

# Planar Open Rectangle-of-Influence Drawings

by

Soroush Hosseini Alamdari

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## Abstract

A straight line drawing of a graph is an *open weak rectangle-of-influence (RI) drawing*, if there is no vertex in the relative interior of the axis parallel rectangle induced by the end points of each edge. Despite recent interest of the graph drawing community in rectangle-of-influence drawings, no algorithm is known to test whether a graph has a planar open weak RI-drawing, not even for inner triangulated graphs.

In this thesis, we have two major contributions. First we study open weak RI-drawings of plane graphs that must have a *non-aligned frame*, i.e., the graph obtained from removing the interior of every filled triangle is drawn such that no two vertices have the same coordinate. We introduce a new way to assign labels to angles, i.e., instances of vertices on faces. Using this labeling, we provide necessary and sufficient conditions characterizing those plane graphs that have open weak RI-drawings with non-aligned frame. We also give a polynomial algorithm to construct such a drawing if one exists.

Our second major result is a negative result: deciding if a planar graph (i.e., one where we can choose the planar embedding) has an open weak RI-drawing is NP-complete. NP-completeness holds even for open weak RI-drawings with non-aligned frames.

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# Chapter 1

## Introduction

A graph is a discrete structure, consisting of vertices (sometimes also called nodes or points) and a set of connections between vertices (sometimes called edges, arcs or links.) We naturally visualize graphs as link-node diagrams. In many applications this link-node diagram is in fact the actual subject of interest. This defines the scope of graph drawings. Producing easily readable diagrams can be very helpful for large and complex graphs. For large scale relations we desire automated construction of such diagrams.

Constructing a diagram for a given graph is not hard, however, in certain applications we would like to add some constraints to the way a diagram is drawn. A common restriction is planarity, that is, no two links may intersect or pass through a node they are not attached to. Another standard restriction is to force links to be straight, defining straight-line drawings. It is easy to see how these two constraints can improve readability of a diagram. Here, we are interested only in diagrams with these two standard properties.

In this thesis, we study the diagrams that have an additional restriction: The axis-aligned rectangle containing a link as its diameter should contain no node in its interior. Figure 1.1(b) depicts an example of such a drawings. This restriction makes the diagrams easier to read and the relative placement of nodes and links more clear. We present algorithms to find such diagrams for some classes of graphs. We also prove hardness of finding such diagram if the input is an abstract graph without a fixed embedding.

Next we review some standard definitions before defining this problem precisely and outlining our results.

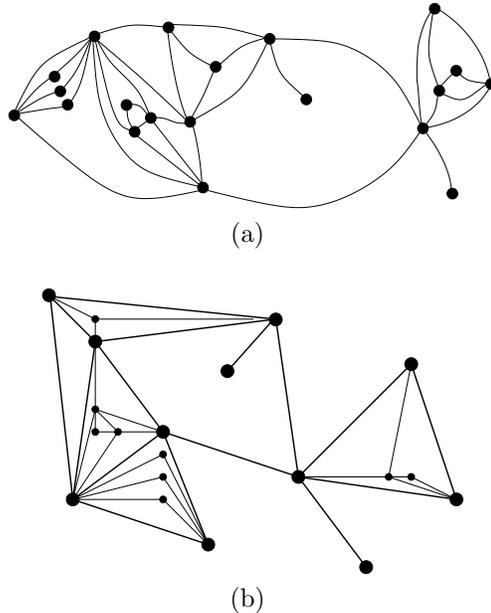


Figure 1.1: A plane graph  $G$  (a) and a open weak rectangle-of-influence drawing of  $G$  (b).

## 1.1 Standard Notation

Here we review a set of standard definitions that are used extensively in the domain of graph theory. These definitions can be found in any standard graph theory textbook (e.g. see [7, 34]).

An *undirected graph*  $G$  is a pair consisting of a set of *vertices*  $V(G)$  and a multiset of *edges*  $E(G)$  where each element  $e$  in  $E(G)$  is a pair  $(u, v)$ , with  $u, v \in V(G)$ . A *directed graph*  $D$  is a pair consisting of a set of *nodes*  $V(D)$  and a multiset of *arcs*  $E(D)$  where each element  $e$  in  $E(D)$  is an ordered pair  $(u, v)$ , with  $u, v \in V(D)$ . We always use  $n$  to denote the number of vertices of a graph  $G$ . The two vertices associated with each edge  $e$  are referred to as the *ends* of  $e$ . For a directed edge  $e = (u, v)$ ,  $u$  and  $v$  are called *tail* and *head* of  $e$  respectively.

A *subgraph*  $H$  of a graph  $G$  is a graph with  $V(H) \subset V(G)$  and  $E(H) \subset E(G)$ . The graph *induced* in  $G$  by a set  $S \subset V(G)$ , denoted as  $G[S]$ , is the maximal subgraph of  $G$  with  $V(G[S]) = S$ . *Subdividing* an edge  $(u, v)$  of a

graph  $G$  is the act of removing  $(u, v)$  from  $G$  and adding a vertex  $z$  and edges  $(u, z)$  and  $(z, v)$  to  $G$ . A graph  $H$  is a *subdivision* of  $G$  if it can be obtained by iteratively subdividing edges of  $G$ .

In a graph  $G$ , a sequence  $(w_1, w_2, \dots, w_k)$  of vertices is a  $w_1 w_k$ -*walk* of length  $k - 1$  if  $(w_i, w_{i+1}) \in E(G)$  for all  $1 \leq i < k$ . A *closed walk* is a walk that begins and ends in the same vertex. A *path* is a walk that visits each vertex at most once. A *cycle* is a closed walk that visits each vertex at most once except that it begins and ends in the same vertex.  $P_i$  and  $C_i$  denote the path of length  $i - 1$  and the cycle of length  $i$ , respectively. A graph is *connected* if for any two vertices  $u$  and  $v$ , there is a  $uv$ -path in  $G$ . A *connected component* of  $G$  is a maximal subgraph  $G[S]$  that is connected. A graph is *disconnected* if it has more than 1 connected component. A set  $S \subset V(G)$  is a *cut set* of the connected graph  $G$  if the graph  $G[V(G) - S]$  is disconnected. A graph is  *$i$ -connected* if it has no cut set of size smaller than  $i$ .

