The Eberlein Compactification of Locally Compact Groups

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

A compact semigroup is, roughly, a semigroup compactification of a locally compact group if it contains a dense homomorphic image of the group. The theory of semigroup compactifications has been developed in connection with subalgebras of continuous bounded functions on locally compact groups.

The Eberlein algebra of a locally compact group is defined to be the uniform closure of its Fourier-Stieltjes algebra. In this thesis, we study the semigroup compactification associated with the Eberlein algebra. It is called the Eberlein compactification and it can be constructed as the spectrum of the Eberlein algebra.

The algebra of weakly almost periodic functions is one of the most important function spaces in the theory of topological semigroups. Both the weakly almost periodic functions and the associated weakly almost periodic compactification have been extensively studied since the 1930s. The Fourier-Stieltjes algebra, and hence its uniform closure, are subalgebras of the weakly almost periodic functions for any locally compact group. As a consequence, the Eberlein compactification is always a semitopological semigroup and a quotient of the weakly almost periodic compactification.

We aim to study the structure and complexity of the Eberlein compactifications. In particular, we prove that for certain Abelian groups, weak*-closed subsemigroups of $L^\infty[0,1]$

may be realized as quotients of their Eberlein compactifications, thus showing that both the Eberlein and weakly almost periodic compactifications are large and complicated in these situations. Moreover, we establish various extension results for the Eberlein algebra and Eberlein compactification and observe that levels of complexity of these structures mimic those of the weakly almost periodic ones. Finally, we investigate the structure of the Eberlein compactification for a certain class of non-Abelian, Heisenberg type locally compact groups and show that aspects of the structure of the Eberlein compactification can be relatively simple.

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Dedication

Dedicated to:

A. Bülent, Ceyhun E. and Hülya.

Contents

Chapter 1

Introduction

The theory of unitary representations of locally compact groups was initiated in 1940s. At first various researchers began looking at the structure of abstract representations and concrete representation theory for specific groups. In [22], Eymard defined the Fourier-Stieltjes algebra $B(G)$ as the space of coefficient functions of unitary representations of a locally compact group, and studied many properties of $B(G)$. Eymard characterized $B(G)$ as the Banach dual of the group C^*-algebra, $C^*(G)$. Equipped with the norm from this duality $B(G)$ becomes a Banach algebra on its own. In fact, $B(G)$ is naturally a subalgebra of $\mathcal{C}_b(G)$, the continuous, bounded, complex valued functions on G. $B(G)$ is a proper subspace of $\mathcal{C}_b(G)$ and fails to be uniformly closed if and only if the locally compact group G is infinite. The uniform closure of $B(G)$ is called the *Eberlein algebra*, and denoted by $\mathcal{E}(G)$.

The Eberlein algebra contains the algebra of almost periodic functions, $AP(G)$, which correspond to the uniformly closed algebra generated by coefficient functions of finite dimensional representations. Furthermore, for a locally compact group G, $\mathcal{E}(G)$ is contained in the algebra of weakly almost periodic functions $WAP(G)$, and hence in the algebra of left uniformly continuous functions $LUC(G)$. The algebras $AP(G)$, $WAP(G)$, and $LUC(G)$ are amongst m-admissible subalgebras of $\mathcal{C}_b(G)$, which are extensively studied for more than 70 years, in connection with right topological semigroup compactifications of G.

The subject of analysis of semigroup compactifications can be traced back to the work of H. Bohr [5, 6, 7] on the almost periodic functions on the real line. In [4], S. Bochner developed a functional analytical approach to the almost periodic functions and his approach led S. Bochner and J. von Neumann to start the theory of almost periodic functions for an arbitrary topological group. Weakly almost periodicity, which is a natural generalization of Bochner's notion of almost periodicity, was first defined and investigated by W. F. Eberlein [20]. Although the algebra of weakly almost periodic functions on groups share many important properties of almost periodic functions, such as admitting an invariant mean and existence of a corresponding universal compactification, there are essential differences between these two algebras of continuous functions.

The definition of semigroup compactifications that we adopt today, is due to Weil (1935-1940), where he generalizes the almost periodic compactification. de Leeuw and Glicksberg [18, 19] expanded the subject by considering the weak almost periodic compactification on a semitopological semigroup. They constructed the weakly almost periodic compactification as the weak operator closure of the semigroup of translations acting on $WAP(G)$. In [2], J. Berglund and K. Hoffmann developed the first categorical approach to semigroup compactifications and produced universal P-compactifications using the coadjoint functor theorem, where P is a property satisfied by a class of semigroup compactifications.

The differences between the almost periodic and weakly almost periodic functions are strongly reflected by the structures of the associated compactifications. For example, the almost periodic compactification of a group is always a topological group, whereas the weakly almost periodic compactification fails to be jointly continuous. in addition to the lack of joint continuity, many subsets that are distinguished in the topological group theory, such as minimal ideals, the set of idempotents, may fail to be closed. Furthermore, producing joint continuity points attracted a great attention and became one of the most important questions of the theory. The first breakthrough in producing joint continuity points in semitopological semigroup compactifications is due to R. Ellis [21] who proved that in any compact semitopological semigroup, the multiplication map is jointly continuous on the group of units. In [34] Lawson extended Ellis's result by proving joint continuity at any point of the form $(x, 1)$ and $(1, x)$ of a separately continuous multiplication of a right topological semigroup with identity 1 when x is an arbitrary element of the semigroup. As a corollary he obtained the fact that if the set of idempotents is closed, then the restriction of the multiplication to the subsemigroup of idempotents is also jointly continuous. After Lawson, the structure of idempotents in semigroup compactifications received special emphasis both in the search for joint continuity points and in the effort to understand the complexity of the compactifications because, as a consequence of their order structure, the set of idempotents is a relatively easier subsemigroup to understand.

Another important property of weakly almost periodic functions is proved by Berglund and Hoffmann in [2]. The algebra of weakly almost periodic functions $WAP(G)$ can be written as the direct sum $AP(G) \oplus W_0(G)$, where $W_0(G)$ consists of the dissipative weakly almost periodic functions, which vanishes under the invariant mean of $WAP(G)$, in a certain sense. This decomposition of a weakly almost periodic function paved the way for many further investigations. However, we still do not know what a general weakly almost periodic function on an arbitrary locally compact group exactly looks like. C. Chou in [16] called a topological group G minimally weakly almost periodic if its weakly almost periodic compactification, G^w, is of the form $G^w = G \cup G^{ap}$, that is a weakly almost periodic function on such a group is the sum of an almost periodic function and a continuous function vanishing at infinity. For connected groups the minimally almost periodicity is characterized by W. Ruppert and M. Mayer in [45, 38, 39]. The question is still open for a general topological group.

As $\mathcal{E}(G)$ is a subalgebra of $WAP(G)$, the corresponding universal compactification G^e is a quotient of G^w. It has been recently proved by Nico Spronk and Ross Stokke in [49] that G^e is the universal compactification amongst those compactifications of G which are representable as contractions on a Hilbert space. A significant amount of the research on the weakly almost periodic compactifications is done in connection with harmonic analysis, which means G^e is one of the most studied quotients of G^w. However, not much attention has been given to the question of explicitly studying the structure of G^e, itself. The first systematic treatment of the Eberlein compactification has been given by [49], where the authors investigate the properties of the compactifications (π, G^π) associated with unitary representations π. In their notation G^e corresponds to the universal representation ω.

In this thesis, we will study the Eberlein compactification of a locally compact group as a quotient of G^w. Our aim is to observe that G^e shares many important properties of G^w. The thesis is organized as follows.

Chapter 2 reviews the necessary background on locally compact groups and semigroup compactifications.

In Chapter 3, we will restrict our attention to Abelian groups. We will construct subsemigroups of the unit ball of $L^\infty[0,1]$ as quotients of G^e, which is a strong indication of the complexity of the structure of G^e in the Abelian setting. For the locally compact Abelain group G, let $\widehat{G}$ denote its (dual) group of characters and $M(\widehat{G})$ be the algebra of bounded regular Borel measures on $\widehat{G}$, endowed with convolution as multiplication.

By a generalized character on $M(\widehat{G})$ (see [50]) we define an element $\chi = \{\chi_\mu\}_{\mu \in M(\widehat{G})} \in \prod_{\mu \in M(\widehat{G})} L^\infty(\mu)$ satisfying

(i) if $\mu \ll \nu$, then $\chi_\mu = \chi_\nu$ (μ *a.e*),

(ii) $\chi_{\mu * \nu}(x+y) = \chi_\mu(x)\chi_\nu(y)$ ($\mu \times \nu$ *a.e.*),

(iii) $\sup_{\mu \in M(\widehat{G})} \|\chi_\mu\|_\infty = 1$.

We equip the set of generalized characters with the topology induced from $\sigma(L^\infty(\mu), L^1(\mu))$-topology on each factor in the product space, and with the multiplication defined by

$$(\chi\psi)_\nu = \chi_\nu \psi_\nu \ \ (\nu \ a.e.) \ \text{ for any } \nu \in M(\widehat{G}).$$

Then the set of generalized characters becomes a compact semitopological semigroup. Furthermore, we identify this compact semigroup with the maximal ideal space $\Delta(\widehat{G})$ of $M(\widehat{G})$, where the action is given by

$$\mu \mapsto \int_{\widehat{G}} \chi_\mu d\mu \text{ for all } \mu \in M(\widehat{G}).$$

We shall write $\Delta(\mu)$ for the set $\{\chi_\mu | \chi \in \Delta(\widehat{G})\}$ for each $\mu \in M(\widehat{G})$. As $\Delta(\widehat{G})$ is a compact separately continuous semigroup, being the continuous homomorphic image of it under the projection map $\chi \mapsto \chi_\mu$, each $\Delta(\mu)$ is also a compact semitopological semigroup. Since $\Delta(\mu) = \Delta(\widehat{G})|_{L^1(\mu)}$ we have that $\Delta(\mu)$ is a compact subset of the unit ball of $L^\infty(\mu)$ for each μ.

We note that if $\gamma \in G$ is a character on $\widehat{G}$, we define for each μ in $M(\widehat{G})$, $\chi_\mu = \gamma$. Then $\chi = (\chi)_{\mu \in M(\widehat{G})}$ is a generalized character, and hence G can be embedded as an open subset of $\Delta(\widehat{G})$. We denote the closure of G in $\Delta(\widehat{G})$ by clG, and furthermore we let $S_\mu(\widehat{G})$ denote the closure of G in $\Delta(\mu)$ for each $\mu \in M(\widehat{G})$. Theorem 3.13 of [49] shows that for any locally compact Abelian group G, its Eberlein compactification G^e is isomorphic to clG. Unfortunately, it is a very difficult task to determine the structure of $\Delta(\widehat{G})$. Most of the research has been done for specific local situations, such as [10, 12, 11, 13, 14].

The aim of Chapter 3 is to consider special measures on a given locally compact Abelian group $\widehat{G}$ and determine the structure of $S_\mu(\widehat{G})$ for this specific measure. The properties of the measures under consideration also allow us to embed $S_\mu(\widehat{G})$ as a subsemigroup of $L^\infty[0, 1]$ which enables us to determine the algebraic and topological properties of $S_\mu(\widehat{G})$.

Chapter 4 deals with the question of determining the structure of the Eberlein compactification of G in connection with its subgroups. We will consider the known results and constructions on G^w and observe that G^e behaves similar to G^w under similar situations. If we let H be a closed subgroup of the locally compact group G, we will consider the relationship between G^e and H^e, in connection with the corresponding Eberlein algebras $\mathcal{E}(G)$ and $\mathcal{E}(H)$, depending on the properties of G and H.

In Chapter 5 we will restrict our attention to locally compact groups G, which have a generalized Heisenberg group structure. Depending on the properties of its subgroups, the structure of both the function algebras, $WAP(G)$ and $\mathcal{E}(G)$, and the corresponding semigroup compactifications G^w and G^e vary drastically. Here our aim is to generalize the Heisenberg group considered in Example 2.1 in [41] to a subclass of locally compact groups of Heisenberg type. We will observe that our assumptions together with uniform continuity forces the functions in $\mathcal{E}(G)$ to have a relatively simple structure.

Chapter 2

Background and Literature

In this chapter we give some basic background necessary for the rest of this thesis. Section 2.1 reviews locally compact groups and Banach algebras associated to them. Section 2.2 introduces the general theory of semitopological semigroup compactifications. The third and fourth sections contain basic properties of two particular semitopological semigroup compactifications of a locally compact group, namely the weakly almost periodic and Eberlein compactifications.

2.1 Locally Compact Groups

A *locally compact group* is a group G equipped with a topology such that

(i) the group operation, $(G \times G \to G : (x, y) \mapsto xy$ (or $x + y$)) is jointly continuous,

(ii) inversion $(G \to G : x \mapsto x^{-1})$ is continuous,

(iii) the identity element e has a neighborhood basis consisting of compact sets.

We will denote the group operation either by multiplication or addition depending on the context.

A *Radon measure* on a locally compact group G is a Borel measure that is finite on all compact sets, outer regular on all Borel sets and inner regular on all open sets. Let $M(G)$ denote the set of all complex valued Radon measures on G. An element μ in $M(G)$ is called *left invariant* if $\mu(xE) = \mu(E)$ for any x in G and any Borel subset E of G.

It is well known that every locally compact group G is equipped with a left invariant Radon measure λ_G, which attains strictly positive values on nonempty open sets. Moreover, λ_G is unique up to multiplication by a positive constant. λ_G is called the *left Haar measure* on G. From now on we will assume that each locally compact group has a fixed left Haar measure. If no confusion arises, we shall write dx for $d\lambda_G(x)$, $\int f dx$ for $\int f d\lambda_G$ for a function f on G.

We denote by $\mathcal{C}_c(G)$ the set of compactly supported continuous functions on G. The left invariance of λ_G means for $f \in \mathcal{C}_c(G)$

$$\int_G f(yx)dx = \int_G f(x)dx$$

for any $y \in G$. However, it is not necessarily true that every left- invariant Haar measure is also right-invariant. As a consequence of the uniqueness of λ_x, there exists a continuous homomorphism $\Delta : G \to (0, \infty)$ such that for any $f \in \mathcal{C}_c(G)$ and $y \in G$

$$\int_G f(xy)dx = \Delta(y^{-1}) \int_G f(x)dx.$$

Δ is called the *modular function* of G. If $\Delta = 1$ on G, then G is called *unimodular*. Examples of unimodular groups are Abelian and compact groups.

We denote by $L^1(G, \lambda_G) = L^1(G)$ the *group algebra* of G. $L^1(G)$ is an involutive Banach algebra when multiplication is defined by *convolution*

$$f * g(x) = \int_G f(y)g(x^{-1}y)dy$$

and the involution is given by

$$f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})}$$

for any $x \in G$.

The Banach dual of $L^1(G)$ is $L^\infty(G)$, the Banach algebra of bounded complex valued functions on G, where the duality is given by

$$\int_G fg d\lambda_G$$

for $f \in L^1(G)$ and $g \in L^\infty(G)$. Note that when G is a compact group, $L^\infty(G)$ can be seen as a subset of $L^1(G)$.

The convolution of two measures $\mu, \nu \in M(G)$ is defined as

$$\int_G f(z) d(\mu * \nu)(z) = \int_G \int_G f(xy) d\mu(x) d\nu(y)$$

for $f \in \mathcal{C}_c(G)$. Any function $f \in L^1(G)$ can be identified with the measure $f d\lambda_G$, and hence $L^1(G)$ can be seen as a closed ideal of $M(G)$.

Furthermore, let $\mathcal{C}_b(G)$ denote the C^*-algebra of continuous, bounded, complex valued functions on G, equipped with the uniform norm, the pointwise operations and complex conjugation as involution.

Finally, we will denote by $C^*(G)$ the *group C^*-algebra*, which is the enveloping C^*-algebra of $L^1(G)$, that is

$$C^*(G) = \overline{L^1(G)}^{\|\cdot\|_{C^*(G)}}.$$

2.2 Semigroup Compactifications

This section introduces semigroups and semigroup compactifications. For further analysis, the reader is referred to [3] or [46]. For the rest of this chapter, we assume that all the locally compact groups are non-compact.

A *semigroup* S is a non-empty set together with an associative operation on S. The semigroup operation will be denoted by multiplication, unless otherwise stated. An element e in S satisfying $ee = e$ is called an *idempotent*. The set of all idempotents of S is denoted by $I(S)$.

We define relations $\leq_l$ and $\leq_r$ on $I(S)$ by

$$e \leq_l f \text{ if and only if } ef = e \text{ and } e \leq_r f \text{ if and only if } fe = e$$

for $e, f \in I(S)$. If e and f commute we omit the indices l and r. A *semilattice* in S is an Abelian semigroup consisting of idempotents. A semilattice is *complete* if every non-empty subset has an infimum and every directed subset has a supremum (with respect to $\leq_l = \leq_r$).

Let s be an element of a semigroup S. The *right translation* by s is the function $R_s : S \to S : t \mapsto ts$, and similarly the *left translation* by s is the function $L_s : S \to S : t \mapsto ts$. If S is also a topological space, it is called *right* (*left*) topological if R_s (L_s) is continuous for each s in S. We define the *topological center* of S as follows:

$$\Lambda(S) = \{s \in S : \text{the translations } R_s \text{ and } L_s \text{ are continuous}\}.$$

S is a *semitopological semigroup* if $\Lambda(S) = S$, and a *topological semigroup* if the multiplication is jointly continuous on S.

Let G be a locally compact group. A *semigroup compactification* of G is a pair (ψ, S) that satisfies

(i) S is a compact, Hausdorff, right topological semigroup,

(ii) $\psi : G \to S$ is a continuous homomorphism,

(iii) $\psi(G)$ is dense in S,

(iv) $\psi(G)$ is contained in $\Lambda(S)$.

The function ψ is called the *compactification map*. We define its *dual* by $\psi^* : \mathcal{C}(S) \to \mathcal{C}_b(G)$ by $\psi^*(g) = g \circ \psi$ for any $g \in \mathcal{C}(S)$.

Given f in $\mathcal{C}_b(G)$, if there exists a function g in $\mathcal{C}(S)$ such that $\psi^*(g) = f$, then g is called an *extension* of f. Since $\psi(G)$ is dense in S, each f in $\mathcal{C}_b(G)$ may have at most one extension to any semigroup compactification of G. We will see that the compactification S is determined up to an isomorphism by the continuous bounded functions extendable to S. To this end, we define an order on the class of semigroup compactifications of a fixed locally compact group G.

Let (ψ, S) and (ϕ, T) be compactifications of G.

(i) A continuous homomorphism σ of S onto T is called a *homorphism of semigroup compactifications* if $\sigma \circ \psi = \phi$. If such a homomorphism exists, then (ϕ, T) is said to be a *factor* of (ψ, S), and (ψ, S) is said to be an *extension* of (ϕ, T).

(ii) If (ψ, S) is both a factor and extension of (ϕ, T), then we say that (ψ, S) is isomorphic to (ϕ, T).

Theorem 2.2.1. *Suppose that (ψ, S) and (ϕ, T) are compactifications of G. Then (ϕ, T) is a factor of (ψ, S) if and only if $\phi^*(\mathcal{C}(T)) \subset \psi^*(\mathcal{C}(S))$.*

For a proof, see [3] Theorem 3.1.9.

Our next result characterizes the subsets of $\mathcal{C}_b(G)$ that permit extensions to some semigroup compactifications of G. Let $F(G)$ denote the set of complex valued functions on G. Let ν be an element of $\mathcal{C}_b(G)^*$. We define the *left (right) introversion operator* determined by ν, $T_\nu : \mathcal{C}_b(G) \to F(G)$ $(U_\nu : \mathcal{C}_b(G) \to F(G))$ by $(T_\nu f)(x) = \nu(L_x f)$ $((U_\nu f)(x) = \nu(R_x f))$.

Theorem 2.2.2. *If (ψ, S) is a semigroup compactification of a locally compact group G, then $\psi^*(\mathcal{C}(S))$ satisfies the following properties:*

(i) $\psi^*(\mathcal{C}(S))$ is a norm closed subalgebra of $\mathcal{C}_b(G)$,

(ii) $\psi^*(\mathcal{C}(S))$ is closed under complex conjugation,

(iii) $\psi^*(\mathcal{C}(S))$ *contains the constant functions,*

(iv) $\psi^*(\mathcal{C}(S))$ *is invariant under translations by elements of* G*,*

(v) $\psi^*(\mathcal{C}(S))$ *is invariant under (left and right) introversion operators determined by multiplicative linear functionals on* $\psi^*(\mathcal{C}(S))$*.*

Conversely, if $\mathcal{F}$ *is a subset of* $\mathcal{C}_b(G)$ *satisfying the properties (i)-(v), then the* $Gelfand$ *spectrum,* $\sigma(\mathcal{F})$*, together with the evaluation map* $\epsilon : G \to \sigma(\mathcal{F})$ *gives a compactification of* G *such that* $\epsilon^*(\mathcal{C}(\sigma(\mathcal{F}))) = \mathcal{A}$*.*

In this situation the product of $\mu, \nu \in \sigma(\mathcal{F})$ *can be defined by* $(\mu\nu)(f) = \mu(T_\nu f)$ *and makes* $(\epsilon, \sigma(\mathcal{F}))$ *a semigroup compactification of* G*.*

The proof can be found in [3] Theorem 3.1.7. A subalgebra of $\mathcal{C}_b(G)$ satisfying conditions (i)-(v) of Theorem 2.2.2 is called an *m-admissible* subalgebra. Let $\mathcal{F}$ be an m-admissible subalgebra of $\mathcal{C}_b(G)$. Furthermore, if (S, ψ) satisfies $\psi^*(\mathcal{C}(S)) = \mathcal{F}$, then (S, ψ) is called an $\mathcal{F}$-$compactification$ of G, and we will denote S by G^f. As a corollary of Theorem 2.2.1 the $\mathcal{F}$-compactification is unique up to isomorphism of semigroups and any semigroup compactification of G satisfying $\psi^*(\mathcal{C}(S)) \subset \mathcal{F}$ can be seen as a quotient of G^f. Therefore, we may consider G^f as the universal semigroup compactification of G corresponding to $\mathcal{F}$.

More generally, let P be a property that is satisfied by a class of semigroup compactifications of G. If there exists (S, ψ) such that (S, ψ) is an extension of every compactification that satisfies the property P, then (S, ψ) is called *the P-compactification* or the *universal P-compactification* of G.

2.3 Weakly Almost Periodic Functions

In this section we will outline the properties of the weakly almost periodic compactification of a locally compact group G. We should note that the weakly almost periodic compactification can be defined for any semitopological semigroup S. The first systematic analysis of weakly almost periodic functions was given by deLeeuw and Glicksberg. A more general and thorough analysis of weak almost periodicity can be found in [3, 15, 46].

