The Cohomology Ring of a Finite Abelian Group

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

The cohomology ring of a finite cyclic group was explicitly computed by Cartan and Eilenberg in their 1956 book on Homological Algebra [8]. It is surprising that the cohomology ring for the next simplest example, that of a finite abelian group, has still not been treated in a systematic way. The results that we do have are combinatorial in nature and have been obtained using "brute force" computations.

In this thesis we will give a systematic method for computing the cohomology ring of a finite abelian group. A major ingredient in this treatment will be the Tate resolution of a commutative ring R (with trivial group action) over the group ring RG, for some finite abelian group G. Using the Tate resolution we will be able to compute the cohomology ring for a finite cyclic group, and confirm that this computation agrees with what is known from [8]. Then we will generalize this technique to compute the cohomology ring for a finite abelian group. The presentation we will give is simpler than what is in the literature to date.

We will then see that a straightforward generalization of the Tate resolution from a group ring to an arbitrary ring defined by monic polynomials will yield a method for computing the Hochschild cohomology algebra of that ring. In particular we will re-prove Theorem 3.2, Lemma 4.1, Lemma 5.1, Theorem 5.2 and Theorem 6.2 from [11], and Theorem 3.9 from [15] in a much more unified way than they were originally proved. We will also be able to prove some new results.

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Dedication

This thesis is dedicated with love to my parents, Donald and Janet Roberts. I would never have finished it without their unfailing support.

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Chapter 1

Introduction

1.1 Overview

The cohomology ring of a finite cyclic group was explicitly computed by Cartan and Eilenberg in their 1956 book on Homological Algebra [8]. It is surprising that the cohomology ring for the next simplest example, that of a finite abelian group, has still not been treated in a systematic way. The results that we do have are combinatorial in nature and have been obtained using "brute force" computations.

In this thesis we will give a systematic treatment for the cohomology ring of a finite abelian group. A major ingredient in this treatment will be the Tate resolution of a commutative ring R (with trivial group action) over the group ring RG, for some finite abelian group G. Using the Tate resolution we will be able to compute the cohomology ring for a finite cyclic group, and confirm that this computation agrees with what is known from [8]. Then we will generalize this technique to compute the cohomology ring for a finite abelian group.

The Tate resolution was given in Theorem 4 of [16]. Immediately after proving this theorem, Tate gave the following application.

> Application 1: Let F be the free abelian group on generators $u_1, \dots, u_n$ and let $R = \mathbb{Z}[u_1, u_1^{-1}, \dots u_n, u_n^{-1}]$ be the group ring of F with integral coefficients. Let $t_i = u_i - 1,\ 1 \leq i \leq n$, and let $M = (t_1, \dots, t_n)$. Let $a_i = u_i^{e_i} - 1,\ 1 \leq i \leq r$, with positive integers $e_1|e_2|\cdots|e_r$, and let $A = (a_1, \dots, a_r)$. Then $\overline{R} = \frac{R}{A}$ is the group ring of the abelian group $\overline{F}$ generated by elements $\overline{u_i}$ with the

> relations $\overline{u_i}^{e_i} = 1,\ 1 \leq i \leq r$, that is, of the direct product of cyclic groups of order $e_i,\ 1 \leq i \leq r$, and $n - r$ infinite cyclic groups. Theorem 4 then yields a free resolution of the $\overline{F}$-module $Z = \frac{R}{M} = \frac{\overline{R}}{\overline{M}}$, a resolution which can be used efficiently to compute the cohomology and homology groups of the finitely generated abelian group $\overline{F}$.

This thesis will give the answer for which Tate asked, for cohomology, when all the generators have finite order (i.e. when $r = n$).

Although we will use some complicated machinery during our analysis, it will turn out that the dualized complex in which we compute our products is a Koszul complex. While the Koszul complex carries a natural algebra structure, this is not the multiplicative structure that we seek. Rather, we will define the cup product of cochains, following the method from [8]. Even though the Koszul complex is fairly simple, its cohomology can still be complicated. This may explain why so much "brute force" has been required to obtain the results that are known to date. In Chapter 6, we will describe an algorithm for computing the cohomology ring for any finite abelian group, and we will explicitly compute the integral cohomology ring for a product of two cyclic groups, and a few more examples. The presentations we will give for the examples in Chapter 6 are simpler than what is in the literature to date. We will then demonstrate that our results verify and complete results in [12], and agree with results in [9], [17] and [6].

In Chapter 7, we will see that a straightforward generalization of the Tate resolution from a group ring to an arbitrary ring defined by monic polynomials will yield a method for computing the Hochschild cohomology algebra of that ring. This will enable us to re-prove some results from the literature in a much more unified way than they were originally proved. In particular, our results will verify and complete results in [11], and agree with a result in [15].

Throughout this thesis, we will see that the most difficult part of the analysis is choosing a correct diagonal approximation for the given setup. Once a correct diagonal approximation has been chosen, the cup product structure is determined. Tate's Theorem is a powerful tool in allowing us to choose a correct diagonal approximation, because the diagonal approximation is determined on the whole resolution once we have defined our maps in degrees zero, one and two. This is why we are able to avoid the combinatorial approach that has been used so often to date.

1.2 Chapter Summaries

Chapter 2 will recall all the foundational machinery that is required.

In Chapter 3 we will begin with a projective resolution of our ring (with trivial group action) over the group algebra. Then we will define a diagonal approximation from the resolution to the tensor product of two copies of the resolution. Next we will dualize our resolution into a trivial representation. We will use our diagonal approximation to define the cup product of cochains in the dual. Last, we will recall that the cup product is homotopic to the Yoneda product on the underlying chain complex, and thus induces the same product in the cohomology ring of the group with coefficients in R, that is, in the *Ext* algebra.

In Chapter 4 we will construct the Tate resolution for a finite cyclic group. Then we will exhibit a diagonal approximation which will enable us to define the cup products. Last, we will verify that the product we obtain agrees with the known results from [8].

In Chapter 5 we will generalize the setup from Chapter 4 to a product of cyclic groups, in other words, to any finite abelian group.

In Chapter 6 we will give the structure of the cohomology ring of a finite abelian group as a fibre product of quotients of polynomial rings. We will also verify that our results agree with what is already known in the literature.

In Chapter 7 we will generalize the setup from Chapter 5 to a ring defined by monic polynomials. This will allow us to obtain some results about Hochschild Cohomology for a hypersurface ring defined by a monic polynomial.

Chapter 2

Preliminaries

In this chapter we recall all the machinery that will be needed throughout the thesis.

2.1 Conventions

We establish the following conventions for use throughout the thesis.

1. Unless otherwise stated, G denotes a finite group and R denotes a commutative ring with unity.
2. The map $\mu : R \otimes_R R \to R$ is the multiplication map.
3. In general, modules over a not necessarily commutative ring have their ring action on the right.
4. If we are in the setting of a graded ring or module, then we use the absolute value bars $|\cdot|$ to denote the degree of an element.
5. When applicable, the notation $\otimes$ denotes the graded tensor product of rings or algebras, which is the usual tensor product, with the multiplication law:

$$(a \otimes b)(a' \otimes b') = (-1)^{|b||a'|}(aa') \otimes (bb')$$

6. We use the following notation for complexes of R-modules:

(a) Every complex is indexed over the integers, potentially with lots of zero terms.

(b) With homological grading $C_i \xrightarrow{d_i} C_{i-1}$, the differential lowers the degree by one.

(c) With cohomological grading $C^j \xrightarrow{d^j} C^{j+1}$, the differential increases the degree by one.

(d) We can freely switch from one to the other by setting $C^j = C_{-j}$.

(e) We denote by $C[m]$ the complex such that $C[m]^j = C^{j+m}$, i.e. $C[m]_i = C_{i-m}$. In either case, we shift against the direction of the differential.

(f) Shifting a complex by 1 degree in either direction reverses the sign of the differential.

7. Whenever we need to, we may view R as the complex $0 \longrightarrow R \longrightarrow 0$, with R in (co)homological degree zero.

8. Many of our complexes, especially the ones we obtain by applying Tate's Theorem, will have a highly useful additional structure, that of a *DG*-algebra (See Definition 2.7.6).

   Suppose that (A, ∂) is a *DG* R-algebra. Then since $\partial^2 = 0$, (A, ∂) is a complex of R-modules.

2.2 The Norm Map Isomorphism

In this section we establish an extremely useful isomorphism, which will be used throughout the thesis. More sophisticated proofs of this result exist in the literature; we present a "down-to-earth" proof here.

Definition 2.2.1. *Let R be a ring and let M be a right R-module. Then the R-dual of M is the left R-module:*

$$M^* = Hom_R(M, R)$$

Definition 2.2.2. *Let R be a ring and let M, N be right R-modules. Define the norm map as*

$$\begin{array}{rccl} \nu & : & N \otimes_R M^* & \rightarrow & Hom_R(M, N) \\ & : & n \otimes \lambda & \mapsto & \varphi_{n\otimes\lambda} : m \mapsto n \cdot \lambda(m) \end{array}$$

Theorem 2.2.3. *The norm map is an isomorphism for all N if and only if M is a finitely generated projective R-module.*

Proof. For the forward direction, assume that ν is an isomorphism for all N. We will prove that M is a finitely generated projective R-module.

Proof that M is projective: We will prove that $Hom_R(M, -)$ is an exact functor. Take any short exact sequence of R-modules

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0\ .$$

We will show that

$$0 \longrightarrow Hom_R(M, A) \xrightarrow{i_*} Hom_R(M, B) \xrightarrow{p_*} Hom_R(M, C) \longrightarrow 0$$

is exact. It suffices to prove that

$$Hom_R(M, B) \xrightarrow{p_*} Hom_R(M, C)$$

is surjective.

Since ν is an isomorphism for all N, we may construct the following diagram in which the vertical maps are isomorphisms:

$$\begin{tikzcd} B \otimes_R M^* \arrow[r, "p\otimes 1_{M^*}"] \arrow[d, "\nu_B", "\cong"'] & C \otimes_R M^* \arrow[d, "\nu_C", "\cong"'] \\ Hom_R(M, B) \arrow[r, "p_*"'] & Hom_R(M, C). \end{tikzcd}$$

I claim that this diagram commutes. Let $b \otimes \lambda \in B \otimes_R M^*$ be arbitrary. The clockwise branch yields

$$\nu_C(p \otimes 1_{M^*})(b \otimes \lambda) = \nu_C(p(b) \otimes \lambda) = [m \mapsto p(b) \cdot \lambda(m)]$$

The counterclockwise branch yields

$$p_*\nu_B(b \otimes \lambda) = p_*[m \mapsto b\lambda(m)] = [m \mapsto p(b\lambda(m)) \underbrace{=}_{p \text{ is } R\text{-linear}} p(b) \cdot \lambda(m)]$$

so the diagram commutes on a generating set of $B \otimes_R M^*$, and thus commutes as claimed.

Let $\alpha \in Hom_R(M, C)$ be arbitrary. Since ν_C is an isomorphism, there exists an element $\beta \in C \otimes_R M^*$ such that $\alpha = \nu_C(\beta)$. Since $(- \otimes_R M^*)$ is right exact, $B \otimes_R M^* \xrightarrow{p \otimes 1_{M^*}} C \otimes_R M^*$ is surjective. Therefore there exists an element $\gamma \in B \otimes_R M^*$ such that $\beta = (p \otimes 1_{M^*})(\gamma)$. The clockwise branch then reads $\alpha = \nu_C(\beta) = \nu_C(p \otimes 1_{M^*})(\gamma)$. Since the diagram commutes, this implies that $\alpha = p_* \nu_B(\gamma)$. Thus the element $\nu_B(\gamma) \in Hom_R(M, B)$ witnesses the fact that $\alpha \in Im\ p_*$, so that p_* is surjective as required.

Since p_* is surjective, $Hom_R(M, -)$ is exact and therefore M is projective as required.

Proof that M is finitely generated:

Letting $N = M$ gives us the isomorphism $\nu_M : M \otimes_R M^* \to Hom_R(M, M)$. Since $1_M \in Hom_R(M, M)$, there exists some element $\delta \in M \otimes_R M^*$ such that $1_M = \nu_M(\delta)$. Write δ as a finite sum $\sum_j m_j \otimes \lambda_j$. I claim that M is generated by the m_j. Let $x \in M$ be arbitrary. Then

$$\begin{aligned} x &= 1_M(x) \\ &= \nu_M(\delta)(x) \\ &= \nu_M\left(\sum_j m_j \otimes \lambda_j\right)(x) \\ &= \sum_j \nu_M(m_j \otimes \lambda_j)(x) \\ &= \sum_j \varphi_{m_j \otimes \lambda_j}(x) \\ &= \sum_j m_j \cdot \underbrace{\lambda_j(x)}_{\in R,\ \forall j} \end{aligned}$$

so since $x \in M$ was arbitrary, M is generated by the m_j, and thus is finitely generated.

For the backwards direction, assume that M is a finitely generated projective R-module. Let N be arbitrary. We must prove that ν as defined above is an isomorphism. We first establish the result for finitely generated free modules, then we show how the result follows for direct summands of these, i.e. for all finitely generated projective modules.

Since M is finitely generated and projective, by the proof of Proposition 7.56 in [14], M is a direct summand of a finitely generated free module F. Let F have a finite R-basis $\{e_j\}$. Then F^* is free with R-basis $\{e_j^*\}$.

It is routine to check that the following function is R-bilinear:

$$\begin{array}{rccl} \eta' & : & N \times F^* & \to & Hom_R(F,N) \\ & & \left(n, \sum_j r_j e_j^*\right) & \mapsto & [e_j \mapsto n \cdot r_j], \end{array}$$

and thus we obtain a homomorphism of abelian groups

$$\begin{array}{rccl} \eta & : & N \otimes_R F^* & \to & Hom_R(F,N) \\ & & n \otimes (\sum_j r_j e_j^*) & \mapsto & [e_j \mapsto n \cdot r_j]. \end{array}$$

We will show that η is a bijection, and thus an isomorphism.

Now define

$$\begin{array}{rcccc} \zeta & : & Hom_R(F,N) & \to & N \otimes_R F^* \\ & & \alpha & \mapsto & \sum_j \alpha(e_j) \otimes e_j^* \end{array}$$

<u>Proof that $\eta\zeta = 1_{Hom_R(F,N)}$</u>: Let $\alpha \in Hom_R(F,N)$ be arbitrary, and take any basis element e_j. Then

$$\eta\zeta(\alpha)(e_j) = \eta\left(\sum_k \alpha(e_k) \otimes e_k^*\right)(e_j) = [e_k \mapsto \alpha(e_k)](e_j) = \alpha(e_j),$$

so that $\eta\zeta(\alpha)$ and α agree on a basis of F, and therefore on all of F. In other words $\eta\zeta(\alpha) = \alpha$ as functions. Since α was arbitrary, therefore $\eta\zeta = 1_{Hom_R(F,N)}$ as claimed.

<u>Proof that $\zeta\eta = 1_{N \otimes_R F^*}$</u>: Let $n \otimes \left(\sum_j r_j \cdot e_j^*\right) \in N \otimes_R F^*$ be arbitrary. Then

$$\zeta\eta\left(n \otimes \left(\sum_j r_j \cdot e_j^*\right)\right) = \zeta\,[e_j \mapsto n \cdot r_j] = \sum_j n \cdot r_j \otimes e_j^* \underbrace{=}_{r_j \in R} \sum_j n \otimes r_j \cdot e_j^* = n \otimes \left(\sum_j r_j \cdot e_j^*\right).$$

Thus $\zeta\eta$ is the identity map on a generating set of $N \otimes_R F^*$, and is therefore the identity on all of $N \otimes_R F^*$.

Now we must show that the result holds for direct summands of finitely generated free modules, i.e. for finitely generated projective modules. Suppose that the norm map is an isomorphism

$$\begin{array}{rcccl} \nu & : & N \otimes_R F^* & \to & Hom_R(F,N) \\ & : & n \otimes \lambda & \mapsto & \varphi_{n \otimes \lambda} : f \mapsto n \cdot \lambda(f) \end{array}$$

Further suppose that F is isomorphic to a direct sum $F \cong F_1 \oplus F_2$. Then we claim that ν restricts to

$$\begin{array}{rcrcl}\nu_1 & : & N \otimes_R F_1^* & \to & Hom_R(F_1, N) \\ & & n \otimes \lambda_1 & \mapsto & \varphi_{n\otimes\lambda_1} : f_1 \mapsto n \cdot \lambda_1(f_1)\end{array}$$

which is an isomorphism.

By Corollary 7.34 in [14], we have

$$\begin{array}{rcl}F^* & = & Hom_R(F, R) \\ & \cong & Hom_R(F_1 \oplus F_2, R) \\ & \cong & Hom_R(F_1, R) \oplus Hom_R(F_2, R) \\ & = & F_1^* \oplus F_2^*, \text{ and} \\ Hom_R(F, N) & \cong & Hom_R(F_1 \oplus F_2, N) \\ & \cong & Hom_R(F_1, N) \oplus Hom_R(F_2, N),\end{array}$$

so we have an isomorphism

$$\nu \;:\; (N \otimes_R F_1^*) \oplus (N \otimes_R F_2^*) \;\to\; Hom_R(F_1, N) \oplus Hom_R(F_2, N).$$

Let $n \otimes \lambda_1 \in N \otimes_R F_1^*$ be arbitrary. Since $F \cong F_1 \oplus F_2$, any $f \in F$ has a unique decomposition $f = f_1 + f_2$, with $f_1 \in F_1,\ f_2 \in F_2$. Thus we may define

$$\begin{array}{rcccl}\lambda & : & F & \to & N \\ & & f_1 + f_2 & \mapsto & \lambda_1(f_1).\end{array}$$

Then λ extends λ_1 to all of F.

Then by the definition of ν we have

$$\nu(n \otimes \lambda) = [f \mapsto n \cdot \lambda(f)] \text{ , and}$$

$$n \cdot \lambda(f) = n \cdot \lambda(f_1 + f_2) = n \cdot \lambda_1(f_1) = \varphi_{n\otimes\lambda_1}(f_1).$$

This shows that the projection of $\varphi_{n\otimes\lambda}$ onto $Hom_R(F_1, N)$ equals $\varphi_{n\otimes\lambda_1} \in Hom_R(F_1, N)$, so that ν restricts to ν_1 as claimed.

Similarly, ν restricts to

$$\begin{array}{rcrcl}\nu_2 & : & N \otimes_R F_2^* & \to & Hom_R(F_2, N) \\ & & n \otimes \lambda_2 & \mapsto & \varphi_{n\otimes\lambda_2} : f_2 \mapsto n \cdot \lambda_2(f_2)\end{array}$$

Now ν_1 is injective, because ν is. For surjectivity of ν_1, let $\alpha \in Hom_R(F_1, N)$ be arbitrary. Then $(\alpha, 0) \in Hom_R(F_1, N) \oplus Hom_R(F_2, N) \cong Hom_R(F, N)$. Since ν is surjective, there exists some element $(x, y) \in (N \otimes_R F_1^*) \oplus (N \otimes_R F_2^*)$ such that

$$(\alpha, 0) = \nu(x, y) = (\nu_1(x), \nu_2(y)).$$

Therefore $\alpha = \nu_1(x)$, and thus ν_1 is surjective, as required.

We have shown that ν restricts to ν_1, which is an isomorphism. Therefore the norm map is an isomorphism whenever M is a finitely generated projective module, and so we are done. $\square$

2.3 The Koszul Complex

Throughout the thesis we will make frequent use of the Koszul complex.

Definition 2.3.1. *If $x \in R$ is central, we let $\mathbb{K}(x)$ denote the chain complex*

$$\begin{tikzcd} & 1 & 0 & \\ 0 \arrow[r] & R \arrow[r, "x"] & R \arrow[r] & 0 \end{tikzcd}$$

concentrated in homological degrees 1 and 0. It is convenient to identify the generator of the degree 1 part of $\mathbb{K}(x)$ as e, so that $d(e) = x$. With this convention, we re-draw the above picture:

$$\begin{tikzcd} & 1 & 0 & \\ 0 \arrow[r] & R \arrow[r, "x"] \arrow[d, equal] & R \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Re \arrow[r, "d"] & R \arrow[r] & 0 \\ & e \arrow[r, mapsto] & x & \end{tikzcd}$$

We may view $\mathbb{K}(x)$ as the DG-algebra $\frac{R[e]}{e^2}$, with $|e| = 1$. Then d is the skew algebra derivation $x\frac{\partial}{\partial e}$.

If $\mathbf{x} = (x_1, \ldots, x_n)$ is a finite sequence of central elements in R, then we define the Koszul complex $\mathbb{K}(\mathbf{x})$ to be the total tensor product complex.

$$\mathbb{K}(x_1) \otimes_R \cdots \otimes_R \mathbb{K}(x_n)$$

The degree p part of $\mathbb{K}(\mathbf{x})$ is a free R-module generated by the symbols

$$e_{i_1} \wedge \cdots \wedge e_{i_p} (i_1 < \cdots < i_p), \text{ where } e_k = 1 \otimes \cdots \otimes 1 \otimes \underbrace{e_k}_{\text{position } k} \otimes 1 \otimes \cdots \otimes 1.$$

In particular, $\mathbb{K}_p(\mathbf{x})$ is isomorphic to the pth exterior power $\bigwedge^p R^n$ of R^n and has rank $\left(\begin{array}{c} n \\ p \end{array} \right)$. The differential is

$$\begin{array}{cccc} d & : & \mathbb{K}_p(\mathbf{x}) & \rightarrow & \mathbb{K}_{p-1}(\mathbf{x}) \\ & & e_{i_1} \wedge \cdots \wedge e_{i_p} & \mapsto & \sum_{k=1}^{p} (-1)^{k+1} x_{i_k} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_p} \end{array}$$

This is an algebra derivation, equal to $\sum_k x_k \frac{\partial}{\partial e_k}$.

2.4 Regular Sequences

Regular sequences are a key ingredient in the statement of Tate's Theorem, and Tate's Theorem is the key ingredient in all our later constructions. Thus we establish the required definitions, and the crucial examples which satisfy the definitions here.

Definition 2.4.1. *Let R be a ring and let M be an R-module. A sequence of elements $x_1, \ldots, x_n \in R$ is called a regular sequence on M (or an M-sequence) if*

1. *$(x_1, \ldots, x_n)M \neq M$, and*
2. *For $i = 1, \ldots, n$, x_i is a non zero divisor on $\frac{M}{(x_1, \ldots, x_{i-1})M}$.*

In particular, for $M = R$, we have the notion of an R-sequence or regular sequence on R. By contrast, we have

Definition 2.4.2. *Any sequence $\mathbf{x} = (x_1, \ldots, x_n)$ such that $H_i(\mathbb{K}(\mathbf{x})) = 0$, for all $i \geq 1$ is called a Koszul regular sequence.*

Remarks:

1. An R-sequence is Koszul regular (see [18], Corollary 4.5.5). A Koszul regular sequence is not necessarily an R-sequence.

2. A Koszul regular sequence of length one is simply a non zero divisor.

   Let $x \in R$ be a non zero divisor. We then require the following sequence to be exact:

   $$0 \longrightarrow R \xrightarrow{x} R ,$$

   in other words, we require multiplication by x to be injective in R. But this is clear because x is a non zero divisor in R, and therefore multiplication by x has a trivial kernel.

   By contrast, an R-sequence of length one is a non zero divisor which is <u>not a unit</u>.

   Thus an example of a sequence which is Koszul regular but not an R-sequence is simply $\{1\}$. As we know, 1 is a non zero divisor, which implies the sequence is Koszul regular. However 1 is a unit, and thus $\{1\}$ is not an R-sequence.

3. The definition of an R-sequence depends on the order in which we write down the elements. For example, let k be a field, and let $R = k[x, y, z]$. Define $a = x(y-1)$, $b = y$ and $c = z(y-1)$. Then $(a, b, c)R = (x, y, z)R \neq R$, and $\{a, b, c\}$ is an R-sequence while $\{a, c, b\}$ is not an R-sequence. However if (R, m, k) is a local noetherian ring, and if $x_1, \ldots, x_n \in m$ form an R-sequence, then any permutation of $x_1, \ldots, x_n$ again form an R-sequence. See [10], Corollary 17.2.

4. Being Koszul regular does not depend on the order in which we write down the elements.

Example 2.4.3. *If R is any ring, and $f(x)$ is any monic polynomial in $R[x]$, then $(f(x))$ is a Koszul regular sequence in $R[x]$.*

This is a simple consequence of Remark 2 above, since $f(x)$ is a non zero divisor if it is monic.

Example 2.4.4. *More generally, $(f_1(x_1), \ldots, , f_r(x_r))$, where each f_i is monic, is a Koszul regular sequence in $R[x_1, \ldots, x_r]$.*

Note that since each $f_i(x_i)$ is monic, it is a non zero divisor in $R[x_1, \ldots, x_r]$.

The proof is by induction on r. For the rest of the proof, unadorned tensor products are over R.

In the base case ($r = 1$), The result follows by Example 2.4.3 above.

For the induction step, assume that $(f_1(x_1), \ldots, , f_k(x_k))$ is Koszul regular, for some $1 \leq k < r$. We have that

$$\mathbb{K}(f_1(x_1), \ldots, , f_k(x_k), f_{k+1}(x_{k+1})) \cong \mathbb{K}(f_1(x_1), \ldots, , f_k(x_k)) \otimes \mathbb{K}(f_{k+1}(x_{k+1}))$$

Let (C, ∂) denote $\mathbb{K}(f_1(x_1), \ldots, , f_k(x_k))$. Then by the induction hypothesis, (C, ∂) is acylic in all degrees ≥ 1.

Since $\mathbb{K}(f_{k+1}(x_{k+1}))$ is concentrated in degrees 0 and 1, the total complex is

$$\mathbf{Tot} := \quad \begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 & C_0 \otimes R[x_{k+1}] \arrow[l] \arrow[d, "1\otimes(f_{k+1}(x_{k+1}))"] & C_1 \otimes R[x_{k+1}] \arrow[l, "\partial\otimes 1"'] \arrow[d, "-1\otimes(f_{k+1}(x_{k+1}))"] & C_2 \otimes R[x_{k+1}] \arrow[l, "\partial\otimes 1"'] \arrow[d, "1\otimes(f_{k+1}(x_{k+1}))"] & \cdots \arrow[l] \\
0 & C_0 \otimes R[x_{k+1}] \arrow[l] \arrow[d] & C_1 \otimes R[x_{k+1}] \arrow[l, "\partial\otimes 1"] \arrow[d] & C_2 \otimes R[x_{k+1}] \arrow[l, "\partial\otimes 1"] \arrow[d] & \cdots \arrow[l] \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The rows are exact in degrees ≥ 1 because $R[x_{k+1}]$ is free, and therefore flat, over R.

The following diagram commutes

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 & C_0 \otimes R[x_{k+1}] \arrow[l] \arrow[d, "1\otimes(f_{k+1}(x_{k+1}))"] & C_1 \otimes R[x_{k+1}] \arrow[l, "\partial\otimes 1"'] \arrow[d, "1\otimes(f_{k+1}(x_{k+1}))"] & C_2 \otimes R[x_{k+1}] \arrow[l, "\partial\otimes 1"'] \arrow[d, "1\otimes(f_{k+1}(x_{k+1}))"] & \cdots \arrow[l] \\
0 & C_0 \otimes R[x_{k+1}] \arrow[l] \arrow[d] & C_1 \otimes R[x_{k+1}] \arrow[l, "\partial\otimes 1"] \arrow[d] & C_2 \otimes R[x_{k+1}] \arrow[l, "\partial\otimes 1"] \arrow[d] & \cdots \arrow[l] \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

and thus the vertical maps assemble into a morphism of complexes. Denote this morphism of complexes by f. We may then construct the complex $cone(f)$, using Definition 1.5.1 of [18].

I claim that $cone(f) \cong \mathbf{Tot}$ as complexes of R-modules. Define

$$\begin{array}{rcl} \Psi \; : \; cone(f) & \to & \mathbf{Tot} \\ (b,c) & \mapsto & \begin{cases} (-b,c) & \text{in odd degrees} \\ (b,c) & \text{in even degrees} \end{cases} \end{array}$$

It is routine to verify that Ψ is an isomorphism of chain complexes of R-modules.

Therefore we will be finished if we can show that $H_i(cone(f)) = 0$, for all $i \geq 1$.

By Lemma 1.5.3 in [18], we have the long exact sequence

$$\begin{tikzcd}
H_0(cone(f)) & H_0(C \otimes R[x_{k+1}]) \arrow[l] & H_0(C \otimes R[x_{k+1}]) \arrow[l, "1\otimes(f_{k+1})"'] & H_1(cone(f)) \arrow[l] \\
& H_1(C \otimes R[x_{k+1}]) \arrow[urr, bend left] & H_1(C \otimes R[x_{k+1}]) \arrow[l, "1\otimes(f_{k+1})"'] & H_2(cone(f)) \arrow[l] \\
& {} \arrow[urr, bend left] & \vdots & {} \\
& H_i(C \otimes R[x_{k+1}]) \arrow[urr, bend left] & H_i(C \otimes R[x_{k+1}]) \arrow[l, "1\otimes(f_{k+1})"'] & H_{i+1}(cone(f)) \arrow[l] \\
& H_{i+1}(C \otimes R[x_{k+1}]) \arrow[urr, bend left] & \cdots \arrow[l] &
\end{tikzcd}$$

and since the rows of the original diagram remain exact in all degrees ≥ 1, we can see that $H_i(cone(f)) = 0$, for all $i \geq 2$. We still need to prove that $H_1(cone(f)) = 0$.

We have the following exact sequence remaining:

$$H_0(cone(f)) \longleftarrow H_0(C \otimes R[x_{k+1}]) \xleftarrow{1\otimes(f_{k+1})} H_0(C \otimes R[x_{k+1}]) \longleftarrow H_1(cone(f)) \longleftarrow 0\,.$$

Because the terms of C are free and therefore flat over R, the Künneth Formula (Theorem 3.6.1 in [18]) implies that

$$H_0(C \otimes R[x_{k+1}]) \cong H_0(C) \otimes R[x_{k+1}].$$

Now observe that

$$H_0(C) \cong \frac{R[x_1]}{(f_1)} \otimes \cdots \otimes \frac{R[x_k]}{(f_k)} \cong \frac{R[x_1, \ldots, x_k]}{(f_1(x_1), \ldots, f_k(x_k))}$$

is free, and therefore flat, over R.

We claim that $H_0(C \otimes R[x_{k+1}]) \xleftarrow{1\otimes(f_{k+1})} H_0(C \otimes R[x_{k+1}])$ is injective. Since $f_{k+1}(x_{k+1})$ is a non-zero divisor, multiplication by $f_{k+1}(x_{k+1})$ is injective on $R[x_{k+1}]$. Therefore, since $H_0(C)$ is flat over R, $1 \otimes (f_{k+1})$ is injective on $H_0(C) \otimes_R R[x_{k+1}] \cong H_0(C \otimes R[x_{k+1}])$, as claimed.

Then the remaining long exact sequence becomes

$$H_0(cone(f)) \longleftarrow H_0(C \otimes R[x_{k+1}]) \xleftarrow{1\otimes(f_{k+1})} H_0(C \otimes R[x_{k+1}]) \xleftarrow{0} H_1(cone(f)) \longleftarrow 0$$

and so it is clear that $H_1(cone(f)) = 0$, as required.

2.5 The *Hom* Complex

By working in the *Hom* complex, some labourious computations can be streamlined. Therefore we establish the needed framework here.

For this section, we work in the category of complexes of R-modules for some commutative ring R. We could do the same construction in any abelian category.

Given two complexes

$$\begin{tikzcd}
C_\bullet \arrow[r, equal] & \cdots \arrow[r] & C_1 \arrow[r, "d_{C1}"] \arrow[dr, dotted] & C_0 \arrow[r, "d_{C0}"] \arrow[dr, dotted] & C_{-1} \arrow[r] & \cdots \\
D_\bullet \arrow[r, equal] & \cdots \arrow[r] & D_1 \arrow[r, "d_{D1}"'] & D_0 \arrow[r, "d_{D0}"'] & D_{-1} \arrow[r] & \cdots
\end{tikzcd}$$

our goal is to construct the <u>*Hom* complex</u> $Hom^\bullet(C_\bullet, D_\bullet)$ such that

1. $H^0(Hom^\bullet(C_\bullet, D_\bullet)) =$ homotopy classes of morphisms of complexes $C_\bullet \to D_\bullet$, and
2. $H^i(Hom^\bullet(C_\bullet, D_\bullet)) =$ homotopy classes of morphisms of complexes $C_\bullet \to D[i]_\bullet$.

Remarks:

1. A morphism of complexes $C_\bullet \to D_\bullet$ is a map of degree 0.
2. The dotted arrows above describe a map of degree 1.
3. We will assemble maps of all degrees here.

Definition 2.5.1. *The ith term of the* <u>*Hom complex*</u> $Hom^\bullet(C_\bullet, D_\bullet)$ *is*

$$Hom^i(C_\bullet, D_\bullet) := \prod_j Hom_R(C_j, D[i]_j)$$

$$\begin{tikzcd}
C_\bullet & \cdots \arrow[r] & C_{j+1} \arrow[r] \arrow[d] & C_j \arrow[r] \arrow[d] & C_{j-1} \arrow[r] \arrow[d] & \cdots \\
D[i]_\bullet & \cdots \arrow[r] & D[i]_{j+1} \arrow[r] \arrow[d, equal] & D[i]_j \arrow[r] \arrow[d, equal] & D[i]_{j-1} \arrow[r] \arrow[d, equal] & \cdots \\
 & & D_{j-i+1} & D_{j-i} & D_{j-i-1} &
\end{tikzcd}$$

Denote an arbitrary element of $Hom^i(C_\bullet, D_\bullet)$ *by* φ*. Then we can write* $\varphi = \{\varphi_j\}_{j\in\mathbb{Z}}$*, where each* $\varphi_j : C_j \to D[i]_j$ *lies in* $Hom_R(C_j, D[i]_j) = Hom_R(C_j, D_{j-i})$*. The differential is*

$$\begin{array}{rcccl} d & : & Hom^i(C_\bullet, D_\bullet) & \to & Hom^{i+1}(C_\bullet, D_\bullet) \\ & & \varphi_j & \mapsto & d_D\varphi_j - (-1)^i \varphi_{j+1} d_C \end{array}$$

This is a differential, since, for any φ_j*, we have*

$$\begin{aligned}
d^2\varphi_j &= d(\underbrace{d_D\varphi_j - (-1)^i\varphi_{j+1}d_C}_{\textit{denote this by } \psi_{j+1}\in Hom_R(C_j, D[i+1]_j)}) \\
&= d_D\psi_{j+1} - (-1)^{i+1}\psi_{j+2}d_C \\
&= d_D[d_D\varphi_j - (-1)^i\varphi_{j+1}d_C] - (-1)^{i+1}[d_D\varphi_{j+1} - (-1)^{i+1}\varphi_{j+2}d_C]d_C \\
&= \underbrace{d_Dd_D}_{=0}\varphi_j - (-1)^i d_D\varphi_{j+1}d_C - (-1)^{i+1}d_D\varphi_{j+1}d_C + (-1)^{2i+2}\varphi_{j+2}\underbrace{d_Cd_C}_{=0} \\
&= 0.
\end{aligned}$$

So we do indeed have a complex.

Remarks:

1. For any i, we have
$$Hom^\bullet(C_\bullet, D_\bullet[i]) \cong Hom^\bullet(C_\bullet, D_\bullet)[i].$$

2. In general for a complex $E^\bullet$, $H^i(E^\bullet) \cong H^0(E[i]^\bullet)$. So we only need to compute $H^0(Hom^\bullet(C_\bullet, D_\bullet))$, and all other degrees are then understood via shifts.

Recall that
$$H^0(Hom^\bullet(C_\bullet, D_\bullet)) = \frac{\ker d^0}{im\ d^{-1}}$$

Let $\varphi \in \ker d^0$ be arbitrary, i.e. $\varphi \in Hom^0(C_\bullet, D_\bullet)$ with $d\varphi = 0$. Then we have, for any j that

$$\begin{aligned} 0 &= d\varphi_j \\ &= d_D\varphi_j - (-1)^0\varphi_{j+1}d_C \\ &= d_D\varphi_j - \varphi_{j+1}d_C \\ \varphi_{j+1}d_C &= d_D\varphi_j \end{aligned}$$

which holds if and only if φ is a morphism of complexes.

Now suppose that $\varphi - \psi \in im\ d^{-1}$ is arbitrary. Then $\varphi - \psi = d^{-1}s$, for some map s of degree -1.

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[r, "d_C"] \arrow[d, dotted, "(\varphi-\psi)_{n+1}"'] & C_n \arrow[r, "d_C"] \arrow[dl, "s_n"] \arrow[d, dotted, "(\varphi-\psi)_n"] & C_{n-1} \arrow[r] \arrow[dl, "s_{n-1}"] \arrow[d, dotted, "(\varphi-\psi)_{n-1}"] & \cdots \\ \cdots \arrow[r] & D_{n+1} \arrow[r, "d_D"'] & D_n \arrow[r, "d_D"'] & D_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

Then

$$\begin{aligned} \varphi - \psi &= d^{-1}s \\ &= d_Ds - (-1)^{-1}sd_C \\ &= d_Ds + sd_C, \end{aligned}$$

i.e. φ and ψ are homotopic.

We have constructed $Hom^\bullet(C_\bullet, D_\bullet)$ such that $H^0(Hom^\bullet(C_\bullet, D_\bullet))$ = homotopy classes of morphisms of complexes $C_\bullet \to D_\bullet$, as desired.

2.6 Divided Powers

Divided powers are another key ingredient in the statement of Tate's Theorem. Thus we establish their important properties before we proceed.

Here we follow the original treatment in [7], as well as [3] and [10].