A graph can be represented in the plane by placing a unique point  $p_w$  for each  $w \in V(G)$  and then drawing a simple curve  $c_e$  from  $p_u$  to  $p_v$  for each  $e \in E(G)$  with  $e = (u, v)$ . Such a representation is called a *drawing* of the graph  $G$ . For simplicity the elements of the graph and the elements of the drawing are mostly referred to in an interchangeable way, that is, references to  $v$  and  $e$  are used instead of  $p_v$  and  $c_e$ , respectively. A *straight-line drawing* of  $G$  is a drawing of  $G$  in which all edges are drawn as straight line segments. A *planar drawing* of  $G$  is a drawing of  $G$  in which no two edges cross anywhere except at their endpoints (see Figure 1.2). A graph is called *planar* if it has a planar drawing. A set of planar drawings of a planar graph  $G$  can be specified by giving for each vertex the cyclic order of edges around it. Such a fixed ordering is called an *embedding* of the graph  $G$ . A *planar embedding* of  $G$  is an embedding such that there exists some planar drawing of  $G$  respecting it. A planar embedding divides the plane into topological regions called *faces*. The unbounded region is called the *outer face*, all other faces are called *inner faces*. Any vertex not on the outer face is called an *inner vertex*. A *plane graph* is a planar graph with a planar embedding and the outer face specified. The set of faces of  $G$  is denoted by  $F(G)$ . An *inner triangulated* graph is a plane graph in which every inner face is a triangle; it is called *triangulated* if the outer face is also a triangle.

Note that in a planar embedding of a graph, each edge is adjacent to exactly two faces or the same face twice. The *dual* of the plane graph  $G$ ,

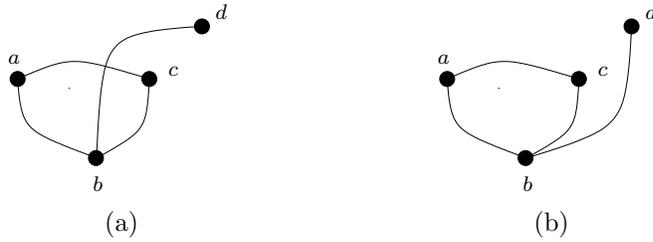


Figure 1.2: A planar drawing (a) and a non-planar drawing (b) of the planar graph  $G$  with  $V(G) = \{a, b, c, d\}$  and  $E(G) = \{(a, b), (a, c), (b, c), (b, d)\}$ .

denoted  $\mathcal{D}(G)$ , is the graph obtained from  $G$  as follows:

- For each face  $f$  of  $G$  there is a vertex  $v_f$  in  $\mathcal{D}(G)$ ,
- For each edge  $e$  in  $E(G)$  that is adjacent with faces  $f_a$  and  $f_b$ , there is a  $(v_{f_a}, v_{f_b})$  edge in  $\mathcal{D}(G)$ .

Note that  $\mathcal{D}(G)$  is not necessarily simple, even if  $G$  is simple (see Figure 1.3). Also, each planar embedding of  $G$  yields a planar embedding of  $\mathcal{D}(G)$ .

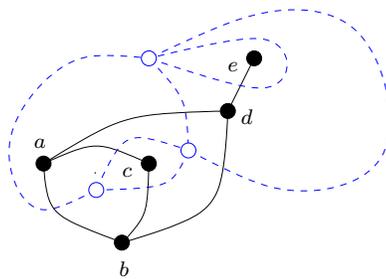


Figure 1.3: A graph  $G$  with 5 vertices,  $a, b, c, d, e$  (solid), and its dual (dashed edges and hollow vertices).

## 1.2 Motivation and Problem Statement

Drawings of graphs have been a matter of interest for almost as long as graphs themselves. As a class of drawings, *proximity drawings* have recently

been studied extensively. In proximity drawings each edge shows a “closeness” relationship between its two ends and for any definition of “closeness” a corresponding type of proximity drawing is defined. A *strong proximity drawing* is a planar straight-line drawing in which two vertices are adjacent *if and only if* they are “close” in some predefined sense. A *weak proximity drawing* is a planar straight-line drawing in which two vertices are adjacent *only if* they are “close”. In many applications two nodes can communicate in some sense, if they are “close” to one another.

A common way to define proximity drawings is assigning a *region of influence* to each edge. Then two vertices are “close” if the corresponding region does not contain any other vertices. For example, the proximity drawing can model a graph modeling a method of communication where existence of another node in the region of influence will cause interference in the communication.

In proximity drawings based on region of influence, if the region assigned to an edge is defined to be the axis-parallel rectangle that has the two ends of the edge as the two ends of its diameter, then the corresponding drawing would be a *rectangle-of-influence drawing*. If the region assigned to a  $(u, v)$  edge is the circle that has  $u$  and  $v$  as the two ends of one of its diagonals, then the drawing would be a *Gabriel drawing*. A list of such types of drawings is given in Table 1.1. In this thesis we will only study rectangle-of-influence drawings.

Most early results were focusing on strong proximity drawings. Battista *et al.* [4] suggested that weak proximity drawings deserve to be studied more.

The constraints that define different versions of proximity drawings arise in computer graphics, computational geometry, pattern recognition, computational morphology, numerical analysis, computational biology, and GIS (e.g. see Chapter “Graph drawing” of [17]). One of the motivations for rectangle-of-influence drawings as stated in [21] is rectangular visibility where two vertices are rectangularly visible if the corresponding rectangle of influence is empty (e.g. see [25, 11]).