Definition 2.3.1. *Let G be a locally compact group. Recall that L_x (R_x) denotes the left translation on G by $x \in G$. Consider the dual map of L_x (R_x)*

$$L_x^*(f)(y) = f \circ L_x(y) = f(xy) \qquad R_x^*(f)(y) = f \circ R_x(y) = f(yx)$$

for any $x, y \in G$. L_x^* (R_x^*) is called the left (right) translation operator determined by x. To simplify our notation we will denote L_x^* (R_x^*) also by L_x (R_x).

In the dual space $\mathcal{C}_b(G)^*$ of $\mathcal{C}_b(G)$, the set of multiplicative functionals is denoted by βG. βG is the Stone-Cech compactification of G. A function $f \in \mathcal{C}_b(G)$ is called *weakly* *almost periodic* provided the set $\{L_x f | x \in G\}$ is weakly compact in $\mathcal{C}_b(G)$, i.e. its closure with respect to the topology $\sigma(\mathcal{C}_b(G), \mathcal{C}_b(G)^*)$ is compact in that topology. We have many characterizations of a weakly almost periodic function.

Theorem 2.3.2. *The following statements about a function $f \in \mathcal{C}_b(G)$ are equivalent.*

(i) f is weakly almost periodic,

(ii) $\{R_x f | x \in G\}$ is relatively weakly compact in $\mathcal{C}_b(G)$,

(iii) $\{L_x f | x \in G\}$ (or $\{R_x f | x \in G\}$) is relatively $\sigma(\mathcal{C}_b(G), \beta G)$-compact in $\mathcal{C}_b(G)$,

(iv) (Grothendieck criterion) Whenever $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are sequences in G such that the iterated limits

$$A = \lim_m \lim_n f(x_n y_m) \text{ and } B = \lim_n \lim_m f(x_n y_m)$$

both exist, then $A = B$.

We note that if we remove the word weakly from the above definition, we get the m-admissible subalgebra of *almost periodic functions* on G, denoted by $AP(G)$. Clearly each almost periodic function is weakly almost periodic. Furthermore, a function f on G is called *left uniformly continuous* if given $\varepsilon > 0$, there is a neighborhood V of e in G such that $|f(x) - f(y)| < \varepsilon$ if either $x^{-1}y \in V$ or $xy^{-1} \in V$. It follows that on G each $f \in WAP(G)$ is both left and right uniformly continuous.

The set of all weakly almost periodic functions on G is denoted by $WAP(G)$, and forms an m-admissible subalgebra of $\mathcal{C}_b(G)$. Its spectrum $\sigma(WAP(G))$ is a compact semitopological semigroup, called the weakly almost periodic compactification of G. We will denote $\sigma(WAP(G))$ by G^w and the compactification map by $w : G \to G^w$.

In addition to being the universal semigroup compactification corresponding to function algebra $WAP(G)$, the weakly almost periodic compactification G^w satisfies many important universal properties:

(i) It is well-known that G^w is the largest semitopological semigroup compactification (see [3] Theorem 4.2.11).

(ii) A semitopological semigroup compactification (S, ψ) is called *involutive* if there is a continuous involution $x \mapsto x^*$ on S such that $\psi(x^{-1}) = \psi(x)^*$ for $x \in G$. It has been proven in [49] that G^w is the universal involutive compactification of G.

(iii) Eberlein [20] proved that for any $x \mapsto U_x$ a weakly continuous representation of G in a uniformly bounded semigroup of linear transformations on a reflexive Banach space X, then the coefficient functions are weakly almost periodic on G. Conversely, it has been proven by Shtern in [48] that G^w is the universal compactification of G amongst all semigroup compactifications that are representable as uniformly bounded linear transformations on reflexive Banach spaces.

Let (ι, G^∞) denote the *one-point-compactification* of the locally compact non-compact group G. Note that $\iota^*(\mathcal{C}(G^\infty)) = \mathbb{C} \oplus \mathcal{C}_0(G)$. Since G^∞ is semitopological, it is a factor of the weakly almost periodic compactification G^w and $\mathbb{C} \oplus \mathcal{C}_0(G) \subset WAP(G)$. A group G is called *minimally weakly almost periodic* if each weakly almost periodic function on G can be written as $g + h$ where $g \in AP(G)$ and $h \in \mathcal{C}_0(G)$.

In [16], Chou proved that the n-dimensional motion group $M(n)$ and the special linear group $SL(2,\mathbb{R})$ are minimally weakly almost periodic. M. Mayer, in [38, 39] extended Chou's result to a larger class of semisimple Lie groups. In fact, $WAP(SL(2,\mathbb{R})) = \mathbb{C} \oplus \mathcal{C}_0(SL(2,\mathbb{R}))$, which implies $SL(2,\mathbb{R})^w \cong SL(2,\mathbb{R}) \cup \{\infty\}$. On the other hand, in Chapter 3, we will observe that when G is a locally compact Abelian group, then G^w has a very complicated structure.

2.4 Eberlein Functions

Let G be a locally compact group. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ be the space of bounded linear operators on $\mathcal{H}$. The *weak operator topology* (WOT) on $\mathcal{B}(\mathcal{H})$ is the topology induced by the seminorms $T \mapsto |\langle T\xi, \eta\rangle|$ for $\xi, \eta \in \mathcal{H}$. We denote by $\mathcal{U}(\mathcal{H})$, the group of unitary operators on $\mathcal{H}$. A *continuous unitary representation* of G on $\mathcal{H}$ is a WOT-continuous group homomorphism $\pi : G \to \mathcal{U}(\mathcal{H})$. So, for every $\xi, \eta \in \mathcal{H}$, the function $f : G \to \mathbb{C}$ for $x \in G$ given by

$$f(x) = \langle \pi(x)\xi, \eta \rangle \tag{2.1}$$

is continuous. Functions of the form 2.1 for $\xi, \eta \in \mathcal{H}$ are called the *coefficient functions associated with* π.

We naturally extend any unitary representation π of G to a norm-decreasing $*$-representation of the group algebra $L^1(G)$ as

$$\langle \pi(f)\xi, \eta \rangle = \int_G f(x)\langle \pi(x)\xi, \eta \rangle dx$$

for $f \in L^1(G)$. We will denote the extension of π, to $L^1(G)$ again by π.

Let $\pi_1 : G \to \mathcal{H}_1$ and $\pi_2 : G \to \mathcal{H}_2$ be two unitary representations of G. We say π_1 and π_2 are *unitarily equivalent* if there exists a unitary operator $T : \mathcal{H}_1 \to \mathcal{H}_2$ such that $T\pi_1(x) = \pi_2(x)T$ for all $x \in G$. For a locally compact group G we denote by $\sum_G$ as the class of equivalence classes of continuous unitary representations of G. The *Fourier Stieltjes algebra* $B(G)$ is the set of all coefficient functions of representatives of elements of $\sum_G$. $B(G)$ is easily seen to be a subalgebra of $\mathcal{C}_b(G)$.

Eymard in [22] defined and studied $B(G)$, and proved that $B(G)$ is the Banach dual space of the group C^*-algebra $C^*(G)$. Equipped with the norm as the dual space $B(G)$ is a translation invariant Banach algebra. However, $B(G)$ fails to be uniformly closed, when

G is infinite. Let $\mathcal{E}(G)$ be the uniform closure of $B(G)$ in $\mathcal{C}_b(G)$, called the *Eberlein algebra* of G. $\mathcal{E}(G)$ is a translation invariant subalgebra of $\mathcal{E}(G)$ which contains the constants and is closed under complex conjugation. Clearly, $\mathcal{E}(G)$ is also a subalgebra of $WAP(G)$, hence by Theorem 2.11(ii) of [49], it is an m-admissible subalgebra of $\mathcal{C}_b(G)$. Therefore, the corresponding universal compactification G^e exists. We will call G^e the *Eberlein compactification* of G. It has been recently discovered in [40] and [49] that G^e is the universal compactification amongst all compactifications (ψ, S) of G, where S is isomorphic to a weak*-closed semigroup of Hilbertian contractions.

The following Theorem in the case of $WAP(G)$ can be found in [15]; and in the case of $B(G)$ can be found in [22]. The case of $\mathcal{E}(G)$ follows from [22] and Proposition 2.10 of [1].

Theorem 2.4.1. *Let G and H be locally compact groups and $f : G \to H$ be a continuous homomorphism. We define the induced map $f^* : C_b(H) \to C_b(G)$ as $f^*(h) = h \circ f$. Then $f^*(\mathcal{E}(H)) \subset \mathcal{E}(G)$.*

Chapter 3

West Semigroups

In the present chapter we restrict our attention to locally compact Abelian groups. Let G be a locally compact Abelian group. Our aim is to construct semitopological compactifications of G via the duality relation with its character group. The origins of our construction may be traced back to an earlier problem concerning idempotents on compact semigroups.

Let S be a compact right topological semigroup. Recall that $I(S)$ denotes its set of idempotents. The question of determining the structure of $I(S)$ naturally arose after Ellis's discovery that every compact right topological semigroup contains an idempotent [46]. For the compact semigroups that are of interest to us, Ellis's theorem is trivial, since the identity of the underlying group is an idempotent of S. As the identity is the only idempotent in G, the structure of idempotents in any semigroup compactification of G is an important tool to understand the algebraic complexity of these semigroups. In particular, the cardinality and the lattice structure of $I(S)$ has been extensively studied.

Furthermore, in [35] Lawson proved that in a semitopological semigroup S, if $I(S)$ is closed, then the multiplication map on $I(S)$ is jointly continuous. Hence, the set of idempotents can be studied in connection with the topological properties of S.

In [51] West produced a semitopological compactification of $\mathbb{Z}$ which contains 2 idempotents. Brown and Moran, in [11, 13] generalized West's idea to produce a number of semitopological compactifications of $\mathbb{Z}$ whose lattices of idempotents satisfy various different properties and all with cardinality at most c. Later, in [9], Bouziad, Lemańzyk and Mentzen characterized the compactifications of the additive group of integers, depending on West's construction, with the largest set of idempotents and observed that their sets of idempotents are not closed. In this chapter we will generalize the above constructions to any noncompact locally compact Abelian group G.

In the first section we will review Pontryagin duality and some of its consequences in connection with the algebraic structure of G. Section 2 is devoted to the construction of the compact semigroups. Next, in section 3 we will consider consequences of this construction on the theory of semitopological semigroup compactifications.

3.1 Background

In this section we will review the duality relationship of G with its group of characters. Our references are the texts [44] and [24].

3.1.1 Dual Group

Let G be a locally compact Abelian group. All the irreducible representations of G are one-dimensional. Such representations are called (*unitary*) *characters* of G, that is, a *character* of G is a continuous group homomorphism on G with values in the multiplicative circle group $\mathbb{T}$. The set of all characters on G is denoted by $\widehat{G}$. $\widehat{G}$ can be made into a locally compact Abelian group, called the *dual group*. Here the group operation is given by pointwise multiplication of functions, the inverse of a character is given by its complex conjugate and the topology on $\widehat{G}$ is the topology of uniform convergence on compact sets (where we consider $\widehat{G}$ as a subset of $\mathcal{C}_b(G)$).

Next we cite the characterizations of the dual group for the locally compact Abelian groups that are of special importance to us:

- $\widehat{\mathbb{R}} = \mathbb{R}$ with the dual pairing $\langle x, \xi \rangle = e^{2\pi i x \xi}$;

- $\widehat{\mathbb{T}} = \mathbb{Z}$ and $\widehat{\mathbb{Z}} = \mathbb{T}$ with the dual pairing $\langle \alpha, n \rangle = \alpha^n$ in both cases;

- If $\mathbb{Z}_k$ is the additive group of integers mod k, where $k \in \mathbb{N}$ and $k \geq 2$, then $\widehat{\mathbb{Z}_k} = \mathbb{Z}_k$ with the pairing $\langle m, n \rangle = e^{2\pi i m n / k}$;

- If $\mathbb{Z}_k^\infty$ is the sum of countably many copies of the finite group $\mathbb{Z}_k$, then $\widehat{\mathbb{Z}_k^\infty}$ is the direct product, with product topology, of countably many copies of $\mathbb{Z}_k$, denoted by $\mathbb{D}_k$.

Pontryagin Duality Theorem 3.1.1. *The map $\alpha : G \to \widehat{\widehat{G}}$, given by $\langle x, \gamma \rangle = \langle \gamma, \alpha(x) \rangle$ is an isomorphism of G onto $\widehat{\widehat{G}}$.*

It follows from Pontryagin Duality theorem that G is compact if and only if $\widehat{G}$ is discrete. Since our aim is to compactify G, we restrict our attention to non-compact G, hence to non-discrete $\widehat{G}$.

3.1.2 Structure Theorem

Our construction and the structure of the resulting compact semigroups depends on the properties of the dual group $\widehat{G}$. We call a locally compact Abelian group G an $I - group$

if every neighborhood of the identity in G contains an element of infinite order. We will first quote a structure theorem on locally compact Abelian groups, which will simplify our construction. We include its proof, which was originally proved in [30] for completeness purposes. The reader is referred to [44], for further analysis.

Theorem 3.1.2. *Let G be a locally compact Abelian group.*

(i) If G is an I-group, then G contains a metric I-group as a closed subgroup.

(ii) If G is not discrete and not an I-group, then G contains $\mathbb{D}_q$ as a closed subgroup, for some $q > 1$.

Proof. The Principal Structure Theorem 2.4.1 of [44] states that any locally compact Abelian group G contains an open subgroup G_1 which is a direct sum of a compact group H and a Euclidean space $\mathbb{R}^n$ for some $n \geq 0$.

First assume that G is an I-group. If $n > 0$, then the result follows. So, suppose that $n = 0$. Then G_1 is a compact I-group. Without loss of generality we will assume that G itself is compact. As G is not of bounded order, it follows that $\widehat{G}$ is also not of bounded order. Now, to prove that G contains a compact metric subgroup H not of bounded order, it is enough to prove that $\widehat{G}$ admits a countable quotient, hence we need to prove that $\widehat{G}$ can be embedded homomorphically onto a countable group which is not of bounded order. $\widehat{G}$ is infinite implies that it contains a countably infinite subgroup Γ, which may be chosen to be not of bounded order. We can embed Γ isomorphically in a countable divisible group D. This isomorphism can be extended to a homomorphism ϕ of $\widehat{G}$ into D ([44] Theorem 2.5.1). Since, $\Gamma = \phi(\Gamma) \subset \phi(\widehat{G}) \subset D$, $\phi(\widehat{G})$ is countable and infinite.

Therefore, G contains a closed metric subgroup H which is not of bounded order. An application of Baire Category theorem, on the compact group H, implies that it must contain a dense set of elements of infinite order. Hence, H is a closed metric I-subgroup of G, as required.

Next assume that G is not discrete and not an I-group. then the compact subgroup G_1 guaranteed by the Principle Structure Theorem is of bounded order and hence its dual $\widehat{G_1}$ is also of bounded order. We can write $\widehat{G_1}$ as a direct sum of infinitely many finite cyclic groups. Some countable subfamily can be chosen to have the same order, say q ([44] Appendix B8). The direct sum of this family is a direct summand of $\widehat{G_1}$, hence is a quotient of $\widehat{G_1}$. Thus, it is the dual of a compact subgroup of G, isomorphic to $\mathbb{D}_q$.

$\square$

3.1.3 On the Unit Ball of $L^\infty[0,1]$

Let G be a locally compact Abelian group. Let $M(G)$ denote the space of bounded regular Borel measures on G. For any μ in $M(G)$, the support of μ is the set of all points $g \in G$

for which $\mu(U) > 0$ for every open set U containing g. Note that the complement of the support of μ is the largest set in G with μ-measure 0. Recall that μ in $M(G)$ is called a *continuous measure* if for each singleton in G, $\mu(\{g\}) = 0$.

A subset K of G is called a *Cantor* set if K is metric, perfect and totally disconnected, or equivalently if K is homeomorphic to the classical Cantor subset, C of the real line. Our present objective is to construct a special Cantor subset for each locally compact Abelian group G. The existence of a compact, perfect subset of G assures the existence of a continuous positive measure μ in $M(G)$. (Note that μ can be chosen to be a probability measure). First, we will observe that for any locally compact Abelian group G, a Cantor subset K of G, together with a continuous measure supported on K, measure theoretically can be considered to be the interval $[0, 1]$, equipped with its Lebesgue measure. Let λ denote the Lebesgue measure on the real line.

Let K be a Cantor set. We will call a subset E of K a $C-$ *open* subset, if E is open with respect to the relative topology of K. Similarly, we define $C-$ *closed* sets, $C-$*neigbourhoods* and if the Cantor set K is a subset of $\mathbb{R}$ then we also define $C-$*intervals*.

The following Theorem is well-known for *Cantor*-subsets of locally compact spaces ([26] Theorem 41.C and [37] Theorem 6.4.2). Let λ denote the Lebesgue measure on the interval $[0, 1]$.

Theorem 3.1.3. *Let G be a locally compact Abelian group. Let μ be a continuous Borel probability measure on a Cantor subset, K of G, with support of μ being K. Then there exists a Borel isomorphism $\phi : K \to [0,1]$ that is measure preserving, with respect to μ and λ, for every Borel subset E of K.*

Proof. First note that as a perfect compact Hausdorff space any Cantor set is uncountable. Let C_1 be a countable subset of C, the classical Cantor set in $[0,1]$. Then there is a measure-preserving Borel isomorphism $\varphi : C \to C \setminus C_1$. Indeed, since C is uncountable, there exists a countably infinite subset C_2 of C such that $C_1 \cap C_2 = \emptyset$. Let $\varphi : C \to C \setminus C_1$ be a function that maps $C_1 \cup C_2$ bijectively onto C_2 and is the identity on $C \setminus (C_1 \cup C_2)$. Then φ satisfies the claim, since $C_1 \cup C_2$ is countable and μ is continuous.

Let $\alpha : K \to C$ be the homeomorphism given by the definition of K. We equip C with the measure, ν defined as follows:

For any Borel subset E of C, let

$$\nu(E) = \mu(\alpha^{-1}(E))$$

Since α is a homeomorphism, ν is a continuous Borel probability measure on C, and α is a measure-preserving Borel isomorphism between (K, μ) and (C, ν). Hence, it suffices to prove that there is a Borel isomorphism $\chi : C \to [0,1]$.

We write the open set $\mathbb{R} \setminus C$ as a countable union of disjoint open intervals: $\mathbb{R} \setminus C = \bigcup_{i=1}^{\infty}(l_k, r_k)$. Put $L = \{l_k : k \in \mathbb{N}\}$. Note that $C \setminus L$ is a disjoint union of half-open intervals of the form $[r_k, l_t)$. By the first paragraph, there exists measure preserving Borel isomorphisms $\varphi_1 : C \to C \setminus L$ and $\varphi_2 : [0, 1) \to [0, 1]$. Therefore, it suffices to find a measure preserving Borel isomorphism from $C \setminus L$ to $[0, 1)$. Define a map $\chi : C \setminus L \to [0, 1)$ by

$$\chi(t) = \nu((-\infty, t] \cap C).$$

χ is well-defined since for every t in $C \setminus L$, $(t, 1) \cap C$ is a non-empty C-open subset of support of ν.

Let $s, t \in C \setminus L$ be such that $s < t$. Note that since s is not in L, we must have $(s, t) \cap C \neq \emptyset$. So,

$$0 < \nu((s, t] \cap C) = \chi(t) - \chi(s)$$

That is, χ is strictly increasing, hence injective.

Next, consider $t \in C \setminus L$, an increasing sequence $\{t_n\}_{n \in \mathbb{N}}$ in $C \setminus L$ such that $t_n \to t$ and a decreasing sequence $\{s_n\}_{n \in \mathbb{N}}$ in $C \setminus L$ such that $s_n \to t$. Observe that

$$\lim_{n \in \mathbb{N}} \nu((-\infty, s_n] \cap C) = \nu(\bigcap_{n \in \mathbb{N}} (-\infty, s_n] \cap C) = \chi(t)$$

and hence

$$\begin{aligned}
\chi(t) - sup_{s<t}\chi(s) &= \lim_{n\in\mathbb{N}} \nu((-\infty, s_n] \cap C) - \lim_{n\in\mathbb{N}} \nu((-\infty, t_n] \cap C) \\
&= \lim_{n\in\mathbb{N}} \nu((t_n, s_n] \cap C) \\
&= \nu(\{t\}) = 0.
\end{aligned}$$

Hence, $\chi(t) = sup_{s<t}\chi(s)$.

Next, we claim that χ is surjective. Indeed, let $x \in [0, 1)$. Put

$$y = inf\{\chi(t) : \chi(t) > x\}.$$

Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence in $C \setminus L$, whose image sequence $\{\chi(t_n)\}_{n\in\mathbb{N}}$ decreases to y. Then $\{t_n\}_{n\in\mathbb{N}}$ is nonincreasing, so it converges to a point $t \in C$. By the choice of L, we observe that $t \in C \setminus L$. As above, we have $y = \chi(t)$. Now, if $y > x$, then there exists u in $C \setminus L$ such that $x < \chi(u) < y$, which contradicts the choice of y. Hence, $x = y = \chi(t)$.

Note that χ maps every C-interval in $C \setminus L$ to some interval in $[0, 1)$ and similarly so does χ^{-1}. Hence, χ is a Borel isomorphism and it remains only to show that χ is measure preserving.

Finally, we observe that both ν and $\lambda \circ \chi$ are Borel probability measures on $C \setminus L$, that agree on the half open C-intervals in $C \setminus L$. Hence, they must agree on the σ-algebra of Borel measures on $C \setminus L$.

Therefore, $\varphi_2 \circ \chi \circ \varphi_1 \circ \alpha : K \to [0,1]$ gives the required measure-preserving Borel isomorphism.

□

Let $L^\infty[0,1]$ denote the Banach algebra of essentially bounded functions with respect to the Lebesgue measure λ on $[0,1]$. We equip $L^\infty[0,1]$ with its weak*-topology. Let $(L^\infty)_1$ denote the norm closed unit ball of $L^\infty[0,1]$. It is well known that with the relative weak*-topology and pointwise multiplication as the operation, $(L^\infty)_1$ is a commutative, compact, and metrizable semitopological semigroup.

Let G be a locally compact Abelian group. Suppose that K is a Cantor subset of G and let μ be a continuous probability measure in $M(G)$ whose support is K. Then $L^\infty(G,\mu)$ is the Banach algebra of all μ-essentially bounded functions on G. Naturally the dual group $\widehat{G}$ can be continuously embedded into $L^\infty(G,\mu)$, when it is equipped with its weak*-topology. Let $(L^\infty(G,\mu))_1$ denote the norm closed unit ball of $L^\infty(G,\mu)$. We define $S_\mu(\widehat{G})$ to be the closure of the $\widehat{G}$ under this embedding in $(L^\infty(G,\mu))_1$. By the Banach-Alaoglu Theorem, $S_\mu(\widehat{G})$ is a compact semitopological semigroup, containing a dense homomorphic image of the dual group, $\widehat{G}$. By the universal properties of both the Eberlein and weakly almost periodic compactifications of $\widehat{G}$, we observe that for any μ in $M(G)$, $S_\mu(\widehat{G})$ is a quotient of both $\widehat{G}^e$ and $\widehat{G}^w$. Note that this observation can be repeated for any μ in $M(G)$.