Definition 2.6.1. *A $\mathbb{Z}$-graded ring $A = \oplus_{i\in\mathbb{Z}} A_i$ is graded commutative, if*

$$xy \;=\; (-1)^{|x||y|} yx \text{ for } x \in A_{|x|};\; y \in A_{|y|}. \tag{2.1}$$

It is strictly graded commutative if further

$$x^2 \;=\; 0 \text{ for } x \in A_{2i+1}.$$

Remark As line (2.1) already implies $2x^2 = 0$ for any odd element $x \in A_- = \oplus_i A_{2i+1}$, an algebra is graded commutative, but not strictly so, exactly when $ann_{A+}(2) \cap A_-^2 \neq 0$. This point becomes particularly relevant when $2 = 0$ in $A_+ = \oplus_i A_{2i}$.

If $2 = 0$ in a graded commutative algebra A, then the re-graded algebra $A'_i = A_{2i}$ is strictly graded commutative, equivalently, it is just commutative in the usual sense. Because of this, some authors allow in the following definition divided powers in *any* degree when $2 = 0$ in A. However, we rather stick to the classical definition with divided powers only in even degrees, re-grading, if we wish to capture the additional freedom in case of characteristic 2.

Definition 2.6.2. *Let $A = \oplus_{i\geq 0} A_i$ be a positively graded algebra that is strictly graded commutative. A system of divided powers on A assigns to each element $x \in A$ of even degree at least 2 a sequence of elements $(\gamma_k(x))_{k\geq 0}$ from A such that*

1. $\gamma_0(x) = 1,\; \gamma_1(x) = x,\; |\gamma_k(x)| = k|x|.$
2. $\gamma_k(x)\gamma_h(x) = \left(\begin{array}{c} k+h \\ k \end{array}\right) \gamma_{k+h}(x).$
3. *(The Binomial or Leibniz Formula)*

$$\gamma_k(x+y) = \sum_{i+j=k} \gamma_i(x)\gamma_j(y).$$

4. For $k \geq 2$,

$$\gamma_k(xy) = \begin{cases} 0 & \text{if } |x|,\ |y| \text{ are odd,} \\ x^k \gamma_k(y) & \text{if } |x|,\ |y| \text{ are even and } |y| \geq 2. \end{cases}$$

The element $\gamma_k(x)$ is called the $\underline{k^{th} \text{ divided power of } x}$.

Remark: It is often typographically more pleasing to write $x^{(k)} = \gamma_k(x)$. This may as well remind the reader that $\gamma_k(x)$ should be thought of as $\frac{x^k}{k!}$, even though in the given algebra one may not be able to divide by $k!$. We use both conventions interchangeably.

In terms of examples, we first give the obligatory trivial one.

Example 2.6.3. *If R is any commutative ring, placing it into degree zero turns it into a strictly graded commutative algebra over itself. As there are no elements of degree greater than 0, it carries the vacuous system of divided powers.*

We next review the two key examples of algebras with divided powers.

Example 2.6.4. *The polynomial ring $\mathbb{Q}[\mathbf{x}] = \mathbb{Q}[x_1, \ldots, x_s]$ over the rational numbers $\mathbb{Q}$ on s variables x_i, placed in even degrees, carries a system of divided powers given by the functional equation*

$$\begin{aligned} \exp(p(\mathbf{x})t) &= \sum_{k\geq 0} \gamma_k(p(\mathbf{x}))t^k, \text{ that is,} \\ \gamma_k(p(\mathbf{x})) &= \frac{1}{k!} p^k(\mathbf{x}). \end{aligned}$$

Its subalgebra

$$\Gamma_{\mathbb{Z}}(x_1, \ldots, x_s) = \mathbb{Z}\left[\frac{x_i^k}{k!}; i = 1, \ldots, s; k \geq 0\right] \subseteq \mathbb{Q}[x_1, \ldots, x_s]$$

is closed under these divided powers.

If R is any commutative ring, then $\gamma_k(r \otimes x_i) = r^k \otimes \gamma_k(x_i)$ gives rise to a unique system of divided powers on

$$\Gamma_R(x_1, \ldots, x_s) = R \otimes_{\mathbb{Z}} \Gamma_{\mathbb{Z}}(x_1, \ldots, x_s).$$

The second key example is provided by exterior algebras. Here, the system of divided powers seems to appear out of "thin air":

Example 2.6.5. *The exterior algebra $\bigwedge_R(y_1, \ldots, y_t)$ over any commutative ring R, with the variables y_j in odd degrees, is strictly graded commutative and carries a system of divided powers uniquely determined by the requirements (1) through (4) above. While condition (4) implies that $\gamma_k(y_{j_1} \cdots y_{j_{2m}}) = 0$, for any $k \geq 2$ and $m \geq 1$, condition (3) makes the structure nontrivial.*

For example, if one identifies the exterior 2-form $\omega = \sum_{1 \leq i < j \leq t} y_i y_j r_{ij}$ with the alternating $(t \times t)$-matrix whose entries from R above the diagonal are the r_{ij}, and with entries $r_{ii} = 0$ on the diagonal, and $r_{ji} = -r_{ij}$ below the diagonal, then the coefficient of $y_{j_1} \cdots y_{j_{2k}}$ in $\gamma_k(\omega)$ is the Pfaffian (see [5], §5.2) of the submatrix cut out by rows and columns $1 \leq j_1 < \cdots < j_{2k} \leq t$, a nontrivial polynomial, homogeneous of degree k in the coefficients r_{ij}. For a concrete example, the reader may readily verify

$$\gamma_2\left(\sum_{1 \leq i < j \leq 4} y_i y_j r_{ij}\right) = y_1 y_2 y_3 y_4 (r_{12} r_{34} - r_{13} r_{24} + r_{14} r_{23}).$$

The following crucial functorial property was also already established in [7], Theorem 2.

Theorem 2.6.6. *If A, B are strictly graded commutative R-algebras, each endowed with a system of divided powers, then $A \otimes_R B$, the graded tensor product algebra over R, carries a unique system of divided powers that extends those on A and B respectively.*

In view of (3) and (4), for $k \geq 2$ it necessarily satisfies

$$\gamma_k(x \otimes y) = \begin{cases} 0 & \text{if } |x|, |y| \text{ are odd,} \\ x^k \otimes \gamma_k(y) & \text{if } |x|, |y| \text{ are even, and } |y| \geq 2, \\ \gamma_k(x) \otimes y^k & \text{if } |x|, |y| \text{ are even, and } |x| \geq 2. \end{cases}$$

The last two cases coincide when both $|x|, |y| \geq 2$, and then, more symmetrically,

$$\gamma_k(x \otimes y) = k! \gamma_k(x) \otimes \gamma_k(y).$$

Definition 2.6.7. *A ring homomorphism $\varphi : A \to B$ between algebras with divided powers is compatible with the systems of divided powers, or is a homomorphism of algebras with divided powers, if further, $\gamma_k(\varphi(a)) = \varphi(\gamma_k(a))$, for all $a \in A$ in even degrees.*

Example: One has isomorphisms of algebras with (systems of) divided powers

$$\begin{aligned}\Gamma_R(x_1,\ldots,x_s) &\cong R\otimes_{\mathbb{Z}}\Gamma_{\mathbb{Z}}(x_1)\otimes_{\mathbb{Z}}\cdots\otimes_{\mathbb{Z}}\Gamma_{\mathbb{Z}}(x_s) \text{ and}\\ \bigwedge_R(y_1,\ldots,y_t) &\cong R\otimes_{\mathbb{Z}}\bigwedge_{\mathbb{Z}}(y_1)\otimes_{\mathbb{Z}}\cdots\otimes_{\mathbb{Z}}\bigwedge_{\mathbb{Z}}(y_t),\end{aligned}$$

where R is viewed as concentrated in degree 0, thus, trivially strictly graded commutative and carrying the vacuous system of divided powers, as pointed out in example 2.6.3 above.

In light of this result, one could have started in examples 2.6.4 and 2.6.5 with the case of just a single variable, then inducing up the structure using the tensor product.

Finally, we note the following.

Theorem 2.6.8. *Let A be a strictly graded commutative R-algebra with a system of divided powers. For any sequence $(a_1,\ldots,a_s)$ of elements of A of even degree at least 2 and any sequence $(b_1,\ldots,b_t)$ of elements of A of odd degree, the assignment $x_i\mapsto a_i, y_j\mapsto b_j$ extends to a unique homomorphism of strictly graded commutative R-algebras with divided powers*

$$\Gamma_R(x_1,\ldots,x_s)\otimes_R\bigwedge_R(y_1,\ldots,y_t)\to A.$$

Thus, $\Gamma_R(x_1,\ldots,x_s)\otimes_R\bigwedge_R(y_1,\ldots,y_t)$ is free within the category of strictly graded commutative R-algebras with divided powers.

Remark 2.6.9. *If $f(x)\in R[x]$ is a polynomial, then we can expand it around any $c\in R$ as*

$$f(x)=\sum_{i\geq 0}f^{(i)}(c)(x-c)^i,$$

for suitable $f^{(i)}(c)\in R$. Now note that $i!f^{(i)}(c)=\frac{\partial^i f}{\partial x^i}(c)$, so provided division by $i!$ is possible in R, we get the usual Taylor expansion. Thus the <u>divided derivatives</u> are analogous to divided powers in that they always exist, regardless of whether all their desired denominators are invertible in R.

2.7 Tate's Theorem

We will use Tate's Theorem to make all of our later constructions. Hence we give a careful statement and proof of the theorem, with a slightly weaker hypothesis (Koszul regularity) than Tate originally used.

John Tate proved the following result in his paper [16].

Theorem 2.7.1. *Let $f_1, \ldots, f_n$ and $g_1, \ldots, g_m$ be Koszul regular sequences such that the ideal $J = (g_1, \ldots, g_m)$ generated by the g_j is contained in the ideal $I = (f_1, \ldots, f_n)$ generated by the f_i. Write $g_j = \sum_{i=1}^n a_{ji} f_i$, $1 \leq j \leq m$, with $a_{ji} \in R$. Let $\overline{R} = \frac{R}{J}$ and $\overline{I} = \frac{I}{J}$, and let $\overline{a}_{ji}$ and $\overline{f}_i$ denote the J-residues of a_{ji} and f_i. Then the DG-algebra (see definition 2.7.6)*

$$\overline{R}\langle \tau_1, \ldots, \tau_n \ ; \ \sigma_1, \ldots, \sigma_m \rangle$$

with exterior variables τ_i of degree 1 and divided power variables σ_j of degree 2, and with algebra differential d defined through

$$\begin{aligned} d\tau_i &= \overline{f}_i \\ d\sigma_j &= \sum_{i=1}^n \overline{a}_{ji} \tau_i \end{aligned}$$

is acyclic, and therefore yields a free resolution of the $\overline{R}$-module $\frac{\overline{R}}{\overline{I}}$.

Remarks:

1. In his original paper, Tate made the stronger assumption that the ideals were generated by R-sequences.

2. It has been known since the publication of [2] that the result is true with the weaker hypothesis of Koszul regularity.

3. We present a "down-to-earth" proof of this improved version of the theorem here.

2.7.1 Preliminaries

Proposition 2.7.2. *Let $B \xrightarrow{g} A$ be a surjective ring homomorphism. Let M be a left A-module. Then the functors $(- \otimes_B M)$ and $(- \otimes_A M)$ (from right A-modules to abelian groups) are naturally isomorphic.*

Proof. This is a consequence of [3], §3.3, Corollary to Proposition 2. $\square$

Proposition 2.7.3. *Let R be a commutative ring and let $I \subset R$ be an ideal. Denote $\frac{R}{I}$ by $\overline{R}$. Let $\psi : M \to N$ be an R-module homomorphism, where I annihilates N. Then ψ factors uniquely through $\overline{R} \otimes_R M$:*

$$\begin{array}{ccc} m & \qquad & \begin{tikzcd} M \arrow[d] \arrow[dr, "\psi"] & \\ \overline{R} \otimes_R M \arrow[r, dotted, "\overline{\psi}"'] & N \end{tikzcd} \\ \downarrow & & \\ (1+I) \otimes m & & \end{array}$$

Proof. This is a consequence of the final remark in [3], §1.3, as follows.

There is a well-defined homomorphism of R-modules

$$\begin{array}{rccc} \alpha & : & \overline{R} \otimes_R M & \to & \frac{M}{IM} \\ & & (r+I) \otimes m & \mapsto & rm + IM \end{array}$$

with inverse

$$\begin{array}{rccc} \beta & : & \frac{M}{IM} & \to & \overline{R} \otimes_R M \\ & & m + IM & \mapsto & (1+I) \otimes m \end{array}$$

and therefore $\overline{R} \otimes_R M \cong \frac{M}{IM}$.

Now the remark applies to the diagram

$$\begin{tikzcd} M \arrow[d, two heads] \arrow[dr, "\psi"] & \\ \frac{M}{IM} \arrow[r, dotted, "\overline{\psi}"'] & N \end{tikzcd}$$

where for any $i \in I$ and $m \in M$, we have

$$\psi(im) = i \underbrace{\psi(m)}_{\in N} = 0$$

so that $IM \subseteq \ker \psi$. □

2.7.2 Derivations and the Tate Construction

We recall general terminology on derivations.

Definition 2.7.4. *Let $A = \oplus_i A_i$ be a graded algebra, not necessarily associative or graded commutative for now.*

A graded derivation ∂ of degree a on A with values in a graded A-bimodule $M = \oplus_{i\in\mathbb{Z}} M_i$ is an additive map from A to M such that $\partial(A_i) \subseteq M_{a+i}$ and the graded Leibniz rule

$$\partial(xy) = \partial(x)y + (-1)^{a|x|} x\partial(y)$$

holds for $x \in A_{|x|}$ homogeneous and $y \in A$.

The kernel of a graded derivation is a graded subalgebra of A and the derivation is then linear over that kernel.

Definition 2.7.5. *If $M = A$ then ∂ is called an algebra derivation, and further if $\partial^2 = \partial \circ \partial = 0$, then it is called an algebra differential. In the latter case, one often assumes that ∂ is of degree ± 1.*

Definition 2.7.6. *A differential graded R-algebra, or DG R-algebra, is a graded R-algebra A, together with an algebra differential ∂.*

Definition 2.7.7. *If A is a strictly graded commutative algebra with divided powers, and ∂ is a derivation into an A-bimodule M, then the derivation is compatible with the system of divided powers if $\partial(\gamma_k(x)) = \partial(x)\gamma_{k-1}(x)$ in M for $k \geq 1$ and any element x of even degree.*

Definition 2.7.8. *Suppose that (A, ∂) is a DG R-algebra, and that A is a strictly graded commutative algebra with divided powers. If ∂ is compatible with the system of divided powers, then we say that (A, ∂) is a DG R-algebra with divided powers.*

We now introduce a useful construction which allows us to efficiently perform computations involving differentials in a *Hom* complex.

Definition 2.7.9. *Consider the Hom complex $Hom^\bullet(C_\bullet, C_\bullet)$. For any homogeneous elements f, g of $Hom^\bullet(C_\bullet, C_\bullet)$, define the graded bracket, or graded commutator*

$$[f, g] := fg - (-1)^{|f||g|} gf.$$

Remark: This bracket satisfies (graded) skew-commutativity

$$[f, g] = -(-1)^{|f||g|}[g, f],$$

and the (signed) Jacobi identity

$$(-1)^{|f||h|}[f, [g, h]] + (-1)^{|g||f|}[g, [h, f]] + (-1)^{|h||g|}[h, [f, g]] = 0,$$

and in this way, the graded bracket defines a graded Lie algebra structure.

Now consider complexes $C_\bullet$, $D_\bullet$ and $E_\bullet$ and the composition map

$$\begin{array}{ccc} Hom^\bullet(D_\bullet, E_\bullet) \times Hom^\bullet(C_\bullet, D_\bullet) & \to & Hom^\bullet(C_\bullet, E_\bullet) \\ (f, g) & \mapsto & fg \end{array}$$

of the indicated *Hom* complexes. One can view these three *Hom* complexes as direct summands in

$$Hom^\bullet(C_\bullet \oplus D_\bullet \oplus E_\bullet,\ C_\bullet \oplus D_\bullet \oplus E_\bullet)$$

Lemma 2.7.10. *Denoting by d the differential in any of these, this graded bracket obeys the Leibniz rule:*

$$[d, fg] = [d, f]g + (-1)^{|f|} f[d, g]$$

Proof. Routine. □

Definition 2.7.11. *The free strictly graded commutative R-algebra with divided powers* $A = \Gamma_R(x_1, \ldots, x_s) \otimes_R \bigwedge_R(y_1, \ldots, y_t)$ *is smooth over R in the following sense.*

First, any R-linear derivation into a graded module over this algebra that is compatible with the system of divided powers is clearly uniquely determined by the values $\partial(x_i),\ \partial(y_j)$.

Conversely, given an integer a and an assignment $x_i \mapsto u_i,\ y_j \mapsto v_j$ *with* $u_i,\ v_j$ *homogeneous elements of that module such that* $a = |u_i| - |x_i| = |v_j| - |y_j|$ *for all* $i,\ j$*, this assignment extends uniquely to a graded derivation* ∂ *of degree a that is compatible with the system of divided powers. This derivation is denoted*

$$\partial = \sum_{i=1}^{s} u_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{t} v_j \frac{\partial}{\partial y_j}$$

Put differently, denote by $Der_R^{\Gamma}(A, M)$*, for any graded module M over this algebra, the graded R-module whose component in degree a consists of those graded derivations of this degree that are compatible with divided powers. One then has*

$$Der_R^{\Gamma}(A, M) = \bigoplus_{i=1}^{s} M \frac{\partial}{\partial x_i} \oplus \bigoplus_{j=1}^{t} M \frac{\partial}{\partial y_j}$$

Example: If we take $M = A$ and consider a derivation $\partial : A \to A$ of odd degree, then $\partial^2 = \partial \circ \partial$ is again a derivation of twice the degree of ∂, given by

$$\partial^2 = \sum_{i=1}^{s} \partial(u_i) \frac{\partial}{\partial x_i} + \sum_{j=1}^{t} \partial(v_j) \frac{\partial}{\partial y_j}.$$

In particular, ∂ is an algebra differential, that is $\partial^2 = 0$, if and only if, each coefficient $u_i,\ v_j$ is a cycle for ∂, that is $\partial(u_i) = 0 = \partial(v_j)$ for all i, j.

A crucial case of this last example occurs when the x_i are situated in degree 2, the y_j in degree 1 and ∂ is of degree -1. In this situation, A is necessarily positively graded.

Moreover, $u_i = \partial(x_i) = \sum_j y_j a_{ji}$ and $v_j = \partial(y_j) = b_j$, with $a_{ji}, b_j \in A_0 = R$. Therefore the v_j are automatically cycles and requiring $\partial^2 = 0$ forces the condition $\partial(u_i) = 0$ to become

$$\partial(u_i) = \sum_j \partial(y_j) a_{ji} = \sum_j b_j a_{ji} = 0$$

Viewing (a_{ji}) as the matrix of an R-linear map $\varphi : \oplus_i R x_i \to \oplus_j R y_j$ and (b_j) as the matrix of an R-linear form $\lambda : \oplus_j R y_j \to R$, we also write in a co-ordinate free way $\partial = \partial_\varphi + \partial_\lambda$ for that derivation as it is independent of a choice of bases. The condition that ∂ is an algebra derivation then simply becomes $\lambda \circ \varphi = 0$.

As this situation is crucial, we single it out by a definition.

Definition 2.7.12. *Let R be a commutative ring, $\varphi : F \to G$ an R-linear map between free modules of finite rank, and $\lambda : G \to R$ an R-linear form.*

If $\lambda \circ \varphi = 0$, then the free divided power algebra $\Gamma_R F \otimes_R \bigwedge_R G$ with differential $\partial = \partial_\varphi + \partial_\lambda$ is the Tate Construction $\mathbb{T}(\varphi, \lambda)$ on the pair (φ, λ).

We call $\mathbb{T}(\varphi, \lambda)$ a Tate resolution if its homology is concentrated in degree zero. In this case, the complex of R-modules underlying $\mathbb{T}(\varphi, \lambda)$ resolves $\frac{R}{Im(\lambda)}$ by finite free R-modules.

Examples:

1. If $F = 0$, then $\mathbb{T}(0, \lambda)$ is nothing but the Koszul complex on the linear form λ. It is a (Tate) resolution, by definition, if $(\lambda(y_1), \ldots, \lambda(y_t))$ is a Koszul regular sequence on R for some (any) basis $\{y_j\}$ of G. (C. f. Definition 2.4.2).

2. If $G = 0$, then necessarily $\varphi = \lambda = 0$, and we are reduced to a free divided power algebra on F.

3. In the tautological example, $\varphi = id$ is the identity map on a free module, and this forces, of course, $\lambda = 0$. It is this example which prompted H. Cartan to introduce systems of divided powers into the mathematical toolbox, as it provides for the minimal graded resolution of the augmentation module of an exterior algebra.

Theorem (Cartan) 2.7.13. *The Tate construction over the identity map on a free R-module F of rank t returns the minimal graded resolution of R, viewed as the graded augmentation module:*

$$\epsilon : \bigwedge_R F \cong \bigwedge_R (y_1, \ldots, y_t) \to R, \ y_j \mapsto 0$$

over the exterior algebra $\bigwedge_R F$. In other words, the differential graded algebra

$$(\mathbb{T}(id_F, 0), \partial) \cong \left(\Gamma_R(x_1, \ldots, x_t) \otimes_R \bigwedge_R (y_1, \ldots, y_t), \sum_{j=1}^{t} y_j \frac{\partial}{\partial x_j} \right)$$

has R as its sole homology, concentrated in degree zero.

Proof: See Example 2.5 in [1]. □

A word on gradings: If one wants to consider $\mathbb{T}(id, 0)$ as a resolution of the augmentation over the exterior algebra in the classical sense, then the divided power degree becomes the homological degree, that is $\Gamma_R^n(F) \otimes_R \bigwedge_R(F)$ is the finite free $\bigwedge_R(F)$-module in homological degree n. The differential is then linear with respect to the (internal) grading on the exterior algebra.

Often, however, it is advantageous to consider $\mathbb{T}(id, 0)$ instead as a differential graded $\bigwedge_R(F)$-module, thus using the total degree, as we did above, where the divided power variables sit in degree two, the exterior variables in degree one.

2.7.3 Why Koszul Regularity is Enough to Apply Tate's Theorem

Now denote $(x_1, \ldots, x_s)$ by $\mathbf{x}$ and $(y_1, \ldots, y_t)$ by $\mathbf{y}$. The notation $\bigwedge_R^\bullet(\mathbf{x}[1])$ denotes the exterior algebra over R with basis $\mathbf{x} = (x_1, \ldots, x_s)$, with $|x_i| = 1$ for all i. Similarly denote by $\bigwedge_R^\bullet(F[1])$ the exterior algebra over R of F, with the elements of F in degree 1.

Lemma 2.7.14. *If $\lambda \circ \varphi = 0$ as in Definition 2.7.12, then the map induced by $\varphi : F \to G$*

$$(\textstyle\bigwedge_R^\bullet(\mathbf{x}[1]), d = 0) \xrightarrow{\bigwedge^\bullet \varphi} (\textstyle\bigwedge_R^\bullet(\mathbf{y}[1]), \partial_\lambda)$$

is a DG-algebra homomorphism, i.e. the following diagram commutes:

$$\begin{tikzcd}
\bigwedge_R^\bullet(\mathbf{x}[1]) \arrow[r, "\bigwedge^\bullet \varphi"] \arrow[d, "d=0"'] & \bigwedge_R^\bullet(\mathbf{y}[1]) \arrow[d, "\partial_\lambda"] \\
\bigwedge_R^\bullet(\mathbf{x}[1]) \arrow[r, "\bigwedge^\bullet \varphi"'] & \bigwedge_R^\bullet(\mathbf{y}[1]).
\end{tikzcd}$$

Proof. Let $\omega \in \bigwedge_R^\bullet(\mathbf{x}[1])$ be an arbitrary basis element. Write $\omega = x_{i_1} \wedge \cdots \wedge x_{i_k}$ for some k. Because the first map in the counterclockwise branch is $d = 0$, the diagram commutes if and only if the clockwise branch is also the zero map. The output from the clockwise branch is

$$\begin{aligned} \partial_\lambda(\wedge^k \varphi(\omega)) &= \partial_\lambda(\varphi(x_{i_1}) \wedge \cdots \wedge \varphi(x_{i_k})) \\ &= \sum_{\nu=1}^{k} \pm \varphi(x_{i_1}) \wedge \cdots \wedge \underbrace{\lambda\varphi(x_{i_\nu})}_{=0} \wedge \cdots \wedge \varphi(x_{i_k}) \\ &= 0. \end{aligned}$$

Since every basis element maps to zero via the clockwise branch, therefore the clockwise branch kills every element. So the diagram commutes as required. $\square$

Remark: The induced map of Lemma 2.7.14 makes $\bigwedge_R^\bullet(\mathbf{y}[1])$ into a module over $\bigwedge_R^\bullet(\mathbf{x}[1])$. Define $\overline{R} = \frac{R}{Im(\lambda)}$. Recall (for example by Exercise 4.5.1 in [18]), that $H_\bullet(\bigwedge_R^\bullet(\mathbf{y}[1]), \partial_\lambda)$ is a strictly graded commutative *DG* R-algebra. We now show that it is necessarily an $\overline{R}$-algebra also. Let

$$R \xrightarrow{\psi} H_\bullet(\textstyle\bigwedge_R^\bullet(\mathbf{y}[1]), \partial_\lambda)$$

witness the fact that $H_\bullet(\bigwedge_R^\bullet(\mathbf{y}[1]), \partial_\lambda)$ is an R-algebra. Then since $Im(\lambda)$ annihilates the target, by Proposition 2.7.3, ψ factors uniquely:

$$\begin{tikzcd} & R \arrow[d] \arrow[drr, "\psi"] & & \\ \overline{R} \arrow[r, equal] & \overline{R} \otimes_R R \arrow[rr, dotted, "\overline{\psi}"'] & & H_\bullet(\bigwedge_R^\bullet(\mathbf{y}[1]), \partial_\lambda) \end{tikzcd}$$

and thus

$$\overline{R} \xrightarrow{\overline{\psi}} H_\bullet(\textstyle\bigwedge_R^\bullet(\mathbf{y}[1]), \partial_\lambda)$$

witnesses the fact that $H_\bullet(\bigwedge_R^\bullet(\mathbf{y}[1]), \partial_\lambda)$ is an $\overline{R}$-algebra.

We see that φ induces an R-linear map

$$\begin{array}{ccccccc} F & = & \bigoplus_{i=1}^s Rx_i & \to & \bigwedge_R^1(\mathbf{y}[1]) & \subset & \bigwedge_R^\bullet(\mathbf{y}[1]) \\ & & x_i & \mapsto & \varphi(x_i) & & \end{array}$$

Note that $\varphi(F)$ consists of cycles, because $\lambda\varphi = 0$. Thus, since there is a surjection from the cycles in degree 1 onto $H_1(\bigwedge_R^\bullet(\mathbf{y}[1]), \partial_\lambda)$, we get a homomorphism of R-modules

$$F = \bigoplus_{i=1}^{s} Rx_i \to H_1\left(\bigwedge_R^\bullet(\mathbf{y}[1]), \partial_\lambda\right).$$

This gives rise to a homomorphism of strictly graded commutative R-algebras

$$(\bigwedge_R^\bullet(\mathbf{x}[1]), 0) \longrightarrow H_\bullet\,(\bigwedge_R^\bullet(\mathbf{y}[1]), \partial_\lambda)$$

Since $Im(\lambda)$ annihilates the target, by Proposition 2.7.3, this map factors uniquely through $\overline{R} \otimes_R (\bigwedge_R^\bullet(\mathbf{x}[1]), 0)$, and thus we get a homomorphism of strictly graded commutative $\overline{R}$-algebras

$$\overline{R} \otimes_R (\bigwedge_R^\bullet(\mathbf{x}[1]), 0) \xrightarrow{\overline{\varphi}} H_\bullet(\bigwedge_R^\bullet(\mathbf{y}[1]), \partial_\lambda)$$

which can be re-written

$$(\bigwedge_{\overline{R}}^\bullet(\mathbf{x}[1]), 0) \xrightarrow{\overline{\varphi}} H_\bullet(\bigwedge_R^\bullet(\mathbf{y}[1]), \partial_\lambda)$$

This map $\overline{\varphi}$ will be the key ingredient in the statement of Theorem 2.7.16.

The following map defined on basis elements in degree 2 extends linearly to a well-defined derivation on $\Gamma_R^\bullet(\mathbf{x}[2]) \otimes_R \bigwedge_R^\bullet(\mathbf{y}[1])$:

$$\begin{array}{cccc} \partial_\varphi & : & \Gamma_R^\bullet(\mathbf{x}[2]) & \to & \bigwedge_R^\bullet(\mathbf{y}[1]) \\ & & x_i & \mapsto & \varphi(x_i) \end{array}$$

Denote by ∂_{can} the canonical differential of Theorem 2.7.13, which, when using the same symbols for the basis elements in degrees 2 and 1, reads as:

$$\begin{array}{ccccc} \partial_{can} & : & \Gamma_R^\bullet(\mathbf{x}[2]) & \to & \bigwedge_R^\bullet(\mathbf{x}[1]) \\ & & x_i & \mapsto & x_i \end{array}$$

Lemma 2.7.15. *The following is a chain of DG-algebra isomorphisms:*

$$\begin{aligned}
& \left(\mathbb{T}_R(id_F,0) \otimes_{\bigwedge_R^\bullet(F[1])} \bigwedge_R^\bullet(\mathbf{y}[1]), \partial_{can} + \partial_\lambda\right) && (2.2)\\
\cong\; & \left(\underbrace{\left(\Gamma_R^\bullet(\mathbf{x}[2]) \otimes_R \bigwedge_R^\bullet(\mathbf{x}[1])\right)}_{\mathbb{T}_R(id_F,0)} \otimes_{\bigwedge_R^\bullet(\mathbf{x}[1])} \bigwedge_R^\bullet(\mathbf{y}[1]), \partial_\varphi + 0 + \partial_\lambda\right) && (2.3)\\
\cong\; & \left(\Gamma_R^\bullet(\mathbf{x}[2]) \otimes_R \bigwedge_R^\bullet(\mathbf{y}[1]), \partial_\varphi + \partial_\lambda\right) && (2.4)\\
\cong\; & \mathbb{T}_R(\varphi,\lambda) && (2.5)
\end{aligned}$$

Proof. We can form the tensor product on line (2.2) because

- the algebra $\bigwedge_R^\bullet(F[1])$ is a subalgebra of $\mathbb{T}_R(id_F,0)$ and thus acts on it in the obvious way, and
- the induced map of Lemma 2.7.14 makes $\bigwedge_R^\bullet(\mathbf{y}[1])$ into a module over $\bigwedge_R^\bullet(F[1])$.

The terms on lines (2.2) and (2.3) agree because we have simply applied the definition of the Tate construction in the tautological example, and because $\bigwedge_R^\bullet(F[1])$ is simply a basis-free version of $\bigwedge_R^\bullet(\mathbf{x}[1])$.

The differentials on lines (2.2) and (2.3) agree because, by the action of $\bigwedge_R^\bullet(\mathbf{x}[1])$ on the third factor of the tensor product on line (2.3), we have

$$1 \otimes_R x_i \otimes_{\bigwedge_R^\bullet(\mathbf{x}[1])} 1 = 1 \otimes_R 1 \otimes_{\bigwedge_R^\bullet(\mathbf{x}[1])} \varphi(x_i)$$

and hence

$$\begin{aligned}
\partial_{can}(x_i \otimes_R 1 \otimes_{\bigwedge_R^\bullet(\mathbf{x}[1])} 1) &= 1 \otimes_R x_i \otimes_{\bigwedge_R^\bullet(\mathbf{x}[1])} 1\\
&= 1 \otimes_R 1 \otimes_{\bigwedge_R^\bullet(\mathbf{x}[1])} \varphi(x_i)\\
&= \partial_\varphi(x_i \otimes_R 1 \otimes_{\bigwedge_R^\bullet(\mathbf{x}[1])} 1)
\end{aligned}$$

so ∂_{can} and ∂_φ agree on arguments in the first factor.

The terms on lines (2.3) and (2.4) agree since $\left(\bigwedge_R^\bullet(\mathbf{x}[1])\right) \otimes_{\bigwedge_R^\bullet(\mathbf{x}[1])} -\right)$ collapses.

The differentials on lines (2.3) and (2.4) agree since, on line (2.4), we have simply collapsed the middle factor, whose differential was already zero.

From line (2.4) to line (2.5) we are simply applying the definition of the Tate construction.

$\square$

Theorem 2.7.16. *If the homomorphism $\overline{\varphi}$ of strictly graded commutative $\overline{R}$-algebras is an isomorphism, then $\mathbb{T}_R(\varphi, \lambda)$ is a Tate resolution of $\overline{R}$ as an R-module.*

Proof. Guided by line (2.2) in the statement of Lemma 2.7.15, we may construct a first quadrant bicomplex M with ∂_{can} as the horizontal differentials and ∂_λ as the vertical differentials. Then we compute the spectral sequence $\{{}^I E^r\}$ arising from the first filtration. By Theorem 11.18 in [13], we have

$$ {}^I E^2_{i,j} = H'_i H''_{i,j}(M) \Rightarrow H_n(Tot(M)) = H_n(\mathbb{T}_R(\varphi, \lambda)) $$

We analyze ${}^I E^2_{i,j} = H'_i H''_{i,j}(M)$. In our notation, it becomes

$$ {}^I E^2_{p,q} = H_i \left(\mathbb{T}_R(id_F, 0) \otimes_{\bigwedge_R^\bullet (F[1])} H_j \left(\bigwedge_R^\bullet (\mathbf{y}[1]), \partial_\lambda \right) \right) $$

We claim that we have the following chain of isomorphisms of DG-algebras:

$$ \begin{aligned} & \mathbb{T}_R(id_F, 0) \otimes_{\bigwedge_R^\bullet (F[1])} H_\bullet \left(\bigwedge_R^\bullet (\mathbf{y}[1]), \partial_\lambda \right) && (2.6) \\ \cong\ & \mathbb{T}_R(id_F, 0) \otimes_{\bigwedge_R^\bullet (F[1])} \overline{R} \otimes_{\overline{R}} H_\bullet \left(\bigwedge_R^\bullet (\mathbf{y}[1]), \partial_\lambda \right) && (2.7) \\ \cong\ & \mathbb{T}_{\overline{R}}(id_{\overline{F}}, 0) \otimes_{\bigwedge_{\overline{R}}^\bullet (\overline{F}[1])} H_\bullet \left(\bigwedge_R^\bullet (\mathbf{y}[1]), \partial_\lambda \right) && (2.8) \end{aligned} $$

As above, $H_\bullet(\bigwedge_R^\bullet(\mathbf{y}[1]), \partial_\lambda)$ is an $\overline{R}$-algebra. Also, the composition of the augmentation with the natural map

$$ \bigwedge_R^\bullet (F[1]) \xrightarrow{\epsilon} R \longrightarrow \overline{R} $$

makes $\overline{R}$ into a module over $\bigwedge_R^\bullet(F[1])$. So we can form the tensor products on line (2.7). The terms on lines (2.6) and (2.7) agree, since $\overline{R} \otimes_{\overline{R}} -$ collapses.

The differentials on lines (2.6) and (2.7) agree by the properties of the tensor product.

For the isomorphism between lines (2.7) and (2.8), we apply Proposition 2.7.2 with the natural map

$$\bigwedge\nolimits_R^\bullet(F[1]) \longrightarrow \bigwedge\nolimits_{\overline{R}}^\bullet(\overline{F}[1])$$

which is a surjective ring homomorphism, and the natural map

$$R \longrightarrow \overline{R}$$

which is also a surjective ring homomorphism. Last, observe that $(\overline{R} \otimes_R -)$ has the effect of taking the quotient of everything modulo $Im(\lambda)$. Since $Im(\lambda)$ annihilates $H_j\left(\bigwedge_R^\bullet(\mathbf{y}[1]), \partial_\lambda\right)$, we have that

$$H_j\left(\bigwedge_R^\bullet(\mathbf{y}[1]), \partial_\lambda\right) \cong H_j\left(\bigwedge_{\overline{R}}^\bullet \overline{\mathbf{y}}[1], \partial_\lambda\right)$$

as DG-algebras.

Now, by Theorem 2.7.13, $\mathbb{T}_{\overline{R}}(id_{\overline{F}}, 0)$ resolves $\overline{R}$ over $\bigwedge_{\overline{R}}^\bullet(\overline{F}[1])$. Therefore we have

$$\begin{aligned} & H_i\left(\mathbb{T}_{\overline{R}}(id_{\overline{F}}, 0) \otimes_{(\bigwedge_{\overline{R}}^\bullet(\overline{F}[1])} H_\bullet\left(\bigwedge_R^\bullet(\mathbf{y}[1]), \partial_\lambda\right)\right) && (2.9) \\ = \;& Tor_i^{\bigwedge_{\overline{R}}^\bullet(\overline{F}[1])}\left(\overline{R}, H_\bullet\left(\bigwedge_R^\bullet(\mathbf{y}[1]), \partial_\lambda\right)\right) && (2.10) \end{aligned}$$

So if

$$\overline{\bigwedge\nolimits_{\overline{R}}^\bullet}(\overline{F}[1]) \xrightarrow{\overline{\varphi}} H_\bullet\left(\bigwedge\nolimits_R^\bullet(\mathbf{y}[1]), \partial_\lambda\right)$$

is an isomorphism, then $H_\bullet(\bigwedge_R^\bullet(\mathbf{y}[1]), \partial_\lambda)$ is a free module over $\overline{\bigwedge_{\overline{R}}^\bullet}(\overline{F}[1])$. Therefore we have that

$$Tor_i^{\bigwedge_{\overline{R}}^\bullet(\overline{F}[1])}\left(\overline{R}, H_\bullet\left(\bigwedge_R^\bullet(\mathbf{y}[1]), \partial_\lambda\right)\right) = \begin{cases} 0 & \text{if } \; i \geq 1 \\ \overline{R} & \text{if } \; i = 0 \end{cases}$$

So the spectral sequence collapses, and we have that $\mathbb{T}_R(\varphi, \lambda)$ has homology $\overline{R}$ concentrated in degree 0. In other words, $\mathbb{T}(\varphi, \lambda)$ resolves $\overline{R}$ over R as claimed. $\square$

Tate pointed out the following particular example.

