Digraph Algebras over Discrete Pre-ordered Groups

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Pure Mathematics

Waterloo, Ontario, Canada, 2013

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

This thesis consists of studies in the separate fields of operator algebras and non-associative algebras.

Two natural operator algebra structures, $\mathcal{A} \otimes_{\max} \mathcal{B}$ and $\mathcal{A} \otimes_{\min} \mathcal{B}$, exist on the tensor product of two given unital operator algebras $\mathcal{A}$ and $\mathcal{B}$. Because of the different properties enjoyed by the two tensor products in connection to dilation theory, it is of interest to know when they coincide (completely isometrically). Motivated by earlier work due to Paulsen and Power, we provide conditions relating an operator algebra $\mathcal{B}$ and another family $\{\mathcal{C}_i\}_i$ of operator algebras under which, for any operator algebra $\mathcal{A}$, the equality $\mathcal{A} \otimes_{\max} \mathcal{B} = \mathcal{A} \otimes_{\min} \mathcal{B}$ either implies, or is implied by, the equalities $\mathcal{A} \otimes_{\max} \mathcal{C}_i = \mathcal{A} \otimes_{\min} \mathcal{C}_i$ for every i. These results can be applied to the setting of a discrete group G pre-ordered by a subsemigroup G^+, where $\mathcal{B} \subset C^*_r(G)$ is the subalgebra of the reduced group C^*-algebra of G generated by G^+, and $\mathcal{C}_i = \mathbb{A}(Q_i)$ are digraph algebras defined by considering certain pre-ordered subsets Q_i of G.

The 16-dimensional algebra $\mathbf{A}_4$ of real sedenions is obtained by applying the Cayley-Dickson doubling process to the real division algebra of octonions. The classification of subalgebras of $\mathbf{A}_4$ up to conjugacy (i.e. by the action of the automorphism group of $\mathbf{A}_4$) was completed in a previous investigation, except for the collection of those subalgebras which are isomorphic to the quaternions. We present a classification of quaternion subalgebras up to conjugacy.

Acknowledgements

First and foremost, I wish to thank my current supervisor Laurent Marcoux and former supervisor Dragomir Đoković for their mentorship throughout my graduate studies. Their equally strong work ethic, as well as their distinct styles of thinking, have been my sources of inspiration. In particular, Professor Marcoux, who sets an example as a serious teacher with a more than healthy sense of humor, has also benefitted my development, both professionally and personally.

I would like to thank my examiners Man-Duen Choi, Ken Davidson, Dan Isaksen, John Lawrence and Nico Spronk for lending their time and expertise. Comments made by Professors Choi, Davidson and Spronk have guided me through various technical points. Special thanks are due to Professors Che-Tat Ng and B. Doug Park for their initial guidance, and to departmental secretaries Lis D'Alessio, Nancy Maloney and Shonn Martin for their constant support in many aspects of my student life.

Over the years, have been deeply blessed by the friendships with new friends Mukto Akash, Clinton Loo, Michael Ka-Shing Ng and Kenneth Emeka Onuma, as well as old friends Dilian Yang, Yanqiao Zhang and Denglin Zhou.

Finally, this work would be impossible without the opportunity and continuous support, financial and otherwise, generously provided by the University of Waterloo and the Natural Sciences and Engineering Research Council of Canada.

Dedication

To my father Siu-Min, mother Lai-Ming, and brother Tony Kai-Bong

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Chapter 1

Tensor Products of Digraph Algebras over Discrete Pre-ordered Groups

1.1 Introduction

In an early investigation of tensor products of non-self-adjoint operator algebras [23], Paulsen and Power studied the maximal and minimal tensor products and their connection to joint dilations of completely contractive representations. The tensor product framework provides an alternative interpretation of Ando's dilation theorem and the Sz.-Nagy-Foias commutant lifting theorem. For instance, Ando's dilation theorem, which says that any two commuting contractions on a Hilbert space admit joint unitary dilations to a larger Hilbert space, can be recast as the claim that

$$A(\mathbb{D}) \otimes_{\max} A(\mathbb{D}) \;=\; A(\mathbb{D}) \otimes_{\min} A(\mathbb{D})$$

completely isometrically, where $A(\mathbb{D})$ is the disk algebra. One of the notable results in that paper is as follows. For each integer $n \geq 1$, let $\mathcal{T}(n)$ denote the upper triangular subalgebra of the matrix algebra M_n.

Theorem 1.1.1 (Theorem 3.2, [23]). *Let $\mathcal{A}$ be a unital operator algebra. The following are equivalent:*

(a) $\mathcal{A} \otimes_{\max} A(\mathbb{D}) = \mathcal{A} \otimes_{\min} A(\mathbb{D})$ *completely isometrically.*

(b) $\mathcal{A} \otimes_{\max} \mathcal{T}(n) = \mathcal{A} \otimes_{\min} \mathcal{T}(n)$ *completely isometrically, for all* $n \geq 1$.

The work that we shall present is motivated by the above result and its proof. The connection between $A(\mathbb{D})$ and the sequence $\mathcal{T}(n)$ becomes clearer if we identify $A(\mathbb{D})$ with the norm-closure of the unital subalgebra of $C(\mathbb{T}) \cong C^*(\mathbb{Z})$ generated by the identity function $z \in C(\mathbb{T})$, and also identify $\mathcal{T}(n)$ with the digraph algebra $\mathbb{A}(C_n)$ associated with the linearly ordered set $C_n = \{1, \cdots, n\} \subset \mathbb{Z}$. (In combinatorics, $\mathbb{A}(C_n)$ is also known as the incidence algebra associated with the partially ordered set C_n.) It turns out that the implication (a) $\Rightarrow$ (b) generalizes quite directly when we have data (G, G^+), where G is a discrete group and G^+ is any unital subsemigroup of G. Then G^+ induces a pre-ordering $\leq$ on G, where $s \leq t$ provided that $ts^{-1} \in G^+$. In this setting, $A(\mathbb{D})$ is replaced by the subalgebra $\mathsf{alg}(G^+)$ of the reduced group C^*-algebra $C^*_r(G)$ generated by the subsemigroup G^+, while the algebras $\mathcal{T}(n)$ are replaced by finite-dimensional digraph algebras $\mathbb{A}(P)$, where P are certain finite "convex" subsets of the and pre-ordered set G. The implication (b) $\Rightarrow$ (a) seems to generalize partially, at least to the case where G is amenable.

1.2 Background

In this section, we recall the concepts of operator spaces and completely bounded maps, leading up to the abstract definition of operator algebras. This standard material can be found in many textbooks, e.g. [2], [14], [22], [24]. Proofs of the cited facts are omitted.

We assume as known the elementary theory of C^*-algebras. In the sequel, we reserve the symbols $\mathcal{H}$ and $\mathcal{K}$ for Hilbert spaces. The inner product of $h_1, h_2 \in \mathcal{H}$ is written as $\langle h_1, h_2 \rangle$. The space of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ is denoted by $\mathbb{B}(\mathcal{H}, \mathcal{K})$. In particular, $\mathbb{B}(\mathcal{H}) = \mathbb{B}(\mathcal{H}, \mathcal{H})$ is a C^*-algebra with identity $I = I_{\mathcal{H}}$. If $\mathcal{H}$ is a subspace of $\mathcal{K}$ and $T \in \mathbb{B}(\mathcal{K})$, the compression of T to $\mathcal{H}$ is the operator $P_{\mathcal{H}} T|_{\mathcal{H}} \in \mathbb{B}(\mathcal{H})$, where $P_{\mathcal{H}} : \mathcal{K} \to \mathcal{H}$ is orthogonal projection of $\mathcal{K}$ to $\mathcal{H}$. The group of unitary operators on $\mathcal{H}$ is denoted by $\mathcal{U}(\mathcal{H})$. The Hilbert space tensor product is simply written as $\mathcal{H} \otimes \mathcal{K}$. As is common, if $h \in \mathcal{H}$ and $k \in \mathcal{K}$, then $h \otimes k \in \mathcal{H} \otimes \mathcal{K}$ is an elementary tensor, whereas $h \otimes k^* \in \mathbb{B}(\mathcal{K}, \mathcal{H})$ is the rank-one operator sending each $x \in \mathcal{K}$ to $\langle x, k \rangle h \in \mathcal{H}$. If A is any set, $\ell_2(A)$ is the Hilbert space of square-summable complex-valued functions on A. The standard orthonormal basis for $\ell_2(A)$ is $\{e_a\}_{a \in A}$, where $e_a(a) = 1$ and $e_a(a') = 0$ for all $a' \in A$ with $a' \neq a$. We shall make use of the standard matrix units $E_{a,a'} = e_a \otimes (e_{a'})^*$ for $a, a' \in A$. If A and B are sets, then $\ell_2(A) \otimes \ell_2(B) \cong \ell_2(A \times B)$ via the canonical unitary isomorphism which sends $e_a \otimes e_b$ to $e_{(a,b)}$. We shall write $\ell_2 = \ell_2(\mathbb{N})$, where $\mathbb{N}$ is the set of non-negative integers.

In what follows, if X is any vector space and $m, n \geq 1$ is an integer, we denote the vector space of $m \times n$ matrices with entries in X by

$$M_{m,n}(X) \;=\; \left\{ \left[x_{i,j}\right]_{(i,j)} \;:\; x_{i,j} \in X,\ \forall\, 1 \leq i \leq m,\ 1 \leq j \leq n \right\}.$$

We write $M_n(X) := M_{n,n}(X)$. If $\phi : X \to Y$ is a linear map between vector spaces X and Y, for each integer $n \geq 1$, we define the induced linear map $\phi_{(n)} : M_n(X) \to M_n(Y)$ by the formula

$$x = \left[x_{i,j}\right]_{(i,j)} \in M_n(X) \quad \mapsto \quad \phi_{(n)}(x) = \left[\phi(x_{i,j})\right]_{(i,j)} \in M_n(Y).$$

1.2.1 Definitions of operator spaces and completely bounded maps

Briefly speaking, an operator space is a subspace X of the C^*-algebra $\mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Note that for each integer $n \geq 1$, the $*$-algebra of $n \times n$ operator matrices

$$M_n\big(\mathbb{B}(\mathcal{H})\big) \;\cong\; \mathbb{B}(\mathcal{H}^n)$$

is canonically identified with the C^*-algebra of bounded operators on the Hilbert space direct sum $\mathcal{H}^n = \mathcal{H} \oplus \cdots \oplus \mathcal{H}$. Consequently, the vector space $M_n(X)$ of $n \times n$ matrices with entries from X inherits a norm $\|\cdot\|_n$ when treated as a subspace of $\mathbb{B}(\mathcal{H}^n)$. Furthermore, if $\phi : X \to Y$ is a linear map between operator spaces X and Y, we consider the induced maps $\phi_{(n)} : M_n(X) \to M_n(Y)$ and their norms. It is clear that if ϕ is bounded, then so is each $\phi_{(n)}$, and $\|\phi_{(n)}\| \leq \|\phi_{(n+1)}\|$ for all n. However, there are many examples in which ϕ is either contractive or isometric, and yet $\phi_{(n)}$ ceases to be contractive or isometric. Alternatively, another possible kind of defect is when $\sup_n \|\phi_{(n)}\| = \infty$. Arveson's work on the dilation theory of representations of non-selfadjoint subalgebras of C^*-algebras indicates the importance of keeping track of the matrix norms $\{\|\cdot\|_n\}_{n\geq 1}$ as well as knowing the finiteness of $\sup_n \|\phi_{(n)}\|$. This motivates the following definition.

Definition 1.2.1. (a) By a ***matrix-normed space***, we mean any vector space X together with a family of norms $\|\cdot\|_n$ on the matrix space $M_n(X)$ for every $n \geq 1$, such that for any $m, n \geq 1$, $\alpha \in M_{m,n}$, $x \in M_n(X)$ and $\beta \in M_{n,m}$,

$$\|\alpha x \beta\|_m \;\leq\; \|\alpha\| \, \|x\|_n \, \|\beta\| \, .$$

(b) By a ***concrete operator space***, we mean a (not necessarily closed) vector subspace $X \subseteq \mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, together with the family of matrix norms $\|\cdot\|_n$ on each space $M_n(X)$ as a normed subspace of $M_n(\mathbb{B}(\mathcal{H})) = \mathbb{B}(\mathcal{H}^n)$.

Definition 1.2.2. Let X and Y be matrix-normed spaces. If $\phi : X \to Y$ is a linear operator, we define

$$\|\phi\|_{\mathrm{cb}} \;:=\; \sup\{\|\phi_{(n)}\| \;:\; n \geq 1\}.$$

If $\|\phi\|_{\mathrm{cb}} < \infty$, ϕ is said to be ***completely bounded*** (c.b.). If $\|\phi\|_{\mathrm{cb}} \leq 1$, ϕ is said to be ***completely contractive*** (c.c.). We call ϕ a ***complete isometry*** if $\phi_{(n)}$ is an isometry for every $n \geq 1$.

The definition of a matrix-normed space implies that the sequence of embeddings

$$X \;=\; M_1(X) \;\hookrightarrow\; M_2(X) \;\hookrightarrow\; \cdots \;\hookrightarrow\; M_n(X) \;\hookrightarrow\; M_{n+1}(X) \;\hookrightarrow \cdots$$

where

$$x \in M_n(X) \;\mapsto\; \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \in M_{n+1}(X)$$

are isometric. The following condition is satisfied by any concrete operator space X:

(R) For any $m, n \geq 1$, $x \in M_m(X)$ and $y \in M_n(X)$,

$$\left\| \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right\|_{m+n} = \max\left\{ \|x\|_m, \|y\|_n \right\}.$$

We can now state Ruan's basic result, which says that conversely, if a matrix-normed space X satisfies condition (R), then X is completely isometrically isomorphic to a concrete operator space. This provides an abstract approach to operator spaces.

Theorem 1.2.1 (Ruan [26])**.** *Let X be a matrix-normed space. The following are equivalent:*

(a) The matrix norms of X satisfy condition (R).

(b) There exists a Hilbert space $\mathcal{H}$ and a complete isometry $i : X \hookrightarrow \mathbb{B}(\mathcal{H})$.

By an ***operator space structure*** (o.s.s.) on a space X, we shall mean a specific choice of matrix norms on X which satisfy condition (R).

Example 1.2.1. A basic example is the class of all (unital) C^*-algebras. By the Gelfand-Naimark Theorem, any C^*-algebra $\mathcal{A}$ admits a faithful (hence isometric) $*$-representation $i : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$. We can thus endow $\mathcal{A}$ with an operator space structure via the embedding i. Since $i_{(n)} : M_n(\mathcal{A}) \to M_n\big(\mathbb{B}(\mathcal{H})\big)$ is an injective $*$-homomorphism for each n, the resulting norm $\|\cdot\|_n$ makes $M_n(\mathcal{A})$ into a C^*-algebra. Knowing the uniqueness of complete C^*-norms, it follows that the resulting norms do not depend on the specific embedding i. Therefore, we may speak of the *canonical operator space structure* on any C^*-algebra.

1.2.2 Operator systems

An operator system is a kind of operator space which retains the order structure of a unital C^*-algebra. By a *unital subspace* of a unital algebra $\mathcal{A}$, we simply mean any subspace of $\mathcal{A}$ which contains the identity of $\mathcal{A}$.

Definition 1.2.3. A (concrete) ***operator system*** is a self-adjoint, unital subspace S of a unital C^*-algebra $\mathcal{A}$. The cone of positive elements in S is denoted by S_+.

Observe that if S is an operator system contained in a unital C^*-algebra $\mathcal{A}$, then $M_n(S)$ is an operator system contained in the C^*-algebra $M_n(\mathcal{A})$ for each positive integer n.

Definition 1.2.4. Let S and T be operator systems. A linear map $\phi : S \to T$ is said to be ***positive*** provided that $\phi(S_+) \subseteq T_+$. Given a positive integer n, we say that ϕ is ***n-positive*** if $\phi_{(n)} : M_n(S) \to M_n(T)$ is positive. If ϕ is n-positive for every $n \geq 1$, we call ϕ ***completely positive***. If ϕ is invertible and both ϕ and ϕ^{-1} are completely positive, we say that ϕ is a ***complete order isomorphism***.

It is a basic fact that any positive linear map $\phi : S \to T$ between operator systems is self-adjoint, i.e. $\phi(x^*) = \phi(x)^*$ for all $x \in S$, and that ϕ is bounded, with $\|\phi\| \leq 2\|\phi(1)\|$. Also, from the fact that for any $x \in S$,

$$\|x\| \leq 1 \quad \Leftrightarrow \quad \begin{bmatrix} 1 & x \\ x^* & 1 \end{bmatrix} \geq 0,$$

it follows that if ϕ is 2-positive, then $\|\phi\| \leq \|\phi(1)\|$. Consequently, if ϕ is completely positive, then $\|\phi\|_{\mathrm{cb}} = \|\phi\| = \|\phi(1)\|$. A similar argument shows that the operator space structure on S is determined by the cones $M_n(S)_+$, since for any $n \geq 1$, $x \in M_n(S)$ and $r > 0$,

$$\|x\|_n \leq r \quad \Leftrightarrow \quad \begin{bmatrix} r & x \\ x^* & r \end{bmatrix} \in M_{2n}(S)_+.$$

In particular, a unital, complete order isomorphism ϕ from S into T is a complete isometry. (Just as operator spaces can be distinguished from matrix-normed spaces by the satisfaction of an additional axiom, it is shown by Choi and Effros [9] that there is an axiomatic characterization of operator systems as a special class of *matrix-ordered spaces*. We shall not pursue this theory here.)

Proposition 1.2.2 (Arveson [1]). *(a) Let $\phi : S \to \mathcal{B}$ be a unital, contractive linear map from an operator system S into a unital C^*-algebra $\mathcal{B}$. Then ϕ is positive.*

(b) Let X be a unital subspace of a unital C^*-algebra $\mathcal{A}$, and let $X^\star = \{x^* : x \in X\}$. Consider the operator system $X + X^\star = \{x + y^* : x, y \in X\}$ generated by X. Any unital, contractive (resp. completely contractive, completely isometric) map $\phi : X \to S$ into an operator system S has a unique positive (resp. completely positive, completely isometric) extension $\tilde{\phi} : X + X^\star \to S$.

As a consequence of Proposition 1.2.2 (a), any unital, contractive homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ between C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ is a $*$-homomorphism, and hence is completely positive as well as completely contractive.

1.2.3 Fundamental results on completely bounded maps

We record below the basic theorems about completely bounded maps on operator algebras.

Theorem 1.2.3 (Arveson's extension theorem [1]). *Let S be an operator system contained in a unital C^*-algebra $\mathcal{A}$, and let $\psi : S \to \mathbb{B}(\mathcal{H})$ be a completely positive map. Then there exists a completely positive map $\phi : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ such that $\phi|_S = \psi$.*

An operator system J is said to be *injective* (in the category of operator systems) provided that, for any pair of operator systems $S \subseteq \tilde{S}$, any completely positive map $\phi : S \to J$ can be extended to a completely positive map $\tilde{\phi} : \tilde{S} \to J$. Arveson's extension theorem amounts to saying that $\mathbb{B}(\mathcal{H})$ is injective.

Definition 1.2.5. Let $\mathcal{A}$ be a unital subalgebra of a unital C^*-algebra $\mathcal{B}$. A ***$\mathcal{B}$-dilation*** of a unital representation $\rho : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ is a pair $(\pi, \mathcal{K})$, where $\mathcal{K}$ is a Hilbert space containing $\mathcal{H}$ as a subspace and $\pi : \mathcal{B} \to \mathbb{B}(\mathcal{K})$ is a unital $*$-representation such that

$$\rho(a) \;=\; P_{\mathcal{H}}\pi(a)|_{\mathcal{H}} \qquad (a \in \mathcal{A}).$$

Theorem 1.2.4 (Arveson's dilation theorem [1]). *Let $\mathcal{A}$ be a unital subalgebra of a C^*-algebra $\mathcal{B}$. Then a contractive representation $\rho : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ has a $\mathcal{B}$-dilation if and only if ρ is completely contractive.*

Theorem 1.2.5 (Wittstock's extension theorem [29]). *Let X be an operator subspace in a unital C^*-algebra $\mathcal{A}$, and let $\psi : X \to \mathbb{B}(\mathcal{H})$ be a completely bounded map. Then there exists a completely bounded map $\phi : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ such that $\phi|_X = \psi$ and $\|\phi\|_{\mathrm{cb}} = \|\psi\|_{\mathrm{cb}}$.*

Theorem 1.2.6 (Wittstock). *Let $\mathcal{A}$ be a unital C^*-algebra, and let $\phi : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ be a completely bounded map. Then there exist a Hilbert space $\mathcal{K}$, a $*$-representation $\pi : \mathcal{A} \to \mathbb{B}(\mathcal{K})$, and bounded operators $V, W : \mathcal{H} \to \mathcal{K}$ such that*

$$\phi(a) \; = \; V^*\pi(a)W \qquad (a \in \mathcal{A})$$

and $\|\phi\|_{\mathrm{cb}} = \|V\| \, \|W\|$. Moreover, if $\|\phi\|_{\mathrm{cb}} = 1$, then V and W can be chosen to be isometries.

1.2.4 Minimal tensor products of operator spaces

We now define the minimal tensor product of operator spaces. Recall that if $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, then for any $S \in \mathbb{B}(\mathcal{H})$ and $T \in \mathbb{B}(\mathcal{K})$, there is a unique bounded operator $S \otimes T$ on the Hilbert space tensor product $\mathcal{H} \otimes \mathcal{K}$ such that $(S \otimes T)(h \otimes k) = Sh \otimes Tk$ for all $h \in \mathcal{H}$ and $k \in \mathcal{K}$, and $\|S \otimes T\| = \|S\| \, \|T\|$. This induces an injective $*$-homomorphism from the algebraic tensor product $\mathbb{B}(\mathcal{H}) \otimes \mathbb{B}(\mathcal{K})$ into $\mathbb{B}(\mathcal{H} \otimes \mathcal{K})$; we shall identify $\mathbb{B}(\mathcal{H}) \otimes \mathbb{B}(\mathcal{K})$ as a normed unital $*$-subalgebra of $\mathbb{B}(\mathcal{H} \otimes \mathcal{K})$.

Let X and Y be operator spaces, and let $i : X \to \mathbb{B}(\mathcal{H})$ and $j : Y \to \mathbb{B}(\mathcal{K})$ be complete isometries. Then the ***minimal tensor product*** or ***spatial tensor product*** $X \otimes_{\min} Y$ is the operator space structure imposed on the tensor product $X \otimes Y$ via pullback by the embedding $i \otimes j : X \otimes Y \to \mathbb{B}(\mathcal{H}) \otimes \mathbb{B}(\mathcal{K})$; that is, for any $n \geq 1$ and $x \in M_n(X \otimes Y)$, we define

$$\|x\|_{\min} \; := \; \big\|(i \otimes j)_{(n)}(x)\big\|.$$

It turns out that this o.s.s. does not depend on the choice of i and j. In fact,

$$\|x\|_{\min} \;=\; \sup\left\{ \left\|(\phi\otimes\theta)_{(n)}(x)\right\| \;:\; \begin{array}{l} \phi : X \to \mathbb{B}(\mathcal{H}') \text{ and } \theta : Y \to \mathbb{B}(\mathcal{K}') \text{ are c.c.,} \\ \mathcal{H}', \mathcal{K}' \text{ are arbitrary Hilbert spaces} \end{array} \right\}.$$

This can be seen as a consequence of Wittstock's Theorems 1.2.5 and 1.2.6. From the definition of the minimal tensor product, we see that if $\phi : X \to X_1$ and $\theta : Y \to Y_1$ are completely bounded maps, then the rule $x \otimes y \mapsto \phi(x) \otimes \theta(y)$ determines a completely bounded map $\phi \otimes_{\min} \theta : X \otimes_{\min} Y \to X_1 \otimes_{\min} Y_1$, and $\|\phi \otimes_{\min} \theta\|_{\mathrm{cb}} = \|\phi\|_{\mathrm{cb}} \|\theta\|_{\mathrm{cb}}$. Moreover, the minimal tensor product is ***injective***, in that if both ϕ and θ are complete isometries, then so is $\phi \otimes_{\min} \theta$.

If X and Y are operator spaces which are not both C^*-algebras, we denote the completion of $X \otimes_{\min} Y$ by $X \bar{\otimes}_{\min} Y$.

If $\mathcal{A} \subset \mathbb{B}(\mathcal{H})$ and $\mathcal{B} \subset \mathbb{B}(\mathcal{K})$ are C^*-algebras, their complete minimal tensor product $\mathcal{A}\bar{\otimes}_{\min}\mathcal{B}$ is a C^*-algebra. Note that for any $n \geq 1$, the canonical $*$-isomorphism

$$M_n \otimes_{\min} \mathbb{B}(\mathcal{K}) \;=\; \mathbb{B}(\mathbf{C}^n) \otimes_{\min} \mathbb{B}(\mathcal{K}) \;=\; \mathbb{B}(\mathbf{C}^n \otimes \mathcal{K}) \;=\; \mathbb{B}(\mathcal{K}^n) \;\cong\; M_n\big(\mathbb{B}(\mathcal{K})\big)$$

restricts to the $*$-isomorphism of C^*-algebras

$$M_n \otimes_{\min} \mathcal{B} \;\cong\; M_n(\mathcal{B}).$$

More generally, if Y is an operator space contained in $\mathbb{B}(\mathcal{K})$, we consider $M_n(Y)$ as the operator space by the identification

$$M_n(Y) \;\cong\; M_n \otimes_{\min} Y.$$

We list some basic properties of the minimal tensor product that will be used in the sequel. If X, Y and Z are operator spaces and $m, n \geq 1$, then there are canonical completely isometric isomorphisms

$$\begin{aligned} (X \otimes_{\min} Y) \otimes_{\min} Z &\cong X \otimes_{\min} (Y \otimes_{\min} Z), \\ X \otimes_{\min} Y &\cong Y \otimes_{\min} X, \\ M_m(X) \otimes_{\min} M_n(Y) &\cong M_{mn}(X \otimes_{\min} Y). \end{aligned}$$

For any $m \geq 1$ and $y \in M_m(Y)$ such that $\|y\|_m = 1$, the map $i_y : X \to X \otimes_{\min} M_m(Y)$ defined by

$$i_y(x) \;=\; x \otimes y \qquad (x \in X)$$

is a complete isometry.

1.2.5 Definition of operator algebras

For our purposes, all algebras will be assumed to be unital. The treatment of non-unital operator algebras is fully discussed in [2].

Definition 1.2.6. A ***concrete operator algebra*** is a unital subalgebra $\mathcal{A}$ of $\mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, together with the operator space structure of $\mathcal{A}$ as a subspace of $\mathbb{B}(\mathcal{H})$.

Note that our definition does not assume that $\mathcal{A}$ is closed in $\mathbb{B}(\mathcal{H})$. Let $\mathcal{A}$ be an operator algebra acting on a Hilbert space $\mathcal{H}$. Then for any $n \geq 1$, $M_n(\mathcal{A})$ is an operator algebra acting on $\mathcal{H}^n$. Since $\|\cdot\|_n$ is the operator norm, for any $a, b \in M_n(\mathcal{A})$, we must have

$$\|ab\|_n \;\leq\; \|a\|_n \|b\|_n \,.$$

It turns out that this extra condition determines the class of concrete operator algebras up to complete isometry, according to the following theorem.

Theorem 1.2.7 (Blecher-Ruan-Sinclair (BRS))**.** *Suppose that $\mathcal{A}$ is a unital algebra as well as an operator space. The following are equivalent:*

(a) For every $n \geq 1$ and $a, b \in M_n(\mathcal{A})$, $\|ab\|_n \leq \|a\|_n \|b\|_n$.

(b) There is a unital, completely isometric homomorphism $\pi : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

1.2.6 Examples of operator algebras

In this section, we present the main examples of operator algebras which will be referred to frequently.

Example 1.2.2 (Nest algebras)**.** A *nest algebra* acting on a Hilbert space $\mathcal{H}$ is a subalgebra of $\mathbb{B}(\mathcal{H})$ of the form

$$\mathcal{T}(\mathscr{N}) \ := \ \mathsf{Alg}\,\mathscr{N} \ = \ \{T \in \mathbb{B}(\mathcal{H}) \ : \ T\mathcal{N} \subseteq \mathcal{N} \text{ for every } \mathcal{N} \in \mathscr{N}\},$$

where $\mathscr{N}$ is a chain of subspaces of $\mathcal{H}$ closed under arbitrary intersections, closed spans, and containing $\{0\}$ and $\mathcal{H}$. This is a unital operator algebra that is closed in the weak operator topology. In particular, if $\mathcal{H} = \mathbf{C}^n$ with standard basis $e_1, \ldots, e_n$ and $\mathscr{N}$ consists of the finite chain $\text{span}\{e_1, \ldots, e_k\}$ $(0 \le k \le n)$, the resulting nest algebra is denoted by $\mathcal{T}(n)$, the algebra of all upper triangular $n \times n$ matrices.

Example 1.2.3 (Group C^*-algebras)**.** Let G be a discrete group. For each unitary representation $(\pi, \mathcal{H})$ of G on a Hilbert space $\mathcal{H}$, the C^*-subalgebra $C^*_\pi(G)$ of $\mathbb{B}(\mathcal{H})$ is defined by

$$C^*_\pi(G) \ := \ \overline{\text{span}}\{\pi(s) \ : \ s \in G\}.$$

Of course, if two unitary representations $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ are unitarily equivalent, in the sense that there is a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $U\pi_1(s) = \pi_2(s)U$ for all $s \in G$, then the assignment $T \mapsto UTU^*$ is a $*$-isomorphism from $C^*_{\pi_1}(G)$ onto $C^*_{\pi_2}(G)$.

The *left regular representation* of G is the unitary representation λ of G on $\ell_2(G)$ given by left translation:

$$\lambda_s f(t) \ = \ f(s^{-1}t),$$

where $f \in \ell_2(G)$ and $s, t \in G$. Likewise, the *right regular representation* of G is the unitary representation ρ of G on $\ell_2(G)$ given by right translation:

$$\rho_s f(t) \ = \ f(ts).$$

Thus G acts on the canonical basis via the permutations

$$\lambda_s e_t \ = \ e_{st}, \qquad \rho_s e_t \ = \ e_{ts^{-1}}.$$

Note that the inversion $U : \ell_2(G) \to \ell_2(G)$ defined by $Uf(t) = f(t^{-1})$ determines a unitary equivalence between λ and ρ. The *reduced group C^*-algebra $C_r^*(G)$* is defined to be the C^*-algebra

$$C_r^*(G) \ := \ C_\lambda^*(G) \ \cong \ C_\rho^*(G).$$

The *canonical trace* on $C_r^*(G)$ is given by

$$\tau(x) \ = \ \langle \lambda(x)e_1, e_1 \rangle \qquad (x \in C_r^*(G)).$$

In fact, the GNS representation for τ is just the left regular representation on $\ell_2(G)$ with cyclic vector e_1.