In the next section we will choose a particular Cantor set and a continuous measure supported on it. First we will study the consequences of Theorem 3.1.3.

Let $\phi : K \to [0,1]$ be a measure preserving Borel isomorphism provided by Theorem 3.1.3. Let $\phi^* : L^1[0,1] \to L^1(G,\mu)$ be given by $\phi^*(f) = f \circ \phi$ for any f in $L^1[0,1]$. As noted in the proof of Theorem 3.1.3 $\lambda \circ \phi = \mu$. As such, for any f in $L^1[0,1]$, we have

$$\begin{aligned} \|\phi^*(f)\| &= \int_K |f \circ \phi(x)| d\mu(x) \\ &= \int_K |f \circ \phi(x)| d(\lambda \circ \phi)(x) \\ &= \int_0^1 |f(y)| d\lambda(y) = \|f\|_1. \end{aligned}$$

We also observe that ϕ^* is a linear isomorphism. Therefore, the Banach spaces $L^1(G,\mu)$ and $L^1[0,1]$ are isometrically isomorphic. Restricted to $L^\infty[0,1]$, ϕ^* also gives an isometric isomorphism of $L^\infty[0,1]$ onto $L^\infty(G,\mu)$. It follows that ψ^* also gives a semigroup isomorphism of $(L^\infty(G,\mu))_1$ onto $(L^\infty)_1$. Throughout this chapter, we will identify the compact semitopological semigroups $(L^\infty)_1$ and $(L^\infty(G,\mu))_1$. Therefore, we will consider $S_\mu(\widehat{G})$ as a subsemigroup of $(L^\infty)_1$.

3.2 Construction of West Semigroups

This section is devoted to the construction and characterization of the West semigroups.

3.2.1 Existence of Cantor K-sets

Let G be a locally compact Abelian group. A subset K of a G is called a *Kronecker set* if K satisfies: to every continuous function $f : K \to \mathbb{T}$ and $\varepsilon > 0$, there exists $\gamma \in \widehat{G}$ such that $\sup_{x \in K} |f(x) - \gamma(x)| < \varepsilon$. Since groups of bounded order contain no non-empty Kronecker sets, we modify the definition to apply to that case. Let $q \in \mathbb{N}$, $q \geq 2$. A subset K of G is said to be a K_q*-set* if K satisfies: for every continuous function $f : K \to \mathbb{Z}_q$ and $\varepsilon > 0$, there exists $\gamma \in \widehat{G}$ such that $\sup_{x \in K} |f(x) - \gamma(x)| < \varepsilon$. We note that this is equivalent to: every continuous function which maps K into $\mathbb{Z}_q$ coincides on K with a continuous character on G. A set which is either a K_q-set or a Kronecker set, will be called a $K - set$.

A subset E of G is called independent if E satisfies the following property: for any $x_1, x_2, \ldots, x_k$ distinct elements of E and integers $n_1, n_2, \ldots, n_k$, either $n_1 x_1 = n_2 x_2 = \ldots = n_k x_k = 0$ or $n_1 x_1 + n_2 x_2 + \ldots + n_k x_k \neq 0$, where $n_i x_i = x_i + x_i + \ldots + x_i$ (n_i *times*).

It follows directly from the above definitions that:

Theorem 3.2.1. *(i) Kronecker sets which contain only elements of infinite order are independent.*

(ii) K_q*-sets in* $\mathbb{D}_q$ *which contain only elements of order* q *are independent subsets.*

For the proof of this theorem the reader is referred to Theorem 5.1.4 of [44]. For finite sets we have a partial converse of the above theorem.

Theorem 3.2.2. *Suppose that E is an independent finite subset of a locally compact Abelian group G.*

(i) If every element of E has infinite order, then E is a Kronecker set.

(ii) If $G = \mathbb{D}_q$ and every element of E has order q, then E is a K_q-set.

For the proof of this theorem the reader is referred to the Corollary of Theorem 5.1.3 of [44]. Next, our aim is to construct Cantor K-sets for any non discrete locally compact Abelian group. First, we will prove that finite Kronecker or K_q-sets exist in abundance. The following Lemma and Theorem are quoted from [44] Chapter 5.

Lemma 3.2.3. *Suppose that G is either a locally compact Abelian I-group or $G = \mathbb{D}_q$. If $V_1, \ldots, V_k$ are disjoint non-empty open sets in G, then there exist x_i in V_i for each i in $\{1, \ldots, k\}$ such that*

(i) if G is an I-group, $\{x_1, \ldots, x_k\}$ is a Kronecker set.

(ii) if $G = \mathbb{D}_q$, $\{x_1, \ldots, x_k\}$ is a K_q-set.

Proof. (i) Assume that G is an I-group. Let $y \in G$ and n be a nonzero integer. Define

$$E_{n,y} = \{x \in G : nx = y\}$$

Clearly $E_{n,y}$ is closed for each n and y. Suppose that the interior of $E_{n,y}$ is not empty. If O is a non-empty open subset of $E_{n,y}$, then there is a neighborhood W of the identity, contained in $O - O \subset E_{n,y} - E_{n,y}$. But for any $x \in W$, x is of the form $x_1 - x_2$ for some $x_1, x_2 \in E_{n,y}$ and hence $nx = n(x_1 - x_2) = y - y = 0$. This contradicts the definition of I-group, so $E_{n,y}$ contains no non-empty open subsets.

Therefore by Baire's Theorem the open set V_1 cannot be covered by the union of the sets $E_{n,0}$, $n \in \{1, 2, \ldots\}$. Hence, V_1 contains an element of infinite order, say x_1.

Suppose that we have chosen $x_i \in V_i$ for $i \in \{1, \ldots, j\}$ for some $j < k$, such that the set $\{x_1, \ldots, x_j\}$ is independent. Let H be the group generated by $\{x_1, \ldots, x_j\}$. Note that H is countable, and hence again by Baire's Theorem, V_{j+1} cannot be covered by the union of the sets $E_{n,y}$, for $n \in \{1, 2, \ldots\}$ and $y \in H$. Hence there is $x_{j+1} \in V_{j+1}$ such that nx_{j+1} is not an element of H for any $n \in \{1, 2, \ldots\}$.

Thus after k steps, we get an independent set $\{x_1, \ldots, x_k\}$ such that $x_i \in V_i$ for each i. It immediately follows from Theorem 3.2.2(i) that $\{x_1, \ldots, x_k\}$ is a Kronecker set.

(ii) Next suppose that $G = \mathbb{D}_q$. Similar to part (i), we define $E_{n,y}$ for any non-zero integer n and $y \in \mathbb{D}_q$. It follows from the same argument that, if $0 < n < q$, then $E_{n,0}$ contains no non-empty open subsets, since each neighborhood of identity in $\mathbb{D}_q$ contains elements of order q. If we have chosen independent elements $\{x_1, \ldots, x_j\}$ with $x_i \in V_i$ of order q, then it follows that V_{j+1} contains an element x_{j+1} such that nx_{j+1} is not in the finite group generated by $x_1, \ldots, x_j$, if q does not divide n. The result now follows from Theorem 3.2.2(ii). □

Theorem 3.2.4. *(i) Every I-group contains a Cantor set which is also a Kronecker set. (ii) Every non-discrete non-I-group contains a Cantor set which is also a K_q set for some $q > 1$.*

Proof. (i) By Theorem 3.1.2(i), G contains a closed metric subgroup, that is an I-group. Since a Kronecker subset of a closed subgroup of G is also a Kronecker subset of G, we will assume that G is itself a metric I-group. Let d denote the metric on G.

By induction we will define a sequence of compact subsets of G. Let P_1^0 be an arbitrary compact subset of G with non-empty interior. Suppose that for a fixed integer $n \geq 1$, we have constructed disjoint compact sets P_i^{n-1} for $i \in \{1, \ldots, 2^{n-1}\}$, which have non-empty interior. Now for each i, let W_{2i-1} and W_i be non-empty disjoint open subsets of P_i^{n-1}. By Lemma 3.2.3(i), there is a Kronecker set $\{x_1^n, \ldots, x_{2^n}^n\}$ with $x_i^n \in W_i$ for each i.

It follows from the independence of $\{x_1^n, \ldots, x_{2^n}^n\}$ that there is a finite set F_n in $\widehat{G}$ satisfying:

For any choice of finite number of elements $e^{i\alpha_1}, \ldots, e^{i\alpha_{2^n}}$ in $\mathbb{T}$, there is $\gamma \in F_n$ such that

$$| e^{i\alpha_j} - \langle \gamma, x_j^n \rangle |< \frac{1}{n} \tag{3.1}$$

for each $j \in \{1, \ldots, 2^n\}$. By the uniform continuity of characters we choose disjoint compact neighborhoods P_i^n of x_i^n for each $i \in \{1, \ldots 2^n\}$ such that $P_i^n \subset W_i$ and

$$| \langle x, \gamma \rangle - \langle x_j^n, \gamma \rangle |< \frac{1}{n} \tag{3.2}$$

for each $x \in P_i^n$ and $\gamma \in F_n$. Note that we may assume $d(x, x_i^n) < \frac{1}{n}$ for all $x \in P_i^n$. This completes the induction.

Define

$$P = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{2^n} P_i^n$$

Clearly P is a Cantor set. Let $f \in \mathcal{C}(P, \mathbb{T})$ and $\varepsilon > 0$. By the uniform continuity of f on the compact set P, there exists n_0 such that f maps each of the sets $P \cap P_i^{n_0}$ for $i \in \{1, \ldots, 2^{n_0}\}$ into a proper connected subset of $\mathbb{T}$. We extend f to a continuous function of $\bigcup_{i=1}^{2^{n_0}} P_i^{n_0}$ into $\mathbb{T}$, by Tietze Extension Theorem. In particular, $f(x_i^n)$ is defined for all $n \geq n_0$.

Let $n > max\{n_0, \frac{3}{\varepsilon}\}$ be such that

$$| f(x) - f(x_i^n) |< \frac{\varepsilon}{3} \tag{3.3}$$

for any $x \in P_i^n$, $i \in \{1, \ldots, 2^n\}$. By definition of F_n, there exists $\gamma \in F_n$ such that

$$| f(x_i^n) - \langle x_i^n, \gamma \rangle | < \frac{1}{n} \tag{3.4}$$

By 3.2, 3.3 and 3.4, we get

$$\begin{aligned} | f(x) - \langle x, \gamma \rangle | &\leq | f(x) - f(x_i^n) | + | f(x_i^n) - \langle x_i^n, \gamma \rangle | + | \langle x, \gamma \rangle - \langle x_i^n, \gamma \rangle | \\ &< \frac{\varepsilon}{3} + \frac{1}{n} + \frac{1}{n} < \varepsilon \end{aligned}$$

for all $x \in \bigcup_{i=1}^{2^{n_0}} P_i^{n_0}$, hence for all $x \in P$. Together with the Lemma 3.2.3(i) this completes the proof of part (i).

(ii) Let G be a non-discrete non-I-group. By Theorem 3.1.2(ii), G contains $\mathbb{D}_q$, for some $q \geq 1$, as a closed subgroup. Similar to part (i), we will assume in the rest of the proof that $G = \mathbb{D}_q$. We will proceed in the same fashion as in part (i). Suppose that for some fixed positive integer n we have constructed disjoint compact sets P_i^{n-1} for $i \in \{1, \ldots, 2^{n-1}\}$, which have non-empty interior, and have chosen disjoint open subsets W_i for $i \in \{1, \ldots, 2^n\}$, as above. Lemma 3.2.3(ii) provides us with a set $\{x_1^n, \ldots, x_{2^n}^n\}$ with $x_i^n \in W_i$ for each i.

There is a finite set F_n in $\widehat{G}$ such that for any choice of numbers $e^{i\alpha_1}, \ldots, e^{i\alpha_{2^n}}$ in $\mathbb{Z}_q$, there is $\gamma \in F_n$ such that

$$e^{i\alpha_j} = \langle \gamma, x_j^n \rangle \tag{3.5}$$

for each $j \in \{1, \ldots, 2^n\}$. By the uniform continuity of characters we choose distinct compact neighborhoods P_i^n of x_i^n for each $i \in \{1, \ldots 2^n\}$ such that $P_i^n \subset W_i$ and $d(x, x_i^n) < \frac{1}{n}$ for all $x \in P_i^n$. Note that since each γ is constant in a neighborhood of x_j^n we may as well assume

$$\langle x, \gamma \rangle = \langle x_j^n, \gamma \rangle \tag{3.6}$$

for each $x \in P_j^n$ and $\gamma \in F_n$. This completes the induction.

Now define

$$P = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{2^n} P_i^n.$$

P is again a Cantor set. Let $f \in \mathcal{C}(P, \mathbb{Z}_q)$, then P can be written as a finite union, say $P = E_1 \cup \ldots \cup E_q$, such that f is constant on each E_i. (Note that here we do not suggest that $E_i \neq \emptyset$.) Then there are closed open sets $K_1, \ldots, K_q$ such that $K_i \supset E_i$ and $\mathbb{D}_q = K_1 \cup \ldots \cup K_q$. Extend f to the continuous function that is constant on each K_i. Then $f \in \mathcal{C}(\mathbb{D}_q, \mathbb{Z}_q)$. Let $n \in \mathbb{N}$ be large enough that

$$f(x) = f(x_i^n) \tag{3.7}$$

for each $x \in P_i^n$, that is f is constant on each of the sets P_i^n. By the definition of F_n, there is $\gamma \in F_n$ such that

$$f(x_i^n) = \langle x_j^n, \gamma \rangle \tag{3.8}$$

by 3.6. We conclude $f(x) = \langle x, \gamma \rangle$ for all x in $\bigcup_{i=1}^{2^n} P_i^n$, hence for all x in P. Together with the Lemma 3.2.3(ii) this completes the proof of the theorem. $\square$

Remark. If we define a measure μ_n supported on the set $\{x_1^n, \ldots, x_{2^n}^n\}$ of the proof of Theorem 3.2.4, by

$$\mu_n(\{x_i^n\}) = 2^{-n}$$

for each i, then the sequence $(\mu_n)_{n\in\mathbb{N}}$ has a weak*-limit $\mu \in M(P)$ such that $\|\mu\| = 1$, $\mu \geq 0$ and μ is continuous. Therefore, there exist non-trivial continuous measures on each Cantor set.

3.2.2 I-group Case

Let G be a locally compact Abelian I-group. By Theorem 3.2.4(i) and the remark following Theorem 3.2.4, we know that G contains a Cantor subset K that is also a Kronecker set, equipped with a positive non-zero continuous probability measure $\mu \in M(G)$ such that the support of μ is exactly K.

Recall that $S_\mu(\widehat{G})$ denotes the weak*-closure of $\widehat{G}$ in $L^\infty(G, \mu)$. As a consequence of Theorem 3.1.3, we identify $L^\infty(K, \mu)$ with $L^\infty[0, 1]$. Since the support of the measure μ is the set K, by almost everywhere equivalence, we can identify $L^\infty(K, \mu)$ with $L^\infty(G, \mu)$.

Hence we consider $S_\mu(\widehat{G})$ as a compact subsemigroup of $(L^\infty)_1$, the norm closed unit ball of $L^\infty[0,1]$. Our aim here is to characterize $S_\mu(\widehat{G})$ in $(L^\infty)_1$ for any locally compact Abelian I-group. First we will present a lemma which uses an idea of [51] on the circle group $\mathbb{T}$. The original proof of the lemma is due to Brown and Moran [11, 14]. In their search for families of idempotents, Brown and Moran used a generalized version of West's construction to produce c-many idempotents in $S_\mu(\widehat{G})$ for any locally compact I-group G. For completeness purposes, we will include the proof of the lemma.

Next, in Theorem 3.2.6, we will use Lemma 3.2.5, to determine the structure of $S_\mu(\widehat{G})$ for any locally compact Abelian I-group. In the case of the circle group, the result is due Bouziad, Lemańczyk and Mentzen [9]. Here we will prove that not only the cardinality of idempotents, but also the structure of the semigroup generalizes to any locally compact Abelian I-group G.

Lemma 3.2.5. *Let G be an I-group. Let $K \subset G$ be a Cantor and Kronecker subset and $\mu \in M(G)$ be a continuous measure whose support is K, as above. Denote by H the set of functions in $L^\infty([0,1],\mu)$ of the form*

$$f_{t,s} = \begin{cases} 1, & \text{on } [0,t) \\ 0, & \text{on } [t,s) \\ 1, & \text{on } [s,1] \end{cases}$$

for some $t, s \in [0,1)$ *such that the interval* $[t, s)$ *has nonempty* C*-interior. Then* H *can be embedded into* $S_\mu(\widehat{G})$.

Proof. Let $\alpha : K \to C$ be the homeomorphism given by the definition of Cantor set, where C is the classical Cantor subset of $[0,1]$. For each $t, s \in C$ with $t < s$, define a continuous function on C by

$$f_{t,s}(x) = \begin{cases} 1, & \text{if } x \in [0,t) \cap C \\ e^{i(t-x)(s-x)}, & \text{if } x \in [t,s) \cap C \\ 1, & \text{if } x \in [s,1] \cap C. \end{cases}$$

Then $g_{t,s} = f_{t,s} \circ \alpha$ is a continuous function on the Kronecker set K, of absolute value 1. Therefore, $g_{t,s}$ is a uniform limit of continuous characters of $\widehat{G}$ restricted to K, i.e. there exist a sequence $\{\gamma_n\}_{n\in\mathbb{N}} \subset G$ with $u - \lim_{n\to\infty} \gamma_n |_K = g_{t,s}$. We will consider the sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ in $L^\infty(K,\mu)$, equipped with its weak*-topology. As $S_\mu(\widehat{G})$ is the closure of characters in this topology, clearly $g_{t,s} \in S_w(G,\mu)$. Write

$$K_{t,s,1} = \alpha^{-1}([0,t)) \cap K, \quad \mu_1 = \mu |_{K_{t,s,1}}$$

$$K_{t,s,2} = \alpha^{-1}([t,s)) \cap K, \quad \mu_2 = \mu |_{K_{t,s,2}}$$

$$K_{t,s,3} = \alpha^{-1}([s,1]) \cap K, \quad \mu_3 = \mu |_{K_{t,s,3}} .$$

It follows that

$$g_{t,s}(x) = \begin{cases} 1, & \text{if } x \in K_{t,s,1} \\ e^{i(t-x)(s-x)}, & \text{if } x \in K_{t,s,2} \\ 1, & \text{if } x \in K_{t,s,3}. \end{cases}$$

Next we define a unitary operator $U_{t,s}$ on $L^2(K,\mu)$ by $U_{t,s}h = g_{t,s}h$. Note that $U_{t,s} = I \oplus V_{t,s} \oplus I$, considered as an operator on $L^2(\mu_1) \oplus L^2(\mu_2) \oplus L^2(\mu_3)$, where $V_{t,s}$ is multiplication by $e^{i(t-.)(s-.)}$. $V_{t,s}$ is unitary and has purely continuous spectrum. It follows from Theorem 4.4 in [51] that 0 is in the weak operator topology closure of the powers of $V_{t,s}$. Therefore, there exists a net of powers of $g_{t,s}$ in the compact semigroup $S_\mu(\widehat{G})$, which converges to $e_{t,s}$, where

$$e_{t,s} = \begin{cases} 1, & \text{on } K_{t,s,1} \\ 0, & \text{on } K_{t,s,2} \\ 1, & \text{on } K_{t,s,3} \end{cases}$$

implying that $e_{t,s} \in S_\mu(\widehat{G})$. Repeating the above construction for any pair $t, s \in C$, $t < s$, we get $H \subset S_\mu(\widehat{G})$ as required.

□

Theorem 3.2.6. *Let G be an I-group. Let $K \subset G$ be a Cantor, Kronecker subset and $\mu \in M(G)$ be a continuous measure whose support is K, as above. Then the compact semitopological semigroup $S_\mu(\widehat{G})$ is isomorphic to $(L^\infty)_1$.*

Proof. By Theorem 3.1.3 it is enough to prove that $S_\mu(\widehat{G}) = (L^\infty(C,\nu))_1$ for a suitable choice of the continuous measure ν. Since K is homeomorphic to the classical Cantor set C, we can (topologically) identify K with C and consider μ as a measure on C. Let $\nu = \mu$. Then any continuous $\mathbb{T}$-valued function on C can be seen as a uniform limit of characters of G, restricted to (the homeomorphic copy of) C. We will prove that any $f \in (L^\infty(C,\mu))_1$ can be approximated by the elements of $S_\mu(\widehat{G})$.

To simplify our notation, we will apply the argument given after Theorem 3.1.3, and replace $(L^\infty(C,\mu))_1$ with $(L^\infty)_1$.

We let $S_1 = \{f = \sum_{i=1}^n a_i\chi_{E_i} : n \in \mathbb{N}, \text{ and } for\ each\ i \in \{1,\dots,n\},\ a_i \in \mathbb{C}\ s.t\ |a_i| \leq 1,$ $E_i\ is\ a\ half\ open\ interval\ of\ [0,1]\}$. Since $[0,1]$ is compact, S_1 is weak* dense in $(L^\infty)_1$. Therefore, it is enough to prove that $S_1 \subset S_\mu(\widehat{G})$.

First, we will prove that any constant function of $(L^\infty)_1$ is in $S_\mu(\widehat{G})$. Let $f = a$, with $|a| < 1$ be given. Considering the constant a on unit disk, let $r = |a|$ and $e^{i\theta}$ be the point where the line joining 0 and a intersects $\mathbb{T}$. Define for each n,

$$g_n = e^{i\theta}\chi_{\bigcup_{k=0}^{2^n-1}[\frac{k}{2^n},\frac{k+r}{2^n})}$$

and put $E_k^n = [\frac{k}{2^n},\frac{k+1}{2^n})$ for $k = 1, 2, \dots, 2^n - 1$.

First, we observe that for each $n \in \mathbb{N}$, $g_n \in S_\mu(\widehat{G})$. Indeed, since K is a Kronecker set, Lemma 3.2.5 guarantees that for all $t, s \in C$, the function

$$f_{t,s} = \begin{cases} 1, & \text{on } [0,t) \\ 0, & \text{on } [t,s) \\ 1, & \text{on } [s,1] \end{cases}$$

is in $S_\mu(\widehat{G})$. Moreover, the Kronecker property clearly implies that any constant $\mathbb{T}$-valued function is also in $S_\mu(\widehat{G})$, which is a semigroup. Hence for any $s, t \in C$ and $e^{i\theta} \in \mathbb{T}$, the function

$$f_{\theta,t,s} = \begin{cases} e^{i\theta}, & \text{on } [0,t) \\ 0, & \text{on } [t,s) \\ e^{i\theta}, & \text{on } [s,1] \end{cases}$$

is in $S_\mu(\widehat{G})$. By multiplying finitely many functions of the above form, we immediately conclude that $g_n \in S_\mu(\widehat{G})$.

