Contributions at the Interface Between

Algebra and Graph Theory

by

Khodakhast Bibak

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

In this thesis, we make some contributions at the interface between 'algebra' and 'graph theory'.

In Chapter 1, we give an overview of the topics and also the definitions and preliminaries.

In Chapter 2, we estimate the number of possible types degree patterns of k-lacunary polynomials of degree $t < p$ which split completely modulo p. The result is based on a rather unusual combination of two techniques: a bound on the number of zeros of lacunary polynomials and a bound on the so-called domination number of a graph.

In Chapter 3, we deal with the determinant of bipartite graphs. The *nullity* of a graph G, denoted by $\eta(G)$, is the multiplicity of 0 in the spectrum of G. Nullity of a (molecular) graph (e.g., a bipartite graph corresponding to an alternant hydrocarbon) has important applications in quantum chemistry and Hückel molecular orbital (HMO) theory. A famous problem, posed by Collatz and Sinogowitz in 1957, asks to characterize all graphs with positive nullity. Clearly, $\det \mathcal{A}(G) = 0$ if and only if $\eta(G) > 0$. So, examining the determinant of a graph is a way to attack this problem. In this Chapter, we show that the determinant of a bipartite graph with at least two perfect matchings and with all cycle lengths divisible by four, is zero.

In Chapter 4, we first introduce an application of spectral graph theory in proving trigonometric identities. This is a very simple double counting argument that gives very short proofs for some of these identities (and perhaps the 'only' existed proof in some cases!). In the rest of Chapter 4, using some properties of the well-known Chebyshev polynomials, we prove some theorems that allow us to evaluate the number of spanning trees in join of graphs, Cartesian product of graphs, and nearly regular graphs. In the last section of Chapter 4, we obtain the number of spanning trees in an (r, s)-semiregular graph and its line graph. Note that the same results, as in the last section, were proved by I. Sato using zeta functions. But our proofs are much shorter based on some well-known facts from spectral graph theory. Besides, we do not use zeta functions in our arguments.

In Chapter 5, we present the conclusion and also some possible projects.

Acknowledgements

Dedication

To my beloved wife Azadeh

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Chapter 1

Introduction

1.1   Overview

In this thesis, we make some contributions at the interface between 'algebra' and 'graph theory'.

Zeros and factorisations of lacunary polynomials has always been a subject of active investigation. Schinzel [95] has obtained a series of important statistical results about the number of k-lacunary irreducible polynomials with prescribed coefficients. (See, the book [96] by Schinzel for further information.)

Canetti et al. [25] proved a strong result in studying a special kind of exponential sums (and from there in studying Diffie-Hellman distribution in cryptography). Their result is an upper bound on the number of zeros of lacunary polynomials over finite field $\mathbb{F}_q$. In Section 2.2 of Chapter 2, using this estimate and some graph theory arguments, we estimate the number of possible types degree patterns of k-lacunary polynomials of degree $t < p$ which split completely modulo p. This section is my joint work with Igor Shparlinski [16]. In the rest of Chapter 2, we discuss about two other applications of Canetti et al.'s estimate.

Our contributions in Chapters 3 and 4, lie in the area of *spectral graph theory* or *spectra of graphs* where 'linear algebra' and 'graph theory' meet together. This area has many applications, in particular, in

- *Additive combinatorics* which studies combinatorial properties of algebraic objects, for example, Abelian groups, rings, or fields, and in fact, focuses on the interplay between combinatorics, number theory, harmonic analysis, ergodic theory, and some other branches. Applications of spectral graph theory in additive combinatorics in-

cludes applications in *sum-product problem*, the *Szemerédi-Trotter type theorems* in finite fields, etc.; see, e.g., the surveys [11, 100] and the references therein.

- Computer science, for example, in expanders and combinatorial optimization, complex networks and the Internet, data mining, computer vision and pattern recognition, Internet search, load balancing and multiprocessor interconnection networks, anti-virus protection vs. spread of knowledge, statistical databases and social networks, and quantum computing; see, e.g., the surveys [34, 60, 101, 102] and the references therein.

- Chemistry, for example, in studying topographical resonance in molecular species, and in mathematical modeling of physico-chemical, pharmacologic, toxicological, and other properties of chemical compounds. Spectral graph theory has also important applications in quantum chemistry and Hückel molecular orbital (HMO) theory; see, e.g., the surveys [54, 83, 93] and the references therein.

In Chapter 3, we deal with the determinant of bipartite graphs. The *nullity* of a graph G, denoted by $\eta(G)$, is the multiplicity of 0 in the spectrum of G. The nullity of a graph is closely related to the minimum rank problem of a family of matrices associated with a graph (see, the survey [41] and the references therein). Nullity of a (molecular) graph (e.g., a bipartite graph corresponding to an alternant hydrocarbon) has also important applications in quantum chemistry and Hückel molecular orbital (HMO) theory (see, the survey [54] and the references therein). A famous problem, posed by Collatz and Sinogowitz in 1957 [31], asks to characterize all graphs with positive nullity. Clearly, $\det \mathcal{A}(G) = 0$ if and only if $\eta(G) > 0$. So, examining the determinant of a graph is a way to attack this problem. In Chapter 3, we show that the determinant of a bipartite graph with at least two perfect matchings and with all cycle lengths divisible by four, is zero. My paper [12] is based on results in this chapter.

Proving trigonometric identities may not be an easy problem. Sometimes, advanced tools may be needed to prove such identities. In Chapter 4, using the number of spanning trees in some classes of graphs and also the determinant of the Cartesian product $P_m \,\square\, P_n$, we prove (and also give very short proofs for) some trigonometric identities. Some of these trigonometric identities also involve Fibonacci and Lucas numbers. As far as I know, the techniques already existed in the literature for proving these identities are mainly analytic with somewhat complicated computations (see, e.g., [46] and the references therein).

Graphs, among other things, may be thought of as representing the connectivity of the atoms that comprise the (microscopic) conjugation network of an unsaturated molecule.

Calculating the number of spanning trees in (i.e., the complexity of) such (labelled) molecular graphs has absorbed much attention (see [65] and the references therein). In fact, enumerating the number of spanning trees in graphs (networks) which comes up with lots of remarkable applications to various sciences, is of great interest among mathematicians, computer scientists, physicists, and chemists. One of the notable applications of spanning trees in mathematics and computer science is that they can be applied to approximately solve the traveling salesman problem [4] (see also [14, 26, 79] for some other applications in mathematics). They also have interesting applications in physics [27, 36, 103, 104]. The number of spanning trees is also a measure of the network reliability. In the rest of Chapter 4, using some properties of the well-known Chebyshev polynomials, we prove some theorems that allow us to evaluate the number of spanning trees in join of graphs, Cartesian product of graphs, and nearly regular graphs. In the last section of Chapter 4, we obtain the number of spanning trees in an (r, s)-semiregular graph and its line graph. Note that the same results, as in the last section, were proved in [94] using zeta functions. But our proofs are much shorter – we do not use zeta functions but instead employ some facts from spectral graph theory. Most of the results in this chapter are my papers [10], and also my joint papers with Shirdareh Haghighi [14, 15].

1.2   Definitions and Preliminaries

Let's start with the some definitions and facts from field theory. We will follow closely the presentation of Dummit and Foote [40].

A *field* $\mathbb{F}$ is a commutative ring with identity in which every non-zero element has an inverse. A *finite field* (or *Galois field*) is a field that contains a finite number of elements. The *order* (i.e., the number of elements) of a finite field is always a prime or a power of a prime. For each prime power, there exists 'exactly one' (up to isomorphism) finite field, usually written as $\mathbb{F}_{p^n}$. If $\mathbb{K}$ is a field containing the subfield $\mathbb{F}$, then $\mathbb{K}$ is said to be an *extension* of $\mathbb{F}$, denoted by $\mathbb{K}$ / $\mathbb{F}$. The *degree* of a field extension $\mathbb{K}$ / $\mathbb{F}$, denoted by $[\mathbb{K} : \mathbb{F}]$, is the dimension of $\mathbb{K}$ as a vector space over $\mathbb{F}$. An *algebraic number field* (a.k.a *number field*) is a finite field extension of $\mathbb{Q}$, the field of rational numbers.

It is well-known that (see, [40, p. 512]) given any field $\mathbb{F}$ and any (irreducible) polynomial $f \in \mathbb{F}[X]$ there exists an extension $\mathbb{K}$ of $\mathbb{F}$ containing a root of f, say α. Equivalently, f has a linear factor $X - \alpha$ in $\mathbb{K}[X]$.

The extension field $\mathbb{K}$ of $\mathbb{F}$ is called a *splitting field* for the polynomial $f \in \mathbb{F}[X]$ if f factors completely into linear factors (or *splits completely*) in $\mathbb{K}[X]$, and f does not factor completely into linear factors over any proper subfield of $\mathbb{K}$ containing $\mathbb{F}$. In fact, a splitting

field of a polynomial with coefficients in a field is a 'smallest' extension field of that field over which the polynomial splits completely into linear factors. It is well-known that (see, [40, p. 536]) if f is of degree n, then f has at most n roots in $\mathbb{F}$ and has precisely n roots, including multiplicities, in $\mathbb{F}$ if and only if f splits completely in $\mathbb{F}[X]$. A well-known fact (see, [40, p. 536]) says that for any field $\mathbb{F}$, if $f \in \mathbb{F}[X]$ then there exists an extension $\mathbb{K}$ of $\mathbb{F}$ which is a splitting field for f. Note that any two splitting fields for f are isomorphic, so we refer to *the* splitting field of a polynomial. For example, the splitting field for $X^2 - 2$ over $\mathbb{Q}$ is just $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$.

We say that a polynomial f over a field $\mathbb{K}$ is k-*lacunary* if it has at most $k+1$ non-zero coefficients, including a non-zero constant term, that is, if $f(0) \neq 0$ and

$$f(X) = a_0 + a_1 X^{t_1} + \ldots + a_k X^{t_k} \in \mathbb{K}[X] \tag{1.1}$$

for some positive integers $t_1 < \ldots < t_k$. Roughly speaking, a *lacunary* (also called *sparse* or *supersparse*) polynomial is a polynomial of high degree with relatively small number of non-zero coefficients.

Now, we give some definitions and facts from graph theory and matrix theory. We use the terminology of Bondy and Murty [22] for graph theory definitions.

All graphs in this thesis are finite, undirected, and simple (i.e., without loops or multiple edges). The *vertex set* (*edge set*) of a graph G is denoted by $V(G)$ (resp., $E(G)$). The *order* (*size*) of a graph G is the number of its vertices (resp., edges), that is, $|V(G)|$ (resp., $|E(G)|$). The *degree*, $d_G(v)$, of a vertex v in a graph G is the number of edges incident to v. We denote by $\delta(G)$ the minimum degree of G. Two vertices which are incident with a common edge are *adjacent*, and two adjacent vertices are *neighbours*. The set of neighbours of a vertex v in a graph G is denoted by $N_G(v)$. An *empty graph* is a graph with zero or more vertices, but no edges. A graph F is called a *subgraph* of a graph G if $V(F) \subseteq V(G)$ and $E(F) \subseteq E(G)$. If e is an edge of a graph G, then by deleting e from G we obtain a graph that is called an *edge-deleted subgraph* of G, and is denoted by $G \setminus e$. Similarly, if v is a vertex of a graph G, then by deleting the vertex v together with all the edges incident with v from G we obtain a graph that is called a *vertex-deleted subgraph* of G, and is denoted by $G - v$.

A path (cycle) on n vertices is denoted by P_n (resp., C_n). The *length* of a path P (cycle C) is denoted by $l(P)$ (resp., $l(C)$). The *complete graph* K_n is a graph on n vertices in which every vertex is adjacent to every other. A *bipartite graph* is a graph whose vertices can be divided into two disjoint sets X and Y such that every edge connects a vertex in X to one in Y. It is well-known that a graph is bipartite if and only if it does not contain an odd cycle. The *complete bipartite graph* on m and n vertices, denoted by $K_{m,n}$,

is the bipartite graph $G = (X, Y, E)$, where X and Y are disjoint sets of size m and n, respectively, and E connects every vertex in X to all vertices in Y. A *multipartite graph* is a graph having a partition of the vertices so that any edge joins vertices in different parts. A *complete multipartite graph* is a graph in which vertices are adjacent if and only if they belong to different partite sets.

There are many ways of combining graphs to produce new graphs. We now describe some binary operations defined on graphs. The *union* of graphs G and H is the graph $G \cup H$ with vertex set $V(G) \cup V(H)$ and edge set $H(G) \cup E(H)$. If G and H are disjoint, we refer to their union as a *disjoint union*, denoted by $G + H$. The *join* of two graphs G and H, $G \vee H$, is obtained from the disjoint union of G and H by additionally joining every vertex of G to every vertex of H. The join $W_n = C_n \vee K_1$ of a cycle C_n and a single vertex is referred to as a *wheel* with n *spokes*. Similarly, the join $\mathcal{F}_n = P_n \vee K_1$ of a path P_n and a single vertex is called a *fan*. The *Cartesian product* of graphs G and H is the graph $G \,\square\, H$ whose vertex set is $V(G) \times V(H)$ and whose edge set is the set of all pairs $(u_1, v_1)(u_2, v_2)$ such that either $u_1 u_2 \in E(G)$ and $v_1 = v_2$, or $v_1 v_2 \in E(H)$ and $u_1 = u_2$. The Cartesian product $P_m \,\square\, P_n$ of two paths is the $(m \times n)$-*grid*. Also the Cartesian product $P_2 \,\square\, P_n$ $(n \geq 2)$ is called a *ladder*, and $P_2 \,\square\, C_n$ $(n \geq 3)$ is referred to as an *n-prism*. A graph is said to be *connected* if there is a path between any two distinct vertices. Every graph G may be expressed uniquely (up to order) as a disjoint union of connected graphs. These graphs are called the *components* of G.

A subgraph H is a *spanning subgraph*, or *factor*, of a graph G if it has the same vertex set as G. For spanning subgraphs $F_1 = (V, E_1)$ and $F_2 = (V, E_2)$ of a graph $G = (V, E)$, the spanning subgraph whose edge set is the symmetric difference $E_1 \,\triangle\, E_2$ is called the *symmetric difference* of F_1 and F_2, and is denoted by $F_1 \,\triangle\, F_2$. A *matching* in a graph is a subset of its edges, no two of which share an end vertex. A vertex is *matched* (or *saturated*) if it is an end of one of the edges in the matching. Otherwise, the vertex is *unmatched* (or *unsaturated*). A *perfect matching* (a.k.a. *1-factor*) is a matching which matches all vertices of the graph. A graph that all of whose components are either K_2's or cycles or combinations thereof is called a *Sachs graph*. There are several other names for this graph in the literature, for example, *sesquivalent graph* and *basic figure*. A spanning subgraph of a graph that has this property (i.e., is a Sachs graph) is called a *perfect 2-matching* (or a *q-factor* by Tutte [106]). A graph is *acyclic* if it contains no cycles. A *tree* is a connected acyclic graph. Clearly, each component of an acyclic graph is a tree. So, an acyclic graph is also called a *forest*. For a graph G, a *spanning tree* in G is a tree which has the same vertex set as G. It is easy to see that a graph is connected if and only if it has a spanning tree. The number of spanning trees in a graph G is denoted by $t(G)$.

The *line graph $L(G)$* of a graph G, is the graph whose vertices correspond to the edges

of G with two vertices of $L(G)$ being adjacent if and only if the corresponding edges in G have a vertex in common. A graph in which every vertex has the same degree is *regular*. It is *k-regular* if every vertex has degree k. A *nearly k-regular* graph is one that all of its vertices except one (referred to as an *exceptional vertex*) have degree k. A graph G is called *(r, s)-semiregular* if it is bipartite with a bipartition $V(G) = X \cup Y$ such that all vertices in X have degree r and all vertices in Y have degree s.

