On the distribution of the time to ruin and related topics

by

Tianxiang Shi

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Abstract

Following the introduction of the discounted penalty function by Gerber and Shiu (1998), significant progress has been made on the analysis of various ruin-related quantities in risk theory. As we know, the discounted penalty function not only provides a systematic platform to jointly analyze various quantities of interest, but also offers the convenience to extract key pieces of information from a risk management perspective. For example, by eliminating the penalty function, the Gerber-Shiu function becomes the Laplace-Stieltjes transform of the time to ruin, inversion of which results in a series expansion for the associated density of the time to ruin (see, e.g., Dickson and Willmot (2005)). In this thesis, we propose to analyze the long-standing finite-time ruin problem by incorporating the number of claims until ruin into the Gerber-Shiu analysis. As will be seen in Chapter 2, many nice analytic properties of the original Gerber-Shiu function are preserved by this generalized analytic tool. For instance, the Gerber-Shiu function still satisfies a defective renewal equation and can be generally expressed in terms of some roots of Lundberg's generalized equation in the Sparre Andersen risk model.

In this thesis, we propose not only to unify previous methodologies on the study of the density of the time to ruin through the use of Lagrange's expansion theorem, but also to provide insight into the nature of the series expansion by identifying the probabilistic contribution of each term in the expansion through analysis involving the distribution of the number of claims until ruin. In Chapter 3, we study the joint generalized density of the time to ruin and the number of claims until ruin in the classical compound Poisson risk model. We also utilize an alternative approach to obtain the density of the time to ruin based on the Lagrange inversion technique introduced by Dickson and Willmot (2005). In Chapter 4, relying on the Lagrange expansion theorem for analytic inversion, the joint density of the time to ruin, the surplus immediately before ruin and the number of claims

until ruin is examined in the Sparre Andersen risk model with exponential claim sizes and arbitrary interclaim times.

To our knowledge, existing results on the finite-time ruin problem in the Sparre Andersen risk model typically involve an exponential assumption on either the interclaim times or the claim sizes (see, e.g., Borovkov and Dickson (2008)). Among the few exceptions, we mention Dickson and Li (2010, 2012) who analyzed the density of the time to ruin for Erlang-n interclaim times. In Chapter 5, we propose a significant breakthrough by utilizing the multivariate version of Lagrange's expansion theorem to obtain a series expansion for the density of the time to ruin under a more general distribution assumption, namely when interclaim times are distributed as a combination of n exponentials. It is worth emphasizing that this technique can also be applied to other areas of applied probability. For instance, the proposed methodology can be used to obtain the distribution of some first passage times for particular stochastic processes. As an illustration, the duration of a busy period in a queueing risk model will be examined.

Interestingly, the proposed technique can also be used to analyze some first passage times for the compound Poisson processes with diffusion. In Chapter 6, we propose an extension to Kendall's identity (see, e.g., Kendall (1957)) by further examining the distribution of the number of jumps before the first passage time. We show that the main result is particularly relevant to enhance our understanding of some problems of interest, such as the finite-time ruin probability of a dual compound Poisson risk model with diffusion and pricing barrier options issued on an insurer's stock price.

Another closely related quantity of interest is the so-called occupation times of the surplus process below zero (also referred to as the duration of negative surplus, see, e.g., Egídio dos Reis (1993)) or in a certain interval (see, e.g., Kolkovska et al. (2005)). Occupation times have been widely used as a contingent characteristic to develop advanced derivatives in financial mathematics. In risk theory, it can be used as an important risk management tool to examine the overall health of an insurer's business. The main subject matter of Chapter 7 is to extend the analysis of occupation times to a class of renewal risk processes. We provide explicit expressions for the duration of negative surplus and the double-barrier occupation time in terms of their Laplace-Stieltjes transform. In the process, we revisit occupation times in the content of the classical compound Poisson risk model and examine some results proposed by Kolkovska et al. (2005). Finally, some concluding remarks and discussion of future research are made in Chapter 8.

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Dedication

To my parents, Yun'ai Feng and Shuimu Shi

&

To Jade and Alex

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Chapter 1

Introduction

1.1 Background

The random nature of an insurer's business and its obligation to fulfill future claim payments have drawn considerable attention of each stakeholder in the society. For instance, to maintain the overall healthiness of insurers, a variety of measures and capital requirements have been imposed by regulators to prevent insurers from insolvency. As an immediate result, the level of risk-based capital to hold becomes the main concern of insurers and has triggered multiple waves of discussion regarding the analytic relationships between the initial capital level, the characteristics of the underlying risk business and the likelihood of an insolvency event. In risk theory, this is partly accomplished by analyzing crucial variables such as the *time of ruin* (the first time that the surplus becomes negative), the *deficit at ruin* (minimum capital injection required to revive the insurer's business) and the *surplus immediately before ruin* interesting from an early warning viewpoint. The main objective of this thesis is to join the effort of the actuarial community to analyze these (and other related) variables and develop efficient risk management tools to enhance our understanding of ruin events. In particular, we focus on the time of ruin and aim to provide insights on the distribution of the time of ruin in various risk models.

In risk theory, the insurer's surplus process $\{U_t, t \geq 0\}$ is commonly modeled as

$$U_t = u + ct - S_t, \tag{1.1}$$

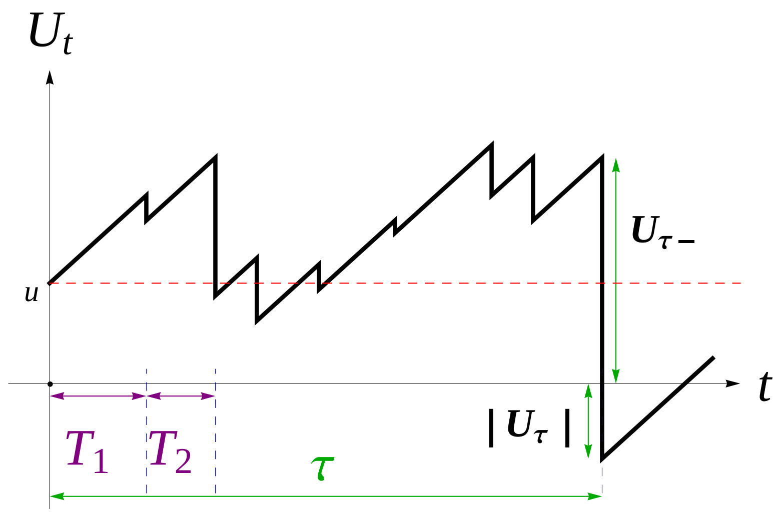
where u ($u \geq 0$) is the initial surplus level, c ($c > 0$) is the level premium rate per unit time and $\{S_t, t \geq 0\}$ is the aggregate claim amount process. We model the aggregate claim amount at time t by

$$S_t = \begin{cases} \sum_{i=1}^{N_t} X_i, & N_t > 0, \\ 0, & N_t = 0, \end{cases} \tag{1.2}$$

where $\{N_t, t \geq 0\}$ is the claim number process defined through the sequence of interclaim time random variables (r.v.'s) $\{T_i\}_{i=1}^{\infty}$ with T_i representing the time between the $(i-1)$-th claim and the i-th claim (and T_1 is the time of the first claim occurrence). Also, we represent by X_i the amount of the i-th claim size. Let $\tau = \inf\{t \geq 0 : U_t < 0\}$ be the time of ruin for the surplus process $\{U_t, t \geq 0\}$ with $\tau = \infty$ if ruin does not occur. Then, $U_{\tau-}$ is the surplus immediately before ruin and $|U_\tau|$ is the deficit at ruin (see Fig 1.1). Note that $X_\tau = U_{\tau-} + |U_\tau|$ is the amount of the claim causing ruin and $N_\tau = \max\{n : \sum_{i=1}^{n} T_i \leq \tau\}$ is the *number of claims until ruin*.

We next introduce one of the most widely studied models in the current actuarial literature by specifying a dependence structure between claim sizes and claim frequencies. In a *Sparre Andersen risk model* (see, e.g., Sparre Andersen (1957)), we assume that the claim size r.v.'s $\{X_i\}_{i=1}^{\infty}$ are independent and identically distributed (i.i.d.) with density p, cumulative distribution function (c.d.f.) $P(x) = 1 - \overline{P}(x)$, Laplace-Stieltjes transform (or

Figure 1.1: Surplus process and ruin quantities



Laplace transform) $\widetilde{p}(s) = \int_0^\infty e^{-sx} p(x)\,dx$, mean μ and k-th moments μ_k (with $\mu_1 = \mu$). Similarly, the interclaim time r.v.'s $\{T_i\}_{i=1}^\infty$ are also i.i.d. with density k, c.d.f. $K(t) = 1 - \overline{K}(t)$, Laplace transform $\widetilde{k}(s) = \int_0^\infty e^{-st} k(t)\,dt$ and mean κ. Furthermore, $\{X_i\}_{i=1}^\infty$ and $\{T_i\}_{i=1}^\infty$ are independent of each other. As a special case, when the r.v.'s $\{T_i\}_{i=1}^\infty$ follow the exponential distribution, (1.1) reduces to the *classical compound Poisson risk model* (see, e.g., Cramér (1955), Gerber (1979) and Grandell (1991)).

While the starting time of the surplus process is not necessarily associated to a claim occurrence, T_1 is not always a "full" interclaim time. It may be proper to assume that T_1 has a different density k_0 than the other interclaim times. In this case, T_1 is called the *delayed period* and the model (1.1) is referred to as the *delayed* Sparre Andersen risk model (see, e.g., Cox (1962)). When $k_0 = k$, the delayed Sparre Andersen risk model becomes the ordinary Sparre Andersen risk model. When $k_0(t) = k_e(t) = \overline{K}(t)/\kappa$, the model (1.1) is referred to as the *stationary* or *equilibrium* Sparre Andersen risk model (see, e.g., Cox

(1962), Karlin and Taylor (1975), Asmussen (2000) and Willmot and Lin (2001)). Readers are also referred to Willmot (2004) for particular distributional assumptions for k_0.

We may also generalize the dependence setting in the aforementioned risk model. In a *dependent* Sparre Andersen risk model, we assume that the bivariate pairs $\{(X_i, T_i)\}_{i=1}^{\infty}$ form a sequence of i.i.d. r.v.'s and consequently $\{cT_i - X_i\}_{i=1}^{\infty}$ also form a sequence of i.i.d. r.v.'s. However, for each pair, T_i and X_i are not necessarily independent. In this case, the surplus process $\{U_t, t \geq 0\}$ still retains the Sparre Andersen random walk structure. This type of process is especially useful to model some insurance events involving dependence between the claim frequency and severity, such as earthquake insurance. Throughout this thesis, we assume a positive security loading θ ($\theta > 0$) such that $c\kappa = (1+\theta)\mu$.

For the most part, recent research on ruin-related quantities can be rooted to the seminal paper of Professors Hans U. Gerber and Elias S.W. Shiu (Gerber and Shiu (1998)). They introduced a unified analytic tool, namely the *Gerber-Shiu expected discounted penalty function* (also referred to as the Gerber-Shiu function), defined as

$$m_\delta(u) \equiv E\left[e^{-\delta\tau} w(U_{\tau-}, |U_\tau|) 1(\tau < \infty) \,|U_0 = u\right], \tag{1.3}$$

where δ ($\delta \geq 0$) can be interpreted as a force of interest or a Laplace transform argument, $w(\cdot)$ is the so-called *penalty function*, and $1(A)$ is the indicator function with value 1 if event A is true and 0 otherwise. A nice feature of the Gerber-Shiu function comes from the flexibility in choosing the penalty function $w(\cdot)$. For instance, if $w(\cdot) = 1$, the Gerber-Shiu function reduces to the Laplace transform of the time to ruin. Further, if $w(x, y) = e^{-sx-zy}$, the Gerber-Shiu function becomes the trivariate Laplace transform of the time to ruin, the surplus immediately before ruin, and the deficit at ruin. Analytic inversion of the Laplace transform naturally leads to the joint density of those three quantities (see, e.g., Landriault and Willmot (2009) in the classical compound Poisson risk model). Therefore, the Gerber-

Shiu function not only provides a systematic way to jointly analyze various quantities of interest, but also provides the convenience to extract key risk management information from a standard Gerber-Shiu type analysis.

The Gerber-Shiu function was first analyzed in the context of the classical compound Poisson risk model by Gerber and Shiu (1998). It was shown that the Gerber-Shiu function $m_\delta(u)$ satisfies a defective renewal structure and can be expressed in terms of the non-negative root of *Lundberg's fundamental equation* (see, e.g., Gerber and Shiu (1998) and Lin and Willmot (1999, 2000)). The analysis has quickly expanded to various generalizations of the classical compound Poisson risk model. In the Sparre Andersen risk model, the interclaim times or the claim sizes are typically assumed to follow a class of distributions, such as Erlang distributions (see, e.g., Dickson and Hipp (2001) and Li and Garrido (2004)), Coxian distributions (see, e.g., Li and Garrido (2005) and Landriault and Willmot (2008)), and combinations of n exponentials (Gerber and Shiu (2005)). In general, the Gerber-Shiu function in these Sparre Andersen risk models still satisfies a defective renewal equation and can be expressed in terms of some solutions to the so-called *Lundberg's generalized equation* for which the dependence on δ appears in a non-trivial manner. Similar Gerber-Shiu type analysis has also been conducted in models perturbed by diffusion (see, e.g., Gerber and Landry (1998) and Tsai and Willmot (2002)), with dividend strategies (see, e.g., Lin et al. (2003), Gerber et al. (2006), Lin and Pavlova (2006) and Lin and Sendova (2008)) or with two-sided jumps (see, e.g., Albrecher et al. (2010) and Zhang et al. (2010)).

On the other hand, the Gerber-Shiu function has also been analyzed in discrete-time risk models (see the review paper by Li et al. (2009)). Typically, discrete-time risk models assume that there are at most one claim in each period and the claim sizes are also integer-valued random variables (see, e.g., Gerber (1988)). The results in discrete-time risk models are generally more explicit and can be used to approximate their continuous analogues (see, e.g., Dickson (1994)). Given that discrete-time risk models are not the focus of this thesis, we refer to Li et al. (2009) for a thorough review of the Gerber-Shiu analysis in the compound binomial model and the discrete-time Sparre Andersen risk model. In the next section, we will continue on the topic of Gerber-Shiu functions and comment on some recent progress related to its generalizations.

1.2 Generalizations of the Gerber-Shiu function

The popularity and ingenuity of the Gerber-Shiu function has led some authors to propose and analyze particular generalizations of this analytic tool. One of them was introduced by Cai et al. (2009). They considered the first passage time of the surplus process down-crossing a given level $d \in \mathbb{R}$, namely the *time to default*, which is defined as $\tau_d = \inf\{t \geq 0 : U_t < d\}$. When $d = 0$, $\tau_0 = \tau$ becomes the time to ruin. By introducing a *cost function* $l(U_t)$, which can be viewed as the operating cost at time t, they define the expectation of the total discounted operating costs up to default as

$$H_\delta(u) \equiv E\left[\int_0^{\tau_d} e^{-\delta t} l(U_t)dt \,|U_0 = u\right]. \tag{1.4}$$

Cai et al. (2009) pointed out that the Gerber-Shiu function is a special case of $H_\delta(u)$ in a general class of surplus processes (namely, the piecewise-deterministic compound Poisson risk model). Moreover, $H_\delta(u)$ can be used to analyze many other time-dependent quantities, such as the expected discounted dividends paid up to ruin,

$$V(u) = E\left[\int_0^{\tau_d} e^{-\delta t} dD(t) \,|U_0 = u\right], \tag{1.5}$$

where $D(t)$ represents the accumulated dividends paid up to time t (see, e.g., Avanzi (2009) and references therein). Cai et al. (2009) further studied the properties of $H_\delta(u)$

and pointed out that $H_\delta(u)$ still satisfies a defective renewal equation for $d = 0$ and a class of function $l(\cdot)$ in the classical compound Poisson risk model. While $H_\delta(u)$ is a very general tool that could meet various interests, further analysis of the properties may rely on some additional assumptions on $l(\cdot)$.

Another class of generalizations consists in adding new quantities of interest into the penalty function $w(\cdot)$ (see, e.g., Cheung et al. (2010a, 2010b) and Biffis and Morales (2010)). Cheung et al. (2010a, 2010b) defined a generalized Gerber-Shiu function,

$$m_\delta(u) \equiv E\left[e^{-\delta\tau} w(U_{\tau-}, |U_\tau|, Y_\tau, R_{N_\tau-1}) 1(\tau < \infty) | U_0 = u\right], \tag{1.6}$$

where $Y_t = \inf_{0\leq s<t} U_s$ is the *minimum surplus before time* t, and $R_n = u + \sum_{i=1}^{n}(cT_i - X_i)$ for $n = 1, 2, ...$ (with $R_0 = u$), which denotes the surplus immediately following the n-th claim. A very important property of this generalization is that the resulting analytic tool still satisfies a defective renewal equation and can be written in terms of the associated compound geometric tail.

Similarly, we can jointly analyze the number of claims until ruin N_τ with the traditional ruin-related variables by introducing a new function,

$$m_{r,\delta}(u) \equiv E\left[r^{N_\tau} e^{-\delta\tau} w(U_{\tau-}, |U_\tau|, Y_\tau, R_{N_\tau-1}) 1(\tau < \infty) | U_0 = u\right] \tag{1.7}$$

for $\delta \geq 0$ and $r \in (0, 1]$. $r = 0$ is excluded, given that $m_{r,\delta}(u) = 0$ for all $u \geq 0$ in this case. We point out that $m_{r,\delta}(u)$ also satisfies a defective renewal equation and general structural properties will be discussed in Section 2.2. It is worth pointing out that the number of claims until ruin has already been examined in some ruin related problems (see, e.g., Stanford and Stroinski (1994) and De Vylder and Goovaerts (1998)). By using probabilistic arguments on the number of claims, recursive formulas were developed to calculate ruin probabilities by Stanford and Stroinski (1994) and Egídio dos Reis (2002).

In this thesis, the proposed analytic function $m_{r,\delta}(u)$ enables us to incorporate the number of claims until ruin into the Gerber-Shiu type analysis in a relatively simple manner, as we will show.

1.3 The finite-time ruin problem

Calculating the finite-time ruin probabilities has been a long-standing problem in risk theory. One fruitful approach proposed in the literature is to develop recursive algorithms. For instance, De Vylder and Goovaerts (1988), and Dickson and Waters (1991) successfully approximated the finite-time ruin probabilities in the classical risk model using the ruin probabilities in the associated discrete-time risk process. By employing the number of claims until ruin, Stanford and Stroinski (1994) and Stanford et al. (2000) proposed a recursive method to calculate the probability of ruin before or on the n-th claim in the classical compound Poisson risk model and some non-Poisson claim processes (e.g. Erlang interclaim times and mixtures of exponentials interclaim times). The advantages of this approach include relatively simple recursive expressions and fast-speed numerical evaluations.

Recently, there has been an accrued interest in the identification of a closed-form expression for the density of the time to ruin, which naturally leads to mathematically tractable expressions for the finite-time ruin probabilities. In the context of the classical compound Poisson risk model, Drekic and Willmot (2003), Dickson and Willmot (2005) and Garcia (2005) have all examined the density of the time to ruin through the analytic inversion of its Laplace transform. The reader is referred to as Picard and Lefèvre (1997) when claim sizes are discrete. The distribution of the time to ruin has also been obtained in the Sparre Andersen risk model with exponential claim sizes; see, e.g., Borovkov and Dickson (2008) and Landriault et al. (2011). The reader is also invited to consult Dickson et al. (2005) for an equivalent representation when further assumptions are made on the interclaim density k. However, results on the distribution of the time to ruin are rather scarce when an exponential assumption is not imposed on either the interclaim times or the claim sizes. Among the few exceptions, we mention Dickson and Li (2010) who obtained an expression for the density of the time to ruin in the Sparre Andersen risk model with Erlang-2 interclaim times and specific claim size distributions. Also, Dickson and Li (2012) used probabilistic arguments to obtain a recursive formula for the calculation of the joint density of the time to ruin and the deficit at ruin for some models with Erlang-n interclaim times. As for some discrete-time risk models, Gerber (1988) and Willmot (1993) provided formulas for the (finite-time) ruin probabilities in the compound binomial model while Cossette et al. (2006) proposed a recursive formula to calculate the finite-time ruin probabilities in the Sparre Andersen risk model.

Although some of the above results are obtained by using probabilistic arguments, analysis of the time to ruin through its Laplace transform, namely

$$\phi_{1,\delta}\left(u\right) \equiv E\left[e^{-\delta\tau}1\left(\tau<\infty\right)|U_0=u\right], \tag{1.8}$$

is generally easier in the sense that it can be achieved through the analysis of the Gerber-Shiu function $m_\delta\left(u\right)$. However, as mentioned earlier, the Laplace transform $m_{1,\delta}\left(u\right)$ depends on δ in a non-straightforward way, which renders its inversion with respect to δ a daunting task even in the most simplistic risk models (see, e.g., Dickson and Willmot (2005) and Landriault et al. (2011) for more details). In general, there are many existing numerical inversion techniques (see, e.g., Abate and Whitt (1992)), but analytic inversion of the Laplace transform remains a very challenging topic.

In this thesis, we propose to unify previous methodology to tackle the finite-time ruin problem via the use of Lagrange's expansion theorem (which will be discussed in Section 1.4.2), as well as provide insight into the nature of the series expansions by identifying the probabilistic contribution of each term in the expansion through analysis involving the distribution of the number of claims until ruin. We define

$$\phi_{r,\delta}(u) \equiv E\left[r^{N_\tau} e^{-\delta\tau} 1(\tau < \infty) | U_0 = u\right], \tag{1.9}$$

for $\delta \geq 0$ and $r \in (0, 1]$. Clearly, this joint Laplace transform/probability generating function (p.g.f.) is a special case of (1.7) when $w(U_{\tau-}, |U_\tau|, Y_\tau, R_{N_\tau-1}) = 1$. The main focus will be to obtain the expression of $\phi_{r,\delta}(u)$ by using the traditional Gerber-Shiu approach and to analytically invert $\phi_{r,\delta}(u)$ w.r.t. r and δ via an application of the multivariate Lagrange expansion theorem.

1.4 Mathematical preliminaries

1.4.1 Defective renewal equation

As mentioned earlier, the defective renewal equation plays an important role in deriving a closed-form expression for the Gerber-Shiu function in Sparre Andersen risk models. In this section, we will briefly discuss the solution of the defective renewal equations (see, e.g., Feller (1971)) and analyze the behavior of the solution in both the general and limiting cases.

Definition 1.4.1 *Suppose that $F(y) = 1 - \overline{F}(y)$ for $y \geq 0$ is a distribution function with $F(0) = 0$ and $v(x) \geq 0$ is a locally bounded continuous function (i.e., $v(x) < \infty$ for $x < \infty$), then $m(x)$ satisfies a defective renewal equation if*

$$m(x) = \phi \int_0^x m(x-y)dF(y) + v(x), \qquad x \geq 0, \tag{1.10}$$

where $0 < \phi < 1$.

In risk theory, $F(y)$ generally represents the claim size distribution with Laplace transform $\widetilde{f}(s) = \int_0^\infty e^{-sy} dF(y)$. The solution to Eq. (1.10) can be expressed as an associated *compound geometric tail* $\overline{G}(y)$, which is defined through $G(y) = 1 - \overline{G}(y) = \Pr(L \le y)$,

$$\overline{G}(y) = \sum_{n=1}^{\infty} (1-\phi)\phi^n \overline{F}^{*n}(y), \qquad y \ge 0. \tag{1.11}$$

Here $\overline{F}^{*n}(y) = 1 - F^{*n}(y)$ is the tail of the distribution of the n-fold convolution of F with itself. It is not difficult to see that $G(y)$ has a mass point of $1-\phi$ at $y = 0$. In fact, if the number of claims is assumed to follow a geometric distribution with parameter ϕ, $G(y)$ can be interpreted as the distribution of the aggregate claim amount L. The Laplace transform of $G(y)$ can be written as

$$\begin{aligned} E\left(e^{-sL}\right) &= \sum_{n=0}^{\infty} (1-\phi)\phi^n \int_0^\infty e^{-sy} f^{*n}(y) dy \\ &= \sum_{n=0}^{\infty} (1-\phi)\phi^n \left\{ \widetilde{f}(s) \right\}^n \\ &= \frac{1-\phi}{1-\phi \widetilde{f}(s)}. \end{aligned} \tag{1.12}$$

Proposition 1.4.2 *The solution to Eq. (1.10) can be expressed as*

$$m(x) = \frac{1}{1-\phi} \int_{0+}^{x} v(x-y) dG(y) + v(x), \qquad x \ge 0. \tag{1.13}$$

Proof. Taking Laplace transform on both sides of (1.10), one finds

$$\widetilde{m}(s) = \phi \widetilde{m}(s) \widetilde{f}(s) + \widetilde{v}(s). \tag{1.14}$$

Combining (1.12) and (1.14) yields

$$\begin{aligned}\widetilde{m}(s) &= \frac{\widetilde{v}(s)}{1-\phi}E\left(e^{-sL}\right)\\ &= \frac{\widetilde{v}(s)}{1-\phi}\left\{E\left(e^{-sL^{+}}\right)+1-\phi\right\}\\ &= \frac{\widetilde{v}(s)E\left(e^{-sL^{+}}\right)}{1-\phi}+\widetilde{v}(s), \end{aligned} \tag{1.15}$$

where L^{+} is the r.v. $L\cdot 1(L>0)$. Inverting (1.15) immediately leads to (1.13). ■

It is worth noting that the compound geometric tail $\overline{G}(y)$ itself is the solution of the defective renewal equation when $v(x)=\phi\overline{F}(x)$ (see, e.g., Willmot and Lin (2001, p156)). In this case,

$$\widetilde{m}(s)=\frac{\widetilde{v}(s)}{1-\phi\widetilde{f}(s)}=\frac{1}{1-\phi\widetilde{f}(s)}\left(\phi\frac{1-\widetilde{f}(s)}{s}\right).$$

Taking the Laplace transform of (1.11) yields

$$\widetilde{\overline{G}}(s)=\frac{1-E\left(e^{-sL}\right)}{s}=\frac{1}{s}\left\{1-\frac{1-\phi}{1-\phi\widetilde{f}(s)}\right\}=\widetilde{m}(s).$$

By the uniqueness property of Laplace transforms, one concludes that $\overline{G}(y)$ satisfies

$$\overline{G}(x)=\phi\int_0^x\overline{G}(x-y)dF(y)+\phi\overline{F}(x),\qquad x\geq 0. \tag{1.16}$$

Eq. (1.16) could be used to obtain the Laplace transform of the time to ruin and will be further discussed in later sections. For more insight on (1.13) with various specifications of $v(x)$, readers are referred to Willmot and Lin (2001, Section 9.1) for a detailed discussion.

Due to the complexity of Eq. (1.13), which involves the convolution between $v(x)$ and the associated compound geometric distribution $G(x)$, asymptotic properties and reliability bounds of $m(x)$ have drawn considerable interest in the literature. Willmot et al. (2001) provided a general approach to obtain different types of bounds by specifying the choice of an F-integrable, nonnegative function $g(x)$ in the following generalized Lundberg adjustment equation

$$\int_0^\infty g(y)dF(y) = \frac{1}{\phi}. \tag{1.17}$$

In particular, if $g(x) = e^{Rx}$ is directly Riemann integrable and $R > 0$ satisfies

$$\int_0^\infty e^{Ry}dF(y) = \frac{1}{\phi},$$

then

$$C_L e^{-Rx} \le m(x) \le C_U e^{-Rx}, \qquad x \ge 0, \tag{1.18}$$

where $C_L = \inf_{z\ge 0} \alpha(z)$, $C_U = \sup_{z\ge 0} \alpha(z)$ and

$$\alpha(z) = \frac{e^{Rz}v(z)}{\phi \int_z^\infty e^{Ry}dF(y)}.$$

Bounds (1.18) are sometimes called *exponential bounds*. When $x \to \infty$, a closely related asymptotic result (also referred to as the Cramér-Lundberg asymptotic result) is readily available (see, e.g., Resnick (1992, Section 3.11)),

$$m(x) \sim Ce^{-Rx}, \qquad x \to \infty, \tag{1.19}$$

where

$$C = \frac{\int_0^\infty e^{Ry}v(y)dy}{\phi \int_0^\infty ye^{Ry}dF(y)}, \tag{1.20}$$

and $a(x) \sim b(x),\ x \to \infty$ denotes that $\lim_{x\to\infty} a(x)/b(x) = 1$.

1.4.2 Multivariate Lagrange expansion theorem

In this section, a brief summary of Lagrange's expansion theorem (see, e.g., Good (1960) and Goulden and Jackson (1983, Section 1.2.9)) in its univariate and multivariate form is presented. Under additional constraints, some of its simplified representations are given which turn out to be of particular interest in the later sections of this thesis. For notational convenience, define

$$h^{(m_1,\ldots,m_n)}(\rho_1, \rho_2, \ldots, \rho_n) \equiv \frac{\partial^{m_1+\ldots+m_n}}{\partial\rho_1^{m_1} \ldots \partial\rho_n^{m_n}} h(\rho_1, \rho_2, \ldots, \rho_n), \qquad m_1, \ldots, m_n \in \mathbb{N}.$$

In what follows, it is convenient to define $\frac{\partial^{-1}}{\partial t^{-1}} f'(t) \equiv f(t)$.

We first state the univariate version of this theorem, also known as the *Lagrange implicit function theorem* (see, e.g., Good (1960, p. 375) and Goulden and Jackson (1983, Section 1.2.4)).

Theorem 1.4.3 *For $h(z)$ an analytic function in a neighborhood of $z = a$, if*

$$\zeta - \alpha = \frac{z - a}{g(z)},$$

with $g(a) \neq 0$, then

$$h(z) = \sum_{m=0}^{\infty} \frac{(\zeta - \alpha)^m}{m!} \frac{d^m}{dt^m} \left\{ h(t)(g(t))^m \left(1 - \frac{(t-a)g'(t)}{g(t)} \right) \right\} \Bigg|_{t=a}. \tag{1.21}$$

It is not difficult to show that (1.21) is equivalent to

$$\begin{aligned} h(z) &= h(a) + \sum_{m=1}^{\infty} \frac{(\zeta - \alpha)^m}{m!} \frac{d^{m-1}}{dt^{m-1}} \left\{ h'(t)(g(t))^m \right\} \Big|_{t=a} \\ &= \sum_{m=0}^{\infty} \frac{(\zeta - \alpha)^m}{m!} \frac{d^{m-1}}{dt^{m-1}} \left\{ h'(t)(g(t))^m \right\} \Big|_{t=a}. \end{aligned} \tag{1.22}$$

In ruin theory, Eq. (1.22) has been central to the inversion of the Laplace transform of the time to ruin and other ruin-related quantities when their functional forms can be expressed in terms of a single solution of the generalized Lundberg equation (see, e.g., Dickson and Willmot (2005)). It has also been used (in an opposite way) to obtain a concise expression when the expansion is readily available (see De Vylder and Goovaerts (1998)). However, for most risk models, the Laplace transform $\phi_{1,\delta}(u)$ is a function of more than one solution of the generalized Lundberg equation, in which case the multivariate version of Lagrange's expansion theorem applies.

Theorem 1.4.4 *For $h(\mathbf{z})$ an analytic function in a neighborhood of $\mathbf{z} = \mathbf{a}$ ($\mathbf{z} = (z_1, z_2, \ldots, z_n)$ and $\mathbf{a} = (a_1, a_2, \ldots, a_n)$), if*

$$\zeta_i - \alpha_i = \frac{z_i - a_i}{g_i(\mathbf{z})}, \tag{1.23}$$

with $g_i(\mathbf{a}) \neq 0$ for $i = 1, \ldots, n$, then

$$h(\mathbf{z}) = \sum_{m_1,\ldots,m_n=0}^{\infty} \prod_{j=1}^{n} \frac{(\zeta_j - \alpha_j)^{m_j}}{m_j!} \frac{\partial^{m_1+\ldots+m_n}}{\partial t_1^{m_1} \ldots \partial t_n^{m_n}} \{H_n(\mathbf{t})(g_1(\mathbf{t}))^{m_1} \ldots (g_n(\mathbf{t}))^{m_n}\}\bigg|_{\mathbf{t}=\mathbf{a}}, \tag{1.24}$$

where

$$H_n(\mathbf{t}) = h(\mathbf{t}) \cdot \det\left(1(i = j) - \frac{t_i - a_i}{g_i(\mathbf{t})} \frac{\partial g_i(\mathbf{t})}{\partial t_j}\right),$$

for $\mathbf{t} = (t_1, t_2, \ldots, t_n)$.

While the theorem is stated in a rather concise way, we propose to work with one of its equivalent representations which turns out to be of more help in Chapter 5 for inversion purposes. For the case $n = 2$, Poincaré (1886) provided this expression which can be viewed as an extension of Eq. (1.22) in the univariate case, namely

$$\begin{aligned} h(\mathbf{z}) = \sum_{m_1,m_2=0}^{\infty} \frac{(\zeta_1 - \alpha_1)^{m_1}(\zeta_2 - \alpha_2)^{m_2}}{m_1! m_2!} & \left\{ \frac{\partial^{m_1+m_2-2}}{\partial t_1^{m_1-1} \partial t_2^{m_2-1}} h^{(1,1)}(\mathbf{t}) g_1^{m_1}(\mathbf{t}) g_2^{m_2}(\mathbf{t}) \right. \\ + \; h^{(1,0)}(\mathbf{t}) \frac{\partial g_1^{m_1}(\mathbf{t})}{\partial t_2} g_2^{m_2}(\mathbf{t}) + h^{(0,1)}(\mathbf{t}) g_1^{m_1}(\mathbf{t}) & \left. \frac{\partial g_2^{m_2}(\mathbf{t})}{\partial t_1} \right\}\bigg|_{\mathbf{t}=\mathbf{a}}. \end{aligned} \tag{1.25}$$

As expected, Poincaré's expansion of (1.24) becomes lengthy even for small values of $n > 2$. However, it is worth pointing out that when $g_i(\mathbf{z})$ is only a function of z_i for $i = 1, \ldots, n$, significant simplifications arise. This is precisely the context of application of Lagrange's expansion theorem in the later chapters of the thesis. The result is stated in the following corollary.

Corollary 1.4.5 *Under the same conditions as in Theorem 1.4.4, together with $g_i(\mathbf{z})$ being functions in z_i only ($i = 1, ..., n$), we have*

$$\begin{aligned} & h(\mathbf{z}) \\ = & \sum_{m_1,...,m_n=0}^{\infty} \prod_{j=1}^{n} \frac{(\zeta_j - \alpha_j)^{m_j}}{m_j!} \frac{\partial^{m_1+...+m_n-n}}{\partial t_1^{m_1-1} \ldots \partial t_n^{m_n-1}} \left\{ h^{(1,...,1)}(\mathbf{t})(g_1(t_1))^{m_1} \ldots (g_n(t_n))^{m_n} \right\} \Big|_{\mathbf{t}=\mathbf{a}} . \end{aligned} \tag{1.26}$$

Proof. To prove Corollary 1.4.5, it is sufficient to verify that

$$\begin{aligned} & \frac{\partial^{m_1+...+m_n}}{\partial t_1^{m_1} \ldots \partial t_n^{m_n}} \{ \mathbf{H}_n(\mathbf{t})(g_1(\mathbf{t}))^{m_1} \ldots (g_n(\mathbf{t}))^{m_n} \} \Big|_{\mathbf{t}=\mathbf{a}} \\ = & \frac{\partial^{m_1+...+m_n-n}}{\partial t_1^{m_1-1} \ldots \partial t_n^{m_n-1}} \left\{ h^{(1,...,1)}(\mathbf{t})(g_1(t_1))^{m_1} \ldots (g_n(t_n))^{m_n} \right\} \Big|_{\mathbf{t}=\mathbf{a}} , \end{aligned} \tag{1.27}$$

for any $m_i \in \mathbb{N}$. We prove (1.27) by induction on n. For $n = 1$, (1.27) becomes (1.22).