To determine the weak* limit of the sequence $\{g_n\}_{n\in\mathbb{N}}$, it suffices to test its action on continuous real valued functions on $[0,1]$. Let $h : [0,1] \to \mathbb{R}$ be continuous, we observe that

$$e^{i\theta}\frac{r}{2^n}\inf_{t\in E_k^n} h(t) \leq \int_{\frac{k}{2^n}}^{\frac{k+r}{2^n}} e^{i\theta}h \leq e^{i\theta}\frac{r}{2^n}\sup_{t\in E_k^n} h(t).$$

We sum over $k = 0, 1, 2, \ldots, 2^n - 1$, to get

$$e^{i\theta} r \sum_{k=0}^{2^n-1} \frac{1}{2^n} \inf_{t \in E_k^n} h(t) \leq \int_{[0,1]} g_n h \leq e^{i\theta} r \sum_{k=0}^{2^n-1} \frac{1}{2^n} \sup_{t \in E_k^n} h(t).$$

Note that the left and right sides of the above inequality are the constant r times the lower and upper Riemann sums of the continuous function h. Hence, by letting $n \to \infty$, we obtain

$$e^{i\theta} r \int_{[0,1]} h \leq \lim_{n\to\infty} \int_{[0,1]} g_n h \leq e^{i\theta} r \int_{[0,1]} h$$

concluding that $w^* - \lim_{n\to\infty} g_n = e^{i\theta} r = a \in S_\mu(\widehat{G})$.

Finally, let $f = \sum_{i=1}^{\ell} a_i \chi_{E_i}$ be a step function such that $0 < |a_i| < 1$ and E_i are disjoint half open intervals in $[0,1]$ for $i = 1, \ldots, \ell$. Without loss of generality assume that $\ell = 2$, that is, $f = a_1\chi_{E_1} + a_2\chi_{E_2}$. Let $r_1 = |a_1|$, $r_2 = |a_2|$ and $e^{i\theta_1}$, $e^{i\theta_2}$ be, respectively, the points on $\mathbb{T}$ chosen as in the previous step. Assume further that the half open intervals are given by $E_1 = [s_1, t_1)$ and $E_2 = [s_2, t_2)$ for $s_1, s_2, t_1, t_2 \in [0,1]$ such that $[s_1, t_1) \cap [s_2, t_2) = \emptyset$. Consider the sequence $\{g_n\}_{n\in\mathbb{N}}$ given by

$$g_n = e^{i\theta_1} \chi_{\bigcup_{k=0}^{2^n-1} [\frac{k(t_1-s_1)}{2^n}, \frac{(k+r_1)(t_1-s_1)}{2^n})} + e^{i\theta_2} \chi_{\bigcup_{k=0}^{2^n-1} [\frac{k(t_2-s_2)}{2^n}, \frac{(k+r_2)(t_2-s_2)}{2^n})}.$$

For each $n \in \mathbb{N}$, note that g_n has only values 0, $e^{i\theta_1}$ or $e^{i\theta_2}$ on half open subintervals of $[0,1]$. Therefore $g_n \in S_\mu(\widehat{G})$ for each n. By a similar computation done in the previous step, we get

$$w^* - \lim_{n\to\infty} g_n = e^{i\theta_1} r\chi_{E_1} + e^{i\theta_2} r\chi_{E_2} = a_1\chi_{E_1} + a_2\chi_{E_2} \in S_\mu(\widehat{G}).$$

Generalizing $\ell = 2$ to $\ell \in \mathbb{N}$, we conclude that $(L^\infty)_1 = S_\mu(\widehat{G})$.

$\square$

3.2.3 Non-discrete Non-I-group Case

Let G be a locally compact non-discrete Abelian non-I-group. As a consequence of Theorem 3.1.2(ii) and Theorem 3.2.4(ii), there exists a continuous measure $\mu \in M(G)$ supported on a Cantor K_q-subset of G. The purpose of this section is to determine the structure of $S_\mu(\widehat{G})$. The class of non-I-groups or particular examples of them have failed to receive as much attention as the class of I-groups. However, in this section we will prove that for non-I-groups, $S_\mu(\widehat{G})$ has a similar structure as with the case of I-groups. As a consequence of Theorem 3.1.2(ii), we know that G contains a closed subgroup isomorphic to $\mathbb{D}_q$ for some integer $q > 1$. Let q be this fixed integer throughout this section. By Theorem 3.2.4(ii) and the remark following Theorem 3.2.4, we get K, a K_q and Cantor subset of G together with a continuous positive probability measure $\mu \in M(G)$, whose support is K. Here we will first device a technique to determine the structure of $S_\mu(\widehat{G})$ when $G = \mathbb{D}_q$, and then for a general non-discrete non-I-group G, we will determine the structure of $S_\mu(\widehat{G})$ as a subsemigroup of $(L^\infty)_1$.

We start by considering $S_\mu(\widehat{G})$ as a compact subsemigroup of $L^\infty(G,\mu)$. By Theorem 3.1.3 applied to the Cantor set K, we identify $S_\mu(\widehat{G})$ as a compact subsemigroup of $L^\infty[0,1]$. For the purposes of the following theorem we define S_q to be the closed convex hull of $\mathbb{Z}_q$ in $\mathbb{C}$. We let $(L^\infty)_{S_q}$ be the subsemigroup of $(L^\infty)_1$ consisting of those $f \in L^\infty[0,1]$ such that there exists a representation of f, whose essential range lies in S_q.

Lemma 3.2.7. *Let $G = \mathbb{D}_q$ and $K \subset G$ be a Cantor, K-set and $\mu \in M(G)$ be a continuous measure with support K, as above. Denote by H the set of functions in $L^\infty([0,1],\lambda)$ of the form*

$$f_{t,s} = \begin{cases} 1, & \text{on } [0,t) \\ 0, & \text{on } [t,s) \\ 1, & \text{on } [s,1] \end{cases}$$

for some $t,s \in [0,1)$ such that the interval $[t,s)$ has nonempty C-interior. Then H can be embedded into $S_\mu(\widehat{\mathbb{D}_q})$.

Proof. Recall that K is a Cantor set. Let $\alpha : K \to C$ denote the homeomorphism onto the classical Cantor subset of $[0,1]$. Replacing K with its image $\alpha(K) = C$, we can consider K as a subset of $[0,1]$. For each $t,s \in C$, note that by the assumption, $[t,s)$ has nonempty

C-interior. We have $\mu([t,s)) \neq 0$, say L. For $x \in K$ let

$$f^1_{t,s}(x) = \begin{cases} 1, & \text{if } x \in [0,t) \cap K \\ (*), & \text{if } x \in [t,s) \cap K \\ 1, & \text{if } x \in [s,1] \cap K \end{cases}$$

where (*) is defined as follows:

Consider the continuous function $F(x) = \mu((t,x])$ for $x \in K \cap [t,s]$. Let $K^1_j = F^{-1}[\frac{L(j-1)}{q}, \frac{Lj}{q}]$ for $j = 1, \ldots, q$, and define $f^1_{t,s} = e^{2\pi ij/q}$ on K^1_j. Since K is totally disconnected, $f^1_{t,s}$ is continuous on K, and as K is a K_q-set, it is a uniform limit of a sequence of continuous characters of $\mathbb{D}_q$, restricted to K.

Suppose that we have defined the functions $\{f^i_{t,s}\}_{i=1}^{n-1}$, we continue with $f^n_{t,s}$ as follows:

$$f^n_{t,s}(x) = \begin{cases} 1, & \text{if } x \in [0,t) \cap K \\ (*)^n_1, & \text{if } x \in K^n_{\alpha_1} \\ \vdots & \vdots \\ (*)^n_{q^n}, & \text{if } x \in K^n_{\alpha_{q^n}} \\ 1, & \text{if } x \in [s,1] \cap K \end{cases}$$

such that each $K^n_{\alpha_i}$ $i = 1, \ldots, q^n$ is an interval subset of K with $\mu(K^n_{\alpha_i}) = \frac{L}{q^n}$. For each i, α_i denotes a sequence of length n. We construct the $K^n_{\alpha_i}$'s by partitioning the $K^{n-1}_{\beta_i}$'s: Given a Cantor interval $K^{n-1}_{\beta_j}$, say $[a,b) \cap K$, with $\mu(K^n_{\beta_j}) = \frac{L}{q^{n-1}}$, we consider the continuous function $F_{\beta_j} = \mu([a,x])$ and put $K^n_{\alpha_i} = F^{-1}_{\beta_i}[\frac{L(k-1)}{q^n}, \frac{Lk}{q^n}]$, where α_i is the sequence whose first $n-1$ coordinates are β_i and n^{th} coordinate is k. In this case, we define $(*)^n_i$ on $K^n_{\alpha_i}$ as the constant function $e^{2\pi ik/q}$. Hence, for each n, $f^n_{t,s}$ is a $\mathbb{Z}_q$-valued continuous function on the K_q set K, so is a uniform limit of a sequences, $\{\gamma^n_m\}_{m\in\mathbb{N}}$ of characters of $\mathbb{D}_q$, i.e $u - \lim_{m\to\infty} \gamma^n_m \mid_K = f^n_{t,s}$ for each $n \in \mathbb{N}$. Therefore, $\{f^n_{t,s}\}_{n\in\mathbb{N}} \subset S_\mu(\widehat{\mathbb{D}_q})$.

Next, we will find the weak*-limit of this sequence in $L^\infty(\mu)$. To this end, it is enough to check its action on the characteristic function of Cantor-intervals, $E \subset K$, say $E = [c,d) \cap K$ with non-empty interior.

Case 1: Assume that there exists $N \in \mathbb{N}$ such that for any $1 \le i \le q^N$, either $K^N_{\alpha_i} \subset E$ or $K^N_{\alpha_i} \cap E = \emptyset$. Note that, for any $n \ge N$, the same statement holds and we observe that the following integral, which we denote by I, satisfies:

$$I = \int_K f^{n+1}_{t,s} \chi_E d\mu = \int_E f^{n+1}_{t,s} d\mu = \int_{[0,t)\cap E} d\mu + \sum_{i=1}^{q^n} \int_{K^n_{\alpha_i}\cap E} f^{n+1}_{t,s} d\mu + \int_{[s,1]\cap E} d\mu$$

Observe that if $K^n_{\alpha_i} \subset E$,

$$\int_{K^n_{\alpha_i}} f^{n+1}_{t,s} d\mu = \int_{K^{n+1}_{\alpha_i,1}} e^{2\pi i/q} d\mu + \ldots + \int_{K^{n+1}_{\alpha_i,q}} e^{2\pi i} d\mu = e^{2\pi i/q}\frac{L}{q^{n+1}} + \ldots + e^{2\pi i}\frac{L}{q^{n+1}} = 0$$

Hence,

$$I = \int_{[0,t)\cap E} d\mu + 0 + \int_{[s,1]\cap E} d\mu$$

Case 2: If no such $N \in \mathbb{N}$ exists, then for each $n \in \mathbb{N}$, there exists (at most 2) $K^n_{\alpha_i}$ such that $K^n_{\alpha_i} \cap E \neq \emptyset$ and $K^n_{\alpha_i} \setminus E \neq \emptyset$, name those as $K^n_{\alpha_l}$ and $K^m_{\alpha_r}$. Let $\varepsilon > 0$ be given. Choose $m \in \mathbb{N}$ such that $\mu(K^m_{\alpha_i}) < \frac{\varepsilon}{2q}$ for each i, then it follows from the observation in Case 1, that

$$\begin{aligned}
|\int_{[t,s)} f^{n+1}_{t,s}\chi_E d\mu| &\leq |\int_{K^n_{\alpha_l}\cap E} f^{n+1}_{t,s} d\mu| + |\int_{K^n_{\alpha_l}\cap E} f^{n+1}_{t,s} d\mu| \\
&\leq |\int_{K^{n+1}_{\alpha_l,1}\cap E} e^{2\pi i/q} d\mu| + \ldots + |\int_{K^{n+1}_{\alpha_l,q}\cap E} e^{2\pi i} d\mu| \\
&\qquad + |\int_{K^{n+1}_{\alpha_r,1}\cap E} e^{2\pi i/q} d\mu| + \ldots + |\int_{K^{n+1}_{\alpha_r,1}\cap E} e^{2\pi i} d\mu| \\
&\leq 2(|\, e^{2\pi i/q}|\frac{\varepsilon}{2q} + \ldots + |e^{2\pi i}|\frac{\varepsilon}{2q}) \\
&= 2q\frac{\varepsilon}{2q} = \varepsilon.
\end{aligned}$$

Hence, in both cases

$$\lim_{n\to\infty} \int_K f^n_{t,s}\chi_E d\mu = \int_{[0,t)\cap E} d\mu + 0 + \int_{[s,1]\cap E} d\mu$$

Therefore,

$$w^* - \lim_{n\to\infty} f^n_{t,s} = e_{t,s} = \begin{cases} 1, & \text{on } [0,t)\cap K \\ 0 & \text{on } [t,s)\cap K \\ 1, & \text{on } [s,1]\cap K \end{cases} \in S_\mu(\widehat{\mathbb{D}_q}).$$

Similar to the I-group case, we repeat the above construction for any pair $t, s \in C$, $t < s$, to get $H \subset S_\mu(\widehat{\mathbb{D}_q})$.

□

Theorem 3.2.8. *Let G be a non-discrete non-I-group. Let q, K and μ be as described above. Then the compact semitopological semigroup $S_\mu(\widehat{G})$, is isomorphic to $(L^\infty)_{S_q}$.*

Proof. As $K \subset \mathbb{D}_q$ by our construction, and K is homeomorphic to the classical Cantor set C, we will again consider K (replacing with its homeomorphic image) as a subset of $[0, 1]$, and we will consider μ as a probability measure on C. Similar to the proof of Theorem 3.2.6, we identify $(L^\infty(C, \mu))_1$ with $(L^\infty)_1$. Furthermore, without loss of generality we will assume that $G = \mathbb{D}_q$. Otherwise $\mathbb{D}_q$ can be identified with a proper closed subgroup of G and a Cantor K_q-subset of this closed subgroup is also a Cantor K_q-subset in G.

We note that any character of $\mathbb{D}_q$ is a $\mathbb{Z}_q$ valued function. Hence under our identification $\widehat{\mathbb{D}_q}$ is a subset of the convex norm-closed semigroup $(L^\infty)_{S_q}$. It follows from Hahn-Banach Theorem that $(L^\infty)_{S_q}$ is weak*-closed. Therefore, the weak*-closure $S_\mu(\widehat{\mathbb{D}_q}) \subset (L^\infty)_{S_q}$. To prove the converse inclusion, we will approximate functions in $(L^\infty)_{S_q}$, by elements of $S_\mu(\widehat{\mathbb{D}_q})$.

We let $S_{S_q} = \{f = \sum_{i=1}^n a_i \chi_{E_i} : n \in \mathbb{N}\ for\ each\ i,\ a_i \in \mathbb{C}\ s.t\ a_i \in S_q$, E_i is a half open interval of $[0, 1]\}$. It is sufficient to prove $S_{S_q} \subset S_\mu(\widehat{G})$.

Let $f = a \in S_{S_q}$ be a constant function. If a is of the form $a = re^{i\theta}$ for some $e^{i\theta} \in \mathbb{Z}_q$ and $0 \le r \le 1$, then from a similar calculation as in the proof of Theorem 3.2.6, as a consequence of Lemma 3.2.7, we see that the sequence $\{g_n\}_{n\in\mathbb{N}}$ given by

$$g_n = e^{i\theta}\chi_{\bigcup_{k=0}^{2^n-1}[\frac{k}{2^n},\frac{k+r}{2^n})}$$

converges in the relative weak*-topology to $re^{i\theta} = a$. If a is a convex combination of two q^{th} roots of unity, i.e. $a = r_1e^{i\theta_1} + r_2e^{i\theta_2}$, where $0 \le r_1, r_2 \le 1$, $r_1 + r_2 = 1$ and $\theta_1, \theta_2 \in \mathbb{Z}_q$, then for each n, we define

$$g_n = e^{i\theta_1}\chi_{\bigcup_{k=0}^{2^n-1}[\frac{k}{2^n},\frac{k+r_1}{2^n})} + e^{i\theta_2}\chi_{\bigcup_{k=0}^{2^n-1}[\frac{k+r_1}{2^n},\frac{k+1}{2^n})}. \ (*)$$

Then the weak*-limit of $\{g_n\}_{n\in\mathbb{N}}$ is $r_1e^{i\theta_1} + r_2e^{i\theta_2} = a$. For a general constant $a = \sum_{j=1}^{\ell} r_je^{i\theta_j}$, with $\sum_{j=1}^{\ell} r_j = 1$, we adapt the sequence given by $(*)$ accordingly. Therefore any constant function $f \in S_{S_q}$ is in $S_\mu(\widehat{\mathbb{D}_q})$.

We let $f = \sum_{i=1}^{n} a_i\chi_{E_i} \in S_{S_q}$ be a step function. We write for each $i = 1, \ldots, n$ the disjoint half open intervals as $E_i = [t_i, s_i)$ the constants (without loss of generality) as $a_i = r_ie^{i\theta_i} + p_ie^{i\phi_i}$, for some $0 \le r_i, p_i \le 1$, $r_i + p_i = 1$ and $\theta_i, \phi_i \in \mathbb{Z}_q$. We consider for each $n \in \mathbb{N}$,

$$g_n = \sum_{i=1}^{n}(e^{i\theta_i}\chi_{\bigcup_{k=0}^{2^n-1}[\frac{k(s_i-t_i)}{2^n},\frac{(k+r_1)(s_i-t_i)}{2^n})} + e^{i\phi_i}\chi_{\bigcup_{k=0}^{2^n-1}[\frac{(k+r_i)(s_i-t_i)}{2^n},\frac{(k+1)(s_i-t_i)}{2^n})}).$$

Hence, we observe that $w^* - \lim_{n\to g_n} = \sum_{i=1}^{n} a_i\chi_{E_i} = f \in S_\mu(\widehat{\mathbb{D}_q})$. Therefore, $S_\mu(\widehat{\mathbb{D}_q})$ is isomorphic to $(L^\infty)_{S_q}$.

For a general non-discrete non-I-group G, as noted above, with $K \subset \mathbb{D}_q \subset G$, and μ supported on K, we get $S_\mu(\widehat{G})$ is isomorphic to $(L^\infty)_{S_q}$, as required. $\square$

3.3 Consequences

We have constructed compact semitopological semigroups $S_\mu(\widehat{G})$, depending on the properties of non-discrete locally compact Abelian groups G. We have already observed that for each G, $S_\mu(\widehat{G})$ is a semitopological semigroup compactification of the dual group $\widehat{G}$, that is a quotient of both the Eberlein compactification $(\widehat{G})^e$ and the weakly almost periodic compactificaton $(\widehat{G})^w$ of $\widehat{G}$. First we will consider the consequences of our construction on the structure of idempotents of $(\widehat{G})^e$ and $(\widehat{G})^w$.

Let $(L^\infty)_{\{0,1\}}$ denote set of all $f \in L^\infty[0,1]$ which have a representation whose essential range is a subset of $\{0,1\}$. The structure Theorem 3.2.6 and Theorem 3.2.8 clearly imply that in both cases the idempotents of $S_\mu(\widehat{G})$ are given by $(L^\infty)_{\{0,1\}}$. Hence, we have:

Corollary 3.3.1. *Let G be a non-discrete locally compact Abelian group. Then $S_\mu(\widehat{G})$ contains uncountably many idempotents.*

Proof. Together with the above observation, it is enough to note that the set $(L^\infty)_{\{0,1\}}$ is the set of characteristic functions of Borel subsets of $[0,1]$, whose cardinality is c. $\square$

The approximation technique used in the proofs of Theorem 3.2.6 and Theorem 3.2.8 allows us determine the closure of the idempotents in the compact semitopological semigroups $S_\mu(\widehat{G})$, for any G. We define $(L^\infty)_{[0,1]}$ to be the set of all $f \in L^\infty[0,1]$ which has a representation whose essential range is a subset of $[0,1]$.

Corollary 3.3.2. *Let G be a non-discrete locally compact Abelian group. The set of idempotents in $S_\mu(\widehat{G})$ is not closed.*

Proof. By the above observation, the closure of idempotents of $S_\mu(\widehat{G})$ is $(L^\infty)_{[0,1]}$, which immediately gives the result. □

Furthermore, we know that the pointwise multiplication on $(L^\infty)_{[0,1]}$ is not jointly continuous. Hence, as a consequence of the above corollary, we get for any locally compact Abelian group $\widehat{G}$, the subsemigroup of idempotents of the semitopological semigroup compactifications $S_\mu(\widehat{G})$ has only separately continuous multiplication.

For a locally compact Abelian group G, a character $\gamma \in \widehat{G}$, can be considered as an element of $M(G)^*$ via the action,

$$\mu \to \int_G \gamma d\mu$$

In fact, this identification gives a character in $M(G)^*$. Furthermore, we can identify the closure of characters in $M(G)^*$ as the closure of unitaries from the universal representation in $W^*(G)$ and also as certain types of multiplication operators. As a corollary of [49] Thm 3.13, we observe that for any locally compact Abelian G, the closure of $\widehat{G}$ in $M(G)^*$ is isomorphic to $(\widehat{G})^e$. Therefore, depending on the structure of the group G, $(L^\infty)_1$ or $(L^\infty)_{S_q}$ embeds as a quotient of $(\widehat{G})^e$.

Remark. We note that if the topology of G is not second countable, then the existence of uncountably many disjoint open sets allows us to repeat the construction for uncountably many $S_w(\mu)$. Therefore, in this case, the closure, clG, of G in $\Delta(G)$ contains $2^{\mathfrak{c}}$ many idempotents.

On the other hand, if G is a σ-compact Abelian group, then the cardinality of the set of Borel subsets of G is $\mathfrak{c}$. Therefore for any $\mu \in M(G)$, the idempotents in $(L^\infty)_1$ is of cardinality at most $\mathfrak{c}$. Therefore, when we restrict our attention to the coordinates of the generalized characters on G, we observe that each coordinate can contain at most $\mathfrak{c}$ idempotents. However, the exact cardinality of $I(clG) = I(G^e)$ is still unknown for a general locally compact Abelian group G.

Chapter 4

Functorial Properties of the Eberlein Compactification

In this chapter we look at the question of constructing the Eberlein compactification (ε, G^e) of a general locally compact group G, from semigroup compactifications of its closed subgroups. In particular, given a closed subgroup H, we ask whether $\overline{\varepsilon(H)} \cong H^e$. We can formulate these questions in terms of the underlying function algebras $\mathcal{E}(G)$ and $\mathcal{E}(H)$. If we can positively answer the second question cited, it immediately follows that $\mathcal{C}(\overline{\varepsilon(H)}) \cong \mathcal{C}(H^e) \cong \mathcal{E}(H)$. Hence, our initial problem can be reformulated to ask whether the restriction map from $\mathcal{E}(G)$ into $\mathcal{E}(H)$ is onto. That is, whether every function in the

Eberlein algebra of H can be extended to an Eberlein function on the whole group G. For a general locally compact group and an arbitrary closed subgroup, it is not always possible. However, $\overline{\varepsilon(H)}$ is always a semigroup compactification of H. It follows from the universal property of H^e that $\overline{\varepsilon(H)}$ is a quotient of H^e. Our aim in this chapter is to study the relation of H^e with its quotient $\overline{\varepsilon(H)}$, and under special conditions construct G^e in terms of H^e.