A *dominating set S* of a graph G is a vertex subset such that any vertex of $V \setminus S$ has a neighbour in S. Intuitively, a dominating set of a graph is a vertex subset whose neighbours, along with themselves, make up the vertex set of the graph. The minimum cardinality of a dominating set of G is called the *domination number $\gamma(G)$* of G. In other words,

$$\gamma(G) = \min_{S \subseteq V(G)} \left\{ |S| \ : \ V(G) \subseteq \bigcup_{v \in S} \hat{N}(v) \right\},$$

where $\hat{N}(v)$ denotes the closed neighbourhood of a vertex v.

A *planar graph* is a graph that can be embedded in the plane, that is, can be drawn on the plane in such a way that there are no edge crossings. In other words, it can be drawn in such a way that its edges intersect only at their endpoints. Such a drawing is called a *plane graph* or a *planar embedding of the graph*. The *dual* of a plane graph G is a graph (in fact, a planar multigraph) G^* that has a vertex corresponding to each plane region of G, and an edge joining two neighboring regions for each edge in G, for a certain embedding of G. The reason for the term 'dual' is that this property is symmetric; that is, if H is a dual of G, then G is a dual of H (if G is connected).

The *adjacency matrix* of a graph G on n vertices is a $(0, 1)$-matrix $\mathcal{A}(G) = (\mathcal{A}_{ij})_{i,j=1}^{n}$, where the off-diagonal entry $\mathcal{A}_{ij}$ is the number of edges from vertex i to vertex j (which is 0 or 1), and the diagonal entry $\mathcal{A}_{ii}$ is 0. The adjacency matrix $\mathcal{A}$ of a bipartite graph $G = (X, Y, E)$ with $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ has the form

$$\mathcal{A} = \begin{pmatrix} O & B \\ B^T & O \end{pmatrix},$$

where B is an m-by-n matrix and O is a zero matrix. The matrix B is called the *biadjacency matrix* of G. In other words, the biadjacency matrix of a bipartite graph $G = (X, Y, E)$ with $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$, is the m-by-n $(0, 1)$-matrix $B = (B_{ij})$ in which $B_{ij} = 1$ if and only if the edge $x_i y_j \in E$.

For an n-by-n matrix $\mathcal{M}$, a number μ is an *eigenvalue* if for some vector $\nu \neq \mathbf{0}$, we have $\mathcal{M}\nu = \mu\nu$. The vector ν is called an *eigenvector corresponding to μ*. It is easy to see that

the eigenvalues of $\mathcal{M}$ are exactly the numbers μ that make the matrix $\mathcal{M} - \mu I$ singular, that is, solutions of $\det(\mathcal{M} - \mu I) = 0$, where I is the identity matrix. It is well-known that the sum (product) of all eigenvalues of matrix $\mathcal{M}$, including multiplicities, is the trace (resp., the determinant) of $\mathcal{M}$. A *symmetric matrix* is a square matrix that is equal to its transpose. A *Hermitian matrix* (also called *self-adjoint matrix*) is a square matrix (with complex entries) that is equal to its conjugate transpose. So, every real symmetric matrices is a Hermitian matrix. It is well-know that the eigenvalues of a Hermitian matrix are real. Thus, the eigenvalues of a real symmetric matrix (e.g., the adjacency matrix of a graph) are also real. The multiset of eigenvalues of a graph G is called the *spectrum* of G. The *nullity* of a graph G of order n, denoted by $\eta(G)$, is the multiplicity of 0 in the spectrum of G. Clearly, $\eta(G) = n - r(\mathcal{A}(G))$, where $r(\mathcal{A}(G))$ is the rank of $\mathcal{A}(G)$.

1.3 Contributions

My new contributions in this thesis are results in Chapters 2, 3, and the last section of Chapter 4.

Chapter 2

Zeros of Lacunary Polynomials over Finite Fields with Applications

2.1 Introduction

Factorisation of polynomials is an important topic in mathematics and computer science with many applications in various disciplines. For example, it is one of the fundamental tools of the *computer algebra systems* (*CAS*); a software program for symbolic computations. Also, as mentioned by Giesbrecht and Roche [49], "the lacunary representation is arguably more intuitive than the standard dense representation, and in fact corresponds to the default linked-list representation of polynomials in modern computer algebra systems such as Maple and Mathematica".

Zeros and factorisations of lacunary polynomials has always been a subject of active investigation, see [28, 42, 48, 49, 64, 72, 73, 95] and the references therein. For example, a classical result of Descartes asserts that a k-lacunary polynomial $f \in \mathbb{R}[X]$ may have at most $2k$ real roots. Furthermore, Lenstra [73] has shown that for an algebraic number field $\mathbb{K}$ of degree m over $\mathbb{Q}$ and a k-lacunary polynomial $f \in \mathbb{K}[X]$, the product g of all irreducible divisors $h \mid f$ of degree at most $\deg h \leq d$ is of degree

$$\deg g = O\left(k^2 2^{md} md \log(2mdk)\right).$$

Schinzel [95] has obtained a series of statistical results about the number of k-lacunary irreducible polynomials with prescribed coefficients. In particular, by [95, Corollary 2], for any algebraic numbers $a_0, \ldots, a_k$ there are at most $O\left(T^{\lfloor (k+1)/2 \rfloor}\right)$ k-tuples of integers

$$\mathbf{t} = (t_1, \ldots, t_k), \qquad 1 \leq t_1 < \ldots < t_k, \tag{2.1}$$

with $t_k \leq T$ and such that the largest non-cyclotomic factor (that is, a factor which does not have roots that are roots of unity) of the k-lacunary polynomial (1.1) is reducible over $\mathbb{K} = \mathbb{Q}(a_1/a_0, \ldots, a_k/a_0)$. (See, the book [96] by Schinzel for further results.)

Canetti et al. [25] proved a strong result in studying a special kind of exponential sums (and from there in studying Diffie-Hellman distribution in cryptography). Their result is an upper bound on the number of zeros of lacunary polynomials over finite field $\mathbb{F}_q$. In the next section, using this estimate and some graph theory arguments (including a bound on the domination number of a graph), we estimate the number of possible types degree patterns of k-lacunary polynomials of degree $t < p$ which split completely modulo p. This section is my joint work with Igor Shparlinski [16].

In the last section, we discuss about two other applications of Canetti et al.'s estimate.

Throughout this chapter, the implied constants in the symbols 'O', '$\ll$' and '$\gg$' may depend on k (we recall that the notations $U \ll V$ and $V \gg U$ are equivalent to $U = O(V)$). Also, $\mathbb{F}_q$ denotes a finite field of q elements. Here p is always a prime.

2.2 An Upper Bound and its Application

Canetti et al. [25] proved the following estimate on the number of zeros of lacunary polynomials over $\mathbb{F}_q$.

Lemma 2.1 *For $k + 1 \geq 2$ elements $a_0, a_1, \ldots, a_k \in \mathbb{F}_q^*$ and integers $0 = t_0 < t_1 < \ldots < t_k < q$, the number of solutions Q to the equation*

$$\sum_{i=0}^{k} a_i x^{t_i} = 0, \qquad x \in \mathbb{F}_q^*,$$

with $t_0 = 0$, satisfies

$$Q \leq 2q^{1-1/k} D^{1/k} + O\left(q^{1-2/k} D^{2/k}\right),$$

where

$$D = \min_{0 \leq i \leq k} \max_{j \neq i} \gcd(t_j - t_i, q - 1).$$

Proof. Let

$$D = \max_{1 \leq j \leq k} \gcd(t_j - t_1, q - 1).$$

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If $D = q - 1$ then the bound is trivial. Thus, we may assume that $D \le (q-1)/2$.

Let g be a primitive root of $\mathbb{F}_q$. Putting $r_i = t_i - t_{k+1}$ we see that Q equals the number of solutions of the equation

$$\sum_{i=0}^{k-1} a_i g^{r_i y} + a_{k+1} = 0, \qquad 0 \le y \le q - 2.$$

Put

$$L = (q-1)/D, \qquad K = \lceil L^{1/k} \rceil - 1, \qquad M = \lfloor (q-1)/K \rfloor.$$

By the pigeonhole principle there exists l with $1 \le l \le L - 1$ and such that the remainders of $s_i \equiv r_i l \pmod{q-1}$, taken in the interval $-(q-1)/2 \le s_i \le q/2$, are all

$$|s_i| \le M, \qquad i = 0, \ldots, k - 1.$$

In fact, for each $l = 1, \ldots, L$ the corresponding vector $(s_1, \ldots, s_k)$ represents a point in the k-dimensional cube with side length $q - 1$. This cube can be split into K^k cubes with side length $h = (q-1)/K$. Since $K^k < L$ then at least one sub-cube contains at least two vectors corresponding to some $1 \le l_1 < l_2 \le L$. Putting $l = l_2 - l_1$ the claim follows.

Let $d = \gcd(l, q-1)$. It is easy to see that for any y, $0 \le y \le q - 2$, there is a unique representation of the form

$$y = dz + \nu, \qquad 0 \le z \le (q-1)/d - 1, \quad 0 \le \nu \le d - 1,$$

and so of the form

$$y \equiv lz + \nu \pmod{q-1}, \qquad 0 \le z \le (q-1)/d - 1, \quad 0 \le \nu \le d - 1.$$

Then

$$Q \le \sum_{\nu=0}^{d-1} Q_\nu,$$

where Q_ν, $\nu = 0, \ldots, d - 1$, is the number of solutions of the equation

$$\sum_{i=0}^{k-1} a_i g^{r_i(lz+\nu)} + a_{k+1} = 0, \qquad 0 \le z \le (q-1)/d - 1.$$

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Let R_ν be the number of solutions of the equation

$$\sum_{i=0}^{k-1} a_i g^{r_i \nu} g^{s_i z} + a_{k+1} = 0, \qquad 0 \le z \le q-2,$$

or of the polynomial equation

$$\sum_{i=0}^{k-1} a_i g^{r_i \nu} x^{s_i + M} + a_{k+1} x^M = 0, \qquad x \in \mathbb{F}_q^*.$$

Clearly,

$$Q_\nu = \frac{1}{d} R_\nu, \qquad \nu = 0, \ldots, d-1.$$

Since $dD \le (L-1)D < q-1$, it is easy to see that for $j = 1, \ldots, k-1$

$$s_1 - s_j \equiv r_1 l - r_j l \equiv (r_1 - r_j) l \equiv (t_1 - t_j) l \not\equiv 0 \pmod{q-1},$$

and

$$s_1 \equiv r_1 l \equiv (t_1 - t_{k+1}) l \not\equiv 0 \pmod{q-1}.$$

Thus, R_ν does not exceed the number of zeros of a non-zero polynomial (in particular, it contains $x^{s_1 + M}$ with a non-zero coefficient) of degree at most

$$2M = 2q^{1-1/k} D^{1/k} + O\left(q^{1-2/k} D^{2/k}\right),$$

and the bound follows. $\qquad\qquad\square$

Now we discuss about an application of Lemma 2.1. We consider a question about estimating the number $N_k(p, t)$ of k-tuples (2.1) such that there is a k-lacunary polynomial of the form (1.1) of degree $t_k = t$ over the finite field $\mathbb{K} = \mathbb{F}_p$ that fully splits over $\mathbb{F}_p$.

Theorem 2.2 *If a positive integer k is fixed then for any prime p and positive integer $t < p$, we have,*

$$N_k(p, t) \le t^{k - k\lceil (k-3)/2 \rceil - 1} p^{(k-1)\lceil (k-3)/2 \rceil + o(1)}$$

as $p \to \infty$.

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Clearly, Theorem 2.2 is nontrivial only for $k > 3$ and for

$$t > p^{1-1/k+\varepsilon},\qquad(2.2)$$

with some fixed $\varepsilon > 0$. Furthermore, for $t \gg p$ we obtain the bound

$$N_k(p, t) \leq t^{\lceil k/2\rceil+1+o(1)}.$$

This result is based on a rather unusual combination of two techniques: the bound on the number of zeros of lacunary polynomials (Lemma 2.1 above) and a bound on the domination number of a graph (Lemma 2.4 below).

We need the following lemma.

Lemma 2.3 *For $k + 1 \geq 2$ elements $a_0, a_1, \ldots, a_k \in \mathbb{F}_p^*$ and integers $0 = t_0 < t_1 < \ldots < t_k < p$, the multiplicity of any root ρ of the polynomial*

$$\sum_{i=0}^{k} a_i X^{t_i} \in \mathbb{F}_p[X]$$

is at most k.

Proof. Let

$$F(X) = \sum_{i=0}^{k} a_i X^{t_i}.$$

Then for the jth derivative $F^{(j)}(X)$ we have

$$F^{(j)}(X)X^j = \sum_{i=0}^{k} \prod_{h=0}^{j-1}(t_i - h)a_i X^{t_i}$$

(where as usual, we set $F^{(0)}(X) = F(X)$). Thus, if $r \neq 0$ is a root of multiplicity at least $k + 1 \leq t_k < p$ in the algebraic closure of $\mathbb{F}_p$, then

$$F^{(j)}(r) = 0, \qquad j = 0, \ldots, k.$$

Therefore, the homogeneous system of equations

$$\sum_{i=0}^{k} \prod_{h=0}^{j-1}(t_i - h)x_i = 0, \qquad j = 0, \ldots, k,$$

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has a non-zero solution $x_i = a_i r^{t_i}$, $i = 0, \ldots, k$. This implies

$$\det \left[\left(\prod_{h=0}^{j-1} (t_i - h) \right)_{i,j=0,\ldots,k} \right] = 0,$$

which is impossible for $0 = t_0 < t_1 < \ldots < t_k < p$ as an easy calculation shows that

$$\det \left[\left(\prod_{h=0}^{j-1} (t_i - h) \right)_{i,j=0,\ldots,k} \right] = \prod_{0 \leq i < j \leq k} (t_j - t_i) \neq 0.$$

The above contradiction implies the desired result. $\square$

Domination is a fundamental concept in graph theory and has many applications not only to graph theory but also to coding theory, web graphs, large-scale wireless ad hoc and sensor networks, biological networks, social networks, parallel and distributed computing, and so on, see [1, 5, 7, 18, 30, 32, 35, 39, 58, 61, 78] and references therein. The problem of determining the size of a minimum dominating set is NP-complete, see [45]. The Minimum Dominating Set (MDS) problem concerns computing a dominating set of minimum cardinality for a given graph. This problem has recently absorbed much interest, and has found a considerable number of seminal applications, see [2, 43, 44, 47, 71, 74, 76, 109] and references therein. In particular, this problem can be applied for backbone structures in communication networks, for example, to get efficient multi-hop routing protocols. The MDS problem has also intriguing applications to computer networks, where we may would like to have a minimum dominating set, which its members provide a service for their neighbours. Another "real-life" application of the same spirit is facility location problems, in which the vertices of a graph correspond to locations, adjacency means accessability, and the goal is to find a subset of locations accessible from all other locations; one can use this subset of locations to build some public facilities such as libraries, bus stops, post offices, hospitals, fire stations, and so on. In this chapter, we introduce another application of domination – in studying zeros of lacunary polynomials over finite fields.