Note that when $g_i(\mathbf{z})$ is only a function of z_i,

$$\begin{aligned} \mathbf{H}_n(\mathbf{t}) &= h(\mathbf{t}) \prod_{i=1}^{n} \left\{ 1 - \frac{t_i - a_i}{g_i(t_i)} \frac{\partial g_i(t_i)}{\partial t_i} \right\} \\ &= \mathbf{H}_{n-1}(\mathbf{t}) \left\{ 1 - \frac{t_n - a_n}{g_n(t_n)} \frac{\partial g_n(t_n)}{\partial t_n} \right\} . \end{aligned}$$

Assume that (1.27) holds for $2, 3, ..., n-1$. Then,

$$
\begin{aligned}
& \frac{\partial^{m_1+\ldots+m_n}}{\partial t_1^{m_1}\ldots\partial t_n^{m_n}}\left\{\mathbf{H}_n(\mathbf{t})(g_1(t_1))^{m_1}\ldots(g_n(t_n))^{m_n}\right\}\Big|_{\mathbf{t}=\mathbf{a}} \\
= & \frac{\partial^{m_n}}{\partial t_n^{m_n}}\left\{\left[1-\frac{t_n-a_n}{g_n(t_n)}\frac{\partial g_n(t_n)}{\partial t_n}\right](g_n(t_n))^{m_n}\right. \\
& \left.\times\frac{\partial^{m_1+\ldots+m_{n-1}}}{\partial t_1^{m_1}\ldots\partial t_n^{m_{n-1}}}\left\{\mathbf{H}_{n-1}(\mathbf{t})\prod_{j=1}^{n-1}(g_j(t_j))^{m_j}\right\}\right\}\Big|_{\mathbf{t}=\mathbf{a}} \\
= & \frac{\partial^{m_n}}{\partial t_n^{m_n}}\left\{\left[1-\frac{t_n-a_n}{g_n(t_n)}\frac{\partial g_n(t_n)}{\partial t_n}\right](g_n(t_n))^{m_n}\right. \\
& \left.\times\frac{\partial^{m_1+\ldots+m_{n-1}-n-1}}{\partial t_1^{m_1-1}\ldots\partial t_n^{m_{n-1}-1}}\left\{h^{(1,\ldots,1,0)}\prod_{j=1}^{n-1}(g_j(t_j))^{m_j}\right\}\right\}\Big|_{\mathbf{t}=\mathbf{a}} \\
= & \frac{\partial^{m_1+\ldots+m_{n-1}-n-1}}{\partial t_1^{m_1-1}\ldots\partial t_n^{m_{n-1}-1}}\left\{\prod_{j=1}^{n-1}(g_j(t_j))^{m_j}\right. \\
& \left.\times\frac{\partial^{m_n}}{\partial t_n^{m_n}}\left\{h^{(1,\ldots,1,0)}\left[1-\frac{t_n-a_n}{g_n(t_n)}\frac{\partial g_n(t_n)}{\partial t_n}\right](g_n(t_n))^{m_n}\right\}\right\}\Big|_{\mathbf{t}=\mathbf{a}} \\
= & \frac{\partial^{m_1+\ldots+m_{n-1}-n-1}}{\partial t_1^{m_1-1}\ldots\partial t_n^{m_{n-1}-1}}\left\{\frac{\partial^{m_n-1}}{\partial t_n^{m_n-1}}\left\{h^{(1,\ldots,1,1)}(g_n(t_n))^{m_n}\right\}\prod_{j=1}^{n-1}(g_j(t_j))^{m_j}\right\}\Big|_{\mathbf{t}=\mathbf{a}} \\
= & \frac{\partial^{m_1+\ldots+m_n-n}}{\partial t_1^{m_1-1}\ldots\partial t_n^{m_n-1}}\left\{h^{(1,\ldots,1)}(\mathbf{t})(g_1(t_1))^{m_1}\ldots(g_n(t_n))^{m_n}\right\}\Big|_{\mathbf{t}=\mathbf{a}}. \qquad (1.28)
\end{aligned}
$$

Substituting (1.27) into (1.24) completes the proof.

■

1.5 Structure of the thesis

The thesis is organized as follows. The general structure of the proposed Gerber-Shiu function (1.7) is first discussed in Chapter 2. We allow the penalty function to only dependent on the surplus immediately before ruin and the deficit at ruin, although the result can be generalized to the four-variable penalty function as discussed in (1.7). In particular, in Section 2.3, we further exploit the structural properties when the interclaim times are exponentially distributed. In Chapter 3, we study the joint generalized density of the time to ruin and the number of claims until ruin in the classical compound Poisson risk model. We present an alternative approach to obtain the density of the time to ruin based on the Lagrange inversion technique introduced by Dickson and Willmot (2005) and then identify the individual contribution in relation with the number of claims until ruin. Our approach recovers the so-called Seal's formula (see, e.g., Prabhu (1961) and Dickson (2007)). In Chapter 4, relying on (1.7) and the Lagrange expansion theorem, the joint density of the time to ruin, the surplus immediately before ruin and the number of claims until ruin is examined in the Sparre Andersen risk model with exponential claim sizes and arbitrary interclaim times. Naturally, the marginal distribution of the number of claims until ruin can be obtained from the resulting joint distribution and this will be the subject matter of Section 4.3. In particular, we consider the case when the interclaim times are mixed Erlang distributed. In Chapter 5, we relax the restriction of exponential assumptions either on the interclaim times or the claim sizes and generalize the results of Chapter 4 by assuming that claim sizes are distributed as a combination of n exponentials. The multivariate Lagrange expansion theorem plays a key role in the ensuing analysis. Also, another application of this general methodology will be considered in a queueing model. A fluid flow process is constructed to build the connection between the Sparre Andersen risk model and the underlying queueing model. In Chapter 6, we propose to analyze the first passage time for the compound Poisson process with diffusion. An interesting connection between Kendall's identity (see, e.g., Kendall (1957)) and the distribution of the first passage time is presented. The main results are used to calculate the finite ruin probabilities in a dual compound Poisson risk model with diffusion and to price some path-dependent exotic options (i.e. barrier options) issued on an insurer's stock. In Chapter 7, we examine another time-related characteristic, namely the occupation time, of an insurer's surplus process in the Sparre Andersen risk model. We provide the Laplace transform of the total duration of negative surplus and the occupation time in an interval $[0, b]$ of an insurer's surplus process. The occupation times in the classical compound Poisson risk model are also revisited. We point out that our approach is based on some regenerative arguments, which are totally different from Kolkovska et al. (2005). More importantly, we show that the Laplace transform of the occupation time is actually much more complicated than the one suggested by Kolkovska et al. (2005, Proposition 3). In Chapter 8, we end the thesis with some concluding remarks and discussion of possible future research.

It is important to remark that most chapters relate to a scientific paper, and therefore were written independently of one another. Although efforts have been made to have consistent notations over the entire thesis, we hope to have been able to accomplish this task to a level acceptable to remove any ambiguity.

Chapter 2

Dependent Sparre Andersen risk model: general structure

2.1 Introduction

In this chapter, we consider analytic properties of the generalized Gerber-Shiu function (1.7) in the dependent Sparre Andersen risk model, in which both the interclaim times $\{T_i\}_{i=1}^{\infty}$ and claim sizes $\{X_i\}_{i=1}^{\infty}$ form a sequence of i.i.d. r.v.'s. To preserve the random walk structure of the surplus process at claim instants, we also assume that $\{cT_i - X_i\}_{i=1}^{\infty}$ forms a sequence of i.i.d. r.v.'s. However, for fixed i, T_i and X_i need not be independent. The positive security loading θ satisfies $c\kappa = (1+\theta)\mu$.

Given that the main focus of the thesis is on the distribution of the time to ruin, in what follows we assume that the penalty function w only depends on $U_{\tau-}$ and $|U_\tau|$ and re-define $m_{r,\delta}(u)$ as

$$m_{r,\delta}(u) \equiv E\left[r^{N_\tau} e^{-\delta\tau} w(U_{\tau-}, |U_\tau|) 1(\tau < \infty) \,|U_0 = u\right], \tag{2.1}$$

for $\delta \geq 0$ and $r \in (0,1]$. We show that (2.1) satisfies a defective renewal equation and general structural properties will be discussed in Section 2.2. Note that this conclusion still holds for the more general Gerber-Shiu function (1.7).

As pointed out in Section 1.3, we are particularly interested in $\phi_{r,\delta}(u)$, the joint Laplace transform (or p.g.f.) of the time to ruin and the number of claims until ruin,

$$\phi_{r,\delta}(u) \equiv E\left[r^{N_\tau} e^{-\delta\tau} 1(\tau < \infty) | U_0 = u\right],$$

which is a special case of (2.1). It can be shown that $\phi_{r,\delta}(u)$ can be expressed as a compound geometric tail in the Sparre Andersen risk model. This is a fundamental result for the remaining chapters of the thesis.

However, it is very challenging to further identify the quantity $m_{r,\delta}(u)$ solely based on the above results. Specification of distributional assumptions for the interclaim times or the claim sizes is generally required. In Section 2.3, we show that $\{m_{r,\delta}(u), u \geq 0\}$ can be expressed in terms of the unique non-negative solution to the so-called *generalized* Lundberg equation in the classical compound Poisson risk model, which is a necessary preparation for the analysis in Chapter 3.

2.2 General structure

In this section, we show that $\{m_{r,\delta}(u), u \geq 0\}$ satisfies a defective renewal equation in the general Sparre Andersen risk model. We employ the "first drop in surplus" argument to demonstrate this result (see, e.g., Gerber and Shiu (1998) and Cheung et al. (2010)), where some slight adjustments will first be required to accommodate our specific needs.

For an initial surplus of u, let $h_1(x, y | u)$ be the joint density of a surplus prior to ruin of x and a deficit at ruin of y for ruin occurring at the time of the first claim. It is immediate that the time to ruin τ is $(x-u)/c$ almost surely in this case (see Landriault and Willmot (2009) for more details). Also, let $h_j(t,x,y)$ be the joint density of a time to ruin of t, a surplus prior to ruin of x and a deficit at ruin of y for ruin occurring at the time of the jth claim ($j=2,3,...$). For convenience, we also define their respective 'discounted' densities as

$$g_{1,\delta}(x,y\,|u) = \begin{cases} e^{-\frac{\delta}{c}(x-u)}h_1(x,y\,|u), & x \geq u, \\ 0, & x < u, \end{cases}$$

and

$$g_{j,\delta}(x,y\,|u) = \int_0^\infty e^{-\delta t} h_j(t,x,y\,|u)\,dt,$$

for $j=2,3,...$ Finally, let

$$\xi_{r,\delta}(x,y\,|u) = \sum_{j=1}^{\infty} r^j g_{j,\delta}(x,y\,|u),$$

for $x \geq 0$ and $y > 0$.

By conditioning on the relevant characteristics of the first drop in surplus, one finds

$$m_{r,\delta}(u) = \int_0^u \int_0^\infty m_{r,\delta}(u-y)\,\xi_{r,\delta}(x,y\,|0)\,dxdy + \omega_{r,\delta}(u), \tag{2.2}$$

where

$$\omega_{r,\delta}(u) = \int_u^\infty \int_0^\infty w(x+u,y-u)\xi_{r,\delta}(x,y\,|0)\,dxdy.$$

Note that $m_{r,\delta}(u)$ can also be expressed as

$$m_{r,\delta}(u) = \int_0^\infty \int_0^\infty w(x,y)\xi_{r,\delta}(x,y\,|u)\,dxdy. \tag{2.3}$$

Let $w(x,y)=1$ and $u=0$ in (2.3), it follows that

$$\phi_{r,\delta}(0) = \int_0^\infty \int_0^\infty \xi_{r,\delta}(x,y\,|0)\,dxdy. \tag{2.4}$$

Utilizing (2.4), (2.2) becomes

$$m_{r,\delta}(u) = \phi_{r,\delta}(0) \int_0^u m_{r,\delta}(u-y)\, k_{r,\delta}(y)\, dy + \omega_{r,\delta}(u), \tag{2.5}$$

where

$$k_{r,\delta}(y) = \frac{\int_0^\infty \xi_{r,\delta}(x, y\,|0)\, dx}{\int_0^\infty \int_0^\infty \xi_{r,\delta}(x, y\,|0)\, dxdy}, \tag{2.6}$$

is a proper density function.

From the definition of $\phi_{r,\delta}(u)$, we conclude that

- for $\delta > 0$ or $r \in (0,1)$,

$$\phi_{r,\delta}(0) < \Pr(\tau < \infty\,|U(0) = 0) \leq 1.$$

- for $\delta = 0$ and $r = 1$, the positive security loading condition $c\kappa > \mu$ ensures

$$\phi_{1,0}(0) = \Pr(\tau < \infty\,|U(0) = 0) < 1.$$

As a result, (2.2) is a defective renewal equation. This is an extremely useful observation, as it has various implications for its solution (see Section 1.4.1), some of which will be used in what follows.

For example, when $w(x,y) = 1$, (2.5) can be simplified to

$$\phi_{r,\delta}(u) = \phi_{r,\delta}(0) \left\{ \int_0^u \phi_{r,\delta}(u-y)\, k_{r,\delta}(y)\, dy + \int_u^\infty k_{r,\delta}(y)\, dy \right\},$$

which implies that $\{\phi_{r,\delta}(u), u \geq 0\}$ is a compound geometric tail, i.e.

$$\phi_{r,\delta}(u) = \sum_{j=1}^{\infty} (1 - \phi_{r,\delta}(0))\, (\phi_{r,\delta}(0))^j\, \overline{K}_{r,\delta}^{*j}(u), \tag{2.7}$$

where $\overline{K}_{r,\delta}^{*j}$ is the survival function associated with the j-fold convolution of the density $k_{r,\delta}$ with itself.

Remark 2.2.1 *When the r.v.'s T and X of the generic pair (T, X) are independent, it follows that*

$$\xi_{r,\delta}(x, y\,|0) = \xi_{r,\delta}(x\,|0)\, p_x(y), \tag{2.8}$$

where p_x is the mean excess loss density

$$p_x(y) = \frac{p(x+y)}{\overline{P}(x)}, \qquad x, y > 0,$$

and

$$\xi_{r,\delta}(x\,|0) = \int_0^\infty \xi_{r,\delta}(x, y\,|0)\, dy.$$

Substituting (2.8) into (2.6), one finds

$$k_{r,\delta}(y) = \int_0^\infty \eta_{r,\delta}(x)\, p_x(y)\, dx, \tag{2.9}$$

where

$$\eta_{r,\delta}(x) = \frac{\xi_{r,\delta}(x\,|0)}{\int_0^\infty \xi_{r,\delta}(w\,|0)\, dw}.$$

One concludes that $k_{r,\delta}(y)$ is a mixture of the mean excess loss densities $\{p_x(y)\}_{x\geq 0}$.

2.3 Exponential interclaim times

To further exploit the structural properties of $\{m_{r,\delta}(u), u \geq 0\}$, we specifically examine the classical compound Poisson risk model, in which the interclaim times are exponentially distributed with mean $1/\lambda$, also independent of the claim sizes.

Conditioning on the time and amount of the first claim, we have

$$m_{r,\delta}(u) = \int_0^\infty re^{-\delta t}\lambda e^{-\lambda t}\left\{\alpha_{r,\delta}(u+ct) + \omega(u+ct)\right\} dt, \tag{2.10}$$

where

$$\omega(x) = \int_x^\infty w(x, y-x) dP(y), \tag{2.11}$$

and

$$\alpha_{r,\delta}(x) = \int_0^x m_{r,\delta}(x-y) dP(y).$$

Changing the variable of integration from t to $x = u + ct$ yields

$$\begin{aligned} m_{r,\delta}(u) &= r\frac{\lambda}{c}\int_u^\infty e^{-\frac{\lambda+\delta}{c}(x-u)} \left\{\alpha_{r,\delta}(x) + \omega(x)\right\} dx \\ &= r\frac{\lambda}{c}\left\{\mathcal{T}_{\frac{\lambda+\delta}{c}}\alpha_{r,\delta}(u) + \mathcal{T}_{\frac{\lambda+\delta}{c}}\omega(u)\right\}, \end{aligned} \tag{2.12}$$

where $\mathcal{T}_\rho f(x)$ is the Dickson-Hipp transform applied to a function f defined as

$$\mathcal{T}_\rho f(x) = \int_x^\infty e^{-\rho(y-x)} f(y)\, dy, \qquad x \geq 0,$$

(see Dickson and Hipp (2001)).

Taking the Laplace transform on both sides of (2.12) and using the properties of Dickson-Hipp operator (see Li and Garrido (2004, Section 3)), one finds that

$$\widetilde{m}_{r,\delta}(s) = r\frac{\lambda}{c}\left\{\frac{\widetilde{\alpha}_{r,\delta}\left(\frac{\lambda+\delta}{c}\right) - \widetilde{\alpha}_{r,\delta}(s)}{s - \frac{\lambda+\delta}{c}} + \frac{\widetilde{\omega}\left(\frac{\lambda+\delta}{c}\right) - \widetilde{\omega}(s)}{s - \frac{\lambda+\delta}{c}}\right\}. \tag{2.13}$$

Substitution by $\widetilde{\alpha}_{r,\delta}(s) = \widetilde{m}_{r,\delta}(s)\widetilde{p}(s)$ into (2.13) yields

$$\begin{aligned} &\widetilde{m}_{r,\delta}(s)\left\{s - \frac{\lambda+\delta}{c} + r\frac{\lambda}{c}\widetilde{p}(s)\right\} \\ = \; & r\frac{\lambda}{c}\left\{\widetilde{\omega}\left(\frac{\lambda+\delta}{c}\right) + \widetilde{m}_{r,\delta}\left(\frac{\lambda+\delta}{c}\right)\widetilde{p}\left(\frac{\lambda+\delta}{c}\right)\right\} - r\frac{\lambda}{c}\widetilde{\omega}(s). \end{aligned} \tag{2.14}$$

To further identify the constant on the right side of (2.14), we consider the following generalized Lundberg equation (in s)

$$s - \frac{\lambda+\delta}{c} + r\frac{\lambda}{c}\widetilde{p}(s) = 0. \tag{2.15}$$

It can be shown by an application of Rouché's theorem that, for $\delta > 0$ or $0 < r < 1$, (2.15) has a unique non-negative solution ρ ($\rho = \rho(r, \delta)$). Specifically speaking, consider a half-circle $\mathcal{K}$ (with non-negative real parts) determined by $|s| = d\left(d > \frac{2(\lambda+\delta)}{c}\right)$ in the complex plane. For the boundary $|s| = d$, we have

$$\left|r\frac{\lambda}{c}\widetilde{p}\,(s)\right| < \frac{\lambda+\delta}{c} < |s| - \frac{\lambda+\delta}{c} \leq \left|s - \frac{\lambda+\delta}{c}\right|;$$

for the part of the contour with real part $\Re\,(s) = 0$ (the imaginary axis), one observes that

$$\left|r\frac{\lambda}{c}\widetilde{p}\,(s)\right| < \frac{\lambda+\delta}{c} \leq \left|s - \frac{\lambda+\delta}{c}\right|.$$

Therefore, $s - \frac{\lambda+\delta}{c} = 0$ has the same number of roots with (2.15) in $\mathcal{K}$. Since $s - \frac{\lambda+\delta}{c} = 0$ only has one positive root in $\mathcal{K}$, and hence (2.15) has a unique solution with non-negative real part. Given that the solutions of real-coefficient equations are conjugate pairs, one concludes that (2.15) has a unique non-negative solution. We remark that when $\delta \to 0^+$ and $r = 1$, $\rho(1, 0^+) = 0$.

Setting $s = \rho$ in (2.14) yields

$$\widetilde{\omega}(\rho) = \widetilde{\omega}\left(\frac{\lambda+\delta}{c}\right) + \widetilde{m}_{r,\delta}\left(\frac{\lambda+\delta}{c}\right)\widetilde{p}\left(\frac{\lambda+\delta}{c}\right),$$

which implies that (2.14) can be rewritten as

$$\widetilde{m}_{r,\delta}(s)\left\{s - \frac{\lambda+\delta}{c} + r\frac{\lambda}{c}\widetilde{p}(s)\right\} = r\frac{\lambda}{c}\left\{\widetilde{\omega}(\rho) - \widetilde{\omega}(s)\right\}. \tag{2.16}$$

Consequently, let $s = \rho$ in (2.15), we have

$$\rho + r\frac{\lambda}{c}\widetilde{p}\,(\rho) = \frac{\lambda+\delta}{c}. \tag{2.17}$$

It follows that

$$\begin{aligned}
& s-\frac{\lambda+\delta}{c}+r\frac{\lambda}{c}\widetilde{p}(s) \\
= \;& s+r\frac{\lambda}{c}\widetilde{p}(s)-\left(\rho+r\frac{\lambda}{c}\widetilde{p}(\rho)\right) \\
= \;& (s-\rho)\left\{1-r\frac{\lambda}{c}\frac{\widetilde{p}(\rho)-\widetilde{p}(s)}{s-\rho}\right\} \\
= \;& (s-\rho)\left\{1-\left(r\frac{\lambda}{c}\frac{1-\widetilde{p}(\rho)}{\rho}\right)\left(\frac{\rho}{s-\rho}\frac{\widetilde{p}(\rho)-\widetilde{p}(s)}{1-\widetilde{p}(\rho)}\right)\right\} \\
= \;& (s-\rho)\left\{1-\phi_{r,\delta}\widetilde{p}_{1,\rho}(s)\right\}, \qquad (2.18)
\end{aligned}$$

where

$$\phi_{r,\delta}=r\frac{\lambda}{c}\left\{\frac{1-\widetilde{p}(\rho)}{\rho}\right\}, \qquad (2.19)$$

and $p_{1,\rho}(y)$ is a proper density defined as

$$p_{1,\rho}(y)\equiv\frac{e^{\rho y}\int_y^\infty e^{-\rho x}dP(x)}{\int_0^\infty e^{-\rho x}\overline{P}(x)dx}, \qquad (2.20)$$

with Laplace transform

$$\widetilde{p}_{1,\rho}(s)=\int_0^\infty e^{-sy}p_{1,\rho}(y)dy=\frac{\rho}{s-\rho}\frac{\widetilde{p}(\rho)-\widetilde{p}(s)}{1-\widetilde{p}(\rho)}.$$

Note that (2.19) can be re-expressed as

$$\phi_{r,\delta}=r\frac{1}{(1+\theta)}\int_0^\infty e^{-\rho y}\frac{\overline{P}(x)}{\mu}dy,$$

which implies that $0<\phi_{r,\delta}<1$.

Substituting (2.18) into (2.16), one arrives at

$$\widetilde{m}_{r,\delta}(s)\left\{1-\phi_{r,\delta}\widetilde{p}_{1,\rho}(s)\right\}=r\frac{\lambda}{c}\left\{\frac{\widetilde{\omega}(\rho)-\widetilde{\omega}(s)}{s-\rho}\right\}. \qquad (2.21)$$

Inverting Laplace transform wrt s, (2.21) becomes

$$m_{r,\delta}(u)=\phi_{r,\delta}\int_0^u m_{r,\delta}(u-y)p_{1,\rho}(y)dy+r\frac{\lambda}{c}\mathcal{T}_\rho\omega(u), \qquad u\geq 0. \qquad (2.22)$$

In conclusion, $m_{r,\delta}(u)$ satisfies the defective renewal equation (2.22). More importantly, from Proposition 1.4.2, the solution to (2.22) can be expressed in terms of the non-negative solution ρ of Eq. (2.15). ■

In what follows, we consider the special case $\phi_{r,\delta}(u)$. When $w(x,y)=1$, we have

$$\begin{aligned} r\frac{\lambda}{c}\mathcal{T}_\rho \omega(u) &= r\frac{\lambda}{c}\int_0^\infty e^{-\rho x}\overline{P}\left(x+u\right)dx \\ &= r\frac{\lambda}{c}\int_0^\infty e^{-\rho x}\int_u^\infty p\left(y+x\right)dydx \\ &= r\frac{\lambda}{c}\int_u^\infty \int_0^\infty e^{-\rho x}p\left(x+y\right)dxdy \\ &= \left\{r\frac{\lambda}{c}\int_0^\infty e^{-\rho x}\overline{P}(x)dx\right\}\int_u^\infty \left(\frac{\int_0^\infty e^{-\rho x}p\left(x+y\right)dx}{\int_0^\infty e^{-\rho x}\overline{P}(x)dx}\right)dy \\ &= \phi_{r,\delta}\int_u^\infty p_{1,\rho}(y)dy. \end{aligned} \tag{2.23}$$

Substituting (2.23) into (2.22) yields

$$\phi_{r,\delta}\left(u\right)=\phi_{r,\delta}\left\{\int_0^u \phi_{r,\delta}(u-y)p_{1,\rho}(y)dy+\int_u^\infty p_{1,\rho}(y)dy\right\}, \qquad u\geq 0. \tag{2.24}$$

It follows that $\phi_{r,\delta}(u)$ can be expressed as a compound geometric tail in terms of ρ, i.e.

$$\phi_{r,\delta}\left(u\right)=\sum_{j=1}^{\infty}\left(1-\phi_{r,\delta}\right)\left(\phi_{r,\delta}\right)^j\overline{P}_{1,\rho}^{*j}\left(u\right), \tag{2.25}$$

where $\overline{P}_{1,\rho}^{*j}$ is the tail of the distribution of the j-fold convolution of the density $p_{1,\rho}$ with itself.

Remark 2.3.1 *When $u=0$, it follows immediately from (2.24) that*

$$\phi_{r,\delta}\left(0\right)=\phi_{r,\delta}\int_0^\infty p_{1,\rho}(y)dy=\phi_{r,\delta}.$$

Thus, (2.25) can be re-expressed as

$$\phi_{r,\delta}\left(u\right)=\sum_{j=1}^{\infty}\left(1-\phi_{r,\delta}\left(0\right)\right)\left(\phi_{r,\delta}\left(0\right)\right)^j\overline{P}_{1,\rho}^{*j}\left(u\right).$$

We end this section by analyzing the reliability bounds and the asymptotic properties of $m_{r,\delta}(u)$ when $u \to \infty$. Consider $R > 0$ satisfies

$$\int_0^\infty e^{Ry} p_{1,\rho}(y)dy = \frac{1}{\phi_{r,\delta}},$$

or

$$\phi_{r,\delta}\widetilde{p}_{1,\rho}(-R) = 1. \tag{2.26}$$

Using (2.18), (2.26) becomes

$$-R - \frac{\lambda+\delta}{c} + r\frac{\lambda}{c}\widetilde{p}(-R) = 0.$$

Therefore, $-R$ is a negative solution of the generalized Lundberg equation (2.15). Note that $-R$ does not always exist given that the moment generating function (m.g.f.) of the claim size distribution may not exist. However, if the m.g.f. of p does exist, it is not difficult to show that Eq. (2.15) has a unique negative solution.

It follows from (1.18) that

$$C_L e^{-Ru} \le m_{r,\delta}(u) \le C_U e^{-Ru}, \tag{2.27}$$

for $u \ge 0$, where $C_L = \inf_{z\ge 0} \alpha(z)$, $C_U = \sup_{z \ge 0} \alpha(z)$ and

$$\alpha(z) = \frac{r\frac{\lambda}{c} e^{Rz} \mathcal{T}_\rho \omega(z)}{\phi_{r,\delta} \int_z^\infty e^{Ry} p_{1,\rho}(y)dy}. \tag{2.28}$$

On the other hand, when $u \to \infty$, (1.19) yields

$$m_{r,\delta}(u) \sim Ce^{-Ru}, \tag{2.29}$$

where

$$C = \frac{r\frac{\lambda}{c}\int_0^\infty e^{Ry}\mathcal{T}_\rho\omega(y)dy}{\phi_{r,\delta}\int_0^\infty ye^{Ry}p_{1,\rho}(y)dy}. \tag{2.30}$$

Generally, it is difficult to further simplify C_L, C_U and C unless further assumptions are made on the claim size distribution. In the following example, we identify the exponential bounds and the asymptotic expression for $\phi_{r,\delta}(u)$ when the claim sizes are exponentially distributed with mean $1/\beta$.

Example 2.3.2 *When the claim sizes are exponentially distributed with mean $1/\beta$, $p_{1,\rho}(y)$ is also exponentially distributed with mean $1/\beta$. Also, the negative solution to Eq. (2.15) satisfies $-\beta < -R < 0$.*

If $w(x, y) = 1$, it follows from (2.23) that

$$\alpha(z) = \frac{e^{Rz}e^{-\beta z}}{\int_z^\infty e^{Ry}\beta e^{-\beta y}dy} = \frac{\beta - R}{\beta},$$

and

$$C = \frac{\int_0^\infty e^{Ry}e^{-\beta y}dy}{\int_0^\infty ye^{Ry}\beta e^{-\beta y}dy} = \frac{\beta - R}{\beta}.$$

Therefore, the bound is exact, i.e.

$$C_L = C_U = C = \frac{\beta - R}{\beta}.$$

We conclude that

$$\phi_{r,\delta}(u) = \frac{\beta - R}{\beta}e^{-Ru}.$$

Chapter 3

Classical compound Poisson risk model

3.1 Density of the time to ruin revisited

In this chapter, we examine with the help of a Gerber-Shiu type analysis the joint distribution of the time of ruin and the number of claims until ruin in the classical compound Poisson risk model. We first briefly review the Lagrange inversion approach proposed by Dickson and Willmot (2005) to obtain the density of the time to ruin. In Section 3.2, we build on the main result of Dickson and Willmot (2005) pertaining to the density of the time to ruin to identify its individual contributions when the number of claims until ruin is also considered. In Section 3.3, we provide an alternative and more compact representation of the density of the time to ruin and show that this expression is consistent with Seal's formula (see, e.g., Prabhu (1961)).

As mentioned in Section 1.1, in a compound Poisson risk model, the interclaim times

$\{T_i\}_{i=1}^{\infty}$ and claim sizes $\{X_i\}_{i=1}^{\infty}$ form two sequences of i.i.d. r.v.'s, mutually independent of one another. Also, $\{X_i\}_{i=1}^{\infty}$ have an arbitrary density p with mean μ and $\{T_i\}_{i=1}^{\infty}$ are exponentially distributed with density

$$k(t) = \lambda e^{-\lambda t}, \qquad t \geq 0.$$

In other words, the claim occurring process $\{N_t, t \geq 0\}$ is a Poisson process with arrival rate λ. The positive security loading θ in this model is determined by $c = (1+\theta)\lambda\mu$.

Let p_1 be the equilibrium density associated with the density p, i.e.

$$p_1(x) = \frac{\overline{P}(x)}{\mu}, \qquad x > 0,$$

with Laplace transform

$$\widetilde{p}_1(s) = \frac{1-\widetilde{p}(s)}{\mu s}.$$

In addition, let $\psi(u,t) = \Pr(\tau \leq t \,|U_0 = u)$ be the ruin probability in the time interval $(0,t]$ with initial surplus level u, then the density of the time to ruin is given by

$$f(t\,|u) = \frac{\partial}{\partial t}\psi(u,t).$$

As a limiting case, the ultimate ruin probability is $\psi(u) = \lim_{t\to\infty}\psi(u,t)$.

In Section 1.1, we pointed out that the Gerber-Shiu function (1.3) in the classical compound Poisson risk model satisfies a defective renewal equation (see Gerber and Shiu (1998)) and can be expressed in terms of the associated compound geometric tail (see Lin and Willmot (1999)). As a special case, the Laplace transform of the time to ruin $\phi_{1,\delta}(u)$ can be expressed as

$$\phi_{1,\delta}(u) = \sum_{n=1}^{\infty}(1-\phi_{1,\delta}(0))(\phi_{1,\delta}(0))^n\overline{P}_{1,\rho}^{*n}(u), \tag{3.1}$$

where $\rho = \rho(1, \delta)$ is the unique non-negative solution (in s) of Lundberg's fundamental equation

$$s - \frac{\lambda + \delta}{c} + \frac{\lambda}{c} \widetilde{p}(s) = 0, \tag{3.2}$$

and

$$\phi_{1,\delta}(0) = \psi(0) \widetilde{p}_1(\rho) = \frac{\lambda \mu}{c} \widetilde{p}_1(\rho), \tag{3.3}$$

In fact, (3.1) is simply the case $r = 1$ of Eq. (2.25). From (3.1) and (3.3), one can easily find that $\phi_{1,\delta}(u)$ only depends (indirectly) on δ through ρ. This implicit dependence structure, as well as the complexity involved in inverting $\overline{P}_{1,\rho}^{*n}(u)$ (even wrt ρ), makes it a very challenging task to analytically invert $\phi_{1,\delta}(u)$ wrt δ.

Yet Dickson and Willmot (2005) solved this problem by first inverting $\phi_{1,\delta}(u)$ term by term wrt ρ and subsequently making use of a relationship between ρ and δ to complete the inversion and obtain the density of the time to ruin. This relationship between ρ and δ is obtained through an application of Lagrange's implicit function theorem (see Theorem 1.4.3 and Eq. (1.22)), namely

$$\begin{aligned} e^{-\rho t} &= e^{-\frac{\delta+\lambda}{c} t} + \sum_{n=1}^{\infty} (-1)^n \frac{(\lambda/c)^n}{n!} \frac{d^{n-1}}{dz^{n-1}} \left\{ \left(-t e^{-zt} \right) \int_0^\infty e^{-zx} p^{*n}(x) dx \right\} \Bigg|_{z = \frac{\delta+\lambda}{c}} \\ &= e^{-\frac{\delta+\lambda}{c} t} + \sum_{n=1}^{\infty} \frac{(\lambda/c)^n}{n!} t \int_0^\infty (x+t)^{n-1} e^{-(\delta+\lambda)(x+t)/c} p^{*n}(x) dx. \end{aligned} \tag{3.4}$$

Substituting (3.4) into

$$\phi_{1,\delta}(u) = \int_0^\infty e^{-\rho t} \xi(t|u) dt = \int_0^\infty e^{-\delta t} f(t|u) dt, \tag{3.5}$$

it follows that

$$f(t|u) = c e^{-\lambda t} \xi(ct|u) + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} t^{n-1} e^{-\lambda t} \int_0^{ct} y p^{*n}(ct-y) \xi(y|u) dy. \tag{3.6}$$

To invert $\phi_{1,\delta}(u)$ wrt ρ, Eq. (3.1) is rewritten as

$$\phi_{1,\delta}(u) = \frac{\lambda\mu}{c}\widetilde{p}_1(\rho) + \sum_{n=1}^{\infty}\left(\frac{\lambda}{c}\right)^n\left(\frac{\lambda\mu}{c}\widetilde{p}_1(\rho)H_\rho^{*n}(u) - H_\rho^{*n}(u)\right), \tag{3.7}$$

(see Dickson and Willmot (2005, Eq. (3))), where

$$H_\rho(u) = \int_0^u\int_z^\infty e^{-\rho(y-z)}p(y)dydz.$$

Clearly, if $H_\rho^{*n}(u)$ can be inverted wrt ρ, i.e.

$$H_\rho^{*n}(u) = \int_0^\infty e^{-\rho t}b_n(u,t)\,dt, \tag{3.8}$$

for $n = 1, 2, ...$, it follows immediately from (3.7) that, for $u > 0$,

$$\xi(t\,|u) = \frac{\lambda}{c}\overline{P}(t) + \sum_{n=1}^{\infty}\left(\frac{\lambda}{c}\right)^n\left\{\frac{\lambda}{c}\int_0^t\overline{P}(x)b_n(u,t-x)\,dx - b_n(u,t)\right\}. \tag{3.9}$$

The following expression for $b_n(u,t)$ was obtained by Dickson and Willmot (2005) following a series of lengthy algebraic manipulations:

$$b_n(u,t) = \sum_{j=0}^{n-1}\binom{n}{j}\frac{(-1)^j}{\Gamma(n)}\int_0^u(u-x)^{n-1}P^{*j}(x)\,p^{*(n-j)}(t+u-x)\,dx, \tag{3.10}$$

for $n = 1, 2, ...$

In conclusion, the density of the time to ruin in the compound Poisson risk model is given by (3.6), where $\xi(t\,|u)$ satisfies (3.9). For the case $u = 0$, the process to obtain $\xi(t\,|0)$ can be significantly simplified. From (3.3), inverting $\phi_{1,\delta}(0)$ wrt ρ yields

$$\xi(t\,|0) = \psi(0)p_1(t) = \frac{\lambda\mu}{c}p_1(t). \tag{3.11}$$

Combining (3.11) with (3.6), one concludes that

$$f(t\,|0) = \frac{\lambda\mu}{c}\left\{ce^{-\lambda t}p_1(ct) + \sum_{n=1}^{\infty}\frac{\lambda^n}{n!}t^{n-1}e^{-\lambda t}\int_0^{ct}yp^{*n}(ct-y)\,p_1(y)dy\right\}. \tag{3.12}$$

3.2 Joint (generalized) density of the time to ruin and the number of claims until ruin

In this section, we re-examine the closed-form expression (3.6) for the density of the time to ruin in connection with the number of claims until ruin, and derive the joint generalized density of the time to ruin and the number of claims until ruin in the context of the compound Poisson risk model. Throughout this thesis, we employ the term *generalized density* (as opposed to density) when at least one random variable of the associated joint distribution is discrete. Note that the general structure of $\phi_{r,\delta}(u)$ has been extensively discussed in Section 2.3.