4.1 Closed Normal Subgroups

Let G be a locally compact group. In this section, we will consider a closed subgroup N, which is also normal in G. In this case, we will first prove that the restrictions to N of functions in $\mathcal{E}(G)$, denoted by $\mathcal{E}(G)|_N$ is a closed subspace of $\mathcal{E}(N)$. Then following the technique of Michael Cowling and Paul Rodway in [17], we will characterize $\mathcal{E}(G)|_N$ as a subset of $\mathcal{E}(N)$. The extensions of functions from $\mathcal{E}(N)$, that will be constructed in the proof come from a variation of a device of [42]. Next, we will restrict our attention to two special cases, namely when G/N is compact, and when N itself is compact.

Lemma 4.1.1. *Let G be a locally compact group with a closed normal subgroup N. Then $\mathcal{E}(G)|_N$ is a (norm) closed subspace of $\mathcal{E}(N)$.*

Proof. Let $H^{\perp}$ denote the closed ideal of $\mathcal{E}(G)$ consisting of the functions that are identically 0 on H. We consider the quotient space $\mathcal{E}(G)/H^{\perp}$. Note that if $f, g \in \mathcal{E}(G)$ such that $f \in g + H^{\perp}$, then for $t \in H$, we have

$$|f(t)| = |g(t)| \leq \|g\|_{\infty}$$

So,

$$|f(t)| \leq \inf_{g \in f + H^{\perp}} \|g\|_{\infty} = \|f + H^{\perp}\|$$

Then $\sup_{t \in H} |f(t)| \leq \|f + H^{\perp}\|$, that is

$$\|f|_H\|_{\infty} \leq \|f + H^{\perp}\| \tag{4.1}$$

By uniform continuity of $f \in \mathcal{E}(G)$, for $\varepsilon > 0$, we find a compact neighborhood V_{ε} of e in G such that if $ts^{-1} \in V_{\varepsilon}$ and $s \in H$, we have

$$|f(t) - f(s)| < \varepsilon$$

So,

$$|f(t)| \leq |f(t) - f(s)| + |f(s)| \leq \|f + H^{\perp}\| + \varepsilon \tag{4.2}$$

Let $\pi : G \to G/H$ be the quotient map, choose f_0 in $C_0(G/H)$ that vanishes off $\pi(V_{\varepsilon})$, is 1 at $\pi(H)$ and is bounded by 1. Then $f_0 \circ \pi$ is also bounded by 1, is identically 1 on H

and vanishes off HV_ε. Then for any $f \in \mathcal{E}(G)$, $f(f_0 \circ \pi) \in f + H^\perp$ so as a consequence of (4.2), we get

$$\|f + H^\perp\| \leq \|f(f_0 \circ \pi)\|_\infty \leq \sup_{t \in HV_\varepsilon} |f(t)| \leq \|f|_H\|_\infty + \varepsilon \tag{4.3}$$

which together with (4.1) implies

$$\|f + H^\perp\| = \|f|_H\|_\infty$$

Since $\mathcal{E}(G)/H^\perp$ is complete, its isometric image $\mathcal{E}(G)|_H$ under the restriction map is complete in $\mathcal{E}(H)$, hence uniformly closed there. □

Throughout this section, we assume that dx, dn and $d\dot{x}$ denote the normalized Haar measures on G, N and G/N, respectively, such that for any compactly supported continuous function w on G, we have

$$\int_{G/N} \int_N w(gn) dn d\dot{x} = \int_G w(g) dx. \tag{4.4}$$

For the purposes of next theorem, given functions u on G, f on N and an element x in G, we define u^x on G and f^x on N by

$$u^x(y) = u(x^{-1}yx) \tag{4.5}$$

$$f^x(n) = f(x^{-1}nx) \tag{4.6}$$

for any $y \in G$ and $n \in N$. Let $x \in G$ be fixed, we define $\phi^x : \mathcal{E}(G) \to \mathcal{C}_b(G)$ by $\phi^x(u) = u^x$. Then ϕ^x is clearly an algebra homomorphism into $\mathcal{C}_b(G)$. Next note that $\|u^x\|_\infty = \|u\|_\infty$, hence ϕ^x is isometric. We also claim that the image of ϕ^x is $\mathcal{E}(G)$. Indeed, first let $u \in B(G)$, and $u(y) = \langle \pi(y)\xi, \eta \rangle$ be a representation of u. Then for any $x \in G$,

$$u^x(y) = u(x^{-1}yx) = \langle \pi(y)\pi(x)\xi, \pi(x)\eta \rangle$$

which implies that $u^x \in B(G)$. Furthermore, if $u \in \mathcal{E}(G) \setminus B(G)$, we take a sequence $\{u_n\}_{n\in\mathbb{N}}$ in $B(G)$ uniformly converging to u. Then

$$\|u_n^x - u^x\|_\infty = \|(u_n - u)^x\|_\infty = \|u_n - u\|_\infty$$

So, $u^x \in \mathcal{E}(G)$ and since $\phi^{x^{-1}}$ is the inverse of ϕ^x, we conclude that ϕ^x is an isometric isomorphism of $\mathcal{E}(G)$ onto itself. Furthermore, let $u \in \mathcal{E}(G)$ be fixed. We put $\varphi^u(x) = u^x$ for any $x \in G$. Then the map φ^u defined on G with values in $\mathcal{E}(G)$ is continuous. Indeed, consider a net $\{x_\alpha\}_{\alpha\in I}$ converging to an element $x \in G$. Let $\varepsilon > 0$ be given. By uniform continuity of u, we choose a neighborhood V_ε of e in G such that for any $x, y \in G$ with $x \in V_\varepsilon y V_\varepsilon$ we have

$$|u(x) - u(y)| < \varepsilon$$

Since both nets $\{x_\alpha x^{-1}\}_{\alpha\in I}$ and $\{x x_\alpha^{-1}\}_{\alpha\in I}$ converge to e, we can choose α_0 sufficiently large so that for any $\alpha \geq \alpha_0$ both $x_\alpha x^{-1}$ and $x x_\alpha^{-1}$ are in V_ε. Hence, $x_\alpha x^{-1} y x x_\alpha^{-1} \in V_\varepsilon y V_\varepsilon$ for any $y \in G$, so $|u(y) - u(x_\alpha x^{-1} y x x_\alpha^{-1})| < \varepsilon$, that is $\|u^{x_\alpha} - u^x\|_\infty \leq \varepsilon$ for any $\alpha \geq \alpha_0$.

The following Theorem is the analogue of Theorem 1 in [17].

Theorem 4.1.2. *Let N be a closed normal subgroup of a locally compact group G. Then*

$$\mathcal{E}(G)|_N = \{f \in \mathcal{E}(N) : \|f^x - f\|_\infty \to 0 \text{ as } x \to e\}$$

and if $f \in \mathcal{E}(G)|_N$, then

$$\|f\|_\infty = inf\{\|u\|_\infty : u \in \mathcal{E}(G) \text{ such that } u|_N = f\}.$$

Proof. By Theorem 2.4.1 applied to the natural injection of N into G, we conclude that $\mathcal{E}(G)|_N \subset \mathcal{E}(N)$. Furthermore, for any $u \in \mathcal{E}(G)$, we clearly have $\|u|_N\|_\infty \leq \|u\|_\infty$. Note that $u^x = [L_{x^{-1}} \circ R_x](u)$. Since $\mathcal{E}(G)$ is invariant under translations, $u \in \mathcal{E}(G)$ implies that $u^x \in \mathcal{E}(G)$ for all $x \in G$. Let $\{x_\alpha\}_{\alpha \in I}$ be a net converging to e in G. Then for $u \in \mathcal{E}(G)$ and each $\alpha \in I$,

$$\|u^{x_\alpha} - u\|_\infty = \|(L_{g_\alpha^{-1}} \circ R_{g_\alpha})(u) - u\|_\infty$$

By the uniform continuity of u and the translation invariance of the algebra of uniformly continuous functions $\|u^{x_\alpha} - u\|_\infty \to 0$ as α tends to infinity. Therefore,

$$\mathcal{E}(G)|_N \subset \{f \in \mathcal{E}(N) : \|f^x - f\|_\infty \to 0 \text{ as } x \to e\}.$$

Conversely, we want to show that any $f \in \mathcal{E}(N)$ satisfying $\|f^x - f\|_\infty \to 0$ as $x \to e$ has an extension to G. We claim that it is enough to prove that for any such $f \in \mathcal{E}(N)$ and $\varepsilon > 0$, there is $u \in \mathcal{E}(G)$ such that $\|u\|_\infty = \|f\|_\infty$ and $\|u|_N - f\|_\infty < \varepsilon$.

Indeed, if we can prove the existence of such u, then f is in the closure of the image (under the restriction map) of the closed ball centered at 0, with radius $\|f\|_\infty$. An application of the proof of Open Mapping Theorem to the restriction map on the Banach space $\mathcal{E}(G)$ implies that f is in the image of the closed ball centered at 0, with radius $\|f\|_\infty$, as required.

Let $f \in \mathcal{E}(N)$ and $\varepsilon > 0$ be given. By uniform continuity of f, we choose a neighborhood U of e in G such that

$$\|f^x - f\|_\infty < \frac{\varepsilon}{2} \quad \text{for all } x \in U \tag{4.7}$$

and a neighborhood O of e in N such that

$$\|L_n f - f\|_\infty < \frac{\varepsilon}{2} \quad \text{for all } n \in O. \tag{4.8}$$

Let V be a compact neighborhood of e such that $V \subset U$ and $V^{-1}V \cap N \subset O$. Furthermore, let v be a nonnegative continuous function on G such that $supp(v) \subset V$ and satisfies the integral equality:

$$1 \;=\; \int_{G/N} [\int_N v(xn)dn]^2 d\dot{x} \tag{4.9}$$

where dn and $d\dot{x}$ are the normalized Haar measures on N and G/N, as noted in (4.4). Let

dx denote the corresponding Haar measure of G and observe that

$$\begin{aligned}
\int_{G/N}[\int_N v(xn)dn]^2 d\dot{x} &= \int_{G/N}(\int_N v(xn)dn)(\int_N v(xn)dn)d\dot{x} \\
&= \int_{G/N}(\int_N v(xn')dn')(\int_N v(xn)dn)d\dot{x} \\
&= \int_{G/N}(\int_N v(xn')dn')(\int_N v(xn'n)dn)d\dot{x} \\
&= \int_{G/N}\int_N(\int_N v(xn')v(xn'n)dn)dn'd\dot{x} \\
&= \int_G\int_N v(x)v(xn)dndx.
\end{aligned}$$

We define a function u on G by

$$u(y) = \int_G\int_N v(yx)v(xn)f(n)dndx. \qquad (4.10)$$

First we study the restriction of u to N and the value of $\|u|_N - f\|_\infty$. Let n' be an element of N.

$$\begin{aligned}
u(n') &= \int_G\int_N v(x)v(n'^{-1}xn)f(n)dndx \\
&= \int_G\int_N v(x)v(x(x^{-1}n'^{-1}x)n)f(n)dndx \\
&= \int_G\int_N v(x)v(xn)f(x^{-1}n'xn)dndx \\
&= \int_G\int_N v(x)v(xn)[L_n f]^{xn}(n')dndx.
\end{aligned}$$

Consider the continuous and compactly supported map $\phi : G$ x $N \to \mathcal{E}(G)$ given by

$(x, n) \mapsto v(x)v(xn)[L_n f]^{xn}$. We immediately see that, its vector-valued integral

$$\int_G \int_N \phi((x, n)) dndx = \int_G \int_N v(x)v(xn)[L_n f]^{xn} dndx$$

exists and by the above calculation it equals $u|_N$.

By the choice of the function v, we have

$$f1 = f \int_G \int_N v(x)v(xn) dndx.$$

Hence,

$$\begin{aligned}
\|u|_N - f\|_\infty &= \| \int_G \int_N v(x)v(xn)[L_n f]^{xn} dndx - f \int_G \int_N v(x)v(xn) dndx \|_\infty \\
&\leq \int_G \int_N v(x)v(xn) \| [L_n f]^{xn} - f \|_\infty dndx \\
&\leq \int_G \int_N v(x)v(xn) [\| [L_n f]^{xn} - f^{xn} \|_\infty + \| f^{xn} - f \|_\infty] dndx \\
&= \int_G \int_N v(x)v(xn) [\| [L_n f] - f \|_\infty + \| f^{xn} - f \|_\infty] dndx.
\end{aligned}$$

Since $supp(v) \subset V$, $v(x)v(xn) \neq 0$ implies that both $x \in V \subset U$ and $xn \in V$, that is, $n \in x^{-1}V \cap N \subset V^{-1}V \cap N \subset O$. Then, by (4.7) and (4.8) we get

$$\|u|_N - f\|_\infty < \int_G \int_N v(x)v(xn)(\frac{\varepsilon}{2} + \frac{\varepsilon}{2}) dndx = \varepsilon.$$

Next, we will show that u is in $\mathcal{E}(G)$ whenever f is in $\mathcal{E}(N)$ and $\|u\|_\infty \leq \|f\|_\infty$. Let

$y \in G$, then

$$\begin{aligned} |u(y)| &= |\int_G \int_N v(yx)v(xn)f(n)dndx| \\ &\leq \int_G \int_N v(yx)v(xn)|f(n)|dndx \\ &\leq \int_G \int_N v(yx)v(xn)\|f\|_\infty dndx = \|f\|_\infty. \end{aligned}$$

Therefore, $\|u\|_\infty \leq \|f\|_\infty$. Finally, we need to prove that $u \in \mathcal{E}(G)$. We divide its proof into two cases: $f \in B(N)$ and $f \in \mathcal{E}(N) \setminus B(N)$.

Case 1: Suppose that $f \in B(N)$. We repeat, for benefit of reader, we adopt the technique in [17] by Michael Cowling and Paul Rodway, where this case, was in fact, proved. Here f is a coefficient function of a unitary representation π of N on a Hilbert space $\mathcal{H}_\pi$, say

$$f(n) = \langle \pi(n)\xi, \eta \rangle$$

for vectors $\xi, \eta \in \mathcal{H}_\pi$ and $n \in N$. Then for any $y \in G$, by definition of u,

$$\begin{aligned}
u(y) &= \int_G \int_N v(yx)v(xn)f(n)dndx \\
&= \int_G \int_N v(x)v(y^{-1}xn)f(n)dndx \\
&= \int_{G/N} \int_N \int_N v(xn')v(y^{-1}xn'n)\langle \pi(n)\xi, \eta\rangle dndn'd\dot{x} \\
&= \int_{G/N} \int_N \int_N v(xn')v(y^{-1}xn)\langle \pi(n'^{-1}n)\xi, \eta\rangle dndn'd\dot{x} \\
&= \int_{G/N} \int_N \int_N v(xn')v(y^{-1}xn)\langle \pi(n)\xi, \pi(n')\eta\rangle dndn'd\dot{x} \\
&= \int_{G/N} \langle \int_N v(y^{-1}xn)\pi(n)\xi dn, \int_N v(xn')\pi(n')\eta dn'\rangle d\dot{x}
\end{aligned}$$

which is a coefficient function of $Ind_G^N \pi$ of G on $\mathcal{H}_\pi$, induced from the representation π on N see [23]. So, $u \in B(G)$.

Case 2: Now suppose that $f \in \mathcal{E}(N) \setminus B(N)$, then there is a sequence $\{f_n\}_{n\in\mathbb{N}} \subset B(N)$ uniformly converging to f. Let $y \in G$, then

$$|u_n(y) - u(y)| \leq \|f_n - f\|_\infty$$

by the above computation, hence, $u_n \to u$ uniformly as $n \to \infty$. Since for each $n \in \mathbb{N}$ $u_n \in B(G)$, we get $u \in \mathcal{E}(G)$.

$\square$

4.1.1 Closed Normal Subgroups with Compact Quotient

Let G be a locally compact group, with a closed subgroup N. Assume that the quotient group G/N is compact. Here our aim is to construct the Eberlein compactification G^e of G, from N^e, under special conditions. The construction of the semigroups depends on an idea of Hahn [25] and has been studied in [32] for many compactifications such as almost periodic, weakly almost periodic and left uniformly continuous compactifications of G. Here we apply their technique to Eberlein compactification of G. We start by constructing a compact semitopological semigroup.

Let (ψ, N^e) denote the Eberlein compactification of N. Let e denote the identity element of G and 1 be $\psi(e)$ in N^e. Consider the (direct product) semigroup $G \times N^e$, with the product topology. We define a relation ρ on $G \times N^e$ by

$$(x, s)\rho(y, t) \text{ if and only if } y^{-1}x \in N \text{ and } \psi(y^{-1}x)s = t \tag{4.11}$$

Then ρ is an equivalence relation. Indeed, let $(x, s), (y, t), (z, u) \in G \times N^e$.

(i) Clearly $x^{-1}x = e \in N$ and $\psi(x^{-1}x)s = s$, that is $(x, s)\rho(x, s)$.

(ii) If $(x, s)\rho(y, t)$, that is $y^{-1}x \in N$ and $\psi(y^{-1}x)s = t$. Hence $x^{-1}y = (y^{-1}x)^{-1} \in N$ and $\psi(x^{-1}y)t = \psi(x^{-1}y)\psi(y^{-1}x)s = \psi(x^{-1}yy^{-1}x)s = s$, since ψ is a homomorphism on N. Hence $(y, t)\rho(x, s)$.

(iii) If $(x, s)\rho(y, t)$ and $(y, t)\rho(z, u)$, then $y^{-1}x, z^{-1}y \in N$ and $\psi(y^{-1}x)s = t$, $\psi(z^{-1}y)t = u$.

Hence, $z^{-1}x = z^{-1}yy^{-1}x \in N$ and $\psi(z^{-1}x)s = \psi(z^{-1}y)\psi(y^{-1}x)s = \psi(z^{-1}y)t = u$. So, $(x,s)\rho(z,u)$.

Let $(x,s) \in G \times N^e$, we denote its equivalence class with respect to ρ as $[(x,s)]$, we observe that it can be written as

$$\begin{aligned}
[(x,s)] &= \{(y,t) \mid (x,s)\rho(y,t)\} \\
&= \{(y,t) \mid y^{-1}x \in N \text{ and } \psi(y^{-1}x)s = t\} \\
&= \{(y,t) \mid y^{-1} = rx^{-1} \text{ for some } r \in N \text{ and } \psi(r)s = t\} \\
&= \{(xr^{-1}, \psi(r)s) \mid r \in N\}.
\end{aligned}$$

Let $\pi : G \times N^e \to (G \times N^e)/\rho : (x,s) \mapsto [(x,s)]$ denote the quotient map. Consider $(e,s),(e,t) \in G \times N^e$, then $(e,s)\rho(e,t)$ implies that $s = \psi(e)s = t$. Hence when restricted to $\{e\} \times N^e$, π is an injection. From now on we will identify N^e with its image $\{e\} \times N^e$, in $(G \times N^e)/\rho$.

We equip $(G \times N^e)/\rho$ with the quotient topology. In the next proposition we study the properties of this topological space.

Proposition 4.1.3. *The quotient space $(G \times N^e)/\rho$ is locally compact and Hausdorff. The quotient map $\pi : G \times N^e \to (G \times N^e)/\rho$ is an open mapping. If G/N is compact then $(G \times N^e)/\rho$ is also compact.*

Proof. First we will prove that the graph of ρ is closed. Take two convergent nets of $G \times N^e$, say $(x_\alpha, s_\alpha) \to (x, s)$ and $(y_\alpha, t_\alpha) \to (y, t)$. Suppose that $(x_\alpha, s_\alpha)\rho(y_\alpha, t_\alpha)$ for each α. That is, $y_\alpha^{-1}x_\alpha \in N$ and $\psi(y_\alpha^{-1}x_\alpha)s_\alpha = t_\alpha$ for each α. Then by continuity of the multiplication and inversion on N, $y_\alpha^{-1}x_\alpha \to y^{-1}x$ and as N is closed, $y^{-1}x \in N$. Now, since the multiplication in N^e is jointly continuous on the image of N, we get $t_\alpha = \psi(y_\alpha^{-1}x_\alpha)s_\alpha \to \psi(y^{-1}x)s$, that is $\psi(y^{-1}x)s = t$, as required.

To prove the second claim, let $O \subset G \times N^e$ be open. We want to show that $\pi(O)$ is open in $(G \times N^e)/\rho$. By definition of the quotient topology, we need to show that $\pi^{-1}(\pi(O))$, namely the union of ρ-classes of elements of O, is open in $G \times N^e$. Let $(y, t) \in \pi^{-1}(\pi(O))$, then $(y, t) = (xr^{-1}, \psi(r)s)$ for some $(x, s) \in O$ and $r \in N$. We choose open neighborhoods $V \subset G$ and $W \subset N^e$ of x and s, respectively such that

$$(x, s) \in V \times W \subset O.$$

Then $Vr^{-1} \times \psi(r)W$ is open in $G \times N^e$, contains (y, t) and is contained in $\pi^{-1}(\pi(O))$, establishing our claim.

Now, we will prove that $(G \times N^e)/\rho$ is Hausdorff. Let $P_i = \pi(x_i, s_i)$, $i = 1, 2$ be points in $(G \times N^e)/\rho$ such that every neighborhood of P_1 intersects every neighborhood of P_2. By the above paragraph, π is open, and hence a neighborhood base for each P_i is given by

$$\{\pi(U) \mid U \text{ is a neighborhood of } (x_i, s_i)\}$$

for $i = 1, 2$. By our assumption we can choose nets $\{(x^i_\alpha, s^i_\alpha)\}_{\alpha \in I}$ converging in $G \times N^e$ to (x_i, s_i), respectively and satisfying $(x^1_\alpha, s^1_\alpha)\rho(x^2_\alpha, s^2_\alpha)$ for each $\alpha \in I$. Since ρ is closed we get $(x_1, s_1)\rho(x_2, s_2)$, which means $P_1 = P_2$ in $(G \times N^e)/\rho$.

Note that by the continuity of π, if V is a compact neighborhood of x in G, then $\pi(V \times N^e)$ is a compact neighborhood of (x, s) for any $s \in N^e$. Hence, the local compactness of $(G \times N^e)/\rho$ follows.

Finally, assume that G/N is a compact group, then there exists a compact subset K of G such that $G = KN$. Since π is continuous $\pi(K \times N^e)$ is also compact. The result will follow once we show that $\pi(K \times N^e) = \pi(G \times N^e) = (G \times N^e)/\rho$. Let $(x, s) \in G \times N^e$, then $x = yr$ for some $y \in K$, $r \in N$ and

$$\pi(x, s) = \pi(y, \psi(r)s) \in \pi(K \times N^e)$$

as required. □

Let $\mu : G \to G \times N^e$ be given by $\mu(x) = (x, 1)$. Consider the composition $\pi \circ \mu : G \to G \times \{1\} \to \pi(G \times \{1\}) : x \mapsto [(x, 1)]$.

