf{d}_{\overline{12}}) \leq M(d_2, d_1, \mathbf{d}_{\overline{12}})$, where $\overline{12} = \{3, \cdots, n\}$.

Note that

$$M(d_1, d_2, \mathbf{d}_{\overline{12}}) - M(d_2, d_2, \mathbf{d}_{\overline{12}}) = \int_{d_2}^{d_1} M_1'^{+}(s, d_2, \mathbf{d}_{\overline{12}}) \mathrm{d}s, \tag{5.2.8}$$

$$M(d_2, d_1, \mathbf{d}_{\overline{12}}) - M(d_2, d_2, \mathbf{d}_{\overline{12}}) = \int_{d_2}^{d_1} M_2'^{+}(d_2, s, \mathbf{d}_{\overline{12}}) \mathrm{d}s. \tag{5.2.9}$$

According to Proposition 5.2.3, we have

$$\begin{aligned} M_1'^{+}(s, d_2, \mathbf{d}_{\overline{12}}) &= \mathbb{E}\left[u'\left(s + X_2 \wedge d_2 + \sum \mathbf{X}_{\overline{12}} \wedge \mathbf{d}_{\overline{12}}\right) \mathbb{I}\{X_1 > s\}\right], \\ M_2'^{+}(d_2, s, \mathbf{d}_{\overline{12}}) &= \mathbb{E}\left[u'\left(s + X_1 \wedge d_2 + \sum \mathbf{X}_{\overline{12}} \wedge \mathbf{d}_{\overline{12}}\right) \mathbb{I}\{X_2 > s\}\right]. \end{aligned}$$

Denote $\mathbf{V} = (V_1, \mathbf{V}_{\overline{12}}) = (s + X_1 \wedge d_2, \mathbf{X}_{\overline{12}} \wedge \mathbf{d}_{\overline{ij}}) \times \mathbb{I}\{X_2 > s\}$ and $\mathbf{W} = (W_1, \mathbf{W}_{\overline{12}} \wedge \mathbf{d}_{\overline{ij}}) = (s + X_2 \wedge d_2, \mathbf{X}_{\overline{12}}) \times \mathbb{I}\{X_1 > s\}$. From Proposition 5.2.1, we know that $\mathbf{V} \geq_{uo} \mathbf{W}$ for all $s \geq d_2$.

Note that for $u(x)$ of the form $e^{\gamma x}$ or x^n, $u'(x)$ keeps the same form as $u(x)$, only differing up to a positive multiplier. From Lemma 1.2.12, we have $\mathbb{E}\left[u'\left(\sum \mathbf{V}\right)\right] \geq \mathbb{E}\left[u'\left(\sum \mathbf{W}\right)\right]$ for all $s \geq d_2$, which is equivalent to

$$M_2'^{+}(d_2, s, \mathbf{d}_{\overline{12}}) + u'(0)\mathbb{P}\{X_2 \leq s\} \geq M_1'^{+}(s, d_2, \mathbf{d}_{\overline{12}}) + u'(0)\mathbb{P}\{X_1 \leq s\}.$$

Since $u'(0) \geq 0$ and $X_1 \leq_{st} X_2$, we have $M_2'^{+}(d_2, s, \mathbf{d}_{\overline{12}}) \geq M_1'^{+}(s, d_2, \mathbf{d}_{\overline{12}})$ for all $s \geq d_2$, which implies $M(d_2, d_1, \mathbf{d}_{\overline{12}}) \geq M(d_1, d_2, \mathbf{d}_{\overline{12}})$ by (5.2.8) and (5.2.9). □

As a matter of fact, for any $u \in \mathcal{U}_{exp}^{+}$, (5.2.7) can be verified through Proposition 4.2.16.

Consider the case $i = 1, j = 2$, then (5.2.7) becomes

$$\begin{aligned}
& \mathbb{E}\left[\exp\left\{\gamma\left(X_1 \wedge d_1 + X_2 \wedge d_2 + \sum_{k=3}^{n} X_k \wedge d_k\right)\right\}\right] \\
\geq\ & \mathbb{E}\left[\exp\left\{\gamma\left(X_1 \wedge d_2 + X_2 \wedge d_1 + \sum_{k=3}^{n} X_k \wedge d_k\right)\right\}\right], \qquad (5.2.10)
\end{aligned}$$

for any $\gamma > 0$ and any $d_1 \leq d_2$.

Define the following function: $g_k(x) = e^{\gamma\,(x \wedge d_k)}$ and $g(x_1, ..., x_n) = \prod_{k=1}^{n} g_k(x_k)$. Noting that $g_k'^{+}(x) = \gamma\ \exp\{\gamma\, x\} \times \mathbb{I}\{x \leq d_k\}$, we have $g_1'^{+}(x) \leq g_2'^{+}(x)$ since $d_1 \leq d_2$. Furthermore, it is easy to verify that

$$\prod_{k=1}^{n} g_k'^{+}(x_k) = \gamma^n\ \exp\left\{\gamma \sum_{k=1}^{n} x_k\right\} \times \mathbb{I}\{x_1 \leq d_1, ..., x_n \leq d_n\} \in \mathcal{G}_{sai}^{12}.$$

Then according to Proposition 4.2.16, we have

$$\mathbb{E}\left[g(X_1, X_2, ..., X_n)\right] \geq \mathbb{E}\left[g(X_2, X_1, ..., X_n)\right],$$

which implies (5.2.10).

In the bivariate case, i.e., $n = 2$, we can derive stronger results from Theorem 5.2.4.

Corollary 5.2.5 Assume that (X_1, X_2) is UOAI and nonnegative. Then for any $d_1 \geq d_2$, we have

$$\mathbb{E}\ \left[u\left(X_1 \wedge d_1 + X_2 \wedge d_2\right)\right] \leq \mathbb{E}\ \left[u\left(X_1 \wedge d_2 + X_2 \wedge d_1\right)\right], \qquad \forall u \in \mathcal{U}_{icx}.$$

Proof. Following the same notations as in the proof for Theorem 5.2.4, the random vectors

$\mathbf{V}$ and $\mathbf{W}$ are reduced to two univariate random variables, which are re-denoted as V and W, and then $V \geq_{uo} W$ means $V \geq_{st} W$. Therefore, for any increasing convex function $u(x)$, we have $\mathbb{E}[u'(V)] \geq \mathbb{E}[u'(W)]$. Following the same argument as in the last paragraph of the proof for Theorem 5.2.4, we have $\mathbb{E}\left[u\left(X_1 \wedge d_1 + X_2 \wedge d_2\right)\right] \leq \mathbb{E}\left[u\left(X_1 \wedge d_2 + X_2 \wedge d_1\right)\right]$. □

Proposition 5.2.6 Assume that random vector (X_1, X_2) is UOAI and nonnegative. Then for any $d_1 \geq d_2$ and any increasing function f, we have

$$\mathbb{E}\left[u\left(f(X_1 \wedge d_1) + f(X_2 \wedge d_2)\right)\right] \leq \mathbb{E}\left[u\left(f(X_1 \wedge d_2) + f(X_2 \wedge d_1)\right)\right], \qquad \forall u \in \mathcal{U}_{icx}.$$

Proof. The conclusion is straightforward by noting that $(f(X_1), f(X_2))$ is also UOAI from Proposition 4.2.5 and the fact that $f(X \wedge d) = f(x) \wedge f(d)$. □

Theorem 5.2.7 Assume that $\mathbf{X}$ is CUOAI and nonnegative. Then for any $i \leq j$ such that $d_i \geq d_j$, we have

$$\mathbb{E}\left[u\left((\mathbf{X} \wedge \mathbf{d}) \cdot \mathbf{e}\right)\right] \leq \mathbb{E}\left[u\left((\mathbf{X} \wedge \pi_{ij}(\mathbf{d})) \cdot \mathbf{e}\right)\right],$$

for any increasing convex function u.

