Maximal ideal space techniques in non-selfadjoint operator algebras

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The following thesis is divided into two main parts. In the first part we study the problem of characterizing algebras of functions living on analytic varieties. Specifically, we consider the restrictions $\mathcal{M}_V$ of the multiplier algebra $\mathcal{M}$ of Drury-Arveson space to a holomorphic subvariety V of the unit ball as well as the algebras $\mathcal{A}_V$ of continuous multipliers under the same restriction.

We find that $\mathcal{M}_V$ is completely isometrically isomorphic to $\mathcal{M}_W$ if and only if W is the image of V under a biholomorphic automorphism of the ball. In this case, the isomorphism is unitarily implemented. Furthermore, when V and W are homogeneous varieties then $\mathcal{A}_V$ is isometrically isomorphic to $\mathcal{A}_W$ if and only if the defining polynomial relations are the same up to a change of variables.

The problem of characterizing when two such algebras are (algebraically) isomorphic is also studied. In the continuous homogeneous case, two algebras are isomorphic if and only if they are similar. However, in the multiplier algebra case the problem is much harder and several examples will be given where no such characterization is possible.

In the second part we study the triangular subalgebras of UHF algebras which provide new examples of algebras with the Dirichlet property and the Ando property. This in turn allows us to describe the semicrossed product by an isometric automorphism. We also study the isometric automorphism group of these algebras and prove that it decomposes into the semidirect product of an abelian group by a torsion free group. Various other structure results are proven as well.

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Dedication

This thesis is dedicated to my dear wife Melissa, our daughter Caroline and the soon to arrive baby Ramsey.

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Chapter 1

Introduction

This thesis is based on three papers which were written during my time at the University of Waterloo. Chapter 2 is on the first two papers [32, 33] written with Ken Davidson and Orr Shalit. The first is published in Advances in Mathematics and the second has been accepted to Transactions of the American Mathematical Society. Chapter 3 is found in [59] which has been accepted to Integral Equations and Operator Theory.

The two main chapters then are self-contained and do not relate to each other in any specific sense. In general they are both in operator algebras and both use the maximal ideal space to provide structure theory and characterization. However, other than this very short gloss, there will be no attempt to bring these studies into a cohesive whole, with apologies if the reader was expecting anything different.

Chapter 2 concerns the study of operator algebras of multipliers on reproducing kernel Hilbert spaces associated to analytic varieties in the unit ball of $\mathbb{C}^d$. Multiplication by coordinate functions form a d-tuple which is the universal model for commuting row contractions [11]. Two natural algebras to look at then, are the weak closed and norm closed algebras generated by these coordinate multipliers. These turn out to be the multiplier algebra $\mathcal{M}_d$ of the Drury-Arveson space and the corresponding algebra of continuous multipliers. The Hilbert space is a reproducing kernel Hilbert space which is a complete Nevanlinna-Pick kernel [31]; and in fact when $d = \infty$ is the universal complete NP kernel [1]. For these reasons, the space and its associated algebras have received a lot of attention in recent years.

An analytic variety V is the joint zero set of a collection of holomorphic functions. When $\mathcal{M}_V$ is the multiplier algebra of the collection of kernel functions coming from V it is proven in [30] that $\mathcal{M}_V$ is a complete quotient of $\mathcal{M}_d$ by a WOT-closed ideal, specifically, the ideal vanishing on V. Under some conditions on the variety it also makes sense to consider the algebra of continuous multipliers on the variety, $\mathcal{A}_V$.

The main question that we address is when two such algebras are isomorphic. We find first that two such algebras $\mathcal{M}_V$ and $\mathcal{M}_W$ are completely isometrically isomorphic if and only if there is a biholomorphic automorphism of the ball that carries V onto W. In this case, the isomorphism is unitarily implemented. The continuous algebra characterization follows similarly and in this case we can determine the C*-envelope.

The question of algebraic isomorphism (which implies continuous algebraic isomorphism because the algebras are semisimple) is much more subtle. In the homogeneous variety setting the algebras $\mathcal{A}_V$ are completely characterized by biholomorphisms of the varieties. This was proven up to some technical assumption on the varieties by Davidson, Shalit and myself and the technical assumption was dispensed with by Michael Hartz in [41].

Outside of this special class, such a characterization problem proves quite difficult. For instance, one can show that biholomorphic Blaschke sequences need not give isomorphic multiplier algebras. However, we will show that what seems to be the easy direction, showing that an isomorphism determines a biholomorphism of V onto W, can be established with some hypotheses. In particular, this will be shown when the varieties are a finite union of irreducible varieties and a discrete variety. The isomorphism is just composition with this biholomorphism.

These methods also allow us to show that an isometric isomorphism is just composition with a conformal automorphism of the ball, and thus is completely isometric and unitarily implemented.

One should note that there are a few other approaches to algebras of functions living on varieties or domains. Arias and Latrémolière [5] have an interesting paper in which they study certain operator algebras of this type in the case where the variety is a countable discrete subset of the unit disc which is the orbit of a point under the action of a Fuchsian group. They establish results akin to ours in the completely isometric case using rather different methods. As well, Popescu has developed an operator theory on noncommutative domains [55].

Chapter 3 studies the automorphisms and dilation theory of triangular uniformly hyperfinite (UHF) algebras. A unital non-selfadjoint operator algebra is a triangular UHF algebra if it is the closed union of a chain of unital subalgebras each isomorphic to a full upper triangular matrix algebra. That is, such an algebra can be thought of as the upper triangular part of a UHF algebra. These were extensively studied by Power [57] and many others in the early 90's.

In their recent paper [26], Davidson and Katsoulis refine various notions of dilation theory, commutant lifting and Ando's theorem for non-selfadjoint operator algebras and show that these notions become simpler when the algebras have the semi-Dirichlet property. Moreover, if the operator algebra has this nice dilation theory then one can describe the C*-envelope of the semicrossed product of the operator algebra by an isometric automorphism. However, almost all examples of such algebras arose from tensor algebras of C*-correspondences, the exception being given recently by E. T. A. Kakariadis in [46], which leads to the question whether other examples exist. While it is unknown (at least to the author) whether a triangular UHF algebra is isomorphic to some tensor algebra it does provide a new example of an operator algebra which has the Dirichlet property and the Ando property.

We also address the isometric automorphism group of such triangular UHF algebras. We prove in Section 3.2 that this group can be decomposed into a semidirect product of approximately inner automorphisms by outer automorphisms and that the outer automorphism group is torsion free. Section 3.3 provides a different proof to that of Power's in [58] showing that the outer automorphism group of the triangular UHF algebra with alternating embeddings is determined by a pair of supernatural numbers associated to the algebra. Section 3.4 develops a method of tensoring the embeddings of two triangular UHF algebras to create a new algebra which combines the automorphic structure of both, giving a slightly richer perspective on what groups one can obtain.

Chapter 2

Operator algebras for analytic varieties

2.1 Algebras, ideals and varieties

2.1.1 Multivariable function theory

For E a finite dimensional Hilbert space with orthonormal basis $e_1, \dots, e_d$, let $\mathcal{F} = \mathcal{F}(E)$ denote the full Fock space

$$\mathcal{F} = \mathbb{C} \oplus E \oplus (E \otimes E) \oplus (E \otimes E \otimes E) \oplus \dots .$$

Consider the left creation operators $L_1, \dots, L_d$ given by

$$L_j e_{i_1} \otimes \dots \otimes e_{i_k} = e_j \otimes e_{i_1} \otimes \dots \otimes e_{i_k} \quad \text{for } 1 \le j \le d.$$

Let $\mathcal{L}_d$ denote the WOT-closed algebra generated by $L_1, \dots, L_d$, called the *non-commutative analytic Toeplitz algebra*, in other words, the algebra generated by the left regular representation of the free semigroup on d generators. It has been established in [29, 30, 31, 55] that $\mathcal{L}_d$ is the appropriate analogue of the analytic Toeplitz algebra in one variable. Moreover, the Bunce-Frazho-Popescu Dilation Theorem [17, 36, 52] establishes that every pure row contraction $T = (T_1, \dots, T_d)$ is the compression of $L^{(\infty)}$ (the direct sum of infinitely many copies of $L = (L_1, \dots, L_d)$) to a covariant subspace. Specifically, there is a unital, completely contractive, surjective homomorphism from the norm closed algebra $\mathfrak{A}_d = \overline{\mathrm{Alg}}\{I, L_1, \dots, L_d\}$, the noncommutative disc algebra, onto $\overline{\mathrm{Alg}}\{I, T_1, \dots, T_d\}$ sending

L_i to T_i. Thus, $\mathfrak{A}_d$ is the universal operator algebra generated by a row contraction [53]. However, it should be noted that there is no equivalent statement in the WOT-closures of these algebras (see [27, 28, 29, 60]).

Again for E a finite dimensional Hilbert space with orthonormal basis $e_1, \ldots, e_d$, we write E^n as the n-fold symmetric tensor product. This is the subspace of $E^{\otimes n}$ that is invariant under all permutations of the basis vectors. The symmetric Fock space then, is defined to be

$$\mathcal{F}_s := \mathbb{C} \oplus E \oplus E^2 \oplus E^3 \oplus \ldots.$$

Let H^2_d be Drury-Arveson space [11] which is defined to be the reproducing kernel Hilbert space on $\mathbb{B}_d$, the unit ball of $\mathbb{C}^d$, with kernel functions

$$k_\lambda(z) = \frac{1}{1 - \langle z, \lambda \rangle} \quad \text{for } z, \lambda \in \mathbb{B}_d.$$

The theory is a bit different in the case of $d = \infty$, in this case $\mathbb{C}^d$ is considered as ℓ^2.

Multiplication by the coordinate functions on H^2_d,

$$(Z_i h)(z) = z_i h(z) \quad \text{for } i = 1, \ldots, d,$$

gives a commuting row contraction $Z = (Z_1, \ldots, Z_d)$. Let $\mathcal{M}_d$ denote the multiplier algebra $\mathrm{Mult}(H^2_d)$ of H^2_d, this is the WOT-closed algebra generated by Z. As well, $\mathcal{A}_d = \overline{\mathrm{Alg}}\{I, Z_1, \ldots, Z_d\}$ is the universal (norm-closed) unital operator algebra generated by a commuting row contraction [11, 54]. When $d = 1$, $H^2_d, \mathcal{M}_d$ and $\mathcal{A}_d$ become the classical Hardy space, H^2, the algebra of bounded analytic functions, $H^\infty(\mathbb{D})$, and the disc algebra, $A(\mathbb{D})$. This setting then, should be thought of as the appropriate multivariable analogue of analytic function theory on the disc. Of course, there are important differences, for example $\mathcal{M}_d \subsetneq H^\infty(\mathbb{B}^d)$ for $d > 1$.

These two settings, noncommutative and commutative, are in fact strongly related. Indeed, consider the commutator ideal $\mathfrak{C}$ generated by $L_iL_j - L_jL_i, 1 \le i < j \le d$. Then, $H^2_d \simeq \mathcal{F}_s = \mathcal{F} \ominus \mathfrak{C}\mathcal{F}$ [11], and moreover,

$$\mathcal{M}_d \simeq \mathcal{L}_d / \overline{\mathfrak{C}}^{w^*} \simeq P_{H^2_d} \mathcal{L}_d |_{H^2_d} \quad \text{and} \quad \mathcal{A}_d \simeq \mathfrak{A}_d / \overline{\mathfrak{C}}^{\|\cdot\|} \simeq P_{H^2_d} \mathfrak{A}_d |_{H^2_d}$$

with $Z_i = P_{H^2_d} L_i |_{H^2_d}$, [31].

2.1.2 Analytic varieties

Let an *analytic variety* be defined as the joint zero set of a family of analytic functions. Specifically, if $F \subset H^2_d$ then

$$V(F) := \{\lambda \in \mathbb{B}_d : f(\lambda) = 0 \text{ for all } f \in F\}.$$

It should be noted that this is a global definition as opposed to the local definition of classical varieties, that around every point λ in a variety V there is a neighbourhood N of λ and analytic functions such that these functions vanish on $N \cap V$, see [39, 68]. However, we will see that the theory goes through with this definition, though ultimately, a local definition may work and would be desirable.

Consider that for $f \in \mathcal{M}_d$ we have $M_f 1 = f \in H^2_d$. Hence, one can define analytic varieties of multipliers. When V is an analytic variety define the WOT-closed ideal

$$J_V = \{f \in \mathcal{M}_d : f(\lambda) = 0 \text{ for all } \lambda \in V\} \subset \mathcal{M}_d.$$

Proposition 2.1.1. *Let F be a subset of H^2_d, and let $V = V(F)$. Then*

$$V = V(J_V) = \{\lambda \in \mathbb{B}_d : f(\lambda) = 0 \textit{ for all } f \in J_V\}.$$

Proof. Obviously $V \subseteq V(J_V)$. For the other inclusion, recall that [2, Theorem 9.27] states that a zero set of an H^2_d function is a weak zero set for $\mathcal{M}_d$ (i.e. the intersection of zero sets of functions in $\mathcal{M}_d$). Since V is the intersection of zero sets for H^2_d, it is a weak zero set for $\mathcal{M}_d$; i.e., there exists a set $S \subseteq \mathcal{M}_d$ such that $V = V(S)$. Now, $S \subseteq J_V$, so $V = V(S) \supseteq V(J_V)$. □

Given the analytic variety V, we define a subspace of H^2_d by

$$\mathcal{F}_V = \overline{\operatorname{span}}\{k_\lambda : \lambda \in V\}.$$

The Hilbert space $\mathcal{F}_V$ is naturally a reproducing kernel Hilbert space of functions on the variety V. One could also consider spaces of the form $\mathcal{F}_S = \overline{\operatorname{span}}\{k_\lambda : \lambda \in S\}$ where S is an arbitrary subset of the ball. The following proposition shows that there is no loss of generality in considering only analytic varieties generated by H^2_d functions.

Proposition 2.1.2. *Let $S \subseteq \mathbb{B}_d$. Let J_S denote the set of multipliers vanishing on S, and let I_S denote the set of all H^2_d functions that vanish on S. Then*

$$\mathcal{F}_S = \mathcal{F}_{V(I_S)} = \mathcal{F}_{V(J_S)}.$$

Proof. Clearly $\mathcal{F}_S \subseteq \mathcal{F}_{V(I_S)}$. Let $f \in \mathcal{F}_S^\perp$. Then $f(x) = 0$ for all $x \in S$; so $f \in I_S$. Hence by definition, $f(z) = 0$ for all $z \in V = V(I_S)$; whence $f \in \mathcal{F}_{V(I_S)}^\perp$. Therefore $\mathcal{F}_S = \mathcal{F}_{V(I_S)}$. The extension to zero sets of multipliers follows again from [2, Theorem 9.27]. □

Remarks 2.1.3. In general, it is not true that $V(I_S)$ is equal to the smallest analytic variety *in the classical sense* containing $S \subseteq \mathbb{B}_d$. In fact, by Weierstrass's Factorization Theorem, every discrete set $Z = \{z_n\}_{n=1}^\infty$ in $\mathbb{D}$ is the zero set of some holomorphic function on $\mathbb{D}$. However, if the sequence Z is not a Blaschke sequence, then there is no nonzero function in H^2 that vanishes on all of it. So here $I_Z = \{0\}$, and therefore $V(I_Z) = \mathbb{D}$.

Returning to the fact that our definition of a variety is not local, one could consider the following variant: V is a variety if for each point $\lambda \in \mathbb{B}_d$, there is an $\varepsilon > 0$ and a finite set $f_1, \dots, f_n$ in $\mathcal{M}_d$ so that

$$b_\varepsilon(\lambda) \cap V = \{z \in b_\varepsilon(\lambda) : 0 = f_1(z) = \cdots = f_n(z)\}.$$

We do not know if every variety of this type is actually the intersection of zero sets.

In particular, we will say that a variety V is *irreducible* if for any regular point $\lambda \in V$, a point around which the variety looks like a manifold, the intersection of zero sets of all multipliers vanishing on a small neighbourhood $V \cap b_\varepsilon(\lambda)$ is exactly V. However we do not know whether an irreducible variety is connected. A local definition of our varieties would presumably clear up this issue.

It is a natural to also look at the norm closed ideal I_V associated to an analytic variety

$$I_V = \{f \in \mathcal{A}_d : f(\lambda) = 0 \text{ for all } \lambda \in V\}.$$

It should be noted that in many situations this ideal will give no useful information, this will be seen later. In general the norm closed ideal is useful for study when its WOT-closure is J_V. This will be seen to happen when

$$[I_V H_d^2] = [J_V H_d^2]. \tag{2.1}$$

In function theoretic terms, this means that every $f \in J_V$ is the bounded pointwise limit of a net of functions in I_V. It is not clear when this happens in general but we do have a fair number of examples.

In dimension $d = 1$, the analytic varieties are sequences of points which are either finite or satisfy the Blaschke condition. For such a sequence V, let us denote $S(V) = \overline{V} \cap \mathbb{T}$. When the Lebesgue measure of $S(V)$ is positive, there is no nonzero $f \in A(\mathbb{D})$ that vanishes on V because a non-zero function in the disk algebra must be non-zero a.e. on the unit circle. So $I_V = 0$. On the other hand, $J_V \neq 0$, because it contains the Blaschke product of the sequence V. So it cannot be the WOT-closure of I_V. In particular, the special assumption (2.1) is not always satisfied. If the Lebesgue measure $|S(V)|$ of $S(V)$ is zero, then the special assumption is valid.

Lemma 2.1.4. *Let V be an analytic variety in $\mathbb{D}$ such that $S(V)$ has zero measure. Then the ideal J_V is the* WOT*-closure of I_V.*

Proof. Let B be the Blaschke product with simple zeros on V. It suffices to construct for every $f \in J_V = BH^\infty$, a bounded sequence in I_V converging pointwise to f. Factor $f = Bh$ with $h \in H^\infty$. By a theorem of Fatou there is an analytic function g with $\operatorname{Re} g \geq 0$ such that e^{-g} is in $A(\mathbb{D})$ and vanishes precisely on $S(V)$. Define

$$f_n(z) = B(z)\, e^{-g(z)/n} h((1 - \tfrac{1}{n})z) \quad \text{for } n \geq 1.$$

This sequence belongs to A($\mathbb{D}$), is bounded by $\|f\|_\infty$, and converges to f uniformly on compact subsets of the disk. Hence it converges to f in the WOT topology. □

For $d > 1$, the canonical example where this special assumption is true is a *homogeneous variety*, $V = V(I)$, where I is a homogeneous ideal of $\mathbb{C}[z]$, where $z = (z_1, \cdots, z_d)$, that is, generated by a finite set of homogeneous polynomials. In this circumstance, V is an affine algebraic variety.

2.1.3 Ideals and invariant subspaces

We will apply some results of Davidson-Pitts [30, Theorem 2.1] and [31, Corollary 2.3] to the commutative context.

In the first paper, a bijective correspondence is established between the collection of WOT-closed ideals J of $\mathcal{L}_d$ and the complete lattice of subspaces which are invariant for both $\mathcal{L}_d$ and its commutant $\mathcal{R}_d$, the algebra of right multipliers. The pairing is just the map taking an ideal J to its closed range $\mu(J) := \overline{J\mathcal{F}}$. The inverse map takes a subspace N to the ideal J of elements with range contained in N.

In [31, Theorem 2.1], it is shown that the quotient algebra $\mathcal{L}_d/J$ is completely isometrically isomorphic and WOT-homeomorphic to the compression of $\mathcal{L}_d$ to $\mu(J)^\perp$. As was mentioned, [31, Corollary 2.3] shows that the multiplier algebra $\mathcal{M}_d$ is completely isometrically isomorphic to $\mathcal{L}_d/\overline{\mathfrak{C}}^{w^*}$, the quotient by the weak-$*$ closure of the commutator ideal. In particular, $\mu(\overline{\mathfrak{C}}^{w^*})^\perp = H^2_d$.

It is easy to see that there is a bijective correspondence between the lattice of WOT-closed ideals $\mathrm{Id}(\mathcal{M}_d)$ of $\mathcal{M}_d$ and the WOT-closed ideals of $\mathcal{L}_d$ which contain $\mathfrak{C}$. Similarly there is a bijective correspondence between invariant subspaces N of $\mathcal{M}_d$ and invariant subspaces of $\mathcal{L}_d$ which contain $\mu(\overline{\mathfrak{C}}^{w^*}) = H_d^{2\perp}$. Since the algebra $\mathcal{M}_d$ is abelian, it is also the quotient of $\mathcal{R}_d$ by its commutator ideal, which also has range $H_d^{2\perp}$. So the subspace $N \oplus H_d^{2\perp}$ is invariant for both $\mathcal{L}_d$ and $\mathcal{R}_d$. Therefore an application of [30, Theorem 2.1] yields the following consequence:

Theorem 2.1.5. *Define the map $\alpha : \mathrm{Id}(\mathcal{M}_d) \to \mathrm{Lat}(\mathcal{M}_d)$ by $\alpha(J) = \overline{J1}$. Then α is a complete lattice isomorphism whose inverse β is given by*

$$\beta(N) = \{f \in \mathcal{M}_d : f \cdot 1 \in N\}.$$

Moreover [31, Theorem 2.1] then yields:

Theorem 2.1.6. *If J is a* WOT*-closed ideal of $\mathcal{M}_d$ with range N, then $\mathcal{M}_d/J$ is completely isometrically isomorphic and* WOT*-homeomorphic to the compression of $\mathcal{M}_d$ to $N^\perp$.*

2.1.4 The multiplier algebra of a variety

The reproducing kernel Hilbert space $\mathcal{F}_V$ comes with its multiplier algebra $\mathcal{M}_V = \mathrm{Mult}(\mathcal{F}_V)$. This is the algebra of all functions f on V such that $fh \in \mathcal{F}_V$ for all $h \in \mathcal{F}_V$. A standard argument shows that each multiplier determines a bounded linear operator $M_f \in \mathcal{B}(\mathcal{F}_V)$ given by $M_f h = fh$. We will usually identify the function f with its multiplication operator M_f. We will also identify the subalgebra of $\mathcal{B}(\mathcal{F}_V)$ consisting of the elements M_f and the algebra of functions $\mathcal{M}_V$ (endowed with the same norm). One reason to distinguish f and M_f is that sometimes we need to consider the adjoints of the operators M_f. The distinguishing property of these adjoints is that $M_f^* k_\lambda = \overline{f(\lambda)} k_\lambda$ for $\lambda \in V$, in the sense that if $A^* k_\lambda = \overline{f(\lambda)} k_\lambda$ for $\lambda \in V$, then f is a multiplier.

The space $\mathcal{F}_V$ is therefore invariant for the adjoints of multipliers; and hence it is the complement of an invariant subspace of $\mathcal{M}_d$. Thus an application of Theorem 2.1.6 and the complete Nevanlinna-Pick property yields:

Proposition 2.1.7. *Let V be an analytic variety in $\mathbb{B}_d$. Then*

$$\mathcal{M}_V = \{f|_V : f \in \mathcal{M}_d\}.$$

Moreover the mapping $\varphi : \mathcal{M}_d \to \mathcal{M}_V$ given by $\varphi(f) = f|_V$ induces a completely isometric isomorphism and WOT*-homeomorphism of $\mathcal{M}_d/J_V$ onto $\mathcal{M}_V$. For any $g \in \mathcal{M}_V$ and any*

$f \in \mathcal{M}_d$ such that $f|_V = g$, we have $M_g = P_{\mathcal{F}_V} M_f|_{\mathcal{F}_V}$. Given any $F \in M_k(\mathcal{M}_V)$, one can choose $\widetilde{F} \in M_k(\mathcal{M})$ so that $\widetilde{F}|_V = F$ and $\|\widetilde{F}\| = \|F\|$.

Proof. Theorem 2.1.6 provides the isomorphism between $\mathcal{M}_d/J_V$ and the restriction of the multipliers to $N^\perp$ where $N = \overline{J_V 1}$. Since J_V vanishes on V, if $f \in J_V$, we have

$$\langle M_f h, k_\lambda \rangle = \langle h, M_f^* k_\lambda \rangle = 0 \quad \text{for all } \lambda \in V \text{ and } h \in H_d^2.$$

So N is orthogonal to $\mathcal{F}_V$. Conversely, if M_f has range orthogonal to $\mathcal{F}_V$, the same calculation shows that $f \in J_V$. Since the pairing between subspaces and ideals is bijective, we deduce that $N = \mathcal{F}_V^\perp$. The mapping of $\mathcal{M}_d/J_V$ into $\mathcal{M}_V$ is given by compression to $\mathcal{F}_V$ by sending f to $P_{\mathcal{F}_V} M_f|_{\mathcal{F}_V}$.

It is now evident that the restriction of a multiplier f in $\mathcal{M}_d$ to V yields a multiplier on $\mathcal{F}_V$, and that the norm is just $\|f + J_V\| = \|P_{\mathcal{F}_V} M_f|_{\mathcal{F}_V}\|$. We need to show that this map is surjective and completely isometric. This follows from the complete Nevanlinna-Pick property as in [31, Corollary 2.3]. Indeed, if $F \in M_k(\mathcal{M}_V)$ with $\|F\| = 1$, then standard computations show that if $\lambda_1, \ldots, \lambda_n$ lie in V, then

$$\Big[\big(I_k - F(\lambda_j)F(\lambda_i)^*\big)\langle k_{\lambda_i}, k_{\lambda_j}\rangle\Big]_{n\times n}$$

is positive semidefinite. By [31], this implies that there is a matrix multiplier $\widetilde{F} \in M_k(\mathcal{M}_d)$ with $\|\widetilde{F}\| = 1$ such that $\widetilde{F}|_V = F$. □

We can argue as in the previous subsection that there is a bijective correspondence between WOT-closed ideals of $\mathcal{M}_V$ and its invariant subspaces:

Corollary 2.1.8. *Define the map $\alpha : \mathrm{Id}(\mathcal{M}_V) \to \mathrm{Lat}(\mathcal{M}_V)$ by $\alpha(J) = \overline{J1}$. Then α is a complete lattice isomorphism whose inverse β is given by*

$$\beta(N) = \{f \in \mathcal{M}_V : f \cdot 1 \in N\}.$$

Remark 2.1.9. By Theorem 4.2 in [1], every irreducible complete Nevanlinna-Pick kernel is equivalent to the restriction of the kernel of Drury-Arveson space to a subset of the ball. It follows from this and from the above discussion that every multiplier algebra of an irreducible complete Nevanlinna-Pick kernel is completely isometrically isomorphic to one of the algebras $\mathcal{M}_V$ that we are considering here.

Remark 2.1.10. By the universality of $Z_1, \dots, Z_d$ [11], for every unital operator algebra $\mathcal{B}$ that is generated by a *pure* commuting row contraction $T = (T_1, \dots, T_d)$, there exists a surjective unital homomorphism $\varphi_T : \mathcal{M}_d \to \mathcal{B}$ that gives rise to a natural functional calculus

$$f(T_1, \dots, T_d) = \varphi_T(f) \quad \text{for } f \in \mathcal{M}_d.$$

So it makes sense to say that a commuting row contraction T annihilates J_V if φ_T vanishes on J_V. By Proposition 2.1.7, we may identify $\mathcal{M}_V$ with the quotient $\mathcal{M}_d/J_V$, thus we may identify $\mathcal{M}_V$ as the universal WOT-closed unital operator algebra generated by a *pure* commuting row contraction $T = (T_1, \dots, T_d)$ that annihilates J_V.

Turning to the continuous case, define $\mathcal{A}_V$ to be the norm closure of the polynomials in $\mathcal{M}_V$. The importance of the special assumption (2.1) is in the following result.

Proposition 2.1.11. *Let V be an ideal such that $[I_V H^2_d] = [J_V H^2_d]$. Then*

1. *For every $f \in \mathcal{A}_d$, the compression of M_f to $\mathcal{F}_V$ is equal to M_g, where $g = f|_V$.*
2. $\mathcal{A}_V = \{f|_V : f \in \mathcal{A}_d\}$.
3. *$\mathcal{A}_d/I_V$ is completely isometrically isomorphic to $\mathcal{A}_V$ via the restriction map $f \mapsto f|_V$ of $\mathcal{A}_d$ into $\mathcal{A}_V$.*
4. *For every $f \in \mathcal{A}_d$,* $\operatorname{dist}(f, I_V) = \operatorname{dist}(f, J_V)$.

Proof. The first item is just a restatement of Proposition 2.1.7. By universality of $\mathcal{A}_d$, $\mathcal{A}_V$ is equal to the compression of $\mathcal{A}_d$ to $\mathcal{F}_V$. Therefore, by (a slight modification of) Popescu's results [55], $\mathcal{A}_V$ is the universal operator algebra generated by a commuting row contraction subject to the relations in $I_V = J_V \cap \mathcal{A}_d$. But so is $\mathcal{A}_d/I_V$. So these two algebras can be naturally identified. Since compression is restriction, (2) and (3) follow. Item (4) follows from the fact that

$$\begin{aligned} \operatorname{dist}(f, I_V) &= \|f + I_V\|_{\mathcal{A}/I_V} = \|P_{\mathcal{F}_V} M_f P_{\mathcal{F}_V}\| \\ &= \|f + J_V\|_{\mathcal{M}/J_V} = \operatorname{dist}(f, J_V). \end{aligned}$$

□

Corollary 2.1.12. *Let V be a homogeneous variety, or a Blaschke sequence in the disc such that $S(V)$ has measure zero. Then $\mathcal{A}_d/I_V$ embeds into $\mathcal{M}_d/J_V$ completely isometrically.*

2.1.5 The character space of $\mathcal{M}_V$

If A is a Banach algebra, denote the set of multiplicative linear functionals on A by $M(A)$; and endow this space with the weak-$*$ topology. We refer to elements of $M(A)$ as *characters*. Note that all characters are automatically unital and continuous with norm one. When A is an operator algebra, characters are completely contractive.