When $\delta(G)$ is big enough, there are very good upper bounds for the domination number of the graph G in terms of $\delta(G)$ and n (see, for example, [29, 59]). However, for small values of $\delta(G)$ the classical result of Ore [88] is stronger and provides an upper bound for the domination number of a graph with no isolated vertices:

Lemma 2.4 *If G is a graph of order n with $\delta(G) \geq 1$, then*

$$\gamma(G) \leq \frac{n}{2}.$$

Proof of Theorem 2.2

Since $p > t_k$, by Lemma 2.3 the multiplicity of each non-zero root of a polynomial of the form (1.1) does not exceed k. Hence, if a polynomial $F(X) \in \mathbb{F}_p[X]$ of the form (1.1) splits completely over $\mathbb{F}_p$ then the equation

$$a_0 + a_1 x^{t_1} + \ldots + a_k x^{t_k} = 0, \qquad x \in \mathbb{F}_p^*,$$

with $1 \le t_1 < \ldots < t_k$ has at least t_k/k solutions. Then, from Lemma 2.1 we have

$$t_k/k = O\left(p^{1-1/k} D_{\mathbf{t}}^{1/k}\right),$$

where

$$D_{\mathbf{t}} = \min_{0 \le i \le k} \max_{j \ne i} \gcd(t_j - t_i, p - 1).$$

Thus $D_{\mathbf{t}} \mid p - 1$ and, since k is fixed,

$$t \ge D_{\mathbf{t}} \gg t_k^k p^{-(k-1)} = t^k p^{-(k-1)}. \tag{2.3}$$

We now fix $D \mid p - 1$, and for each $\mathbf{t} = (t_1, \ldots, t_k)$ construct a graph $G_{\mathbf{t}}(D)$ on $k + 1$ vertices $0, \ldots, k$, connecting i and j if and only if $\gcd(t_i - t_j, p - 1) \ge D$ (where, as before $t_0 = 0$).

Clearly, if $D_{\mathbf{t}} = D$ and $G_{\mathbf{t}}(D) = G$ then $\delta(G) \ge 1$.

Now, for a fixed positive integer $D \le t < p$ and a graph G with $k + 1$ vertices and $\delta(G) \ge 1$, we estimate the number $M_p(D, G, t)$ of vectors $\mathbf{t} = (t_1, \ldots, t_k) \in \mathbb{Z}^k$ with $1 \le t_1 < \ldots < t_k$ and $t_k = t$ such that $G_{\mathbf{t}}(D) = G$. Summing over all graphs G (since k is fixed there are only finitely many graphs) and admissible values of D, that is, with $t \ge D \gg t^k p^{-(k-1)}$, see (2.3), leads to the desired estimate.

Given a graph G with $k + 1$ vertices and $\delta(G) \ge 1$, we now fix a dominating set S in G of cardinality $\#S = \lfloor (k+1)/2 \rfloor$, which exists by Lemma 2.4 (obviously, we can always add more vertices to S if necessary to guarantee $\#S = \lfloor (k+1)/2 \rfloor$). So for each $j \notin S$ with $j \ne 0, k$, there is $i \in S$ such that $\gcd(t_i - t_j, p - 1) \ge D$. So if t_i is fixed, then t_j can take at most

$$\sum_{\substack{d \mid p-1 \\ d \ge D}} \frac{t}{d} \ll \frac{t}{D} \sum_{d \mid p-1} 1 = \frac{t}{D} p^{o(1)} \tag{2.4}$$

values, where we have used the known bound on the divisor function, (see [57, Theorem 320]). Finally, when $t_k = t$ is fixed, each t_i, $i \in S$, can take at most t values.

Furthermore, if both $0, k \in S$ then there are only

$$\#S - 2 \leq \lfloor (k+1)/2 \rfloor - 2 = \lfloor (k-3)/2 \rfloor$$

elements t_i with $i \in S \setminus \{0, k\}$ to be chosen. After all values of t_i with $i \in S$ are fixed, we see from (2.4) that the remaining

$$k + 1 - \#S = \lceil (k+1)/2 \rceil$$

elements t_j, $j \notin S$, can be chosen in at most $(tp^{o(1)}/D)^{\lceil (k+1)/2 \rceil}$ ways. So in this case

$$M_p(D, G, t) \leq t^{\lfloor (k-3)/2 \rfloor} (t/D)^{\lceil (k+1)/2 \rceil} p^{o(1)} = t^{k-1} D^{-\lceil (k+1)/2 \rceil} p^{o(1)}. \tag{2.5}$$

If $0 \in S$ but $k \notin S$, or $0 \notin S$ but $k \in S$, then the same argument implies:

$$M_p(D, G, t) \leq t^{\lfloor (k-1)/2 \rfloor} (t/D)^{\lceil (k-1)/2 \rceil} p^{o(1)} = t^{k-1} D^{-\lceil (k-1)/2 \rceil} p^{o(1)}. \tag{2.6}$$

Finally, if both $0, k \notin S$ then we get

$$M_p(D, G, t) \leq t^{\lfloor (k+1)/2 \rfloor} (t/D)^{\lceil (k-3)/2 \rceil} p^{o(1)} = t^{k-1} D^{-\lceil (k-3)/2 \rceil} p^{o(1)}. \tag{2.7}$$

Clearly, bound (2.7) dominates the bounds (2.5) and (2.6). In particular, for $t \geq D \gg t^k p^{-(k-1)}$ we obtain

$$M_p(D, G, t) \leq t^{k-1-k\lceil (k-3)/2 \rceil} p^{(k-1)\lceil (k-3)/2 \rceil + o(1)}.$$

Since, as we have mentioned, there are only finitely many possibilities for the graphs $G_{\mathbf{t}}(D)$, recalling (2.3) and the bound on the divisor function (see [57, Theorem 320]), we obtain the desired result.

2.3   More Applications

Let us mention two other applications of Lemma 2.1. Let γ be an integer of multiplicative order s modulo a prime number $p \geq 3$, that is,

$$\gamma^x \not\equiv 1 \pmod{p}, \ x = 1, \ldots, s-1, \quad \gamma^s \equiv 1 \pmod{p}.$$

Employing Lemma 2.1, Canetti et al. [25] studied the exponential sums

$$W_{a,c}(s) = \sum_{y=1}^{s} \left| \sum_{x=1}^{s} \mathbf{e}_p(a\gamma^x + c\gamma^{xy}) \right|,$$

where

$$\mathbf{e}_n(z) = \exp(2\pi i z/n).$$

Then they applied these exponential sums in studying Diffie-Hellman distribution which is of significant importance in cryptography. In fact, they derived the following bound. Here, $\tau(n)$ denotes the number of integer divisors of $n \geq 1$, that is,

$$\tau(n) = \sum_{d|n} 1.$$

Theorem 2.5 *For any integers a, c such that $\gcd(a, c, p) = 1$, we have*

$$W_{a,c}(s) = \begin{cases} O\left(sp^{1/2}\tau(s)\right), & \text{if } a \equiv 0 \pmod{p}; \\ O\left(s^{5/3}p^{1/4}\right), & \text{otherwise.} \end{cases}$$

Now we state another application of Lemma 2.1. Using this lemma, Shparlinski [99] obtained the following result.

Lemma 2.6 *Let $\alpha \in \mathbb{F}_q$ be a fix primitive root. For $k+1 \geq 2$ elements $a_0, a_1, \ldots, a_k \in \mathbb{F}_q^*$ and integers $t_0, t_1, \ldots, t_k$, the number of solutions Q to the equation*

$$\sum_{i=0}^{k} a_i \alpha^{t_i u} = 0, \qquad u \in [0, q-2],$$

satisfies

$$Q \leq 3(q-1)^{1-1/k} D^{1/k},$$

where

$$D = \min_{0 \leq i \leq k} \max_{j \neq i} \gcd(t_j - t_i, q-1).$$

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Proof. Putting $z = \alpha^u$ we see that Q equals the number of solutions of the equation

$$\sum_{i=0}^{k} a_i z^{t_i} = 0, \qquad z \in \mathbb{F}_q^*.$$

By Lemma 2.1,

$$Q \le 2 \left\lfloor \frac{q-1}{\lceil L^{1/k} \rceil - 1} \right\rfloor,$$

where $L = (q-1)/D$.

If $L \le 3^k$ then

$$3(q-1)^{1-1/k} D^{1/k} \ge 3(q-1)L^{-1/k} \ge q-1 > Q.$$

Otherwise $\lceil L^{1/k} \rceil - 1 \ge 2L^{-1/k}/3$ and the result follows. $\qquad\square$

Let us mention an application of Lemma 2.6. Fix $\eta_1, \ldots, \eta_n \in \mathbb{F}_q^*$. Look at the matrices of the form $(\eta_i^{m_j})_{i,j=1}^n$ with integers $m_1, \ldots, m_n \in [0, q-2]$. Putting $m_j = j-1$, $j = 1, \ldots, n$, we get the Vandermonde matrix, which is known to be nonsingular, provided that $\eta_1, \ldots, \eta_n$ are pairwise distinct. Using Lemma 2.6, Shparlinski [99] proved that for n fixed almost all matrices of the above form are nonsingular. Let us state this result more precisely. Denote by $V(\eta_1, \ldots, \eta_n)$ the number of n-tuples

$$(m_1, \ldots, m_n) \in [0, q-2]^n,$$

for which

$$\det(\eta_i^{m_j})_{i,j=1}^n = 0.$$

In fact, Shparlinski [99] proved the following results. Recall that ord η (multiplicative order of η) is the smallest positive integer s with $\eta^s = 1$.

Theorem 2.7 *For any $n \ge 2$ we have*

$$V(\eta_1, \ldots, \eta_n) \le 3(n-1)(q-1)^n T^{-1/(n-1)},$$

where

$$T = \min_{1 \leq i \leq n} \min_{j \neq i} \operatorname{ord} \eta_j/\eta_i.$$

Theorem 2.8 *For any $n \geq 2$ and any $\varepsilon > 0$ we have*

$$\frac{1}{(q-1)^n} \sum_{\eta_1,\ldots,\eta_n \in \mathbb{F}_q^*} V(\eta_1,\ldots,\eta_n) = O(q^{n-1/(n-1)+\varepsilon}).$$

Chapter 3

On the Determinant of Bipartite Graphs

3.1 Introduction

The nullity of a graph is closely related to the minimum rank problem of a family of matrices associated with a graph (see, the survey [41] and the references therein). Nullity of a (molecular) graph (e.g., a bipartite graph corresponding to an alternant hydrocarbon) has also important applications in quantum chemistry and Hückel molecular orbital (HMO) theory (see, the survey [54] and the references therein). For example, in [54] the authors mention that "if the nullity of the molecular graph of an alternant unsaturated conjugated hydrocarbon is greater than zero, then the respective molecule is predicted to have an unstable, open-shell, electron configuration and the respective compound is expected to be highly reactive, chemically unstable and often not capable of existence". A famous problem, posed by Collatz and Sinogowitz in 1957 [31], asks to characterize all graphs with positive nullity. Clearly, $\det \mathcal{A}(G) = 0$ if and only if $\eta(G) > 0$. So, examining the determinant of a graph is a way to attack this problem. But there seems to be little published on calculating the determinant of a graph (see, e.g., [17, 53, 63, 90, 91, 97] for some results on the determinant of graphs).

Gutman [53] (see, also, [87]) proved that if G is a bipartite graph on n vertices with no cycle of length divisible by four, then $\det \mathcal{A}(G) = (-1)^{n/2} t^2$, where t is the number of perfect matchings in G. (Note that if n is odd then G has no perfect matching, and so as we will see later, $\det \mathcal{A}(G) = 0$). It is natural to ask what happens for the determinant of a graph in which some cycle lengths are divisible by four. Very recently, in [63] the

authors using some results of [91], proved that the determinant of a plane graph, with every face-boundary is a cycle of size divisible by four, is equal to −1, 0, or 1 provided that the inner dual of the graph is a tree. In this chapter, using a new method, we settle this problem for (all) graphs with all cycle lengths divisible by four. Our approach or its modifications may lead to further work.

We denote the set of all perfect matchings and the set of all perfect 2-matchings of a graph G on n vertices by $M = \{M_1, \ldots, M_t\}$, and $N = \mathcal{U}_n = \{N_1, \ldots, N_s\}$, respectively. Clearly, a bipartite graph has a perfect 2-matching if and only if it has a perfect matching. So, in the case of bipartite graphs we have $s = t$.

The *characteristic polynomial* of a graph G on n vertices is

$$\mathcal{P}_G(\lambda) = \det(\lambda I - \mathcal{A}(G)) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n.$$

The following expression for the coefficients of the characteristic polynomial is well-known (see, e.g., [33, p. 32] and [93, p. 1973])

$$a_i = \sum_{\mathcal{U} \in \mathcal{U}_i} (-1)^{|\mathcal{U}|} 2^{c(\mathcal{U})}, \tag{3.1}$$

where $\mathcal{U}$ belongs to the set of Sachs graphs $\mathcal{U}_i$ on i vertices, and $|\mathcal{U}|$ and $c(\mathcal{U})$ are the number of components and the number of cycles in $\mathcal{U}$, respectively.

From (3.1), we simply get the following expression for the determinant of the adjacency matrix of G

$$\det \mathcal{A}(G) = (-1)^n a_n = (-1)^n \sum_{i=1}^{s} (-1)^{|N_i|} 2^{c(N_i)}. \tag{3.2}$$

From (3.2), we immediately conclude that the determinant of the adjacency matrix of a bipartite graph with no perfect matching is zero. (This is a well-known result; see, e.g., [33].)

Given a graph G on n vertices, the *Tutte matrix* of G is a skew-symmetric, symbolic matrix $T = (T_{ij})_{i,j=1}^n$ with

$$T_{ij} = \begin{cases} x_{ij} & \text{if } i \sim j, \ i > j; \\ -x_{ji} & \text{if } i \sim j, \ i < j; \\ 0 & \text{otherwise}, \end{cases} \tag{3.3}$$

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where $\sim$ means vertices i and j are adjacent in G, and the x_{ij}'s are variables.

Let G be a graph with the Tutte matrix T. A celebrated result by Tutte [105] says that $\det T \neq 0$ if and only if G has a perfect matching.

Let's recall the Leibniz formula (see, e.g., [9]) for the determinant of a matrix, as we need in the next section. For the matrix $A = (A_{ij})_{i,j=1}^n$, we have

$$\det A = \sum_{\sigma} (-1)^{N_\sigma} \prod_{i=1}^n A_{i\sigma(i)}, \tag{3.4}$$

where the sum is taken over all $n!$ permutations $\sigma = (\sigma(1), \ldots, \sigma(n))$ of the column indices $1, \ldots, n$, and where N_σ is the minimal number of pairwise transpositions needed to transform $\sigma(1), \ldots, \sigma(n)$ to $1, \ldots, n$.

In the next section, we show that the determinant of the adjacency matrix of a bipartite graph on $2n$ vertices with a unique perfect matching is equal to $(-1)^n$. In the third section, we show that the determinant of a bipartite graph with at least two perfect matchings and with all cycle lengths divisible by four, is zero. Summarizing these results, in the last section, we show that the determinant of a graph G with all cycle lengths divisible by four, is 0 or $(-1)^{|V(G)|/2}$. My paper [12] is based on this chapter.

3.2 Unique Perfect Matching

In this section, we show that the determinant of the adjacency matrix of a bipartite graph on $2n$ vertices with a unique perfect matching is equal to $(-1)^n$. We present two different proofs for this result. Clearly, if $G = (X, Y, E)$ is a bipartite graph with $|X| \neq |Y|$ then there is no perfect matching in G (and so the determinant of G is zero).

The determinant of the biadjacency matrix of a bipartite graph with a unique perfect matching is known (see, [69]).

Theorem 3.1 *Suppose $G = (X, Y, E)$ is a bipartite graph with $|X| = |Y| = n$. If G has a unique perfect matching, then $\det B = \pm 1$, where B is the biadjacency matrix of G.*

Proof. We follow the presentation of [69] closely. Let $B = (B_{ij})_{i,j=1}^n$ be the biadjacency matrix of G. For a permutation σ on $\{1, 2, \ldots, n\}$, define

$$f(\sigma) = \prod_{i=1}^n B_{i\sigma(i)}.$$

From this definition, we see that the permutation σ corresponds to a perfect matching in G if and only if $f(\sigma) = 1$. Using the Leibniz formula, we have

$$\det B = \sum_{\sigma} (-1)^{N_\sigma} f(\sigma).$$

Therefore, if G has a unique perfect matching, then $\det B = \pm 1$, and if G has no perfect matching, then $\det B = 0$. $\qquad\square$

Note that in [69] the authors presented an NC algorithm for testing if a given bipartite graph has a unique perfect matching.