Proof. Without loss of generality, assume $i = 1, j = 2$. Since $\mathbf{X}$ is CUOAI, we have $(X_1, X_2)|\mathbf{X}_{\overline{12}} = \mathbf{x}_{\overline{12}}$ is UOAI, and thus $(X_1 + c, X_2 + c)|\mathbf{X}_{\overline{12}} = \mathbf{x}_{\overline{12}}$ is UOAI for any $c \in \mathbb{R}$. For any $\mathbf{x}_{\overline{12}} \in S(\mathbf{X}_{\overline{12}})$, let $c = \frac{1}{2}\sum \mathbf{x}_{\overline{12}} \wedge \mathbf{d}_{\overline{12}}$, and $d_1' = d_1 + c, d_2' = d_2 + c$, then $d_1' \geq d_2'$. According to Corollary 5.2.5, we have

$$\begin{aligned} & \mathbb{E}\left[u\left((X_1 + c) \wedge d_1' + (X_2 + c) \wedge d_2'\right) \mid \mathbf{X}_{\overline{12}} = \mathbf{x}_{\overline{12}}\right] \\ \leq\ & \mathbb{E}\left[u\left((X_1 + c) \wedge d_2' + (X_2 + c) \wedge d_1'\right) \mid \mathbf{X}_{\overline{12}} = \mathbf{x}_{\overline{12}}\right], \end{aligned}$$

or

$$\begin{aligned} & \mathbb{E}\left[u\left(X_1 \wedge d_1 + X_2 \wedge d_2 + \sum_{i=3}^{n} X_i \wedge d_i\right) \middle| \mathbf{X}_{\overline{12}}\right] \\ \leq\ & \mathbb{E}\left[u\left(X_1 \wedge d_2 + X_2 \wedge d_1 + \sum_{i=3}^{n} X_i \wedge d_i\right) \middle| \mathbf{X}_{\overline{12}}\right]. \end{aligned}$$

It completes the proof by taking expectation on $\mathbf{X}_{\overline{12}}$. □

Theorem 5.2.4 and Theorem 5.2.7 indicate that the solutions to problem (5.1.7) with different criteria should satisfy $d_1^* \geq ... \geq d_n^*$. Ordering the components of the solution to problem (5.1.8) needs a stronger assumption of CTDAI, as shown in the following theorem.

Theorem 5.2.8 If random vector $\mathbf{X} = (X_1, ..., X_n)$ is CTDAI, then

$$\mathbb{E}\left[u\left((\mathbf{X} - \mathbf{d})_+ \cdot \mathbf{e}\right)\right] \leq \mathbb{E}\left[u\left((\mathbf{X} - \pi_{ij}(\mathbf{d}))_+ \cdot \mathbf{e}\right)\right] \tag{5.2.11}$$

for any $u \in \mathcal{U}_2$ and any $1 \leq i < j \leq n$ such that $d_i \leq d_j$.

Proof. For any $d_i \leq d_j$, consider the function $g(\mathbf{x}) = g(x_1, ..., x_n) = (\mathbf{x} - \pi_{ij}(\mathbf{d}))_+ \cdot \mathbf{e}$. Then $\Delta_{ij} h(\mathbf{x}) = h(\mathbf{x}) - h(\pi_{ij}(\mathbf{x})) = h(x_j) - h(x_i)$, where $h(x) = (x - d_i)_+ - (x - d_j)_+$ is an increasing function. Therefore $g(x_1, ..., x_n) \geq g(\pi_{ij}(x_1, ..., x_n))$ for any $x_i \leq x_j$. We shall show that $u(g(x_1, ..., x_n)) \in \mathcal{G}_{ctdai}^{ij}$, and then (5.2.11) follows immediately from Proposition 4.3.8.

Noting that $\frac{\partial^+}{\partial x_j} g(x_1, ..., x_n) = \mathbb{I}\{x_j \geq d_i\}$ and $\frac{\partial^+}{\partial x_i} g(\pi_{ij}(x_1, ..., x_n)) = \mathbb{I}\{x_j \geq d_j\}$, we have

$$\begin{aligned}
\frac{\partial^+}{\partial x_j} u(g(x_1,...,x_n)) &= u'^+(g(x_1,...,x_n)) \times \mathbb{I}\{x_j \geq d_i\}, \\
\frac{\partial^+}{\partial x_j} u(g(\pi_{ij}(x_1,...,x_n))) &= u'^+(g(\pi_{ij}(x_1,...,x_n))) \times \mathbb{I}\{x_j \geq d_j\}.
\end{aligned}$$

Since $u'(t)$ is nonnegative and increasing and $g(x_1,...,x_n) \geq g(\pi_{ij}(x_1,...,x_n))$ for any $x_i \leq x_j$, we conclude that $\frac{\partial^+}{\partial x_j} u(g(x_1,...,x_n)) \geq \frac{\partial^+}{\partial x_j} u(g(\pi_{ij}(x_1,...,x_n)))$ for any $x_i \leq x_j$, which means $u(g(x_1,...,x_n)) \in \mathcal{G}^{ij}_{ctdai}$. □

5.3 Optimal Allocation of Policy Deductibles/Limits with Discount Factors

Theorem 5.3.1 If $\mathbf{X} = (X_1, ..., X_n)$ is SAI, then the solution to (5.1.8) satisfies: $d_1^* \leq \cdots \leq d_n^*$.

Proof. It is sufficient to show that $\mathbb{E}\left[u\left((\mathbf{X}-\mathbf{d})_+ \cdot \mathbf{e}\right)\right] \leq \mathbb{E}\left[u\left((\mathbf{X}-\pi_{12}(\mathbf{d}))_+ \cdot \mathbf{e}\right)\right]$ for all $\mathbf{d} \in \mathbb{R}^n$ such that $d_1 \leq d_2$.

Consider the function $g(\mathbf{x}) = (\mathbf{x} - \pi_{12}(\mathbf{d}))_+ \cdot \mathbf{e}$. Note that

$$g(\mathbf{x}) - g(\pi_{12}(\mathbf{x})) = (x_1 - d_2)_+ + (x_2 - d_1)_+ - (x_1 - d_1)_+ - (x_2 - d_2)_+ = h(x_2) - h(x_1),$$

where $h(x) = (x - d_1)_+ - (x - d_2)_+$ is increasing. Then we have $g(\mathbf{x}) - g(\pi_{12}(\mathbf{x})) \geq 0$ for all $x_1 \leq x_2$, which means that $g(\mathbf{x})$ is $\{12\}-$PAI. Since $u(x)$ is increasing, we know that $u \circ g(\mathbf{x})$ is also $\{12\}-$PAI. According to the definition of SAI, we have $\mathbb{E}\left[g(\mathbf{X})\right] \geq \mathbb{E}\left[g(\pi_{12}(\mathbf{X}))\right]$, i.e., $\mathbb{E}\left[u\left((\mathbf{X}-\mathbf{d})_+ \cdot \mathbf{e}\right)\right] \leq \mathbb{E}\left[u\left((\mathbf{X}-\pi_{12}(\mathbf{d}))_+ \cdot \mathbf{e}\right)\right]$. □

To study the model with discount factor, we need to extend some existing results. Righter and Shanthikumar (1992) have systematically studied the characterizations of bivariate stochastic orders throng functionals.

For any bivariate functions $g_1(x,y)$ and $g_2(x,y)$, we denote $\Delta_{21}g(x,y) = g_2(x,y) - g_1(x,y)$. We derive an analogue of Theorem 1(i) in Righter and Shanthikumar (1992) below.

Proposition 5.3.2 (X,Y) is SAI if and only if $\mathbb{E}[g_1(X,Y)] \le \mathbb{E}[g_2(X,Y)]$ for all $g_1(x,y)$ and $g_2(x,y)$ such that $\Delta g_{21}(x,y) \ge 0$ and $\Delta g_{21}(x,y) + \Delta g_{21}(y,x) \ge 0$ for all $x \le y$.

Proof. "$\Rightarrow$" For any arrangement increasing function $g(x,y)$, let $g_1(x,y) = g(y,x)$ and $g_2(x,y) = g(x,y)$. Then $\Delta g_{21}(x,y) \ge 0$ and $\Delta g_{21}(x,y) + \Delta g_{21}(y,x) \ge 0$ for all $x \le y$, thus $\mathbb{E}[g(X,Y)] \ge \mathbb{E}[g(Y,X)]$.