We now turn to the full group C^*-algebra. Let $\mathbf{C}G$ denote the group algebra, with canonical basis $\{\delta_s : s \in G\}$. The $*$-algebra structure on $\mathbf{C}G$ is determined by $\delta_s\delta_t = \delta_{st}$ and $(\delta_s)^* = \delta_{s^{-1}}$. This ensures that every unitary representation $(\pi, \mathcal{H})$ of G extends by linearity to a unital $*$-representation $\tilde{\pi} : \mathbf{C}G \to \mathbb{B}(\mathcal{H})$; conversely, any unital $*$-representation of $\mathbf{C}G$ on a Hilbert space restricts to a unitary representation of G. Consequently, such a π induces a C^*-seminorm $\|x\|_\pi := \|\tilde{\pi}(x)\|$ on $\mathbf{C}G$, and the completion of $\mathbf{C}G$ with respect to this seminorm is a C^*-algebra isomorphic with the concrete algebra $C_\pi^*(G)$. Since the basis elements δ_s are unitary, any C^*-seminorm on $\mathbf{C}G$ is dominated by the ℓ_1-norm $\|\sum_{s\in G} x_s\delta_s\|_1 := \sum_{s\in G} |x_s|$. As a result, the supremum of an arbitrary family of C^*-seminorms on $\mathbf{C}G$ remains a well-defined C^*-seminorm. In particular, we obtain the maximal C^*-norm

$$\|x\|_* \ := \ \sup_\pi \|\tilde{\pi}(x)\|,$$

where the supremum is taken over all C^*-seminorms induced by unitary representations π. (Note that $\|\cdot\|_*$ is a norm, since $\|x\|_\lambda \leq \|x\|_* \leq \|x\|_1$ and $\|\cdot\|_\lambda$ is a norm.) It can be shown that

$$\|x\|_* \ = \ \sup_{\pi \in \widehat{G}} \|\tilde{\pi}(x)\|,$$

where $\widehat{G}$ denotes the set of unitary equivalence classes of irreducible unitary representations of

G. The completion of $\mathbf{C}G$ with respect to the maximal C^*-norm is called the *full group C^*-algebra* and is denoted by $C^*(G)$. We denote the canonical image of δ_s ($s \in G$) in the unitary group of $C^*(G)$ by u_s, and call $u : G \to C^*(G)$ the *universal representation* of G. In general, there is a canonical surjective $*$-homomorphism $C^*(G) \to C^*_r(G)$ sending u_s to λ_s. This homomorphism becomes an isomorphism (i.e. $C^*(G) = C^*_r(G)$) precisely when G is amenable; see Section 1.2.7 for more on amenability.

Example 1.2.4. If G is a discrete abelian group, its dual group $\widehat{G}$ is an abelian, compact topological group under the topology of pointwise convergence. Also, $\widehat{G}$ is homeomorphic to the spectrum of the full group C^*-algebra $C^*(G)$. Hence the Gelfand transform provides a $*$-isomorphism $C^*(G) \cong C(\widehat{G})$. This isomorphism sends each $u_s \in C^*(G)$ (where $s \in G$) to the character $\check{s} \in C(\widehat{G})$ of $\widehat{G}$, where $\check{s}(\gamma) = \gamma(s)$ for $\gamma \in \widehat{G}$. Let μ be the normalized Haar measure on $\widehat{G}$, and let $L^2(\widehat{G}) = L^2(\widehat{G}, \mu)$. It is well-known that $\{\check{s} : s \in G\}$ is an orthonormal basis for $L^2(\widehat{G})$, and that the Fourier transform

$$f \;=\; \sum_{s \in G} f(s) e_s \quad \mapsto \quad \mathscr{F}(f) \;=\; \sum_{s \in G} f(s) \check{s}$$

extends to a unitary operator $\mathscr{F} : \ell_2(G) \to L^2(\widehat{G})$. Let $M : C(\widehat{G}) \to \mathbb{B}\big(L^2(\widehat{G})\big)$ denote the multiplication representation, which is an injective $*$-representation. Since $\mathscr{F} \lambda_s \mathscr{F}^{-1} = M_{\check{s}}$ for all $s \in G$, it follows that unitary equivalence by $\mathscr{F}$ induces a $*$-isomorphism $C^*_r(G) \cong M\big[C(\widehat{G})\big] \cong C(\widehat{G})$. Thus, we have the $*$-isomorphisms

$$C^*(G) \;\cong\; C(\widehat{G}) \;\cong\; C^*_r(G).$$

Example 1.2.5 (Disk and polydisk algebras)**.** Let $\mathbb{T} = \{z \in \mathbf{C} : |z| = 1\}$ be the torus. It is well-known that $\widehat{\mathbb{Z}} \cong \mathbb{T}$ as topological groups. More generally, the dual group of $\mathbb{Z}^n$ is $\widehat{\mathbb{Z}^n} \cong \mathbb{T}^n$ for each $n \geq 1$. It follows that $C(\mathbb{T}^n) \cong C^*(\mathbb{Z}^n)$ as C^*-algebras.

Let $\mathbb{D} = \{z \in \mathbf{C} : |z| < 1\}$ be the open unit disk, with closure $\overline{\mathbb{D}}$, the closed unit disk in $\mathbf{C}$. For each $n \geq 1$, the open n-polydisk is the subset $\mathbb{D}^n$ of $\mathbf{C}^n$. The *polydisk algebra* $A(\mathbb{D}^n)$ is the algebra of all continuous functions on $\overline{\mathbb{D}}^n$ which are analytic on $\mathbb{D}^n$. In particular, $A(\mathbb{D})$ is called the *disk algebra*. For each $1 \leq j \leq n$, let $z_j \in C(\mathbb{T}^n)$ denote the j-th coordinate function. For each

$\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$, let $z^{\mathbf{k}} = z_1^{k_1} \ldots z_n^{k_n}$. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the set of non-negative integers. Then the unital subalgebra $\mathcal{P}(\mathbb{D}^n) = \mathrm{span}\{z^{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^n\} \cong \mathbf{C}[z_1, \ldots, z_n]$ is dense in $A(\mathbb{D}^n)$; it is completely isometrically isomorphic to the subalgebra $\mathsf{alg}(\mathbb{N}^n) = \mathrm{span}\{u_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^n\}$ of $C^*(\mathbb{Z}^n)$.

Example 1.2.6. *Sz.-Nagy's dilation theorem* states that any contractive operator T on a Hilbert space $\mathcal{H}$ admits a unitary dilation. Specifically, there is an explicit construction of a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a subspace as well as a unitary operator U on $\mathcal{K}$ such that

$$T^n \;=\; P_{\mathcal{H}} U^n|_{\mathcal{H}} \qquad (n \in \mathbb{N}).$$

Let $\rho_T : \mathcal{P}(\mathbb{D}) \to \mathbb{B}(\mathcal{H})$ be the unital representation of the polynomial algebra $\mathcal{P}(\mathbb{D}) \cong \mathbf{C}[z]$ on $\mathcal{H}$ such that $\rho_T(z) = T$, and let $\pi : C(\mathbb{T}) \to \mathbb{B}(\mathcal{K})$ be the unital $*$-representation of $C(\mathbb{T})$ on $\mathcal{K}$ such that $\pi(z) = U$. Since $\rho_T(p) = P_{\mathcal{H}} \pi_U(p)|_{\mathcal{H}}$ for all $p \in \mathcal{P}(\mathbb{D})$, it follows that $\|\rho_T\| = 1$ and ρ_T extends to a completely contractive homomorphism on $A(\mathbb{D})$. Conversely, if $\rho : A(\mathbb{D}) \to \mathbb{B}(\mathcal{H})$ is a unital contractive representation, then $\rho = \rho_T$, where $T = \rho(z)$ is a contraction on $\mathcal{H}$. We conclude that $T \leftrightarrow \rho_T$ is a bijective correspondence between the contractions on $\mathcal{H}$ and the set of unital, completely contractive representations of $A(\mathbb{D})$ on $\mathcal{H}$. In particular, all unital contractive representations of $A(\mathbb{D})$ are completely contractive.

Using Arveson's dilation theorem, we see that Sz.-Nagy's theorem is equivalent to the assertion that every unital contractive representation of $A(\mathbb{D})$ is completely contractive.

Example 1.2.7. *Ando's theorem* states that any pair of commuting contractions T_1, T_2 on a Hilbert space $\mathcal{H}$ admit a joint unitary dilation. This means that there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a subspace, together with a pair of commuting unitary operators U_1, U_2 on $\mathcal{K}$ such that

$$T_1^m T_2^n \;=\; P_{\mathcal{H}} U_1^m U_2^n|_{\mathcal{H}} \qquad (m, n \in \mathbb{N}.)$$

Let $\rho = \rho_{T_1, T_2} : \mathcal{P}(\mathbb{D}^2) \to \mathbb{B}(\mathcal{H})$ be the unital representation of $\mathcal{P}(\mathbb{D}^2) \cong \mathbf{C}[z_1, z_2]$ such that $\rho(z_j) = T_j$ for $j = 1, 2$. It follows from Ando's theorem that ρ admits a dilation to the $*$-representation $\pi : C(\mathbb{T}^2) \to \mathbb{B}(\mathcal{K})$ such that $\pi(z_j) = U_j$ for $j = 1, 2$, and therefore $\|\rho\|_{\mathrm{cb}} = 1$.

We conclude that there is a bijective correspondence between commuting pairs of contractions (T_1, T_2) on $\mathcal{H}$ and unital, completely contractive representations ρ_{T_1,T_2} of the bidisk algebra $A(\mathbb{D}^2)$ on $\mathcal{H}$. Once again, Arveson's dilation theorem implies that Ando's theorem is equivalent to the assertion that every unital, contractive representation of $A(\mathbb{D}^2)$ is completely contractive.

1.2.7 Tensor products of operator algebras

The maximal and minimal tensor products of operator algebras are defined is a way that is analogous with their C^*-counterparts. While the minimal tensor product is essentially the familiar spatial tensor product together with the inherited operator space structure, the maximal tensor product is related to joint completely contractive representations.

If $\mathcal{A}$ and $\mathcal{B}$ are unital algebras with respective identities $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$, we denote their algebraic tensor product by $\mathcal{A} \otimes \mathcal{B}$. It is a unital algebra whose multiplication is determined by

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) \;=\; (a_1 a_2 \otimes b_1 b_2)$$

for all $a_1, a_2 \in \mathcal{A}$ and $b_1, b_2 \in \mathcal{B}$. There are canonical injective homomorphisms $\iota_{\mathcal{A}} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}$ and $\iota_{\mathcal{B}} : \mathcal{B} \to \mathcal{A} \otimes \mathcal{B}$, given by $\iota_{\mathcal{A}}(a) = a \otimes 1_{\mathcal{B}}$ and $\iota_{\mathcal{B}}(b) = 1_{\mathcal{A}} \otimes b$ respectively. If $\rho : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ and $\sigma : \mathcal{B} \to \mathbb{B}(\mathcal{H})$ are unital representations of $\mathcal{A}$ and $\mathcal{B}$ on the same Hilbert space $\mathcal{H}$, we say that ρ and σ *have commuting ranges* (or simply that ρ and σ *commute*) whenever

$$\rho(a)\sigma(b) \;=\; \sigma(b)\rho(a)$$

for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. In this case, the linear map

$$\rho \odot \sigma \;:\; \mathcal{A} \otimes \mathcal{B} \to \mathbb{B}(\mathcal{H})$$

determined by

$$\rho \odot \sigma(a \otimes b) \;:=\; \rho(a)\sigma(b)$$

is a unital representation of $\mathcal{A} \otimes \mathcal{B}$. Conversely, any unital representation

$$\pi \;:\; \mathcal{A} \otimes \mathcal{B} \to \mathbb{B}(\mathcal{H})$$

is of the form $\pi = \pi_{\mathcal{A}} \odot \pi_{\mathcal{B}}$ for a unique pair $(\pi_{\mathcal{A}}, \pi_{\mathcal{B}})$ of commuting representations of $\mathcal{A}$ and $\mathcal{B}$ on $\mathcal{H}$; indeed, $\pi_{\mathcal{A}} = \pi \circ \iota_{\mathcal{A}}$ and $\pi_{\mathcal{B}} = \pi \circ \iota_{\mathcal{B}}$. Thus, there is a bijective correspondence $(\rho, \sigma) \leftrightarrow \rho \odot \sigma$ between commuting pairs of unital representations of $\mathcal{A}$ and $\mathcal{B}$ and unital representations of $\mathcal{A} \otimes \mathcal{B}$.

The maximal tensor product $\mathcal{A} \otimes_{\max} \mathcal{B}$ of unital operator algebras $\mathcal{A}$ and $\mathcal{B}$ is the algebra $\mathcal{A} \otimes \mathcal{B}$ endowed with the unique o.s.s. which ensures that the correspondence $(\rho, \sigma) \leftrightarrow \rho \odot \sigma$ converts pairs of completely contractive representations on $\mathcal{A}$ and $\mathcal{B}$ to completely contractive representations of $\mathcal{A} \otimes_{\max} \mathcal{B}$. Note that if the representations ρ and σ are contractive, then for any $x = \sum_{i=1}^{k} a_i \otimes b_i \in \mathcal{A} \otimes \mathcal{B}$,

$$\|\rho \odot \sigma(x)\| \;\leq\; \sum_{i=1}^{k} \|\rho(a_i)\| \, \|\sigma(b_i)\| \;\leq\; \sum_{i=1}^{k} \|a_i\| \, \|b_i\| \, .$$

Hence, $\|\rho \odot \sigma(x)\|$ is bounded by $\|x\|_{\gamma}$, the projective tensor norm of x. Now, let $\mathscr{R} = \mathscr{R}(\mathcal{A}, \mathcal{B})$ denote the class of pairs (ρ, σ) of unital, completely contractive (u.c.c.) representations $\rho : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ and $\sigma : \mathcal{B} \to \mathbb{B}(\mathcal{H})$ on the same Hilbert space $\mathcal{H}$ having commuting ranges. Then for any $\left[x_{i,j}\right]_{(i,j)} \in M_n(\mathcal{A} \otimes \mathcal{B})$, we have the crude estimate

$$\sup_{(\rho,\sigma)\in\mathscr{R}} \left\| \left[\rho \odot \sigma(x_{i,j})\right]_{(i,j)} \right\| \;\leq\; \sum_{i,j} \|x_{i,j}\|_{\gamma} \;<\; \infty.$$

Definition 1.2.7. Let $\mathcal{A}$ and $\mathcal{B}$ be unital operator algebras. The ***maximal tensor product*** $\mathcal{A} \otimes_{\max} \mathcal{B}$ is the algebra $\mathcal{A} \otimes \mathcal{B}$ whose operator space structure is defined by the formula

$$\|x\|_{\max} \;:=\; \sup_{(\rho,\sigma)\in\mathscr{R}} \left\| (\rho \odot \sigma)_{(n)}(x) \right\|$$

for any $n \geq 1$ and $x \in M_n(\mathcal{A} \otimes \mathcal{B})$. We denote the completion of $\mathcal{A} \otimes_{\max} \mathcal{B}$ by $\mathcal{A} \bar{\otimes}_{\max} \mathcal{B}$.

One may apply the BRS characterization to confirm that the above defines an operator algebra. Alternatively, we can realize $\mathcal{A} \otimes_{\max} \mathcal{B}$ as a concrete operator algebra by considering the direct sum $\bigoplus_{(\rho,\sigma)\in\mathscr{S}} (\rho \odot \sigma)(\mathcal{A} \otimes \mathcal{B})$, where $\mathscr{S}$ is a sufficiently large subset of $\mathscr{R}$. Note that the canonical embeddings $\iota_{\mathcal{A}} : \mathcal{A} \to \mathcal{A} \otimes_{\max} \mathcal{B}$ and $\iota_{\mathcal{B}} : \mathcal{B} \to \mathcal{A} \otimes_{\max} \mathcal{B}$ are complete isometries. For if $n \geq 1$ and $a \in M_n(\mathcal{A})$, then

$$\left\|(\iota_{\mathcal{A}})_{(n)}(a)\right\|_{\max} = \sup_{(\rho,\sigma)\in\mathscr{R}} \left\|\left[\rho(a_{i,j})\sigma(1_{\mathcal{B}})\right]_{(i,j)}\right\| = \sup_{(\rho,\sigma)\in\mathscr{R}} \left\|\rho_{(n)}(a)\right\| = \|a\|_n .$$

If $\phi_j : \mathcal{A}_j \to \mathcal{B}_j$ ($j = 1, 2$) are unital, completely contractive homomorphisms of operator algebras, then the linear map $\phi_1 \otimes \phi_2 : \mathcal{A}_1 \otimes_{\max} \mathcal{A}_2 \to \mathcal{B}_1 \otimes_{\max} \mathcal{B}_2$ is a completely contractive homomorphism. Indeed, if $n \geq 1$ and $x \in M_n(\mathcal{A}_1 \otimes \mathcal{A}_2)$, then

$$\begin{aligned}
\left\|(\phi_1 \otimes \phi_2)_{(n)}(x)\right\|_{\max} &= \sup_{(\sigma_1,\sigma_2)\in\mathscr{R}(\mathcal{B}_1,\mathcal{B}_2)} \left\|(\sigma_1 \odot \sigma_2)_{(n)} \circ (\phi_1 \otimes \phi_2)_{(n)}(x)\right\| \\
&= \sup_{(\sigma_1,\sigma_2)\in\mathscr{R}(\mathcal{B}_1,\mathcal{B}_2)} \left\|\left((\sigma_1 \circ \phi_1) \odot (\sigma_2 \circ \phi_2)\right)_{(n)}(x)\right\| \\
&\leq \sup_{(\rho_1,\rho_2)\in\mathscr{R}(\mathcal{A}_1,\mathcal{A}_2)} \left\|(\rho_1 \odot \rho_2)_{(n)}(x)\right\| \\
&= \|x\|_{\max} .
\end{aligned}$$

Finally, note that if $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are unital operator algebras, then there are completely isometric isomorphisms

$$\begin{aligned}
(\mathcal{A} \otimes_{\max} \mathcal{B}) \otimes_{\max} \mathcal{C} &\cong \mathcal{A} \otimes_{\max} (\mathcal{B} \otimes_{\max} \mathcal{C}), \\
\mathcal{A} \otimes_{\max} \mathcal{B} &\cong \mathcal{B} \otimes_{\max} \mathcal{A}.
\end{aligned}$$

The ***minimal tensor product*** of unital operator algebras $\mathcal{A}$ and $\mathcal{B}$ is defined as $\mathcal{A} \otimes_{\min} \mathcal{B}$, which we recall is the minimal tensor product of the operator spaces $\mathcal{A}$ and $\mathcal{B}$. Since the o.s.s. on $\mathcal{A} \otimes_{\min} \mathcal{B}$ depends only on the o.s.s. of $\mathcal{A}$ and $\mathcal{B}$ up to complete isometry, we choose unital, completely isometric representations $i : \mathcal{A} \hookrightarrow \mathbb{B}(\mathcal{H})$ and $j : \mathcal{B} \hookrightarrow \mathbb{B}(\mathcal{K})$. Then the map $i \otimes j : \mathcal{A} \otimes_{\min} \mathcal{B} \hookrightarrow \mathbb{B}(\mathcal{H} \otimes \mathcal{K})$ is a unital, completely isometric representation. Thus, $\mathcal{A} \otimes_{\min} \mathcal{B}$ is naturally an operator algebra. Furthermore, the canonical homomorphisms $\iota_{\mathcal{A}} : \mathcal{A} \to \mathcal{A} \otimes_{\min} \mathcal{B}$ and $\iota_{\mathcal{B}} : \mathcal{B} \to \mathcal{A} \otimes_{\min} \mathcal{B}$ are complete isometries. Since $\iota_{\mathcal{A}}$ and $\iota_{\mathcal{B}}$ have commuting ranges, it follows that the identity map $\mathrm{id} = \iota_{\mathcal{A}} \odot \iota_{\mathcal{B}} : \mathcal{A} \otimes_{\max} \mathcal{B} \to \mathcal{A} \otimes_{\min} \mathcal{B}$ is completely contractive.

The two tensor norms are *cross-norms*, in the sense that for any $m, n \geq 1$, $a \in M_m(\mathcal{A})$ and $b \in M_n(\mathcal{B})$,

$$\|a \otimes b\|_{\max} \;=\; \|a \otimes b\|_{\min} \;=\; \|a\|_m \, \|b\|_n \,.$$

(Here, $a \otimes b \in M_m(\mathcal{A}) \otimes M_n(\mathcal{B})$ is identified with an element of $M_{mn}(\mathcal{A} \otimes \mathcal{B})$ by the canonical algebra isomorphism $M_m(\mathcal{A}) \otimes M_n(\mathcal{B}) \cong M_{mn}(\mathcal{A} \otimes \mathcal{B})$.) To see this, first note that

$$\|a \otimes b\|_{\max} \;\leq\; \left\|a \otimes 1_{M_n(\mathcal{B})}\right\|_{\max} \cdot \left\|1_{M_m(\mathcal{A})} \otimes b\right\|_{\max}$$

where

$$\left\|a \otimes 1_{M_n(\mathcal{B})}\right\|_{\max} \;=\; \sup_{(\rho,\sigma) \in \mathscr{R}} \left\|\rho_{(m)}(a) \otimes I\right\| \;=\; \sup_{(\rho,\sigma) \in \mathscr{R}} \left\|\rho_{(m)}(a)\right\| \;=\; \|a\|_m \,,$$

and similarly $\left\|1_{M_n(\mathcal{A})} \otimes b\right\|_{\max} = \|b\|_n$. On the other hand, since $\|\cdot\|_{\min}$ coincides with the spatial tensor norm, we obtain

$$\|a \otimes b\|_{\min} \;=\; \|a\|_m \, \|b\|_n \,.$$

The proof is completed by invoking the fact that $\|\cdot\|_{\max} \geq \|\cdot\|_{\min}$.

Note that completely contractive representations of C^*-algebras are the same as $*$-representations. Hence, when both $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras, $\mathcal{A} \bar{\otimes}_{\max} \mathcal{B}$ coincides with the C^*-algebraic maximal tensor product of $\mathcal{A}$ and $\mathcal{B}$. It is elementary that any C^*-seminorm on the algebraic tensor product $\mathcal{A} \otimes \mathcal{B}$ is dominated by the maximal tensor norm $\|\cdot\|_{\max}$. *Takesaki's theorem*, which states that the minimal tensor norm $\|\cdot\|_{\min}$ is the least C^*-norm on $\mathcal{A} \otimes \mathcal{B}$, is much deeper. Proofs can be found in [5] and [28] .

Example 1.2.8. We shall take for granted the fact that if X and Y are compact Hausdorff spaces, then

$$C(X) \bar{\otimes}_{\min} C(Y) \;\cong\; C(X \times Y),$$

where an elementary tensor $f \otimes g$ is identified with the function $(x, y) \mapsto f(x)g(y)$ on $X \times Y$. Now let $X = Y = \mathbb{T}$, and consider the subalgebras $A(\mathbb{D}) = \overline{\operatorname{span}}\{z^n : n \in \mathbb{N}\}$ of $C(\mathbb{T})$ and $A(\mathbb{D}^2) = \overline{\operatorname{span}}\{z_1^{n_1} z_2^{n_2} : n_1, n_2 \in \mathbb{N}\}$ of $C(\mathbb{T}^2)$. By the injectivity of $\bar{\otimes}_{\min}$, the $*$-isomorphism $C(T) \bar{\otimes}_{\min} C(\mathbb{T}) \cong C(\mathbb{T}^2)$ restricts to the completely isometric isomorphism

$$A(\mathbb{D}) \bar{\otimes}_{\min} A(\mathbb{D}) \;\cong\; A(\mathbb{D}^2).$$

More generally, for any $n \geq 1$, there is a completely isometric isomorphism of operator algebras $A(\mathbb{D}^n) \cong A(\mathbb{D})\bar{\otimes}_{\min} \ldots \bar{\otimes}_{\min} A(\mathbb{D})$ (n times).

Example 1.2.9. A well-known construction of S. Parrott [20] consists of three commuting contractions T_1, T_2, T_3 on a Hilbert space $\mathcal{H}$ which induce a contractive representation ρ of $A(\mathbb{D}^3)$ that is not completely contractive. We may write $\rho = \rho_1 \odot \rho_2$, where ρ_1 is the representation of $A(\mathbb{D}^2)$ determined by T_1 and T_2, and ρ_2 is the representation of $A(\mathbb{D})$ determined by T_3. Since both ρ_1 and ρ_2 are completely contractive due to the theorems of Ando and of Sz.-Nagy, whereas $\rho = \rho_1 \odot \rho_2$ is not so, we conclude that

$$A(\mathbb{D}^2) \otimes_{\min} A(\mathbb{D}) \neq A(\mathbb{D}^2) \otimes_{\max} A(\mathbb{D}).$$

Example 1.2.10. Let G_1 and G_2 be discrete groups. Then the canonical unitary isomorphism $\ell_2(G_1) \otimes \ell_2(G_2) \to \ell_2(G_1 \times G_2)$ intertwines the spatial tensor product of representations

$$\lambda_{G_1} \otimes \lambda_{G_2} \; : \; C_r^*(G_1)\bar{\otimes}_{\min} C_r^*(G_2) \to \mathbb{B}(\ell_2(G_1) \otimes \ell_2(G_2))$$

and the left regular representation

$$\lambda_{G_1 \times G_2} \; : \; C_r^*(G_1 \times G_2) \to \mathbb{B}\big(\ell_2(G_1 \times G_2)\big),$$

thereby inducing the canonical $*$-isomorphism

$$C_r^*(G_1)\bar{\otimes}_{\min} C_r^*(G_2) \cong C_r^*(G_1 \times G_2).$$

On the other hand, since unitary representations $\pi : G_1 \times G_2 \to \mathscr{U}(\mathcal{H})$ are in bijective correspondence with pairs of unitary representations $\pi_1 : G_1 \to \mathscr{U}(\mathcal{H})$ and $\pi_2 : G_2 \to \mathscr{U}(\mathcal{H})$ having commuting ranges, which in turn correspond to pairs of commuting $*$-representations of $C^*(G_1)$ and $C^*(G_2)$ on $\mathcal{H}$, it follows that there is a canonical $*$-isomorphism

$$C^*(G_1)\bar{\otimes}_{\max} C^*(G_2) \cong C^*(G_1 \times G_2).$$

A C^*-algebra $\mathcal{A}$ is said to be *nuclear* provided that for any C^*-algebra $\mathcal{B}$, there is just one

C^*-norm on $\mathcal{A} \otimes \mathcal{B}$, or equivalently, $\mathcal{A} \otimes_{\max} \mathcal{B} = \mathcal{A} \otimes_{\min} \mathcal{B}$. It is known that abelian C^*-algebras are nuclear. Also, for each integer $n \geq 1$, M_n is nuclear, since for any C^*-algebra $\mathcal{A}$, the (minimal) C^*-norm on $M_n(\mathcal{A}) = M_n \otimes \mathcal{A}$ is already complete, so that $\|\cdot\|_{\max} = \|\cdot\|_{\min}$ on $M_n \otimes \mathcal{A}$. Since inductive limits of nuclear C^*-algebras are nuclear, the algebra $\mathfrak{K}$ of compact operators is nuclear. We refer to [5] for an extensive discussion of nuclearity and its numerous equivalent formulations. The following result follows quite easily from such results. It can also be proved via the theory of maximal tensor products of operator systems as laid out in [22, Chapter 12] and considering the operator system $\mathcal{B} + \mathcal{B}^\star$. We omit its proof.

Proposition 1.2.8 (Proposition 2.9, [23])**.** *If $\mathcal{A}$ is a nuclear C^*-algebra, then $\mathcal{A} \otimes_{\max} \mathcal{B} = \mathcal{A} \otimes_{\min} \mathcal{B}$ completely isometrically for any operator algebra $\mathcal{B}$.*

Let G be a discrete group. A *mean* on G is a state on the algebra $\ell_\infty(G)$. The left translation action of G on $\ell_\infty(G)$ (namely, $(s \cdot f)(t) = f(s^{-1}t)$ for $f \in \ell_\infty(G)$ and $s, t \in G$) induces via duality an action of G on the set of means. We say that G is *amenable* if a left invariant mean exists. It is well-known that the class of discrete amenable groups is closed under taking subgroups, quotients, extensions and inductive limits. A standard reference of amenable groups is Paterson's book [21]. We shall take for granted a few well-known characterizations of (discrete) amenable groups [5, Theorem 2.6.8]:

(a) $C^*(G) = C^*_r(G)$.

(b) $C^*_r(G)$ is nuclear.

(c) The trivial character on G extends to a $*$-homomorphism $\varepsilon : C^*_r(G) \to \mathbf{C}$.

In the sequel, we shall adopt the following convention. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are operator algebras, we shall write

$$\mathcal{A}_1 \otimes_{\min} \mathcal{A}_2 \; = \; \mathcal{A}_1 \otimes_{\max} \mathcal{A}_2$$

whenever the identity map $\mathrm{id} : \mathcal{A}_1 \otimes_{\max} \mathcal{A}_2 \to \mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$ is a complete isometry. This statement is equivalent to the claim that for any pair $(\rho_1, \rho_2) \in \mathscr{R}(\mathcal{A}_1, \mathcal{A}_2)$ of commuting u.c.c.

representations on a Hilbert space $\mathcal{H}$, their joint representation $\rho_1 \odot \rho_2$ is completely contractive on $\mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$. Similarly, we shall write

$$\mathcal{A}_1 \otimes_{\min} \mathcal{A}_2 \underset{1}{=} \mathcal{A}_1 \otimes_{\max} \mathcal{A}_2 \tag{1.1}$$

if the canonical map id : $\mathcal{A}_1 \otimes_{\max} \mathcal{A}_2 \to \mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$ is an isometry. This means that $\rho_1 \odot \rho_2$ is contractive on $\mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$ for any pair $(\rho_1, \rho_2) \in \mathscr{R}(\mathcal{A}_1, \mathcal{A}_2)$. As is known already in the case of C^*-algebras, this weaker condition does not always hold.

The following result is a simple consequence of Arveson's dilation theorem.

Proposition 1.2.9 (Proposition 2.5, [23])**.** *For $j = 1, 2$, let $\mathcal{A}_j$ be a unital subalgebra of a unital C^*-algebra $\mathcal{B}_j$. The following are equivalent:*

(i) $\mathcal{A}_1 \otimes_{\min} \mathcal{A}_2 = \mathcal{A}_1 \otimes_{\max} \mathcal{A}_2$*;*

(ii) for any pair (ρ_1, ρ_2) of unital, completely contractive representations $\rho_j : \mathcal{A}_j \to \mathbb{B}(\mathcal{H})$ having commuting ranges, there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a unital $*$-representation $\pi : \mathcal{B}_1 \otimes_{\min} \mathcal{B}_2 \to \mathbb{B}(\mathcal{K})$ such that

$$\rho_1(a_1)\rho_2(a_2) \;=\; P_{\mathcal{H}}\pi(a_1 \otimes a_2)|_{\mathcal{H}}$$

for all $a_1 \in \mathcal{A}_1$ and $a_2 \in \mathcal{A}_2$.

Example 1.2.11. Let $\mathcal{A}$ be a unital subalgebra of a unital C^*-algebra $\mathcal{B}$. Then $\mathcal{A} \otimes_{\min} A(\mathbb{D}) = \mathcal{A} \otimes_{\max} A(\mathbb{D})$ if and only if for any u.c.c. representation $\rho : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ and any contraction $T \in \mathbb{B}(\mathcal{H})$ such that $\rho(a)T = T\rho(a)$ for all $a \in \mathcal{A}$, there is a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$, a unital $*$-representation $\pi : \mathcal{B} \to \mathbb{B}(\mathcal{K})$, and a unitary operator U on $\mathcal{K}$ such that $U\pi(b) = \pi(b)U$ for all $b \in \mathcal{B}$ and

$$\rho(a)T^n \;=\; P_{\mathcal{H}}\pi(a)U^n|_{\mathcal{H}} \qquad (a \in \mathcal{A}, n \geq 0).$$

Example 1.2.12. Let $\mathcal{A}_1 = \mathcal{A}_2 = A(\mathbb{D})$. We show that Ando's theorem is equivalent to the equality $A(\mathbb{D}) \otimes_{\min} A(\mathbb{D}) = A(\mathbb{D}) \otimes_{\max} A(\mathbb{D})$. Suppose that ρ_1 and ρ_2 are commuting u.c.c. representations of $A(\mathbb{D})$ on a Hilbert space $\mathcal{H}$, and for each $j = 1, 2$, let $T_j = \rho_j(z)$. Then T_1 and T_2 are commuting contractions on $\mathcal{H}$. Ando's theorem implies that there is a $*$-representation $\pi : C(\mathbb{T}^2) \to \mathbb{B}(\mathcal{K})$ on a Hilbert space $\mathcal{K} \supset \mathcal{H}$ such that $\rho_1 \odot \rho_2(x) = P_{\mathcal{H}}\pi(x)|_{\mathcal{H}}$ for all $x \in A(\mathbb{D}) \otimes A(\mathbb{D})$. Since $C(\mathbb{T}^2) = C(\mathbb{T})\bar{\otimes}_{\min} C(\mathbb{T})$, it follows that the joint representation $\rho_1 \odot \rho_2 : A(\mathbb{D}) \otimes_{\min} A(\mathbb{D}) \to \mathbb{B}(\mathcal{H})$ is completely contractive. Therefore, by Proposition 1.2.9, $A(\mathbb{D}) \otimes_{\min} A(\mathbb{D}) = A(\mathbb{D}) \otimes_{\max} A(\mathbb{D})$.