Using virtually identical arguments that derived (3.7) in Dickson and Willmot (2005), it is not difficult to deduce from (2.25) that

$$\phi_{r,\delta}(u) = \frac{\lambda}{c}\mu r\widetilde{p}_1(\rho) + \sum_{n=1}^{\infty}\left(\frac{\lambda}{c}r\right)^n\left(\frac{\lambda}{c}\mu r\widetilde{p}_1(\rho)H_\rho^{*n}(u) - H_\rho^{*n}(u)\right), \tag{3.13}$$

Utilizing (3.8), (3.13) becomes

$$\begin{aligned}\phi_{r,\delta}(u) &= \int_0^\infty e^{-\rho t}\left\{\frac{\lambda}{c}r\overline{P}(t) + \sum_{n=1}^{\infty}\left(\frac{\lambda}{c}r\right)^n\left(\frac{\lambda}{c}r\int_0^t \overline{P}(x)\,b_n(u,t-x)\,dx - b_n(u,t)\right)\right\}dt \\ &= \sum_{n=1}^{\infty}\int_0^\infty \left\{r^n e^{-\rho t}\right\}\xi(t,n\,|u)\,dt, \end{aligned}\tag{3.14}$$

where

$$\xi(t,n\,|u) = \begin{cases} \frac{\lambda}{c}\left(\overline{P}(t) - b_1(u,t)\right), & n=1, \\ \left(\frac{\lambda}{c}\right)^n\left(\int_0^t \overline{P}(x)\,b_{n-1}(u,t-x)\,dx - b_n(u,t)\right), & n=2,3,... \end{cases}$$

From the definition of ρ as the unique non-negative solution of (2.15), the use of Lagrange's expansion theorem (Eq. (1.22)) yields

$$e^{-\rho t} = e^{-\frac{\lambda+\delta}{c}t} + \sum_{m=1}^{\infty} r^m\frac{\left(\frac{\lambda}{c}\right)^m}{m!}t\int_0^\infty (x+t)^{m-1}e^{-\frac{\lambda+\delta}{c}(x+t)}p^{*m}(x)\,dx. \tag{3.15}$$

Substituting (3.15) into (3.14) yields

$$\begin{aligned}
\phi_{r,\delta}(u) &= \sum_{n=1}^{\infty}\int_0^{\infty} r^n\left\{e^{-\frac{\lambda+\delta}{c}t}\right\}\xi(t,n\,|u)\,dt \\
&\quad+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\int_0^{\infty} r^{m+n}\left\{\frac{\left(\frac{\lambda}{c}\right)^m}{m!}t\int_0^{\infty}(x+t)^{m-1}e^{-\frac{\lambda+\delta}{c}(x+t)}p^{*m}(x)\,dx\right\}\xi(t,n\,|u)\,dt \\
&=\sum_{n=1}^{\infty}\int_0^{\infty}\left\{r^n e^{-\delta t}\right\}ce^{-\lambda t}\xi(ct,n\,|u)\,dt \\
&\quad+\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}\int_0^{\infty} r^m\left\{\frac{\left(\frac{\lambda}{c}\right)^{m-n}}{(m-n)!}t\int_t^{\infty}x^{m-n-1}e^{-\frac{\lambda+\delta}{c}x}p^{*(m-n)}(x-t)\,dx\right\}\xi(t,n\,|u)\,dt.
\end{aligned} \tag{3.16}$$

Interchanging the order of both summations followed by a similar manipulation of the two integrals, (3.16) becomes

$$\begin{aligned}
\phi_{r,\delta}(u) &= \sum_{n=1}^{\infty}\int_0^{\infty}\left\{r^n e^{-\delta t}\right\}ce^{-\lambda t}\xi(ct,n\,|u)\,dt \\
&\quad+\sum_{m=2}^{\infty}\sum_{n=1}^{m-1}\int_0^{\infty}\left\{r^m e^{-\frac{\delta}{c}x}\right\}\left(\frac{\gamma_{\frac{\lambda}{c},m-n}(x)}{m-n}\int_0^x p^{*(m-n)}(x-t)\,t\xi(t,n\,|u)\,dt\right)dx \\
&=\sum_{n=1}^{\infty}\int_0^{\infty}\left\{r^n e^{-\delta t}\right\}ce^{-\lambda t}\xi(ct,n\,|u)\,dt \\
&\quad+\sum_{m=2}^{\infty}\int_0^{\infty}\left\{r^m e^{-\delta x}\right\}\left(\sum_{n=1}^{m-1}\frac{\gamma_{\lambda,m-n}(x)}{m-n}\int_0^{cx}p^{*(m-n)}(cx-t)\left\{t\xi(t,n\,|u)\right\}dt\right)dx,
\end{aligned}$$

where $\gamma_{\beta,n}$ is the Erlang density

$$\gamma_{\beta,n}(y)=\frac{\beta^n y^{n-1}e^{-\beta y}}{(n-1)!},\qquad y\geq 0.$$

In conclusion,

$$\phi_{r,\delta}(u)=\sum_{n=1}^{\infty}\int_0^{\infty}\left\{r^n e^{-\delta t}\right\}f(t,n\,|u)\,dt,$$

where the joint generalized density of the time to ruin, and the number of claims until ruin is

$$f\left(t,n\left|u\right.\right)=ce^{-\lambda t}\xi\left(ct,n\left|u\right.\right)+\sum_{m=1}^{n-1}\frac{\gamma_{\lambda,n-m}\left(t\right)}{n-m}\int_{0}^{ct}p^{*(n-m)}\left(ct-x\right)\left\{x\xi\left(x,m\left|u\right.\right)\right\}dx, \quad (3.17)$$

for $t>0$ and $n=1,2,...$

With a zero initial surplus, (3.17) can be further simplified. Indeed, for $u=0$, one easily deduces from (3.10) that $b_n\left(0,t\right)=0$ for all $t\geq 0$ and $n=1,2,...$, which in turn implies

$$\xi\left(t,n\left|0\right.\right)=\begin{cases}\frac{\lambda}{c}\overline{P}\left(t\right), & n=1,\\ 0, & n=2,3,...\end{cases}$$

for $t\geq 0$. Thus, for $u=0$, one concludes that

$$f\left(t,n\left|0\right.\right)=\begin{cases}\lambda e^{-\lambda t}\overline{P}\left(ct\right), & n=1,\\ \frac{\lambda}{c}\frac{\gamma_{n-1,\lambda}(t)}{n-1}\int_{0}^{ct}p^{*(n-1)}\left(ct-x\right)\left\{x\overline{P}\left(x\right)\right\}dx, & n=2,3,...\end{cases} \quad (3.18)$$

Remark 3.2.1 *An alternative approach to obtain $f\left(t,n\left|u\right.\right)$ (see Dickson (2012)) is to use the probabilistic arguments proposed by Prabhu (1961). By first deriving expressions for $f\left(t,n\left|0\right.\right)$, $f\left(t,n\left|u\right.\right)$ can be calculated recursively from $f\left(t,n\left|0\right.\right)$, i.e.*

$$\begin{aligned}f\left(t,n+1\left|u\right.\right) &= e^{-\lambda t}\frac{\left(\lambda t\right)^{n}}{n!}\int_{0}^{u+ct}p^{*n}\left(x\right)\lambda\overline{P}\left(u+ct-x\right)dx\\ &\quad -c\sum_{j=1}^{n}\int_{0}^{t}e^{-\lambda z}\frac{\left(\lambda z\right)^{j}}{j!}p^{*j}(u+cz)f(t-z,n+1-j\left|0\right.)dz,\end{aligned}$$

for $n=1,2,...$

Remark 3.2.2 *Using Eq. (3.17), the covariance of N_τ and τ given that ruin occurs can be calculated. Note that an alternative and possibly more clever way to calculate the covariance is to take the derivatives of $\widetilde{\phi}_{r,\delta}\left(s\right)$ w.r.t. δ and r and subsequently invert the Laplace* *transform w.r.t. s. For some light-tailed claim sizes, a strong positive correlation between N_τ and τ is observed. However, it would be difficult to make such a conclusion in general.*

In theory, an explicit expression for the (marginal) probability mass function (p.m.f.) of the number of claims until ruin

$$p(n|u) \equiv \Pr(N_\tau = n|U_0 = u) \tag{3.19}$$

can be obtained from (3.17) (or (3.18) when $u = 0$) by integrating out the joint generalized density $f(t, n|u)$ over the (time) variable t from 0 to infinity. For instance, in the context of a zero initial surplus,

$$p(n|0) = \begin{cases} \lambda \int_0^\infty e^{-\lambda t}\overline{P}(ct)\, dt, & n = 1, \\ \frac{\lambda}{c}\int_0^\infty \frac{\gamma_{\lambda,n-1}(t)}{n-1}\left(\int_0^{ct} p^{*(n-1)}(ct-x)\left\{x\overline{P}(x)\right\}dx\right)dt, & n = 2, 3, ... \end{cases} \tag{3.20}$$

It is not difficult to show the consistency of this result with Egídio dos Reis (2002, Eqs. (14) and (15)), which states that

$$p(n|0) = \begin{cases} \widetilde{g}(\frac{\lambda}{c}|0), & n = 1, \\ -\frac{\lambda}{c}\widetilde{p}(\frac{\lambda}{c})\widetilde{g}'(\frac{\lambda}{c}|0), & n = 2, \\ \frac{\left(-\frac{\lambda}{c}\right)^{n-1}}{(n-1)!}\frac{d^{n-2}}{ds^{n-2}}\left\{(\widetilde{p}(s))^{n-1}\widetilde{g}'(s|0)\right\}\Big|_{s=\frac{\lambda}{c}}, & n = 3, 4, ... \end{cases} \tag{3.21}$$

where

$$g(y|0) = \frac{\lambda}{c}\overline{P}(y), \tag{3.22}$$

is the (defective) density of the deficit at ruin for a zero initial surplus level.

To verify this, we start from the case $n = 1$. Substituting (3.22) into (3.21) and changing the order of integration, one immediately arrives at

$$\begin{aligned} p(1|0) &= \int_0^\infty e^{-\frac{\lambda}{c}t}\left(\frac{\lambda}{c}\overline{P}(t)\right)dt \\ &= \lambda\int_0^\infty e^{-\lambda t}\overline{P}(ct)\, dt. \end{aligned}$$

For $n=2$, we have

$$\begin{aligned} p(2\,|0) &= -\frac{\lambda}{c}\left(\int_0^\infty e^{-\frac{\lambda}{c}t}p(t)\,dt\right)\left(\int_0^\infty e^{-\frac{\lambda}{c}t}\left(-\frac{\lambda}{c}t\overline{P}(t)\right)dt\right) \\ &= \left(\frac{\lambda}{c}\right)^2\int_0^\infty e^{-\frac{\lambda}{c}t}\left(\int_0^t p(t-x)\left\{x\overline{P}(x)\right\}dx\right)dt \\ &= \frac{\lambda}{c}\int_0^\infty \lambda e^{-\lambda t}\left(\int_0^{ct} p(ct-x)\left\{x\overline{P}(x)\right\}dx\right)dt. \end{aligned}$$

For $n=3,4,...$, we notice that

$$(\widetilde{p}(s))^{n-1}\,\widetilde{g}'(s\,|0) = -\frac{\lambda}{c}\int_0^\infty e^{-st}\left(\int_0^t p^{*(n-1)}(t-x)\left\{x\overline{P}(x)\right\}dx\right)dt. \tag{3.23}$$

Combining (3.23) and (3.21), one concludes that

$$\begin{aligned} & p(n\,|0) \\ =\; & \frac{\left(-\frac{\lambda}{c}\right)^n}{(n-1)!}\frac{d^{n-2}}{ds^{n-2}}\left(\int_0^\infty e^{-st}\left(\int_0^t p^{*(n-1)}(t-x)\left\{x\overline{P}(x)\right\}dx\right)dt\right)\Bigg|_{s=\frac{\lambda}{c}} \\ =\; & \frac{\left(\frac{\lambda}{c}\right)^n}{(n-1)!}\int_0^\infty e^{-\frac{\lambda}{c}t}\left(t^{n-2}\int_0^t p^{*(n-1)}(t-x)\left\{x\overline{P}(x)\right\}dx\right)dt \\ =\; & \frac{\lambda}{c}\int_0^\infty \frac{\lambda^{n-1}t^{n-2}}{(n-1)!}e^{-\lambda t}\left(\int_0^{ct} p^{*(n-1)}(ct-x)\left\{x\overline{P}(x)\right\}dx\right)dt. \qquad \blacksquare \end{aligned}$$

In general, it appears doubtful that the resulting expression for the p.m.f. of the number of claims until ruin (e.g., Eq. (3.20)) will allow for much simplifications unless some distributional assumptions are imposed on the claim size density p. For illustrative purposes, we derive an explicit expression of $p(n\,|0)$ for mixed Erlang claim sizes in the following example.

Example 3.2.3 *Assume that the claim sizes follow a Mixed Erlang density with Laplace transform*

$$\widetilde{p}(s) = Q\left(\frac{\beta}{\beta+s}\right),$$

where

$$Q(s) = \sum_{j=1}^{\infty} q_j s^j,$$

with $\{q_j\}_{j=1}^{\infty}$ *being a discrete probability measure. Therefore,*

$$(\widetilde{p}(s))^n = \sum_{j=1}^{\infty} q_j^{*n} s^j,$$

where q^{*n}*, the* n*-fold convolution of the p.m.f.* q*, is obtained through*

$$\sum_{j=1}^{\infty} q_j^{*n} s^j = \left(\sum_{j=1}^{\infty} q_j s^j \right)^n .$$

We also notice that

$$\overline{P}(x) = e^{-\beta x} \sum_{k=0}^{\infty} \overline{Q}_k \frac{(\beta x)^k}{k!}, \tag{3.24}$$

where $\overline{Q}_k = \sum_{i=k+1}^{\infty} q_i$.

For $n = 1$*, using (3.21), we have*

$$\begin{aligned} p(1\,|0) &= \frac{\lambda}{c} \left. \frac{1 - \widetilde{p}(s)}{s} \right|_{s=\frac{\lambda}{c}} \\ &= 1 - Q\left(\frac{c\beta}{\lambda + c\beta} \right). \end{aligned}$$

For $n = 2, 3, ...,$ *using (3.24), one finds*

$$\begin{aligned} & \int_0^t p^{*(n-1)}(t - x) \left\{ x\overline{P}(x) \right\} dx \\ = & \int_0^t \left\{ \sum_{j=0}^{\infty} q_j^{*(n-1)} \frac{\beta^j (t-x)^{j-1}}{(j-1)!} e^{-\beta(t-x)} \right\} \left\{ \sum_{k=0}^{\infty} \overline{Q}_k \frac{\beta^k x^{k+1}}{k!} e^{-\beta x} \right\} dx \\ = & \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} q_j^{*(n-1)} \overline{Q}_k \frac{(k+1)}{\beta^2} \frac{\beta^{j+k+2} t^{j+k+1}}{(j+k+1)!} e^{-\beta t}. \end{aligned} \tag{3.25}$$

Substituting (3.25) into (3.20), one arrives at

$$
\begin{aligned}
& p\left(n\,|0\right) \\
= \; & \frac{\lambda}{c}\int_0^\infty \frac{\lambda^{n-1}t^{n-2}}{(n-1)!}e^{-\lambda t}\left(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} q_j^{*(n-1)}\overline{Q}_k \frac{(k+1)}{\beta^2}\frac{\beta^{j+k+2}\,(ct)^{j+k+1}}{(j+k+1)!}e^{-c\beta t}\right)dt \\
= \; & \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} q_j^{*(n-1)}\overline{Q}_k \frac{(k+1)\,\lambda^n\,(c\beta)^{j+k}}{(n-1)!(j+k+1)!}\int_0^\infty \left(t^{n+j+k-1}e^{-(\lambda+c\beta)t}\right)dt \\
= \; & \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} q_j^{*(n-1)}\overline{Q}_k \frac{(k+1)\,\lambda^n\,(c\beta)^{j+k}}{(n-1)!(j+k+1)!}\frac{(n+j+k-1)!}{(\lambda+c\beta)^{n+j+k}} \\
= \; & \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} q_j^{*(n-1)}\overline{Q}_k \frac{k+1}{n+j+k}\binom{n+j+k}{n-1}\left(\frac{\lambda}{\lambda+c\beta}\right)^n\left(\frac{c\beta}{\lambda+c\beta}\right)^{j+k}.
\end{aligned}
$$

3.3 Alternative approach

In this section, we present an alternative analytic approach to obtain the joint generalized density of the time to ruin and the number of claims until ruin. Let $w(x,y)=1$ in Eq. (2.16), it follows that the Laplace transform of $\phi_{r,\delta}(u)$ satisfies

$$
\begin{aligned}
\widetilde{\phi}_{r,\delta}(s) &= \frac{r\lambda\left(\widetilde{\widetilde{P}}\left(\rho\right)-\widetilde{\widetilde{P}}\left(s\right)\right)}{cs-(\delta+\lambda)+r\lambda\widetilde{p}\left(s\right)} \\
&= r\lambda\widetilde{v}_{r,\delta}(s)\left(\widetilde{\widetilde{P}}\left(\rho\right)-\widetilde{\widetilde{P}}\left(s\right)\right), \qquad (3.26)
\end{aligned}
$$

where

$$
\widetilde{v}_{r,\delta}(s) = \frac{1}{cs-\lambda(1-r\widetilde{p}\left(s\right))-\delta}. \qquad (3.27)
$$

We remark that $v_{1,\delta}(x)$ is the δ-scale function of a compound Poisson risk process defined through its Laplace transform

$$
\widetilde{v}_{1,\delta}(s) = 1/\left(\varphi(s)-\delta\right),
$$

with $\varphi(s)$ the Laplace exponent defined as

$$\varphi(s) = cs - \lambda(1 - \widetilde{p}\,(s)),$$

(see, e.g., Kyprianou (2006)).

In order to obtain the joint generalized density of the time to ruin and the number of claims until ruin with an initial surplus level u, we need to invert (3.26) wrt s, r and δ. Note that in Section 3.2, (3.26) was first inverted wrt s and then in r and δ. In what follows, we invert $\widetilde{v}_{r,\delta}(s)\widetilde{\overline{P}}\,(s)$ and $\widetilde{v}_{r,\delta}(s)\widetilde{\overline{P}}\,(\rho)$ wrt s, r and δ simultaneously, which turns out to be relatively easier.

3.3.1 Inversion of $\widetilde{v}_{r,\delta}(s)\widetilde{\overline{P}}\,(s)$

Clearly, the key of the inversion is to invert $\widetilde{v}_{r,\delta}(s)$ wrt s, r and δ. From (3.27), we have

$$\begin{aligned} -\widetilde{v}_{r,\delta}(s) &= \frac{1}{\delta + \lambda(1 - r\widetilde{p}\,(s)) - cs} \\ &= \int_0^\infty e^{-\delta t} \left\{ e^{-\lambda t + cst} e^{r\lambda t \widetilde{p}(s)} \right\} dt, \end{aligned} \tag{3.28}$$

(see, e.g., Panjer and Willmot (1992, Section 11.7)). Given that

$$\begin{aligned} e^{r\lambda t\widetilde{p}(s)} &= 1 + \sum_{n=1}^{\infty} r^n \frac{(\lambda t)^n}{n!} \left(\widetilde{p}\,(s)\right)^n \\ &= 1 + \sum_{n=1}^{\infty} r^n \frac{(\lambda t)^n}{n!} \widetilde{p}^{*n}\,(s)\,, \end{aligned}$$

(3.28) becomes

$$
\begin{aligned}
-\widetilde{v}_{r,\delta}(s) &= \int_0^\infty e^{-\delta t}\left\{e^{-\lambda t+cst}\left(1+\sum_{n=1}^\infty r^n\frac{(\lambda t)^n}{n!}\widetilde{p}^{*n}(s)\right)\right\}dt \\
&= \int_0^\infty e^{-\delta t}\left\{e^{-\lambda t+cst}+e^{-\lambda t}\sum_{n=1}^\infty r^n\frac{(\lambda t)^n}{n!}\int_0^\infty e^{-s(y-ct)}p^{*n}(y)\,dy\right\}dt \\
&= \int_0^\infty e^{-\delta t}\left\{e^{-\lambda t+cst}+\sum_{n=1}^\infty r^n\int_{-ct}^\infty e^{-sy}f_n(y+ct,t)dy\right\}dt \\
&= \int_0^\infty e^{-\delta t}\left\{e^{-\lambda t+cst}+\int_{-ct}^\infty e^{-sy}f(y+ct,t;r)dy\right\}dt, \qquad (3.29)
\end{aligned}
$$

where

$$
f_n(y,t)=\frac{(\lambda t)^n e^{-\lambda t}}{n!}p^{*n}(y), \qquad (3.30)
$$

is the density (at y) of the total amount of n claims incurred by time t and

$$
f(y,t;r)=\sum_{n=1}^\infty r^n f_n(y,t).
$$

Clearly,

$$
f(y,t;1)=\sum_{n=1}^\infty\frac{(\lambda t)^n e^{-\lambda t}}{n!}p^{*n}(y),
$$

is the density (at y) of the aggregate claim amount S_t.

It follows that

$$
\begin{aligned}
&-\widetilde{v}_{r,\delta}(s)\widetilde{\overline{P}}(s) \\
&= \int_0^\infty e^{-sx}\overline{P}(x)\,dx\int_0^\infty e^{-\delta t}\left\{e^{-\lambda t+cst}+\sum_{n=1}^\infty r^n\int_{-ct}^\infty e^{-sy}f_n(y+ct,t)dy\right\}dt \\
&= \int_0^\infty e^{-\delta t}e^{-\lambda t}\int_0^\infty e^{-s(x-ct)}\overline{P}(x)\,dxdt \\
&\quad +\sum_{n=1}^\infty r^n\int_0^\infty e^{-\delta t}\int_{-ct}^\infty\int_0^\infty e^{-s(x+y)}\overline{P}(x)\,f_n(y+ct,t)dxdydt. \qquad (3.31)
\end{aligned}
$$

Changing the integration variables in (3.31), one arrives at

$$\begin{aligned}
-\widetilde{v}_{r,\delta}(s)\widetilde{\overline{P}}(s) &= \int_0^\infty e^{-\delta t}\int_{-ct}^\infty e^{-su}e^{-\lambda t}\overline{P}(u+ct)\,dudt \\
&\quad +\sum_{n=1}^\infty r^n\int_0^\infty e^{-\delta t}\int_{-ct}^\infty\int_0^{u+ct} e^{-su}\overline{P}(x)\,f_n(u-x+ct,t)dxdudt. \\
&= \int_0^\infty e^{-\delta t}\int_{-ct}^\infty e^{-su}e^{-\lambda t}\overline{P}(u+ct)\,dudt \\
&\quad +\sum_{n=1}^\infty r^n\int_0^\infty e^{-\delta t}\int_{-ct}^\infty e^{-su}h_n(u,t)dudt, \qquad (3.32)
\end{aligned}$$

where

$$h_n(u,t)=\int_0^{u+ct} f_n(x,t)\overline{P}(u+ct-x)\,dx.$$

3.3.2 Inversion of $\widetilde{v}_{r,\delta}(s)\widetilde{\overline{P}}(\rho)$

To invert $\widetilde{\overline{P}}(\rho)$ wrt r and δ, we first rewrite the Lagrange identity (3.15) as

$$\begin{aligned}
e^{-\rho y} &= e^{-\frac{\delta+\lambda}{c}y}+\sum_{n=1}^\infty r^n\frac{(\lambda/c)^n}{n!}y\int_0^\infty (x+y)^{n-1}e^{-(\delta+\lambda)(x+y)/c}p^{*n}(x)dx \\
&= e^{-\frac{\delta+\lambda}{c}y}+\sum_{n=1}^\infty r^n\frac{(\lambda/c)^n}{n!}y\int_{\frac{y}{c}}^\infty e^{-(\delta+\lambda)z}c\,(cz)^{n-1}p^{*n}(cz-y)dz \\
&= e^{-\frac{\delta+\lambda}{c}y}+\sum_{n=1}^\infty r^n\int_{\frac{y}{c}}^\infty e^{-\delta z}\left\{\frac{y}{z}f_n(cz-y,y)\right\}dz, \qquad (3.33)
\end{aligned}$$

where $f_n(cz-y,y)$ is the density (3.30) at $cz-y$.

Using (3.33), we have

$$\begin{aligned}
\widetilde{\overline{P}}(\rho) &= \int_0^\infty e^{-\rho y}\overline{P}(y)\,dy \\
&= \int_0^\infty e^{-\frac{\delta+\lambda}{c}y}\overline{P}(y)\,dy + \int_0^\infty \left\{\sum_{n=1}^\infty r^n \int_{\frac{y}{c}}^\infty e^{-\delta z}\left\{\frac{y}{z}f_n(cz-y,y)\right\}dz\right\}\overline{P}(y)\,dy \\
&= \int_0^\infty e^{-\delta z} c e^{-\lambda z}\overline{P}(cz)\,dz + \sum_{n=1}^\infty r^n \int_0^\infty \int_0^{cz} e^{-\delta z}\frac{y}{z}f_n(cz-y,y)\overline{P}(y)\,dydz \\
&= \int_0^\infty e^{-\delta z} c e^{-\lambda z}\overline{P}(cz)\,dz + \sum_{n=1}^\infty r^n \int_0^\infty e^{-\delta z} g_n(z)dz \\
&= \int_0^\infty e^{-\delta z} g(z;r)dz, \qquad (3.34)
\end{aligned}$$

where

$$g_n(z) = \int_0^{cz} \frac{y}{z} f_n(cz-y,y)\overline{P}(y)\,dy, \qquad (3.35)$$

and

$$g(z;r) = ce^{-\lambda z}\overline{P}(cz) + \sum_{n=1}^\infty r^n g_n(z).$$

Combining (3.29) and (3.34) yields

$$\begin{aligned}
&-\widetilde{v}_{r,\delta}(s)\widetilde{\overline{P}}(\rho) \\
&= \left\{\int_0^\infty e^{-\delta z} g(z;r)dz\right\}\int_0^\infty e^{-\delta t}\left\{e^{-\lambda t+cst} + \int_{-ct}^\infty e^{-sy} f(y+ct,t;r)dy\right\}dt \\
&= \int_0^\infty e^{-\delta t}\int_{-ct}^0 e^{-su}\left\{\frac{1}{c}g(t+\frac{u}{c};r)e^{-\frac{\lambda}{c}u}\right\}dudt \\
&\quad + \int_0^\infty e^{-\delta t}\int_{-ct}^0 e^{-su}\int_{-\frac{y}{c}}^t g(t-z;r)f(u+cz,z;r)dzdudt \\
&\quad + \int_0^\infty e^{-\delta t}\int_0^\infty e^{-su}\int_0^t g(t-z;r)f(u+cz,z;r)dzdudt. \qquad (3.36)
\end{aligned}$$

Substituting (3.32) and (3.36) into (3.26), one arrives at

$$\begin{aligned}
\widetilde{\phi}_{r,\delta}(s) &= r\lambda \int_0^\infty e^{-\delta t} \int_{-ct}^\infty e^{-su} e^{-\lambda t} \overline{P}\left(u+ct\right) dudt \\
&\quad + \lambda \sum_{n=1}^\infty r^{n+1} \int_0^\infty e^{-\delta t} \int_{-ct}^\infty e^{-su} h_n(u,t) dudt \\
&\quad - r\lambda \int_0^\infty e^{-\delta t} \int_{-ct}^0 e^{-su} \left\{ \frac{1}{c} g(t+\frac{u}{c};r) e^{-\frac{\lambda}{c}u} \right\} dudt \\
&\quad - r\lambda \int_0^\infty e^{-\delta t} \int_{-ct}^0 e^{-su} \int_{-\frac{y}{c}}^t g(t-z;r) f(u+cz,z;r) dzdudt \\
&\quad - r\lambda \int_0^\infty e^{-\delta t} \int_0^\infty e^{-su} \int_0^t g(t-z;r) f(u+cz,z;r) dzdudt. \qquad (3.37)
\end{aligned}$$

Given that

$$\widetilde{\phi}_{r,\delta}\left(s\right) = \int_0^\infty e^{-su} \phi_{r,\delta}\left(u\right) du,$$

(3.37) can be simplified to

$$\begin{aligned}
\widetilde{\phi}_{r,\delta}(s) &= r \int_0^\infty e^{-\delta t} \int_0^\infty e^{-su} \left\{ \lambda e^{-\lambda t} \overline{P}\left(u+ct\right) \right\} dudt \\
&\quad + \sum_{n=1}^\infty r^{n+1} \int_0^\infty e^{-\delta t} \int_0^\infty e^{-su} \left\{ \lambda h_n(u,t) \right\} dudt \\
&\quad - r\lambda \int_0^\infty e^{-\delta t} \int_0^\infty e^{-su} \int_0^t g(t-z;r) f(u+cz,z;r) dzdudt \\
&= r \int_0^\infty e^{-\delta t} \int_0^\infty e^{-su} \left\{ \lambda e^{-\lambda t} \overline{P}\left(u+ct\right) \right\} dudt \\
&\quad + \sum_{n=1}^\infty r^{n+1} \int_0^\infty e^{-\delta t} \int_0^\infty e^{-su} \left\{ \lambda h_n(u,t) - \lambda \zeta_n\left(u,t\right) \right\} dudt, \qquad (3.38)
\end{aligned}$$

where

$$\zeta_n\left(u,t\right) = \begin{cases} \int_0^t \left\{ c e^{-\lambda(t-z)} \overline{P}\left(c\left(t-z\right)\right) f_n(u+cz,z) \right\} dz, & n=1, \\ \int_0^t k_n\left(u,z\right) dz, & n=2,3,..., \end{cases}$$

and

$$k_n(u,z) = ce^{-\lambda(t-z)}\overline{P}(c(t-z))f_n(u+cz,z) + \sum_{m=1}^{n-1} g_m(t-z) f_{n-m}(u+cz,z),$$

for $n = 2, 3, ...$

From (3.38), one concludes that the joint generalized density of the time to ruin and the number of claims is given by

$$f(t,n|u) = \begin{cases} \lambda e^{-\lambda t}\overline{P}(u+ct), & n=1, \\ \lambda h_{n-1}(u,t) - \lambda\zeta_{n-1}(u,t), & n=2,3,... \end{cases} \tag{3.39}$$

When $u = 0$, an application of the initial value theorem on (3.26) yields

$$\phi_{r,\delta}(0) = \lim_{s\to\infty} s\widetilde{\phi}_{r,\delta}(s) = r\frac{\lambda}{c}\widetilde{\widetilde{P}}(\rho).$$

Using (3.34), it follows that

$$\begin{aligned} \phi_{r,\delta}(0) &= r\frac{\lambda}{c}\left\{\int_0^\infty e^{-\delta z} ce^{-\lambda z}\overline{P}(cz)\,dz + \sum_{n=1}^\infty r^n \int_0^\infty e^{-\delta z} g_n(z)dz\right\} \\ &= r\int_0^\infty e^{-\delta z}\left\{\lambda e^{-\lambda z}\overline{P}(cz)\right\}dz + \sum_{n=2}^\infty r^n \int_0^\infty e^{-\delta z}\left\{\frac{\lambda}{c} g_{n-1}(z)\right\}dz. \end{aligned}$$

The uniqueness property of Laplace transform implies that

$$f(t,n|0) = \begin{cases} \lambda e^{-\lambda t}\overline{P}(ct), & n=1, \\ \frac{\lambda}{c} g_{n-1}(t), & n=2,3,... \end{cases} \tag{3.40}$$

By simple algebraic manipulations, it is immediate that (3.40) is consistent with (3.18).

Remark 3.3.1 *Substituting (3.40) into (3.39), we have, for* $n = 1, 2, 3, ...,$

$$\begin{aligned} f\left(t, n+1\,|u\right) &= \lambda h_n(u,t) - c\int_0^t f\left(t, 1\,|0\right) f_n(u+cz, z)dz \\ &\quad -c\int_0^t \left\{\sum_{m=1}^{n-1} f\left(t-z, m+1\,|0\right) f_{n-m}(u+cz, z)\right\} dz \\ &= \lambda h_n(u,t) - c\int_0^t \left\{\sum_{m=1}^{n} f\left(t-z, m\,|0\right) f_{n+1-m}(u+cz, z)\right\} dz, \end{aligned}$$

which is precisely the recursive formula in Dickson (2012) (see Remark 3.2.1).

Remark 3.3.2 *Letting* $r = 1$ *in (3.37), one finds that the density of the time to ruin is given by*

$$f(t\,|u) = \lambda e^{-\lambda t}\overline{P}\left(u+ct\right) + \lambda\sum_{n=1}^{\infty} h_n(u,t) - \lambda\int_0^t g(t-z;1)f(u+cz,z;1)dz. \tag{3.41}$$

Also, from (3.40), it can be easily shown that for $u = 0$,

$$\begin{aligned} f(t\,|0) &= \lambda e^{-\lambda t}\overline{P}\left(ct\right) + \frac{\lambda}{c}\sum_{n=1}^{\infty} g_n(t) \\ &= \frac{\lambda}{c} g(t;1). \end{aligned} \tag{3.42}$$

Substituting (3.42) into (3.41), we have

$$\begin{aligned} f(t\,|u) &= \lambda\left\{e^{-\lambda t}\overline{P}\left(u+ct\right) + \int_0^{u+ct} f(x,t;1)\overline{P}\left(u+ct-x\right)dx\right\} \\ &\quad -c\int_0^t f(u+cz,z;1)f(t-z\,|0)dz. \end{aligned} \tag{3.43}$$

Eq. (3.43) is consistent with Seal's formula (see, e.g., Prabhu (1961) and Dickson (2007)).

Chapter 4

Sparre Andersen risk model: exponential claim sizes

4.1 Introduction

In this chapter, we consider the finite-time ruin problem in the ordinary Sparre Andersen risk model, in which the interclaim times $\{T_i\}_{i=1}^{\infty}$ and claim sizes $\{X_i\}_{i=1}^{\infty}$ are i.i.d. r.v.'s. The claim size r.v.'s $\{X_i\}_{i=1}^{\infty}$ are assumed to be mutually independent of the interclaim times $\{T_i\}_{i=1}^{\infty}$. We further assume that the claim size r.v.'s $\{X_i\}_{i=1}^{\infty}$ are exponentially distributed with mean $1/\beta$.

As pointed out in Section 1.3, Borovkov and Dickson (2008) used a duality argument to derive an infinite series representation for the density of the time to ruin in a Sparre Andersen risk model with exponential claims. We consider an alternative approach to that of Borovkov and Dickson (2008) which involves the use of Lagrange's expansion theorem. The advantage of this latter approach is that it allows for the probabilistic interpretation of each term in the series expansion which results for the density of the time of ruin. To be more precise, we incorporate the random variable representing the number of claims until ruin into the analysis, and interpret each term in the expansion in terms of the distribution of this new variable. From the point of view of risk management, such analysis provides additional insight for planning purposes by virtue of the fact that the distribution of quantities associated with the event of ruin (including the time of ruin and the number of claims until ruin) may be ascertained in advance of the event of ruin itself (even though the realized values of the associated variables obviously cannot be). Consequently, the proposed new Gerber-Shiu function (1.9)

$$\phi_{r,\delta}(u) \equiv E\left[r^{N_\tau} e^{-\delta\tau} 1(\tau < \infty) | U_0 = u\right], \qquad \delta \geq 0, \ r \in (0,1],$$

allows the identification of the joint distribution of the time to ruin and the number of claims until ruin. In turn, this joint distribution of the time to ruin and the number of claims until ruin will play an integral part in the interpretation of the joint generalized density of the time to ruin, the surplus prior to ruin, and the number of claims until ruin.

The main result of this chapter is obtained in Section 4.4 where a closed-form expression for the trivariate distribution of the time to ruin, the surplus prior to ruin and the number of claims until ruin is derived. Due to the intrinsic structure of this trivariate distribution, the simpler case involving the bivariate distribution of the time to ruin and the number of claims until ruin is analyzed first in Section 4.2. The marginal distribution of the number of claims until ruin is the subject matter of Section 4.3.

4.2 Joint distribution of the time to ruin and the number of claims until ruin

In this section, we analyze the joint distribution of the time to ruin and the number of claims until ruin. We start from the compound geometric tail expression (2.7) for $\{\phi_{r,\delta}(u), u \geq 0\}$.

When the claim sizes are exponentially distributed with mean $1/\beta$, from (2.9), it is immediate that

$$k_{r,\delta}(y) = \beta e^{-\beta y}, \qquad y \geq 0. \tag{4.1}$$

Utilizing the fact that $k_{r,\delta}$ is an exponential density with mean $1/\beta$, (2.7) becomes

$$\begin{aligned}
\phi_{r,\delta}(u) &= (1-\phi_{r,\delta}(0)) \sum_{j=1}^{\infty} (\phi_{r,\delta}(0))^j \sum_{i=0}^{j-1} \frac{(\beta u)^i e^{-\beta u}}{i!} \\
&= (1-\phi_{r,\delta}(0)) e^{-\beta u} \sum_{i=0}^{\infty} \frac{(\beta u)^i}{i!} \sum_{j=i}^{\infty} (\phi_{r,\delta}(0))^{j+1} \\
&= \phi_{r,\delta}(0) e^{-\beta u \left(1-\phi_{r,\delta}(0)\right)},
\end{aligned} \tag{4.2}$$

for $u \geq 0$. Note that (4.2) is a generalization of Eq. (3.16) in Willmot (2007).

Also, by conditioning on the time and the amount of the first claim, we have

$$\phi_{r,\delta}(u) = \int_0^\infty r e^{-\delta t} k(t) \left\{ \int_0^{u+ct} \phi_{r,\delta}(y) \beta e^{-\beta(u+ct-y)} dy + e^{-\beta(u+ct)} \right\} dt. \tag{4.3}$$

Using the defective renewal equation (2.5) with $k_{r,\delta}(y) = \beta e^{-\beta y}$ for $y \geq 0$, (4.3) becomes

$$\phi_{r,\delta}(u) = \int_0^\infty r e^{-\delta t} k(t) \frac{\phi_{r,\delta}(u+ct)}{\phi_{r,\delta}(0)} dt. \tag{4.4}$$

Then, substituting (4.2) into (4.4) yields

$$\phi_{r,\delta}(0) e^{-\beta u \left(1-\phi_{r,\delta}(0)\right)} = r e^{-\beta u \left(1-\phi_{r,\delta}(0)\right)} \widetilde{k}\left(\delta + c\beta\left(1-\phi_{r,\delta}(0)\right)\right). \tag{4.5}$$

Equating the coefficients of $e^{-\beta u\left(1-\phi_{r,\delta}(0)\right)}$ on both sides of (4.5) implies that $\phi_{r,\delta}(0)$ is a solution (in z) of

$$z = r\widetilde{k}\left(\delta + c\beta\left(1-z\right)\right). \tag{4.6}$$

Using Rouché's theorem, it can be proved that (4.6) has exactly one solution in the unit circle whenever $r < 1$ or $\delta > 0$.