"$\Leftarrow$" Define $h(x,y) = (g_2(x,y) - g_1(x,y)) \times \mathbb{I}\{x < y\}$, then $h(y,x) = (g_2(y,x) - g_1(y,x)) \times \mathbb{I}\{y < x\}$, thus $h(x,y) \ge 0 = h(y,x)$ for all $x \le y$, which means that $h(x,y)$ is arrangement increasing. According to the definition of SAI, we have $\mathbb{E}[h(X,Y)] \ge \mathbb{E}[h(Y,X)]$, or

$$\mathbb{E}[(g_2(X,Y) - g_1(X,Y))\,\mathbb{I}\{X < Y\}] \ge \mathbb{E}[(g_2(Y,X) - g_1(Y,X))\,\mathbb{I}\{Y < X\}]. \quad (5.3.1)$$

Since $\Delta g_{21}(y,x) \ge -\Delta g_{21}(x,y)$ for all $y < x$, we have

$$\mathbb{E}[(g_2(Y,X) - g_1(Y,X))\,\mathbb{I}\{Y < X\}] \ge \mathbb{E}[(g_1(X,Y) - g_2(X,Y))\,\mathbb{I}\{Y < X\}]. \quad (5.3.2)$$

Combining (5.3.1), (5.3.2) and the fact that $\mathbb{E}[(g_2(X,Y) - g_1(X,Y)) \times \mathbb{I}\{X = Y\}] \ge 0$, we get $\mathbb{E}[(g_2(X,Y) - g_1(X,Y))] \ge 0$. □

The following lemma is a generalization of Lemma 4.6 in Zhuang et al. (2009).

Lemma 5.3.3 Let (X_1, X_2) be UOAI and $X_1, X_2 \geq 0$. Assume that ϕ is an increasing convex function. For any $d_1 \geq d_2$, define two functions:

$$g_1(\omega_1, \omega_2) = \mathbb{E}\,[\phi(\omega_1(X_1 \wedge d_1) + \omega_2(X_2 \wedge d_2))],$$
$$g_2(\omega_1, \omega_2) = \mathbb{E}\,[\phi(\omega_1(X_1 \wedge d_2) + \omega_2(X_2 \wedge d_1))].$$

Then (i) $\Delta g_{21}(x, y) \geq 0$ for all $x \leq y$; and (ii) $\Delta g_{21}(x, y) \geq -\Delta g_{21}(y, x)$ for all $x \leq y$.

Proof. Denote $g(\omega_1, \omega_2, d_1, d_2) = \mathbb{E}\,[\phi(\omega_1(X_1 \wedge d_1) + \omega_2(X_2 \wedge d_2))]$, then

$$g(\omega_1, \omega_2, d_1, d_2) - g(\omega_1, \omega_2, d_2, d_2) = \int_{d_2}^{d_1} \frac{\partial^+}{\partial s} g(\omega_1, \omega_2, s, d_2) ds, \tag{5.3.3}$$
$$g(\omega_1, \omega_2, d_2, d_1) - g(\omega_1, \omega_2, d_2, d_2) = \int_{d_2}^{d_1} \frac{\partial^+}{\partial s} g(\omega_1, \omega_2, d_2, s) ds. \tag{5.3.4}$$

According to Proposition 5.2.3, we have

$$\frac{\partial^+}{\partial s} g(\omega_1, \omega_2, s, d_2) = \mathbb{E}\,[\phi'^+(\omega_1\, s + \omega_2(X_2 \wedge d_2)) \times \omega_1\, \mathbb{I}\{X_1 > s\}], \tag{5.3.5}$$
$$\frac{\partial^+}{\partial s} g(\omega_1, \omega_2, d_2, s) = \mathbb{E}\,[\phi'^+(\omega_2\, s + \omega_1(X_1 \wedge d_2)) \times \omega_2\, \mathbb{I}\{X_2 > s\}]. \tag{5.3.6}$$

Consider the following two functions: $f_1(x) = \omega_1\, s + \omega_2\, x$ and $f_2(x) = \omega_2\, s + \omega_1\, x$. It is easy to verify that f_1, and f_2 satisfy the conditions in Proposition 5.2.2 for any $s \geq d_2$, and thus,

$$f_1(X_2 \wedge d_2) \times \mathbb{I}\{X_1 > s\} \leq_{st} f_2(X_2 \wedge d_2) \times \mathbb{I}\{X_2 > s\}.$$

Therefore,

$$\begin{aligned}
& \mathbb{E}\,[\phi'^{+}(\omega_1\, s + \omega_2(X_2 \wedge d_2)) \times \mathbb{I}\{X_1 > s\}] \\
= \;& \mathbb{E}\,[\phi'^{+}(\omega_1\, s + \omega_2(X_2 \wedge d_2) \times \mathbb{I}\{X_1 > s\})] - \phi'^{+}(0)\,\mathbb{P}\{X_1 \le s\} \\
\le \;& \mathbb{E}\,[\phi'^{+}(\omega_2\, s + \omega_1(X_1 \wedge d_2) \times \mathbb{I}\{X_2 > s\})] - \phi'^{+}(0)\,\mathbb{P}\{X_2 \le s\} \\
= \;& \mathbb{E}\,[\phi'^{+}(\omega_2\, s + \omega_1(X_1 \wedge d_2)) \times \mathbb{I}\{X_2 > s\}], \qquad (5.3.7)
\end{aligned}$$

which implies $\frac{\partial^+}{\partial s} g(\omega_1, \omega_2, s, d_2) \le \frac{\partial^+}{\partial s} g(\omega_1, \omega_2, d_2, s)$ if $\omega_1 \le \omega_2$, and thus it completes the proof of (i) by (5.3.3) and (5.3.4).

As for (ii), note that

$$\begin{aligned}
& \Delta g_{21}(\omega_1, \omega_2) + \Delta g_{21}(\omega_2, \omega_1) = g_2(\omega_1, \omega_2) - g_1(\omega_1, \omega_2) + g_2(\omega_2, \omega_1) - g_1(\omega_2, \omega_1) \\
= \;& g(\omega_1, \omega_2, d_2, d_1) - g(\omega_1, \omega_2, d_1, d_2) + g(\omega_2, \omega_1, d_2, d_1) - g(\omega_2, \omega_1, d_1, d_2) \\
= \;& \int_{d_2}^{d_1} \frac{\partial^+}{\partial s} g(\omega_1, \omega_2, d_2, s) ds - \int_{d_2}^{d_1} \frac{\partial^+}{\partial s} g(\omega_1, \omega_2, s, d_2) ds \\
& + \int_{d_2}^{d_1} \frac{\partial^+}{\partial s} g(\omega_2, \omega_1, d_2, s) ds - \int_{d_2}^{d_1} \frac{\partial^+}{\partial s} g(\omega_2, \omega_1, s, d_2) ds. \qquad (5.3.8)
\end{aligned}$$

Recalling (5.3.5) and (5.3.6), we have

$$\begin{aligned}
\frac{\partial^+}{\partial s} g(\omega_1, \omega_2, d_2, s) &= \mathbb{E}\,[\phi'^{+}(\omega_2\, s + \omega_1(X_1 \wedge d_2)) \times \omega_2\, \mathbb{I}\{X_2 > s\}], \\
\frac{\partial^+}{\partial s} g(\omega_2, \omega_1, s, d_2) &= \mathbb{E}\,[\phi'^{+}(\omega_2\, s + \omega_1(X_2 \wedge d_2)) \times \omega_2\, \mathbb{I}\{X_1 > s\}].
\end{aligned}$$

According to Corollary 5.2.2, for any $s \ge d_2$, we have

$$(\omega_2\, s + \omega_1(X_1 \wedge d_2)) \times \mathbb{I}\{X_2 > s\} \ge_{st} (\omega_2\, s + \omega_1(X_2 \wedge d_2)) \times \mathbb{I}\{X_1 > s\},$$

which, by the similar argument as in (5.3.7), implies $\frac{\partial^+}{\partial s} g(\omega_1, \omega_2, d_2, s) \geq \frac{\partial^+}{\partial s} g(\omega_2, \omega_1, s, d_2)$. Similarly, we can prove $\frac{\partial^+}{\partial s} g(\omega_2, \omega_1, d_2, s) \geq \frac{\partial^+}{\partial s} g(\omega_1, \omega_2, s, d_2)$. Combining with (5.3.8), we get $\Delta g_{21}(\omega_1, \omega_2) + \Delta g_{21}(\omega_2, \omega_1) \geq 0$. □

Theorem 5.3.4 Assume that $\mathbf{X} = (X_1, ..., X_n)$ is CUOAI and nonnegative, and $-\mathbf{S} = (-S_1, ..., -S_n)$ is SAI. Then the solution to (5.1.5) with $u \in \mathcal{U}_{icx}$, $(d_1^*, ..., d_n^*)$, satisfies: $d_1^* \geq \cdots \geq d_n^*$.