When V is an analytic variety in $\mathbb{B}_d$, we will abuse notation and let $Z_1, \dots, Z_d$ also denote the images of the coordinate functions $Z_1, \dots, Z_d$ of $\mathcal{M}_d$ in $\mathcal{M}_V$. Since $\begin{bmatrix} Z_1, \dots, Z_d \end{bmatrix}$ is a row contraction,

$$\big\|\big(\rho(Z_1), \dots, \rho(Z_d)\big)\big\| \leq 1 \quad \text{for all } \rho \in M(\mathcal{M}_V).$$

The map $\pi : M(\mathcal{M}_V) \to \overline{\mathbb{B}}_d$ given by

$$\pi(\rho) = (\rho(Z_1), \dots, \rho(Z_d))$$

is continuous as a map from $M(\mathcal{M}_V)$, with the weak-$*$ topology, into $\overline{\mathbb{B}}_d$ (endowed with the weak topology in the case $d = \infty$). We define

$$\overline{V}^{\mathcal{M}} = \pi((M(\mathcal{M}_V)).$$

Since π is continuous, $\overline{V}^{\mathcal{M}}$ is a (weakly) compact subset of $\overline{\mathbb{B}_d}$. For every $\lambda \in \overline{V}^{\mathcal{M}}$, the *fiber* over λ is defined to be the set $\pi^{-1}(\lambda)$ in $M(\mathcal{M}_V)$. We will see below that $V \subseteq \overline{V}^{\mathcal{M}}$, and that over every $\lambda \in V$ the fiber is a singleton.

Every unital homomorphism $\varphi : A \to B$ between Banach algebras induces a mapping $\varphi^* : M(B) \to M(A)$ by $\varphi^* \rho = \rho \circ \varphi$. If φ is a continuous isomorphism, then φ^* is a homeomorphism. We will see below that in many cases a homomorphism $\varphi : \mathcal{M}_V \to \mathcal{M}_W$ gives rise to an induced map $\varphi^* : M(\mathcal{M}_W) \to M(\mathcal{M}_V)$ which has additional structure. The most important aspect is that φ^* restricts to a holomorphic map from W into V.

The weak-$*$ continuous characters of $\mathcal{M}_V$

In the case of $\mathcal{M}_d$, the weak-$*$ continuous characters coincide with the point evaluations at points in the open ball [6, 29]

$$\rho_\lambda(f) = f(\lambda) = \langle f\nu_\lambda, \nu_\lambda\rangle \quad \text{for } \lambda \in \mathbb{B}_d,$$

where $\nu_\lambda = k_\lambda/\|k_\lambda\|$. The fibers over points in the boundary sphere are at least as complicated as the fibers in $M(H^\infty)$ [30], which are known to be extremely large [43].

As a quotient of a dual algebra by a weak-$*$ closed ideal, the algebra $\mathcal{M}_V$ inherits a weak-$*$ topology. As an operator algebra concretely represented on a reproducing kernel Hilbert space, $\mathcal{M}_V$ also has the weak-operator topology (WOT). In $\mathcal{L}_d$ these topologies coincide [29], which leads to the following:

Lemma 2.1.13. *The weak-$*$ and weak-operator topologies on $\mathcal{M}_V$ coincide.*

Proof. By [7, Proposition 1.2] (see also [25, Theorem 5.2]), $\mathcal{L}_d/J_V$ has property $\mathbb{A}_1(1)$. This means that for every ρ in the open unit ball of $(\mathcal{L}_d/J_V)_*$, there are $x, y \in \mathcal{F}_V$ with $\|x\|\|y\| < 1$ such that

$$\rho(T) = \langle Tx, y\rangle \ , \ T \in \mathcal{L}_d/J_V.$$

The conclusion immediately follows from this because the commutator ideal $\mathfrak{C}$ is a subset of J_V as an ideal of $\mathcal{L}_d$ and so $\mathcal{L}_d/J_V \simeq \mathcal{M}_V$. □

Proposition 2.1.14. *The* WOT*-continuous characters of $\mathcal{M}_V$ can be identified with V. Moreover, $\overline{V}^{\mathcal{M}} \cap \mathbb{B}_d = V$. The restriction of each $f \in \mathcal{M}_V$ to V is a bounded holomorphic function.*

Proof. As $\mathcal{M}_V$ is the multiplier algebra of a reproducing kernel Hilbert space on V, it is clear that for each $\lambda \in V$, the evaluation functional

$$\rho_\lambda(f) = f(\lambda) = \langle f\nu_\lambda, \nu_\lambda\rangle$$

is a WOT-continuous character.

On the other hand, the quotient map from the free semigroup algebra $\mathcal{L}_d$ onto $\mathcal{M}_V$ is weak-operator continuous. Thus, if ρ is a WOT-continuous character of $\mathcal{M}_V$, then it induces a WOT-continuous character on $\mathcal{L}_d$ by composition. Therefore, using [30, Theorem 2.3], we find that ρ must be equal to the evaluation functional ρ_λ at some point $\lambda \in \mathbb{B}_d$. Moreover ρ_λ annihilates J_V. By Proposition 2.1.1, the point λ lies in V.

If ρ is a character on $\mathcal{M}_V$ such that $\pi(\rho) = \lambda \in \mathbb{B}_d$, then again it induces a character $\tilde{\rho}$ on $\mathcal{L}_d$ with the property that $\tilde{\rho}(L_1, \ldots, L_d) = \lambda$. By [30, Theorem 3.3], it follows that $\tilde{\rho}$ is WOT-continuous and coincides with point evaluation. Hence by the previous paragraph, λ belongs to V. So $\overline{V}^{\mathcal{M}} \cap \mathbb{B}_d = V$.

Therefore $\pi : \pi^{-1}(V) \to V$ is seen to be a homeomorphism between $\pi^{-1}(V)$ endowed with the weak-$*$ topology and V with the (weak) topology induced from $\mathbb{B}_d$.

By Proposition 2.1.7, $\mathcal{M}_V$ is a quotient of $\mathcal{M}_d$, and the map is given by restriction to V. Hence f is a bounded holomorphic function by [30, Theorem 3.3] or [11, Proposition 2.2]. $\square$

Thus the character space $M(\mathcal{M}_V)$ consists of V and $M(\mathcal{M}_V) \setminus V$, which we call the *corona*. By definition, the corona is fibered over $\overline{V}^{\mathcal{M}} \setminus V$, and by the above proposition this latter set is contained in $\partial\mathbb{B}_d$.

The continuous characters

Define

$$\overline{V}^{\mathcal{A}} = \{\lambda \in \overline{\mathbb{B}_d} : f(\lambda) = 0 \text{ for all } f \in I_V\}.$$

Clearly $\overline{V}^{\mathcal{A}}$ contains the closure of V in $\mathbb{B}_d$. But it is not clear exactly what else it contains. However, it seems most reasonable to restrict our attention to the algebras $\mathcal{A}_V$ such that $V = V(I_V)$, so that the variety V is determined by functions in $\mathcal{A}_d$. In this case, we obtain

$$\mathbb{B}_d \cap \overline{V}^{\mathcal{A}} = V. \tag{2.2}$$

The proof is the same as that of Proposition 2.1.1. It is not clear whether this holds for arbitrary varieties. This identity does hold when $V \subseteq \mathbb{D}$ is a Blaschke sequence and $|S(V)| = 0$.

Proposition 2.1.15. *Let V be a variety satisfying condition* (2.1)*. Then the character space $M(\mathcal{A}_V)$ of $\mathcal{A}_V$ can be identified with $\overline{V}^{\mathcal{A}}$.*

Proof. Let $\lambda \in \overline{V}^{\mathcal{A}}$. Then the evaluation functional ρ_λ given by $\rho_\lambda(f) = f(\lambda)$ is a character of $\mathcal{A}_d$ with kernel equal to $I_{\{\lambda\}} \supseteq I_V$. Thus ρ_λ can be promoted to a character of $\mathcal{A}_V = \mathcal{A}_d/I_V$.

Denote by $Z_1, \ldots, Z_d$ the images of the coordinate functions in $\mathcal{A}_V$. If ρ is a character of $\mathcal{A}_V$, let

$$\lambda = (\lambda_1, \ldots, \lambda_d) = (\rho(Z_1), \ldots, \rho(Z_d)).$$

Then $\lambda \in \overline{\mathbb{B}}_d$ because ρ is completely contractive. For every $f \in I_V$, $f(Z_1, \ldots, Z_d) = 0$. Thus

$$\rho(f(Z_1, \ldots, Z_d)) = f(\lambda_1, \ldots, \lambda_d) = 0.$$

So λ lies in the set of all points in $\overline{\mathbb{B}}_d$ that annihilate I_V, which is $\overline{V}^{\mathcal{A}}$.

This identification is easily seen to be a homeomorphism. $\square$

2.1.6 Nullstellensatz for homogeneous ideals

Our goal in this section is to obtain a (projective) Nullstellensatz for a large class of operator algebras, including $\mathcal{M}_d$, $\mathcal{A}_d$ and the "ball algebra" $A(\mathbb{B}_d)$.

Let $\Omega \subseteq \mathbb{C}^d$ be an open bounded domain that is the union of polydiscs centered at 0. Then Ω has the following property:

$$\lambda \in \Omega \Rightarrow t\lambda \in \Omega \ , \text{ for all } t \in \overline{\mathbb{D}}$$

and Ω also the property that every function f holomorphic in Ω has a Taylor series that converges in Ω.

Let $\mathcal{H}$ be a reproducing kernel Hilbert space of analytic functions in Ω containing the polynomials with the additional property that $f(z) \mapsto f(e^{it}z)$ is a unitary operator on $\mathcal{H}$ for all $t \in \mathbb{R}$. It follows that if $p, q \in \mathcal{H}$ are homogeneous polynomials of different total degrees, then $\langle p, q \rangle = 0$.

In the discussion below $\mathcal{A}_\mathcal{H}$ will denote the closure of the polynomials in the multiplier algebra $\mathcal{M}_\mathcal{H} = \text{Mult}(\mathcal{H})$. If $\mathcal{H} = H^2_d$, then $\mathcal{M}_\mathcal{H} = \mathcal{M}_d$, which is the case of principal interest. If $\mathcal{H}$ is taken to be the Bergman space on Ω, then $\mathcal{A}_\mathcal{H}$ is $A(\Omega)$, the space of continuous functions on $\overline{\Omega}$ which are analytic on Ω, with the sup norm. As is always the case with algebras of multipliers, the norm of $\mathcal{M}_\mathcal{H}$, which will be denoted simply by $\|\cdot\|$, satisfies $\|f\|_\infty \leq \|f\|$ (see [2, Chapter 2]).

Every $f \in \mathcal{M}_\mathcal{H}$ has a Taylor series in Ω, $f(z) = \sum_\alpha a_\alpha z^\alpha$. We write

$$f = \sum_{n=0}^{\infty} f_n \tag{2.3}$$

where $f_n(z) = \sum_{|\alpha|=n} a_\alpha z^\alpha$ is the nth homogeneous component of f. The series (2.3) converges locally uniformly in Ω.

Lemma 2.1.16. *For all n, the map $P_n : \mathcal{M}_\mathcal{H} \to \mathbb{C}[z] \subseteq \mathcal{M}_\mathcal{H}$ given by $P_n(f) = f_n$ is contractive. Furthermore, the series (2.3) is Cesàro norm convergent to f in the norm of $\mathcal{A}_\mathcal{H}$ if $f \in \mathcal{A}_\mathcal{H}$ and the Cesàro means converge weakly otherwise.*

Proof. Consider the gauge automorphisms on $\mathcal{M}_\mathcal{H}$:

$$[\gamma_t(f)](z) = f(e^{it}z).$$

The unitary group given by $[U_t(h)](z) = h(e^{it}z)$ is continuous in the strong operator topology, and $\gamma_t = \operatorname{ad} U_t$. Hence the path $t \mapsto \gamma_t(f)$ is continuous with respect to the strong operator topology. One sees therefore that the integral

$$P_n(f) = \frac{1}{2\pi}\int_0^{2\pi} \gamma_t(f)e^{-int}dt$$

converges in the strong operator topology to an element of $B(\mathcal{H})$. The operator P_n is a complete contraction, as it is an average of complete contractions. Note that P_n maps $\mathbb{C}[z]$ onto the space H_n of homogeneous polynomial of degree n. This fact follows from the simple identity $U_s P_n(f) = e^{ins}P_n(f)$. Therefore, P_n maps $\mathcal{M}_{\mathcal{H}} = \overline{\mathbb{C}[z]}^{\|\cdot\|}$ onto H_n. A standard argument using the Fejér kernel shows that the Cesàro means $\Sigma_n(f)$ are completely contractive and converge weakly to f, and in norm if $f \in \mathcal{A}_{\mathcal{H}}$, and that $P_n(f) = f_n$. □

In particular, we see that f is in the closed linear span of its homogeneous components. This will be used repeatedly below.

Definition 2.1.17. *An ideal $I \subseteq \mathcal{M}_{\mathcal{H}}$ is said to be* homogeneous *if $f_n \in I$ for all $n \in \mathbb{N}$ and all $f \in I$.*

Proposition 2.1.18. *A closed ideal $I \subseteq \mathcal{A}_{\mathcal{H}}$ is homogeneous if and only if for all $t \in \mathbb{D}$ and all $f \in I$, one has $f(tz) \in I$.*

Proof. Assume that I is homogeneous, and let $f(z) = \sum_n f_n(z) \in I$. By the previous lemma $\|f_n\| \leq \|f\|$, so for all $t \in \mathbb{D}$, $f(tz) = \sum_n t^n f_n(z)$ is a norm convergent series of elements in I. Hence $f(tz) \in I$.

Conversely, let $f \in I$, and assume that for all $t \in \mathbb{D}$, $f(tz) \in I$. Assuming that I is proper, $f_0 = 0$ follows from taking $t = 0$. But then

$$\frac{f(tz)}{t} = \sum_{n=0}^{\infty} t^n f_{n+1} \in I.$$

Taking $t \to 0$ we find that $f_1(z) \in I$. Now we consider

$$\frac{f(tz) - f_1(tz)}{t^2} = \sum_{n=0}^{\infty} t^n f_{n+2}(z) \in I,$$

taking the limit as $t \to 0$ we find that $f_2(z) \in I$. The result follows by recursion. □

Lemma 2.1.19. *Let $I \subseteq \mathbb{C}[z]$ be a homogeneous ideal. Then the closure of I in $\mathcal{A}_\mathcal{H}$ and the weak-$*$ closure in $\mathcal{M}_\mathcal{H}$ are homogeneous. If p is a homogeneous polynomial in $\overline{I}^{w^*}$, then $p \in I$.*

Proof. This follows easily from the continuity of P_n. □

Lemma 2.1.20. *Let J be a homogeneous ideal in $\mathcal{A}_\mathcal{H}$. Then the ideal $I = \mathbb{C}[z] \cap J$ of $\mathbb{C}[z]$ satisfies $I \subseteq J \subseteq \overline{I}$, and it is the unique homogeneous ideal in $\mathbb{C}[z]$ with this property.*

Proof. Clearly $I \subseteq J$, and that $J \subseteq \overline{I}$ follows from Lemma 2.1.16. If K is another homogeneous ideal in $\mathbb{C}[z]$ such that $K \subseteq J \subseteq \overline{K}$, then we have $I \subseteq \overline{K}$ and $K \subseteq \overline{I}$. From Lemma 2.1.19, $I = K$. □

Corollary 2.1.21. *Let J be a homogeneous ideal in $\mathcal{M}_\mathcal{H}$. The the ideal $I = J \cap \mathbb{C}[z]$ satisfies $I \subseteq J \subseteq \overline{I}^{w^*}$, and it is the unique homogeneous ideal in $\mathbb{C}[z]$ with this property.*

Proof. The P_n from Lemma 2.1.16 extend to be WOT-continuous on $\mathcal{M}_\mathcal{H}$ and recall that this corresponds to the weak-$*$ topology. □

Corollary 2.1.22. *Every weak-$*$ closed homogeneous ideal in $\mathcal{M}_\mathcal{H}$ and every norm closed homogeneous ideal in $\mathcal{A}_\mathcal{H}$ is finitely generated (as a closed ideal).*

Remark 2.1.23. There do exist closed ideals in $A(\mathbb{B}_d)$ which are not finitely generated (one may adjust the example in [61, Proposition 4.4.2]).

For a weak-$*$ closed ideal in $J \subset \mathcal{M}_\mathcal{H}$, *the radical of* J is defined to be the ideal $\sqrt{J}$ given by

$$\sqrt{J} = \{f \in \mathcal{M}_\mathcal{H} : f^n \in J \text{ for some } n \geq 1\}.$$

Note that we will also be working with the radical of ideals in $\mathcal{A}_\mathcal{H}$ and $\mathbb{C}[z]$ as well.

Lemma 2.1.24. *The radical of a weak-$*$ closed homogeneous ideal J of $\mathcal{M}_\mathcal{H}$ (resp. a norm closed ideal in $\mathcal{A}_\mathcal{H}$) is homogeneous.*

Proof. Let f and m be such that $f^m \in J$. Write the homogeneous decomposition of f as $f(z) = \sum_{n\geq k} f_n(z)$, where $f_k(z)$ is the lowest non-vanishing homogeneous term. Then $f^m(z) = f_k(z)^m + \dots$. Since J is homogeneous, $f_k^m \in J$, so $f_k \in \sqrt{J}$. Proceeding recursively, we find that $f_j \in \sqrt{J}$ for all j. □

Theorem 2.1.25. *Let $J \subseteq \mathcal{M}_{\mathcal{H}}$ be a weak-$*$ closed homogeneous ideal (resp. a norm closed ideal in $\mathcal{A}_{\mathcal{H}}$). Then there exists $N \in \mathbb{N}$ such that $f^N \in J$ for all $f \in \sqrt{J}$.*

Proof. By the effective Nullstellensatz [49, Theorem 1.5] there is an $N \in \mathbb{N}$ such that $p^N \in J \cap \mathbb{C}[z]$ for all $p \in \sqrt{J \cap \mathbb{C}[z]} = \sqrt{J} \cap \mathbb{C}[z]$. If $f \in \sqrt{J}$, then $f \in \overline{\sqrt{J} \cap \mathbb{C}[z]}$ by Lemma 2.1.20 and Corollary 2.1.21. If $\{f_n\}$ is a sequence in $\sqrt{J} \cap \mathbb{C}[z]$ converging to f, then $f_n^N \in J$ for all n, thus $f^N = w^* - \lim_n f_n^N \in J$. □

Corollary 2.1.26. *The radical of a weak-$*$ closed homogeneous ideal $J \subseteq \mathcal{M}_{\mathcal{H}}$ (resp. a norm closed ideal in $\mathcal{A}_{\mathcal{H}}$) is weak-$*$ closed (resp. norm closed).*

Proposition 2.1.27. *If $I \subseteq \mathbb{C}[z]$ is radical, $I = \sqrt{I}$, then $\overline{I}$ is radical in $\mathcal{A}_{\mathcal{H}}$ and $\overline{I}^{w^*}$ is radical in $\mathcal{M}_{\mathcal{H}}$.*

Proof. Put $J = \overline{I}$. Then $\overline{\sqrt{J} \cap \mathbb{C}[z]}$ is the unique homogeneous ideal in $\mathbb{C}[z]$ with closure equal to $\sqrt{J}$. But $\sqrt{J} \cap \mathbb{C}[z] = \sqrt{J \cap \mathbb{C}[z]} = I$, so $\sqrt{J} = \overline{I} = J$. The weak-$*$ proof follows identically. □

The main result of this section is a projective Nullstellensatz for closed ideals in $\mathcal{A}_{\mathcal{H}}$ and $\mathcal{M}_{\mathcal{H}}$. We shall need the following notation. For an ideal J in some algebra B and a set X, recall the notation for varieties and ideals

$$V_X(J) = \{z \in X : f(z) = 0 \text{ for all } f \in J\}.$$

and

$$I_B(X) = \{f \in B : f(\lambda) = 0 \text{ for all } \lambda \in X\}.$$

First we prove the Nullstellensatz in our context in for ideals of $\mathbb{C}[z]$.

Lemma 2.1.28. *Let I be a radical ideal in $\mathbb{C}[z]$ such that all the irreducible components of $V_{\mathbb{C}^d}(I)$ intersect $\mathbb{B}_d$. Then $I_{\mathbb{C}[z]}(V_{\mathbb{C}^d} \cap \mathbb{B}_d) = I$.*

Proof. This is an exercise in algebraic geometry. Assume first that $V_{\mathbb{C}^d}(I)$ is irreducible. Let $f \in \mathbb{C}[z]$ such that $f(\lambda) = 0$ for all $\lambda \in V_{\mathbb{C}^d}(I) \cap \mathbb{B}_d$. Denote $W = V_{\mathbb{C}^d}(f)$. By assumption, $W \cap \mathbb{B}_d \supseteq V_{\mathbb{C}^d}(I) \cap \mathbb{B}_d$, therefore $\dim W \cap V_{\mathbb{C}^d}(I) = \dim V_{\mathbb{C}^d}(I)$. It follows from [50, Proposition 1.4] that $W \cap V_{\mathbb{C}^d}(I) = V_{\mathbb{C}^d}(I)$, therefore $f \in I_{\mathbb{C}[z]}(V_{\mathbb{C}^d}(I)) = I$.

Finally, if $V_{\mathbb{C}^d}(I)$ is reducible then we apply this argument to each irreducible component. □

Corollary 2.1.29. *If I is a homogeneous ideal in $\mathbb{C}[z]$, then*

$$\sqrt{I} = I_{\mathbb{C}[z]}(V_{\mathbb{C}^d}(I)) = I_{\mathbb{C}[z]}(V_{\mathbb{B}_d}(I)).$$

Now we prove the main Nullstellensatz result:

Theorem 2.1.30. *Let $I \subseteq \mathcal{A}_{\mathcal{H}}$ be a closed homogeneous ideal and $J \subseteq \mathcal{M}_{\mathcal{H}}$ be a weak-$*$ closed homogeneous ideal. Then*

$$\sqrt{I} = I_{\mathcal{A}_{\mathcal{H}}}(V_\Omega(I)) \quad \text{and} \quad \sqrt{J} = I_{\mathcal{M}_{\mathcal{H}}}(V_\Omega(J)). \tag{2.4}$$

Proof. Define $K = I_{\mathcal{A}_{\mathcal{H}}}(V_\Omega(I))$. First, note that K is closed. Next we show that K is homogeneous. Notice that $V_\Omega(I) = V_\Omega(I \cap \mathbb{C}[z])$, so $tV_\Omega(I) \subseteq V_\Omega(I)$ for all $t \in \mathbb{D}$. Thus if $f \in K$, then for all $\lambda \in V_\Omega(I)$ it follows that $f(t\lambda) = 0$. By Proposition 2.1.18, K is homogeneous.

Finally, $K \cap \mathbb{C}[z]$ is the set of all polynomials vanishing on

$$V_\Omega(I) = V_\Omega(I \cap \mathbb{C}[z]) = V(I \cap \mathbb{C}[z]) \cap \Omega.$$

So by an easy extension of Corollary 2.1.29, we find

$$K \cap \mathbb{C}[z] = \sqrt{I \cap \mathbb{C}[z]} = \sqrt{I} \cap \mathbb{C}[z].$$

By Lemma 2.1.20 and Corollary 2.1.26,

$$K = \overline{K \cap \mathbb{C}[z]} = \overline{\sqrt{I} \cap \mathbb{C}[z]} = \sqrt{I}.$$

The weak-$*$ proof follows similarly. □

Corollary 2.1.31. *Let $I \subseteq \mathbb{C}[z]$ be a radical homogeneous ideal, and let $f \in \mathcal{A}_{\mathcal{H}}$ (resp. $\mathcal{M}_{\mathcal{H}}$) be a function that vanishes on $V_{\mathbb{C}^d}(I) \cap \Omega$. Then $f \in \overline{I}$ (resp. $\overline{I}^{w^*}$).*

Proof. Define $J = \overline{I}$. Then, using Theorem 2.1.30 and then Proposition 2.1.27,

$$f \in I_{\mathcal{A}_{\mathcal{H}}}(V_\Omega(I)) = I_{\mathcal{A}_{\mathcal{H}}}(V_\Omega(J)) = \sqrt{J} = J = \overline{I}.$$ □

2.2 Completely isometric isomorphisms

2.2.1 The general case

The automorphisms of $\mathcal{M}$ arise as composition with an automorphism of the ball (i.e., a biholomorphism of the ball onto itself). This can be deduced from [30, Section 4], or alternatively from Theorems 3.5 and 3.10 in [56]. In fact we can say more than this, specifically that the Voiculescu unitaries, when restricted to symmetric Fock space, are just composition with the conformal map followed by an appropriate multiplier.

Theorem 2.2.1. *Let $\varphi \in \mathrm{Aut}(\mathbb{B}_d)$. Then there is a completely isometric automorphism Θ_φ of $\mathcal{M}_d$ (and $\mathcal{A}_d$) given by $\Theta_\varphi(f) = f \circ \varphi = UfU^*$, where the unitary $U : H^2_d \to H^2_d$ is*

$$Uf = \left(1 - |\varphi^{-1}(0)|^2\right)^{1/2} k_{\varphi^{-1}(0)} (f \circ \varphi).$$

Proof. We begin with Voiculescu's construction of automorphisms of the Cuntz algebra [67]. Consider the Lie group $U(1,d)$ consisting of $(d+1) \times (d+1)$ matrices X satisfying $X^* J X = J$, where $J = \left[\begin{smallmatrix} -1 & 0 \\ 0 & I_d \end{smallmatrix}\right]$. When X is of the form $X = \left[\begin{smallmatrix} x_0 & \eta_1^* \\ \eta_2 & X_1 \end{smallmatrix}\right]$ it must have the following relations:

1. $\|\eta_1\|^2 = \|\eta_2\|^2 = |x_0|^2 - 1$
2. $X_1 \eta_1 = \overline{x}_0 \eta_2$ and $X_1^* \eta_2 = x_0 \eta_1$
3. $X_1^* X_1 = I_d + \eta_1 \eta_1^*$ and $X_1 X_1^* = I_d + \eta_2 \eta_2^*$.

Furthermore, if $X \in U(1,d)$ then $JX^T J \in U(1,d)$ since

$$(JX^T J)^* J (JX^T J) = J(X^*)^T J X^T J = (XJX^*J)^T J = I_{d+1} J = J.$$

It follows from Voiculescu's work that the map $U(1,d) \to \mathrm{Aut}(\mathbb{B}_d)$ given by

$$X \mapsto \varphi_X(z) := \frac{X_1 z + \eta_2}{x_0 + \langle z, \eta_1 \rangle}$$

is a surjective homomorphism. Thus, fix $X \in U(1,d)$ such that $\varphi = \varphi_{JX^T J}$ which makes

$$\varphi_{\overline{X}} = \varphi^{-1}_{JX^*J} = \varphi^{-1}_{JX^T J} = \varphi^{-1}.$$

There is a unique automorphism of $\mathcal{L}_d$ which preserves $\mathfrak{A}_d$ defined by

$$\Theta_\varphi(L_\zeta) = (x_0 I - L_{\eta_2})^{-1}(L_{X_1\zeta} - \langle \zeta, \eta_1\rangle I),$$

where we use the convention that $L_\zeta = \sum_{i=1}^n \zeta_i L_i$ for $\zeta \in \mathbb{C}^d$. This extends to an automorphism of the Cuntz-Toeplitz algebra. As well, Voiculescu defined a unitary $U \in \mathcal{U}(\mathcal{F}(\mathbb{C}^d))$ by

$$U(A\Omega) = \Theta_\varphi(A)(x_0 I - L_{\eta_2})^{-1}\Omega, \quad \text{for all } A \in \mathcal{L}_d,$$

establishing that the automorphism $\Theta_\varphi(A) = UAU^*$ is unitarily implemented. It is easy to see that H^2_d is an invariant subspace of U and so Θ_φ also yields an automorphism of $\mathcal{M}_d$ which preserves $\mathcal{A}_d$ implemented by the restriction of U. We will show that U has the desired form.

For $w \in \mathbb{F}_d^+, |w| = m$, we have

$$\begin{aligned} U(z_w) = U\big(\frac{1}{m!}\sum_{\sigma\in S_m} \xi_{\sigma(w)}\big) &= P_{H^2_d} U\Big(\big(\frac{1}{m!}\sum_{\sigma\in S_m} L_{\sigma(w)}\big)\Omega\Big) \\ &= P_{H^2_d}\Theta_\varphi(M_{z_w})P_{H^2_d}(x_0 I - L_{\eta_2})^{-1}\Omega. \end{aligned}$$

As noted above, because H^2_d must reduce U, we obtain $P_{H^2_d}\Theta_\varphi(A) = P_{H^2_d}\Theta_\varphi(A)P_{H^2_d}$. Suppose that $\zeta \in \mathbb{C}^d$. Then

$$P_{H^2_d}(L_\zeta)(z) = \sum_{i=1}^d \zeta_i z_i(z) = \sum_{i=1}^d \zeta_i \langle z, e_i\rangle = \langle z, \overline{\zeta}\rangle.$$

Now with $\overline{x_0^{-1}\eta_2} = \varphi_{\overline{X}}(0) = \varphi^{-1}(0)$, we have that

$$P_{H^2_d}(x_0 I - L_{\eta_2})^{-1}\Omega = \frac{1}{x_0 - \langle z, \overline{\eta_2}\rangle} = x_0^{-1} k_{\varphi^{-1}(0)}.$$

Note that if $|\theta| = 1$, then θX implements φ_X as well. So we may assume that $x_0 \geq 0$. As well, $X \in U(1,d)$ implies that $|x_0|^2 - |\eta_2|^2 = 1$. Hence,

$$|\varphi^{-1}(0)|^2 = |\varphi_X(0)|^2 = \frac{|\eta_2|^2}{|x_0|^2} = \frac{|x_0|^2 - 1}{|x_0|^2}.$$

Thus $x_0 = (1 - |\varphi^{-1}(0)|^2)^{-1/2}$.