Using Lagrange's expansion theorem with $f(x) = xe^{-\beta u(1-x)}$ (see Eq. (1.22)), we obtain

$$\begin{aligned}
\phi_{r,\delta}(0)\, e^{-\beta\left(1-\phi_{r,\delta}(0)\right)u} &= \sum_{n=1}^{\infty} \frac{r^n}{n!} \frac{d^{n-1}}{dx^{n-1}} \left[(1+\beta ux)\, e^{-\beta u(1-x)} \int_0^{\infty} e^{-(\delta + c\beta(1-x))z} k^{*n}(z)\, dz \right] \Bigg|_{x=0} \\
&= \sum_{n=1}^{\infty} \frac{r^n}{n!} \int_0^{\infty} e^{-\delta z} \left\{ \frac{d^{n-1}}{dx^{n-1}} (1+\beta ux)\, e^{-\beta(u+cz)(1-x)} \right\} k^{*n}(z)\, dz \Bigg|_{x=0} \\
&= re^{-\beta u}\widetilde{k}(\delta + c\beta) \\
&\quad + \sum_{n=2}^{\infty} \frac{r^n}{n!} \int_0^{\infty} e^{-\delta z} \left\{ \frac{d^{n-1}}{dx^{n-1}} (1+\beta ux)\, e^{-\beta(u+cz)(1-x)} \right\} k^{*n}(z)\, dz \Bigg|_{x=0},
\end{aligned}$$

where k^{*n} is the n-fold convolution of the density k with itself. Given that

$$\begin{aligned}
&\frac{d^{n-1}}{dx^{n-1}} (1+\beta ux)\, e^{-\beta(u+cz)(1-x)} \\
&= \sum_{j=0}^{n-1} \binom{n-1}{j} \left[\frac{d^j}{dx^j} (1+\beta ux) \right] \left[\frac{d^{n-1-j}}{dx^{n-1-j}} e^{-\beta(u+cz)(1-x)} \right] \\
&= \sum_{j=0}^{n-1} \binom{n-1}{j} \left[\frac{d^j}{dx^j} (1+\beta ux) \right] \left[\left(\beta\left(u+cz\right)\right)^{n-1-j} e^{-\beta(u+cz)(1-x)} \right] \\
&= \left((1+\beta ux)\, \beta\left(u+cz\right) + (n-1)\,\beta u \right) \left(\beta\left(u+cz\right)\right)^{n-2} e^{-\beta(u+cz)(1-x)},
\end{aligned}$$

for $n \geq 2$, it follows that

$$\phi_{r,\delta}(0)\, e^{-\beta u\left(1-\phi_{r,\delta}(0)\right)} = r\int_0^\infty e^{-\delta z}\left\{e^{-\beta(u+cz)}k(z)\right\}dz + \sum_{n=2}^{\infty} r^n \int_0^\infty e^{-\delta z}\left\{\frac{\beta^{n-1}(nu+cz)(u+cz)^{n-2}}{n!}e^{-\beta(u+cz)}k^{*n}(z)\right\}dz. \tag{4.7}$$

One concludes that the joint generalized density of the time to ruin and the number of claim until ruin is given by

$$f(t,n\,|u) = \begin{cases} e^{-\beta(u+ct)}k(t), & n=1, \\ \frac{nu+ct}{n(n-1)}\gamma_{\beta,n-1}(u+ct)\,k^{*n}(t), & n=2,3,..., \end{cases} \tag{4.8}$$

for $t \geq 0$, where $\gamma_{\beta,n}$ is the Erlang density

$$\gamma_{\beta,n}(y) = \frac{\beta^n y^{n-1} e^{-\beta y}}{(n-1)!}, \qquad y \geq 0.$$

It is immediate that the density of the time to ruin is given by

$$f(t\,|u) = e^{-\beta(u+ct)}k(t) + \sum_{n=2}^{\infty}\frac{nu+ct}{n(n-1)}\gamma_{\beta,n-1}(u+ct)\,k^{*n}(t), \tag{4.9}$$

which corresponds to Eq. (3) in Borovkov and Dickson (2008). As a result, the individual terms of the infinite-sum representation (4.9) correspond to the "density of the time to ruin" contributions with respect to the number of claims until ruin.

Remark 4.2.1 *An alternative approach to solve for $f(t,n\,|u)$ is to use the method proposed by Chan and Zhang (2006). By conditioning on the time and the amount of the first claim, we have*

$$f(t,1\,|u) = k(t)\,e^{-\beta(u+ct)}, \tag{4.10}$$

and

$$f(t,n\,|u) = \int_0^t k(t-x)\left[\int_0^{u+c(t-x)} \beta e^{-\beta(u+c(t-x)-y)} f(x,n-1\,|y)\,dy\right]dx, \tag{4.11}$$

for $n = 2, 3, ...$ With the help of (4.10) and (4.11), it is easy to find an explicit expression for $f(t,n\,|u)$ for the first few values of n. Then, one infers the general form of the solution of $f(t,n\,|u)$, and uses an inductive argument to verify its validity. However, although this technique is straightforward and involves only simple algebraic manipulations in the present context, it largely relies on the identification of a general solution which may turn out to be a challenging task in some (other) cases. For instance, the application of this technique in Section 4.4 to obtain the distribution of the time to ruin, the surplus prior to ruin, and the number of claims at ruin is definitely not a trivial one. For this reason (and others), we believe that the Gerber-Shiu approach proposed above is deductive rather than inductive and can be employed to solve more general problems (see Section 4.4 for more details).

In Borovkov and Dickson (2008), an expression for the density of the time to ruin is also derived in the context of the delayed renewal risk model. We extend this result to the joint generalized density of the time to ruin and the number of claims until ruin, namely $f_0(t,n\,|u)$. By conditioning on the time of the first interclaim time with density k_0, we have

$$f_0(t,1\,|u) = k_0(t)\,e^{-\beta(u+ct)}.$$

In general, for $n = 2, 3, ...$, we condition on the time and the amount of the first claim to obtain

$$f_0(t,n\,|u) = \int_0^t k_0(t-x)\left[\int_0^{u+c(t-x)} \beta e^{-\beta(u+c(t-x)-y)} f(x,n-1\,|y)\,dy\right]dx. \tag{4.12}$$

Substituting (4.8) into (4.12), when $n=2$, results in

$$\begin{aligned} f_0(t,2\,|u) &= \int_0^t k_0(t-x)\left[\int_0^{u+c(t-x)} \beta e^{-\beta(u+c(t-x)-y)}\left\{e^{-\beta(y+cx)}k(x)\right\}dy\right]dx \\ &= e^{-\beta(u+ct)}\int_0^t k_0(t-x)\,\beta\,(u+c(t-x))\,k(x)\,dx \\ &= e^{-\beta(u+ct)}\int_0^t k_0(x)\,\beta\,(u+cx)\,k(t-x)\,dx \\ &= e^{-\beta(u+ct)}\beta\,(u\,(k_0*k(t))+c\,(k_1*k(t))), \end{aligned}$$

where $k_1(x)=xk_0(x)$, and $*$ denotes convolution, e.g., $a*b(t)=\int_0^t a(x)b(t-x)dx$.

Similarly, for $n=3,4,...$, we have

$$\begin{aligned} &f_0(t,n\,|u) \\ &= \int_0^t k_0(t-x)\left[\int_0^{u+c(t-x)} \beta e^{-\beta(u+c(t-x)-y)} f(x,n-1\,|y)\,dy\right]dx \\ &= e^{-\beta(u+ct)}\frac{\beta^{n-1}}{(n-1)!}\int_0^t k_0(t-x)\,k^{*(n-1)}(x)\left[\int_0^{u+c(t-x)}((n-1)\,y+cx)\,(y+cx)^{n-3}\,dy\right]dx, \end{aligned}$$

where

$$\begin{aligned} &\int_0^{u+c(t-x)}((n-1)\,y+cx)\,(y+cx)^{n-3}\,dy \\ &= \int_{cx}^{u+ct}((n-1)\,(y-cx)+cx)\,y^{n-3}dy \\ &= \int_{cx}^{u+ct}(n-1)\,y^{n-2}dy - cx\int_{cx}^{u+ct}(n-2)\,y^{n-3}dy \\ &= (u+ct)^{n-1} - cx\,(u+ct)^{n-2} \\ &= (u+ct)^{n-2}\,(u+c\,(t-x)). \end{aligned}$$

One concludes that

$$f_0(t, n|u) = \frac{\beta^{n-1}(u+ct)^{n-2}e^{-\beta(u+ct)}}{(n-1)!}\int_0^t k_0(t-x)\,k^{*(n-1)}(x)\,((u+c(t-x)))\,dx$$
$$= \frac{\gamma_{n-1,\beta}(u+ct)}{n-1}\left(u\left(k_0 * k^{*(n-1)}(t)\right) + c\left(k_1 * k^{*(n-1)}(t)\right)\right),$$

for $n = 2, 3, ...$

4.3 Marginal distribution of the number of claims until ruin

Recall from (3.19) that the p.m.f. of the number of claims until ruin for an initial surplus of $U_0 = u$ is denoted as $p(n|u)$ for $n = 1, 2, ...$ From (4.8), it is easy to see that

$$p(n|u) = \begin{cases} e^{-\beta u}\widetilde{k}(\beta c), & n = 1, \\ \int_0^\infty \frac{nu+ct}{n(n-1)}\gamma_{\beta,n-1}(u+ct)\,k^{*n}(t)\,dt, & n = 2, 3, ... \end{cases} \tag{4.13}$$

Using (4.13), a closed-form expression for $p(n|u)$ is first derived when the interclaim time distribution is mixed Erlang. For that purpose, we assume that the interclaim time density k has a Laplace transform of the form

$$\widetilde{k}(s) = Q\left(\frac{\lambda}{\lambda+s}\right), \tag{4.14}$$

where

$$Q(s) = \sum_{j=1}^{\infty} q_j s^j,$$

with $\{q_j\}_{j=1}^{\infty}$ being a probability measure.

We refer the readers to Willmot and Woo (2007) and Willmot and Lin (2011) for the scope of distributions that belong to this class of mixed Erlang distributions. For the mixed

Erlang distribution (4.14), (4.13) yields

$$p\left(1\left|u\right.\right)=e^{-\beta u}Q\left(\frac{\lambda}{\lambda+c\beta}\right),$$

and

$$\begin{aligned}&p\left(n\left|u\right.\right)\\&=\frac{\beta^{n-1}}{n!}e^{-\beta u}\sum_{j=1}^{\infty}q_j^{*n}\frac{\lambda^j}{(j-1)!}\int_0^{\infty}\left(nu+ct\right)\left(u+ct\right)^{n-2}t^{j-1}e^{-(\lambda+\beta c)t}dt\\&=\frac{\beta^{n-1}}{n!}e^{-\beta u}\sum_{j=1}^{\infty}q_j^{*n}\frac{\lambda^j}{(j-1)!}\sum_{i=0}^{n-2}\binom{n-2}{i}u^{n-2-i}c^i\int_0^{\infty}\left(nu+ct\right)t^{i+j-1}e^{-(\lambda+\beta c)t}dt\\&=\frac{\beta^{n-1}}{n!}e^{-\beta u}\sum_{j=1}^{\infty}q_j^{*n}\frac{\left(\frac{\lambda}{\lambda+c\beta}\right)^j}{(j-1)!}\sum_{i=0}^{n-2}\binom{n-2}{i}u^{n-2-i}c^i\left(nu\frac{(i+j-1)!}{\left(\lambda+\beta c\right)^i}+c\frac{(i+j)!}{\left(\lambda+\beta c\right)^{i+1}}\right),\end{aligned}\tag{4.15}$$

where q^{*n} is the n-fold convolution of the p.m.f. q. In particular, when $u=0$, we have

$$\begin{aligned}p\left(n\left|0\right.\right)&=\sum_{j=1}^{\infty}q_j^{*n}\frac{\left(c\beta\right)^{n-1}}{n!}\frac{\lambda^j}{(j-1)!}\int_0^{\infty}t^{n+j-2}e^{-(\lambda+\beta c)t}dt\\&=\sum_{j=1}^{\infty}q_j^{*n}\frac{(n+j-2)!}{n!\,(j-1)!}\left(\frac{\lambda}{\lambda+c\beta}\right)^j\left(\frac{c\beta}{\lambda+c\beta}\right)^{n-1},\end{aligned}$$

for $n=1,2,...$

We remark that (4.15) can be viewed as an alternative to the recursive formula given by Stanford et al. (2000, Theorem 3.1 and Eq. (3.6)) for the calculation of $p\left(n\left|u\right.\right)$. In the following example, we identify the (marginal) distribution of the number of claims until ruin when the interclaim times are exponentially distributed.

Example 4.3.1 *(Exponential interclaim times) We assume that the Laplace transform of the interclaim time density k is of the form (4.14) with $Q\left(s\right)=s$. Under this distributional* *assumption, it follows that*

$$p(n|0) = \frac{(2n-2)!}{n!(n-1)!}\left(\frac{\lambda}{\lambda+c\beta}\right)^{n}\left(\frac{c\beta}{\lambda+c\beta}\right)^{n-1}, \tag{4.16}$$

for $n = 1, 2, ...$ *for which the form of this p.m.f. can be linked to the extended-truncated negative binomial (ETNB) distribution. Indeed, by making use of Gauss' multiplication formula*

$$\Gamma(2n-2) = \frac{2^{2(n-1)-1}\Gamma(n-1)\Gamma\left(n-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)},$$

(4.16) becomes

$$\begin{aligned} p(n|0) &= \frac{2(n-1)\Gamma(2n-2)}{n!\Gamma(n)}\left(\frac{\lambda}{\lambda+c\beta}\right)^{n}\left(\frac{c\beta}{\lambda+c\beta}\right)^{n-1} \\ &= \frac{4^{n-1}\Gamma\left(n-\frac{1}{2}\right)}{n!\Gamma\left(\frac{1}{2}\right)}\left(\frac{\lambda}{\lambda+c\beta}\right)^{n}\left(\frac{c\beta}{\lambda+c\beta}\right)^{n-1} \\ &= \frac{4^{n-1}\Gamma\left(n-\frac{1}{2}\right)}{n!\left(-\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\right)}\left(\frac{\lambda}{\lambda+c\beta}\right)^{n}\left(\frac{c\beta}{\lambda+c\beta}\right)^{n-1}. \end{aligned}$$

Let

$$\binom{n+r-1}{n} = \frac{\Gamma(n+r)}{n!\Gamma(r)},$$

be the generalized binomial coefficient for a non-negative integer n. *It follows that*

$$\begin{aligned} p(n|0) &= \frac{\binom{n-\frac{1}{2}-1}{n}\left(\frac{4c\beta\lambda}{(\lambda+c\beta)^{2}}\right)^{n}}{\frac{-2c\beta}{\lambda+c\beta}} \\ &= \frac{\binom{n-\frac{1}{2}-1}{n}\left(\frac{(\lambda+c\beta)^{2}-(\lambda-c\beta)^{2}}{(\lambda+c\beta)^{2}}\right)^{n}}{\frac{-2c\beta}{\lambda+c\beta}} \\ &= \frac{\binom{n-\frac{1}{2}-1}{n}\left(1-\left(\frac{\lambda-c\beta}{\lambda+c\beta}\right)^{2}\right)^{n}}{\frac{-2c\beta}{\lambda+c\beta}}, \end{aligned}$$

for $n = 1, 2, ...$

Whenever the positive security loading $c\beta > \lambda$ is satisfied, the p.m.f. of the number of claims until ruin is defective. We have

$$p(n\,|0) = \frac{\frac{c\beta-\lambda}{\lambda+c\beta}-1}{\frac{-2c\beta}{\lambda+c\beta}} \frac{\binom{n-\frac{1}{2}-1}{n}\left(1-\left(\frac{\lambda-c\beta}{\lambda+c\beta}\right)^2\right)^n}{\frac{c\beta-\lambda}{\lambda+c\beta}-1}$$
$$= \frac{\lambda}{c\beta} \frac{\binom{n-\frac{1}{2}-1}{n}\left(1-\left(\frac{\lambda-c\beta}{\lambda+c\beta}\right)^2\right)^n}{\frac{c\beta-\lambda}{\lambda+c\beta}-1}$$
$$= \frac{1}{1+\theta} g_n,$$

where θ is the positive security loading which satisfies

$$c\beta = \lambda(1+\theta),$$

and

$$g_n = \frac{\binom{n-\frac{1}{2}-1}{n}\left(1-\left(\frac{\lambda-c\beta}{\lambda+c\beta}\right)^2\right)^n}{\frac{c\beta-\lambda}{\lambda+c\beta}-1},$$

is the p.m.f. of an extended truncated negative binomial (ETNB) r.v.'s with parameters $\alpha = -\frac{1}{2}$ and $p = 1-\left(\frac{\lambda-c\beta}{\lambda+c\beta}\right)^2$ (see Willmot (1988, Eq. 3.1)).

Whenever $c\beta < \lambda$, $\{p(n\,|0)\}_{n\geq 1}$ is a proper p.m.f. with

$$p(n\,|0) = \frac{\binom{n-\frac{1}{2}-1}{n}\left(1-\left(\frac{\lambda-c\beta}{\lambda+c\beta}\right)^2\right)^n}{\frac{\lambda-c\beta}{\lambda+c\beta}-1}.$$

We point out that

$$\phi_{r,0}(u) = C(\phi_{r,0}(0)),$$

where

$$C(z) = ze^{\beta u(z-1)},$$

is the p.g.f. of a shifted (by one unit) Poisson r.v. with mean βu. The secondary distribution has p.g.f. $\phi_{r,0}(0)$.

4.4 Joint distribution of the time to ruin, the surplus prior to ruin and the number of claims until ruin

In this section, the joint Laplace transform/p.g.f. of the number of claims until ruin, the time to ruin and the surplus prior to ruin is presented, generalizing a result of Willmot (2007). Further, this new analytic tool is explicitly inverted (with the help of (4.8)) to identify the joint generalized density of the time to ruin, the surplus prior to ruin, and the number of claims until ruin.

If we let $w(U_{\tau-}, |U_\tau|) = e^{-sU_{\tau-}}$ in Eq. (2.1), the Gerber-Shiu function becomes

$$\phi_{r,\delta,s}(u) \equiv E\left[r^{N_\tau} e^{-\delta\tau - sU_{\tau-}} 1(\tau < \infty) | U_0 = u\right],$$

for $s \geq 0$, $\delta \geq 0$ and $r \in (0,1]$. By conditioning on the first drop in initial surplus, we have

$$\begin{aligned}\phi_{r,\delta,s}(u) &= \int_0^u \int_0^\infty \phi_{r,\delta,s}(u-y)\, \xi_{r,\delta}(x,y|0)\, dxdy + \int_u^\infty \int_0^\infty e^{-s(u+x)} \xi_{r,\delta}(x,y|0)\, dxdy \\ &= \phi_{r,\delta}(0) \int_0^u \phi_{r,\delta,s}(u-y)\, k_{r,\delta}(y)\, dy + \int_u^\infty \int_0^\infty e^{-s(u+x)} \xi_{r,\delta}(x,y|0)\, dxdy, \quad (4.17)\end{aligned}$$

for $u \geq 0$. Using (4.1), (4.17) can be rewritten as

$$\begin{aligned}\phi_{r,\delta,s}(u) &= \phi_{r,\delta}(0) \int_0^u \phi_{r,\delta,s}(u-y)\, \beta e^{-\beta y} dy + \int_u^\infty \int_0^\infty e^{-s(u+x)} \xi_{r,\delta}(x|0)\, \beta e^{-\beta y} dxdy \\ &= \phi_{r,\delta}(0) \int_0^u \phi_{r,\delta,s}(u-y)\, \beta e^{-\beta y} dy + \widetilde{b}_{r,\delta}(s) e^{-(s+\beta)u}, \quad (4.18)\end{aligned}$$

where

$$\widetilde{b}_{r,\delta}(s) = \int_0^\infty e^{-sx} \xi_{r,\delta}(x|0)\, dx.$$

Let $\widetilde{\phi}_{r,\delta,s}\left(z\right)=\int_0^\infty e^{-zu}\phi_{r,\delta,s}\left(u\right)du$. From (4.18), one deduces that

$$\widetilde{\phi}_{r,\delta,s}\left(z\right)=\phi_{r,\delta}\left(0\right)\widetilde{\phi}_{r,\delta,s}\left(z\right)\frac{\beta}{\beta+z}+\frac{\widetilde{b}_{r,\delta}(s)}{s+\beta+z}.$$

Solving for $\widetilde{\phi}_{r,\delta,s}\left(z\right)$ and subsequently expanding into a partial fraction form, one obtains

$$\begin{aligned}\widetilde{\phi}_{r,\delta,s}\left(z\right)&=\frac{\frac{\widetilde{b}_{r,\delta}(s)}{s+\beta+z}}{1-\phi_{r,\delta}(0)\frac{\beta}{\beta+z}}\\&=\frac{\widetilde{b}_{r,\delta}(s)(\beta+z)}{(s+\beta+z)(\beta(1-\phi_{r,\delta}\left(0\right))+z)}\\&=\frac{\widetilde{b}_{r,\delta}(s)}{s+\beta\phi_{r,\delta}\left(0\right)}\left\{\frac{s}{s+\beta+z}+\frac{\beta\phi_{r,\delta}\left(0\right)}{z+\beta(1-\phi_{r,\delta}\left(0\right))}\right\}.\end{aligned}\qquad(4.19)$$

Inversion of (4.19) w.r.t. z yields

$$\begin{aligned}\phi_{r,\delta,s}\left(u\right)&=\frac{\widetilde{b}_{r,\delta}(s)}{s+\beta\phi_{r,\delta}\left(0\right)}\left\{se^{-(s+\beta)u}+\beta\phi_{r,\delta}\left(0\right)e^{-\beta u\left(1-\phi_{r,\delta}(0)\right)}\right\}\\&=\frac{\widetilde{b}_{r,\delta}(s)}{s+\beta\phi_{r,\delta}\left(0\right)}\left\{se^{-(s+\beta)u}+\beta\phi_{r,\delta}\left(u\right)\right\},\end{aligned}\qquad(4.20)$$

for $u\geq 0$.

To complete the characterization of $\phi_{r,\delta,s}\left(u\right)$ given in (4.20), we shall find an explicit expression for $\widetilde{b}_{r,\delta}(s)$ (to ultimately allow its inversion w.r.t. the probability generating function argument r as well as the Laplace transform arguments δ and s). Indeed, by conditioning on the time and the amount of the first claim,

$$\phi_{r,\delta,s}\left(u\right)=\int_0^\infty re^{-\delta t}k\left(t\right)\left\{\int_0^{u+ct}\phi_{r,\delta,s}\left(y\right)\beta e^{-\beta(u+ct-y)}dy+e^{-(s+\beta)(u+ct)}\right\}dt.\qquad(4.21)$$

Substituting (4.20) into (4.21) and using (2.5) and (4.2), (4.21) becomes

$$
\begin{aligned}
&\phi_{r,\delta,s}(u) \\
&= \int_0^\infty re^{-\delta t}k(t)\left\{\frac{\widetilde{b}_{r,\delta}(s)}{s+\beta\phi_{r,\delta}(0)}\beta e^{-\beta(u+ct)}(1-e^{-s(u+ct)})\right. \\
&\left. +\frac{\widetilde{b}_{r,\delta}(s)}{s+\beta\phi_{r,\delta}(0)}\beta\left[\frac{\phi_{r,\delta}(u+ct)}{\phi_{r,\delta}(0)}-e^{-\beta(u+ct)}\right]+e^{-(s+\beta)(u+ct)}\right\}dt \\
&= \int_0^\infty re^{-\delta t}k(t)\left\{\frac{\beta\widetilde{b}_{r,\delta}(s)}{s+\beta\phi_{r,\delta}(0)}\frac{\phi_{r,\delta}(u+ct)}{\phi_{r,\delta}(0)}+\left[1-\frac{\beta\widetilde{b}_{r,\delta}(s)}{s+\beta\phi_{r,\delta}(0)}\right]e^{-(s+\beta)(u+ct)}\right\}dt \\
&= \frac{\beta\widetilde{b}_{r,\delta}(s)}{s+\beta\phi_{r,\delta}(0)}\int_0^\infty re^{-\delta t}k(t)\frac{\phi_{r,\delta}(u+ct)}{\phi_{r,\delta}(0)}dt \\
&+\left[1-\frac{\beta\widetilde{b}_{r,\delta}(s)}{s+\beta\phi_{r,\delta}(0)}\right]\int_0^\infty re^{-\delta t}k(t)e^{-(s+\beta)(u+ct)}dt \\
&= \frac{\beta\widetilde{b}_{r,\delta}(s)}{s+\beta\phi_{r,\delta}(0)}\phi_{r,\delta}(u)+\left[1-\frac{\beta\widetilde{b}_{r,\delta}(s)}{s+\beta\phi_{r,\delta}(0)}\right]r\widetilde{k}(\delta+c(s+\beta))e^{-(s+\beta)u}. \qquad (4.22)
\end{aligned}
$$

Equating the coefficients of $e^{-(s+\beta)u}$ on both sides of (4.22), one concludes, using (4.20), that

$$\frac{\widetilde{b}_{r,\delta}(s)}{s+\beta\phi_{r,\delta}(0)}=\frac{r\widetilde{k}(\delta+c(s+\beta))}{s+r\beta\widetilde{k}(\delta+c(s+\beta))},$$

which implies that

$$
\begin{aligned}
\phi_{r,\delta,s}(u) &= \frac{r\widetilde{k}(\delta+c(s+\beta))}{s+r\beta\widetilde{k}(\delta+c(s+\beta))}\left\{se^{-(s+\beta)u}+\beta\phi_{r,\delta}(u)\right\} \\
&= \frac{r\beta\widetilde{k}(\delta+c(s+\beta))}{s+r\beta\widetilde{k}(\delta+c(s+\beta))}\left\{\frac{s}{\beta}e^{-(s+\beta)u}+\phi_{r,\delta}(u)\right\}. \qquad (4.23)
\end{aligned}
$$

It remains to invert (4.23) w.r.t. the arguments r, δ, and s. We point out that $\phi_{r,\delta,s}(u)$ is explicitly expressed in terms of $\phi_{r,\delta}(u)$ which explains the order of presentation in this section.

Capitalizing on the well-known identity

$$\frac{x}{1+x} = \sum_{n=1}^{\infty} (-1)^{n-1} x^n, \qquad |x| < 1,$$

the first term on the right-hand side of (4.23) can be expanded into the following power series in r (for $s > \beta$):

$$\begin{aligned}
\frac{r\beta\widetilde{k}\left(\delta + c(s+\beta)\right)}{s + r\beta\widetilde{k}\left(\delta + c(s+\beta)\right)} &= \sum_{n=1}^{\infty} r^n \left\{ (-1)^{n-1} \left(\frac{\beta\widetilde{k}\left(\delta + c(s+\beta)\right)}{s} \right)^n \right\} \\
&= \sum_{n=1}^{\infty} r^n \int_0^{\infty} e^{-\delta t} \left\{ \frac{e^{-cst}}{s^n} (-1)^{n-1} \beta^n e^{-c\beta t} k^{*n}(t) \right\} dt. \qquad (4.24)
\end{aligned}$$

It remains to invert (4.24) w.r.t. the Laplace transform argument s.

Given that

$$\begin{aligned}
\frac{e^{-cst}}{s^n} &= e^{-cst} \int_0^{\infty} \frac{1}{(n-1)!} x^{n-1} e^{-sx} dx \\
&= \int_{ct}^{\infty} \frac{1}{(n-1)!} (x - ct)^{n-1} e^{-sx} dx, \qquad (4.25)
\end{aligned}$$

for $n = 1, 2, ...,$ (4.24) becomes

$$\begin{aligned}
&\frac{r\beta\widetilde{k}\left(\delta + c(s+\beta)\right)}{s + r\beta\widetilde{k}\left(\delta + c(s+\beta)\right)} \\
&= \sum_{n=1}^{\infty} r^n \int_0^{\infty} e^{-\delta t} \left\{ (-1)^{n-1} \beta^n e^{-c\beta t} k^{*n}(t) \int_{ct}^{\infty} \frac{1}{(n-1)!} (x - ct)^{n-1} e^{-sx} dx \right\} dt \\
&= \sum_{n=1}^{\infty} r^n \int_0^{\infty} \int_{ct}^{\infty} e^{-\delta t} e^{-sx} \left\{ \frac{\beta^n (ct - x)^{n-1} e^{-c\beta t}}{(n-1)!} k^{*n}(t) \right\} dx dt. \qquad (4.26)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\frac{r\beta\widetilde{k}\left(\delta+c(s+\beta)\right)}{s+r\beta\widetilde{k}\left(\delta+c(s+\beta)\right)}\frac{s}{\beta}e^{-(s+\beta)u} \\
&= se^{-(s+\beta)u}\left\{\sum_{n=1}^{\infty}r^{n}\int_{0}^{\infty}e^{-\delta t}\left\{\frac{e^{-cst}}{s^{n}}(-1)^{n-1}\beta^{n-1}e^{-c\beta t}k^{*n}\left(t\right)\right\}dt\right\} \\
&= \sum_{n=1}^{\infty}r^{n}\int_{0}^{\infty}e^{-\delta t}\left\{\frac{e^{-s(u+ct)}}{s^{n-1}}(-1)^{n-1}\beta^{n-1}e^{-\beta(u+ct)}k^{*n}\left(t\right)\right\}dt \\
&= r\int_{0}^{\infty}e^{-\delta t}e^{-s(u+ct)}\left\{e^{-\beta(u+ct)}k\left(t\right)\right\}dt \\
&-\sum_{n=2}^{\infty}r^{n}\int_{0}^{\infty}\int_{u+ct}^{\infty}e^{-\delta t}e^{-sx}\left\{\frac{\beta^{n-1}(u+ct-x)^{n-2}e^{-\beta(u+ct)}}{(n-2)!}k^{*n}\left(t\right)\right\}dxdt. \qquad (4.27)
\end{aligned}$$

Using (4.26) and (4.7), we also have

$$\begin{aligned}
&\frac{r\beta\widetilde{k}\left(\delta+c(s+\beta)\right)}{s+r\beta\widetilde{k}\left(\delta+c(s+\beta)\right)}\phi_{r,\delta}\left(u\right) \\
&= \left(\sum_{n=1}^{\infty}r^{n}\int_{0}^{\infty}\int_{ct}^{\infty}e^{-\delta t}e^{-sx}\left\{\frac{\beta^{n}(ct-x)^{n-1}e^{-c\beta t}}{(n-1)!}k^{*n}\left(t\right)\right\}dxdt\right) \\
&\times\left(\sum_{n=1}^{\infty}r^{n}\int_{0}^{\infty}e^{-\delta t}f\left(t,n\left|u\right.\right)dt\right). \qquad (4.28)
\end{aligned}$$

Through convolutions, (4.28) becomes

$$\begin{aligned}
&\frac{r\beta\widetilde{k}\left(\delta+c(s+\beta)\right)}{s+r\beta\widetilde{k}\left(\delta+c(s+\beta)\right)}\phi_{r,\delta}\left(u\right) \\
&= \sum_{n=2}^{\infty}r^{n}\int_{0}^{\infty}e^{-\delta t}\left\{\sum_{m=1}^{n-1}\int_{0}^{t}\int_{cz}^{\infty}e^{-sx}g_{n}\left(t,z,m\left|u\right.\right)dxdz\right\}dt,
\end{aligned}$$

where

$$g_{n}\left(t,z,m\left|u\right.\right)=\frac{\beta^{m}\left(cz-x\right)^{m-1}e^{-c\beta z}}{(m-1)!}k^{*m}\left(z\right)f\left(t-z,n-m\left|u\right.\right).$$

Changing the order of integration of the two inner integrals, one arrives at

$$\frac{r\beta\widetilde{k}\left(\delta+c(s+\beta)\right)}{s+r\beta\widetilde{k}\left(\delta+c(s+\beta)\right)}\phi_{r,\delta}\left(u\right)$$
$$=\sum_{n=2}^{\infty}r^{n}\int_{0}^{\infty}\int_{0}^{\infty}e^{-\delta t}e^{-sx}\left\{\sum_{m=1}^{n-1}\int_{0}^{\min\left(\frac{x}{c},t\right)}g_{n}\left(t,z,m\left|u\right.\right)dz\right\}dxdt, \qquad (4.29)$$

Substituting (4.27) and (4.29) into (4.23), one concludes that

$$\phi_{r,\delta,s}\left(u\right)$$
$$=\sum_{n=2}^{\infty}r^{n}\int_{0}^{\infty}\int_{0}^{\infty}e^{-\delta t}e^{-sx}\left\{\sum_{m=1}^{n-1}\int_{0}^{\min\left(\frac{x}{c},t\right)}g_{n}\left(t,z,m\left|u\right.\right)dz\right\}dxdt$$
$$+r\int_{0}^{\infty}e^{-\delta t}e^{-s(u+ct)}\left\{e^{-\beta(u+ct)}k\left(t\right)\right\}dt$$
$$-\sum_{n=2}^{\infty}r^{n}\int_{0}^{\infty}\int_{u+ct}^{\infty}e^{-\delta t}e^{-sx}\left\{\frac{\beta^{n-1}(u+ct-x)^{n-2}e^{-\beta(u+ct)}}{(n-2)!}k^{*n}\left(t\right)\right\}dxdt. \qquad (4.30)$$

Given that the surplus prior to ruin is at most $u+ct$ for a time to ruin of t, (4.30) simplifies to

$$\phi_{r,\delta,s}\left(u\right)=r\int_{0}^{\infty}e^{-\delta t}e^{-s(u+ct)}\left\{e^{-\beta(u+ct)}k\left(t\right)\right\}dt$$
$$+\sum_{n=2}^{\infty}r^{n}\int_{0}^{\infty}\int_{0}^{u+ct}e^{-\delta t}e^{-sx}\left\{\sum_{m=1}^{n-1}\int_{0}^{\min\left(\frac{x}{c},t\right)}g_{n}\left(t,z,m\left|u\right.\right)dz\right\}dxdt.$$

One concludes that the joint generalized density of the time to ruin, the surplus prior to ruin, and the number of claims until ruin is

$$f\left(t,x,n\left|u\right.\right)$$
$$=\begin{cases} e^{-\beta(u+ct)}k\left(t\right), & n=1,\ x=u+ct,\\ \sum_{m=1}^{n-1}\int_{0}^{\min\left(\frac{x}{c},t\right)}\frac{\beta^{m}(cz-x)^{m-1}e^{-c\beta z}}{(m-1)!}k^{*m}\left(z\right)f\left(t-z,n-m\left|u\right.\right)dz, & n\geq 2,\ x\in[0,u+ct],\\ 0, & \text{elsewhere,}\end{cases} \qquad (4.31)$$

for $t, u \geq 0$.

Formula (4.31) can be further simplified when the interclaim time distribution is specified. In the following example, we assume that interclaim times are mixed Erlang distributed.

Example 4.4.1 *In the case that the interclaim time distribution is mixed Erlang with Laplace transform (4.14), (4.31) becomes*

$$f\left(t, u+ct, 1\left|u\right.\right) = e^{-\beta(u+ct)} \sum_{j=1}^{\infty} q_j^{*n} \frac{\lambda^j t^{j-1} e^{-\lambda t}}{(j-1)!},$$

and

$$\begin{aligned}
& f\left(t, x, n\left|u\right.\right) \\
&= \sum_{m=1}^{n-1} \int_0^{\min\left(\frac{x}{c}, t\right)} \frac{\beta^m (cz-x)^{m-1} e^{-c\beta z}}{(m-1)!} \left(\sum_{k=1}^{\infty} q_k^{*m} \frac{\lambda^k z^{k-1} e^{-\lambda z}}{(k-1)!} \right) f\left(t-z, n-m\left|u\right.\right) dz \\
&= \sum_{m=1}^{n-1} \frac{\beta^m}{(m-1)!} \sum_{k=1}^{\infty} \frac{q_k^{*m} \lambda^k}{(k-1)!} \int_0^{\min\left(\frac{x}{c}, t\right)} \left((cz-x)^{m-1} z^{k-1} e^{-(\lambda+\beta c)z} \right) f\left(t-z, n-m\left|u\right.\right) dz,
\end{aligned} \tag{4.32}$$

for $n = 2, 3, \ldots$ and $x \in [0, u+ct]$.

Note that

$$\begin{aligned}
& f\left(t, n\left|u\right.\right) \\
= \;& \frac{(nu+ct)(u+ct)^{n-2} \beta^{n-1}}{n!} \sum_{j=1}^{\infty} q_j^{*n} \frac{\lambda^j t^{j-1} e^{-\lambda t}}{(j-1)!} \\
= \;& \frac{\beta^{n-1}}{n!} e^{-\beta u} \sum_{j=1}^{\infty} q_j^{*n} \frac{\lambda^j}{(j-1)!} (nu+ct)(u+ct)^{n-2} t^{j-1} e^{-(\lambda+\beta c)t}.
\end{aligned} \tag{4.33}$$

Substituting (4.33) into (4.32), one arrives at

$$f(t,x,n\,|u) = \sum_{m=1}^{n-1} \frac{\beta^{n-1} e^{-\beta u} e^{-(\lambda+\beta c)t}}{(m-1)!\,(n-m)!} \sum_{k,j=1}^{\infty} \frac{q_k^{*m} q_j^{*(n-m)} \lambda^{k+j}}{(k-1)!\,(j-1)!} I_{k,j}(t,x,m,n\,|u), \tag{4.34}$$

where

$$\begin{aligned} & I_{k,j}(t,x,m,n\,|u) \\ = & \int_0^{\min\left(\frac{x}{c},t\right)} (cz-x)^{m-1} z^{k-1} (t-z)^{j-1} ((n-m)u + c(t-z))(u + c(t-z))^{n-m-2}\, dz \\ = & (n-m-1) \sum_{i=0}^{n-m-2} \binom{n-m-2}{i} u^{n-m-1-i} H_i^m(t,x;k,j) \\ & + \sum_{i=0}^{n-m-1} \binom{n-m-1}{i} u^{n-m-1-i} H_i^m(t,x;k,j), \end{aligned} \tag{4.35}$$

and

$$H_i^m(t,x;k,j) = c^i \int_0^{\min\left(\frac{x}{c},t\right)} (cz-x)^{m-1} z^{k-1} (t-z)^{i+j-1}\, dz. \tag{4.36}$$

Using the kernel of the beta distribution, one finds

(i) for $0 \le x \le ct$*, (4.36) becomes*

$$\begin{aligned} & H_i^m(t,x;k,j) \\ = & c^i \int_0^{\frac{x}{c}} (cz-x)^{m-1} z^{k-1} (t-z)^{i+j-1}\, dz \\ = & \sum_{l=0}^{i+j-1} \binom{i+j-1}{l} (-1)^{m-1+l} c^{i-k-l} t^{i+j-l-1} x^{m+k+l-1} \int_0^1 (1-z)^{m-1} z^{k+l-1} dz \\ = & \sum_{l=0}^{i+j-1} \binom{i+j-1}{l} \frac{\Gamma(m)\Gamma(k+l)}{\Gamma(m+k+l)} (-1)^{m-1+l} c^{i-k-l} t^{i+j-l-1} x^{m+k+l-1}; \end{aligned}$$

(ii) for $ct < x \leq u + ct$, (4.36) reduces to

$$\begin{aligned} & H_i^m (t, x; k, j) \\ = \ & c^i \int_0^t (cz - x)^{m-1} z^{k-1} (t - z)^{i+j-1} dz \\ = \ & \sum_{l=0}^{m-1} \binom{m-1}{l} (-x)^{m-1-l} c^{i+l} \int_0^t z^{k+l-1} (t - z)^{i+j-1} dz \\ = \ & \sum_{l=0}^{m-1} \binom{m-1}{l} \frac{\Gamma(i+j)\Gamma(k+l)}{\Gamma(i+j+k+l)} (-x)^{m-1-l} c^{i+l} t^{i+j+k+l-1}. \end{aligned}$$

As a special case, when the initial surplus level $u = 0$, (4.35) becomes

$$\begin{aligned} I_{k,j} (t, x, m, n \,|0\,) &= c^{n-m-1} \int_0^{\min\left(\frac{x}{c}, t\right)} (cz - x)^{m-1} z^{k-1} (t - z)^{j+n-m-2} dz \\ &= H_{n-m-1}^m (t, x; k, j) . \end{aligned} \tag{4.37}$$

Substituting (4.37) back into (4.34), it follows that

$$f (t, x, n \,|0\,) = \sum_{m=1}^{n-1} \frac{\beta^{n-1} e^{-(\lambda+\beta c)t}}{(m-1)! \, (n-m)!} \sum_{k,j=1}^{\infty} \frac{q_k^{*m} q_j^{*(n-m)} \lambda^{k+j}}{(k-1)! \, (j-1)!} H_{n-m-1}^m (t, x; k, j) ,$$

for $n = 2, 3, ...$ and $x \in [0, ct]$. ■

For the delayed Sparre Andersen risk model (with exponential claims), the joint generalized density of the time to ruin, the surplus prior to ruin, and the number of claims until ruin follows quite naturally from (4.31) (e.g. by conditioning on the time and the amount of the first claim and recognizing that the process restarts in its non-delayed (ordinary) form). Therefore, we omit the details here.