Lemma 4.1.4. *$\pi \circ \mu$ is a continuous map onto $\pi(G \times \psi(N))$. As $\psi : N \to N^e$ is a homeomorphism it follows that $\pi \circ \mu$ is also a homeomorphism. Moreover, G is homeomorphic to an open subset of $(G \times N^e)/\rho$.*

Proof. Being the composition of continuous maps, $\pi \circ \mu$ is continuous. To, prove that $\pi \circ \mu$ is onto, let $(x, \psi(n)) \in G \times \psi(N)$. Then $n^{-1} = n^{-1}x^{-1}x \in N$ and $\psi(n^{-1})\psi(n) = 1$, so $(x, \psi(n))\rho(xn, 1)$. That is, $(x, \psi(n)) \in \pi \circ \mu(G)$.

Note that injectivity of ψ implies that if $(x, 1)\rho(y, 1)$ for some $x, y \in G$, that is $\psi(y^{-1}x) = 1$, we must have $y = x$. So, $\pi \circ \mu$ is also injective.

Next, we claim that $\pi \circ \mu$ is open. To this end, let $V \subset G$ be open. We need to show that $\pi^{-1}(\pi(\mu(V)))$ is open in $G \times N^e$. But

$$\begin{aligned} \pi^{-1}(\pi(\mu(V))) &= \{(x, s) \mid (x, s)\rho(y, 1) \text{ for some } y \in V\} \\ &= \{(x, s) \mid y^{-1}x \in N,\ \ \psi(y^{-1}x)s = 1 \text{ for some } y \in V\}. \end{aligned}$$

Since $\psi(y^{-1}x)s = 1$ means $\psi(x^{-1}y) = s$, we get

$$\pi^{-1}(\pi(\mu(V))) = \{(x, \psi(r)) \mid r \in N \text{ and } xr \in V\}$$

which is open in $G \times N^e$, since the compactification map ψ is open. □

We have constructed a compact Hausdorff space as a quotient of $G \times N^e$, given that N is a closed normal subgroup of G, with compact quotient. Furthermore, we have embedded G homeomorphically into it, with dense image $(G \times \psi(N))/\rho$. Next, we want to extend the group operation to $(G \times N^e)/\rho$ to make it into a semigroup. However, it is not always possible, we need the Eberlein compactification N^e of N, to be compatible with the action of G on N. We say, N^e is compatible with G if for each $x \in G$, the function $\sigma_x : N \to N : n \mapsto x^{-1}nx$, extends to a continuous function $\tilde{\sigma_x} : N^e \to N^e$. This condition can be reformulated as: if a net $\{\psi(r_\alpha)\}_{\alpha \in I}$ of elements in the image of N converges, then also the net $\{\psi(\sigma_x(r_\alpha))\}_{\alpha \in I}$ converges in N^e. The compatibility of N^e with G implies that each $\widetilde{\sigma_x}$, determines a continuous transformation of N^e.

Lemma 4.1.5. *If N^e is compatible with G, then for any $x \in G$, $\widetilde{\sigma_x}$ is a continuous automorphism of N^e.*

Proof. Let $x \in G$ be fixed. It is easily seen that $\widetilde{\sigma_x}$ is a homeomorphism of N^e onto itself, where the inverse map is given by $\widetilde{\sigma_{x^{-1}}}$. Furthermore, since σ_x is a homomorphism on N, when restricted to $\psi(N)$, $\widetilde{\sigma_x}$ is also multiplicative. Since N^e is semitopological, we first observe that $\widetilde{\sigma_x}$ satisfies $\widetilde{\sigma_x}(st) = \widetilde{\sigma_x}(s)\widetilde{\sigma_x}(t)$ for any $s \in \psi(N)$ and $t \in N^e$, and next conclude that $\widetilde{\sigma_x}(st) = \widetilde{\sigma_x}(s)\widetilde{\sigma_x}(t)$ for any $s, t \in N^e$, as required. □

In the rest of this section we further assume that for G and N as above, N^e is compatible with G. To simplify our notation for each $x \in G$, we will denote the extension $\widetilde{\sigma_x}$ also by σ_x. We define a semidirect multiplication on $G \times N^e$ by

$$(x, s)(y, t) = (xy, \sigma_y(s)t). \tag{4.12}$$

Lemma 4.1.6. *Let G, N, N^e be as above. Suppose that the map $x \mapsto \sigma_x(s) : G \to N^e$ is continuous for all $s \in N^e$. Then $G \times N^e$ is a semitopological semigroup and the multiplication (4.12) restricted to*

$$(G \times N) \times (G \times N^e) \to G \times N^e$$

is jointly continuous. Furthermore ρ is a congruence relation with respect to the multiplication (4.12).

Proof. The continuity results are consequences of Ellis' Theorem ([21] or [46] Chapter 2), together with the fact that N^e is a compact semitopological semigroup. Let $(x, s), (y, t)$ and $(z, u) \in G \times N^e$. Assume that $(x, s)\rho(y, t)$, which means $y^{-1}x \in N$ and $\psi(y^{-1}x)s = t$. So, by (4.12)

$$(x, s)(z, u) = (xz, \sigma_z(s)u) \text{ and } (y, t)(z, u) = (yz, \sigma_z(t)u).$$

By normality of N, we have $z^{-1}y^{-1}xz \in N$ and

$$\psi(z^{-1}y^{-1}xz)\sigma_z(s)u = \psi(\sigma_z(y^{-1}x))\sigma_z(s)u = \sigma_z(\psi(y^{-1}x)s)u = \sigma_z(t)u.$$

Hence $(x, s)(z, u)\rho(y, t)(z, u)$.

On the other hand, let $\{u_\alpha\}_{\alpha \in I}$ be a net in N such that $\psi(u_\alpha) \to u$ in N^e, then again by (4.12)

$$(z, u)(x, s) = (zx, \sigma_x(u)s) \text{ and } (z, u)(y, t) = (zy, \sigma_y(u)t).$$

Hence, trivially $y^{-1}z^{-1}zx = y^{-1}x \in N$ and by our continuity assumptions

$$\begin{aligned}\psi(y^{-1}x)\sigma_x(u)s &= \lim_\alpha \psi(y^{-1}x)\sigma_x(u_\alpha)s = \lim_\alpha \psi(y^{-1}xx^{-1}u_\alpha x)s \\ &= \lim_\alpha \psi(y^{-1}u_\alpha x)s = \lim_\alpha \psi(y^{-1}u_\alpha yy^{-1}x)s \\ &= \lim_\alpha \psi(y^{-1}u_\alpha y)\psi(y^{-1}x)s = \lim_\alpha \sigma_y(u_\alpha)t = \sigma_y(u)t.\end{aligned}$$

Hence $(z,u)(x,s)\rho(z,u)(y,t)$, and ρ is a congruence with respect to (4.12). □

Theorem 4.1.7. *Let G, N, N^e and $(G\times N^e)/\rho$ be as above. Suppose that G/N is compact, N^e is compatible with G and $x \mapsto \sigma_x(s) : G \to N^e$ is continuous for all $s \in N^e$. Then $(G\times N^e)/\rho$ equipped with the quotient map of (4.12) is a compact, Hausdorff semitopological semigroup and a semigroup compactification of G. $(G\times N^e)/\rho$ satisfies the following universal property:*

Let (φ, X) be a semigroup compactification of G such that $\varphi|_N$ extends to a continuous homomorphism $\tilde{\varphi} : N^e \to X$ in such a way that for each $x \in G$ and $s \in N^e$

$$\tilde{\varphi}(\sigma_x(s)) = \varphi(x^{-1})\tilde{\varphi}(s)\varphi(x).$$

Then there is a unique homomorphism $\vartheta : (G\times N^e)/\rho \to X$ such that $\vartheta\circ\pi\circ\mu = \varphi$.

Proof. Note that $\pi\circ\mu : G \to \pi(G\times\psi(N))$ is onto which implies that $\pi\circ\mu(G)$ is dense in $(G\times N^e)/\rho$. The first statement is now a corollary of Proposition 4.1.3 and Lemmas 4.1.4 and 4.1.6.

To prove the universal property, let (φ, X) be as stated. First, we define $\vartheta_0 : G \times N^e \to X$ by

$$\vartheta_0(x, s) = \varphi(x)\tilde{\varphi}(s). \tag{4.13}$$

Then ϑ_0 is a continuous homomorphism. Finally, we obtain the required homomorphism ϑ as the quotient map of ϑ_0, by noting that ϑ_0 is constant on ρ-classes of $G \times N^e$. Indeed, let $(x, s), (y, t)$ be ρ-related elements of $G \times N^e$, then

$$\begin{aligned}
\vartheta_0(y, t) &= \varphi(y)\tilde{\varphi}(t) = \varphi(y)\tilde{\varphi}(\psi(y^{-1}x)s) \\
&= \varphi(y)\tilde{\varphi}(\psi(y^{-1}x))\tilde{\varphi}(s) = \varphi(y)\varphi(y^{-1}x)\tilde{\varphi}(s) \\
&= \varphi(x)\tilde{\varphi}(s) = \vartheta_0(x, s).
\end{aligned}$$

□

Theorem 4.1.8. *Let N be a closed normal subgroup of G with G/N compact. Suppose that N^e is compatible with G. If $(G \times N^e)/\rho$ is an Eberlein compactification of G, then $(G \times N^e)/\rho \cong G^e$.*

Proof. We will prove that under the hypotheses of the theorem $(G \times N^e)/\rho$ satisfies the universal mapping property for the Eberlein compactifications of G. Let (φ, X) be an Eberlein compactification of G. Then by Theorem 2.4.1 $(\varphi|_N, \overline{\varphi(N)})$ is an Eberlein compactification of N, it follows from the universal property of N^e that $\varphi|_N$ extends to a continuous homomorphism

$$\tilde{\varphi} : N^e \to X. \tag{4.14}$$

We will show that for $x \in G$, $s \in N^e$

$$\tilde{\varphi}(\sigma_x(s)) = \varphi(x^{-1})\tilde{\varphi}(s)\varphi(x). \tag{4.15}$$

Indeed, for any $x \in G$, both sides of (4.15) give continuous homomorphisms of N^e into X, and coincide on the dense subset N, on which $\tilde{\varphi}$ is just φ. So, the map $\varphi \times \tilde{\varphi} : G \times N^e \to X$ given by $(x, s) \mapsto \varphi(x)\tilde{\varphi}(s)$ is clearly continuous and satisfies

$$\begin{aligned}
\varphi \times \tilde{\varphi}((x, s)(y, t)) &= \varphi \times \tilde{\varphi}((xy, \sigma_y(s)t)) = \varphi(xy)\tilde{\varphi}(\sigma_y(s)t) \\
&= \varphi(x)\varphi(y)\tilde{\varphi}(\sigma_y(s))\tilde{\varphi}(t) = \varphi(x)\varphi(y)\varphi(y^{-1})\tilde{\varphi}(s)\varphi(y)\tilde{\varphi}(t) \\
&= \varphi \times \tilde{\varphi}(x, s)\varphi \times \tilde{\varphi}(y, t).
\end{aligned}$$

Finally, we observe that $\varphi\times\tilde{\varphi}$ is constant on ρ-classes of $G\times N^e$. Indeed, let $(x, s), (y, t) \in G \times N^e$ satisfy $(x, s)\rho(y, t)$, then

$$\begin{aligned}
\varphi \times \tilde{\varphi}(x, s) &= \varphi(x)\tilde{\varphi}(s) = \varphi(y)\varphi(y^{-1}x)\tilde{\varphi}(s) \\
&= \varphi(y)(\tilde{\varphi} \circ \psi)(y^{-1}x)\tilde{\varphi}(s) = \varphi(y)\tilde{\varphi}(t) \\
&= \varphi \times \tilde{\varphi}(y, t).
\end{aligned}$$

Thus the quotient map of $\varphi \times \tilde{\varphi}$ gives a continuous homomorphism of $(G \times N^e)/\rho$ into X. □

Theorem 4.1.9. *Let N be a closed normal subgroup of a locally compact group G, with the quotient group G/N compact. Then if the map $G \to N^e : x \mapsto \sigma_x(s)$ is continuous for any $s \in G^e$, then*

$$(G \times N^e)/\rho \cong G^e.$$

Proof. Consider the compactification map $\psi : N \to N^e$ as the evaluation mapping $\psi(n)(f) = f(n)$ for $n \in N$ and $f \in \mathcal{E}(N)$. Fix $x \in G$, then $\psi \circ \sigma_x$ is a continuous homomorphism of N into a compact semitopological semigroup $\psi \circ \sigma_x : N \to N \to N^e$, which is a quotient of N^e. By the universal property of N^e, $\psi \circ \sigma_x$ factors through N^e, that is there is a continuous homomorphism $\nu : N^e \to N^e$ such that

$$\psi \circ \sigma_x = \nu \circ \psi.$$

Therefore, ν is a continuous homomorphism on N^e, extending σ_x, giving the compatibility of N^e with G. Now, the result follows from Theorem 4.1.8. □

The compact group G/N is clearly an Eberlein compactification of G together with the quotient map $\pi_1 : G \to G/N$, so that there is a canonical extension

$$\tilde{\pi_1} : (G \times N^e)/\rho \to G/N$$

(under the assumptions of Theorem 4.1.9) such that $\pi_1 = \psi \circ \pi \circ \mu$, that is π_1 factors through both $G \times N^e$ and $(G \times N^e)/\rho$. Since $\pi_1^{-1}(1) = N$ in G, it follows from the definition of $\pi \circ \mu$ that $(\psi \circ \pi)^{-1}(1)$ is the closure of the union of the ρ-classes of members of $\mu(N)$. Hence we have

$$(\psi \circ \pi)^{-1} = N \times N^e.$$

Therefore,

$$\psi^{-1}(1) = \pi(N \times N^e) = \pi(\{e\} \times N^e).$$

Proposition 4.1.10. *Suppose that G, N, N^e and $(G \times N^e)/\rho$ are as in the above paragraph. Then the set of idempotents $I(G^e)$ and $I(N^e)$ of the compactifications G^e and N^e, respectively, are isomorphic semigroups.*

Proof. The result immediately follows from the fact that the homomorphism $\tilde{\pi}_1$ of the above argument must map all the idempotents to 1, the only idempotent in the group G/N. □

We have already observed that if (ε, G^e) is the Eberlein compactification of G, then $(\varepsilon|_N, \overline{\varepsilon(N)})$ is always an Eberlein compactification of N. We can clearly repeat the above construction for any closed quotient of N^e. If we let (ε_1, N^f) be a compactification of

N that appears as a quotient of N^e and if we denote by $\mathcal{F}$ the subalgebra of $\mathcal{C}_b(G)$ that corresponds to N^f, that is $\mathcal{F} = \mathcal{C}(N^f)|_N$, then the compact semitopological semigroup $(G \times N^f)/\rho$ yields the following generalization of Theorem 4.1.9:

Theorem 4.1.11. *Let G, N, (ε_1, N^f), $\mathcal{F}$ and $(G \times N^f)/\rho$ be as above. Then the following statements are equivalent.*

(i) N^f is compatible with G and $G \times N^f/\rho \cong G^e$;

(ii) $\mathcal{E}(G)|_N = \mathcal{F}$;

(iii) There is a topological isomorphism $\psi_1 : N^f \to G^e$ such that $\varepsilon|_N = \psi_1 \circ \varepsilon_1$.

Proof. The equivalence of (ii) and (iii) is a consequence of the constructions of G^e and N^f as the Gelfand spectrums of $\mathcal{E}(G)$ and $\mathcal{F}$, respectively.

(i) $\Rightarrow$ (iii) Recall that the quotient map $\pi : (G \times N^f) \to (G \times N^f)/\rho$ is injective on $N^f \cong \{e\} \times N^f$, hence gives the required topological isomorphism of N^f into G^e.

(iii) $\Rightarrow$ (i) Since (iii) implies (ii), we use the extensions in $\mathcal{E}(G)$ of functions of $\mathcal{F}$ to define σ_x by

$$\sigma_x(s)(f) = s(g \circ \sigma_x|_N)$$

for $x \in G$, $s \in N^f$, $f \in \mathcal{F}$ and where $g \in \mathcal{E}(G)$ is such that $g|_N = f$. Since every such g, extending f, should agree on N, and hence on N^f, $\sigma_x(s)$ is well-defined for $s \in N^f$ and N^f

is compatible with G. Now $(G \times N^f)/\rho \cong G^e$ follows from the construction of $(G \times N^f)/\rho$ and (iii). $\square$

Theorem 4.1.11 immediately implies the following well-known fact in the special case under our consideration.

Corollary 4.1.12. *Let N be a closed normal subgroup of G with G/N compact. Then $(G \times N^e)/\rho \cong G^e$ if and only if $\mathcal{E}(G)|_N = \mathcal{E}(N)$.*

4.1.2 Compact Normal Subgroups

In this section we will restrict our attention to locally compact groups G and their compact normal subgroups, which we will denote by K. Our aim is to consider the quotient group G/K, study the structure of $\mathcal{E}(G/K)$ in terms of $\mathcal{E}(G)$ and construct the Eberlein compactification of G/K as a quotient of G^e.

We start by characterizing the Eberlein functions on G/K as a subset of $\mathcal{E}(G)$. Let $\mathcal{E}(G:K)$ be the subset of $\mathcal{E}(G)$ which consists of functions that are constant on each coset of K.

Proposition 4.1.13. *Let G be a locally compact group, K a compact normal subgroup of G. Then $\mathcal{E}(G/K) \cong \mathcal{E}(G:K)$.*

Proof. Let $\pi : G \to G/K$ be the quotient map, then its dual $\pi^* : \mathcal{E}(G/K) \to \mathcal{E}(G)$ is an isometric isomorphism by Theorem 2.4.1 onto its image. But given $f \in \mathcal{E}(G/K)$ and $x, y \in G$ with $x \in yK$, we have

$$\pi^*(f)(x) = f \circ \pi(x) = f \circ \pi(y) = \pi^*(f)(y).$$

Hence, $\pi^*(f) \in \mathcal{E}(G : K)$. □

Assume that the Haar measure dk on the compact group K is normalized. We define a map $P : \mathcal{E}(G) \to \mathcal{E}(G : K)$ by

$$(Pf)(x) = \int_K f(xk)dk. \tag{4.16}$$

We immediately observe that for any $f \in \mathcal{E}(G)$ and $x \in G$

$$|(Pf)(x)| \leq \int_K |f(xk)dk| \leq \|f\|_\infty \int_K dk = \|f\|_\infty$$

which implies that P is a contraction.

Next, we observe that if $f \in B(G)$, then also $Pf \in B(G)$ (see [22]). Indeed, let f be represented as $f(x) = \langle \pi(x)\xi, \eta \rangle$. Then

$$\begin{aligned}
(Pf)(x) &= \int_K f(xk)dk = \int_K \langle \pi(xk)\xi, \eta \rangle dk \\
&= \int_K \langle \pi(k)\xi, \pi(x^{-1})\eta \rangle dk = \int_K \langle \pi|_K(k)\xi, \pi(x^{-1})\eta \rangle dk \\
&= \langle \pi|_K(k)(\chi_K)\xi, \pi(x^{-1})\eta \rangle = \langle \pi(x)\pi|_K(k)(\chi_K)\xi, \eta \rangle
\end{aligned}$$

where χ_K denotes the characteristic function of K and hence $Pf \in B(G)$.

Furthermore, note that

$$\begin{aligned}(P^2f)(x) &= \int_K (Pf)(xk)dk = \int_K \int_K f(xkt)dtdk \\ &= \int_K f(xk)dk \int_K dt = (Pf)(x)\end{aligned}$$

for any $x \in G$ and $f \in \mathcal{E}(G)$, as the Haar measure on K is normalized. Hence $P^2 = P$. Since $\mathcal{E}(G : K) \subset \mathcal{E}(G)$, it follows that P is a surjection.

In the rest of this section for a compact normal subgroup K, of G, given the Eberlein compactification (ϵ, G^e) of G, we will construct the Eberlein compactification of the quotient G/K as a quotient semigroup of G^e. The construction for the case of weakly almost periodic compactification is due to Ruppert ([46] page 106). It was generalized to a larger class of semigroup compactifications in [33]. Here we will prove that the construction is also valid for the Eberlein compactification of locally compact groups.

Lemma 4.1.14. *Let G be a locally compact group and K a compact normal subgroup of G. Suppose that (ϵ, G^e) is the Eberlein compactification of G. Then for any $\mu \in G^e$, $\mu\epsilon(K) \subset \epsilon(K)\mu$.*

Proof. Let $k \in K$ and $\{x_\alpha\}_{\alpha\in I}$ be a net in G such that $x_\alpha \to \mu$ in G^e. By normality of K for each α, $x_\alpha^{-1}kx_\alpha \in K$, say $x_\alpha^{-1}kx_\alpha = t_\alpha \in K$, that is $kx_\alpha = t_\alpha x_\alpha$ for each α.

By compactness of K, we choose a limit point $t \in K$ of $\{t_\alpha\}_{\alpha\in I}$, and assume that the net itself is convergent by passing to a subnet if necessary. Since the multiplication of G^e is jointly continuous on $G \times G^e$, the net $\{\epsilon(t_\alpha)\epsilon(s_\alpha)\}_{\alpha\in I}$ converges to $\epsilon(t)\mu$. On the other hand, by right translation invariance of $\mathcal{E}(G)$, we have $\epsilon(s_\alpha)\epsilon(s) \to \mu\epsilon(s)$. Therefore, $\mu\epsilon(s) = \epsilon(t)\mu \in \epsilon(K)\mu$. □

On G^e we define a relation $\sim$ by

$$\mu \sim \nu \text{ if and only if } \mu \in \epsilon(K)\nu \tag{4.17}$$

It is easy to see that $\sim$ gives an equivalence relation on G^e. We will denote the set of equivalence classes of $\sim$ by G^e/K and equip this space with the quotient topology.

Lemma 4.1.15. *The equivalence relation $\sim$ on G^e is closed and the projection map $\pi : G^e \to G^e/K$ is open. Hence the quotient space, G^e/K is compact and Hausdorff.*

Proof. Let $\mu_\alpha \to \mu$ and $\nu_\alpha \to \nu$ in G^e satisfy $\mu_\alpha \sim \nu_\alpha$ for each α. Then $\mu_\alpha \in \epsilon(K)\nu_\alpha$. That is there exist $t_\alpha \in K$ with $\mu_\alpha = \epsilon(t_\alpha)\nu_\alpha$ for each α. By compactness of K, we get a limit point $t \in K$ and suppose that $t_\alpha \to t$. By joint continuity property of the action of G on G^e, we have $\epsilon(t_\alpha)\nu_\alpha \to \epsilon(t)\nu$. So, $\mu = \epsilon(t)\nu \in \epsilon(K)\nu$, that is $\mu \sim \nu$, implying that $\sim$ is a closed relation on G^e.