Next we compute

$$\begin{aligned}
P_{H^2_d}\Theta_\varphi(M_{z_w}) &= P_{H^2_d}\Theta_\varphi\big(\frac{1}{m!}\sum_{\sigma\in S_m} L_{\sigma(w)}\big) \\
&= \frac{1}{m!}\sum_{\sigma\in S_m}\prod_{j=1}^m P_{H^2_d}\Theta_\varphi(L_{\sigma(w)_j}) \\
&= \prod_{j=1}^m P_{H^2_d}\Theta_\varphi(L_{w_j}) \\
&= \prod_{j=1}^m P_{H^2_d}\frac{L_{X_1 e_{w_j}} - \langle e_{w_j}, \eta_1\rangle I}{x_0 I - L_{\eta_2}}.
\end{aligned}$$

Observe that

$$JX^T J = \begin{bmatrix} x_0 & -\overline{\eta_2}^* \\ -\overline{\eta_1} & X_1^T \end{bmatrix}.$$

Consequently,

$$\begin{aligned}
P_{H^2_d}\Theta_\varphi(M_{z_w})(z) &= \prod_{j=1}^m \frac{P_{H^2_d}L_{X_1 e_{w_j}}(z) - \langle e_{w_j}, \eta_1\rangle}{x_0 - P_{H^2_d}L_{\eta_2}(z)} \\
&= \prod_{j=1}^m \frac{\langle z, \overline{X_1 e_{w_j}}\rangle - \langle \overline{\eta_1}, e_{w_j}\rangle}{x_0 - \langle z, \overline{\eta_2}\rangle} = \prod_{j=1}^m \frac{\langle X_1^T z, e_{w_j}\rangle + \langle -\overline{\eta_1}, e_{w_j}\rangle}{x_0 + \langle z, -\overline{\eta_2}\rangle} \\
&= \prod_{j=1}^m z_{w_j}\left(\frac{X_1^T z + -\overline{\eta_1}}{x_0 + \langle z, -\overline{\eta_2}\rangle}\right) = \prod_{j=1}^m z_{w_j}(\varphi_{JX^TJ}(z)) \\
&= \prod_{j=1}^m z_{w_j}(\varphi(z)) = (z_w \circ \varphi)(z).
\end{aligned}$$

Combining these equations, we get that

$$\begin{aligned}
U(z_w) &= \Big(\prod_{j=1}^m z_{w_j}\circ\varphi\Big)(1 - |\varphi^{-1}(0)|^2)^{1/2} k_{\varphi^{-1}(0)} \\
&= (z_w\circ\varphi)(1 - |\varphi^{-1}(0)|^2)^{1/2} k_{\varphi^{-1}(0)}.
\end{aligned}$$

Extending this to the span, we have that

$$Uf = (1 - |\varphi^{-1}(0)|^2)^{1/2} k_{\varphi^{-1}(0)}(f\circ\varphi)$$

for all $f \in \mathcal{M}_d$. □

Proposition 2.2.2. *Let V and W be varieties in $\mathbb{B}_d$. Let F be an automorphism of $\mathbb{B}_d$ that maps W onto V. Then $f \mapsto f \circ F$ is a unitarily implemented completely isometric isomorphism of $\mathcal{M}_V$ onto $\mathcal{M}_W$; i.e. $M_{f\circ F} = UM_fU^*$. The unitary U^* is the linear extension of the map*

$$U^*k_w = c_w k_{F(w)} \quad \textit{for } w \in W,$$

where $c_w = (1 - \|F^{-1}(0)\|^2)^{1/2}\overline{k_{F^{-1}(0)}(w)}$.

Proof. Let F be such an automorphism, and set $\alpha = F^{-1}(0)$. By the previous theorem, the unitary map $U \in \mathcal{B}(H^2_d)$ is given by

$$Uh = (1 - \|\alpha\|^2)^{1/2} k_\alpha (h \circ F) \quad \text{for } h \in H^2_d.$$

As $F(W) = V$, U takes the functions in H^2_d that vanish on V to the functions in H^2_d that vanish on W. Therefore it takes $\mathcal{F}_V$ onto $\mathcal{F}_W$.

Let us compute U^*. For $h \in H^2_d$ and $w \in W$, we have

$$\begin{aligned}\langle h, U^*k_w\rangle &= \langle Uh, k_w\rangle \\ &= \left\langle (1 - \|\alpha\|^2)^{1/2} k_\alpha (h\circ F), k_w \right\rangle \\ &= (1 - \|\alpha\|^2)^{1/2} k_\alpha(w)\, h(F(w)) \\ &= \left\langle h, c_w k_{F(w)} \right\rangle,\end{aligned}$$

where $c_w = (1 - \|F^{-1}(0)\|^2)^{1/2}\overline{k_{F^{-1}(0)}(w)}$. Thus $U^*k_w = c_w k_{F(w)}$. Note that since U^* is a unitary, $|c_w| = \|k_w\|/\|k_{F(w)}\|$.

Finally, we show that conjugation by U implements the isomorphism between $\mathcal{M}_V$ and $\mathcal{M}_W$ given by composition with F. Observe that $Uc_wk_{F(w)} = k_w$. For $f \in \mathcal{M}_V$ and $w \in W$,

$$UM_f^*U^*k_w = UM_f^*c_wk_{F(w)} = \overline{f(F(w))}Uc_wk_{F(w))} = \overline{(f\circ F)(w)}k_w.$$

Therefore $f \circ F$ is a multiplier on $\mathcal{F}_W$ and $M_{f\circ F} = UM_fU^*$. ∎

Now we turn to the converse.

Lemma 2.2.3. *Let $V \subseteq \mathbb{B}_d$ and $W \subseteq \mathbb{B}_{d'}$ be varieties. Let φ be a unital, completely contractive algebra isomorphism of $\mathcal{M}_V$ into $\mathcal{M}_W$. Then there exists a holomorphic map $F : \mathbb{B}_{d'} \to \mathbb{B}_d$ such that*

1. *$F(W) \subseteq V$.*

2. $F|_W = \varphi^*|_W$.

3. the components $f_1, \dots, f_d$ *of* F *form a row contraction of operators in* $\mathcal{M}_{d'}$.

4. φ *is given by composition with* F*, that is*

$$\varphi(f) = f \circ F \quad \textit{for } f \in \mathcal{M}_V.$$

Proof. Consider the image of the coordinate functions Z_i in $\mathcal{M}_V$. As φ is completely contractive, Proposition 2.1.7 shows that $\begin{bmatrix}\varphi(Z_1) & \dots & \varphi(Z_d)\end{bmatrix}$ is the restriction to W of a row contractive multiplier $F = \begin{bmatrix}f_1, \dots, f_d\end{bmatrix}$ with coefficients in $\mathcal{M}_{d'}$. As F is contractive as a multiplier, it is also contractive in the sup norm. Moreover, since φ is injective, the f_i and F are non-constant holomorphic functions. Therefore F must have range in the open ball $\mathbb{B}_d$.

Fix $\lambda \in W$, and let ρ_λ be the evaluation functional at λ on $\mathcal{M}_W$. Then $\varphi^*(\rho_\lambda)$ is a character in $M(\mathcal{M}_V)$. We want to show that it is also an evaluation functional. Compute

$$[\varphi^*(\rho_\lambda)](Z_i) = Z_i(\varphi^*(\rho_\lambda)) = \rho_\lambda(\varphi(Z_i)) = \varphi(Z_i)(\lambda).$$

So $\varphi^*(\rho_\lambda)$ lies in the fiber over $(\varphi(Z_1)(\lambda), \dots, \varphi(Z_d)(\lambda)) = F(\lambda)$. This is in the interior of the ball. By Proposition 2.1.14, $\varphi^*(\rho_\lambda)$ is the point evaluation functional $\rho_{F(\lambda)}$ and $F(\lambda) \in V$. We abuse notation by saying that $\varphi^*(\rho_\lambda) \in V$.

Finally, for every $f \in \mathcal{M}_V$ and every $\lambda \in W$,

$$\begin{aligned}\varphi(f)(\lambda) &= \rho_\lambda(\varphi(f)) = \varphi^*(\rho_\lambda)(f) \\ &= \rho_{F(\lambda)}(f) = (f \circ F)(\lambda).\end{aligned}$$

Therefore $\varphi(f) = f \circ F$. □

Lemma 2.2.4. *Let* $0 \in V \subseteq \mathbb{B}_d$ *and* $0 \in W \subseteq \mathbb{B}_{d'}$ *be varieties. Let* $\varphi : \mathcal{M}_V \to \mathcal{M}_W$ *be a completely isometric isomorphism such that* $\varphi^*\rho_0 = \rho_0$*. Then there exists an isometric linear map* F *of* $\mathbb{B}_{d'} \cap \operatorname{span} W$ *onto* $\mathbb{B}_d \cap \operatorname{span} V$ *such that* $F(W) = V$*,* $F(0) = 0$ *and* $F|_W = \varphi^*$*.*

Proof. By making d smaller, we may assume that $\mathbb{C}^d = \operatorname{span} V$. Similarly, we may assume $\mathbb{C}^{d'} = \operatorname{span} W$.

By Lemma 2.2.3 applied to φ, there is a holomorphic map F of $\mathbb{B}_{d'}$ into $\mathbb{B}_d$ that implements φ^*. Thus $F(W) \subseteq V$ and $F(0) = 0$. By the same lemma applied to φ^{-1}, there is a holomorphic map G of $\mathbb{B}_d$ into $\mathbb{B}_{d'}$ that implements $(\varphi^{-1})^*$. Hence $G(V) \subseteq W$ and $G(0) = 0$. Now, φ^* and $(\varphi^{-1})^*$ are inverses of each other. Therefore $F \circ G|_V$ and $G \circ F|_W$ are the identity maps.

Let $H = F \circ G$. Then H is a holomorphic map of $\mathbb{B}_d$ into itself, such that $H|_V$ is the identity. In particular $H(0) = 0$. By [62, Theorem 8.2.2], the fixed point set of H is an affine set equal to the fixed point set of $H'(0)$ in $\mathbb{B}_d$. Therefore H is the identity on $\mathbb{B}_d$ since $\mathbb{C}^d = \operatorname{span} V$. Applying the same reasoning to $G \circ F$, we see that F is a biholomorphism of $\mathbb{B}_{d'}$ onto $\mathbb{B}_d$ such that $F(W) = V$. In particular, $d' = d$. It now follows from a theorem of Cartan [62, Theorem 2.1.3] that F is a unitary linear map. □

Now we combine these lemmas to obtain the main result of this section.

If $V \subseteq \mathbb{B}_d$ and $W \subseteq \mathbb{B}_{d'}$ are varieties, then we can consider them both as varieties in $\mathbb{B}_{\max(d,d')}$. This does not change the operator algebras. Therefore, we may assume that $d = d'$.

Theorem 2.2.5. *Let V and W be varieties in $\mathbb{B}_d$. Then $\mathcal{M}_V$ is completely isometrically isomorphic to $\mathcal{M}_W$ if and only if there exists an automorphism F of $\mathbb{B}_d$ such that $F(W) = V$.*

In fact, every completely isometric isomorphism $\varphi : \mathcal{M}_V \to \mathcal{M}_W$ arises as composition $\varphi(f) = f \circ F$ where F is such an automorphism. In this case, φ is unitarily implemented by the unitary sending the kernel function $k_w \in \mathcal{F}_W$ to a scalar multiple of the kernel function $k_{F(w)} \in \mathcal{F}_V$.

Proof. If there is such an automorphism, then the two algebras are completely isometrically isomorphic by Proposition 2.2.2; and the unitary is given explicitly there.

Conversely, assume that φ is a completely isometric isomorphism of $\mathcal{M}_V$ onto $\mathcal{M}_W$. By Lemma 2.2.3, φ^* maps W into V. Pick a point $w_0 \in W$ and set $v_0 = \varphi^*(w_0)$. By applying automorphisms of $\mathbb{B}_d$ that move v_0 and w_0 to 0 respectively, and applying Proposition 2.2.2, we may assume that $0 \in V$ and $0 \in W$ and $\varphi^*(0) = 0$.

Now we apply Lemma 2.2.4 to obtain an isometric linear map F of the ball $\mathbb{B}_d \cap \operatorname{span} W$ onto the ball $\mathbb{B}_d \cap \operatorname{span} V$ such that $F|_W = \varphi^*$. In particular, $\operatorname{span} W$ and $\operatorname{span} V$ have the same dimension. (Caveat: this is only true in the case that both V and W contain 0.) We may extend the definition of F to a unitary map on $\mathbb{C}^d$, and so it extends to a biholomorphism of $\mathbb{B}_d$.

Now Proposition 2.2.2 yields a unitary which implements composition by φ^*. By Lemma 2.2.3, every completely isometric isomorphism φ is given as a composition by φ^*. So all maps have the form described. □

There is a converse to Lemma 2.2.3, which may provide an alternative proof for one half of Theorem 2.2.5. Arguments like the following are not uncommon in the theory of RKHS; see for example [44, Theorem 5].

Proposition 2.2.6. *Let $V \subseteq \mathbb{B}_d$ and $W \subseteq \mathbb{B}_{d'}$ be varieties. Suppose that there exists a holomorphic map $F : \mathbb{B}_{d'} \to \mathbb{B}_d$ that satisfies $F(W) \subseteq V$, such that the components $f_1, \dots, f_d$ of F form a row contraction of operators in $\mathcal{M}_{d'}$. Then the map given by composition with F*

$$\varphi(f) = f \circ F \quad \text{for } f \in \mathcal{M}_V$$

yields a unital, completely contractive algebra homomorphism of $\mathcal{M}_V$ into $\mathcal{M}_W$.

Proof. Composition obviously gives rise to a unital homomorphism, so all we have to demonstrate is that φ is completely contractive. We make use of the complete NP property of these kernels.

Let $G \in M_k(\mathcal{M}_V)$ with $\|G\| \leq 1$. Then for any N points $w_1, \dots, w_N$ in W, we get N points $F(w_1), \dots, F(w_N)$ in V. The fact that $\|G\| \leq 1$ implies that the $N \times N$ matrix with $k \times k$ matrix entries

$$\left[\frac{I_k - (G \circ F)(w_i)(G \circ F)(w_j)^*}{1 - \langle F(w_i), F(w_j) \rangle} \right]_{N \times N} \geq 0.$$

Also, since $\|F\| \leq 1$ as a multiplier on $\mathcal{F}_W$, we have that

$$\left[\frac{1 - \langle F(w_i), F(w_j) \rangle}{1 - \langle w_i, w_j \rangle} \right]_{N \times N} \geq 0.$$

Therefore the Schur product of these two positive matrices is positive:

$$\left[\frac{I_k - (G \circ F)(w_i)(G \circ F)(w_j)^*}{1 - \langle w_i, w_j \rangle} \right]_{N \times N} \geq 0.$$

Now the complete NP property yields that $G \circ F$ is a contractive multiplier in $M_k(\mathcal{M}_W)$. □

2.2.2 The continuous case

If the varieties satisfy condition 2.1, i.e. $[I_V H^2_d] = [J_V H^2_d]$, then the general multiplier results drop down to the continuous multiplier algebras.

Proposition 2.2.7. *Let $V \subseteq \mathbb{B}_d$ and $W \subseteq \mathbb{B}_{d'}$ be varieties which satisfy condition* (2.1)*. Let $\varphi : \mathcal{A}_V \to \mathcal{A}_W$ be a unital algebra homomorphism. Then there exists a holomorphic map $F : \mathbb{B}_{d'} \to \mathbb{C}^d$ that extends continuously to $\overline{\mathbb{B}}_{d'}$ such that*

$$F|_{\overline{W}^{\mathcal{A}}} = \varphi^*.$$

The components of F are in $\mathcal{A}_{d'}$, and norm of F as a row of multipliers is less than or equal to the cb-norm of φ. Moreover, φ is given by composition with F, that is

$$\varphi(f) = f \circ F \quad \text{for } f \in \mathcal{A}_V.$$

Proof. Every character in $M(\mathcal{A}_W)$ is an evaluation functional at some point $\lambda \in \overline{W}^{\mathcal{A}}$. Identifying $\overline{W}^{\mathcal{A}}$ and $M(\mathcal{A}_W)$, we find, as in Lemma 2.2.3, that the mapping φ^* is given by

$$\varphi^*(\lambda) = (\varphi(Z_1)(\lambda), \ldots, \varphi(Z_d)(\lambda)) \quad \text{for all } \lambda \in \overline{W}^{\mathcal{A}}.$$

Proposition 2.1.11 implies that $\varphi(Z_1), \ldots, \varphi(Z_d)$ are restrictions to W of functions $f_1, \ldots, f_d$ in $\mathcal{A}_{d'}$. (This is only true under our special assumption (2.1). Otherwise we only get $f_1, \ldots, f_d$ in $\mathcal{M}_{d'}$.) Defining

$$F(z) = (f_1(z), \ldots, f_d(z)),$$

we obtain the required map F. Finally, for every $\lambda \in \overline{W}^{\mathcal{A}}$,

$$\varphi(f)(\lambda) = \rho_\lambda(\varphi(f)) = \varphi^*(\rho_\lambda)(f) = \rho_{F(\lambda)}(f) = f(F(\lambda)).$$

Therefore $\varphi(f) = f \circ F$. □

This immediately yields:

Corollary 2.2.8. *Let $V \subseteq \mathbb{B}_d$ and $W \subseteq \mathbb{B}_{d'}$ be varieties satisfying condition* (2.1)*. If $\mathcal{A}_V$ and $\mathcal{A}_W$ are isomorphic, then there are two holomorphic maps $F : \mathbb{B}_{d'} \to \mathbb{C}^d$ and $G : \mathbb{B}_d \to \mathbb{C}^{d'}$ which extend continuously to the closed balls, such that $F(\overline{W}^{\mathcal{A}}) = \overline{V}^{\mathcal{A}}$, $G(\overline{V}^{\mathcal{A}}) = \overline{W}^{\mathcal{A}}$, and $F|_{\overline{W}^{\mathcal{A}}}$ and $G|_{\overline{V}^{\mathcal{A}}}$ are inverses of each other. If V and W satisfy the condition* (2.2)*, then $F(W) = V$ and $G(V) = W$.*

From these results and the techniques of Lemma 2.2.4, we also get if $\mathcal{A}_V$ and $\mathcal{A}_W$ are completely isometrically isomorphic, then there exists an automorphism $F \in \mathrm{Aut}(\mathbb{B}_d)$ such that $F(V) = W$. On the other hand, the completely isometric isomorphisms of Proposition 2.2.2 are easily seen to respect the norm closures of the polynomials in $\mathcal{M}_V$ and $\mathcal{M}_W$. Together with the above corollary we obtain the following analogue to Theorem 2.2.5.

Theorem 2.2.9. *Let V and W be varieties in $\mathbb{B}_d$ satisfying* (2.1)*. Then $\mathcal{A}_V$ is completely isometrically isomorphic to $\mathcal{A}_W$ if and only if there exists an automorphism F of $\mathbb{B}_d$ such that $F(W) = V$.*

Every completely isometric isomorphism $\varphi : \mathcal{A}_V \to \mathcal{A}_W$ arises as composition $\varphi(f) = f \circ F$ where F is such an automorphism. In this case φ is unitarily implemented by a unitary sending the kernel function $k_w \in \mathcal{F}_W$ to a scalar multiple of the kernel function $k_{F(w)} \in \mathcal{F}_V$.

Remark 2.2.10. In the first paper with Davidson and Shalit [32], this theory was established by working with subproduct systems [14, 63, 66], which is a subject of much interest in and of itself. In fact, it provided similar results in the non-commutative setting.

2.2.3 Toeplitz algebras and C*-envelopes

Central to the theory of non-selfadjoint operator algebras is the notion of a C*-envelope [12, 34, 40, 45], which can be thought of as the smallest C*-algebra that contains the operator algebra.

In this section we consider the *Toeplitz algebra* of V, defined as $\mathcal{T}_V = \mathrm{C}^*(\mathcal{A}_V)$. Theorem 2.2.9 tells us that every completely isometric isomorphism between continuous multiplier algebras is unitarily implemented. This gives us that:

Proposition 2.2.11. *Let V and W be varieties satisfying condition 2.1. If $\mathcal{A}_V$ and $\mathcal{A}_W$ are completely isometrically isomorphic then $\mathcal{T}_V$ and $\mathcal{T}_W$ are $*$-isomorphic.*

One can say some things in general without the special assumption. In particular, [19, Proposition 6.4.6] tells us that $\mathcal{T}_V$ contains all compact operators on $\mathcal{F}_V$. This is due to the fact that the compression of the compact operator $I - \sum_{i=1}^d Z_i^* Z_i$ to $\mathcal{F}_V$ is still non-zero. In light of this, call $\mathcal{O}_V = \mathcal{T}_V / \mathcal{K}(\mathcal{F}_V)$, the *Cuntz-Toeplitz algebra*. A variant of another part of the same proposition in [19] gives:

Lemma 2.2.12. *If $d > 1$ then the quotient map $q : \mathcal{T}_V \to \mathcal{O}_V$ is not a complete isometry on $\mathcal{A}_V$.*

By [10, Theorem 2.1.1], the identity representation is a boundary representation if and only if the quotient map $q : \mathcal{T}_V \to \mathcal{O}_V$ is not a complete isometry. Thus the above lemma gives immediately:

Corollary 2.2.13. *The identity representation of $\mathcal{T}_V$ is a boundary representation for $\mathcal{A}_V$.*

Since the Silov boundary ideal is contained in the kernel of any boundary representation, we find that the Silov ideal of $\mathcal{A}_V$ in $\mathcal{T}_V$ is $\{0\}$. Thus we obtain:

Theorem 2.2.14. *The C*-envelope of $\mathcal{A}_V$ is $\mathcal{T}_V$.*

Having brought C*-algebras into our discussion about universal operator algebras, one might wonder whether our methods give any handle on the universal unital C*-algebra generated by a row contraction subject to homogeneous polynomial relations. Unfortunately, these universal C*-algebras are out of our reach. All we can say is that $\mathcal{T}_V$ is *not*, in general, the universal unital C*-algebra generated by a row contraction subject to the relations in V. One can see this by considering the case $d = 1$ and no relations. Then $\mathcal{T}_V$ is the ordinary Toeplitz algebra, which is not the universal unital C*-algebra generated by a contraction.

2.3 Algebraic isomorphisms

2.3.1 The general case

We turn now to the question: *when does there exist an (algebraic) isomorphism between $\mathcal{M}_V$ and $\mathcal{M}_W$?* This problem is more subtle, and we frequently need to assume that the variety sits inside a finite dimensional ambient space. Even the construction of the biholomorphism seems to rely on some delicate facts about complex varieties.

We begin with a well-known automatic continuity result. Recall that a commutative Banach algebra is *semi-simple* if the Gelfand transform is injective.

Lemma 2.3.1. *Let V and W be varieties in $\mathbb{B}_d$. Every homomorphism from $\mathcal{M}_V$ to $\mathcal{M}_W$ is norm continuous.*

Proof. The algebras that we are considering are easily seen to be semi-simple. A general result in the theory of commutative Banach algebras says that every homomorphism into a semi-simple algebra is automatically continuous (see [22, Prop. 4.2]). □

Lemma 2.3.2. *Let V and W be varieties in $\mathbb{B}_d$ and $\mathbb{B}_{d'}$, respectively, with $d' < \infty$. Let $\varphi : \mathcal{M}_V \to \mathcal{M}_W$ be an algebra isomorphism. Suppose that λ is an isolated point in W. Then $\varphi^*(\rho_\lambda)$ is an evaluation functional at a point in V.*

Proof. The character ρ_λ is an isolated point in $M(\mathcal{M}_W)$. (Here is where we need $d' < \infty$.) Since φ^* is a homeomorphism, $\varphi^*(\rho_\lambda)$ must also be an isolated point in $M(\mathcal{M}_V)$. By Shilov's idempotent theorem (see [15, Theorem 21.5]), the characteristic function $\chi_{\varphi^*(\rho_\lambda)}$ of $\varphi^*(\rho_\lambda)$ belongs to $\mathcal{M}_V$. Now suppose that $\varphi^*(\rho_\lambda)$ is in the corona $M(\mathcal{M}_V) \setminus V$. Then $\chi_{\varphi^*(\rho_\lambda)}$ vanishes on V. Therefore, as an element of a multiplier algebra, this means that $\chi_{\varphi^*(\rho_\lambda)} = 0$. Therefore $\chi_{\varphi^*(\rho_\lambda)}$ must vanish on the entire maximal ideal space, which is a contradiction. Thus $\varphi^*(\rho_\lambda)$ lies in V. $\square$

Next we want to show that any algebra isomorphism φ between $\mathcal{M}_V$ and $\mathcal{M}_W$ must induce a biholomorphism between W and V. This identification will be the restriction of φ^* to the characters of evaluation at points of W. In order to achieve this, we need to make some additional assumption.

Our difficulty is basically that we do not have enough information about varieties. In the classical case, if one takes a regular point $\lambda \in V$, takes the connected component of λ in the set of all regular points of V, and closes it up (in $\mathbb{B}_d$), then one obtains a subvariety. Moreover the closure of the complement of this component is also a variety [68, ch.3, Theorem 1G].

However our varieties are the intersections of zero sets of a family of multipliers. Let us say that a variety V is *irreducible* if for any regular point $\lambda \in V$, the intersection of zero sets of all multipliers vanishing on a small neighbourhood $V \cap b_\varepsilon(\lambda)$ is exactly V. We do not know, for example, whether an irreducible variety in our sense is connected. Nor do we know that if we take an irreducible subvariety of a variety, then there is a complementary subvariety as in the classical case.

A variety V is said to be *discrete* if it has no accumulation points in $\mathbb{B}_d$.

We will resolve this in two situations. The first is the case of a finite union of irreducible varieties and a discrete variety. The second is the case of an isometric isomorphism. In the latter case, the isomorphism will turn out to be completely isometric. This yields a different approach to the results of the previous section.

We need some information about the maximal ideal space $M(\mathcal{M}_V)$. Recall that there is a canonical projection π into $\overline{\mathbb{B}}_d$ obtained by evaluation at $[Z_1, \dots, Z_d]$. For any point μ in the unit sphere, $\pi^{-1}(\mu)$ is the fiber of $M(\mathcal{M}_V)$ over μ. We saw in Proposition 2.1.14 that for $\lambda \in \mathbb{B}_d$, $\pi^{-1}(\lambda)$ is the singleton $\{\rho_\lambda\}$, the point evaluation at λ. The following lemma is analogous to results about Gleason parts for function algebras (see [13]). However part (2) shows that this is different from Gleason parts, as disjoint subvarieties of V will be at a distance of less than 2 apart. This is because $\mathcal{M}_V$ is a (complete) quotient of $\mathcal{M}_d$, and thus the difference $\|\rho_\lambda - \rho_\mu\|$ is the same whether evaluated as functionals on $\mathcal{M}_V$ or $\mathcal{M}_d$. In the latter algebra, λ and ν do lie in the same Gleason part.

Lemma 2.3.3. *Let V be a variety in $\mathbb{B}_d$.*

1. *Let $\varphi \in \pi^{-1}(\mu)$ for $\mu \in \partial\mathbb{B}_d$. Suppose that $\psi \in M(\mathcal{M}_V)$ satisfies $\|\psi - \varphi\| < 2$. Then ψ also belongs to $\pi^{-1}(\mu)$.*
2. *If λ and μ belong to V, then $\|\rho_\mu - \rho_\lambda\| \le 2r < 2$, where r is the pseudohyperbolic distance between μ and λ.*

Proof. If $\psi \in \pi^{-1}(\nu)$ for $\nu \ne \mu$ in the sphere, then there is an automorphism of $\mathbb{B}_d$ that takes μ to $(1, 0, \dots, 0)$ and ν to $(-1, 0, \dots, 0)$. Proposition 2.2.2 shows that composition by this automorphism is a completely isometric automorphism. So we may suppose that $\mu = (1, 0, \dots, 0)$ and $\nu = (-1, 0, \dots, 0)$. But then

$$\|\psi - \varphi\| \ge |(\psi - \varphi)(Z_1)| = 2.$$

Similarly, if $\psi = \rho_\lambda$ for some $\lambda \in V$, then for any $0 < \varepsilon < 1$, there is an automorphism of $\mathbb{B}_d$ that takes μ to $(1, 0, \dots, 0)$ and ν to $(-1+\varepsilon, 0, \dots, 0)$. The same conclusion is reached by letting ε decrease to 0.

If λ and μ belong to V, then there is an automorphism γ of $\mathbb{B}_d$ sending λ to 0 and μ to some $v := (r, 0, \dots, 0)$ where $0 < r < 1$ is the pseudohyperbolic distance between λ and μ. Given any multiplier $f \in \mathcal{M}_V$ with $\|f\| = 1$, Proposition 2.1.7 provides a multiplier $\tilde{f}$ in $\mathcal{M}_d$ so that $\tilde{f}|_V = f$ and $\|\tilde{f}\| = 1$. In particular, $\tilde{f} \circ \gamma^{-1}$ is holomorphic on $\mathbb{B}_d$ and $\|\tilde{f} \circ \gamma^{-1}\|_\infty \le 1$. Hence the Schwarz Lemma [62, Theorem 8.1.4] shows that

$$\left| \frac{f(\mu) - f(\lambda)}{1 - f(\mu)\overline{f(\lambda)}} \right| = \left| \frac{\tilde{f} \circ \gamma^{-1}(v) - \tilde{f} \circ \gamma^{-1}(0)}{1 - \tilde{f} \circ \gamma^{-1}(v)\overline{\tilde{f} \circ \gamma^{-1}(0)}} \right| \le r.$$

Hence

$$\|\rho_\mu - \rho_\lambda\| = \sup_{\|f\| \le 1} |(\rho_\mu - \rho_\lambda)(f)| \le r \sup_{\|f\| \le 1} |1 - f(\mu)\overline{f(\lambda)}| \le 2r. \qquad \square$$

This provides some immediate information about norm continuous maps between these maximal ideal spaces.

Corollary 2.3.4. *Suppose that φ is a homomorphism of $\mathcal{M}_V$ into $\mathcal{M}_W$.*

1. *Then φ^* maps each irreducible subvariety of W into V or into a single fiber of the corona.*

2. *If φ is an isomorphism, and V and W are the disjoint union of finitely many irreducible subvarieties, then φ^* must map W onto V.*

3. *If φ is an isometric isomorphism, then φ^* maps W onto V and preserves the pseudohyperbolic distance.*

Proof. (1) Let W_1 be an irreducible subvariety of W, and let λ be any regular point of W_1. We do not assert that W_1 is connected.

Suppose that $\varphi^*(\rho_\lambda)$ is a point evaluation at some point μ in $\mathbb{B}_d$. Then by Proposition 2.1.14, μ belongs to V. Since φ is norm continuous, by Lemma 2.3.3 it must map the connected component of λ into a connected component of V.