The following result is essentially Proposition 2.6 of [23], which indicates the connection between tensor products with $\mathcal{T}(n)$ and dilation theory. (That paper proves the result for $n = 2$, but the proof extends to arbitrary n easily. See also Proposition 6.3.8 of [2].)

Proposition 1.2.10. *Let $\mathcal{A}$ be a unital subalgebra of a unital C^*-algebra $\mathcal{B}$, and let $n \geq 1$. The following are equivalent:*

(i) $\mathcal{A} \otimes_{\max} \mathcal{T}(n) = \mathcal{A} \otimes_{\min} \mathcal{T}(n)$.

(ii) For any family of Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$, u.c.c. representations $\rho_i : \mathcal{A} \to \mathbb{B}(\mathcal{H}_i)$ and contractions $T_i : \mathcal{H}_{i+1} \to \mathcal{H}_i$ such that $\rho_i(a)T_i = T_i\rho_{i+1}(a)$ for all $a \in \mathcal{A}$ and $1 \leq i \leq n-1$, there exist Hilbert spaces $\mathcal{K}_i \supseteq \mathcal{H}_i$, unital $*$-representations $\pi_i : \mathcal{B} \to \mathbb{B}(\mathcal{K}_i)$ dilating ρ_i, and unitary operators $U_i : \mathcal{K}_{i+1} \to \mathcal{K}_i$ such that

$$\rho_i(a)T_i \ldots T_{j-1} = P_{\mathcal{H}_i}\pi_i(a)U_i \ldots U_{j-1}|_{\mathcal{H}_j} \qquad (a \in \mathcal{A}, 1 \leq i < j \leq n.)$$

1.3 General results on tensor products

Let $\mathcal{B}$ and $\mathcal{C}$ be unital operator algebras. We shall provide a general condition relating $\mathcal{B}$ and $\mathcal{C}$ under which, for any unital operator algebra $\mathcal{A}$, the completely isometric isomorphism $\mathcal{A} \otimes_{\max} \mathcal{B} = \mathcal{A} \otimes_{\min} \mathcal{B}$ implies that $\mathcal{A} \otimes_{\max} \mathcal{C} = \mathcal{A} \otimes_{\min} \mathcal{C}$ completely isometrically. Let $\{\mathcal{C}_i\}_{i \in I}$ be a family of unital operator algebras. We shall also give a general condition relating $\mathcal{B}$ and

$\{\mathcal{C}_i\}_i$ which guarantees that if $\mathcal{A} \otimes_{\max} \mathcal{C}_i = \mathcal{A} \otimes_{\min} \mathcal{C}_i$ completely isometrically for all i, then $\mathcal{A} \otimes_{\max} \mathcal{B} = \mathcal{A} \otimes_{\min} \mathcal{B}$ completely isometrically. Later, these results will be applied to specific pairs of operator algebras $\mathcal{B}$ and $\{\mathcal{C}_i\}_i$ in the context of pre-ordered groups.

Proposition 1.3.1. *Suppose that $\mathcal{B}$ and $\mathcal{C}$ are unital operator algebras having u.c.c. homomorphisms $\phi : \mathcal{B} \to \mathcal{C}$ and $\psi : \mathcal{C} \to \mathcal{B}\bar{\otimes}_{\min}\mathcal{C}$. Moreover, assume that for every u.c.c. representation $\gamma : \mathcal{C} \to \mathbb{B}(\mathcal{H})$, there is a completely contractive map $\Theta : \mathbb{B}(\mathcal{H})\bar{\otimes}_{\min}\mathcal{C} \to \mathbb{B}(\mathcal{H})$ such that*

$$\Theta\big[(T \otimes 1_{\mathcal{C}}) \cdot \tilde{\gamma}(c)\big] \;=\; T\gamma(c) \tag{1.2}$$

for all $c \in \mathcal{C}$ and $T \in [\gamma(\mathcal{C})]'$, where $\tilde{\gamma} = \big((\gamma \circ \phi) \otimes \mathrm{id}_{\mathcal{C}}\big) \circ \psi : \mathcal{C} \to \mathbb{B}(\mathcal{H})\bar{\otimes}_{\min}\mathcal{C}$. Then for any unital operator algebra $\mathcal{A}$, $\mathcal{A} \otimes_{\max} \mathcal{B} = \mathcal{A} \otimes_{\min} \mathcal{B}$ implies that $\mathcal{A} \otimes_{\max} \mathcal{C} = \mathcal{A} \otimes_{\min} \mathcal{C}$.

Proof. Suppose that $\mathcal{A}$ satisfies $\mathcal{A} \otimes_{\max} \mathcal{B} = \mathcal{A} \otimes_{\min} \mathcal{B}$. Let $\alpha : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ and $\gamma : \mathcal{C} \to \mathbb{B}(\mathcal{H})$ be a pair of u.c.c. representations having commuting ranges. Then $\alpha \odot (\gamma \circ \phi)$ is a u.c.c. representation of $\mathcal{A} \otimes_{\min} \mathcal{B}$. Let

$$\tilde{\alpha} \;:\; \mathcal{A} \to \mathbb{B}(\mathcal{H})\bar{\otimes}_{\min}\mathcal{C}$$

be the u.c.c. homomorphism given by

$$\tilde{\alpha}(a) \;=\; \alpha(a) \otimes 1_{\mathcal{C}},$$

and consider the u.c.c. homomorphism

$$\tilde{\gamma} \;=\; \big((\gamma \circ \phi) \otimes \mathrm{id}_{\mathcal{C}}\big) \circ \psi \;:\; \mathcal{C} \to \mathbb{B}(\mathcal{H})\bar{\otimes}_{\min}\mathcal{C}.$$

Then $\tilde{\alpha}$ and $\tilde{\gamma}$ have commuting ranges. Also, it is easy to check that

$$\tilde{\alpha} \odot \tilde{\gamma} \;=\; \big[\big(\alpha \odot (\gamma \circ \phi)\big) \otimes \mathrm{id}_{\mathcal{C}}\big] \circ (\mathrm{id}_{\mathcal{A}} \otimes \psi).$$

This formula implies that $\tilde{\alpha} \odot \tilde{\gamma}$ is c.c. on $\mathcal{A} \otimes_{\min} \mathcal{C}$. Finally, by hypothesis, there is a c.c. map $\Theta : \mathbb{B}(\mathcal{H})\bar{\otimes}_{\min}\mathbf{C} \to \mathbb{B}(\mathcal{H})$ such that $\Theta\big((T \otimes 1_{\mathcal{C}}) \cdot \tilde{\gamma}(c)\big) = T\gamma(c)$ for any $c \in \mathcal{C}$ and $T \in [\gamma(\mathcal{C})]'$. Then

$$\Theta \circ (\tilde{\alpha} \odot \tilde{\gamma}) \;=\; \alpha \odot \gamma,$$

as can be seen by putting $T = \alpha(a)$ where $a \in \mathcal{A}$. Therefore, $\alpha \odot \gamma$ is c.c. on $\mathcal{A} \otimes_{\min} \mathcal{C}$. As this is true for all $(\alpha, \gamma) \in \mathscr{R}(\mathcal{A}, \mathcal{C})$, we conclude that $\mathcal{A} \otimes_{\max} \mathcal{C} = \mathcal{A} \otimes_{\min} \mathcal{C}$. □

Remark 1.3.1. It is easy to see that the hypotheses of Proposition 1.3.1 force ψ to be a complete isometry. Indeed, suppose that $\mathcal{C}$ acts faithfully on a Hilbert space $\mathcal{K}$, and let $\gamma : \mathcal{C} \to \mathbb{B}(\mathcal{K})$ be the completely isometric embedding. Choose $T = I_{\mathcal{K}}$. Then we obtain a c.c. map Θ such that $\Theta \circ (\phi \otimes \mathrm{id}_{\mathcal{C}}) \circ \psi = \gamma$, which implies that ψ is a complete isometry.

In the next lemma, we refer to page 21 (see (1.1)) for the notation.

Lemma 1.3.2. *Let $\{\mathcal{B}_i\}_{i \in I}$ and $\{\mathcal{C}_j\}_{j \in J}$ be two families of unital operator algebras. Consider the following conditions regarding unital operator algebras $\mathcal{A}$:*

(1) $\mathcal{A} \otimes_{\max} \mathcal{B}_i \ = \ \mathcal{A} \otimes_{\min} \mathcal{B}_i$ *for all* $i \in I$.

(2) $\mathcal{A} \otimes_{\max} \mathcal{C}_j \ \underset{1}{=} \ \mathcal{A} \otimes_{\min} \mathcal{C}_j$ *for all* $j \in J$.

(2′) $\mathcal{A} \otimes_{\max} \mathcal{C}_j \ = \ \mathcal{A} \otimes_{\min} \mathcal{C}_j$ *for all* $j \in J$.

Suppose that, for every unital operator algebra $\mathcal{A}$, condition (1) implies condition (2). Then, for every unital operator algebra $\mathcal{A}$, condition (1) implies condition (2′).

Proof. This is a simple consequence of the nuclearity of M_n. In more detail, suppose that (1) implies (2) for all unital operator algebras $\mathcal{A}$. Now, take any $\mathcal{A}$ for which (1) holds. Then for any integer $n \geq 1$ and $i \in I$, by (1),

$$M_n(\mathcal{A}) \otimes_{\max} \mathcal{B}_i \ = \ M_n(\mathcal{A} \otimes_{\max} \mathcal{B}_i) \ = \ M_n(\mathcal{A} \otimes_{\min} \mathcal{B}_i) \ = \ M_n(\mathcal{A}) \otimes_{\min} \mathcal{B}_i$$

completely isometrically. Applying (2) to $M_n(\mathcal{A})$ instead of $\mathcal{A}$, for each $j \in J$ we have

$$M_n(\mathcal{A} \otimes_{\max} \mathcal{C}_j) \ = \ M_n(\mathcal{A}) \otimes_{\max} \mathcal{C}_j \ = \ M_n(\mathcal{A}) \otimes_{\min} \mathcal{C}_j \ = \ M_n(\mathcal{A} \otimes_{\min} \mathcal{C}_j)$$

isometrically. Since this is true for all $n \geq 1$, (2′) holds for $\mathcal{A}$. □

Remark 1.3.2. Suppose that $\{\phi_i : X \to Y_i\}_{i\in I}$ are c.c. maps between operator spaces such that for any $x \in \mathbb{B}(\ell_2) \otimes X$,

$$\lim_i \left\| (\mathrm{id}_{\mathbb{B}(\ell_2)} \otimes \phi_i)(x) \right\|_{\min} \;=\; \|x\|_{\min}.$$

Then for any Hilbert space $\mathcal{H}$ and any $x \in \mathbb{B}(\mathcal{H}) \otimes X$,

$$\lim_i \left\| (\mathrm{id}_{\mathbb{B}(\mathcal{H})} \otimes \phi_i)(x) \right\|_{\min} \;=\; \|x\|_{\min}.$$

To see this, let $x = \sum_{k=1}^n T_k \otimes x_k \in \mathbb{B}(\mathcal{H}) \otimes X$. Since the C*-algebra $\mathcal{A} = C^*(T_1, \ldots, T_n)$ generated by $T_1, \ldots, T_n$ is separable, there is a faithful $*$-representation $\pi : \mathcal{A} \to \mathbb{B}(\ell_2)$. Then $\pi \otimes \mathrm{id}_X$ and $\pi \otimes \mathrm{id}_Y$ are complete isometries. Hence,

$$\begin{aligned}
\lim_i \|(\mathrm{id} \otimes \phi_i)(x)\|_{\min} &= \lim_i \|(\pi \otimes \mathrm{id}_Y) \circ (\mathrm{id} \otimes \phi_i)(x)\|_{\min} \\
&= \lim_i \|(\mathrm{id} \otimes \phi_i) \circ (\pi \otimes \mathrm{id}_X)(x)\|_{\min} \\
&= \|(\pi \otimes \mathrm{id}_X)(x)\|_{\min} \\
&= \|x\|_{\min}.
\end{aligned}$$

The following result will be applied to the proof of Theorem 1.6.2 below.

Proposition 1.3.3. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}_i$ $(i \in I)$ be unital operator algebras. Suppose that there exist u.c.c. homomorphisms $\varepsilon : \mathcal{B} \to \mathbf{C}$, $\delta : \mathcal{B} \to \mathcal{B}\bar{\otimes}_{\min}\mathcal{B}$ and $\phi_i : \mathcal{B} \to \mathcal{C}_i$, $\psi_i : \mathcal{C}_i \to \mathcal{B}\bar{\otimes}_{\min}\mathcal{C}_i$ $(i \in I)$ such that the following conditions hold:*

(i) $(\mathrm{id}_{\mathcal{B}} \otimes \varepsilon) \circ \delta = \mathrm{id}_{\mathcal{B}}$.

(ii) *For each $i \in I$, $(\mathrm{id}_{\mathcal{B}} \otimes \phi_i) \circ \delta = \psi_i \circ \phi_i$.*

(iii) *For any $x \in \mathbb{B}(\ell_2) \otimes \mathcal{B}$,*

$$\lim_i \left\| (\mathrm{id}_{\mathbb{B}(\ell_2)} \otimes \phi_i)(x) \right\|_{\mathbb{B}(\ell_2)\otimes_{\min}\mathcal{C}_i} \;=\; \|x\|_{\mathbb{B}(\ell_2)\otimes_{\min}\mathcal{B}}.$$

If $\mathcal{A} \otimes_{\max} \mathcal{C}_i = \mathcal{A} \otimes_{\min} \mathcal{C}_i$ for all $i \in I$, then $\mathcal{A} \otimes_{\max} \mathcal{B} = \mathcal{A} \otimes_{\min} \mathcal{B}$.

Proof. By Lemma 1.3.2, it is enough to show that if $\mathcal{A} \otimes_{\max} \mathcal{C}_i = \mathcal{A} \otimes_{\min} \mathcal{C}_i$ for all $i \in I$, then $\mathcal{A} \otimes_{\max} \mathcal{B} \underset{1}{=} \mathcal{A} \otimes_{\min} \mathcal{B}$. That is, we shall prove that $\alpha \odot \beta$ is a contractive representation of $\mathcal{A} \otimes_{\min} \mathcal{B}$ for any pair of u.c.c. representations $\alpha : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ and $\beta : \mathcal{B} \to \mathbb{B}(\mathcal{H})$ whose ranges commute.

We begin by defining some homomorphisms derived from α and β and making a few observations concerning them. Let $i \in I$. We denote by $j : \mathcal{A} \to \mathcal{A} \otimes_{\min} \mathcal{B}$ and $j_i : \mathcal{A} \to \mathcal{A} \otimes_{\min} \mathcal{C}_i$ the canonical embeddings $j(a) = a \otimes 1_{\mathcal{B}}$ and $j_i(a) = a \otimes 1_{\mathcal{C}_i}$. Then define the u.c.c. homomorphisms

$$\begin{aligned}
\tilde{\alpha} &:= (\alpha \otimes \mathrm{id}_{\mathcal{B}}) \circ j &:& \quad \mathcal{A} \to \mathbb{B}(\mathcal{H}) \bar{\otimes}_{\min} \mathcal{B}, \\
\tilde{\beta} &:= (\beta \otimes \mathrm{id}_{\mathcal{B}}) \circ \delta &:& \quad \mathcal{B} \to \mathbb{B}(\mathcal{H}) \bar{\otimes}_{\min} \mathcal{B}, \\
\alpha_i &:= (\alpha \otimes \mathrm{id}_{\mathcal{C}_i}) \circ j_i &:& \quad \mathcal{A} \to \mathbb{B}(\mathcal{H}) \bar{\otimes}_{\min} \mathcal{C}_i, \\
\beta_i &:= (\beta \otimes \mathrm{id}_{\mathcal{C}_i}) \circ \psi_i &:& \quad \mathcal{C}_i \to \mathbb{B}(\mathcal{H}) \bar{\otimes}_{\min} \mathcal{C}_i.
\end{aligned}$$

We claim that $\tilde{\alpha}$ and $\tilde{\beta}$ have commuting ranges, and that

$$(\mathrm{id}_{\mathbb{B}(\mathcal{H})} \otimes \varepsilon) \circ (\tilde{\alpha} \odot \tilde{\beta}) = \alpha \odot \beta. \tag{1.3}$$

First, let $a \in \mathcal{A}$ and $x \in \mathcal{B} \otimes \mathcal{B}$. Since $\alpha(a)$ and $\beta(b)$ commute for every $b \in \mathcal{B}$, so do $\tilde{\alpha}(a) = \alpha(a) \otimes 1_{\mathcal{B}}$ and $(\beta \otimes \mathrm{id}_{\mathcal{B}})(x)$. Thus, $\beta(\mathcal{B}) \otimes_{\min} \mathcal{B} \subset [\tilde{\alpha}(\mathcal{A})]'$, and as $[\tilde{\alpha}(\mathcal{A})]'$ is norm-closed, we have $\beta(\mathcal{B}) \bar{\otimes}_{\min} \mathcal{B} \subset [\tilde{\alpha}(\mathcal{A})]'$. Therefore, $\tilde{\alpha}$ and $\tilde{\beta} = (\beta \otimes \mathrm{id}_{\mathcal{B}}) \circ \delta$ have commuting ranges. Next, if $a \in \mathcal{A}$ and $b \in \mathcal{B}$, then

$$\tilde{\alpha}(a)\tilde{\beta}(b) = (\alpha(a) \otimes 1_{\mathcal{B}}) \cdot \big[\big((\beta \otimes \mathrm{id}_{\mathcal{B}}) \circ \delta\big)(b)\big].$$

Noting that $\mathrm{id}_{\mathbb{B}(\mathcal{H})} \otimes \varepsilon$ is a homomorphism and that $(\mathrm{id}_{\mathcal{B}} \otimes \varepsilon) \circ \delta = \mathrm{id}_{\mathcal{B}}$, we get

$$\begin{aligned}
(\mathrm{id}_{\mathbb{B}(\mathcal{H})} \otimes \varepsilon)\big(\tilde{\alpha}(a)\tilde{\beta}(b)\big) &= (\alpha(a)\varepsilon(1_{\mathcal{B}})) \cdot \big((\beta \otimes \varepsilon) \circ \delta\big)(b) \\
&= \alpha(a)\big(\beta \circ (\mathrm{id}_{\mathcal{B}} \otimes \varepsilon) \circ \delta\big)(b) \\
&= \alpha(a)\beta(b).
\end{aligned}$$

Hence (1.3) follows from linearity.

With virtually the same argument, from the equality $(\mathrm{id}_{\mathcal{B}} \otimes \phi_i) \circ \delta = \psi_i \circ \phi_i$, we see that α_i and β_i have commuting ranges and that

$$(\mathrm{id}_{\mathbb{B}(\mathcal{H})} \otimes \phi_i) \circ (\alpha_i \odot \beta_i) \;=\; (\alpha_i \odot \beta_i) \circ (\mathrm{id}_{\mathcal{A}} \otimes \phi_i). \tag{1.4}$$

We are ready to prove that $\alpha \odot \beta$ is contractive on $\mathcal{A} \otimes_{\min} \mathcal{B}$. Take any $x \in \mathcal{A} \otimes \mathcal{B}$. Define

$$A \;:=\; (\tilde{\alpha} \odot \tilde{\beta})(x) \;\in\; \mathbb{B}(\mathcal{H}) \bar{\otimes}_{\min} \mathcal{B}$$

and, for each $i \in I$,

$$A_i \;:=\; (\mathrm{id}_{\mathbb{B}(\mathcal{H})} \otimes \phi_i)(A) \;\in\; \mathbb{B}(\mathcal{H}) \bar{\otimes}_{\min} \mathcal{C}_i.$$

By (1.3),

$$(\mathrm{id}_{\mathbb{B}(\mathcal{H})} \otimes \varepsilon)(A) \;=\; (\alpha \odot \beta)(x).$$

Since $\mathrm{id}_{\mathbb{B}(\mathcal{H})} \otimes \varepsilon$ is completely contractive, it follows that

$$\|(\alpha \odot \beta)(x)\| \;\leq\; \|A\|_{\min}.$$

On the other hand, by (1.4),

$$A_i \;=\; (\alpha_i \odot \beta_i)\left[(\mathrm{id}_{\mathcal{A}} \otimes \phi_i)(x)\right].$$

Since $\mathcal{A} \otimes_{\max} \mathcal{C}_i = \mathcal{A} \otimes_{\min} \mathcal{C}_i$ by assumption, $\alpha_i \odot \beta_i$ is contractive on $\mathcal{A} \otimes_{\min} \mathcal{C}_i$. It follows that

$$\|A_i\|_{\min} \;\leq\; \|(\mathrm{id}_{\mathcal{A}} \otimes \phi_i)(x)\|_{\min} \;\leq\; \|x\|_{\min}.$$

Finally, by condition (iii) and Remark 1.3.2, we have

$$\begin{aligned} \|A\|_{\min} \;&=\; \lim_i \left\|(\mathrm{id}_{\mathbb{B}(\mathcal{H})} \otimes \phi_i)(A)\right\|_{\min} \\ &=\; \lim_i \|A_i\|_{\min}. \end{aligned}$$

Combining all these pieces,

$$\|(\alpha \odot \beta)(x)\| \;\leq\; \|A\|_{\min} \;=\; \lim_i \|A_i\|_{\min} \;\leq\; \|x\|_{\min},$$

and we are done. $\square$

The last result of this section goes along the same theme as the preceding ones. It will not be used later. Note that its hypotheses mean that $\mathcal{C}$ is embedded completely isometrically into $\mathcal{B}$ and there is a completely contractive expectation of $\mathcal{B}$ onto $\mathcal{C}$.

Proposition 1.3.4. *Let $\mathcal{B}$ and $\mathcal{C}$ be unital operator algebras. Suppose that there exist completely contractive homomorphisms $\sigma : \mathcal{C} \to \mathcal{B}$ and $\theta : \mathcal{B} \to \mathcal{C}$ such that $\theta \circ \sigma = \mathrm{id}_{\mathcal{C}}$ and $\theta(1_{\mathcal{B}}) = 1_{\mathcal{C}}$. Then for any unital operator algebra $\mathcal{A}$, $\mathcal{A} \otimes_{\max} \mathcal{B} = \mathcal{A} \otimes_{\min} \mathcal{B}$ implies that $\mathcal{A} \otimes_{\max} \mathcal{C} = \mathcal{A} \otimes_{\min} \mathcal{C}$.*

Proof. Recall that the statement $\mathcal{A} \otimes_{\max} \mathcal{C} = \mathcal{A} \otimes_{\min} \mathcal{C}$ is equivalent to saying that, for every pair of unital, completely contractive representations $\alpha : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ and $\gamma : \mathcal{C} \to \mathbb{B}(\mathcal{H})$ with commuting ranges, their joint representation $\alpha \odot \gamma$ is completely contractive on $\mathcal{A} \otimes_{\min} \mathcal{C}$.

Let α and γ be as above. Then $\beta = \gamma \circ \theta : \mathcal{B} \to \mathbb{B}(\mathcal{H})$ is a unital, completely contractive representation whose range commutes with $\alpha(\mathcal{A})$. Since $\mathcal{A} \otimes_{\max} \mathcal{B} = \mathcal{A} \otimes_{\min} \mathcal{B}$, it follows that $\alpha \odot \beta : \mathcal{A} \otimes_{\min} \mathcal{B} \to \mathbb{B}(\mathcal{H})$ is completely contractive. Also, $\mathrm{id}_{\mathcal{A}} \otimes \sigma : \mathcal{A} \otimes_{\min} \mathcal{C} \to \mathcal{A} \otimes_{\min} \mathcal{B}$ is completely contractive. Since $\theta \circ \sigma = \mathrm{id}_{\mathcal{C}}$, it follows that $(\alpha \odot \beta) \circ (\mathrm{id}_{\mathcal{A}} \otimes \sigma) = \alpha \odot \gamma$ is completely contractive. Hence $\mathcal{A} \otimes_{\max} \mathcal{C} = \mathcal{A} \otimes_{\min} \mathcal{C}$. $\square$

1.4 Digraph algebras $\mathbb{A}(P)$ for pre-ordered sets P

We begin by recalling some notions concerning pre-ordered sets. Let $(P, \leq)$ be any pre-ordered set; that is, $\leq$ is a reflexive, transitive relation on the set P. For each $k \in P$, we define the right-open interval

$$P_{\geq k} := \{ j \in P : j \geq k \}.$$

It is clear that for any $i, j \in P$, $i \leq j$ if and only if $P_{\geq j} \subseteq P_{\geq i}$. The intervals $P_{\geq k}$ are examples of increasing subsets. A subset S of P is said to be **increasing** provided that for any $j \in P$, if $i \leq j$ for some $i \in S$, then $j \in S$. The *increasing subset of P generated by S* is given by

$$\overrightarrow{S} := \{ j \in P : i \leq j \text{ for some } i \in S \} = \bigcup_{i \in S} P_{\geq i}.$$

We say that a subset Q of P is **convex** if for every $i, j \in Q$ and $k \in P$, $i \leq k \leq j$ implies that $k \in Q$. We shall have occasion to use the following description of convexity.

Lemma 1.4.1. *Let P be a pre-ordered set and $Q \subseteq P$. The following are equivalent:*

(i) Q *is convex.*

(ii) $\overrightarrow{Q} \setminus Q$ *is increasing.*

(iii) $Q = S_1 \setminus S_0$*, where* S_0 *and* S_1 *are increasing subsets with* $S_0 \subseteq S_1$*.*

Proof. [(i) $\Rightarrow$ (ii)] Suppose that Q is convex. Let $i \in \overrightarrow{Q} \setminus Q$ and $i \leq j$. Then $j \in \overrightarrow{Q}$, and there is $k \in Q$ such that $k \leq i$. If j were in Q, then since $k \leq i \leq j$ and Q is convex, this would give $i \in Q$, a contradiction. Hence $j \in \overrightarrow{Q} \setminus Q$, proving that $\overrightarrow{Q} \setminus Q$ is increasing.

[(ii) $\Rightarrow$ (iii)] We have $Q = S_1 \setminus S_0$, where $S_1 = \overrightarrow{Q}$ and $S_0 = \overrightarrow{Q} \setminus Q$ are increasing.

[(iii) $\Rightarrow$ (i)] Suppose that $Q = S_1 \setminus S_0$, where $S_0 \subseteq S_1$ are increasing subsets. Suppose that $i \leq j \leq k$ in P, where $i, k \in Q$. Since $i \in S_1$ and S_1 is increasing, we have $j \in S_1$. If j were in S_0, then since S_0 is also increasing, this would make $k \in S_0$, contrary to $k \in Q = S_1 \setminus S_0$. Hence $j \in S_1 \setminus S_0 = Q$, which proves that Q is convex. $\square$

We now define the **digraph algebra** $\mathbb{A}(P)$ for a pre-ordered set P. Let $\ell_2(P)$ denote the Hilbert space with standard orthonormal basis $\{e_k : k \in P\}$. Each subset Q of P corresponds to the subspace $\ell_2(Q) := \overline{\operatorname{span}}\{e_k : k \in Q\}$ of $\ell_2(P)$. Define the collection of subspaces

$$\mathscr{N}(P) := \{\ell_2(P_{\geq k}) \ : \ k \in P\}.$$

Then we define the reflexive operator algebra

$$\begin{aligned}
\mathbb{A}(P) \quad &:= \quad \mathsf{Alg}\,\mathscr{N}(P) \\
&= \quad \{T \in \mathbb{B}(\ell_2(P)) \ : \ T\mathcal{N} \subseteq \mathcal{N} \ \text{ for every } \ \mathcal{N} \in \mathscr{N}(P)\} \\
&= \quad \{T \in \mathbb{B}(\ell_2(P)) \ : \ Te_k \in \ell_2(P_{\geq k}) \ \text{ for every } \ k \in P\}.
\end{aligned}$$

In terms of matrix entries $t_{i,j} = \langle Te_j, e_i \rangle$, a bounded operator $T = [t_{i,j}]_{(i,j)}$ on $\ell_2(P)$ belongs to $\mathbb{A}(P)$ if and only if $t_{i,j} \neq 0$ only when $i \geq j$. For each $(i,j) \in P \times P$, we have the matrix unit $E_{i,j} = e_i \otimes e_j^* \in \mathfrak{K}(\ell_2(P))$. It is easy to see that $\mathbb{A}(P)$ is the WOT-closure of the subalgebra

$$\mathbb{A}_{00}(P) \;=\; \operatorname{span}\{E_{i,j} \;:\; (i,j) \in P \times P, \, i \geq j\}.$$

When P is infinite, the norm-closed subalgebra

$$\mathbb{A}_0(P) \;:=\; \mathbb{A}(P) \cap \mathfrak{K}(\ell_2(P)) \;=\; \overline{\mathbb{A}_{00}(P)}^{\|\cdot\|}$$

is a proper subalgebra of $\mathbb{A}(P)$.

Proposition 1.4.2. *Let P be a pre-ordered set.*

(a) The subspaces of $\ell_2(P)$ that are invariant for $\mathbb{A}(P)$ are of the form $\ell_2(Q)$, where $Q \subseteq P$ is increasing.

(b) The subspaces of $\ell_2(P)$ that are semi-invariant for $\mathbb{A}(P)$ are of the form $\ell_2(Q)$, where $Q \subseteq P$ is convex.

Proof. (a) For any subset $\mathcal{X}$ of $\ell_2(P)$, one can check that the $\mathbb{A}(P)$-invariant subspace generated by $\mathcal{X}$ is

$$\overline{\operatorname{span}}(\mathbb{A}(P)\mathcal{X}) \;=\; \ell_2(\overrightarrow{S_{\mathcal{X}}}),$$

where $S_{\mathcal{X}} = \{k \in P : e_k \notin \mathcal{X}^{\perp}\}$.

(b) If $\mathcal{M} \subseteq \ell_2(P)$ is any semi-invariant subspace for $\mathbb{A}(P)$, then by a well-known characterization of such subspaces, we have $\mathcal{M} = \mathcal{N}_1 \ominus \mathcal{N}_0$, where $\mathcal{N}_0 \subseteq \mathcal{N}_1$ are invariant subspaces for $\mathbb{A}(P)$. By (a), $\mathcal{N}_i = \ell_2(S_i)$ for $i = 0, 1$, where $S_0 \subseteq S_1$ are increasing subsets of P. Hence $\mathcal{M} = \ell_2(S_1) \ominus \ell_2(S_0) = \ell_2(S_1 \setminus S_0)$. By Lemma 1.4.1, $\mathcal{M} = \ell_2(Q)$ where $Q = S_1 \setminus S_0$ is convex. $\square$

Fix a subset Q of P. We denote by $V_Q : \ell_2(Q) \hookrightarrow \ell_2(P)$ the inclusion map. Then for any

$T \in \mathbb{A}(P)$, its compression $V_Q^* T V_Q$ to $\ell_2(Q)$ belongs to $\mathbb{A}(Q)$. Indeed, if $i, j \in Q$ and $i \not\geq j$, then

$$\langle (V_Q^* T V_Q) e_j, e_i \rangle \ = \ \langle T e_j, e_i \rangle \ = \ t_{i,j} \ = \ 0.$$

Therefore, compression to $\ell_2(Q)$ restricts to a unital, completely contractive map

$$\theta_Q : \mathbb{A}(P) \to \mathbb{A}(Q), \qquad \theta_Q(T) := V_Q^* T V_Q.$$

It is a homomorphism precisely when $\ell_2(Q)$ is a semi-invariant subspace for $\mathbb{A}(P)$. By Proposition 1.4.2 (b), this happens precisely when Q is convex. On the other hand, since V_Q is an isometry, the map

$$\sigma_Q : \mathbb{A}(Q) \to \mathbb{A}(P), \qquad \sigma_Q(S) := V_Q S V_Q^*$$

is a completely isometric homomorphism. Indeed, σ_Q is a section of θ_Q; that is,

$$\theta_Q \circ \sigma_Q \ = \ \mathrm{id}_{\mathbb{A}(Q)}.$$

If $I \in \mathbb{A}(Q)$ is the identity operator, then $p_Q \ := \ \sigma_Q(I) \ = \ V_Q V_Q^*$ is the projection onto $\ell_2(Q)$. It is easy to see that

$$\mathrm{ran}\,(\sigma_Q) \ = \ p_Q\, \mathbb{A}(P)\, p_Q.$$

We may therefore identify $\mathbb{A}(Q)$ with the corner $p_Q\, \mathbb{A}(P)\, p_Q$ of $\mathbb{A}(P)$.