Theorem 2.7.17. *Let $\tilde{R}$ be a commutative ring and $\Phi : \tilde{F} \to \tilde{G}$ an $\tilde{R}$-linear map between free modules of finite rank, $\Lambda : \tilde{G} \to \tilde{R}$ an $\tilde{R}$-linear form.*

If the Koszul complexes over Λ, respectively over $\Lambda\Phi$, have homology only in (homological) degree zero, then the Tate construction over $R = \frac{\tilde{R}}{Im(\Lambda\Phi)}$ on $\varphi = \Phi \otimes_{\tilde{R}} R : F = \tilde{F} \otimes_{\tilde{R}} R \to G = \tilde{G} \otimes_{\tilde{R}} R$ and $\lambda = \Lambda \otimes_{\tilde{R}} R : G \to \tilde{R}$ has its only homology in degree zero.

In other words, $\mathbb{T}(\varphi, \lambda)$ is a Tate resolution of the cyclic R-module $\overline{R} = \frac{R}{Im(\lambda)}$.

Proof. Indeed, with the notation as above, one shows that $\overline{\varphi}$ is an isomorphism. This is a consequence of calculating $Tor^{\tilde{R}}(\overline{R}, R)$ through the given projective resolutions by Koszul complexes in each argument. With

$$\begin{array}{ccccccccc} \epsilon_{\tilde{F}} & : & \bigwedge_{\tilde{R}}(\tilde{F}) & \to & H_0\left(\bigwedge_{\tilde{R}}(\tilde{F}), \partial_{\Lambda\Phi}\right) & \cong & \frac{\tilde{R}}{Im(\Lambda\Phi)} & = & R \\ \epsilon_{\tilde{G}} & : & \bigwedge_{\tilde{R}}(\tilde{G}) & \to & H_0\left(\bigwedge_{\tilde{R}}(\tilde{G}), \partial_{\Lambda}\right) & \cong & \frac{\tilde{R}}{Im(\Lambda)} & = & \overline{R} \end{array}$$

the respective augmentations, one has the following diagram of quasi-isomorphisms of DG-algebras,

$$\begin{tikzcd} & \bigwedge_{\tilde{R}}(\tilde{G}) \otimes_{\tilde{R}} \bigwedge_{\tilde{R}}(\tilde{F}) \arrow[dl, "\bigwedge_{\tilde{R}}(-id_{\tilde{G}})\otimes\epsilon_{\tilde{F}}"', "\simeq"] \arrow[dr, "\epsilon_{\tilde{G}}\otimes\bigwedge_{\tilde{R}}(id_{\tilde{F}})", "\simeq"'] & \\ \bigwedge_{\tilde{R}}(\tilde{G}) \otimes_{\tilde{R}} \frac{\tilde{R}}{Im(\Lambda\Phi)} \arrow[d, "\cong"'] & & \frac{\tilde{R}}{Im(\Lambda)} \otimes_{\tilde{R}} \bigwedge_{\tilde{R}}(\tilde{F}) \arrow[d, "\cong"] \\ \bigwedge_R(G) & & \overline{R} \otimes_R \bigwedge_R(F) \end{tikzcd}$$

Each of these five algebras has $Tor^{\tilde{R}}(\overline{R}, R)$ as its homology, and the morphisms induce algebra isomorphisms in that homology.

With unadorned tensor products taken over $\tilde{R}$, for $\tilde{x} \in \tilde{F}$, the element $1 \otimes \tilde{x} - \Phi(\tilde{x}) \otimes 1$ in the top term is a cycle, as

$$\begin{aligned} (\partial_\Lambda \otimes 1 + 1 \otimes \partial_{\Lambda\Phi})(1 \otimes \tilde{x} - \Phi(\tilde{x}) \otimes 1) &= 1 \otimes \Lambda\Phi(\tilde{x}) + \Lambda(-\Phi(\tilde{x})) \otimes 1 \\ &= 0 \end{aligned}$$

in $\bigwedge^0_{\tilde R}(\tilde G)\otimes_{\tilde R}\bigwedge^0_{\tilde R}(\tilde F)\cong \tilde R\otimes_{\tilde R}\tilde R\cong \tilde R$.

Because $\epsilon_{\tilde F}(\tilde x)=0$, the image in the bottom left term is

$$\begin{aligned}\left(\bigwedge_{\tilde R}(-id_{\tilde G})\otimes\epsilon_{\tilde F}\right)(1\otimes\tilde x-\Phi(\tilde x)\otimes 1) &= \Phi(\tilde x)\otimes_{\tilde R}(1 \mod Im(\Lambda\Phi))\\ &= \varphi(x)\in\bigwedge_R(G)\end{aligned}$$

where x is the image of $\tilde x$ in $G\cong\frac{\tilde R}{Im(\Lambda\Phi)}\otimes_{\tilde R}\tilde G$. This says that $\varphi(x)$ is a cycle for $(\bigwedge_R(G),\partial_\lambda)$.

In the $\overline{R}$-module $H\left(\bigwedge_R(G),\partial_\lambda\right)$, $\varphi(x)$ represents $\overline{\varphi}(\overline{x})$, where $\overline{x}$ is the class of x in $\frac{\tilde F}{\tilde F Im(\Lambda)}\cong \overline{R}\otimes_R F=\overline{F}$.

Mapping to the bottom right, because $\epsilon_{\tilde G}\Phi(\tilde x)=0$ one finds

$$\begin{aligned}(\epsilon_{\tilde G}\otimes id_{\tilde F})(1\otimes\tilde x-\Phi(\tilde x)\otimes 1) &= (1 \mod Im(\Lambda))\otimes\tilde x\\ &= \overline{x}.\end{aligned}$$

This establishes that passing to homology the diagram of algebra homomorphisms

$$\begin{tikzcd} & Tor^{\tilde R}\left(\frac{\tilde R}{Im(\Lambda\Phi)},\frac{\tilde R}{Im(\Lambda)}\right) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\ H\left(\bigwedge_R(G),\partial_\lambda\right) & & \overline{R}\otimes_R\bigwedge_R(F) \arrow[ll, "\overline{\varphi}"'] \end{tikzcd}$$

commutes on the generating set, isomorphic to $\overline{R}\otimes_R F$, and so in general. In particular, $\overline{\varphi}$ is an isomorphism, as claimed. $\square$

2.8 Applications of Tate's Theorem

2.8.1 The General Setup

Let P be a (polynomial) ring and let $J = (g_1, \dots, g_m) \subseteq I = (f_1, \dots, f_n) \subseteq P$ be ideals of P generated by Koszul regular sequences. Let

$$\begin{aligned} g_j &= \sum_{i=1}^{n} a_{ji} f_i \\ A &= (a_{ji}),\ a_{ji} \in P \\ \sigma &= (\sigma_1, \dots, \sigma_m) \\ \tau &= (\tau_1, \dots, \tau_n). \end{aligned}$$

Then Tate's Theorem 2.7.1 provides a projective resolution of $\frac{P}{I}$ over $\frac{P}{J}$:

$$\left(\Gamma^{\bullet}_{\frac{P}{J}}(\sigma) \otimes_{\frac{P}{J}} \bigwedge_{\frac{P}{J}}^{\bullet}(\tau), \partial = \sum_i f_i \frac{\partial}{\partial \tau_i} + \sum_{i,j} a_{ji} \tau_i \frac{\partial}{\partial \sigma_j} \right)$$

where

$$\begin{aligned} |R| &= 0 \\ |\tau_i| &= 1, \text{ exterior variables} \\ |\sigma_j| &= 2, \text{ divided power variables.} \end{aligned}$$

2.8.2 Examples

Example 2.8.1. *Let K be a field. Let $P = K[x_1, \dots, x_n]$. Define $I = (x_1, \dots, x_n) \subseteq P$, so that $\frac{P}{I} \cong K$.*

Let $J = (g_1, \dots, g_m) \subseteq I^2$ be generated by a Koszul regular sequence. We claim that with this setup, we have

$$Ext^{\bullet}_{\frac{P}{J}}(K,\ K) \cong \mathbb{S}^{\bullet}_K(\mathbf{s}) \otimes_K \bigwedge_K^{\bullet}(\mathbf{t})$$

as graded K-modules, where

$$\begin{aligned}\mathbf{s} &= (s_1, \ldots, s_m), \textit{ and } |s_j| = 2,\\ \mathbf{t} &= (t_1, \ldots, t_n), \textit{ and } |t_i| = 1.\end{aligned}$$

Since $J \subseteq I^2$, necessarily $J \subseteq I$, so the hypotheses of Tate's Theorem are satisfied. Observe that each g_j is a polynomial in $K[x_1, \ldots, x_n]$ in which each monomial has degree ≥ 2. Therefore the entries a_{ji} of the coefficient matrix A from Tate's Theorem all lie in I.

Applying Tate's Theorem 2.7.1 gives a projective resolution of $\frac{P}{I} \cong K$ over $\frac{P}{J}$:

$$\mathbb{F} = \left(\Gamma^{\bullet}_{\frac{P}{J}}(\sigma) \otimes_{\frac{P}{J}} \bigwedge^{\bullet}_{\frac{P}{J}}(\tau), \partial = \sum_{i}^{n} x_i \frac{\partial}{\partial \tau_i} + \sum_{i,j} a_{ji}\tau_i \frac{\partial}{\partial \sigma_j} \right).$$

where

$$\begin{aligned}\sigma &= (\sigma_1, \ldots, \sigma_m)\\ \tau &= (\tau_1, \ldots, \tau_n).\end{aligned}$$

It is now easy to see that when we apply $Hom_{\frac{P}{J}}(-, K) = Hom_{\frac{P}{J}}\left(-, \frac{P}{I}\right)$ to this resolution, the dualized differential ∂^* will be zero. Therefore the homology of the dualized complex will simply be the individual terms in each degree. By Proposition A2.7 in [10], we know that dualizing turns divided powers into symmetric powers, and that the exterior algebra is self-dual.

Putting it all together, and defining

$$\begin{aligned}s_j &= (\sigma_j)^*,\\ t_i &= (\tau_i)^*,\end{aligned}$$

the desired result is now established.

Example 2.8.2. *Here we use Tate's Theorem 2.7.1 to compute the cohomology groups of a finite cyclic group.*

Let $G = \langle x \mid x^h = 1\rangle$ be the cyclic group of order h. Let R be a commutative ring. Let A be a G-module.

By Exercise 6.1.2 in [18], we know that

$$H^{\bullet}(G, A) \cong Ext^{\bullet}_{RG}(R, A),$$

so we can get what we want by computing these Exts.

We use Tate's Theorem to obtain the required projective resolution of R over RG. Let N denote the norm element of RG, i.e. $N = \sum_{i=0}^{h-1} x^i$. Then define

$$\begin{aligned} P &= R[x] \\ I &= (x-1) \subset P \\ J &= (x^h - 1) \subset P \end{aligned}$$

Then since $x^h - 1 = (x-1)N$ in P, we have that $J \subset I$ as required. Since $x-1$ and x^h-1 are monic, they are non zero divisors. Thus I and J are generated by Koszul regular sequences. Moreover, the above definitions give us that

$$\begin{aligned} \frac{P}{I} &= \frac{R[x]}{(x-1)} \\ &\cong R, \\ \frac{P}{J} &= \frac{R[x]}{(x^h-1)} \\ &\cong RG. \end{aligned}$$

Also note that the matrix of coefficients which Tate's Theorem requires is simply $A = (N)$.

Applying Tate's Theorem 2.7.1 gives a canonical projective resolution of $\frac{P}{I} \cong R$ over $\frac{P}{J} \cong RG$:

$$\mathbb{F} = \left(\Gamma^{\bullet}_{\frac{R}{J}}(\sigma) \otimes_{\frac{R}{J}} \bigwedge^{\bullet}_{\frac{R}{J}}(\tau), \partial = (x-1)\frac{\partial}{\partial \tau} + N\tau\frac{\partial}{\partial \sigma} \right)$$

where

$$\begin{aligned} |R| &= 0 \\ |\tau| &= 1, \text{ an exterior variable} \\ |\sigma| &= 2, \text{ a divided power variable.} \end{aligned}$$

The free resolution that we get from applying Tate's Theorem is therefore the 2-periodic complex:

$$0 \longleftarrow RG \xleftarrow{x-1} RG\tau \xleftarrow{N} RG\sigma \xleftarrow{x-1} RG\tau\sigma \xleftarrow{N} RG\sigma^{(2)} \longleftarrow \cdots \tag{2.11}$$

Remarks:

1. In detail, our augmentation is:

$$\begin{array}{cccll} \epsilon & : & \mathbb{F} & \to & R \\ & & r & \mapsto & r, \quad r \in R \\ & & x & \mapsto & 1 \\ & & \tau & \mapsto & 0 \\ & & \sigma & \mapsto & 0 \end{array}$$

2. The resolution $\mathbb{F}$ carries an algebra structure with a system of divided powers coming from the Tate construction.

3. The map ∂ is an algebra differential, per Definition 2.7.7, and therefore $\mathbb{F}$ is a *DG* R-algebra with a system of divided powers.

Now apply $Hom_{RG}(-, A)$ to the resolution on line (2.11) to get

$$0 \longrightarrow A \xrightarrow{x-1} A\tau^* \xrightarrow{N} A\sigma^* \xrightarrow{x-1} A\sigma^*\tau^* \xrightarrow{N} A(\sigma^*)^{(2)} \longrightarrow \cdots \tag{2.12}$$

Computing homology of this complex proves the well-known results (for example see [14], Theorem 10.112 and Corollary 10.113):

Proposition 2.8.3. *The cohomology groups $H^\bullet(G, A)$ of the finite cyclic group $G = \langle x \mid x^h = 1\rangle$ with coefficients in a G-module A are given by*

$$\begin{array}{rcl} H^0(G,A) & = & A^G, \\ H^i(G,A) & = & \dfrac{\ker N}{(x-1)A}, \text{ for } i \geq 1 \text{ odd,} \\ H^i(G,A) & = & \dfrac{A^G}{NA}, \text{ for } i \geq 2 \text{ even.} \end{array}$$

Corollary 2.8.4. *The cohomology groups $H^*(G, A)$ of the finite cyclic group $G = \langle x \mid x^h = 1\rangle$ with coefficients in a trivial G-module A are given by*

$$\begin{array}{rcl} H^0(G,A) & = & A, \\ H^i(G,A) & = & A[h], \text{ the } h\text{-torsion elements of } A, \text{ for } i \geq 1 \text{ odd,} \\ H^i(G,A) & = & \dfrac{A}{hA}, \text{ for } i \geq 2 \text{ even.} \end{array}$$

Corollary 2.8.5. *The cohomology groups $H^*(G, A)$ of the finite cyclic group $G = \langle x \mid x^h = 1\rangle$ with coefficients in the trivial G-module $\mathbb{Z}$ are given by*

$$\begin{aligned} H^0(G, \mathbb{Z}) &= \mathbb{Z}, \\ H^i(G, \mathbb{Z}) &= 0, \textit{ for } i \geq 1 \textit{ odd}, \\ H^i(G, \mathbb{Z}) &= \frac{\mathbb{Z}}{h\mathbb{Z}}, \textit{ for } i \geq 2 \textit{ even}. \end{aligned}$$

Chapter 3

The Cup Product

3.1 Introduction

In this chapter, we will start with a finite group G and a projective resolution $\mathbb{F}$ of the ring R as a trivial G-module. We will then define a diagonal approximation from $\mathbb{F}$ to $\mathbb{F} \otimes_R \mathbb{F}$. Then we will define the cup product of cochains in the dual of $\mathbb{F}$, and show that the cup product of cochains is homotopic to the Yoneda product, which implies that the cup product and the Yoneda product coincide once we pass to cohomology.

3.2 Supplemented Algebras

The most general setting in which we can make our constructions is that of a supplemented algebra. We shall begin with the special case of a group algebra. In Chapter 7 we will again use the more general setting of a supplemented algebra.

Definition 3.2.1. *An R-algebra $\eta : R \to \Lambda$ together with an R-algebra homomorphism $\epsilon : \Lambda \to R$ such that $\epsilon\eta = id_R$ is called a supplemented algebra, and ϵ is called the augmentation.*

$$\begin{tikzcd}
& \Lambda \arrow[dr, "\epsilon"] & \\
R \arrow[ur, "\eta"] \arrow[rr, equal] & & R
\end{tikzcd}$$

Let R be a commutative ring. Let G be a finite group. Then the group ring RG is a supplemented algebra, with

$$\begin{array}{cccll} \eta & : & R & \to & RG \\ & & r & \mapsto & r \cdot 1 \\ \epsilon & : & RG & \to & R \\ & & \sum_{x \in G} a_x x & \mapsto & \sum_{x \in G} a_x. \end{array}$$

We make the following definitions in the special case of a group ring. We will later use the definitions in the more general setting of a supplemented algebra.

3.3 Definition of a Diagonal Approximation

Let R be a commutative ring. Let G be a finite group. Form the group ring RG.

We will shortly apply Tate's Theorem 2.7.1 as explained earlier to obtain a projective resolution $\mathbb{F}$ of R over RG, where G is abelian.

Then we can make $\mathbb{F} \otimes_R \mathbb{F}$ into a resolution of R over RG, provided we can turn $\mathbb{F} \otimes_R \mathbb{F}$, a complex of $RG \otimes_R RG$-modules, into a complex of free, therefore projective RG-modules. The needed ingredient to do this is an R-algebra homomorphism

$$\Phi_0 : RG \to RG \otimes_R RG$$

such that the following diagram commutes:

$$\begin{tikzcd}
 & & RG & & R \otimes_R RG \arrow[ll, "\cong"'] \\
RG \arrow[urr, equal] \arrow[drr, equal] \arrow[r, "\Phi_0"] & RG \otimes_R RG \arrow[urrr, "\epsilon \otimes 1"] \arrow[drrr, "1 \otimes \epsilon"'] & & & \\
 & & RG & & RG \otimes_R R. \arrow[ll, "\cong"]
\end{tikzcd}$$

Define Φ_0 to be the diagonal map:

$$\begin{array}{cccll} \Phi_0 & : & RG & \to & RG \otimes_R RG \cong R(G \times G) \\ & & \sum_{x \in G} a_x x & \mapsto & \sum_{x \in G} a_x x \otimes x \cong \sum_{x \in G} a_x (x, x). \end{array}$$

Then it is clear that Φ_0 is an R-algebra homomorphism which makes the diagram commute.

Remarks:

1. Observe that $\mathbb{F} \otimes_R \mathbb{F}$ is again a DG-algebra over R.
2. We have initially defined our augmentation $\epsilon : RG \to R$. We abuse notation and also write $\epsilon : \mathbb{F} \to R$ for the map which comes from extending the original ϵ to all of $\mathbb{F}$.
3. The R-algebra homomorphism Φ_0 turns $\mathbb{F} \otimes_R \mathbb{F}$ into a complex of RG-modules, and makes $1 \otimes \epsilon$ and $\epsilon \otimes 1$ into morphisms of complexes of RG-modules.
4. The augmentation $\epsilon : RG \to R$ gives rise to two homomorphisms of DG-algebras, which we name ϵ_1 and ϵ_2, defined by the following compositions:

$$\begin{tikzcd}
\mathbb{F} \otimes_R \mathbb{F} \arrow[r, "\epsilon\otimes 1"] \arrow[rr, bend right, "\epsilon_1"'] & R \otimes_R \mathbb{F} \arrow[r, "\cong"] & \mathbb{F}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{F} \otimes_R \mathbb{F} \arrow[r, "1\otimes\epsilon"] \arrow[rr, bend right, "\epsilon_2"'] & \mathbb{F} \otimes_R R \arrow[r, "\cong"] & \mathbb{F}
\end{tikzcd}$$

5. By construction, $\epsilon \otimes 1$ is RG-linear in the first factor of $RG \otimes_R RG$ and $1 \otimes \epsilon$ is RG-linear in the second factor. The map Φ_0 as defined above makes ϵ_1 and ϵ_2 RG-linear over the same single copy of RG.

Definition 3.3.1. *Let R be a commutative ring and let G be a finite group. Given a projective resolution $\mathbb{F} \xrightarrow{\epsilon} R \longrightarrow 0$ over the group ring RG, a* <u>*diagonal approximation*</u> *is a map of complexes of RG-modules*

$$\Phi : \mathbb{F} \to \mathbb{F} \otimes_R \mathbb{F},$$

(where $\mathbb{F} \otimes_R \mathbb{F}$ is considered as a complex of RG-modules via Φ_0) which is compatible with the augmentation ϵ, in that the following diagram of complexes of RG-modules commutes:

$$\begin{tikzcd}
 & \mathbb{F} & R \otimes_R \mathbb{F} \arrow[l, "\cong"'] \\
\mathbb{F} \arrow[ur, equal] \arrow[r, "\Phi"] \arrow[dr, equal] & \mathbb{F} \otimes_R \mathbb{F} \arrow[u, "\epsilon_1"] \arrow[d, "\epsilon_2"'] \arrow[ur, "\epsilon\otimes 1"'] \arrow[dr, "1\otimes\epsilon"] & \\
 & \mathbb{F} & \mathbb{F} \otimes_R R. \arrow[l, "\cong"]
\end{tikzcd}$$

In other words, identifying $\mathbb{F}$ with $R \otimes_R \mathbb{F}$ and $\mathbb{F} \otimes_R R$ via the canonical isomorphisms, we have

$$\begin{array}{ccccc} (\epsilon \otimes 1)\Phi & = & id_{\mathbb{F}} & = & (1 \otimes \epsilon)\Phi \quad , \textit{ or equivalently,} \\ \epsilon_1 \Phi & = & id_{\mathbb{F}} & = & \epsilon_2 \Phi. \end{array}$$

3.4 Definition of the Cup Product

We will use the resolution $\mathbb{F}$ and a suitable diagonal approximation $\Phi : \mathbb{F} \to \mathbb{F} \otimes_R \mathbb{F}$ to determine the products in cohomology, taking coefficients in some trivial representation A.

Definition 3.4.1. *Take $A = R$. With the above Φ in hand, we define the* <u>*cup product*</u> *of cochains $f \in Hom_{RG}(\mathbb{F}_p, R)$ and $g \in Hom_{RG}(\mathbb{F}_q, R)$ as*

$$f \cup g := \mu \circ (f \otimes g) \circ \Phi.$$

This product is R-bilinear in f and g, and thus can be viewed as a product

$$- \cup - : Hom_{RG}(\mathbb{F}_p, R) \otimes Hom_{RG}(\mathbb{F}_q, R) \to Hom_{RG}(\mathbb{F}_{p+q}, R).$$

Lemma 3.4.2. *With the above notation, and denoting by d the differential in the Hom complex $Hom^\bullet(\mathbb{F}, R)$, we have the* <u>*Leibniz rule*</u>*:*

$$[d, f \cup g] = [d, f] \cup g + (-1)^{|f|} f \cup [d, g]$$

Proof.

$$\begin{array}{cl} & [d, f \cup g] \\ = & [d, \mu(f \otimes g)\Phi] \\ \underbrace{=}_{\text{Lemma 2.7.10}} & \underbrace{[d, \mu]}_{=0}(f \otimes g)\Phi + (-1)^{|\mu|}\mu \left([d, (f \otimes g)]\Phi + (-1)^{|f \otimes g|}\mu(f \otimes g) \underbrace{[d, \Phi]}_{=0} \right) \\ = & \mu\left([d, f] \otimes g + (-1)^{|f|} f \otimes [d, g]\right) \Phi \\ = & [d, f] \cup g + (-1)^{|f|} f \cup [d, g]. \end{array}$$

□

Remarks:

1. Lemma 3.4.2 shows that the cup product induces a well-defined product in cohomology.

2. It is desirable that the cup product be associative. However, we are not guaranteed that the cup product will be associative without an additional assumption. Applying the definition of the cup product gives

$$\begin{aligned} f \cup (g \cup h) &= \mu(\mu \otimes 1)(f \otimes g \otimes h)(1 \otimes \Phi)\Phi, \text{ and} \\ (f \cup g) \cup h &= \mu(1 \otimes \mu)(f \otimes g \otimes h)(\Phi \otimes 1)\Phi. \end{aligned}$$

   These two expressions will be equal if μ is associative and Φ is co-associative. Since μ is simply multiplication in the ring R, μ is always associative. However, Φ need not be co-associative. Indeed the particular Φs which we will construct will fail to be co-associative unless we take our coefficients to have trivial G-action. With trivial G-action, we will have that Φ is co-associative and therefore the cup product will be associative.

3. Observe that $\mathbb{F} \to R$ is an RG-resolution, and an R-homotopy equivalence, as $\mathbb{F}$ and R itself are R-projective resolutions of R. This implies that

$$\mathbb{F} \otimes_R \mathbb{F} \sim_{\text{homotopy equivalence over } R} R \otimes_R R \cong R$$

   and therefore $\mathbb{F} \otimes_R \mathbb{F} \to R$ is also an RG-resolution of R, where the RG-module structure on $\mathbb{F} \otimes_R \mathbb{F}$ is determined by Φ_0.

We will now recall a key fact about the cup product.

Theorem 3.4.3. *The cup product is homotopic to the Yoneda product.*

Proof. For this proof, all tensor products are over R. Write

$$\mathbb{F} = \cdots \longrightarrow P_{p+2} \longrightarrow P_{p+1} \longrightarrow P_p \longrightarrow P_{p-1} \longrightarrow P_{p-2} \longrightarrow \cdots$$

Let $f \in Hom_{RG}(P_p, R)$ and $g \in Hom_{RG}(P_q, R)$ be arbitrary.

Since P_p is projective, and $P_0 \xrightarrow{\epsilon} R$ is surjective, we can obtain a map $\tilde{f}_p$ which makes the following diagram commute:

$$\begin{tikzcd} & & & P_p \arrow[dl, dotted, "\tilde{f}_p"'] \arrow[d, "f"] & \\ \cdots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r, "\epsilon"'] & R \arrow[r] & 0 \end{tikzcd}$$

Then using the Comparison Theorem (Theorem 10.46 in [14]), we may lift f to a map $\tilde{f} : \mathbb{F} \to \mathbb{F}$ of complexes (of degree p) by filling in the following diagram.

$$\begin{tikzcd}
\cdots \arrow[r] & P_{p+2} \arrow[r] \arrow[d, dotted, "\tilde{f}_{p+2}"] & P_{p+1} \arrow[r] \arrow[d, dotted, "\tilde{f}_{p+1}"] & P_p \arrow[r] \arrow[d, "\tilde{f}_p"] \arrow[dd, bend left, "f"] & P_{p-1} \arrow[r] \arrow[d] & P_{p-2} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d, "\epsilon"] & 0 \arrow[r] & 0 \arrow[r] & \cdots \\
 & & 0 \arrow[r] & R \arrow[r] & 0 & &
\end{tikzcd}$$

Similarly, lift g to a map $\tilde{g} : \mathbb{F} \to \mathbb{F}$ of complexes (of degree q).

Recall that the Yoneda product is defined as $f \circ \tilde{g}$. Also note that $f \circ \tilde{g}$ and $\tilde{f} \circ \tilde{g}$ induce the same product in cohomology.

We will show that the left and right portions of the following diagram commute, where h denotes a homotopy from $\epsilon \otimes 1$ to $1 \otimes \epsilon$:

$$\begin{tikzcd}
\mathbb{F} \arrow[r, "\Phi"] \arrow[ddr, equal] & \mathbb{F} \otimes \mathbb{F} \arrow[rr, "1\otimes\tilde{g}"] \arrow[d, "\epsilon\otimes 1"'] \arrow[dd, bend left, "\epsilon_1"] & & \mathbb{F} \otimes \mathbb{F} \arrow[rr, "\tilde{f}\otimes 1"] \arrow[dl, "\epsilon\otimes 1"'] \arrow[dr, "1\otimes\epsilon"] \arrow[ddl, bend right, "\epsilon_1"] \arrow[ddr, bend left, "\epsilon_2"'] & & \mathbb{F} \otimes \mathbb{F} \arrow[r, "\epsilon\otimes\epsilon"] \arrow[d, "1\otimes\epsilon"] \arrow[dd, bend right, "\epsilon_2"'] & R \otimes R \, . \arrow[dd, "\mu"', "\cong"] \\
 & R \otimes \mathbb{F} \arrow[d, "\cong"] & R \otimes \mathbb{F} \arrow[d, "\cong"] & & \mathbb{F} \otimes R \arrow[d, "\cong"] & \mathbb{F} \otimes R \arrow[d, "\cong"] & \\
 & \mathbb{F} \arrow[r, "\tilde{g}"'] & \mathbb{F} \arrow[rr, dotted, bend right, "\sim", "h"'] & & \mathbb{F} \arrow[r, "\tilde{f}"'] \arrow[rr, bend right, "f"'] & \mathbb{F} \arrow[r, "\epsilon"'] & R
\end{tikzcd}$$

This implies the desired result, as follows.

Lemma 2.7.10 implies that, for any differential d and homotopy h, and morphisms of complexes a and b, we have

$$\begin{aligned}
[d, ahb] &= \underbrace{[d, a]}_{=0} hb + (-1)^{|a|} a [d, h] b \pm ah \underbrace{[d, b]}_{=0} \\
&= (-1)^{|a|} a [d, h] b \qquad (3.1)
\end{aligned}$$

and this shows that composing a homotopy with morphisms of complexes always returns a new homotopy.

So, letting d denote the differential in the appropriate complex, we now have

$$\begin{array}{rcl}
f \cup g & = & \mu(f \otimes g)\Phi \\
& = & \mu(\epsilon\tilde{f} \otimes \epsilon\tilde{g})\Phi \\
& = & \mu(\epsilon \otimes \epsilon)(\tilde{f} \otimes 1)(1 \otimes \tilde{g})\Phi \\
& = & \underbrace{\mu(\epsilon\tilde{f} \otimes 1)}_{=f} \ \underbrace{(1 \otimes \epsilon)}_{=(\epsilon\otimes 1+[d,h])} \ (1 \otimes \tilde{g})\Phi \\
& = & f(\epsilon \otimes 1 + [d,h])(1 \otimes \tilde{g})\Phi \\
& = & f\underbrace{(\epsilon \otimes 1)(1 \otimes \tilde{g})\Phi}_{=\tilde{g}} + f[d,h](1 \otimes \tilde{g})\Phi \\
& \underbrace{=}_{\text{by line (3.1)}} & f\tilde{g} + (-1)^p[d, fh(1 \otimes \tilde{g})\Phi] \\
& = & f\tilde{g} + (-1)^p[d,H], \text{ letting } H = fh(1 \otimes \tilde{g})\Phi \\
\Rightarrow f \cup g & \sim & f\tilde{g}.
\end{array}$$

So now it remains to prove what he have claimed about the above diagram.

The commuting diagram

$$\begin{tikzcd}
\mathbb{F} \otimes \mathbb{F} \arrow[r, "\epsilon\otimes\epsilon"] \arrow[d, "1\otimes\epsilon"'] & R \otimes R \arrow[r] \arrow[d, "\mu"', "\cong"] & 0 \\
\mathbb{F} \otimes R \cong \mathbb{F} \arrow[r, "\epsilon"'] & R \arrow[r] & 0
\end{tikzcd}$$

shows that $1 \otimes \epsilon$ lifts μ.

The commuting diagram

$$\begin{tikzcd}
\mathbb{F} \otimes \mathbb{F} \arrow[r, "\epsilon\otimes\epsilon"] \arrow[d, "\epsilon\otimes 1"'] & R \otimes R \arrow[r] \arrow[d, "\mu"', "\cong"] & 0 \\
R \otimes \mathbb{F} \cong \mathbb{F} \arrow[r, "\epsilon"'] & R \arrow[r] & 0
\end{tikzcd}$$

shows that $\epsilon \otimes 1$ lifts μ.

Therefore again by the Comparison Theorem, $1 \otimes \epsilon \sim \epsilon \otimes 1$. There exists a homotopy h such that $[d,h] = dh + hd = 1 \otimes \epsilon - \epsilon \otimes 1$, writing d for the differential in both complexes.

Each piece of the diagram commutes as follows.

- The left hand triangle commutes by the construction of Φ.

- The left hand trapezoid commutes as follows. Let $x \otimes y$ be an arbitrary elementary tensor in the top left copy of $\mathbb{F} \otimes \mathbb{F}$. Then the clockwise branch yields

$$x \otimes y \mapsto (-1)^{|\tilde{g}||x|} x \otimes \tilde{g}(y) \mapsto (-1)^{|\tilde{g}||x|} \epsilon(x)\tilde{g}(y) \underbrace{=}_{\epsilon(x)=0 \text{ if } |x|>0} \epsilon(x)\tilde{g}(y)$$

and the counter-clockwise branch yields

$$x \otimes y \mapsto \epsilon(x) \otimes y \mapsto \epsilon(x)y \mapsto \tilde{g}(\epsilon(x)y) \underbrace{=}_{\tilde{g} \text{ is } RG-\text{linear}} \epsilon(x)\tilde{g}(y).$$

- The right hand trapezoid commutes as follows. Let $z \otimes w$ be an arbitrary elementary tensor in the top central copy of $\mathbb{F} \otimes \mathbb{F}$. Then the clockwise branch yields

$$z \otimes w \mapsto \tilde{f}(z) \otimes w \mapsto \tilde{f}(z) \otimes \epsilon(w) \mapsto \tilde{f}(z)\epsilon(w)$$

and the counter-clockwise branch yields

$$z \otimes w \mapsto z \otimes \epsilon(w) \mapsto z\epsilon(w) \mapsto \tilde{f}(z\epsilon(w)) \underbrace{=}_{\tilde{f} \text{ is } RG-\text{linear}} \tilde{f}(z)\epsilon(w).$$

- The right hand square commutes as follows. Let $a \otimes b$ be an arbitrary elementary tensor in the top right copy of $\mathbb{F} \otimes \mathbb{F}$. Then the clockwise branch yields

$$a \otimes b \mapsto \epsilon(a) \otimes \epsilon(b) \mapsto \epsilon(a)\epsilon(b)$$

and the counter-clockwise branch yields

$$a \otimes b \mapsto a \otimes \epsilon(b) \mapsto a\epsilon(b) \mapsto \epsilon(a\epsilon(b)) \underbrace{=}_{\epsilon \text{ is } RG-\text{linear}} \epsilon(a)\epsilon(b).$$

The diagram behaves as claimed, and so we are done. $\square$

Chapter 4

The Tate Resolution for a Finite Cyclic Group

4.1 Introduction

In this chapter, we will apply Tate's Theorem to compute the cohomology ring of a finite cyclic group. We begin with a finite cyclic group G, and recall the Tate resolution $\mathbb{F}$ for the trivial G-module R over RG which we computed earlier. We then construct a diagonal approximation $\Phi : \mathbb{F} \to \mathbb{F} \otimes_R \mathbb{F}$. We finish by computing the dualized differential ∂^* on $Hom_{RG}(\mathbb{F}, R)$, and the products of cochains in $Hom_{RG}(\mathbb{F}, R)$. We confirm that our results agree with the known results from [8].

4.2 The Tate Resolution for a Finite Cyclic Group

Theorem 4.2.1. *Let $G = \langle x \mid x^h = 1 \rangle$, the cyclic group of order h, and let R be a commutative ring. Form the group ring RG and view R as an RG-module with the trivial G-action. Let $N = \sum_{i=0}^{h-1} x^i$ be the* <u>*norm element*</u> *of RG. Then the Tate resolution of R*

over RG is given by:

$$
\begin{tikzcd}
 & 0 \\
 & R \arrow[u] \\
0 & RG \cdot 1 \arrow[u, "x\mapsto 1"'] \\
1 & RG \cdot \tau \arrow[u, "(x-1)"'] \\
2 & RG \cdot \sigma \arrow[u, "N"'] \\
3 & RG \cdot \tau\sigma \arrow[u, "(x-1)"'] \\
 & \vdots \arrow[u, "N"'] \\
2i & RG \cdot \sigma^{(i)} \arrow[u] \\
2i+1 & RG \cdot \tau\sigma^{(i)} \arrow[u, "(x-1)"'] \\
2i+2 & RG \cdot \sigma^{(i+1)} \arrow[u, "N"'] \\
 & \vdots \arrow[u] \\
 & \mathbb{F} \arrow[u, equal]
\end{tikzcd}
$$

with the (compact) differential coming from Tate:

$$\partial = (x-1)\frac{\partial}{\partial \tau} + N\tau\frac{\partial}{\partial \sigma}.$$

Proof. This is precisely the resolution obtained on line (2.11) in Example 2.8.2. ☐

Remarks:

1. Recall that $\frac{\partial}{\partial\sigma}$ is compatible with divided powers per Definition 2.7.7, so that

$$\frac{\partial\sigma^{(i)}}{\partial\sigma} = \sigma^{(i-1)}.$$

2. As in Example 2.8.2,

   (a) $\mathbb{F}$ is a resolution of R over RG.

   (b) $\mathbb{F}$ is a DG R-algebra with divided powers.