Similarly, suppose that $\varphi^*(\rho_\lambda)$ is mapped into a fiber of the corona. Without loss of generality, we may suppose that it is the fiber over $(1, 0, \ldots, 0)$. Since φ is norm continuous, by Lemma 2.3.3 it must map the connected component of λ into this fiber as well. Suppose that there is some point μ in W_1 mapped into V or into another fiber. So the whole connected component of μ is also mapped into V or another fiber. Then the function $h = \varphi(Z_1) - 1$ vanishes on the component of λ but does not vanish on the component containing μ. This contradicts the fact that W_1 is irreducible. Thus the whole subvariety must map entirely into a single fiber or entirely into V.

(2) Suppose that W is the union of irreducible subvarieties $W_1, \ldots, W_n$. Fix a point $\lambda \in W_1$. For each $2 \leq i \leq n$, there is a multiplier $h_i \in \mathcal{M}_{d'}$ which vanishes on W_i but $h_i(\lambda) \neq 0$. Hence $h = h_2 h_3 \cdots h_k|_W$ belongs to $\mathcal{M}_W$ and vanishes on $\cup_{i=2}^k W_i$ but not on W_1. Therefore $\varphi^{-1}(h) = f$ is a non-zero element of $\mathcal{M}_V$. Suppose that $\varphi^*(W_1)$ is contained in a fiber over a point in the boundary of the sphere, say $(1, 0, \ldots, 0)$. Since $Z_1 - 1$ is non-zero on V, we see that $(Z_1 - 1)f$ is not the zero function. However, $(Z_1 - 1)f$ vanishes on $\varphi^*(W_1)$. Therefore $\varphi((Z_1 - 1)f)$ vanishes on W_1 and on $\cup_{i=2}^k W_i$. Hence $\varphi((Z_1 - 1)f) = 0$, contradicting injectivity. We deduce that W_1 is mapped into V.

By interchanging the roles of V and W, we deduce that φ^* must map W onto V.

(3) In the isometric case, we can make use of Lemma 2.3.3(2) because then φ^* is also isometric. Therefore all of W is mapped by φ^* either into V or into a single fiber. In the latter case, we may suppose that the fiber is over $(1, 0, \ldots, 0)$. Then $\varphi(Z_1 - 1)$ will vanish on all of W, and hence $\varphi(Z_1 - 1) = 0$, contradicting injectivity. Thus W is mapped into V. Reversing the role of V and W shows that this map is also onto V.

The proof of Lemma 2.3.3(2) actually yields more information, namely that $\|\rho_\lambda - \rho_\mu\|$ is a function of the pseudohyperbolic distance r,

$$\|\rho_\lambda - \rho_\mu\| = r \sup_{\|f\| \le 1} |1 - f(\mu)\overline{f(\lambda)}|.$$

In the proof of that lemma we only used that the left hand side is less than or equal to the right hand side, but it is easy to see that one obtains equality by choosing a particular f. So the fact that the quantities $\|\rho_\lambda - \rho_\mu\|$ and $\sup_{\|f\|\le 1} |1 - f(\mu)\overline{f(\lambda)}|$ are preserved by an isometric isomorphism implies that the pseudohyperbolic distance r is also preserved. □

Remarks 2.3.5. (1) In a previous version of [33], we claimed incorrectly that if φ is a surjective continuous homomorphism of $\mathcal{M}_V$ onto $\mathcal{M}_W$, then φ^* must map W into V. This is false, and we thank Michael Hartz for pointing this out. This follows from Hoffman's theory [42] of analytic disks in the corona of H^∞. There is an analytic map L of the unit disk $\mathbb{D}$ into the corona of $M(H^\infty)$, mapping onto a Gleason part, with the property that $\varphi(h)(z) = h(L(z))$ is a homomorphism of H^∞ onto itself [37, ch.X§1]. Therefore the map φ^* maps the disk into the corona via L.

(2) The main obstacle preventing us from establishing part (2) of the corollary in greater generality is that we do not know that if $\lambda \in W$, then there is an irreducible subvariety $W_1 \subset W$ containing λ and another subvariety $W_2 \subset W$ so that $\lambda \notin W_2$ and $W = W_1 \cup W_2$. As mentioned in the introduction, for any classical analytic variety this is possible [68, ch.3, Theorem 1G]. But our definition requires these subvarieties to be the intersection of zero sets of multipliers. Moreover our proof makes significant use of these functions. So we cannot just redefine our varieties to have a local definition as in the classical case even if we impose the restriction that all functions are multipliers. A better understanding of varieties in our context is needed.

(3) Costea, Sawyer and Wick [20] establish a corona theorem for the algebra $\mathcal{M}_d$. That is, the closure of the ball $\mathbb{B}_d$ in $M(\mathcal{M}_d)$ is the entire maximal ideal space. This result may also hold for the quotients $\mathcal{M}_V$, but we are not aware of any direct proof deducing this from the result for the whole ball.

A corona theorem for $\mathcal{M}_V$ would resolve the difficulties in case (2). The topology on $V = \mathbb{B}_d \cap M(\mathcal{M}_V)$ coincides with the usual one. In particular, each component has closed complement. The corona theorem would establish that every open subset of any fiber is in the closure of its complement. Thus any homeomorphism φ^* of $M(\mathcal{M}_W)$ onto $M(\mathcal{M}_V)$ must take W onto V. However it is likely that the corona theorem for $\mathcal{M}_V$ is much more difficult than our problem.

Now we can deal with the case in which our variety is a finite union of nice subvarieties, where nice will mean either irreducible or discrete.

Theorem 2.3.6. *Let V and W be varieties in $\mathbb{B}_d$, with $d < \infty$, which are the union of finitely many irreducible varieties and a discrete variety. Let φ be a unital algebra isomorphism of $\mathcal{M}_V$ onto $\mathcal{M}_W$. Then there exist holomorphic maps F and G from $\mathbb{B}_d$ into $\mathbb{C}^d$ with coefficients in $\mathcal{M}_d$ such that*

1. $F|_W = \varphi^*|_W$ *and* $G|_V = (\varphi^{-1})^*|_V$
2. $G \circ F|_W = \mathrm{id}_W$ *and* $F \circ G|_V = \mathrm{id}_V$
3. $\varphi(f) = f \circ F$ *for* $f \in \mathcal{M}_V$, *and*
4. $\varphi^{-1}(g) = g \circ G$ *for* $g \in \mathcal{M}_W$.

Proof. First we show that φ^* maps W into V. Write

$$W = D \cup W_1 \cup \cdots \cup W_n$$

where D is discrete and each W_i is an irreducible variety. The points in D are isolated, and thus are mapped into V by Lemma 2.3.2. A minor modification of Corollary 2.3.4(2) deals with the irreducible subvarieties. Since D is a variety, there is a multiplier $k \in \mathcal{M}_d$ which vanishes on D and is non-zero at a regular point $\lambda \in W_1$. Proceed as in the proof of the lemma, but define $f = h_2 \dots h_n k$. Then the argument is completed in the same manner. Reversing the roles of V and W shows that φ^* maps W onto V.

We have observed that $\varphi^*(\rho_\lambda)$ lies in the fiber over the point

$$F(\lambda) = (\varphi(Z_1)(\lambda), \dots, \varphi(Z_d)(\lambda)).$$

Since we now know that φ^* maps W into V, we see (with a slight abuse of notation) that $F = \varphi^*|_W$. In particular, the coefficients of F are multipliers. Thus by Proposition 2.1.7, each f_i is the restriction to W of a multiplier in $\mathcal{M}_d$, which we also denote by f_i. In particular, each f_i is holomorphic on the entire ball $\mathbb{B}_d$. Thus (since $d < \infty$), F is a bounded holomorphic function of the ball into $\mathbb{C}^d$. It may not carry $\mathbb{B}_d$ into itself, but we do have $F(W) = V$.

A similar argument applied to φ^{-1} shows that $G(V) \subset W$ and $G|_V = (\varphi^{-1})^*|_V$. Since $(\varphi^{-1})^* = (\varphi^*)^{-1}$, we obtain that $G \circ F|_W = \mathrm{id}_W$ and $F \circ G|_V = \mathrm{id}_V$. The last two statements follow as in Lemma 2.2.3. □

Remark 2.3.7. Note that in the above theorem, the map F can be chosen to be a polynomial if and only if the algebra homomorphism φ takes the coordinate functions to (restrictions of) polynomials; and hence takes polynomials to polynomials. Likewise, F can be chosen to have components which are continuous multipliers if and only if φ takes the coordinate functions to continuous multipliers; and hence takes all continuous multipliers to continuous multipliers.

Remark 2.3.8. When $d = \infty$, there is no guarantee that the map F constructed in our proof would actually have values in ℓ^2. However if we assume that φ is completely bounded, then we can argue as follows. The row operator $Z = \begin{bmatrix} Z_1 \ Z_2 \ Z_3 \ \dots \end{bmatrix}$ is a contraction. Thus $\varphi(Z) = \begin{bmatrix} \varphi(Z_1) \ \varphi(Z_2) \ \varphi(Z_3) \ \dots \end{bmatrix}$ is bounded by $\|\varphi\|_{cb}$. By Proposition 2.1.7, there are functions $f_i \in \mathcal{M}_{d'}$ so that $f_i|_W = \varphi(Z_i)$ and

$$\left\| \begin{bmatrix} M_{f_1} \ M_{f_2} \ M_{f_3} \ \dots \end{bmatrix} \right\| \leq \|\varphi\|_{cb}.$$

In particular, $F = \begin{bmatrix} f_1 \ f_2 \ f_3 \ \dots \end{bmatrix}$ is bounded by $\|\varphi\|_{cb}$ in the sup norm. Theorem 2.3.6 can then be modified to apply in the case $d = \infty$. However these hypotheses are very strong.

Corollary 2.3.9. *Every algebraic automorphism of $\mathcal{M}_d$ for d finite is completely isometric, and is unitarily implemented.*

Proof. The previous theorem shows that every automorphism is implemented as composition by a biholomorphic map of the ball onto itself, i.e. a conformal automorphism of $\mathbb{B}_d$. Proposition 2.2.2 shows that these automorphisms are completely isometric and unitarily implemented. □

Now we consider the isometric case.

Theorem 2.3.10. *Let V and W be varieties in $\mathbb{B}_d$, with $d < \infty$. Every isometric isomorphism of $\mathcal{M}_V$ onto $\mathcal{M}_W$ is completely isometric, and thus is unitarily implemented.*

Proof. Let φ be an isometric isomorphism of $\mathcal{M}_V$ onto $\mathcal{M}_W$. By Corollary 2.3.4(3), φ^* maps W onto V and preserves the pseudohyperbolic distance. Let F be the function constructed as in Theorem 2.3.6. As in Lemma 2.2.3 and Theorem 2.3.6, F is a biholomorphism of W onto V and $\varphi(h) = h \circ F$.

After modifying both V and W by a conformal automorphism of the ball, we may assume that 0 belongs to both V and W, and that $F(0) = 0$. Set $w_0 = 0$ and choose a basis $w_1, \dots, w_k$ for $\operatorname{span} W$. Let $v_p = F(w_p)$ for $1 \leq p \leq k$.

Suppose that $\|w_p\| = r_p$. This is the pseudohyperbolic distance to $w_0 = 0 = v_0$, so $\|v_p\| = r_p$ as well. Write $v_p/r_p = \sum_{j=1}^d c_j e_j$. Let $h_p(z) = \langle z, v_p/r_p\rangle = \sum_{j=1}^d \overline{c}_j Z_j(z)$. This is a linear function on V, and thus lies in $\mathcal{M}_V$. Since Z is a row contraction, f has norm at most one. Therefore $k_p := \varphi(h_p) = h_p \circ F$ has norm at most one in $\mathcal{M}_W$.

Now let $w_{k+1} = w$ be an arbitrary point in W, and set $v_{k+1} = v = F(w) \in V$. By a standard necessary condition for interpolation [2, Theorem 5.2], the fact that $\|k_p\| \le 1$ means that in particular interpolating at the points $w_0, \dots, w_k, w_{k+1}$, we obtain

$$0 \le \left[\tfrac{1-h_p(v_i)\overline{h_p(v_j)}}{1-\langle w_i, w_j\rangle}\right]_{0\le i,j\le k+1}.$$

In particular, look at the 3×3 minor using rows $0, p, k+1$ to obtain

$$0 \le \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & \frac{1-\overline{\langle v, v_p\rangle}}{1-\langle w_p, w\rangle} \\ 1 & \frac{1-\langle v, v_p\rangle}{1-\langle w, w_p\rangle} & \frac{1-|\langle v, v_p/r_p\rangle)|^2}{1-\|w\|^2} \end{bmatrix}$$

By the Cholesky algorithm, we find that $\frac{1-\langle v, v_p\rangle}{1-\langle w, w_p\rangle} = 1$. Therefore

$$\langle v, v_p\rangle = \langle w, w_p\rangle \quad \text{for } 1 \le p \le k.$$

In particular, we obtain

$$\langle v_i, v_j\rangle = \langle w_i, w_j\rangle \quad \text{for } 1 \le i, j \le k.$$

Therefore there is a unitary operator U acting on $\mathbb{C}^d$ such that $Uw_i = v_i$ for $1 \le i \le k$. Now since $w \in W$ lies in $\operatorname{span}\{w_1, \dots, w_k\}$, it is uniquely determined by the inner products $\langle w, w_i\rangle$ for $1 \le i \le k$. Since v has the same inner products with $v_1, \dots, v_k$, we find that $Uw = P_N v$ where $N = \operatorname{span}\{v_1, \dots, v_k\}$. However we also have

$$\|v\| = \|w\| = \|Uw\| = \|P_N v\|;$$

whence $v = Uw$.

Therefore F agrees with the unitary U, and hence φ is implemented by an automorphism of the ball. So by Proposition 2.2.2, φ is completely isometric and is unitarily implemented. □

Lastly for this section, we conclude that every isomorphism is automatically continuous with respect to the weak-operator and weak-$*$ topologies.

Lemma 2.3.11. *A bounded net $\{M_{f_n}\}$ in $\mathcal{M}_V$ converges in the weak-operator topology to M_f if and only if for all $\lambda \in V$, $f_n(\lambda) \to f(\lambda)$.*

Proof. If $M_{f_n} \xrightarrow{\text{WOT}} M_f$, then for all $\lambda \in V$,

$$\frac{f_n(\lambda)}{1-\|\lambda\|^2} = \left\langle k_\lambda, \overline{f_n(\lambda)} k_\lambda \right\rangle = \langle M_{f_n} k_\lambda, k_\lambda \rangle \to \langle M_f k_\lambda, k_\lambda \rangle = \frac{f(\lambda)}{1-\|\lambda\|^2}.$$

Conversely, suppose $\{M_{f_n}\} \subset \mathcal{M}_V$ is a bounded net such that $\{f_n\}$ converges pointwise to f. Since $\{M_{f_n}\}$ is bounded, it suffices to show that $\langle M_{f_n} k_\lambda, k_\mu \rangle \to \langle M_f k_\lambda, k_\mu \rangle$ for all $\lambda, \mu \in V$, because $\operatorname{span}\{k_\lambda : \lambda \in V\}$ is dense in $\mathcal{F}_V$. But

$$\langle M_{f_n} k_\lambda, k_\mu \rangle = \frac{f_n(\mu)}{1-\langle \mu, \lambda \rangle} \to \frac{f(\mu)}{1-\langle \mu, \lambda \rangle} = \langle M_f k_\lambda, k_\mu \rangle . \qquad \square$$

Theorem 2.3.12. *Let $\varphi : \mathcal{M}_V \to \mathcal{M}_W$, for $d < \infty$, be a unital algebra isomorphism given by composition: $\varphi(h) = h \circ F$ where F is a holomorphic map of W onto V whose coefficients are multipliers. Then φ is continuous with respect to the weak-operator and the weak-$*$ topologies.*

Proof. By Lemma 2.1.13 together with the Krein-Šmulian Theorem (Theorem 7, Section V.5, [35]), it is enough to show that φ is WOT-continuous on bounded sets.

Let $\{M_{f_n}\}$ be a bounded net in $\mathcal{M}_V$ converging to M_f in the weak-operator topology. By Lemma 2.3.1, $\{\varphi(M_{f_n})\} = \{M_{f_n \circ F}\}$ is a bounded net in $\mathcal{M}_W$. Therefore, by Lemma 2.3.11, it suffices to show that $f_n \circ F$ converges pointwise to $f \circ F$. But since f_n converges pointwise to f (by the same lemma), this is evident. $\square$

2.3.2 The homogeneous case

We have already seen in Corollary 2.2.8 that an algebraic isomorphism of norm closed algebras induces a biholomorphism of their associated varieties. What is much harder to establish is the converse. However, homogeneous varieties behave like classical varieties and so we will have none of the difficulties of the previous section.

We first wish to establish that if $\mathcal{A}_V$ and $\mathcal{A}_W$ are isomorphic then there exists another isomorphism $\varphi : \mathcal{A}_V \to \mathcal{A}_W$ such that $\varphi^*(0) = 0$.

Lemma 2.3.13. *Let V be a homogeneous variety. Then either V has singular points, or V is a linear subspace.*

Proof. If V is reducible, then by (iv) of Theorem 8 in [21, Section 9.6] the origin is in the singular set. So we may assume that V is irreducible.

Let $f_1, \dots, f_k$ be a generating set for I_V, and assume the dimension of V is m. By the theorem on page 88, [64], the singular locus of V is the common zero set of polynomials obtained from the $(d-m) \times (d-m)$ minors of the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_d} \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial z_1} & \cdots & \frac{\partial f_k}{\partial z_d} \end{pmatrix}.$$

But since $f_1, \dots, f_k$ are homogeneous, all these minors will vanish at the point 0 unless at least $d-m$ of the f_i's are linearly independent linear forms. But then V lies inside m dimensional subspace. Being an m-dimensional variety, V must be that subspace. □

Let V be a homogenous variety in $\mathbb{C}^d$. Then by the lemma, either V is a subspace of $\mathbb{C}^d$, or the singular locus $\mathrm{Sing}(V)$ is nonempty. Now $\mathrm{Sing}(V)$ is also a homogeneous variety, so either $\mathrm{Sing}(V)$ is a subspace or $\mathrm{Sing}(\mathrm{Sing}(V))$ is not empty. Since the dimension of the singular locus is strictly less than the dimension of a variety, we eventually arrive at a subspace $N(V) = \mathrm{Sing}(\cdots(\mathrm{Sing}(V)\cdots)$ which we call *the singular nucleus of* V. Note that $N(V) = \{0\}$ might happen, as well as $N(V) = V$.

In what follows we will need to consider the group $\mathrm{Aut}(\mathbb{B}_n)$ of automorphisms of $\mathbb{B}_n$, that is, the biholomorphisms of the unit ball. We will use well known properties of these fractional linear maps (see [62, Section 2.2]). For $a \in \mathbb{B}_n$, we define

$$\varphi_a(z) = \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a\rangle}, \tag{2.5}$$

where P_a is the orthogonal projection onto $\mathrm{span}\{a\}$, $Q_a = I_n - P_a$ and $s_a = (1-|a|^2)^{1/2}$. Then φ_a is an automorphism of $\overline{\mathbb{B}}_n$ that maps 0 to a and satisfies $\varphi_a^2 = \mathbf{id}$. For every $\psi \in \mathrm{Aut}(\mathbb{B}_n)$ there exists a unique unitary U and $a \in \mathbb{B}_n$ such that $\psi = U \circ \varphi_a$.

By a *disc* in $\mathbb{B}_n$ we shall mean a set D of the form $D = \mathbb{B}_n \cap L$, where $L \subseteq \mathbb{C}^n$ is a one dimensional subspace.

Lemma 2.3.14. *Let $\psi \in \mathrm{Aut}(\mathbb{B}_n)$. Then there are two discs D_1, D_2 in $\mathbb{B}_n$ such that $\psi(D_1) = D_2$.*

Proof. If $\psi = U \circ \varphi_a$ and $a \neq 0$, take $D_1 = \mathrm{span}\{a\} \cap \mathbb{B}_n$. Then $\varphi_a|_{D_1}$ is a Möbius map of D_1 onto itself. Take $D_2 = UD_1$. If $a = 0$, take $D_1 = D_2$ to be $\mathbb{B}_n \cap L$ where L is any one-dimensional eigenspace of U. □

Proposition 2.3.15. *Let V and W be homogeneous varieties and assume that there exists an isomorphism $\varphi : \mathcal{A}_V \to \mathcal{A}_W$. Then there exists another isomorphism $\psi : \mathcal{A}_V \to \mathcal{A}_W$ such that $\psi^*(0) = 0$.*

Proof. By the discussion following Lemma 2.3.13, the singular nucleus of W must be mapped biholomorphically by φ^* onto the singular nucleus of V. If these nuclei are both $\{0\}$, we are done. Otherwise, by rotating the coordinate systems we may assume that $N(V) = N(W) = B$, a complex ball.

Now, $\varphi^*\big|_B \in \mathrm{Aut}(B)$, thus by Lemma 2.3.14 there are two discs $D_1, D_2 \subseteq B$ such that $\varphi^*(D_2) = D_1$.

Let us introduce the notation

$$\mathcal{O}(0; V, W) = \{z \in D_1 : z = \psi^*(0) \text{ for some isomorphism } \psi : \mathcal{A}_V \to \mathcal{A}_W\},$$

and

$$\mathcal{O}(0; W) = \{z \in D_2 : z = \psi^*(0) \text{ for some automorphism } \psi \text{ of } \mathcal{A}_W\}.$$

Claim: *The sets $\mathcal{O}(0; V, W)$ and $\mathcal{O}(0; W)$ are invariant under rotations about 0.*
Proof of claim: For λ with $|\lambda| = 1$, write φ_λ for the isometric automorphism mapping Z_i to λZ_i $(i = 1, \ldots, d)$. Let $b = \varphi^*(0) \in \mathcal{O}(0; V, W)$. Recall that $b = (b_1, \ldots, b_d)$ is identified with a character $\rho_b \in M(\mathcal{A}_V) \cap \mathbb{B}_d$ such that $\rho_b(Z_i) = b_i$ for $i = 1, \ldots, d$. Consider $\varphi \circ \varphi_\lambda$. We have

$$\rho_0((\varphi \circ \varphi_\lambda)(Z_i)) = \rho_0(\varphi(\lambda Z_i)) = \lambda \rho_0(\varphi(Z_i)) = \lambda b_i.$$

Thus $\lambda b = (\varphi \circ \varphi_\lambda)^*(\rho_0) \in \mathcal{O}(0; V, W)$. The proof for $\mathcal{O}(0; W)$ is the same. This proves the claim.

We can now show the existence of a vacuum preserving isomorphism. Let $b = \varphi^*(0)$. If $b = 0$ then we are done, so assume that $b \neq 0$. By definition, $b \in \mathcal{O}(0; V, W)$. Denote $C := \{z \in D_1 : |z| = |b|\}$. By the above claim, $C \subseteq \mathcal{O}(0; V, W)$. Consider $C' := (\varphi^*)^{-1}(C)$. We have that $C' \subseteq \mathcal{O}(0; W)$. Now C' is a circle in D_2 that goes through the origin. By the claim, the interior of C', $\mathrm{int}(C')$, is in $\mathcal{O}(0; W)$. But then $\varphi^*(\mathrm{int}(C'))$ is the interior of C, and it is in $\mathcal{O}(0; V, W)$. Thus $0 \in \mathcal{O}(0; V, W)$, as required. □

We now follow the discussion in [62, Chapter 2] to obtain some rigidity results for biholomorphisms between varieties. These rigidity results will help us determine the possibilities for isomorphisms between the various algebras $\mathcal{A}_V$.

Lemma 2.3.16. *Let V be a homogeneous variety in $\mathbb{C}^d$. Let $F : \overline{\mathbb{B}}_d \to \mathbb{C}^d$ be a continuous map, holomorphic on $\mathbb{B}_d$, such that $F|_{\overline{V}}$ is a bijection of $\overline{V}$. If $F(0) = 0$ and $\frac{d}{dt}F(tz)\Big|_{t=0} = z$ for all $z \in \overline{V}$, then $F|_{\overline{V}}$ is the identity.*

Proof. It seems that a careful variation of the proof for "Cartan's Uniqueness Theorem" given in [62] (page 23) will work. One only needs to use the facts that $\overline{V}$ is circular and bounded. The reason one must be careful is that $\overline{V}$ typically has empty interior.

Let's make sure that it all works. We write the homogeneous expansion of F:

$$F(z) = Az + \sum_{n \geq 2} F_n(z), \tag{2.6}$$

where $A = F'(0)$. First let us show that, without loss of generality, we may assume

$$F(z) = z + \sum_{n \geq 2} F_n(z). \tag{2.7}$$

Let W be the linear span of $\overline{V}$, and let $W^\perp$ be its orthogonal complement in $\mathbb{C}^d$. By the assumption $\frac{d}{dt}F(tz)\Big|_{t=0} = z$ for $z \in \overline{V}$, so the matrix A can be written as

$$A = \begin{pmatrix} I & B \\ 0 & C \end{pmatrix}$$

with respect to the decomposition $\mathbb{C}^d = W \oplus W^\perp$. Replacing F by $F + I - A$ we obtain a function that is continuous on $\overline{\mathbb{B}}_d$, analytic on $\mathbb{B}_d$, agrees with F on $\overline{V}$, and has homogeneous decomposition as in (2.7).

Following Rudin [62, bottom of page 23], we consider the kth iterate F^k of F:

$$F^k(z) = z + kF_2(z) + \ldots.$$

Since $\overline{V}$ is circular and since F^k maps $\overline{V}$ onto itself, we find that for all $z \in V$

$$kF_2(z) = \frac{1}{2\pi} \int_0^{2\pi} F^k(e^{i\theta} z) e^{-2i\theta} d\theta,$$

from which it follows that $\|kF_2(z)\| \leq 1$ for all k and all $z \in V$. This implies that $F_2(z) = 0$ for all $z \in V$. Therefore there exists a continuous function $G : \overline{\mathbb{B}}_d \to \mathbb{C}^d$ that is holomorphic on $\mathbb{B}_d$ and agrees with F on $\overline{V}$, that has homogeneous expansion

$$G(z) = z + \sum_{n \geq 3} G_n(z),$$

(namely, one takes $G = F - F_2$). Note that $G_n = F_n$ for all $n > 2$. This last observation allows us to repeat the argument inductively and deduce that $F(z) = z$ for all $z \in V$. By continuity, $F|_{\overline{V}}$ equals the identity. □

We now obtain the desired analogue of Cartan's uniqueness theorem.

Theorem 2.3.17. *Let $V \subset \mathbb{B}_d$ and $W \subset \mathbb{B}_{d'}$ be homogeneous varieties. Let $F : \overline{\mathbb{B}}_{d'} \to \mathbb{C}^d$ be a continuous map that is holomorphic on $\mathbb{B}_{d'}$ and maps 0 to 0. Assume that there exists a continuous map $G : \overline{\mathbb{B}}_d \to \mathbb{C}^{d'}$ that is holomorphic on $\mathbb{B}_d$ such that $F \circ G|_{\overline{V}}$ and $G \circ F|_{\overline{W}}$ are the identity maps. Then there exists a linear map $A : \mathbb{C}^{d'} \to \mathbb{C}^d$ such that $F|_{\overline{W}} = A$.*

Proof. Again we adjust the proof of [62, Theorem 2.1.3] to the current setting. The derivatives $F'(0)$ and $G'(0)$ might not be inverses of each other, but from $G \circ F(z) = z$, we find that $G'(0)F'(0)z = z$ for all $z \in \overline{W}$.

Fix $\theta \in [0, 2\pi]$, and define $H : \overline{\mathbb{B}}_{d'} \to \mathbb{C}^{d'}$ by

$$H(z) = G(e^{-i\theta} F(e^{i\theta} z)).$$

Then $H(0) = 0$ and

$$\frac{d}{dt} H(tz)\Big|_{t=0} = G'(0)e^{-i\theta} F'(0)e^{i\theta} z = z.$$

By the previous lemma

$$H(z) = z$$

for $z \in \overline{W}$. After replacing z by $e^{-i\theta} z$ and applying F to both sides we find that

$$F(e^{-i\theta} z) = e^{-i\theta} F(z) \quad \text{for all } z \in \overline{W}.$$

Integrating over θ, this implies that if (2.6) is the homogeneous expansion of F, then $F_n(z) = 0$ for all $z \in W$ and all $n \geq 2$. Thus $F|_{\overline{W}} = A$. □

The following easy result is a straightforward consequence of homogeneity.

Lemma 2.3.18. *Let $V \subset \mathbb{B}_d$ and $W \subset \mathbb{B}_{d'}$ be homogeneous varieties. If a linear map $A : \mathbb{C}^{d'} \to \mathbb{C}^d$ carries $\overline{W}$ bijectively onto $\overline{V}$, then A is isometric on W.*

Proof. Each unit vector $w \in \overline{W}$ determines a disc $\overline{\mathbb{D}} w = \mathbb{C} w \cap \overline{\mathbb{B}_{d'}}$ in $\overline{W}$. Observe that A carries $\mathbb{C} w$ onto $\mathbb{C} A w$, and must take the intersection with the ball to the corresponding intersection with the ball $\overline{\mathbb{B}_d}$. Thus it takes $\overline{\mathbb{D}} w$ onto $\overline{\mathbb{D}} A w$. Therefore $\|Aw\| = \|w\|$. □

This lemma can be significantly strengthened to obtain a rigidity result which will be useful for the algebraic classification of the algebras $\mathcal{A}_V$. Note that $\mathrm{Sing}(V)$ denotes the set of singular points of V.

Proposition 2.3.19. *Let V be a homogeneous variety in $\mathbb{B}^d$, and let A be a linear map on $\mathbb{C}^d$ such that $\|Az\| = \|z\|$ for all $z \in V$. If $V = V_1 \cup \cdots \cup V_k$ is the decomposition of V into irreducible components, then A is isometric on* $\mathrm{span}(V_i)$ *for $1 \leq i \leq k$.*

Proof. It is enough to prove the proposition for an irreducible variety V. The idea of the proof is to produce a sequence of algebraic varieties $V \subseteq V_1' \subseteq V_2' \subseteq \ldots$ such that $\|Az\| = \|z\|$ for all $z \in V_i'$ and all i, where either $\dim V_i' < \dim V_{i+1}'$, or V_i' is a subspace (and then it is the subspace spanned by V).