Now we prove a result that relates the determinant of the 'adjacency matrix' of a bipartite graph to the determinant of the 'biadjacency matrix' of that graph. This is a quite simple result and probably is well-known, but we present its proof.

First, we state the following lemma (see, [9, p. 135]).

Lemma 3.2 *Let A, B, C, and D be square matrices of order n such that $AC = CA$. Then*

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - CB).$$

Since the adjacency matrix $\mathcal{A}$ of a bipartite graph $G = (X, Y, E)$ with $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ has the form

$$\mathcal{A} = \begin{pmatrix} O & B \\ B^T & O \end{pmatrix},$$

where B is the biadjacency matrix of G, so we have

Corollary 3.3 *Let $G = (X, Y, E)$ be a bipartite graph with $|X| = |Y| = n$. Then*

$$\det \mathcal{A} = (-1)^n (\det B)^2,$$

where $\mathcal{A}$ and B are the adjacency matrix and the biadjacency matrix of G, respectively.

Using Theorem 3.1 and Corollary 3.3 we get

Theorem 3.4 *Suppose $G = (X, Y, E)$ is a bipartite graph with $|X| = |Y| = n$. If G has a unique perfect matching, then $\det \mathcal{A}(G) = (-1)^n$.*

Now we give another proof for Theorem 3.4.

First, we state the following lemma (see, [91]) which gives a relation for the determinant of the adjacency matrix of a graph with a *pendant vertex* (that is, a vertex of degree one).

Lemma 3.5 *Let G be a graph and v be any vertex of G. Denote by G^* the graph obtained from G by joining v to a new vertex u. Then $\det \mathcal{A}(G^*) = - \det \mathcal{A}(G - v)$.*

Hall [55] proved the following result on the number of perfect matchings in bipartite graphs.

Theorem 3.6 *If a bipartite graph $G = (X, Y, E)$ has a perfect matching and $\deg(x) \geq k$ for every $x \in X$, then G has at least $k!$ perfect matchings.*

Therefore, a bipartite graph with a perfect matching and with minimum degree at least two (in a part), has at least two perfect matchings.

Second Proof of Theorem 3.4. Clearly, we have $\delta(G) \geq 1$. If $\delta(G) \geq 2$, then by Theorem 3.6 there are at least two perfect matchings in G, which is a contradiction. Thus, G has a pendant vertex. Delete that vertex and its neighbour to get a graph G'. By Lemma 3.5, we have $\det \mathcal{A}(G) = - \det \mathcal{A}(G')$. By the same argument, G' has a pendant vertex. Delete that vertex and its neighbour to get a graph G''. By Lemma 3.5, we have $\det \mathcal{A}(G) = \det \mathcal{A}(G'')$. Since G has a unique perfect matching, by $(n - 1)$ times continuing this process (in X) we eventually get K_2. Clearly, $\det \mathcal{A}(K_2) = -1$. Consequently, we have $\det \mathcal{A}(G) = (-1)^n$. $\square$

3.3 At Least Two Perfect Matchings

In this section, we show that the determinant of a bipartite graph with at least two perfect matchings and with all cycle lengths divisible by four, is zero.

Let $G = (V, E)$ be a graph. For spanning subgraphs $F_1 = (V, E_1)$ and $F_2 = (V, E_2)$ of G, the spanning subgraph whose edge set is the symmetric difference $E_1 \triangle E_2$ is called the *symmetric difference* of F_1 and F_2, and is denoted by $F_1 \triangle F_2$. It is easy to see that the symmetric difference of two perfect matchings is a spanning subgraph in which each vertex has degree 2 or 0. Note that the vertices of degree 2 form an alternating cycle, that is, a cycle of even length which contains edges from the two perfect matchings alternately.

Suppose that $M = \{M_1, \ldots, M_t\}$ is the set of all perfect matchings of a graph G, and $\mathcal{C} = \{\mathcal{C}_1, \ldots, \mathcal{C}_k\}$ is the set of cycles that are components of $M_i \triangle M_j$, where $1 \le i < j \le t$.

Lemma 3.7 *Let G be a graph in which all cycle lengths are divisible by four. Also, let G have at least two perfect matchings. Then any two distinct elements of $\mathcal{C}$ (defined as above) are vertex-disjoint.*

Proof. Suppose there are p, q, $1 \le p < q \le k$, such that $\mathcal{C}_p$ and $\mathcal{C}_q$ are not vertex-disjoint. So, there are vertices u, v, w, with $v \ne w$, such that $uv \in \mathcal{C}_p$ and $uw \in \mathcal{C}_q$. Let $\mathcal{C}_p$ and $\mathcal{C}_q$ be the cycles formed by $M_i \triangle M_j$ and $M_r \triangle M_s$, respectively, for some i, j, r, s, where $1 \le i < j \le t$ and $1 \le r < s \le t$. Without loss of generality, assume that $uv \in M_i$ and $uw \in M_r$. Thus, uv and uw are two edges of the cycle, say $\mathcal{C}_m$, $1 \le m \le k$, formed by $M_i \triangle M_r$. Since $uw \in \mathcal{C}_q \cap \mathcal{C}_m$, so traversing $\mathcal{C}_m$, starting from u and then moving to v, ultimately intersects $\mathcal{C}_q$. Let y be the first vertex of this intersection. Let $P : uv \cdots xy \subset \mathcal{C}_m$ be a path joining u and y in $\mathcal{C}_m$. Also, let $Q : uw \cdots y \subset \mathcal{C}_q$, and $R : u \cdots y \subset \mathcal{C}_q$, with $uw \notin R$, be the paths joining u and y in $\mathcal{C}_q$. Since $P : uv \cdots xy \subset \mathcal{C}_m$, and $\mathcal{C}_m$ is formed by $M_i \triangle M_r$, so we have $xy \in M_i$ or $xy \in M_r$. But $xy \in M_r$ is not the case otherwise, as $Q, R \subset \mathcal{C}_q$ and $\mathcal{C}_q$ is formed by $M_r \triangle M_s$, so by looking at the edges incident with y (in $\mathcal{C}_q$ and $\mathcal{C}_m$) we see that this contradicts the definition of a matching. Thus, $xy \in M_i$. Now since the first and the last edge of P are in M_i, so the length of P is odd. That is, $l(P) \equiv 1 \pmod 4$ or $l(P) \equiv 3 \pmod 4$. Note that $P \cup Q = \mathcal{C}_m$ with $l(\mathcal{C}_m) \equiv 0 \pmod 4$, and $Q \cup R = \mathcal{C}_q$ with $l(\mathcal{C}_q) \equiv 0 \pmod 4$. Therefore, if $l(P) \equiv 1 \pmod 4$ then $l(R) \equiv 1 \pmod 4$, and if $l(P) \equiv 3 \pmod 4$ then $l(R) \equiv 3 \pmod 4$. Thus, $l(P) + l(R) \equiv 2 \pmod 4$. But we know that $P \cup R$ is a cycle, and so we must have $l(P) + l(R) \equiv 0 \pmod 4$. Hence, the lemma follows. $\square$

Suppose that $N = \{N_1, \ldots, N_s\}$ is the set of all perfect 2-matchings of a graph G, and define $\mathcal{D} := \{C \mid C$ is a cycle, and $\exists i, 1 \le i \le s$, s.t. $C \subset N_i\}$.

The next lemma shows that for the graphs of interest, $\mathcal{C}$ and $\mathcal{D}$ are in fact the same sets.

Lemma 3.8 *Let G be a graph in which all cycle lengths are divisible by four. Also, let G have at least two perfect matchings. Then for the sets $\mathcal{C}$ and $\mathcal{D}$ defined as above, we have $\mathcal{C} = \mathcal{D}$.*

Proof. Let $\mathcal{C}_i \in \mathcal{C}, 1 \le i \le k$. Then there exist $M_i, M_j \in M$, $1 \le i < j \le t$, such that $\mathcal{C}_i$ is formed by $M_i \triangle M_j$. Clearly, $\mathcal{C}_i \cup (M_i \cap M_j)$ is a perfect 2-matching of G. So, $\mathcal{C}_i \in \mathcal{D}$. Thus, $\mathcal{C} \subset \mathcal{D}$. Now, let $C \in \mathcal{D}$. So, there exists $i, 1 \le i \le t$, such that $C \subset N_i$. Since N_i is

a perfect 2-matching of G, so it consists of some disjoint cycles (of lengths divisible by 4), including C, and possibly some disjoint K_2's. Color the edges of each cycle in N_i by blue and red, alternately. Now the set of all blue edges (of all cycles in N_i) plus disjoint K_2's form a perfect matching of G, say M_p, $1 \le p \le t$. Also, the set of all red edges of C and all blue edges (of all other cycles in N_i) plus disjoint K_2's again form a perfect matching of G, say M_q, $1 \le q \le t$. Now, we see that C is formed by $M_p \triangle M_q$. So, $C \in \mathcal{C}$. Thus, $\mathcal{D} \subset \mathcal{C}$. Consequently, we have $\mathcal{C} = \mathcal{D}$. $\qquad\square$

Now we are ready to prove the main result of this section.

Theorem 3.9 *Let G be a graph on $2n$ vertices in which all cycle lengths are divisible by four. Also, let G have at least two perfect matchings. Then $\det \mathcal{A}(G) = 0$.*

Proof. Obviously, each edge of G that lies in a perfect 2-matching, must also lie in a perfect matching. So, those edges of G that lie in no perfect matching (we call them edges of 'Class A'), do not lie in any perfect 2-matching. Also, by Lemma 3.8, we conclude that those edges of G that lie in every perfect matching (we call them edges of 'Class B'), their roles in perfect 2-matchings of G (i.e., N_i's, $1 \le i \le t$) are just disjoint K_2's. If we delete all edges of Class A (not their end vertices) and also all edges of Class B (and their end vertices), then we get a new graph which we denote by G'. Note that G' is the union of the cycles in $\mathcal{C}$, and by Lemma 3.7, these cycles are vertex-disjoint. It is well-known that the determinant of a cycle of length divisible by four, is zero (just look at its eigenvalues!). Thus, $\det \mathcal{A}(G') = 0$. Also note that if from each N_i, we delete all edges of Class B (and their end vertices), that is, all disjoint K_2's, then we get the set of all perfect 2-matchings of G'. Therefore, by writing the expression (3.2) for G and G', we see that $\det \mathcal{A}(G) = (-1)^{2n+\alpha} \det \mathcal{A}(G')$, where α is the number of edges of Class B (that is, the number of disjoint K_2's). Consequently, we have $\det \mathcal{A}(G) = 0$. $\qquad\square$

3.4 Conclusion

In this section, we summarize the results proved in previous sections. For a graph G, we define the *matching core* of G to be the graph obtained from G by successively deleting each pendant vertex along with its neighbour.

Theorem 3.10 *The determinant of a graph G with all cycle lengths divisible by four, is 0 or $(-1)^{|V(G)|/2}$. Furthermore, the determinant is 0 if and only if the matching core of G is nonempty.*

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Proof. Since each cycle in G is divisible by four, so G is bipartite, say $G = (X, Y, E)$. If $|X| \neq |Y|$, then there is no perfect matching in G and so $\det \mathcal{A}(G) = 0$. Now, we assume that $|X| = |Y| = n$. Again, if G has no perfect matching then $\det \mathcal{A}(G) = 0$. So, we assume that G has at least one perfect matching. Thus, $\delta(G) \geq 1$. If $\delta(G) \geq 2$, then by Theorem 3.6 there are at least two perfect matchings in G, and so by Theorem 3.9 we have $\det \mathcal{A}(G) = 0$. So, let G have a pendant vertex. Delete that vertex and its neighbour to get a graph G'. By Lemma 3.5, we have $\det \mathcal{A}(G) = -\det \mathcal{A}(G')$. If G' has not a pendant vertex, then by Theorem 3.6 there are at least two perfect matchings in G', and so by Theorem 3.9 we have $\det \mathcal{A}(G') = 0$, which gives $\det \mathcal{A}(G) = 0$. So, let G' have a pendant vertex. Delete that vertex and its neighbour to get a graph G''. By continuing this process, we eventually arrive at one of the following cases:

- We get a graph H with $\delta(H) \geq 2$. (Note that the matching core of G is nonempty in this case.) Then by Theorems 3.6 and 3.9, we have $\det \mathcal{A}(H) = 0$. Thus, by Lemma 3.5, $\det \mathcal{A}(G) = 0$.

- We get K_2. (Note that the matching core of G is empty in this case.) This implies that $\det \mathcal{A}(G) = (-1)^n$. Note that in this case, G has a unique perfect matching.

$\square$

Chapter 4

The Number of Spanning Trees in Some Classes of Graphs

4.1 Introduction

Graphs, among other things, may be thought of as representing the connectivity of the atoms that comprise the (microscopic) conjugation network of an unsaturated molecule. Calculating the number of spanning trees in (i.e., the complexity of) such (labelled) molecular graphs has absorbed much attention (see [65] and the references therein). In fact, enumerating the number of spanning trees in graphs (networks) which comes up with lots of remarkable applications to various sciences, is of great interest among mathematicians, computer scientists, physicists, and chemists.

One of the notable applications of spanning trees in mathematics and computer science is that they can be applied to approximately solve the traveling salesman problem [4] (see also [14, 26, 79] for some other applications in mathematics). They also have interesting applications in physics [27, 36, 103, 104]. The number of spanning trees is also a measure of the network reliability.

Since intercommunication between all nodes of a network requires that its corresponding graph contain a spanning tree, one way to maximize the reliability of the network is to maximize the number of spanning trees in the graph [19, 21, 77, 85]. In chemistry, spanning trees of the molecular graph are of particular interest as they are related with physical properties of the corresponding chemical compound, for example, for counting certain chemical isomers [24]. They also arise in magnetic properties of conjugated systems [80, 81], in some concepts related to chemical nomenclature [52], and several other places [62].

Spanning trees are also used to study the complexity of reaction mechanisms of molecular graphs, in calculating the magnetic properties of conjugated polycyclic molecules by means of the ring-current model within the framework of the π-electron molecular orbital theory (see [83] and the references therein). In quantum field theory, it is well-known that the value of a Feynman integral can be written in terms of spanning trees [86]. Finally, they have interesting applications in high-dimensional structured tensor contractions arising in quantum chemistry [6, 66].

In [65] the authors prove a result for enumerating the number of spanning trees in general chemical graphs and give some applications to toroidal fullerenes. Also, in [50] the authors obtain some bounds on the number of spanning trees by using standard machinery from quantum information theory. However, determining the exact value or even the asymptotic number of spanning trees of a given graph is a difficult problem, in general. Therefore, researchers concentrate on enumerating the number of spanning trees in some special classes of graphs.

The *Laplacian matrix* (also called *Kirchhoff matrix*) of a graph G is defined as $\mathcal{L}(G) = \mathcal{D}(G) - \mathcal{A}(G)$, where $\mathcal{D}(G)$ and $\mathcal{A}(G)$ are the degree matrix and the adjacency matrix of G, respectively. The *characteristic polynomial* of a graph G is $\mathcal{P}_G(\lambda) = \det(\lambda I - \mathcal{A})$. Define

$$\mathcal{C}_G(\lambda) = \det\big(\lambda I - \mathcal{L}(G)\big).$$

Also, define

$$\mathcal{Q}_G(\lambda) = \det\big(\lambda I - \mathcal{D}^{-1}\mathcal{A}\big) = \det\big(\lambda I - \mathcal{A}\mathcal{D}^{-1}\big).$$

This can be rewritten as

$$\mathcal{Q}_G(\lambda) = \det\big(\lambda I - \mathcal{D}^{\frac{-1}{2}}\mathcal{A}\mathcal{D}^{\frac{-1}{2}}\big), \tag{4.1}$$

where $\mathcal{D}^{\frac{-1}{2}}$ is obtained by raising the entries on the principal diagonal of $\mathcal{D}$, to power $-\frac{1}{2}$.