Proof. We first give the proof for the case $n = 2$. Denote $(W_1, W_2) = (e^{-\delta S_1}, e^{-\delta S_2})$, then (W_1, W_2) is SAI according to Proposition 4.4.14. It is sufficient to show that for any increasing convex function $u(x)$, the following holds:

$$\mathbb{E}\left[u((X_1 \wedge d_1)W_1 + (X_2 \wedge d_2)W_2)\right] \leq \mathbb{E}\left[u((X_1 \wedge d_2)W_1 + (X_2 \wedge d_1)W_2)\right], \tag{5.3.9}$$

for all $d_1 \geq d_2$.

Using the notation in Lemma 5.3.3, we have

$$\begin{aligned} & \mathbb{E}\left[u((X_1 \wedge d_1)W_1 + (X_2 \wedge d_2)W_2)\right] \\ = \; & \mathbb{E}\left[\mathbb{E}\left[u((X_1 \wedge d_1)W_1 + (X_2 \wedge d_2)W_2)\right] | (W_1, W_2)\right] = \mathbb{E}\left[g_1(W_1, W_2)\right], \\ & \mathbb{E}\left[u((X_1 \wedge d_2)W_1 + (X_2 \wedge d_1)W_2)\right] \\ = \; & \mathbb{E}\left[\mathbb{E}\left[u((X_1 \wedge d_2)W_1 + (X_2 \wedge d_1)W_2)\right] | (W_1, W_2)\right] = \mathbb{E}\left[g_2(W_1, W_2)\right]. \end{aligned}$$

Combining Lemma 5.3.3 and Proposition 5.3.2, we get $\mathbb{E}\left[g_1(W_1, W_2)\right] \leq \mathbb{E}\left[g_2(W_1, W_2)\right]$.

As for the case $n \geq 3$, it is sufficient to show that $\mathbb{E}\left[u(I_{\mathbf{X},\mathbf{S}}(\mathbf{d}))\right] \leq \mathbb{E}\left[u(I_{\mathbf{X},\mathbf{S}}(\pi_{ij}\mathbf{d}))\right]$ for any $1 \leq i < j \leq n$ and $d_i \geq d_j$. Without loss of generality, assume $i = 1, j = 2$ and

$d_1 \geq d_2$. Denote $\mathbf{W} = (e^{-\delta S_1}, ..., e^{-\delta S_n})$. Note that

$$\begin{aligned} & \mathbb{E}\left[u(I_{\mathbf{X},\mathbf{S}}(\mathbf{d}))\right] = \mathbb{E}\left[\mathbb{E}\left[u(I_{\mathbf{X},\mathbf{S}}(\mathbf{d}))|(\mathbf{X}_{\overline{12}}, \mathbf{S}_{\overline{12}})\right]\right] \\ = \; & \mathbb{E}\left[\mathbb{E}\left[u((X_1 \wedge d_1)W_1 + (X_2 \wedge d_2)W_2 + \mathbf{X}_{\overline{12}} \wedge \mathbf{d}_{\overline{12}} \star \mathbf{W}_{\overline{12}}|(\mathbf{X}_{\overline{12}}, \mathbf{S}_{\overline{12}})\right]\right]. \end{aligned} \quad (5.3.10)$$

For any fixed $(\mathbf{x}_{\overline{12}}, \mathbf{s}_{\overline{12}}) \in S(\mathbf{X}_{\overline{12}}, \mathbf{S}_{\overline{12}})$, we have $(X_1, X_2)|(\mathbf{X}_{\overline{12}}, \mathbf{S}_{\overline{12}}) = (\mathbf{x}_{\overline{12}}, \mathbf{s}_{\overline{12}})$ is UOAI, $(W_1, W_2)|(\mathbf{X}_{\overline{12}}, \mathbf{S}_{\overline{12}}) = (\mathbf{x}_{\overline{12}}, \mathbf{s}_{\overline{12}})$ is SAI and they are independent. Consider the increasing convex function $u_1(x) = u\left(x + \mathbf{X}_{\overline{12}} \wedge \mathbf{d}_{\overline{12}} \star \mathbf{W}_{\overline{12}}\right)$, according to (5.3.9) we have

$$\begin{aligned} & \mathbb{E}\left[u_1((X_1 \wedge d_1)W_1 + (X_2 \wedge d_2)W_2)|(\mathbf{X}_{\overline{12}}, \mathbf{S}_{\overline{12}})\right] \\ \leq_{a.s.} \; & \mathbb{E}\left[u_1((X_1 \wedge d_2)W_1 + (X_2 \wedge d_1)W_2)|(\mathbf{X}_{\overline{12}}, \mathbf{S}_{\overline{12}})\right]. \end{aligned}$$

Taking expectation on $(\mathbf{X}_{\overline{12}}, \mathbf{S}_{\overline{12}})$ and combing with (5.3.10), we have $\mathbb{E}\left[u(I_{\mathbf{X},\mathbf{S}}(\mathbf{d}))\right] \leq \mathbb{E}\left[u(I_{\mathbf{X},\mathbf{S}}(\pi_{12}(\mathbf{d})))\right]$. □

Theorem 5.3.5 Assume that $\mathbf{X} = (X_1, ..., X_n)$ is SAI and nonnegative, and $-\mathbf{S} = (-S_1, ..., -S_n)$ is SAI. Then the solution to (5.1.6) with $u \in \mathcal{U}_{icx}$, $(d_1^*, ..., d_n^*)$, satisfies: $d_1^* \leq \cdots \leq d_n^*$.

Proof. Let $\phi(x)$ be any increasing convex function. For any $d_1 \leq d_2$ and $\omega_1 \leq \omega_2$, define two functions:

$$\begin{aligned} \phi_1(x, y) &= \phi(\omega_1(x - d_1)_+ + \omega_2(y - d_2)_+), \\ \phi_2(x, y) &= \phi(\omega_1(x - d_2)_+ + \omega_2(y - d_1)_+). \end{aligned}$$

Then according to Lemma 4.1 in Zhuang et al. (2009), we have $\Delta\phi_{21}(x, y) \geq 0$ and $\Delta\phi_{21}(x, y) \geq -\Delta\phi_{21}(y, x)$ for all $x \leq y$. Since (X_1, X_2) is SAI, according to Proposition 5.3.2, we have $\mathbb{E}[\phi_1(X_1, X_2)] \leq \mathbb{E}[\phi_2(X_1, X_2)]$. Define the following two functions:

$$g_1(\omega_1, \omega_2) = \mathbb{E}[\phi(\omega_1(X_1 - d_1)_+ + \omega_2(X_2 - d_2)_+)],$$
$$g_2(\omega_1, \omega_2) = \mathbb{E}[\phi(\omega_1(X_1 - d_2)_+ + \omega_2(X_2 - d_1)_+)].$$

Following the same argument in Lemma 4.2 in Zhuang et al (2009), we have $\Delta g_{21}(x, y) \geq 0$ and $\Delta g_{21}(x, y) \geq -\Delta g_{21}(y, x)$ for all $x \leq y$. Since (W_1, W_2) is SAI and independent of (X_1, X_2), applying Proposition 5.3.2 again, we have

$$\mathbb{E}[u((X_1 - d_1)_+ W_1 + (X_2 - d_2)_+ W_2)] = \mathbb{E}[g_1(W_1, W_2)]$$
$$\leq \mathbb{E}[g_2(W_1, W_2)] = \mathbb{E}[u((X_1 - d_2)_+ W_1 + (X_2 - d_1)_+ W_2)],$$

for all increasing convex function u and $d_1 \leq d_2$.

The proof for multivariate case is similar as the proof in Theorem 5.3.4. □

Theorem 5.3.4 and Theorem 5.3.5 show that main results in Zhuang et al. (2009) still hold under the generalized dependence structures proposed in Chapter 4.

5.4 Optimal Allocation Problems in Finance

In this section, we shall study the optimal investment problem and optimal allocation of risk capitals. Consider n different assets in the market, and denote the returns of one dollar over the investment period by random variables $R_1, ..., R_n$. Assume that the initial capital is m, and the investment ratio on each asset is $a_1, ..., a_n$, where $a_k \geq 0, k = 1, ..., n$ and $\sum_{k=1}^n a_k = 1$. At the end of the investment term, the total wealth is $W = \sum_{k=1}^n m a_k \times R_k$, and the total return rate is $R = W/m = \sum_{k=1}^n a_k R_k$. One of an investor's objective is to maximize the expected utility of the total return rate. The optimization problem is formulated as

$$\max_{\sum_{k=1}^n a_k=1} \mathbb{E}\left[u\left(\sum_{k=1}^n a_k R_k\right)\right], \qquad \text{for all } u \in \mathcal{U},$$

where $\mathcal{U}$ is a class of utility functions, which is to be determined.