Planar rectangle-of-influence drawings have another useful property: In a rectangle-of-influence drawing moving vertices without changing the relative order of vertices cannot cause a vertex to enter the rectangle-of-influence of an edge. This implies that a planar rectangle-of-influence drawing stays planar as long as the relative order of vertices with respect to both coordinates is

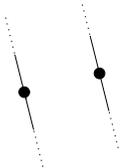
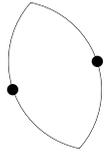
Name	Region of influence corresponding to $e = (u, v)$	Illustration	Ref.
Gabriel drawing	The circle with $(u, v)$ as a diagonal		[14]
Rectangle-of-Influence drawing	The axis-aligned rectangle with $(u, v)$ as a diagonal		[12]
Strip drawing	The strip formed by the locus of lines that are perpendicular to $(u, v)$		[8]
Relative neighborhood drawing	The intersection of the two circles that are centered at $u$ and $v$ and have radius $d(u, v)$		[32]

Table 1.1: Some possible ways to define the region of influence

preserved. For example, using this property one can shift vertices and make space for inserting drawings of other elements (e.g. labels) without worrying about violating planarity.

Aside from known applications of rectangle-of-influence drawings, in this thesis we show a strong structural relation between open planar rectangle-of-influence drawings and rectangle-contact drawings, which in turn leads to applications in VLSI design and architectural planning (e.g. see Chapter “rectangular drawing algorithms” of [30]). We will introduce rectangle-contact drawings in Section 2.3.

### 1.3 Prior Work

The rectangle-of-influence (RI for short) drawability problem was introduced by Liotta *et al.* [21]. Recall that in a strong RI drawing of a graph, there is an edge between two vertices of the graph if and only if there is no other vertex in the axis-parallel rectangle defined by the two ends of every edge (see Figure 1.4(a)). There are two variants of RI-drawings: In a *closed RI-drawing*, the rectangle required to be empty is closed, whereas in an *open RI-drawing*, only the relative interior of the rectangle is required to be empty. Liotta *et al.* [21] gave complete characterization of cycles, wheels, outerplanar graphs, and triangle-free graphs that are strong open RI drawable. For strong closed RI drawable graphs, they presented complete characterization of cycles, wheels and trees.



Figure 1.4: A strong RI-drawing (a) and a weak RI-drawing (b).

Biedl *et al.* [5] were the first to study *weak RI drawings*. Recall that this means that graphs are drawn such that for any edge the corresponding axis-aligned rectangle is empty, but for all such empty rectangles the edge is not necessarily present (see Figure 1.4(b)). They proved that a plane

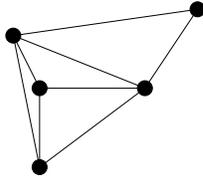


Figure 1.5: A graph with an open weak planar RI-drawing. One can show that it has no open strong planar RI-drawing.

graph has a planar weak closed RI drawing if and only if it has no *filled triangle* (i.e., a triangle that has vertices in its interior.) Furthermore, they presented an algorithm to find such a drawing in an  $(n - 1) \times (n - 1)$  grid in linear time. Recently, Sadasivam and Zhang [29] improved this grid size to  $(n - 3) \times (n - 3)$ . They also show that there are infinitely many weak closed RI drawable graphs that admit no drawing on  $W \times H$  grid with  $H < n - 3$  or  $W < n - 3$ . In the case of quadrangulations, Barrière and Huemer [2] proved that all quadrangulated graphs admit weak closed RI drawings. They presented an algorithm to find such a drawing in an  $(n - 2) \times (n - 2)$  grid.

For open RI drawings, better bounds for the size of the drawing are known. Miura and Nishizeki [24] presented an algorithm to find a small weak open RI drawing of a given 4-connected graph, if the input graph has more than 3 vertices on the outer face. Note that these two conditions imply that there is no filled triangle in the input graph. Their grid size is  $W \times H$  where  $W + H \leq n$ . Zhang and Vaidya [35] also provided small weak open RI drawings for inner triangulated 4-connected graphs with quadrangular outer face. They do this by proving that the drawing presented by Fusy [13] is a weak open RI drawing.

In this thesis, we focus on weak open RI-drawings. No necessary and sufficient conditions or testing algorithms are known for the existence of (weak planar) open RI-drawings, even for inner triangulated graphs. This study was initiated by Miura, Matsuno and Nishizeki [23]. We briefly review their results here to motivate our work, and give a detailed review in Chapter 4. Miura *et al.* first gave necessary and sufficient conditions for planar weak open RI-drawability of triangulated planar graphs. Here all faces including the outer-face are triangles, so the outer-face is a filled triangle, which severely restricts the placement of the vertices not on the outer-face and hence makes testing existence of a weak open RI-drawing easy.

	<b>Strong RI-drawings</b>	<b>Weak RI-drawings</b>	<b>Weak RI-drawings with non-aligned frame</b>
<b>Closed</b>	cycles, wheels and trees [21]	All graphs [5]	All graphs [5]
<b>Open</b>	cycles, wheels, outerplanar graphs, and triangle-free graphs [21]	Triangulated graphs and special cases of inner triangulated graphs [23]	All graphs (Chapter 4)

Table 1.2: Classes of graphs for which existence of planar RI-drawings can be tested.

Miura *et al.* [23] also aimed to develop necessary and sufficient conditions for existence of a weak open RI-drawing for inner triangulated graphs, but did not succeed. Such a drawing imposes conditions on how filled triangles are drawn; a natural first step is hence to remove the interior of all filled triangles and try to draw the resulting *frame graph* while satisfying these conditions. Miura *et al.* then changed their model a bit and only considered what they called *oblique drawings* where no edges of the frame graph are drawn horizontally or vertically. They still could not give necessary and sufficient conditions for oblique drawings, but they gave one set of conditions that are clearly necessary, and showed that adding one condition made them sufficient. We review their result in more detail in Section 4.1. Table 1.2 contains a summary of known results on recognition of RI drawable plane graphs.