A famous and classic result on the study of $t(G)$, the number of spanning trees in a graph G, is the following theorem, known as the *matrix-tree theorem* [67].

Theorem 4.1 (Matrix-tree Theorem) *For every connected graph G, $t(G)$ is equal to any cofactor of $\mathcal{L}(G)$.*

It is worth mentioning that in [50] the authors by reinterpreting the matrix-tree theorem in the context of quantum information theory, gave an exact formula to count spanning

trees based on the notion of quantum relative entropy, which is the quantum mechanical analog of the relative entropy.

The number of spanning trees of a connected graph G can be expressed in terms of the eigenvalues of $\mathcal{L}(G)$. Since by the definition, $\mathcal{L}(G)$ is a real symmetric matrix, it therefore has n non-negative real eigenvalues, which n is the number of vertices of G. Anderson and Morley [3, Theorem 1] proved that the multiplicity of 0 as an eigenvalue of $\mathcal{L}(G)$ equals the number of components of G. Therefore, the Laplacian matrix of a connected graph G has 0 as an eigenvalue with multiplicity one. Now, we state the following theorem from [33] which is very useful in this chapter.

Theorem 4.2 *Suppose G is a connected graph with n vertices. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\mathcal{L}(G)$, with $\lambda_n = 0$. Then*

$$t(G) = \frac{(-1)^{n-1}}{n} \mathcal{C}'_G(\lambda) \bigg|_{\lambda=0} = \frac{1}{n} \lambda_1 \ldots \lambda_{n-1}.$$

In the next section, using the number of spanning trees in some classes of graphs and also the determinant of the Cartesian product $P_m \,\square\, P_n$, we prove (and also give very short proofs for) some trigonometric identities. Some of these trigonometric identities also involve Fibonacci and Lucas numbers. As far as I know, the techniques already existed in the literature for proving these identities are mainly analytic with somewhat complicated computations (see, e.g., [46] and the references therein). In sections three and four, using some properties of the well-known Chebyshev polynomials, we prove some theorems that allow us to evaluate the number of spanning trees in join of graphs, Cartesian product of graphs, and nearly regular graphs. In the last section, we obtain the number of spanning trees in an (r, s)-semiregular graph and its line graph. Note that the same results, as in the last section, were proved in [94] using zeta functions. But our proofs are much shorter – we do not use zeta functions but instead employ some facts from Spectral Graph Theory. Most of the results in this chapter are my papers [10], and also my joint papers with Shirdareh Haghighi [14, 15].

4.2 On Some Trigonometric Identities Using Ideas from Graph Theory

Proving trigonometric identities may not be an easy problem. Sometimes, advanced tools may be needed to prove such identities. In this section we present an application of spectral

graph theory in proving trigonometric identities. This technique is a very simple double counting argument.

In this section, we derive the identities

$$F_n = \frac{2^{n-1}}{n} \sqrt{\prod_{k=1}^{n-1} \left(1 - \cos\frac{k\pi}{n}\cos\frac{3k\pi}{n}\right)}, \qquad n \geq 2, \tag{4.2}$$

$$\prod_{k=0}^{n-1} \left(1 + 4\sin^2\frac{k\pi}{n}\right) = L_{2n} - 2 = F_{2n+2} - F_{2n-2} - 2, \qquad n \geq 1, \tag{4.3}$$

$$\prod_{i=1}^{m}\prod_{j=1}^{n} \left(\cos\frac{i\pi}{m+1} + \cos\frac{j\pi}{n+1}\right) = \begin{cases} (-1)^{mn/2}/2^{mn} & \text{if } \gcd(m+1, n+1) = 1; \\ 0 & \text{otherwise,} \end{cases} \tag{4.4}$$

where F_n and L_n denote the Fibonacci and Lucas numbers, respectively. That is, $F_{n+2} = F_{n+1} + F_n$, for $n \geq 1$ with $F_1 = F_2 = 1$, and $L_{n+2} = L_{n+1} + L_n$, for $n \geq 1$ with $L_1 = 1$ and $L_2 = 3$.

To prove Identity (4.2), we apply the number of spanning trees in a special class of graphs known as circulant graphs. Identity (4.3) is derived from the number of spanning trees in a wheel. Also, we derive Identity (4.4) from the determinant of the Cartesian product $P_m \,\Box\, P_n$.

Applying the same technique to a fan gives a very short proof for the following identity

$$F_n = \prod_{k=1}^{\lfloor\frac{n-1}{2}\rfloor} \left(1 + 4\sin^2\frac{k\pi}{n}\right) = \prod_{k=1}^{\lfloor\frac{n-1}{2}\rfloor} \left(1 + 4\cos^2\frac{k\pi}{n}\right), \qquad n \geq 4, \tag{4.5}$$

which already was proved in [46] (and its corresponding references) by somewhat complicated analytic arguments.

Also, applying this technique to the path P_n and the cycle C_n, respectively, gives a new proof for the well-known identities:

$$\prod_{k=1}^{n-1} \left(2 - 2\cos\frac{k\pi}{n}\right) = n, \qquad n \geq 2, \tag{4.6}$$

$$\prod_{k=1}^{n-1} \left(2 - 2\cos\frac{2k\pi}{n}\right) = n^2, \qquad n \geq 2. \tag{4.7}$$

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Clearly, we can also write Identities (4.6) and (4.7) as the following, respectively:

$$\prod_{k=1}^{n-1} \sin \frac{k\pi}{2n} = \frac{\sqrt{n}}{2^{n-1}}, \qquad n \geq 2,$$

$$\prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = \frac{n}{2^{n-1}}, \qquad n \geq 2.$$

As the first application of Theorem 4.2, we prove Identity (4.6).

Proof of Identity (4.6). Consider the path P_n. It is well-known that the eigenvalues of the Laplacian matrix of P_n are $2 - 2\cos\frac{k\pi}{n}$ ($0 \leq k \leq n-1$) (see, e.g., [23]). On the other hand, we know that $t(P_n) = 1$. Therefore, by using Theorem 4.2 we obtain Identity (4.6). $\square$

For any given complex numbers $a_0, a_1, \ldots, a_{n-1}$, the *circulant matrix* $C = (c_{ij})_{i,j=1}^n$ is defined by $c_{ij} := a_{j-i \ (\mathrm{mod}\ n)}$. The following well-known result gives the eigenvalues of circulant matrices; see, e.g., [108, Theorem 4.8].

Theorem 4.3 *All eigenvalues of the circulant matrix C defined as above, are*

$$\mu_\omega = \sum_{j=0}^{n-1} a_j \omega^j,$$

where $\omega = e^{\frac{2\pi i}{n}}$.

Let $1 \leq s_1 < s_2 < \cdots < s_k < \frac{n}{2}$, where n and s_i ($1 \leq i \leq k$) are positive integers. A *circulant graph* $C_n(s_1, s_2, \ldots, s_k)$ is a $2k$-regular graph with vertex set $V = \{0, 1, \ldots, n-1\}$ and the edge set

$$E = \big\{\{i, i + s_j \quad (\mathrm{mod}\ n)\} \mid i = 0, 1, \ldots, n-1, \ j = 1, 2, \ldots, k\big\}.$$

The Laplacian matrix of circulant graph $C_n(s_1, s_2, \ldots, s_k)$ is clearly a circulant matrix. Thus, using Theorem 4.3 we get

Lemma 4.4 *The non-zero eigenvalues of $\mathcal{L}\big(C_n(s_1, s_2, \ldots, s_k)\big)$ are*

$$2k - \omega^{s_1 j} - \cdots - \omega^{s_k j} - \omega^{-s_1 j} - \cdots - \omega^{-s_k j}, \ \ 1 \leq j \leq n-1,$$

where $\omega = e^{\frac{2\pi i}{n}}$.

By combining Theorem 4.2 and the lemma above, we obtain the following corollary:

Corollary 4.5 *The number of spanning trees in $G = C_n(s_1, s_2, \ldots, s_k)$ is equal to*

$$t(G) = \frac{1}{n} \prod_{j=1}^{n-1} \Big(\sum_{i=1}^{k} \big(2 - 2\cos\frac{2js_i\pi}{n} \big) \Big).$$

Now, we are ready to prove Identities (4.2) and (4.7).

Proof of Identity (4.2). Consider the *square cycle* $C_n(1, 2)$. We can use Corollary 4.5 to obtain the number of spanning trees of $C_n(1, 2)$. On the other hand, Kleitman and Golden [68] proved that $t(C_n(1, 2)) = nF_n^2$. Now, with a little additional algebraic manipulation, Identity (4.2) follows. □

Proof of Identity (4.7). Look at the cycle $C_n(1) = C_n$. We know that $t(C_n) = n$. Therefore, by applying Corollary 4.5 to it, Identity (4.7) follows. □

Sedlacek [98] and later Myers [84] showed that

$$t(W_n) = L_{2n} - 2 = F_{2n+2} - F_{2n-2} - 2, \qquad n \geq 1.$$

Also, in [13, 15] we proved that

$$t(\mathcal{F}_n) = F_{2n}, \qquad n \geq 1.$$

Now, we find the number of spanning trees in W_n and $\mathcal{F}_n$ by applying Theorem 4.2. We first need to determine the eigenvalues of $\mathcal{L}(W_n)$ and $\mathcal{L}(\mathcal{F}_n)$.

Theorem 4.6 ([82]) *Let G_1 and G_2 be graphs on disjoint sets of r and s vertices, respectively. If $S(G_1) = (\lambda_1, \ldots, \lambda_r)$ and $S(G_2) = (\nu_1, \ldots, \nu_s)$ are the eigenvalues of $\mathcal{L}(G_1)$ and $\mathcal{L}(G_2)$ arranged in non-increasing order, then the eigenvalues of $\mathcal{L}(G_1 \vee G_2)$ are $n = r + s$; $\lambda_1 + s, \ldots, \lambda_{r-1} + s$; $\nu_1 + r, \ldots, \nu_{s-1} + r$; and 0.*

Since the eigenvalues of $\mathcal{L}(C_n)$ are $2 - 2\cos\frac{2k\pi}{n}$ ($0 \leq k \leq n-1$) (by Lemma 4.4), and the eigenvalues of $\mathcal{L}(P_n)$ are $2 - 2\cos\frac{k\pi}{n}$ ($0 \leq k \leq n-1$), therefore, by Theorem 4.6 we can determine the eigenvalues of $\mathcal{L}(W_n)$ and $\mathcal{L}(\mathcal{F}_n)$.

Theorem 4.7 *The nonzero eigenvalues of $\mathcal{L}(W_n)$ are $n + 1$, and $1 + 4\sin^2\frac{k\pi}{n}$ ($1 \leq k \leq n-1$). Also, the nonzero eigenvalues of $\mathcal{L}(\mathcal{F}_n)$ are $n + 1$, and $1 + 4\sin^2\frac{k\pi}{2n}$ ($1 \leq k \leq n-1$) (or $n + 1$, and $1 + 4\cos^2\frac{k\pi}{2n}$ ($1 \leq k \leq n-1$)).*

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Proofs of the Identities (4.3) and (4.5). By Theorems 4.2 and 4.7, the number of spanning trees of W_n and $\mathcal{F}_n$ are, respectively,

$$t(W_n) = \prod_{k=0}^{n-1} (1 + 4\sin^2 \frac{k\pi}{n}), \qquad n \geq 1,$$

$$t(\mathcal{F}_n) = \prod_{k=1}^{n-1} \left(1 + 4\sin^2 \frac{k\pi}{2n}\right) = \prod_{k=1}^{n-1} \left(1 + 4\cos^2 \frac{k\pi}{2n}\right), \qquad n \geq 2.$$

On the other hand, as we already referred, $t(W_n) = L_{2n} - 2 = F_{2n+2} - F_{2n-2} - 2$, $n \geq 1$, and $t(\mathcal{F}_n) = F_{2n}$, $n \geq 1$. Therefore, we obtain Identities (4.3) and (4.5). $\qquad\square$

Proving Identity (4.4), directly, seems to be challenging! But we prove it by combining a well-known result from spectral graph theory that gives the eigenvalues of the Cartesian product of graphs (see, e.g., [92, p. 587]), and a recently obtained result that gives the determinant of the adjacency matrix of the Cartesian product of paths [17, 90].

Theorem 4.8 *Let G be a graph of order m with eigenvalues $\mu_1(G), \ldots, \mu_m(G)$, and H be a graph of order n with eigenvalues $\mu_1(H), \ldots, \mu_n(H)$. Then the eigenvalues of the Cartesian product $G \square H$, are precisely the numbers $\mu_i(G) + \mu_j(H)$, for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.*

It is well-known that the eigenvalues of the path P_n are $2\cos\frac{j\pi}{n+1}$, where $1 \leq j \leq n$ (see, e.g., [92, p. 588]). Thus, by Theorem 4.8 the eigenvalues of $P_m \square P_n$ are obtained. Since the determinant of a matrix is equal to the product of its eigenvalues, so we easily get the determinant of the adjacency matrix of $P_m \square P_n$

$$\det A(P_m \square P_n) = 2^{mn} \prod_{i=1}^{m} \prod_{j=1}^{n} \left(\cos\frac{i\pi}{m+1} + \cos\frac{j\pi}{n+1} \right). \tag{4.8}$$

On the other hand, very recently, in [17, 90] the authors obtained the following explicit formula (using different methods) for the determinant of the adjacency matrix of $P_m \square P_n$.

Theorem 4.9 *Let m and n be positive integers. Then*

$$\det A(P_m \square P_n) = \begin{cases} (-1)^{mn/2} & \text{if } \gcd(m+1, n+1) = 1; \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Identity (4.4). This identity follows from (4.8) and Theorem 4.9. $\qquad\square$

Remark 4.10 *If Identity (4.4) can be proved, directly, then this gives another proof (perhaps even shorter) for the main result of [17, 90].*

4.3 Joins and Cartesian Products

In this section, using properties of Chebyshev polynomials, we give explicit formulas for the number of spanning trees in join and Cartesian product of some classes of graphs.

The starting point of our calculations is the following theorem.

Theorem 4.11 ([33]) *Suppose $G_1, \ldots, G_k$, are graphs of order $n_1, \ldots, n_k$, respectively, and let $n_1 + \cdots + n_k = n$. For the disjoint union $G_1 + \cdots + G_k$, and the join $G_1 \vee \cdots \vee G_k$, we have*

$$\mathcal{C}_{G_1 + \cdots + G_k}(\lambda) = \prod_{i=1}^{k} \mathcal{C}_{G_i}(\lambda),$$

$$\mathcal{C}_{G_1 \vee \cdots \vee G_k}(\lambda) = \lambda(\lambda - n)^{k-1} \prod_{i=1}^{k} \frac{\mathcal{C}_{G_i}(\lambda - n + n_i)}{\lambda - n + n_i}.$$

Now, by applying Theorems 4.2 and 4.11 we evaluate the number of spanning trees of the complete multipartite (or complete k-partite) graph $K_{n_1,\ldots,n_k}$, which is the main result of [75] and also studied in [89].