Chapter 5

Sparre Andersen risk model: combination of n exponentials claim sizes

5.1 Introduction

In this chapter, we capitalize on the most recent advances in connection with the time to ruin of an insurer's surplus process, and identify a closed-form expression for the distribution of the time to ruin in some Sparre Andersen risk models. Again, we do so through the analytic inversion of the Laplace transform of the time to ruin (see Eq.(1.8))

$$\phi_{1,\delta}(u) \equiv E\left[e^{-\delta\tau}1(\tau<\infty)\,|U_0=u\right],$$

and ruin-related quantity known in a variety of Sparre Andersen risk models (see, e.g., Li and Garrido (2005) and Gerber and Shiu (2005)). As pointed out in Section 1.3, most of the results on the distribution of the time to ruin so far are based on an exponential assumption imposed on either the interclaim times or the claim sizes with very few exceptions.

By relaxing the exponential assumption, we propose to build on the recent contributions of Dickson and Li (2010, 2012) and provide an analytic expression to the density of the time to ruin in some Sparre Andersen risk models through the use of Lagrange's expansion theorem in its multivariate form (see Theorem 1.4.4 and Corollary 1.4.5). To our knowledge, these results are the first of their kind for the class of surplus processes of interest in this chapter. Also, we would like to point out that our proposed methodology to tackle the density of the time to ruin can also be applied to study other first passage times of interest in applied probability. One particular queueing application will be considered in Section 5.3.

As a general setting, we assume that the surplus process $\{U_t, t \geq 0\}$ follows a Sparre Andersen risk process exactly as described in Section 1.1, where the interclaim times $\{T_i\}_{i=1}^{\infty}$ and claim sizes $\{X_i\}_{i=1}^{\infty}$ form a sequence of i.i.d. r.v.'s. It is further assumed that the interclaim times $\{T_i\}_{i=1}^{\infty}$ and the claim sizes $\{X_i\}_{i=1}^{\infty}$ are mutually independent. In this chapter, we show that, when the claim sizes have a combination of n exponential distributions with Laplace transform

$$\widetilde{p}(s) = \sum_{i=1}^{n} \alpha_i \frac{\beta_i}{\beta_i + s}, \qquad s \geq 0, \tag{5.1}$$

for $\sum_{i=1}^{n} \alpha_i = 1$, $\beta_i > 0$, and $\beta_i \neq \beta_j$ for $i \neq j$, the analytic inversion of $\phi_{1,\delta}(u)$ can be performed through the use of the multivariate Lagrange expansion theorem. We recall that one of the important properties of this class of distributions is that they are dense among the set of continuous distributions with support on $[0, \infty)$ (see, e.g., Dufresne (2007)).

The rest of this chapter is structured as follows: in Section 5.2, we obtain a closed-form expression for the density of the time to ruin in the Sparre Andersen risk model when claim sizes have Laplace transform (5.1). We also discuss how our proposed methodology can be used to solve other finite-time ruin problems. In Section 5.3, we consider another application of the multivariate Lagrange expansion theorem to obtain the distribution of the duration of a busy period in a subclass of the $K_m/G/1$ queueing model. Finally, numerical examples are considered in Section 5.4.

5.2 Density of the time to ruin

In this section, we propose to make use of Eq. (1.26) to derive the density of the time to ruin in the Sparre Andersen risk model with claim sizes having Laplace transform (5.1). In a more general risk model, Landriault and Willmot (2008) showed that the Laplace transform of the time to ruin is of the form

$$\phi_{1,\delta}(u) = \sum_{i=1}^{n} C_i e^{-\rho_i u}, \tag{5.2}$$

where $\{\rho_i\}_{i=1}^n$ are the n solutions in the right-half of the complex plane (i.e. $\mathrm{Re}(\rho_i) \geq 0$) of the generalized Lundberg equation

$$\widetilde{k}(\delta + c\rho_i)\,\widetilde{p}(-\rho_i) = 1. \tag{5.3}$$

In the sequel, we assume that the solutions $\{\rho_i\}_{i=1}^n$ are distinct, i.e. $\rho_i \neq \rho_j$ for $i \neq j$. For claim sizes with Laplace transform (5.1), the coefficients $\{C_i\}_{i=1}^n$ are the solution to the system of linear equations

$$\sum_{i=1}^{n} C_i \frac{\beta_j}{\beta_j - \rho_i} = 1, \tag{5.4}$$

for $j = 1, \ldots, n$. Using a matrix representation, (5.4) can be rewritten as $\mathbf{AC} = \mathbf{B}$ where $\mathbf{A} = \{a_{ij}\}_{i,j=1}^n$ with $a_{ij} = \frac{1}{\beta_i - \rho_j}$, $\mathbf{B} = \left((\beta_1)^{-1}, \ldots, (\beta_n)^{-1}\right)^\top$, and $\mathbf{C} = (C_1, \ldots, C_n)^\top$. It is worth pointing out that $\mathbf{A}$ is a Cauchy matrix with determinant

$$\det \mathbf{A} = \frac{\prod_{k=2}^{n}\prod_{s=1}^{k-1}(\beta_k - \beta_s)(\rho_s - \rho_k)}{\prod_{s,k=1}^{n}(\beta_s - \rho_k)}.$$

Using Cramer's rule for the solution of a system of linear equations, we have

$$C_i = \left\{\prod_{s=1}^{n}\frac{\beta_s - \rho_i}{\beta_s}\right\}\left\{\prod_{\substack{k=1\\k\neq i}}^{n}\frac{\rho_k}{\rho_k - \rho_i}\right\}. \tag{5.5}$$

Given that $\phi_{1,\delta}(u)$ has now been expressed in terms of the n solutions $\{\rho_i\}_{i=1}^{n}$ of the generalized Lundberg equation (5.3), we propose to invert the Laplace transform of the time to ruin (5.2) wrt δ with the help of the multivariate Lagrange expansion theorem. To this end, we first rewrite the generalized Lundberg equation (5.3) as

$$1 = \frac{\rho_i - \beta_i}{l_i(\rho_i)}, \tag{5.6}$$

where

$$l_i(\rho_i) = f_i(\rho_i)\widetilde{k}\left(\delta + c\rho_i\right),$$

and

$$f_i(\rho_i) = -\alpha_i\beta_i + \sum_{\substack{j=1\\j\neq i}}^{n}\frac{\alpha_j\beta_j(\rho_i - \beta_i)}{\beta_j - \rho_i}. \tag{5.7}$$

Note that (5.6) is of the form (1.23) with $a_i = \beta_i$, $z_i = \rho_i$, $\zeta_i - \alpha_i = 1$, and $g_i(\mathbf{z}) = l_i(z_i)$. It is not difficult to verify that $l_i(\beta_i) \neq 0$ for $i = 1, \ldots, n$ which is a necessary condition for the application of Lagrange's expansion theorem. Also, given that $g_i(\mathbf{z})$ is only a function in z_i for $i = 1, \ldots, n$, this allows us to make use of the simplified version (1.26) of Lagrange's expansion theorem. Thus, letting

$$h(\rho_1, \ldots, \rho_n; u) = \sum_{i=1}^{n} C_i e^{-\rho_i u},$$

it follows that

$$\phi_{1,\delta}(u) = \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{1}{\prod_{j=1}^{n} m_j!} \times \frac{\partial^{m_1+\ldots+m_n-n}}{\partial\rho_1^{m_1-1}\ldots\partial\rho_n^{m_n-1}} \left\{ h^{(1,\ldots,1)}(\rho_1,\ldots,\rho_n;u) \prod_{i=1}^{n} \left(f_i(\rho_i)\widetilde{k}(\delta+c\rho_i) \right)^{m_i} \right\}\Bigg|_{\rho_i=\beta_i}. \tag{5.8}$$

Recall that we define $\frac{\partial^{-1}}{\partial t^{-1}} f'(t) \equiv f(t)$. When $m_i = 0$, we simply evaluate $h(\ldots,\rho_i,\ldots)$ at $\rho_i = \beta_i$. For a given vector $\mathbf{m} = (m_1,\ldots,m_n)$, we define $\Lambda_{\mathbf{m}} = \{i \in \{1,2,\ldots,n\} : m_i \neq 0\}$. Using the chain rule for differentiation, (5.8) becomes

$$\phi_{1,\delta}(u) = \sum_{m_1,\ldots,m_n=0}^{\infty} \sum_{k_1=0}^{\max(m_1-1,0)} \cdots \sum_{k_n=0}^{\max(m_n-1,0)} \chi_{\mathbf{k},\mathbf{m}}(u) \prod_{i\in\Lambda_{\mathbf{m}}} \rho_{k_i,\delta}^{m_i}, \tag{5.9}$$

where

$$\chi_{\mathbf{k},\mathbf{m}}(u) = \frac{\prod_{j\in\Lambda_{\mathbf{m}}} \binom{m_j-1}{k_j}}{\prod_{j=1}^{n} m_j!} \frac{\partial^{\sum_{j=1}^{n}(m_j-k_j-1)}}{\partial\rho_1^{m_1-k_1-1}\ldots\partial\rho_n^{m_n-k_n-1}} \left\{ h^{(1,\ldots,1)}(\rho_1,\ldots,\rho_n;u) \prod_{i=1}^{n} f_i(\rho_i)^{m_i} \right\}\Bigg|_{\rho_i=\beta_i}, \tag{5.10}$$

and

$$\begin{aligned} \rho_{k_i,\delta}^{m_i} &= \frac{\partial^{k_i}}{\partial\rho_i^{k_i}} \widetilde{k}(\delta+c\rho_i)^{m_i}\Bigg|_{\rho_i=\beta_i} \\ &= \int_0^\infty e^{-\delta t} \left\{ (-ct)^{k_i} e^{-c\beta_i t} k^{*m_i}(t) \right\} dt. \end{aligned}$$

We point out that the term $\chi_{\mathbf{k},\mathbf{m}}(u)$ does not depend on δ which implies that the inversion of (5.9) wrt δ only concerns the terms $\left\{\rho_{k_i,\delta}^{m_i}\right\}_{i=1}^{n}$. More precisely, we have

$$\begin{aligned} \prod_{i\in\Lambda_{\mathbf{m}}} \rho_{k_i,\delta}^{m_i} &= \prod_{i\in\Lambda_{\mathbf{m}}} \int_0^\infty e^{-\delta t} \left\{ (-ct)^{k_i} e^{-c\beta_i t} k^{*m_i}(t) \right\} dt \\ &= \int_0^\infty e^{-\delta t} g_{\mathbf{k},\mathbf{m}}(t) dt, \end{aligned} \tag{5.11}$$

where $g_{\mathbf{k},\mathbf{m}}(t)$ corresponds to the convolution (in t) of the terms $(-ct)^{k_i}e^{-c\beta_i t}k^{*m_i}(t)$ for $i \in \Lambda_{\mathbf{m}}$.

Substituting (5.11) into (5.9), it follows that

$$\phi_{1,\delta}(u) = \int_0^\infty e^{-\delta t}\left\{\sum_{m_1,\ldots,m_n=0}^{\infty}\sum_{k_1=0}^{\max(m_1-1,0)}\cdots\sum_{k_n=0}^{\max(m_n-1,0)}\chi_{\mathbf{k},\mathbf{m}}(u)g_{\mathbf{k},\mathbf{m}}(t)\right\}dt. \tag{5.12}$$

In conclusion, the density of the time to ruin when claim sizes have a Laplace transform of the form (5.1) is given by

$$f_\tau(t|\,u) = \sum_{m_1,\ldots,m_n=0}^{\infty}\sum_{k_1=0}^{\max(m_1-1,0)}\cdots\sum_{k_n=0}^{\max(m_n-1,0)}\chi_{\mathbf{k},\mathbf{m}}(u)g_{\mathbf{k},\mathbf{m}}(t). \tag{5.13}$$

Remark 5.2.1 *The explicit expression for $\chi_{\mathbf{k},\mathbf{m}}(u)$ involves n partial derivatives and as such, yields a lengthy expression for given $h(\rho_1,\ldots,\rho_n;u)$ and $f_i(\rho_i)$ in general. However, it can be treated as a constant for a given initial surplus level and therefore will not affect the structural form of the density of the time to ruin as a function of t. For higher order derivatives, $\chi_{\mathbf{k},\mathbf{m}}(u)$ may be cumbersome to obtain in an explicit manner; in such cases, the evaluation of $\chi_{\mathbf{k},\mathbf{m}}(u)$ can be done via numerical algorithms such as the finite difference method (see, e.g., Khan and Ohba (2003)).*

Remark 5.2.2 *When $n = 1$, the claim sizes become exponentially distributed and our approach naturally reduces to the application of the univariate Lagrange expansion theorem, which has been extensively analyzed by Landriault et al. (2011). The results were shown to be consistent with Borovkov and Dickson (2008).*

Remark 5.2.3 *The present analysis can readily be extended to include the number of claims until ruin N_τ. Indeed, recall that*

$$\phi_{r,\delta}(u) = E[r^{N_\tau}e^{-\delta\tau}I(\tau<\infty)|U_0=u],$$

for $\delta \geq 0$ and $r \in (0,1]$ (see Eq. (1.9)). It turns out that the only difference between $\phi_{1,\delta}(u)$ and $\phi_{r,\delta}(u)$ arises in the generalized Lundberg equation (5.6) where the "1" is replaced by the probability generating function argument r. As such, (5.9) can be generalized for $\phi_{r,\delta}(u)$ to

$$\phi_{r,\delta}(u) = \sum_{m_1,\ldots,m_n=0}^{\infty} r^{m_1+\ldots+m_n} \sum_{k_1=0}^{\max(m_1-1,0)} \cdots \sum_{k_n=0}^{\max(m_n-1,0)} \chi_{\mathbf{k},\mathbf{m}}(u) \prod_{i\in\Lambda_{\mathbf{m}}} \rho_{k_i,\delta}^{m_i}.$$

One immediately concludes that the joint generalized density of the time to ruin (at t) and the number of claim until ruin (at l) is given by

$$f_{\tau,N_\tau}(t,l\,|u) = \sum_{\substack{m_1+\ldots+m_n=l\\ m_i\geq 0}} \sum_{k_1=0}^{\max(m_1-1,0)} \cdots \sum_{k_n=0}^{\max(m_n-1,0)} \chi_{\mathbf{k},\mathbf{m}}(u) g_{\mathbf{k},\mathbf{m}}(t), \tag{5.14}$$

for $t \geq 0$ and $l = 1, 2, \ldots$

In what follows, we show that a tractable expression for $g_{\mathbf{k},\mathbf{m}}(t)$ can be found when the density k is of a particular form. We examine the class of Erlang distributions, i.e. a density k with Laplace transform

$$\widetilde{k}(s) = \left(\frac{\theta}{\theta+s}\right)^l,$$

for $\theta > 0$ and l is a positive integer. By routine manipulations, we have

$$\begin{aligned} e^{-c\beta_i t}(ct)^{k_i} k^{*m_i}(t) &= e^{-c\beta_i t}(ct)^{k_i} \frac{(\theta t)^{m_i l-1}\theta e^{-\theta t}}{\Gamma(m_i l)} \\ &= \frac{\Gamma(k_i+m_i l)c^{k_i}\theta^{m_i l}}{\Gamma(m_i l)(c\beta_i+\theta)^{k_i+m_i l}} \tau_{k_i+m_i l, c\beta_i+\theta}(t), \end{aligned}$$

where $\tau_{n,\beta}(t)$ is the Erlang density with mean $n\beta^{-1}$ and variance $n\beta^{-2}$. Therefore, the Laplace transform of $g_{\mathbf{k},\mathbf{m}}(t)$ can be rewritten as

$$\widetilde{g}_{\mathbf{k},\mathbf{m}}(s) = \varrho \prod_{i\in\Lambda_{\mathbf{m}}} \left(\frac{c\beta_i+\theta}{c\beta_i+\theta+s}\right)^{k_i+m_i l}, \tag{5.15}$$

where

$$\varrho = \prod_{i\in\Lambda_{\mathbf{m}}} (-1)^{k_i} \frac{\Gamma(k_i + m_i l) c^{k_i} \theta^{m_i l}}{\Gamma(m_i l)(c\beta_i + \theta)^{k_i + m_i l}}.$$

Laplace transforms of the form (5.15) have been studied extensively (see, e.g., Willmot and Woo (2007) and Willmot and Lin (2011)). Without loss of generality, we assume $m_i \geq 1$ for $i = 1, 2, \ldots, n$ and $\beta_i < \beta_n$ for any $i < n$ (for the case when some $m_i = 0$, the method still applies). Using results of Willmot and Woo (2007, Section 2.3), one obtains that

$$g_{\mathbf{k},\mathbf{m}}(t) = \varrho \sum_{j=1}^{\infty} q_j \tau_{j, c\beta_n + \theta}(t),$$

where $q_j = 0$ for $j < M = \sum_{i=1}^{n} (k_i + m_i l)$, and

$$q_M = \prod_{i=1}^{n-1} \left(\frac{c\beta_i + \theta}{c\beta_n + \theta} \right)^{k_i + m_i l}.$$

For $j > M$, q_j can be calculated recursively via

$$q_j = \frac{1}{j - M} \sum_{p=1}^{j-M} \left\{ \sum_{i=1}^{n-1} (k_i + m_i l) \left(1 - \frac{c\beta_i + \theta}{c\beta_n + \theta} \right)^p \right\} q_{j-p}.$$

The inversion of the Laplace transform of the time to ruin through the multivariate Lagrange expansion theorem largely relies on the generalized Lundberg equation (5.3) to be of the form (1.23) with specific $g_i(\mathbf{a}) \neq 0$ for $i = 1, \ldots, n$. As shown earlier, this is satisfied when the claim sizes are a combination of exponentials. It is not difficult to show that this condition is also satisfied for other claim size distributions (such as Erlangs and mixtures/combinations of Erlangs with appropriately chosen $\mathbf{a}$), but the ensuing inversion of Lagrange's expansion wrt δ is expected to be challenging. This is less of a concern given that the class of combinations of exponentials is dense in the set of positive continuous distributions.

Remark 5.2.4 *By a similar use of the multivariate Lagrange expansion theorem, one could derive a closed-form expression for the density of the time to ruin in the SA risk model when the interclaim times are assumed to be a combination of n exponentials and claim sizes have an arbitrary distribution. However, we point out that such expression is expected to be particularly lengthy given that the Laplace transform of the time to ruin has a more complicated form in this context (see, e.g., Li and Garrido (2005)).*

In the next section, we consider another application of the multivariate Lagrange expansion theorem in applied probability, more precisely to invert the Laplace transform of the duration of a busy period in a queueing system.

5.3 Duration of a busy period in a queueing system

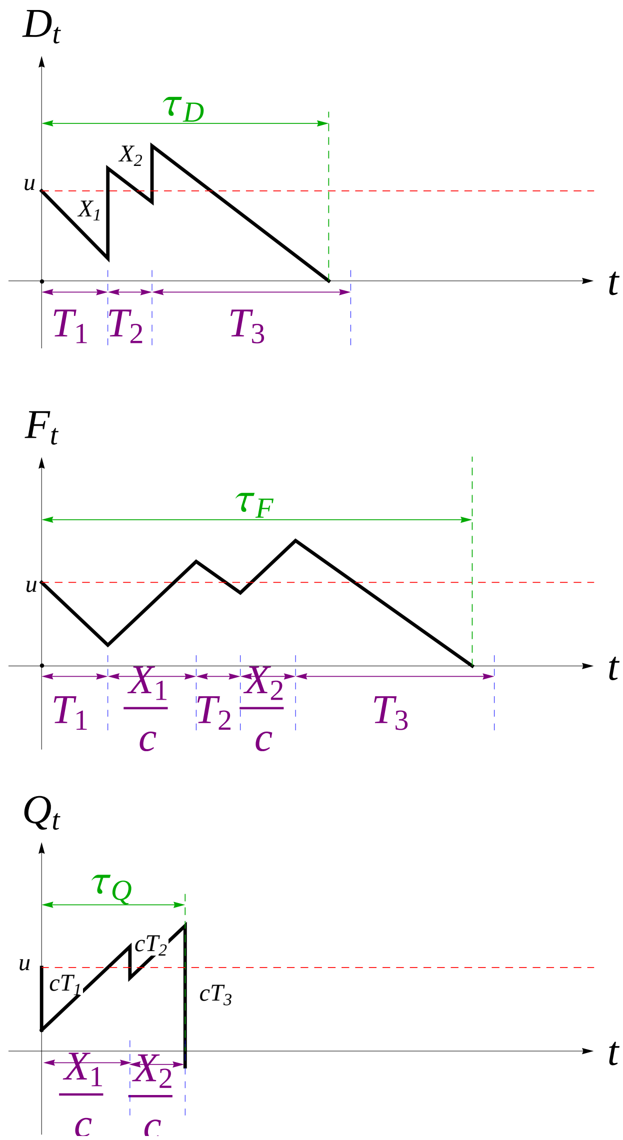
Until the system becomes empty, the workload process $\{D_t, t \geq 0\}$ of the queueing system $G/G/1$ (see, e.g., Cohen (1982) and Kleinrock (1975)) is defined as

$$D_t = u - ct + S_t, \tag{5.16}$$

where $u > 0$ is the time-0 workload, $c > 0$ is the service speed, and $\{S_t, t \geq 0\}$ is a compound renewal process defined as in (1.2). Here, S_t represents the total workload of all customers' arrivals in $(0, t]$. In this context, the r.v.'s $\{T_i\}_{i=1}^{\infty}$ represent the interarrival times of customers in the queue while $\{X_i\}_{i=1}^{\infty}$ are their associated service times. Let $\tau_D = \inf \{t \geq 0 : D_t = 0\}$ be the duration of the busy period with initial workload u. Define its Laplace transform as

$$\Phi_\delta(u) \equiv E\left[e^{-\delta \tau_D} 1(\tau_D < \infty) | D_0 = u\right].$$

Figure 5.1: Connections of D_t, F_t and Q_t



We point that τ_D can also be interpreted as the time to ruin in the dual risk model in ruin theory (see, e.g., Avanzi et al. (2007) and Takács (1967)).

Our first objective is to obtain the Laplace transform of the length of the busy period τ_D. We propose to do so by connecting the queueing system (5.16) with its corresponding Sparre Andersen risk model through the intermediary of a fluid flow process (see, e.g., Asmussen (1995)). Similar duality arguments were used by Frostig (2004). For the process $\{D_t, t \geq 0\}$, its corresponding fluid flow process $\{F_t, t \geq 0\}$ is constructed by replacing the upward jumps of workload by periods of ascent of the fluid flow. More precisely, a workload jump of size x is substituted by a period of ascent of the fluid flow at a rate of c over a period of length x/c (see Fig 5.1). Let $\tau_F = \inf\{t \geq 0 : F_t < 0\}$ be the first passage time to 0 of the fluid flow $\{F_t, t \geq 0\}$. Associated to the fluid flow $\{F_t, t \geq 0\}$ is the risk model $\{Q_t, t \geq 0\}$ constructed by replacing the downward linear paths of the fluid flow by downward jumps of appropriate size (see, e.g., Ramaswami (2006)). In the risk process $\{Q_t, t \geq 0\}$, $\{cT_i\}_{i=1}^{\infty}$ corresponds to the claim sizes, $\{X_i/c\}_{i=1}^{\infty}$ are the interclaim times. The premium rate is c and the first claim cT_1 is assumed to occur at time 0. Let $\tau_Q = \inf\{t \geq 0 : Q_t < 0\}$ be the time to ruin for the risk process $\{Q_t, t \geq 0\}$. By construction, the first passage time τ_D of the queueing process $\{D_t, t \geq 0\}$ dominates the first passage time τ_Q of the risk process $\{Q_t, t \geq 0\}$ by u/c. This is because the time of descent of the fluid flow process (aka τ_D) exceeds its time of ascent (aka τ_Q) until the first passage time τ_F by a factor of u/c. Given that a claim of size cT_1 occurs at time 0, we have

$$\Phi_\delta(u) = E\left[E\left[e^{-\delta\left(\tau_Q + \frac{u}{c}\right)} 1(\tau_Q < \infty) \,|Q_0 = u - cT_1\right]\right].$$

To use the results of Section 5.2, we assume that the r.v.'s $\{cT_i\}_{i=1}^{\infty}$ follow a combination of n exponentials distribution with Laplace transform (5.1). Also, the r.v.'s $\{X_i/c\}_{i=1}^{\infty}$ are assumed to have density k. Given that combinations of exponentials are a subclass of the

K_m family of distributions, the queueing system of interest is a special case of the $K_m/G/1$ queueing model (see Kleinrock (1975)). By considering whether the first claim will cause ruin at time 0 or not, one deduces

$$\Phi_\delta(u) = e^{-\delta\frac{u}{c}}\left(\int_0^u E\left[e^{-\delta\tau_Q}1(\tau_Q<\infty)\,|Q_{0^+}=u-y\right]p(y)\,dy + \int_u^\infty p(y)\,dy\right),$$

where Q_{0^+} is the initial surplus level after the claim payment at time 0. Under the above distributional assumptions, it is not difficult to see that

$$E\left[e^{-\delta\tau_Q}1(\tau_Q<\infty)\,|Q_{0^+}=u\right] = \phi_{1,\delta}(u),$$

which implies that

$$\Phi_\delta(u) = e^{-\delta\frac{u}{c}}\left(\int_0^u \phi_{1,\delta}(u-y)\,p(y)\,dy + \int_u^\infty p(y)\,dy\right). \tag{5.17}$$

Substituting (5.2) into (5.17) followed by the use of (5.1), one finds that

$$\begin{aligned}
\Phi_\delta(u) &= e^{-\delta\frac{u}{c}}\left\{\int_0^u\left(\sum_{i=1}^n C_i e^{-\rho_i(u-y)}\right)p(y)\,dy + \overline{P}(u)\right\}\\
&= e^{-\delta\frac{u}{c}}\left\{\sum_{i=1}^n C_i\sum_{j=1}^n \alpha_j\beta_j\int_0^u e^{-\rho_i(u-y)}e^{-\beta_j y}dy + \sum_{j=1}^n \alpha_j e^{-\beta_j u}\right\}\\
&= e^{-\delta\frac{u}{c}}\left\{\sum_{i=1}^n C_i\sum_{j=1}^n \alpha_j\beta_j\frac{e^{-\rho_i u}-e^{-\beta_j u}}{\beta_j-\rho_i} + \sum_{j=1}^n \alpha_j e^{-\beta_j u}\right\}\\
&= e^{-\delta\frac{u}{c}}\left\{\sum_{i=1}^n C_i e^{-\rho_i u}\widetilde{p}(-\rho_i) - \sum_{j=1}^n \alpha_j e^{-\beta_j u}\sum_{i=1}^n C_i\frac{\beta_j}{\beta_j-\rho_i} + \sum_{j=1}^n \alpha_j e^{-\beta_j u}\right\}.
\end{aligned} \tag{5.18}$$

Using (5.4), (5.18) becomes

$$\Phi_\delta(u) = e^{-\delta\frac{u}{c}}\sum_{i=1}^n \eta_i e^{-\rho_i u} \tag{5.19}$$

where

$$\eta_i = C_i \widetilde{p}(-\rho_i)$$
$$= \left\{ \sum_{j=1}^{n} \alpha_j \prod_{\substack{s=1 \\ s \neq j}}^{n} \frac{\beta_s - \rho_i}{\beta_s} \right\} \left\{ \prod_{\substack{k=1 \\ k \neq i}}^{n} \frac{\rho_k}{\rho_k - \rho_i} \right\}.$$

Here again, an application of the multivariate Lagrange expansion theorem with

$$h^*(\rho_1, \ldots, \rho_n; u) = \sum_{i=1}^{n} \eta_i e^{-\rho_i u},$$

yields

$$\Phi_\delta(u) = e^{-\frac{\delta}{c}u} h^*(\rho_1, \ldots, \rho_n; u)$$
$$= e^{-\frac{\delta}{c}u} \sum_{m_1,\ldots,m_n=0}^{\infty} \sum_{k_1=0}^{\max(m_1-1,0)} \cdots \sum_{k_n=0}^{\max(m_n-1,0)} \chi^*_{\mathbf{k},\mathbf{m}}(u) \prod_{i \in \Lambda_{\mathbf{m}}} \rho_{k_i,\delta}^{m_i}.$$

Note that the symbol $*$ is added to the functions h and χ to emphasize that χ is as defined in (5.10), but with the function h replaced by h^*. Using (5.11), it follows that

$$\Phi_\delta(u)$$
$$= e^{-\frac{\delta}{c}u} \overline{P}(u) + e^{-\frac{\delta}{c}u} \int_0^\infty e^{-\delta t} \left\{ \sum_{\substack{m_1,\ldots,m_n=0 \\ m_1+\ldots+m_n \neq 0}}^{\infty} \sum_{k_1=0}^{\max(m_1-1,0)} \cdots \sum_{k_n=0}^{\max(m_n-1,0)} \chi^*_{\mathbf{k},\mathbf{m}}(u) g_{\mathbf{k},\mathbf{m}}(t) \right\} dt$$
$$= e^{-\frac{\delta}{c}u} \overline{P}(u) + \int_{\frac{u}{c}}^\infty e^{-\delta t} \left\{ \sum_{\substack{m_1,\ldots,m_n=0 \\ m_1+\ldots+m_n \neq 0}}^{\infty} \sum_{k_1=0}^{\max(m_1-1,0)} \cdots \sum_{k_n=0}^{\max(m_n-1,0)} \chi^*_{\mathbf{k},\mathbf{m}}(u) g_{\mathbf{k},\mathbf{m}}\left(t - \frac{u}{c}\right) \right\} dt$$

One concludes that τ_D is a mixed r.v. with a mass point at u/c of size $\overline{P}(u)$ associated to the arrival of no customers (i.e. $N_{\tau_D} = 0$) until the initial workload of u is completed.

With at least one customer arrival before τ_D, the duration of the busy period with initial workload u has density given by

$$f_{\tau_D}(t|\, u) = \sum_{\substack{m_1,\ldots,m_n=0 \\ m_1+\ldots+m_n\neq 0}}^{\infty} \sum_{k_1=0}^{\max(m_1-1,0)} \cdots \sum_{k_n=0}^{\max(m_n-1,0)} \chi^*_{\mathbf{k},\mathbf{m}}(u) g_{\mathbf{k},\mathbf{m}}\left(t-\frac{u}{c}\right), \qquad t > \frac{u}{c}. \quad (5.20)$$

Note that the combination of Eqs. (5.12) and (5.17) leads to the following alternative expression for $\chi^*_{\mathbf{k},\mathbf{m}}(u)$:

$$\chi^*_{\mathbf{k},\mathbf{m}}(u) = \int_0^u \chi_{\mathbf{k},\mathbf{m}}(u-y)p\,(y)\,dy.$$

We conclude this section with the generalized Erlang-2 example for which an explicit expression for $\chi^*_{\mathbf{k},\mathbf{m}}(u)$ is identified.

Example 5.3.1 *We assume that* $\{cT_i\}_{i=1}^{\infty}$ *follows the generalized Erlang-2 distribution with Laplace transform* $\widetilde{p}(s) = \beta_1\beta_2/\left\{(\beta_1+s)\,(\beta_2+s)\right\}$*. In this case, (5.19) reduces to*

$$\phi_\delta\,(u) = e^{-\frac{\delta}{c}u}\left(\frac{\rho_2}{\rho_2-\rho_1}e^{-\rho_1 u} + \frac{\rho_1}{\rho_1-\rho_2}e^{-\rho_2 u}\right),$$

and (5.7) becomes

$$f_i(\rho_i) = \frac{\beta_1\beta_2}{\beta_{3-i}-\rho_i}, \qquad i = 1, 2.$$

To identify $\chi^*_{\mathbf{k},\mathbf{m}}(u)$*, the following two preliminary results turn out to be useful:*

(a) for $k \le m-1$*, the* $(m-k-1)$*-th derivative of* $(\rho+s)^{-2}\,(\beta_1-\rho)^{-m}$ *evaluated at* $\rho = \beta_2$ *is given by*

$$\frac{d^{m-k-1}}{d\rho^{m-k-1}}\left\{\frac{1}{(\rho+s)^2\,(\beta_1-\rho)^m}\right\}\Bigg|_{\rho=\beta_2} = \sum_{j=0}^{m-k-1} \vartheta_{k,m}(j;\beta_1,\beta_2)\frac{1}{(\beta_2+s)^{j+2}}, \qquad (5.21)$$

where

$$\vartheta_{k,m}(j;\beta_1,\beta_2) = \binom{m-k-1}{j}\frac{(-1)^j\,(j+1)!(2m-k-j-2)!}{(m-1)!\,(\beta_1-\beta_2)^{2m-k-j-1}}.$$

This result is immediate from Leibniz's chain rule.