3. What is $\mathbb{F} \otimes_R \mathbb{F}$?

   Recalling the notation for $\mathbb{F}$, we now write a single prime to denote an element of the first factor of $\mathbb{F} \otimes_R \mathbb{F}$, and a double prime to denote an element of the second factor, e.g.

$$\begin{aligned} x' &= x \otimes 1 \\ x'' &= 1 \otimes x, \end{aligned}$$

   and similarly for τ and σ.

   Generally, $A^{ev} = A^{op} \otimes_R A$, where $R \longrightarrow A$ is an R-algebra. Here, $A = RG$ is commutative, thus $A^{op} = A$. Therefore we may interpret $RG \otimes_R RG$ as RG^{ev}, equally well as

$$\begin{aligned} R(G \times G) &= \frac{R[x', x'']}{((x')^h - 1, (x'')^h - 1)} \\ &\cong \frac{R[x']}{((x')^h - 1)} \otimes_R \frac{R[x'']}{((x'')^h - 1)} \end{aligned}$$

4. The differentials in $\mathbb{F} \otimes_R \mathbb{F}$ can be computed from the differentials in $\mathbb{F}$, using

   (a) the compact form of the differential in $\mathbb{F}$,

   (b) the fact that ∂ is a derivation and thus satisfies the Leibniz rule,

(c) the fact that $(\tau')^2 = 0 = (\tau'')^2$.

5. $\mathbb{F} \otimes_R \mathbb{F}$ is a *DG* R-algebra with divided powers.

Theorem 4.2.2. *The tensor product*

$$\mathbb{F} \otimes_R \mathbb{F} \cong (RG^{ev}\langle \tau', \tau''; \sigma', \sigma'' \rangle, \partial) \to R$$

is an RG*-resolution of* R*.*

Proof. The fact that $\mathbb{F} \otimes_R \mathbb{F} \to R$ is an RG-resolution of R has been explained in earlier comments. We just need to establish the above isomorphism. We have

$$\begin{aligned}
& \mathbb{F} \otimes_R \mathbb{F} \\
= \quad & (RG\langle \tau',\ \sigma' \rangle, \partial') \otimes_R (RG\langle \tau'',\ \sigma'' \rangle, \partial'') \\
= \quad & \left(\frac{R[x']}{((x')^h - 1)} \langle \tau',\ \sigma' \rangle, \partial' \right) \otimes_R \left(\frac{R[x'']}{((x'')^h - 1)} \langle \tau'',\ \sigma'' \rangle, \partial'' \right) \\
\cong \quad & \left(\underbrace{\left(\frac{R[x']}{((x')^h - 1)} \otimes_R \frac{R[x'']}{((x'')^h - 1)} \right)}_{R(G \times G) =: RG^{ev}, \text{ as above}} \langle \tau',\ \sigma'; \tau'',\ \sigma'' \rangle, \partial \right) \\
\cong \quad & (RG^{ev}\langle \tau', \tau''; \sigma', \sigma'' \rangle, \partial)
\end{aligned}$$

□

Remark: Up to this point we have been essentially recalling known results; from here onwards we will present new results.

4.3 A Diagonal Approximation

Now we re-draw the earlier diagram, to show a diagonal approximation Φ. Viewing both resolutions as *DG*-algebras, we want Φ to be a homomorphism of *DG*-algebras. Then it will be enough to specify how Φ acts on the algebra generators.

First define augmentation maps

$$\begin{array}{rcccc} \epsilon & : & RG & \to & R \\ & & x & \mapsto & 1 \\ \tilde{\epsilon} = \epsilon \otimes \epsilon & : & RG^{ev} & \to & R \otimes R \\ & & x' & \mapsto & 1 \\ & & x'' & \mapsto & 1 \end{array}$$

Next define the diagonal map

$$\begin{array}{rcccc} \Phi_0 & : & RG & \to & RG^{ev} \\ & & x & \mapsto & x'x'' \end{array}$$

and note that this is well-defined because

$$\begin{array}{rcl} \Phi_0(x^h - 1) & = & (x'x'')^h - 1 \\ & \equiv & 0 \mod ((x')^h - 1, (x'')^h - 1). \end{array}$$

Then this diagram commutes

$$\begin{tikzcd} R \arrow[r, "\cong"] & R \otimes_R R \\ RG \arrow[u, "\epsilon"] \arrow[r, "\Phi_0"] & RG^{ev} \arrow[u, "\epsilon\otimes\epsilon"'] \end{tikzcd}$$

as we have

$$\begin{tikzcd} 1 \arrow[r, equal] & 1 \\ x \arrow[u, mapsto] \arrow[r, mapsto] & x'x'' \arrow[u, mapsto] \end{tikzcd}$$

To streamline the notation in the following theorem, we make this definition (recalling that $N(x) = \sum_{j=0}^{h-1} x^j$).

Definition 4.3.1. *Define*

$$\nabla_N(x', x'') := \frac{N(x'x'') - N(x'')}{x' - 1}$$

Note that substituting $x' = 1$ kills the numerator, and thus $(x' - 1)$ divides the numerator. Therefore $\nabla_N(x', x'')$ is a polynomial in $R[x', x'']$, and then also in RG^{ev}.

The following identity will be useful later.

Lemma 4.3.2.

$$\begin{aligned}
\nabla_N(x', x'') &:= \frac{N(x'x'') - N(x'')}{x' - 1} && (4.1)\\
&= \sum_{0 \le m < n \le h-1} (x')^m (x'')^n && (4.2)\\
&= \sum_{0 \le m < n \le h-1} x^m \otimes x^n, \textit{ in the notation from [8]} && (4.3)
\end{aligned}$$

Proof. We have

$$\begin{aligned}
& \nabla_N(x', x'')\\
:= \;& \frac{N(x'x'') - N(x'')}{x' - 1}\\
= \;& \frac{(1 + x'x'' + (x')^2(x'')^2 + \cdots + (x')^{h-1}(x'')^{h-1}) - (1 + x'' + \cdots + (x'')^{h-1})}{x' - 1}\\
= \;& \frac{(1 - 1) + (x'x'' - x'') + \cdots + (x')^{h-1}(x'')^{h-1} - (x'')^{h-1})}{x' - 1}\\
= \;& \frac{(x' - 1)x'' + ((x')^2 - 1)(x'')^2 + \cdots + ((x')^{h-1} - 1)(x'')^{h-1}}{x' - 1}\\
= \;& \frac{(x' - 1)x'' + (x' - 1)(1 + x')(x'')^2 + \cdots + (x' - 1)((1 + x' + \cdots + (x')^{h-2})(x'')^{h-1}}{x' - 1}\\
= \;& x'' + (1 + x')(x'')^2 + \cdots + (1 + x' + \cdots + (x')^{h-2})(x'')^{h-1}\\
= \;& \sum_{0 \le m < n \le h-1} (x')^m (x'')^n,
\end{aligned}$$

as required. □

Remark: Lemma 4.3.2 implies that augmentation sends $\nabla_N(x', x'')$ to $\begin{pmatrix} h \\ 2 \end{pmatrix}$.

Theorem 4.3.3. *A diagonal approximation Φ is given by the following diagram, where the maps in higher degrees are determined by the maps in degrees zero, one and two.*

$$
\begin{tikzcd}[column sep=large]
 & 0 & 0 \\
 & R \arrow[u] \arrow[r, equal] & R \otimes R \arrow[u] \\
0 & RG \cdot 1 \arrow[u, "\epsilon : x \mapsto 1"] \arrow[r, "\Phi_0", "x \mapsto x'x''"'] & RG^{ev} \arrow[u, "{\tilde{\epsilon} = \epsilon \otimes \epsilon :\ \begin{array}{l} x' \mapsto 1 \\ x'' \mapsto 1 \end{array}}"'] \\
1 & RG \cdot \tau \arrow[u, "(x-1)"] \arrow[r, "\Phi_1", "\tau \mapsto x''\tau' + \tau''"'] & RG^{ev} \cdot (\tau' \bigoplus \tau'') \arrow[u, "{\begin{array}{l} \tau' \mapsto x' - 1 \\ \tau'' \mapsto x'' - 1 \end{array}}"'] \\
2 & RG \cdot \sigma \arrow[u, "N"] \arrow[r, "\Phi_2", "\sigma \mapsto \sigma' + \sigma'' + \nabla_N(x', x'')\tau'\tau''"'] & RG^{ev} \cdot (\sigma' \bigoplus \tau'\tau'' \bigoplus \sigma'') \arrow[u, "{\begin{array}{l} \sigma' \mapsto N(x')\tau' \\ \sigma'' \mapsto N(x'')\tau'' \\ \tau'\tau'' \mapsto (x'-1)\tau'' - \tau'(x''-1) \end{array}}"'] \\
3 & RG \cdot \tau\sigma \arrow[u, "(x-1)"] \arrow[r, "\Phi_3"] & RG^{ev} \cdot (\tau'\sigma' \bigoplus \tau''\sigma' \bigoplus \tau'\sigma'' \bigoplus \tau''\sigma'') \arrow[u] \\
 & \vdots \arrow[u, "N"] & \vdots \arrow[u] \\
2i & RG \cdot \sigma^{(i)} \arrow[u] \arrow[r, "\Phi_{2i}"] & RG^{ev} \cdot \big((\bigoplus_{k+l=i} (\sigma')^{(k)} (\sigma')^{(l)}) \bigoplus (\bigoplus_{k+l=i-1} \tau'\tau''(\sigma')^{(k)}(\sigma')^{(l)}) \big) \arrow[u] \\
2i+1 & RG \cdot \tau\sigma^{(i)} \arrow[u, "(x-1)"] \arrow[r, "\Phi_{2i+1}"] & RG^{ev} \cdot \big(\bigoplus_{k+l=i} (\tau'(\sigma')^{(k)}(\sigma')^{(l)} \bigoplus \tau''(\sigma')^{(k)}(\sigma')^{(l)}) \big) \arrow[u] \\
2i+2 & RG \cdot \sigma^{(i+1)} \arrow[u, "N"] \arrow[r, "\Phi_{2i+2}"] & RG^{ev} \cdot \big((\bigoplus_{k+l=i+1} (\sigma')^{(k)} (\sigma')^{(l)}) \bigoplus (\bigoplus_{k+l=i} \tau'\tau''(\sigma')^{(k)}(\sigma')^{(l)}) \big) \arrow[u] \\
 & \vdots \arrow[u] & \vdots \arrow[u] \\
 & \mathbb{F} \arrow[u, equal] \arrow[r, "\Phi"] & \mathbb{F} \otimes_R \mathbb{F} \arrow[u, equal]
\end{tikzcd}
$$

The rest of this section will give the proof of this Theorem.

<u>How do we choose $\Phi_1(\tau)$?</u> The following square must commute.

$$\begin{tikzcd}
(x-1) \arrow[r, mapsto, "\Phi_0"] & x'x''-1 \\
\tau \arrow[u, mapsto] \arrow[r, mapsto, "\Phi_1"'] & ? \arrow[u, mapsto]
\end{tikzcd}$$

where the unknown is unique up to any boundary. By writing $(x'x''-1)$ in terms of $(x'-1)$ and $(x''-1)$, we obtain

$$\begin{aligned}
(x'-1)(x''-1) &= x'x''-x'-x''+1 \\
&= (x'x''-1)-(x'-1)-(x''-1)
\end{aligned}$$

Therefore

$$\begin{aligned}
x'x''-1 &= (x'-1)(x''-1)+(x'-1)+(x''-1) \\
&= (x'-1)[(x''-1)+1]+(x''-1) \\
&= \underbrace{(x'-1)}_{\partial(\tau')}x''+\underbrace{(x''-1)}_{\partial(\tau'')}
\end{aligned}$$

so that one choice which works is $\Phi_1(\tau) = x''\tau'+\tau''$.

This choice for $\Phi_1(\tau)$ implies

$$\Phi_1(N(x)\tau) = N(x'x'')(x''\tau'+\tau'')$$

<u>How do we choose $\Phi_2(\sigma)$?</u>

The context for the following explanation comes from Definition 3.3.1. A general element of $\mathbb{F}\otimes_R\mathbb{F}$ of degree 2 has the form $a\sigma'+b\tau'\tau''+c\sigma''$, for some $a,b,c,\in RG^{ev}$. So letting $\Phi_2(\sigma) = a\sigma'+b\tau'\tau''+c\sigma''$, then following the expression through both branches of the given diagram gives

$$\begin{tikzcd}
 & & c\sigma'' \arrow[r, mapsto] & c\sigma \\
\sigma \arrow[r, mapsto, "\Phi"] & a\sigma'+b\tau'\tau''+c\sigma'' \arrow[ur, mapsto] \arrow[dr, mapsto] & & \\
 & & a\sigma' \arrow[r, mapsto] & a\sigma
\end{tikzcd}$$

so we want $a = c = 1$. It remains to determine the coefficient b.

The following square must commute:

$$\begin{tikzcd} N(x)\tau \arrow[r, mapsto, "\Phi_1"] & N(x'x'')(x''\tau' + \tau'') \\ \sigma \arrow[u, mapsto] \arrow[r, mapsto, "\Phi_2"'] & \sigma' + \sigma'' + g(x', x'')\tau'\tau'' \arrow[u, mapsto, "\partial"'] \end{tikzcd}$$

for some $b = g(x', x'')$.

The following Lemma will show that the choice of Φ_2 in Theorem 4.3.3 is correct. After that, it still remains to show that everything in higher degrees is determined by the choices we have made in degrees zero, one and two.

Lemma 4.3.4. *$g(x', x'') = \nabla_N(x', x'') := \frac{N(x'x'') - N(x'')}{x'-1} \in R\frac{[x', x'']}{((x'')^h - 1)}$ makes the required square commute. Thus the diagram still commutes when we pass to*

$$RG^{ev} \cong \frac{R[x', x'']}{((x')^h - 1, (x'')^h - 1)}.$$

Proof. We work in the ring $\frac{R[x', x'']}{((x'')^h - 1)}$, as there $(x' - 1)$ is still a non zero divisor, so that we can "divide" a class $p(x', x'')$ by this element, as long as $p(1, x'') = 0$.

We must prove that

$$\partial \left(\frac{N(x'x'') - N(x'')}{x' - 1}(\tau'\tau'') + \sigma' + \sigma'' \right) = N(x'x'')(x''\tau' + \tau'')$$

As it will come up later in the computation, we claim that

$$\frac{(x'' - 1)[N(x'x'') - N(x'')]}{x' - 1} = N(x') - x''N(x'x'')$$

We have the following chain of equalities:

$$
\begin{aligned}
&\frac{(x''-1)[N(x'x'')-N(x'')]}{x'-1}\\
= \quad & \frac{(x''-1)N(x'x'')-\overbrace{(x''-1)N(x'')}^{=0}}{x'-1}\\
= \quad & \frac{(x''-1)N(x'x'')}{x'-1}\\
= \quad & \frac{1}{x'-1}\left[\begin{array}{l} x''+x'(x'')^2+(x')^2(x'')^3+\cdots+(x')^{h-2}(x'')^{h-1}+(x')^{h-1}\\ -(1+x'x''+(x')^2(x'')^2+\cdots+(x')^{h-1}(x'')^{h-1}) \end{array}\right]\\
= \quad & \frac{1}{x'-1}\left[\begin{array}{l} (1-x')x''\\ +(x'-(x')^2)(x'')^2+\cdots+((x')^{h-2}-(x')^{h-1})(x'')^{h-1}\\ +((x')^{h-1}-1) \end{array}\right]\\
= \quad & \frac{1}{x'-1}\left[\begin{array}{l} -(x'-1)x''\\ -((x')^2-x')(x'')^2-\cdots-((x')^{h-1}-(x')^{h-2})(x'')^{h-1}\\ +((x')-1)(N(x')-(x')^{h-1}) \end{array}\right]\\
= \quad & (-x''-x'(x'')^2-\cdots-(x')^{h-2}(x'')^{h-1})+N(x')-(x')^{h-1}\\
= \quad & -(x''N(x'x'')-(x')^{h-1})+N(x')-(x')^{h-1}\\
= \quad & N(x')-x''N(x'x'')
\end{aligned}
$$

as claimed.

Now for our main result we have

$$
\begin{aligned}
&\partial\left(\frac{N(x'x'')-N(x'')}{x'-1}(\tau'\tau'')+\sigma'+\sigma''\right)\\
= \quad & \frac{N(x'x'')-N(x'')}{x'-1}\left((x'-1)\tau''-\tau'(x''-1)\right)+N(x')\tau'+N(x'')\tau''\\
= \quad & (N(x'x'')-N(x''))\tau''-\tau'\frac{(x''-1)[N(x'x'')-N(x'')]}{x'-1}+N(x')\tau'+N(x'')\tau''\\
= \quad & N(x'x'')\tau''-N(x'')\tau''-\tau'\left[N(x')-x''N(x'x'')\right]+N(x')\tau'+N(x'')\tau''\\
= \quad & N(x'x'')(x''\tau'+\tau'')
\end{aligned}
$$

as required. □

We now show that everything in higher degrees is determined by the choices in degrees zero, one and two. We want an algebra homomorphism, compatible with divided powers (c.f. Definition 2.6.7), so this forces

$$\Phi\left(\tau^j\sigma^{(i)}\right) = \Phi(\tau)^j\Phi\left(\sigma\right)^{(i)}.$$

We know $\Phi(\tau)$, and we determine $\Phi\left(\sigma^{(i)}\right)$ as follows. Start with our earlier definition $\Phi(\sigma) = \sigma' + \sigma'' + \nabla_N(x', x'')\tau'\tau''$, so that, by the definition of the divided powers on $\mathbb{F}$, we have

$$\begin{aligned}
\Phi\left(\sigma^{(i)}\right) &= (\Phi(\sigma))^{(i)} \\
&= ((\sigma' + \sigma'') + \nabla_N(x', x'')\tau'\tau'')^{(i)} \\
&= \sum_{j=0}^{i}(\sigma' + \sigma'')^{(i-j)}\nabla_N(x', x'')^j(\tau'\tau'')^{(j)} \\
&\underbrace{=}_{(\tau'\tau'')^{(j)}=0 \text{ for } j\geq 2} (\sigma' + \sigma'')^{(i)} + (\sigma' + \sigma'')^{(i-1)}\nabla_N(x', x'')\tau'\tau''. \qquad (4.4)
\end{aligned}$$

So we see that, in higher degrees, everything is already determined by the choices we have made in degrees 1 and 2. This completes the proof of Theorem 4.3.3.

We now record two identities which will be useful later. The computation ending on line (4.4) gives us that

$$\begin{aligned}
\Phi(\tau\sigma^{(i)}) &= \Phi(\tau)\Phi\left(\sigma\right)^{(i)} \\
&= (x''\tau' + \tau'')((\sigma' + \sigma'')^{(i)} + (\sigma' + \sigma'')^{(i-1)}\nabla_N(x', x'')\tau'\tau'') \\
&\underbrace{=}_{\tau'\tau'=\tau''\tau''=0} (x''\tau' + \tau'')(\sigma' + \sigma'')^{(i)}, \text{ in particular} \qquad (4.5) \\
\Phi(\tau\sigma) &= (x''\tau' + \tau'')(\sigma' + \sigma'') \qquad (4.6)
\end{aligned}$$

Theorem 4.3.3 has exhibited one correct diagonal approximation. The following Corollary describing all possible choices for correct diagonal approximations is now clear.

Corollary 4.3.5. *All choices for Φ are defined by*

$$\begin{array}{cccl}
\Phi & : & \mathbb{F} & \to \mathbb{F}\otimes_R\mathbb{F} \\
\Phi_0 & : & x & \mapsto x'x'' \\
\Phi_1 & : & \tau & \mapsto x''\tau' + \tau'' + \partial(\omega) \\
\Phi_2 & : & \sigma & \mapsto \sigma' + \sigma'' + \nabla_N(x', x'')\tau'\tau'' + N(x'x'')\omega + \partial(\eta)
\end{array}$$

where $\omega \in (\mathbb{F} \otimes_R \mathbb{F})_2$, the degree 2 part of $\mathbb{F} \otimes_R \mathbb{F}$, satisfies $\epsilon_1(\omega) = 0 = \epsilon_2(\omega)$, and $\eta \in (\mathbb{F} \otimes_R \mathbb{F})_3$ satisfies $\epsilon_1(\eta) = 0 = \epsilon_2(\eta)$, both conditions being imposed to preserve the diagram in Definition 3.3.1. All of these choices for Φ induce the same product when we pass to cohomology.

4.4 The Dual of the Tate Resolution

We now dualize and analyze the resulting cohomology.

We have the resolution $\mathbb{F} \longrightarrow R$ over RG and a diagonal approximation $\Phi : \mathbb{F} \to \mathbb{F} \otimes_R \mathbb{F}$:

$$\begin{aligned}
\mathbb{F} &= RG\langle \tau;\ \sigma\rangle,\ |\tau| = 1;\ |\sigma| = 2 \\
\partial &= (x-1)\frac{\partial}{\partial \tau} + N(x)\tau\frac{\partial}{\partial \sigma}, \text{ where } N(x) = \frac{x^h - 1}{x-1} = \sum_{j=0}^{h-1} x^j \\
\Phi(x) &= x'x'' \\
\Phi(\tau) &= x''\tau' + \tau'' + \partial(\omega) \\
\Phi(\sigma) &= \sigma' + \sigma'' + \nabla_N(x', x'')\tau'\tau'' + N(x'x'')\omega + \partial(\eta)
\end{aligned}$$

where $\omega \in (\mathbb{F} \otimes_R \mathbb{F})_2$ satisfies $\epsilon_1(\omega) = 0 = \epsilon_2(\omega)$, and $\eta \in (\mathbb{F} \otimes_R \mathbb{F})_3$ satisfies $\epsilon_1(\eta) = 0 = \epsilon_2(\eta)$. These assignments determine a unique homomorphism of algebras with divided powers.

Remark: The flexibility of modifying Φ by a boundary will be very useful later.

Because we are interested in cohomology with trivial coefficients, we choose Φ so that it becomes particularly simple when evaluating modulo

$$I = (x' - 1,\ x'' - 1) \subset RG \otimes_R RG.$$

We want to compute the cohomology products, so we start by analyzing $Hom_{RG}(\mathbb{F}, R)$.

Dualizing $\mathbb{F}$ into R via $Hom_{RG}(-,R)$ (and denoting $Hom_{RG}(\mathbb{F},RG)$ by $\mathbb{F}^*$) gives

$$
\begin{aligned}
& Hom_{RG}(\mathbb{F},R) && (4.7)\\
\underbrace{\cong}_{\text{by } 2.2.3} \quad & R \otimes_{RG} \mathbb{F}^* && (4.8)\\
\cong \quad & R \otimes_{RG} RG[S] \otimes_{RG} \bigwedge_{RG} \langle T \rangle && (4.9)\\
\underbrace{\cong}_{G \text{ acts trivially on } R} \quad & R[S] \otimes_R \bigwedge_{R} \langle T \rangle && (4.10)
\end{aligned}
$$

where S is a polynomial variable dual to σ, and T is dual to τ.

Warning: Although line (4.10) above carries an algebra structure, it is not the algebra structure that we are seeking. We must use the definition of the cup product (which uses our chosen Φ) to work out the products.

We now determine the dualized differential ∂^*.

4.5 The Action of ∂^*

Note that $\mathbb{F}^* = Hom_{RG}(\mathbb{F},R)$ is a Hom complex as in Definition 2.5.1.

Let $\omega = \tau^k\sigma^{(n)} \in \mathbb{F}$ be an arbitrary monomial, for $n \geq 0,\ k \in \{0,1\}$. Since the $\tau^k\sigma^{(n)}$ form an RG-basis for $\mathbb{F}$, we define the dual R-basis elements for $Hom_{RG}(\mathbb{F},R)$ to be S^LT_M, for $L, M \geq 0$. In detail, S^LT_M evaluates to 1 on $\tau^M\sigma^{(L)}$, and evaluates to 0 on all other basis elements of $\mathbb{F}$.

Now to determine the effect of ∂^* on an arbitrary S^LT_M, we evaluate

$$
\begin{aligned}
\partial^*(S^LT_M)(\omega) &= \underbrace{d_R}_{=0}(S^LT_M)(\tau^k\sigma^{(n)}) - (-1)^{|S^LT_M|}S^LT_M d_{\mathbb{F}}(\tau^k\sigma^{(n)}) && (4.11)\\
&= -(-1)^M S^LT_M(h\tau^{k+1}\sigma^{(n-1)}), && (4.12)
\end{aligned}
$$

as $\partial_{\mathbb{F}}(\tau^k\sigma^{(n)}) = ((x-1)\frac{\partial}{\partial\tau} + N(x)\tau\frac{\partial}{\partial\sigma})(\tau^k\sigma^{(n)}) \equiv h\tau\frac{\partial}{\partial\sigma}(\tau^k\sigma^{(n)}) \mod (x-1)$. By the definition of S^LT_M, line (4.12) evaluates to zero unless

- $n = L+1$, and

- $k = 0$ and $M = 1$ (we know that $k \in \{0,1\}$ and if $k = 1$ then $\tau^{k+1} = 0$),

in which case it evaluates to h.

Thus, formally
$$\partial^*(S^L T_M)(\omega) = S^{L+1}\frac{\partial T_M}{\partial T}(\omega),$$
or for short,
$$\begin{aligned} \partial^*(S^L T_M) &= hS^{L+1}\frac{\partial T_M}{\partial T} \\ &= \begin{cases} hS^{L+1} & \text{if} \quad M = 1 \\ 0 & \text{if} \quad M = 0. \end{cases} \end{aligned}$$

So, in compact form,
$$\partial^* = hS\frac{\partial}{\partial T},$$
when evaluated on monomials $S^L T_M$.

We have shown that (temporarily, using the algebra structure of the Koszul complex) the dualized complex becomes
$$\left(R[S] \otimes_R \bigwedge_R (T),\ \partial = hS\frac{\partial}{\partial T}\right)$$
This is just the Koszul complex
$$\mathbb{K}(hS;\ R[S])$$
in the linear sense, i.e. as a complex of R-modules.

4.6 Cochain Products

For this section, unadorned tensor products are over R.

Theorem 4.6.1. *The choice of* Φ *in Theorem 4.3.3 defines the following cup product structure on* $Hom_{RG}(\mathbb{F}, R)$*, which makes it into a DG-algebra. With* t *for* T *and* s *for* S*, we have*

$$t \cup t = -\binom{h}{2} s \tag{4.13}$$
$$t \cup s = s \cup t \tag{4.14}$$

and the elements t and s generate $Hom_{RG}(\mathbb{F}, R)$ with respect to the cup product, subject only to these relations.

Proof. 1. $\underline{t \cup t = -\begin{pmatrix} h \\ 2 \end{pmatrix} s}$: We have $t \cup t = \mu(t \otimes t)\Phi_{1,1}$, and $\Phi_{1,1} : \mathbb{F}_2 \to \mathbb{F}_1 \otimes \mathbb{F}_1$. In degree 2, it suffices to examine the effect of $\mu(t \otimes t)\Phi_{1,1}$ on σ. Recall that

$$\begin{aligned} \Phi(\sigma) &= \sigma' + \sigma'' + \nabla_N(x', x'')\tau'\tau'' \\ \Rightarrow \Phi_{1,1}(\sigma) &= \nabla_N(x', x'')\tau'\tau'' \\ &= \nabla_N(x', x'')\tau \otimes \tau. \end{aligned}$$

Applying $\mu(t \otimes t)$ gives

$$\begin{aligned} & \mu(t \otimes t)(\nabla_N(x', x'')\tau \otimes \tau) \\ \underbrace{=}_{\text{by Lemma 4.3.2}} & \begin{pmatrix} h \\ 2 \end{pmatrix} \mu(t \otimes t)(\tau \otimes \tau) \\ = & \begin{pmatrix} h \\ 2 \end{pmatrix} \mu(-1)(t(\tau) \otimes t(\tau)) \\ = & -\begin{pmatrix} h \\ 2 \end{pmatrix} \mu(1 \otimes 1) \\ = & -\begin{pmatrix} h \\ 2 \end{pmatrix} \end{aligned}$$

So since $t \cup t$ evaluates to 0 on all basis elements except σ, on which it evaluates to $-\begin{pmatrix} h \\ 2 \end{pmatrix}$, we can express this compactly as $t \cup t = -\begin{pmatrix} h \\ 2 \end{pmatrix} s$, as required.

2. $\underline{t \cup s = s \cup t}$: We have $s \cup t = \mu(s \otimes t)\Phi_{2,1}$ and $\Phi_{2,1} : \mathbb{F}_3 \to \mathbb{F}_2 \otimes \mathbb{F}_1$. Also, $t \cup s = \mu(t \otimes s)\Phi_{1,2}$ and $\Phi_{1,2} : \mathbb{F}_3 \to \mathbb{F}_1 \otimes \mathbb{F}_2$. The element $\tau\sigma$ is a basis of $(\mathbb{F} \otimes_R \mathbb{F})_3$. We therefore use the example on line (4.6):

$$\Phi(\tau\sigma) = (x''\tau' + \tau'')(\sigma' + \sigma'').$$

As $(t \otimes s)$ vanishes on all occurring monomials except $\tau'\sigma''$, applying $\mu(t \otimes s)$ gives

$$\begin{aligned} & \mu(t \otimes s)(x''\tau'\sigma'') \\ = & \mu(t \otimes s)(x''\tau \otimes \sigma) \\ = & \mu(t(\tau) \otimes s(\sigma)) \\ = & \mu(1 \otimes 1) \\ = & 1 \end{aligned}$$

Similarly, applying $\mu(s \otimes t)$ gives

$$\begin{array}{rl} & \mu(s \otimes t)(\tau''\sigma') \\ = & \mu(s \otimes t)(\sigma \otimes \tau) \\ = & \mu(s(\sigma) \otimes (\tau)) \\ = & \mu(1 \otimes 1) \\ = & 1 \end{array}$$

Thus the relation $t \cup s = s \cup t$ is proved.

We now know how the generators in degrees one and two interact with each other. We still need to argue that this is enough to determine the algebra structure.

Let $S^L T_M \in Hom_{RG}(\mathbb{F}, R)$ be an arbitrary dual basis element as in our earlier notation. It suffices to construct a product of copies of s and t which has the same effect as $S^L T_M$ on an arbitrary monomial $\omega = \tau^k \sigma^{(n)} \in \mathbb{F}$.

We denote the cup product of L copies of s by $s^{\cup L} := \underbrace{s \cup \cdots \cup s}_{L \text{ copies}}$, and similarly for the cup product of M copies of t, $t^{\cup M} := \underbrace{t \cup \cdots \cup t}_{M \text{ copies}}$. We claim that $s^{\cup L} t^{\cup M}$ has the same effect as $S^L T_M$ on $\omega = \tau^k \sigma^{(n)}$. We need to prove a Lemma before we can proceed.

Lemma 4.6.2. *With the above notation,*

$$s^{\cup L}(\tau^k \sigma^{(n)}) = S^L(\tau^k \sigma^{(n)}) = \begin{cases} 1 & \textit{if } k = 0 \textit{ and } n = L \\ 0 & \textit{otherwise} \end{cases}$$

for all $L \geq 1$.

Proof. The proof is by induction on L.

In the base case ($L = 1$), the result is clear from the definitions.

Now assume the result holds for $L = a$, for some $1 \leq a$. Then since $k \in \{0, 1\}$, the following two cases are exhaustive.

1. <u>If $k = 0$</u>, then

$$\begin{array}{rcl} s^{\cup a+1}(\sigma^{(n)}) & = & (s \cup s^{\cup a})(\sigma^{(n)}) \\ & = & \mu(s \otimes (s^{\cup a}))\Phi(\sigma^{(n)}) \\ & = & \mu(s \otimes (s^{\cup a}))((\sigma' + \sigma'')^{(n)} + (\sigma' + \sigma'')^{(n-1)} \nabla_N(x', x'')\tau'\tau'') \\ & = & \begin{cases} 1 & \text{if } i = a+1 \\ 0 & \text{otherwise} \end{cases} \end{array}$$

using the induction hypothesis in the second factor, as $(s \otimes (s^{\cup a}))$ evaluates to 1 on $\sigma'(\sigma'')^{(a)}$, and to 0 on all other monomials.

2. If $k = 1$, then

$$\begin{aligned} s^{\cup a+1}(\sigma^{(n)}) &= (s \cup s^{\cup a})(\tau\sigma^{(n)}) \\ &= \mu(s \otimes (s^{\cup a}))\Phi(\tau\sigma^{(n)}) \\ &= \mu(s \otimes (s^{\cup a}))(x''\tau' + \tau'')((\sigma' + \sigma'')^{(n)} + (\sigma' + \sigma'')^{(n-1)}\nabla_N(x', x'')\tau'\tau'') \\ &= 0 \end{aligned}$$

using the induction hypothesis in the second factor, as $(s \otimes (s^{\cup a}))$ evaluates to 0 on any term involving τ' or τ''.

This completes the induction, and the proof of the Lemma. $\square$

Now by definition we have that

$$S^L T_M \left(\tau^k \sigma^{(n)}\right) = \begin{cases} 1 & \text{if } M = k \text{ and } L = n \\ 0 & \text{otherwise} \end{cases}$$

Again, since $k \in \{0, 1\}$, the following cases are exhaustive.

1. If $k = 0$:

$$\begin{aligned} & (s^{\cup L} \cup t^{\cup M})(\sigma^{(n)}) \\ = \quad & \mu((s^{\cup L}) \otimes (t^{\cup M}))\Phi(\sigma^{(n)}) \\ \underbrace{=}_{\text{line (4.4)}} \quad & \mu((s^{\cup L}) \otimes (t^{\cup M}))((\sigma' + \sigma'')^{(n)} + (\sigma' + \sigma'')^{(n-1)}\nabla_N(x', x'')\tau'\tau'') \\ \underbrace{=}_{\text{Lemma 4.6.2}} \quad & \begin{cases} 1 & \text{if } M = 0 \text{ and } L = n \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

2. If $k = 1$:

$$\begin{aligned} & (s^{\cup L} \cup t^{\cup M})(\sigma^{(n)}) \\ = \quad & \mu((s^{\cup L}) \otimes (t^{\cup M}))\Phi(\tau\sigma^{(n)}) \\ \underbrace{=}_{\text{line (4.5)}} \quad & \mu((s^{\cup L}) \otimes (t^{\cup M}))((x''\tau' + \tau'')(\sigma' + \sigma'')^{(n)}) \\ \underbrace{=}_{\text{Lemma 4.6.2}} \quad & \begin{cases} 1 & \text{if } M = 1 \text{ and } L = n \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

We have shown that

$$s^L t^M \left(\tau^k \sigma^{(n)}\right) = \begin{cases} 1 & \text{if } M = j \text{ and } L = i \\ 0 & \text{otherwise} \end{cases}$$

The monomials have the same effect, as claimed.

Therefore $Hom_{RG}(\mathbb{F}, R)$ admits the monomials $s^{\cup L} t^{\cup M}$ as an R-basis, and on those basis elements, the multiplication is uniquely determined by the relations. Conversely, given the relations, any word in s and t can be recorded uniquely as a scalar times $s^{\cup L} t^{\cup M}$, for some L, M. The dualized differential ∂^* obeys the Leibniz rule by Lemma 2.7.10, so we do have a DG R-algebra. □

Remarks: Using our graded bracket notation, we have

$$[\partial^*, s^{\cup L} \cup t^{\cup M}] = \partial^*(s^{\cup L} \cup t^{\cup M}) = \begin{cases} hs^{\cup L+1} & \text{if} \quad M = 1 \\ 0 & \text{if} \quad M = 0. \end{cases}$$

and therefore

$$\begin{aligned} [\partial^*, t \cup t] &= [\partial^*, t] \cup t - t \cup [\partial^*, t] \\ &= (hs) \cup t - t \cup (hs) \\ &\underbrace{=}_{s \cup t = t \cup s} h(s \cup t - s \cup t) \\ &= 0. \end{aligned} \tag{4.15}$$

Similarly,

$$[\partial^*, s] = 0. \tag{4.16}$$

The above identities give us that

$$\begin{aligned}
& \left[\partial^*, t \cup t + \binom{h}{2} s\right] \\
= \quad & \underbrace{[\partial^*, t \cup t]}_{=0 \text{ by } (4.15)} + \binom{h}{2} \underbrace{[\partial^*, s]}_{=0 \text{ by } (4.16)} \\
= \quad & 0, \text{ and} \qquad (4.17) \\
& [\partial^*, s \cup t - t \cup s] \\
\underbrace{=}_{t \cup s = s \cup t} \quad & [\partial^*, s \cup t] - [\partial^*, s \cup t] \\
= \quad & 0. \qquad (4.18)
\end{aligned}$$

We can analyze the product on line (4.13) further, as the authors do in [8]. Notice that

$$-\binom{h}{2} = -\frac{h(h-1)}{2}.$$

Recall that $\lceil \frac{h-1}{2} \rceil$ is defined to be the least integer which is $\geq \frac{h-1}{2}$, whence

$$\left\lceil \frac{h-1}{2} \right\rceil h - \frac{h(h-1)}{2} = \begin{cases} \frac{h}{2} & \text{if } h \text{ is even} \\ 0 & \text{if } h \text{ is odd.} \end{cases}$$

This implies that

$$-\binom{h}{2} \equiv_{\bmod h} \begin{cases} \frac{h}{2} & \text{if} \quad h \text{ is even} \\ 0 & \text{if} \quad h \text{ is odd} \end{cases} \qquad (4.19)$$

so that we can simplify using the following result.