Theorem 4.12 *The number of spanning trees in the complete multipartite graph $K_{n_1,\ldots,n_k}$ of order n, is equal to*

$$t(K_{n_1,\ldots,n_k}) = n^{k-2} \prod_{i=1}^{k} \left(n - n_i\right)^{n_i - 1}.$$

Proof. Let $\mathcal{N}_m$ denote the empty graph of order m. Since $\mathcal{N}_m$ is the disjoint union of m copies of a single vertex, therefore $\mathcal{C}_{\mathcal{N}_m}(\lambda) = \lambda^m$. The complete multipartite graph $K_{n_1,\ldots,n_k}$ is the join of graphs $\mathcal{N}_{n_1}, \ldots, \mathcal{N}_{n_k}$. Now, Theorem 4.11 implies that

$$\mathcal{C}_{K_{n_1,\ldots,n_k}}(\lambda) = \lambda(\lambda - n)^{k-1} \prod_{i=1}^{k} \left(\lambda - n + n_i\right)^{n_i-1}.$$

Therefore, by Theorem 4.2,

$$
\begin{aligned}
t(K_{n_1,\ldots,n_k}) &= \left.\frac{(-1)^{n-1}}{n}\mathcal{C}'_{K_{n_1,\ldots,n_k}}(\lambda)\right|_{\lambda=0} \\
&= \left.\frac{(-1)^{n-1}}{n}(\lambda - n)^{k-1}\prod_{i=1}^{k}\left(\lambda - n + n_i\right)^{n_i-1}\right|_{\lambda=0} \\
&= n^{k-2}\prod_{i=1}^{k}\left(n - n_i\right)^{n_i-1}.
\end{aligned}
$$

$\square$

The function $\cos n\theta$ is a Chebyshev polynomial function of $\cos\theta$. Specifically, for $n \geq 0$, $\cos n\theta = T_n(\cos\theta)$, where T_n is the *Chebyshev polynomial of the first kind*, defined by $T_0(x) = 1$, $T_1(x) = x$, and for $n \geq 2$,

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x).$$

If we change the initial conditions to be $U_0(x) = 1$ and $U_1(x) = 2x$, but keep the same recurrence

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x),$$

we get the *Chebyshev polynomials of the second kind*.

Now, we list a few intriguing identities satisfied by the Chebyshev polynomials (taken from [56]), that help us to derive explicit formulas for the number of spanning trees in some other classes of graphs.

It is easy to show that for all $n \geq 0$, $T_n(1) = 1$ and $U_n(1) = n + 1$, $T_n(-1) = (-1)^n$, $U_n(-1) = (-1)^n(n + 1)$.

$$T_n(x) = \frac{1}{2}\Big(\big(x + \sqrt{x^2-1}\big)^n + \big(x - \sqrt{x^2-1}\big)^n\Big), \tag{4.9}$$

$$T_n(-x) = (-1)^n T_n(x), \tag{4.10}$$

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}\Big(\big(x + \sqrt{x^2-1}\big)^{n+1} - \big(x - \sqrt{x^2-1}\big)^{n+1}\Big), \quad |x| \neq 1, \tag{4.11}$$

$$U_n(-x) = (-1)^n U_n(x), \tag{4.12}$$

$$U_n(x) = \prod_{k=1}^{n}\Big(2x \pm 2\cos\frac{k\pi}{n+1}\Big), \tag{4.13}$$

$$T_n(x) = U_n(x) - xU_{n-1}(x). \tag{4.14}$$

The lemma below gives us the characteristic polynomial of the path P_n and the cycle C_n in terms of Chebyshev polynomials.

Lemma 4.13 ([33]) *For the path P_n, the cycle C_n, and the complete graph K_n, we have*

$$\mathcal{P}_{P_n}(\lambda) = U_n(\frac{\lambda}{2}), \tag{4.15}$$

$$\mathcal{P}_{C_n}(\lambda) = 2\big(T_n(\frac{\lambda}{2}) - 1\big), \tag{4.16}$$

$$\mathcal{P}_{K_n}(\lambda) = (\lambda - n + 1)(\lambda + 1)^{n-1}. \tag{4.17}$$

Suppose G is a k-regular graph of order n. It is easy to see that

$$\mathcal{C}_G(\lambda) = (-1)^n \mathcal{P}_G(k - \lambda).$$

Thus, by using the lemma above we can evaluate $\mathcal{C}_{C_n}(\lambda)$ and $\mathcal{C}_{K_n}(\lambda)$. The eigenvalues of $\mathcal{L}(P_n)$, as we have mentioned, are $2 - 2\cos\frac{k\pi}{n}$ $(0 \leq k \leq n-1)$, then by applying Identity (4.13), $\mathcal{C}_{P_n}(\lambda)$ also follows.

Lemma 4.14 *For the path P_n, the cycle C_n, and the complete graph K_n, we have*

$$\mathcal{C}_{P_n}(\lambda) = \lambda U_{n-1}(\frac{\lambda - 2}{2}), \tag{4.18}$$

$$\mathcal{C}_{C_n}(\lambda) = 2\Big(T_n(\frac{\lambda - 2}{2}) - (-1)^n\Big), \tag{4.19}$$

$$\mathcal{C}_{K_n}(\lambda) = \lambda(\lambda - n)^{n-1}. \tag{4.20}$$

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Now, we calculate the number of spanning trees in some special graphs.

Theorem 4.15

$$t\big(K_m \vee P_n\big) = (m+n)^{m-1}U_{n-1}\big(\frac{m+2}{2}\big).$$

Proof. By Theorem 4.11 and Lemma 4.14,

$$\mathcal{C}_{K_m \vee P_n}(\lambda) = \lambda(\lambda - m - n)^m U_{n-1}\big(\frac{\lambda - m - 2}{2}\big).$$

Now Theorem 4.2 gives

$$
\begin{aligned}
t\big(K_m \vee P_n\big) &= \frac{(-1)^{m+n-1}}{m+n}\mathcal{C}'_{K_m \vee P_n}(\lambda)\Big|_{\lambda=0} \\
&= \frac{(-1)^{m+n-1}}{m+n}(\lambda - m - n)^m U_{n-1}\big(\frac{\lambda - m - 2}{2}\big)\Big|_{\lambda=0} \\
&= (m+n)^{m-1}U_{n-1}\big(\frac{m+2}{2}\big).
\end{aligned}
$$

$\square$

By similar calculations, we can enumerate the number of spanning trees in some more cases.

Theorem 4.16

$$
\begin{aligned}
t\big(K_m \vee C_n\big) &= \frac{2}{m}(m+n)^{m-1}\Big(T_n\big(\frac{m+2}{2}\big) - 1\Big), \\
t\big(P_m \vee C_n\big) &= \frac{2}{m}U_{m-1}\big(\frac{n+2}{2}\big)\Big(T_n\big(\frac{m+2}{2}\big) - 1\Big), \\
t\big(P_m \vee P_n\big) &= U_{m-1}\big(\frac{n+2}{2}\big)U_{n-1}\big(\frac{m+2}{2}\big), \\
t\big(C_m \vee C_n\big) &= \frac{4}{mn}\Big(T_m\big(\frac{n+2}{2}\big) - 1\Big)\Big(T_n\big(\frac{m+2}{2}\big) - 1\Big).
\end{aligned}
$$

Proof. To prove the first formula, by Theorem 4.11 and Lemma 4.14, we have

$$\mathcal{C}_{K_m \vee C_n}(\lambda) = \frac{2\lambda}{\lambda - m}(\lambda - m - n)^m\Big(T_n\big(\frac{\lambda - m - 2}{2}\big) - (-1)^n\Big).$$

37

Now Theorem [4.2] gives

$$
\begin{aligned}
t\big(K_m \vee C_n\big) &= \left. \frac{(-1)^{m+n-1}}{m+n} \mathcal{C}'_{K_m \vee C_n}(\lambda) \right|_{\lambda=0} \\
&= \left. \frac{2(-1)^{m+n-1}}{(m+n)(\lambda-m)}(\lambda-m-n)^m \Big(T_n\big(\frac{\lambda-m-2}{2}\big) - (-1)^n \Big) \right|_{\lambda=0} \\
&= \frac{2}{m}(m+n)^{m-1} \Big(T_n\big(\frac{m+2}{2}\big) - 1 \Big).
\end{aligned}
$$

In order to prove the second formula, by Theorem [4.11] and Lemma [4.14], again, we have

$$
\mathcal{C}_{P_m \vee C_n}(\lambda) = \frac{2\lambda(\lambda-m-n)}{\lambda-m} U_{m-1}\big(\frac{\lambda-n-2}{2}\big)\Big(T_n\big(\frac{\lambda-m-2}{2}\big) - (-1)^n \Big).
$$

Now Theorem [4.2] gives

$$
\begin{aligned}
t\big(P_m \vee C_n\big) &= \left. \frac{(-1)^{m+n-1}}{m+n} \mathcal{C}'_{P_m \vee C_n}(\lambda) \right|_{\lambda=0} \\
&= \left. \frac{2(-1)^{m+n-1}(\lambda-m-n)}{(m+n)(\lambda-m)} U_{m-1}\big(\frac{\lambda-n-2}{2}\big)\Big(T_n\big(\frac{\lambda-m-2}{2}\big) - (-1)^n \Big) \right|_{\lambda=0} \\
&= \frac{2}{m} U_{m-1}\big(\frac{n+2}{2}\big)\Big(T_n\big(\frac{m+2}{2}\big) - 1 \Big).
\end{aligned}
$$

The proofs of the other formulas are similar. □

By the same method, we get $t\big(K_m \vee K_n\big) = (m+n)^{m+n-2}$, which is nothing but the Cayley's formula. Our machinery gives the formulas in the corollary below which have also appeared in [20].

Corollary 4.17 *The number of spanning trees of fan $\mathcal{F}_n$ and wheel W_n are*

$$
t(\mathcal{F}_n) = U_{n-1}\big(\frac{3}{2}\big) = F_{2n} = \frac{1}{\sqrt{5}}\Big(\big(\frac{3+\sqrt{5}}{2}\big)^n - \big(\frac{3-\sqrt{5}}{2}\big)^n \Big),
$$

$$
t(W_n) = 2\big(T_n\big(\frac{3}{2}\big) - 1 \big) = L_{2n} - 2 = \big(\frac{3+\sqrt{5}}{2}\big)^n + \big(\frac{3-\sqrt{5}}{2}\big)^n - 2,
$$

Proof. By Theorem 4.16, we have

$$t(\mathcal{F}_n) = t(P_n \vee P_1) = U_0\left(\frac{n+2}{2}\right)U_{n-1}\left(\frac{3}{2}\right) = U_{n-1}\left(\frac{3}{2}\right) = F_{2n},$$

$$t(W_n) = t(C_n \vee P_1) = 2U_0\left(\frac{n+2}{2}\right)\left(T_n\left(\frac{3}{2}\right) - 1\right) = 2\left(T_n\left(\frac{3}{2}\right) - 1\right) = L_{2n} - 2.$$

$\square$

Now, we study the number of spanning trees in Cartesian products of graphs. The key theorem here, is the following:

Theorem 4.18 ([8]) *The Laplacian eigenvalues of the Cartesian product $G \,\square\, H$, are precisely the numbers*

$$\lambda_i(G) + \lambda_j(H),$$

for $i = 1, 2, \ldots, |V(G)|$ and $j = 1, 2, \ldots, |V(H)|$.

Now we get the number of spanning trees of the *complete prism* $K_n \,\square\, P_m$.

Theorem 4.19 *For any $m, n \geq 2$,*

$$t\left(K_n \,\square\, P_m\right) = n^{n-2}\left(U_{m-1}\left(\frac{n+2}{2}\right)\right)^{n-1}.$$

Proof. Since the eigenvalues of $\mathcal{L}(K_n)$ by Lemma 4.14 are $0, n, n, \ldots, n$; and the eigenvalues of $\mathcal{L}(P_m)$ are $2 - 2\cos\frac{k\pi}{m}$ ($0 \leq k \leq m-1$), therefore, by Theorems 4.2 and 4.18,

$$t\left(K_n \,\square\, P_m\right) = \frac{1}{mn}\left(\prod_{k=1}^{m-1}\left(2 - 2\cos\frac{k\pi}{m}\right)\right)\left(\prod_{k=0}^{m-1}\left(n + 2 - 2\cos\frac{k\pi}{m}\right)\right)^{n-1}.$$

By Identity (4.6)

$$\prod_{k=1}^{m-1}\left(2 - 2\cos\frac{k\pi}{m}\right) = m.$$

Now applying Identity (4.13) implies the theorem. $\square$

Similarly, we obtain the number of spanning trees of the $(m \times n)$-grid $P_m \,\square\, P_n$, and *complete cyclic prism* $K_n \,\square\, C_m$.

Theorem 4.20

$$t(P_m \,\square\, P_n) = 4^{(m-1)(n-1)} \prod_{i=1}^{m-1} \prod_{j=1}^{n-1} \left(\sin^2 \frac{i\pi}{2m} + \sin^2 \frac{j\pi}{2n} \right),$$

$$t(C_m \,\square\, C_n) = mn4^{(m-1)(n-1)} \prod_{i=1}^{m-1} \prod_{j=1}^{n-1} \left(\sin^2 \frac{i\pi}{m} + \sin^2 \frac{j\pi}{n} \right),$$

$$t(P_m \,\square\, C_n) = n4^{(m-1)(n-1)} \prod_{i=1}^{m-1} \prod_{j=1}^{n-1} \left(\sin^2 \frac{i\pi}{2m} + \sin^2 \frac{j\pi}{n} \right),$$

$$t(K_m \,\square\, K_n) = m^{m-2} n^{n-2} (m+n)^{(m-1)(n-1)},$$

$$t(K_n \,\square\, C_m) = \frac{m2^{n-1}}{n} \left(T_m\left(\frac{n+2}{2}\right) - 1 \right)^{n-1}.$$

Proof. The eigenvalues of $\mathcal{L}(K_n)$, by Lemma 4.14, are $0, n, n, \ldots, n$; and the eigenvalues of $\mathcal{L}(P_m)$ and $\mathcal{L}(C_m)$ are $2 - 2\cos\frac{k\pi}{m}$ ($0 \le k \le m-1$), and $2 - 2\cos\frac{2k\pi}{m}$ ($0 \le k \le m-1$), respectively. Therefore, by a direct application of Theorems 4.2 and 4.18, and Identities (4.6) and (4.7), the proofs of these formulas follows easily. $\square$

The first and latter formulas also appeared in [70], and [20], respectively.

We now derive the number of spanning trees of the ladder $P_2 \,\square\, P_n$, and the n-prism $P_2 \,\square\, C_n$, which was also proved in [20].

Corollary 4.21 *The number of spanning trees of the ladder $P_2 \,\square\, P_n$, and the n-prism $P_2 \,\square\, C_n$ are*

$$t\big(P_2 \,\square\, P_n\big) = U_{n-1}(2) = \frac{\sqrt{3}}{6}\big((2 + \sqrt{3})^n - (2 - \sqrt{3})^n\big),$$

$$t\big(P_2 \,\square\, C_n\big) = n\big(T_n(2) - 1\big) = \frac{n}{2}\big((2 + \sqrt{3})^n + (2 - \sqrt{3})^n - 2\big).$$

Proof. Using Theorem 4.20 we have

$$t\big(P_2 \,\square\, P_n\big) \;=\; 4^{n-1} \prod_{j=1}^{n-1} \Big(\frac{1}{2} + \sin^2 \frac{j\pi}{2n}\Big) = U_{n-1}(2)$$

$$\;=\; \frac{\sqrt{3}}{6}\big((2+\sqrt{3})^n - (2-\sqrt{3})^n\big),$$

$$t\big(P_2 \,\square\, C_n\big) \;=\; t\big(K_2 \,\square\, C_n\big) = n\big(T_n(2) - 1\big)$$

$$\;=\; \frac{n}{2}\big((2+\sqrt{3})^n + (2-\sqrt{3})^n - 2\big).$$

$\square$

4.4 Nearly Regular Graphs

A *nearly k-regular* graph is one that all of its vertices except one (referred to as an *exceptional vertex*) have degree k. In this section, we prove a theorem for enumerating the number of spanning trees in nearly regular graphs. First, a theorem for k-regular graphs.