Remark 1.4.1. If P and Q are pre-ordered sets, we provide the Cartesian product $P \times Q$ the canonical pre-ordering

$$(i, j) \ \leq \ (k, l) \qquad \Leftrightarrow \qquad i \leq j \ \text{ and } \ k \leq l.$$

The following result is likely well-known.

Proposition 1.4.3. *Let P and Q be pre-ordered sets. Then the canonical unitary isomorphism $\ell_2(P) \otimes \ell_2(Q) = \ell_2(P \times Q)$ induces a completely isometric embedding*

$$\mathbb{A}(P) \bar{\otimes}_{\min} \mathbb{A}(Q) \ \hookrightarrow \ \mathbb{A}(P \times Q)$$

as well as the completely isometric equality

$$\mathbb{A}_0(P)\bar{\otimes}_{\min}\mathbb{A}_0(Q) \;=\; \mathbb{A}_0(P \times Q).$$

If one of P and Q is finite, then $\mathbb{A}(P) \otimes_{\min} \mathbb{A}(Q) = \mathbb{A}(P \times Q)$ holds.

Proof. The identification $\ell_2(P) \otimes \ell_2(Q) = \ell_2(P \times Q)$ given by $e_i \otimes e_j = e_{(i,j)}$ induces a canonical embedding of C^*-algebras

$$\iota \;:\; \mathbb{B}(\ell_2(P))\bar{\otimes}_{\min}\mathbb{B}(\ell_2(Q)) \;\hookrightarrow\; \mathbb{B}(\ell_2(P \times Q)).$$

We prove the first claim by verifying that ι maps $\mathbb{A}(P) \otimes_{\min} \mathbb{A}(Q)$ into $\mathbb{A}(P \times Q)$. Given $S \in \mathbb{A}(P)$ and $T \in \mathbb{A}(Q)$, for any $(i_1, i_2), (j_1, j_2) \in P \times Q$,

$$\left\langle \iota(S \otimes T)e_{(j_1,j_2)}, e_{(i_1,i_2)} \right\rangle \;=\; \langle Se_{j_1}, e_{i_1}\rangle\langle Te_{j_2}, e_{i_2}\rangle \;=\; s_{i_1,j_1}t_{i_2,j_2},$$

which is non-zero only when $i_1 \geq j_1$ and $i_2 \geq j_2$, that is, $(i_1, i_2) \geq (j_1, j_2)$. Thus, $\iota(S \otimes T) \in \mathbb{A}(P \times Q)$, as required.

Since both algebras $\mathbb{A}_0(P) \otimes_{\min} \mathbb{A}_0(Q)$ and $\mathbb{A}_0(P \times Q)$ contain the dense subalgebra

$$\mathbb{A}_{00}(P \times Q) \;=\; \mathrm{span}\{E_{i,j} \otimes E_{k,l} \;:\; i, j \in P,\; k, l \in Q,\; i \geq j,\; k \geq l\},$$

we see that $\mathbb{A}_0(P)\bar{\otimes}_{\min}\mathbb{A}_0(Q) = \mathbb{A}_0(P \times Q)$.

To prove the final claim, we may assume that P is finite. Then

$$\begin{aligned}\mathbb{B}(\ell_2(P \times Q)) &= \mathbb{B}(\ell_2(P)) \otimes_{\min} \mathbb{B}(\ell_2(Q)) \\ &= \mathrm{span}\{E_{i,j} \otimes T \;:\; (i, j) \in P \times P,\; T \in \mathbb{B}(\ell_2(Q))\}.\end{aligned}$$

Given $T \in \mathbb{A}(P \times Q)$, we may write $T = \sum_{i,j \in P} E_{i,j} \otimes T_{i,j}$, where $T_{i,j} \in \mathbb{B}(\ell_2(Q))$. For any $i, j \in P$ and $k, l \in Q$,

$$\langle Te_{(j,l)}, e_{(i,k)}\rangle \;=\; \langle T_{i,j}e_l, e_k\rangle.$$

Since $T \in \mathbb{A}(P \times Q)$, the above is non-zero only when $i \geq j$ and $k \geq l$. In other words, $T_{i,j} \neq 0$

only when $i \geq j$, in which case $T_{i,j} \in \mathbb{A}(Q)$. Hence,

$$T \;=\; \sum_{(i,j)\in P\times P,\; i\geq j} E_{i,j} \otimes T_{i,j} \;\in\; \mathbb{A}(P) \otimes_{\min} \mathbb{A}(Q),$$

proving the claim. □

Remark 1.4.2. If $\mathcal{A} \subseteq \mathbb{B}(\mathcal{H})$ and $\mathcal{B} \subseteq \mathbb{B}(\mathcal{K})$ are operator algebras, their *normal spatial tensor product* $\mathcal{A}\overline{\otimes}^{\sigma}_{\min}\mathcal{B}$ is defined as the weak*-closure of $\mathcal{A} \otimes_{\min} \mathcal{B}$ in $\mathbb{B}(\mathcal{H} \otimes \mathcal{K})$. We have

$$\mathbb{A}(P)\overline{\otimes}^{\sigma}_{\min}\mathbb{A}(Q) \;=\; \mathbb{A}(P \times Q).$$

1.5 Shift operators on pre-ordered groups

Let G be a discrete group with identity 1. We do not assume that G is abelian. Before discussing the left regular representation of a pre-ordered group, we gather a few general facts regarding the reduced group C^*-algebra $C^*_r(G)$, and also fix notation.

For a fixed unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$, recall that the C^*-subalgebra of $\mathbb{B}(\mathcal{H})$ generated by the range $\pi(G)$ is

$$C^*_\pi(G) \;=\; \overline{\operatorname{span}}\{\pi(g) \;:\; g \in G\}.$$

Consider the unitary operator $W = W_G$ on $\mathcal{H} \otimes \ell_2(G)$ defined by

$$W(h \otimes e_s) \;:=\; \pi(s)h \otimes e_s$$

for $h \in \mathcal{H}$ and $s \in G$. Alternatively, under the identification $\mathcal{H} \otimes \ell_2(G) = \ell_2(G, \mathcal{H})$, for any square-summable function $\xi \in \ell_2(G, \mathcal{H})$, we have

$$(W\xi)(s) \;=\; \pi(s).\xi(s) \qquad\qquad (s \in G).$$

With respect to the decomposition $\mathcal{H}\otimes\ell_2(G)=\oplus_{s\in G}(\mathcal{H}\otimes\mathbf{C}e_s)$, we have $W=\oplus_{s\in G}\pi(s)$. Note that for each subset Q of G, $\mathcal{H}\otimes\ell_2(Q)$ is a reducing subspace for W. Let $W_Q\in\mathcal{U}(\mathcal{H}\otimes\ell_2(Q))$ denote the restriction of W to $\mathcal{H}\otimes\ell_2(Q)$. Then define the injective $*$-homomorphism

$$\begin{aligned}\Psi_Q\ :\ \mathbb{B}(\ell_2(Q)) &\hookrightarrow \mathbb{B}(\mathcal{H}\otimes\ell_2(Q)),\\ \Psi_Q(A) &:= W_Q(I_{\mathcal{H}}\otimes A)W_Q^*.\end{aligned}$$

If $A\in\mathbb{B}(\ell_2(Q))$ has the matrix representation $A=[\alpha_{s,t}]_{(s,t\in Q)}$ with respect to the standard basis $\{e_s\}_{s\in Q}$, then $\Psi_Q(A)$ has the block matrix representation

$$\Psi_Q(A)\ =\ \left[\alpha_{s,t}\,\pi(st^{-1})\right]_{(s,t)}.$$

In particular, for any $s,t\in G$,

$$\Psi_Q(E_{s,t})\ =\ \pi(st^{-1})\otimes E_{s,t}. \tag{1.5}$$

Therefore,

$$\Psi_Q\left[\mathfrak{K}(\ell_2(Q))\right]\ \subset\ C^*_\pi(G)\bar{\otimes}_{\min}\mathfrak{K}(\ell_2(Q)).$$

Proposition 1.5.1. *Let $\pi: G\to\mathcal{U}(\mathcal{H})$ be a unitary representation. Given $Q\subseteq G$, let Ψ_Q be the $*$-homomorphism defined as above.*

(a) (Fell's absorption principle) *For every $g\in G$,*

$$\Psi_G(\lambda_g)\ =\ \pi(g)\otimes\lambda_g.$$

Hence Ψ_G identifies $C^*_r(G)$ with the C^*-subalgebra $C^*_{\pi\otimes\lambda}(G)$ of $C^*_\pi(G)\bar{\otimes}_{\min}C^*_r(G)$.

(b) Let $\theta_Q:\mathbb{B}(\ell_2(G))\to\mathbb{B}(\ell_2(Q))$ and $\Theta_Q:\mathbb{B}(\mathcal{H}\otimes\ell_2(G))\to\mathbb{B}(\mathcal{H}\otimes\ell_2(Q))$ denote the respective compressions. Then $\Psi_Q\circ\theta_Q=\Theta_Q\circ\Psi_G$. Moreover, for any $g\in G$,

$$\Psi_Q(\theta_Q(\lambda_g))\ =\ \pi(g)\otimes\theta_Q(\lambda_g).$$

Proof. (a) For any $g, s \in G$ and $h \in \mathcal{H}$,

$$\begin{aligned} \Psi_G(\lambda_g)(h \otimes e_s) &= W(I \otimes \lambda_g)W^*(h \otimes e_s) \\ &= W(I \otimes \lambda_g)(\pi(s^{-1})h \otimes e_s) \\ &= W(\pi(s^{-1})h \otimes e_{gs}) \\ &= \pi(gs)\pi(s^{-1})h \otimes e_{gs} \\ &= \pi(g)h \otimes \lambda_g e_s. \end{aligned}$$

Hence $\Psi_G(\lambda_g) = \pi(g) \otimes \lambda_g$. Thus, Ψ_G maps $C_r^*(G)$ onto $C^*_{\pi\otimes\lambda}(G)$.

(b) Let $A \in \mathbb{B}(\ell_2(G))$ have matrix representation $[\alpha_{s,t}]_{(s,t\in G)}$. Then its compression to $\ell_2(Q)$ is $\theta_Q(A) = [\alpha_{s,t}]_{(s,t\in Q)}$. On the other hand, $\Psi_G(A) = [\alpha_{s,t}\, \pi(st^{-1})]_{(s,t\in G)}$, whose compression to $\mathcal{H} \otimes \ell_2(Q)$ is

$$\Theta_Q(\Psi_G(A)) \;=\; \left[\alpha_{s,t}\, \pi(st^{-1})\right]_{(s,t\in Q)} \;=\; \Psi_Q(\theta_Q(A)).$$

This proves the first claim in (b). Setting $A = \lambda_g$ where $g \in G$, we obtain

$$\begin{aligned} \Psi_Q\big(\theta_Q(\lambda_g)\big) &= \Theta_Q\big(\Psi_G(\lambda_g)\big) \\ &= \Theta_Q(\pi(g) \otimes \lambda_g) \qquad\qquad \text{by (a)} \\ &= \pi(g) \otimes \theta_Q(\lambda_g), \end{aligned}$$

proving the second claim. $\square$

Applying Proposition 1.5.1 (a) to the left regular representation $\pi = \lambda$, we obtain the injective diagonal homomorphism

$$\delta_G \;:\; C_r^*(G) \;\to\; C_r^*(G)\bar{\otimes}_{\min} C_r^*(G) \tag{1.6}$$

such that $\delta_G(\lambda_s) = \lambda_s \otimes \lambda_s$ for all $s \in G$.

We now turn to pre-ordered groups. For the rest of the section, let G^+ be a fixed unital subsemigroup of the group G. Then G^+ induces the **left pre-ordering** $\leq$ on G, defined as follows. For any $s, t \in G$,

$$s \;\leq\; t \qquad \Leftrightarrow \qquad ts^{-1} \;\in\; G^+.$$

As such, $G^+ = \{s \in G : s \geq 1\}$. We call the pair (G, G^+) a **pre-ordered group**. Note that right translations are monotone with respect to $\leq$; that is, for $r, s, t \in G$, $s \leq t$ implies that $sr \leq tr$. Also, $G_{\geq s} = G^+ s$ for any $s \in G$.

Remark 1.5.1. Beware that the usual definition of ordered groups requires that the partial ordering $\leq$ is invariant under both left and right translations. This is equivalent to saying that $sG^+s^{-1} = G^+$ for all $s \in G$. We will not enforce such a strong condition.

If S is any unital subsemigroup of G, then

$$\mathsf{alg}(S) \ := \ \mathrm{span}\{\lambda_s \ : \ s \in S\}$$

is a unital subalgebra of $C_r^*(G)$. If we set $S = G^+$, then $\mathsf{alg}(G^+)$ is a subalgebra of the digraph algebra $\mathbb{A}(G)$. Indeed, for any $s \in G^+$ and $g \in G$, $sg \in G^+g = G_{\geq g}$ and so $\lambda_s e_g \in \ell_2(G_{\geq g})$. Thus $\lambda_s \in \mathbb{A}(G)$ for $s \in G^+$.

Fix any subset Q of G. We now discuss the subalgebra $\mathbb{A}(Q)$ of $\mathbb{A}(G)$ in connection to the left pre-ordering. Recall that compression to $\ell_2(Q)$ induces a unital, completely contractive surjection $\theta_Q \ : \ \mathbb{A}(G) \ \to \ \mathbb{A}(Q)$, which is a homomorphism precisely when Q is convex. Consequently, restricting θ_Q to $\mathsf{alg}(G^+)$ gives us the unital, completely contractive map

$$\phi_Q \ : \ \mathsf{alg}(G^+) \ \to \ \mathbb{A}(Q), \tag{1.7}$$

which is a homomorphism whenever Q is convex. In fact, the converse also holds, and we prove this as follows. As ϕ_Q is given by compression of $\mathsf{alg}(G^+)$ to $\ell_2(Q)$, we must show that $\ell_2(Q)$ is semi-invariant for $\mathsf{alg}(G^+)$ exactly when Q is convex. The $\mathsf{alg}(G^+)$-invariant subspace generated by $\ell_2(Q)$ is

$$\overline{\mathrm{span}}\big(\mathsf{alg}(G^+).\ell_2(Q)\big) \ = \ \ell_2(G^+Q) \ = \ \ell_2(\overrightarrow{Q}).$$

In particular, $\ell_2(Q)$ is $\mathsf{alg}(G^+)$-invariant if and only if Q is increasing. But $\ell_2(Q)$ is semi-invariant for $\mathsf{alg}(G^+)$ if and only if

$$\overline{\mathrm{span}}\big(\mathsf{alg}(G^+).\ell_2(Q)\big) \ominus \ell_2(Q) \ = \ \ell_2(\overrightarrow{Q}) \ominus \ell_2(Q) \ = \ \ell_2(\overrightarrow{Q} \setminus Q)$$

is invariant for $\mathsf{alg}(G^+)$. Combining these facts, we conclude that $\ell_2(Q)$ is semi-invariant for $\mathsf{alg}(G^+)$ precisely when $\overrightarrow{Q} \setminus Q$ is increasing, i.e., Q is convex.

Whenever Q is a convex subset of G, the range

$$\mathcal{R}_Q \;:=\; \phi_Q\big[\mathsf{alg}(G^+)\big] \tag{1.8}$$

of the homomorphism ϕ_Q is a subalgebra of $\mathbb{A}(Q)$. We may call $\mathcal{R}_Q$ an algebra of **analytic Toeplitz operators**, in view of the classical case where $(G, G^+) = (\mathbb{Z}, \mathbb{N})$ and either $Q = \mathbb{N}$ or $Q = \{1, 2, \ldots, n\}$ for finite n.

Finally, we examine the restriction of the unital $*$-homomorphism

$$\Psi_Q \;:\; \mathbb{B}(\ell_2(Q)) \;\hookrightarrow\; \mathbb{B}(\ell_2(G) \otimes \ell_2(Q)), \qquad \Psi_Q(T) \;=\; W_Q(I \otimes T)W_Q^*$$

to the subalgebra $\mathbb{A}_0(Q) = \mathbb{A}(Q) \cap \mathfrak{K}(\ell_2(Q))$. By (1.5), for any $s, t \in Q$,

$$\Psi_Q(E_{s,t}) \;=\; \lambda_{st^{-1}} \otimes E_{s,t}.$$

When $s \geq t$, this gives $\Psi_Q(E_{s,t}) \in \mathsf{alg}(G^+) \otimes_{\min} \mathbb{A}_0(Q)$. As a result, Ψ_Q restricts to a unital, completely isometric homomorphism

$$\psi_Q \;:\; \mathbb{A}_0(Q) \;\hookrightarrow\; \mathsf{alg}(G^+) \otimes_{\min} \mathbb{A}_0(Q) \tag{1.9}$$

such that $\psi_Q(E_{s,t}) = \lambda_{st^{-1}} \otimes E_{s,t}$ for all $s \geq t$ in Q. Applying Proposition 1.5.1 (b) to the left regular representation $\pi = \lambda$, we find that

$$\psi_Q(\phi_Q(\lambda_s)) \;=\; \lambda_s \otimes \phi_Q(\lambda_s) \;=\; (\mathrm{id} \otimes \phi_Q) \circ \delta_G(\lambda_s) \qquad (s \in G^+). \tag{1.10}$$

This implies, in particular, that

$$\psi_Q(\mathcal{R}_Q) \;\subset\; \mathsf{alg}(G^+) \otimes_{\min} \mathcal{R}_Q. \tag{1.11}$$

1.6 Main Results

We now prove the two results which generalize Theorem 1.1.1.

Theorem 1.6.1. *Let (G, G^+) be a discrete pre-ordered group, and let $\mathcal{A}$ be a unital operator algebra. If $\mathcal{A} \otimes_{\max} \mathsf{alg}(G^+) = \mathcal{A} \otimes_{\min} \mathsf{alg}(G^+)$, then $\mathcal{A} \otimes_{\max} \mathbb{A}(Q) = \mathcal{A} \otimes_{\min} \mathbb{A}(Q)$ for every finite, convex subset Q of G.*

Proof. Let $Q \subseteq G$ be a finite convex subset. We shall apply Proposition 1.3.1 to $\mathcal{B} = \mathsf{alg}(G^+)$ and $\mathcal{C} = \mathbb{A}(Q)$. We consider the homomorphisms $\phi = \phi_Q : \mathsf{alg}(G^+) \to \mathbb{A}(Q)$ and $\psi = \psi_Q : \mathbb{A}(Q) \to \mathsf{alg}(G^+) \otimes_{\min} \mathbb{A}(Q)$, where ϕ_Q is given by (1.7) and ψ_Q is defined by (1.9).

Suppose that $\gamma : \mathbb{A}(Q) \to \mathbb{B}(\mathcal{H})$ is a u.c.c. representation. In this case, the homomorphism $\tilde{\gamma} = \big((\gamma \circ \phi_Q) \otimes \mathrm{id}_{\mathbb{A}(Q)}\big) \circ \psi_Q : \mathbb{A}(Q) \to \mathbb{B}(\mathcal{H}) \otimes_{\min} \mathbb{A}(Q)$ is determined by

$$\tilde{\gamma}(E_{s,t}) \ = \ \gamma(\phi(\lambda_{st^{-1}})) \otimes E_{s,t} \qquad (s, t \in Q,\ s \geq t).$$

Since γ is unital and contractive, it follows that $\{\gamma(E_{s,s}) : s \in Q\}$ is a finite set of mutually orthogonal projections which add up to $I_{\mathcal{H}}$. Thus we have the decomposition $\mathcal{H} = \oplus_{s \in Q} \mathcal{H}_s$, where $\mathcal{H}_s = \gamma(E_{s,s})\mathcal{H}$. Consider the isometry $V : \mathcal{H} \hookrightarrow \mathcal{H} \otimes \ell_2(Q)$ defined by

$$Vh \ := \ \sum_{s \in Q} \gamma(E_{s,s})h \otimes e_s \qquad (h \in \mathcal{H}).$$

We define $\Theta : \mathbb{B}(\mathcal{H}) \otimes_{\min} \mathbb{A}(Q) \to \mathbb{B}(\mathcal{H})$ to be compression to $\mathcal{H}$; that is,

$$\Theta(A) \ = \ V^* A V \qquad (A \in \mathbb{B}(\mathcal{H}) \otimes_{\min} \mathbb{A}(Q)).$$

We claim that the c.c. map Θ fulfills the condition (1.2) required by Proposition 1.3.1. Fix $T \in [\gamma(\mathbb{A}(Q))]'$. For any $s, t \in Q$ with $s \geq t$,

$$\Theta\big[(T \otimes I) \cdot \tilde{\gamma}(E_{s,t})\big] \ = \ V^* \big(T \cdot \gamma(\phi_Q(\lambda_{st^{-1}})) \otimes E_{s,t}\big) V.$$

Observe that V can be written as $V = \sum_{s \in Q} V_s$, where $V_s h = \gamma(E_{s,s})h \otimes e_s$. So,

$$\begin{aligned} V^*\big(T \cdot \gamma(\phi_Q(\lambda_{st^{-1}})) \otimes E_{s,t}\big)V &= V_s^*.\big(T \cdot \gamma[\phi_Q(\lambda_{st^{-1}})] \otimes E_{s,t}\big).V_t \\ &= \gamma(E_{s,s})T\gamma[\phi_Q(\lambda_{st^{-1}})]\gamma(E_{t,t}) \\ &= T\gamma[E_{s,s}\phi_Q(\lambda_{st^{-1}})E_{t,t}]. \end{aligned}$$

But $E_{s,s}\phi_Q(\lambda_{st^{-1}})E_{t,t} = E_{s,t}$, because

$$\begin{aligned} E_{s,s}\phi_Q(\lambda_{st^{-1}})E_{t,t} &= (e_s \otimes e_s^*)\lambda_{st^{-1}}(e_t \otimes e_t^*) \\ &= (e_s \otimes e_s^*)(e_t \otimes e_t^*) \\ &= E_{s,t}. \end{aligned}$$

Combining the above, we obtain

$$\Theta\big((T \otimes I) \cdot \tilde{\gamma}(E_{s,t})\big) \;=\; T\gamma(E_{s,t}).$$

Since $\mathbb{A}(Q) = \operatorname{span}\{E_{s,t} : s, t \in Q,\ s \geq t\}$, we have proved (1.2) indeed holds. By Proposition 1.3.1, we are done. □

Our second goal is to obtain the converse of Theorem 1.6.1 under additional assumptions on G or G^+. Recall that a family $\mathcal{F}$ of subsets of a set X is *upward-directed* provided that for any $A, B \in \mathcal{F}$, there exists $C \in \mathcal{F}$ such that $A \cup B \subseteq C$.

Theorem 1.6.2. *Let (G, G^+) be a discrete pre-ordered group, satisfying the following conditions:*

(i) There is a c.c. homomorphism $\varepsilon : \mathsf{alg}(G^+) \to \mathbf{C}$ such that $\varepsilon(\lambda_s) = 1$ for all $s \in G^+$.

(ii) There exists an upward-directed collection $\{P_i\}_{i \in I}$ of finite, convex subsets of G such that $\bigcup_{i \in I} P_i = G$.

Let the family $\{\mathcal{C}_i\}_{i \in I}$ of operator algebras be either $\{\mathbb{A}(P_i)\}_{i \in I}$ or $\{\mathcal{R}_{P_i}\}_{i \in I}$. For any unital operator algebra $\mathcal{A}$, if $\mathcal{A} \otimes_{\max} \mathcal{C}_i = \mathcal{A} \otimes_{\min} \mathcal{C}_i$ completely isometrically for all $i \in I$, then $\mathcal{A} \otimes_{\max} \mathsf{alg}(G^+) = \mathcal{A} \otimes_{\min} \mathsf{alg}(G^+)$ completely isometrically.

Proof. We apply Proposition 1.3.3, where $\mathcal{B} = \mathsf{alg}(G^+)$, and $\{\mathcal{C}_i\}_i$ is chosen as in the statement of the theorem. The injective diagonal homomorphism δ_G of $C_r^*(G)$ from (1.6) restricts to a u.c.c. (in fact, completely isometric) homomorphism

$$\delta \ : \ \mathsf{alg}(G^+) \ \to \ \mathsf{alg}(G^+) \otimes_{\min} \mathsf{alg}(G^+).$$

The homomorphism ε clearly satisfies $(\mathrm{id}_{\mathcal{B}} \otimes \varepsilon) \circ \delta \ = \ \mathrm{id}_{\mathcal{B}}$. Next, for each $i \in I$, we take $\phi_i = \phi_{P_i} : \mathsf{alg}(G^+) \to \mathcal{R}_{P_i}(\subseteq \mathcal{C}_i)$. If $\mathcal{C}_i = \mathbb{A}(P_i)$, we simply take the map

$$\psi_i = \psi_{P_i} \ : \ \mathbb{A}(P_i) \to \mathsf{alg}(G^+) \otimes_{\min} \mathbb{A}(P_i)$$

given in (1.9). If $\mathcal{C}_i = \mathcal{R}_{P_i}$, it follows from (1.11) that restricting ψ_{P_i} to $\mathcal{R}_{P_i}$ gives the well-defined u.c.c. homomorphism

$$\psi_i \ = \ \psi_{P_i}|_{\mathcal{R}_{P_i}} \ : \ \mathcal{R}_{P_i} \ \to \ \mathsf{alg}(G^+) \otimes_{\min} \mathcal{R}_{P_i}.$$

For either choice of $\mathcal{C}_i$, the equality $(\mathrm{id}_{\mathcal{B}} \otimes \phi_i) \circ \delta \ = \ \psi_i \circ \phi_i$ now follows from (1.10). Finally, since the family $\{P_i\}_{i \in I}$ is upward-directed and covers G, the collection of subspaces $\{\ell_2(\mathbb{N}) \otimes \ell_2(P_i)\}_{i \in I}$ is also upward-directed and has dense union in $\ell_2(\mathbb{N}) \otimes \ell_2(G)$. It follows that for any $x \in \mathbb{B}(\ell_2) \otimes \mathsf{alg}(G^+)$,

$$\lim_i \|(\mathrm{id} \otimes \phi_i)(x)\|_{\min} \ = \ \|x\|_{\min}.$$

Having verified the requisite conditions on the various maps, Proposition 1.3.3 leads to the desired conclusion. □

Remark 1.6.1. If G is an amenable group, then the trivial character on G extends to a $*$-homomorphism $\varepsilon : C_r^*(G) \to \mathbf{C}$ such that $\varepsilon(\lambda_s) = 1$ for all $s \in G$. For every subsemigroup S of G, restriction of ε to the subalgebra $\mathsf{alg}(S)$ remains completely contractive. It follows that condition (i) of Theorem 1.6.2 is automatically satisfied.

1.7 Applications

We first give a straightforward application of the above results, from which Theorem 1.1.1 follows as a special case. For any integer $n \geq 1$, consider the free abelian group $\mathbb{Z}^n$, which is partially ordered by the subsemigroup $\mathbb{N}^n$ of non-negative n-tuples. For any $k \geq 1$, let C_k denote the chain $\{1, \ldots, k\}$. Define the n-fold product poset $C_k^n = C_k \times \cdots \times C_k$ (see Remark 1.4.1). Since $\mathbb{A}(C_k) = \mathcal{T}(k)$, it follows from Proposition 1.4.3 that $\mathbb{A}(C_k^n)$ is completely isometric with the n-fold minimal tensor product $\mathcal{T}(k) \otimes_{\min} \cdots \otimes_{\min} \mathcal{T}(k)$.

Corollary 1.7.1. *For any integer $n \geq 1$, if $\mathcal{A}$ is any unital operator algebra, then $\mathcal{A} \otimes_{\max} A(\mathbb{D}^n) = \mathcal{A} \otimes_{\min} A(\mathbb{D}^n)$ if and only if $\mathcal{A} \otimes_{\max} \mathbb{A}(C_k^n) = \mathcal{A} \otimes_{\min} \mathbb{A}(C_k^n)$ for all $k \geq 1$.*

Proof. First of all, since $C^*(\mathbb{Z}^n) = C(\mathbb{T}^n)$, the norm closure of $\mathrm{alg}(\mathbb{N}^n)$ in $C^*(\mathbb{Z}^n)$ coincides with $A(\mathbb{D}^n)$. Second, the posets C_k^n ($k \geq 1$) can be embedded as convex subsets $Q_k^{(n)}$ of the ordered group $\mathbb{Z}^n$, in such a way that $\{Q_k^{(n)}\}_{k\geq 1}$ is a chain whose union is $\mathbb{Z}^n$. For instance, for each $m \geq 1$ we may select

$$\begin{aligned} Q_{2m+1}^{(n)} &= \{(x_1, \ldots, x_n) \in \mathbb{Z}^n \ : \ -m \leq x_j \leq m \text{ for all } 1 \leq j \leq n\}, \\ Q_{2m}^{(n)} &= \{(x_1, \ldots, x_n) \in \mathbb{Z}^n \ : \ -(m-1) \leq x_j \leq m \text{ for all } 1 \leq j \leq n\}. \end{aligned}$$

Hence Theorems 1.6.1 and 1.6.2 apply. □

Remark 1.7.1. Due to Parrott's Example 1.2.9, it is clear that the conditions of Corollary 1.7.1 are not universally met by all unital operator algebras $\mathcal{A}$ (take $\mathcal{A}$ to be the disk algebra and $n \geq 2$). However, we recall from Proposition 1.2.8 that the condition $\mathcal{A} \otimes_{\max} A(\mathbb{D}^n) = \mathcal{A} \otimes_{\min} A(\mathbb{D}^n)$ always holds if $\mathcal{A}$ is a nuclear C^*-algebra.

Example 1.7.1. The *discrete Heisenberg group*

$$H_3(\mathbb{Z}) = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \ : \ x, y, z \in \mathbb{Z} \right\}$$

provides an example of a non-abelian ordered group to which Theorem 1.6.2 applies. Being nilpotent, $H_3(\mathbb{Z})$ is solvable and hence amenable. To save space, we identify $G = H_3(\mathbb{Z})$ with the set $\mathbb{Z}^3$. Then the identity is $1 = (0,0,0)$, and the product and inverse operations are expressed by the formulas

$$\begin{aligned} (x_1, y_1, z_1)(x_2, y_2, z_2) &= (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2), \\ (x, y, z)^{-1} &= (-x, -y, -z + xy). \end{aligned}$$

We choose as positive cone the subsemigroup

$$G^+ = \{(x, y, z) \in G : x, y, z \geq 0\}.$$

For each integer $n \geq 1$, consider the elements

$$s_n := (-n, -n, -n^2), \qquad t_n := (n, n, n^2).$$

Direct computation yields, for $1 \leq m \leq n$,

$$\begin{aligned} s_n^{-1} &= (n, n, 2n^2), \\ t_n^{-1} &= (-n, -n, 0), \\ t_n t_m^{-1} &= \big(n - m, n - m, n(n - m)\big) \in G^+, \\ s_m s_n^{-1} &= \big(n - m, n - m, (n - m)(2n + m)\big) \in G^+ \end{aligned}$$

so that $s_n \leq s_m \leq 1 \leq t_m \leq t_n$. Consider the order intervals

$$P_n = [s_n, t_n] := \{g \in G : s_n \leq g \leq t_n\}.$$

The above shows that $P_m \subseteq P_n$ whenever $m \leq n$. For any $g = (x, y, z) \in G$,

$$\begin{aligned} t_n g^{-1} &= (n - x, n - y, n^2 - ny + xy - z), \\ g\, s_n^{-1} &= (n + x, n + y, 2n^2 + nx + z). \end{aligned}$$

Hence $g \in P_n$ precisely when

$$|x| \;\leq\; n, \qquad |y| \;\leq\; n, \qquad -2n^2 - nx \leq \; z \; \leq \; n^2 - ny + xy.$$

This shows that P_n is finite and that $g \in P_n$ provided $n \geq (|x|+1)(|y|+1)+|z|$. Therefore, the chain $\{P_n\}_{n\geq 1}$ is a covering of G by finite convex subsets. The author is unable to determine the structure of the posets P_n.