To prove that π is open, let $O \subset G^e$ be an open subset. Then

$$\epsilon(K)O = \bigcap \{\epsilon(k)O \mid k \in K\}$$

is a union of open sets since translations by elements of G are homeomorphisms on G^e. Hence $\{\epsilon(K)\mu \mid \mu \in O\} = \pi(O)$ is open in G^e/K.

Now the second statement follows from [31] Theorem 11 on page 98. $\square$

Lemma 4.1.16. *Following the above notation, we have $(\epsilon(K)\mu)(\epsilon(K)\nu) = \epsilon(K)\mu\nu$ for all $\mu, \nu \in G^e$. Hence, $\sim$ is a congruence with respect to the multiplication of G^e.*

Proof. Let $k_1 \in K$, then

$$\begin{aligned} \epsilon(k_1)\mu\nu &= \epsilon(k_1)\mu\epsilon(k_2)\epsilon(k_2^{-1})\nu \\ &= \epsilon(k_1)\epsilon(k_3)\mu\epsilon(k_2^{-1})\nu \in \epsilon(K)\mu\epsilon(K)\nu \end{aligned}$$

where k_2 is an arbitrary element of K and k_3 is given by Lemma 4.1.14. Furthermore, since K is a group there exists $k_4 \in K$ such that

$$\begin{aligned} \epsilon(k_1)\epsilon(k_3)\mu\epsilon(k_2^{-1})\nu &= \epsilon(k_4)(\mu\epsilon(k_2^{-1}))\nu \\ &= \epsilon(k_4)\epsilon(k_5)\mu\nu\epsilon(K)\mu\nu \end{aligned}$$

again by Lemma 4.1.14. $\square$

Let $\mu \in G^e$, we write $[\mu]$ for the equivalence class of μ under $\sim$. As a corollary of Lemma 4.1.16, the quotient space G^e/K becomes a semigroup, if for $\mu, \nu \in G^e$, we define

$$[\mu][\nu] = [\mu\nu]. \tag{4.18}$$

Proposition 4.1.17. *G^e/K is a compact Hausdorff semitopological semigroup, which is a semitopological compactification of G/K.*

Proof. By Lemma 4.1.16, $\sim$ is a closed congruence relation on G^e. Hence by [3] Chapter 1, 3.8(ii) and (iii) G^e/K is a compact semitopological semigroup. The proof of G^e/K being Hausdorff is similar to the corresponding part of Proposition 4.1.3.

Let $x \in G$, then by [31] Theorem 9 on page 95, $R_{\epsilon(x)}$ and $R_{\epsilon(x)}$ on G^e, the right and left translations by the image of x, are continuous with respect to the quotient topology. Put

$$\psi : G/K \to G^e/K : xK \mapsto [\epsilon(x)]$$

We easily see that ψ is a continuous homomorphism of G/K onto a dense subset of the compact semitopological semigroup G^e/K. □

Theorem 4.1.18. *Let K be a compact normal subgroup of a locally compact group G. Then $(G/K)^e \cong G^e/K$.*

Proof. By the previous proposition, G^e/K is a compact semitopological semigroup, which is a factor of the universal compactification G^e of G amongst all compactifications representable as Hilbertian contractions. Hence G^e/K is a semitopological compactification of G/K, which is also representable as Hilbertian contractions. So, by the universal property of $(\theta, (G/K)^e)$, there exists a continuous homomorphism ϕ_1 of $(G/K)^e$ onto G^e/K such that $\theta \circ \phi_1 = \psi$.

Also, by the universal property of G^e, the quotient map $\pi : G^e \to G^e/K$ composed with the $\epsilon : G \to G^e$ gives a compactification map $\phi_2 : G \to G^e/K$. We observe that ϕ_2 preserves $\sim$-classes. Indeed, let $\mu, \nu \in G^e$ satisfy $\nu = \epsilon(k)\mu \in \epsilon(K)\mu$ for some $k \in K$. Let $\{x_\alpha\}_{\alpha \in I} \subset G$ be a net converging to μ, so, $\epsilon(xx_\alpha) = \epsilon(x)\epsilon(x_\alpha) \to \nu$. Hence,

$$\begin{aligned} \phi_2(\nu) &= \lim_\alpha \phi_2 \circ \epsilon(xx_\alpha) = \lim_\alpha \theta(x)\theta(x_\alpha) \\ &= \lim_\alpha \phi_2 \circ \epsilon(x_\alpha) = \phi_2(\mu). \end{aligned}$$

Therefore, ϕ_2 factors through G^e/K. If ϕ_3 is the resulting continuous homomorphism of G^e/K onto $(G/K)^e$, then $\phi_3 \circ \phi_1$ is the identity map on $(G/K)^e$, implying that $(G/K)^e \cong G^e/K$. $\square$

4.2 Closed Subgroups of SIN Groups

The purpose of this section is to consider the extension problem for the Eberlein algebra in the case of locally compact SIN-groups and their closed subgroups. The extension problem for the Fourier-Stieltjes algebra and for the algebra weakly almost periodic functions has been positively answered by [17]. After reviewing properties of SIN-groups, we will prove that the restriction map from $\mathcal{E}(G)$ is a surjection onto $\mathcal{E}(H)$, for any closed subgroup H of G. We adopt the technique of [17].

4.2.1 Properties of SIN Groups

Let G be a locally compact group. G is said to *have small invariant neighborhoods*, denoted by $G \in [SIN]$, if the identity element of G has a neighborhood basis invariant under inner automorphisms.

A function v on G is called *central* if for all x, y in G it satisfies $v(xy) = v(yx)$.

Proposition 4.2.1. *Let $G \in [SIN]$. Then*

(i) The identity element, e of G has a neighborhood base consisting of compact sets whose characteristic functions are central.

(ii) For every neighborhood V of e in G, there is a nonnegative continuous central function v with $supp(v) \subset V$.

Proof. Let $\{U_\alpha\}_{\alpha \in I}$ be an invariant neighborhood base of e in G. Then for any $x \in G$ and $\alpha \in I$, $x^{-1}U_\alpha x \subset U_\alpha$. Let χ_α denote the characteristic function of U_α for each α. Choose $x, y \in G$ and $\alpha \in I$, then $\chi_\alpha(xy) = 1$ if and only if $xy \in U_\alpha$ if and only if $yx = y(xy)y^{-1} \in U_\alpha$ if and only if $\chi_\alpha(yx) = 1$. Therefore, $\{\chi_\alpha\}_{\alpha \in I}$ is a collection of central functions, establishing (i).

Suppose in addition that $\{U_\alpha\}_{\alpha \in I}$ is a family of relatively compact open neighborhoods. Let U_α, U_β be chosen such that $U_\alpha U_\alpha^{-1} \subset U_\beta \subset V$. Define ϕ_α on G by

$$\phi_\alpha(y) = \int_G \chi_\alpha(x)\chi_\alpha(yx)dx.$$

Then $supp(\phi_\alpha) \subset U_\beta$ and we can write ϕ_α as the convolution $\check{\chi_\alpha} * \chi_\alpha$. Hence it is an element of the Fourier algebra $A(G)$, and is therefore continuous.

Finally, we will prove that ϕ_α is central. Let $y, z \in G$, put $a = y^{-1}$ and $b = zy$. Then

$$\begin{aligned}
\phi_\alpha(yz) &= \phi_\alpha(a^{-1}ba) = \int_G \chi_\alpha(x)\chi_\alpha(a^{-1}bax)dx \\
&= \int_G \chi_\alpha(a^{-1}x)\chi_\alpha(a^{-1}bx)dx = \int_G \chi_\alpha(xa^{-1})\chi_\alpha(bxa^{-1})dx \\
&= \int_G \chi_\alpha(x)\chi_\alpha(bx)dx = \phi_\alpha(b) \\
&= \phi_\alpha(zy)
\end{aligned}$$

as required. $\square$

Proposition 4.2.2. *Let $G \in [SIN]$. Then G is unimodular.*

Proof. Let V be an invariant neighborhood of e in G and v be a central function supported on V provided by Proposition 4.2.1(ii). Observe that

$$\begin{aligned}\int_G v(x)dx &= \int_G v(yx)dx = \int_G v(xy)dx \\ &= \Delta(y)\int_G v(x)dx\end{aligned}$$

So, $\Delta(y) = 1$ for all $y \in G$. $\square$

4.2.2 Surjectivity Theorem

Theorem 4.2.3. *Let H be a closed subgroup of a $[SIN]$ group G. Then*

$$\mathcal{E}(G)|_H = \mathcal{E}(H)$$

and if $f \in \mathcal{E}(H)$, then

$$\|f\|_\infty = inf\{\|u\|_\infty : u \in \mathcal{E}(G) \textit{ such that } u|_N = f\}.$$

Proof. Clearly $\mathcal{E}(G)|_H \subset \mathcal{E}(H)$ and for any $u \in \mathcal{E}(G)$, $\|u\|_\infty \geq \|u|_N\|$. Conversely, it is enough to show that for any $f \in \mathcal{E}(H)$ and $\varepsilon > 0$, there is $u \in \mathcal{E}(G)$ such that $\|u\|_\infty \leq \|f\|_\infty$ and $\|u|_H - f\|_\infty < \varepsilon$.

First, for any invariant neighborhood base $\{U_\alpha\}_{\alpha\in I}$ of e in G, $\{U_\alpha \cap H\}_{\alpha\in I}$ is an invariant neighborhood base for e in H. Hence, we also have $H \in [SIN]$ and both G and H are unimodular. Recall that we denote by dx and dh the Haar measures of G and H, respectively. By [27] there exists a G-invariant measure $d\dot{x}$ on the quotient space G/H. Furthermore, we assume that the Haar measures dx and dh are normalized so that for any compactly supported continuous g on G, we have

$$\int_{G/N}\int_H g(xh)dnd\dot{x} \;=\; \int_G g(x)dx.$$

Let $f \in \mathcal{E}(H)$ and $\varepsilon > 0$ be given. By uniform continuity of f, choose a compact neighborhood V of e in G such that

$$\|L_h f - f\|_\infty < \varepsilon \tag{4.19}$$

if $h \in V^{-1}V\cap H$. Similar to the proof of Theorem 4.1.2, we choose a nonnegative continuous function v on G such that $supp(v) \subset V$, v is a central function and

$$\int_{G/N}[\int_H v(xn)dh]^2 d\dot{x} \;=\; 1.$$

We define a function u for $y \in G$ by

$$u(y) = \int_G\int_H v(yx)v(xh)f(h)dhdx.$$

Consider the restriction of u to H, for $h" \in H$, we observe

$$\begin{aligned} u(h') &= \int_G \int_H v(xh)v(h'x)f(h)dhdx \\ &= \int_G \int_H v(x)v(h'^{-1}xh)f(h)dhdx \\ &= \int_G \int_H v(x)v(xhh'^{-1})f(h)dhdx \\ &= \int_G \int_H v(x)v(xh)[L_h f](h')dhdx \end{aligned}$$

where we used the facts that v is central and H is unimodular. Similar to the calculation in the proof of Proposition 4.1.2, from (4.18) and the definition of v, we conclude that $\|u|_H - f\|_\infty < \varepsilon$.

The arguments in the proof of Proposition 4.1.2 applied to u and f conclude that $f \in B(H)$ implies $u \in B(G)$ and $f \in \mathcal{E}(H)$ implies $u \in \mathcal{E}(G)$ together with $\|u\|_\infty \leq \|f\|_\infty$, as required. □

4.3 Special Subgroups

In this section, we will consider three types of special subgroups of a locally compact group G, namely open subgroups, central subgroups, and the connected component of the identity. For these classes of subgroups, the restriction map from the Fourier-Stieltjes algebra of G is proven to be surjective by Liukonen and Mislove [36]. Our aim here is to apply their techniques to the Eberlein algebra.

Theorem 4.3.1. *Let G be a locally compact group and H an open subgroup. Then the restriction map from $\mathcal{E}(G)$ into $\mathcal{E}(H)$ is a surjection.*

Proof. First we will restrict our attention to the Fourier-Stieltjes algebra $B(G)$. Let r denote the restriction map on $C^*(G)$ into $C^*(H)$. By [43], we know that r is norm-decreasing. Moreover, the map $\iota : C^*(H) \to C^*(G)$ given by $f \mapsto \tilde{f}$, where

$$\tilde{f} = \begin{cases} f \text{ on } H \\ 0 \text{ on } G \setminus H \end{cases}$$

gives an injection on $L^1(H)$ and as $r \circ \iota$ is the identity on $L^1(H)$, it follows that r is a surjection on $C^*(H)$. Let r^* be the dual map of r, then $r^* : B(H) \to B(G)$ is a $*$-homomorphism. Therefore, when H is an open subgroup of G, $B(H)$ can be considered as a subalgebra of $B(G)$.

When we consider the uniform closures of $B(H)$ and $B(G)$, it easily follows that $\mathcal{E}(H)$ is a norm-closed subalgebra of $\mathcal{E}(G)$. □

Recall that the *center* of a group G, denoted by $Z(G)$ is the set of all $x \in G$ such that $xy = yx$ for all $y \in G$.

Theorem 4.3.2. *Let G be a locally compact group and $Z(G)$ be the center of G. Then the restriction map from $\mathcal{E}(G)$ into $\mathcal{E}(H)$ is a surjection for any closed subgroup H of $Z(G)$.*

Proof. We easily see that $Z(G)$ is normal in G and it follows from the continuity of multiplication that $Z(G)$ is closed. In the notation of Theorem 4.1.2, given any $f \in \mathcal{E}(Z(G))$, $x \in Z(G)$ and $z \in Z(G)$, we have

$$f^x(z) = f(x^{-1}zx) = f(z)$$

that is $f^x = f$. Therefore, by Theorem 4.1.2, the restriction map from $\mathcal{E}(G)$ is surjective onto $\mathcal{E}(Z(G))$.

Next, let H be a closed subgroup of $Z(G)$. Since $Z(G)$ is commutative, $Z(G) \in [SIN]$. Hence, by Theorem 4.2.3, the restriction map from $\mathcal{E}(Z(G))$ is surjective onto $\mathcal{E}(H)$. Therefore, the restriction map is surjective from $\mathcal{E}(G)$ onto $\mathcal{E}(H)$. □

Before proceeding with our final case, we recall the definitions required for the proof. Let $\mathfrak{A}$ be a directed set by a partial ordering $\preceq$. For every $\alpha \in \mathfrak{A}$, let G_α be a topological group. Suppose that for every $\alpha, \beta \in \mathfrak{A}$ such that $\alpha \prec \beta$, there is an open continuous homomorphism $f_{\beta\alpha}$ of G_β into G_α. Suppose finally that if $\alpha \prec \beta \prec \gamma \in \mathfrak{A}$, then $f_{\gamma\alpha} = f_{\beta\alpha} \circ f_{\gamma\beta}$. The object consisting of $\mathfrak{A}$, the groups G_α and the functions $f_{\beta\alpha}$ is called an *inverse mapping system.*

Let G be the direct product, $G = \prod_{\alpha \in \mathfrak{A}} G_\alpha$. Let G_p be the subset of G consisting of all (x_α) such that if $\alpha \prec \beta$, then $x_\alpha = f_{\beta\alpha}(x_\beta)$. This subset is called the *projective limit* of the given inverse mapping system. It is well-known that G_p is a subgroup of G and if all the groups G_α are T_0 groups, then the projective limit is a closed subgroup of the direct product G. We denote the the projective limit G_p as $\varprojlim G_\alpha$.

Let K be a compact normal subgroup of a locally compact group G. Suppose that π is a unitary representation of G on a Hilbert space $\mathcal{H}_\pi$. Recall that dk denotes the normalized Haar measure on K. We define an operator $Q : \mathcal{H}_\pi \to \mathcal{H}_\pi$ by

$$\langle Q\xi, \eta \rangle = \int_K \langle \pi(k)\xi, \eta \rangle dk \tag{4.20}$$

for $\xi, \eta \in \mathcal{H}_\pi$. Recall that we defined a map, $P : \mathcal{E}(G) \to \mathcal{E}(G : K)$ in (4.16), by the following formula:

$$(Pf)(x) = \int_K f(xk)dk.$$

Proposition 4.3.3. *Let G, K, π, $\mathcal{H}_\pi$ and Q be as defined. Then*

(i) Q is a projection.

(ii) For $x \in G$, $Q\pi(x) = \pi(x)Q$.

(iii) Let P_1 be the map P defined in (4.16) restricted to $B(G)$. If $\xi \in \mathcal{H}_\pi$ and $f(x) = \langle \pi(x)\xi, \xi \rangle$ is a positive definite element of $B(G)$, then $P(f)(x) = \langle \pi(x)Q\xi, Q\xi \rangle$.

(iv) If $\xi \in \mathcal{H}_\pi$, then the function $f(x) = \langle \pi(x)\xi, \xi\rangle$ satisfies $f(x) = \langle \pi(x)Q\xi, Q\xi\rangle + \langle \pi(x)(I - Q)\xi, (I-Q)\xi\rangle$, where I is the identity operator on $\mathcal{H}_\pi$.

Proof. Q is clearly a linear map and its boundedness follows from compactness of K. To prove that Q is a projection, let $\xi, \eta \in \mathcal{H}_\pi$, then by noting that dk is unimodular, we have

$$\begin{aligned}
\langle Q^*\xi, \eta\rangle &= \langle \xi, Q\eta\rangle = \overline{\langle Q\eta, \xi\rangle} \\
&= \overline{\int_K \langle \pi(k)\eta, \xi\rangle dk} = \int_K \langle \xi, \pi(k)\eta\rangle dk \\
&= \int_K \langle \pi(k^{-1})\xi, \eta\rangle dk = \int_K \langle \pi(k)\xi, \eta\rangle dk \\
&= \langle \pi(k)\xi, \eta\rangle
\end{aligned}$$

Hence $Q^* = Q$. Furthermore, for $\xi, \eta \in \mathcal{H}_\pi$

$$\begin{aligned}
\langle Q^2\xi, \eta\rangle &= \int_K \langle \pi(k)Q\xi, \eta\rangle dk = \int_K \int_K \langle \pi(k)\pi(t)\xi, \eta\rangle dt dt k \\
&= \int_K \int_K \langle \pi(kt)\xi, \eta\rangle dt dk = \int_K \int_K \langle \pi(t)\xi, \eta\rangle dt dk \\
&= \int_K \langle \pi(t)\xi, \eta\rangle dk \int_K dk = \langle Q\xi, \eta\rangle
\end{aligned}$$

Hence $Q^2 = Q^* = Q$, that is Q is a projection.

Let $x \in G$, since K is compact and normal, dk is invariant under inner automorphisms, hence for $\xi, \eta \in \mathcal{H}_\pi$, we have

$$\begin{aligned}
\langle \pi(x)Q\xi, \eta\rangle &= \int_K \langle \pi(x)\pi(k)\xi, \eta\rangle dk = \int_K \langle \pi(xk)\xi, \eta\rangle dk \\
&= \langle Q\pi(x)\xi, \eta\rangle
\end{aligned}$$

as required by (ii). Next, let $x \in G$, and consider $f(x) = \langle \pi(x)\xi, \xi \rangle$ for some $\xi \in \mathcal{H}_\pi$. Then

$$\begin{aligned} P(f)(x) &= \int_K f(xk)dk = \int_K \langle \pi(xk)\xi, \xi \rangle dk \\ &= \int_K \langle \pi(k)\xi, \pi(x^{-1})\xi \rangle dk = \langle \pi(x)Q\xi, \xi \rangle \\ &= \langle Q\pi(x)Q\xi, \xi \rangle = \langle \pi(x)Q\xi, Q\xi \rangle \end{aligned}$$

establishing (iii). Finally, let $x \in G$ and $f(x) = \langle \pi(x)\xi, \xi \rangle$ for some $\xi \in \mathcal{H}_\pi$. To prove the required identity, we need to show that the function $g(x) = \langle \pi(x)Q\xi, (I - Q)\xi \rangle = 0$. Observe that

$$\begin{aligned} g(x) &= \langle \pi(x)Q\xi, (I - Q)\xi \rangle = \langle Q\pi(x)\xi, (I - Q)\xi \rangle \\ &= \langle \pi(x)\xi, Q^*(I - Q)\xi \rangle = 0 \end{aligned}$$

as required. □

Let G be a locally compact group and G_0 be the connected component of the identity. G is called *almost connected* if the quotient group G/G_0 is compact.

Theorem 4.3.4. *Let G be a locally compact group and G_0 be the connected component of the identity. Then the restriction map from $\mathcal{E}(G)$ into $\mathcal{E}(G_0)$ is a surjection.*

Proof. Note that the connected component G_0 is a closed normal subgroup of G and the quotient group G/G_0 is totally disconnected. There is a compact open subgroup H/G_0 of

G/G_0, and the almost connected subgroup H is open in G ([27], Theorem II.7.7). Hence by Theorem 4.3.1, $\mathcal{E}(H)$ can be seen as a subalgebra of $\mathcal{E}(G)$. Therefore, it is enough to prove that the restriction map from $\mathcal{E}(H)$ is onto $\mathcal{E}(G_0)$.

Let $\{K_i\}_{i\in I}$ be a net of compact normal subgroups of H such that $H_i = H/K_i$ is an almost connected Lie group for each i and $H = \varprojlim H_i$.

First, consider a positive definite function ϕ in $B(G_0)$. Let $\varepsilon > 0$ be given. Let $\phi(x) = \langle \pi(x)\xi, \xi\rangle$ be a representation of ϕ. For each $i \in I$, let μ_i denote the Haar measure of $G_0 \cap K_i$. Since the function ϕ is continuous at the identity of H, we can find $i \in I$ such that

$$|\phi(e) - \phi * \mu_i(e)| < \varepsilon$$

Since each K_i is compact, $G_0 \cap K_i$ is compact and normal in H, hence the modular function of G_0 restricted to $G_0 \cap K_i$ is identically 1. Then we observe that

$$\begin{aligned}(\phi * \mu_i)(x) &= \int_{G_0} \phi(xy^{-1})\Delta(y^{-1})d\mu_i(y) = \int_{G_0} \phi(xy^{-1})d\mu_i(y) \\ &= \int_{G_0} \phi(xy)d\mu_i(y) = P\phi(x(G_0 \cap K_i))\end{aligned}$$

Therefore by Proposition 4.3.3, the function $\phi * \mu_i$ is a positive definite function on the quotient space $G_0/G_0 \cap K_i \cong G_0K_i/K_i$.