The optimal portfolio selection problems have been extensively studied with different assumptions. For example, with the assumption of exchangeability of the return rates random variables, Denuit and Vermandele (1998) have concluded that the capital should be equally invested on each asset so as to maximize the concave utility. As intuitive as the result seems, it is not trivial to prove the result. In the independent case, Landsberger and Meilijson (1990) and Kijima and Ohnishi (1996) have studied this problem with different assumptions on the return rates $R_1, ..., R_n$. Recently, Hua and Cheung (2008a) have introduced discount factor into this optimization problem. All the studies come to a conclusion that the optimal weight should be arranged in a certain order.

In the following, we focus on increasing convex utility function and study the following problem:

$$\max_{\sum_{k=1}^n a_k=1} \mathbb{E}\left[u\left(\sum_{k=1}^n a_k R_k\right)\right], \qquad \text{for all } u \in \mathcal{U}_{icx}. \tag{5.4.1}$$

Theorem 5.4.1 If the random vector $(R_1, ..., R_n)$ is CTDAI, then the optimal solution to Problem (5.4.1), $(a_1^*, ..., a_n^*)$, should satisfy $a_1^* \leq ... \leq a_n^*$.

Proof. It is sufficient to show that

$$\mathbb{E}\left[u\left(\sum_{k=1}^n a_k Y_k\right)\right] \geq \mathbb{E}\left[u\left(a_i Y_j + a_j Y_i + \sum_{k \neq i,j} a_k Y_k\right)\right],$$

for any $u \in \mathcal{U}_{icx}$ and any $1 \le i < j \le n$ and $a_i \le a_j$, which follows directly from Proposition 4.3.11. $\square$

We now study the threshold model proposed by Cheung and Yang (2004) with the new dependence structures proposed in Chapter 4.

Let $R_1, ..., R_n$ be return rates of n different assets. Each return rate is not realized only if the rate reaches a certain threshold, otherwise the actual return rate is 0. Mathematically, denote the actual return rates as $R_k^{act}, k = 1, ..., n$, and then $R_k^{act} = R_k \times \mathbb{I}\{R_k > l_k\}, k = 1, ..., n$, where $l_1, ..., l_n$ are predetermined thresholds. The investor's target is to maximize the expected utility of the total return rate, i.e.,

$$\max_{\sum_{k=1}^{n} a_k = 1} \mathbb{E}\left[u\left(\sum_{k=1}^{n} a_k R_k \, \mathbb{I}\{R_k > l_k\}\right)\right], \qquad \text{for all } u \in \mathcal{U}, \tag{5.4.2}$$

where $\mathcal{U}$ is to be determined.

We consider the case that $l_1 = \cdots = l_n = l$. Note that $R_k^{act} = f_l(R_k)$ where $f_l(x) = x\mathbb{I}\{x > l\}$ is an increasing function. According to Proposition 4.2.22 and Proposition 4.3.9, we know that LOAI and CTDAI are invariant under uniform increasing transformations. Therefore, following Theorem 5.4.1, we have

Corollary 5.4.2 Consider Problem (5.4.2) with $l_1 = ... = l_n = l$ and $\mathcal{U} = \mathcal{U}_{st}$. If $(R_1, ..., R_n)$ is CTDAI, then the optimal solution, $(a_1^*, ..., a_n^*)$, should satisfy $a_1^* \le ... \le a_n^*$. $\square$

Now we consider the allocation problem of risk capitals. Let $X_1, ..., X_n$ be n random variables, which represent losses or profits from n lines of business. From the viewpoint of the regulator, the investor is required to reserve certain amount of risk capital for each line of business to cope with the future uncertainty. A commonly used principle is Euler's principle, from which the risk capital for each line of business is determined by

$$\rho_i = \mathbb{E}\left[X_i \,|\, S > \mathrm{VaR}_\alpha S\right], \qquad i = 1, ..., n, \tag{5.4.3}$$

where $S = \sum_{k=1}^{n} X_k$ is the aggregate risk.

One interesting topic is to determine the amount of risk capitals. However, it is difficult to calculate the conditional expectation without full information about the joint distribution of $(X_1, ..., X_n)$. In that case, qualitative analysis is needed. Asimit et al. (2011) have derived some asymptotic results. In this section, we order the risk capitals for different lines of business under CTDAI dependence structure.

Proposition 5.4.3 If the risk vector $(X_1, ..., X_n)$ is CTDAI, then $\rho_1 \leq ... \leq \rho_n$, where $\rho_i, i = 1, ..., n$, are given by (5.4.3).

Proof. Define $g(x_1, ..., x_n) = x_i \times \mathbb{I}\{\sum_{k=1}^{n} x_k > s\}$ for any fixed s and j. Note that, for any $1 \leq i < j \leq n$, $\Delta g_{ij}(x_1, ..., x_n) = (x_j - x_i) \times \mathbb{I}\{\sum_{k=1}^{n} x_k > s\}$ is increasing in $x_j \geq x_i$, which means $g \in \mathcal{G}_{ctdai}^{ij}$. Therefore, according to Proposition 4.3.8(ii), we have $\mathbb{E}[g(X_1, ..., X_n)] \geq \mathbb{E}[g(\pi_{ij}(X_1, ..., X_n))]$, i.e., $\mathbb{E}\left[X_j \times \mathbb{I}\{S > s\}\right] \geq \mathbb{E}\left[X_i \times \mathbb{I}\{S > s\}\right]$. Let $s = \mathrm{VaR}_\alpha S$. Then we have

$$\mathbb{E}\left[X_j \times \mathbb{I}\{S > \mathrm{VaR}_\alpha S\}\right] \geq \mathbb{E}\left[X_j \times \mathbb{I}\{S > \mathrm{VaR}_\alpha S\}\right],$$

for any $1 \leq i < j \leq n$. Therefore, $\mathbb{E}\left[X_i \,|\, S > \mathrm{VaR}_\alpha S\right] \leq \mathbb{E}\left[X_j \,|\, S > \mathrm{VaR}_\alpha S\right]$ for any $1 \leq i < j \leq n$. $\square$

5.5 Stochastic Scheduling Problems

In Shanthikumar and Yao (1991), many stochastic scheduling problems were proposed and studied with the dependence structures developed therein. In this section, we show that, we can extend most of their results with the dependence notions proposed in Chapter 4.

Consider a queuing system. There are n individuals in the queue. The processing time for the individual k is modeled by a nonnegative random variable X_k, $k = 1, ..., n$. Let π be a schedule (permutation), under which the individual $\pi(k)$ is scheduled at the k^{th} position. Then, the waiting time of this individual $\pi(k)$ is $T_k^{\pi} = \sum_{l=1}^{k} X_{\pi(l)}$, and the total waiting time of all individuals in the queue is

$$T^{\pi} = \sum_{k=1}^{n} T_k^{\pi} = \sum_{k=1}^{n} \sum_{l=1}^{k} X_{\pi(l)} = \sum_{l=1}^{n} (n - l + 1) X_{\pi(l)}.$$

In this scheduling problem, one of our interests is to find a schedule so as to minimize the total waiting time in some stochastic sense. With the assumption of independence of $(X_1, ..., X_n)$ and $X_1 \leq_{hr} ... \leq_{hr} X_n$, Shanthikumar and Yao (1991) have proved that the identical permutation $\pi(1, ..., n) = (1, ..., n)$ minimize the total waiting time in the sense of increasing convex order. With the dependence structures proposed in Chapter 4, we can generalize the optimal queuing problem.

Proposition 5.5.1 If random vector $(X_1, ..., X_n)$ is UOAI, then $T^{\pi^*} \leq_{mom} T^{\pi}$ for any permutation π, where $\pi^*(1, ..., n) = (1, ..., n)$.