(b) Using partial fractions, the generalized Erlang-($n_1 + n_2$) transform can be expressed as

$$\left(\frac{\beta_1}{\beta_1 + s}\right)^{n_1} \left(\frac{\beta_2}{\beta_2 + s}\right)^{n_2} = \sum_{i=1}^{2} \sum_{j=1}^{n_i} \frac{w_{i,j}\left(\beta_1, \beta_2\right)}{\left(\beta_i + s\right)^j}, \tag{5.22}$$

where

$$w_{i,j}\left(\beta_1, \beta_2\right) = \left(\beta_1\right)^{n_1} \left(\beta_2\right)^{n_2} \binom{n_1 + n_2 - j - 1}{n_i - j} \frac{(-1)^{n_i - j}}{\left(\beta_i - \beta_{3-i}\right)^{n_1 + n_2 - j}}$$

for $i = 1, 2$ and $j = 1, \ldots, n_i$ (see, e.g., Li and Garrido (2005, p.841)). Inverting (5.22) yields

$$\left(\frac{\beta_1}{\beta_1 + s}\right)^{n_1} \left(\frac{\beta_2}{\beta_2 + s}\right)^{n_2} = \int_0^\infty e^{-su} z_{n_1,n_2}(u; \beta_1, \beta_2) du, \tag{5.23}$$

where

$$z_{n_1,n_2}(u; \beta_1, \beta_2) = \sum_{i=1}^{2} \sum_{j=1}^{n_i} \frac{w_{i,j}\left(\beta_1, \beta_2\right)}{\left(\beta_i\right)^j} \tau_{j,\beta_i}(u).$$

In what follows, we consider $\chi^*_{\mathbf{k},\mathbf{m}}(u)$ in four cases:

(i) for $\mathbf{m} = (0, 0)$:

$$\chi^*_{\mathbf{k},\mathbf{m}}(u) = h^*(\beta_1, \beta_2; u) = \frac{\beta_2}{\beta_2 - \beta_1} e^{-\beta_1 u} + \frac{\beta_1}{\beta_1 - \beta_2} e^{-\beta_2 u} = \overline{P}(u).$$

(ii) for $m_1 = 0$ and $m_2 \geq 1$: (5.10) becomes

$$\chi^*_{\mathbf{k},\mathbf{m}}(u) = \frac{\binom{m_2 - 1}{k_2}}{m_2!} \frac{\partial^{m_2 - k_2 - 1}}{\partial \rho_2^{m_2 - k_2 - 1}} \left\{ h^{*(0,1)}(\beta_1, \rho_2; u) \left(\frac{\beta_1 \beta_2}{\beta_1 - \rho_2}\right)^{m_2} \right\} \Bigg|_{\rho_2 = \beta_2} \tag{5.24}$$

with

$$\widetilde{h}^{*(0,1)}(\beta_1, \rho_2; s) = \frac{-\beta_1}{\left(\beta_1 + s\right)\left(\rho_2 + s\right)^2}.$$

Using (5.21), the Laplace transform of (5.24) yields

$$\widetilde{\chi}^*_{\mathbf{k},\mathbf{m}}(s) = \sum_{j=0}^{m_2 - k_2 - 1} \gamma_{k_2,m_2}(j; \beta_1, \beta_2) \left(\frac{\beta_1}{\beta_1 + s}\right) \left(\frac{\beta_2}{\beta_2 + s}\right)^{j+2}, \tag{5.25}$$

where

$$\gamma_{k_2,m_2}(j;\beta_1,\beta_2) = -\frac{\binom{m_2-1}{k_2}(\beta_1\beta_2)^{m_2}}{m_2!(\beta_2)^{j+2}}\vartheta_{k_2,m_2}(j;\beta_1,\beta_2).$$

Using (5.23) with $n_1 = 1$ and $n_2 = j+2$, one immediately arrives at

$$\chi^*_{\mathbf{k},\mathbf{m}}(u) = \sum_{j=0}^{m_2-k_2-1} \gamma_{k_2,m_2}(j;\beta_1,\beta_2) z_{1,j+2}(u;\beta_1,\beta_2).$$

(iii) for $m_1 \geq 1$ and $m_2 = 0$: by symmetry to (ii), one finds

$$\begin{aligned}\chi^*_{\mathbf{k},\mathbf{m}}(u) &= \frac{\binom{m_1-1}{k_1}}{m_1!}\frac{\partial^{m_1-k_1-1}}{\partial\rho_1^{m_1-k_1-1}}\left\{h^{*(1,0)}(\rho_1,\beta_2;u)\left(\frac{\beta_1\beta_2}{\beta_2-\rho_1}\right)^{m_1}\right\}\Bigg|_{\rho_1=\beta_1} \\ &= \sum_{j=0}^{m_1-k_1-1}\gamma_{k_1,m_1}(j;\beta_2,\beta_1)z_{j+2,1}(u;\beta_1,\beta_2).\end{aligned}$$

(iv) for $m_1 \geq 1$ and $m_2 \geq 1$: given that

$$\widetilde{h}^{*(1,1)}(\rho_1,\rho_2;s) = \frac{s}{(\rho_1+s)^2(\rho_2+s)^2},$$

and using (5.21) twice, the Laplace transform of $\chi^*_{\mathbf{k},\mathbf{m}}(u)$ can be expressed as

$$\begin{aligned}&\widetilde{\chi}^*_{\mathbf{k},\mathbf{m}}(s) \\ &= \left(\prod_{i=1}^{2}\frac{\binom{m_i-1}{k_i}}{m_i!}\right)\frac{\partial^{m_1-k_1-1+m_2-k_2-1}}{\partial\rho_1^{m_1-k_1-1}\partial\rho_2^{m_2-k_2-1}}\left\{\widetilde{h}^{*(1,1)}(\rho_1,\rho_2;s)\frac{(\beta_1\beta_2)^{m_1+m_2}}{(\beta_2-\rho_1)^{m_1}(\beta_1-\rho_2)^{m_2}}\right\}\Bigg|_{\rho_i=\beta_i} \\ &= \sum_{j_1=0}^{m_1-k_1-1}\sum_{j_2=0}^{m_2-k_2-1}\left(\prod_{i=1}^{2}\gamma_{k_i,m_i}(j_i;\beta_{3-i},\beta_i)\right)s\left(\frac{\beta_1}{\beta_1+s}\right)^{2+j_1}\left(\frac{\beta_2}{\beta_2+s}\right)^{2+j_2}. \qquad (5.26)\end{aligned}$$

Using the fact that

$$\begin{aligned}&s\left(\frac{\beta_1}{\beta_1+s}\right)^{n_1}\left(\frac{\beta_2}{\beta_2+s}\right)^{n_2} \\ &= \beta_1\left\{\left(\frac{\beta_1}{\beta_1+s}\right)^{n_1-1}\left(\frac{\beta_2}{\beta_2+s}\right)^{n_2} - \left(\frac{\beta_1}{\beta_1+s}\right)^{n_1}\left(\frac{\beta_2}{\beta_2+s}\right)^{n_2}\right\},\end{aligned}$$

the inversion of (5.26) with the help of (5.23) results in

$$\chi^*_{\mathbf{k},\mathbf{m}}(u) = \sum_{j_1=0}^{m_1-k_1-1} \sum_{j_2=0}^{m_2-k_2-1} \left(\prod_{i=1}^{2} \gamma_{k_i,m_i}(j_i;\beta_{3-i},\beta_i) \right) \varsigma_{2+j_1,2+j_2}(u;\beta_1,\beta_2),$$

where

$$\varsigma_{n_1,n_2}(u;\beta_1,\beta_2) = \beta_1 \left\{ z_{n_1-1,n_2}(u;\beta_1,\beta_2) - z_{n_1,n_2}(u;\beta_1,\beta_2) \right\}.$$

5.4 Numerical examples

5.4.1 The finite-time ruin probability

In this section, we provide a numerical example to calculate the finite-time ruin probabilities in the Sparre Andersen risk model of Section 5.2. We assume that the claim sizes follow a combination of 3 exponentials with mean $\mu = 12$ and density

$$p(y) = \frac{3}{200} e^{-\frac{y}{20}} + \frac{1}{20} e^{-\frac{y}{10}} + \frac{1}{25} e^{-\frac{y}{5}}, \qquad y > 0.$$

The interclaim times are assumed to follow an Erlang-3 distribution with mean $\kappa = 3/0.45$, and Laplace transform $\widetilde{k}(s) = (1 + s/0.45)^{-3}$. The premium rate is payable continuously at a rate of $c = 2$ which implies that the loading factor θ defined as $c = (1+\theta)\mu\kappa^{-1}$ is $11.\bar{1}\%$.

Formulas (5.13) and (5.14) are used to calculate the finite-time ruin probabilities in Tables 5.1 and 5.2. Note that for our choice of claim size distribution, the derivatives $\chi_{\mathbf{k},\mathbf{m}}(u)$ were analytically evaluated with the help of *Maple*. Tables 5.1 and 5.2 present the values of the finite-time ruin probabilities up to n claims for a time horizon of $T = 10$ and $T = 50$, respectively. The initial surplus level u varies from 0, 5, 10 and 20. As expected, for lower initial surplus levels, ruin is more likely to happen. Fig 5.2 and Fig 5.3 present

Figure 5.2: The density of the time to ruin when $u = 0$

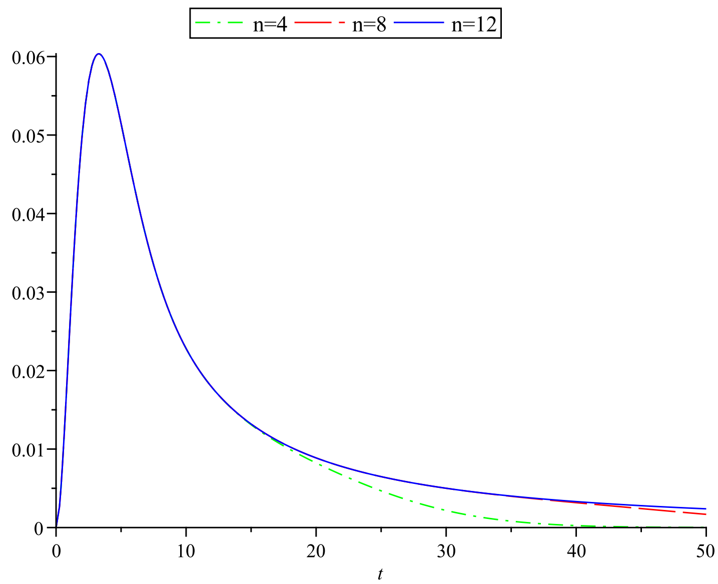


Figure 5.3: The density of the time to ruin when $u = 10$

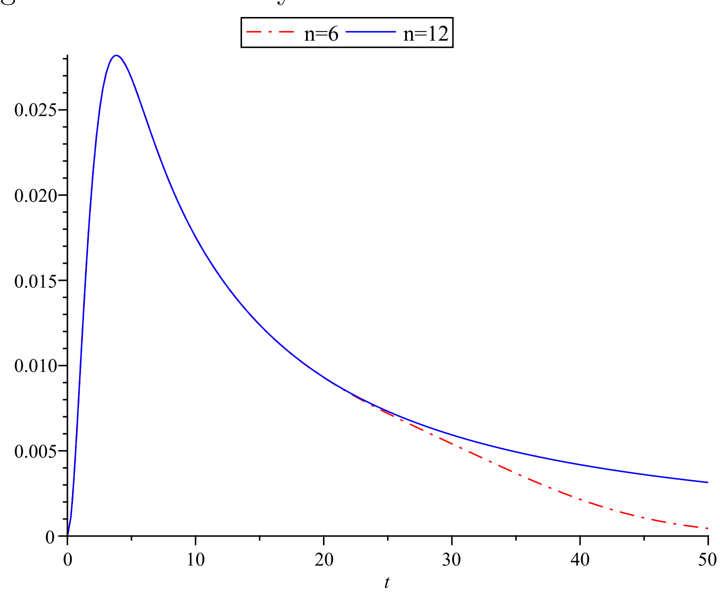


Table 5.1: Finite-time ruin probabilities with no more than n claims ($T = 10$)

$u \setminus n$	1	2	3	4	5	6
0	0.33796	0.38866	0.39215	0.39224	0.39224	0.39224
5	0.22377	0.27843	0.28364	0.28383	0.28383	0.28383
10	0.15324	0.20308	0.20895	0.20921	0.20922	0.20922
20	0.07713	0.11222	0.11768	0.11799	0.11800	0.11800

Table 5.2: Finite-time ruin probabilities with no more than n claims ($T = 50$)

$u \setminus n$	1	2	4	6	8	10	12
0	0.35990	0.48107	0.58966	0.64154	0.66312	0.66719	0.66746
5	0.23951	0.35832	0.47919	0.54119	0.56875	0.57448	0.57490
10	0.16471	0.26958	0.39055	0.45753	0.48906	0.49612	0.49668
20	0.08344	0.15602	0.25993	0.32670	0.36159	0.37041	0.37120

the density of the time to ruin (with no more than n claims) when the initial surplus level is $u = 0$ and $u = 10$, respectively. We point out that the density of the time to ruin for the case $n = 12$ (solid line) has demonstrated convergence of its numerical values for $t \leq 50$. As shown in Table 5.1, the finite-time ruin probabilities converge very fast for a relatively shorter time horizon. However, as the time horizon T gets larger, contributions of ruin from larger n becomes more significant and the speed of convergence is slower (as expected). This can also be seen in Fig 5.2 (or Fig 5.3). The densities for $n = 4$, $n = 8$ and $n = 12$ all coincide for $t \leq 15$. For larger T, we see the differences (which are the portion of ruin probabilities from larger number of claims until ruin) arises from $n = 4$, and later from $n = 8$ and $n = 12$.

5.4.2 The duration of a busy period

Similarly, we can use (5.20) (with a joint analysis of the number of customers waiting in the queue) to identify the distribution of the duration of a busy period. We assume that the interarrival times $\{T_i\}_{i=1}^{\infty}$ of customers in the queue follow a combination of 2 exponentials with mean $\mu_0 = 7.5$ and density

$$p_0(y) = \frac{1}{2}\left(\frac{1}{10}e^{-\frac{y}{10}} + \frac{1}{5}e^{-\frac{y}{5}}\right), \qquad y > 0.$$

The associated service times $\{X_i\}_{i=1}^{\infty}$ follow an Erlang-3 distribution with mean $\kappa_0 = 12$. We also assume that the service speed of the system is $c = 2$. Therefore, $\{cT_i\}_{i=1}^{\infty}$ has mean $\mu = 15$ with density

$$p(y) = \frac{1}{2}\left(\frac{1}{20}e^{-\frac{y}{20}} + \frac{1}{10}e^{-\frac{y}{10}}\right), \qquad y > 0,$$

and $\{X_i/c\}_{i=1}^{\infty}$ follow an Erlang-3 distribution with mean $\kappa = 6$, and Laplace transform $\widetilde{k}(s) = (1 + s/0.5)^{-3}$.

Table 5.3: Duration of a busy period with no more than n customers ($T = 10$)

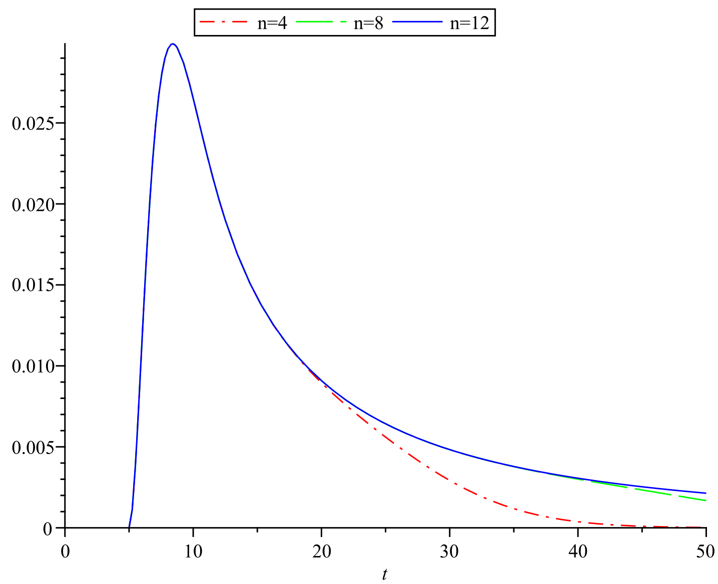
$u \setminus n$	0	1	2	3	4	5	6
5	0.69267	0.79591	0.80752	0.80794	0.80794	0.80794	0.80794
10	0.48721	0.58987	0.59520	0.59526	0.59527	0.59527	0.59527
20	0.25161	0.25161	0.25161	0.25161	0.25161	0.25161	0.25161

Table 5.4: Duration of a busy period with no more than n customers ($T = 50$)

$u \setminus n$	0	2	4	6	8	10	12
5	0.69267	0.86330	0.90911	0.93102	0.93999	0.94160	0.94170
10	0.48721	0.73770	0.82087	0.86174	0.87694	0.87916	0.87926
20	0.25161	0.52612	0.65575	0.72175	0.74056	0.74224	0.74228
30	0.13646	0.36694	0.50996	0.57987	0.59327	0.59393	0.59393

Tables 5.3 and 5.4 present the probabilities that the duration of a busy period is no more than $T = 10$ and $T = 50$, respectively. The initial work loading u varies from 5 to 30. The columns with $n = 0$ show the probabilities of the mass point at $\tau_D = u/c$. Note that in Table 5.3, the lowest duration of a busy period for an initial work loading of $u = 20$ is $T = 10$. Fig 5.4 presents the (defective) density of the duration of a busy period for $t > u/c$ (with no more than n customers) when the initial work loading u equals 10.

Figure 5.4: (Defective) density of the duration of a busy period for $t > u/c$ ($u = 10$)



Chapter 6

The compound Poisson processes with diffusion

6.1 Introduction

In this chapter, first passage times in a compound Poisson process perturbed by diffusion are studied through a generalization of an identity derived by Kendall (1957) (commonly referred to as *Kendall's identity*). Our main result directly leads to an infinite-series expression for the density of the first passage time, which could be used to calculate finite-ruin probabilities in risk theory and to price barrier options in finance, among others. In comparison to the traditional numerical approaches to invert Laplace transforms, the proposed series expansion not only improves the accuracy, but also provides a probabilistic interpretation to each term of the resulting expansion.

Let $\underline{R} = \{R_t, t \geq 0\}$ be a spectrally negative Lévy process with $R_0 = 0$. Such a process is known to have independent and stationary increments with no positive jumps. For $b > 0$, define $\tau_b = \inf\{t \geq 0 : R_t = b\}$. From Kendall (1957), it is known that the distribution of the first passage time τ_b satisfies the following identity:

$$\int_x^{\infty} \Pr(\tau_b \leq t) \frac{db}{b} = \int_0^t \Pr(R_s > x) \frac{ds}{s}, \tag{6.1}$$

for $t, x > 0$ (see also Borovkov and Burq (2001) and references therein for further discussion on the distribution of this first passage time). In a ruin theoretical context, Eq. (6.1) leads to the determination of the distribution of various first hitting times of interest. Among them, we mention the distribution of the time to ruin in the dual risk model with exponential inter-innovation times and with or without a diffusion component (see, e.g., Avanzi et al. (2007) and Avanzi and Gerber (2008)), as well as the distribution of the time to reach a given surplus level in the classical compound Poisson risk model (see, e.g., Gerber (1990)).

We propose to extend (6.1) by further incorporating one particular property of this first passage, namely the number of negative jumps before τ_b. For this to be of interest, we shall restrict the class of spectrally negative Lévy processes to those which takes the form of a compound Poisson process perturbed by a Brownian motion with drift, that is

$$R_t = ct + \sigma W_t - M_t, \tag{6.2}$$

with $c \neq 0$, $\sigma > 0$, $\underline{W} = \{W_t, t \geq 0\}$ a standard Brownian motion, and $\underline{M} = \{M_t, t \geq 0\}$ an independent (of $\underline{W}$) compound Poisson process. More precisely, we define the process $\underline{M}$ as

$$M_t = \begin{cases} \sum_{i=1}^{N_t} X_i, & N_t > 0, \\ 0, & N_t = 0, \end{cases}$$

where $\{N_t, t \geq 0\}$ is a Poisson process with rate $\lambda > 0$ and is defined via the sequence of i.i.d. interclaim time r.v.'s $\{T_i\}_{i=1}^{\infty}$ with density $k(t) = \lambda e^{-\lambda t}$. $\{X_i\}_{i=1}^{\infty}$ also form a sequence of i.i.d. r.v.'s with density p, and Laplace transform $\widetilde{p}(s) = \int_0^\infty e^{-sx} p(x)\, dx$. We assume that the r.v.'s $\{X_i\}_{i=1}^\infty$ are also independent of the Poisson process $\{N_t, t \geq 0\}$. For spectrally negative Lévy processes not of the form (6.2), it is well known that these processes have infinitely many jumps in every interval (i.e. infinite intensity of infinitely small activity). Thus, the present analysis is not applicable to this class of spectrally negative Lévy processes.

Our main objective is to generalize (6.1) for the class of compound Poisson processes with diffusion by jointly analyzing the number of jumps before the first passage time. Our approach makes use of Lagrange's expansion theorem and establishes an interesting connection with Kendall's identity. In the process, an alternative proof of Kendall's identity which is both relatively straightforward and only involves simple algebraic manipulations is provided for spectrally negative Lévy processes of the form (6.2). An implied result of the generalization is the joint generalized density of the first passage time and the number of negative jumps until this first passage time. In Section 6.4, we point out that our result can be directly used to find the finite-time ruin probability in a dual risk model with diffusion (see, e.g., Avanzi and Gerber (2008)). It is worth mentioning that this dual risk model can also be applied in a fluid flow context (see, e.g., Asmussen 1995). Finally, we show that our main result can be used to price path-dependent options on an insurer's stock price. Numerical examples are provided for illustrations.

6.2 Kendall's identity: revisited

6.2.1 Preamble

In this section, the joint Laplace transform of the first passage time to level b and the number of jumps until this first passage time is derived. Consider the process

$$\left\{\gamma^{N_t} e^{-\delta t+sR_t}, t \geq 0\right\} \tag{6.3}$$

for $\gamma \in (0,1]$ and $\delta > 0$. Under the condition

$$\gamma E\left[e^{-\delta T_1} e^{s\left(cT_1+\sigma W_{T_1}-X_1\right)}\right] = 1, \tag{6.4}$$

it is easy to verify that (6.3) is a martingale. Eq. (6.4) has a unique positive solution (see the case $\gamma = 1$ in Gerber and Landry (1998)). Indeed, routine calculations yield

$$\begin{aligned} E\left[e^{-\delta T_1} e^{s\left(cT_1+\sigma W_{T_1}-X_1\right)}\right] &= \left(\int_0^\infty \lambda e^{-(\lambda+\delta-cs)t} E\left[e^{s\sigma W_t}\right] dt\right) \widetilde{p}\left(s\right) \\ &= \frac{\lambda}{\lambda+\delta-cs-\frac{1}{2}\sigma^2 s^2} \widetilde{p}\left(s\right). \end{aligned} \tag{6.5}$$

Substituting (6.5) into (6.4), we have

$$y\left(s\right) = \frac{1}{2}\sigma^2 s^2 + cs - (\lambda+\delta) + \gamma\lambda\widetilde{p}\left(s\right) = 0.$$

Since $y''\left(s\right) = \sigma^2 + \gamma\lambda \int_0^\infty e^{-sx} x^2 p\left(x\right) dx > 0$, $y\left(s\right)$ is convex. Also, $y\left(0\right) = \lambda\left(\gamma-1\right) - \delta$ and $y\left(s\right) \to \infty$ as $s \to \infty$. It follows that (6.4) has a unique positive solution ξ when $\gamma < 1$ or $\delta > 0$. When $s = \xi$, the stopped process $\left\{\gamma^{N_{t\wedge\tau_b}} e^{-\delta(t\wedge\tau_b)+\xi R_{t\wedge\tau_b}}, t \geq 0\right\}$ is bounded. By the optional sampling theorem, we have

$$E\left[\gamma^{N_{\tau_b}} e^{-\delta\tau_b} 1\left(\tau_b < \infty\right)\right] = e^{-\xi b}. \tag{6.6}$$

Next, we develop an identity particularly relevant to the use of Lagrange's expansion theorem in the next section. We start from the left side of (6.5). Using partial fractions,

$$\frac{\lambda}{\lambda+\delta-cs-\frac{\sigma^2}{2}s^2}=\frac{\frac{-2\lambda}{\sigma^2}}{\left(s-\rho_1\right)\left(s-\rho_2\right)}, \tag{6.7}$$

where

$$\begin{aligned}\rho_i &= \frac{-\frac{2c}{\sigma^2}\pm\sqrt{\left(\frac{2c}{\sigma^2}\right)^2+8\frac{\lambda+\delta}{\sigma^2}}}{2}\\ &= -\frac{c}{\sigma^2}\pm\sqrt{\left(\frac{c}{\sigma^2}\right)^2+2\frac{\lambda+\delta}{\sigma^2}}\\ &= \frac{1}{\sigma^2}\left(-c\pm\sqrt{c^2+2\sigma^2\left(\lambda+\delta\right)}\right).\end{aligned}$$

In the sequel, we assume (without loss of generality) that

$$\rho_1=-\frac{1}{\sigma^2}\left(c-\sqrt{c^2+2\sigma^2\left(\lambda+\delta\right)}\right), \tag{6.8}$$

and

$$\rho_2=-\frac{1}{\sigma^2}\left(c+\sqrt{c^2+2\sigma^2\left(\lambda+\delta\right)}\right). \tag{6.9}$$

It is worth pointing out that $\rho_1>0$ and $\rho_2<0$ for $c\neq 0$. Finally, substituting (6.5) and (6.7) into (6.4) yields

$$s-\rho_1+\frac{2\lambda\gamma}{\sigma^2}\frac{1}{s-\rho_2}\widetilde{p}\left(s\right)=0. \tag{6.10}$$

6.2.2 Main result

In this section, a generalization of Kendall's identity is proposed for the compound Poisson process with diffusion.

Theorem 6.2.1 *For a compound Poisson process with diffusion, we have*

$$\int_x^\infty \Pr\left(\tau_b\leq t, N_{\tau_b}=n\right)\frac{db}{b}=\int_0^t \Pr\left(R_s>x, N_s=n\right)\frac{ds}{s}, \tag{6.11}$$

for $t, x > 0$ *and* $n = 0, 1, ...$

Proof: We employ a transform-based approach to derive (6.11). Indeed, using (6.6),

$$\sum_{n=0}^{\infty} \gamma^n \int_0^{\infty} e^{-\delta t} \left\{ \int_x^{\infty} \Pr\left(\tau_b \leq t, N_{\tau_b} = n\right) \frac{db}{b} \right\} dt$$
$$= \int_x^{\infty} \frac{e^{-\xi b}}{\delta} \frac{db}{b}. \tag{6.12}$$

The existence of the integral on the right side of Eq. (6.12) ensures that the left side is also integrable. From the representation (6.10), the use of Lagrange's expansion theorem (see Eq. (1.22)) allows to re-write $e^{-\xi b}$ as

$$\begin{aligned} e^{-\xi b} &= e^{-\rho_1 b} + \sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \left(\frac{-2\lambda}{\sigma^2} \right)^n \frac{d^{n-1}}{dx^{n-1}} \left\{ (-b)\, e^{-xb} \left(\frac{\widetilde{p}(x)}{x - \rho_2} \right)^n \right\} \Bigg|_{x=\rho_1} \\ &= e^{-\rho_1 b} + \sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \left(\frac{-2\lambda}{\sigma^2} \right)^n (-b) \frac{d^{n-1}}{dx^{n-1}} \left\{ \frac{\int_0^{\infty} e^{-x(b+y)} p^{*n}(y)\, dy}{(x - \rho_2)^n} \right\} \Bigg|_{x=\rho_1} \\ &= e^{-\rho_1 b} + \sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \left(\frac{2\lambda}{\sigma^2} \right)^n b \sum_{j=0}^{n-1} \frac{(2n-j-2)!}{(n-j-1)!j!} \left\{ \frac{\int_b^{\infty} y^j e^{-\rho_1 y} p^{*n}(y-b)\, dy}{(\rho_1 - \rho_2)^{2n-1-j}} \right\} \end{aligned} \tag{6.13}$$

From (6.8) and (6.9), we know that

$$\begin{aligned} \rho_1 - \rho_2 &= \frac{2}{\sigma^2} \sqrt{c^2 + 2\sigma^2 (\lambda + \delta)} \\ &= 2 \left(\rho_1 + \frac{c}{\sigma^2} \right), \end{aligned}$$

which implies that (6.13) becomes

$$\begin{aligned} e^{-\xi b} \;=\; & e^{-\rho_1 b} + \sum_{n=1}^{\infty} \frac{\gamma^n}{n!} b \\ & \times \sum_{j=0}^{n-1} \zeta_{j,n} \left\{ (2n-j-2)! \left(\rho_1 + \frac{c}{\sigma^2} \right)^{-(2n-1-j)} \right\} \left\{ \int_b^{\infty} e^{-\rho_1 y} \left\{ y^j p^{*n}(y-b) \right\} dy \right\}, \end{aligned} \tag{6.14}$$

where

$$\zeta_{j,n} = \frac{1}{(n-j-1)!j!}\left(\frac{\lambda}{2\sigma^2}\right)^n 2^{j+1}.$$

Given that

$$\left(\rho_1 + \frac{c}{\sigma^2}\right)^{-n} = \int_0^\infty e^{-\rho_1 x}\left\{\frac{x^{n-1}e^{-\frac{c}{\sigma^2}x}}{(n-1)!}\right\}dx,$$

for $n = 1, 2, ...$, it follows that

$$e^{-\xi b} = e^{-\rho_1 b} + \sum_{n=1}^{\infty}\int_b^\infty \left\{\gamma^n e^{-\rho_1 y}\right\}\varphi_n\left(y\left|b\right.\right)dy, \tag{6.15}$$

where

$$\begin{aligned}
&\varphi_n\left(y\left|b\right.\right) \\
&= \frac{b}{n!}\sum_{j=0}^{n-1}\zeta_{j,n}\int_b^y x^j\left(y-x\right)^{2(n-1)-j}e^{-\frac{c}{\sigma^2}(y-x)}p^{*n}\left(x-b\right)dx \\
&= \frac{2b}{n!\left(n-1\right)!}\left(\frac{\lambda}{2\sigma^2}\right)^n\int_b^y\left(y+x\right)^{n-1}\left(y-x\right)^{n-1}e^{-\frac{c}{\sigma^2}(y-x)}p^{*n}\left(x-b\right)dx. \qquad (6.16)
\end{aligned}$$

From (6.8), it is also clear that

$$e^{-\rho_1 b} = e^{\frac{1}{\sigma^2}(c-|c|)b}e^{\frac{|c|}{\sigma^2}\left(1-\sqrt{1+\frac{2\sigma^2}{c^2}(\lambda+\delta)}\right)b}. \tag{6.17}$$

Given that an inverse Gaussian r.v. with parameters κ and μ $(\kappa, \mu > 0)$ and density

$$q\left(t\right) = \left(\frac{\kappa}{2\pi t^3}\right)^{\frac{1}{2}}e^{-\frac{\kappa(t-\mu)^2}{2\mu^2 t}}, \qquad x > 0,$$

has Laplace transform

$$\widetilde{q}\left(s\right) = e^{\frac{\kappa}{\mu}\left[1-\sqrt{1+\frac{2\mu^2 s}{\kappa}}\right]}, \tag{6.18}$$

(6.17) can be rewritten as

$$\begin{aligned}
e^{-\rho_1 b} &= \int_0^\infty e^{-\delta t}\left\{\frac{e^{-\lambda t}}{\sqrt{2\pi\sigma^2 t^3}}be^{\frac{1}{\sigma^2}(c-|c|)b}e^{-\frac{1}{2\sigma^2 t}(|c|t-b)^2}\right\}dt \\
&= \int_0^\infty e^{-\delta t}\left\{\frac{e^{-\lambda t}}{\sqrt{2\pi\sigma^2 t^3}}be^{-\frac{1}{2\sigma^2 t}(ct-b)^2}\right\}dt. \qquad (6.19)
\end{aligned}$$

Substituting (6.19) into (6.15) yields

$$e^{-\xi b} = \sum_{n=0}^{\infty} \int_0^{\infty} \left\{ \gamma^n e^{-\delta t} \right\} f_n \left(t \left| b \right. \right) dt, \tag{6.20}$$

where

$$f_n \left(t \left| b \right. \right) = \begin{cases} \frac{e^{-\lambda t}}{\sqrt{2\pi\sigma^2 t^3}} b e^{-\frac{1}{2\sigma^2 t}(ct-b)^2}, & n = 0, \\ \frac{e^{-\lambda t}}{\sqrt{2\pi\sigma^2 t^3}} \int_b^{\infty} y e^{-\frac{1}{2\sigma^2 t}(ct-y)^2} \varphi_n \left(y \left| b \right. \right) dy, & n = 1, 2, ... \end{cases} \tag{6.21}$$

Interchanging the order of integration, (6.21) becomes

$$f_n \left(t \left| b \right. \right) = \frac{e^{-\lambda t}}{\sqrt{2\pi\sigma^2 t^3}} \frac{2b}{n! \left(n-1 \right)!} \left(\frac{\lambda}{2\sigma^2} \right)^n \int_b^{\infty} p^{*n} \left(x - b \right) I_n \left(x, t \right) dx, \tag{6.22}$$

for $n = 1, 2, ...$ where

$$\begin{aligned} I_n \left(x, t \right) &= \int_x^{\infty} y \left(y + x \right)^{n-1} \left(y - x \right)^{n-1} e^{-\frac{c}{\sigma^2}(y-x)} e^{-\frac{1}{2\sigma^2 t}(ct-y)^2} dy \\ &= e^{-\frac{1}{2\sigma^2 t}(ct-x)^2} \int_x^{\infty} y \left(y^2 - x^2 \right)^{n-1} e^{-\frac{1}{2\sigma^2 t}\left(y^2 - x^2\right)} dy. \end{aligned}$$

Using integration by parts, it is not difficult to show that

$$\begin{aligned} I_n \left(x, t \right) &= e^{-\frac{1}{2\sigma^2 t}(ct-x)^2} \int_x^{\infty} \left\{ \frac{\left(y^2 - x^2 \right)^n}{2n} \right\} \left\{ \frac{y}{\sigma^2 t} e^{-\frac{1}{2\sigma^2 t}\left(y^2 - x^2\right)} \right\} dy \\ &= \frac{1}{2n\sigma^2 t} I_{n+1} \left(x, t \right), \end{aligned} \tag{6.23}$$

for $n = 1, 2, ...$ Through a recursive use of (6.23) together with its starting point

$$\begin{aligned} I_1 \left(x, t \right) &= e^{-\frac{1}{2\sigma^2 t}(ct-x)^2} \int_x^{\infty} y e^{-\frac{1}{2\sigma^2 t}\left(y^2 - x^2\right)} dy \\ &= \frac{e^{-\frac{1}{2\sigma^2 t}(ct-x)^2}}{2} \int_0^{\infty} e^{-\frac{1}{2\sigma^2 t} w} dw \\ &= \left(\sigma^2 t \right) e^{-\frac{1}{2\sigma^2 t}(ct-x)^2}, \end{aligned}$$

one concludes that

$$I_n \left(x, t \right) = \frac{(2\sigma^2 t)^n}{2} (n-1)! \; e^{-\frac{1}{2\sigma^2 t}(ct-x)^2},$$

for $n = 1, 2, ...$ Substituting into (6.22) yields

$$f_n(t\,|b) = \begin{cases} \frac{b}{t} e^{-\lambda t} \left\{ \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{1}{2\sigma^2 t}(ct-b)^2} \right\}, & n = 0, \\ \frac{b}{t} \frac{(\lambda t)^n e^{-\lambda t}}{n!} \int_b^\infty p^{*n}(x-b) \left\{ \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{1}{2\sigma^2 t}(ct-x)^2} \right\} dx, & n = 1, 2, ... \end{cases} . \tag{6.24}$$

Note that

$$f_n(t\,|b) = \frac{b}{t} \frac{(\lambda t)^n e^{-\lambda t}}{n!} f_{R_t|N_t}(b\,|n), \tag{6.25}$$

for $n = 0, 1, 2, ...$

Substituting (6.25) and (6.20) into (6.12) followed by some simple manipulations, one arrives at

$$\begin{aligned} &\sum_{n=0}^{\infty} \gamma^n \int_0^\infty e^{-\delta t} \left\{ \int_x^\infty \Pr(\tau_b \le t, N_{\tau_b} = n) \frac{db}{b} \right\} dt \\ &= \sum_{n=0}^{\infty} \gamma^n \int_x^\infty \int_0^\infty \frac{e^{-\delta t}}{\delta} \left\{ \frac{b}{t} \frac{(\lambda t)^n e^{-\lambda t}}{n!} f_{R_t|N_t}(b\,|n) \right\} dt \frac{db}{b} \\ &= \sum_{n=0}^{\infty} \gamma^n \int_0^\infty \frac{e^{-\delta t}}{\delta} \left\{ \frac{(\lambda t)^n e^{-\lambda t}}{n!} \Pr(R_t > x\,|N_t = n) \right\} \frac{dt}{t} \\ &= \sum_{n=0}^{\infty} \gamma^n \int_0^\infty \frac{e^{-\delta t}}{\delta} \Pr(R_t > x, N_t = n) \frac{dt}{t}. \end{aligned}$$

Using integration by parts, one concludes that

$$\begin{aligned} &\sum_{n=0}^{\infty} \gamma^n \int_0^\infty \frac{e^{-\delta t}}{\delta} \Pr(R_t > x, N_t = n) \frac{dt}{t} \\ &= \sum_{n=0}^{\infty} \gamma^n \int_0^\infty e^{-\delta t} \left\{ \int_0^t \Pr(R_s > x, N_s = n) \frac{ds}{s} \right\} dt. \end{aligned}$$

By the uniqueness property of Laplace transforms and probability generating functions, the result follows. ■

Remark 6.2.2 *Given that R_s $(s > 0)$ has a density at $x \in \mathbb{R}$, it is immediate from Theorem 6.2.1 that the (defective) joint generalized density of the first passage time τ_b and the jumps until the first passage time N_{τ_b} at (t, n) is given by $f_n(t\,|b)$.*

Remark 6.2.3 *As expected, Theorem 6.2.1 still holds when $\sigma = 0$. (note that the proof has to be modified accordingly). We point out that the marginal distribution of the first passage time τ_b was discussed by, e.g., Gerber and Shiu (1998, Eq. (5.15)) together with its connection with a particular version of the Ballot theorem. In our context, Theorem 6.2.1 provides a generalization of this result: the first passage time τ_b is a mixed random variable with a mass point at b/c of $\Pr(\tau_b = b/c, N_{\tau_b} = 0) = e^{-\frac{\lambda}{c}b}$. When at least one claim occurs before the first passage time, the joint generalized density of (τ_b, N_{τ_b}) at (t, n) is given by*

$$\varkappa_n(t\,|b) = \frac{b}{t}\frac{(\lambda t)^n e^{-\lambda t}}{n!} p^{*n}(ct - b),$$

for $t > b/c$ and $n = 1, 2, ...$

In the following section, we consider a large class of distributions which leads to a mathematically tractable expression for $f_n(t\,|b)$.

6.3 Mixed Erlang distributed jumps

In this section, we assume that the Laplace transform of the negative jumps is of the form

$$\widetilde{p}(s) = C\left(\frac{\beta}{\beta + s}\right), \tag{6.26}$$

where

$$C(z) = \sum_{j=1}^{\infty} c_j z^j,$$

with $c_j \geq 0$ for $j = 1, 2, ...$ and $\sum_{j=1}^{\infty} c_j = 1$. The reader is referred to Tijms (1994, p.163) for a proof that any continuous and positive random variable can be approximated arbitrary accurately by a mixed Erlang density and to Willmot and Woo (2007) and Willmot and Lin (2011) for an extensive analysis of this class of distributions.