First, we prove that $\|Ax\| = \|x\|$ for all x lying in the tangent space $T_z(V)$ for every $z \in V \setminus \mathrm{Sing}(V)$. Since z is nonsingular, for every such x there is a complex analytic curve $\gamma : \mathbb{D} \to V$ such that $\gamma(0) = z$ and $\gamma'(0) = x$. By the polar decomposition, we may assume that A is a diagonal matrix with nonnegative entries $a_1, \ldots, a_d$. Since A is isometric on V,

$$\sum_{i=1}^{d} a_i^2 |\gamma_i(z)|^2 = \sum_{i=1}^{d} |\gamma_i(z)|^2 \quad \text{for } z \in \mathbb{D}.$$

Applying the Laplacian to both sides of the above equation, and evaluating at 0, we obtain

$$\sum_{i=1}^{d} a_i^2 |\gamma_i'(0)|^2 = \sum_{i=1}^{d} |\gamma_i'(0)|^2.$$

Thus, $\|Ax\| = \|x\|$ for all $x \in T_z(V)$ and all nonsingular $z \in V$.

Consider now the set

$$X_0 = \bigcup_{z \in V \setminus \mathrm{Sing}(V)} \{z\} \times T_z(V) \subseteq \mathbb{C}^d \times \mathbb{C}^d.$$

Let X denote the Zariski closure of X_0, that is, $X = V(I(X_0))$. As X sits inside the tangent bundle $\bigcup_{z \in V} \{z\} \times T_z(V)$, X_0 is equal to $X \setminus \left(\mathrm{Sing}(V) \times \mathbb{C}^d \right)$. Therefore X_0 is Zariski open in X. By Proposition 7 of Section 7, Chapter 9 in [21], the closure (in the usual topology of $\mathbb{C}^{2d}$) of X_0 is X. Letting π denote the projection onto the last d variables, we have $\pi(X) \subseteq \overline{\pi(X_0)}$. But $\pi(X_0) = \bigcup_{z \in V \setminus \mathrm{Sing}(V)} T_z(V)$, therefore $\|Ax\| = \|x\|$ for all $x \in \pi(X)$. Now, $\pi(X)$ might not be an algebraic variety, but by Theorem 3 of Section 2, Chapter 3 in [21], there is an algebraic variety W in which $\pi(X)$ is dense. Observe that W must be a homogeneous variety, and $\|Az\| = \|z\|$ for every $z \in W$.

Being irreducible, V must lie completely in one of the irreducible components of W. We denote this irreducible component by V_1', and let $W_2, \dots, W_m$ be the other irreducible components of W. We claim: if V itself is not a linear subspace, then $\dim V_1' > \dim V$. We prove this claim by contradiction. If $\dim V_1 = \dim V$ then $V = V_1'$, because $V \subseteq V_1'$ and both are irreducible. Let $z \in V = V_1'$ be a regular point. Since $\dim T_z(V) = \dim V$, and $T_z(V)$ is irreducible, $T_z(V)$ is not contained in V_1'. But $T_z(V)$ is contained in W, thus $T_z(V) \subseteq W_i$ for some i. But $z \in T_z(V)$ by homogeneity. What we have shown is that, under the assumption $\dim V_1' = \dim V$, every regular point $z \in V$ is contained in $\bigcup_{i=2}^m W_i$. Thus $V_1' \subseteq \cup_i W_i$. That contradicts the assumed irreducible decomposition.

If V is not a linear subspace then we are now in the situation in which we started, with V_1' instead of V, and with $\dim V_1' > \dim V$. Continue this procedure finitely many times to obtain a sequence of irreducible varieties $V_1' \subseteq \ldots \subseteq V_n'$ that terminates at a subspace on which A is isometric. V_n must be $\operatorname{span} V$. Indeed, it certainly contains V. On the other hand, every V_i' lies in $\operatorname{span} V_{i-1}'$ and hence in $\operatorname{span} V$. □

When the variety V is a hypersurface we sketch a more elementary proof which provides somewhat more information.

Proposition 2.3.20. *Let $f \in \mathbb{C}[z_1, \dots, z_d]$ be a homogeneous polynomial, and let $V = V(f)$. Let A be a linear map on $\mathbb{C}^d$ such that $\|Az\| = \|z\|$ for all $z \in V$. Let $A = UP$ be the polar decomposition of A with U unitary and P positive. Then one of the following possibilities hold:*

1. *$P = I$;*
2. *P has precisely one eigenvalue different from 1 and $V(f)$ is a hyperplane;*
3. *P has precisely two eigenvalues not equal to 1 (one larger and one smaller), and in this case V is the union of hyperplanes which all intersect in a common $d-2$-dimensional subspace.*

Proof. After a unitary change of variables, we may assume that A is a positive diagonal matrix $A = \operatorname{diag}(a_1, \dots, a_d)$ with $a_i \geq a_{i+1}$ for $1 \leq i < d$. Now A takes the role of P in the statement.

We first show that $a_2 = \cdots = a_{d-1} = 1$. For if $a_1 \geq a_2 > 1$, there is a non-zero solution to $f = 0$ and $z_3 = \cdots = z_d = 0$, say $v = (z_1, z_2, 0, \dots, 0)$. But $\|Av\| > \|v\|$, contrary to the hypothesis. Hence $a_2 \leq 1$. Similarly one shows that $a_{d-1} \geq 1$. Hence all singular values equal 1 except possibly $a_1 > 1$ and $a_d < 1$.

If $A = I$ then we have (1). When there is precisely one eigenvalue different from 1, A is only isometric on the hyperplane $\ker(A - I)$; thus (2) holds. So we may assume that there are precisely two singular values different from 1, $a_1 > 1 > a_d$. Then f must have the form $f = \alpha z_1^m + \dots$ for some $\alpha \neq 0$. Indeed, otherwise (if z_1 appears only in mixed terms) there is non-zero solution $v = (1, 0, \dots, 0)$ to $f = 0$, and $\|Av\| > \|v\|$, contrary to the hypothesis. Now there are two cases:

Case 1: f does not depend on $z_2, \dots, z_{d-1}$. In this case f is essentially a polynomial in two variables, and can therefore be factored as $f = \prod_i (\alpha_i z_1 + \beta_i z_d)$, from which case (3) follows.

Case 2: f depends on $z_2, \dots, z_{d-1}$. Say f depends on z_2. Fix $z_3, \dots, z_d$ such that the polynomial $f(\cdot, \cdot, z_3, \dots, z_d)$ still depends on z_2. For every z_2 there is a solution z_1 to the equation $f(z_1, z_2, \dots, z_d) = 0$. As z_2 tends to ∞, the form of f forces z_1 to tend to ∞ as well. But since $(z_1, \dots, z_d)$ is a solution and A is isometric on $V(f)$, one has

$$a_1^2 |z_1|^2 + a_d^2 |z_d|^2 = |z_1|^2 + |z_d|^2.$$

This cannot hold when z_d is fixed and z_1 tends to ∞. So this case does not occur. $\square$

Example 2.3.21. Let us show that arbitrarily many hyperplanes can appear in case (3) above. Let $a, b > 0$ be such that $a^2 + b^2 = 2$, and let $\lambda_1, \dots, \lambda_k \in \mathbb{T}$. Let $V = \ell_1 \cup \dots \cup \ell_k$, where $\ell_i = \mathbb{C}(\lambda_i/\sqrt{2}, 1/\sqrt{2})$. Then $A = \operatorname{diag}(a, b)$ is isometric on V.

Example 2.3.22. Propositions 2.3.19 and 2.3.20 depend on the fact that we are working over $\mathbb{C}$. Indeed, consider the cone $V = V(x^2 + y^2 - z^2)$ over $\mathbb{R}$. With a and b as in the previous example, one sees that $A = \operatorname{diag}(a, a, b)$ is isometric on V, but it is clearly not an isometry on $\mathbb{R}^3 = \operatorname{span}(V)$.

Let V be a homogeneous variety in $\mathbb{B}^d$ and let $V = V_1 \cup \dots \cup V_k$ be the decomposition of V into irreducible components. Then we call

$$S(V) := \operatorname{span}(V_1) \cup \dots \cup \operatorname{span}(V_k)$$

the *minimal subspace span of* V. By Proposition 2.3.19, the linear map A must be isometric on $S(V)$. Note that $V = S(V)$ if and only if V is already the union of subspaces.

Our goal is to establish that A induces a bounded linear isomorphism $\tilde{A}$ between the spaces $\mathcal{F}_W$ and $\mathcal{F}_V$ given by $\tilde{A}f = f \circ A^*$. This is evidently linear (provided it is defined) and satisfies

$$\tilde{A}\nu_\lambda = \nu_{A\lambda} \quad \text{for } \lambda \in W. \tag{2.8}$$

Conversely, $\tilde{A}$ is determined by (2.8) because the kernel functions span $\mathcal{F}_W$.

On a single subspace $\tilde{A}$ defines an isometric map from $\mathcal{F}_{W_i}$ onto $\mathcal{F}_{V_i}$ because the map $A|_{W_i}$ can be extended to an automorphism of $\mathbb{B}_d$, so any difficulties will appear in the closure

$$\mathcal{F}_W = \overline{\mathcal{F}_{W_1} + \cdots + \mathcal{F}_{W_k}}.$$

However, Hartz proved the following impressive automatic closure result:

Theorem 2.3.23 ([41], Corollary 5.8)**.** *Let $S_1, \cdots, S_r \subset \mathbb{C}^d$ be subspaces. Then the algebraic sum*

$$\mathcal{F}_{S_1} + \cdots + \mathcal{F}_{S_k} \subset \mathcal{F}(\mathbb{C}^d)$$

is closed.

Theorem 2.3.24. *Let $V \subset \mathbb{B}_d$ and $W \subset \mathbb{B}_{d'}$ be homogeneous varieties. If there is a linear map $A : \mathbb{C}^{d'} \to \mathbb{C}^d$ that maps W bijectively onto V, then the map $\tilde{A} : \mathcal{F}_W \to \mathcal{F}_V$ given by* (2.8) :

$$\tilde{A}\nu_\lambda = \nu_{A\lambda} \quad \text{for } \lambda \in W$$

is a bounded linear map of $\mathcal{F}_W$ into $\mathcal{F}_V$.

Proof. Suppose $V = V_1 \cup \cdots \cup V_k$ and $W = W_1 \cup \cdots \cup W_k$ are the respective decompositions into irreducible varieties, and assume that A maps W_i to V_i for $1 \le i \le k$. Proposition 2.3.19 tells us that A sends $\text{span}(W_i)$ isometrically onto $\text{span}(V_i)$.

On each subspace $\text{span}(W_i)$, since A acts isometrically, it is clear that $\tilde{A}$ from $\mathcal{F}_{\text{span}(W_i)}$ to $\mathcal{F}_{\text{span}(V_i)}$ is a isometric linear map. It is a straightforward application of the open mapping theorem that this implies that $\tilde{A}$ is a well defined map on the algebraic sum $\mathcal{F}_{\text{span}(W_1)} + \cdots + \mathcal{F}_{\text{span}(W_k)}$ which by the previous Theorem is all of $\mathcal{F}_{S(W)}$ and hence, $\tilde{A}$ is bounded (cf. [41, Proposition 2.5]).

Finally, since $\tilde{A}$ is a bounded linear map of $\mathcal{F}_{S(W)}$ into $\mathcal{F}_{S(V)}$, restriction to W yields the desired result. □

Corollary 2.3.25. *Let $V \subset \mathbb{B}_d$ and $W \subset \mathbb{B}_{d'}$ be homogeneous varieties. Let $A : \mathbb{C}^{d'} \to \mathbb{C}^d$ and $B : \mathbb{C}^d \to \mathbb{C}^{d'}$ be linear maps such that $AB|_V = \text{id}_V$ and $BA|_W = \text{id}_W$. Then the map $\tilde{A}$, $\nu_\lambda \mapsto \nu_{A\lambda}, \lambda \in W$ is invertible, and the map*

$$\varphi : f \to f \circ A$$

is a completely bounded isomorphism from $\mathcal{A}_V$ onto $\mathcal{A}_W$, and it is given by conjugation with $\tilde{A}^*$:

$$\varphi(f) = \tilde{A}^* f (\tilde{A}^{-1})^*.$$

Proof. By Theorem 2.3.24, $\tilde{A}$ and $\tilde{B}$ are bounded. By checking the products on the kernel functions, it follows easily that $\tilde{B} = \tilde{A}^{-1}$. So these maps are linear isomorphisms.

Let $f \in \mathcal{A}_V$ and $\lambda \in W$. Denote by M_f the operator of multiplication by f on $\mathcal{F}_V$. Then

$$\tilde{A}^{-1} M_f^* \tilde{A} \nu_\lambda = \tilde{A}^{-1} M_f^* \nu_{A\lambda} = \tilde{A}^{-1} \overline{f \circ A(\lambda)} \nu_{A\lambda} = \overline{f \circ A(\lambda)} \nu_\lambda.$$

Thus $(\tilde{A}^{-1} M_f^* \tilde{A})^* = \tilde{A}^* M_f (\tilde{A}^{-1})^*$ is the operator on $\mathcal{F}_W$ given by multiplication by $f \circ A$. □

Thus, by the previous corollary and Corollary 2.2.8 we have the desired characterization for homogeneous varieties:

Theorem 2.3.26. *Let $V \subset \mathbb{B}_d$ and $W \subset \mathbb{B}_{d'}$ be homogeneous varieties. Then $\mathcal{A}_V$ is algebraically isomorphic to $\mathcal{A}_W$ if and only if V and W are biholomorphic.*

Remark 2.3.27. Originally in [32], we proved Theorem 2.3.24 under some conditions on the varieties. In particular, it was true for:

1. Any irreducible variety V because $S(V)$ is a subspace.
2. $V = V_1 \cup V_2$, the union of two irreducible varieties.
3. $V = V_1 \cup \cdots \cup V_k$ where V_i are irreducible and $S(V_i) \cap S(V_j) = E$, a fixed subspace, for all $i \neq j$.
4. $V = V_1 \cup \cdots \cup V_k$ where $\dim S(V_1) \geq d - 1$.
5. Any variety in $\mathbb{C}^3$.

With our method of proof, it became difficult to see whether we could extend our results to the general case. Consequently, we were very pleased to see Michael Hartz's clever paper which completed the characterization with respect to homogeneous varieties.

Remark 2.3.28. The various lemmas established above only require that A be length preserving on V. It need not be invertible on $\operatorname{span}(V)$ in order to show that the map $\tilde{A}$ is bounded. However, if A is singular on $\operatorname{span}(V)$, then $\tilde{A}$ is not injective because the homogeneous part of order one, $M_1 := \operatorname{span}\{\nu_\lambda^1 : \lambda \in Z^o(V)\} \simeq \operatorname{span}(V)$ and $\tilde{A}|_{M_1} \simeq A$.

For example, if $V = \mathbb{C}e_1 \cup \mathbb{C}e_2 \cup \mathbb{C}e_3$ and $A = \begin{bmatrix} 1 & 0 & 1/\sqrt{2} \\ 0 & 1 & 1/\sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}$, then one can see that A is isometric on V and maps $\mathbb{C}^3$ into $\operatorname{span}\{e_1, e_2\}$, taking V to the union of three lines in

2-space. The map $\tilde{A}$ is bounded, and satisfies $\tilde{A}\nu_\lambda = \nu_{A\lambda}$ for $\lambda \in Z^o(V)$. But for the reasons mentioned in the previous paragraph, it is not injective.

On the other hand, if A is bounded below by $\delta > 0$ on $\operatorname{span} V$, one can argue in each of the various lemmas that $A^{\otimes_s n}$ is bounded below by δ^n for $n \leq N$ and use the original arguments for upper and lower bounds on the higher degree terms. In this way, one sees directly that $\tilde{A}$ is an isomorphism.

Although the following example does not disprove Theorem 2.3.24 for arbitrary complex algebraic varieties, it does illustrate some of the difficulties one must overcome.

Example 2.3.29. In this example we identify $\mathbb{C}^2$ with $\mathbb{R}^4$. Let

$$V = \{(w, x, y, z) : w^2 + x^2 = y^2 + z^2\}.$$

Then V is a real algebraic variety in $\mathbb{R}^4$, but is not a complex algebraic variety in $\mathbb{C}^2$ because it has odd real dimension. Note that

$$V = \bigcup_{\theta \in \mathbb{T}} \left\{ \lambda \left(\frac{1}{\sqrt{2}}, \frac{\theta}{\sqrt{2}} \right) : \lambda \in \mathbb{C} \right\}.$$

Let $A = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$, where $a > 1 > b > 0$ satisfy $a^2 + b^2 = 2$. Then A is an invertible linear map that preserves the lengths of vectors in V. Put $V' = AV$. We will now show that the densely defined operator given by $\tilde{A}\nu_\lambda = \nu_{A_\lambda}$ does not extend to a bounded map taking $\overline{\operatorname{span}}\{\nu_\lambda : \lambda \in V \cap \mathbb{B}_2\}$ into $\overline{\operatorname{span}}\{\nu_\lambda : \lambda \in V' \cap \mathbb{B}_2\}$. Let $\alpha, \beta > 0$, and consider

$$\sum_{j=1}^{n} (\alpha e_1 + \theta_j \beta e_2)^n \in (\mathbb{C}^2)^n,$$

where $\theta_j = \exp(\frac{2\pi i}{n} j)$. We find

$$\begin{aligned}
\sum_{j=1}^{n} (\alpha e_1 + \theta_j \beta e_2)^n &= \sum_{j=1}^{n} \sum_{k=0}^{n} (\alpha e_1)^k (\theta_j \beta e_2)^{n-k} \\
&= \sum_{k=0}^{n} \alpha^k \beta^{n-k} (e_1)^k \Big(\sum_{j=1}^{n} \theta_j^{n-k} (e_2)^{n-k} \Big) \\
&= \beta^n n e_2^n + \alpha^n n e_1^n,
\end{aligned}$$

because $\sum_{j=1}^{n} \exp(\frac{2\pi i}{n}(n-k)j)$ is equal to 0 for $1 \leq k \leq n-1$, and equal to n for $k=0$ and n. Thus,

$$\| \sum_{j=1}^{n} (\alpha e_1 + \theta_j \beta e_2)^n \|^2 = (\alpha^{2n} + \beta^{2n}) n^2.$$

Comparing this norm for $(\alpha, \beta) = (a, b)$ and $(\alpha, \beta) = (1, 1)$ we find that the densely defined $\tilde{A}$ is unbounded.

2.4 Examples

In this section, we examine a possible converse to Theorem 2.3.6 in the context of a number of examples. What we find is that the desired converse is not always true. That is, suppose that V and W are varieties in $\mathbb{B}_d$ and F and G are holomorphic functions on the ball satisfying the conclusions of Theorem 2.3.6. We are interested in when this implies that the algebras $\mathcal{M}_V$ and $\mathcal{M}_W$ are isomorphic.

2.4.1 Finitely many points in the ball

Let $V = \{v_1, \ldots, v_n\} \subseteq \mathbb{B}_d$. Then $\mathcal{A}_V = \mathcal{M}_V$ and they are both isomorphic to $\ell_n^\infty = C(V)$. The characters are evaluations at points of V. If W is another n point set in $\mathbb{B}_{d'}$, then $\mathcal{M}_W$ is isomorphic to $\mathcal{M}_V$. Also, there are (polynomial) maps $f : \mathbb{B}_d \to \mathbb{C}^{d'}$ and $g : \mathbb{B}_{d'} \to \mathbb{C}^d$ which are inverses of one another when restricted to V and W. And if W is an m point set, $m \neq n$, then obviously $\mathcal{M}_V$ is not isomorphic to $\mathcal{M}_W$, and there also exists no biholomorphism. In this simple case we see that $\mathcal{M}_V \cong \mathcal{M}_W$ if and only if there exists a biholomorphism, and this happens if and only if $|W| = |V|$.

Nevertheless, the situation for finite sets is not ideal. Let V and W be finite subsets of the ball, and let $F : W \to V$ be a biholomorphism. It is natural to hope that the norm of the induced isomorphism can be bounded in terms of the multiplier norm of F. The following example shows that this is not possible.

Example 2.4.1. Fix $n \in \mathbb{N}$ and $r \in (0, 1)$. Put $\xi = \exp(\frac{2\pi i}{n})$ and let

$$V = \{0\} \cup \{r\xi^j\}_{j=1}^n,$$

and

$$W = \{0\} \cup \{\frac{r}{2}\xi^j\}_{j=1}^n.$$

The map $F(z) = 2z$ is a biholomorphism of W onto V that extends to an H^∞ function of multiplier norm 2. We will show that the norm of the induced isomorphism $\mathcal{M}_V \to \mathcal{M}_W$, given by $f \mapsto f \circ F$, is at least 2^n.

Consider the following function in $\mathcal{M}_V$:

$$f(0) = 0 \quad \text{and} \quad f(r\xi^j) = r^n \quad \text{for } 1 \leq j \leq n.$$

We claim that the multiplier norm of f is 1. By Proposition 2.1.7, $\|f\|$ is the minimal norm of an H^∞ function that interpolates f. The function $g(z) = z^n$ certainly interpolates and has norm 1. We will show that it is of minimal norm.

The Pick matrix associated to the problem of interpolating f on V by an H^∞ function of norm 1 is

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \frac{1-r^{2n}}{1-r^2\xi\overline{\xi}} & \frac{1-r^{2n}}{1-r^2\xi\overline{\xi^2}} & \cdots & \frac{1-r^{2n}}{1-r^2\xi\overline{\xi^n}} \\ 1 & \frac{1-r^{2n}}{1-r^2\xi^2\overline{\xi}} & \frac{1-r^{2n}}{1-r^2\xi^2\overline{\xi^2}} & \cdots & \frac{1-r^{2n}}{1-r^2\xi^2\overline{\xi^n}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{1-r^{2n}}{1-r^2\xi^n\overline{\xi}} & \frac{1-r^{2n}}{1-r^2\xi^n\overline{\xi^2}} & \cdots & \frac{1-r^{2n}}{1-r^2\xi^n\overline{\xi^n}} \end{bmatrix}.$$

To show that g is the (unique) function of minimal norm that interpolates f, it suffices to show that this matrix is singular. (We are using well known facts about Pick interpolation. See Chapter 6 in [2]).

We will show that the lower right principal sub-matrix

$$A = \left[\frac{1-r^{2n}}{1-r^2\xi^i\overline{\xi^j}}\right]_{i,j=1}^{n}$$

has the vector $(1, \ldots, 1)^t$ as an eigenvector with eigenvalue n. Therefore it will follow that $(n, -1, -1, \ldots, -1)^t$ is in the kernel of the Pick matrix. The matrix A is invertible, so the Pick matrix has rank n.

Indeed, for any i,

$$\begin{aligned}\sum_{j=1}^{n} \frac{1-r^{2n}}{1-r^2\xi^i\overline{\xi^j}} &= (1-r^{2n}) \sum_{j=1}^{n}\sum_{k=0}^{\infty}(r^2\xi^i\overline{\xi^j})^k \\ &= (1-r^{2n}) \sum_{k=0}^{\infty}\sum_{j=1}^{n} r^{2k}\xi^{ik}\overline{\xi^{jk}} \\ &= (1-r^{2n}) \sum_{m=0}^{\infty} n r^{2mn}\xi^{imn} \\ &= n\frac{1-r^{2n}}{1-r^{2n}} = n.\end{aligned}$$

We used the familiar fact that $\sum_{j=1}^{n} \xi^{jk}$ is equal to n for $k \equiv 0 \pmod{n}$ and equal to 0 otherwise. Therefore $\|f\| = 1$.

Now we will show that $f \circ F \in \mathcal{M}_W$ has norm 2^n, where $F(z) = 2z$. The function $f \circ F$ is given by

$$f \circ F(0) = 0 \quad \text{and} \quad f \circ F(\tfrac{r}{2}\xi^j) = r^n \quad \text{for } 1 \le j \le n.$$

The unique H^∞ function of minimal norm that interpolates $f \circ F$ is $h(z) = 2^n z^n$. This follows from precisely the same reasoning as above. Therefore the isomorphism has norm at least 2^n.

2.4.2 Blaschke sequences

We will now provide an example of two discrete varieties which are biholomorphic but yield non-isomorphic algebras.

Example 2.4.2. Let

$$v_n = 1 - 1/n^2 \quad \text{and} \quad w_n = 1 - e^{-n^2} \quad \text{for } n \ge 1.$$

Set $V = \{v_n\}_{n=1}^{\infty}$ and $W = \{w_n\}_{n=1}^{\infty}$. Both V and W satisfy the Blaschke condition so they are analytic varieties in $\mathbb{D}$. Let $B(z)$ be the Blaschke product with simple zeros at points in W. Define

$$h(z) = 1 - e^{\frac{1}{z-1}},$$

and

$$g(z) = \frac{\log(1-z)+1}{\log(1-z)}\Big(1 - \frac{B(z)}{B(0)}\Big).$$

Then $g, h \in H^\infty$ and they satisfy

$$h \circ g|_W = \mathrm{id}_W \quad \text{and} \quad g \circ h|_V = \mathrm{id}_V .$$

However, by the corollary in [43, p.204], W is an interpolating sequence and V is not. Thus the algebras $\mathcal{M}_V$ and $\mathcal{M}_W$ cannot be similar by a map sending normalized kernel functions to normalized kernel functions. The reason is that the normalized kernel functions corresponding to an interpolating sequence form a Riesz system, while those corresponding to a non-interpolating sequence do not. In fact, $\mathcal{M}_V$ and $\mathcal{M}_W$ cannot be isomorphic via *any* isomorphism, as we see below.

Theorem 2.4.3. *Let $V = \{v_n\}_{n=1}^\infty \subseteq \mathbb{B}_d$, with $d < \infty$, be a sequence satisfying the Blaschke condition $\sum(1 - \|v_n\|) < \infty$. Then $\mathcal{M}_V$ is isomorphic to ℓ^∞ if and only if V is interpolating.*

Proof. By definition, V is interpolating if and only if $\mathcal{M}_V$ is isomorphic to ℓ^∞ via the restriction map. It remains to prove that if V is not an interpolating sequence, then $\mathcal{M}_V$ cannot be isomorphic to ℓ^∞ via any other isomorphism.

Let V be a non-interpolating sequence, and let W be any interpolating sequence. If $\mathcal{M}_V$ is isomorphic to ℓ^∞, then it is isomorphic to $\mathcal{M}_W$. But by Lemma 2.3.2, this isomorphism must be implemented by composition with a holomorphic map, showing that $\mathcal{M}_V$ is isomorphic to ℓ^∞ via the restriction map. This is a contradiction. □

Remark 2.4.4. We require the Blaschke condition to insure that V is a variety of the type we consider, i.e., a zero set of an ideal of multipliers (see [5, Theorem 1.11]). Any discrete variety in $\mathbb{D}$ satisfies this condition.

2.4.3 Curves

Let V be a variety in $\mathbb{B}_d$. If $\mathcal{M}_V$ is isomorphic to $H^\infty(\mathbb{D})$, then by Theorem 2.3.6 we know that V must be biholomorphic to the disc. To study the converse implication, we shall start with a disc biholomorphically embedded in a ball and try to establish a relationship between the associated algebras $\mathcal{M}_V$ and its reproducing kernel Hilbert space $\mathcal{F}_V$ and $H^\infty(\mathbb{D})$ and $H^2(\mathbb{D})$.

Suppose that h is a holomorphic map from the disc $\mathbb{D}$ into $\mathbb{B}_d$ such that $h(\mathbb{D}) = V$, and that there exists a holomorphic map $g : \mathbb{B}_d \to \mathbb{C}$ such that $g|_V = h^{-1}$.

The following result shows that in many cases, the desired isomorphism exists [3]. See [4, §2.3.6] for a strengthening to planar domains, and a technical correction.

Theorem 2.4.5 (Alpay-Putinar-Vinnikov)**.** *Suppose that h is an injective holomorphic function of $\mathbb{D}$ onto $V \subset \mathbb{B}_d$ such that*

1. *h extends to a C^2 function on $\overline{\mathbb{D}}$,*
2. *$\|h(z)\| = 1$ if and only if $|z| = 1$,*
3. *$\langle h(z), h'(z)\rangle \neq 0$ when $|z| = 1$.*

Then $\mathcal{M}_V$ is isomorphic to H^∞.

Condition (3) should be seen as saying that V meets the boundary of the ball non-tangentially. We do not know whether such a condition is necessary.

The authors of [3] were concerned with extending multipliers on V to multipliers on the ball. This extension follows from Proposition 2.1.7.

By the results of Section 2.2, there is no loss of generality in assuming that $h(0) = 0$, and we do so. Define a kernel $\tilde{k}$ on $\mathbb{D}$ by

$$\tilde{k}(z, w) = k(h(z), h(w)) = \frac{1}{1 - \langle h(z), h(w)\rangle}.$$

Let $\mathcal{H}$ be the RKHS determined by $\tilde{k}$. Write $\tilde{k}_w$ for the function $\tilde{k}(\cdot, w)$.

The following routine lemma shows that we can consider this new kernel on the disc instead of V.

Lemma 2.4.6. *The map $\tilde{k}_z \mapsto k_{h(z)}$ extends to a unitary map U of $\mathcal{H}$ onto $\mathcal{F}_V$. Hence, the multiplier algebra $\mathrm{Mult}(\mathcal{H})$ is unitarily equivalent to $\mathcal{M}_V$. This equivalence is implemented by composition with h:*

$$U^* M_f U = M_{f \circ h} \quad \text{for } f \in \mathcal{M}_V.$$

Proof. A simple computation shows that

$$\Big\| \sum_i c_i \tilde{k}_{z_i} \Big\|^2 = \sum_{i,j} \frac{c_i \overline{c_j}}{1 - \langle h(z_i), h(z_j)\rangle} = \Big\| \sum_i c_i k_{h(z_i)} \Big\|^2.$$

So we get a unitary $U : \mathcal{H} \to \mathcal{F}_V$. As in the proof of Proposition 2.2.2, for all $f \in \mathcal{M}_V$ we have $U^* M_f U = M_{f \circ h}$. □

Our goal in this section is to study conditions on h which yield a natural isomorphism of the RKHSs $\mathcal{H}$ and $H^2(\mathbb{D})$. The first result is that the Szego kernel k_z dominates the kernel $\tilde{k}_z$.