Proof. Consider any permutation π, and denote $\mathbf{a}^{\pi} = (n + 1 - \pi(1), ..., n + 1 - \pi(n))$. Assume that there exists $1 \leq i < j \leq n$ such that $\pi(i) > \pi(j)$. According to Corollary 4.2.19, we have $\mathbf{a}^{\pi^{ij}(\pi)} \cdot \mathbf{X} \leq_{mom} \mathbf{a}^{\pi} \cdot \mathbf{X}$. Therefore, the optimal schedule π^* that minimizes

T^{π} in the moment order should satisfy $\pi^*(i) < \pi^*(j)$ for any $1 \le i < j \le n$, which means $\pi^*(1, ..., n) = (1, ..., n)$. □

Proposition 5.5.2 If random vector $(X_1, ..., X_n)$ is CTDAI, then $T^{\pi^*} \le_{icx} T^{\pi}$ for any permutation π, where $\pi^*(1, ..., n) = (1, ..., n)$.

Proof. The desired conclusion follows from Proposition 4.3.11 with the same argument of the proof for Proposition 5.5.1. □

According to Proposition 4.3.4, if mutually independent random variables $X_1, ..., X_n$ satisfy $X_1 \le_{hr} ... \le_{hr} X_n$, then $(X_1, ..., X_n)$ is CTDAI. In this sense, Proposition 5.5.2 generalizes the result in Shanthukumar and Yao (1991).

The results of Proposition 5.5.1 and Proposition 5.5.2 are intuitive. Roughly speaking, in a queuing system, the individual with "shorter" (in the stochastic sense) processing time should be arranged prior to the one with "longer" processing time so as to save the total waiting time.

Brown and Solomon (1973) proposed the problem of optimal issuing policies. The model setting is as follows. There are n components kept in stock with their lifetimes modeled by nonnegative random variables $X_1, ..., X_n$. The components are to be issued to a system one by one upon the failure of the preceding component. If component k is issued to the system at time t, the actual lifetime of this component is $d(t)X_k$, where $d(t)$ is a positive function.

The function $d(t)$ has two different interpretations. It can be explained as either the "amplification" or the "decay" factor of the component lifetime. As an "amplification factor", $d(t)$ is assumed to be increasing and convex; as a "decay" factor, $d(t)$ is assumed to be decreasing and concave. Shanthikumar and Yao (1991) focuses on the model with increasing convex $d(t)$. In this section, we shall generalize their results and study the model with decreasing concave $d(t)$ as well.

Denote by $L_k, k = 1, ..., n$, the system duration up to the failure of the k^{th} component. The total duration of the system is L_n. Denote $L_0 = 0$, then $\{L_k, k = 0, 1, ..., n\}$ satisfy the following recursive formulas.

$$L_k = L_{k-1} + d(L_{k-1})\, X_k, \qquad k = 1, ..., n.$$

We are particularly interested in how to issue the components so as to maximize the total system duration. With the assumption of SAI $(X_1, ..., X_n)$, Shanthikumar and Yao (1991) concluded that the issuing sequence $(n, n-1, ..., 1)$ (or $(X_n, ..., X_1)$) is optimal, in the sense of maximizing the total system duration in the usual stochastic order. In the following, we derive a similar result with decreasing concave $d(t)$. They have also studied a special case: $(X_1, ..., X_n)$ is independent and $X_1 \geq_{hr} ... \geq_{hr} X_n$. In that case, the issuing sequence $(n, n-1, ..., 1)$ is optimal in the sense of increasing convex order. We extend this result to the case that $(X_1, ..., X_n)$ follows CTDAI dependence structure.

Before we study the optimal issuing problem, we first introduce some notations. Recall that in Section 4.3, we define the functional class $\mathcal{G}^{ij}_{ctdai}$ and focus the case that $1 \leq i < j \leq n$ to characterize the notion of CTDAI. In the following, we allow the case $i > j$. We also define the following functions recursively.

$$l_1(x_1) = d(0)x_1,\ l_k(x_1, ..., x_k) = l_{k-1}(x_1, ..., x_{k-1}) + d(l_{k-1}(x_1, ..., x_{k-1}))x_k, \qquad k = 2, ..., n.$$

Obviously, $L_k = l_k(X_1, ..., X_k)$ for all $k = 1, ..., n$. It is easy to verify by induction that

$l_k(x_1, ..., x_k)$ is increasing in each argument for all $k = 1, ..., n$, as long as $d(t) \geq 0$.

Lemma 5.5.3 The multivariate function $l_n(x_1, ..., x_n)$ has the following properties:

(i) If $d(t)$ is increasing convex, then

$$l_n(x_1, ..., x_n) \in \mathcal{G}_{ctdai}^{\{m+1,m\}}, \qquad \text{for all } m = 1, ..., n-1, \tag{5.5.1}$$

(ii) If $d(t)$ is decreasing concave, then

$$l_n(x_1, ..., x_n) \in \mathcal{G}_{ctdai}^{\{m,m+1\}}, \qquad \text{for all } m = 1, ..., n-1. \tag{5.5.2}$$

Proof. We shall prove by induction that the relations (5.5.1) and (5.5.2) hold for all $l_k(x_1, ..., x_k)$.

(i) Step 1. We first verify that (5.5.1) holds for $l_2(x_1, x_2)$. Assume that $d(t)$ is differentiable for simplicity. Noting that $l_2(x_1, x_2) = d(0)x_1 + d(d(0)x_1)x_2$, we have

$$\Delta_{12} l_2 = d(0)(x_1 - x_2) + d(d(0)x_1)x_2 - d(d(0)x_2)x_1,$$
$$\frac{\partial}{\partial x_1} \Delta_{12} l_2(x_1, x_2) = d(0) + d'(d(0)x_1)d(0)x_2 - d(d(0)x_2).$$

Recall that $d(t)$ is convex. Then for any $x_1 \geq x_2$, we have

$$d(d(0)x_2) - d(0) \leq d'(d(0)x_2) \times d(0)x_1 \leq d'(d(0)x_1) \times d(0)x_1,$$

which means that $\frac{\partial}{\partial x_1} \Delta_{12} l_2(x_1, x_2) \leq 0$ for any $x_1 \geq x_2$. Then $l_2(x_1, x_2) \in \mathcal{G}_{ctdai}^{21}$.

Step 2. Now consider the case $k \geq 3$, we assume (5.5.1) holds for l_{k-1}. We want to show that $l_k \in \mathcal{G}_{ctdai}^{\{m+1,m\}}$ for all $m = 1, ..., k-1$. Note that $l_k = l_{k-1} + d(l_{k-1})x_k = u(l_{k-1})$, where $u(t) = t + x_k d(t)$ is increasing convex. According to Lemma 4.3.7 and the fact that l_{k-1} is increasing, we know that $l_k(x_1, ..., x_k) \in \mathcal{G}_{ctdai}^{\{m+1,m\}}$ for all $m = 1, ..., k-2$. The proof is completed if we can show that $l_k(x_1, ..., x_k) \in \mathcal{G}_{ctdai}^{\{k,k-1\}}$.

Step 3. For convenience, we use l_m to denote $l_m(x_1, ..., x_m)$ for $m = 1, ..., k$. Particularly, we denote $l'_{k-1} = l(x_1, ..., x_{k-2}, x_k)$, then $l'_{k-1} \le l_{k-1}$ if $x_{k-1} \ge x_k$, and $l'_{k-1} - l_{k-2} = d(l_{k-2})x_k$. Since $l_k = l_{k-1} + d(l_{k-1})x_k$, we have $\frac{\partial}{\partial x_k} l_k = d(l_{k-1})$ and

$$\frac{\partial}{\partial x_{k-1}} l_k = (1 + d'(l_{k-1})x_k) \frac{\partial}{\partial x_{k-1}} l_{k-1} = (1 + d'(l_{k-1})x_k) d(l_{k-2}).$$

On the other hand, noting that $l_k(\pi_{k-1,k}(x_1, ..., x_k)) = l_k(x_1, ..., x_k, x_{k-1})$, we have

$$\frac{\partial}{\partial x_{k-1}} l_k(\pi_{k-1,k}(x_1, ..., x_k)) = d(l_{k-1}(x_1, ..., x_{k-2}, x_k)) = d(l'_{k-1}).$$

Therefore,

$$\begin{aligned}\frac{\partial}{\partial x_{k-1}} \Delta_{k-1,k} l_k &= \frac{\partial}{\partial x_{k-1}} l_k(x_1, ..., x_k) - \frac{\partial}{\partial x_{k-1}} l_k(\pi_{k-1,k}(x_1, ..., x_k)) \\ &= d'(l_{k-1}) d(l_{k-2}) x_{k-1} - (d(l'_{k-1}) - d(l_{k-2})). \qquad (5.5.3)\end{aligned}$$

Recalling that $u(t)$ is convex, we have $d'(t)$ is increasing. For any $x_{k-1} \ge x_k$, we have $d'(l'_{k-1}) \le d'(l_{k-1})$. Then

$$d(l'_{k-1}) - d(l_{k-2}) \le d'(l'_{k-1})(l'_{k-1} - l_{k-2}) = d'(l_{k-1}) d(l_{k-2}) x_k \le d(l_{k-1}) d(l_{k-2}) x_k,$$

which, combining with (5.5.3), implies that $\frac{\partial}{\partial x_{k-1}} \Delta_{k-1,k} l_k \ge 0$ for all $x_{k-1} \ge x_k$, i.e., $l_k(x_1, ..., x_k) \in \mathcal{G}_{ctdai}^{\{k,k-1\}}$.