Under this distributional assumption, (6.24) becomes

$$
\begin{aligned}
f_n\left(t\,|b\right) &= \frac{b}{t}\frac{\left(\lambda t\right)^n e^{-\lambda t}}{n!}\int_0^\infty \left\{\sum_{j=1}^{\infty} c_j^{*n}\frac{\beta^j x^{j-1}e^{-\beta x}}{(j-1)!}\right\}\frac{1}{\sqrt{2\pi\sigma^2 t}}e^{-\frac{1}{2\sigma^2 t}(ct-b-x)^2}dx \\
&= \frac{b}{t}\frac{\left(\lambda t\right)^n e^{-\lambda t}}{n!}\frac{1}{\sqrt{2\pi\sigma^2 t}}\sum_{j=1}^{\infty} c_j^{*n}\frac{\beta^j}{(j-1)!}\int_0^\infty x^{j-1}e^{-\beta x}e^{-\frac{1}{2\sigma^2 t}(ct-b-x)^2}dx \\
&= \frac{b}{t}\frac{\left(\lambda t\right)^n e^{-\lambda t}}{n!}\frac{e^{-\beta\left(ct-b-\frac{\beta\sigma^2}{2}t\right)}}{\sqrt{2\pi\sigma^2 t}}\sum_{j=1}^{\infty} c_j^{*n}\frac{\beta^j}{(j-1)!}\int_0^\infty x^{j-1}e^{-\frac{1}{2\sigma^2 t}\left(x-\left(\left(c-\beta\sigma^2\right)t-b\right)\right)^2}dx,
\end{aligned}
\tag{6.27}
$$

where c_j^{*n} are defined via the transform relationship

$$
\left(C\left(z\right)\right)^n = \sum_{j=1}^{\infty} c_j^{*n} z^j.
$$

Simple modifications of the integrand in (6.27) results in

$$
f_n\left(t\,|b\right) = \frac{b}{t}\frac{\left(\lambda t\right)^n e^{-\lambda t}}{n!}\frac{e^{-\beta\left(z_t+\frac{\beta\sigma^2}{2}t\right)}}{\sqrt{2\pi\sigma^2 t}}\sum_{j=1}^{\infty} c_j^{*n}\beta^j\sum_{k=0}^{j-1}\frac{\left(z_t\right)^{j-1-k}}{(j-1-k)!k!}\alpha_k\left(z_t\right),
\tag{6.28}
$$

where

$$
\alpha_k\left(z\right) = \int_{-z}^{\infty} x^k e^{-\frac{1}{2\sigma^2 t}x^2}dx,
$$

and

$$
z_t = \left(c-\beta\sigma^2\right)t - b.
$$

For k odd (say $k = 2i + 1$), we have

$$\alpha_{2i+1}(z) = \int_{|z|}^{\infty} x^{2i+1} e^{-\frac{1}{2\sigma^2 t}x^2} dx$$
$$= \frac{1}{2} \int_{z^2}^{\infty} y^i e^{-\frac{1}{2\sigma^2 t}y} dy$$
$$= \frac{1}{2}(i!)\left(2\sigma^2 t\right)^{i+1} \sum_{l=0}^{i} \frac{\left(\frac{z^2}{2\sigma^2 t}\right)^l e^{-\frac{z^2}{2\sigma^2 t}}}{l!}. \tag{6.29}$$

For k even (say $k = 2i$), using integration by parts, one finds that

$$\alpha_{2i}(z) = \sigma^2 t \left\{ (-z)^{2i-1} e^{-\frac{1}{2\sigma^2 t}z^2} + (2i-1) \int_{-z}^{\infty} y^{2(i-1)} e^{-\frac{1}{2\sigma^2 t}y^2} dy \right\}.$$

By repeating this argument, one finds

$$\alpha_{2i}(z) = \sum_{j=1}^{i} \frac{\gamma_i}{\gamma_j} \left(\sigma^2 t\right)^{i-j+1} \left\{ (-z)^{2j-1} e^{-\frac{1}{2\sigma^2 t}z^2} \right\} + \gamma_i \left(\sigma^2 t\right)^i \int_{-z}^{\infty} e^{-\frac{1}{2\sigma^2 t}y^2} dy$$
$$= \sum_{j=1}^{i} \frac{\gamma_i}{\gamma_j} \left(\sigma^2 t\right)^{i-j+1} \left\{ (-z)^{2j-1} e^{-\frac{1}{2\sigma^2 t}z^2} \right\} + \gamma_i \left(\sigma^2 t\right)^{i+\frac{1}{2}} \sqrt{2\pi} \left(1 - \Phi\left(\frac{-z}{\sigma\sqrt{t}}\right)\right), \tag{6.30}$$

where Φ is the cumulative distribution function of a Normal random variable with mean 0 and variance 1, and

$$\gamma_i = \prod_{k=1}^{i} (2k-1).$$

6.4 Applications

6.4.1 First passage times in dual risk model with diffusion and in fluid flow model

The dual risk process with diffusion $\underline{U}^d = \left\{U_t^d, t \geq 0\right\}$ is defined as

$$\begin{aligned} U_t^d &= u - ct + \sigma W_t + M_t \\ &= u - R_t, \end{aligned} \tag{6.31}$$

where u is the initial surplus level, and c is the non-negative expense rate (see, e.g., Grandell (1991, p.8) and Avanzi and Gerber (2008)). As pointed out by Avanzi and Gerber (2008), the dual risk model is well suited to model the cash flow dynamics of a portfolio of life annuities, or of companies specializing in inventions and discoveries. Let $\varsigma = \inf\left\{t \geq 0 : U_t^d < 0\right\}$ be the time to ruin for this surplus process. By a reflective argument, one can easily conclude that the time to ruin ς with an initial surplus u in the dual risk process $\underline{U}^d$ corresponds to the first passage time to level u, namely τ_u, of the process $\{R_t, t \geq 0\}$. Therefore, one can directly use the main result of Section 6.2 to obtain finite-time ruin probabilities in the dual risk process with no more than a given number of claims in the interim.

Many researchers have analyzed the ruin probability in the dual risk process (see, e.g., Cramér (1955, Section 5.13) and Mazza and Rulliére (2004)). A traditional way to obtain the distribution of the time to ruin is through the numerical inversion of its Laplace transform (which is known under various distributional assumptions). The explicit expression (6.24) can be considered as an alternative to calculate finite-time ruin probabilities. More interestingly, it enables one to break the contribution to the finite-time ruin probability by the number of claims until ruin, which by itself is of interest. Moreover, the resulting approximative quantity provides an insightful ruin related quantity.

The dual risk process (6.31) can also be considered as a second-order fluid flow queue, where u is the initial fluid in the system and M_t is the non-decreasing fluid arrivals into the queue. We assume a linear service rate c and use the Brownian motion to represent the traffic noise. A more general definition of a second-order fluid queue can be found in Kulkarni (1997), Rabehasaina and Sericola (2004), as well as references therein. Fluid flow models are widely used in engineering to analyze the behavior of telecommunication flow, whereby the fluid represents the signals temporarily stored in a buffer. The first passage time to level 0 is the duration of a busy period of the buffer. Similarly, Eq. (6.24) can be directly used to obtain the joint (defective) distribution of the busy period and the number of signal arrivals.

In the following example, we will illustrate the usage of Eq. (6.24) to obtain the finite ruin probabilities in a dual risk model.

Example 6.4.1 *We assume that the jump size distribution follows a mixture of* 3 *Erlangs with Laplace transform*

$$\widetilde{p}(s) = 0.3 \cdot \frac{0.05}{0.05+s} + 0.5 \cdot \left(\frac{0.05}{0.05+s}\right)^2 + 0.2 \cdot \left(\frac{0.05}{0.05+s}\right)^3, \qquad s \geq 0.$$

Also, let $\lambda = 0.15$, $\sigma = 0.2$, *and the expense rate* $c = 5$. *Tables 6.1 and 6.2 present the values of finite-time ruin probabilities with no more than n jumps (i.e.* $\Pr(\varsigma \leq T, N_\varsigma \leq n)$*) for a time horizon of* $T = 10$ *and* $T = 50$, *respectively.*

As expected, longer is the time horizon, more jumps are required to observe the convergence of the joint cumulative distribution function of the time to ruin and the number of

Table 6.1: Finite-time ruin probability with no more than n jumps ($T = 10$)

$u \setminus n$	0	1	2	3	4	5	6
5	0.8607	0.9107	0.9176	0.9183	0.9183	0.9183	0.9183
10	0.7408	0.8235	0.8349	0.8359	0.8359	0.8359	0.8359
25	0.4724	0.5765	0.5882	0.5888	0.5889	0.5889	0.5889
50	0.1138	0.1151	0.1151	0.1151	0.1151	0.1151	0.1151

Table 6.2: Finite-time ruin probability with no more than n jumps ($T = 50$)

$u \setminus n$	0	1	3	5	7	8	9
5	0.8607	0.9164	0.9421	0.9512	0.9549	0.9556	0.9558
10	0.7408	0.8368	0.8860	0.9036	0.9107	0.9119	0.9124
25	0.4724	0.6253	0.7288	0.7686	0.7837	0.7861	0.7870
50	0.2232	0.3676	0.5089	0.5706	0.5920	0.5948	0.5957

claims at ruin to the finite-time ruin probability (as $n \to \infty$). Theoretically, we know that

$$\lim_{n\to\infty} \Pr\left(\varsigma \leq T, N_\varsigma \leq n\right) = \Pr\left(\varsigma \leq T\right).$$

6.4.2 Pricing path-dependent exotic options

In recent years, jump-diffusion processes have been widely used to model financial assets. In general, researchers are using two-sided jumps to represent random gains and losses of a company (see, e.g., Kou and Wang (2004)). However, many authors have argued the relevance of using one-sided jump-diffusion processes to model the stock price of insurance companies, where the one-sided negative jumps represent the impact of catastrophic losses (see, e.g., Gerber and Landry (1998), Cox et al. (2004) and Lin and Wang (2009)). Rather than investigating catastrophe-linked securities (in which the catastrophe loss is the exercise trigger, as in, e.g., Cox et al. (2004) and Lin and Wang (2009)), we are interested here in analyzing general path-dependent options issued on the insurer's stock. Specifically speaking, we will use the main result of Section 6.2 to price the up-and-in call option of an insurer. Other up-and-in, up-and-out call (put) options can be obtained in a similar fashion. An up-and-in call is a regular call option that will be activated only if the price of the underlying asset rises above a certain price level (see, e.g., Hull (2010)).

The underlying asset process of an insurance company $\{S(t), t \geq 0\}$ is assumed to have the form

$$S(t) = S(0) \exp\left(ct + \sigma W_t - M_t\right),$$

where $S(0)$ is the initial stock price. The process is further assumed to be under the risk-neutral probability measure Q with a continuous risk-free rate $r > 0$. To ensure that the discounted stock price process $\{e^{-rt}S(t), t \geq 0\}$ is a martingale under Q, c is assumed to be given by

$$c = r - \frac{\sigma^2}{2} - \lambda(\widetilde{p}(1) - 1). \tag{6.32}$$

Under the risk-neutral probability measure, the price of an up-and-in call option with time to maturity T, strike price K, barrier level H ($H > S(0)$) and up to n negative jumps is given by

$$\begin{aligned} C_n(T, H, K) &= E^Q\left[e^{-rT}(S(T) - K)^+ 1\left(\max_{0\leq t\leq T} S(t) \geq H,\ N_T \leq n\right)\right] \\ &= \Lambda_n - Ke^{-rT}\Pi_n, \end{aligned}$$

where

$$\Lambda_n = E^Q\left[e^{-rT}S(T) 1\left(S(T) \geq K, \max_{0\leq t\leq T} S(t) \geq H,\ N_T \leq n\right)\right], \tag{6.33}$$

and

$$\Pi_n = P^Q\left(S(T) \geq K, \max_{0\leq t\leq T} S(t) \geq H,\ N_T \leq n\right). \tag{6.34}$$

Naturally, the price of an up-and-in call option is the limit of $C_n(T,H,K)$ as $n \to \infty$, i.e.

$$C(T,H,K) = \lim_{n\to\infty} C_n(T,H,K).$$

Let $b = \ln \frac{H}{S(0)}$. By conditioning on the first passage time of the process $\underline{R}$ to level b and on the number of jumps, (6.34) becomes

$$\begin{aligned}\Pi_n &= P^Q\left(S(T) \geq K, \tau_b \leq T,\ N_T \leq n\right)\\ &= \sum_{k=0}^{n} \int_0^T P^Q(S(T) \geq K,\ N_T \leq n |\, \tau_b = t,\ N_{\tau_b} = k) f_k\left(t\,|b\right) dt.\end{aligned}$$

Noting that $R_{\tau_b} = b$ and $S_{\tau_b} = H$, and making use of the strong Markov property of the process $\underline{R}$, one deduces

$$\begin{aligned}&P^Q\left(S(T) \geq K,\ N_T \leq n |\, \tau_b = t, N_{\tau_b} = k\right)\\ &= \sum_{j=k}^{n} P^Q(S_{\tau_b} e^{R_T - b} \geq K,\ N_{T-\tau_b} = j-k \big|\, \tau_b = t)\\ &= \sum_{j=k}^{n} P^Q(R_{T-t} \geq \ln\frac{K}{H},\ N_{T-t} = j-k)\\ &= \sum_{j=k}^{n} \int_{\ln\frac{K}{H}}^{\infty} g_{j-k}\left(y; T-t\right) dy,\end{aligned}$$

where $g_n\left(y;t\right)$ is the density of R_t with n jumps, namely

$$g_n\left(y;t\right) = \begin{cases} e^{-\lambda t}\left\{\frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{1}{2\sigma^2 t}(ct-y)^2}\right\}, & n=0,\\ \frac{(\lambda t)^n e^{-\lambda t}}{n!} \int_0^\infty p^{*n}\left(x\right)\left\{\frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{1}{2\sigma^2 t}(ct-y-x)^2}\right\} dx, & n = 1,2,...\end{cases}$$

Therefore,

$$\Pi_n = \sum_{k=0}^{n}\sum_{j=k}^{n} \int_0^T f_k\left(t\,|b\right) \int_{\ln\frac{K}{H}}^{\infty} g_{j-k}\left(y; T-t\right) dydt. \tag{6.35}$$

Using the same change of numeraire arguments as in Kou and Wang (2004), an expression for Λ_n can be obtained. Indeed, define a new probability measure $\widetilde{Q}$ such that

$$\frac{d\widetilde{Q}}{dQ} = e^{-rt}\frac{S(t)}{S(0)} = \exp\left(\left(-\frac{\sigma^2}{2} - \lambda(\widetilde{p}(1)-1)\right)t + \sigma W_t - M_t\right).$$

Under the $\widetilde{Q}$ measure, R_t is a jump-diffusion process

$$R_t = c^{\widetilde{Q}}t + \sigma W_t^{\widetilde{Q}} - M_t^{\widetilde{Q}},$$

where $c^{\widetilde{Q}} = c + \sigma^2$, and $\{W_t^{\widetilde{Q}},\ t \geq 0\}$ defined as $W_t^{\widetilde{Q}} = W_t - \sigma t$ is a standard Brownian motion, and $\{M_t^{\widetilde{Q}},\ t \geq 0\}$ is a compound Poisson process with Poisson arrival rate $\lambda^{\widetilde{Q}} = \lambda\widetilde{p}(1)$ and secondary distribution having density $p^{\widetilde{Q}}(x) = \frac{e^{-x}}{\widetilde{p}(1)}p(x)$ for $x \geq 0$.

Furthermore, (6.34) under the $\widetilde{Q}$ measure is given by

$$\begin{aligned}\Lambda_n &= S(0)E^{\widetilde{Q}}\left[1\left(S(T) \geq K, \max_{0\leq t\leq T} S(t) \geq H,\ N_T \leq n\right)\right]\\ &= S(0)\widetilde{Q}\left(S(T) \geq K, \tau_b \leq T,\ N_T \leq n\right)\\ &= S(0)\Pi_n^{\widetilde{Q}},\end{aligned}$$

where $\Pi_n^{\widetilde{Q}}$ is the value of Π_n replacing c, λ and $p(x)$ by $c^{\widetilde{Q}}$, $\lambda^{\widetilde{Q}}$ and $p^{\widetilde{Q}}(x)$ respectively.

Example 6.4.2 *Assume $S(0) = 100$, $H = 120$, $T = 1$, $r = 0.05$, $\sigma = 0.2$, and the jump sizes have Laplace transform $\widetilde{p}(s) = 0.9\frac{50}{50+s} + 0.1\left(\frac{50}{50+s}\right)^2$. We point out that this example is identical to Kou and Wang (2004, Section 4.3) except for the jump size density. Indeed, Kou and Wang (2004) assume a jump-diffusion process which allows for both positive and negative exponential jumps, whereas only one-sided jumps are considered in this paper. However, for comparative purposes, the mean of the jump size was preserved. Tables 6.3 and 6.4 contain the price for up-and-in call options with no more than n negative jumps*

Table 6.3: Prices of the up-and-in call option with no more than n jumps ($\lambda = 0.01$)

$K \setminus n$	0	1	2	3	4	5	6	B-S Price
80	16.7702	16.9191	16.9198	16.9198	16.9198	16.9198	16.9198	16.9182
90	12.9167	13.0303	13.0308	13.0308	13.0308	13.0308	13.0308	13.0296
100	9.1955	9.2752	9.2756	9.2756	9.2756	9.2756	9.2756	9.2746
110	5.8394	5.8891	5.8893	5.8893	5.8893	5.8893	5.8893	5.8885
120	3.2213	3.2481	3.2482	3.2482	3.2482	3.2482	3.2482	3.2477
130	1.6269	1.6400	1.6401	1.6401	1.6401	1.6401	1.6401	1.6396

when $\lambda = 0.01$ and $\lambda = 3$, respectively. Note that the drift c of the Brownian motion (as defined in (6.32)) is 0.0302 (0.0946) when $\lambda = 0.01$ ($\lambda = 3$).

Remark that the last column of Table 6.3 gives the price of up-and-in call options under the Black-Scholes model ($\lambda = 0$). We observe that the values of $C_n(T, H, K)$ for $\lambda = 0.01$ and n relatively large are very close to the Black-Scholes price of these up-and-in call options, as anticipated. Also, as the Poisson arrival rate λ of the jump-diffusion processes gets larger, the speed of convergence (in n) of $C_n(T, H, K)$ to $C(T, H, K)$ gets slower. Indeed, for a jump-diffusion process with a large value of λ, more jumps are expected (on average) within a given time horizon.

When the strike price is no less than the barrier level, the up-and-in call option becomes a regular call option. This can be formally proven as highlighted in the following remark.

Remark 6.4.3 *When $K \geq H$, (6.35) is consistent with the expression for regular call options. Indeed, for $y > 0$, by using Laplace transform arguments, it can be shown that the*

Table 6.4: Prices of the up-and-in call option with no more than n jumps ($\lambda = 3$)

$K \setminus n$	0	1	3	5	7	9	11	12
80	1.1824	4.3804	12.5362	16.4717	17.3013	17.3973	17.4043	17.4045
90	0.9338	3.4380	9.7395	12.7252	13.3417	13.4115	13.4164	13.4166
100	0.6906	2.5192	7.0325	9.1159	9.5341	9.5800	9.5831	9.5833
110	0.4642	1.6717	4.5727	5.8652	6.1150	6.1413	6.1430	6.1431
120	0.2774	0.9822	2.6175	3.3135	3.4418	3.4547	3.4555	3.4555
130	0.1530	0.5322	1.3803	1.7243	1.7847	1.7904	1.7908	1.7908

convolution of $g_j\,(y;t)$ and $f_k\,(t\,|b)$ satisfies

$$\sum_{k=0}^{j}\int_0^T g_{j-k}\,(y;T-t)\,f_k\,(t\,|b)\,dt = g_j(y+b;T),$$

which, substituted into (6.35), yields

$$\Pi_n = \sum_{j=0}^{n}\int_{\ln\frac{K}{H}}^{\infty} g_j(y+b;T)dy = \sum_{j=0}^{n}\int_{\ln\frac{K}{S(0)}}^{\infty} g_j(y;T)dy = P^Q(S(T)\geq K, N_T\leq n).$$

Chapter 7

Occupation times

7.1 Introduction

In this chapter, we examine the occupation time of an insurer's surplus process within an interval of the form $[a, b]$ for $a < b$. This includes some quantities which have already drawn some attention in the ruin theory community, notably the time spent by the surplus process in the negative half-plane, also referred to as the total duration of negative surplus or the *time in red*. In the context of some Sparre Andersen risk models, we propose to utilize renewal techniques and some probabilistic arguments to obtain an expression for the Laplace transform of the occupation time.

In finance, occupation times have been widely used as a contingent characteristic in the development of financial derivatives, such as step options (see, e.g., Linetsky (1999)) and corridor options (see, e.g., Fusai (2000)). Researchers have derived the Laplace transform of the occupation time for various stochastic processes. For instance, Linetsky (1999) and Davydov and Linetsky (2002) analyzed the Laplace transform of the single-barrier (intervals of the form $(-\infty, 0)$ or $(0, \infty)$) and double-barrier (intervals of the form $[a, b]$ for a, b finite) occupation time in a geometric Brownian motion model. Cai et al. (2010) generalized their results to the jump-diffusion process with double exponential jumps.

In risk theory, occupation times can be utilized as an enhanced risk management tool for insurers. Gerber (1990) pointed out that the recovery time of a ruin event can help insurers to determine whether to continue or terminate the business in the case of ruin. As a generalization of the recovery time, the duration of negative surplus can also be used as an alternative risk management tool for insurers to examine the health of the insurance business. On the other hand, by properly choosing the barrier level b, the occupation time in $[0, b]$ is an indicator of the time the insurer's surplus remains in critical levels, which also provides valuable risk management information for both the insurers and regulators.

In the context of the classical compound Poisson risk model, Egídio dos Reis (1993) and Dickson and Egídio dos Reis (1996) derived the distribution of the duration of negative surplus by analyzing the recovery time of ruin and the number of negative excursions. Similar studies have also been conducted in the dual compound Poisson risk model (see Song et al. (2008)), the compound Poisson risk model perturbed by diffusion (see Zhang and Wu (2002)) and the compound Poisson risk model with dependent premium structures (see Chiu and Yin (2002) and He et al. (2009)). More recently, Landriault et al. (2011) investigated the single-barrier occupation time in a spectrally negative Lévy process. In the two-barrier case, Kolkovska et al. (2005) studied the local time and occupation measure in the compound Poisson risk model. Loeffen et al. (2013) obtained the Laplace transform of occupation times of intervals until some first passage times for spectrally negative Lévy processes. However, as far as we know, the literature is rather scarce on the analysis of the duration of negative surplus and other occupation times in the Sparre Andersen risk model, which is a very common modeling assumption for an insurer's surplus process. One of the few exceptions is the recent paper by Dickson and Li (2013) who studied the duration of negative surplus in the Erlang-2 risk model.

The main subject matter of this chapter is to study the occupation time in a class of Sparre Andersen risk processes, where the interclaim times r.v.'s $\{T_i\}_{i=1}^{\infty}$ and the claim sizes $\{X_i\}_{i=1}^{\infty}$ form a sequence of i.i.d. r.v.'s. We also assume that the interclaim times $\{T_i\}_{i=1}^{\infty}$ and the claim sizes $\{X_i\}_{i=1}^{\infty}$ are mutually independent and such that $c\kappa > \mu$, which implies that in the long run the surplus process goes to infinity with probability 1.

In what follows, the interclaim times are assumed to be phase-type distributed (see, e.g., Bladt(2005)) with representation $(\boldsymbol{\alpha}, \mathbf{B})$, where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$ is the initial probability vector with $\sum_{i=1}^{n} \alpha_i = 1$ and the generator $\mathbf{B} = [b_{ij}]_{ij}$ is an $n \times n$ matrix with $b_{ii} < 0$, $b_{ij} \geq 0$ for $i \neq j$ and $\sum_{j=1}^{n} b_{ij} \leq 0$ for $i = 1, 2, ..., n$. Then the density function and Laplace transform are expressed as

$$k(t) = \boldsymbol{\alpha} e^{t\mathbf{B}} \mathbf{b}^{\top},$$

for $t \geq 0$ and

$$\widetilde{k}(s) = \boldsymbol{\alpha} (s\mathbf{I} - \mathbf{B})^{-1} \mathbf{b}^{\top}, \tag{7.1}$$

respectively, where $\mathbf{I}$ is the $n \times n$ identity matrix and $\mathbf{b}^{\top} = -\mathbf{B}\mathbf{e}^{\top}$ with $\mathbf{e}^{\top}$ a column vector of 1's.

In the double-barrier context, the occupation time of $\{U_t, t \geq 0\}$ in the interval $[a, b]$ $(a < b)$ can be written in an integral form as

$$\widetilde{T}_{a,b} = \int_0^{\infty} 1 (a \leq U_s \leq b) \, ds. \tag{7.2}$$

Without loss of generality, we assume $a = 0$. Define the Laplace transform of $\widetilde{T}_{0,b}$ as

$$V_{\delta}(u; b) \equiv E\left[e^{-\delta \widetilde{T}_{0,b}} | U_0 = u\right],$$

for $\delta \geq 0$. We also define the total duration of negative surplus by

$$\widetilde{T}_{-\infty} \equiv \int_0^\infty 1\left(U_s \leq 0\right) ds,$$

with Laplace transform

$$V_\delta\left(u\right) \equiv E\left[e^{-\delta \widetilde{T}_{-\infty}} | U_0 = u\right], \qquad u \geq 0.$$

In comparison to the Poisson claim arrival process and the general Lévy process, the challenge for analyzing $V_\delta\left(u\right)$ and $V_\delta\left(u; b\right)$ in the present context arises from the non-regenerative property of the renewal risk model at all times. However, this challenge can be overcome in the Sparre Andersen risk model with interclaim claims of distributional form (7.1) by introducing a continuous-time Markov process (CTMC) and tracking the state of the CTMC at some first passage times. Consider an $(n+1)$-state continuous time-homogeneous Markov process $\underline{J} = \{J_t, t \geq 0\}$ with $J_t = i$ $(i = 0, 1, 2, ..., n)$. We construct $\underline{J}$ as follows:

1. Let $\underline{Z} = \{Z_t, t \geq 0\}$ be an $(n+1)$-state terminating CTMC with $Z_t = i$ $(i = 0, 1, 2, ..., n)$ and initial probability vector $\boldsymbol{\alpha}$. We assume that state 0 is the absorbing state. Then, the interclaim times $\{T_i\}_{i=1}^\infty$ can be viewed as the time until absorption of this terminating CTMC. Claim instants only occur when state 0 is reached. The infinitesimal generator of the CTMC is given by

$$G = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{b}^\top & \mathbf{B} \end{bmatrix}.$$

2. The process $\underline{J}$ is constructed by pasting sample paths of the terminating CTMC $\underline{Z}$ up to the absorption time. That is, whenever state 0 is reached, the process $\underline{J}$ immediately restarts as $\underline{Z}$ with initial probability vector $\boldsymbol{\alpha}$.

Or equivalently, the process $\underline{J}$ (excluding the absorbing state 0) is a CTMC with the infinitesimal generator

$$G_J = \mathbf{B} + \mathbf{b}^{\top}\boldsymbol{\alpha}.$$

In this setting, the bivariate process $\{(U_t, J_t), t \geq 0\}$ is a Markov process.

Let

$$V_{\delta,i}(u) \equiv E\left[e^{-\delta \widetilde{T}_{-\infty}} | U_0 = u, J_0 = i\right],$$

for $i = 1, 2, ..., n$, be the Laplace transform of the duration of negative surplus for the surplus process $\{U_t, t \geq 0\}$ with an initial state $J_0 = i$. Its unconditional version is given by

$$V_{\delta}(u) = \sum_{i=1}^{n} \alpha_i V_{\delta,i}(u) = \boldsymbol{\alpha}\mathbf{V}_{\delta}(u), \tag{7.3}$$

where $\mathbf{V}_{\delta}(u) = (V_{\delta,1}(u), V_{\delta,2}(u), ..., V_{\delta,n}(u))^{\top}$ and the initial probability vector of J_0 is $\boldsymbol{\alpha}$. Therefore, the main task is to find an expression for $\mathbf{V}_{\delta}(u)$.

Similarly, the Laplace transform of $\widetilde{T}_{0,b}$ satisfies

$$V_{\delta}(u; b) = \boldsymbol{\alpha}\mathbf{V}_{\delta}(u; b), \tag{7.4}$$

where $\mathbf{V}_{\delta}(u; b) = (V_{\delta,1}(u; b), V_{\delta,2}(u; b), ..., V_{\delta,n}(u; b))^{\top}$ and

$$V_{\delta,i}(u; b) \equiv E\left[e^{-\delta \widetilde{T}_{0,b}} | U_0 = u, J_0 = i\right],$$

for $i = 1, 2, ..., n$.

The remaining sections of this chapter are structured as follows: in Section 7.2, we revisit the one-sided and two-sided exit problems for the aforementioned class of Sparre Andersen risk models. These quantities are essential components to develop the Laplace transform of the one-barrier and two-barrier occupation times in the later sections. In Section 7.3, the Laplace transform of the duration of negative surplus is derived. The results are natural extensions of those obtained by Egídio dos Reis (1993). In Section 7.4, the Laplace transform of the occupation time in $[0, b]$ of the surplus process $\{U_t, t \geq 0\}$ is obtained. As a special case, we revisit the occupation time in the classical compound Poisson risk model and further examine some results of Kolkovska et al. (2005). An example to calculate the mean occupation time will also be considered.

7.2 The one-sided and two-sided exit problem

In this section, we examine the time to ruin, the time to reach a certain level b, and the two-sided exit times for the process $\{U_t, t \geq 0\}$. Most results are obtained in vector or matrix form given that these first passage times are analyzed jointly with the initial and ending states of the process $\{J_t, t \geq 0\}$. Many of these exit times for phase-type interclaim times have been extensively analyzed in the literature (see, e.g., Albrecher and Boxma (2005), Ren (2007) and Li (2008b)) and will be reviewed here for completeness purposes. Some results will also be derived to accommodate the occupation time analysis in the later sections.

7.2.1 The time to ruin and other first passage times

As mentioned in Chapter 1, a systematic approach to analyze the time to ruin τ is proposed by Gerber and Shiu (1998) through the discounted penalty function (1.3). It is well known that in a Sparre Andersen risk model, the Gerber-Shiu function $m_\delta(u)$ satisfies a defective renewal equation and can generally be expressed in terms of the solution(s) of *Lundberg's generalized equation*

$$\widetilde{k}\,(\delta - cs)\,\widetilde{p}\,(s) = 1, \tag{7.5}$$

(see, e.g., Gerber and Shiu (2005) and Landriault and Willmot (2008)). For a phase-type interclaim times, Eq. (7.5) is known to have n solutions $\{\rho_k\}_{k=1}^n$ with non-negative real parts (see Ren (2007)). We assume that $\rho_i \neq \rho_j$ for $i \neq j$.

First, we define the Gerber-Shiu function conditional on the initial state $J_0 = i$ by

$$m_{\delta,i}(u) \equiv E\left[e^{-\delta\tau} w(U_{\tau^-}, |U_\tau|) 1\left(\tau < \infty\right) | U_0 = u, J_0 = i\right], \tag{7.6}$$

and aim to obtain an expression for $\mathbf{m}_\delta\left(u\right) = \left(m_{\delta,1}\left(u\right), m_{\delta,2}\left(u\right), ..., m_{\delta,n}\left(u\right)\right)^\top$. Note that when $w(U_{\tau^-}, |U_\tau|) = 1$ in (7.6), the Gerber-Shiu function becomes the Laplace transform of the time to ruin, i.e.

$$\phi_{\delta,i}(u) \equiv E\left[e^{-\delta\tau} 1\left(\tau < \infty\right) | U_0 = u, J_0 = i\right], \tag{7.7}$$

for $i = 1, 2, ..., n$ with vector form $\mathbf{\Phi}_\delta\left(u\right) = \left(\phi_{\delta,1}\left(u\right), \phi_{\delta,2}\left(u\right), ..., \phi_{\delta,n}\left(u\right)\right)^\top$.

Clearly, the (unconditional) Gerber-Shiu function $m_\delta(u)$ is given by

$$m_\delta(u) = \sum_{i=1}^n \alpha_i m_{\delta,i}(u) \equiv \boldsymbol{\alpha}\mathbf{m}_\delta\left(u\right). \tag{7.8}$$

It was shown by Schmidli (2005) that $\mathbf{m}_\delta\left(u\right)$ satisfies the following integro-differential equation

$$c\mathbf{m}_\delta'\left(u\right) = \delta\mathbf{m}_\delta\left(u\right) - \mathbf{B}\mathbf{m}_\delta\left(u\right) - E\left[m_\delta(u - X)\right]\mathbf{b}^\top, \tag{7.9}$$

where

$$E\left[m_\delta(y - X)\right] = \int_0^y \boldsymbol{\alpha}\mathbf{m}_\delta\left(y - x\right) p(x)dx + \omega(y),$$

and

$$\omega(y) = \int_y^\infty w(y, x - y)p(x)dx,$$

(see also Albrecher and Boxma (2005)). By taking the Laplace transform of (7.9), Ren (2007) showed that

$$\mathbf{L}_\delta\left(s\right)\widetilde{\mathbf{m}}_\delta\left(s\right) = c\mathbf{m}_\delta\left(0\right) - \widetilde{\omega}(s)\mathbf{b}^\top, \tag{7.10}$$

where

$$\mathbf{L}_\delta(s) = (cs-\delta)\,\mathbf{I}+\mathbf{B}+\mathbf{b}^\top\boldsymbol{\alpha}\widetilde{p}(s).$$

Note that $\det\mathbf{L}_\delta(s)=0$ is an equivalent representation of Lundberg's generalized equation (7.5) and thus, has n solutions $\{\rho_k\}_{k=1}^n$. Moreover, Ren (2007) demonstrated that $\mathbf{m}_\delta(0)$ can be expressed as

$$\mathbf{m}_\delta(0)=\frac{1}{c}\mathbf{V}^{-1}\widetilde{\omega}(\boldsymbol{\rho})\mathbf{V}\mathbf{b}^\top=\frac{1}{c}\widetilde{\omega}(\mathbf{Q})\mathbf{b}^\top,$$

where $\mathbf{V}=\left(\mathbf{v}_1^\top,\mathbf{v}_2^\top,...,\mathbf{v}_n^\top\right)^\top$ with the row vector $\mathbf{v}_i=\boldsymbol{\alpha}\left((\delta-c\rho_i)\,\mathbf{I}-\mathbf{B}\right)^{-1}$ satisfying $\mathbf{v}_i\mathbf{L}_\delta(\rho_i)=\mathbf{0}$ for $i=1,2,...,n$, $\boldsymbol{\rho}=diag\,(\rho_1,\rho_2,...,\rho_n)$ and $\mathbf{Q}=\mathbf{V}^{-1}\boldsymbol{\rho}\mathbf{V}$. It is worthy to mention that $\mathbf{v}_i$ is the left eigenvector corresponding to the eigenvalue ρ_i.

Next, for $b\ge u$, let $\tau_b=\inf\{t\ge 0: U_t=b\}$ be the first passage time of the surplus process to level b. Note that τ_b is equivalent to the time of ruin in the dual risk model (see, e.g., Grandell (1991, p.8)). We are interested in the Laplace transform of τ_b with an initial state $J_0=i$ and a hitting state $J_{\tau_b}=j$, which is defined as

$$R_{\delta,ij}(u;b)\equiv E\left[e^{-\delta\tau_b}1\left(J_{\tau_b}=j\right)|U_0=u,J_0=i\right],$$

with matrix representation $\mathbf{R}_\delta(u;b)=\left[R_{\delta,ij}(u;b)\right]_{ij}$.

By the skip-free upward property of the surplus process, one immediately notices that

$$\mathbf{R}_\delta(u_1;b)=\mathbf{R}_\delta(u_1;u_2)\,\mathbf{R}_\delta(u_2;b) \tag{7.11}$$

for any $0\le u_1<u_2\le b$ with boundary condition $\mathbf{R}_\delta(b;b)=\mathbf{I}$. Thus, the solution to (7.11) can be expressed as

$$\mathbf{R}_\delta(u;b)=e^{\mathbf{K}(u-b)}, \tag{7.12}$$

where $\mathbf{K}$ is an $n\times n$ matrix to be determined.

It was shown by Li (2008a) that $\mathbf{K} = \mathbf{H}\boldsymbol{\rho}\mathbf{H}^{-1}$, where $\mathbf{H} = (\mathbf{h}_1, \mathbf{h}_2, ..., \mathbf{h}_n)$ with the column vector $\mathbf{h}_i$ satisfying $\mathbf{L}_\delta(\rho_i)\mathbf{h}_i = \mathbf{0}$ for $i = 1, 2, ..., n$. Note that $\mathbf{h}_i$ is the right eigenvector corresponding to the eigenvalue ρ_i. We also remark that another possible expression for $\mathbf{h}_i$ is given by

$$\mathbf{h}_i = ((\delta - c\rho_i)\mathbf{I} - \mathbf{B})^{-1}\mathbf{b}^\top\boldsymbol{\alpha}$$

Indeed, using (7.1) and (7.5), we have

$$\begin{aligned}\mathbf{L}_\delta(\rho_i)\mathbf{h}_i &= \left\{(c\rho_i - \delta)\mathbf{I} + \mathbf{B} + \mathbf{b}^\top\boldsymbol{\alpha}\widetilde{p}(\rho_i)\right\}((\delta - c\rho_i)\mathbf{I} - \mathbf{B})^{-1}\mathbf{b}^\top\boldsymbol{\alpha} \\ &= -\mathbf{b}^\top\boldsymbol{\alpha} + \mathbf{b}^\top\left(\boldsymbol{\alpha}((\delta - c\rho_i)\mathbf{I} - \mathbf{B})^{-1}\mathbf{b}^\top\widetilde{p}(\rho_i)\right)\boldsymbol{\alpha} \\ &= \mathbf{0}.\end{aligned}$$

It follows that

$$\mathbf{R}_\delta(u; b) = \mathbf{H}e^{\boldsymbol{\rho}(u-b)}\mathbf{H}^{-\mathbf{1}}. \tag{7.13}$$

7.2.2 First passage times in the two-sided exit problem

Finally, we will analyze the Laplace transform of the two-sided exit time in the interval $[0, b]$ of the surplus process $\{U_t, t \geq 0\}$. Define the Laplace transform of the first passage time to reach level b before ruin occurs, with an initial state $J_0 = i$ and a hitting state $J_{\tau_b} = j$, as

$$\varphi_{\delta,ij}(u; b) \equiv E\left[e^{-\delta\tau_b}1(\tau_b < \tau, J_{\tau_b} = j)\,|U_0 = u, J_0 = i\right],$$

for $i, j = 1, 2, ..., n$ with matrix representation $\boldsymbol{\Psi}_\delta(u; b) = [\varphi_{\delta,ij}(u; b)]_{ij}$. Using the same arguments as in Ko (2007), it is not difficult to show that $\boldsymbol{\Psi}_\delta(u; b)$ satisfies

$$c\boldsymbol{\Psi}'_\delta(u; b) = (\delta\mathbf{I} - \mathbf{B})\boldsymbol{\Psi}_\delta(u; b) - \mathbf{b}^\top\boldsymbol{\alpha}\int_0^u \boldsymbol{\Psi}_\delta(y; b)\,p(u - y)dy, \tag{7.14}$$

with boundary condition $\boldsymbol{\Psi}_\delta(b;b) = \mathbf{I}$. Taking the Laplace transform of (7.14) wrt u, one arrives at

$$c\left(s\widetilde{\boldsymbol{\Psi}}_\delta(s;b) - \boldsymbol{\Psi}_\delta(0;b)\right) = (\delta\mathbf{I} - \mathbf{B})\,\widetilde{\boldsymbol{\Psi}}_\delta(s;b) - \mathbf{b}^\top\boldsymbol{\alpha}\widetilde{\boldsymbol{\Psi}}_\delta(s;b)\,\widetilde{p}(s)\,. \tag{7.15}$$

Rearranging the terms of (7.15) yields

$$\mathbf{L}_\delta(s)\,\widetilde{\boldsymbol{\Psi}}_\delta(s;b) = c\boldsymbol{\Psi}_\delta(0;b)\,.$$

Define $\mathbf{v}_\delta(x)$ through its Laplace transform as

$$\mathbf{L}_\delta^{-1}(s) = \widetilde{\mathbf{v}}_\delta(s) = \int_0^\infty e^{-sx}\mathbf{v}_\delta(x)\,dx.$$

It follows from the boundary condition that

$$\boldsymbol{\Psi}_\delta(u;b) = \mathbf{v}_\delta(u)\mathbf{v}_\delta^{-1}(b). \tag{7.16}$$

Next, we define a generalized Gerber-Shiu function $m_{\delta,i}(u;b)$ for the surplus process $\{U_t, t \geq 0\}$ killed by reaching level b, i.e.

$$m_{\delta,i}(u;b) \equiv E\left[e^{-\delta\tau}w(U_{\tau-}, |U_\tau|)1(\tau < \tau_b)\,|U_0 = u, J_0 = i\right],$$

with vector form $\mathbf{m}_\delta(u;b) = (m_{\delta,1}(u;b), m_{\delta,2}(u;b), ..., m_{\delta,n}(u;b))^\top$. In the limit, $m_{\delta,i}(u) = \lim_{b\to\infty} m_{\delta,i}(u;b)$. Furthermore, if $w(U_{\tau-}, |U_\tau|) = 1$, $m_{\delta,i}(u;b)$ becomes the Laplace transform of the exit time from level 0 before hitting level b with an initial state $J_0 = i$, i.e.

$$\xi_{\delta,i}(u;b) \equiv E\left[e^{-\delta\tau}1(\tau < \tau_b)\,|U_0 = u, J_0 = i\right],$$

with vector representation $\boldsymbol{\xi}_\delta(u;b) = (\xi_{\delta,1}(u;b), \xi_{\delta,2}(u;b), ..., \xi_{\delta,n}(u;b))^\top$.