1.8 The case of free groups

Recall that if G is amenable, then it is enough to find an upward-directed covering of G by finite convex subsets in order to apply Theorem 1.6.2. Free groups provide the simplest non-amenable groups where a covering by particularly nice convex subsets can be found. For a fixed positive integer $N \geq 2$, $\mathbb{F}_N$ denotes the free group of rank N, with free generators $E_N = \{a_j\}_{1\leq j\leq N}$. As is well-known, $\mathbb{F}_N$ is non-amenable for $N \geq 2$. Let $\mathbb{F}_N^+$ denote the unital subsemigroup of $\mathbb{F}_N$ generated by E_N. In other words, $\mathbb{F}_N^+$ is the free unital semigroup of rank N, consisting of all words over the alphabet E_N. We shall consider the partially ordered group $(\mathbb{F}_N, \mathbb{F}_N^+)$. Thus, for any $v, w \in \mathbb{F}_N$, $v \leq w$ if and only if $w = uv$ for some $u \in \mathbb{F}_N^+$. It is an elementary fact that the Hasse diagram of the poset $(\mathbb{F}_N, \leq)$ is a tree. (For any reduced word $w \in \mathbb{F}_N$, the unique path from 1 to w can be read off from w itself.)

We shall apply a result of Davidson, Paulsen and Power concerning the class of finite-dimensional tree algebras. A *tree algebra* is a diagraph algebra $\mathbb{A}(P)$ over a poset P whose Hasse diagram is a tree. We state Theorem 4.2 of [12] in the following equivalent form:

Theorem 1.8.1 ([12])**.** *If $\mathbb{A}(P)$ is a finite-dimensional tree algebra, then*

$$\mathbb{A}(P) \otimes_{\max} A(\mathbb{D}) \;=\; \mathbb{A}(P) \otimes_{\min} A(\mathbb{D}).$$

Equivalently, if $\mathbb{A}(P)$ is a finite-dimensional tree algebra, then

$$\mathbb{A}(P) \otimes_{\max} \mathcal{T}(n) \;=\; \mathbb{A}(P) \otimes_{\min} \mathcal{T}(n) \qquad \text{for all } \; n \geq 2.$$

If $w \in \mathbb{F}_N$ is a reduced word, we denote the length of w by $|w|$. For each integer $n \geq 0$, consider

$$P_n \;:=\; \{w \in \mathbb{F}_N : |w| \leq n\}.$$

Clearly, $\{P_n\}_{n\geq 0}$ is an increasing chain of finite subtrees whose union is $\mathbb{F}_N$. It follows from Theorem 1.8.1 and the subsequent remark that $\mathbb{A}(P_k) \otimes_{\max} \mathcal{T}(2) = \mathbb{A}(P_k) \otimes_{\min} \mathcal{T}(2)$ for every k. Thus, with $\mathcal{A} = \mathcal{T}(2)$, part of the hypothesis of Theorem 1.6.2 is fulfilled.

In connection to the hypothesis (i) of Theorem 1.6.2, we point out that when $N \geq 2$, the homomorphism $\varepsilon : \mathsf{alg}(\mathbb{F}_N^+) \to \mathbf{C}$ for which $\varepsilon(\lambda_s) = 1$ for all $s \in \mathbb{F}_N^+$ is not bounded, let alone completely contractive. We thank K. R. Davidson for showing us the following argument. It applies norm estimates of operators in $C_r^*(\mathbb{F}_N)$ due to Haagerup [15]. For each integer $m \geq 1$, let A_m denote the set of all words in $\mathbb{F}_N^+$ of length m, so that $|A_m| = N^m$. Let $x_m = \sum_{w\in A_m} \lambda_w$. Then by Lemma 1.4 of [15], $\|x_m\| \leq (m+1)N^{m/2}$ whereas $\varepsilon(x_m) = N^m$. It follows that

$$\lim_{m\to\infty} \frac{\varepsilon(x_m)}{\|x_m\|} \;=\; \infty.$$

We shall provide partial evidence to support the possibility that

$$\mathsf{alg}(\mathbb{F}_2^+) \otimes_{\max} \mathcal{T}(2) \;\neq\; \mathsf{alg}(\mathbb{F}_2^+) \otimes_{\min} \mathcal{T}(2).$$

Let a and b denote the free generators of $\mathbb{F}_2$. The subspace $\ell_2(\mathbb{F}_2^+)$ of $\ell_2(\mathbb{F}_2)$ is clearly invariant under λ_a, λ_b, $\rho_{a^{-1}}$ and $\rho_{b^{-1}}$. Define $R_a = \rho_{a^{-1}}|_{\ell_2(\mathbb{F}_2^+)}$ and $R_b = \rho_{b^{-1}}|_{\ell_2(\mathbb{F}_2^+)}$. Since R_a and R_b are isometries having orthogonal ranges, the operator

$$T \;:=\; \frac{1}{\sqrt{2}}(R_a + R_b)$$

is an isometry. Let $L : \mathsf{alg}(\mathbb{F}_2^+) \to \mathbb{B}(\ell_2(\mathbb{F}_2^+))$ be the representation obtained by restriction of $\mathsf{alg}(\mathbb{F}_2^+) \subset C_r^*(\mathbb{F}_2)$ to $\ell_2(\mathbb{F}_2^+)$. Clearly, T commutes with L. Also, the identity representation of $C_r^*(\mathbb{F}_2)$ is a minimal $*$-dilation of L. If we could show that, for any pair of

$*$-dilations $\pi_j : C_r^*(\mathbb{F}_2) \to \mathbb{B}(\mathcal{K}_j)$ $(j = 1, 2)$ of L, there is no unitary operator $U : \mathcal{K}_2 \to \mathcal{K}_1$ such that $\pi_1(x)U = U\pi_2(x)$ and $P_{\mathcal{H}_1}\pi_1(x)U|_{\mathcal{H}_2} = L(x)T$ for all $x \in C_2^*(\mathbb{F}_2)$, then Proposition 1.2.10 would imply that $\mathsf{alg}(\mathbb{F}_2^+) \otimes_{\max} \mathcal{T}(2)$ and $\mathsf{alg}(\mathbb{F}_2^+) \otimes_{\min} \mathcal{T}(2)$ are not completely isometric. So far, we are unable to demonstrate this. On the other hand, it is not hard to show that T does not dilate to a contraction $\widetilde{T}$ on $\ell_2(\mathbb{F}_2)$ that commutes with $C_\lambda^*(\mathbb{F}_2)$. Suppose that such $\widetilde{T}$ exists. Since T is an isometry and $\widetilde{T}$ is a contraction, $\widetilde{T}$ is an extension of T. Since $\widetilde{T}$ commutes with λ_w for any $w \in \mathbb{F}_2$, we see that

$$\widetilde{T}(e_w) \ = \ \widetilde{T}\lambda_w(e_1) \ = \ \lambda_w\widetilde{T}(e_1) \ = \ \frac{1}{\sqrt{2}}\lambda_w(e_a + e_b) \ = \ \frac{1}{\sqrt{2}}(e_{wa} + e_{wb}),$$

which means that

$$\widetilde{T} \ = \ \frac{1}{\sqrt{2}}\big(\rho_{a^{-1}} + \rho_{b^{-1}}\big).$$

However, it turns out that

$$\|\rho_{a^{-1}} + \rho_{b^{-1}}\| \ = \ 2,$$

so that $\|\widetilde{T}\| = \sqrt{2} > 1$, contrary to assumption. For completeness, here we compute the norm of $S = \rho_{a^{-1}} + \rho_{b^{-1}}$. Clearly, $\|S\| \le 2$. For each $m \ge 1$, define the vector

$$\xi_m \ = \ \sum_{j=0}^{m} \big(e_{a^{-1}(ba^{-1})^j} + e_{b^{-1}(ab^{-1})^j}\big) \ \in \ \ell_2(\mathbb{F}_2).$$

Then

$$S(\xi_m) \ = \ 2e_1 + \sum_{j=1}^{m} 2\big(e_{(a^{-1}b)^j} + e_{(b^{-1}a)^j}\big) + e_{(a^{-1}b)^{m+1}} + e_{(b^{-1}a)^{m+1}}.$$

Hence

$$\frac{\|S(\xi_m)\|_2}{\|\xi_m\|_2} \ = \ \Big(\frac{8m+6}{2(m+1)}\Big)^{\frac{1}{2}} \to 2 \qquad \text{as } \ m \to \infty.$$

Chapter 2

Classification of Quaternion Subalgebras of the Real Sedenions

2.1 Introduction

The Cayley-Dickson doubling process generates a sequence of finite-dimensional algebras over $\mathbf{R}$:

$$\mathbf{A}_0 \subset \mathbf{A}_1 \subset \mathbf{A}_2 \subset \mathbf{A}_3 \subset \mathbf{A}_4 \subset \cdots .$$

For each $n \geq 0$, the Cayley-Dickson algebra $\mathbf{A}_n$ has dimension 2^n, is unital and carries an involution $x \mapsto \overline{x}$ such that $x + \overline{x}$ is a scalar and $x\overline{x} = \|x\|^2$ gives the Euclidean norm on the underlying vector space. Also, $\mathbf{A}_n$ is a unital involutive subalgebra of $\mathbf{A}_{n+1}$. The first four Cayley-Dickson algebras are the well-known division algebras $\mathbf{A}_0 := \mathbf{R}$, $\mathbf{A}_1 := \mathbf{C}$ (the field of complex numbers), $\mathbf{A}_2 := \mathbf{H}$ (the associative algebra of Hamilton *quaternions*) and $\mathbf{A}_3 := \mathbf{O}$ (the algebra of Cayley *octonions*). While not associative, $\mathbf{O}$ is *alternative* in the sense that $x^2y = x(xy)$ and $xy^2 = (xy)y$ for all $x, y \in \mathbf{O}$. These four algebras satisfy the equation

$$\|xy\| = \|x\| \, \|y\| \qquad (x, y \in \mathbf{A}_n, \; n \leq 3);$$

as such, they are sometimes called *composition algebras*. The next algebra $\mathbf{A}_4$ is known as *sedenions*. It departs from its better known predecessors in that $\mathbf{A}_4$ not only fails to be a composition algebra, it is also not a division algebra, i.e. there are non-zero elements $x, y \in \mathbf{A}_4$ such that $xy = 0$.

Let G denote the automorphism group of $\mathbf{A}_4$. Two subalgebras S and S' of $\mathbf{A}_4$ are said to be *G-conjugate* (or simply *conjugate*) if $S' = \varphi(S)$ for some $\varphi \in G$. The subalgebras of $\mathbf{A}_4$ were classified in [8] up to conjugacy, except for the single case of *quaternion subalgebras*, i.e. those subalgebras of $\mathbf{A}_4$ which are isomorphic to $\mathbf{H}$. An attempt is made in [7] to classify the quaternion subalgebras, but it contains some errors. This part of the thesis is devoted to presenting a corrected classification of quaternion subalgebras, thereby completing the work of [8].

2.1.1 Background on group actions

We provide here the basic terminology of transformation groups for convenience; see for instance [3] for details.

Let G be a compact topological group. If $\varphi_1, \ldots, \varphi_k \in G$, then $\langle \varphi_1, \ldots, \varphi_k \rangle$ denotes the subgroup they generate in G. Let X be a topological space which carries a continuous left G-action, i.e. X is a topological G-space. The G-stabilizer of a point $x \in X$ is the subgroup $G_x = \{g \in G : g \cdot x = x\}$, while its orbit is denoted by $Gx = [x]_G = \{g \cdot x : g \in G\}$. Two points x, x' in X are said to be G-conjugate, written $x \sim_G x'$, whenever they belong to the same orbit. The orbit space $X/G = \{Gx : x \in X\}$ carries the quotient topology induced by the canonical map $q : X \to X/G$ which sends $x \in X$ to its orbit Gx. If Y is another topological G-space, a continuous map $f : X \to Y$ is said to be G-equivariant (and f is said to be a G-map) if $f(g \cdot x) = g \cdot f(x)$ for all $x \in X$ and $g \in G$. We say that f is G-invariant if, instead, $f(x) = f(x')$ whenever $x \sim_G x'$. Note that if $f : X \to Y$ is a continuous surjection such that for any $x, x' \in G$, $f(x) = f(x')$ if and only if $x \sim_G x'$, then f induces a continuous bijection $\bar{f} : X/G \to Y$ such that $f = \bar{f} \circ q$. A subset R of X is called a set of representatives for X/G if the composite $R \hookrightarrow X \to X/G$ is a bijection. A cross section of a continuous surjection $f : X \to Y$ is a continuous map $s : Y \to X$ such that $f \circ s = \mathrm{id}_Y$. In this case, the subset $s(Y)$ of X is also referred to as a cross section of f.

For any closed subgroup K of the compact group G, the associated homogeneous space is $G/K = \{gK : g \in G\}$. If H is another closed subgroup of G which acts on G/K by left translation, then the resulting orbit space $(G/K)/H$ is canonically homeomorphic to the double coset space $H\backslash G/K$.

Let V be a real vector space. If $v_1, \dots, v_k \in V$, we denote their linear span by $\langle v_1, \dots, v_k \rangle$. (This should not cause confusion with the notation chosen for generated subgroups.) The projective space associated to V is $\mathbb{P}(V) = \{\langle v \rangle : v \in V - \{0\}\}$. (If V is a vector space over a field $\Bbbk$ other than $\mathbf{R}$, the corresponding projective space is denoted by $\mathbb{P}_{\Bbbk}(V)$.)

Let W be an n-dimensional real inner product space. The unit sphere of W is denoted by $\mathcal{S}(W) = \mathcal{S}^{n-1}$. For $1 \leq m \leq n$, the *Stiefel manifold* $V_m(W) = V_m(\mathbf{R}^n)$ of orthonormal m-frames is the closed set of m-tuples $(x_1, \dots, x_m) \subset \mathcal{S}(W)^m$ satisfying $(x_p|x_q) = \delta_{p,q}$ for all $1 \leq p, q \leq m$. The *Grassmannian* $\mathrm{Gr}_m(W) = \mathrm{Gr}_m(\mathbf{R}^n)$ is the set of m-dimensional subspaces of W. We give $\mathrm{Gr}_m(W)$ the quotient topology induced by the canonical surjection $V_m(W) \to \mathrm{Gr}_m(W)$ which sends $(x_1, \dots, x_m)$ to $\langle x_1, \dots, x_m \rangle$. Both $V_m(W)$ and $\mathrm{Gr}_m(W)$ are compact homogeneous spaces.

2.1.2 The algebra of sedenions $\mathbf{A}_4$

We begin by defining the real *Cayley-Dickson algebras* $\mathbf{A}_n$ for $n \geq 0$. Some of their basic properties are stated here without proof. As a real vector space, we may take $\mathbf{A}_n = \mathbf{R}^{2^n}$, the Euclidean space with standard orthonormal basis $\{e_i : 0 \leq i < 2^n\}$. Its algebra structure is obtained by iterating the *Cayley-Dickson process*. To begin, we endow the field of real numbers $\mathbf{A}_0 = \mathbf{R}$ with the trivial involution and the identity $e_0 = 1$. Given the involutive algebra $\mathbf{A}_n = \mathrm{span}\{e_i : 0 \leq i < 2^n\}$, we identify $\mathbf{A}_{n+1}$ with $\mathbf{A}_n \times \mathbf{A}_n$ and set $e_{i+2^n} = (0, e_i)$ for $0 \leq i < 2^n$. We can write $x, y \in \mathbf{A}_{n+1}$ as $x = (x_1, x_2)$ and $y = (y_1, y_2)$ where $x_1, x_2, y_1, y_2 \in \mathbf{A}_n$. Then the product and the involution in $\mathbf{A}_{n+1}$ are defined by

$$\begin{aligned} xy &:= (x_1 y_1 - \overline{y}_2 x_2, y_2 x_1 + x_2 \overline{y}_1), \\ \overline{x} &:= (\overline{x}_1, -x_2). \end{aligned}$$

In particular, $\overline{e_i} = -e_i$ and $e_i^2 = -1$ for $i > 0$, and $x = (x_1, x_2) \in \mathbf{A}_{n+1}$ can be written as $x = x_1 + x_2 e_{2^n}$. Hence $\mathbf{A}_{n+1} = \mathbf{A}_n + \mathbf{A}_n e_{2^n}$ contains $\mathbf{A}_n$ as a subalgebra, where $x \in \mathbf{A}_n$ is identified with $(x, 0) \in \mathbf{A}_{n+1}$. We shall write $\mathbf{A}_0 = \mathbf{R}$, $\mathbf{A}_1 = \mathbf{C}$, $\mathbf{A}_2 = \mathbf{H}$ and $\mathbf{A}_3 = \mathbf{O}$ in the sequel.

There is a close connection between the algebraic and geometric structures on $\mathbf{A}_n$. Let $x = \sum_i \xi_i e_i$, $y = \sum_i \eta_i e_i \in \mathbf{A}_n$, where $\xi_i, \eta_i \in \mathbf{R}$. The *trace* of x is defined by

$$\mathrm{tr}(x) \;:=\; x + \overline{x} \;=\; 2\xi_0.$$

The standard inner product $(x|y)$ in $\mathbf{A}_n$ can be expressed as

$$(x|y) \;=\; \frac{1}{2}\mathrm{tr}(x\overline{y}) \;=\; \sum_i \xi_i \eta_i.$$

In particular, $\mathrm{tr}(x) = 2(x|1)$, and the square of the norm of x is given by

$$\|x\|^2 \;=\; x\overline{x} \;=\; \sum_i \xi_i^2.$$

Since every $x \in \mathbf{A}_n$ satisfies a quadratic equation over $\mathbf{R}$:

$$x^2 - \mathrm{tr}(x)x + \|x\|^2 \;=\; 0,$$

it follows that every algebra automorphism of $\mathbf{A}_n$ is an isometry and commutes with the involution. We say that $x \in \mathbf{A}_n$ is *pure* if $\mathrm{tr}(x) = 0$. The subspace of pure elements of $\mathbf{A}_n$ is denoted by

$$\mathbf{A}_n^{\mathrm{pu}} \;:=\; \mathbf{A}_n \cap \mathbf{R}^{\perp}.$$

Every pure unit vector a satisfies $a^2 = -1$, and the plane

$$\mathbf{C}_a \;:=\; \langle 1, a \rangle$$

is a 2-dimensional subalgebra isomorphic to $\mathbf{C}$. Thus, $\mathbf{A}_n$ is *power-associative* in the sense that the subalgebra generated by any single element is associative. It is worth noting that any subalgebra generated by two linearly independent pure octonions $x, y \in \mathbf{O}$ is isomorphic to $\mathbf{H}$.

All Cayley-Dickson algebras satisfy the *flexible law*

$$(xy)x \;=\; x(yx).$$

This implies that $(xy)\overline{x} = x(y\overline{x})$ for all $x, y \in \mathbf{A}_n$. In particular, if $\|x\| = 1$, then the well-defined linear map

$$\gamma_x(y) \; := \; xy\overline{x}$$

is called *conjugation* by x.

For any $x, y, z \in \mathbf{A}_n$,

$$(xy|z) \; = \; (y|\overline{x}z) \; = \; (x|z\overline{y}). \tag{2.1}$$

This can be seen as a consequence of the flexible law. It follows that the linear operators L_x and R_x induced by left and right multiplications by $x \in \mathbf{A}_n^{\mathrm{pu}}$ are skew-symmetric. Moreover, for any subalgebra S of $\mathbf{A}_n$, we have $SS^\perp \subseteq S^\perp$ and $S^\perp S \subseteq S^\perp$.

We shall use the following alternative notation for some of the standard basis vectors of $\mathbf{A}_4$:

$$i \; := \; e_1, \quad j \; := \; e_2, \quad k \; := \; e_3, \quad l \; := \; e_4, \quad e \; := \; e_8.$$

Hence $\mathbf{O} \; = \; \mathbf{H} + \mathbf{H}l$ and $\mathbf{A}_4 \; = \; \mathbf{O} + \mathbf{O}e$. For convenience, we restate the definition of multiplication in $\mathbf{A}_4$ as follows. For any $x, y \in \mathbf{O}$,

$$(xe)y \; = \; (x\overline{y})e, \quad x(ye) \; = \; (yx)e, \quad (xe)(ye) \; = \; -\overline{y}x. \tag{2.2}$$

The subalgebra $\mathbf{C}_e \; = \; \langle 1, e \rangle$ will play a distinguished role in the sequel. We reserve the familiar notation $|z| \; = \; \|z\|$ for $z \; \in \; \mathbf{C}_e$. An element $x \; = \; (x_1, x_2) \; \in \; \mathbf{A}_4$ is *doubly pure* if $x_1, x_2 \in \mathbf{O}^{\mathrm{pu}}$. The subspace of doubly pure elements of $\mathbf{A}_4$ is denoted by

$$\mathbf{A}_4^{\mathrm{pp}} \; := \; \mathbf{A}_4 \cap (\mathbf{C}_e)^\perp \; = \; \mathbf{O}^{\mathrm{pu}} + \mathbf{O}^{\mathrm{pu}}e.$$

For a subspace V of $\mathbf{A}_4$, we write

$$V^{\mathrm{pu}} \; := \; V \cap \mathbf{A}_4^{\mathrm{pu}}, \qquad V^{\mathrm{pp}} \; := \; V \cap \mathbf{A}_4^{\mathrm{pp}}.$$

Both $\mathbf{A}_4$ and $\mathbf{A}_4^{\mathrm{pp}}$ are $\mathbf{C}_e$-bimodules, i.e. for any $a \in \mathbf{A}_4$ and $z, w \in \mathbf{C}_e$,

$$(zw)a \; = \; z(wa), \quad (za)w \; = \; z(aw), \quad (az)w \; = \; a(zw).$$

Moreover, $az = \overline{z}a$ if $a \in \mathbf{A}_4^{\mathrm{pp}}$. We shall treat $\mathbf{A}_4$ as a right vector space over $\mathbf{C}_e$.

At the end of this section, we recall the notion of alternative elements and zero-divisors in $\mathbf{A}_4$, and record their simple geometric characterizations. These two types of elements, in a sense, lie on diametrically opposite sides of the spectrum among unit vectors in $\mathbf{A}_4^{\mathrm{pp}}$. (We refer to Remark 2.2.1 below for support of this claim.)

That much interest was drawn to $\mathbf{A}_4$ is partly because of the fact, proved by Moreno [19], that the set of pairs of normalized zero-divisors

$$\{(x,y) \in \mathbf{A}_4 \times \mathbf{A}_4 \; : \; \|x\| = \|y\| = 1, \; xy = 0\}$$

is diffeomorphic to the exceptional compact Lie group of type G_2. More precisely, the natural action of G_2 on this set is free and transitive.

An element $a \in \mathbf{A}_4$ is *alternative* (with respect to $\mathbf{A}_4$) if $a^2 x = a(ax)$ for all $x \in \mathbf{A}_4$.

Theorem 2.1.1. *(a) [Khalil & Yiu [18]] If $a \in \mathbf{A}_4$ is written as $a = z + x$, where $z \in \mathbf{C}_e$ and $x \in \mathbf{A}_4^{\mathrm{pp}}$, then a is alternative precisely when x is alternative. Moreover, a doubly pure element $x = (x_1, x_2) \in \mathbf{A}_4^{\mathrm{pp}}$ is alternative if and only if x_1 and x_2 are linearly dependent.*

(b) [Moreno [19]] A non-zero element $x = (x_1, x_2) \in \mathbf{A}_4$ is a zero-divisor if and only if $x \in \mathbf{A}_4^{\mathrm{pp}}$, $\|x_1\| = \|x_2\|$, and $(x_1|x_2) = 0$.

It is significant that $\mathbf{C}_e$ consists of alternative elements. Also, it is clear from (a) that a doubly pure element x is alternative if and only if $x = az$ for some $a \in \mathbf{O}^{\mathrm{pu}}$ and $z \in \mathbf{C}_e$. As for (b) above, if $x = (x_1, x_2)$ where x_1 and x_2 are orthogonal pure unit vectors of $\mathbf{O}$, note that $H = \langle 1, x_1, x_2, x_1x_2 \rangle$ is a quaternion subalgebra of $\mathbf{O}$. Then

$$\ker(L_x) \;=\; \{\big(v, -(x_1x_2)v\big) \; : \; v \in \mathbf{O} \cap H^{\perp}\}.$$

2.1.3 Automorphisms of $\mathbf{A}_4$

The automorphism groups of $\mathbf{A}_n$ for $n \leq 3$ are much studied. It is known that $\mathrm{Aut}(\mathbf{H})$ is isomorphic to $\mathrm{SO}(3)$, and $\mathrm{Aut}(\mathbf{O})$ is the exceptional simple compact Lie group of type G_2. For simplicity, we shall write $K := \mathrm{Aut}(\mathbf{H})$, $G_2 := \mathrm{Aut}(\mathbf{O})$ and $G := \mathrm{Aut}(\mathbf{A}_4)$.

For each $n \geq 1$, any automorphism φ of $\mathbf{A}_n$ induces a automorphism $\bar{\varphi}$ of $\mathbf{A}_{n+1}$ which sends $x = (x_1, x_2) \in \mathbf{A}_{n+1}$ to $(\varphi(x_1), \varphi(x_2))$, i.e.

$$\bar{\varphi}(x_1 + x_2 e_{2^n}) \;:=\; \varphi(x_1) + \varphi(x_2) e_{2^n}.$$

In other words, $\bar{\varphi}$ extends φ and fixes e_{2^n}. By identifying $\bar{\varphi}$ with φ, $\mathrm{Aut}(\mathbf{A}_n)$ is embedded into $\mathrm{Aut}(\mathbf{A}_{n+1})$ as the subgroup

$$\mathrm{Aut}(\mathbf{A}_n) \;=\; \{\varphi \in \mathrm{Aut}(\mathbf{A}_{n+1}) \;:\; \varphi(\mathbf{A}_n) = \mathbf{A}_n,\ \varphi(e_{2^n}) = e_{2^n}\}.$$

In particular, we have the chain of subgroups $K \subset G_2 \subset G$.

Now let

$$\zeta \;:=\; \frac{1}{2}(-1 + \sqrt{3}e) \;\in\; \mathbf{C}_e,$$

so that $\zeta^3 = 1$. Any $a \in \mathbf{A}_4$ can be written as $a = z + b = x + ye$, where $z \in \mathbf{C}_e$, $b \in \mathbf{A}_4^{\mathrm{pp}}$, and $x, y \in \mathbf{O}$. Then the maps μ and τ on $\mathbf{A}_4$, defined by

$$\begin{aligned} \mu(a) &\;:=\; \zeta a \bar{\zeta} \;=\; z + b\zeta, \\ \tau(a) &\;:=\; x - ye, \end{aligned}$$

are automorphisms. The subgroup $\Sigma_3 := \langle \mu, \tau \rangle$ is isomorphic to the symmetric group on three symbols. A basic result of R. B. Brown [6] states that

$$G \;=\; G_2 \times \Sigma_3.$$

It follows that G leaves invariant both $\mathbf{C}_e$ and $\mathbf{A}_4^{\mathrm{pp}}$. As is well-known, for $n = 1, 2, 3$, $\mathrm{Aut}(\mathbf{A}_n)$ acts transitively on $\mathcal{S}(\mathbf{A}_n^{\mathrm{pu}})$, but we see that this is no longer the case for $n = 4$. Eakin and Sathaye [13] determined $\mathrm{Aut}(\mathbf{A}_n)$ for arbitrary n, thereby proving a conjecture of Brown. The book by Conway and Smith [10] delivers a fascinating discussion of the algebras $\mathbf{H}$ and $\mathbf{O}$ and their automorphism groups.

Remark 2.1.1. There is a unique homomorphism $\mathrm{sgn} : G \to \{\pm 1\}$ such that for any $\varphi \in G$,

$$\varphi(e) \;=\; \mathrm{sgn}(\varphi) e.$$

It follows from the structure of G that $\ker(\mathrm{sgn}) = G_2 \times \langle \mu \rangle$.

Remark 2.1.2. An orthonormal triple of pure octonions (a, b, c) such that $(c|ab) = 0$ is sometimes termed a *special triple*. (There seems to have no standard terminology in the literature.) It is known that any special triple (a, b, c) determines a unique automorphism φ of $\mathbf{O}$ such that $\varphi(i) = a$, $\varphi(j) = b$ and $\varphi(l) = c$. Hence, if H is any quaternion subalgebra of $\mathbf{O}$ and $c \in \mathbf{O}$ is any unit vector perpendicular to H, then $\mathbf{O} \cap H^{\perp} = Hc$, and multiplication in $\mathbf{O} = H + Hc$ can just as well be given by

$$(x_1 + y_1 c)(x_2 + y_2 c) \;=\; (x_1 x_2 - \overline{y_2} y_1) + (y_2 x_1 + y_1 \overline{x_2})c$$

for all $x_1, x_2, y_1, y_2 \in H$. Put another way, the group $G_2 = \mathrm{Aut}(\mathbf{O})$ acts freely and transitively on the set of special triples.

Remark 2.1.3. The group $K = \mathrm{Aut}(\mathbf{H})$ is no doubt well understood. Because of its importance in this work, we record for convenience some standard facts which are used frequently. First of all, the 3-sphere $\mathcal{S}(\mathbf{H})$ is a subgroup of the group of units of $\mathbf{H}$. Each $w \in \mathcal{S}(\mathbf{H})$ induces an automorphism $\gamma_w \in K$ by conjugation $x \mapsto wx\overline{w}$. The assignment $w \mapsto \gamma_w$ defines a homomorphism $\gamma : \mathcal{S}(\mathbf{H}) \to K$ with kernel $\{\pm 1\}$. Next, note that K acts faithfully by isometries on the Euclidean 3-space $\mathbf{H}^{\mathrm{pu}}$. Since multiplication on $\mathbf{H}$ involves a fixed orientation of $\mathbf{H}^{\mathrm{pu}}$, it follows that each $\gamma \in K$ acts as a rotation about a fixed line in $\mathbf{H}^{\mathrm{pu}}$. For each unit vector $u \in \mathbf{H}^{\mathrm{pu}}$, if $w \in \mathbf{C}_u$ is given by $w = \cos\theta + u \sin\theta$, then γ_w fixes u, while for each $x \in \mathbf{H}^{\mathrm{pu}} \cap \langle u \rangle^{\perp}$,

$$\gamma_w(x) \;=\; wx\overline{w} \;=\; w^2 x \;=\; x\cos(2\theta) + ux\sin(2\theta).$$

This means that γ_w is a rotation about u. Thus, K contains all rotations, so that $K \cong \mathrm{SO}(3)$. Surjectivity of γ indicates the short exact sequence

$$1 \;\to\; \{\pm 1\} \;\to\; \mathcal{S}(\mathbf{H}) \;\to\; K \;\to\; 1.$$

Moreover, the circle $\mathbf{T}_u := \mathcal{S}(\mathbf{C}_u)$ is a one-parameter subgroup of $\mathcal{S}(\mathbf{H})$. The K-stabilizer of u is $\gamma(\mathbf{T}_u)$, i.e.

$$K_u \;=\; \{\gamma_w \;:\; w \in \mathbf{T}_u\} \;\cong\; \mathrm{SO}(2).$$

Remark 2.1.4. Let $G_2(\mathbf{H})$ denote the G_2-stabilizer of $\mathbf{H}$. The following description of $G_2(\mathbf{H})$ is taken from [10]. Restriction to $\mathbf{H}$ gives an epimorphism $r : G_2(\mathbf{H}) \to \mathrm{Aut}(\mathbf{H})$ which is a left inverse of the embedding $\mathrm{Aut}(\mathbf{H}) \to K \subset G_2(\mathbf{H})$. Let $U := \ker(r)$, so that

$$G_2(\mathbf{H}) \;=\; U \rtimes K.$$

For each $v \in \mathcal{S}(\mathbf{H})$, the automorphism θ_v on $\mathbf{O}$ given by

$$\theta_v(x + yl) \;=\; x + y(vl) \;=\; x + (vy)l \qquad (x, y \in \mathbf{H})$$

belongs to U, and that the assignment $v \mapsto \theta_v$ defines a isomorphism $\theta : \mathcal{S}(\mathbf{H}) \to U$. See [10] for a proof that $G_2(\mathbf{H}) \cong \mathrm{SO}(4)$.