Theorem 4.22 ([33]) *Suppose G is a connected k-regular graph with n vertices. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of G, with $\lambda_n = k$. Then*

$$t(G) = \frac{1}{n} \prod_{i=1}^{n-1} (k - \lambda_i) = \frac{1}{n} \mathcal{P}'_G(k).$$

Theorem 4.23 *Suppose G is a connected nearly k-regular graph. Then*

$$t(G) = \mathcal{P}_H(k),$$

where H is the subgraph of G obtained by removing the exceptional vertex.

Proof. By the matrix-tree theorem, $t(G)$ is equal to any cofactor of $\mathcal{L}(G)$. Now we take the cofactor of the diagonal element corresponding to the exceptional vertex of G. Hence, the theorem follows. $\square$

Example 4.24 A wheel W_n is a nearly 3-regular graph. If we remove the exceptional vertex (called *hub*), we obtain the cycle C_n. The characteristic polynomial of the cycle C_n, by Lemma 4.13, is

$$\mathcal{P}_{C_n}(\lambda) = 2\big(T_n(\frac{\lambda}{2}) - 1\big).$$

Therefore, by Theorem 4.23

$$t(W_n) = 2\big(T_n(\frac{3}{2}) - 1\big),$$

as we already obtained.

The following lemma [51, Lemma 14.3.3] relates the number of spanning trees of a plane graph and its dual.

Lemma 4.25 *Let G be a connected plane graph. Then the graphs G and G^* have the same number of spanning trees.*

Example 4.26 Consider the fan $\mathcal{F}_n$. Replace any edge on the *rim* by the path P_{k+1} ($k \geq 1$), and denote the obtained graph by $\mathcal{F}_{n,k}$. The dual $\mathcal{F}_{n,k}^*$ is nearly $(k + 2)$-regular. If we remove the exceptional vertex of $\mathcal{F}_{n,k}^*$, then we obtain the path P_{n-1}. The characteristic polynomial of the path P_n by Lemma 4.13 is

$$\mathcal{P}_{P_n}(\lambda) = U_n(\frac{\lambda}{2}).$$

Consequently, by Theorem 4.23 and Lemma 4.25

$$t(\mathcal{F}_{n,k}) = t(\mathcal{F}_{n,k}^*) = U_{n-1}\big(\frac{k+2}{2}\big).$$

Example 4.27 Consider the wheel W_n. Replace any edge on the rim by the path P_{k+1} ($k \geq 1$), and denote the obtained graph by $W_{n,k}$. The dual $W_{n,k}^*$ is nearly $(k + 2)$-regular. If we remove the exceptional vertex of $W_{n,k}^*$, then we obtain the cycle C_n. Similar to the example above,

$$t(W_{n,k}) = t(W_{n,k}^*) = 2\Big(T_n\big(\frac{k+2}{2}\big) - 1\Big).$$

Example 4.28 Place n k-gons in a row, such that each two consecutive k-gons have a side in common. Denote this graph by $G_{n,k}$. The dual $G^*_{n,k}$ is nearly k-regular. If we remove the exceptional vertex of $G^*_{n,k}$, then we obtain the path P_n. As above,

$$t(G_{n,k}) = t(G^*_{n,k}) = U_n(\frac{k}{2}).$$

4.5 Spanning Trees in Line Graphs

In [94], using zeta functions, the number of spanning trees in an (r, s)-semiregular graph and its line graph are obtained (also, look at [107], in which the authors study the asymptotic behavior of the number of spanning trees and the Kirchhoff index of iterated line graphs and iterated para-line graphs of a regular graph). In this section, we give short proofs for the formulas presented in [94] without using zeta functions. Furthermore, by applying a formula that enumerates the number of spanning trees in the line graph of an (r, s)-semiregular graph, we give a new proof of Cayley's Theorem.

It is easy to see that if G is a k-regular graph, then its line graph $L(G)$ is $2(k-1)$-regular, and if G is (r, s)-semiregular, then its line graph is $(r+s-2)$-regular. In fact, the line graph of a graph G is regular if and only if G is regular or semiregular.

Let G be an (r, s)-semiregular graph of order n. By looking at $\mathcal{A}(G)$ and $\mathcal{D}(G)$, and in view of (4.1), one can easily show that

$$\mathcal{Q}_G(\lambda) = (rs)^{-\frac{n}{2}} \mathcal{P}_G\big(\sqrt{rs}\lambda\big). \tag{4.21}$$

First, we seek a formula for enumerating the number of spanning trees in a connected (r, s)-semiregular graph G. We need the following well-known result (see, [9, Lemma 4.4.8]).

Theorem 4.29 *Let A be a matrix of order n. Then*

$$\mathcal{P}'(\lambda) = \sum_{i=1}^{n} \det(\lambda I - A_i),$$

where A_i is the submatrix of A of order $n-1$ obtained by striking out row and column i of A.

By putting $\lambda = 0$ in Theorem 4.29, we get

Corollary 4.30 *Let G be a graph of order n, with adjacency matrix $\mathcal{A}$. Then*

$$\mathcal{P}'_G(\lambda)\Big|_{\lambda=0} = (-1)^{n-1} \sum_{i=1}^{n} \det \mathcal{A}_i,$$

where $\mathcal{A}_i$ is the submatrix of $\mathcal{A}$ of order $n-1$ obtained by striking out row and column i of $\mathcal{A}$.

By applying Corollary 4.30, we express the number of spanning trees in a graph G, in terms of $\mathcal{Q}'_G(\lambda)$.

Theorem 4.31 *Suppose G is a connected graph of order n and size m with vertex set $\{v_1, \ldots, v_n\}$. Then*

$$t(G) = \frac{1}{2m}\left(\prod_{i=1}^{n} d_G(v_i)\right)\mathcal{Q}'_G(\lambda)\Big|_{\lambda=1},$$

where $d_G(v_i)$ denotes the degree of vertex v_i.

Proof. Let $\mathcal{A}$ and $\mathcal{D}$ be the adjacency matrix and the degree matrix of G, respectively. One can easily write

$$
\begin{aligned}
\mathcal{Q}'_G(\lambda)\Big|_{\lambda=1} &= \left(\det\left(\lambda I - \mathcal{D}^{-1}\mathcal{A}\right)\right)'\Big|_{\lambda=1} \\
&= \left(\det\left((\lambda+1)I - \mathcal{D}^{-1}\mathcal{A}\right)\right)'\Big|_{\lambda=0} \\
&= \left(\det\left(\lambda I + \mathcal{D}^{-1}\mathcal{L}\right)\right)'\Big|_{\lambda=0} \\
&= \mathcal{P}'_H(\lambda)\Big|_{\lambda=0},
\end{aligned}
$$

where H is the graph with adjacency matrix $-\mathcal{D}^{-1}\mathcal{L}$. Now using Corollary 4.30, the matrix-tree theorem, and with a little additional algebraic manipulation, the theorem follows. $\square$

By (4.21) and the theorem above, we obtain a formula for enumerating the number of spanning trees in a connected (r, s)-semiregular graph G.

Theorem 4.32 *Suppose G is a connected (r, s)-semiregular graph of order n and size m, where the number of vertices of degrees r and s, are n_1 and n_2, respectively. Then*

$$t(G) = \frac{\sqrt{rs}}{2m} \left(\frac{r}{s}\right)^{\frac{n_1-n_2}{2}} \mathcal{P}'_G(\lambda)\Big|_{\lambda=\sqrt{rs}}.$$

The following theorem gives the characteristic polynomial of the line graph of an (r, s)-semiregular graph.

Theorem 4.33 ([33]) *Let G be an (r, s)-semiregular graph of order n and size m, where the number of vertices of degrees r and s, are n_1 and n_2, respectively. Then the characteristic polynomial of its line graph $L(G)$ is*

$$\mathcal{P}_{L(G)}(\lambda) = (\lambda + 2)^{m-n} \left(\frac{x}{y}\right)^{\frac{n_1-n_2}{2}} \mathcal{P}_G(\sqrt{xy}),$$

where $x = \lambda - r + 2$, and $y = \lambda - s + 2$.

Now we are ready to calculate the number of spanning trees in the line graph of a connected (r, s)-semiregular graph.

Theorem 4.34 *Let G be a connected (r, s)-semiregular graph of order n and size m, where the number of vertices of degrees r and s, are n_1 and n_2, respectively. Then*

$$t(L(G)) = \frac{(r + s)^{m-n+1}}{rs} \left(\frac{r}{s}\right)^{n_2-n_1} t(G).$$

Proof. Since G is (r, s)-semiregular, then its line graph $L(G)$ is $(r + s - 2)$-regular. Also, we know that $\mathcal{P}_G(\sqrt{rs}) = 0$. Thus, using Theorem 4.33,

$$\mathcal{P}'_{L(G)}(\lambda)\Big|_{\lambda=r+s-2} = (\lambda + 2)^{m-n} \left(\frac{x}{y}\right)^{\frac{n_1-n_2}{2}} \frac{x+y}{2\sqrt{xy}} \mathcal{P}'_G(\sqrt{xy})\Big|_{\lambda=r+s-2}$$

$$= \frac{(r + s)^{m-n+1}}{2\sqrt{rs}} \left(\frac{s}{r}\right)^{\frac{n_1-n_2}{2}} \mathcal{P}'_G(\sqrt{rs}),$$

After applying Theorems 4.22 and 4.32, the theorem follows. $\qquad\square$

This theorem gives a new proof of Cayley's Theorem.

Theorem 4.35 (Cayley's Theorem) *Assume that K_n is the complete graph of order n. Then*

$$t(K_n) = n^{n-2}.$$

Proof. The complete graph K_n is the line graph of the star S_n (i.e., $K_{1,n}$). Now, Theorem 4.34 implies that

$$t(K_n) = \frac{(n+1)^0}{n}(\frac{n}{1})^{n-1}t(S_n) = n^{n-2}.$$

$\square$

Chapter 5

Conclusions

In Chapter 2, we estimated the number of possible types degree patterns of k-lacunary polynomials of degree $t < p$ which split completely modulo p. The result was based on a rather unusual combination of an upper bound on the number of zeros of lacunary polynomials with some graph theory arguments. A slight modification of our approach can easily produce a nontrivial bound for $1 \le k \le 3$ as well, however we do not know how to relax the condition (2.2). It is certainly an interesting question to show that almost all k-lacunary polynomials of a large degree are irreducible over $\mathbb{F}_p$. In fact, as a first step one can try to get a lower bound on the degree over $\mathbb{F}_p$ of the splitting field of a "random" k-lacunary polynomial.

In Chapter 3, we showed that the determinant of a bipartite graph with at least two perfect matchings and with all cycle lengths divisible by four, is zero. Giving (or estimating) the determinant of bipartite graphs with only 'some' cycle lengths divisible by four is an interesting problem. Also, evaluating the determinant of other classes of graphs (e.g., join and Cartesian product of graphs), and characterizing other graphs with positive nullity, towards attacking the Collatz and Sinogowitz problem, are good projects for future research.

In Chapter 4, we first introduced an application of spectral graph theory in proving trigonometric identities. It is an interesting problem to develop this technique, to cover more identities and also derive new identities. Also, it is an interesting problem to give a direct proof for Identity (4.4) – as this gives another proof (perhaps even shorter) for the main result of [17, 90]. In the rest of Chapter 4, using some properties of the well-known Chebyshev polynomials, we calculated the number of spanning trees in several classes of graphs. Considering more classes of graphs and also giving combinatorial proofs for

these results are nice projects. In the last section of Chapter 4, we obtained the number of spanning trees in an (r, s)-semiregular graph and its line graph, without using zeta functions, which gives a very short proof for the result obtained by Sato [94]. Then we gave a new proof of Cayley's Theorem.

Note that the operation of a line graph produces many new types of graphs. For example, the line graph of a tetrahedron (respectively, cube and dodecahedron) graph is an octahedron (respectively, cuboctahedron and icosidodecahedron) graph. The key idea which connects a molecular graph to its embedding in three dimensional space is that of the line graph. For the chemical interpretation of (iterated) line graphs (i.e., where the graph is associated to a molecule), the reader is referred to [37, 38], in which the author, among other things, gives an application of iterated line graphs to biomolecular conformation. Also, the paper [38] discusses more applications of (iterated) line graphs and spanning trees in studying Z-matrices (which is a way to represent a system built of atoms in chemistry) and computational quantum chemistry. Because of these wide-ranging applications, it is important and is a nice project to develop useful techniques for enumerating the number of spanning trees in line graphs of various classes of graphs.

Note that many techniques work only for enumerating the number of 'labeled' spanning trees and do not apply to the number of 'non-isomorphic' spanning trees. Thus, while some methods exist which address non-isomorphic spanning trees, finding more suitable tools in this matter would be very helpful.

Bibliography

[1] J. Alber, N. Betzler and R. Niedermeier, Experiments on data reduction for optimal domination in networks, *Ann. Oper. Res.* 146 (2006), 105–117.

[2] V. E. Alekseev, D. V. Korobitsyn and V. V. Lozin, Boundary classes of graphs for the dominating set problem, *Discrete Math.* 285 (2004), 1–6.

[3] W. N. Anderson, JR. and T. D. Morley, Eigenvalues of the Laplacian of a graph, *Linear Multilinear Algebra* 18 (1985), 141–145.

[4] D. L. Applegate, R. E. Bixby, V. Chvátal, and W. J. Cook, *The Traveling Salesman Problem: A Computational Study*, Princeton University Press, (2006).

[5] Y. Ashir, B. Bose, B. Broeg and Y. Kwon, Lee distance and topological properties of k-ary n-cubes, *IEEE Trans. Comput.* 44 (1995), 1021–1030.

[6] A. Auer, G. Baumgartner, D. E. Bernholdt, A. Bibireata, V. Choppella, D. Cociorva, X. Gao, R. J. Harrison, S. Hirata, S. Krishnamoorthy, S. Krishnan, C.-C. Lam, Q. Lu, M. Nooijen, R. M. Pitzer, J. Ramanujam, P. Sadayappan and A. Sibiryakov, Synthesis of high-performance parallel programs for a class of ab initio quantum chemistry models, *Proc. IEEE* 93 (2005), 276–292.

[7] B. Balasundaram and S. Butenko, Graph domination, coloring and cliques in telecommunications, *Handbook of Optimization in Telecommunications*, Springer, 2006, 865–890.

[8] L. W. Beineke and R. J. Wilson, *Topics in algebraic graph theory*, Cambridge University Press, New York, (2004).

[9] D. S. Bernstein, *Matrix mathematics: theory, facts, and formulas*, second ed., Princeton University Press, (2009).

[10] K. Bibak, The number of spanning trees in an (r, s)-semiregular graph and its line graph, *Int. J. Quant. Chem.*, to appear; DOI: http://dx.doi.org/10.1002/qua.24252.

[11] K. Bibak, Additive combinatorics with a view towards computer science and cryptography: An exposition, In *Proc. Int. Number Theory Conf. in Memory of Alf van der Poorten*, to appear; http://cacr.uwaterloo.ca/techreports/2011/cacr2011-31.pdf.

[12] K. Bibak, On the determinant of bipartite graphs, submitted.

[13] K. Bibak and M. H. Shirdareh Haghighi, Recursive relations for the number of spanning trees, *Appl. Math. Sci.* 3 (2009), 2263–2269.

[14] K. Bibak and M. H. Shirdareh Haghighi, Some trigonometric identities involving Fibonacci and Lucas numbers, *J. Integer Seq.* 12 (2009), Article 09.8.4.