## 1.4 Our Results

In Chapter 2 we give some definitions that will be used throughout the thesis. In Chapter 3 we introduce a new way to assign labels to the angles of a plane graph. This way of labeling is used in the following chapters extensively. In Chapters 4 and 5 we study RI drawings of plane and planar graphs, respectively. These two chapters contain the core of our contribution. We will conclude in Chapter 6.

### 1.4.1 Plane Graphs

In chapter 4 we study a slight variant of oblique drawings that we call drawings with *non-aligned frame*, which means that no two vertices of the frame graph have the same  $x$ -coordinate or the same  $y$ -coordinate. We give necessary and sufficient conditions for a plane graph to have a planar weak open RI-drawing with a non-aligned frames. Our results applies to any plane graph (whereas Miura *et al.* required an inner triangulated graph). Having a non-aligned frame is crucial for our algorithm to work, but of course is a very artificial restriction. We view this as a first step towards to the exact characterization of weak open RI-drawable plane graphs.

Note that every drawing with non-aligned frame is oblique, but the other direction does not hold (and in fact, in Chapter 6 we give an example of a graph that has a RI drawing with an oblique frame, but no RI drawing with a non-aligned frame). The two concepts are the same if the outer-face is nice in some sense; this is the additional condition by Miura *et al.*. So the thesis supersedes the results by Miura *et al.*, but improves it in that we exactly characterize the graphs that have a non-aligned frame drawing and can handle graphs that are not inner triangulated.

Our proof is algorithmic and yields a test of whether a plane graph has a planar weak open RI-drawing with non-aligned frame; it also constructs such a drawing if one exists. Also, the algorithm is developed via a detour into rectangular drawings and proves a correspondence between RI-drawings and rectangular drawings that may be of independent interest.

### 1.4.2 Planar Graphs

Our drawability result crucially relies on using the dual graph, hence it cannot be applied if the planar embedding is not fixed (i.e. the graph is planar, rather than plane).

Liotta in Chapter “proximity drawings” of [30] points out that “it would be interesting to characterize which planar graphs have a weak open rectangle of influence drawing”. In chapter 5, we show that most likely there is no quickly verifiable characterization of planar graphs that admit weak open RI drawings. To be exact, we prove NP-completeness of the problem of deciding if an open RI-drawing for a given planar graph exists by a reduction

from NOT-ALL-EQUAL-3-SAT. To the best of our knowledge, this is the first hardness result regarding RI drawings. The graphs that we construct are 2-connected and the truth assignment is made based on flipping 3-connected components of this constructed graph, very similar (and based on) a proof by Garg and Tamassia [15].

# Chapter 2

## Preliminaries

In this chapter we present some more definitions and a few well-known results. These definitions will be used extensively in the next chapters.

### 2.1 More definitions

Assume  $G$  is a plane graph. A triangle of  $G$  is called *filled* if there is at least one vertex inside the triangle. Crucial for our study is the *frame graph*, which is the graph obtained by removing the inside of every filled triangle (see Figure 2.1). In order to create RI-drawings, we will create drawings of the frame graph and then “paste in” drawings of the filled triangles. Recall that filled triangles play an important role in RI-drawings, as Biedl *et al.*[5] showed that all graphs that do not have any filled triangles are closed RI-drawable and hence also open RI-drawable.

Also crucial is the concept of angles of a plane graph. In an embedding of a graph  $G$ , each instance of a vertex  $v$  appearing in a face is called an *angle*. Note that each angle can also be identified by a vertex  $v$  and two edges (or one edge twice) that appear consecutively around  $v$ . For an angle  $\alpha$ , let  $f(\alpha)$  and  $v(\alpha)$  denote the corresponding face and vertex, respectively. The angles on the outer face are *outer angles* and the angles on the inner faces are called *inner angles*. Two angles are *adjacent* if they share an edge and are on the same face. A *chain*  $C = (\alpha_0, \alpha_1, \dots, \alpha_{k-1})$  of angles is a sequence of distinct adjacent angles along a face. We will denote the number of angles in  $C$  by  $|C|$  (see Figure 2.2).

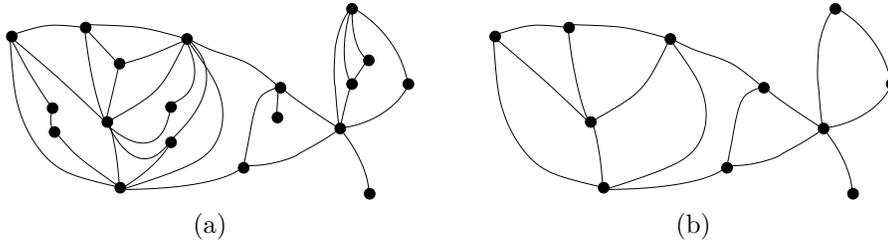


Figure 2.1: A graph (a) and the corresponding frame graph (b).

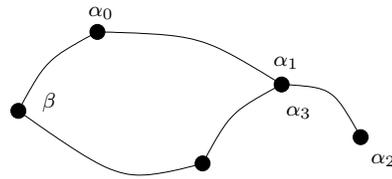


Figure 2.2: A chain of outer angles  $C = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$  with  $|C| = 4$  and a chain of inner angles  $B = (\beta)$  with  $|B| = 1$ .

### 2.1.1 Duals and Surrounding-dual

Given a plane graph  $G$ , the *dual* graph  $\mathcal{D}(G)$  is obtained by creating a vertex  $v_f$  for every face  $f$ , and adding an edge  $(v_f, v_g)$  whenever faces  $f$  and  $g$  share an edge. The vertex of  $\mathcal{D}(G)$  corresponding to the outer face of  $G$  is called the *outer face vertex*. The *inner-dual* graph  $\mathcal{D}^-(G)$  is obtained by removing the outer face vertex of  $\mathcal{D}(G)$ . The angles in  $\mathcal{D}(G)$  are in natural 1-1 correspondence with the angles of  $G$ : The angle at vertex  $v$  in face  $f$  corresponds in the dual graph to the angle at vertex  $v_f$  in the face formed where  $v$  used to be (see Figure 2.3(a)).