2.1.4 Types of quaternion subalgebras

The subalgebra generated by i, j and e is denoted by

$$\mathbf{O}_e \;:=\; \mathbf{H} + \mathbf{H}e.$$

(We remark that this algebra is named $\mathbf{O}_{i,j,e}$ in [8].) The algebra $\mathbf{O}_e$ is isomorphic but not conjugate to $\mathbf{O}$. As is well known, the G_2-action on orthonormal pairs in $\mathbf{O}^{\mathrm{pu}}$ is transitive. It follows easily that any $x \in \mathbf{A}_4$ is G_2-conjugate to an element of $\mathbf{O}_e$. This simple observation is subsumed by the following

Theorem 2.1.2. *[8, Thm 7.1] Every quaternion subalgebra of $\mathbf{A}_4$ is G_2-conjugate to a subalgebra of $\mathbf{O}_e$.*

Definition 2.1.1. We denote by

$$\mathscr{H} \;:=\; \{S \subset \mathbf{O}_e \;:\; S \cong \mathbf{H}\}$$

the set of quaternion subalgebras in $\mathbf{O}_e$. This set is partitioned as $\mathscr{H} = \mathscr{H}_{\mathrm{I}} \cup \mathscr{H}_{\mathrm{II}}$, where

$$\mathscr{H}_{\mathrm{I}} \;:=\; \{S \in \mathscr{H} \;:\; \dim S^{\mathrm{pp}} = 2\}, \qquad \mathscr{H}_{\mathrm{II}} \;:=\; \{S \in \mathscr{H} \;:\; \dim S^{\mathrm{pp}} = 3\}.$$

A subalgebra $S \in \mathscr{H}_{\mathrm{I}}$ (resp. $\mathscr{H}_{\mathrm{II}}$) is said to be of *type I* (resp. *type II*).

As mentioned earlier, any orthonormal pair of pure elements in an octonion algebra generates a quaternion subalgebra. Consider a 2-dimensional subspace $U = \langle v, w \rangle$ of $\mathbf{O}_e^{\mathrm{pp}}$ with orthonormal basis $\{v, w\}$. Let $a = vw$, and let $S = \langle 1, a, v, w \rangle = \mathbf{C}_a + U$ be the quaternion subalgebra generated by U. Then S is of type I if and only if $(a|e) \neq 0$. In this case, as $U = S^{\mathrm{pp}} = \mathbf{C}_a v$ and $\mathbf{C}_a = S \cap U^{\perp}$, the subalgebra $\mathbf{C}_a$ is in a sense 'rigid' under the action of automorphisms. Our technique for classifying type I subalgebras up to conjugacy is based on this simple observation. In contrast, due to the seemingly stronger symmetry within a type II subalgebra, the treatment of the type II case involves different methods.

Let $G(\mathbf{O}_e)$ denote the G-stabilizer of $\mathbf{O}_e$, which has the subgroup

$$\bar{K} \;:=\; K \times \Sigma_3.$$

It is not hard to see that

$$G(\mathbf{O}_e) \;=\; U \rtimes \bar{K},$$

where $U = \{\varphi \in G_2 : \varphi|_{\mathbf{H}} = \mathrm{id}\}$ is the kernel of the homomorphism $G(\mathbf{O}_e) \to \mathrm{Aut}(\mathbf{O}_e)$ sending φ to the restriction $\varphi|_{\mathbf{O}_e}$ (see also Remark 2.1.4).

Lemma 2.1.3. *Let $S, S' \in \mathscr{H}$. If $S \sim_G S'$, then $S \sim_{\bar{K}} S'$.*

Proof. Suppose that $S' = \varphi(S)$ where $\varphi \in G$. Write $\varphi = \nu\psi$, where $\nu \in \Sigma_3$ and $\psi \in G_2$. Take an orthonormal pair $x, y \in S^{\mathrm{pp}}$. Write $x = x_1 + x_2 e$ and $y = y_1 + y_2 e$, where $x_1, x_2, y_1, y_2 \in \mathbf{H}^{\mathrm{pu}}$. Note that ψ maps the subspace $V = \langle x_1, x_2, y_1, y_2 \rangle \subseteq \mathbf{H}^{\mathrm{pu}}$ into $\mathbf{H}^{\mathrm{pu}}$. If $\dim V = 2$, then V generates $\mathbf{H}$, and so $\psi \in G_2(\mathbf{H})$, so that there is a unique $\gamma \in K$ which agrees with ψ on V. If $\dim V = 1$, then by transitivity of $K \cong \mathrm{SO}(3)$ on $\mathbf{H}^{\mathrm{pu}}$, we can choose $\gamma \in K$ which agrees with ψ on V. In either case, $\varphi_1 = \nu\gamma \in \bar{K}$ agrees with φ on S, and hence $\varphi_1(S) = S'$. $\square$

Thus the classification of the G-conjugacy classes of quaternion subalgebras of $\mathbf{A}_4$ is reduced to the classification of $\bar{K}$-orbits of $\mathscr{H}_{\mathrm{I}}$ and $\mathscr{H}_{\mathrm{II}}$.

Remark 2.1.5. As an aside, we briefly mention the topology of $\mathscr{H}$. The correspondence $S \leftrightarrow S^{\mathrm{pu}}$ allows us to identify $\mathscr{H}$ with a closed (hence compact) subset of the Grassmannian $\mathrm{Gr}_3(\mathbf{O}_e^{\mathrm{pu}})$. For, if $V_2(\mathbf{O}_e^{\mathrm{pu}})$ is the Stiefel manifold of orthonormal 2-frames in $\mathbf{O}_e^{\mathrm{pu}}$, then the composite of continuous maps

$$V_2(\mathbf{O}_e^{\mathrm{pu}}) \ \to \ V_3(\mathbf{O}_e^{\mathrm{pu}}) \ \to \ \mathrm{Gr}_3(\mathbf{O}_e^{\mathrm{pu}})$$

given by

$$(x_1, x_2) \ \mapsto \ (x_1, x_2, x_1x_2) \ \mapsto \ \langle x_1, x_2, x_1x_2 \rangle$$

has range $\mathscr{H}$. A similar argument shows that $\mathscr{H}_{\mathrm{II}}$ is also closed. Indeed, a 2-frame (x_1, x_2) in $\mathbf{O}_e^{\mathrm{pu}}$ generates a type II subalgebra if and only if x_1, x_2 and x_1x_2 are all doubly pure. Define the subset

$$\begin{aligned} SV_2 \ &:= \ \{(x_1, x_2) \in V_2(\mathbf{O}_e^{\mathrm{pp}}) \ : \ x_1x_2 \in \mathbf{O}_e^{\mathrm{pp}}\} \\ &= \ \{(x_1, x_2) \in V_2(\mathbf{O}_e^{\mathrm{pp}}) \ : \ (x_1x_2|e) = 0\}, \end{aligned}$$

which is clearly closed. The continuous map $SV_2 \to \mathrm{Gr}_3(\mathbf{O}_e^{\mathrm{pp}})$ given by $(x_1, x_2) \mapsto \langle x_1, x_2, x_1x_2 \rangle$ has range $\mathscr{H}_{\mathrm{II}}$. As a consequence, $\mathscr{H}_{\mathrm{I}}$ is open in $\mathscr{H}$. It would be interesting to analyze the geometry of these sets.

2.1.5 Synopsis

We now outline the steps towards solving the classification problem. In Section 2.2 we describe the actions of certain subgroups of $\bar{K}$ on the unit spheres $\mathcal{S}^5 := \mathcal{S}(\mathbf{O}_e^{\mathrm{pp}})$ and $\mathcal{S}^6 := \mathcal{S}(\mathbf{O}_e^{\mathrm{pu}})$. Then we treat the $\bar{K}$-conjugacy classes of type I subalgebras in Section 2.3. In the type I case, the collection of those subalgebras which contain e is exceptional, and their classification is dealt with rather easily. A type I subalgebra S with $e \notin S$ decomposes into the orthogonal sum $S = \mathbf{C}_a + S^{\mathrm{pp}}$, where $a \in \mathbf{O}_e$ is a pure unit vector with $(a|e) > 0$. The subspace S^{pp}, being of the form $v\mathbf{C}_a$ for some $v \in S^{\mathrm{pp}}$, is treated as a point of a projective space $\mathbb{P}_a$ over the field $\mathbf{C}_a$ (see (2.14)). We first note (Lemma 2.3.1) that the action of $\bar{K}$ brings a into a unique element of a particular subset $\mathcal{B} \subset \mathbf{O}_e$. Let $\bar{K}(\mathbf{C}_a)$ be the $\bar{K}$-stabilizer of $\mathbf{C}_a$. Our problem reduces to classifying the $\bar{K}(\mathbf{C}_a)$-orbits of $\mathbb{P}_a$. Finally, we provide a set of representatives of type I subalgebras (Theorem 2.3.7).

In Section 2.4 we shall identify $\mathscr{H}_{\mathrm{II}}$ with the homogeneous space $\mathrm{SU}(3)/\mathrm{SO}(3)$ (see (2.21).) By transferring the $\bar{K}$-action on $\mathscr{H}_{\mathrm{II}}$ to $\mathrm{SU}(3)/\mathrm{SO}(3)$, we construct (Theorem 2.4.2) a bijection between $\mathscr{H}_{\mathrm{II}}/\bar{K}$ and a subset of the orbit space T/W, where T is the standard maximal torus of $\mathrm{SU}(3)$ and W is the Weyl group of $\mathrm{SU}(3)$ with respect to T.

2.2 Preliminaries

2.2.1 Complex structure of $\mathbf{A}_4$

We introduce a well-known $\mathbf{C}_e$-valued inner product on $\mathbf{A}_4$. For any $x, y \in \mathbf{A}_4$, let $\langle x|y\rangle$ denote the orthogonal projection of $\overline{x}y$ in $\mathbf{C}_e$. Explicitly,

$$\langle x|y\rangle \;\; := \;\; \mathrm{proj}_{\mathbf{C}_e}(\overline{x}y) \;\; = \;\; (x|y) + (xe|y)e. \tag{2.3}$$

The resulting form $\langle\cdot|\cdot\rangle : \mathbf{A}_4 \times \mathbf{A}_4 \to \mathbf{C}_e$ satisfies $\langle x|x\rangle = (x|x)$ and

$$\langle xz|yw\rangle \;=\; \overline{z}\,\langle x|y\rangle\, w$$

for $x, y \in \mathbf{A}_4$ and $z, w \in \mathbf{C}_e$, and can therefore be thought of as a complex inner product. Note that the subgroup $K \times \langle\mu\rangle$ acts as unitary operators on $\mathbf{A}_4$, while τ satisfies $\langle\tau(x)|\tau(y)\rangle = \overline{\langle x|y\rangle}$ for all $x, y \in \mathbf{A}_4^{\mathrm{pp}}$.

There exists a quadratic form $Q : \mathbf{A}_4 \to \mathbf{C}_e$ that provides a key invariant for the action of G_2. For $x = (x_1, x_2) \in \mathbf{A}_4$, $Q(x)$ is given by

$$Q(x) \;\; := \;\; \langle\tau(x)|x\rangle \;\; = \;\; \|x_1\|^2 - \|x_2\|^2 + 2(x_1|x_2)e, \tag{2.4}$$

Since G_2 acts by isometries on $\mathbf{O}$, (2.4) confirms that Q is G_2-invariant.

We are interested in the behaviour of Q on doubly pure unit vectors. Since the G_2-orbit of any element of $\mathbf{A}_4^{\mathrm{pp}}$ meets $\mathbf{O}_e^{\mathrm{pp}}$, there is no loss of information by treating Q as a function on the 5-sphere $\mathcal{S}^5 := \mathcal{S}(\mathbf{O}_e^{\mathrm{pp}})$. Let Δ stand for the closed unit disk in $\mathbf{C}_e$.

Lemma 2.2.1. *(a) The map $Q : \mathcal{S}^5 \to \Delta$ is surjective.*

(b) An element $x \in \mathcal{S}^5$ is alternative (resp. a zero-divisor) if and only if $|Q(x)| = 1$ (resp. $Q(x) = 0$).

Proof. (a) Let $x = (x_1, x_2) \in \mathcal{S}^5$. Note that

$$|Q(x)|^2 \;=\; 1 - 4(\|x_1\|^2\,\|x_2\|^2 - (x_1|x_2)^2). \tag{2.5}$$

It follows from the Cauchy-Schwarz inequality that $Q(x) \in \Delta$. Next, consider $x_\theta = i\cos\theta + je\sin\theta \in \mathcal{S}^5$. Then $Q(x_\theta) = \cos 2\theta$. Since $Q(x_\theta z) = Q(x_\theta)z^2$ for $z \in \mathbf{C}_e$, $Q(x_\theta z)$ covers the whole disk Δ as z runs through the unit circle of $\mathbf{C}_e$ while θ runs through $[0, \pi/4]$.

(b) By (2.5), we see that $|Q(x)| = 1$ precisely when x_1 and x_2 are linearly dependent. The condition $Q(x) = 0$ means that $\|x_1\| = \|x_2\|$ and $(x_1|x_2) = 0$. It remains to recall the comments made in Section 2.1.2. □

Next, we examine the transformation of Q under the action of Σ_3. For $u \in \mathbf{A}_4^{\mathrm{pp}}$, we have

$$Q(\mu(u)) \;=\; Q(u\zeta) \;=\; Q(u)\zeta^2, \qquad Q(\tau(u)) \;=\; \overline{Q(u)}. \tag{2.6}$$

This suggests that we impose a Σ_3-action on Δ such that $Q : \mathcal{S}^5 \to \Delta$ is Σ_3-equivariant; namely, for $z \in \mathbf{C}_e$, we set

$$\mu \cdot z \;:=\; z\zeta^2, \qquad \tau \cdot z \;:=\; \bar{z}.$$

Let (r, θ) be the usual polar coordinates of $z_{r,\theta} := r(\cos\theta + e\sin\theta) \in \mathbf{C}_e$. Then the sector Λ of Δ given by

$$\Lambda \;:=\; \{z_{r,\theta} \in \Delta \;:\; 0 \le \theta \le \pi/3\} \tag{2.7}$$

is a fundamental region for this Σ_3-action, in the sense that Λ is a connected cross section of Δ/Σ_3.

For $x, y \in \mathbf{A}_4^{\mathrm{pp}}$, let $x \times y := \mathrm{proj}_{\mathbf{A}_4^{\mathrm{pp}}}(xy)$ stand for the projection of xy to $\mathbf{A}_4^{\mathrm{pp}}$. Since $\langle x|y\rangle = \mathrm{proj}_{\mathbf{C}_e}(\bar{x}y) = -\mathrm{proj}_{\mathbf{C}_e}(xy)$, we see that

$$x \times y \;=\; xy + \langle x|y\rangle.$$

The resulting operation $\times : \mathbf{A}_4^{\mathrm{pp}} \times \mathbf{A}_4^{\mathrm{pp}} \to \mathbf{A}_4^{\mathrm{pp}}$ is bilinear over $\mathbf{R}$. It is skew-symmetric, since $x \times x = 0$. It turns out that $\times$ is conjugate-linear in both variables:

$$(xz) \times (yw) \;=\; (x \times y)\bar{z}\,\overline{w} \qquad (x, y \in \mathbf{A}_4^{\mathrm{pp}},\; z, w \in \mathbf{C}_e). \tag{2.8}$$

To see this, note that by (2.2), we have

$$(xe)y \;=\; -(xy)e \;=\; -(x \times y - \langle x|y\rangle)e.$$

Taking projections to $\mathbf{A}_4^{\mathrm{pp}}$ gives $(xe) \times y = -(x \times y)e$. The general equality (2.8) follows from skew-symmetry and linearity over $\mathbf{R}$.

For $u, v, w \in \mathbf{A}_4^{\mathrm{pp}}$, define $\omega(u, v, w) \in \mathbf{C}_e$ by

$$\omega(u, v, w) \;:=\; \langle uv|w\rangle \;=\; \langle u \times v|w\rangle. \tag{2.9}$$

It is clear that ω is a $\mathbf{C}_e$-trilinear form. We point out that ω is alternating. For this, it suffices to show that $\omega(u, u, v) = \omega(u, v, v) = 0$ for all $u, v \in \mathbf{A}_4^{\mathrm{pp}}$. The first equality is obvious, as $\omega(u, u, v) = \langle u \times u|v\rangle = \langle 0|v\rangle = 0$. We prove the second equality by direct computation. By (2.2), $(ve)v = (v\bar{v})e = \|v\|^2 e$. Hence, using (2.1),

$$\begin{aligned}
\omega(u, v, v) &= \langle uv|v\rangle \\
&= (uv|v) + ((uv)e|v)e \\
&= -(u|v^2) + (u|(ve)v)e \\
&= 0 + 0e \;=\; 0.
\end{aligned}$$

This 3-form was used by Jacobson [16] in his study of octonion algebras over arbitrary fields and their automorphism groups.

Remark 2.2.1. Recall that a pure element x defines a skew-symmetric linear operator L_x on $\mathbf{A}_4$ given by left multiplication. A doubly pure unit vector x is alternative if and only if $L_x^2 = -I$. If x is not alternative, then the symmetric operator L_x^2 has the three eigenvalues -1 and $-(1 \pm \sqrt{1 - |Q(x)|^2})$. The eigenspace structure of L_x^2 is as follows. Write $x = (x_1, x_2)$, where $x_1, x_2 \in \mathbf{O}^{\mathrm{pu}}$. Let $\mathbf{H}_x$ be the quaternion subalgebra of $\mathbf{O}$ generated by x_1 and x_2, let

$$x_3 \;=\; \|x_1 \times x_2\|^{-1}\,(x_1 \times x_2) \;\in\; \mathbf{H}_x,$$

and let $\mathbf{O}_{x,e} = \mathbf{H}_x + \mathbf{H}_x e$ be the octonion subalgebra of $\mathbf{A}_4$ generated by x_1, x_2 and e. Then $\mathbf{O}_{x,e}$ is the eigenspace of L_x^2 corresponding to -1. The two eigenspaces corresponding to $-(1+\sqrt{1-|Q(x)|^2})$ and $-(1-\sqrt{1-|Q(x)|^2})$ are given respectively by

$$\begin{aligned} E_+ &= \{v + (x_3 v)e \;:\; v \in \mathbf{O} \cap \mathbf{H}_x^{\perp}\}, \\ E_- &= \{v - (x_3 v)e \;:\; v \in \mathbf{O} \cap \mathbf{H}_x^{\perp}\}. \end{aligned}$$

Both of these subspaces consist entirely of zero-divisors.

2.2.2 Actions of $\bar{K}$ on spheres

We study the actions of the group $\bar{K} = K \times \Sigma_3$ on the unit spheres $\mathcal{S}^5 = \mathcal{S}(\mathbf{O}_e^{\mathrm{pp}})$ and $\mathcal{S}^6 = \mathcal{S}(\mathbf{O}_e^{\mathrm{pu}})$, and this leads to the classification of $\bar{K}$-conjugacy classes of the 2-dimensional subalgebras of $\mathbf{O}_e$. To begin, recall that the surjection $Q : \mathcal{S}^5 \to \Delta$ given by (2.4) is a K-invariant Σ_3-map. The following result reformulates Theorem 5.1 (a) and Proposition 5.2 of [8]. We give their proofs for completeness.

Lemma 2.2.2. *(a) Two points $x, y \in \mathcal{S}^5$ are K-conjugate if and only if $Q(x) = Q(y)$. As a result, Q induces homeomorphisms $\mathcal{S}^5/K \cong \Delta$ and $\mathcal{S}^5/\bar{K} \cong \Lambda$.*

(b) The map $Q : \mathcal{S}^5 \to \Delta$ has a cross section $\sigma : \Delta \to \mathcal{S}^5$ whose image is a cap of the unit sphere of $\langle i + je, i - je, j + ie\rangle$.

Proof. (a) Write $x = (x_1, x_2)$ and $y = (y_1, y_2)$, where $x_1, x_2, y_1, y_2 \in \mathbf{H}^{\mathrm{pu}}$. The equality $Q(x) = Q(y)$ means that $\|x_a\| = \|y_a\|$ for $a = 1, 2$ and $(x_1|x_2) = (y_1|y_2)$, i.e. the ordered vector pairs (x_1, x_2) and (y_1, y_2) are congruent. Since K effectively acts on $\mathbf{H}^{\mathrm{pu}}$ as $\mathrm{SO}(3)$, the above condition is equivalent to the K-conjugacy of such pairs. This proves the first claim, which leads to the homeomorphism $\mathcal{S}^5/K \cong \Delta$. The second homeomorphism $\mathcal{S}^5/\bar{K} \cong \Lambda$ follows from the Σ_3-equivariance of Q.

(b) For $r \in [0,1]$, define the real numbers

$$\alpha_r \ := \ \sqrt{\frac{1+r}{2}}, \qquad \beta_r \ := \ \sqrt{\frac{1-r}{2}},$$

and the unit vector

$$u_r \ := \ \alpha_r i + \beta_r je.$$

Then $Q(u_r) = r$. The K-stabilizer of the vector $k \in \mathbf{H}$ can be expressed as the one-parameter subgroup

$$K_k \ = \ \{\varphi_\theta \ : \ \theta \in \mathbf{R}\},$$

where φ_θ acts on $\mathbf{H}$ as conjugation by $w_\theta = \cos(\theta/4) + k\sin(\theta/4)$. Note that

$$\begin{aligned} \varphi_\theta(i) &= i\cos(\theta/2) + j\sin(\theta/2), \\ \varphi_\theta(j) &= -i\sin(\theta/2) + j\cos(\theta/2). \end{aligned}$$

Consider a point $z_{r,\theta} \in \Delta$ with polar coordinates (r,θ). We define $\sigma : \Delta \to \mathcal{S}^5$ by

$$\sigma(z_{r,\theta}) \ := \ \varphi_\theta(u_r)(\cos(\theta/2) + e\sin(\theta/2)). \tag{2.10}$$

It is easy to see that $Q \circ \sigma$ is the identity on Δ. Expansion of the right-hand side gives

$$\sigma(z_{r,\theta}) \ = \ \frac{\alpha_r + \beta_r}{2} \cdot (i + je) + \frac{\alpha_r - \beta_r}{2} \cdot \big[(i\cos\theta + j\sin\theta) + (i\sin\theta - j\cos\theta)e\big].$$

This also shows that σ is continuous on Δ. □

Next, we describe the orbit space $\mathcal{S}^6/\bar{K}$. Let $\mathcal{D}$ denote the suspension of Δ, namely, the quotient space of the cylinder $[-1,1] \times \Delta$ obtained by collapsing the top $\{1\} \times \Delta$ and bottom $\{-1\} \times \Delta$ to the two points $(1,0)$ and $(-1,0)$ respectively. We identify Δ with $\{0\} \times \Delta \subset \mathcal{D}$ in the obvious way. The Σ_3-action on Δ extends to $\mathcal{D}$ as follows. For any $\nu \in \Sigma_3$, $\alpha \in [-1,1]$ and $w \in \Delta$,

$$\nu \cdot (\alpha, w) \ := \ (\mathrm{sgn}(\nu)\alpha, \nu \cdot w). \tag{2.11}$$

It is evident that the subset

$$\mathcal{L} \ := \ \{(\alpha, w) \in \mathcal{D} \ : \ \alpha \in [-1, 1], \, w \in \Lambda\}$$

is a fundamental region for $\mathcal{D}$ under this action. We define the surjective Σ_3-map $f : \mathcal{S}^6 \to \mathcal{D}$ by

$$f(\alpha e + \sqrt{1-\alpha^2} u) \ = \ (\alpha, Q(u)), \tag{2.12}$$

where $u \in \mathcal{S}^5$ and $\alpha \in [-1, 1]$. It follows from Proposition 2.2.2 that f induces homeomorphisms $\mathcal{S}^6/K \to \mathcal{D}$ and $\mathcal{S}^6/\bar{K} \to \mathcal{L}$. Furthermore, the cross section $\sigma : \Delta \to \mathcal{S}^5$ readily extends to a cross section $\tilde{\sigma} : \mathcal{D} \to \mathcal{S}^6$ of f, defined by

$$\tilde{\sigma}(\alpha, w) \ := \ \alpha e + \sqrt{1-\alpha^2} \sigma(w) \tag{2.13}$$

for $\alpha \in [-1, 1]$ and $w \in \Delta$.

Using the above results, we provide a set of representatives of the $\bar{K}$-conjugacy classes of 2-dimensional subalgebras, which is needed later when classifying the type I quaternion subalgebras. Define

$$\mathcal{L}_+ \ := \ \{(\alpha, w) \in \mathcal{L} \ : \ \alpha \geq 0\}.$$

Lemma 2.2.3. *[8, Prop. 6.1] The subalgebras $\{\mathbf{C}_a : a \in \tilde{\sigma}(\mathcal{L}_+)\}$ form a set of representatives of the $\bar{K}$-orbits of 2-dimensional subalgebras of $\mathbf{O}_e$.*

Proof. Any 2-dimensional subalgebra of $\mathbf{O}_e$ is of the form $\mathbf{C}_a = \mathbf{C}_{-a}$, where $a \in \mathcal{S}^6$. As such, the collection $\mathscr{C}$ of 2-dimensional subalgebras of $\mathbf{O}_e$ is naturally identified with the real projective space $\mathbb{P}(\mathbf{O}_e^{\mathrm{pu}}) = \mathcal{S}^6/\mathbb{Z}_2$. The antipodal map $a \mapsto -a$ on $\mathcal{S}^6$ is obviously a $\bar{K}$-map. Likewise, the reflection $(\alpha, w) \mapsto (-\alpha, w)$ on $\mathcal{D}$ is a Σ_3-map. If we let the cyclic group $\mathbb{Z}_2 = \{1, \rho\}$ act on both $\mathcal{S}^6$ and $\mathcal{D}$ by $\rho \cdot a = -a$ and $\rho \cdot (\alpha, w) = (-\alpha, w)$ respectively, then it is easily checked that $f(\rho \cdot a) = \rho \cdot f(a)$ for all $a \in \mathcal{S}^6$. This means that the homeomorphism $\mathcal{S}^6/\bar{K} \to \mathcal{L}$ induced by f is $\mathbb{Z}_2$-equivariant. Therefore,

$$\mathscr{C}/\bar{K} \ = \ \mathbb{P}(\mathbf{O}_e^{\mathrm{pu}})/\bar{K} \ = \ (\mathcal{S}^6/\mathbb{Z}_2)/\bar{K} \ = \ (\mathcal{S}^6/\bar{K})/\mathbb{Z}_2 \ \cong \ \mathcal{L}/\mathbb{Z}_2 \ \cong \ \mathcal{L}_+.$$

The claim now follows from the fact that $\tilde{\sigma} : \mathcal{L} \to \mathcal{S}^6$ is a cross section of f. $\square$

2.3 Type I subalgebras

2.3.1 Construction of type I subalgebras

We begin by constructing type I quaternion subalgebras. Fix a pure unit vector $a \in \mathbf{O}_e^{\mathrm{pu}}$. We denote by $\mathbf{T}_a = \mathcal{S}(\mathbf{C}_a)$ the subgroup of unit vectors in $\mathbf{C}_a$. Let $\bar{K}(\mathbf{C}_a)$ denote the $\bar{K}$-stabilizer of $\mathbf{C}_a$. It is clear that $\bar{K}(\mathbf{C}_a) = \{\varphi \in \bar{K} : \varphi(a) = \{a, -a\}\}$.

We define two subspaces of $\mathbf{O}_e$:

$$H_a \ := \ \langle 1, e, a, ae \rangle \qquad \text{and} \qquad V_a \ := \ \mathbf{O}_e \cap H_a^{\perp}.$$

Clearly, $H_a = H_{-a}$ and $V_a = V_{-a}$. Some simple observations are in order:

(a) H_a is a quaternion subalgebra whenever $a \neq \pm e$. In addition, if $a = \alpha e + \sqrt{1-\alpha^2}u$ where $\alpha \in (-1, 1)$ and $u \in \mathcal{S}^5$, then $H_a = H_u$. On the other hand, $H_e = \mathbf{C}_e$.

(b) V_a is closed under multiplication on both sides by elements of H_a. We may treat V_a as a (left or right) vector space over $\mathbf{C}_a$. Then $\dim_{\mathbf{C}_a} V_a = 2$ if $a \neq \pm e$, and $V_e = \mathbf{O}_e^{\mathrm{pp}}$. Note that for any $v \in V_a$, because $av = -va$, it is clear that $v\mathbf{C}_a = \mathbf{C}_a v$. We define the projective space

$$\mathbb{P}_a \ := \ \mathbb{P}_{\mathbf{C}_a}(V_a) \ = \ \{\mathbf{C}_a v \ : \ v \in \mathcal{S}(V_a)\} \tag{2.14}$$

Note that $\mathbb{P}_a \cong \mathbb{P}^1(\mathbf{C})$ if $a \neq \pm e$, and $\mathbb{P}_e \cong \mathbb{P}^2(\mathbf{C})$. Clearly, both H_a and V_a are stable under $\bar{K}(\mathbf{C}_a)$. Hence, $\bar{K}(\mathbf{C}_a)$ naturally acts on $\mathbb{P}_a$.

(c) To each $\mathbf{C}_a v \in \mathbb{P}_a$, we associate the quaternion subalgebra

$$S_{\mathrm{I}}(\mathbf{C}_a v) \ := \ \mathbf{C}_a + \mathbf{C}_a v \ = \ \langle 1, a, v, av \rangle, \tag{2.15}$$

which is of type I if and only if a is not doubly pure.

(d) Conversely, any type I subalgebra S is of the form (2.15). For, given any orthonormal basis v, w of S^{pp}, one may choose $a = vw \in S^{\mathrm{pu}}$. Then $w = -va = av$, and hence $S = \langle 1, a, v, av \rangle = S_{\mathrm{I}}(\mathbf{C}_a v)$.

Define the following subsets of $\mathcal{S}^6$:

$$\mathcal{B} \ := \ \{a \in \tilde{\sigma}(\mathcal{L}_+) \ : \ (a|e) > 0\}, \qquad \mathcal{B}^* \ := \ \mathcal{B} - \{e\}. \tag{2.16}$$

The following lemma reduces the classification problem of type I subalgebras to one of figuring out the orbit space $\mathbb{P}_a/\bar{K}(\mathbf{C}_a)$ for each $a \in \mathcal{B}$. Therefore, the geometric object that preoccupies us is, loosely speaking, a 'bundle' of complex projective spaces having $\mathcal{B}$ as base space. Except for the point $e \in \mathcal{B}$ whose fibre is $\mathbb{P}_e \cong \mathbb{P}^2(\mathbf{C})$, the 'generic' fibres $\mathbb{P}_a$ based at $a \in \mathcal{B}^*$ are simply $\mathbb{P}^1(\mathbf{C}) \cong \mathcal{S}^2$. As we shall see, the structure of the group $\bar{K}(\mathbf{C}_a)$ varies. This is the major source of complication for the classification problem, which makes understanding the topology of the orbit space $\mathscr{H}_{\mathrm{I}}/\bar{K}$ much more difficult. We shall be content with constructing representatives of type I subalgebras by selecting a cross section of $\mathbb{P}_a/\bar{K}(\mathbf{C}_a)$ for each $a \in \mathcal{B}$. In contrast, the orbit space $\mathscr{H}_{\mathrm{II}}/\bar{K}$ of type II subalgebras is much better understood (see Theorem 2.4.2).