[15] K. Bibak and M. H. Shirdareh Haghighi, The number of spanning trees in some classes of graphs, *Rocky Mountain J. Math.* 42(4) (2012), 1183–1195.

[16] K. Bibak and I. E. Shparlinski, On fully split lacunary polynomials in finite fields, *Bull. Pol. Acad. Sci. Math.* 59 (2011), 197–202.

[17] A. Bień, The problem of singularity for planar grids, *Discrete Math.* 311 (2011), 921–931.

[18] N. Biggs, Perfect codes in graphs, *J. Comb. Theory Ser. B* 15 (1973), 289–296.

[19] F. Boesch, L. Petingi and C. Suffel, On the characterization of graphs with maximum number of spanning trees, *Discrete Math.* 179 (1998), 155–166.

[20] F. Boesch and H. Prodinger, Spanning tree formulas and chebyshev polynomials, *Graphs Combin.* 2 (1986), 191–200.

[21] F. Boesch, A. Satyanarayana, and C. L. Suffel, A survey of some network reliability analysis and synthesis results, *Networks* 54(2) (2009), 99–107.

[22] J. A. Bondy and U.S.R. Murty, *Graph theory*, Springer-Verlag, New York (2008).

[23] A. E. Brouwer, A. M. Cohen, and A. Neumaier, *Distance-regular graphs*, Springer-Verlag, New York, (1989).

[24] T. J. N. Brown, R. B. Mallion, P. Pollak and A. Roth, Some methods for counting the spanning trees in labelled molecular graphs, examined in relation to certain fullerenes, *Discrete Appl. Math.* 67 (1996), 51–66.

[25] R. Canetti, J. B. Friedlander, S. V. Konyagin, M. Larsen, D. Lieman and I. E. Shparlinski, On the statistical properties of Diffie–Hellman distributions, *Israel J. Math.* 120 (2000), 23–46.

[26] K.-M. Chao and B. Y. Wu, *Spanning trees and optimization problems*, CRC Press, (2004).

[27] G. Chen, B. Wu and Z. Zhang, Properties and applications of Laplacian spectra for Koch networks, *J. Phys. A: Math. Theor.* 45 (2012), 025102.

[28] Q. Cheng, S. Tarasov and M. Vyalyi, Efficient algorithms for sparse cyclotomic integer zero testing, *Theory Comput. Syst.* 46 (2010), 120–142.

[29] W. E. Clark, B. Shekhtman, S. Suen and D. C. Fisher, Upper bounds for the domination number of a graph, *Congr. Numer.* 132 (1998), 99–123.

[30] E. J. Cockayne and S. T. Hedetniemi, Towards a theory of domination in graphs, *Networks* 7 (1977), 211–219.

[31] L. Collatz and U. Sinogowitz, Spektren endlicher Grafen, *Abh. Math. Sem. Univ. Hamburg* 21 (1957), 63–77.

[32] C. Cooper, R. Klasing and M. Zito, Lower bounds and algorithms for dominating sets in web graphs, *Internet Math.* 2 (2005), 275–300.

[33] D. Cvetkovič, M. Doob and H. Sachs, *Spectra of graphs: theory and applications*, third ed., Johann Ambrosius Barth, (1995).

[34] D. Cvetković and S. Simić, Graph spectra in computer science, *Linear Algebra Appl.* 434 (2011), 1545–1562.

[35] F. Dai and J. Wu, On constructing k-connected k-dominating set in wireless ad hoc and sensor networks, *J. Parallel Distrib. Comput.*, 66 (2006), 947–958.

[36] D. Dhar, Theoretical studies of self-organized criticality, *Physica A* 369 (2006), 29–70.

[37] D. B. Dix, An application of iterated line graphs to biomolecular conformation, preprint.

[38] D. B. Dix, Polyspherical coordinate systems on orbit spaces with applications to biomolecular shape, *Acta Appl. Math.* 90 (2006), 247–306.

[39] W. Duckworth and M. Zito, Sparse hypercube 3-spanners, *Discrete Appl. Math.*, 103 (2000), 289–295.

[40] D. S. Dummit and R. M. Foote, *Abstract algebra*, 3rd ed., John Wiley and Sons, Inc., 2003.

[41] S. M. Fallat and L. Hogben, The minimum rank of symmetric matrices described by a graph: A survey, *Linear Algebra Appl.* 426 (2007), 558–582.

[42] M. Filaseta, A. Granville and A. Schinzel, Irreducibility and greatest common divisor algorithms for sparse polynomials, *Number Theory and Polynomials*, LMS Lecture Note Series 352, Cambridge Univ. Press, 2008, 155–176.

[43] F. V. Fomin, F. Grandoni and D. Kratsch, Measure and conquer: Domination – a case study, In *Proc. Internat. Colloq. Autom. Lang. Program.*, 2005, 191–203.

[44] F. V. Fomin, F. Grandoni, A. V. Pyatkin and A. A. Stepanov, Combinatorial bounds via measure and conquer: Bounding minimal dominating sets and applications, *ACM Trans. Algorithms* 5 (2008), 1–17.

[45] M. R. Garey and D. S. Johnson, *Computers and intracability: A guide to the theory of NP-Completeness*, W. H. Freeman & Co., 1979.

[46] N. Garnier and O. Ramaré, Fibonacci numbers and trigonometric identities, *Fibonacci Quart.* 46/47(1) (2008/2009), 56–61.

[47] S. Gaspers, D. Kratsch, M. Liedloff and I. Todinca, Exponential time algorithms for the minimum dominating set problem on some graph classes, *ACM Trans. Algorithms* 6 (2009), 1–21.

[48] M. Giesbrecht and D. Roche, Interpolation of shifted-lacunary polynomials, *Comput. Complexity* 19(3) (2010), 333–354.

[49] M. Giesbrecht and D. Roche, Detecting lacunary perfect powers and computing their roots, *J. Symbolic Comput.* 46 (2011), 1242–1259.

[50] V. Giovannetti and S. Severini, The Kirchhoff's matrix-tree theorem revisited: counting spanning trees with the quantum relative entropy, *Advances in Network Complexity*, A. Mowshowitz and M. Dehmer (Eds.), Wiley-VCH, to appear.

[51] C. Godsil and G. Royle, *Algebraic graph theory* Springer-Verlag, New York, (2001).

[52] A. L. Goodson, Nomenclature of chemical compounds, In D. Bonchev and D. H. Rouvray, eds., *Chemical graph theory: Introduction and fundamentals*, Abacus Press/Gordon & Breach, 1991, pp. 97–132.

[53] I. Gutman, On permutations in bipartite graphs, *Kragujevac J. Math.* 22 (2000), 57–66.

[54] I. Gutman and B. Borovićanin, Nullity of graphs: An updated survey, in *Selected topics on applications of graph spectra*, Math. Inst., Belgrade, 2011, pp. 137–154.

[55] M. Hall Jr., Distinct representatives of subsets, *Bull. Amer. Math. Soc.* 54 (1948), 922–926.

[56] D. C. Handscomb and J. C. Mason, *Chebyshev polynomials*, CRC Press, New York, (2002).

[57] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Oxford Univ. Press, Oxford, 1979.

[58] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Domination in graphs: Advanced topics*, Marcel Dekker Inc., 1998.

[59] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of domination in graphs*, Marcel Dekker Inc., 1998.

[60] S. Hoory, N. Linial, and A. Wigderson, Expander graphs and their applications, *Bull. Amer. Math. Soc.* 43 (2006), 439–561.

[61] J. D. Horten, J. G. Kalbfleisch and R. G. Stanton, On covering sets and error-correcting codes, *J. Comb. Theory, Ser. A*, 11 (1971), 233–250.

[62] S.-Y. Huan, J.-H. Jiang, C.-X. Jiao, G.-L. Shen, Q. Shen and R.-Q. Yu, Optimized partition of minimum spanning tree for piecewise modeling by particle swarm algorithm. QSAR studies of antagonism of angiotensin II antagonists, *J. Chem. Inf. Comput. Sci.* 44(6) (2004), 2027–2031.

[63] L. Huang and W. Yan, On the determinant of the adjacency matrix of a type of plane bipartite graphs, *MATCH Commun. Math. Comput. Chem.* 68 (2012), 931–938.

[64] E. Kaltofen and P. Koiran, Finding small degree factors of multivariate supersparse (lacunary) polynomials over algebraic number fields, In *Proc. 2006 Int. Symp. Symb. and Alg. Comput. (ISSAC '06)*, pp. 162–168.

[65] E. C. Kirby, D. J. Klein, R. B. Mallion, P. Pollak, and H. Sachs, A theorem for counting spanning trees in general chemical graphs and its particular application to toroidal fullerenes, *Croat. Chem. Acta* 77 (2004), 263–278.

[66] R. C. Kirby and L. R. Scott, Geometric optimization of the evaluation of finite element matrices, *SIAM J. Sci. Comput.* 29(2) (2007), 827–841.

[67] G. Kirchhoff, Über die auflösung der gleichungen auf, welche man bei der untersuchung der linearen verteilung galvanischer ströme geführt wird, *Ann. Phy. Chem.*, 72 (1847), 497–508.

[68] D. J. Kleitman and B. Golden, Counting trees in a certain class of graphs, *Amer. Math. Monthly* 82 (1975), 40–44.

[69] D. Kozen, U. V. Vazirani, and V. V. Vazirani, NC algorithms for comparability graphs, interval graphs, and testing for unique perfect matching, In *Foundations of software technology and theoretical computer science*, Lecture Notes in Computer Science, 1985, Vol. 206, pp. 496–503.

[70] G. Kreweras, Complexité et circuits eulériens dans les sommes tensorielles de graphs, *J. Combin. Theory Ser. B* 24 (1978), 202–212.

[71] F. Kuhn and R. Wattenhofer, Constant-time distributed dominating set approximation, In *Proc. 22nd Annual ACM Symp. Principles Distrib. Comput.*, 2003, 25–32.

[72] H. W. Lenstra, Finding small degree factors of lacunary polynomials, In *Number Theory in Progress, Vol. 1, Zakopane, 1997*, Walter de Gruyter, Berlin, 1999, 267–276.

[73] H. W. Lenstra, On the factorization of lacunary polynomials, In *Number Theory in Progress, Vol. 1, Zakopane, 1997*, Walter de Gruyter, Berlin, 1999, 277–291.

[74] C. Lenzen and R. Wattenhofer, Minimum dominating set approximation in graphs of bounded arboricity, In *Proc. of the 24th Internat. Conf. on Distrib. Comput. (DISC '10)*, 2010, 510–524.

[75] R. P. Lewis, The number of spanning trees of a complete multipartite graph, *Discrete Math.* 197/198 (1999), 537–541.

[76] J. Li, C. H. Foh, L. L. H. Andrew and M. Zukerman, Sizes of minimum connected dominating sets of a class of wireless sensor networks, *IEEE Internat. Conf. Commun. (ICC '08)*, 2008, 332–336.

[77] Z. Lonc, K. Parol, J. Wojcieckowski, On the number of spanning trees in directed circulant graphs, *Networks* 36 (2001), 129–133.

[78] M. Lu, H. Liu and F. Tian, Bounds of Laplacian spectrum of graphs based on the domination number, *Linear Algebra Appl.*, 402 (2005), 390–396.

[79] R. Lyons, Asymptotic enumeration of spanning trees, *Combin. Probab. Comput.* 14(4) (2005), 491–522.

[80] R. B. Mallion, On the number of spanning trees in a molecular graph, *Chem. Phys. Lett.* 36 (1975), 170–174.

[81] R. B. Mallion, Some graph-theoretical aspects of simple ring-current calculations on conjugated systems, *Proc. R. Soc. Lond. A* 341 (1975), 429–449.

[82] R. Merris, Laplacian graph eigenvectors, *Linear Algebra Appl.* 278 (1998), 221–236.

[83] A. Miličević and N. Trinajstić, Combinatorial enumeration in chemistry, In *Chemical modelling: applications and theory,* Vol. 4 (2006), pp. 405–469.

[84] B. R. Myers, Number of spanning trees in a wheel, *IEEE Trans. Circuit Theory* 18 (1971), 280–282.

[85] W. Myrvold, Reliable network synthesis: some recent developments, In *Proc. 8th Intern. Conf. on Graph Theory, Combinatorics, Algorithms and Applications* 2, 1999, pp. 650–660.

[86] N. Nakanishi, *Graph theory and Feynman integrals*, Gordon and Breach, New York, (1971).

[87] G. R. Omidi, On the nullity of bipartite graphs, *Graphs Combin.* 25 (2009), 111–114.

[88] O. Ore, *Theory of graphs*, Amer. Math. Soc. Colloq. Publ., Vol. 38, 1962.

[89] T. D. Porter, Generating the list of spanning trees in $K_{s,t}$, *J. Comb. Math. Comb. Comput.* 50 (2004), 17–32.

[90] D. Pragel, Determinants of box products of paths, *Discrete Math.* 312 (2012), 1844–1847.

[91] H. M. Rara, Reduction procedures for calculating the determinant of the adjacency matrix of some graphs and the singularity of square planar grids, *Discrete Math.* 151 (1996), 213–219.

[92] K. H. Rosen, *Handbook of Discrete and Combinatorial Mathematics*, CRC Press, (1999).

[93] D. H. Rouvray, Combinatorics in chemistry, in *Handbook of combinatorics, vol. II*, (1996), pp. 1955–1981.

[94] I. Sato, Zeta functions and complexities of a semiregular bipartite graph and its line graph, *Discrete Math.* 307 (2007), 237–245.

[95] A. Schinzel, Reducibility of lacunary polynomial, XII, *Acta Arith.* 90 (1999), 273–289.

[96] A. Schinzel, *Polynomials with special regard to reducibility*, Cambridge University Press, 2000.

[97] I. Sciriha, A characterization of singular graphs, *Electron. J. Linear Algebra* 16 (2007), 451–462.

[98] J. Sedlacek, On the skeletons of a graph or digraph, *In Proc. Calgary International Conference on Combinatorial Structures and their Applications*, Gordon and Breach, 1970, pp. 387–391.

[99] I. E. Shparlinski, On the singularity of generalised Vandermonde matrices over finite fields, *Finite Fields Appl.* 11 (2005), 193–199.

[100] I. E. Shparlinski, Additive combinatorics over finite fields: New results and applications, In *Proc. RICAM-Workshop on Finite Fields and Their Applications: Character Sums and Polynomials*, De Gruyter, to appear.

[101] D. Spielman, Spectral graph theory, In *Combinatorial scientific computing*, Chapman and Hall/CRC Press, 2012, pp. 495–524.

[102] D. Stevanović, Applications of graph spectra in quantum physics, In *Selected topics on applications of graph spectra*, 2011, Mathematical Institute SANU, pp. 85–111.

[103] E. Teufl and S. Wagner, On the number of spanning trees on various lattices, *J. Phys. A: Math. Theor.* 43 (2010) 415001.

[104] E. Teufl and S. Wagner, Resistance scaling and the number of spanning trees in self-similar lattices, *J. Stat. Phys.* 142(4) (2011), 879–897.

[105] W. T. Tutte, The factorization of linear graphs, *J. London Math. Soc.* 22(2) (1947), 107–111.

[106] W. T. Tutte, The 1-factors of oriented graphs, *Proc. Amer. Math. Soc.* 4 (1953), 922–931.

[107] W. Yan, Y.-N. Yeh and F. Zhang, The asymptotic behavior of some indices of iterated line graphs of regular graphs, *Discrete Appl. Math.* 160 (2012), 1232–1239.

[108] F. Zhang, *Matrix theory: Basic results and techniques*, Springer-Verlag, 1999.

[109] J. Zhang and C. Jia, Calculation of minimal dominating set in wireless sensor network with host switch-on/off, *Trans. Tianjin Univ.* 16 (2010), 279–283.