Lemma 2.4.7. *Suppose that h is a holomorphic map of $\mathbb{D}$ into $\mathbb{B}_d$. Then for any finite subset $\{z_1, \ldots, z_n\} \subset \mathbb{D}$,*

$$\left[\frac{1}{1 - \langle h(z_j), h(z_i)\rangle}\right] \leq \left[\frac{1}{1 - z_j\overline{z_i}}\right].$$

Proof. Observe that $h(z)/z$ maps $\mathbb{D}$ into $\overline{\mathbb{B}_d}$ by Schwarz's Lemma [62, Theorem 8.1.2]. Thus by the matrix version of the Nevanlinna-Pick Theorem for the unit disk, we obtain that

$$0 \leq \left[\frac{1 - \langle h(z_j)/z_j, h(z_i)/z_i\rangle}{1 - z_j\overline{z_i}}\right] = \left[\frac{1}{z_j\overline{z_i}}\right] \circ \left[\frac{1 - \langle h(z_j), h(z_i)\rangle}{1 - z_j\overline{z_i}} - 1\right].$$

Here $\circ$ represents the Schur product. But $\left[\frac{1}{z_j\overline{z_i}}\right]$ and its Schur inverse $\left[z_j\overline{z_i}\right]$ are positive. Therefore the second matrix on the right is positive. This can be rewritten as

$$\left[\ 1\ \right] \leq \left[\frac{1 - \langle h(z_j), h(z_i)\rangle}{1 - z_j\overline{z_i}}\right]$$

where $\left[1\right]$ represents an $n \times n$ matrix of all 1's. Now

$$\left[\frac{1}{1 - \langle h(z_j), h(z_i)\rangle}\right] = \left[\langle \tilde{k}_{z_i}, \tilde{k}_{z_j}\rangle\right] \geq 0.$$

So the Schur multiplication by this operator to the previous inequality yields

$$\left[\frac{1}{1 - \langle h(z_j), h(z_i)\rangle}\right] \leq \left[\frac{1}{1 - z_j\overline{z_i}}\right]. \qquad \square$$

We obtain the well-known consequence that there is a contractive map of H^2 into $\mathcal{H}$.

Proposition 2.4.8. *The linear map R, defined by $Rk_z = \tilde{k}_z$ for $z \in \mathbb{D}$, from* $\mathrm{span}\{k_z : z \in \mathbb{D}\}$ *to* $\mathrm{span}\{\tilde{k}_z : z \in \mathbb{D}\}$ *extends to a contractive map from H^2 into $\mathcal{H}$.*

Proof. This follows from an application of Lemma 2.4.7. Given $a_i \in \mathbb{C}$, let $\mathbf{a} = (a_1, \dots, a_n)^t$. Observe that

$$\begin{aligned}\|R\sum_{i=1}^n a_i k_{z_i}\|^2 &= \|\sum_{i=1}^n a_i \tilde{k}_{z_i}\|^2 = \sum_{i,j=1}^n a_i\overline{a_j}\langle \tilde{k}_{z_i}, \tilde{k}_{z_j}\rangle \\ &= \Big\langle \Big[\langle \tilde{k}_{z_i}, \tilde{k}_{z_j}\rangle\Big]\mathbf{a}, \mathbf{a}\Big\rangle \le \Big\langle \Big[\langle k_{z_i}, k_{z_j}\rangle\Big]\mathbf{a}, \mathbf{a}\Big\rangle \\ &= \sum_{i,j=1}^n a_i\overline{a_j}\langle k_{z_i}, k_{z_j}\rangle = \|\sum_{i=1}^n a_i k_{z_i}\|^2\end{aligned}$$

Hence R is contractive, and extends to H^2 by continuity. □

Example 2.4.9. Let $h : \mathbb{D} \to \mathbb{B}_d$ be given by

$$h(z) = (a_1 z, a_2 z^{n_2}, \dots, a_d z^{n_d}),$$

where $a_1 \neq 0$ and $\sum_{l=1}^d |a_l|^2 = 1$. Let $V = h(\mathbb{D})$. Then $\mathcal{M}_V$ is similar to $H^\infty(\mathbb{D})$, and $\mathcal{M}_V = H^\infty(V)$. Moreover, $\mathcal{A}_V$ is similar to $\mathrm{A}(\mathbb{D})$. This follows from Theorem 2.4.5, but we will provide a direct argument.

First observe that for $p \ge N = \max\{n_l : 1 \le l \le d\}$, we have

$$\begin{aligned}\frac{\langle h(z), h(w)\rangle - z^p\bar{w}^p}{1 - z\bar{w}} &= \sum_{l=1}^d |a_l|^2\left(\frac{z^{n_l}\bar{w}^{n_l} - z^p\bar{w}^p}{1 - z\bar{w}}\right) \\ &= \sum_{l=1}^d |a_l|^2 z^{n_l}\bar{w}^{n_l}\left(\frac{1 - z^{p-n_l}\bar{w}^{p-n_l}}{1 - z\bar{w}}\right)\end{aligned}$$

Therefore if $z_1, \dots, z_k$ are distinct points in $\mathbb{D}$, the $k \times k$ matrix

$$A_p := \left[\frac{\langle h(z_i), h(z_j)\rangle - z_i^p\bar{z}_j^p}{1 - z_i\bar{z}_j}\right] = \sum_{l=1}^d |a_l|^2\left[z_i^{n_l}\bar{z}_j^{n_l}\right] \circ \left[\frac{1 - z_i^{p-n_l}\bar{z}_j^{p-n_l}}{1 - z\bar{w}}\right]$$

is positive definite because the second matrix on the right is positive by Pick's condition, and the Schur product of positive matrices is positive.

Since the first coordinate of h is injective, we see that h is injective. Moreover,

$$\|h^{-1}\|_{\mathcal{M}_V} \le \|a_1^{-1}z_1\|_{\mathcal{M}} = |a_1|^{-1} =: C.$$

Since the kernel for $\mathcal{F}_V$ is a complete NP kernel, applying this to $(h^{-1})^{2^{n-1}}$ yields the positivity of the matrices

$$\left[\frac{C^{2^n} - z_i^{2^{n-1}} \bar{z}_j^{2^{n-1}}}{1 - \langle h(z_i), h(z_j) \rangle}\right].$$

Since $z^{2^{n-1}}$ has norm one, the Pick condition shows that

$$\left[\frac{1 - z_i^{2^{n-1}} \bar{z}_j^{2^{n-1}}}{1 - z_i \bar{z}_j}\right] \geq 0.$$

Thus we obtain positive matrices

$$\begin{aligned} H_n &:= \left[\frac{C^{2^n} - z_i^{2^{n-1}} \bar{z}_j^{2^{n-1}}}{1 - \langle h(z_i), h(z_j) \rangle}\right] \circ \left[\frac{1 - z_i^{2^{n-1}} \bar{z}_j^{2^{n-1}}}{1 - z_i \bar{z}_j}\right] \\ &= \left[\frac{C^{2^n} - (C^{2^n} + 1) z_i^{2^{n-1}} \bar{z}_j^{2^{n-1}} + z_i^{2^n} \bar{z}_j^{2^n}}{(1 - \langle h(z_i), h(z_j) \rangle)(1 - z_i \bar{z}_j)}\right] \end{aligned}$$

Choose M so that $2^M \geq N$. We form a telescoping sum of positive multiples of the H_n's:

$$0 \leq \sum_{n=1}^{M} b_n H_n = \left[\frac{(D-1) - D z_i \bar{z}_j + z_i^{2^M} \bar{z}_j^{2^M}}{(1 - \langle h(z_i), h(z_j) \rangle)(1 - z_i \bar{z}_j)}\right] =: H$$

where $b_M = 1$, $b_n = \prod_{k=n+1}^{M} (C^{2^k} + 1)$ for $1 \leq n < M$ and $D = \prod_{k=1}^{M} (C^{2^k} + 1)$. Thus

$$\begin{aligned} \left[\frac{D}{1 - \langle h(z_i), h(z_j) \rangle}\right] - \left[\frac{1}{1 - z_i \bar{z}_j}\right] &= \left[\frac{(D-1) - D z_i \bar{z}_j + \langle h(z_i), h(z_j) \rangle}{(1 - \langle h(z_i), h(z_j) \rangle)(1 - z_i \bar{z}_j)}\right] \\ &= H + A_{2^M} \circ \left[\frac{1}{1 - \langle h(z_i), h(z_j) \rangle}\right] \geq 0. \end{aligned}$$

This inequality shows that the two kernels k_z and $\tilde{k}_z$ are comparable. The argument of Proposition 2.4.8 shows that $\|R^{-1}\| \leq D$. In particular, R yields an isomorphism of the two RKHSs H^2 and $\mathcal{H}$. This yields the desired isomorphism of H^∞ and $\mathcal{M}_V$.

This isomorphism is not isometric. Indeed, if it were, then we would have $\|h^{-1}\|_{\mathcal{M}_V} = \|z\|_\infty = 1$. This would imply that

$$0 \leq \left[\frac{1 - z_i \bar{z}_j}{1 - \langle h(z_i), h(z_j) \rangle}\right].$$

Thus arguing as in Lemma 2.4.7, we obtain

$$\left[\frac{1}{1 - z_j\overline{z_i}}\right] \leq \left[\frac{1}{1 - \langle h(z_j), h(z_i)\rangle}\right].$$

But then the map R would be unitary, and the algebras would be completely isometric. So by Lemma 2.2.4, the map h would map onto an affine disk—which it does not do. $\square$

Remark 2.4.10. Kerr, McCarthy and Shalit in [48] have recently extended Theorem 2.4.5. They proved that when V is a finite Riemann surface that is sufficiently nice with a biholomorphic map $h : V \to W$ that extends to be C^2, one-to-one and transversal on ∂V, then $\mathcal{M}_V$ and $\mathcal{M}_W$ are isomorphic.

2.4.4 A class of examples in $\mathbb{B}_\infty$

We will now exhibit biholomorphisms of $\mathbb{D}$ into $\mathbb{B}_\infty$, some of which yield an isomorphism and some which do not.

Let $\{b_n\}_{n=1}^\infty$ be a sequence of complex numbers with $\sum |b_n|^2 = 1$ and $b_1 \neq 0$. Let $h : \mathbb{D} \to \mathbb{B}_\infty$ be given by

$$h(z) = (b_1 z, b_2 z^2, b_3 z^3, \ldots).$$

Note that h is analytic (because it is given by a power series in the disc), with the analytic inverse:

$$g(z_1, z_2, z_3, \ldots) = z_1/b_1.$$

The set $V = h(\mathbb{D})$ is the variety in $\mathbb{B}_\infty$ determined by the equations

$$z_k = \frac{b_k}{b_1} z_1^k \quad \text{for } k \geq 2.$$

As above let

$$\tilde{k}(z, w) = \frac{1}{1 - \langle h(z), h(w)\rangle},$$

and let $\mathcal{H}$ be the RKHS determined by $\tilde{k}$. By Lemma 2.4.6, $\mathcal{H}$ is equivalent to $\mathcal{F}_V$. The special form of h allows us to write

$$\frac{1}{1 - \langle h(z), h(w)\rangle} = \sum_{n=0}^{\infty}\Big(\sum_{i=1}^{\infty} |b_i|^2 z^i \overline{w}^i\Big)^n = \sum_{n=0}^{\infty} a_n (z\overline{w})^n.$$

By a basic result in RKHSs, $\tilde{k}(z,w) = \sum e_n(z)\overline{e_n(w)}$ where $\{e_n\}$ is an orthonormal basis for $\tilde{k}$ (see Proposition 2.18 of [2]). Hence $\mathcal{H}$ is the space of holomorphic functions on $\mathbb{D}$ with orthonormal basis $\{\sqrt{a_n}z^n\}_{n=0}^{\infty}$.

The map R defined in Proposition 2.4.8 is a contraction. Observe that $R^* : \mathcal{H} \to H^2$ is given by composition with the identity mapping because

$$(R^*f)(z) = \langle R^*f, k_z\rangle = \langle f, Rk_z\rangle = \langle f, \tilde{k}_z\rangle = f(z).$$

It is easy to see that the issue is whether R is bounded below. Since $\|z^n\|_{H^2} = 1$ and $\|z^n\|_{\mathcal{H}} = 1/\sqrt{a_n}$, we get:

Proposition 2.4.11. *$\mathcal{H}$ is equivalent to H^2 via R if and only if there are constants $0 < c < C$ so that $c \le a_n \le C$ for $n \ge 0$.*

The coefficients a_n are determined by the sequence $\{|b_n|\}_{n=1}^{\infty}$, and can be found recursively by the formulae

$$a_0 = 1 \quad \text{and} \quad a_n = |b_1|^2 a_{n-1} + \ldots + |b_n|^2 a_0 \quad \text{for } n \ge 1. \tag{2.9}$$

The logic behind this recursion is that the term $a_n(z\overline{w})^n$ gets contributions from the sum

$$\sum_{k=1}^{n}\Big(\sum_{i=1}^{n}|b_i|^2 z^i\overline{w}^i\Big)^k = \Big(\sum_{i=1}^{n}|b_i|^2 z^i\overline{w}^i\Big)\sum_{k=1}^{n}\Big(\sum_{i=1}^{n}|b_i|^2 z^i\overline{w}^i\Big)^{k-1}.$$

Every $|b_i|^2 z^i\overline{w}^i$ from the factor $\sum_{i=1}^{n}|b_i|^2 z^i\overline{w}^i$ needs to be matched with the $(z\overline{w})^{n-i}$ term from the factor $\sum_{k=1}^{n}(\sum_{i=1}^{n}|b_i|^2 z^i\overline{w}^i)^{k-1}$, which has coefficient precisely a_{n-i}. It follows by induction from equation (2.9) that $a_n \le 1$. This provides an alternative proof of Proposition 2.4.8 in this special case.

We will now construct a sequence $\{b_n\}_{n=1}^{\infty}$ that makes $\liminf a_n > 0$, and another sequence that makes $\liminf a_n = 0$. By Proposition 2.4.11, this will show that there are choices of $\{b_n\}_{n=1}^{\infty}$ for which $\mathcal{H}$ and H^2 are naturally isomorphic, and there are choices for which they are not.

Example 2.4.12. Define $b_n = (1/2)^{n/2}$ for $n \ge 1$. It follows from the recursion relation (2.9) that $a_n = 1/2$ for $n > 1$. Thus R^* is bounded below, showing that $\mathcal{H}$ and H^2 are naturally isomorphic.

Example 2.4.13. We will choose a rapidly increasing sequence $\{n_k\}_{k=1}^{\infty}$ with $n_1 = 1$ and define the sequence $\{b_n\}_{n=1}^{\infty}$ by

$$b_m = \begin{cases} (1/2)^{k/2} & \text{if } m = n_k \\ 0 & \text{otherwise .} \end{cases}$$

The sequence $\{n_k\}_{k=1}^{\infty}$ will be defined recursively so that $a_{n_k-1} \leq 1/k$.

We begin with $n_1 = 1$ and $a_0 = 1$. Suppose that we have already chosen $n_1, \ldots, n_k$. This means that we have already determined the sequence $b_1, \ldots, b_{n_k}$, but the tail $b_{n_k+1}, b_{n_k+2}, \ldots$ is yet to be determined. We compute

$$\sum_{m=1}^{n_k} b_m^2 = \sum_{j=1}^{k} b_{n_j}^2 = \sum_{j=1}^{k} 1/2^j = r < 1.$$

Thus, if $b_{n_k+1} = b_{n_k+2} = \ldots = b_{(N+1)n_k} = 0$, then it follows from (2.9) that $a_{(N+1)n_k} \leq r^N$ (recall that $a_n \leq 1$ for all n). Therefore we may choose N so large that $a_{(N+1)n_k} \leq (k+1)^{-1}$, and we set $n_{k+1} = (N+1)n_k + 1$.

Our construction yields a sequence $\{b_n\}_{n=1}^{\infty}$ so that $\liminf a_n = 0$. Thus the kernel for the analytic disk V so defined is not similar to H^2.

We do not know whether $\mathcal{M}_V$ is isomorphic to H^∞ or not. We suspect that it isn't.

Example 2.4.14. It should be noted that there are many reproducing kernel Hilbert spaces on the disk whose multiplier algebras are non-isomorphic to H^∞. One such example is Dirichlet space, denoted $\mathcal{D}$, consisting of all holomorphic functions on the disk with derivative in the Bergman space. The canonical norm is given by

$$\|f\|_{\mathcal{D}}^2 = \sum_{n=0}^{\infty} (n+1)|\hat{f}(n)|^2,$$

where $\hat{f}(n)$ denotes the n^{th} Taylor coefficient of f at 0.

Dirichlet space has the complete Pick property and so is there is an embedding function $b : \mathbb{D} \to \mathbb{B}^\infty$ such that $\mathcal{D}$ can be identified with H^2_∞ restricted to the range of b [2, Theorem 7.33]. In fact, [2, Example 8.8] gives a precise formula for this embedding

$$b(z) = (b_n z^n)_{n=1}^{\infty}$$

where the $b_n \geq 0$ come from the equation

$$\sum_{n=1}^{\infty} b_n^2 \bar{w}^n z^n = 1 - \frac{1}{k_{\mathcal{D}}(z,w)}$$

$$= 1 - \frac{1}{\sum_{n=0}^{\infty}(n+1)^{-1}\bar{w}^n z^n} = 1 + \frac{\bar{w}z}{\ln(1-\bar{w}z)}.$$

Hence, by letting w and z approach 1 we see that $\sum_{n=1}^{\infty} b_n^2 = 1$ and so this map b is an example of our construction, with credit to Michael Hartz for noticing this. However, $\mathrm{Mult}(\mathcal{D})$ is not isomorphic to H^∞ [65].

Remark 2.4.15. Suppose that there is some N such that $b_n = 0$ for all $n > N$. Then the mapping $h : \mathbb{D} \to \mathbb{B}_\infty$ given by

$$h(z) = (b_1 z, b_2 z^2, b_3 z^3, \dots)$$

can be considered as a mapping into $\mathbb{B}_N$. Equation (2.9) implies that for $n > N$, a_n will always remain between the minimum and the maximum of $a_0, a_1, \dots, a_N$. Therefore, the conditions of Proposition 2.4.11 are fulfilled, and $\mathcal{H}$ is equivalent to H^2 via R. This is an alternate argument to obtain Example 2.4.9.

2.4.5 Quotients of $\mathrm{A}(\mathbb{D})$

Let $V = \{z_n : n \geq 1\}$ be a Blaschke sequence in the disk. Write B_V for the Blaschke product with simple zeros at the points in V. Observe that $J_V = B_V H^\infty$ and $I_V = J_V \cap \mathrm{A}(\mathbb{D})$. By Lemma 2.1.4 and Proposition 2.1.11, if the measure $|S(V)|$ of $S(V) = \overline{V} \cap \mathbb{T}$ is zero, then $\mathcal{A}_V = \mathrm{A}(\mathbb{D})|_V \cong \mathrm{A}(\mathbb{D})/I_V$.

The interpolating sequences play a special role.

Theorem 2.4.16. *Let $S(V) = \overline{V} \cap \mathbb{T}$.*

1. *If $|S(V)| > 0$ then $I_V = \{0\}$.*
2. *If V is interpolating and $|S(V)| = 0$, then $\mathcal{A}_V$ is isomorphic to $\mathrm{C}(\overline{V})$ by the restriction map.*
3. *If $\mathcal{A}_V$ is isomorphic to $\mathrm{C}(\overline{V})$ via the restriction map, then V is an interpolating sequence.*

Proof. (1) If $|S(V)| > 0$, then any $f \in I_V$ must vanish on $S(V)$, and hence is 0.

(2) The map taking $f \in \mathrm{A}(\mathbb{D})$ to $f|_{\overline{V}}$ is clearly a contractive homomorphism of $\mathrm{A}(\mathbb{D})$ into $\mathrm{C}(\overline{V})$ with kernel I_V. So it factors through $\mathcal{A}_V$, and induces an injection of $\mathcal{A}_V$ into $\mathrm{C}(\overline{V})$. It suffices to show that this map is surjective, for the result then follows from the open mapping theorem.

Fix $h \in \mathrm{C}(\overline{V})$. By Rudin's Theorem (see [43, p.81]), there is a function $f \in \mathrm{A}(\mathbb{D})$ such that $f|_{S(V)} = h|_{S(V)}$. By replacing h with $h-f$, we may suppose that $h|_{S(V)} = 0$. Hence $h|_V$ is a function that $\lim_{n\to\infty} h(z_n) = 0$. Now it suffices to show that if $h(z_n) = 0$ for all $n > N$, then there is a function $f \in \mathrm{A}(\mathbb{D})$ with $f|_V = h|_V$ and $\|f\| \leq C\|h\|_\infty$ for a constant C which is independent of N. Surjectivity will follow from a routine approximation argument. Let c be the interpolation constant for V.

Fix N. Take $h \in \mathrm{C}(\overline{V})$ with $h(z_n) = 0$ for all $n > N$ and $\|h\|_\infty \leq 1$. By a theorem of Fatou [43, p.81], there is an analytic function g on $\mathbb{D}$ such that $\operatorname{Re} g \geq 0$ and $e^{-g} \in \mathrm{A}(\mathbb{D})$ vanishes precisely on $S(V)$. There is an integer $m > 0$ so that $|e^{-g/m}(z_n)| > .5$ for $1 \leq n \leq N$. Set $V_N = \{z_n : n > N\}$. Since V is interpolating,

$$\min\{|B_{V_N}(z_n)| : 1 \leq n \leq N\} \geq 1/c.$$

We will look for a function f of the form $f = B_{V_N} e^{-g/m} f_0$. By the arguments for Rudin's theorem, this will lie in $\mathrm{A}(\mathbb{D})$. Clearly it vanishes on $V_N \cup S(V)$, and we require

$$h(z_n) = f(z_n) = B_{V_N}(z_n) e^{-g/m}(z_n) f_0(z_n).$$

So we need to find $f_0 \in \mathrm{A}(\mathbb{D})$ with $\|f_0\| \leq C$ and

$$f_0(z_n) = a_n := h(z_n) e^{g(z_n)/m} / B_{V_N}(z_n) \quad \text{for } 1 \leq n \leq N.$$

The estimates made show that $|a_n| \leq 2c$. The interpolation constant for $\{z_n : 1 \leq n \leq N\}$ is at most c, and since this is a finite set, we can interpolate using functions in $\mathrm{A}(\mathbb{D})$ which are arbitrarily close to the optimal norm. Thus we can find an f_0 with $\|f_0\| \leq 3c^2$. Hence we obtain f with the same norm bound.

(3) If $\mathcal{A}_V$ is isomorphic to $\mathrm{C}(\overline{V})$ via the restriction map, then by the open mapping theorem, there is a constant c so that for any $h \in \mathrm{C}(\overline{V})$, there is an $f \in \mathrm{A}(\mathbb{D})$ with $f|_V = h|_V$ and $\|f\| \leq c\|h\|$. In particular, for any bounded sequence (a_n) and $N \geq 1$, there is an $f_N \in \mathrm{A}(\mathbb{D})$ such that $\|f_N\| \leq c\|(a_n)\|_\infty$ and

$$f_N(z_n) = \begin{cases} a_n & \text{if } 1 \leq n \leq N \\ 0 & \text{if } n > N \end{cases}.$$

Take a weak-$*$ cluster point f of this sequence in H^∞. Then $\|f\| \le c$ and f interpolates the sequence (a_n) on V. So V is interpolating. $\square$

We can now strengthen Example 2.4.2, showing that there are discrete varieties giving rise to non-isomorphic algebras which are biholomorphic with a biholomorphism that extends continuously to the boundary.

Example 2.4.17. We will show that there is a Blaschke sequence V which is not interpolating and an interpolating sequence W and functions f and g in $\mathrm{A}(\mathbb{D})$ so that $f|_V$ is a bijection of V onto W and $g|_W$ is its inverse. Take

$$V = \{z_n := 1 - n^{-2} : n \ge 1\} \text{ and } W = \{w_n := 1 - n^{-2}e^{-n^2} : n \ge 1\}.$$

Then W is an interpolating sequence, and V is not. Let

$$f(z) = 1 + (z-1)e^{1/(z-1)}.$$

Then since $1/(z-1)$ takes $\mathbb{D}$ conformally onto $\{z : \operatorname{Re} z < -1/2\}$, it is easy to see that $e^{1/(z-1)}$ is bounded and continuous on $\overline{\mathbb{D}} \setminus \{1\}$. Hence f is continuous, so lies in $\mathrm{A}(\mathbb{D})$. Clearly, $f(z_n) = 1 - n^{-2}e^{-n^2} = w_n$ for $n \ge 1$. The inverse of $f|_V$ is the map $h(w_n) = z_n$. Since

$$\lim_{n\to\infty} h(w_n) = \lim_{n\to\infty} z_n = 1,$$

this extends to be a continuous function on $\overline{W} = W \cup \{1\}$. By Theorem 2.4.16, there is a function $g \in \mathrm{A}(\mathbb{D})$ such that $g|_W = h$.

Remark 2.4.18. Let $V = \{v_n\}$ and $W = \{w_n\}$ be two interpolating sequences in $\mathbb{D}$ with $\lim v_n = \lim w_n = 1$. Then the algebras $\mathcal{A}_V$ and $\mathcal{A}_W$ are both isomorphic to c, the space of convergent sequences. As in our counterexamples using Blaschke products, we can find biholomorphisms carrying one sequence onto the other. However there is no reason for the rates at which they approach the boundary to be comparable.

We now give a strengthening of Theorem 2.4.16.

Theorem 2.4.19. *Let $V = \{v_n\}_{n=1}^\infty$ be a Blaschke sequence in $\mathbb{D}$, such that $|S(V)| = 0$. Then $\mathcal{A}_V$ is isomorphic to $\mathrm{C}(\overline{V})$ if and only if V is interpolating.*

Proof. Theorem 2.4.16 says that $\mathcal{A}_V$ is isomorphic to $\mathrm{C}(\overline{V})$ via the restriction map if and only if V is interpolating. All that remains to prove is that if V is not an interpolating sequence, then it cannot be isomorphic via any other isomorphism.

Suppose that V is a non-interpolating sequence and define $w_n = (1-e^{-n})v_n/|v_n|$. Then $W = \{w_n\}$ is an interpolating sequence with $S(W) = S(V)$, and $\overline{V}$ is homeomorphic to $\overline{W}$ via the map that continuously extends $v_n \mapsto (1-e^{-n})v_n/|v_n|$. Therefore, $\mathcal{A}_W$ is isomorphic to $\mathrm{C}(\overline{V})$ via the restriction map. Now assume that $\mathcal{A}_V$ is isomorphic to $\mathrm{C}(\overline{V})$ by any isomorphism. Then it is isomorphic to $\mathcal{A}_W$. But by Corollary 2.2.8, this isomorphism is given by composition with a holomorphic map. Therefore $\mathcal{A}_V$ is isomorphic to $\mathrm{C}(\overline{V})$ via the restriction map—a contradiction. □

Remark 2.4.20. In [32], now improved by [41], we saw that in the case of homogeneous varieties V and W, the algebras $\mathcal{A}_V$ and $\mathcal{A}_W$ are isomorphic if and only if the algebras $\mathcal{M}_V$ and $\mathcal{M}_W$ are isomorphic. The above discussion shows that this is not true in general. If V and W are two interpolating sequences in $\mathbb{D}$, then $\mathcal{M}_V$ and $\mathcal{M}_W$ are both isomorphic to ℓ^∞, whereas the isomorphism classes of $\mathcal{A}_V$ and $\mathcal{A}_W$ depend on the structure of the limit sets.

Chapter 3

Triangular UHF algebras

3.1 Definitions and Notation

A C*-algebra is called *uniformly hyperfinite* (*UHF*) (or a *Glimm algebra*) if it is the closed union of a chain of unital subalgebras each isomorphic to a full matrix algebra. In other words, suppose we have integers $k_n, n \in \mathbb{N}$ such that $k_n | k_{n+1}$, for all n, and unital C*-algebra embeddings $\varphi_n : M_{k_n} \to M_{k_{n+1}}$. Then $\mathfrak{A}_\varphi = \overline{\bigcup_n M_{k_n}}$ is a UHF algebra. Such a sequence of integers $k_n | k_{n+1}$ (and setting $k_0 = 1$) defines a formal product $\delta(\mathfrak{A}_\varphi) := \prod_{n \geq 0} \frac{k_{n+1}}{k_n} = \prod_{p \text{ prime}} p^{\delta_p}$, where $\delta_p \in \mathbb{N} \cup \{\infty\}$, called a *supernatural number* or *generalized integer*. $\delta(\mathfrak{A}_\varphi)$ can also be thought of as the least common multiple of the set $\{k_1, k_2, \cdots\}$. A famous theorem of Glimm's [38] states that two UHF algebras are isomorphic if and only if they have the same generalized integers. In particular, the choice of unital embeddings does not make a difference. See [24, 57] for more on UHF algebras and approximately finite-dimensional (AF) C*-algebras, where such algebras are defined to be closed unions of a chain of finite dimensional subalgebras.

Let $\mathcal{T}_k$ be the upper triangular matrices of M_k. We have the following definition:

Definition 3.1.1. *Consider a UHF algebra $\mathfrak{A}_\varphi = \overline{\bigcup_n M_{k_n}}$ where $\varphi_n : M_{k_n} \to M_{k_{n+1}}$ are unital $*$-embeddings and assume that $\varphi_n(\mathcal{T}_{k_n}) \subset \mathcal{T}_{k_{n+1}}$. Then $\mathcal{T}_\varphi = \overline{\bigcup_n \mathcal{T}_{k_n}}$ is called a* triangular UHF (TUHF) algebra*.*

In contrast to Glimm's theorem we must take note of the embeddings as different embeddings lead to non-isomorphic algebras [57]. Hence, in the above definition $\varphi = \{\varphi_1, \varphi_2, \cdots\}$ is the collection of embeddings. Two of the simplest embeddings are:

Definition 3.1.2. *The* standard *embedding of $\mathcal{T}_k$ into $\mathcal{T}_{k'}$ when $k|k'$ is*

$$A \in \mathcal{T}_k \;\mapsto\; I_{k'/k} \otimes A = \begin{bmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{bmatrix} \in \mathcal{T}_{k'}$$

Definition 3.1.3. *The* nest *or* refinement *embedding of $\mathcal{T}_k$ into $\mathcal{T}_{k'}$ when $k|k'$ is*

$$A \in \mathcal{T}_k \;\mapsto\; A \otimes I_{k'/k} \in \mathcal{T}_k'$$

or in other words

$$\begin{bmatrix} a_{11} & \cdots & \cdots & a_{1k} \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{kk} \end{bmatrix} \mapsto \begin{bmatrix} a_{11} \cdot I_{k'/k} & \cdots & \cdots & a_{1k} \cdot I_{k'/k} \\ 0 \cdot I_{k'/k} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 \cdot I_{k'/k} & \cdots & 0 \cdot I_{k'/k} & a_{kk} \cdot I_{k'/k} \end{bmatrix}.$$

As was mentioned in Section 2, an important object for the study of non-selfadjoint operator algebras is the C*-envelope [12, 34, 40, 45]. It is immediate in this case that the C*-envelope, $C_e^*(\mathcal{T}_\varphi)$, is equal to $C^*(\mathcal{T}_\varphi) = \mathfrak{A}_\varphi$ because all UHF algebras are simple.