Lemma 2.3.1. *Every type I subalgebra S is $\bar{K}$-conjugate to a subalgebra of the form $S_{\mathrm{I}}(\mathbf{C}_a v)$ for a unique $a \in \mathcal{B}$ and some $\mathbf{C}_a v \in \mathbb{P}_a$. Moreover, for any $\mathbf{C}_a v, \mathbf{C}_a w \in \mathbb{P}_a$ and $\varphi \in \bar{K}$, if $\varphi\big(S_{\mathrm{I}}(\mathbf{C}_a v)\big) = S_{\mathrm{I}}(\mathbf{C}_a w)$, then $\varphi \in \bar{K}(\mathbf{C}_a)$ and $\varphi(\mathbf{C}_a v) = \mathbf{C}_a w$.*

Proof. We write $S = \mathbf{C}_{a'} + \mathbf{C}_{a'} v$, where $a' \in \mathcal{S}^6 - \mathcal{S}^5$. By Lemma 2.2.3, there exists $\varphi \in \bar{K}$ such that $\varphi(\mathbf{C}_{a'}) = \mathbf{C}_a$ for a unique $a \in \mathcal{B}$. This proves the first claim.

Next, let $\mathbf{C}_a v, \mathbf{C}_a w \in \mathbb{P}_a$ and $\varphi \in \bar{K}$, and write $S_1 = S_{\mathrm{I}}(\mathbf{C}_a v)$ and $S_2 = S_{\mathrm{I}}(\mathbf{C}_a w)$. Suppose that $\varphi(S_1) = S_2$. Since $\mathbf{C}_a v = S_1^{\mathrm{pp}}$ and $\mathbf{C}_a w = S_2^{\mathrm{pp}}$, it follows that $\mathbf{C}_a = S_j \cap (S_j^{\mathrm{pp}})^\perp$ for $j = 1, 2$. Hence $\varphi(\mathbf{C}_a) = \mathbf{C}_a$ and $\varphi(\mathbf{C}_a v) = \mathbf{C}_a w$. In particular, $\varphi \in \bar{K}(\mathbf{C}_a)$, proving the second claim. $\square$

2.3.2 Type I subalgebras containing e

We begin our study of the $\bar{K}(\mathbf{C}_a)$-action on $\mathbb{P}_a$ for each $a \in \mathcal{B}$. For convenience, we first consider the simplest, 'exceptional' case $a = e$. Clearly, $\bar{K}(\mathbf{C}_e) = \bar{K}$.

Proposition 2.3.2. *Two points $\mathbf{C}_e v, \mathbf{C}_e v' \in \mathbb{P}_e$ (where $v, v' \in \mathcal{S}^5$) are $\bar{K}$-conjugate if and only if $|Q(v)| = |Q(v')|$. Consequently, the family*

$$\mathcal{R}_e \;:=\; \{S_{\mathrm{I}}(\mathbf{C}_e v) \;:\; v = i\cos\theta + je\sin\theta,\; 0 \le \theta \le \pi/4\}$$

is a set of representatives of the conjugacy classes of type I subalgebras containing e.

Proof. First, recall that the quadratic form Q restricts to a Σ_3-equivariant, K-invariant surjection $Q : \mathcal{S}^5 \to \Delta$. It follows that the function $\mathbb{P}_e \to [0,1]$, $\mathbf{C}_e v \mapsto |Q(v)|$, is well-defined, surjective and invariant under $\bar{K}$.

Next, we show that this function distinguishes the $\bar{K}$-orbits in $\mathbb{P}_e$. First, suppose that $\varphi(\mathbf{C}_e v) = \mathbf{C}_e v'$ for some $\varphi \in \bar{K}$. Then $\varphi(v) = v'z$ for some $z \in \mathbf{T}_e$, and it is easy to see that $|Q(v)| = |Q(v')|$. Conversely, if $|Q(v)| = |Q(v')|$, then there exists $z \in \mathbf{T}_e$ so that $Q(v) = Q(v')z^2 = Q(v'z)$. Hence v and $v'z$ are K-conjugate by Lemma 2.2.2 (a). This implies that $\mathbf{C}_e v$ and $\mathbf{C}_e v'$ are K-conjugate.

Finally, Let $v_\theta = i\cos\theta + je\sin\theta$, where $\theta \in [0, \pi/4]$. Then $Q(v_\theta) = \cos(2\theta)$ takes all possible values in $[0,1]$. By the preceding paragraph, we are done. □

2.3.3 The group $\bar{K}(\mathbf{C}_a)$ for $a \in \mathcal{B}^*$

To better study the 'generic case' of those type I subalgebras, i.e. those which do not contain e, we need to understand the group $\bar{K}(\mathbf{C}_a)$ for $a \in \mathcal{B}^*$. Throughout this section, we shall write

$$a \;=\; \alpha e + \sqrt{1-\alpha^2}u,$$

for unique $\alpha \in (0,1)$ and $u \in \sigma(\Lambda)$. Note that for any $\varphi \in \bar{K}$, $\varphi(a) = a$ if and only if $\varphi(e) = e$ and $\varphi(u) = u$; similarly, $\varphi(a) = -a$ if and only if $\varphi(e) = -e$ and $\varphi(u) = -u$. It follows that the following statements are equivalent:

(a) $\varphi \in \bar{K}(\mathbf{C}_a)$.

(b) $\varphi(a) = \mathrm{sgn}(\varphi)a$.

(c) $\varphi(u) = \operatorname{sgn}(\varphi)u$.

(d) $\varphi(ue) = ue$.

Clearly, (d) means that $\bar{K}(\mathbf{C}_a)$ is just the $\bar{K}$-stabilizer of ue. However, we shall find conditions (b) and (c) more useful below.

Let K_u be the K-stabilizer of u. It is clear that $K_u \subset \bar{K}(\mathbf{C}_a)$. Write $u = u_1 + u_2 e$, where $u_1, u_2 \in \mathbf{H}^{\mathrm{pu}}$. Note that an element $\gamma \in K$ fixes u precisely when γ fixes both u_1 and u_2. If u is not alternative, then u_1 and u_2 generate $\mathbf{H}$ as a subalgebra, which forces $\gamma = \mathrm{id}$. Hence, $K_u = \{\mathrm{id}\}$ if u is not alternative. On the other hand, if u is alternative, then $u = bz$ where $b \in \mathcal{S}^5$ and $z \in \mathbf{T}_e$. In this case, $K_u = K_b = \{\gamma_w : w \in \mathbf{T}_b\} \cong \mathrm{SO}(2)$.

Recall that Σ_3 acts on the closed unit disk Δ and that $Q : \mathcal{S}^5 \to \Delta$ is Σ_3-equivariant. Let $(\Sigma_3)_w$ denote the Σ_3-stabilizer of $w \in \Delta$.

Lemma 2.3.3. *(a) Let $u \in \mathcal{S}^5$. The group K_u is given as follows:*

i) If $|Q(u)| < 1$, then $K_u = \{\mathrm{id}\}$.

ii) If $|Q(u)| = 1$, in which case $u = bz$ where $b \in \mathcal{S}(\mathbf{H}^{\mathrm{pu}})$ and $z \in \mathbf{T}_e$, then

$$K_u \;=\; K_b \;\cong\; \mathrm{SO}(2).$$

(b) Let $w \in \Delta$. The group $(\Sigma_3)_w$ is given as follows:

i) If $w^3 \notin \mathbf{R}$, then $(\Sigma_3)_w = \{\mathrm{id}\}$.

ii) If $w^3 \in \mathbf{R} - \{0\}$, then $(\Sigma_3)_w$ is one of the three conjugates of $\langle \tau \rangle$ in Σ_3. In fact, if $w \in \mathbf{R}\zeta^s - \{0\}$ for some $s \in \{0,1,2\}$, then $(\Sigma_3)_w = \langle \mu^s \tau \rangle$.

iii) If $w = 0$, then $(\Sigma_3)_w = \Sigma_3$.

Proof. We have already proved (a); recall that u is alternative if and only if $|Q(u)| = 1$. As for (b), the claims are obvious in view of the Σ_3-action on Δ. We point out that the condition $w^3 \in \mathbf{R} - \{0\}$ means that $w \in \mathbf{R}\zeta^s - \{0\}$ for some $s \in \{0,1,2\}$, in which case $(\Sigma_3)_w = \langle \mu^s \tau \rangle$. $\square$

Suppose that $\varphi \in \bar{K}(\mathbf{C}_a)$ is given by $\varphi = \gamma\nu$, where $\gamma \in K$ and $\nu \in \Sigma_3$. Then $\varphi(u) = \operatorname{sgn}(\varphi)u$, so that

$$\nu \cdot Q(u) \;=\; Q(\nu(u)) \;=\; Q(\varphi(u)) \;=\; Q\left(\operatorname{sgn}(\varphi)u\right) \;=\; Q(u),$$

that is, $\nu \in (\Sigma_3)_{Q(u)}$. Thus, the projection homomorphism from $\bar{K} = K \times \Sigma_3$ to Σ_3 restricts to a homomorphism $\eta : \bar{K}(\mathbf{C}_a) \to (\Sigma_3)_{Q(u)}$. It is easy to see that $\ker(\eta) = K_u$. To see that η is surjective, note that for any $\nu \in (\Sigma_3)_{Q(u)}$, we have

$$Q\big(\nu(u)\big) \;=\; \nu \cdot Q(u) \;=\; Q(u) \;=\; Q\big(\operatorname{sgn}(\nu)u\big),$$

so that $\nu(u) \sim_K \operatorname{sgn}(\nu)u$. Take any $\gamma \in K$ such that $\gamma\nu(u) = \operatorname{sgn}(\nu)u$. Setting $\varphi = \gamma\nu$, we see that $\varphi \in \bar{K}(\mathbf{C}_a)$ and $\eta(\varphi) = \nu$. In summary, there is a short exact sequence of groups

$$1 \;\to\; K_u \;\to\; \bar{K}(\mathbf{C}_a) \;\xrightarrow{\eta}\; (\Sigma_3)_{Q(u)} \;\to\; 1. \tag{2.17}$$

We are ready to compute $\bar{K}(\mathbf{C}_a)$.

Proposition 2.3.4. *Let $a \in \mathcal{B}^*$ be given by $\alpha e + \sqrt{1-\alpha^2}u$, where $\alpha \in (0,1)$ and $u \in \sigma(\Lambda)$.*

(a) If $|Q(u)| < 1$, then $\eta : \bar{K}(\mathbf{C}_a) \to (\Sigma_3)_{Q(u)}$ is an isomorphism. Moreover, for each $\nu \in (\Sigma_3)_{Q(u)}$, the element $\nu' = \eta^{-1}(\nu) \in \bar{K}(\mathbf{C}_a)$ satisfies $\nu'(k) = \pm\nu(k)$. In particular:

- *i) If $Q(u)^3 \notin \mathbf{R}$, then $\bar{K}(\mathbf{C}_a) = \{\mathrm{id}\}$.*
- *ii) If $Q(u)^3 \in (0,1)$, then $\bar{K}(\mathbf{C}_a) = \langle \tau' \rangle \cong \mathbb{Z}_2$.*
- *iii) If $Q(u)^3 \in (-1,0)$, then $\bar{K}(\mathbf{C}_a) = \langle (\mu^2\tau)' \rangle \cong \mathbb{Z}_2$.*
- *iv) If $Q(u) = 0$, then $\bar{K}(\mathbf{C}_a) = \langle \mu', \tau' \rangle \cong \Sigma_3$.*

(b) Suppose that $|Q(u)| = 1$. In ii) below, $\gamma_k \in K$ denotes conjugation by k.

- *i) If $Q(u)^3 \notin \{\pm 1\}$, then $\bar{K}(\mathbf{C}_a) = K_u$.*
- *ii) If $Q(u)^3 = 1$, then $\bar{K}(\mathbf{C}_a) = K_u \rtimes \langle \gamma_k\tau \rangle \cong \mathrm{O}(2)$.*
- *iii) If $Q(u)^3 = -1$, then $\bar{K}(\mathbf{C}_a) = K_u \times \langle \mu^2\tau \rangle \cong \mathrm{SO}(2) \times \mathbb{Z}_2$.*

Proof. As preparation, recall from Lemma 2.2.2 (b) that

$$u \in \sigma(\Lambda) \subset \langle i+je, i-je, j+ie\rangle \subset H_k^\perp = \langle i,j\rangle + \langle i,j\rangle e,$$

where $H_k = \langle 1, e, k, ke\rangle$. Note that H_k and $H_k^\perp$ are invariant under Σ_3. We shall write $u = u_1 + u_2 e$, where $u_1, u_2 \in \langle i, j\rangle$.

(a) Suppose that $|Q(u)| < 1$. Since $K_u = \{\mathrm{id}\}$ by Lemma 2.3.3 (a), it follows from (2.17) that η is an isomorphism. Since $Q(u) \in \Lambda$, if $Q(u)^3 \in (0,1)$, then $Q(u) \in (0,1)$, whereas if $Q(u)^3 \in (-1,0)$, then $Q(u) \in \{-r\zeta^2 : r \in (0,1)\}$. We may then use Lemma 2.3.3 (b) to determine $(\Sigma_3)_{Q(u)}$ and subsequently $\bar{K}(\mathbf{C}_a)$. It remains to consider the action of $\bar{K}(\mathbf{C}_a)$ at k. For fixed $\nu \in (\Sigma_3)_{Q(u)}$, the element $\nu' = \eta^{-1}(\nu) \in \bar{K}(\mathbf{C}_a)$ has the form $\nu' = \gamma\nu$ for a unique $\gamma \in K$. To show that $\nu'(k) = \pm\nu(k)$, it suffices to show that $\gamma(k) = \pm k$. Since $\gamma\nu(u) = \mathrm{sgn}(\nu)u$, we have $\gamma^{-1}(u) = \mathrm{sgn}(\nu)\nu(u) \in H_k^\perp$. In other words, $\gamma^{-1}(u_1), \gamma^{-1}(u_2) \in \langle i, j\rangle$. Since $|Q(u)| < 1$ means that u_1 and u_2 form a basis of $\langle i, j\rangle$, this implies that $\langle i, j\rangle$ is invariant under γ. Hence $\langle k\rangle = \mathbf{H}^{\mathrm{pu}} \cap \langle i, j\rangle^\perp$ is also invariant under γ, which means that $\gamma(k) = \pm k$.

(b) Suppose that $|Q(u)| = 1$. If $Q(u)^3 \notin \{\pm 1\}$, then from Lemma 2.3.3 (b), we have $(\Sigma_3)_{Q(u)} = \{\mathrm{id}\}$, and we obtain $\bar{K}(\mathbf{C}_a) = K_u = K_b$ from (2.17). This proves Case i).

For the remaining cases where $Q(u)^3 = \pm 1$, we write $u = bz$, where $b \in \langle i, j\rangle$ and $z \in \mathbf{T}_e$. Then $(\Sigma_3)_{Q(u)} = \langle \nu\rangle \cong \mathbb{Z}_2$, where $\nu = \mu^s\tau$ for some $s \in \{0,1,2\}$. (See the proof of Lemma 2.3.3 (b) for the choice of s.) By assumption, $Q(u) = z^2$ lies in Λ. Hence, when $Q(u)^3 = 1$, we have in fact $z^2 = 1$ and $s = 0$, whereas when $Q(u)^3 = -1$, we have $z^2 = \cos(\pi/3) + e\sin(\pi/3) = -\zeta^2$ and $s = 2$. Next, since $\bar{K}(\mathbf{C}_a)$ is an extension of $K_u = K_b$ by $\mathbb{Z}_2$, we have $\bar{K}(\mathbf{C}_a) = K_u \cup K_u\tau'$, where $\tau' \in \bar{K}(\mathbf{C}_a) - K$ is of the form $\tau' = \delta\nu$ for some $\delta \in K$. We now determine the possible choice of δ:

$$\begin{aligned} \delta\nu \in \bar{K}(\mathbf{C}_a) \;&\Leftrightarrow\; \delta\nu(u) = -u \\ &\Leftrightarrow\; \delta\mu^s\tau(bz) = -bz \\ &\Leftrightarrow\; \delta(b)\zeta^s\bar{z} = -bz \\ &\Leftrightarrow\; \delta(b) = -bz^2\zeta^{-s}. \end{aligned}$$

From earlier remarks, this condition simplifies to $\delta(b) = -b$ when $Q(u)^3 = 1$, while $\delta(b) = b$

when $Q(u)^3 = -1$. Among those $\delta \in K$ for which $\delta(b) = -b$, we take $\delta = \gamma_k$. Since any element of $K_u = K_b$ is of the form γ_w for some $w \in \mathbf{T}_b$, we get $\delta\gamma_w\delta^{-1} = \gamma_{kw\bar{k}} = \gamma_w^{-1}$. This shows that when $Q(u)^3 = 1$, $\bar{K}(\mathbf{C}_a)$ is the semidirect product $K_b \rtimes \langle \gamma_k \tau \rangle \cong \mathrm{O}(2)$. Similarly, among those $\delta \in K$ for which $\delta(b) = b$, we take $\delta = \mathrm{id}$. Hence, when $Q(u)^3 = -1$, we find that $\bar{K}(\mathbf{C}_a) = K_b \times \langle \mu^2 \tau \rangle \cong \mathrm{SO}(2) \times \mathbb{Z}_2$. $\square$

2.3.4 Type I subalgebras not containing e

We continue to fix $a \in \mathcal{B}^*$, which can be written as $a = \alpha e + \sqrt{1-\alpha^2}u$ for unique $0 < \alpha < 1$ and $u \in \sigma(\Lambda)$. Consider the 3-sphere $\mathcal{S}_a^3$ and the 2-sphere $\mathcal{S}_a^2$, defined as

$$\mathcal{S}_a^3 \;:=\; \mathcal{S}(H_a), \qquad \mathcal{S}_a^2 \;:=\; \mathcal{S}(H_a^{\mathrm{pu}}).$$

In order to determine representatives of the $\bar{K}(\mathbf{C}_a)$-orbits of $\mathbb{P}_a$ visually, we exploit the well-known fact that $\mathbb{P}^1(\mathbf{C}) \cong \mathcal{S}^2$. Our plan is to specify a homeomorphism ρ_a from $\mathbb{P}_a$ to $\mathcal{S}_a^2$, and then transfer the $\bar{K}(\mathbf{C}_a)$-action from $\mathbb{P}_a$ to $\mathcal{S}_a^2$ via ρ_a. The representatives of type I subalgebras containing a will then be chosen implicitly as representatives on the sphere.

Since $u \in \sigma(\Lambda) \subset \langle i+je, i-je, j+ie \rangle \subset \langle k \rangle^\perp$, we have $k \in V_a$ and so, by Remark 2.1.2, it follows that $V_a = H_a k$. Moreover,

$$x(yk) \;=\; (yx)k, \qquad (xk)y \;=\; (x\overline{y})k, \qquad (xk)(yk) \;=\; -\overline{y}x,$$

for $x, y \in H_a$. In particular, for any $\mathbf{C}_a v \in \mathbb{P}_a$ where v is a unit vector in V_a, we may write $v = xk$ for some $x \in \mathcal{S}_a^3$, so that

$$\mathbf{C}_a v \;=\; \mathbf{C}_a(xk) \;=\; (x\mathbf{C}_a)k.$$

Conversely, each $x \in \mathcal{S}_a^3$ determines an element $\mathbf{C}_a(xk) = (x\mathbf{C}_a)k \in \mathbb{P}_a$. In other words,

$$\mathbb{P}_a \;=\; \{(x\mathbf{C}_a)k \;:\; x \in \mathcal{S}_a^3\} \;\cong\; \mathbb{P}_{\mathbf{C}_a}(H_a).$$

Observe that for any $x, y \in \mathcal{S}_a^3$, $(x\mathbf{C}_a)k = (y\mathbf{C}_a)k$ just when $x\mathbf{T}_a = y\mathbf{T}_a$, and the latter holds if and only if $xa\overline{x} = ya\overline{y}$. Since $xa\overline{x} \in \mathcal{S}_a^2$, we obtain a function

$$\rho_a \ : \ \mathbb{P}_a \to \mathcal{S}_a^2, \qquad (x\mathbf{C}_a)k \ \mapsto \ xa\overline{x}. \tag{2.18}$$

In fact, ρ_a is a homeomorphism, as can be seen by considering the well-known construction of the Hopf fibration $\mathcal{S}^1 \ \to \ \mathcal{S}^3 \ \to \ \mathcal{S}^2$ by means of unit quaternions. Recall that $\mathcal{S}_a^3$ is the group of unit vectors of H_a which acts on $\mathcal{S}_a^2$ by conjugation, i.e. $x \cdot p = xp\overline{x}$ for $x \in \mathcal{S}_a^3$ and $p \in \mathcal{S}_a^2$. The $\mathcal{S}_a^3$-stabilizer of the point $a \in \mathcal{S}_a^2$ is the subgroup $\mathbf{T}_a$. Hence $\mathcal{S}_a^3/\mathbf{T}_a \cong \mathcal{S}_a^2$. On the other hand, $\mathcal{S}_a^3/\mathbf{T}_a$ is clearly homeomorphic to $\mathbb{P}_a$ via the map $x\mathbf{T}_a \mapsto (x\mathbf{C}_a)k$. Thus, we see that ρ_a is the composite of the homeomorphisms $\mathbb{P}_a \to \mathcal{S}_a^3/\mathbf{T}_a$ and $\mathcal{S}_a^3/\mathbf{T}_a \to \mathcal{S}_a^2$.

The next step is to consider the unique action of $\bar{K}(\mathbf{C}_a)$ on $\mathcal{S}_a^2$ for which ρ_a is equivariant. In more detail, given $p \in \mathcal{S}_a^2$, we write $p = xa\overline{x} = \rho_a\big((x\mathbf{C}_a)k\big)$ for some $x \in \mathcal{S}_a^3$. For any $\varphi \in \bar{K}(\mathbf{C}_a)$, we define

$$\varphi \cdot p \ = \ \rho_a\big[\varphi\big((x\mathbf{C}_a)k\big)\big].$$

Note that the orthonormal triple $\{e, u, ue\}$ is always contained in $\mathcal{S}_a^2$. Moreover, if $|Q(u)| = 1$, in which case $u = bz$ where $b \in \mathcal{S}(\mathbf{H}^{\mathrm{pu}})$ and $z \in \mathbf{T}_e$, then $\{e, b, be\}$ is also an orthonormal triple contained in $\mathcal{S}_a^2$.

Proposition 2.3.5. *Let $a \in \mathcal{B}^*$ be given by $a = \alpha e + \sqrt{1-\alpha^2}u$, where $\alpha \in (0,1)$ and $u \in \sigma(\Lambda)$. The action of $\bar{K}(\mathbf{C}_a)$ on $\mathcal{S}_a^2$ is described as follows.*

(a) When $|Q(u)| < 1$, let $\nu \mapsto \nu'$ denote the inverse of the isomorphism $\eta : \bar{K}(\mathbf{C}_a) \to (\Sigma_3)_{Q(u)}$ from Proposition 2.3.4. Then $\bar{K}(\mathbf{C}_a)$ fixes $e \in \mathcal{S}_a^2$. Moreover, if $\mu \in (\Sigma_3)_{Q(u)}$, then μ' acts as the rotation on the plane $\langle u, ue\rangle$ such that $\mu \cdot u = u\zeta^2$. For $s = 0, 1, 2$, if $\mu^s\tau \in (\Sigma_3)_{Q(u)}$, then $(\mu^s\tau)'$ acts on $\mathcal{S}_a^2$ as the reflection in the plane $\langle e, u\zeta^s\rangle$.

(b) If $|Q(u)| = 1$, write $u = bz$ with $b \in \mathcal{S}(\mathbf{H}^{\mathrm{pu}})$ and $z \in \mathbf{T}_e$. Then $K_u = K_b$ acts on $\mathcal{S}_a^2$ by rotations about b. Specifically, for each $w \in \mathbf{T}_b$ and $p \in \mathcal{S}_a^2$, $\gamma_w \cdot p = w^2pw^{-2}$. If $Q(u)^3 = 1$, then $\gamma_k\tau$ acts as the reflection in the plane $\langle b, e\rangle$. If $Q(u)^3 = -1$, then $\mu^2\tau$ acts as the reflection in the plane $\langle e, be\rangle$.

Proof. For a fixed $p \in \mathcal{S}_a^2$, we can write $p = \rho_a\big((x\mathbf{C}_a)k\big) = xa\overline{x}$, where $x \in \mathcal{S}_a^3$.

(a) Assume that $|Q(u)| < 1$. Let $\nu \in (\Sigma_3)_{Q(u)}$, and assume that either $\nu = \mu^s$ or $\nu = \mu^s\tau$ for some $s \in \{0,1,2\}$. Let $\nu' = \eta^{-1}(\nu) \in \bar{K}(\mathbf{C}_a)$. We claim that

$$\nu' \cdot p \;=\; \mathrm{sgn}(\nu).\mu^{-s}\nu'(p) \qquad (p \in \mathcal{S}_a^2). \tag{2.19}$$

Once this is proved, it follows that $\nu' \cdot e = e$ for every ν. Moreover, since $\nu'(u) = \mathrm{sgn}(\nu)u$, we have $(\mu^s)' \cdot u = \mu^{-s}(u) = u\zeta^{-s}$, and $(\mu^s\tau)' \cdot (u\zeta^t) = \mu^{-s}(u)\zeta^{-t} = u\zeta^{-s-t}$ for $t \in \{0,1,2\}$, i.e. $(\mu^s\tau)'$ fixes $u\zeta^s$ and interchanges $u\zeta^{s-1}$ and $u\zeta^{s+1}$.

To prove (2.19), first note that $\nu'(a) = \mathrm{sgn}(\nu)a$, and by Proposition 2.3.4, we get

$$\nu'(k) \;=\; \varepsilon\nu(k) \;=\; \varepsilon k\zeta^s \;=\; \varepsilon\zeta^{-s}k$$

with $\varepsilon = \pm 1$. Consider

$$\begin{aligned}
\nu'\big[(x\mathbf{C}_a)k\big] &= (\nu'(x)\mathbf{C}_a)\nu'(k) \\
&= (\nu'(x)\mathbf{C}_a)(\varepsilon\zeta^{-s}k) \\
&= \big[(\varepsilon\zeta^{-s}\nu'(x))\mathbf{C}_a\big]k.
\end{aligned}$$

Using this expression, we can compute $\nu' \cdot p$ as follows:

$$\begin{aligned}
\nu' \cdot p &= \nu' \cdot \rho_a\big[(x\mathbf{C}_a)k\big] \\
&= \rho_a\big[\nu'\big((x\mathbf{C}_a)k\big)\big] \\
&= \rho_a\big[\big(\varepsilon\zeta^{-s}\nu'(x)\mathbf{C}_a\big)k\big] \\
&= (\varepsilon\zeta^{-s}\nu'(x))a\overline{(\varepsilon\zeta^{-s}\nu'(x))} \\
&= \zeta^{-s}\nu'(x)\big(\mathrm{sgn}(\nu)\nu'(a)\big)\nu'(\overline{x})\zeta^s \\
&= \mathrm{sgn}(\nu)\mu^{-s}\nu'(xa\overline{x}) \\
&= \mathrm{sgn}(\nu)\mu^{-s}\nu'(p).
\end{aligned}$$

(b) For any $w \in \mathbf{T}_b$, let $\gamma_w \in K_u = K_b$ be conjugation by w. Then $\gamma_w(k) = wk\overline{w} = w^2k$ and γ_w fixes $H_a = \langle 1, e, u, ue\rangle$ pointwise. So,

$$\gamma_w\big[(x\mathbf{C}_a)k\big] \;=\; (x\mathbf{C}_a)(w^2k) \;=\; \big((w^2x)\mathbf{C}_a\big)k.$$

It follows that

$$\gamma_w \cdot p \;=\; (w^2x)a\overline{(w^2x)} \;=\; w^2(xa\bar{x})\overline{w^2} \;=\; w^2pw^{-2}.$$

Thus, K_b fixes $b \in \mathcal{S}_a^2$ while acting as rotation on the plane $\langle b, be\rangle$.

In the case where $Q(u)^3 = 1$, recall that $\bar{K}(\mathbf{C}_a) = K_b \rtimes \langle \gamma_k\tau\rangle$. A similar direct calculation gives $(\gamma_k\tau)\cdot p = -(\gamma_k\tau)(p)$ for $p \in \mathcal{S}_a^2$. From this and the fact that $\gamma_k(b) = -b$ (see the proof of Proposition 2.3.4), we deduce that $\gamma_k\tau$ acts as the reflection in $\langle b, e\rangle$.

Finally, in the case where $Q(u)^3 = -1$, recall that $\bar{K}(\mathbf{C}_a) = K_b \rtimes \langle \mu^2\tau\rangle$. One can check that $(\mu^2\tau)\cdot p = -\tau(p)$ for all $p \in \mathcal{S}_a^2$. From this and the fact that $\tau(b) = b$ and $\tau(e) = e$, it is clear that $\mu^2\tau$ acts as the reflection in $\langle e, be\rangle$. □

Proposition 2.3.6. *For $a \in \mathcal{B}^*$, write $a = \alpha e + \sqrt{1-\alpha^2}u$ with $\alpha \in (0,1)$ and $u \in \sigma(\Lambda)$. A set $\mathcal{X}_a \subset \mathcal{S}_a^2$ of representatives of the orbit space $\mathcal{S}_a^2/\bar{K}(\mathbf{C}_a)$ can be chosen as follows.*

(a) Assume that $|Q(u)| < 1$.

- *i) If $Q(u)^3 \notin \mathbf{R}$, then $\mathcal{X}_a := \mathcal{S}_a^2$.*
- *ii) If $Q(u)^3 \in (0,1)$, then $\mathcal{X}_a := \{p \in \mathcal{S}_a^2 \;:\; (p|ue) \geq 0\}$.*
- *iii) If $Q(u)^3 \in (-1,0)$, then $\mathcal{X}_a := \{p \in \mathcal{S}_a^2 \;:\; (p|u\zeta^2 e) \geq 0\}$.*
- *iv) If $Q(u) = 0$, then*

$$\mathcal{X}_a \;:=\; \{\beta e + \sqrt{1-\beta^2}(u\cos\theta + ue\sin\theta) \;:\; -1 \leq \beta \leq 1,\; 0 \leq \theta \leq \pi/3\}.$$

(b) Suppose that $|Q(u)| = 1$, in which case $u = bz$ for some $b \in \mathcal{S}(\mathbf{H}^{\mathrm{pu}})$ and $z \in \mathbf{T}_e$.

- *i) If $Q(u)^3 \notin \{\pm 1\}$, then $\mathcal{X}_a := \{b\cos\theta + e\sin\theta \;:\; 0 \leq \theta \leq \pi\}$.*
- *ii) If $Q(u)^3 = 1$, then $\mathcal{X}_a := \{b\cos\theta + be\sin\theta \;:\; 0 \leq \theta \leq \pi\}$.*
- *iii) If $Q(u)^3 = -1$, then $\mathcal{X}_a := \{b\cos\theta + e\sin\theta \;:\; 0 \leq \theta \leq \pi/2\}$.*

Proof. This follows easily from the structure of $\bar{K}(\mathbf{C}_a)$ (Proposition 2.3.4) and the geometric description of the $\bar{K}(\mathbf{C}_a)$-action on $\mathcal{S}_a^2$ (Proposition 2.3.5). We omit the details. □

Theorem 2.3.7. *For each $a \in \mathcal{B}^*$, define*

$$\mathcal{R}_a \;:=\; \{S_{\mathrm{I}}(\mathbf{C}_a v) \;:\; \mathbf{C}_a v \;\in\; \rho_a^{-1}(\mathcal{X}_a)\}.$$

A set of representatives of the conjugacy classes of type I subalgebras is given by $\bigcup_{a\in\mathcal{B}} \mathcal{R}_a$.