Here we define a modification to the dual graph. The *surrounding-dual* of a graph  $G$ , denoted  $\mathcal{D}^*(G)$ , is obtained from  $\mathcal{D}(G)$  by replacing the outer face vertex and any other vertex  $v$  that has degree  $k > 3$  with a cycle of length  $k$  such that each edge of  $v$  in  $\mathcal{D}(G)$  is incident to one the vertices of this cycle (see Figures 2.3 and 2.4). More formally,  $\mathcal{D}^*(G)$  is obtained from  $G$  as follows:

- (i) Let  $G^+$  be the graph obtained from  $G$  by adding one vertex  $v_f$  in the outer face and each non-triangular inner face  $f$ .
- (ii) For each angle  $\alpha$  at an instance of a vertex  $u$  on a non-triangular face

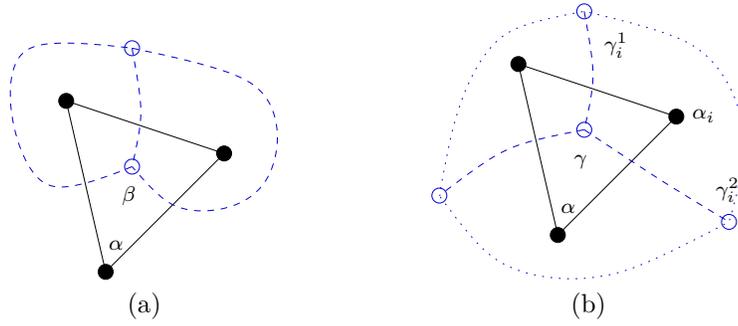


Figure 2.3: An angle  $\alpha$  of  $G$  and the corresponding angle  $\beta$  in  $\mathcal{D}(G)$  and  $\gamma$  in  $\mathcal{D}^*(G)$ .

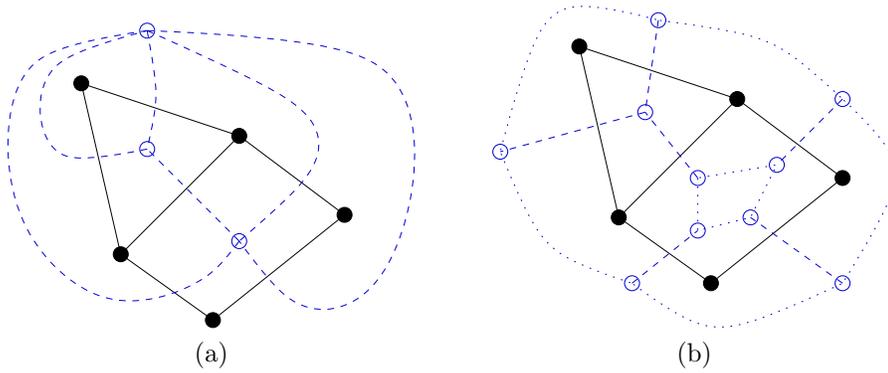


Figure 2.4: A graph  $G$  (solid) along with its dual  $\mathcal{D}(G)$  (hollow vertices and dashed edges) (a) and its surrounding dual  $\mathcal{D}^*(G)$  (hollow vertices and dashed edges) (b).

or outer face  $f$ , add an edge from  $u$  to  $v_f$  in  $G^+$  at the place (in the cyclic order around  $u$ ) where  $\alpha$  was.

- (iii) Then  $\mathcal{D}^*(G)$  is the graph dual of  $G^+$  with the outer face vertex removed, i.e.  $\mathcal{D}^-(G^+)$ .

Observe that there is a one-to-many correspondence between angles of  $G$  and the angles of  $\mathcal{D}^*(G)$  (see angles  $\alpha$  and  $\alpha_i$  in Figure 2.3(b)).

## 2.1.2 Flipping a Graph

Recall that a plane graph has a fixed cyclic order of edges at every vertex and a fixed outer face. If we reverse the order at all vertices, then we obtain a different planar embedding, which corresponds to having flipped (mirrored) the planar drawing (see Figure 2.5). We will sometimes need to do this for subgraphs. So let  $G$  be a planar graph with a fixed planar embedding and let  $G'$  be a subgraph of  $G$  such that for all vertices  $v$  in  $G$ , the edges of  $v$  in  $G'$  form a consecutive set in the ordering of edges of  $v$  in the planar embedding of  $G$ . The planar embedding obtained by *flipping*  $G'$  is the embedding resulting by reversing the order of edges in  $G'$  at each vertex in  $G$ . We will only apply this operation at subgraphs for which the resulting embedding is again planar.

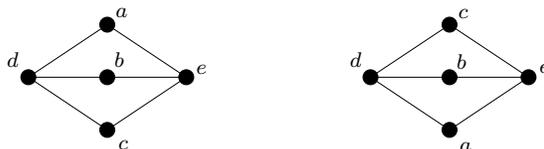


Figure 2.5: Two drawings of the same planar graph with embeddings that are flipped versions of each other.

## 2.2 RI-drawings

Recall that a *planar straight-line drawing* of a planar graph is a drawing without edge crossing where all edges are straight line segments. Such a drawing is called a *planar weak open rectangle-of-influence* (RI for short) drawing if for every edge  $(v, w)$ , the relative interior of the axis-parallel rectangle defined by  $v$  and  $w$  contains no other vertex. The drawings in Figure 2.6(a) and Figure 2.6(b) are both planar weak open RI drawings. Since we will rarely consider any other type of RI-drawing, we omit the classifiers “planar”, “weak” and “open” occasionally.

A straight-line drawing of a graph is *oblique* if no edge in the drawing is axis parallel. It is *non-aligned* if no axis parallel line intersects two or more vertices of the graph. Every non-aligned drawing is oblique, but not vice versa.