As G_0K_i/K_i is open in H/K_i, we can extend $\phi * \mu_i$ to a positive definite function ϕ_i in $B(G/K_i)$ by Theorem 4.3.1. Let $\phi_2 = \phi_1 \circ \pi_i$, where π_i is the quotient map from H onto

H/K_i. then ϕ_2 is a positive definite element in $B(H)$. Note that

$$\|\phi|_{G_0} - \phi\|_\infty \leq \|\phi * \mu_i - \phi\|_\infty = |\phi * \mu_i(e) - \phi(e)| < \varepsilon$$

Therefore, ϕ is a limit point of $\mathcal{E}(H)|_{G_0}$, which is closed in $\mathcal{E}(G_0)$, that is $\phi \in \mathcal{E}(H)|_{G_0}$. Hence, $\mathcal{E}(H)|_{G_0}$ is a norm-closed invariant subspace of $\mathcal{E}(G_0)$, that contains every positive definite function in $B(G_0)$, so $\mathcal{E}(H)|_{G_0} = \mathcal{E}(G_0)$, as required. $\square$

Remark. Let H be a closed subgroup of a locally compact group G. Then the surjectivity results, considered in the cases above (Theorems 4.2.3, 4.3.1, 4.3.2, 4.3.3 and 4.3.4) are consequences of the corresponding results on the Fourier-Stieltjes algebras of G and H, together with Proposition 2.10 in [1].

Chapter 5

Locally Compact Groups of Heisenberg Type

Let $(H_1, +)$, $(H_2, +)$ be additive locally compact Abelian groups and $(N, .)$ be a multiplicative locally compact Abelian group. We consider the direct product group $H_1 \times H_2$ and assume that there exists a continuous $bi-additive$ map φ from $H_1 \times H_2$ into N. That is for any $x, x' \in H_1$ and $y, y' \in H_2$, the continuous map φ satisfies:

$$\varphi(x + x', y) = \varphi(x, y)\varphi(x', y) \text{ and } \varphi(x, y + y') = \varphi(x, y)\varphi(x, y').$$

In particular, if we denote the identity elements of the additive groups H_1 and H_2 by 0_1 and 0_2, respectively, for $x \in H_1$ and $y \in H_2$, we have

$$\varphi(x, 0_2) = 1 = \varphi(0_1, y),$$

where 1 denotes the identity element of N.

On the cartesian product $G = H_1 \times H_2 \times N$, we define an operation by

$$(x, y, n)(x', y', n') = (x + x', y + y', nn'\varphi(x, y'))$$

for any $x, x' \in H_1$, $y, y' \in H_2$ and $n, n' \in N$. We observe that this definition is associative. Indeed, let $x_1, x_2, x_3 \in H_1$, $y_1, y_2, y_3 \in H_2$ and $n_1, n_2, n_3 \in N$. Then

$$\begin{aligned}
((x_1, y_1, n_1)(x_2, y_2, n_2))(x_3, y_3, n_3) &= (x_1 + x_2, y_1 + y_2, n_1 n_2 \varphi(x_1, y_2))(x_3, y_3, n_3) \\
&= ((x_1 + x_2) + x_3, (y_1 + y_2) + y_3, n_1 n_2 \varphi(x_1, y_2) n_3 \varphi(x_1 + x_2, y_3)) \\
&= ((x_1 + x_2) + x_3, (y_1 + y_2) + y_3, n_1 n_2 \varphi(x_1, y_2) n_3 \varphi(x_1, y_3) \varphi(x_2, y_3)) \\
&= (x_1 + (x_2 + x_3), y_1 + (y_2 + y_3), n_1 (n_2 n_3 \varphi(x_2, y_3)) \varphi(x_1, y_2 + y_3)) \\
&= (x_1, y_1, n_1)(x_2 + x_3, y_2 + y_3, n_2 n_3 \varphi(x_2, y_3)) \\
&= (x_1, y_1, n_1)((x_2, y_2, n_2)(x_3, y_3, n_3)).
\end{aligned}$$

The identity element of G is given by $(0_1, 0_2, 1)$ and if (x, y, n) is an element of G, the inverse with respect to this operation is given by $(-x, -y, n^{-1}\varphi(x, y))$.

Therefore, equipped with the product topology, G is a locally compact group. A basic neighborhood of the identity $(0_1, 0_2, 1)$ is given by sets of the form

$$W = \{(x, y, n) \in G \mid x \in U, y \in V \text{ and } n \in O\} \tag{5.1}$$

where U, V and O runs through compact symmetric neighborhoods of 0_1, 0_2 and 1, respectively.

We note that $N \cong \{0_1\} \times \{0_2\} \times N$ is a closed normal subgroup of G. Indeed, let $n, n' \in N$, then

$$\begin{aligned}(0_1, 0_2, n)(0_1, 0_2, n') &= (0_1, 0_2, nn'\varphi(0_1, 0_2)) \\ &= (0_1, 0_2, nn') \in N.\end{aligned}$$

In addition, for $x \in H_1$, $y \in H_2$, $n, n' \in N$, we observe

$$\begin{aligned}(x, y, n)(0_1, 0_2, n')(x, y, n)^{-1} &= (x, y, nn'\varphi(x, 0_2))(-x, -y, n^{-1}\varphi(x, y)) \\ &= (0_1, 0_2, nn'1n^{-1}\varphi(x, y)\varphi(x, -y)) \\ &= (0_1, 0_2, n'nn^{-1}\varphi(x, y - y)) \\ &= (0_1, 0_2, n') \in N,\end{aligned}$$

as required.

Furthermore, the quotient group G/N is isomorphic to the direct product group $H_1 \times H_2$

which is not necessarily a subgroup of G. Under these circumstances, we will call G a *locally compact group of Heisenberg type*.

Here is the notation on the topology of G we shall be using. Let P be a property, that may be satisfied by the elements of the direct product group $H_1 \times H_2$. By the statement *as* $(x, y) \to \infty$ $P(x, y)$, we mean that there are increasing chanis of compact subsets $\{C_\alpha\}_{\alpha \in I}$ in H_1 and $\{K_\alpha\}_{\alpha \in I}$ in H_2 such that $\bigcup_{\alpha \in I} C_\alpha \times K_\alpha = H_1 \times H_2$ and eventually for each $x \in H_1 \setminus C_\alpha$ and $y \in H_2 \setminus K_\alpha$ $P(x, y)$ is satisfied.

We say a locally compact group of Heisenberg type G satisfies the *small transitivity condition on* H_1 if: for any $n \in N$, any compact neighborhood V of 0_2 in H_2, there exists a compact subset C of H_1 such that for any $x \in H_1 \setminus C$, we have $\varphi(x, V) \cap nO \neq \emptyset$ for any neighborhood O of 1 in N.

Similarly, we say that G satisfies the *small transitivity condition on* H_2 if: for any compact neighborhood U of 0_1 in H_1, there exists a compact subset K in H_2 such that for any $y \in H_2 \setminus K$, we have $\varphi(U, y) = N$.

Furthermore, we say G satisfies the $small\ bi-transitivity\ condition$ if G satisfies the small transitivity conditions on both H_1 and H_2.

In this section our purpose is to determine the structure of the Eberlein compactification G^e and the weakly almost periodic compactification G^w, in terms of the corresponding compactifications of H_1, H_2 and N, when G is a locally compact group of Heisenberg type satisfying the small bi-transitivity condition. Our techniques are generalizations of the example considered in [41] Section 2.1.

Lemma 5.0.5. *Let $G = H_1 \times H_2 \times N$ be a Heisenberg type semidirect product that satisfies the small bi-transitivity condition. Then for any $f \in WAP(G)$, we have*

$$\lim_{(x,y)\to\infty} \sup\{|f(x,y,n) - f(x,y,n')| \mid n, n' \in N\} = 0.$$

Proof. Let $f \in WAP(G)$ and $\varepsilon > 0$. Since f is uniformly continuous in G, there exists a neighborhood W of $(0_1, 0_2, 1)$ in G such that for any $(x, y, n), (x', y', n') \in G$ whenever

$$(x', y', n')(x, y, n)^{-1} \in W \text{ or } (x, y, n)^{-1}(x', y', n') \in W \tag{5.2}$$

we have

$$|f(x, y, n) - f(x', y', n')| < \varepsilon.$$

We assume that W is of the form given in (5.1), that is

$$W = \{(x, y, n) \in G \mid x \in U, y \in V \text{ and } n \in O\} \tag{5.3}$$

where U, V and O are compact symmetric neighborhoods of 0_1, 0_2 and 1, respectively.

Let n, n' be fixed. We will apply the small transitivity condition on H_2 of G to $(n'n^{-1})^{-1} \in N$, together with the neighborhood U of 0_1 given by (5.3). Then there is a compact subset K in H_2 such that for any $y \in H_2 \setminus K$ we have

$$\varphi(U, y) \cap (n'n^{-1})^{-1}O \neq \emptyset \tag{5.4}$$

for any neighborhood O of 1 in N.

Next, we assume that O is the neighborhood of 1 given by (5.3). Hence, if we fix an element $y \in H_2 \setminus K$, by (5.4), there exists x_1 in the symmetric neighborhood U such that

$$\varphi(-x_1, y) \in (n'n^{-1})^{-1}O. \tag{5.5}$$

Now, we observe that for any $x \in H_1$,

$$\begin{aligned}(x - x_1, y, n)^{-1}(x, y, n) &= (-x + x_1, -y, n^{-1}\varphi(x - x_1, y))(x, y, n) \\ &= (x_1, 0_2, n^{-1}\varphi(x - x_1, y)n\varphi(-x + x_1, y)) \\ &= (x_1, 0_2, n^{-1}n\varphi(x - x_1 - x + x_1, y)) \\ &= (x_1, 0_2, 1) \in W.\end{aligned}$$

Hence, by (5.2),

$$|f(x, y, n) - f(x - x_1, y, n)| < \varepsilon. \tag{5.6}$$

Furthermore, we also observe that

$$\begin{aligned}(x, y, n')(x - x_1, y, n)^{-1} &= (x, y, n')(-x + x_1, -y, n^{-1}\varphi(x - x_1, y)) \\ &= (x + x_1 - x, y - y, n'n^{-1}\varphi(x - x_1, y)\varphi(x, -y)) \\ &= (x_1, 0_2, n'n^{-1}\varphi(x - x_1 - x, y)) \\ &= (x_1, 0_2, n'n^{-1}\varphi(-x_1, y)).\end{aligned}$$

By the choice of $x_1 \in U$ we have $\varphi(-x_1, y) \in (n'n^{-1})^{-1}O$, so

$$(n'n^{-1})\varphi(-x_1, y) \in (n'n^{-1})(n'n^{-1})^{-1}O = O.$$

Therefore, by (5.2), we get

$$|f(x - x_1, y, n) - f(x, y, n')| < \varepsilon. \tag{5.7}$$

We conclude, by (5.6) and (5.7) that

$$\begin{aligned}|f(x, y, n) - f(x, y, n')| &\leq |f(x, y, n) - f(x - x_1, y, n)| + |f(x - x_1, y, n) - f(x, y, n')| \\ &< \varepsilon + \varepsilon = 2\varepsilon.\end{aligned}$$

Similarly, an application of small transitivity condition of G on H_1 implies that for any $n, n' \in N$, and the neighborhood V of 0_2 in H_2, there exists a compact subset C of H_1 such that for any $x \in H_1 \setminus C$, $y \in H_2$ and the neighborhood O of 1, we have

$$|f(x, y, n) - f(x, y, n')| < 2\varepsilon.$$

Therefore, the result follows. □

Theorem 5.0.6. *Under the conditions of Lemma 5.0.5, we have*

$$\mathcal{E}(G) \cong \mathcal{E}(H_1 \times H_2) + \mathcal{C}_0(G)$$

and

$$WAP(G) \cong WAP(H_1 \times H_2) + \mathcal{C}_0(G).$$

Proof. First we identify the the direct product group $H_1 \times H_2$ with the direct product group $H_1 \times H_2 \times \{1\}$. Let $f \in \mathcal{E}(G)$ (or $f \in WAP(G)$), then the function on the direct product group $H_1 \times H_2$ defined by

$$h(x, y) = f(x, y, 1)$$

is in $\mathcal{E}(H_1 \times H_2)$ (or in $WAP(H_1 \times H_2)$). Then by Lemma 5.0.5, the function g defined on G by

$$g(x, y, w) = f(x, y, w) - f(x, y, 1)$$

is in $\mathcal{C}_0(G)$. Hence, $f = g + h \in \mathcal{E}(H_1 \times H_2) + \mathcal{C}_0(G)$ (or $f = g + h \in WAP(H_1 \times H_2) + \mathcal{C}_0(G)$). □

As a consequence of Theorem 5.0.6, we conclude that only linear combinations of constant functions and functions in $\mathcal{C}_0(N)$ extend to functions in $\mathcal{E}(G)$ (and also to $WAP(G)$).

However, we observe that N is a central subgroup of G. Indeed, let $x \in H_1$, $y \in H_2$ and $n, n' \in N$, then

$$(x, y, n)(0_1, 0_2, n') = (x, y, nn'\varphi(x, 0_2)) = (x, y, nn')$$
$$= (x, y, n'n\varphi(0_1, y)) = (0_1, 0_2, n')(x, y, n).$$

Hence, by Theorem 4.3.2, the restriction map from $\mathcal{E}(G)$ into $\mathcal{E}(N)$ is a surjection. That is, $WAP(N) = \mathcal{E}(N) = \mathcal{C}_0(N) + \mathbb{C}$. Since N is also a locally compact Abelian group, we have:

Corollary 5.0.7. *If $G = H_1 \times H_2 \times N$ is a group of Heisenberg type which satisfies the small bi-transitivity condition, then N is a compact group.*

Proof. The result follows immediately from the fact that for any locally compact Abelian group N, $WAP(N) = \mathcal{E}(N) = \mathcal{C}_0(N)$ if and only if N is compact. □

Remark. The sum in Theorem 5.0.6 is not a direct sum since $\mathcal{E}(H_1 \times H_2) \cap \mathcal{C}_0(G) \neq \{0\}$.

For any locally compact group G, since $\mathcal{C}_0(G)$ is a subalgebra of $\mathcal{E}(G)$, by Theorem 3.6 of [15] G is an open subgroup in G^e. Assume that $G = H_1 \times H_2 \times N$ is a locally compact group of Heisenberg type, satisfying the small bi-transitivity condition. Let $q : G \to G/N \cong H_1 \times H_2$ be the quotient map. Let $(\psi, (H_1 \times H_2)^e)$ be the Eberlein compactification of the direct product group $H_1 \times H_2$. First, we observe that q composed with the compactification map ψ gives a compactification of G. Let $C = (H_1 \times H_2)^e \setminus (H_1 \times H_2)$. Then C is an ideal in $(H_1 \times H_2)^e$. We define the set S to be the disjoint union of the group G with the compact semigroup $(H_1 \times H_2)^e$, that is $S = G \sqcup (H_1 \times H_2)^e$. Then S is a semigroup with the following multiplication:

$$s_1 s_2 = \begin{cases} s_1 s_2, & \text{if } s_1 s_2 \in G \text{ or } s_1 s_2 \in (H_1 \times H_2)^e \\ \psi(q(s_1)) s_2, & \text{if } s_1 \in G \text{ and } s_2 \in (H_1 \times H_2)^e \\ s_1 \psi(q(s_2)), & \text{if } s_1 \in (H_1 \times H_2)^e \text{ and } s_2 \in G \end{cases}$$

for any $s_1, s_2 \in S$. Furthermore, we equip S with a compact topology where G is open and for any $x \in (H_1 \times H_2)^e$ a neighborhood base is given by sets of the form $(G \setminus K) \sqcup V$, where K is a compact subset of G and V is a neighborhood of x in the compact semigroup $(H_1 \times H_2)^e$.

We define an equivalence relation $\sim$ on S by $s_1 \sim s_2$ if $s_1 \in G, s_2 \in S$ and $s_2 = \psi(q(s_1))$ if $s_2 \in (H_1 \times H_2)^e$ or $s_1 = s_2$ if $s_2 \in G$. Then $\sim$ is a closed congruence on S and as a consequence of Theorem 5.0.6, $G^e \cong S/\sim$. It follows that C is also an ideal of G^e and we can describe the topological structure of G^e as follows: $G^e = G \sqcup (H_1 \times H_2)^e \setminus H_1 \times H_2$, where a neighborhood of a point $(x, y, z) \in G$ is given by

$$V \setminus (H_1 \times H_2) \cup \{(x', y', z') \in G : (x', y') \in V\}$$

where V is a neighborhood of (x, y) in $(H_1 \times H_2)^e$.

Example 5.0.8. ([41] Example 2.1) We will consider the group $G = \mathbb{R} \times \mathbb{R} \times \mathbb{T}$, where the continuous bi-additive map $\varphi(x, y) = \exp(ixy)$ and the product on G is given by

$$(x, y, \exp(i\theta))(x', y', \exp(i\theta')) = (x + x', y + y', \exp(i(\theta + \theta' + xy')).$$

First, we will prove that G satisfies the small transitivity condition on $\mathbb{R}$:

Let $V = [-a, a] \subset \mathbb{R}$ be a symmetric compact interval for some $a > 0$. Then there exists a positive integer M such that $\varphi(M, V) = \mathbb{T}$. Let $K = [-M, M]$. Then for any $y \in \mathbb{R} \setminus K$, we have

$$\varphi(y, V) = \mathbb{T}.$$

By symmetry, the small transitivity condition holds on both copies of $\mathbb{R}$, and hence we conclude that G satisfies the small bi-transitivity condition and the conclusion of Theorem 5.0.6, holds for $G = \mathbb{R} \times \mathbb{R} \times \mathbb{T}$.

Example 5.0.9. More generally, we consider $G = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{T}$, where the multiplication is given by

$$(x, y, \exp(i\theta))(a, b, \exp(i\gamma)) = (x + a, y + b, \exp(i(\theta + \gamma + x_1 b_1 + \ldots + x_n b_n)).$$

Let $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{T}$ denote the function $\varphi(x, y) = \exp(ix_1 y_1 + \ldots + x_n y_n)$. We want to verify that G satisfies the small transitivity condition on $\mathbb{R}^n$.

Let V be the closed ball $B_a(0)$ with radius $a > 0$ and center 0 in $\mathbb{R}^n$. There exists a positive integer M such that $\exp(i[-a,a]M) = \mathbb{T}$. Let K be the closed ball $B_M(0)$ with radius M and center 0. Consider an element y in $\mathbb{R}^n \setminus K$. Then $y_1^2 + \ldots + y_n^2 > M^2$. Hence, there exists $j \in \{1, \ldots, n\}$ such that $y_j > M$.

Let $\exp(i\theta)$ be an arbitrary element of $\mathbb{T}$. Since $\exp(i[-a,a]y_j) = \mathbb{T}$, there exists $t \in [-a, a]$ such that

$$\exp(ity_j) = \exp(i\theta).$$

Consider $x \in \mathbb{R}^n$ with $x_j = t$ and for any $k \in \{1, \ldots, j-1, j = 1, \ldots, n\}$, $x_k = 0$. Then, $\varphi(x, y) = \exp(ity_j) = \exp(i\theta)$ and $x \in V$. Therefore, $\varphi(V, y) = \mathbb{T}$, concluding that G satisfies the small transitivity condition on $\mathbb{R}^n$. By symmetry, G satisfies the small bi-transitivity condition and hence the conclusion of Theorem 5.0.6 holds for $G = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{T}$.

Example 5.0.10. Let $G = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ be the Heisenberg group on the real line. By Corollary 5.0.8, we observe that G cannot satisfy the small bi-transitivity condition. Furthermore, [46] Chapter 3 example 6.9, proves that the structure of G^w is not isomorphic with the conclusion of Corollary 5.0.7.

Example 5.0.11. Let $G = H_1 \times H_2 \times N = \mathbb{Z} \times \mathbb{T} \times \mathbb{T}$, together with the continuous bi-additive map $\varphi(k, w) = w^k$. Let P be a property that may be satisfied by the elements of the direct product group $H_1 \times H_2 = \mathbb{Z} \times \mathbb{T}$. Since $H_2 = \mathbb{T}$ is compact, the statement

$(x, y) \to \infty$ $P(x, y)$ reduces to *as* $x \to \infty$ in $\mathbb{Z}$, for any $y \in \mathbb{T}$, we eventually have $P(x, y)$.

For $G = \mathbb{Z} \times \mathbb{T} \times \mathbb{T}$, we can consider only one-sided small transitivity of G, namely the small transitivity condition on $\mathbb{Z}$. First, we will prove that G satisfies this condition: Indeed let $V = \{z \in \mathbb{T} \mid |z - 1| \leq \delta'\}$ be a neighborhood of 1 in $H_2 = \mathbb{T}$, for some $\delta' > 0$. We write $V = exp(i(-\delta, \delta))$ for some $\delta > 0$. Let M be the smallest integer that is greater than $\frac{1}{\delta}$, that is M is chosen to be the integer ceiling of $\frac{1}{\delta}$. Next, let C be the compact set $\{-M, \ldots, M\}$ in $\mathbb{Z}$. Then for any $x \in \mathbb{Z} \setminus C$, we have

$$\varphi(x, V) = V^x = \mathbb{T} = N.$$

In this case, the statement of Lemma 5.0.5 implies: For any $f \in WAP(G)$, we have

$$\lim_{x \to \infty} \max\{|f(x, y, n) - f(x, y, n')| \mid n, n' \in \mathbb{T} = N, y \in \mathbb{T} = H_2\} = 0. \tag{5.8}$$

Moreover, the symmetry in the proof of the Lemma together with the small transitivity of G on $\mathbb{Z}$, concludes the above version of Lemma 5.0.5. Therefore Theorem 5.0.6 can be applied to $G = \mathbb{Z} \times \mathbb{T} \times \mathbb{T}$ to give $\mathcal{E}(G) = \mathcal{E}(\mathbb{Z} \times \mathbb{T}) + \mathcal{C}_0(G)$ and $WAP(G) = WAP(\mathbb{Z} \times \mathbb{T}) + \mathcal{C}_0(G)$.

Example 5.0.12. Let $G = H_1 \times H_2 \times N = \mathbb{Z} \times H \times H$, where H is a connected compact Abelian group. We define $\varphi : \mathbb{Z} \times H \to H$ by $\varphi(n, h) = h^n$.

Since H is connected and compact for any symmetric neighborhood V of identity e in

$H = H_2$, for all sufficiently large integers M, we have

$$\varphi(M, V) = V^M = H = N.$$

Therefore, the argument of Example 4 implies that $G = \mathbb{Z} \times H \times H$ satisfies the small transitivity condition on $\mathbb{Z}$.

Example 5.0.13. Let $G = H_1 \times H_2 \times N = \mathbb{Z} \times \mathbb{R} \times \mathbb{T}$, where $\varphi : \mathbb{Z} \times \mathbb{R} \to \mathbb{T}$ is given by $\varphi(n, s) = e^{ins}$. By a similar argument to the Examples 5.0.11 and 5.0.12, we observe that G satisfies the small transitivity condition on $\mathbb{Z}$. On the other hand, G fails to satisfy the small transitivity condition on $\mathbb{R}$: Let U be the trivial neighborhood, $\{0\}$, of the identity in $\mathbb{Z}$. Then $\varphi(U, \mathbb{R}) = e^{i0\mathbb{R}} = \{1\}$ in $\mathbb{T}$.

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