(ii) In the case that $d(t)$ is decreasing concave, the proof for (5.5.2) is similar as above, only except that we need to verify the following two facts.

(a) $d(l'_{k-1}) - d(l_{k-2}) \geq d(l_{k-1})d(l_{k-2})x_k$;

(b) $g(x_1, ..., x_n) \in \mathcal{G}^{ij}_{ctdai}$ for any $1 \leq i < j \leq n$ implies that $u(g(x_1, ..., x_n)) \in \mathcal{G}^{ij}_{ctdai}$, where $u(t) = t + d(t)y$ for any fixed $y > 0$.

Fact (a) guarantees the validness of Step 1 and Step 3 of the proof of (i), and Fact (b) guarantees the validness of Step 2.

Fact (a) is true since $d'(t)$ is negative and decreasing and

$$d(l'_{k-1}) - d(l_{k-2}) \geq d'(l'_{k-1})(l'_{k-1} - l_{k-2}) = d'(l'_{k-1})d(l_{k-2})x_k \geq d'(l_{k-1})d(l_{k-2})x_k.$$

Fact (b) can be easily verified. □

Lemma 5.5.4 If (X, Y) is TDAI, then

$$\mathbb{E}[g(X, Y)] \leq \mathbb{E}[g(Y, X)] \qquad \text{for any } g(x, y) \in \mathcal{G}^{21}_{ctdai}.$$

Proof. Define $h(x, y) = g(y, x)$. Then $h(x, y) \in \mathcal{G}^{12}_{ctdai}$. According to Proposition 4.3.8, we have $\mathbb{E}[h(X, Y)] \geq \mathbb{E}[h(Y, X)]$, i.e., $\mathbb{E}[g(X, Y)] \leq \mathbb{E}[g(Y, X)]$. □

Proposition 5.5.5 Assume that $(X_1, ..., X_n)$ is CTDAI and $d(t)$ is increasing and convex. Then for any permutation π, we have

$$L_n(X_n, ..., X_1) \geq_{icx} L_n(X_{\pi(1)}, ..., X_{\pi(n)}).$$

Proof. For any permutation π, if there exists $1 \le i < j \le n$ such that $\pi(i) < \pi(j)$, we shall show that the issuing sequence $\pi_{ij} \circ \pi$ is superior to π, i.e.,

$$L_n(X_{\pi(1)}, ..., X_{\pi(j)}, ..., X_{\pi(i)}, ..., X_{\pi(n)}) \ge_{icx} L_n(X_{\pi(1)}, ..., X_{\pi(j)}, ..., X_{\pi(i)}, ..., X_{\pi(n)}).$$

Therefore, for any issuing sequence π, we can derive a superior issuing sequence by exchanging two consecutive components, which results in a decreasing arrangement of the index of the two exchanged components. For example, $L_3(X_2, X_1, X_3) \le_{icx} L_3(X_2, X_3, X_1) \le_{icx} L_3(X_3, X_2, X_1)$. In this way, we show that the issuing sequence $(n, n-1, ..., 1)$ is optimal.

For simplicity, we assume $\pi(1) < \pi(2)$. We are going to show that

$$\mathbb{E}\left[u(L_n(X_{\pi(2)}, X_{\pi(1)}, ..., X_{\pi(n)}))\right] \ge \mathbb{E}\left[u(L_n(X_{\pi(1)}, X_{\pi(2)}, ..., X_{\pi(n)}))\right], \tag{5.5.4}$$

for any $u \in \mathcal{U}_{icx}$.

Since $(X_1, ..., X_n)$ is CTDAI, we know that $(X_{\pi(1)}, X_{\pi(2)})|\mathbf{X}_{\pi(3,...,n)} = \mathbf{x}_{\pi(3,...,n)}$ is TDAI for any $\mathbf{x}_{\pi(3,...,n)} \in S(\mathbf{X}_{\pi(3,...,n)})$. From Lemma 5.5.3, we have $L_n(y_1, ..., y_n) \in \mathcal{G}^{21}_{ctdai}$. For fixed $y_3, ..., y_n$, $L_n(y_1, y_2, y_3, ..., y_n)$ is a bivariate function which also satisfies that $L_n(y_1, y_2, y_3, ..., y_n) \in \mathcal{G}^{21}_{ctdai}$. Then $u(L_n(y_1, y_2, y_3, ..., y_n)) \in \mathcal{G}^{21}_{ctdai}$ for any $u \in \mathcal{U}_{icx}$. According to Lemma 5.5.4, we have

$$\begin{aligned} & \mathbb{E}\left[u(L_n(X_{\pi(2)}, X_{\pi(1)}, ..., X_{\pi(n)})) \,\middle|\, \mathbf{X}_{\pi(3,...,n)} = \mathbf{x}_{\pi(3,...,n)}\right] \\ \ge \ & \mathbb{E}\left[u(L_n(X_{\pi(1)}, X_{\pi(2)}, ..., X_{\pi(n)})) \,\middle|\, \mathbf{X}_{\pi(3,...,n)} = \mathbf{x}_{\pi(3,...,n)}\right]. \end{aligned}$$

Taking expectation with respect to $\mathbf{X}_{\pi(3,...,n)}$ on both sides of the above inequality, we get (5.5.4). □

Proposition 5.5.6 Assume that $(X_1, ..., X_n)$ is CTDAI and $d(t)$ is decreasing and concave. Then for any permutation π, we have

$$L_n(X_1, ..., X_n) \geq_{icx} L_n(X_{\pi(1)}, ..., X_{\pi(n)}).$$

Proof. The conclusion follows from Lemma 5.5.3 (ii) and similar argument as proof for Proposition 5.5.5. □

Proposition 5.5.7 Assume that $(X_1, ..., X_n)$ is SAI and $d(t)$ is decreasing and concave. Then for any permutation π, we have

$$L_n(X_1, ..., X_n) \geq_{st} L_n(X_{\pi(1)}, ..., X_{\pi(n)}).$$

Proof. Note that $\mathcal{G}^{ij}_{ctdai} \subset \mathcal{G}^{ij}_{sai}$. Then Lemma 5.5.3 (ii) implies that $L_n(x_1, ..., X_n) \in \mathcal{G}^{m,m+1}_{sai}$ for all $m = 1, ..., n-1$. Since $(X_1, ..., X_n)$ is SAI, we know that $(X_i, X_j)|\mathbf{X}_{\overline{ij}} = \mathbf{x}_{\overline{ij}}$ is SAI for any fixed $\mathbf{x}_{\overline{ij}} \in S(\mathbf{X}_{\overline{ij}})$.

Following the same argument as the proof for Proposition 5.5.5, we obtain that

$$\mathbb{E}\left[u(L_n(\pi_{ij} \circ (X_{\pi(1)}, ..., X_{\pi(n)})))\right] \geq \mathbb{E}\left[u(L_n(\pi_{ij} \circ (X_{\pi(1)}, ..., X_{\pi(n)})))\right],$$

for any $u \in \mathcal{U}_{st}$ and any $1 \leq i < j \leq n$ such that $\pi(i) > \pi(j)$. □

Conclusion

The main contribution of this thesis is that it proposes some new dependence structures and explores their applications in different areas. The new dependence structures proposed in this thesis not only provide a general framework for the studies of optimization problems in the fields of insurance, finance and operations research, but also show their own interests in the studies of probability theory and statistics.