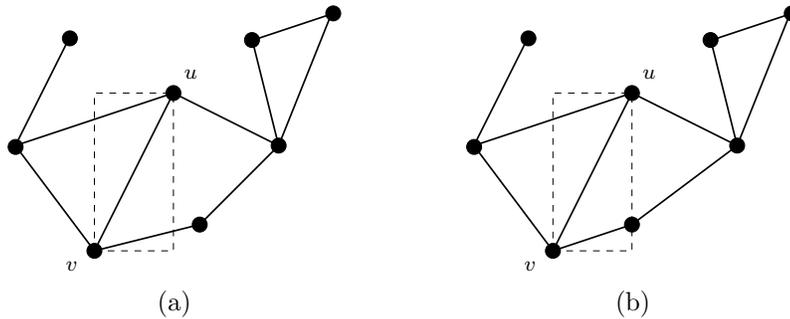


Figure 2.6: A drawing that is both a weak open RI drawing and a weak closed RI drawing (a) and another weak open RI drawing of the same plane graph that is not a weak closed RI drawing (b).

Recall Biedl *et al.* [5] proved that a graph has a closed RI-drawing if and only if it has no filled triangle. Also, a non-aligned open RI-drawing by definition of non-aligned has no vertices on the boundary of any axis-aligned rectangle induced by an edge, except for the two ends of that edge. This means any non-aligned RI-drawing is a closed RI-drawing. Also, it is easy to see (and we will formally show it in Chapter 4) that any closed RI-drawing can be made into a non-aligned RI-drawing by moving some of the vertices by a small amount. Therefore, a graph has a non-aligned open RI-drawing if and only if it has no filled triangle.

## 2.3 Orthogonal Drawings and Relatives

In this section we introduce several types of drawings that are in contrast with oblique drawings: A straight-line drawing of a graph is called an *orthogonal drawing* if all edges are axis aligned.<sup>1</sup> Figures 2.7(a and b) show two different orthogonal drawings of the same plane graph. We will be using orthogonal drawings in both our major results, showing a strong connection between RI-drawings and special cases of orthogonal drawings. The following types of drawings are all orthogonal drawings.

---

<sup>1</sup>In the graph drawing literature sometimes the term “orthogonal drawing” is used to refer to a drawing that has edges drawn as a series of axis-aligned line segments. Such a drawing need not be a straight-line drawing and might introduce bends on edges. Here, we do not consider such drawings: for us an orthogonal drawing has no bends.

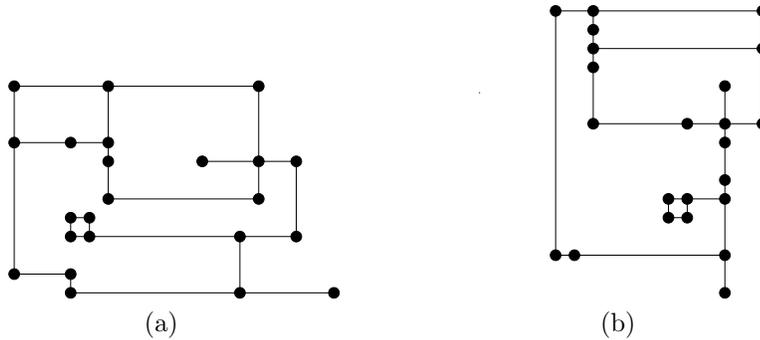


Figure 2.7: Two orthogonal drawings of the same graph.

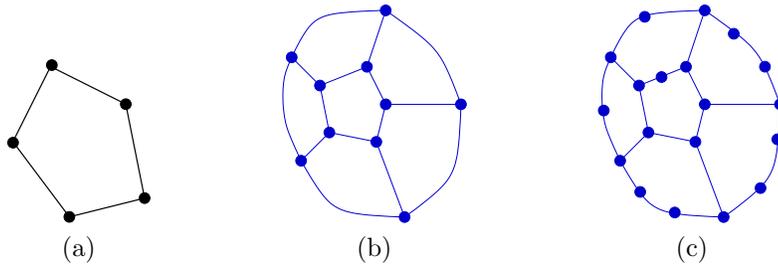


Figure 2.8: A graph  $G$  (a), its surrounding-dual  $\mathcal{D}^*(G)$  (b), and a subdivision of  $\mathcal{D}^*(G)$  (c).

A planar orthogonal drawing of a plane graph  $G$  is said to be a *rectangle-contact drawing* of  $G$  with respect to a subset of faces  $F' \subset F(G)$ , if:

- each face in  $F'$  has a rectangular boundary, and
- each edge of  $G$  is adjacent with at least one of the faces in  $F'$ .

(For example, see Figures 2.9(a,b, and c).)

A planar orthogonal drawing of a graph  $G$  is called an *inner-rectangular* drawing if all inner faces in the drawing have a rectangular boundary (e.g. see Figures 2.9(b and c)). In other words, it is a rectangle-contact drawing of  $G$  with respect to  $F'$  where  $F'$  is the set of all inner faces of  $G$ .

An inner-rectangular drawing of a graph  $G$  is a *rectangular* drawing if the outer face is also a rectangle. In other words, a rectangular drawing is

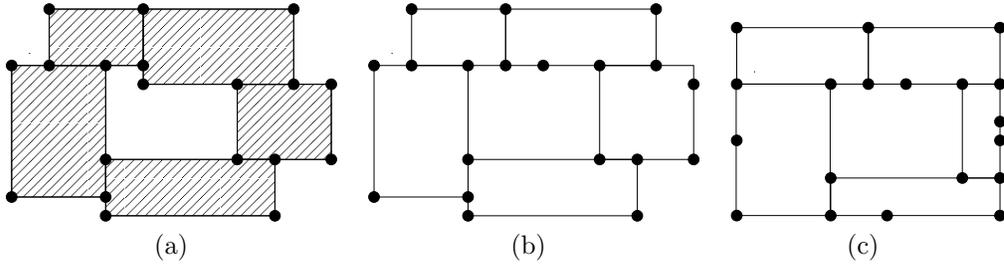


Figure 2.9: A rectangle-contact drawing (a, faces of  $F'$  are hatched), an inner-rectangular drawing (b), and a rectangular drawing (c) of the graph depicted in Figure 2.8(c).

a rectangle-contact drawing with respect to all faces  $F(G)$  (e.g. see Figure 2.9(c)).