Distinct from the theory of UHF algebras is that there is a partial order on $\mathrm{Proj}(\mathcal{T}_\varphi)$ which is not the subprojection partial order.

Definition 3.1.4. *If $p, q \in \mathcal{T}$ are projections then we say $p \preceq q$ if there is a partial isometry $v \in \mathcal{T}$ such that $vv^* = p$ and $v^*v = q$.*

We will use $e_i^{k_n}$ to denote $e_{i,i} \in \mathcal{T}_{k_n}$, the minimal projections at each level, and similarly $e_{i,j}^{k_n}$ to denote $e_{i,j} \in \mathcal{T}_{k_n}$. From the previous definition we have $e_i^{k_n} \preceq e_j^{k_n}$ and $e_j^{k_n} \not\preceq e_i^{k_n}$ for $i < j$.

A subalgebra $\mathcal{T}$ of a UHF algebra is *triangular* if $\mathcal{T} \cap \mathcal{T}^*$ is abelian. In the terminology of [57] our TUHF algebras are *maximal triangular* in that there is no other triangular algebra sitting strictly between $\mathcal{T}_\varphi$ and $\mathfrak{A}_\varphi$. Observe that $\varphi_n(\mathcal{T}_{k_n} \cap \mathcal{T}_{k_n}^*) \subset \mathcal{T}_{k_{n+1}} \cap \mathcal{T}_{k_{n+1}}^*$, that is, the diagonal is mapped to the diagonal. So there is a *maximal abelian self-adjoint subalgebra* (masa) $C_\varphi \subset \mathcal{T}_\varphi$ defined as

$$C_\varphi \;=\; \mathcal{T}_\varphi \cap \mathcal{T}_\varphi^* \;=\; \overline{\bigcup_n \mathcal{T}_{k_n} \cap \mathcal{T}_{k_n}^*} \;\simeq\; \overline{\bigcup_n C_n} \;:=\; \overline{\bigcup_n \mathbb{C}^{k_n}}.$$

Hence, C_φ is an AF C*-algebra and $C_\varphi \simeq C(X)$ where the Gelfand space is a generalized Cantor set:

$$M(C_\varphi) = X = \prod_{n\geq 1} \left[\frac{k_n}{k_{n-1}}\right],$$

with $k_0 = 1$ to make the formula work and where $[k] = \{0, 1, \cdots, k-1\}$. We will often refer to C_φ as the diagonal of $\mathcal{T}_\varphi$.

Define the *normalizer* of C_n in $\mathcal{T}_{k_n}$ as

$$N_{C_n}(\mathcal{T}_{k_n}) \;=\; \{v \in \mathcal{T}_{k_n} \text{ partial isometry } : vC_nv^* \subset C_n, v^*C_nv \subset C_n\}.$$

It is not hard to see that any element of $N_{C_n}(\mathcal{T}_{k_n})$ is the multiplication of a diagonal unitary by a partial permutation matrix, that is, where there is at most one 1 in each row and column. We say that an embedding $\varphi : \mathcal{T}_{k_n} \to \mathcal{T}_{k_{n+1}}$ is *regular* if $\varphi(N_{C_n}(\mathcal{T}_{k_n})) \subset N_{C_{n+1}}(\mathcal{T}_{k_{n+1}})$. Note that the standard and nest embeddings are regular embeddings. We will say $\mathcal{T}_\varphi$ is a regular TUHF if it has regular embeddings. In the same way, define the normalizer of C_φ in $\mathcal{T}_\varphi$:

$$N_{C_\varphi}(\mathcal{T}_\varphi) \;=\; \{v \in \mathcal{T}_\varphi \text{ partial isometry } : vC_\varphi v^* \subset C_\varphi, v^*C_\varphi v \subset C_\varphi\}.$$

The following lemma by Power gives a decomposition of any element in the normalizer into a product of a unitary and a partial permuation matrix. Note that $U(C_\varphi)$ denotes the unitary group of C_φ.

Lemma 3.1.5 ([57], Lemma 5.5)**.** *Let $\mathcal{T}_\varphi$ be a regular TUHF algebra. Then $v \in N_{C_\varphi}(\mathcal{T}_\varphi)$ if and only if $v = dw$ where $w \in N_{C_n}(\mathcal{T}_{k_n})$, for some n, and $d \in U(C_\varphi)$, a diagonal unitary. Moreover, w can be chosen to be a partial permutation matrix which makes the decomposition unique.*

Finally, when we have a regular TUHF algebra for each point $x \in X$ there is a unique sequence of projections

$$e^{k_1}_{i_1} \geq e^{k_2}_{i_2} \geq e^{k_3}_{i_3} \geq \cdots$$

where now we refer to the subprojection partial order, with $x(e^{k_n}_{i_n}) = 1$ for all $n \geq 1$. Define a partial order on X by letting the following be equivalent for $x = (x_n)_{n\geq 1}, y = (y_n)_{n\geq 1} \in \prod_{n\geq 1}\left[\frac{k_n}{k_{n-1}}\right] = X$ which have sequences of projections $e^{k_n}_{i_n}$ and $e^{k_n}_{j_n}$ respectively:

1) $x \leq y$,

2) $\exists\, n$ such that $e^{k_n}_{i_n} \preceq e^{k_n}_{j_n}$ and $e^{k_{n'}}_{i_{n'}} = e^{k_n}_{i_n,j_n} e^{k_{n'}}_{j_{n'}} (e^{k_n}_{i_n,j_n})^*$ for all $n' > n$.

Note this is a partial order. Let $E_{ij}^{k_n}$ be all such pairs $(x, y) \in X \times X$ that depend on $i = i_n, j = j_n$ and n in the above definition.

Definition 3.1.6. *The* topological binary relation of $\mathcal{T}_\varphi$ relative to C_φ *is*

$$R(\mathcal{T}_\varphi) = \bigcup \{E_{ij}^{k_n} : e_{i,j}^{k_n} \in \mathcal{T}_\varphi, n \geq 1\},$$

equipped with the topology defined by basic clopen sets

$$\{x \in X : x(e_i^{k_n}) = 1\}, \quad n \geq 1, 1 \leq i \leq k_n.$$

3.2 Isometric automorphisms

Let $\mathcal{T}_\varphi$ be a regular TUHF algebra and $\mathrm{Aut}(\mathcal{T}_\varphi)$ denote the isometric automorphism group. Such an automorphism will preserve the masa, the partial order on projections and the normalizer.

Theorem 3.2.1 ([57], Theorem 7.5)**.** *Let $C_\varphi \subset \mathcal{T}_\varphi \subset \mathfrak{A}_\varphi$ and $C_\psi \subset \mathcal{T}_\psi \subset \mathfrak{A}_\psi$ be the algebras defined for two sequences of regular embeddings φ and ψ. Then the following are equivalent:*

1. *There is an isometric isomorphism $\theta : \mathcal{T}_\varphi \to \mathcal{T}_\psi$ with $\theta(C_\varphi) = C_\psi$.*
2. *The topological binary relations $R(\mathcal{T}_\varphi)$ and $R(\mathcal{T}_\psi)$ are isomorphic as topological relations.*
3. *There is a $*$-isomorphism $\tilde{\theta} : \mathfrak{A}_\varphi \to \mathfrak{A}_\psi$ with $\tilde{\theta}(\mathcal{T}_\varphi) = \mathcal{T}_\psi$ and $\tilde{\theta}(C_\varphi) = C_\psi$.*

Furthermore, by [24, Corollary IV.5.8] all automorphisms of $\mathfrak{A}_\varphi$ are approximately inner, i.e. the pointwise limit of inner automorphisms. Hence, by the previous theorem the automorphisms in $\mathrm{Aut}(\mathcal{T}_\varphi)$ are just restrictions of approximately inner automorphisms. Consider now, that the only unitaries in $\mathcal{T}_\varphi$ live in the masa, that is $U(\mathcal{T}_\varphi) = U(C_\varphi)$. Since we refer to C_φ as the diagonal of $\mathcal{T}_\varphi$ this leads us to the following definition:

Definition 3.2.2. *An approximately inner (or just inner) automorphism of $\mathcal{T}_\varphi$ is called an* approximately diagonal *automorphism. We denote this group by $\overline{\mathrm{Inn}}(\mathcal{T}_\varphi)$. More specifically, $\gamma \in \overline{\mathrm{Inn}}(\mathcal{T}_\varphi)$ if there exists $U_n \in U(C_\varphi)$ such that*

$$\lim_{n \to \infty} U_n A U_n^* = \gamma(A), \quad \forall A \in \mathcal{T}_\varphi.$$

Now because $U(C_\varphi)$ is commutative we immediately get that $\overline{\text{Inn}}(\mathcal{T}_\varphi)$ is commutative as well.

Define as well the outer automorphism group:

$$\text{Out}(\mathcal{T}_\varphi) := \text{Aut}(\mathcal{T}_\varphi)/\overline{\text{Inn}}(\mathcal{T}_\varphi).$$

Lemma 3.2.3. *Let $\theta \in \text{Aut}(\mathcal{T}_\varphi)$ for a regular $\mathcal{T}_\varphi$. Then there exists $\gamma \in \overline{\text{Inn}}(\mathcal{T}_\varphi)$ such that*

$$\gamma \circ \theta(\cup_{n\geq 1}\mathcal{T}_{k_n}) = \cup_{n\geq 1}\mathcal{T}_{k_n}.$$

Proof. Let $n_1 \geq 1$ be big enough such that $\theta(\text{proj}(\mathcal{T}_{k_1})) \subset \text{proj}(\mathcal{T}_{k_{n_1}})$ and using Lemma 3.1.5, $\theta(e^{k_1}_{i,i+1}) = d_i w_i \in N_{C_\varphi}(\mathcal{T}_\varphi)$ with $d_i \in U(C_\varphi)$ and $w_i \in N_{C_{n_1}}(\mathcal{T}_{k_{n_1}})$, a partial permutation matrix, for $1 \leq i < k_1$.

Set $u_1 = I \in C_\varphi$ and $u_2 \in U(C_\varphi)$ such that $u_2 = w_1^* d_1^* w_1$. Now, recursively define $u_i \in U(C_\varphi)$ by

$$u_i = w_{i-1}^* d_{i-1}^* u_{i-1} w_{i-1}, \quad \text{for } \ 2 < i \leq k_1.$$

Set $U_1 = \sum_{i=1}^{k_1} \theta(e_i^{k_1}) u_i \in U(C_\varphi)$ and notice that

$$U_1^* \theta(e^{k_1}_{i,i+1}) U_1 = u_i^* \theta(e^{k_1}_{i,i+1}) u_{i+1} = u_i^*(d_i w_i) u_{i+1} = w_i \in \mathcal{T}_{k_{n_1}}.$$

Thus, $U_1^* \theta(\mathcal{T}_{k_1}) U_1 \subset \mathcal{T}_{k_{n_1}}$.

In the same way there exists $n_2 \geq n_1$ and $\tilde{U}_2 \in U(C_\varphi)$ such that $\tilde{U}_2^* \theta(\mathcal{T}_{k_{n_1}}) \tilde{U}_2 \subset \mathcal{T}_{k_{n_2}}$. Since the following are both regular embeddings they must differ by a unitary V in $\mathcal{T}_{k_{n_2}}$:

$$V^* \tilde{U}_2^* \theta(\varphi_{k_{n_1}-1} \circ \cdots \circ \varphi_1(\mathcal{T}_{k_1})) \tilde{U}_2 V = \varphi_{k_{n_2}-1} \circ \cdots \circ \varphi_{k_{n_1}}(U_1^* \theta(\mathcal{T}_{k_1}) U_1).$$

Thus, define $U_2 = \tilde{U}_2 V$. Repeating this we recursively get $n_{m+1} \geq n_m$ and $U_{m+1} \in U(C_\varphi)$ such that $U_{m+1}^* \theta(\mathcal{T}_{k_{n_m}}) U_{m+1} \subset \mathcal{T}_{k_{n+1}}$ with $U_{m+1} U_m^*|_{\theta(\mathcal{T}_{k_m})} = I$.

Therefore, the sequence U_m defines an approximately inner automorphism $\gamma \in \overline{\text{Inn}}(\mathcal{T}_\varphi)$ and $\gamma \circ \theta(\cup_{n\geq 1}\mathcal{T}_{k_n}) = \cup_{n\geq 1}\mathcal{T}_{k_n}$. Furthermore, for every $n \geq 1$, $\gamma \circ \theta|_{\mathcal{T}_{k_n}}$ is a regular embedding into some $\mathcal{T}_{k_{n'}}$. $\square$

Proposition 3.2.4. *Let $\theta \in \text{Aut}(\mathcal{T}_\varphi)$ for a regular $\mathcal{T}_\varphi$ and $\theta(p) = p$, for all $p \in \text{Proj}(\mathcal{T}_\varphi)$. Then θ is an approximately diagonal automorphism.*

Proof. By the previous Lemma there exists $\gamma \in \overline{\mathrm{Inn}}(\mathcal{T}_\varphi)$ such that $\tilde{\theta} := \gamma \circ \theta$ preserves the unclosed union and from the end of the proof we may assume that $\tilde{\theta}|_{\mathcal{T}_{k_n}}$ is a regular embedding into $\mathcal{T}_{k_{n'}}$.

Hence, for $1 \leq i < j \leq k_n$ and $\lambda_l \in \mathbb{T}$,

$$\varphi_{n'-1} \circ \cdots \circ \varphi_n(e_{i,j}^{k_n}) = \sum_{l=1}^{k_{n'}/k_n} \lambda_l e_{i_l,j_l}^{k_{n'}}.$$

and so

$$\sum_{l=1}^{k_{n'}/k_n} e_{i_l}^{k_{n'}} \tilde{\theta}(e_{i_l,j_l}^{k_{n'}}) e_{j_l}^{k_{n'}} = \tilde{\theta}(\sum_{l=1}^{k_{n'}/k_n} e_{i_l,j_l}^{k_{n'}}) = \tilde{\theta}(e_{i,j}^{k_n}) \in \mathcal{T}_{k_{n'}}$$

because $\tilde{\theta}(p) = p$ for all projections p. However, $\tilde{\theta}|_{\mathcal{T}_{k_n}}$ is a regular embedding so there is no other option than to have $\tilde{\theta}(e_{i_l,j_l}^{k_{n'}}) = \mu_l e_{i_l,j_l}^{k_{n'}}$ where $\mu_l \in U(C_{n''})$ where $\tilde{\theta}|_{\mathcal{T}_{k_{n'}}} \subset \mathcal{T}_{k_{n''}}$ and so $\tilde{\theta}(e_{i,j}^{k_n})$ differs from $\varphi_{n'-1} \circ \cdots \circ \varphi_n(e_{i,j}^{k_n})$ by a unitary conjugation.

Therefore, $\tilde{\theta} = \overline{\mathrm{Inn}}(\mathcal{T}_\varphi)$ and so θ. ☐

Corollary 3.2.5. *Let $\mathcal{T}_\varphi$ have regular embeddings. Then $\mathcal{T}_\varphi$ is isomorphic to a TUHF algebra where the embeddings are not only regular but map partial permutation martrices to partial permutation matrices.*

Proof. Define $\psi_n : \mathcal{T}_{k_n} \to \mathcal{T}_{k_{n+1}}$ by mapping $\psi_n(e_i^{k_n}) = \varphi_n(e_i^{k_n})$ and defining

$$\psi_n(e_{i,j}^{k_n}) = \sum_{l=1}^{k_{n+1}} e_{i_l,j_l}^{k_{n+1}}, \quad \text{where} \quad \varphi_n(e_{i,j}^{k_n}) = \sum_{l=1}^{k_{n+1}} \lambda_l e_{i_l,j_l}^{k_{n+1}}$$

because φ_n was regular. The topological binary relations of $\mathcal{T}_\varphi$ and $\mathcal{T}_\psi$ are the same and thus the algebras are isometrically isomorphic by Theorem 3.2.1. ☐

Theorem 3.2.6. *For a regular $\mathcal{T}_\varphi$ we have* $\mathrm{Aut}(\mathcal{T}_\varphi) \simeq \overline{\mathrm{Inn}}(\mathcal{T}_\varphi) \rtimes \mathrm{Out}(\mathcal{T}_\varphi)$.

Proof. By the Corollary above assume that the embeddings have this stronger form. Proposition 3.2.4 tells us that each coset of $\mathrm{Out}(\mathcal{T}_\varphi)$ maps the projections of $\mathcal{T}_\varphi$ in a unique way. Then the end of the proof of Lemma 3.2.3 tells us that one can choose a representative of the coset uniquely, specifically by choosing the automorphism that acts as a regular embedding at each finite level sending partial permutation matrices to partial permutation matrices. We denote the collection of these representatives as $\mathcal{O} \subset \mathrm{Aut}(\mathcal{T}_\varphi)$. Composition of regular embeddings gives a regular embedding so it is immediate that if $\theta, \tilde{\theta} \in \mathcal{O}$ then $\theta \circ \tilde{\theta} \in \mathcal{O}$. Finally, θ^{-1} must send partial permutation matrices to partial permutation matrices because $\theta \in \mathcal{O}$. But then $\theta^{-1}|_{\mathcal{T}_{k_n}}$ must be a regular embedding and so $\theta^{-1} \in \mathcal{O}$ as well. Therefore, $\mathcal{O}$ is a group and is isomorphic to $\mathrm{Out}(\mathcal{T}_\varphi)$.

Furthermore, for $\theta \in \mathcal{O}$ and $\gamma \in \overline{\mathrm{Inn}}(\mathcal{T}_\varphi)$ we have that for any $p \in \mathrm{proj}(\mathcal{T}_\varphi)$

$$\theta^{-1} \circ \gamma \circ \theta(p) = \theta^{-1}(\theta(p)) = p$$

because approximately diagonal automorphisms preserve projections. By Proposition 3.2.4 this implies that $\theta^{-1} \circ \gamma \circ \theta \in \overline{\mathrm{Inn}}(\mathcal{T}_\varphi)$, which gives an action of $\mathrm{Out}(\mathcal{T}_\varphi)$ on $\overline{\mathrm{Inn}}(\mathcal{T}_\varphi)$. Therefore the result follows. $\square$

A set of totally ordered projections $e_1 \preceq \cdots \preceq e_n \in \mathcal{T}_n$ when embedded into $\mathcal{T}_m$ becomes a partition $A_1 \dot\cup \cdots \dot\cup A_n$ of $\{1, \cdots, m\}$ where $|A_i| = |A_{i'}| = m/n$ and $A_i \le A_{i+1}$ in the sense that the jth smallest element of A_i is smaller than the jth smallest element of A_{i+1}. We will call A an *ordered partition*.

Suppose we have two such ordered partitions $A = \dot\cup A_i$ and $B = \dot\cup B_i$ then we say $A \preceq B$ if for some $1 \le j \le m$, $j' \in A_i$ if and only if $j' \in B_i$ for all $1 \le j' < j$ and $j \in A_i, j \in B_{i'}$ with $i < i'$. In other words, the element where they differ occurs in an earlier set. Hence, this is a total order on ordered partitions of the same set.

Lemma 3.2.7. *Let $A = \dot\cup_{i=1}^n A_i$ and $B = \dot\cup_{i=1}^n B_i$ be ordered partitions of $\{1, \cdots, m\}$ and suppose that $\varphi : \mathcal{T}_m \to \mathcal{T}_{m'}$ is a unital embedding. If $A \preceq B$ then $\varphi(A) \preceq \varphi(B)$.*

Proof. Let $j \in A_i, j \in B_{i'}, i < i'$ be the first element that differs in the two partitions. Consider the first elementary projection of $\varphi(e_j) \in \mathcal{T}_{m'}$, say $e_{j_1} \le \varphi(e_j)$ then $j_1 \in \varphi(A_i)$ and $j_1 \in \varphi(B_{i'})$. Now let $j' < j_1$. Then $e_{j'} \preceq e_{j_1}$ which implies that $e_{j'} \le \varphi(e_{j''})$ with $j'' < j$ but then $j'' \in A_i$ if and only if $j'' \in B_i$ and so $j' \in \varphi(A_i)$ if and only if $j' \in \varphi(B_i)$. Therefore, $\varphi(A) \preceq \varphi(B)$. $\square$

Consider two embeddings $\varphi, \psi : \mathcal{T}_k \to \mathcal{T}_{k'}$. We say that $\varphi \preceq \psi$ if and only if $\varphi(\{1\} \cup \cdots \cup \{k\}) \preceq \psi(\{1\} \cup \cdots \cup \{k\})$. By the previous proposition if $\varphi' : \mathcal{T}_{k'} \to \mathcal{T}_{k''}$ is another embedding then $\varphi \preceq \psi$ implies that $\varphi' \circ \varphi \preceq \varphi' \circ \psi$. Note that if $\varphi \preceq \psi$ and $\psi \preceq \varphi$ then they agree on projections and furthermore, that two such embeddings are always comparable in this way.

Proposition 3.2.8. *For $\mathcal{T}_\varphi$ regular $\mathrm{Out}(\mathcal{T}_\varphi)$ is torsion free.*

Proof. Let $\theta \in \mathrm{Aut}(\mathcal{T}_\varphi)$ such that it preserves the unclosed union and $\theta^m = \mathrm{id}$ for some $m \geq 1$. For any choice of $n_1 \geq 1$ there exist $n_{m+1} \geq \cdots \geq n_2 \geq n_1$ such that

$$\theta(\mathcal{T}_{k_{n_i}}) \subset \mathcal{T}_{k_{n_{i+1}}}, \quad \text{for} \quad 1 \leq i \leq m.$$

For ease of notation let $k_i := k_{n_i}$, $\varphi_i := \varphi_{n_i}$ and $\theta_i := \theta|_{\mathcal{T}_{k_i}}$. This gives us the following identities:

$$\varphi_m \circ \cdots \circ \varphi_1 = \theta_m \circ \cdots \circ \theta_1 \quad \text{and} \quad \theta_{i+1} \circ \varphi_i = \varphi_{i+1} \circ \theta_i.$$

If $\varphi_1 \preceq \theta_1$ then by the previous lemma

$$\begin{aligned}
\varphi_m \circ \cdots \circ \varphi_1 &\preceq \varphi_m \circ \cdots \circ \varphi_3 \circ \varphi_2 \circ \theta_1 \\
&= \varphi_m \circ \cdots \circ \varphi_3 \circ \theta_2 \circ \varphi_1 \\
&\preceq \varphi_m \circ \cdots \circ \varphi_3 \circ \theta_2 \circ \theta_1 \\
&= \varphi_m \circ \cdots \circ \varphi_4 \circ \theta_3 \circ \theta_2 \circ \varphi_1 \\
&\preceq \cdots \\
&\preceq \varphi_m \circ \cdots \circ \varphi_i \circ \theta_{i-1} \circ \cdots \circ \theta_1 \\
&= \varphi_m \circ \cdots \circ \varphi_{i+1} \circ \theta_i \cdots \circ \theta_2 \circ \varphi_1 \\
&\preceq \cdots \\
&\preceq \varphi_m \circ \theta_{m-1} \circ \cdots \circ \theta_1 \\
&= \theta_m \circ \cdots \circ \theta_2 \circ \varphi_1 \\
&\preceq \theta_m \circ \cdots \circ \theta_1 \\
&= \varphi_m \circ \cdots \circ \varphi_1.
\end{aligned}$$

Hence, all of the inequalities are equalities which gives us that $\varphi_1 = \theta_1$ on $\mathrm{proj}(\mathcal{T}_{k_1})$. The same holds true if we assume $\theta_1 \preceq \varphi_1$ and thus, $\theta(p) = p$ for all projections $p \in \mathcal{T}_\varphi$ and by Proposition 3.2.4 $\theta \in \overline{\mathrm{Inn}}(\mathcal{T}_\varphi)$. Therefore, $\mathrm{Out}(\mathcal{T}_\varphi)$ is torsion free. □

3.3 The alternating embedding

Definition 3.3.1. *We say that* φ *is an* alternating embedding *if* $k_n = s_n t_n, n \geq 1$ *with* $s_n | s_{n+1}$ *and* $t_n | t_{n+1}$ *and*

$$\varphi_n(A) = I_{s_{n+1}/s_n} \otimes A \otimes I_{t_{n+1}/t_n}.$$

This is called alternating because φ_n is a standard embedding of size s_{n+1}/s_n followed by a nest embedding of size t_{n+1}/t_n, though the order does not matter as tensoring is associative. To each such embedding associate a pair of supernatural numbers (s_φ, t_φ)

where $s_\varphi = \prod_{n\geq 1} \frac{s_{n+1}}{s_n}$ and $t_\varphi = \prod_{n\geq 1} \frac{t_{n+1}}{t_n}$, the supernatural numbers of the standard and nest embeddings treated separately.

For these algebras there is a version of Glimm's theorem, that an alternating TUHF is characterized by a pair of supernatural numbers up to finite rearranging:

Proposition 3.3.2 ([57], Theorem 9.6). *Let $\mathcal{T}_\varphi$ and $\mathcal{T}_\psi$ have alternating embeddings. Then $\mathcal{T}_\varphi$ is isometrically isomorphic to $\mathcal{T}_\psi$ if and only if there exists $r \in \mathbb{Q}$ such that $s_\varphi = r \cdot s_\psi$ and $t_\varphi = r^{-1} \cdot t_\psi$.*

Proposition 3.3.3. *Let $\mathcal{T}_\varphi$ have an alternating embedding. To every prime p that infinitely divides both s_φ and t_φ there is a non-diagonal automorphism of $\mathcal{T}_\varphi$, called a* shift automorphism *and denoted θ_p.*

Proof. Without loss of generality, by dropping to a subsequence of the k_n, we may assume that $p|\frac{s_{n+1}}{s_n}$ and $p|\frac{t_{n+1}}{t_n}$. Define a map $\theta_p : \bigcup_{n\geq 1} \mathcal{T}_{k_n} \to \bigcup_{n\geq 1} \mathcal{T}_{k_n}$ by

$$A \in \mathcal{T}_{k_n} \mapsto \theta_p(A) = I_{\frac{ps_{n+1}}{s_n}} \otimes A \otimes I_{\frac{t_{n+1}}{pt_n}} \in \mathcal{T}_{k_{n+1}}.$$

First off, θ_p is well-defined:

$$\theta_p(\varphi_n(A)) = I_{\frac{ps_{n+2}}{s_{n+1}}} \otimes \left(I_{\frac{s_{n+1}}{s_n}} \otimes A \otimes I_{\frac{t_{n+1}}{t_n}} \right) \otimes I_{\frac{t_{n+2}}{pt_{n+1}}}$$

$$= I_{\frac{s_{n+2}}{s_{n+1}}} \otimes \left(I_{\frac{ps_{n+1}}{s_n}} \otimes A \otimes I_{\frac{t_{n+1}}{pt_n}} \right) \otimes I_{\frac{t_{n+2}}{t_{n+1}}} = \varphi_{n+1}(\theta_p(A)).$$

Note that $\theta_p(e_1^{(k_n)}) \neq \varphi_n(e_1^{(k_n)})$ and so if this extends to an automorphism it will not be approximately diagonal. Second, θ_p^{-1} is defined in the most obvious way:

$$\theta_p^{-1}(\theta_p(A)) = I_{\frac{s_{n+2}}{ps_{n+1}}} \otimes \left(I_{\frac{ps_{n+1}}{s_n}} \otimes A \otimes I_{\frac{t_{n+1}}{pt_n}} \right) \otimes I_{\frac{pt_{n+2}}{t_{n+1}}}$$

$$= I_{\frac{s_{n+2}}{s_n}} \otimes A \otimes I_{\frac{t_{n+2}}{t_n}} = \varphi_{n+1}(\varphi_n(A)).$$

Similarly, $\theta_p(\theta_p^{-1}(A)) = \varphi_{n+1}(\varphi_n(A))$ as well. Hence, θ_p is an isometric automorphism on the unclosed union and so extends to be an isometric automorphism of $\mathcal{T}_\varphi$. □

Let $p_1, \cdots, p_m$ be distinct primes that infinitely divide s_φ and t_φ and $\delta_1, \cdots, \delta_m \in \mathbb{N}$. For $u = \prod_{i=1}^m p_i^{\delta_i}$ define $\theta_u \in \mathrm{Aut}(\mathcal{T}_\varphi)$ to be

$$\theta_u = \theta_{p_1}^{\delta_1} \circ \cdots \circ \theta_{p_m}^{\delta_m}.$$

Note that the order of the p_i does not matter as all of these automorphisms commute.

We shift focus now back to ordered partitions. Before proving the main theorem of the section we first need two definitions and two technical lemmas.

Recall that $P = \dot\cup_{i=1}^n P_i$ is an ordered partition if $|P_1| = \cdots = |P_n| = m$ and $P_1 \leq P_2 \leq \cdots \leq P_n$. This ordering can also be given by letting $P_i = \{p_{1,i}, \cdots, p_{m,i}\}$ with $p_{1,i} < p_{2,i} < \cdots < p_{m,i}$ and then $P_i \leq P_j$ gives $p_{k,i} < p_{k,j}$ for every $1 \leq k \leq m$.