Proposition 4.6.3. *We may choose a new Φ which is homotopic to the original choice, by adding a suitable boundary to the original $\Phi_2(\sigma)$. Having made this new choice, we can rewrite line (4.13) above as*

$$t \cup t = \begin{cases} \frac{h}{2} \cdot s & \textit{if} \quad h \textit{ is even} \\ 0 & \textit{if} \quad h \textit{ is odd} \end{cases} \qquad (4.20)$$

Proof. From the above analysis, it is clear that we will get what we want if we take

$$\Phi(\sigma) = \sigma' + \sigma'' + \left[\nabla_N(x'x'') + \left\lceil \frac{h-1}{2} \right\rceil h\right] \tau'\tau''$$

Thus we will be finished if we can express the correction term, $\left\lceil \frac{h-1}{2} \right\rceil h\tau'\tau''$ as $\partial(\eta)$ for some $\eta \in (\mathbb{F} \otimes_R \mathbb{F})_3$, satisfying $\epsilon_1(\eta) = 0 = \epsilon_2(\eta)$.

A general element of $(\mathbb{F} \otimes_R \mathbb{F})_3$ looks like

$$a\tau'\sigma' + b\tau''\sigma' + c\tau'\sigma'' + d\tau''\sigma''$$

for some coefficients a, b, c, d. It suffices to look at the monomials. Recall that

$$I = (x' - 1, x'' - 1) \subset RG \otimes_R RG.$$

Thus we may compute

$$\begin{aligned}
\partial\left(\left\lceil \frac{h-1}{2} \right\rceil \tau''\sigma'\right) &= \left\lceil \frac{h-1}{2} \right\rceil (\partial(\tau'')\sigma' + (-1)^{|\tau''|}\tau''\partial(\sigma')) \\
&= \left\lceil \frac{h-1}{2} \right\rceil ((x'' - 1)\sigma' - \tau''N(x')\tau') \\
&= \left\lceil \frac{h-1}{2} \right\rceil ((x'' - 1)\sigma' + N(x')\tau'\tau'') \\
&\equiv \left\lceil \frac{h-1}{2} \right\rceil h\tau'\tau'' \bmod I. \qquad (4.21)
\end{aligned}$$

So line (4.21) shows that we may choose $\eta = \left\lceil \frac{h-1}{2} \right\rceil \tau''\sigma'$.

We now verify that $\epsilon_1(\eta) = 0 = \epsilon_2(\eta)$. We have

$$\begin{aligned}
&\epsilon_1\left(\left\lceil \frac{h-1}{2} \right\rceil \tau''\sigma'\right) \\
= &\left\lceil \frac{h-1}{2} \right\rceil \epsilon_1(\tau''\sigma') \\
= &0,
\end{aligned}$$

since ϵ_1 evaluates to 0 on all elements in the first factor in degree higher than 0 (in particular, on σ).

Similarly,

$$\begin{aligned} & \epsilon_2 \left(\left\lceil \frac{h-1}{2} \right\rceil \tau'' \sigma' \right) \\ = \; & \left\lceil \frac{h-1}{2} \right\rceil \epsilon_2 \left(\tau'' \sigma' \right) \\ = \; & 0, \end{aligned}$$

since ϵ_2 evaluates to 0 on all elements in the second factor in degree higher than 0 (in particular, on τ).

So we have our required correction term, and we are done. $\square$

Remark: These multiplication rules agree with the known results from [8].

Chapter 5

The Tate Resolution for a Finite Abelian Group

5.1 Introduction

In this chapter, we will apply the results of the previous chapter to compute the cup products for any finite abelian group.

5.2 The Tate Resolution for a Finite Abelian Group

We can handle any finite abelian group by building on the case of a finite cyclic group. As is known (for example by Corollary 9.13 in [14]), we may write any finite abelian group as $G = \mu_{h_1} \times \cdots \times \mu_{h_r}$, where μ_{h_i} denotes a multiplicatively written cyclic group of order h_i. Then

$$\begin{aligned} RG &\cong \frac{R[x_1, \ldots, x_r]}{(x_i^{h_i} - 1\ ;\ 1 \leq i \leq r)} \\ &\cong R\mu_{h_1} \otimes_R \cdots \otimes_R R\mu_{h_r}. \end{aligned}$$

Analogously to Definition 4.3.1, we make this definition (recalling that $N_i(x_i) = \sum_{j=0}^{h_i-1} x_i^j$).

Definition 5.2.1. *Define*

$$\nabla_{N_i}(x_i', x_i'') := \frac{N_i(x_i' x_i'') - N_i(x_i'')}{x_i' - 1}$$

Note that substituting $x_i' = 1$ kills the numerator, and thus $(x_i' - 1)$ divides the numerator. Therefore $\nabla_{N_i}(x_i', x_i'')$ is a polynomial in $R[x_i', x_i'']$, and then also in RG^{ev}.

5.3 A Diagonal Approximation

Analogously to Chapter 4, we get the resolution $\mathbb{F} \longrightarrow R$ over RG and a diagonal approximation $\Phi : \mathbb{F} \to \mathbb{F} \otimes_R \mathbb{F}$:

$$\begin{aligned}
\mathbb{F} &= RG\langle \tau_1, \ldots, \tau_r \ ; \ \sigma_1, \ldots, \sigma_r \rangle, \ |\tau_i| = 1; \ |\sigma_i| = 2 \\
\partial &= \sum_{i=1}^{r} \left[(x_i - 1)\frac{\partial}{\partial \tau_i} + N_i(x_i)\tau_i \frac{\partial}{\partial \sigma_i} \right], \text{ where } N_i(x_i) = \frac{x_i^{h_i} - 1}{x_i - 1} = \sum_{j=0}^{h_i - 1} x_i^j \\
\Phi(x_i) &= x_i' x_i'' \\
\Phi(\tau_i) &= x_i'' \tau_i' + \tau_i'' + \partial \omega_i \\
\Phi(\sigma_i) &= \sigma_i' + \sigma_i'' + \nabla_{N_i}(x_i', x_i'')\tau_i'\tau_i'' + N_i(x_i' x_i'')\omega_i + \partial \eta_i
\end{aligned}$$

where $\omega_i \in (\mathbb{F} \otimes_R \mathbb{F})_2$ satisfies $\epsilon_1(\omega) = 0 = \epsilon_2(\omega)$, and $\eta_i \in (\mathbb{F} \otimes_R \mathbb{F})_3$ satisfies $\epsilon_1(\eta) = 0 = \epsilon_2(\eta)$. These assignments determine a unique homomorphism of algebras with divided powers.

Remark: The flexibility of modifying Φ by a boundary will be very useful later.

Because we are interested in cohomology with trivial coefficients, we choose the most convenient Φ when evaluating these formulas modulo the ideal

$$I = (x_i' - 1, \ x_i'' - 1 \ : \ i = 1, \ldots, r) \subset RG \otimes_R RG.$$

We want to compute the cohomology products, so we start by analyzing $Hom_{RG}(\mathbb{F}, R)$.

5.4 The Dual of the Tate Resolution

Dualizing $\mathbb{F}$ into any R via $Hom_{RG}(-, R)$ (and denoting $Hom_{RG}(\mathbb{F}, RG)$ by $\mathbb{F}^*$) gives

$$
\begin{aligned}
& & & Hom_{RG}(\mathbb{F}, R) && (5.1)\\
&\underbrace{\cong}_{\text{by } 2.2.3} & & R \otimes_{RG} \mathbb{F}^* && (5.2)\\
&\cong & & R \otimes_{RG} RG[s_1, \ldots, s_r] \otimes_{RG} \bigwedge_{RG}\langle t_1, \ldots, t_r\rangle && (5.3)\\
&\underbrace{\cong}_{G \text{ acts trivially on } R} & & R[s_1, \ldots, s_r] \otimes_R \bigwedge_{R}\langle t_1, \ldots, t_r\rangle && (5.4)
\end{aligned}
$$

where s_i is a polynomial variable dual to σ_i, and t_j is dual to τ_j.

Warning: Although line (5.4) above carries an algebra structure, it is not the algebra structure that we are seeking. We must use the definition of the cup product (which uses our chosen Φ) to work out the products.

5.5 The Action of ∂^*

As in the previous chapter, $\mathbb{F}^* = Hom_{RG}(\mathbb{F}, R)$ is a Hom complex. Therefore its differential is determined by Definition 2.5.1, and obeys the Leibniz rule.

An RG-basis for $\mathbb{F}$ is given by monomials $\omega = \tau^K \sigma^{(N)}$, where

- $K = (K_1, \ldots, K_r)$ records the exterior powers of the τs which are present, i.e. $\tau^K = \tau_1^{K_1} \cdots \tau_r^{K_r}$. Note that $K_n \in \{0, 1\}$ for all n.
- $N = (N_1, \ldots, N_r) \in \mathbb{N}^r$ records the divided powers of the σs which are present, i.e. $\sigma^{(N)} = \sigma_1^{(N_1)} \cdots \sigma_r^{(N_r)}$.

Analogously to Chapter 4, we define the R-dual basis elements for $Hom_{RG}(\mathbb{F}, R)$ to be $S^L T_M$, where

$$
\begin{aligned}
L &= (L_1, \ldots, L_r),\\
M &= (M_1, \ldots, M_r),
\end{aligned}
$$

and $S^L T_M$ evaluates to 1 on $\tau^M \sigma^{(L)}$, and evaluates to 0 on all other basis elements of $\mathbb{F}$. Note that each $M_n \in \{0,1\}$ for all n, since these are the only occurring exponents for the corresponding τs.

Now to determine the effect of ∂^* on an arbitrary $S^L T_M$, we evaluate

$$\begin{aligned}
& \partial^*(S^L T_M)(\tau^K \sigma^{(N)}) \\
= \; & \underbrace{d_R}_{=0}(S^L T_M)(\tau^K \sigma^{(N)}) - (-1)^{|S^L T_M|} S^L T_M d_{\mathbb{F}}(\tau^K \sigma^{(N)}) && (5.5) \\
= \; & -(-1)^{|T_M|} S^L T_M \left(\sum_{i=1}^{r} (x_i - 1) \frac{\partial \tau^K}{\partial \tau_i} \sigma^{(N)} + N_i(x_i) \tau_i \tau^K \frac{\partial \sigma^{(N)}}{\partial \sigma_i} \right) && (5.6) \\
= \; & -(-1)^{|T_M|} S^L T_M \left(\sum_{i=1}^{r} (x_i - 1) \frac{\partial \tau^K}{\partial \tau_i} \sigma^{(N)} + (-1)^{\sum_{\nu<i} K_\nu} N_i(x_i) \tau^{K_i'} \frac{\partial \sigma^{(N)}}{\partial \sigma_i} \right) && (5.7)
\end{aligned}$$

where we define

$$K_i' = K + (0, \ldots, 0, \underbrace{1}_{\text{position } i}, 0, \ldots, 0)$$

The expression on line (5.7) is congruent, modulo I, to

$$-(-1)^{|T_M|} S^L T_M \left(\sum_{i=1}^{r} (-1)^{\sum_{\nu<i} K_\nu} h_i \tau^{K_i'} \frac{\partial \sigma^{(N)}}{\partial \sigma_i} \right) \tag{5.8}$$

By the definition of $S^L T_M$, the i^{th} term of the sum in (5.8) evaluates to 0 unless

- $K_i' = K + (0, \ldots, 0, \underbrace{1}_{\text{position } k}, 0, \ldots, 0) = M$, and
- $N' = N - (0, \ldots, 0, \underbrace{1}_{\text{position } k}, 0, \ldots, 0) = L$,

in which case it evaluates to $-(-1)^{|T_M|}(-1)^{\sum_{\nu<i} K_\nu} h_i$.

Therefore we have

$$\begin{aligned}
& \partial^*(S^L T_M) \\
= \; & -(-1)^{|T_M|} \left(\sum_{i=1}^{r} (-1)^{\sum_{\nu<i} K_\nu} h_i S^{L+(0,\ldots,0,\underbrace{1}_{i},0,\ldots,0)} T_{M-(0,\ldots,0,\underbrace{1}_{i},0,\ldots,0)} \right) && (5.9)
\end{aligned}$$

We may, temporarily using the algebra structure of the Koszul complex, rewrite the differential from line (5.9) in compact form as

$$\begin{aligned} & \partial^*(S^L T_M) \\ = \; & -(-1)^{|T_M|} S^L \sum_{i=1}^{r} h_i S_i \frac{\partial T_M}{\partial T_i} \end{aligned} \tag{5.10}$$

The next Theorem says that we can replace the above differential with a simpler one, and preserve the original cohomology groups.

Theorem 5.5.1. *If we change the differential to*

$$\begin{aligned} & \partial'(S^L T_M) \\ = \; & S^L \sum_{i=1}^{r} h_i S_i \frac{\partial T_M}{\partial T_i} \end{aligned} \tag{5.11}$$

then we will still have the same cohomology groups.

Proof. Consider the following diagram.

$$\begin{tikzcd} 0 & R[\mathbf{s}] \arrow[l] \arrow[d, equal] & R[\mathbf{s}] \bigwedge^1(\mathbf{t}) \arrow[l, "\partial^*"'] & R[\mathbf{s}] \bigwedge^2(\mathbf{t}) \arrow[l, "\partial^*"'] & R[\mathbf{s}] \bigwedge^3(\mathbf{t}) \arrow[l, "\partial^*"'] & \cdots \arrow[l] \\ 0 & R[\mathbf{s}] \arrow[l] & R[\mathbf{s}] \bigwedge^1(\mathbf{t}) \arrow[l, "\partial'=-\partial^*"] \arrow[u, "(-1)^1"] & R[\mathbf{s}] \bigwedge^2(\mathbf{t}) \arrow[l, "\partial'=(-1)^2\partial^*"] \arrow[u, "(-1)^{1+2}"] & R[\mathbf{s}] \bigwedge^3(\mathbf{t}) \arrow[l, "\partial'=(-1)^3\partial^*"] \arrow[u, "(-1)^{1+2+3}"] & \cdots \arrow[l] \end{tikzcd}$$

It is clear from the construction that the vertical maps assemble into an isomorphism of complexes. Therefore the rows have equal cohomology groups, and we are done. □

5.6 Cochain Products

For this section, unadorned tensor products are over R.

Theorem 5.6.1. *The above choice of* Φ *defines the following cup product structure on* $Hom_{RG}(\mathbb{F}, R)$*, which makes it into a DG-algebra, where* s_i *is a polynomial variable dual* *to σ_i, and t_j is dual to τ_i:*

$$\begin{aligned} t_i \cup t_i &= -\binom{h_i}{2} s_i && (5.12)\\ t_i \cup t_j + t_j \cup t_i &= 0, \text{ when } i \neq j && (5.13)\\ t_j \cup s_i &= s_i \cup t_j && (5.14)\\ s_j \cup s_i &= s_i \cup s_j && (5.15) \end{aligned}$$

and the elements t_j and s_i generate $Hom_{RG}(\mathbb{F}, R)$ with respect to the cup product, subject only to these relations.

Proof. 1. $\underline{t_i \cup t_i = -\binom{h_i}{2} s_i}$: We have $t_i \cup t_i = \mu(t_i \otimes t_i)\Phi_{1,1}$, and $\Phi_{1,1} : \mathbb{F}_2 \to \mathbb{F}_1 \otimes \mathbb{F}_1$. We only need to look at index i, since applying $t_i \otimes t_i$ will kill all other indices. So in degree 2, the only basis element in the domain that we need to look at is σ_i. Recall that

$$\begin{aligned} \Phi(\sigma_i) &= \sigma_i' + \sigma_i'' + \nabla_{N_i}(x_i', x_i'')\tau_i'\tau_i'' \\ \Rightarrow \Phi_{1,1}(\sigma_i) &= \nabla_{N_i}(x_i', x_i'')\tau_i'\tau_i'' \end{aligned}$$

Applying $\mu(t_i \otimes t_i)$ gives

$$\begin{aligned} & \mu(t_i \otimes t_i)(\nabla_{N_i}(x_i', x_i'')\tau_i \otimes \tau_i) \\ = & \binom{h_i}{2} \mu(-1)(t_i(\tau_i) \otimes t_i(\tau_i)) \\ = & -\binom{h_i}{2} \mu(1 \otimes 1) \\ = & -\binom{h_i}{2} \end{aligned}$$

So since $t_i \cup t_i$ evaluates to 0 on all basis elements except σ_i, on which it evaluates to $-\binom{h_i}{2}$, we can express this compactly as $t_i \cup t_i = -\binom{h_i}{2} s_i$, as required.

2. $\underline{t_i \cup t_j + t_j \cup t_i = 0}$: We have $t_i \cup t_j = \mu(t_i \otimes t_j)\Phi_{1,1}$, $t_j \cup t_i = \mu(t_j \otimes t_i)\Phi_{1,1}$ and $\Phi_{1,1} : \mathbb{F}_2 \to \mathbb{F}_1 \otimes \mathbb{F}_1$. The only basis elements for which this can evaluate to something

non-zero are $\tau_i\tau_j$ and $\tau_j\tau_i$. Since $\tau_i\tau_j = -\tau_j\tau_i$, it suffices to determine the effect of $\mu(t_i \otimes t_j)$ and $\mu(t_j \otimes t_i)$ on $\Phi(\tau_i\tau_j)$. So we compute:

$$\begin{aligned} & \Phi_{1,1}(\tau_i\tau_j) \\ = \; & \Phi_{1,1}(\tau_i)\Phi_{1,1}(\tau_j) \\ = \; & (x_i''\tau_i' + \tau_i'')(x_j''\tau_j' + \tau_j'') \\ = \; & x_i''x_j''\tau_i'\tau_j' + x_i''\tau_i'\tau_j'' + x_j''\tau_i''\tau_j' + \tau_i''\tau_j'' \end{aligned}$$

As $(t_i \otimes t_j)$ vanishes on all occurring monomials except $\tau_i'\tau_j''$, applying $\mu(t_i \otimes t_j)$ gives

$$\begin{aligned} & \mu(t_i \otimes t_j)(x_i''\tau_i \otimes \tau_j) \\ = \; & \mu(-1)(t_i(\tau_i) \otimes t_j(\tau_j)) \\ = \; & -1 \end{aligned}$$

Similarly, applying $\mu(t_j \otimes t_i)$ gives

$$\begin{aligned} & \mu(t_j \otimes t_i)(-x_j''\tau_j'\tau_i'') \\ = \; & -\mu(-1)(t_j(\tau_j) \otimes t_i(\tau_i)) \\ = \; & 1 \end{aligned}$$

Thus the relation $t_i \cup t_j + t_j \cup t_i = 0$ is proved.

3. $\underline{t_j \cup s_i = s_i \cup t_j}$: We have $s_i \cup t_j = \mu(s_i \otimes t_j)\Phi_{2,1}$ and $\Phi_{2,1} : \mathbb{F}_3 \to \mathbb{F}_2 \otimes \mathbb{F}_1$. Also, $t_j \cup s_i = \mu(t_j \otimes s_i)\Phi_{1,2}$ and $\Phi_{1,2} : \mathbb{F}_3 \to \mathbb{F}_1 \otimes \mathbb{F}_2$. The elements $\tau_j\sigma_i$ form a basis for $\mathbb{F}_3$. Therefore it suffices to determine the effect of $\mu(t_j \otimes s_i)$ and $\mu(s_i \otimes t_j)$ on $\Phi(\tau_j\sigma_i)$.

   First we compute:

$$\begin{aligned} & \Phi(\tau_j\sigma_i) \\ = \; & \Phi(\tau_j)\Phi(\sigma_i) \\ = \; & (x_j''\tau_j' + \tau_j'')\,(\sigma_i' + \sigma_i'' + \nabla_{N_i}(x_i', x_i'')\tau_i'\tau_i'') \\ = \; & x_j''\tau_j'\sigma_i' + x_j''\tau_j'\sigma_i'' + x_j''\nabla_{N_i}(x_i', x_i'')\tau_j'\tau_i'\tau_i'' \\ & + \tau_j''\sigma_i' + \tau_j''\sigma_i'' + \nabla_{N_i}(x_i', x_i'')\tau_j''\tau_i'\tau_i'' \end{aligned}$$

   As $(t_j \otimes s_i)$ vanishes on all occurring monomials except $\tau_j'\sigma_i''$, applying $\mu(t_j \otimes s_i)$ gives

$$\begin{aligned} & \mu(t_j \otimes s_i)(x_j''\tau_j \otimes \sigma_i) \\ = \; & \mu(t_j(\tau_j) \otimes s_i(\sigma_i)) \\ = \; & \mu(1 \otimes 1) \\ = \; & 1 \end{aligned}$$

Similarly, applying $\mu(s_i \otimes t_j)$ gives

$$\begin{aligned} & \mu(s_i \otimes t_j)(\sigma_i \otimes \tau_j) \\ = \ & \mu(s_i(\sigma_i) \otimes t_j(\tau_j)) \\ = \ & \mu(1 \otimes 1) \\ = \ & 1 \end{aligned}$$

Thus the relation $t_j \cup s_i = s_i \cup t_j$ is proved.

4. $\underline{s_j \cup s_i = s_i \cup s_j}$: We have $s_i \cup s_j = \mu(s_i \otimes s_j)\Phi_{2,2}$ and $\Phi_{2,2} : \mathbb{F}_4 \to \mathbb{F}_2 \otimes \mathbb{F}_2$. Also $s_j \cup s_i = \mu(s_j \otimes s_i)\Phi_{2,2}$. The elements $\sigma_j\sigma_i$ and $\tau_j\tau_k\sigma_i$ form a basis for $\mathbb{F}_4$. We do not need to consider elements of the form $\tau_j\tau_k\sigma_i$, since $\Phi(\tau_j\tau_k) = \Phi(\tau_j)\Phi(\tau_k)$, and each of these factors will involve τ_j', τ_j'', τ_k' or τ_k''. As we will apply s_i or s_j, we may consider the following computation modulo the ideal

$$T = (\tau_k', \tau_k'' \ : \ 1 \leq k \leq r) \subset \mathbb{F} \otimes \mathbb{F}.$$

We begin by computing:

$$\begin{aligned} & \Phi(\sigma_j\sigma_i) \\ = \ & \Phi(\sigma_j)\Phi(\sigma_i) \\ \equiv \ & \left(\sigma_j' + \sigma_j''\right)(\sigma_i' + \sigma_i'') \bmod T \\ = \ & \sigma_j'\sigma_i' + \sigma_j'\sigma_i'' + \sigma_j''\sigma_i' + \sigma_j''\sigma_i'' \end{aligned}$$

As $(s_j \otimes s_i)$ vanishes on all occurring monomials except $\sigma_j'\sigma_i''$, applying $\mu(s_j \otimes s_i)$ gives

$$\begin{aligned} & \mu(s_j \otimes s_i)(\sigma_j \otimes \sigma_i) \\ = \ & \mu(s_j(\sigma_j) \otimes s_i(\sigma_i)) \\ = \ & \mu(1 \otimes 1) \\ = \ & 1 \end{aligned}$$

Similarly, applying $\mu(s_i \otimes s_j)$ gives

$$\begin{aligned} & \mu(s_i \otimes s_j)(\sigma_i \otimes \sigma_j) \\ = \ & \mu(s_i(\sigma_i) \otimes s_j(\sigma_j)) \\ = \ & \mu(1 \otimes 1) \\ = \ & 1 \end{aligned}$$

Thus the relation $s_j \cup s_i = s_i \cup s_j$ is proved.

We now know how the generators in degrees one and two interact with each other. We still need to argue that this is enough to determine the algebra structure.

Let $S^L T_M \in Hom_{RG}(\mathbb{F}, R)$ be an arbitrary dual basis element as in our earlier notation. It suffices to construct a product of copies of s_i and t_j which has the same effect as $S^L T_M$ on an arbitrary monomial $\omega = \tau^K \sigma^{(N)} \in \mathbb{F}$.

Analogously to Chapter 4, we denote the cup product of L_i copies of s_i by $s_i^{\cup L_i} := \underbrace{s_i \cup \cdots \cup s_i}_{L_i \text{ copies}}$, and similarly for the cup product of M_j copies of t_j, $t_j^{\cup M_j} := \underbrace{t_j \cup \cdots \cup t_j}_{M_j \text{ copies}}$. We claim that $(s_r^{\cup L_r} \cup \cdots \cup s_1^{\cup L_1}) \cup (t_r^{\cup M_r} \cup \cdots \cup t_1^{\cup M_1})$ has the same effect as $S^L T_M$ on $\omega = \tau^K \sigma^{(N)}$. We need two Lemmas before we can proceed.

Lemma 5.6.2. *With the above notation,*

$$(s_r^{\cup L_r} \cup \cdots \cup s_1^{\cup L_1})(\sigma^{(N)}) = \begin{cases} 1 & \textit{if } N = L \\ 0 & \textit{otherwise} \end{cases}$$

for all $L \in \mathbb{N}^r$.

Proof. This is proved in a way which is completely analogous to the proof of Lemma 4.6.2. □

Lemma 5.6.3. *With the above notation,*

$$(t_r^{\cup M_r} \cup \cdots \cup t_1^{\cup M_1})(\tau^K) = \begin{cases} 1 & \textit{if } K = M \\ 0 & \textit{otherwise} \end{cases}$$

for all $L \in \mathbb{N}^r$.

Proof. This is proved in a way which is completely analogous to the proof of Lemma 4.6.2. □

Now by the definition of $S^L T_M$, we have that

$$S^L T_M \left(\tau^K \sigma^{(N)}\right) = \begin{cases} 1 & \text{if } M = K \text{ and } L = N \\ 0 & \text{otherwise} \end{cases}$$

We can now evaluate

$$\begin{array}{rl} & ((s_r^{\cup L_r} \cup \cdots \cup s_1^{\cup L_1}) \cup (t_r^{\cup M_r} \cup \cdots \cup t_1^{\cup M_1}))(\tau^K \sigma^{(N)}) \\ = & \mu((s_r^{\cup L_r} \cup \cdots \cup s_1^{\cup L_1}) \otimes (t_r^{\cup M_r} \cup \cdots \cup t_1^{\cup M_1}))\Phi(\tau^K \sigma^{(N)}) \\ = & \mu((s_r^{\cup L_r} \cup \cdots \cup s_1^{\cup L_1}) \otimes (t_r^{\cup M_r} \cup \cdots \cup t_1^{\cup M_1}))\Phi(\tau^K)\Phi(\sigma^{(N)}) \\ \underbrace{=}_{\text{(Line 4.4)}} & \mu((s^{\cup L}) \otimes (t^{\cup M})) \prod_{i=1}^{r} (x_i'' \tau_i' + \tau_i'')^{K_i} \prod_{j=1}^{r} \left[(\sigma' + \sigma'')^{(N_j)} + (\sigma' + \sigma'')^{(N_j - 1)} \nabla_{N_j}(x_j', x_j'') \tau_j' \tau_j'' \right] \end{array}$$

This expression evaluates to 1 if and only if $L_1 = N_1, \ldots, L_r = N_r, M_1 = K_1, \ldots, M_r = K_r$, in other words, if and only if $N = L$ and $K = M$.

The monomials have the same effect, as claimed.

Therefore $Hom_{RG}(\mathbb{F}, R)$ admits the monomials $s^{\cup L} t^{\cup M}$ as an R-basis, and on those basis elements, the multiplication is uniquely determined by the relations. Conversely, given the relations, any word in s_i and t_j can be reordered uniquely as a scalar times $s^{\cup L} t^{\cup M}$, for some L, M.

The dualized differential ∂^* obeys the Leibniz rule by Lemma 2.7.10, so we do have a *DG* R-algebra. $\square$

Now ∂' from line (5.11) is an algebra differential, and satisfies

$$\begin{array}{rcl} (\partial')^2 & = & 0 \\ \partial'(t_i \cup t_i) & = & \partial'(t_i) \cup t_i - t_i \cup \partial'(t_i) \\ & = & 0 \\ \partial' \left(\begin{pmatrix} h_i \\ 2 \end{pmatrix} s_i \right) & = & 0 \end{array}$$

In exact analogy to Proposition 4.6.3 in Chapter 4, we can analyze the product on line (5.12) further. We can simplify using the following result.

Proposition 5.6.4. *We may choose a new Φ which is homotopic to the original choice, by adding a suitable boundary to the original $\Phi_2(\sigma)$. Having made this new choice, we can rewrite line (5.12) above as*

$$t_i \cup t_i = \begin{cases} \frac{h_i}{2} \cdot s_i & \text{if } h_i \text{ is even} \\ 0 & \text{if } h_i \text{ is odd} \end{cases} \tag{5.16}$$

Proof. This proof is completely analogous to the proof of Proposition 4.6.3.

□

Remark: These results agree with [8], in the special case when $r = 1$.

Chapter 6

The Cohomology Ring for a Finite Abelian Group

6.1 Introduction

In this chapter, we will describe the cohomology ring of a finite abelian group as a fibre product of quotients of polynomial rings. This will lead us to cleaner presentations than those that exist in the literature to date.

6.2 The Structure of the Cohomology Ring

6.2.1 Introduction

For this chapter, all products of cochains are understood to be the cup products of Chapter 5.

We start from the following result already established:

Theorem 6.2.1. *Let $G = \mu_{n_1} \times \cdots \times \mu_{n_r}$ be a finite abelian group, written as the product of $r > 1$ cyclic groups of orders n_1 through n_r. Whenever it is convenient, we may assume that the n_i are the elementary divisors of G, so that $2 \leq n_1 | n_2 | \cdots | n_r$. The cohomology of G with coefficients in a commutative ring R, on which G acts trivially, is then the* *cohomology of the DG R-algebra*

$$\mathbb{K} = \mathbb{S}ym_R \left(\bigoplus_{i=1}^{r} Rs_i \right) \otimes_R \bigwedge_R \left(\bigoplus_{i=1}^{r} Rt_i \right)$$

with each t_i in (cohomological) degree 1, each s_i in degree 2, and with its differential the algebra derivation

$$\partial = \sum_{i=1}^{r} n_i s_i \frac{\partial}{\partial t_i}.$$

However, the multiplicative structure is not the "obvious" one, it is rather deformed in that the polynomial ring $\mathbb{S}ym_R(\bigoplus_{i=1}^{r} Rs_i)$, concentrated in even degrees, is contained in the centre, while the t_i satisfy

$$\begin{aligned} t_j^2 &= \begin{cases} \frac{n_j}{2} \cdot s_j & \text{if } n_j \text{ is even} \\ 0 & \text{if } n_j \text{ is odd} \end{cases} \\ t_i t_j + t_j t_i &= 0, \text{ when } i \neq j \\ t_j s_i &= s_i t_j \end{aligned}$$

Proof. Refer to Theorem 5.6.1, Proposition 5.6.4 and Theorem 5.5.1. □

The aim of this chapter is to determine the structure of the cohomology of this DG-algebra. Not to overlook the trivial cases, we state right away the following.

Corollary 6.2.2. *Assume that each n_i is zero in the ring R. Then the differential in the above DG R-algebra is identically zero and the algebra $H^\bullet(G, R)$ is isomorphic to the Clifford algebra over the polynomial ring $P = R[s_1, \ldots, s_r]$ on the quadratic form*

$$q : P^r \to P,\ q(p_1, \ldots, p_r) = \sum_{n_i\, even} \frac{n_i}{2} s_i p_i^2$$

that takes its values in the 2-torsion of P.

If for each even n_i we also have $\frac{n_i}{2}$ is zero in R, then the algebra structure is the ordinary, strictly graded commutative one on the Koszul complex.

Proof. It is clear that we get the polynomial ring $R[\mathbf{s}]$, from the copy of $\mathbb{S}ym_R\left(\bigoplus_{i=1}^{r} Rs_i\right)$ in the DG-algebra.

The multiplication of the t_js comes from the known rules. All that survives is for the even n_j. Treat $\{t_1, \ldots, t_r\}$ as a basis of P^r, then we must have

$$\begin{array}{rccc} q & : & P^r & \to & P \\ & & \sum_{j=1}^r p_j t_j & \mapsto & \sum_{n_j \text{ even}} p_j^2 \frac{n_j}{2} s_i. \end{array}$$

Since $n_j \geq 0$ and $n_j s_j = 0, \forall j$, it is clear that this sum lies in the 2-torsion of P. □

Example 6.2.3. *The corollary applies in particular to the case when $R = \mathbb{F}$ is a field of characteristic p and G is a p-group. If $n_1|n_2|\cdots|n_r$ are the elementary divisors, then*

$$H^\bullet(G, \mathbb{F}) \cong \begin{cases} \mathbb{F}[s_1, \ldots, s_r] \otimes_{\mathbb{F}} \bigwedge_{\mathbb{F}} (\otimes_{i=1}^r \mathbb{F} t_i) & \text{if } p \text{ is odd} \\ \frac{\mathbb{F}[t_1, \ldots, t_r, s_a, \ldots, s_r]}{(t_a^2, \ldots, t_r^2)} & \text{if } p = 2 \text{ and } 2 = n_{a-1} < n_a. \end{cases}$$

Proof. Since G is a p-group, then $p|n_i$, $\forall i$. Thus each n_i equals 0 in $\mathbb{F}$, so Corollary 6.2.2 applies.

If p is odd, then 2 is invertible in $\mathbb{F}$, and $\frac{n_i}{2}$ is still zero in $\mathbb{F}$.

If p is even, then suppose $2 = n_{a-1} < n_a$. For $i < a$, we have that $\frac{n_i}{2} = 1$ in $\mathbb{F}$, implying $t_i^2 = s_i$ for those $i < a$. So $s_1, \ldots, s_{a-1}$ can be obtained from $t_1, \ldots, t_{a-1}$ and can be omitted from the list of variables. Also, 2^2 divides $n_a, \ldots, n_r$, implying $\frac{n_a}{2} = \cdots = \frac{n_r}{2} = 0$ in $\mathbb{F}$. Thus $t_a^2 = \cdots = t_r^2 = 0$. □

Remark: This result agrees with Proposition 4.5.4 in [6].

6.2.2 Preliminaries

Symmetric Powers of Direct Sums of Cyclic Modules

The following result can easily be deduced, say from Proposition A2.2 in [10].

Lemma 6.2.4. *Let R be a commutative ring and $I_1, \ldots, I_r \subseteq R$ ideals. The symmetric algebra on $\oplus_{i=1}^r \frac{R}{I_i}$ over R has then the following structure.*

$$\begin{aligned} \mathbb{S}ym_R\left(\bigoplus_{i=1}^r \frac{R}{I_i}\right) &\cong \mathbb{S}ym_R\left(\frac{R}{I_1}\right) \otimes_R \cdots \otimes_R \mathbb{S}ym_R\left(\frac{R}{I_r}\right) && (6.1)\\ &\cong \frac{R[x_1]}{x_1 I_1 R[x_1]} \otimes_R \cdots \otimes_R \frac{R[x_r]}{x_r I_r R[x_r]} && (6.2)\\ &\cong \frac{R[x_1, \ldots, x_r]}{(\sum_{i=1}^r x_i I_i)}, && (6.3) \end{aligned}$$

where $x_1, \ldots, x_r$ are independent variables.

Assigning x_i the multi degree $e_i \in \oplus_{i=1}^r \mathbb{Z}e_i = \mathbb{Z}^r$, this symmetric algebra becomes $\mathbb{N}^r$-graded and its homogeneous component of multi degree $N = (N(1), \ldots, N(r)) \in \mathbb{N}^r$ is the R-module

$$\mathbb{S}ym_R^N\left(\bigoplus_{i=1}^r \frac{R}{I_i}\right) \cong \frac{R}{\sum_{N(i)\neq 0} I_i}\mathbf{x}^N. \tag{6.4}$$

Alternatively,

$$\mathbb{S}ym_R\left(\bigoplus_{i=1}^r \frac{R}{I_i}\right) \cong \bigoplus_{S\subseteq\{1,\ldots,r\}} \frac{R}{\sum_{i\in S} I_i}[x_i \mid i \in S]\mathbf{x}^S \tag{6.5}$$

as an $R[x_1, \ldots, x_r]$-module, where we have abbreviated $\mathbf{x}^S = \prod_{i\in S} x_i$.

Proof. The equality on line (6.1) is clear from the fact that $\mathbb{S}ym_R(M \oplus N) \cong \mathbb{S}ym_R(M) \otimes_R \mathbb{S}ym_R(N)$.

Consider $I_i \subset R$. R is a rank 1 free R-module. Let x_i be a basis element. Then we have a short exact sequence

$$0 \longrightarrow x_iI_i \longrightarrow x_iR \longrightarrow \tfrac{x_iR}{x_iI_i} \longrightarrow 0$$

Now, passing to $\mathbb{S}ym$, we obtain a new short exact sequence

$$0 \longrightarrow (x_iI_i) \otimes_R \mathbb{S}ym_R(x_iR) \longrightarrow \mathbb{S}ym_R(x_iR) \longrightarrow \mathbb{S}ym_R\left(\tfrac{x_iR}{x_iI_i}\right) \longrightarrow 0$$

which gives us that

$$\mathbb{S}ym_R\left(\frac{R}{I_i}\right) \cong \frac{\mathbb{S}ym_R(x_iR)}{(x_iI_i) \otimes_R \mathbb{S}ym_R(x_iR)} \cong \frac{R[x_i]}{x_iI_iR[x_i]}$$

which establishes the equality on line (6.2).

Line (6.3) is clear from line (6.2).

For (6.4), observe that, on line (6.2), we reduce the polynomial ring involving x_i by the ideal x_iI_i. Extending this, we reduce the coefficients of every monomial by the sum of the ideals corresponding to the variables involved in that monomial.

Line (6.5) is clear from line (6.4). $\square$

Next we specialize to the case that the ideals in the preceding Lemma form a chain.