Proof. This follows from Lemma 2.3.1 together with Propositions 2.3.2 and 2.3.6. $\square$

Remark 2.3.1. We record without proof a formula for computing the inverse of the homeomorphism $f_a : \mathcal{S}_a^2 \to \mathcal{S}_a^3/\mathbf{T}_a$ which sends $x\mathbf{T}_a$ to $xa\bar{x}$. This leads to a concrete expression for $\rho_a^{-1} : \mathcal{S}_a^2 \to \mathbb{P}_a$.

First of all, $f_a^{-1}(a) = \mathbf{T}_a$, and

$$f_a^{-1}(-a) \;=\; \mathcal{S}_a^2 \cap \langle a\rangle^{\perp} \;=\; b\mathbf{T}_a$$

where b is any element of $\mathcal{S}_a^2 \cap \langle a\rangle^{\perp}$. Next, for any $p \in \mathcal{S}_a^2 - \{a, -a\}$, set

$$x \;=\; \sqrt{\frac{1+(a|p)}{2}} + \frac{a\times p}{\|a\times p\|}\sqrt{\frac{1-(a|p)}{2}} \;=\; \frac{1}{\sqrt{2}}\left(\sqrt{1+(a|p)} + \frac{a\times p}{\sqrt{1+(a|p)}}\right).$$

Then $f_a^{-1}(p) \;=\; x\mathbf{T}_a$. As an aside, the second expression for x is a continuous function of $p \in \mathcal{S}_a^2 - \{-a\}$.

2.4 Type II subalgebras

2.4.1 Construction of type II subalgebras

Recall that $\mathbf{O}_e^{\mathrm{pp}} = i\mathbf{C}_e + j\mathbf{C}_e + k\mathbf{C}_e$ is a right inner product vector space over $\mathbf{C}_e$. Let $\mathcal{E} = (i, j, k)$ be the standard complex orthonormal basis of $\mathbf{O}_e^{\mathrm{pp}}$. A (unitary) *3-frame* in $\mathbf{O}_e^{\mathrm{pp}}$ is an ordered complex orthonormal basis (u_1, u_2, u_3) of $\mathbf{O}_e^{\mathrm{pp}}$. The complex Stiefel manifold $W_3 \cong V_3(\mathbf{C}^3)$ is defined as the set of all 3-frames in $\mathbf{O}_e^{\mathrm{pp}}$. We shall find it convenient to treat the unitary group $\mathrm{U}(3)$ as a subgroup of $\mathrm{GL}_3(\mathbf{C}_e) \equiv \mathrm{GL}_3(\mathbf{C})$. If we denote the $\mathcal{E}$-coordinates of

$$x \;=\; iz_1 + jz_2 + kz_3 \;\in\; \mathbf{O}_e^{\mathrm{pp}}$$

by the column vector

$$[x]_{\mathcal{E}} \;=\; [z_1\ z_2\ z_3]^t,$$

then the assignment

$$u \;=\; (u_1, u_2, u_3) \qquad \mapsto \qquad \Xi(u) \;:=\; \Big[[u_1]_{\mathcal{E}}\ [u_2]_{\mathcal{E}}\ [u_3]_{\mathcal{E}}\Big]$$

sets up a homeomorphism $\Xi : W_3 \to \mathrm{U}(3)$. Under this correspondence, the action of K on W_3 corresponds to left translations on $\mathrm{U}(3)$ by $\mathrm{SO}(3)$. Specifically, given $\gamma \in K$, let

$$a_\gamma \;:=\; \Big[[\gamma(i)]_{\mathcal{E}}\ [\gamma(j)]_{\mathcal{E}}\ [\gamma(k)]_{\mathcal{E}}\Big] \;\in\; \mathrm{SO}(3).$$

If γ carries $u = (u_1, u_2, u_3)$ to $\gamma \cdot u = (\gamma(u_1), \gamma(u_2), \gamma(u_3))$, then $\Xi(\gamma \cdot u) = a_\gamma \Xi(u)$. Also, since $\mu(x) = x\zeta$ for $x \in \mathbf{O}_e^{\mathrm{pp}}$ and $\tau(iz_1 + jz_2 + kz_3) = i\overline{z_1} + j\overline{z_2} + k\overline{z_3}$, obviously we have

$$\Xi(\mu \cdot u) \;=\; \Xi(u)\zeta, \qquad \Xi(\tau \cdot u) \;=\; \overline{\Xi(u)}. \tag{2.20}$$

Using these formulas, we impose a Σ_3-action on $\mathrm{U}(3)$ such that Ξ is Σ_3-equivariant. Thus, we obtain a $\bar{K}$-action on W_3 such that Ξ is a $\bar{K}$-map. On the other hand, if $u \in W_3$ and $a = [\alpha_{p,q}]_{(p,q)} \in \mathrm{O}(3)$, then the triple

$$u \cdot a \;:=\; \Big(\sum_p u_p \alpha_{p,1},\ \sum_p u_p \alpha_{p,2},\ \sum_p u_p \alpha_{p,3}\Big)$$

defines another 3-frame such that $\mathrm{span}(u) = \mathrm{span}(u \cdot a)$. This defines a right $\mathrm{O}(3)$-action on W_3 that commutes with the above left $\bar{K}$-action. It is clear that $\Xi(u \cdot a) = \Xi(u)a$. Also, two 3-frames $u, v \in W_3$ span the same real 3-dimensional subspace of $\mathbf{O}_e^{\mathrm{pp}}$ if and only if $u = v \cdot a$ for a unique $a \in \mathrm{O}(3)$. It follows that W_3 is the total space of a principal $\mathrm{O}(3)$-bundle $W_3 \xrightarrow{\pi} \mathrm{Gr}_3(\mathbf{O}_e^{\mathrm{pp}})$, whose projection map π sends u to $\mathrm{span}(u)$.

A 3-frame $u = (u_1, u_2, u_3) \in W_3$ is said to be *special* if

$$u_1 u_2 \;=\; u_3.$$

For instance, $\mathcal{E} = (i, j, k)$ is a special 3-frame. The point is that each special 3-frame u determines a type II subalgebra $\langle 1, u_1, u_2, u_3 \rangle$. Conversely, every type II subalgebra arises in this way. Indeed, given $S \in \mathscr{H}_{\mathrm{II}}$, choose any orthonormal pair u_1, u_2 in S^{pp}, and set $u_3 = u_1 u_2 \in S^{\mathrm{pp}}$. Then (u_1, u_2, u_3) is a special 3-frame.

Let $SW_3 \subset W_3$ be the closed subset of special 3-frames. We point out that Ξ maps SW_3 onto the special unitary group $\mathrm{SU}(3)$. To see this, consider an arbitrary 3-frame $u = (u_1, u_2, u_3)$, such that

$$[u_q]_{\mathcal{E}} \;=\; [z_{1q}\ z_{2q}\ z_{3q}]^t \qquad (q = 1, 2, 3)$$

where $z_{p,q} \in \mathbf{C}_e$ for $p, q \in \{1, 2, 3\}$. Then $\Xi(u) = \left[z_{p,q}\right]_{(p,q)}$. Recall from Section 2.2.1 that there is an alternating 3-form ω on $\mathbf{A}_4^{\mathrm{pp}}$ given by $\omega(v, w, x) = \langle vw | x \rangle$. Hence,

$$\langle u_1 u_2 | u_3 \rangle \;=\; \omega(u_1, u_2, u_3) \;=\; \omega(i, j, k) \cdot \det \Xi(u) \;=\; \det \Xi(u).$$

On the other hand, as $u_1 u_2$ and u_3 are unit vectors, $\langle u_1 u_2 | u_3 \rangle = 1$ just when $u_1 u_2 = u_3$. Therefore, u is special if and only if $\Xi(u) \in \mathrm{SU}(3)$. From now on, by a slight abuse of notation, we shall simply identify any special 3-frame $u = (u_1, u_2, u_3) \in SW_3$ with the matrix $u \equiv \Xi(u) = [u_1\ u_2\ u_3] \in \mathrm{SU}(3)$.

For any $u = [u_1\ u_2\ u_3] \in \mathrm{SU}(3)$, we define the associated type II subalgebra

$$S_{\mathrm{II}}(u) \;:=\; \langle 1, u_1, u_2, u_3 \rangle. \tag{2.21}$$

For any $u, v \in \mathrm{SU}(3)$, $S_{\mathrm{II}}(u) = S_{\mathrm{II}}(v)$ if and only if $u = va$ for some $a \in \mathrm{SO}(3)$. It follows that the map S_{II} is the projection map of a principal $\mathrm{SO}(3)$-bundle $\mathrm{SU}(3) \to \mathscr{H}_{\mathrm{II}}$. This also induces a homeomorphism from the homogeneous space $\mathrm{SU}(3)/\mathrm{SO}(3)$ onto $\mathscr{H}_{\mathrm{II}}$.

Note that $\mathrm{SU}(3)$ is stable under the left $\bar{K}$-action on $\mathrm{U}(3)$. Thus the homogeneous space $\mathrm{SU}(3)/\mathrm{SO}(3)$ inherits a natural $\bar{K}$-action, such that the above homeomorphism $\mathrm{SU}(3)/\mathrm{SO}(3) \cong \mathscr{H}_{\mathrm{II}}$ is $\bar{K}$-equivariant. By identifying the $\mathrm{SO}(3)$-orbit space of $\mathrm{SU}(3)/\mathrm{SO}(3)$

with the double coset space $\mathrm{SO}(3)\backslash\mathrm{SU}(3)/\mathrm{SO}(3)$, we obtain the homeomorphism

$$f \;:\; \mathrm{SO}(3)\backslash\mathrm{SU}(3)/\mathrm{SO}(3) \to \mathscr{H}_{\mathrm{II}}/K, \qquad \mathrm{SO}(3)\,u\,\mathrm{SO}(3) \;\mapsto\; [\,S_{\mathrm{II}}(u)\,]_K\,. \tag{2.22}$$

Thus, in order to provide representatives of $\mathscr{H}_{\mathrm{II}}/\bar{K} \;=\; (\mathscr{H}_{\mathrm{II}}/K)/\Sigma_3$, we shall pullback the Σ_3-action to $\mathrm{SO}(3)\backslash\mathrm{SU}(3)/\mathrm{SO}(3)$ via f. Obviously, what we need is given by

$$\mu\cdot\mathrm{SO}(3)\,u\,\mathrm{SO}(3) \;=\; \mathrm{SO}(3)\,\zeta u\,\mathrm{SO}(3), \qquad \tau\cdot\mathrm{SO}(3)\,u\,\mathrm{SO}(3) \;=\; \mathrm{SO}(3)\,\overline{u}\,\mathrm{SO}(3).$$

We will study $\mathrm{SO}(3)\backslash\mathrm{SU}(3)/\mathrm{SO}(3)$ and determine its Σ_3-orbit space in the next two sections.

2.4.2 Lie-theoretic preliminaries

We briefly recall a few facts about compact connected Lie groups and their Weyl groups which can be found in many textbooks, e.g. [4]. Our reference on the affine Weyl group and alcoves is [17]. Although we are interested in the single group $G = \mathrm{SU}(3)$ where the claims can be proved by linear algebra, we omit such a proof for reason of space.

Let G be a compact connected Lie group and let T be a maximal torus in G. The Weyl group $W = W(G,T)$ of G with respect to T is defined as $W = N/T$, where $N = N_G(T)$ is the normalizer of T in G. It is known that W is a finite group, and that conjugation by N induces a faithful action of W on T by automorphisms. Every conjugacy class in G meets T, and any two elements of T are G-conjugate if and only if they belong to the same W-orbit. Consequently, the compact space $\mathrm{Conj}(G) = \{[g]_G : g \in G\}$ of conjugacy classes in G is canonically homeomorphic to the orbit space $T/W = \{[t]_W : t \in T\}$.

Now let $G = \mathrm{SU}(n)$, where $n \geq 2$. The standard maximal torus T consists of the diagonal matrices in $\mathrm{SU}(n)$. In this case, $N = N_G(T)$ can be expressed as $N = S\Sigma_n^{\pm}\cdot T$, where $S\Sigma_n^{\pm} \subset \mathrm{SO}(n)$ is the group of signed permutation matrices with determinant 1. The Weyl group W is isomorphic to the symmetric group Σ_n.

A complete description of T/W is provided by the root system of $G = \mathrm{SU}(n)$. Consider the Euclidean space $\mathbf{R}^n$ with the standard orthonormal basis $\{\varepsilon_1,\ldots,\varepsilon_n\}$. By identifying each ε_p

with $i \cdot E_{p,p}$, where $\{E_{p,q} : 1 \leq p, q \leq n\}$ is the set of standard matrix units of $M_n(\mathbf{C})$, we identify the Lie algebra $\mathrm{L}T$ of T with the following hyperplane in $\mathbf{R}^n$:

$$\mathrm{L}T \;=\; \{(\theta_1, \ldots, \theta_n) \in \mathbf{R}^n \;:\; \theta_1 + \ldots + \theta_n = 0\}.$$

Note that we also identify $\mathrm{L}T$ with its dual space $\mathrm{L}T^*$ by means of the inner product on $\mathbf{R}^n$. The exponential map $\exp : \mathrm{L}T \to T$ is given by

$$x \;=\; (\theta_1, \ldots, \theta_n) \;\in\; \mathrm{L}T \quad\mapsto\quad \exp(x) \;=\; (e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_n}) \;\in\; T.$$

The *integral lattice* is $I := \ker(\exp)$. Thus, $\mathrm{L}T/I \cong T$. Moreover, the adjoint representation of G on its Lie algebra $\mathrm{L}G$ induces a faithful action of W on $\mathrm{L}T$. As such, the exponential map is W-equivariant, and I is invariant under W.

The root system R of G with respect to T is obtained from the root space decomposition of $\mathbf{C} \otimes \mathrm{L}G$ as a T-module. For $G = \mathrm{SU}(n)$, we have

$$R \;=\; \{\varepsilon_k - \varepsilon_l \;:\; 1 \leq k \neq l \leq n\},$$

which is of type A_{n-1}. The usual choice of positive roots in R is $R_+ = \{\varepsilon_k - \varepsilon_l \;:\; 1 \leq k < l \leq n\}$. Then the set of simple roots in R_+ is given by

$$\alpha_j \;=\; \varepsilon_j - \varepsilon_{j+1} \qquad (1 \leq j \leq n-1).$$

The highest root of R (with respect to the partial ordering induced by R_+) is

$$\alpha_0 \;=\; \alpha_1 + \ldots + \alpha_{n-1} \;=\; \varepsilon_1 - \varepsilon_n.$$

Since all roots $\alpha \in R$ have length 2, R is equal to the *coroot system* $R^* = \{\alpha^* : \alpha \in R\}$, where $\alpha^* = 2\alpha/(\alpha|\alpha)$. It follows that the abelian group Γ generated by R^* (also known as the *coroot lattice*) coincides with I. The Weyl group $W(R)$ is generated by the reflections $\{s_\alpha : \alpha \in R_+\}$, where s_α fixes the hyperplane $H_\alpha = \alpha^\perp$ and maps α to $-\alpha$. The action of $W(R)$ is the same as the adjoint action of the Weyl group $W = W(G, T)$. Thus, $W(R) \cong W$, and we obtain the homeomorphism

$$T/W \;\cong\; (\mathrm{L}T/I)/W \;\cong\; T/(I \rtimes W).$$

The *Stiefel diagram* of G is the union $\mathrm{St}(G) = \bigcup\{H_{\alpha,k} : \alpha \in R_+,\ k \in \mathbb{Z}\}$ of the affine hyperplanes

$$H_{\alpha,k} = \{x \in \mathrm{L}T : (\alpha|x) = k\} \qquad (\alpha \in R_+, k \in \mathbb{Z}).$$

Any connected component of the complement $\mathrm{L}T - \mathrm{St}(G)$ is called an *alcove*. The *fundamental alcove* is defined by

$$\begin{aligned}\mathcal{A}_0 &= \{x \in \mathrm{L}T : 0 < (\alpha|x) < 1 \text{ for all } \alpha \in R_+\}\\ &= \{x \in \mathrm{L}T : (\alpha_j|x) > 0 \text{ for all } 1 \leq j \leq n-1,\ (\alpha_0|x) < 1\}\\ &= \{(\theta_1, \ldots, \theta_n) \in \mathrm{L}T : \theta_1 > \theta_2 > \ldots > \theta_n,\ \theta_1 - \theta_n < 1\}.\end{aligned}$$

The *affine Weyl group* $\widetilde{W}$ is the group of affine isometries on $\mathrm{L}T$ generated by the reflections $s_{\alpha,k}$ in the hyperplanes $H_{\alpha,k}$. In general, $\widetilde{W} = \Gamma \rtimes W$. Hence for $R = A_{n-1}$, $\widetilde{W} = I \rtimes W$. The affine Weyl group acts freely and transitively on the set of alcoves. Furthermore, each $\widetilde{W}$-orbit of $\mathrm{L}T$ meets $\overline{\mathcal{A}_0}$ at exactly one point. Therefore, we have the homeomorphism

$$\overline{\mathcal{A}_0} \cong \mathrm{L}T/\widetilde{W} \cong T/W.$$

The explicit homeomorphism

$$e : \overline{\mathcal{A}_0} \to T/W \tag{2.23}$$

sends each $x \in \overline{\mathcal{A}_0}$ to $e(x) := [\exp(x)]_W$.

There is a homeomorphism between the homogeneous space $\mathrm{SU}(n)/\mathrm{SO}(n)$ and the compact space $\mathcal{M}$ of symmetric matrices in $\mathrm{SU}(n)$, namely,

$$\mathcal{M} := \{x \in \mathrm{SU}(n) : x^t = x\},$$

where $x \mapsto x^t$ is the transpose map. We give a proof of this fact below. Note that $T \subset \mathcal{M}$. Also, $\mathrm{SO}(n)$ acts on $\mathrm{SU}(n)/\mathrm{SO}(n)$ by left translation, and on $\mathcal{M}$ by conjugation. The map

$$\rho : \mathrm{SU}(n)/\mathrm{SO}(n) \to \mathcal{M}, \qquad u\,\mathrm{SO}(n) \mapsto uu^t$$

is continuous and $\mathrm{SO}(n)$-equivariant.

Lemma 2.4.1. *(a) Any $x \in \mathcal{M}$ is* $\mathrm{SO}(n)$*-conjugate to some element of T. Consequently, there is a homeomorphism*

$$k : T/W \to \mathcal{M}/\mathrm{SO}(n)$$

which sends $[t]_W$ to $[t]_{\mathrm{SO}(n)}$ for each $t \in T$.

(b) The continuous $\mathrm{SO}(n)$*-map $\rho : \mathrm{SU}(n)/\mathrm{SO}(n) \to \mathcal{M}$ given by $\rho\big(u\,\mathrm{SO}(n)\big) = uu^t$ is a homeomorphism.*

(c) Every double coset in $\mathrm{SO}(n)\backslash\mathrm{SU}(n)/\mathrm{SO}(n)$ *meets T. Consequently, there is a homeomorphism*

$$g : \mathrm{SO}(n)\backslash\mathrm{SU}(n)/\mathrm{SO}(n) \to T/W$$

such that for each $s \in T$, $g\big(\mathrm{SO}(n)\,s\,\mathrm{SO}(n)\big) = [s^2]_W$.

Proof. (a) Given $x \in \mathcal{M}$, write $x = x_1 + ix_2$ where $x_1, x_2 \in M_n(\mathbf{R})$. Since x is normal and symmetric, x_1 and x_2 are commuting real symmetric matrices and as such are simultaneously orthogonally diagonalizable. Thus, conjugation by some $a \in \mathrm{O}(n)$ brings x into T; clearly, such an a can be chosen from $\mathrm{SO}(n)$. This proves the first claim of (a), which in turn implies that $\mathcal{M}/\mathrm{SO}(n) = \{[t]_{\mathrm{SO}(n)} : t \in T\}$. If two elements $s, t \in T$ are $\mathrm{SO}(n)$-conjugate, then s and t are W-conjugate. Conversely, if s and t are W-conjugate, then they are identical up to permutation of the diagonal entries. Hence there exists $a \in S\Sigma_n^{\pm}$ such that $s = ata^{-1}$. We conclude that $[s]_{\mathrm{SO}(n)} = [t]_{\mathrm{SO}(n)}$ precisely when $[s]_W = [t]_W$. This proves the existence of the homeomorphism $k : T/W \to \mathcal{M}/\mathrm{SO}(n)$ such that $k\left([t]_W\right) = [t]_{\mathrm{SO}(n)}$ for $t \in T$.

(b) By the first claim in (a), any $x \in \mathcal{M}$ can be written as $x = ata^{-1}$, where $a \in \mathrm{SO}(n)$ and $t \in T$. There exists $s \in T$ such that $s^2 = t$. Setting $u := asa^{-1}$, we obtain $x = uu^t = \rho\big(u\,\mathrm{SO}(n)\big)$. Next, suppose that for $u, v \in \mathrm{SU}(n)$ we have $\rho\big(u\,\mathrm{SO}(n)\big) = \rho\big(v\,\mathrm{SO}(n)\big)$, i.e. $uu^t = vv^t$. Then $v^{-1}u = v^t(u^t)^{-1} = (\overline{v})^{-1}\overline{u} = \overline{v^{-1}u}$. Therefore, $v^{-1}u \in \mathrm{SO}(n)$, proving that $u\,\mathrm{SO}(n) = v\,\mathrm{SO}(n)$. We see that ρ is bijective and hence is a homeomorphism.

(c) Given any $u \in \mathrm{SU}(n)$, by (a), we can write $uu^t = ata^{-1}$ for some $a \in \mathrm{SO}(n)$ and $t \in T$. Writing $t = s^2$ where $s \in T$, we get $uu^t = (asa^{-1})(asa^{-1})^t$. Hence by (b), there exists $b \in \mathrm{SO}(n)$ such that $u = asa^{-1}b$. Hence, $\mathrm{SO}(n)\,u\,\mathrm{SO}(n) = \mathrm{SO}(n)\,s\,\mathrm{SO}(n)$, proving the first claim of (c).

Next, by factoring the $\mathrm{SO}(n)$-map ρ through the orbit spaces, we obtain a homeomorphism $\bar{\rho}$ from $\mathrm{SO}(n)\backslash\mathrm{SU}(n)/\mathrm{SO}(n)$ onto $\mathcal{M}/\mathrm{SO}(n)$ which sends $\mathrm{SO}(n)\,u\,\mathrm{SO}(n)$ to $[uu^t]_{\mathrm{SO}(n)}$. Finally, $g := k^{-1}\circ\bar{\rho} : \mathrm{SO}(n)\backslash\mathrm{SU}(n)/\mathrm{SO}(n) \to T/W$ is a homeomorphism which sends $\mathrm{SO}(n)\,s\,\mathrm{SO}(n)$ to $[s^2]_W$ for every $s \in T$. $\square$

It follows from Lemma 2.4.1 (c) and (2.23) that

$$\mathrm{SO}(n)\backslash\mathrm{SU}(n)/\mathrm{SO}(n) \;\cong\; T/W \;\cong\; \overline{\mathcal{A}_0}. \tag{2.24}$$

For later use, we record the explicit form of the homeomorphism

$$h = g^{-1}\circ e \;:\; \overline{\mathcal{A}_0} \;\to\; \mathrm{SO}(n)\backslash\mathrm{SU}(n)/\mathrm{SO}(n).$$

For any $x \in \overline{\mathcal{A}_0}$, $h(x)$ is given by the double coset

$$h(x) \;:=\; \mathrm{SO}(n)\,\exp(x/2)\,\mathrm{SO}(n). \tag{2.25}$$

2.4.3 $\bar{K}$-orbits of type II subalgebras

We return to the classification of type II subalgebras. As shown in the previous sections, we have the homeomorphism

$$\mathscr{H}_{\mathrm{II}}/K \;\cong\; \mathrm{SO}(3)\backslash\mathrm{SU}(3)/\mathrm{SO}(3) \;\cong\; \mathrm{L}T/\widetilde{W} \;\cong\; \overline{\mathcal{A}_0},$$

where

$$\mathrm{L}T \;=\; \{(\theta_1,\theta_2,\theta_3)\in\mathbf{R}^3 \;:\; \theta_1+\theta_2+\theta_3=0\},$$

$\widetilde{W}$ is the affine Weyl group of the root system R of type A_2, and

$$\overline{\mathcal{A}_0} \;=\; \{(\theta_1,\theta_2,\theta_3)\in \mathrm{L}T \;:\; \theta_1\ge\theta_2\ge\theta_3,\ \theta_1-\theta_3\le 1\}$$

is the closure of the fundamental alcove $\mathcal{A}_0$. More explicitly, the composite of the maps $h : \overline{\mathcal{A}_0} \to \mathrm{SO}(3)\backslash\mathrm{SU}(3)/\mathrm{SO}(3)$ and $f : \mathrm{SO}(3)\backslash\mathrm{SU}(3)/\mathrm{SO}(3) \to \mathscr{H}_{\mathrm{II}}/K$ defined in (2.25) and(2.22) gives the desired homeomorphism

$$\bar{h} \,:\, \overline{\mathcal{A}_0} \;\to\; \mathscr{H}_{\mathrm{II}}/K \tag{2.26}$$

which sends $x = (\theta_1, \theta_2, \theta_3) \in \overline{\mathcal{A}_0}$ to the K-conjugacy class of

$$S_{\mathrm{II}}\big(\exp(x/2)\big) \;=\; \langle 1, iz_1, jz_2, kz_3 \rangle,$$

where $z_p = \cos(\theta_p/2) + e \sin(\theta_p/2)$ for $p = 1, 2, 3$. Note that $\bar{h}$ is the composite of maps

$$\overline{\mathcal{A}_0} \;\xrightarrow{e}\; T/W \;\xrightarrow{g^{-1}}\; \mathrm{SO}(3)\backslash\mathrm{SU}(3)/\mathrm{SO}(3) \;\xrightarrow{f}\; \mathscr{H}_{\mathrm{II}}/K.$$

In order to find representatives of the orbits space $\mathscr{H}_{\mathrm{II}}/\bar{K} \;=\; (\mathscr{H}_{\mathrm{II}}/K)\,/\Sigma_3$, our final task is to transfer the Σ_3-action on $\mathscr{H}_{\mathrm{II}}/K$ to $\overline{\mathcal{A}_0}$ via $\bar{h}$. Then a natural set of representatives of $\overline{\mathcal{A}_0}/\Sigma_3$ can be found by inspection, and this corresponds to the desired set of representatives of $\mathscr{H}_{\mathrm{II}}/\bar{K}$.

We have already defined a Σ_3-action on $\mathrm{SO}(3)\backslash\mathrm{SU}(3)/\mathrm{SO}(3)$ by pullback via f. For any $s \in T$, we have $g\big(\mathrm{SO}(3)\, s\, \mathrm{SO}(3)\big) = [\, s^2\,]_W$ and

$$\begin{aligned} g\big(\mu \cdot \mathrm{SO}(3)\, s\, \mathrm{SO}(3)\big) \;&=\; g\big(\mathrm{SO}(3)\, \zeta s\, \mathrm{SO}(3)\big) \;=\; [\, \zeta^2 s^2\,]_W, \\ g\big(\tau \cdot \mathrm{SO}(3)\, s\, \mathrm{SO}(3)\big) \;&=\; g\big(\mathrm{SO}(3)\, \bar{s}\, \mathrm{SO}(3)\big) \;=\; [\, \overline{s^2}\,]_W. \end{aligned}$$

Therefore, the homeomorphism $g : \mathrm{SO}(3)\backslash\mathrm{SU}(3)/\mathrm{SO}(3) \to T/W$ becomes a Σ_3-map provided that Σ_3 acts on T/W by the rules

$$\mu \cdot [\, t\,]_W \;=\; [\, \zeta^2 t\,]_W, \qquad \tau \cdot [\, t\,]_W \;=\; [\, \bar{t}\,]_W,$$

where $t \in T$. To continue, we now transfer the Σ_3-action from T/W to $\mathrm{L}T/\widetilde{W}$. For any $x = (\theta_1, \theta_2, \theta_3) \in \mathrm{L}T$, we define

$$\begin{aligned} \mu \cdot x \;&:=\; (\theta_2 + 1/3, \theta_3 + 1/3, \theta_1 - 2/3), \\ \tau \cdot x \;&:=\; (-\theta_3, -\theta_2, -\theta_1). \end{aligned}$$

It is straightforward to check that the above indeed defines an action of Σ_3 on $\mathrm{L}T$. This action is by affine isometries; indeed,

$$\begin{aligned}\mu \cdot x &= s_{\alpha_1} \circ s_{\alpha_0}\big(x + (-2/3, 1/3, 1/3)\big),\\ \tau \cdot x &= s_{\alpha_0}(-x),\end{aligned}$$

where $\alpha_1 = (1, -1, 0)$ and $\alpha_0 = (1, 0, -1)$ are positive roots in the root system R. Note that the closed alcove $\overline{\mathcal{A}_0}$ is stable under this Σ_3-action. Essentially, μ rotates $\overline{\mathcal{A}_0}$ by an angle of $2\pi/3$, whereas τ acts as a reflection which fixes $(0, 0, 0)$. Let

$$q \;:\; \mathrm{L}T/\widetilde{W} \to T/W$$

be the homeomorphism $[x]_{\widetilde{W}} \mapsto [\exp(x)]_W$ for $x \in \mathrm{L}T$. Then q is a Σ_3-map, because

$$\begin{aligned} q\left([\,\mu \cdot x\,]_{\widetilde{W}}\right) &= q\left([\,x + (-2/3, 1/3, 1/3)\,]_{\widetilde{W}}\right)\\ &= [\,\exp(x) \cdot \mathrm{diag}(\zeta^2, -\zeta^2, -\zeta^2)\,]_W\\ &= [\,\exp(x)\zeta^2\,]_W\\ &= \mu \cdot [\,\exp(x)\,]_W\\ &= \mu \cdot q\left([\,x\,]_{\widetilde{W}}\right)\end{aligned}$$

and

$$\begin{aligned} q\left([\,\tau \cdot x\,]_{\widetilde{W}}\right) &= q\left([\,-x\,]_{\widetilde{W}}\right)\\ &= [\,\exp(-x)\,]_W\\ &= \left[\,\overline{\exp(x)}\,\right]_W\\ &= \tau \cdot [\,\exp(x)\,]_W\\ &= \tau \cdot q\left([\,x\,]_{\widetilde{W}}\right).\end{aligned}$$

We conclude that the map $e : \overline{\mathcal{A}_0} \to T/W$ is Σ_3-equivariant.

Consider the subset

$$\Psi_0 \;:=\; \{(\theta_1, \theta_2, \theta_3) \in \mathrm{L}T \;:\; \theta_1 \geq \theta_2 \geq 0,\; \theta_3 \geq -1/3\},$$

obtained from $\overline{\mathcal{A}_0}$ by barycentric subdivision. Then Ψ_0 is a cross section of $\overline{\mathcal{A}_0}/\Sigma_3$. In view of the Σ_3-equivariant homeomorphism $\mathscr{H}_{\mathrm{II}}/K \cong \overline{\mathcal{A}_0}$ from (2.26), we have completed the classification of the type II subalgebras.

Theorem 2.4.2. *A set of representatives of the $\bar{K}$-conjugacy classes of $\mathscr{H}_{\mathrm{II}}$ is given by*

$$\{S_{\mathrm{II}}\big(\exp(x/2)\big) \ : \ x \in \Psi_0\}.$$

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