The next four types of orthogonal drawings corresponding to a graph  $G$  are not drawings of  $G$  itself, but drawings of some subdivision of  $\mathcal{D}^*(G)$ , i.e. the dual graph of  $G$  with vertices of high degree replaced by cycles.

A *rectangle-contact dual drawing* of  $G$  is a rectangle-contact drawing of a subdivision of  $\mathcal{D}^*(G)$  with respect to the set of faces of  $\mathcal{D}^*(G)$  that correspond to vertices of  $G$ . Figures 2.9(a,b, and c) depict rectangle-contact dual drawings of the graph in Figure 2.8(a). In fact, a rectangle-contact dual drawing of a graph gives us a representation of the vertices of the graph by a set of non-overlapping rectangles, where two rectangles touch if and only if there is an edge between their corresponding vertices. Drawing planar graphs as contact graphs is a topic of its own that we will not review in detail here. Rectangle-contact dual drawings are crucial for our arguments in Chapter 4.

An *inner-rectangular dual drawing* of  $G$  is an inner-rectangular drawing of a subdivision of  $\mathcal{D}^*(G)$ . Figures 2.9(b, and c) depict inner-rectangular dual drawings of the graph in Figure 2.8(a).<sup>2</sup>

A *rectangular dual drawing* of  $G$  is a rectangular drawing of a subdivision of  $\mathcal{D}^*(G)$ . Figure 2.9(c) depicts a rectangular dual drawing of the graph in Figure 2.8(a).

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<sup>2</sup>In the literature, “inner-rectangular dual drawing” usually refers to a drawing of a subdivision of  $\mathcal{D}(G)$ , not  $\mathcal{D}^*(G)$ . Our drawing type perhaps should be called “inner-rectangular surround-dual drawing”, but we drop “surround” to keep things somewhat shorter.

## 2.4 Networks Flows and Circulations

In the next chapters, we will be referring to the concept of network flows. In Chapter 4 we will use some network flow algorithms to construct restricted orthogonal drawings. In Chapter 5 we will use some hardness result on network flows. Here we briefly recall some definitions regarding network flows and circulations.

Let  $G$  be a directed graph with a capacity set  $s(e) \subseteq \mathbb{R}$  assigned to each edge  $e \in E(G)$  and a consumption  $s(v) \in \mathbb{R}$  assigned to each vertex  $v \in V(G)$ . A *valid flow assignment* is an assignment of a flow  $f(e) \in \mathbb{R}$  to each edge  $e \in E(G)$  such that:

- $f(e) \in s(e)$  for any  $e \in E$ .<sup>3</sup>
- The sum of the flow of the incoming edges of  $v$  minus the sum of the flow of the outgoing edges of  $v$  is equal to the consumption of  $v$ .

In chapter 4 we will be using network flow to find orthogonal drawing of a graph that will help us find an RI-drawing with a non-aligned frame of a given graph.

A *circulation* is a special case of a network flow in which all vertices have consumption  $f(v) = 0$ . Garg *et al.* [15] showed that, given an undirected graph  $G$  with a discrete capacity set for each edge, it is NP-hard to assign directions to the edges of  $G$  so that the resulting graph has a valid circulation. In chapter 5 we will present a reduction from a special case of this problem to prove NP-hardness of the problem of finding an open RI-drawing of a given planar graph.

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<sup>3</sup>Note that this definition is different from the usual definition of network flows and in fact is a generalization of “normal” network flow, where  $s(e) = [0, c]$  for some capacity  $c$ .

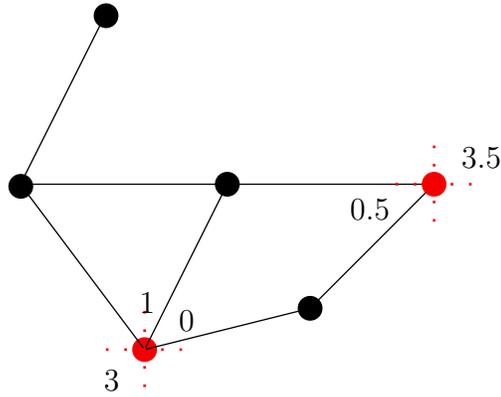
# Chapter 3

## Axis-Count Labelings

In this chapter, we will define a new way of assigning labels to angles of a graph drawing that captures the combinatorial structure of the drawing. The definition is such that it includes the labeling methods defined in the RI drawing literature [23] as well as the labeling methods defined for rectilinear drawings (e.g. see [30, 3]). In Chapter 4 we use this labeling to construct open weak RI-drawings with non-aligned frames. In Chapter 5 we solve a given instance of NOT-ALL-EQUAL-3-SAT based on such labeling of an open RI-drawing of a planar graph. We believe this new type of labeling to be of independent interest.

A *labeling* is an assignment of real numbers to each angle  $\alpha$  of a plane graph  $G$ . For any drawing of  $G$ , the *free rays* of an angle  $\alpha$  are axis-aligned rays with apex at  $v(\alpha)$  that extend into  $f(\alpha)$ ; e.g. any right angle that is not axis-aligned has exactly one free ray. An *Axis-Count labeling* (AC for short) for a straight-line drawing of a graph is a labeling defined as follows: Let  $\Gamma_G$  be a straight-line drawing of a graph  $G$ . Based on  $\Gamma_G$ , to each angle  $\alpha$  of  $G$  we assign a label  $\ell(\alpha) \in \{0, .5, 1, 1.5, 2, 2.5, 3, 3.5, 4\}$  that counts how many coordinate axes are covered by this angle (see Figure 3.1). More precisely, let  $0 \leq a \leq 2$  be the number of axis-aligned edges of  $\alpha$  in  $\Gamma_G$ . Let  $b$  be the number of free rays of  $\alpha$ , that is, the number of coordinate axes contained in the strict interior of the angle  $\alpha$  in  $\Gamma_G$ . Then  $\ell(\alpha) = \frac{a}{2} + b$ . A few examples of drawings and corresponding AC labelings are given in Figure 3.2.