Proposition 6.2.5. *Assume we are given $R \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I_r \supseteq I_{r+1} = (0)$, a descending chain of ideals in the commutative ring R. The final description in Lemma (6.2.4) can then be simplified to*

$$\mathbb{S}ym_R\left(\bigoplus_{i=1}^{r}\frac{R}{I_i}\right) \quad\cong\quad R\oplus\bigoplus_{i=1}^{r}\frac{R}{I_i}[x_i,x_{i+1},\ldots,x_r]x_i. \tag{6.6}$$

As a ring, even as an $R[x_1,\ldots,x_r]$-algebra, it is the fibre product

$$\mathbb{S}ym_R\left(\bigoplus_{i=1}^{r}\frac{R}{I_i}\right) \quad\cong\quad R_1\times_{\frac{R_1}{(x_1)}} R_2\times_{\frac{R_2}{(x_2)}}\cdots\times_{\frac{R_r}{(x_r)}} R_{r+1}, \tag{6.7}$$

where we have set

$$R_i \quad=\quad \left(\frac{R}{I_i}\right)[x_i,\ldots,x_r]\cong\frac{R[x_1,\ldots,x_r]}{(I_i)+(x_1,\ldots,x_{i-1})} \tag{6.8}$$

for $i=1,\ldots,r+1$, so that, in particular, $R_{r+1}\cong R$.

Note that the ring homomorphisms used in the formation of the fibre product are the natural epimorphisms from

$$R_i=\left(\frac{R}{I_i}\right)[x_i,\ldots,x_r],$$

respectively from R_{i+1}, onto

$$\frac{R_i}{(x_i)}\cong\left(\frac{R}{I_i}\right)[x_{i+1},\ldots,x_r],$$

for $i=1,\ldots,r$.

Proof. Since the ideals form a chain, the only choices for S that we need to consider are $S=\emptyset,\{1\},\{2\},\ldots,\{r\}$. These choices yield the form on line (6.6).

Rewrite line (6.6) as

$$\frac{R}{I_1}[x_1,\ldots,x_r]x_1\oplus\frac{R}{I_2}[x_2,\ldots,x_r]x_2\oplus\cdots\oplus\frac{R}{I_{r-1}}[x_{r-1},x_r]x_{r-1}\oplus\frac{R}{I_r}[x_r]x_r\oplus R$$

Rewrite line (6.7) as

$$\frac{R}{I_1}[x_1,\ldots,x_r]\times_{\frac{R}{I_1}[x_2,\ldots,x_r]}\frac{R}{I_2}[x_2,\ldots,x_r]\times_{\frac{R}{I_2}[x_3,\ldots,x_r]}\cdots\times_{\frac{R}{I_{r-1}}[x_r]}\frac{R}{I_r}[x_r]\times_{\frac{R}{I_r}} R$$

So we can see that there is an obvious map $(f_1,\ldots,f_{r+1})\mapsto(f_1,\ldots,f_{r+1})$ from the first set of $(r+1)$-tuples to the second set of $(r+1)$-tuples. It is clear that this map will be a morphism of $R[x_1,\ldots,x_r]$-algebras provided it is a well-defined function. To see that this map takes values in the fibre product, let $i\in\{1,\ldots,r-1\}$ be arbitrary, and consider (f_i,f_{i+1}). For (f_i,f_{i+1}) to lie in the fibre product, we require

$$\overline{f_{i+1}} \;=\; f_i|_{x_i=0}$$

By our setup, $x_i|f_i\Rightarrow f_i|_{x_i=0}=0$, so that $\overline{f_{i+1}}=0$ i.e. all coefficients of f_{i+1} lie in I_i. But since $I_i\supseteq I_{i+1}$, we already have this condition satisfied. We have shown that the map does take values in the fibre product for any $(r+1)$-tuple.

We still need to argue why this map is a bijection. We will exhibit an inverse. Consider any f_i in a tuple in the fibre product. To show that f_i lies in $\frac{R}{I_i}[x_i,\ldots,x_r]x_i$, we need to show that $x_i|f_i$. Since f_i is in a tuple in the fibre product, there is some f_{i+1} such that

$$\overline{f_{i+1}} \;=\; f_i|_{x_i=0}$$

Since $I_i\supseteq I_{i+1}$, therefore $\overline{f_{i+1}}=0$. Therefore $f_i|_{x_i=0}=0$, so that $x_i|f_i$, as required. □

Geometrically, Spec $\mathbb{S}ym_R\left(\bigoplus_{i=1}^r\frac{R}{I_i}\right)$ is thus the union of the affine spaces

$$\mathbb{A}^{r+1-i}_{\frac{R}{Ii}} = Spec\ \frac{R}{I_i}[x_i,\ldots,x_r]$$

of (relative) dimension $r+1-i$ over the rings $\frac{R}{I_i}$ that in turn become larger as i increases. This linear arrangement of sorts can be viewed as a closed subscheme of $\mathbb{A}^r_R = SpecR[x_1,\ldots,x_r]$.

Remark: If the chain of ideals is not proper, then the indicated fibre product contains redundant factors. Namely, if $I_i=I_{i+1}$, for some $i=1,\ldots,r$, then the natural surjection $R_{i+1}\to\frac{R_i}{(x_i)}$ is an isomorphism and $\frac{R_{i+1}}{(x_{i+1})}\cong\frac{R_i}{(x_i,x_{i+1})}$.

Thus, the part $\times_{\frac{R_i}{(x_i)}}R_{i+1}\times_{\frac{R_{i+1}}{(x_{i+1})}}$ in the fibre product can be replaced with $\times_{\frac{R_i}{(x_i,x_{i+1})}}$, and similarly when more of the ideals are equal.

For example, if all ideals are zero, then all factors but the first can be dropped and we regain the fact that the symmetric algebra on a free module is the polynomial ring, $\mathbb{S}ym_R(\oplus_{i=1}^r R) \cong R_r = R[x_1, \dots, x_r]$.

As a slightly less extreme case that will concern us below, if $I_1 = \cdots = I_r = I$, then all factors but the first and last can be dropped and one finds

$$\mathbb{S}ym_R\left(\bigoplus_{i=1}^r \frac{R}{I}\right) \cong \left(\frac{R}{I}\right)[x_1, \dots, x_r] \times_{\frac{R}{I}} R.$$

Now we apply this investigation of symmetric algebras to the determination of the cohomology of the finite abelian group G from above.

Ignoring the degrees of the elements s_i in $\mathbb{K}$, this complex can be viewed as the Koszul complex on the sequence $(n_1 s_1, \dots, n_r s_r) \subseteq R[s_1, \dots, s_r]$. We denote by $H_j(\mathbb{K})$ the resulting Koszul homology, where the index j refers to the degree in the t_i. These homology groups are naturally $R[s_1, \dots, s_r]$-modules.

Lemma 6.2.6. *Assume, as we may, that the n_i are the elementary divisors of the finite abelian group G in that $2 \leq n_1 | n_2 | \cdots | n_r$. One then has*

$$\begin{aligned} H_0(\mathbb{K}) &\cong \mathbb{S}ym_R\left(\bigoplus_{i=1}^r \frac{R}{(n_i)}\right) \\ &\cong R \oplus \bigoplus_{i=1}^r \frac{R}{(n_i)}[s_i, s_{i+1}, \dots, s_r] s_i \text{ as an } R[s_1, \dots, s_r]\text{-module} \\ &\cong R_1 \times_{\frac{R_1}{(x_1)}} R_2 \times_{\frac{R_2}{(x_2)}} \cdots \times_{\frac{R_r}{(x_r)}} R_{r+1} \text{ as an } R[s_1, \dots, s_r]\text{-algebra,} \end{aligned}$$

where now $R_i = \frac{R}{(n_i)}[s_i, \dots, s_r]$, for $i = 1, \dots, r$, still with the convention that $R_{r+1} = R$.

Proof. Just note that $H_0(\mathbb{K}) \cong \frac{R[s_1, \dots, s_r]}{(n_1 s_1, \dots, n_r s_r)}$ can be identified as the indicated symmetric algebra by Lemma 6.2.4 and that the ideals $(n_1) \supseteq \cdots \supseteq (n_r)$ form a descending chain in R, so Proposition 6.2.5 applies. $\square$

As always for a Koszul complex, the homology groups $H_j(\mathbb{K})$ are modules over the ring $H_0(\mathbb{K})$ that we just described. To give a concise presentation of the homology, we next identify the cycles in that Koszul complex in two cases.

The Cycles in the DG-Algebra

For a multi index $N \in \mathbb{N}^r$ and a subset $S \subseteq \{1,\dots,r\}$, we let $N+S$ denote the multi index given by the r-vector $N+\sum_{i\in S} e_i \in \oplus_{i=1}^r \mathbb{N}e_i$. In particular, $N+\emptyset = N$. In this way we also view S as the multi index whose component at i is 1, if $i\in S$, and 0 otherwise.

We set $\ell(N+S) = \min\{i=1,\dots,r \mid (N+S)(i)\neq 0\}$, and think of it as the leading index of the multi index $N+S$.

Proposition 6.2.7. *Let G be a finite abelian group as before, with elementary divisors $n_1|n_2|\cdots|n_r$.*

1. *If R is any commutative ring, then for $I=\{i_1<\cdots<i_a\}$ a non-empty subset of $\{1,\dots,r\}$, the element*

$$\begin{aligned}\partial'(\mathbf{t}_I) &:= \frac{1}{n_{\ell(I)}}\partial(\mathbf{t}_I)\\ &:= \sum_{\nu=1}^{a}(-1)^{\nu-1}\frac{n_{i_\nu}}{n_{\ell(I)}} s_{i_\nu} t_{i_1}\wedge\cdots\wedge\widehat{t_{i_\nu}}\wedge\cdots\wedge t_{i_a}\end{aligned}$$

   is well defined in $\mathbb{K}$. It is a cycle of degree $|I|-1$ in the t_i, and of cohomological degree $|I|+1$. Its class in $H_{|I|-1}(\mathbb{K})$ is annihilated by $n_{\ell(I)}$.

2. *If $m\in R$ is a non-zero-divisor in R that is a multiple of the largest elementary divisor n_r (equal to the exponent of the group), then in the Koszul complex with coefficients in $\overline{R}=\frac{R}{(m)}$ the elements*

$$\mathbf{t}'_J := \frac{m}{n_{\ell(J)}}\mathbf{t}_J$$

   are cycles as well, for any subset $J\subseteq\{1,\dots,r\}$.

   If $J=\emptyset$, then interpret $\mathbf{t}_\emptyset = 1$ and $n_{\ell(\emptyset)}=m$ to regain $\mathbf{t}'_\emptyset = 1$ as a cycle.

3. *With assumptions as in 2, in the long exact homology sequence that results from applying $\mathbb{K}\otimes_R -$ to the short exact sequence*

$$0\longrightarrow R\xrightarrow{m} R\longrightarrow \overline{R}\longrightarrow 0$$

   the connecting homomorphism $H_j(\mathbb{K}\otimes_R\overline{R})\to H_{j-1}(\mathbb{K})$ sends the class of the cycle $\mathbf{t}'_J$, with $|J|=j>0$, to $\partial'(\mathbf{t}_J)$.

Proof.

1. For assertion 1, note that the differential on $\mathbb{K}$ yields

$$\partial(\mathbf{t}_I) = \sum_{\nu=1}^{a} (-1)^{\nu-1} n_{i_\nu} s_{i_\nu} t_{i_1} \wedge \cdots \wedge \widehat{t_{i_\nu}} \wedge \cdots \wedge t_{i_a},$$

   whence

$$\frac{1}{n_{\min I}} \partial(\mathbf{t}_I) := \sum_{\nu=1}^{a} (-1)^{\nu-1} \frac{n_{i_\nu}}{n_{\ell(I)}} s_{i_\nu} t_{i_1} \wedge \cdots \wedge \widehat{t_{i_\nu}} \wedge \cdots \wedge t_{i_a}$$

   is an element of that algebra as the integer $n_{\ell(I)}$ divides each of the n_i for $i \in I$. It follows further immediately that this element is a cycle, as $\partial^2 = 0$, and that $n_{\ell(I)}$ annihilates this cycle in cohomology - after all, $\partial(\mathbf{t}_I)$ is a true boundary.

2. Assertion 2 follows immediately from the explicit form of the differential on the Koszul complex as just recalled.

3. Finally, assertion 3 is a simple consequence of the snake lemma, in that $\mathbf{t}'_J$, viewed as an element in $\mathbb{K}$ lifts that same element from $\mathbb{K} \otimes_R \overline{R}$ and is then sent by the differential to $\partial(\mathbf{t}'_J) = \frac{m}{n_{\ell(J)}} \partial(\mathbf{t}_J)$, which in turn is the image of $\partial'(\mathbf{t}_J)$ under multiplication by m.

□

Remark 6.2.8. *Note that in case $I = \{i\}$ is a singleton, then $\partial'(t_i) = s_i$.*

With these preparations, we can now formulate our main result.

6.2.3 The Structure of the Cohomology of Finite Abelian Groups

Theorem 6.2.9. *We keep the notation from Proposition 6.2.7. The cohomology of the group G with coefficients in a commutative ring R, in which the order of G, equivalently, its exponent n_r, is a non-zero-divisor, is given by*

$$H^\bullet(G, R) \cong R \oplus \bigoplus_{\emptyset \neq I \subset \{1,\ldots,r\}} \bigoplus_{N \in \mathbb{N}^r} \frac{R}{(n_{\ell(N+I)})} \mathbf{s}^N \partial'(\mathbf{t}_I)$$

where we denote the cohomology classes of the cycles s_i and $\partial'(\mathbf{t}_I)$ by the same symbols. The cohomology class $\mathbf{s}^N \partial'(\mathbf{t}_I)$ sits in $H^{2\sum_{i=1}^{r} N(i)+|I|+1}(G, R)$, by Proposition 6.2.7 (1) .

In particular, we regain classical results for the low-dimensional cohomology groups,

$$\begin{aligned} H^0(G,R) &\cong R,\\ H^1(G,R) &\cong 0,\\ H^2(G,R) &\cong \bigoplus_{i=1}^{r} \frac{R}{(n_i)} s_i. \end{aligned}$$

Proof. Use induction on the number of elementary divisors. If $r = 1$, so that $G \cong \mu_n$ is cyclic of order $n_1 = n$, then the claimed description simplifies, with $s,\ t$ for $s_1,\ t_1$, to

$$H^\bullet(\mu_n, R) \cong R \oplus \bigoplus_{N\in\mathbb{N}} \frac{R}{(n)} s^N \partial'(t) \cong R \oplus \bigoplus_{N\in\mathbb{N}} \frac{R}{(n)} s^{N+1},$$

as $\partial'(t) = s$, and this is the correct result as one sees immediately from the (periodic) resolution of R over $RG \cong \frac{R[x]}{(x^n-1)}$. Note also that as a ring,

$$H^\bullet(\mu_n, R) \cong \frac{R}{(n)}[s] \times_{\frac{R}{(n)}} R.$$

Now assume by induction that the result has been established for abelian groups with $r - 1 \geq 1$ elementary divisors. If G is then a group with r elementary divisors, write $G \cong G' \times \mu_{n_r}$ with n_r the largest elementary divisor.

Write temporarily $\mathbb{K}_{r-1}$ for the Koszul complex for G' and note that the Koszul complex $\mathbb{K}$ for G over R can be realized as a tensor product of complexes

$$\mathbb{K} \cong \mathbb{K}_{r-1} \otimes_R (\, 0 \longrightarrow R[s_r]t_r \xrightarrow{n_r s_r} R[s_r] \longrightarrow 0\,).$$

with $R[s_r]$ in complex degree 0. This gives rise to a short exact sequence of complexes of R-modules

$$0 \longrightarrow \mathbb{K}_{r-1}[s_r] \xrightarrow{i} \mathbb{K} \xrightarrow{p} \mathbb{K}_{r-1}[s_r]t_r[1] \longrightarrow 0$$

where we abbreviate $\mathbb{K}_{r-1}[s_r] = \mathbb{K}_{r-1} \otimes_R R[s_r]$ and the translation [1] refers to the homological degree (in the t_i) of the Koszul complexes.

The map i is the natural inclusion of $\mathbb{K}_{r-1}[s_r]$ as a sub complex of $\mathbb{K}$, in that a typical element ω in $\mathbb{K}$ can be written uniquely as

$$\omega = \omega_1 + \omega_2 t_r$$

with $\omega_1 = i(\omega_1)$ and ω_2 elements from $\mathbb{K}_{r-1}[s_r]$. In these terms, $p(\omega) = \omega_2 t_r$.

If we now pass to the long exact homology sequence,

$$H_j(\mathbb{K}_{r-1}[s_r]) \longrightarrow H_j(\mathbb{K}_r) \longrightarrow H_{j-1}(\mathbb{K}_{r-1}[s_r]t_r) \xrightarrow{n_r s_r} H_{j-1}(\mathbb{K}_{r-1}[s_r])$$

then the rightmost map is zero except for $j-1=0$, when the sequence ends in

$$H_1(\mathbb{K}_r) \longrightarrow H_0(\mathbb{K}_{r-1}[s_r]t_r) \xrightarrow{n_r s_r} H_0(\mathbb{K}_{r-1}[s_r]) \longrightarrow H_0(\mathbb{K}_r) \longrightarrow 0\ .$$

As taking the tensor product with $R[s_r]$ over R is exact ($R[s_r]$ being free and thus flat over R), Lemma 6.2.6 shows that

$$\begin{aligned} H_0(\mathbb{K}_{r-1}[s_r]) &\cong H_0(\mathbb{K}_{r-1})[s_r] \\ &\cong R[s_r] \oplus \bigoplus_{i=1}^{r-1} \frac{R}{(n_i)}[s_i, s_{i+1}, \ldots, s_r]s_i. \end{aligned}$$

Now multiplication with $n_r s_r$ is injective on the first summand $R[s_r]$, but annihilates the remaining summands, as n_r is the largest elementary divisor. If we therefore set

$$\begin{aligned} \tilde{H}_0 &= \bigoplus_{i=1}^{r-1} \frac{R}{(n_i)}[s_i, s_{i+1}, \ldots, s_r]s_i \\ &\cong \bigoplus_{I\subseteq\{1,\ldots,r-1\},|I|=1} \bigoplus_{N\in\mathbb{N}^r} \frac{R}{(n_{\ell(N+I)})}\mathbf{s}^N \partial'(\mathbf{t}_I) \end{aligned}$$

and write $\tilde{H}_j = H_j(\mathbb{K}_{r-1}[s_r]) \cong H_j(\mathbb{K}_{r-1})[s_r]$ for $j > 0$, then the long exact homology sequence breaks into the short exact sequences

$$0 \longrightarrow \tilde{H}_j \xrightarrow{i} H_j(\mathbb{K}) \xrightarrow{p} \tilde{H}_{j-1}t_r \longrightarrow 0 \tag{6.9}$$

of R-modules for $j \geq 1$. Now by induction we already know that for $j \geq 1$ we have

$$\tilde{H}_j \cong \bigoplus_{I\subseteq\{1,\ldots,r-1\},|I|=j+1} \bigoplus_{N\in\mathbb{N}^r} \frac{R}{(n_{\ell(N+I)})}\mathbf{s}^N \partial'(\mathbf{t}_I)$$

as a direct summand of the homology of $\mathbb{K}_{r-1}[s_r]$, and as we just showed, this description is valid as well for $j = 0$.

On the other hand, we deduce from Proposition 6.2.7 that for $j > 1$ the direct sum

$$\begin{aligned}\tilde{H}_j \oplus \tilde{H}_{j-1}t_r \quad &\cong \quad \bigoplus_{\emptyset \neq I \subseteq \{1,\ldots,r-1\}, |I|=j+1} \bigoplus_{N\in\mathbb{N}^r} \frac{R}{(n_{\ell(N+I)})}\mathbf{s}^N\partial'(\mathbf{t}_I) \\ &\oplus \bigoplus_{\emptyset \neq I \subseteq \{1,\ldots,r-1\}, |I|=j} \bigoplus_{N\in\mathbb{N}^r} \frac{R}{(n_{\ell(N+I)})}\mathbf{s}^N\partial'(\mathbf{t}_I)t_r\end{aligned}$$

maps to $H_j(\mathbb{K})$, equivalently, $p : H_j(\mathbb{K}) \to \tilde{H}_{j-1}t_r$ admits a section s, given by $s(\partial'(\mathbf{t}_I)t_r) = \partial'(\mathbf{t}_{I\cup\{r\}})$ as indeed $\partial'(\mathbf{t}_I)t_r = p(\partial'(\mathbf{t}_{I\cup\{r\}}))$ for $I \neq \emptyset$ and we saw in Proposition 6.2.7 (1) that each monomial $\mathbf{s}^N\partial'(\mathbf{t}_J)$, with $J \subseteq \{1,\ldots,r\}$ is a cycle in $\mathbb{K}$ whose class in homology is annihilated by $n_{\ell(N+J)}$. Therefore, each short exact sequence (6.9) splits and the middle terms is identified as the direct sum just displayed. Summing up over all j yields the result. □

Remark: The cohomology ring is a finitely generated module over the ring

$$H_0(\mathbb{K}) \cong \mathbb{S}ym_R\left(\bigoplus_{i=1}^{r}\frac{R}{(n_i)}\right)$$

discussed above, generated by the classes $\partial'(\mathbf{t}_I)$ with $|I| \geq 2$. While this symmetric algebra is naturally $\mathbb{N}^r$-graded, that is not so for the cohomology ring, as the elements $\partial'(\mathbf{t}_I)$ are only homogeneous for the total, cohomological, degree.

However, a closer inspection of the result gives some more information on the module structure, in that

$$H_j(\mathbb{K}) \cong \bigoplus_{|I|=j} H_0(\mathbb{K}) \otimes_R \frac{R}{(n_{\ell(I)})}\partial'(\mathbf{t}_I)$$

is a direct sum of cyclic $H_0(\mathbb{K})$-modules as indicated for any $j \geq 1$.

Remark: As concerns the algebra structure, we know already that $H_0(\mathbb{K})$ is central in $H^\bullet(G,R)$, whence, by the previous remark, it suffices to understand the products $\partial'(\mathbf{t}_I)\partial'(\mathbf{t}_{I'})$ for subsets $I, I' \subseteq \{1,\ldots,r\}$.

As ∂ is an algebra differential and $\partial'(\mathbf{t}_{I'})$ is a cycle, we have

$$\partial'(\mathbf{t}_I)\partial'(\mathbf{t}_{I'}) = \frac{1}{n_{\ell(I)}}\partial(\mathbf{t}_I\partial'(\mathbf{t}_{I'}))$$

and from there one can work out the product explicitly.

Using now the preceding theorem together with Proposition 6.2.7 (2), the same arguments prove the following result.

Theorem 6.2.10. *With the assumptions and notation of Proposition 6.2.7 (2), the group cohomology of G with values in $\overline{R}$ has the following form,*

$$H^\bullet(G,\overline{R}) \cong H^\bullet(G,R)\otimes_R \overline{R} \oplus H^{\bullet+1}(G,R),$$

with $H^\bullet(G,R)\otimes_R \overline{R}$ a sub algebra, and in the second summand the cycles $\mathbf{t}'_I$ as defined in Proposition 6.2.7 (2) replacing the $\partial'(\mathbf{t}_I)$. Note that $H^i(G,R)\otimes_R\overline{R} \cong H^i(G,R)$, for $i>0$. In other words, only the direct summand R in $H^\bullet(G,R)$ gets changed to $\overline{R}$, the remaining direct summands stay unchanged under the tensor product with $\overline{R}$ over R.

The $H^\bullet(G,R)\otimes_R\overline{R}$-linear map that sends

$$\mathbf{t}'_I \in H^{|I|}(G,\overline{R}) \mapsto \partial'(\mathbf{t}_I) \in H^{|I|+1}(G,R) \subseteq H^{|I|+1}(G,\overline{R})$$

defines the Böckstein derivation on $H^\bullet(G,\overline{R})$ with kernel $H^\bullet(G,R)\otimes_R\overline{R}$.

Remark: Note that, despite appearances, the above direct sum decomposition of $H^\bullet(G,\overline{R})$ is not one of $H^\bullet(G,R)\otimes_R\overline{R}$-modules. If we call the latter ring S, and denote by $S^+(1)$ its irrelevant ideal generated by the elements of strictly positive degree and shifted in degree by 1, so that $S^+(1)^i = (S^+)^{i+1}$, then there is rather a short exact sequence of graded S-modules,

$$0 \longrightarrow S \longrightarrow H^\bullet(G,\overline{R}) \longrightarrow S^+(1) \longrightarrow 0\,,$$

the direct sum over the short exact sequences of $\overline{R}$-modules

$$0 \longrightarrow H^i(G,R)\otimes_R\overline{R} \longrightarrow H^i(G,\overline{R}) \longrightarrow H^{i+1}(G,R)\otimes_R\overline{R} \longrightarrow 0\,.$$

However, this sequence is not split in general. For example, if $G=V$ is the Kleinian four-group and $R=\mathbb{Z}$, $\overline{R}=\mathbb{F}_2$, then

$$S = \frac{\mathbb{F}_2[a,b,c]}{(c^2-ab(a+b))},$$

with a,b of degree 2 and c of degree 3. The embedding of S as a sub algebra of

$$H^\bullet(V,\mathbb{F}_2) \cong \mathbb{F}_2[t_1,t_2],$$

with t_1,t_2 in degree 1, sends

$$\begin{array}{rcl} a & \mapsto & t_1^2\\ b & \mapsto & t_2^2\\ c & \mapsto & t_1t_2(t_1+t_2). \end{array}$$

In particular, S is a domain as a subring of the polynomial ring, whence its depth is at least 1. It follows that $S^+(1)$ has depth exactly 1, as, up to degree shift, it is the first syzygy module of $\mathbb{F}_2 \cong \frac{S}{S^+}$ as an S-module, and that quotient has depth 0, being annihilated by S^+. As a module of given depth cannot occur as a direct summand of a module of larger depth, $S^+(1)$ is not a direct S-summand of $\mathbb{F}_2[t_1, t_2]$ as that module has depth 2.

The same argument shows that for an elementary abelian 2-group G of rank $r \geq 2$ the ring

$$S = H^\bullet(G, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_2$$

has depth exactly 2, as

$$H^\bullet(G, \mathbb{F}_2) \cong \mathbb{F}_2[t_1, \ldots, t_r]$$

has depth r, but the quotient $S^+(1)$ still has only depth 1, being the first syzygy module of $\mathbb{F}_2(1)$.

Remark: While the R-module structure of $H^i(G, \overline{R})$, for $i > 0$, is independent of the choice of the non-zero-divisor m that defines $\overline{R}$, this is not true in general for the multiplicative structure when even elementary divisors are present. Indeed, this is already in evidence for cyclic 2-groups.

Example 6.2.11. *Fix a prime number p and consider the elementary abelian p-group of rank r, that is, $G = \mu_p^r$, isomorphic to the additive group underlying the r-dimensional vector space over the field $\mathbb{F}_p$ with p elements. If $R = \mathbb{F}_p$, or more generally, if R is a field of characteristic p, then the cohomology ring $H^\bullet(G, R)$ was described in Corollary 6.2.2.*

Example 6.2.12. *Now let us consider the integral cohomology for $G = \mu_p^r$. Additively it is given by*

$$H^\bullet(G, \mathbb{Z}) \cong \mathbb{Z} \times_{\mathbb{F}_p} (\mathbb{F}_p[s_1, \ldots, s_r]\langle \partial'(\mathbf{t}_I) \rangle),$$

where I runs over those subsets of $\{1, \ldots, r\}$ with at least 2 elements.

If p is odd, this is indeed an isomorphism of strictly graded algebras, where the $\partial'(\mathbf{t}_I)$ are multiplied among themselves as elements of the corresponding Koszul complex $\mathbb{K}$.

For $p = 2$, one has

$$H^\bullet(G, \mathbb{Z}) \cong \mathbb{Z} \times_{\mathbb{F}_2} \mathbb{F}_2 \left[t_1^2, \ldots, t_r^2 \; ; \; \mathbf{t}_I \sum_{i \in I} t_i \right] \; , \; \text{where } \mathbf{t}_I = \prod_{i \in I} t_i$$

and the second factor is considered as a sub algebra of the polynomial ring $\mathbb{F}_2[t_1, \ldots, t_r]$. Indeed, that polynomial ring is isomorphic to $H^\bullet(G, \mathbb{F}_2)$ by Corollary 6.2.2, and the Böckstein *homomorphism sends* $\mathbf{t}'_I = \mathbf{t}_I$ *to*

$$\begin{aligned}\partial'(\mathbf{t}_I) &= \sum_\nu (-1)^{\nu-1} s_\nu t_{i_1} \cdots \widehat{t_\nu} \cdots t_{i_a} \\ &= \sum_\nu t_\nu^2 t_{i_1} \cdots \widehat{t_\nu} \cdots t_{i_a} \\ &= \mathbf{t}_I \sum_{i \in I} t_i,\end{aligned}$$

as the signs disappear because we are in characteristic 2*, and* $s_i = t_i^2$*.*

In compact form, the Böckstein derivation is $\sum_i t_i^2 \frac{\partial}{\partial t_i}$*.*

6.3 Examples and Earlier Results

6.3.1 The Integral Cohomology Ring for a Product of Two Cyclic Groups

Theorem 6.3.1. *Suppose that we have a finite abelian group* G *with elementary divisors* $n_1|n_2$*. Write* $n_2 = mn_1$*. Then we have*

$$H^\bullet(G,\mathbb{Z}) \cong \frac{\mathbb{Z}[a,b,c]}{(n_1 a,\ n_2 b,\ n_1 c,\ c^2)} \text{ where } |a| = |b| = 2;\ |c| = 3$$

in all cases except when n_1 *and* n_2 *are both* even *with* m odd*, in which case we get*

$$H^\bullet(G,\mathbb{Z}) \cong \frac{\mathbb{Z}[a,b,c]}{\left(n_1 a,\ n_2 b,\ n_1 c,\ c^2 - \left(\frac{n_1}{2}\right) ab\,(a+mb)\right)} \text{ where } |a| = |b| = 2;\ |c| = 3.$$

Proof. Since we have taken our coefficients in $\mathbb{Z}$, Theorem 6.2.9 applies. The legal choices for I are $\{1\}$, $\{2\}$ and $\{1,2\}$. These choices for I give us the following generators:

1. $\underline{I = \{1\}}$: $\partial'(t_1) = s_1$, by Remark 6.2.8.
2. $\underline{I = \{2\}}$: $\partial'(t_2) = s_2$, by Remark 6.2.8.

3. $\underline{I = \{1,2\}}$:

$$\begin{aligned}
\partial'(\mathbf{t}_I) &= \frac{1}{n_{\ell(I)}}\partial(\mathbf{t}_I) \\
&= \frac{1}{n_1}\partial(t_1t_2) \\
&= \frac{1}{n_1}(\partial(t_1)t_2 - t_1\partial(t_2)) \\
&= \frac{1}{n_1}(n_1s_1t_2 - t_1n_2s_2) \\
&= s_1t_2 - ms_2t_1
\end{aligned}$$

So setting $a = s_1,\ b = s_2$ and $c = s_1t_2 - ms_2t_1$, we have that $|a| = |b| = 2;\ |c| = 3$, and

$$\begin{aligned}
n_1a &= n_1s_1 \\
&= 0 \\
n_2b &= n_2s_2 \\
&= 0 \\
n_1c &= n_1(s_1t_2 - ms_2t_1) \\
&= \underbrace{n_1s_1}_{=0}t_2 - \underbrace{n_2s_2}_{=0}t_1 \\
&= 0
\end{aligned}$$

We also see that

$$\begin{aligned}
c^2 &= (s_1t_2 - ms_2t_1)(s_1t_2 - ms_2t_1) \\
&= s_1^2t_2^2 - ms_1s_2t_2t_1 - ms_1s_2t_1t_2 + m^2s_2^2t_1^2 \\
&= s_1^2t_2^2 + ms_1s_2t_1t_2 - ms_1s_2t_1t_2 + m^2s_2^2t_1^2 \\
&= s_1^2t_2^2 + m^2s_2^2t_1^2
\end{aligned}$$

Now we have the following cases.

1. $\underline{n_1 \text{ is even} \Rightarrow n_2 \text{ is also even}}$: Then $t_1^2 = \frac{n_1}{2}s_1$ and $t_2^2 = \frac{n_2}{2}s_2$, and so we have

$$\begin{aligned} c^2 &= \frac{n_2}{2}s_1^2 s_2 + m^2\left(\frac{n_1}{2}\right)s_1 s_2^2 \\ &= \frac{n_2}{2}s_1^2 s_2 + \left(\frac{m^2 n_1}{2}\right)s_1 s_2^2 \\ &= \frac{n_2}{2}s_1^2 s_2 + \left(\frac{m n_2}{2}\right)s_1 s_2^2 \\ &= \left(\frac{n_2}{2}\right)ab\,(a+mb) \\ &= \left(\frac{m n_1}{2}\right)ab\,(a+mb) \end{aligned}$$

   (a) $\underline{\text{If } m \text{ is even}}$, then $\frac{m}{2} \in \mathbb{Z}$, and since $n_1 a = 0$, we get $c^2 = 0$ in this case. Therefore

$$H^\bullet(G,\mathbb{Z}) \cong \frac{\mathbb{Z}[a,b,c]}{(n_1 a,\ n_2 b,\ n_1 c,\ c^2)} \text{ where } |a| = |b| = 2;\ |c| = 3.$$

   (b) $\underline{\text{If } m \text{ is odd}}$, then writing $m = 2k+1$, we have

$$\begin{aligned} \frac{m n_1}{2}a &= \frac{(2k+1)n_1}{2}a \\ &= k\underbrace{n_1 a}_{=0} + \frac{n_1}{2}a \\ &= \frac{n_1}{2}a \end{aligned}$$

   so the above equation simplifies to $c^2 = \left(\frac{n_1}{2}\right)ab\,(a+mb)$, and therefore

$$H^\bullet(G,\mathbb{Z}) \cong \frac{\mathbb{Z}[a,b,c]}{\left(n_1 a,\ n_2 b,\ n_1 c,\ c^2 - \left(\frac{n_1}{2}\right)ab\,(a+mb)\right)} \text{ where } |a| = |b| = 2;\ |c| = 3.$$

2. $\underline{n_1 \text{ is odd and } n_2 \text{ is even}}$: Then m is even, $t_1^2 = 0$ and $t_2^2 = \frac{n_2}{2}s_2$, and so we have

$$\begin{aligned} c^2 &= \frac{n_2}{2}s_1^2 s_2 \\ &= \frac{m n_1}{2}a^2 b \\ &= \underbrace{\frac{m}{2}}_{\in\mathbb{Z}}\underbrace{n_1 a^2}_{=0} b \\ &= 0 \end{aligned}$$

Therefore we have that

$$H^\bullet(G,\mathbb{Z}) \cong \frac{\mathbb{Z}[a,b,c]}{(n_1a,\ n_2b,\ n_1c,\ c^2)} \text{ where } |a|=|b|=2;\ |c|=3.$$

3. <u>n_1 and n_2 are odd:</u> Then $t_1^2 = 0 = t_2^2$. Thus $c^2 = 0$, and we have

$$H^\bullet(G,\mathbb{Z}) \cong \frac{\mathbb{Z}[a,b,c]}{(n_1a,\ n_2b,\ n_1c,\ c^2)} \text{ where } |a|=|b|=2;\ |c|=3.$$

□

We can now apply this result to compare with the known results from [9], [17] and [12].

Example 6.3.2. *Let p be an odd prime. Let $\nu_1 \leq \nu_2$ be positive integers. Let $G = \mathbb{Z}_{p^{\nu_1}} \oplus \mathbb{Z}_{p^{\nu_2}}$. Then $p^{\nu_2-\nu_1}$ is odd, so applying Theorem 6.3.1 gives*

$$H^\bullet(G,\mathbb{Z}) \cong \frac{\mathbb{Z}[a,b,c]}{(p^{\nu_1}a,\ p^{\nu_2}b,\ p^{\nu_1}c,\ c^2)} \text{ where } |a|=|b|=2;\ |c|=3.$$

This agrees with Corollary 1 from [9], with our relation c^2 coming from the author's (2c).

Example 6.3.3. *Let $\nu_1 \leq \nu_2$ be positive integers. Let $G = \mathbb{Z}_{2^{\nu_1}} \oplus \mathbb{Z}_{2^{\nu_2}}$. Then we have two cases:*

1. *<u>If $\nu_1 < \nu_2$</u>, then applying Theorem 6.3.1 gives*

$$H^\bullet(G,\mathbb{Z}) \cong \frac{\mathbb{Z}[a,b,c]}{(2^{\nu_1}a,\ 2^{\nu_2}b,\ 2^{\nu_1}c,\ c^2)} \text{ where } |a|=|b|=2;\ |c|=3.$$

   This agrees with Corollary 7.4 from [17], with our relation c^2 coming from the author's $(2c_1)$.

2. *<u>If $\nu_1 = \nu_2$</u>, then applying Theorem 6.3.1 gives*

$$H^\bullet(G,\mathbb{Z}) \cong \frac{\mathbb{Z}[a,b,c]}{(2^{\nu_1}a,\ 2^{\nu_1}b,\ 2^{\nu_1}c,\ c^2 - 2^{\nu_1-1}ab(a+b))} \text{ where } |a|=|b|=2;\ |c|=3.$$

   This agrees with Corollary 7.4 from [17], with our relation $c^2 - 2^{\nu_1-1}ab(a+b)$ coming from the author's $(2c_1)$.

Example 6.3.4. *Let $G = \mathbb{Z}_2 \oplus \mathbb{Z}_4$. Then applying Theorem 6.3.1 gives*

$$H^\bullet(G, \mathbb{Z}) \cong \frac{\mathbb{Z}[a, b, c]}{(2a,\ 4b,\ 2c,\ c^2)} \text{ where } |a| = |b| = 2;\ |c| = 3.$$

This agrees with Proposition 3.7 in [12], except that the author omits the relation $2c$.

Example 6.3.5. *Let p be a prime. Let $G = \mathbb{Z}_p \oplus \mathbb{Z}_p$.*

1. *If p is odd, then applying Theorem 6.3.1 gives*

$$H^\bullet(G, \mathbb{Z}) \cong \frac{\mathbb{Z}[a, b, c]}{(pa,\ pb,\ pc,\ c^2)} \text{ where } |a| = |b| = 2;\ |c| = 3.$$

   This agrees with Proposition 4.1 in [12], except that the author omits the annihilator relations.