We will call $P = \dot\cup_{i=1}^n P_i$ an *ordered subpartition* if $|P_1| \geq |P_2| \geq \cdots |P_n|$ and $P_i \leq P_j$ for $1 \leq i < j \leq n$, meaning that $p_{l,i} < p_{l,j}$ for all $1 \leq l \leq |P_j|$.

Lemma 3.3.4. *Let $P = \dot\cup_{i=1}^n P_i = \{1, \cdots, m\}$ be an ordered partition. Then for $1 \leq m' \leq m$ we have that*

$$P \cap \{1, \cdots, m'\} = \dot\cup_{i=1}^n (P_i \cap \{1, \cdots, m'\})$$

is an ordered subpartition.

Proof. If $P_i \leq P_j$ then the kth smallest element of P_i precedes the kth smallest element of P_j. Hence, if the latter is in $\{1, \cdots, m'\}$ then the former will be as well, and so, $P_i \cap \{1, \cdots, m'\} \leq P_j \cap \{1, \cdots, m'\}$. □

A subset R is called a *run* if whenever $i < j < k$ and $i, k \in R$ then $j \in R$. If R and S are runs we say that $R < S$ if $r < s$ for all $r \in R$ and $s \in S$.

Lemma 3.3.5. *Let $R_1 < R_2 < \cdots < R_n$ be runs in $\{1, \cdots, r\}$ and $S_1 < \cdots < S_n < S_{n+1}$ be runs in $\{1, \cdots, s\}$ with $|S_1| = \cdots = |S_n| \geq 1$. If θ is a unital embedding of $\mathcal{T}_r$ into $\mathcal{T}_s$ such that $\theta(R) = S$ as sets and $\theta(R_i) \supset S_i$ then $|R_1| \leq \cdots \leq |R_n|$.*

Proof. Let $R_i = \{r_1^i, \cdots, r_{m_i}^i\}$ for $1 \leq i \leq n$. Because θ is a unital embedding we know that it takes the indices

$$r_1^1 < r_2^1 < \cdots < r_{m_1}^1 < r_1^2 < r_2^2 < \cdots < r_{m_n}^n$$

to the ordered partition

$$\theta(r_1^1) \leq \theta(r_2^1) \leq \cdots \leq \theta(r_{m_1}^1) \leq \theta(r_1^2) \leq \cdots \leq \theta(r_{m_n}^n).$$

In particular, they all have the same size, $|\theta(r_j^i)| = s/r$. By the previous lemma this order is maintained when considering only the first part of S, leading to the ordered subpartition

$$\theta(r_1^1) \cap (S_1 \cup \cdots \cup S_n) \leq \cdots \leq \theta(r_{m_n}^n) \cap (S_1 \cup \cdots \cup S_n).$$

Since $\theta(R_i) \supset S_i$ the ordered subpartition becomes

$$\theta(r_1^1) \cap S_1 \leq \cdots \leq \theta(r_{n_1}^1) \cap S_1 \leq \theta(r_1^2) \cap S_2 \leq \cdots \leq \theta(r_{m_n}^n) \cap S_n.$$

This implies that

$$|\theta(r_1^1) \cap S_1| \geq \cdots \geq |\theta(r_{n_1}^1) \cap S_1| \geq |\theta(r_1^2) \cap S_2| \geq \cdots \geq |\theta(r_{m_n}^n) \cap S_n|.$$

However, if $i < i'$

$$\sum_{k=1}^{m_i} |\theta(r_k^i) \cap S_i| = |S_i| = |S_{i'}| = \sum_{k=1}^{m_{i'}} |\theta(r_k^{i'}) \cap S_{i'}|$$

with every summand on the left being greater than every summand on the right, and so we must have $m_i \leq m_{i'}$. In other words,

$$|R_1| \leq |R_2| \leq \cdots \leq |R_n|.$$

□

Theorem 3.3.6. *Let $\mathcal{T}_\varphi$ have an alternating embedding for $k_n = s_n t_n$ and $\theta \in \mathrm{Aut}(\mathcal{T}_\varphi)$. Then there exists an approximately diagonal automorphism ψ and $u, v \in \mathbb{N}$ such that $\theta = \theta_u \circ \theta_v^{-1} \circ \psi$. Moreover, this factorization is unique if* $\gcd(u, v) = 1$.

Proof. Let $m \geq 1$ then there exist $m' \geq n \geq m$ such that

$$\theta^{-1}(\mathrm{proj}(\mathcal{T}_{k_m})) \subset \mathrm{proj}(\mathcal{T}_{k_n}), \quad \text{and} \quad \theta(\mathrm{proj}(\mathcal{T}_{k_n})) \subset \mathrm{proj}(\mathcal{T}_{k_{m'}}).$$

We will use the language of ordered partitions. In particular, let

$$P = \dot{\bigcup}_{i=1}^{k_m} P_i = \varphi_{m'-1} \circ \cdots \circ \varphi_m(\{1\} \cup \cdots \cup \{k_m\}),$$

that is the image in $k_{m'}$ of the elementary projections in k_m. Writing these as the disjoint union of runs we get

$$P_i = \dot{\bigcup}_{j=1}^{s_{m'}/s_m} P_{j,i} \quad \text{and} \quad P_{1,1} < P_{1,2} < \cdots < P_{1,k_m} < P_{2,1} \cdots < P_{s_{m'}/s_m, k_m}$$

with $|P_{j,i}| = t_{m'}/t_m$, which is obvious from the alternating embedding. Similarly, let

$$Q = \dot{\bigcup}_{i=1}^{k_m} Q_i = \theta^{-1}(\{1\} \cup \cdots \cup \{k_m\}), \quad \text{that is} \quad \theta^{-1}(e_i^{k_m}) = \sum_{j \in Q_i} e_j^{k_n}.$$

Also decompose this into runs

$$Q_i = \dot{\bigcup}_{j=1}^{s} Q_{j,i} \text{ and } Q_{1,1} < Q_{1,2} < \cdots < Q_{s,k_m}$$

where many of the $Q_{j,i}$ may be empty, but there are never $k_m - 1$ empty $Q_{j,i}$ all in a row because if this was not so then we could represent the partition as a shorter sequence. Note that $Q_{1,1}$ and Q_{s,k_m} are nonempty.

Claim: $|Q_{1,1}| = |Q_{1,2}| = \cdots = |Q_{1,k_m}|$.

Proof of Claim:
First, we know that

$$P_{1,i} = P_i \cap P_{1,i} = \theta(\theta^{-1}(e_i^{k_m})) \cap P_{1,i} = \theta(Q_i) \cap P_{1,i} = \bigcup_{j=1}^{k_n/k_m} \theta(Q_{j,i}) \cap P_{1,i}.$$

By Lemma 3.3.4 we get an ordered subpartition by intersecting with $P_{1,1}$,

$$(\theta(Q_{1,1}) \leq \theta(Q_{1,2}) \cdots \leq \theta(Q_{s,k_m})) \bigcap P_{1,1}$$

$$= \theta(Q_{1,1}) \cap P_{1,1} \leq \emptyset \leq \cdots \leq \emptyset \leq \theta(Q_{2,1}) \cap P_{1,1} \leq \emptyset \leq \cdots$$
$$\cdots \leq \emptyset \leq \theta(Q_{3,1}) \cap P_{1,1} \leq \emptyset \leq \cdots \leq \emptyset \leq \theta(Q_{s,1}) \cap P_{1,1} \leq \emptyset \leq \cdots \leq \emptyset$$

which implies that if any $\theta(Q_{j,1}) \cap P_{1,1}$ is nonempty then all the intermediate $Q_{1,1} < Q_{j',i'} < Q_{j,1}$ must be empty to remain an ordered subpartition under the above restriction, but this contradicts the requirement that there cannot be $k_m - 1$ empty $Q_{j',i'}$ in a row. Therefore, $\theta(Q_{1,1}) \cap P_{1,1} = P_{1,1}$.

Again

$$(\theta(Q_{1,1}) \leq \theta(Q_{1,2}) \leq \cdots \leq \theta(Q_{s,k_m})) \bigcap (P_{1,1} \cup P_{1,2})$$
$$= \theta(Q_{1,1}) \cap P_{1,1} \leq \theta(Q_{1,2}) \cap P_{1,2} \leq \emptyset \leq \cdots \leq \emptyset \leq \theta(Q_{2,2}) \cap P_{1,2} \leq \emptyset \leq \cdots$$

to get that $\theta(Q_{1,2}) \cap P_{1,2} = P_{1,2}$. Repeat this recursively to get that $\theta(Q_{1,i}) \cap P_{1,i} = P_{1,i}$. Noting that all $|P_{1,i}| = |P_{1,i'}|$ we have satisfied the hypotheses of Lemma 3.3.5. Hence, $|Q_{1,1}| \leq \cdots \leq |Q_{1,k_m}|$. The reverse direction is given by the fact that $Q_{1,1} < \cdots < Q_{1,k_m}$ is the first part of an ordered partition. Therefore, $|Q_{1,1}| = \cdots = |Q_{1,k_m}|$ and the claim has been verified.

This tells us that any isometric automorphism of an alternating embedding TUHF sends the elementary projections from a finite level to a partition with a specific starting pattern, that is, one iteration of equal runs. We apply this to the elementary projections of $\mathcal{T}_{k_n}$ to get that there exist runs

$$R_1 \leq R_2 \leq R_3 \leq \cdots \leq R_{k_n}$$

such that $|R_i| = |R_j| = r \geq 1$, $\cup R_i = \{1, \cdots, k\}$, $k \leq k_{m'}$ and $\theta(e_i^{k_n}) \supset R_i$.

Let $Q'_{j,i} = \cup_{l \in Q_{j,i}} R_l$ giving us runs with $|Q'_{j,i}| = |Q_{j,i}| \cdot r$ and $\theta(Q_{j,i}) \supset Q'_{j,i}$. Then the following partitions

$$P \cap \{1, \cdots, k\} = \theta(\theta^{-1}(\{1, \cdots, k_m\})) \cap \{1, \cdots, k\} = \theta(Q) \cap \{1, \cdots, k\}$$

must be equal. Which implies that

$$\cup P_{j,i} \cap \{1, \cdots, k\} = Q'_{1,1} < Q'_{1,2} < Q'_{1,3} < \cdots < Q'_{j,i} < \cdots < Q'_{s,k_m},$$

where both are decompositions into runs. Hence, $P_{j,i} = Q'_{j,i}$ which implies that $t = |Q_{j,i}| = |Q'_{j,i}|/r = |P_{j,i}|/r = \frac{t_{m'}}{t_m r}$, they are all the same size. Therefore, for $A \in \text{proj}(\mathcal{T}_{k_m})$

$$\theta^{-1}|_{\mathcal{T}_{k_m}}(A) = I_s \otimes A \otimes I_t.$$

We have then, that $t \cdot s \cdot k_m = k_n$. Let $\frac{s}{s_n/s_m} = \frac{u}{v}$ where $u = \prod_{i=1}^{l} p_i^{\delta_i}$ and $v = \prod_{j=1}^{k} q_j^{\epsilon_j}$ with $p_1, \cdots, p_l, q_1, \cdots, q_k$ distinct primes and $\delta_1, \cdots, \delta_l, \epsilon_1, \cdots, \epsilon_k \in \mathbb{N}$. Because $st = \frac{k_n}{k_m} = \frac{s_n}{s_m}\frac{t_n}{t_m}$ then $\frac{t}{t_n/t_m} = \frac{v}{u}$. This gives us that $v|\frac{s_n}{s_m}$ and $u|\frac{t_n}{t_m}$. Hence, for $A \in \text{proj}(\mathcal{T}_{k_m})$

$$\begin{aligned}\theta^{-1}|_{\mathcal{T}_{k_m}}(A) &= I_s \otimes A \otimes I_t = I_{\frac{s_n}{s_m}\frac{u}{v}} \otimes A \otimes I_{\frac{t_n}{t_m}\frac{v}{u}}. \\ &= \theta_{p_1}^{\delta_1} \circ \cdots \circ \theta_{p_l}^{\delta_l} \circ \theta_{q_1}^{-\epsilon_1} \circ \cdots \circ \theta_{q_k}^{-\epsilon_k}(A).\end{aligned}$$

Repeat this argument for any $\theta^{-1}(\text{proj}(\mathcal{T}_{k_{m'}})) \subset \text{proj}(\mathcal{T}_{k_{n'}})$, getting a similar result,

$$\theta^{-1}|_{\mathcal{T}_{k_{m'}}}(A) = \theta_{p'_1}^{\delta'_1} \circ \cdots \circ \theta_{p'_{l'}}^{\delta'_{l'}} \circ \theta_{q'_1}^{-\epsilon'_1} \circ \cdots \circ \theta_{q'_{k'}}^{-\epsilon'_{k'}}(A).$$

However, these two descriptions must agree on $\mathcal{T}_{k_m} \hookrightarrow \mathcal{T}_{k_{m'}}$ and so $u = u', v = v'$ and note that $v|\frac{s_{n'}}{s_{m'}}$ and $u|\frac{t_{n'}}{t_{m'}}$. In this way we see that $\theta^{-1} = \theta_u \circ \theta_v^{-1}$ on the projections of $\mathcal{T}_\varphi$ and that $v^\infty|s_\varphi, u^\infty|t_\varphi$. Finally then, by Proposition 3.2.4 there exists a approximately diagonal automorphism ψ such that $\theta = \theta_u^{-1} \circ \theta_v \circ \psi$ which also gives that $u^\infty|s_\varphi, v^\infty|t_\varphi$. Uniqueness of the factorization when $\gcd(u, v) = 1$ is obvious since we have seen that shift automorphisms and their inverses commute with other such automorphisms. Therefore, the result is established. $\square$

Corollary 3.3.7 (cf. [58], Theorem 1). *Let $\mathcal{T}_\varphi$ have an alternating embedding. Then* $\mathrm{Out}(\mathcal{T}_\varphi) \simeq \mathbb{Z}^d$ *where d is the number of common prime factors that infinitely divide both s_φ and t_φ.*

3.4 Tensoring TUHF algebras

The following section provides a technique to create new automorphism groups from old. To this end, suppose that $\mathcal{T}_\varphi = \overline{\cup_{n=1}^\infty \mathcal{T}_{k_n}}$ and $\mathcal{T}_\psi = \overline{\cup_{n=1}^\infty \mathcal{T}_{j_n}}$ are TUHF algebras.

We can create a new TUHF algebra

$$\mathcal{T}_{\varphi\otimes\psi} = \overline{\cup_{n=1}^\infty \mathcal{T}_{k_n j_n}}$$

with unital embeddings $\varphi_n \otimes \psi_n : \mathcal{T}_{k_n j_n} \to \mathcal{T}_{k_{n+1} j_{n+1}}$ defined by tensoring the old embeddings

$$\varphi_n \otimes \psi_n(A) = \varphi_n \otimes \psi_n([A_{i,i'}]_{i,i'=1}^{k_n}) = (\varphi_n \otimes I_{j_{n+1}})([\psi_n(A_{i,i'})]_{i,i'=1}^{k_n}).$$

Note that the ψ_n are $*$-extendable to all of M_{j_n}, meaning that ψ_n is the restriction of a unital C*-embedding from M_{j_n} into $M_{j_{n+1}}$, which is used when $i < i'$ in the block matrix. Therefore,

$$\mathcal{T}_{\varphi\otimes\psi} = \overline{\cup_{n=1}^\infty \mathcal{T}_{k_n j_n}} \supsetneq \overline{\cup_{n=1}^\infty \mathcal{T}_{k_n} \otimes \mathcal{T}_{j_n}} = \mathcal{T}_\varphi \otimes \mathcal{T}_\psi.$$

The new TUHF algebra is thus strictly bigger than the tensor product of the two previous algebras, but it inherits the automorphic structure of the two. It should be noted that this tensor operation is not commutative. That is, $\mathcal{T}_{\varphi\otimes\psi}$ and $\mathcal{T}_{\psi\otimes\varphi}$ need not be isomorphic.

This new embedding gives that $M(\mathcal{T}_{\varphi\otimes\psi}) = M(\mathcal{T}_\varphi) \times M(\mathcal{T}_\psi)$ with the order $((x_1, x_2), (y_1, y_2)) \in R(\mathcal{T}_{\varphi\otimes\psi})$ if and only if $(x_1, y_1) \in R(\mathcal{T}_\varphi)$ and $(x_2, y_2) \in R(\mathcal{T}_\psi)$ if $x_1 = y_1$.

In the following, $G^{\oplus\infty}$ refers to the infinite direct sum of a group G, a subgroup of the infinite direct product where elements are infinite tuples with all but a finite number of entries equal to the identity.

Theorem 3.4.1. *Let $\mathcal{T}_\varphi$ and $\mathcal{T}_\psi$ be regular TUHF algebras then*

$$\mathrm{Aut}(\mathcal{T}_\psi)^{\oplus\infty} \rtimes \mathrm{Aut}(\mathcal{T}_\varphi) \quad \subseteq \quad \mathrm{Aut}(\mathcal{T}_{\varphi\otimes\psi}).$$

Proof. Clearly $\mathrm{Aut}(\mathcal{T}_\varphi) \hookrightarrow \mathrm{Aut}(\mathcal{T}_{\varphi\otimes\psi})$ since if θ is an order preserving homeomorphism of $M(\mathcal{T}_\varphi)$ then $\theta \times \mathrm{id}$ is an order preserving homeomorphism of $M(\mathcal{T}_{\varphi\otimes\psi}) = M(\mathcal{T}_\varphi) \times M(\mathcal{T}_\psi)$; and so by Theorem 3.2.1 we get an induced automorphism on $\mathcal{T}_{\varphi\otimes\psi}$. The same argument works for the embedding $\mathrm{Aut}(\mathcal{T}_\psi) \hookrightarrow \mathrm{Aut}(\mathcal{T}_{\varphi\otimes\psi})$ as well.

Moreover, we see that if $X \subset M(\mathcal{T}_\varphi)$ is a clopen subset and θ is an order preserving homeomorphism of $M(\mathcal{T}_\psi)$ then

$$\mathrm{id}_X \times \theta + \mathrm{id}_{X^C} \times \mathrm{id}$$

is an also order preserving homeomorphism of $M(\mathcal{T}_{\varphi\otimes\psi})$. Since clopen subsets of $M(\mathcal{T}_\varphi)$ are in bijective correspondence with the projections of $\mathcal{T}_\varphi$ then for each $n \geq 1$ we see that

$$\mathrm{id}_{X_1} \times \theta_1 + \cdots + \mathrm{id}_{X_{k_n}} \times \theta_{k_n}$$

is an order preserving homeomorphism where X_j is the clopen subset associated with $e_j^{(k_n)} \in \mathcal{T}_{k_n}$ and θ_j is an order preserving homeomorphism on $M(\mathcal{T}_\psi)$. Thus, $\mathrm{Aut}(\mathcal{T}_\psi)^{k_n} \hookrightarrow \mathrm{Aut}(\mathcal{T}_{\varphi\otimes\psi})$.

Therefore, we have that $\varinjlim \mathrm{Aut}(\mathcal{T}_\psi)^{\oplus k_n} \subset \mathrm{Aut}(\mathcal{T}_{\varphi\otimes\psi})$ where the direct limit has the following injective homomorphisms: $\tilde{\varphi}_n : \mathrm{Aut}(\mathcal{T}_\psi)^{\oplus k_n} \to \mathrm{Aut}(\mathcal{T}_\psi)^{\oplus k_{n+1}}$ where

$$\tilde{\varphi}_n(\gamma_1, \cdots, \gamma_{k_n}) \;=\; (\gamma_{i_1}, \gamma_{i_2}, \cdots, \gamma_{i_{k_{n+1}}}),$$

with $e_j^{(k_{n+1})} \leq \varphi_n(e_{i_j}^{(k_n)})$, for $1 \leq j \leq k_{n+1}$. Note that the direct limit $\varinjlim \mathrm{Aut}(\mathcal{T}_\psi)^{\oplus k_n}$ is equal to the infinite direct sum $\mathrm{Aut}(\mathcal{T}_\psi)^{\oplus\infty}$.

Finally, we need to describe the action of $\mathrm{Aut}(\mathcal{T}_\varphi)$ on the direct limit. Taking θ and γ as order preserving homeomorphisms in $M(\mathcal{T}_\psi)$ and $M(\mathcal{T}_\varphi)$ respectively, and X clopen in $M(\mathcal{T}_\varphi)$ we get that

$$(\gamma \times \mathrm{id}) \circ (\mathrm{id}_X \times \theta + \mathrm{id}_{X^C} \times \mathrm{id}) \circ (\gamma^{-1} \times \mathrm{id}) = \mathrm{id}_{\gamma(X)} \times \theta + \mathrm{id}_{\gamma(X)^C} \times \mathrm{id}\,.$$

Therefore, $\mathrm{Aut}(\mathcal{T}_\psi)^{\oplus\infty} \rtimes \mathrm{Aut}(\mathcal{T}_\varphi) \subseteq \mathrm{Aut}(\mathcal{T}_{\varphi\otimes\psi})$. □

Corollary 3.4.2. $\mathrm{Out}(\mathcal{T}_\psi)^{\oplus\infty} \rtimes \mathrm{Out}(\mathcal{T}_\varphi) \subseteq \mathrm{Out}(\mathcal{T}_{\varphi\otimes\psi})$

Proof. By Theorem 3.2.6 the outer automorphisms of both $\mathcal{T}_\varphi$ and $\mathcal{T}_\psi$ are well defined subgroups given by those automorphisms which are regular embeddings when restricted to a finite level. This property is clearly preserved in the proof of the last theorem and so the result follows. □

This implies that there are non-abelian outer automorphism groups. However, these groups may not be equal as in the following example:

Example 3.4.3. Let $\mathcal{T}_\varphi$ be the standard embedding algebra for 2^∞ and $\mathcal{T}_\psi$ be the nest embedding algebra for 2^∞. Then $\mathcal{T}_{\varphi\otimes\psi}$ is the alternating algebra for 2^∞. Hence, $\mathrm{Out}(\mathcal{T}_{\varphi\otimes\psi}) = \mathbb{Z} \neq \{0\} = \mathrm{Out}(\mathcal{T}_\psi)^{\oplus\infty} \rtimes \mathrm{Out}(\mathcal{T}_\varphi)$.

3.5 Dilation theory

All the definitions in this last section come from the paper of Davidson and Katsoulis [26]. An operator algebra $\mathcal{A}$ is said to be *semi-Dirichlet* if $\mathcal{A}^*\mathcal{A} \subset \overline{\mathcal{A} + \mathcal{A}^*}$ when $\mathcal{A}$ is considered as a subspace of its C*-envelope. Moreover, a unital operator algebra $\mathcal{A}$ is *Dirichlet* if $\mathcal{A} + \mathcal{A}^*$ is norm dense in its C*-envelope, $C_e^*(\mathcal{A})$.

Lemma 3.5.1. *Triangular UHF algebras are Dirichlet.*

Proof. For a TUHF algebra $\mathcal{T}_\varphi$ we have the much stronger condition that $\mathfrak{A}_\varphi = \overline{\mathcal{T}_\varphi + \mathcal{T}_\varphi^*}$. Therefore, because the UHF algebra is simple we immediately get the desired result. □

A unital operator algebra $\mathcal{A}$ is said to have the *Fuglede property* if for every faithful unital $*$-representation π of $C_e^*(\mathcal{A})$ we have $\pi(\mathcal{A})' = \pi(C_e^*(\mathcal{A}))'$.

Lemma 3.5.2. *Triangular UHF algebras have the Fuglede property.*

Proof. Suppose π is a faithful unital $*$-representation of $C_e^*(\mathcal{T}_\varphi) = \overline{\cup_{k_n} M_{k_n}}$. Then $\pi(\mathcal{T}_{k_n})' = \pi(M_{k_n})'$ and so $\pi(\mathcal{T}_\varphi)' = \pi(C_e^*(\mathcal{T}_\varphi))'$. □

An operator algebra $\mathcal{A}$ has *isometric commutant lifting (ICLT)* if whenever there is a completely contractive representation $\rho : \mathcal{A} \to B(\mathcal{H})$ commuting with a contraction X, there is a coextension σ of ρ and an isometric coextension V of X on a common Hilbert space $\mathcal{K}$ so that $\sigma(\mathcal{A})$ and V commute.

Proposition 3.5.3. *Triangular UHF algebras have isometric commutant lifting.*

Proof. Let ρ be a contractive representation of $\mathcal{T}_\varphi$ on $\mathcal{H}$ commuting with a contraction X. Without loss of generality assume that ρ is also unital. Now ρ is completely contractive when restricted to any $\mathcal{T}_{k_n}$ and thus on a dense set of $\mathcal{T}_\varphi$. Hence, ρ is a completely contractive representation. By Arveson's Extension Theorem and Stinespring's Dilation Theorem there is a $*$-homomorphism π and an isometry $V : \mathcal{H} \to \mathcal{K}$ such that $\rho(a) = V^*\pi(a)V, \forall a \in \mathcal{T}_\varphi$. This argument was given by Paulsen and Power in [51] but can also be found in [23].

For each $n \geq 1$ we know that X commutes with $\rho(\mathcal{T}_{k_n})$ and so by [23, Corollary 20.23] there is an operator Y_n on $\mathcal{K}$ commuting with $\pi|_{M_{k_n}}$ such that $\|Y_n\| = \|X\|$ and

$$P(\mathcal{H})Y_n^m\pi(A)|\mathcal{H} = X^m\rho(A), \quad \forall m \geq 0, A \in \mathcal{T}_{k_n}.$$

Since all the Y_n are bounded by $\|X\| \leq 1$ there is a subsequence converging in the weak operator topology to $Y \in B(\mathcal{K})$ which clearly commutes with π. Now, dilate Y to a lower triangular unitary V on $\mathcal{K}^{(\infty)}$ which commutes with $\pi^{(\infty)}$ because π commutes with Y^* as well. Thus, by restricting to the coextension part of the dilation we see that we have a coextension of ρ which commutes with an isometric coextension of X. Therefore, $\mathcal{T}_\varphi$ has property ICLT. $\square$

Let ρ be a representation of a unital operator algebra $\mathcal{A}$. Then a coextension σ of ρ is called *fully extremal* if whenever π is a dilation of σ which is also a coextension of ρ then π is just a direct sum, $\pi = \sigma \oplus \sigma'$.

Definition 3.5.4. *A unital operator algebra $\mathcal{A}$ has the* Ando property *if whenever ρ is a representation of $\mathcal{A}$ and X is a contraction commuting with $\rho(\mathcal{A})$, then there is a fully extremal coextension σ of ρ commuting with an isometric coextension of X.*

Theorem 3.5.5. *Triangular UHF algebras have the Ando property.*

Proof. The following commutant lifting properties are all listed in [26] and will not be defined as they only are used as stepping stones in the proof below.

[26, Corollary 7.4] gives that ICLT implies MCLT and [26, Corollary 5.18] gives that being Dirichlet and having MCLT implies CLT and CLT*. Lastly, by [26, Corollary 9.12] having the Fuglede property, CLT and CLT* implies that triangular UHF algebras have the Ando property. $\square$

If $\mathcal{A}$ is an operator algebra and θ is an automorphism, the semicrossed product is the operator algebra

$$\mathcal{A} \times_\theta \mathbb{Z}_+$$

that encapsulates the dynamical system $(\mathcal{A}, \theta)$. This first occurs in the work of Arveson [8] with a more modern treatment given by [47]. In particular, this is the universal operator algebra generated by all covariant representations (ρ, T) where ρ is a completely contractive representation of $\mathcal{A}$ and a contraction T such that

$$\rho(a)T = T\rho(\theta(a)), \quad \forall a \in \mathcal{A}.$$

The following corollary says that the C*-envelope of a semicrossed product of a TUHF algebra with an automorphism is in fact a full crossed product algebra.

Corollary 3.5.6. *Let $\mathcal{T}_\varphi$ be a TUHF algebra and $\theta \in \mathrm{Aut}(\mathcal{T}_\varphi)$ then*

$$C_e^*(\mathcal{T}_\varphi \times_\theta \mathbb{Z}_+) = C_e^*(\mathcal{T}_\varphi) \times_\theta \mathbb{Z} = \mathfrak{A}_\varphi \times_\theta \mathbb{Z}.$$

Proof. By [26, Theorem 12.3] if θ is an isometric automorphism of $\mathcal{T}_\varphi$ then because TUHF algebras have the Ando property $C_e^*(\mathcal{T}_\varphi \times_\theta \mathbb{Z}_+) = C_e^*(\mathcal{T}_\varphi) \times_\theta \mathbb{Z}$. Lastly, recall that $C_e(\mathcal{T}_\varphi) \simeq \mathfrak{A}_\varphi$. □

We end with the following example:

Example 3.5.7. Suppose $\mathcal{T}_\varphi$ is a TUHF algebra with the 2^∞ alternating embedding and consider the shift automorphism θ_2. Now $\mathcal{T}_\varphi$ is a non-selfadjoint subalgebra of the CAR algebra, $M_{2^\infty} = \bigotimes_{-\infty}^{\infty} M_2$. In this form θ_2 extends to the so called Bernoulli shift on the CAR algebra, taking a tensor in $\bigotimes_{-\infty}^{\infty} M_2$ and shifting it to the right.

Bratteli, Kishimoto, Rørdam and Størmer show in [16] that

$$M_{2^\infty} \times_{\theta_2} \mathbb{Z} \simeq \varinjlim M_{4^n} \otimes C(\mathbb{T}),$$

a limit circle algebra with embeddings being two copies of the twice-around embedding. Moreover, this AT algebra is isomorphic to $M_{2^\infty} \otimes \mathfrak{B}$ where $\mathfrak{B} = \varinjlim M_{2^n} \otimes C(\mathbb{T})$ is the Bunce-Deddens algebra [18]. Many thanks to Mikael Rørdam for pointing this last isomorphism out. Among other things, this implies that the crossed product is a unital simple C*-algebra which falls into Elliott's classification.

Therefore, by the above Corollary:

$$C_e^*(\mathcal{T}_\varphi \times_{\theta_2} \mathbb{Z}_+) \simeq M_{2^\infty} \otimes \mathfrak{B}.$$

This leads to the question of whether the semicrossed product is itself isomorphic to a "nice" subalgebra of $M_{2^\infty} \otimes \mathfrak{B}$, for instance a tensor of two non-selfadjoint operator algebras sitting in the CAR algebra and the Bunce-Deddens algebra.

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