By excluding the paths of the surplus process hitting level b before ruin, we have

$$\mathbf{m}_\delta(u;b) = \mathbf{m}_\delta(u) - \boldsymbol{\Psi}_\delta(u;b)\,\mathbf{m}_\delta(b)\,. \tag{7.17}$$

Eq. (7.17) implies that $\mathbf{m}_\delta(u;b)$ is fully characterized by $\boldsymbol{\Psi}_\delta(u;b)$ given in (7.16) and $\mathbf{m}_\delta(u)$ whose Laplace transform was obtained in (7.10).

Remark 7.2.1 *In the classical compound Poisson risk model with exponential interclaim times of mean* $1/\lambda$, $\mathbf{v}_\delta(x)$ *reduces to the* δ*-scale function* $v_{1,\delta}(x)$ *whose Laplace transform is given by (3.27). Using similar arguments as in Section 3.3.1 for inversion yields*

$$v_{1,\delta}(x) = \frac{1}{c}e^{\frac{\lambda+\delta}{c}x} + \int_0^{x/c}\left(\sum_{n=1}^{\infty}\frac{(-\lambda t)^n e^{(\lambda+\delta)t}}{n!}p^{*n}(x-ct)\right)dt.$$

In this case, the Laplace transform of the first passage time to level b *before ruin occurs is given by*

$$\varphi_\delta(u;b) \equiv E\left[e^{-\delta\tau_b}1(\tau_b<\tau)\,|U_0=u\right] = \frac{v_{1,\delta}(u)}{v_{1,\delta}(b)}, \tag{7.18}$$

and the Laplace transform of the exit time from level 0 *before hitting level* b *is*

$$\xi_\delta(u;b) \equiv E\left[e^{-\delta\tau}1(\tau<\tau_b)\,|U_0=u\right] = \phi_\delta(u) - \phi_\delta(b)\frac{v_{1,\delta}(u)}{v_{1,\delta}(b)}, \tag{7.19}$$

where $\phi_\delta(u)$ *is the Laplace transform of the time to ruin (see, e.g., Kyprianou (2006, Theorem 8.1)).*

Remark 7.2.2 *An alternative expression of* $\boldsymbol{\Psi}_\delta(u;b)$ *is obtained by using similar arguments as in Gerber and Shiu (1998) and Li (2008b). Similar to Eqs. (6.22 and 6.23) in Gerber and Shiu (1998), we have*

$$\boldsymbol{\Phi}_\delta(u) = \boldsymbol{\xi}_\delta(u;b) + \boldsymbol{\Psi}_\delta(u;b)\,\boldsymbol{\Phi}_\delta(b), \tag{7.20}$$

and

$$\mathbf{R}_\delta(u;b) = \boldsymbol{\xi}_\delta(u;b)\,\mathbf{R}_\delta(0;b) + \boldsymbol{\Psi}_\delta(u;b). \tag{7.21}$$

Solving Eqs. (7.20) and (7.21) yields

$$\begin{aligned}\boldsymbol{\Psi}_\delta(u;b) &= \left[\mathbf{R}_\delta(u;b) - \boldsymbol{\Phi}_\delta(u)\,\mathbf{R}_\delta(0;b)\right]\left[\mathbf{I} - \boldsymbol{\Phi}_\delta(b)\,\mathbf{R}_\delta(0;b)\right]^{-1}\\ &= \left[e^{\mathbf{K}u} - \boldsymbol{\Phi}_\delta(u)\right]\left[e^{\mathbf{K}b} - \boldsymbol{\Phi}_\delta(b)\right]^{-1}.\end{aligned}$$

7.3 The duration of negative surplus

7.3.1 Laplace transform

Equipped with the intermediary results of Section 7.2, we are now in position to derive an expression for the Laplace transform of the duration of negative surplus. Let

$$\psi_i(u) = \Pr\left(\tau < \infty \left| U_0 = u, J_0 = i \right.\right),$$

and define $\mathbf{g}_d(y|u) = \left(\psi_1(u)g_1\left(y\left|u\right.\right), \psi_2(u)g_2\left(y\left|u\right.\right), ..., \psi_n(u)g_n\left(y\left|u\right.\right)\right)^\top$, where $g_i\left(y\left|u\right.\right)$ is the proper density of the deficit at ruin (given that ruin occurs) for an initial surplus level u with an initial state $J_0 = i$.

By conditioning on whether ruin occurs and on the time to recovery if it does, one finds

$$\begin{aligned}\mathbf{V}_\delta\left(u\right) &= \boldsymbol{\Upsilon}(u) + \int_0^\infty \mathbf{g}_d(y|u)\left\{\boldsymbol{\alpha}\mathbf{R}_\delta\left(-y;0\right)\mathbf{V}_\delta\left(0\right)\right\}dy \\ &= \boldsymbol{\Upsilon}(u) + \mathbf{A}_\delta(u)\mathbf{V}_\delta\left(0\right),\end{aligned} \tag{7.22}$$

where

$$\boldsymbol{\Upsilon}(u) = (1-\psi_1(u), 1-\psi_2(u), ..., 1-\psi_n(u))^\top,$$

and

$$\mathbf{A}_\delta(u) = \int_0^\infty \mathbf{g}_d(y|u)\left\{\boldsymbol{\alpha}\mathbf{R}_\delta\left(0;y\right)\right\}dy. \tag{7.23}$$

Substituting (7.13) into (7.23), we have

$$\begin{aligned}\mathbf{A}_\delta(u) &= \int_0^\infty \mathbf{g}_d(y|u)\left\{\boldsymbol{\alpha}\mathbf{H}e^{-\boldsymbol{\rho}y}\mathbf{H}^{-\mathbf{1}}\right\}dy \\ &= \left(\int_0^\infty \mathbf{g}_d(y|u)\left(\boldsymbol{\alpha}\mathbf{h}_1e^{-\rho_1 y}, \boldsymbol{\alpha}\mathbf{h}_2e^{-\rho_2 y}, ..., \boldsymbol{\alpha}\mathbf{h}_ne^{-\rho_n y}\right)dy\right)\mathbf{H}^{-\mathbf{1}} \\ &= \left(\boldsymbol{\alpha}\mathbf{h}_1\mathbf{m}_0\left(u,\rho_1\right), \boldsymbol{\alpha}\mathbf{h}_2\mathbf{m}_0\left(u,\rho_2\right), ..., \boldsymbol{\alpha}\mathbf{h}_n\mathbf{m}_0\left(u,\rho_n\right)\right)\mathbf{H}^{-\mathbf{1}},\end{aligned}$$

where $\mathbf{m}_0(u,z) = (m_{0,1}(u,z), m_{0,2}(u,z), ..., m_{0,n}(u,z))^\top$ with

$$m_{0,i}(u,z) \equiv E\left[e^{-z|U_\tau|}1(\tau<\infty)\,|U_0=u, J_0=i\right].$$

Note that $m_{0,i}(u,z)$ is the Laplace transform of the deficit at ruin (i.e., a special case of the Gerber-Shiu function (7.6) with $\delta=0$ and $w(x,y)=e^{-zy}$).

To solve (7.22), letting $u=0$, one obtains

$$\mathbf{V}_\delta(0) = \mathbf{A}_\delta(0)\mathbf{V}_\delta(0) + \boldsymbol{\Upsilon}(0),$$

which implies that

$$\mathbf{V}_\delta(0) = (\mathbf{I}-\mathbf{A}_\delta(0))^{-1}\boldsymbol{\Upsilon}(0). \tag{7.24}$$

Substituting (7.24) into (7.22), one arrives at

$$\mathbf{V}_\delta(u) = \boldsymbol{\Upsilon}(u) + \mathbf{A}_\delta(u)(\mathbf{I}-\mathbf{A}_\delta(0))^{-1}\boldsymbol{\Upsilon}(0). \tag{7.25}$$

Multiplying both sides of (7.25) by $\boldsymbol{\alpha}$, one concludes that the Laplace transform of the duration of negative surplus for the phase-type interclaim times is given by

$$V_\delta(u) = \boldsymbol{\alpha}\left(\boldsymbol{\Upsilon}(u) + \mathbf{A}_\delta(u)(\mathbf{I}-\mathbf{A}_\delta(0))^{-1}\boldsymbol{\Upsilon}(0)\right). \tag{7.26}$$

Remark 7.3.1 *In the classical compound Poisson risk model, Eq. (7.26) becomes*

$$V_\delta(u) = 1-\psi(u) + \frac{a_\delta(u)(1-\psi(0))}{1-a_\delta(0)}, \tag{7.27}$$

where $\psi(u)$ is the ruin probability with an initial surplus level u, and $a_\delta(u)$ satisfies

$$a_\delta(u) = \psi(u)\int_0^\infty e^{-\rho_0 y} g(y\,|u)\,dy.$$

Here, ρ_0 is the unique non-negative solution of Lundberg's fundamental equation (7.5) and $g(y\,|u)$ is the proper density of the deficit at ruin. That is, $g(y\,|u)$ is given by

$$g(y\,|u) = \frac{\lambda}{c\psi(u)}\int_0^\infty \alpha_0(x,u)\,p(x+y)\,dx,$$

for $y \geq 0$ and

$$\alpha_0(x,u) = \begin{cases} \frac{\psi(u-x)-\psi(u)}{1-\psi(0)}, & x \leq u, \\ \frac{1-\psi(u)}{1-\psi(0)}, & x > u, \end{cases}$$

(see, e.g., Landriault and Willmot (2009)). When $u = 0$, $g(y|0) = \frac{1}{\mu}\overline{P}(y)$. Eq. (7.27) is consistent with Egídio dos Reis (1993, Section 5.2.2).

7.3.2 Number of claims with negative surplus level

Let $N_{\widetilde{T}_{-\infty}}$ be the number of claims with a negative surplus value following the claim settlement, i.e.

$$N_{\widetilde{T}_{-\infty}} \equiv \sum_{k=1}^{\infty} 1\left(U_{V_k} < 0\right),$$

where $V_k = \sum_{i=1}^{k} T_i$ is the occurring time of the kth claim ($k = 1, 2, ...$). Note that the positive security loading ensures that $N_{\widetilde{T}_{-\infty}}$ is finite with probability 1. In what follows, we jointly analyze $\widetilde{T}_{-\infty}$ and $N_{\widetilde{T}_{-\infty}}$ in the classical compound Poisson risk model through the generalized analytic tool

$$V_{r,\delta}(u) \equiv E\left[r^{N_{\widetilde{T}_{-\infty}}} e^{-\delta \widetilde{T}_{-\infty}} | U_0 = u\right],$$

for $u, \delta \geq 0$ and $r \in (0, 1]$. For a given deficit level $-y$ ($y > 0$), using the same arguments as in Chapter 6, it can be shown that the joint Laplace transform of the first passage time to reach level 0 and the number of claims before the first passage is given by

$$R_{r,\delta}(y) = e^{-\rho y}, \tag{7.28}$$

where ρ is the unique non-negative solution of

$$s - \frac{\lambda + \delta}{c} + r\frac{\lambda}{c}\widetilde{p}(s) = 0.$$

The inversion of (7.28) yields

$$e^{-\rho y}=e^{-\frac{\lambda+\delta}{c}y}+\sum_{n=1}^{\infty}r^{n}\int_{\frac{y}{c}}^{\infty}e^{-\delta t}\varkappa_{n}\left(t\left|y\right.\right)dt, \tag{7.29}$$

where

$$\varkappa_{n}\left(t\left|y\right.\right)=\frac{y}{t}\frac{\left(\lambda t\right)^{n}e^{-\lambda t}}{n!}p^{*n}\left(ct-y\right),$$

(see Remark 6.2.3).

By conditioning on whether ruin occurs and on the time to recovery if it does, it follows that

$$V_{r,\delta}\left(u\right)=1-\psi(u)+r\psi(u)\int_{0}^{\infty}g\left(y\left|u\right.\right)\left\{R_{r,\delta}\left(y\right)V_{r,\delta}\left(0\right)\right\}dy. \tag{7.30}$$

Substituting (7.28) into (7.30) followed by some simple manipulations, one arrives at

$$V_{r,\delta}\left(u\right)=1-\psi(u)+\frac{ra_{r,\delta}(u)\left(1-\psi(0)\right)}{1-ra_{r,\delta}(0)}, \tag{7.31}$$

where

$$a_{r,\delta}(u)=\psi(u)\int_{0}^{\infty}e^{-\rho y}g\left(y\left|u\right.\right)dy.$$

To invert $V_{r,\delta}\left(u\right)$, we rewrite (7.31) as

$$\begin{aligned}V_{r,\delta}\left(u\right)&=1-\psi(u)+\left(1-\psi(0)\right)\sum_{m=0}^{\infty}r^{m+1}a_{r,\delta}(u)\left(a_{r,\delta}(0)\right)^{m}\\&=1-\psi(u)+\left(1-\psi(0)\right)\psi(u)r\int_{0}^{\infty}e^{-\rho y}g\left(y\left|u\right.\right)dy\\&+\psi(u)\left(1-\psi(0)\right)\sum_{m=1}^{\infty}r^{m+1}\left(\psi(0)\right)^{m}\left(\int_{0}^{\infty}e^{-\rho y}g\left(y\left|u\right.\right)dy\right)\left(\int_{0}^{\infty}e^{-\rho y}g^{*m}\left(y\left|0\right.\right)dy\right)\\&=1-\psi(u)+\psi(u)\sum_{m=1}^{\infty}r^{m}\left(1-\psi(0)\right)\left(\psi(0)\right)^{m-1}\int_{0}^{\infty}e^{-\rho y}g_{m}\left(y\left|u\right.\right)dy,\end{aligned} \tag{7.32}$$

where

$$g_{m}\left(y\left|u\right.\right)=\begin{cases}g\left(y\left|u\right.\right), & m=1,\\ \int_{0}^{y}g\left(y-x\left|u\right.\right)g^{*(m-1)}\left(x\left|0\right.\right)dx, & m=2,3,...\end{cases}$$

Finally, substituting the Lagrangian identity (7.29) into (7.32), one finds that

$$
\begin{aligned}
V_{r,\delta}(u) &= 1-\psi(u) \\
&\quad + \psi(u)\sum_{m=1}^{\infty} r^{m}\left(1-\psi(0)\right)\left(\psi(0)\right)^{m-1}\int_{0}^{\infty} e^{-\frac{\lambda+\delta}{c}y} g_{m}\left(y\,|u\right)dy \\
&\quad + \psi(u)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} r^{n+m}\left(1-\psi(0)\right)\left(\psi(0)\right)^{m-1}\int_{0}^{\infty} e^{-\delta t}\left\{\int_{0}^{ct}\varkappa_{n}\left(t\,|y\right)g_{m}\left(y\,|u\right)dy\right\}dt \\
&= 1-\psi(u) \\
&\quad + \sum_{n=1}^{\infty} r^{n}\int_{0}^{\infty} e^{-\delta t}\left\{\psi(u)\left(1-\psi(0)\right)\left(\psi(0)\right)^{n-1} ce^{-\lambda t} g_{n}\left(ct\,|u\right)\right\}dt \\
&\quad + \sum_{n=2}^{\infty} r^{n}\int_{0}^{\infty} e^{-\delta t}\left\{\psi(u)\sum_{m=1}^{n-1}\left(1-\psi(0)\right)\left(\psi(0)\right)^{m-1}\int_{0}^{ct}\varkappa_{n-m}\left(t\,|y\right)g_{m}\left(y\,|u\right)dy\right\}dt.
\end{aligned}
$$

In conclusion, if there is no excursion below 0, the total duration of negative surplus has a mass point at 0 with

$$\Pr\left(\widetilde{T}_{-\infty}=0, N_{\widetilde{T}_{-\infty}}=0\middle|\, U_0=u\right)=1-\psi\left(u\right).$$

Otherwise, the joint generalized density of $\widetilde{T}_{-\infty}$ and $N_{\widetilde{T}_{-\infty}}$ is given by

$$h(t,n\,|u)=\begin{cases}\psi(u)\left(1-\psi(0)\right)ce^{-\lambda t}g\left(ct\,|u\right), & n=1,\\ z(t,n\,|u), & n=2,3,...,\end{cases} \tag{7.33}$$

for $t>0$, where

$$
\begin{aligned}
z(t,n\,|u) &= \psi(u)\left(1-\psi(0)\right)\left(\psi(0)\right)^{n-1} ce^{-\lambda t} g_{n}\left(ct\,|u\right) \\
&\quad + \psi(u)\sum_{m=1}^{n-1}\left(1-\psi(0)\right)\left(\psi(0)\right)^{m-1}\int_{0}^{ct}\varkappa_{n-m}\left(t\,|y\right)g_{m}\left(y\,|u\right)dy.
\end{aligned}
$$

Remark 7.3.2 *As an immediate corollary, the total duration of negative surplus has a mass point at* 0 *with*

$$\Pr\left(\widetilde{T}_{-\infty}=0\middle|\, U_0=u\right)=1-\psi(u),$$

and for $t > 0$, the marginal density of the total duration of negative surplus can be expressed as

$$\begin{aligned} h(t\,|u) &= \psi(u)\,(1-\psi(0))\,ce^{-\lambda t}g\,(ct\,|u) + \sum_{n=2}^{\infty} z(t,n\,|u) \\ &= \psi(u)\sum_{n=1}^{\infty}(1-\psi(0))\,(\psi(0))^{n-1}\left\{ce^{-\lambda t}g_n\,(ct\,|u) + \int_0^{ct}\left(\sum_{m=1}^{\infty}\varkappa_m\,(t\,|y)\right)g_n\,(y\,|u)\,dy\right\}. \end{aligned}$$

This expression is an explicit solution to the recursive equation (3.1) in Dickson and Egídio dos Reis (1996).

7.4 Occupation time in $[0, b]$

7.4.1 Laplace transform

In this section, we analyze the double-barrier occupation time for the assumed Sparre Andersen risk model. For notational convenience, we write

$$V_\delta\,(u;b) = \begin{cases} V_\delta^1\,(u;b)\,, & u < 0, \\ V_\delta^2\,(u;b)\,, & 0 \le u < b, \\ V_\delta^3\,(u;b)\,, & u \ge b, \end{cases}$$

and

$$V_{\delta,i}\,(u;b) = \begin{cases} V_{\delta,i}^1\,(u;b)\,, & u < 0, \\ V_{\delta,i}^2\,(u;b)\,, & 0 \le u < b, \\ V_{\delta,i}^3\,(u;b)\,, & u \ge b. \end{cases}$$

From (7.4), we have

$$V_\delta^k\,(u;b) = \boldsymbol{\alpha}\mathbf{V}_\delta^k\,(u;b)\,, \tag{7.34}$$

where $\mathbf{V}_\delta^k\,(u;b) = (V_{\delta,1}^k\,(u;b)\,, V_{\delta,2}^k\,(u;b)\,, ..., V_{\delta,n}^k\,(u;b))^\top$, for $k = 1, 2, 3$.

For $u < 0$, the skip-free upward property of the surplus process leads to

$$V_{\delta}^{1}(u;b) = \boldsymbol{\alpha}\mathbf{R}_0(u;0)\,\mathbf{V}_{\delta}^{2}(0;b), \tag{7.35}$$

where $\mathbf{R}_0(u;0)$ is given by (7.13).

For $0 \le u < b$, conditioning on whether the surplus process exits the interval $[0,b]$ from the lower or upper boundary, one deduces

$$\mathbf{V}_{\delta}^{2}(u;b) = \boldsymbol{\Psi}_{\delta}(u;b)\,\mathbf{V}_{\delta}^{3}(b;b) + \mathbf{m}_{\delta}^{*}(u;b)\,\mathbf{V}_{\delta}^{2}(0;b), \tag{7.36}$$

where $\mathbf{m}_{\delta}^{*}(u;b) = \left(\left(\mathbf{m}_{\delta,1}^{\mathbf{R}}(u;b)\right)^{\top}, \left(\mathbf{m}_{\delta,2}^{\mathbf{R}}(u;b)\right)^{\top}, ..., \left(\mathbf{m}_{\delta,3}^{\mathbf{R}}(u;b)\right)^{\top}\right)^{\top}$ is an $n \times n$ matrix with the row vector $\mathbf{m}_{\delta,i}^{\mathbf{R}}(u;b)$ defined as

$$\mathbf{m}_{\delta,i}^{\mathbf{R}}(u;b) \equiv E\left[e^{-\delta\tau}\boldsymbol{\alpha}\mathbf{R}_0(U_{\tau};0)\,1(\tau < \tau_b)\,|U_0 = u, J_0 = i\right],$$

for $i = 1, 2, ..., n$. The expression for the elements in $\mathbf{m}_{\delta,i}^{\mathbf{R}}(u;b)$ can be obtained using Eq. (7.17). In particular, letting $u = 0$, one finds

$$\mathbf{V}_{\delta}^{3}(b;b) = \boldsymbol{\Psi}_{\delta}^{-1}(0;b)\left(\mathbf{I} - \mathbf{m}_{\delta}^{*}(0;b)\right)\mathbf{V}_{\delta}^{2}(0;b). \tag{7.37}$$

Also, for $u \ge b$, conditioning on the first drop below level b, one arrives at

$$\mathbf{V}_{\delta}^{3}(u;b) = \boldsymbol{\Upsilon}(u-b) + \int_0^b \mathbf{g}_d(y|u-b)V_{\delta}^{2}(b-y;b)\,dy + \int_b^{\infty} \mathbf{g}_d(y|u-b)V_{\delta}^{1}(b-y;b)\,dy. \tag{7.38}$$

Define a row vector $\mathbf{w}_{\delta}(x)$ as

$$\mathbf{w}_{\delta}(x) \equiv \begin{cases} \boldsymbol{\alpha}\left[\boldsymbol{\Psi}_{\delta}(b-x;b)\,\boldsymbol{\Psi}_{\delta}^{-1}(0;b)\left(\mathbf{I} - \mathbf{m}_{\delta}^{*}(0;b)\right) + \mathbf{m}_{\delta}^{*}(b-x;b)\right], & 0 < x \le b, \\ \boldsymbol{\alpha}\mathbf{R}_0(b-y;0), & x > b. \end{cases}$$

Substituting (7.36) and (7.37) into (7.38), it follows that

$$\mathbf{V}_{\delta}^{3}(u;b) = \boldsymbol{\Upsilon}(u-b) + \mathbf{m}_0^{\mathbf{w}_{\delta}}(u-b)\mathbf{V}_{\delta}^{2}(0;b), \tag{7.39}$$

where $\mathbf{m}_0^{\mathbf{w}_\delta}(u) = \left(\left(\mathbf{m}_{0,1}^{\mathbf{w}_\delta}(u)\right)^\top, \left(\mathbf{m}_{0,2}^{\mathbf{w}_\delta}(u)\right)^\top, ..., \left(\mathbf{m}_{0,n}^{\mathbf{w}_\delta}(u)\right)^\top \right)^\top$ is an $n \times n$ matrix with the row vector $\mathbf{m}_{0,i}^{\mathbf{w}_\delta}(u)$ defined as

$$\mathbf{m}_{0,i}^{\mathbf{w}_\delta}(u) \equiv E\left[\mathbf{w}_\delta\left(|U_\tau|\right) 1\left(\tau < \infty\right) | U_0 = u, J_0 = i\right].$$

The Laplace transform of the elements in $\mathbf{m}_{0,i}^{\mathbf{w}_\delta}(u)$ is given by (7.10). As a special case, letting $u = b$ in (7.39), we have

$$\mathbf{V}_\delta^3\left(b; b\right) = \boldsymbol{\Upsilon}(0) + \mathbf{m}_0^{\mathbf{w}_\delta}(0)\mathbf{V}_\delta^2\left(0; b\right). \tag{7.40}$$

Solving (7.37) and (7.40), one obtains

$$\mathbf{V}_\delta^2\left(0; b\right) = \left[\boldsymbol{\Psi}_\delta^{-1}\left(0; b\right)\left(\mathbf{I} - \mathbf{m}_\delta^*\left(0; b\right)\right) - \mathbf{m}_0^{\mathbf{w}_\delta}(0)\right]^{-1} \boldsymbol{\Upsilon}(0). \tag{7.41}$$

Finally, combining (7.34) with (7.35), (7.36) and (7.39), one concludes that

$$V_\delta\left(u; b\right) = \begin{cases} \boldsymbol{\alpha}\mathbf{R}_0\left(u; 0\right)\mathbf{V}_\delta^2\left(0; b\right), & u < 0, \\ \boldsymbol{\alpha}\left(\boldsymbol{\Psi}_\delta\left(u; b\right)\mathbf{V}_\delta^3\left(b; b\right) + \mathbf{m}_\delta^*\left(u; b\right)\mathbf{V}_\delta^2\left(0; b\right)\right), & 0 \leq u < b, \\ \boldsymbol{\alpha}\left(\boldsymbol{\Upsilon}(u - b) + \mathbf{m}_0^{\mathbf{w}_\delta}(u - b)\mathbf{V}_\delta^2\left(0; b\right)\right), & u \geq b, \end{cases}$$

where $\mathbf{V}_\delta^2\left(0; b\right)$ and $\mathbf{V}_\delta^3\left(b; b\right)$ are as given in (7.41) and (7.37) respectively.

7.4.2 Classical compound Poisson risk model revisited

When the claim arrival process is Poisson, the above analysis can be significantly simplified. In this section, we further examine the occupation time in the classical compound Poisson risk model and revisit some results proposed by Kolkovska et al. (2005).

For $u < 0$, capitalizing on the strong Markov property and skip-free upward properties of the surplus process with a positive security loading, we have

$$V_\delta^1\left(u; b\right) = V_\delta^2\left(0; b\right). \tag{7.42}$$

For $0 \leq u < b$ and $u \geq b$, using the same properties, (7.36) and (7.38) respectively become

$$V_\delta^2(u;b) = \varphi_\delta(u;b)\, V_\delta^3(b;b) + \xi_\delta(u;b)\, V_\delta^2(0;b), \tag{7.43}$$

and

$$\begin{aligned} V_\delta^3(u;b) &= 1 - \psi(u-b) + \int_0^b g(y|u-b) V_\delta^2(b-y;b)\, dy + \int_b^\infty g(y|u-b) V_\delta^1(b-y;b)\, dy \\ &= 1 - \psi(u-b) + \int_0^b g(y|u-b) V_\delta^2(b-y;b)\, dy + V_\delta^2(0;b) \int_b^\infty g(y|u-b) dy. \end{aligned} \tag{7.44}$$

Combining (7.18), (7.19) and (7.39), (7.44) becomes

$$V_\delta^3(u;b) = 1 - \psi(u-b) + W_\delta(u;b)\, V_\delta^2(0;b), \tag{7.45}$$

where

$$W_\delta(u;b) = \int_0^b \left\{ \frac{v_\delta(y)}{v_\delta(0)} (1 - \phi_\delta(0)) + \phi_\delta(y) \right\} g(b-y|u-b) dy + \int_b^\infty g(y|u-b) dy.$$

Eqs. (7.43) and (7.45) at $u = 0$ and $u = b$ respectively yield

$$V_\delta^2(0;b) = \frac{(1 - \psi(0))\, \varphi_\delta(0;b)}{1 - \xi_\delta(0;b) - \varphi_\delta(0;b)\, W_\delta(b;b)}, \tag{7.46}$$

where $\psi(0) = \lambda\mu/c$. Substituting (7.46) back into (7.42), (7.43) and (7.45) yields the Laplace transform of the occupation time for different initial surplus levels.

Remark 7.4.1 *For $u \leq 0$, the Laplace transform of the occupation time is directly given by (7.46) and does not depend on the initial deficit level. Eq. (7.46) is in contradiction with the result of Kolkovska et al. (2005, Proposition 3). Note that the Laplace transform obtained in Kolkovska et al. (2005) can easily be inverted and results in*

$$\Pr\left(\widetilde{T}_{0,b} = \frac{(n+1)b}{c} \right) = \left(1 - \frac{\lambda\mu}{c}\right) \left(\frac{\lambda\mu}{c}\right)^n, \qquad n = 0, 1, 2, \ldots$$

However, it is clear that $\widetilde{T}_{0,b}$ cannot be a discrete r.v.

Generally speaking, it would be tedious to calculate the moments of the occupation time by taking derivatives wrt δ of its Laplace transform. However, the first moment can be easily calculated through the duration of negative surplus, i.e.

$$E\left[\widetilde{T}_{0,b}\middle| u\right] = E\left[\widetilde{T}_{-\infty,b}\middle| u\right] - E\left[\widetilde{T}_{-\infty}\middle| u\right]. \tag{7.47}$$

The mean of the duration of negative surplus $E\left[\widetilde{T}_{-\infty}\middle| u\right]$ is readily available from Egídio dos Reis (1993):

$$E\left[\widetilde{T}_{-\infty}\middle| u\right] = \begin{cases} \frac{-u}{c-\lambda\mu} + E\left[\widetilde{T}_{-\infty}\middle| 0\right], & u < 0, \\ \psi(u)\left(\frac{E[Y|u]}{c-\lambda\mu} + E\left[\widetilde{T}_{-\infty}\middle| 0\right]\right), & u \geq 0, \end{cases} \tag{7.48}$$

in which $E\left[Y\middle| u\right] \equiv E\left[|U_\tau|\middle| \tau < \infty, U_0 = u\right]$ is the mean of the deficit at ruin given that ruin occurs for an initial surplus u and $E\left[\widetilde{T}_{-\infty}\middle| 0\right]$ is given by

$$E\left[\widetilde{T}_{-\infty}\middle| 0\right] = \frac{\lambda\mu_2}{2\left(c-\lambda\mu\right)^2}.$$

Substituting (7.48) into (7.47), it follows that:

(i) for $u < 0$,

$$\begin{aligned} E\left[\widetilde{T}_{0,b}\middle| u\right] &= \left(\frac{b-u}{c-\lambda\mu} + E\left[\widetilde{T}_{-\infty}\middle| 0\right]\right) - \left(\frac{-u}{c-\lambda\mu} + E\left[\widetilde{T}_{-\infty}\middle| 0\right]\right) \\ &= \frac{b}{c-\lambda\mu}, \end{aligned}$$

(ii) for $0 \leq u < b$,

$$\begin{aligned} E\left[\widetilde{T}_{0,b}\middle| u\right] &= \left(\frac{b-u}{c-\lambda\mu} + E\left[\widetilde{T}_{-\infty}\middle| 0\right]\right) - \psi(u)\left(\frac{E\left[Y\middle| u\right]}{c-\lambda\mu} + E\left[\widetilde{T}_{-\infty}\middle| 0\right]\right) \\ &= \frac{b-u}{c-\lambda\mu} + \left(1-\psi(u)\right)E\left[\widetilde{T}_{-\infty}\middle| 0\right] - \psi(u)\frac{E\left[Y\middle| u\right]}{c-\lambda\mu}, \end{aligned}$$

(iii) and for $u \geq b$,

$$E\left[\widetilde{T}_{0,b}\middle| u\right] = \psi(u-b)\left(\frac{E\left[Y\middle| u-b\right]}{c-\lambda\mu} + E\left[\widetilde{T}_{-\infty}\middle| 0\right]\right) - \psi(u)\left(\frac{E\left[Y\middle| u\right]}{c-\lambda\mu} + E\left[\widetilde{T}_{-\infty}\middle| 0\right]\right).$$

Example 7.4.2 *When the claim size X_i follows an exponential distribution with mean $1/\beta$, explicit expressions can be derived for $E\left[\widetilde{T}_{0,b}\middle| u\right]$. In this case, the ruin probability is*

$$\psi(u) = \frac{\lambda}{c\beta} e^{-\left(\beta-\frac{\lambda}{c}\right)u},$$

the mean of the deficit at ruin given that ruin occurs is $E\left[Y\middle| u\right] = 1/\beta$ and

$$E\left[\widetilde{T}_{-\infty}\middle| 0\right] = \frac{\lambda}{(c\beta-\lambda)^2}.$$

Therefore,

$$E\left[\widetilde{T}_{0,b}\middle| u\right] = \begin{cases} \frac{b}{c-\lambda/\beta}, & u < 0, \\ \frac{b-u}{c-\lambda/\beta} + \frac{\lambda}{(c\beta-\lambda)^2}\left(1 - e^{-\left(\beta-\frac{\lambda}{c}\right)u}\right), & 0 \leq u < b, \\ \frac{\lambda}{(c\beta-\lambda)^2}\left(e^{-\left(\beta-\frac{\lambda}{c}\right)(u-b)} - e^{-\left(\beta-\frac{\lambda}{c}\right)u}\right), & u \geq b. \end{cases}$$

(see also Kolkovska et al. (2005, Proposition 4)).

To conclude, we derived in this chapter the Laplace transform of the occupation time in the class of Sparre Andersen risk models with phase-type distributed interclaim times. A similar analysis can even lead to the determination of the Laplace transform of these occupation times in the generalized MAP risk model (see, e.g., Ahn and Badescu (2007)). However, we chose the present model over the MAP risk model for the simplicity of presentation.

Chapter 8

Concluding Remarks

The main topic of this thesis is the distribution of the time until ruin in risk theory. We first introduce a generalized Gerber-Shiu function in Chapter 1 by incorporating the number of claims until ruin into the analysis. As a result, by deriving the joint distribution of the time until ruin and the number of claims until ruin, we are not only able to obtain the marginal density of the time to ruin but also to identify the individual contribution of ruin from each claim. We show in Chapter 2 that the proposed Gerber-Shiu function still satisfies a defective renewal equation and can be generally expressed in terms of an associated compound geometric tail. In Chapters 3 and 4, we start the analysis by imposing an exponential distribution assumption on the interclaim times and the claim sizes, respectively. In this case, the joint Laplace transform/p.g.f. of the time to ruin and the number of claims until ruin can be expressed in terms of the unique non-negative solution of Lundberg's (generalized) equation. We employ Lagrange's implicit function theorem for inversion to obtain the joint generalized density of these two quantities.

In Chapter 5, we extend the analysis of the time until ruin into the Sparre Andersen risk model under the assumption of a combination of n exponentials distributed claim sizes. The multivariate version of Lagrange's expansion theorem plays a key role in the inversion. We point out that it remains a challenging research problem to obtain an explicit expression for the density of the time to ruin in the mixed Erlang claim size risk model, in which case the Lagrange inversion approach might not be the most appropriate methodology to employ. In our proposed methodology, we also demonstrate that other quantities of interest in applied probability can be obtained, such as the duration of the busy period in a $K_m/G/1$ queue. It is the author's belief that the proposed technique has further applications in obtaining the density of the duration of the busy period in more general queueing systems. To further illustrate the proposed unified inversion approach, in Chapter 6, we analyze the first passage time in the compound Poisson risk model with diffusion. A generalization of Kendall's identity is derived and several applications of the results in ruin theory and financial mathematics are extensively discussed. One potential direction for future research is to study the density of the time to ruin in the compound Poisson risk model with diffusion or to study general first passage times in a diffusion model with two-sided jumps. The density of the first passage times in the latter model has wide applications in pricing financial options. However, it is expected that the inversion of the Laplace transform would be quite tedious and it would be interesting to compare the inversion approach suggested in this thesis with existing numerical Laplace transform inversion techniques.

In Chapter 7, we study the single-barrier and double-barrier occupation times in some Sparre Andersen risk models with phase-type interclaim times. This is a more advanced topic in the sense that the aforementioned exit times are essential components of the analysis of these occupation times. We obtain the Laplace transform of the occupation time in this model and also derive the density of the total duration of negative surplus in the classical compound Poisson risk model. Another possible direction for future research consists of deriving the density of the occupation time in more general cases, such as the mixture of exponentials interclaim time risk model or even the mixed Erlang interclaim time risk model.

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