2. *If $p = 2$, then applying Theorem 6.3.1 gives*

$$H^\bullet(G, \mathbb{Z}) \cong \frac{\mathbb{Z}[a, b, c]}{(2a,\ 2b,\ 2c,\ c^2 - ab(a + b))} \text{ where } |a| = |b| = 2;\ |c| = 3.$$

   This agrees with Proposition 4.1 in [12], except that the author omits the annihilator relations.

Example 6.3.6. *Let p be a prime and let $G = \mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$. We may assume that p is odd, since the $p = 2$ case has been handled above in Proposition 3.7. Then applying Theorem 6.3.1 gives*

$$H^\bullet(G, \mathbb{Z}) \cong \frac{\mathbb{Z}[a, b, c]}{(pa,\ p^2b,\ pc,\ c^2)} \text{ where } |a| = |b| = 2;\ |c| = 3.$$

This agrees with Proposition 4.3 in [12].

6.3.2 Comparison With a Result from [12] for $G = \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$

Here we compare Example 6.2.12 with the case of Proposition 4.2 in [12] in which p is odd. The $p = 2$ case is analogous, but requires some more work since we no longer have the relations $t_j^2 = 0$.

Example 6.3.7. *Let p be an odd prime. Let $G = \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$. Then we obtain the following generators for $H^\bullet(G, \mathbb{Z})$ over $\mathbb{Z}$.*

$$
\begin{aligned}
\alpha &:= \partial'(t_1) \\
&= s_1 \\
\beta &:= \partial'(t_2) \\
&= s_2 \\
\gamma &:= \partial'(t_3) \\
&= s_3 \\
\mu &:= \partial'(t_{\{12\}}) \\
&= \frac{1}{p}\partial(t_1 t_2) \\
&= \frac{1}{p}(ps_1 t_2 - pt_1 s_2) \\
&= s_1 t_2 - t_1 s_2 \\
\chi &:= \partial'(t_{\{31\}}) \\
&= \frac{1}{p}\partial(t_3 t_1) \\
&= \frac{1}{p}(ps_3 t_1 - pt_3 s_1) \\
&= s_3 t_1 - t_3 s_1 \\
\nu &:= \partial'(t_{\{23\}}) \\
&= \frac{1}{p}\partial(t_2 t_3) \\
&= \frac{1}{p}(ps_2 t_3 - pt_2 s_3) \\
&= s_2 t_3 - t_2 s_3 \\
\xi &:= \partial'(t_{\{123\}}) \\
&= \frac{1}{p}\partial(t_1 t_2 t_3) \\
&= \frac{1}{p}(ps_1 t_2 t_3 - pt_1 s_2 t_3 + pt_1 t_2 s_3) \\
&= s_1 t_2 t_3 - t_1 s_2 t_3 + t_1 t_2 s_3
\end{aligned}
$$

We verify one of each type of relation. the verifications of the other relations of the same type are completely analogous.

$$
\begin{aligned}
\mu^2 &= (s_1t_2 - t_1s_2)(s_1t_2 - t_1s_2) \\
&= s_1^2t_2^2 - s_1s_2t_2t_1 - s_1s_2t_1t_2 + t_1^2s_2^2 \\
&= 0, \text{ since } t_1^2 = t_2^2 = 0 \text{ and } t_2t_1 = -t_1t_2 \\
\xi^2 &= (s_1t_2t_3 - t_1s_2t_3 + t_1t_2s_3)(s_1t_2t_3 - t_1s_2t_3 + t_1t_2s_3) \\
&= 0, \text{ since } t_1^2 = t_2^2 = t_3^2 = 0 \\
\nu\chi &= (s_2t_3 - t_2s_3)(s_3t_1 - t_3s_1) \\
&= s_2s_3t_3t_1 - s_1s_2t_3^2 - s_3^2t_2t_1 + s_1s_3t_2t_3 \\
&= s_3(s_1t_2t_3 - t_1s_2t_3 + t_1t_2s_3), \text{ since } t_3^2 = 0 \\
&= \gamma\xi \\
\mu\xi &= (s_1t_2 - t_1s_2)(s_1t_2t_3 - t_1s_2t_3 + t_1t_2s_3) \\
&= s_1^2t_2^2t_3 + s_1s_2t_1t_2t_3 - s_1s_3t_1t_2^2 - s_1s_2t_1t_2t_3 + s_2^2t_1^2t_3 - s_2s_3t_1^2t_2 \\
&= 0, \text{ since } t_1^2 = t_2^2 = t_3^2 = 0 \\
\alpha\nu + \beta\chi + \gamma\mu &= s_1(s_2t_3 - t_2s_3) + s_2(s_3t_1 - t_3s_1) + s_3(s_1t_2 - t_1s_2) \\
&= s_1s_2t_3 - s_1s_3t_2 + s_2s_3t_1 - s_1s_2t_3 + s_1s_3t_2 - s_2s_3t_1 \\
&= 0
\end{aligned}
$$

So putting it all together, we have

$$
H^\bullet(G,\mathbb{Z}) \cong \frac{\mathbb{Z}[\alpha,\beta,\gamma,\mu,\chi,\nu,\xi]}{\left(\begin{array}{l} p\alpha, p\beta, p\gamma, p\mu, p\chi, p\nu, p\xi, \mu^2, \nu^2, \chi^2, \xi^2, \\ \nu\mu - \alpha\xi, \nu\chi - \gamma\xi, \mu\nu - \beta\xi, \mu\xi, \chi\xi, \nu\xi, \alpha\nu + \beta\chi + \gamma\mu \end{array}\right)}
$$

where $|\alpha| = |\beta| = |\gamma| = 2,\ |\mu| = |\chi| = |\nu| = 3,\ |\xi| = 4.$

The author again omits the annihilation relations.

Chapter 7

The Tate Resolution and Hochschild Cohomology for Monic Polynomials

7.1 Introduction

In this chapter, we will generalize the setup from Chapter 5. This will enable us to obtain some results on Hochschild Cohomology. In particular we will improve on a result from [11] on the multiplicative structure of the Hochschild cohomology ring of a hypersurface ring $\frac{R[x]}{(f(x))}$, where $f(x)$ is monic.

7.2 Preliminaries

In our earlier setup we have

$$\begin{tikzcd} & RG \arrow[dr, "\epsilon"] & \\ R \arrow[ur, "\eta"] \arrow[rr, equal] & & R \end{tikzcd}$$

with

$$\begin{array}{rccl} \eta & : & R & \rightarrow & RG \\ & & r & \mapsto & r \cdot 1 \\ \epsilon & : & RG & \rightarrow & R \\ & & \sum_{x \in G} a_x x & \mapsto & \sum_{x \in G} a_x \end{array}$$

More generally, for a supplemented algebra A over K (see Definition 3.2.1), we have

$$\begin{tikzcd} & A \arrow[dr, "\epsilon"] & \\ K \arrow[ur, "\eta=\text{include}"] \arrow[rr, equal] & & K \end{tikzcd}$$

where A is projective over K. With this setup, we have

$$Ext_A(K,K) \cong HH(A/K,K) \cong Ext_{A^{ev}}(A,K)$$

Remarks:

1. The augmentation ϵ makes K into an A-module, so that it makes sense to write down the expression $Ext_A(K,K)$.
2. The outer terms in the above line are isomorphic by Theorem 2.8a on p167 of [8], with $\Lambda = A^{op}, \Gamma = A, \Sigma = K, B = K, C = K$.

7.3 One Monic Polynomial

7.3.1 Preliminaries

In this chapter we will put the preceding work into a more general framework, which easily specializes to the desired case of the group ring for a finite abelian group. This more general framework will allow us to obtain some new results on Hochschild cohomology.

Let $f(x) \in R[x]$ be monic. Define

$$\begin{aligned} R_a &:= \frac{R[x]}{(f(x))} \\ R_b &:= R_a \otimes_R R_a \\ &\cong \frac{R[x', x'']}{(f(x'), f(x''))} \\ R_c &:= R_b \otimes_{R_a} R_b \\ &= (R_a \otimes_R R_a) \otimes_{R_a} (R_a \otimes_R R_a) \\ &\cong R_a \otimes_R R_a \otimes_R R_a =: R_a^{\otimes 3} \\ &\cong \frac{R[x', x, x'']}{(f(x'), f(x), f(x''))} \end{aligned}$$

where our notation means

1. in R_b:

$$\begin{aligned} x' &= x \otimes 1 \\ x'' &= 1 \otimes x \end{aligned}$$

2. in R_c:

$$\begin{aligned} x' &= x \otimes 1 \otimes 1 \\ x &= 1 \otimes x \otimes 1 \\ x'' &= 1 \otimes 1 \otimes x \end{aligned}$$

   i.e. we identify x' with the leftmost copy of R_a, x'' with the rightmost copy of R_a, and x with both of the middle copies of R_a.

We will need to turn R_b into a bimodule over R_a. It will agree best with the above notation if we make the following definitions:

As a right R_a-module: Write $R_b \cong \frac{R[x',x]}{(f(x'),f(x))}$. Then define the R-algebra homomorphism

$$\begin{array}{cccc} \alpha & : & \frac{R[x]}{(f(x))} & \rightarrow & \frac{R[x',x]}{(f(x'),f(x))} \\ & & x & \mapsto & x. \end{array}$$

In this way, $R_b \cong \frac{R_a[x']}{(f(x'))}$ becomes a free R_a-module.

As a left R_a-module: Write $R_b \cong \frac{R[x,x'']}{(f(x),f(x''))}$. Then define the R-algebra homomorphism

$$\begin{array}{cccc} \beta & : & \frac{R[x]}{(f(x))} & \rightarrow & \frac{R[x,x'']}{(f(x),f(x''))} \\ & & x & \mapsto & x. \end{array}$$

In this way, $R_b \cong \frac{R_a[x'']}{(f(x''))}$ becomes a free R_a-module.

Specializing to the defining polynomial $f(x) = x^h - 1$ for the group ring of a finite cyclic group of order h gives

$$\begin{aligned} R_a &= RG, \\ R_b &= RG^{ev}, \\ R_c &= RG^{\otimes^3}, \\ &\cong \frac{R[x',x,x'']}{(f(x'),f(x),f(x''))}, \\ &\cong R[G \times G \times G]. \end{aligned}$$

In §7.5.1, we will show that our more general setup does indeed specialize to our earlier setup.

7.3.2 Difference Quotients

To streamline what follows, we introduce the following notation:

$$\begin{aligned} \Delta x &:= x'' - x' \in R[x'', x']; \\ \Delta f &:= f(x'') - f(x') \in R[x'', x']. \end{aligned}$$

Observe that $\Delta x = x'' - x'$ is monic, and therefore it is a non zero divisor.

Now let us consider the polynomial $\Delta f = f(x'') - f(x')$. If we set $x'' = x'$, then the polynomial Δf evaluates to zero. Therefore $\Delta x = x'' - x'$ divides Δf, and we can write

$$\Delta f = (\Delta x)(\Delta(x'', x')), \text{ for some } \Delta(x'', x') \in R[x'', x'].$$

We may think of the expression $\Delta(x'', x')$ as the quotient

$$\Delta(x'', x') = \frac{f(x'') - f(x')}{x'' - x'}.$$

This leads us to the following definition.

Definition 7.3.1. *For independent variables z and y, define the difference quotient*

$$\Delta(z, y) := \frac{f(z) - f(y)}{z - y}.$$

Remark: Whenever we write such a quotient of polynomials, we of course mean the polynomial which multiplies with the denominator to yield the numerator. For this to be well-defined, the denominator must be a non zero divisor which divides the numerator. We have explained why this holds for the particular difference quotient $\Delta(x'', x')$. Similar observations hold for all the difference quotients throughout this chapter.

We record some key properties of these difference quotients for later use.

First, it is easy to see that Δ is symmetric, i.e. that $\Delta(z, y) = \Delta(y, z)$ for any variables y and z.

Second, we record a form for $\Delta(z, y)$ which we will need later.

Lemma 7.3.2. *For independent variables y and z, we have*

$$\Delta(z,y) = \sum_{i\geq 1} f^{(i)}(y)(z-y)^{i-1}.$$

Proof. By definition, we have

$$\Delta(z,y) = \frac{f(z)-f(y)}{z-y}$$

Define $g(z) = f(z) - f(y) \in (R[y])[z]$. We view g as a polynomial in the variable z with coefficients from the ring $R[y]$. Writing the Taylor expansion for $g(z)$ about $z = y$ (using the divided derivatives as in Remark 2.6.9) gives

$$\begin{aligned}
g(z) &= \frac{g(y)}{0!} + \frac{g'(y)}{1!}(z-y) + \frac{g''(y)}{2!}(z-y)^2 + \frac{g'''(y)}{3!}(z-y)^3 + \cdots \\
&= f^{(1)}(y)(z-y) + f^{(2)}(y)(z-y)^2 + f^{(3)}(y)(z-y)^3 + \cdots \\
\Rightarrow \Delta(z,y) = \frac{g(z)}{z-y} &= f^{(1)}(y) + f^{(2)}(y)(z-y) + f^{(3)}(y)(z-y)^2 + \cdots \\
&= \sum_{i\geq 1} f^{(i)}(y)(z-y)^{i-1}.
\end{aligned}$$

□

7.3.3 The Tate Resolution

We are now able to give the Tate resolution on which we will base the rest of the results of this chapter.

Theorem 7.3.3. *With the above setup, the Tate resolution for R_a over R_b is given by*

$$\begin{tikzcd}[row sep=large]
 & 0 \\
 & R_a \arrow[u] \\
0 & R_b \cdot 1 \arrow[u, "\epsilon\colon\ \begin{smallmatrix} x' \mapsto x \\ x'' \mapsto x \end{smallmatrix}"'] \\
1 & R_b \cdot \tau \arrow[u, "\Delta x"'] \\
2 & R_b \cdot \sigma \arrow[u, "\Delta(x'',x')"'] \\
3 & R_b \cdot \tau\sigma \arrow[u, "\Delta x"'] \\
 & \vdots \arrow[u, "\Delta(x'',x')"'] \\
2i & R_b \cdot \sigma^{(i)} \arrow[u, "\Delta(x'',x')"'] \\
2i+1 & R_b \cdot \tau\sigma^{(i)} \arrow[u, "\Delta x"'] \\
2i+2 & R_b \cdot \sigma^{(i+1)} \arrow[u, "\Delta(x'',x')"'] \\
 & \vdots \arrow[u] \\
 & \mathbb{F} \arrow[u, equal]
\end{tikzcd}$$

Proof. Take

$$\begin{aligned} P &= \frac{R_a[x',\ x'']}{(f(x'))} \\ &\cong R_b[x''] \\ I &= (x'' - x') \subset P \\ J &= (f(x'') - f(x')) \subset P \end{aligned}$$

As above, $(x'' - x')$ is a non zero divisor which divides $f(x'') - f(x')$. Therefore $f(x'') - f(x') \in (x'' - x')$, so that $J \subseteq I$ as required.

Since f is monic, $f(x'') - f(x')$ is a monic non zero divisor. So as per our second example of a Koszul regular sequence, I and J are generated by Koszul regular sequences.

Then we have:

$$\begin{aligned} \frac{P}{J} &= \frac{\frac{R_a[x',\ x'']}{(f(x'))}}{(f(x'') - f(x'))} \\ &\cong \frac{R_a[x',\ x'']}{(f(x''), f(x'))} \\ &\cong R_b \\ \frac{P}{I} &= \frac{\frac{R_a[x',\ x'']}{(f(x'))}}{(x'' - x')} \\ &\cong \frac{R_a[x']}{(f(x'))} \\ &\cong R_a \end{aligned}$$

<u>What is $A = (a_{ij})$?</u>

$$\begin{aligned} f(x'') - f(x') &= \Delta f \\ &= (\Delta(x'', x'))(\Delta x) \end{aligned}$$

So that $A = (a_{ii}) = (\Delta(x'', x'))$.

The hypotheses of Tate's Theorem are satisfied, and so we get a resolution of $\frac{P}{I} \cong R_a$ over $\frac{P}{J} \cong R_b$:

$$\begin{aligned} \mathbb{F} &= R_b\langle \tau, \sigma \rangle \\ \partial(\tau) &= \Delta x \\ \partial(\sigma) &= (\Delta(x'', x'))\tau. \end{aligned}$$

$\square$

We will define the cup product using a diagonal approximation here just as we did earlier. We will also take advantage of the fact that, just as before, the cup product is homotopic to the Yoneda product.

7.3.4 A Diagonal Approximation

As $\mathbb{F}$ is a *DG* R_b-algebra, we may use the maps α and β to form the tensor product $\mathbb{F} \otimes_{R_a} \mathbb{F}$. By construction, this will be a complex of free R_c-modules, as $R_b \otimes_{R_a} R_b \cong R_c$.

However, we wish to turn $\mathbb{F} \otimes_{R_a} \mathbb{F}$ into a resolution of R_a over R_b. The needed ingredient to do this is an R-algebra homomorphism $\Phi_0 : R_b \to R_b \otimes_{R_a} R_b$ such that the following diagram commutes:

$$\begin{tikzcd}
 & R_b & R_a \otimes_{R_a} R_b \arrow[l, "\cong"'] \\
R_b \arrow[ur, equal] \arrow[dr, equal] \arrow[r, "\Phi_0"] & R_b \otimes_{R_a} R_b \arrow[ur, "\epsilon\otimes 1"] \arrow[dr, "1\otimes\epsilon"] & \\
 & R_b & R_b \otimes_{R_a} R_a. \arrow[l, "\cong"]
\end{tikzcd}$$

Define

$$\begin{array}{ccccl} \Phi_0 & : & R_b & \to & R_b \otimes_{R_a} R_b = R_c \\ & & x' & \mapsto & x' \\ & & x'' & \mapsto & x'' \end{array}$$

Proposition 7.3.4. *The map Φ_0 is an R-algebra homomorphism that makes the diagram commute, and makes R_c into a finite free R_b-module. In this way, $\mathbb{F} \otimes_{R_a} \mathbb{F}$ becomes a DG R_b-algebra with divided powers whose terms are free R_b-modules and whose sole homology is R_a in degree 0.*

Proof. It is clear from construction that Φ_0 is an R-algebra homomorphism, which makes the diagram commute, and which endows R_c with an R_b-module structure. It is also clear that $R_c \cong R_b \otimes_{R_a} R_b$ is a finite free R_b-module, based on $\{1, x, \ldots, x^{d-1}\}$, where $d = \deg(f)$ is the degree of the polynomial f.

Now observe that $\mathbb{F} \to R_a$ is an R_b-resolution, and an R_a-homotopy equivalence, as $\mathbb{F}$ (via either α or β) and R_a itself are R_a-projective resolutions of R_a. This implies that

$$\mathbb{F} \otimes_{R_a} \mathbb{F} \sim_{\text{homotopy equivalence over } R_a} R_a \otimes_{R_a} R_a \cong R_a$$

and therefore $\mathbb{F} \otimes_{R_a} \mathbb{F} \to R_a$ is also an R_b-resolution of R_a, where the R_b-module structure on $\mathbb{F} \otimes_{R_a} \mathbb{F}$ is induced by Φ_0. $\square$

With this setup, we require this modified definition.

Definition 7.3.5. *Given a projective resolution* $\mathbb{F} \xrightarrow{\epsilon} R_a \longrightarrow 0$ *over the ring* R_b*, a* *diagonal approximation* *is a map of complexes of* R_b*-modules*

$$\Phi : \mathbb{F} \to \mathbb{F} \otimes_{R_a} \mathbb{F},$$

(where $\mathbb{F} \otimes_{R_a} \mathbb{F}$ *is considered as a complex of* R_b*-modules via* Φ_0*) that induces an isomorphism in homology, and which is compatible with the augmentation* ϵ*, in that the following diagram of complexes of* R_b*-modules commutes:*

$$\begin{tikzcd}
& \mathbb{F} & R_a \otimes_{R_a} \mathbb{F} \arrow[l, "\cong"'] \\
\mathbb{F} \arrow[ur, equal] \arrow[r, "\Phi"] \arrow[dr, equal] & \mathbb{F} \otimes_{R_a} \mathbb{F} \arrow[u, "\epsilon_1"] \arrow[ur, "\epsilon\otimes 1"'] \arrow[d, "\epsilon_2"'] \arrow[dr, "1\otimes\epsilon"] & \\
& \mathbb{F} & \mathbb{F} \otimes_{R_a} R_a. \arrow[l, "\cong"]
\end{tikzcd}$$

In other words, identifying $\mathbb{F}$ *with* $R_a \otimes_{R_a} \mathbb{F}$ *and* $\mathbb{F} \otimes_{R_a} R_a$ *via the canonical isomorphisms, we have*

$$\begin{array}{ccccc} (\epsilon \otimes 1)\Phi & = & id_{\mathbb{F}} & = & (1 \otimes \epsilon)\Phi \quad \text{, or equivalently,} \\ \epsilon_1 \Phi & = & id_{\mathbb{F}} & = & \epsilon_2 \Phi. \end{array}$$

Now in complete analogy with Chapter 4, we make the following remarks.

1. The Tate resolution $\mathbb{F}$ is a *DG* R_a-algebra with divided powers.

2. The resolution $\mathbb{F} \otimes_{R_a} \mathbb{F}$ is also a *DG* R_a-algebra with divided powers.

Define the ideal

$$I = (x - x', x'' - x) \subset R_c.$$

Theorem 7.3.6. *The following diagram defines a DG-algebra homomorphism* $\Phi : \mathbb{F} \to \mathbb{F} \otimes_{R_a} \mathbb{F}$ *which is a diagonal approximation, where the maps in higher degrees are determined by the maps in degrees zero, one and two.*

$$
\begin{tikzcd}
 & 0 & 0 \\
 & R_a \arrow[u] \arrow[r, equal] & R_a \arrow[u] \\
0 & R_b \cdot 1 \arrow[u, "{\epsilon:\ \begin{array}{l} x' \mapsto x \\ x'' \mapsto x \end{array}}"] \arrow[r, "\Phi_0", "{x' \mapsto x';\, x'' \mapsto x''}"'] & R_c \arrow[u, "{\tilde{\epsilon}:\ \begin{array}{l} x' \mapsto x \\ x \mapsto x \\ x'' \mapsto x \end{array}}"'] \\
1 & R_b \cdot \tau \arrow[u, "\Delta x"] \arrow[r, "\Phi_1", "{\tau \mapsto \tau' + \tau''}"'] & R_c(\tau' \bigoplus \tau'') \arrow[u, "{\partial:\ \begin{array}{l} \tau' \mapsto x - x' \\ \tau'' \mapsto x'' - x \end{array}}"'] \\
2 & R_b \cdot \sigma \arrow[u, "{\Delta(x'', x')}"] \arrow[r, "\Phi_2", "{\begin{array}{c} \sigma \mapsto \sigma' + \sigma'' - (f^{(2)}(x) + y)\tau'\tau'', \\ \textit{where } y \in I \end{array}}"'] & R_c(\sigma' \bigoplus \tau'\tau'' \bigoplus \sigma'') \arrow[u, "{\partial:\ \begin{array}{l} \sigma' \mapsto \frac{f(x) - f(x')}{x - x'}\tau' \\ \sigma'' \mapsto \frac{f(x'') - f(x)}{x'' - x}\tau'' \end{array}}"'] \\
3 & R_b \cdot \tau\sigma \arrow[u, "\Delta x"] \arrow[r, "\Phi_3"] & R_c(\tau'\sigma' \bigoplus \tau''\sigma' \bigoplus \tau'\sigma'' \bigoplus \tau''\sigma'') \arrow[u, "\partial"'] \\
 & \vdots \arrow[u, "{\Delta(x'', x')}"] & \vdots \arrow[u] \\
2i & R_b \cdot \sigma^{(i)} \arrow[u, "{\Delta(x'', x')}"] \arrow[r, "\Phi_{2i}"] & R_c\left((\bigoplus_{k+l=i} (\sigma')^{(k)} (\sigma'')^{(l)}) \bigoplus (\bigoplus_{k+l=i-1} \tau'\tau''(\sigma')^{(k)}(\sigma'')^{(l)}) \right) \arrow[u] \\
2i+1 & R_b \cdot \tau\sigma^{(i)} \arrow[u, "\Delta x"] \arrow[r, "\Phi_{2i+1}"] & R_c\left(\bigoplus_{k+l=i} (\tau'(\sigma')^{(k)}(\sigma'')^{(l)}) \bigoplus (\tau''(\sigma')^{(k)}(\sigma'')^{(l)}) \right) \arrow[u, "\partial"'] \\
2i+2 & R_b \cdot \sigma^{(i+1)} \arrow[u, "{\Delta(x'', x')}"] \arrow[r, "\Phi_{2i+2}"] & R_c\left((\bigoplus_{k+l=i+1} (\sigma')^{(k)}(\sigma'')^{(l)}) \bigoplus (\bigoplus_{k+l=i} \tau'\tau''(\sigma')^{(k)}(\sigma'')^{(l)}) \right) \arrow[u, "\partial"'] \\
 & \vdots \arrow[u] & \vdots \arrow[u] \\
 & \mathbb{F} \arrow[u, equal] \arrow[r, "\Phi"] & \mathbb{F} \otimes_{R_a} \mathbb{F} \arrow[u, equal]
\end{tikzcd}
$$

Remark: We intend to dualize into a trivial representation in order to compute the cup products on $Hom_{R_b}(\mathbb{F}, A)$, for some R_a-algebra A. Therefore it is enough to know Φ modulo $I \cdot \mathbb{F}$, because our augmentation sends $x' \mapsto x; x'' \mapsto x$, whence everything in $I \cdot \mathbb{F}$ is killed.

The remainder of this section will complete the proof of this Theorem. While it is fairly easy to obtain the maps Φ_0 and Φ_1, the real work lies in making a correct choice for Φ_2.

Define

$$\begin{array}{cccc} \Phi_0 & : & R_b & \rightarrow & R_c \\ & & x' & \mapsto & x' \\ & & x'' & \mapsto & x'' \end{array}$$

As before, Φ_0 is a ring homomorphism which allows us to restrict scalars from R_c to R_b.

How do we choose $\Phi_1(\tau)$? The following square must commute:

$$\begin{tikzcd} \Delta x \arrow[r, mapsto, "\Phi_0"] & \Delta x \\ \tau \arrow[u, mapsto] \arrow[r, mapsto, "\Phi_1"'] & ?. \arrow[u, mapsto, "\partial_{\mathbb{F}\otimes\mathbb{F}}"'] \end{tikzcd}$$

We check explicitly that $\Phi_1(\tau) = \tau' + \tau''$ works.

$$\partial : \tau' + \tau'' \mapsto (x - x') + (x'' - x) = x'' - x' = \Delta x$$

How do we choose $\Phi_2(\sigma)$? The square

$$\begin{tikzcd} & 0 \\ \Delta(x'', x')\tau \arrow[r, mapsto] & \Delta(x'', x')(\tau' + \tau'') \arrow[u, "\partial_{\mathbb{F}\otimes\mathbb{F}}"'] \\ \sigma \arrow[u, mapsto] \arrow[r, mapsto] & ? \arrow[u, mapsto, "\partial_{\mathbb{F}\otimes\mathbb{F}}"'] \end{tikzcd}$$

must commute.

We first show that some choice of $\Phi_2(\sigma)$ exists which makes the required square commute. Because $\partial_{\mathbb{F}\otimes\mathbb{F}}$ is exact, it is enough to prove that $\partial_{\mathbb{F}\otimes\mathbb{F}}(\Delta(x'',x')(\tau'+\tau''))=0$. We have

$$\begin{aligned}
& \partial_{\mathbb{F}\otimes\mathbb{F}}(\Delta(x'',x')(\tau'+\tau'')) \\
= \;& \Delta(x'',x')\partial_{\mathbb{F}\otimes\mathbb{F}}(\tau'+\tau'') \\
= \;& \Delta(x'',x')((x-x')+(x''-x)) \\
= \;& \Delta(x'',x')(x''-x') \\
= \;& f(x'')-f(x') \\
= \;& 0 \text{ in } R_c.
\end{aligned}$$

We have proved that some choice of $\Phi_2(\sigma)$ exists which makes the square commute. Next, we prove that we can choose $\Phi_2(\sigma)$ of a certain convenient form.

Lemma 7.3.7. *There exists a choice for $\Phi_2(\sigma)$ of the form*

$$\Phi_2(\sigma)=\sigma'+\sigma''+a(x',x,x'')\tau'\tau''$$

for some $a(x',x,x'')\in R_c$.

Proof. The element $\Phi_2(\sigma)$ lies in $(\mathbb{F}\otimes_{R_a}\mathbb{F})_2$, and $\{\sigma',\sigma'',\tau'\tau''\}$ is an R_c-basis of $(\mathbb{F}\otimes_{R_a}\mathbb{F})_2$. So we may write $\Phi_2(\sigma)=u\sigma'+v\sigma''+w\tau'\tau''$, for some $u,v,w\in R_c$. Applying $\partial_{\mathbb{F}\otimes_{R_a}\mathbb{F}}$ to this expression gives

$$\begin{aligned}
& \partial_{\mathbb{F}\otimes_{R_a}\mathbb{F}}(u\sigma'+v\sigma''+w\tau'\tau'') \\
= \;& u\Delta(x,x')\tau'+v\Delta(x'',x)\tau''+w(x-x')\tau''-w(x''-x)\tau'
\end{aligned}$$

and since the square commutes, we must have

$$u\Delta(x,x')\tau'+v\Delta(x'',x)\tau''+w(x-x')\tau''-w(x''-x)\tau' \;=\; \Delta(x'',x')(\tau'+\tau''). \tag{7.1}$$

Now, equating coefficients of τ' and τ'' in equation (7.1) yields the two equations

$$u\Delta(x,x')-w(x''-x) \;=\; \Delta(x'',x') \tag{7.2}$$
$$v\Delta(x'',x)+w(x-x') \;=\; \Delta(x'',x') \tag{7.3}$$

Considering equation (7.2) modulo $(x''-x)$, and equation (7.3) modulo $(x-x')$ yields

$$(1-u)\Delta(x,x') \;\equiv\; 0 \bmod (x''-x) \tag{7.4}$$
$$(1-v)\Delta(x'',x) \;\equiv\; 0 \bmod (x-x') \tag{7.5}$$

Equation (7.4) is an equation in $\frac{R_c}{((x''-x))} \cong R_b$. The Tate resolution of R_a over R_b shows that $ann_{R_b}\Delta(x,x')$ is $(x-x')$, which implies that $1-u \in (x-x', x''-x)R_c$. So we can write $u = 1+\tilde{u}$, for some $\tilde{u} \in (x-x', x''-x)R_c$. Write $\tilde{u} = u_1(x-x') + u_2(x''-x)$, for some $u_1, u_2 \in R_c$. Then

$$\begin{aligned}
& \partial(u_1\tau'\sigma' + u_2\tau''\sigma') \\
= \ & u_1[(x-x')\sigma' - \tau'\Delta(x,x')\tau'] + u_2[(x''-x)\sigma' - \tau''\Delta(x,x')\tau'] \\
= \ & [u_1(x-x') + u_2(x''-x)]\sigma' + u_2\Delta(x,x')\tau'\tau'' \\
= \ & \tilde{u}\sigma' + u_2\Delta(x,x')\tau'\tau''
\end{aligned}$$

which implies that

$$\tilde{u}\sigma' \quad = \quad \partial(u_1\tau'\sigma' + u_2\tau''\sigma') - u_2\Delta(x,x')\tau'\tau''. \tag{7.6}$$

From our earlier setup, we have

$$\begin{aligned}
u\sigma' \quad & = \quad (1+\tilde{u})\sigma' \\
& = \quad \sigma' + \tilde{u}\sigma' \\
& \underbrace{=}_{\text{equation 7.6}} \quad \sigma' + (\partial(u_1\tau'\sigma' + u_2\tau''\sigma') - u_2\Delta(x,x')\tau'\tau'')
\end{aligned}$$

so we can replace $u\sigma'$ with σ' at the cost of adding a boundary and modifying the coefficient of $\tau'\tau''$.

Similarly by analyzing equation (7.5) we can replace $v\sigma''$ with σ'' at the cost of adding another boundary and further modifying the coefficient of $\tau'\tau''$.

We have shown that we may choose $\Phi_2(\sigma)$ in the desired form, and we are finished. $\square$

The determination of the coefficient of $\tau'\tau''$ is achieved by the following Lemma.

Lemma 7.3.8. *The diagram*

$$\begin{tikzcd}
\Delta(x'',x')\tau \arrow[r, mapsto] & \Delta(x'',x')(\tau'+\tau'') \\
\sigma \arrow[u] \arrow[r, mapsto] & \sigma' + \sigma'' + a(x',x,x'')\tau'\tau'' \arrow[u, "\partial_{\mathbb{F}\otimes\mathbb{F}}"']
\end{tikzcd}$$

commutes modulo $I^2 \cdot \mathbb{F}$ if and only if $a \equiv -f^{(2)}(x) \mod I$.

Proof. All congruences in this proof are modulo $I^2 \cdot \mathbb{F}$ unless otherwise stated.

First, assume that $a + f^{(2)}(x) \in I$. Then $(a + f^{(2)}(x))I \subseteq I^2$. To prove that the diagram commutes modulo $I^2 \cdot \mathbb{F}$, we must show that

$$\begin{aligned} & \Delta(x'', x')(\tau' + \tau'') \\ \equiv \;& \partial_{\mathbb{F}\otimes\mathbb{F}}(\sigma' + \sigma'' - f^{(2)}(x)\tau'\tau'') \\ = \;& \Delta(x, x')\tau' + \Delta(x'', x)\tau'' - f^{(2)}(x)[(x - x')\tau'' - \tau'(x'' - x)] \end{aligned} \tag{7.7}$$

so, equating coefficients of τ' and τ'' in equation (7.7), we are finished if we can prove both of

$$\Delta(x'', x') \equiv \Delta(x, x') + f^{(2)}(x)(x'' - x) \tag{7.8}$$

$$\Delta(x'', x') \equiv \Delta(x'', x) - f^{(2)}(x)(x - x') \tag{7.9}$$

<u>Proof of (7.8):</u> Applying Lemma 7.3.2 gives

$$\begin{aligned} \Delta(x'', x') &= f^{(1)}(x') + f^{(2)}(x')(x'' - x') + \text{ terms in } I^2 \\ \Delta(x, x') &= f^{(1)}(x') + f^{(2)}(x')(x - x') + \text{ terms in } \cdot \mathbb{F} \\ f^{(2)}(x)(x'' - x) &= f^{(2)}(x' + (x - x'))(x'' - x) \\ &= [f^{(2)}(x') + \text{ terms in } (x - x')](x'' - x) \\ &\equiv f^{(2)}(x')(x'' - x) \end{aligned}$$

so the RHS of (7.8) is congruent to

$$\begin{aligned} &= f^{(1)}(x') + f^{(2)}(x')(x - x') + f^{(2)}(x')(x'' - x) \\ &= f^{(1)}(x') + f^{(2)}(x')(x'' - x') \\ &\equiv \Delta(x'', x') \end{aligned}$$

as required.

<u>Proof of (7.9):</u> Applying Lemma 7.3.2 gives

$$\begin{aligned} \Delta(x'', x') &= \Delta(x', x'') \\ &= f^{(1)}(x'') + f^{(2)}(x'')(x' - x'') + \text{ terms in } I^2 \\ \Delta(x'', x) &= \Delta(x, x'') \\ &= f^{(1)}(x'') + f^{(2)}(x'')(x - x'') + \text{ terms in } I^2 \\ f^{(2)}(x)(x - x') &= f^{(2)}(x'' + (x - x''))(x - x') \\ &= [f^{(2)}(x'') + \text{ terms in } (x - x'')](x - x') \\ &\equiv f^{(2)}(x'')(x - x') \end{aligned}$$

so the RHS of (7.9) is congruent to

$$\begin{aligned} &= f^{(1)}(x'') + f^{(2)}(x'')(x - x'') - f^{(2)}(x'')(x - x') \\ &= f^{(1)}(x'') + f^{(2)}(x'')(x' - x'') \\ &\equiv \Delta(x'', x') \end{aligned}$$

as required.

We have shown that the diagram commutes modulo $I^2 \cdot \mathbb{F}$, as required.

Now assume that we have made a choice for the coefficient a which makes the diagram commute modulo $I^2 \cdot \mathbb{F}$. We will show that this requires $a \equiv -f^{(2)}(x) \bmod I$.

Applying the Leibniz rule gives

$$\partial(\tau'\tau'') \;=\; (x - x')\tau'' - \tau'(x'' - x) \tag{7.10}$$

Then by all the earlier definitions, for the diagram to commute modulo $I^2 \cdot \mathbb{F}$ we require

$$\begin{aligned} \Delta(x'', x')(\tau' + \tau'') &\equiv \partial(\sigma' + \sigma'' + a(x', x, x'')\tau'\tau'') \\ &= \Delta(x, x')\tau' + \Delta(x'', x)\tau'' + a(x - x')\tau'' - a\tau'(x'' - x) \end{aligned} \tag{7.11}$$

Equating the coefficients of τ' and τ'' in equation (7.11) gives the two equations

$$\Delta(x, x') - \Delta(x'', x') \;\equiv\; a(x'' - x) \tag{7.12}$$

$$\Delta(x'', x') - \Delta(x'', x) \;\equiv\; a(x - x') \tag{7.13}$$

In equation (7.12), setting $x'' = x$ kills the left hand side, thus we can find an a to satisfy the equation. In equation (7.13), setting $x' = x$ kills the left hand side, thus we can find an a to satisfy the equation. But these two choices of a might not agree with each other.