The studies in this thesis are motivated by the optimal reinsurance problems. In order to study the optimal reinsurance problems with multiple risks, we examine the notion of PDS and also propose the notion of PDUO in Chapter 2. We derive some important properties of these two notions and investigate their relations with copulas. In Chapter 3, we apply these notions of dependence to model insurance risks and identify the optimal reinsurance form. The introduction of PDS and PDUO provides a new approach to address the optimal reinsurance problem.

Chapter 3 identify the individualized excess-of-loss form as the optimal reinsurance strategy, leaving the parameters of the optimal form to be determined. Due to the difficulty of the problem, we focus on qualitative analysis of the optimal solutions, and this triggers the studies of the optimal allocation problems.

In Chapter 4, in order to study the optimal allocation problems, we improve the existing dependence notion of SAI and propose several new dependence notions of UOAI, CUOAI, TDAI and CTDAI. We systematically study these new notions of dependence. We develop probabilistic and functional characterizations for these notions. These characterizations help understanding the nature of these dependence structures. These new notions are of their own interests in the sense that they have close relations to other concepts in probability theory, such as copulas, positive dependence and multivariate stochastic orders. Furthermore, we provide a uniform way to construct these dependence structures, which enhances their applicability in practice.

In Chapter 5, we show applications of the new dependence structures in solving optimal allocation problems. The studies in Chapter 5 unify and greatly extend the existing studies in the literature of optimal allocation problems.

By the introduction of the new dependence structures, this thesis also opens a door to many new topics. We believe we can extend the studies in this thesis by developing more dependence structures and applying them in more fields.

Bibliography

Aboudi, R. and Thon, D. (1995). Second-degree stochastic dominance decisions and random initial wealth with applications to the economics of insurance. *Journal of Risk and Insurance*, 62(1):30–49.

Aly, E. and Kochar, S. C. (1993). On hazard rate ordering of dependent-variables. *Advances in Applied Probability*, 25(2):477–482.

Asimit, A. V., Furman, E., Tang, Q., and Vernic, R. (2011). Asymptotics for risk capital allocations based on conditional tail expectation. *Insurance: Mathematics and Economics*, 49(3):310–324.

Bäuerle, N. and Müller, A. (2006). Stochastic orders and risk measures: Consistency and bounds. *Insurance Mathematics and Economics*, 38(1):132–148.

Block, H. W., Savits, T. H., and Shaked, M. (1982). Some concepts of negative dependence. *Annals of Probability*, 10(3):765–772.

Block, H. W., Savits, T. H., and Shaked, M. (1985). A concept of negative dependence using stochastic ordering. *Statistics and Probability Letters*, 3(2):81–86.

Block, H. W. and Ting, M. (1981). Some concepts of multivariate dependence. *Communications in Statistics Part A-Theory and Methods*, 10(8):749–762.

Brown, M. and Solomon, H. (1973). Optimal issuing policies under stochastic field lives. *Journal of Applied Probability*, 10(4):761–768.

Cai, J., Tan, K. S., Weng, C., and Zhang, Y. (2008). Optimal reinsurance under var and cte risk measures. *Insurance: Mathematics and Economics*, 43(1):185–196.

Cherubini, U., Luciano, E., and Vecchiato, W. (2004). *Copula methods in finance*. Wiley, Hoboken.

Cheung, K. C. (2007). Optimal allocation of policy limits and deductibles. *Insurance Mathematics and Economics*, 41(3):382–391.

Cheung, K. C. and Yang, H. (2004). Ordering optimal proprtions in the asset allocation problem with dependent default risks. *Insurance Mathematics and Economics*, 35(3):595–609.

Delbaen, F. (2000). Coherent risk measures. *Blätter der DGVFM*, 24(4):733–739.

Denuit, M., Dhaene, J., Goovaerts, M., and Kaas, R. (2006). *Actuarial theory for dependent risks: measures, orders and models*. Wiley, Hoboken.

Denuit, M. and Vermandele, C. (1998). Optimal reinsurance and stop-loss order. *Insurance: Mathematics and Economics*, 22(3):229–233.

Denuit, M. and Vermandele, C. (1999). Lorenz and excess wealth orders, with applications in reinsurance theory. *Scandinavian Actuarial Journal*, 1999(2):170–185.

Fernández-Ponce, J. M., Pellerey, F., and Rodriguez-Grinolo, M. R. (2011). A characterization of the multivariate excess wealth ordering. *Insurance Mathematics and Economics*, 49(3):410–417.

Gerber, H. U. and Shiu, E. S. (1998). On the time value of ruin. *North American Actuarial Journal*, 2(1):48–72.

Hong, S. K., Lew, K. O., MacMinn, R., and Brockett, P. (2011). Mossin's theorem given random initial wealth. *Journal of Risk and Insurance*, 78(2):309–324.

Hu, F. and Wang, R. (2010). Optimal allocation of policy limits and deductibles in a model with mixture risks and discount factors. *Journal of Computational and Applied Mathematics*, 234(10):2953–2961.

Hua, L. and Cheung, K. C. (2008a). Stochastic orders of scalar products with applications. *Insurance Mathematics and Economics*, 42(3):865–872.

Hua, L. and Cheung, K. C. (2008b). Worst allocations of policy limits and deductibles. *Insurance Mathematics and Economics*, 43(1):93–98.

Joe, H. (1997). *Multivariate models and dependence concepts*. Chapman and Hall, London; New York.

Kijima, M. and Ohnishi, M. (1996). Portfolio selection problems via the bivariate characterization of stochastic dominance relations. *Mathematical Finance*, 6(3):237–277.

Klugman, S. A., Panjer, H. H., and Willmot, G. E. (2012). *Loss models: from data to decisions*. Wiley, Chichester.

Kusuoka, S. (2001). On law invariant coherent risk measures. *Advances in mathematical economics*, 3(1):83–95.

Landsberger, M. and Meilijson, I. (1990). Demand for risky financial assets - a portfolio analysis. *Journal of Economic Theory*, 50(1):204–213.

Lehmann, E. (1966). Some concepts of dependence. *Annals of Mathematical Statistics*, 37(5):1137–1153.

Li, X. and You, Y. (2012). On allocation of upper limits and deductibles with dependent frequencies and comonotonic severities. *Insurance Mathematics and Economics*, 50(3):423–429.

Marshall, A. W. and Olkin, I. (1979). *Inequalities: theory of majorization and its applications.* Academic Press, New York.

McNeil, A. J., Frey, R., and Embrechts, P. (2005). *Quantitative risk management : concepts, techniques and tools.* Princeton University Press, Princeton.

Müeller, A. and Stoyan, D. (2002). *Comparison methods for stochastic models and risks.* Wiley, Chichester.

Müller, A. and Scarsini, M. (2005). Archimedean copulae and positive dependence. *Journal of Multivariate Analysis*, 93(2):434–445.

Nelsen, R. B. (1999). *An introduction to copulas.* Springer, New York.

Ohlin, J. (1969). On a class of measures of dispersion with application to optimal reinsurance. *Astin Bulletin*, 5(2):249–266.

Righter, R. and Shanthikumar, J. (1992). Extension of the bivariate characterization for stochastic orders. *Advances in Applied Probability*, 24(2):506–508.

Seal, H. (1978). From aggregate claim distribution to probability of ruin. *Astin Bulletin*, 10(1):47–53.

Shaked, M. (1977). A family of concepts of dependence for bivariate distributions. *Journal of the American Statistical Association*, 72(359):642–650.

Shaked, M. and Shanthikumar, G. J. (2007). *Stochastic orders.* Springer, New York.

Shanthikumar, J. and Yao, D. (1991). Bivariate characterization of some stochastic order relations. *Advances in Applied Probability*, 23(3):642–659.

Tan, K. S., Weng, C., and Zhang, Y. (2011). Optimality of general reinsurance contracts under cte risk measure. *Insurance: Mathematics and Economics*, 49(2):175–187.

Vanheerwaarden, A. E., Kaas, R., and Goovaerts, M. J. (1989). Optimal reinsurance in relation to ordering of risks. *Insurance Mathematics and Economics*, 8(1):11–17.

von Neumann, J. and Morgenstern, O. (1947). *The Theory of Games and Economic Behavior.* Princeton University Press, Princeton.

Willmot, G. E. and Yang, H. (1996). Martingales and ruin probability. *Actuarial Clearing House*, 1:521–527.

Zhuang, W., Chen, Z., and Hu, T. (2009). Optimal allocation of policy limits and deductibles under distortion risk measures. *Insurance Mathematics and Economics*, 44(3):409–414.