Applying Lemma 7.3.2 to the LHS of equation (7.12) gives

$$\begin{aligned} \Delta(x, x') &= f^{(1)}(x') + f^{(2)}(x')(x - x') + \text{ terms in } I^2 \\ \Delta(x'', x') &= f^{(1)}(x') + f^{(2)}(x')(x'' - x') + \text{ terms in } I^2 \\ \Rightarrow \Delta(x, x') - \Delta(x'', x') &\equiv f^{(2)}(x')(x - x'') \\ &= f^{(2)}(x + (x' - x))(x - x'') \\ &= [f^{(2)}(x) + \text{ terms in } (x' - x)](x - x'') \\ &\equiv f^{(2)}(x)(x - x'') \\ &\equiv -f^{(2)}(x)(x'' - x) \bmod I^2 \end{aligned} \tag{7.14}$$

Applying Lemma 7.3.2 to the LHS of equation (7.13) gives

$$\begin{aligned}
\Delta(x'', x') &= \Delta(x', x'') \\
&= f^{(1)}(x'') + f^{(2)}(x'')(x' - x'') + \text{ terms in } I^2 \\
\Delta(x'', x) &= \Delta(x, x'') \\
&= f^{(1)}(x'') + f^{(2)}(x'')(x - x'') + \text{ terms in } I^2 \\
\Rightarrow \Delta(x'', x') - \Delta(x'', x) &\equiv f^{(2)}(x'')(x' - x) \\
&= f^{(2)}(x + (x'' - x))(x' - x) \\
&= [f^{(2)}(x) + \text{ terms in } (x'' - x)](x' - x) \\
&\equiv f^{(2)}(x)(x' - x) \\
&\equiv -f^{(2)}(x)(x - x') \bmod I^2 \qquad (7.15)
\end{aligned}$$

Now comparing lines (7.12), (7.13) (7.14) and (7.15), we have the identities:

$$(a + f^{(2)}(x))(x - x') \equiv 0 \mod I^2 \qquad (7.16)$$

$$(a + f^{(2)}(x))(x'' - x) \equiv 0 \mod I^2 \qquad (7.17)$$

Define

$$y := a + f^{(2)}(x)$$

We are finished if we can prove that $y \in I$. We want to determine all the possible choices for y modulo I which simultaneously satisfy

$$y(x - x') \equiv 0 \mod I^2 \qquad (7.18)$$

$$y(x'' - x) \equiv 0 \mod I^2 \qquad (7.19)$$

We claim that any such y is of the form

$$y = u(x)(x - x') + v(x)(x'' - x) \mod I^2 \qquad (7.20)$$

for some $u(x), v(x)$. Using the two-variable Taylor expansion in variables (x', x'') centered at (x, x), we obtain

$$y = y_0(x) + y_1(x)(x - x') + y_2(x)(x'' - x) + \text{ terms in } I^2$$

With this notation, we must have that

$$y_0(x) = 0 \qquad (7.21)$$

If not, then notice that augmentation sends $x' \mapsto x; x'' \mapsto x; x \mapsto x$, so if $y_0(x)$ was not zero before augmentation, it will remain non-zero after augmentation. But then equations (7.18) and (7.19) would not hold. Thus $y_0(x) = 0$ must be true. Therefore the claim on line (7.20) also holds. But this says that

$$\begin{aligned} y &\in I && (7.22)\\ \Rightarrow a &\equiv -f^{(2)}(x) \mod I && (7.23)\end{aligned}$$

as required. □

Theorem 7.3.6 has exhibited one correct diagonal approximation. It is clear that the following corollary gives all the correct choices.

Corollary 7.3.9. *All choices for Φ are defined by:*

$$\begin{array}{lclcl} \Phi_0 & : & x' & \mapsto & x' \\ \Phi_0 & : & x'' & \mapsto & x'' \\ \Phi_1 & : & \tau & \mapsto & \tau' + \tau'' + \Delta(x'', x')\partial(\omega) \\ \Phi_2 & : & \sigma & \mapsto & \sigma' + \sigma'' - (f^{(2)}(x) + y)\tau'\tau'' + \Delta(x'', x')\omega + \partial(\eta) \end{array}$$

where $\omega \in (\mathbb{F} \otimes_{R_a} \mathbb{F})_2$ satisfies $\epsilon_1(\omega) = 0 = \epsilon_2(\omega)$, $\eta \in (\mathbb{F} \otimes_{R_a} \mathbb{F})_3$ satisfies $\epsilon_1(\eta) = 0 = \epsilon_2(\eta)$ (to ensure the diagram in Definition 7.3.5 will still commute in degrees 1 and 2), and $y \in I$.

Now we have established a diagonal approximation and can use it as before to determine the multiplication in the *Ext* algebra.

7.4 Several Monic Polynomials

We now generalize the setup from the previous section to several monic polynomials. We will later specialize to the case of the group ring for a finite abelian group $G = \mu_{h_1} \times \cdots \times \mu_{h_r}$.

7.4.1 A Diagonal Approximation

To generalize the definitions of R_a, R_b and R_c from one variable to r-many variables, we now define

$$\begin{aligned}
S_a &= \frac{R[x_1,\ldots,x_r]}{(f_i(x_i); 1 \leq i \leq r)}, \text{ where each } f_i(x_i) \text{ is monic} \\
S_b &= S_a \otimes_R S_a \\
&\cong \frac{R[x_1', x_1'', \ldots, x_r', x_r'']}{(f_1(x_1'), f_1(x_1''), \ldots, f_r(x_r'), f_r(x_r''))} \\
S_c &= S_b \otimes_{S_a} S_b \\
&\cong \frac{R[x_1', x_1, x_1'', \ldots, x_r', x_r, x_r'']}{(f_1(x_1'), f_1(x_1), f_1(x_1''), \ldots, f_r(x_r'), f_r(x_r), f_r(x_r''))}
\end{aligned}$$

Now for $1 \leq i \leq r$, define the ideals

$$I_i = (x_i - x_i', x_i'' - x_i) \subset S_c.$$

In generalization of the results from the previous section we will obtain a Tate resolution $\mathbb{F} \longrightarrow S_a$ over S_b, and a diagonal approximation $\Phi : \mathbb{F} \to \mathbb{F} \otimes_{S_a} \mathbb{F}$. (In exact analogy to the previous section, we can form the tensor product $\mathbb{F} \otimes_{S_a} \mathbb{F}$.) That diagonal approximation will be a DG-algebra homomorphism, so that we get an analogous commutative diagram of complexes of S_b-modules to that in Theorem 7.3.6.

$$\begin{aligned}
\mathbb{F} &= S_b\langle \tau_1, \ldots, \tau_r \; ; \; \sigma_1, \ldots, \sigma_r\rangle, \; |\tau_i| = 1; \; |\sigma_i| = 2 \\
\partial &= \sum_{i=1}^{r} \left[(x_i'' - x_i') \frac{\partial}{\partial \tau_i} + \Delta_i(x_i'', x_i')\tau_i \frac{\partial}{\partial \sigma_i} \right], \text{ where } \Delta_i(x_i'', x_i') = \frac{f_i(x_i'') - f_i(x_i')}{x_i'' - x_i'} \\
\Phi_0(x_i') &= x_i' \\
\Phi_0(x_i'') &= x_i'' \\
\Phi_1(\tau_i) &= \tau_i'' + \tau_i' \\
\Phi_2(\sigma_i) &= \sigma_i' + \sigma_i'' - (f_i^{(2)}(x_i) + y_i)\tau_i'\tau_i'' \text{ where } y_i \in I_i
\end{aligned}$$

and these assignments determine a unique homomorphism of algebras with divided powers.

7.4.2 The Dual of the Tate Resolution

Let A be an S_b-module on which every $(x''_i - x'_i)$ acts as 0, equivalently, A is a symmetric S_b-module, equivalently, A is an S_a-module. Then dualizing $\mathbb{F}$ into A via $Hom_{S_b}(-, A)$ (and denoting $Hom_{S_b}(\mathbb{F}, S_b)$ by $\mathbb{F}^*$) gives

$$
\begin{aligned}
& Hom_{S_b}(\mathbb{F}, A) && (7.24)\\
\underbrace{\cong}_{\text{by } 2.2.3}\quad & A \otimes_{S_b} \mathbb{F}^* && (7.25)\\
\cong\quad & \left(A \otimes_{S_b} S_b[s_1, \ldots, s_r] \otimes_{S_b} \bigwedge_{S_b} \langle t_1, \ldots, t_r\rangle, \partial = \sum_{i=1}^{r} f'_i(x_i) s_i \frac{\partial}{\partial t_i} \right) && (7.26)\\
\underbrace{\cong}_{A \text{ is an } S_a\text{-module}}\quad & \left(A \otimes_{S_a} S_a[s_1, \ldots, s_r] \otimes_{S_a} \bigwedge_{S_a} \langle t_1, \ldots, t_r\rangle, \partial = \sum_{i=1}^{r} f'_i(x_i) s_i \frac{\partial}{\partial t_i} \right) && (7.27)
\end{aligned}
$$

where s_i is a polynomial variable dual to σ_i, and t_i is dual to τ_i.

The dualized differential is correct because, under $Hom_{S_b}(-, A)$,

- $(x''_i - x'_i) \mapsto 0$, and
- $\Delta_i(x''_i, x'_i) \mapsto f'_i(x_i)$.

The actual action of ∂^* is determined in exactly the same way as in the case of group algebras. As there, we may, temporarily, think of $A \otimes_{S_a} S_a(\mathbf{s}) \otimes_{S_a} \bigwedge_{S_a}(\mathbf{t})$ as a Koszul complex, so that the differential can be written in this compact form.

7.4.3 The Action of ∂^*

As in Chapter 5, $\mathbb{F}^* = Hom_{S_b}(\mathbb{F}, S_a)$ is a Hom complex. Therefore its differential is determined by Definition 2.5.1, and obeys the Leibniz rule.

An S_b-basis for $\mathbb{F}$ is given by monomials $\omega = \tau^K \sigma^{(N)}$, where

- $K = (K_1, \ldots, K_r)$ records the exterior powers of the τs which are present, i.e. $\tau^K = \tau_1^{K_1} \cdots \tau_r^{K_r}$. Note that $K_n \in \{0, 1\}$ for all n.
- $N = (N_1, \ldots, N_r) \in \mathbb{N}^r$ records the divided powers of the σs which are present, i.e. $\sigma^{(N)} = \sigma_1^{(N_1)} \cdots \sigma_r^{(N_r)}$.

Analogously to Chapter 5, we define the S_a-dual basis elements for $Hom_{S_b}(\mathbb{F}, A)$ to be $S^L T_M$, where

$$\begin{aligned} L &= (L_1, \ldots, L_r), \\ M &= (M_1, \ldots, M_r), \end{aligned}$$

and $S^L T_M$ evaluates to 1 on $\tau^M \sigma^{(L)}$, and evaluates to 0 on all other basis elements of $\mathbb{F}$. Note that each $M_n \in \{0, 1\}$ for all n, since there are no other possibilities for the corresponding τs.

Now to determine the effect of ∂^* on an arbitrary $S^L T_M$, we evaluate

$$\begin{aligned} & \partial^*(S^L T_M)(\tau^K \sigma^{(N)}) \\ = \; & \underbrace{d_R}_{=0}(S^L T_M)(\tau^K \sigma^{(N)}) - (-1)^{|S^L T_M|} S^L T_M d_{\mathbb{F}}(\tau^K \sigma^{(N)}) && (7.28) \\ = \; & -(-1)^{|T_M|} S^L T_M \left(\sum_{i=1}^{r} (x_i'' - x_i') \frac{\partial \tau^K}{\partial \tau_i} \sigma^{(N)} + \Delta_i(x_i'', x_i') \tau_i \tau^K \frac{\partial \sigma^{(N)}}{\partial \sigma_i} \right) && (7.29) \\ = \; & -(-1)^{\sum_{n=1}^{r} M_n} S^L T_M \left(\sum_{i=1}^{r} (x_i'' - x_i') \frac{\partial \tau^K}{\partial \tau_i} \sigma^{(N)} + (-1)^{\sum_{\nu<i} K_\nu} \Delta_i(x_i'', x_i') \tau^{K_i'} \frac{\partial \sigma^{(N)}}{\partial \sigma_i} \right) && (7.30) \end{aligned}$$

where we define

$$K_i' = K + (0, \ldots, 0, \underbrace{1}_{\text{position } i}, 0, \ldots, 0)$$

The expression on line (7.30) is congruent, modulo $I \cdot \mathbb{F}$, to

$$-(-1)^{\sum_{n=1}^{r} M_n} S^L T_M \left(\sum_{i=1}^{r} (-1)^{\sum_{\nu<i} K_\nu} f_i'(x_i) \tau^{K_i'} \frac{\partial \sigma^{(N)}}{\partial \sigma_i} \right) \tag{7.31}$$

By the definition of $S^L T_M$, The i^{th} term of expression (7.31) evaluates to 0 unless

- $K_i' = K + (0, \ldots, 0, \underbrace{1}_{\text{position } i}, 0, \ldots, 0) = M$, and
- $N' = N - (0, \ldots, 0, \underbrace{1}_{\text{position } i}, 0, \ldots, 0) = L$,

in which case it evaluates to $-(-1)^{\sum_{n=1}^r M_n}(-1)^{\sum_{\nu<i} K_\nu} f_i'(x_i)$.

Therefore we have

$$\begin{aligned} & \partial^*(S^L T_M) \\ = \; & -(-1)^{\sum_{n=1}^r M_n}\left(\sum_{i=1}^r (-1)^{\sum_{\nu<i} K_\nu} f_i'(x_i) S^{L+(0,\ldots,0,\underbrace{1}_{i},0,\ldots,0)} T_{M-(0,\ldots,0,\underbrace{1}_{i},0,\ldots,0)}\right) \end{aligned} \tag{7.32}$$

We may, temporarily using the algebra structure of the Koszul complex, rewrite the differential from line (7.32) in compact form as

$$\begin{aligned} & \partial^*(S^L T_M) \\ = \; & -(-1)^{\sum_{n=1}^r M_n} S^L \sum_{i=1}^r f_i'(x_i) S_i \frac{\partial T_M}{\partial T_i} \end{aligned} \tag{7.33}$$

The next Theorem says that we can replace the above differential with a simpler one, and preserve the original cohomology groups.

Theorem 7.4.1. *If we change the differential to*

$$\begin{aligned} & \partial'(S^L T_M) \\ = \; & S^L \sum_{i=1}^r f_i'(x_i) S_i \frac{\partial T_M}{\partial T_i} \end{aligned} \tag{7.34}$$

then we will still have the same cohomology groups.

Proof. This is proved in a way which is exactly analogous to the proof of Theorem 5.5.1. □

7.4.4 Cochain Products

Theorem 7.4.2. *Now let A be an S_a-algebra. The above choice of Φ defines the following multiplicative structure on $Hom_{S_b}(\mathbb{F}, A)$, which makes it into a DG-algebra, where s_i is a polynomial variable dual to σ_i, and t_j is dual to τ_j:*

$$t_i \cup t_i = f_i^{(2)}(x_i) s_i \tag{7.35}$$

$$t_i \cup t_j + t_j \cup t_i = 0, \text{ when } i \neq j \tag{7.36}$$

$$t_j \cup s_i = s_i \cup t_j \tag{7.37}$$

$$s_j \cup s_i = s_i \cup s_j \tag{7.38}$$

(where $f_i^{(2)}(x_i)$ represents now the image of that element from S_a in A) and the elements t_j and s_i generate $Hom_{S_b}(\mathbb{F}, A)$ with respect to the cup product, subject only to these relations.

Proof. For this section, unadorned tensor products are over S_a.

In complete analogy to Chapter 5, $Hom_{S_b}(\mathbb{F}, A)$ is a DG-algebra.

1. $\underline{t_i \cup t_i = f_i^{(2)}(x_i)s_i}$: We have $t_i \cup t_i = \mu(t_i \otimes t_i)\Phi_{1,1}$, and $\Phi_{1,1} : \mathbb{F}_2 \to \mathbb{F}_1 \otimes \mathbb{F}_1$. We only need to look at index i here, since applying $(t_i \otimes t_i)$ will kill all other indices. So in degree 2, the only basis element in the domain that we need to look at is σ_i. Recall that

$$\begin{aligned} \Phi(\sigma_i) &= \sigma_i' + \sigma_i'' - f_i^{(2)}(x_i)\tau_i'\tau_i'' \\ \Rightarrow \Phi_{1,1}(\sigma_i) &= -f_i^{(2)}(x_i)(\tau_i \otimes \tau_i). \end{aligned}$$

   Applying $\mu(t_i \otimes t_i)$ gives

$$\begin{aligned} & \mu(t_i \otimes t_i)(-f_i^{(2)}(x_i)(\tau_i \otimes \tau_i)) \\ = & -f_i^{(2)}(x_i)\mu(t_i \otimes t_i)(\tau_i \otimes \tau_i) \\ = & -f_i^{(2)}(x_i)(-1)\mu(t_i(\tau_i) \otimes t_i(\tau_i)) \\ = & f_i^{(2)}(x_i)\mu(1 \otimes 1) \\ = & f_i^{(2)}(x_i) \end{aligned}$$

   Thus $t_i \cup t_i$ evaluates to 0 on every basis element except σ_i, on which it evaluates to $f_i^{(2)}(x_i)$. Therefore $t_i \cup t_i = f_i^{(2)}(x_i)s_i$, as required.

2. $\underline{t_i \cup t_j + t_j \cup t_i = 0, \text{ when } i \neq j}$: We have $t_i \cup t_j = \mu(t_i \otimes t_j)\Phi_{1,1}$, $t_j \cup t_i = \mu(t_j \otimes t_i)\Phi_{1,1}$ and $\Phi_{1,1} : \mathbb{F}_2 \to \mathbb{F}_1 \otimes \mathbb{F}_1$. The only basis elements for which this can evaluate to something non-zero are $\tau_i\tau_j$ and $\tau_j\tau_i$. Since $\tau_i\tau_j = -\tau_j\tau_i$, it suffices to examine the effect of $\mu(t_i \otimes t_j)$ and $\mu(t_j \otimes t_i)$ on $\Phi(\tau_i\tau_j)$. So we compute:

$$\begin{aligned} & \Phi_{1,1}(\tau_i\tau_j) \\ = & \Phi_{1,1}(\tau_i)\Phi_{1,1}(\tau_j) \\ = & (\tau_i' + \tau_i'')(\tau_j' + \tau_j'') \\ = & \tau_i'\tau_j' + \tau_i'\tau_j'' + \tau_i''\tau_j' + \tau_i''\tau_j'' \end{aligned}$$

As $(t_i \otimes t_j)$ vanishes on all occurring monomials except $\tau_i'\tau_j''$, applying $\mu(t_i \otimes t_j)$ gives

$$\begin{aligned}
& \mu(t_i \otimes t_j)(\tau_i'\tau_j'') \\
= \; & \mu(-1)(t_i(\tau_i) \otimes t_j(\tau_j)) \\
= \; & -\mu(1 \otimes 1) \\
= \; & -1
\end{aligned}$$

Similarly, applying $\mu(t_j \otimes t_i)$ gives

$$\begin{aligned}
& \mu(t_j \otimes t_i)(-\tau_j'\tau_i'') \\
= \; & -\mu(-1)(t_j(\tau_j) \otimes t_i(\tau_i)) \\
= \; & 1
\end{aligned}$$

Thus the relation $t_i \cup t_j + t_j \cup t_i = 0$ is proved.

3. $\underline{t_j \cup s_i = s_i \cup t_j}$: We have $s_i \cup t_j = \mu(s_i \otimes t_j)\Phi_{2,1}$ and $\Phi_{2,1} : \mathbb{F}_3 \to \mathbb{F}_2 \otimes \mathbb{F}_1$. Also, $t_j \cup s_i = \mu(t_j \otimes s_i)\Phi_{1,2}$ and $\Phi_{1,2} : \mathbb{F}_3 \to \mathbb{F}_1 \otimes \mathbb{F}_2$. The only basis elements for which this can evaluate to something non-zero are $\tau_j'\sigma_i''$ and $\sigma_i'\tau_j''$. Since $\sigma_i\tau_j = \tau_j\sigma_i$, it suffices to examine the effect of $\mu(t_j \otimes s_i)$ and $\mu(s_i \otimes t_j)$ on $\Phi(\tau_j\sigma_i)$. So we compute:

$$\begin{aligned}
& \Phi(\tau_j\sigma_i) \\
= \; & \Phi(\tau_j)\Phi(\sigma_i) \\
= \; & (\tau_j' + \tau_j'')\left(\sigma_i' + \sigma_i'' - f_i^{(2)}(x_i)\tau_i'\tau_i''\right) \\
= \; & \tau_j'\sigma_i' + \tau_j'\sigma_i'' - f_i^{(2)}(x_i)\tau_j'\tau_i'\tau_i'' + \tau_j''\sigma_i' + \tau_j''\sigma_i'' - f_i^{(2)}(x_i)\tau_j''\tau_i'\tau_i''
\end{aligned}$$

As $(t_j \otimes s_i)$ vanishes on all occurring monomials except $\tau_j'\sigma_i''$, applying $\mu(t_j \otimes s_i)$ gives

$$\begin{aligned}
& \mu(t_j \otimes s_i)(\tau_j \otimes \sigma_i) \\
= \; & \mu(t_j(\tau_j) \otimes s_i(\sigma_i)) \\
= \; & \mu(1 \otimes 1) \\
= \; & 1
\end{aligned}$$

Similarly, applying $\mu(s_i \otimes t_j)$ gives

$$\begin{aligned}
& \mu(s_i \otimes t_j)(\sigma_i \otimes \tau_j) \\
= \; & \mu(s_i(\sigma_i) \otimes t_j(\tau_j)) \\
= \; & \mu(1 \otimes 1) \\
= \; & 1
\end{aligned}$$

Thus the relation $t_j \cup s_i = s_i \cup t_j$ is proved.

4. $\underline{s_j \cup s_i = s_i \cup s_j}$: We have $s_i \cup s_j = \mu(s_i \otimes s_j)\Phi_{2,2}$ and $\Phi_{2,2} : \mathbb{F}_4 \to \mathbb{F}_2 \otimes \mathbb{F}_2$. Also $s_j \cup s_i = \mu(s_j \otimes s_i)\Phi_{2,2}$. The only basis elements for which this can evaluate to something non-zero are $\sigma'_j\sigma''_i$ and $\sigma'_i\sigma''_j$. Since $\sigma_j\sigma_i = \sigma_i\sigma_j$, it suffices to examine the effect of $(s_j \otimes s_i)$ and $(s_i \otimes s_j)$ on $\Phi(\sigma_j\sigma_i)$. So we compute:

$$\begin{aligned}
& \Phi(\sigma_j\sigma_i) \\
= \; & \Phi(\sigma_j)\Phi(\sigma_i) \\
= \; & \left(\sigma'_j + \sigma''_j - f_j^{(2)}(x_j)\tau'_j\tau''_j\right)\left(\sigma'_i + \sigma''_i - f_i^{(2)}(x_i)\tau'_i\tau''_i\right) \\
= \; & \sigma'_j\sigma'_i + \sigma'_j\sigma''_i - f_i^{(2)}(x_i)\sigma'_j\tau'_i\tau''_i \\
& + \sigma''_j\sigma'_i + \sigma''_j\sigma''_i - f_i^{(2)}(x_i)\sigma''_j\tau'_i\tau''_i \\
& - f_j^{(2)}(x_j)\tau'_j\tau''_j\sigma'_i - f_j^{(2)}(x_j)\tau'_j\tau''_j\sigma''_i + \left(f_j^{(2)}(x_j)\right)\left(f_i^{(2)}(x_i)\right)\tau'_j\tau''_j\tau'_i\tau''_i
\end{aligned}$$

As $(s_j \otimes s_i)$ vanishes on all occurring monomials except $\sigma'_j\sigma''_i$, applying $\mu(s_j \otimes s_i)$ gives

$$\begin{aligned}
& \mu(s_j \otimes s_i)(\sigma_j \otimes \sigma_i) \\
= \; & \mu(s_j(\sigma_j) \otimes s_i(\sigma_i)) \\
= \; & \mu(1 \otimes 1) \\
= \; & 1
\end{aligned}$$

Similarly, applying $\mu(s_i \otimes s_j)$ gives

$$\begin{aligned}
& \mu(s_i \otimes s_j)(\sigma_i \otimes \sigma_j) \\
= \; & \mu(s_i(\sigma_i) \otimes s_j(\sigma_j)) \\
= \; & \mu(1 \otimes 1) \\
= \; & 1
\end{aligned}$$

Thus the relation $s_j \cup s_i = s_i \cup s_j$ is proved.

The proof that these relations completely determine the algebra structure is analogous to the proof of this fact given for Theorem 5.6.1. □

Remark: Theorem 7.4.2 shows in particular that

$$Hom_{S_b}(\mathbb{F}, A) \cong Hom_{S_b}(\mathbb{F}, S_a) \otimes_{S_a} A \text{ ,as } DG\text{-algebras.}$$

We make one further observation about this setup.

The Algebra Structure Is The Tensor Product of Individual Algebra Structures

Define

$$\begin{aligned} R_i &= \frac{R[x_i]}{(f_i(x_i))},\ 1 \leq i \leq r \\ \mathbb{F}_i &= R_i^{ev}\langle \tau_i, \sigma_i \rangle,\ 1 \leq i \leq r \end{aligned}$$

Then we have

$$\begin{aligned} \mathbb{F} &= \mathbb{F}_1 \otimes_R \cdots \otimes_R \mathbb{F}_r \\ S_a &= \frac{R[x_1, \ldots, x_r]}{(f_i(x_i); 1 \leq i \leq r)}, \text{ where each } f_i(x_i) \text{ is monic} \\ &\cong R_1 \otimes_R \cdots \otimes_R R_r \\ \Phi &= \Phi_1 \otimes \cdots \otimes \Phi_r \end{aligned}$$

where Φ_i denotes the part of Φ that lives in factor i.

Theorem 7.4.3. *As algebras, we have*

$$Hom_{S_b}(\mathbb{F}, R) \cong Hom_{R_1^{ev}}(\mathbb{F}_1, R) \otimes_R \cdots \otimes_R Hom_{R_r^{ev}}(\mathbb{F}_r, R)$$

Proof. The proof is by induction on r. All tensor products are over R.

<u>Base ($r = 1$):</u> There is nothing to prove.

<u>Induction:</u> Assume, for some $1 \leq k < r$, that we have, as algebras:

$$Hom_{S_{b_k}}(\mathbb{F}_1 \otimes \cdots \otimes \mathbb{F}_k, R) \cong Hom_{R_1^{ev}}(\mathbb{F}_1, R) \otimes \cdots \otimes Hom_{R_k^{ev}}(\mathbb{F}_k, R)$$

and we want to prove that

$$Hom_{S_{b_{k+1}}}(\mathbb{F}_1 \otimes \cdots \otimes \mathbb{F}_{k+1}, R) \cong Hom_{R_1^{ev}}(\mathbb{F}_1, R) \otimes \cdots \otimes Hom_{R_{k+1}^{ev}}(\mathbb{F}_{k+1}, R)$$

Define

$$\begin{aligned} Hom_{S_{b_k}}(\mathbb{F}_1 \otimes \cdots \otimes \mathbb{F}_k, R) \times Hom_{R_{k+1}^{ev}}(\mathbb{F}_{k+1}, R) &\xrightarrow{\Psi} Hom_{S_{b_{k+1}}}(\mathbb{F}_1 \otimes \cdots \otimes \mathbb{F}_{k+1}, R) \\ (f, g) &\longmapsto \mu \circ (f \otimes g) \end{aligned}$$

Then Ψ is R-bilinear, so we get a map

$$\begin{array}{ccc} Hom_{S_{b_k}}(\mathbb{F}_1 \otimes \cdots \otimes \mathbb{F}_k, R) \otimes Hom_{R^{ev}_{k+1}}(\mathbb{F}_{k+1}, R) & \xrightarrow{\tilde{\Psi}} & Hom_{S_{b_{k+1}}}(\mathbb{F}_1 \otimes \cdots \otimes \mathbb{F}_{k+1}, R) \\ f \otimes g & \longmapsto & \mu \circ (f \otimes g) \end{array}$$

From the setup, $\tilde{\Psi}$ is a bijection, so we have the isomorphism of modules.

Now because the algebra structure is defined by Φ, which decomposes by index as above, we can see that $\tilde{\Psi}$ is also an isomorphism of algebras, as required. $\square$

7.5 Applications

7.5.1 Comparison With Chapter 5

We now show that Theorem 7.4.2 is a generalization of Theorem 5.6.1 with the refinement given in Proposition 5.6.4. Let $G = \mu_{h_1} \times \cdots \times \mu_{h_r}$ be a product of cyclic groups of orders $h_1, \ldots, h_r$. Then define the corresponding monic polynomials

$$f_i(x_i) = x_i^{h_i} - 1, \ 1 \leq i \leq r.$$

With these defining polynomials, we have that $S_a \cong RG$ and $S_b \cong RG^{ev}$. Recall that the resolution $\mathbb{F}$ of Theorem 7.4.2 resolves $S_a \cong RG$ over $S_b \cong RG^{ev}$. Let $A = R$. We may regard R as a module over S_a, where each x_i acts as 1. Thus Theorem 7.4.2 applies and gives the multiplicative structure on $Hom_{RG^{ev}}(\mathbb{F}, R)$.

Now recall that:

1. We may regard R as an RG-module with trivial G-action.
2. Similarly, we may regard R as an RG^{ev}-module with trivial G-action from each copy of RG.
3. The ring homomorphism defined by

$$\begin{array}{cccl} \varphi & : & RG^{ev} & \to & RG \\ & & x'_i & \mapsto & x_i, \ 1 \leq i \leq r \\ & & x''_i & \mapsto & x_i, \ 1 \leq i \leq r \end{array}$$

turns RG into an RG^{ev}-module.

4. Using the above ring homomorphism, we see that the terms of the complex $RG \otimes_{RG^{ev}} \mathbb{F}$ are free RG-modules, and thus this complex resolves R over RG.

By the Adjoint Isomorphism (e.g. Theorem 8.99 in [14]), we have

$$\begin{aligned} Hom_{RG^{ev}}(\mathbb{F}, Hom_{RG}(RG, R)) &\cong Hom_{RG}(RG \otimes_{RG^{ev}} \mathbb{F}, R), \text{ which can be re-written} \\ Hom_{RG^{ev}}(\mathbb{F}, R) &\cong Hom_{RG}(RG \otimes_{RG^{ev}} \mathbb{F}, R) \end{aligned}$$

Since the RHS computes $Ext^{\bullet}_{RG}(R, R)$, then so does the LHS. In this way we can interpret the multiplicative structure given by Theorem 7.4.2 in $Ext^{\bullet}_{RG}(R, R)$.

Now apply Theorem 5.6.1 to obtain the multiplicative structure of $Ext^{\bullet}_{RG}(R, R)$ directly. It is already clear that the multiplicative structures coming from the two theorems agree, except possibly in the rule for $t_i \cup t_i$.

From the polynomials defined above, we have

$$f_i^{(2)}(x_i) = \binom{h_i}{2} x_i^{h_i - 2}, \ 1 \leq i \leq r.$$

So by Theorem 7.4.2, we have the cup product $t_i \cup t_i = \binom{h_i}{2}$ in $Hom_{RG^{ev}}(\mathbb{F}, R) \cong Hom_{RG}(RG \otimes_{RG^{ev}} \mathbb{F}, R)$, since our S_a-module structure for R comes from letting each x_i act as 1.

Now, analogously to Proposition 4.6.3, we may apply the correction term $-\left\lfloor \frac{h_i - 1}{2} \right\rfloor \tau_i' \tau_i''$ to $\Phi_2(\sigma_i)$. By doing this, we obtain the modified multiplication rule

$$t_i \cup t_i = \begin{cases} \frac{h_i}{2} \cdot s_i & \text{if} \quad h_i \text{ is even} \\ 0 & \text{if} \quad h_i \text{ is odd,} \end{cases}$$

and we see that the two multiplicative structures coincide.

So we recover Theorem 5.6.1 with the refinement given in Proposition 5.6.4, as claimed.

7.5.2 Extending Results of Holm

We are now able to extend some results of Holm from [11].

Theorem 7.5.1. *Let R be a commutative ring. Let $f(x) \in R[x]$ be monic. Define $d(x) = gcd(f(x), f'(x))$, computed in $R[x]$. Let $q(x) \in R[x]$ satisfy $f = qd$.*

Then

$$HH^\bullet\left(\frac{R[x]}{(f(x))}\right) \cong \frac{R[x,\lambda,s]}{(f(x),\ d(x)\lambda,\ f'(x)s,\ \lambda^2 - q(x)^2 f^{(2)}(x)s)},\ \textit{where } |x| = 0, |\lambda| = 1, |s| = 2.$$

Proof. In the special case where $r = 1$, we recall our original notation:

$$\begin{aligned} R_a &:= \frac{R[x]}{(f(x))} \\ R_b &:= R_a \otimes_R R_a, \end{aligned}$$

and we get the additive structure

$$\begin{aligned} & Hom_{R_b}(\mathbb{F}, R_a) \\ \underbrace{\cong}_{\text{by } 2.2.3} \quad & R_a \otimes_{R_b} \mathbb{F}^* \\ \cong \quad & \left(R_a \otimes_{R_b} R_b[s] \otimes_{R_b} \bigwedge_{R_b}\langle t\rangle, \partial^* = f'(x)s\frac{\partial}{\partial t}\right) \\ \underbrace{\cong}_{R_b \text{ is an } R_a\text{-module}} \quad & \left(R_a[s] \otimes_{R_a} \bigwedge_{R_a}\langle t\rangle, \partial^* = f'(x)s\frac{\partial}{\partial t}\right). \end{aligned}$$

We compute the kernel of ∂^* in degree 1. We write square brackets to denote a class in $\frac{R[x]}{(f(x))}$. Let $[h]t$ be a cycle in degree 1, for some $[h] \in R_a = \frac{R[x]}{(f(x))}$. Then, modulo (f), we have

$$\begin{aligned} 0 &\equiv \partial^*([h]t) \\ &= [h]f's, \end{aligned}$$

therefore, f must divide hf', and since $d = gcd(f, f')$, therefore q must divide h. Thus the cycles in degree 1 are generated by $[q] \in \frac{R[x]}{(f(x))}$. So we may choose $\lambda = [q]t$ as our generator in degree 1. Then it is clear that $[d]$ generates the annihilator of λ in R_a. We have the cochain product defined by $t^2 = f^{(2)}s$. With the above choice for λ, this becomes

$$\begin{aligned} \lambda^2 &= q^2t^2 \\ &= q^2 f^{(2)}s \end{aligned}$$

whence the relation $\lambda^2 - q(x)^2 f^{(2)}(x)s$ is established.

In degree 2, the kernel of ∂^* is everything since the differential is zero. So we may choose s as our generator in degree 2. We have the relation $f'(x)s$ because $\partial^*(t) = f'(x)s$.

Thus the desired structure is established. $\square$

Remarks:

1. Theorem 7.5.1 implies Theorem 3.2, Lemma 4.1, Lemma 5.1, Theorem 5.2 and Theorem 6.2 from [11].
2. Theorem 7.5.1 also completes the characteristic 2 case which Holm did not handle.
3. Theorem 7.5.1 implies Theorem 3.9 from [15].
4. Often it will happen that 2 is a non zero divisor in $\frac{R[x]}{(f(x))}$. When this happens the presentation above can be simplified, as the next Lemma and Corollary show.

Lemma 7.5.2. *With the above notation,* $2q(x)^2 f^{(2)}(x) \equiv 0 \bmod (f(x))$*, in other words* $q(x)^2 f^{(2)}(x)$ *lies in the 2-torsion of* $\frac{R[x]}{(f(x))}$.

Proof. Recall that $f = qd$, so that

$$qf' \;=\; f\left(\frac{f'}{d}\right). \tag{7.39}$$

Also, we have that $2f^{(2)} = f''$, so that

$$\begin{aligned}
2q^2 f^{(2)} &= q^2 f'' \\
&= q(qf'') \\
&= q((qf')' - q'f'), \text{ by the product rule} \\
&= q\left(\left(f\left(\frac{f'}{d}\right)\right)' - q'f'\right), \text{ by } 7.39 \\
&= q\left(f'\left(\frac{f'}{d}\right) + f\left(\frac{f'}{d}\right)' - q'f'\right) \\
&= f\left(\frac{f'}{d}\left(\frac{f'}{d}\right) + q\left(\frac{f'}{d}\right)' - q'\frac{f'}{d}\right), \text{ again by } 7.39 \\
&\equiv 0 \bmod (f)
\end{aligned}$$

□

Remark: Lemma 7.5.2 implies that the sign of the $q(x)^2 f^{(2)}(x)$ term in the statement of Theorem 7.5.1 does not matter.

Corollary 7.5.3. *If* 2 *is a non zero divisor in* $\frac{R[x]}{(f(x))}$, *then*

$$HH^\bullet\left(\frac{R[x]}{(f(x))}\right) \cong \frac{R[x,\lambda,s]}{(f(x),\ d(x)\lambda,\ f'(x)s,\ \lambda^2)},\ \textit{where}\ |x|=0, |\lambda|=1, |s|=2.$$

Proof. Start with the result of Theorem 7.5.1. By Lemma 7.5.2, $2q(x)^2 f^{(2)}(x) = 0$ in $\frac{R[x]}{(f(x))}$. Since 2 is a non zero divisor in $\frac{R[x]}{(f(x))}$, therefore $q(x)^2 f^{(2)}(x) = 0$ in $\frac{R[x]}{(f(x))}$. Thus the relation $\lambda^2 - q(x)^2 f^{(2)}(x)$ simplifies to λ^2 and we are done. □

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