Next, we show some properties of AC labels of drawings:



(a)

Figure 3.1: The AC labels of the angles around two vertices of a straight-line drawing of a plane graph.

**Lemma 3.1** *In any AC labeling of a connected graph, the following properties hold:*

- a) *The sum of labels around each vertex is 4.*
- b) *Let  $\alpha_1$  and  $\alpha_2$  be two consecutive angles on vertex  $v$ , sharing the edge  $e$  (see Figure 3.3). Let  $\alpha$  be the angle that replaces  $\alpha_1$  and  $\alpha_2$  when removing  $e$  from the drawing. Then  $\ell(\alpha) = \ell(\alpha_1) + \ell(\alpha_2)$ .*
- c) *The sum of labels on an inner face with  $k$  angles is  $2k - 4$ .*
- d) *The sum of labels on the outer face with  $k$  angles is  $2k + 4$ .*

**Proof.** We prove each property separately:

- (a) This property holds since there are exactly four axis-aligned rays with apex at  $v$  and each of them contributes exactly 1 to the sum of the AC labels around  $v$ : It either is in the interior of exactly one angle or it belongs to two angles if there is an edge drawn on it, and then adds  $1/2$  to the label of both angles.

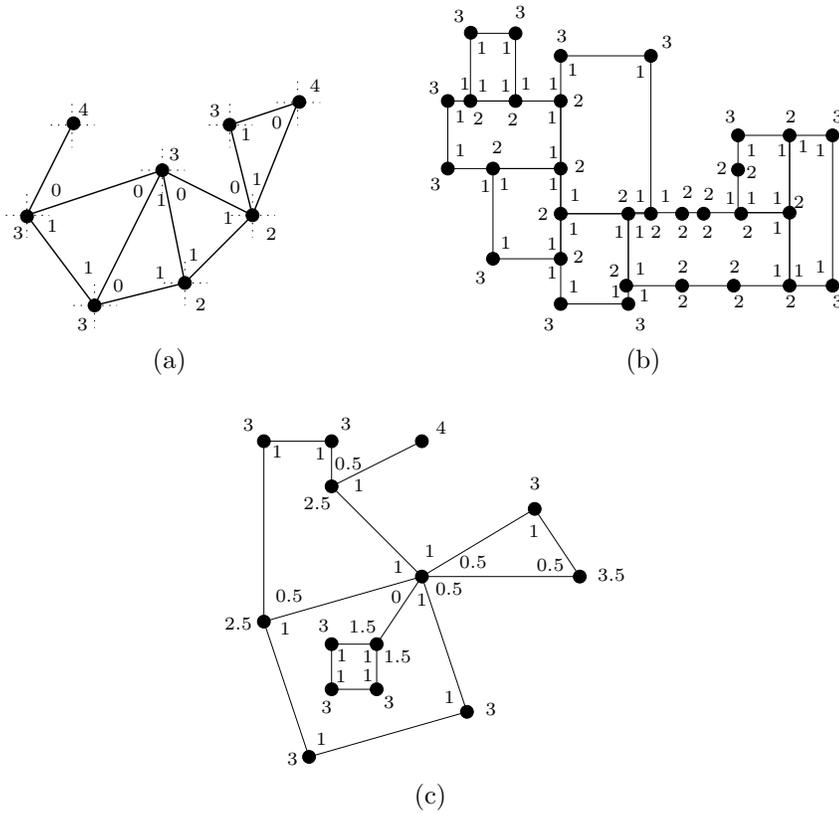


Figure 3.2: An oblique RI-drawing (a), an orthogonal drawing (b) and a drawing with both oblique and axis aligned edges (c) with corresponding AC labelings.

- (b) This is straightforward to check by definition of AC labels.
- (c) We prove this property using induction. It is easy to check that the claim holds for triangles (Figure 3.4 illustrates some of these cases). Now consider the case of a face  $f$  with  $k > 3$  angles. It is well known that there is a pair of vertices  $\alpha$  and  $\beta$  on the boundary of  $f$  such that the line segment  $\alpha\beta$  is completely inside  $f$  and intersects  $f$  only at its endpoints. Such line segment is called a *chord* of a polygon (e.g. see [17]). Let  $f_1$  and  $f_2$  be the faces obtained from subdividing  $f$  by drawing this  $(\alpha, \beta)$  edge. Let the number of the angles in  $f_1$  and  $f_2$  be  $k_1$  and  $k_2$ , respectively. Since two angles were replaced by four new angles, we have

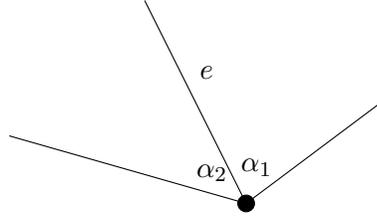


Figure 3.3: Label  $\alpha$  with  $\ell(\alpha) = \ell(\alpha_1) + \ell(\alpha_2)$  depicting Property b of AC labels.

$k_1 + k_2 = k + 2$ . By induction, the sum of inner angles of  $f_1$  and  $f_2$  is  $2k_1 - 4$  and  $2k_2 - 4$  respectively. By property (b) of AC labelings, the sum of the inner angles of  $f$  is the sum of AC labels of  $f_1$  and  $f_2$ . Therefore, the sum of the AC labels of  $f$  is  $2k_1 + 2k_2 - 8 = 2(k_1 + k_2 - 2) - 4 = 2k - 4$ .

- (d) The proof here is similar to the proof of property (c) except that the base case occurs when all outer angles are convex, and hence there is no chord in the outer face.

So we must prove that the property holds if the outer face is convex. If  $G$  is a path with all vertices on a straight line, then it is straightforward to verify that the property holds. Let us assume that  $G$  is not a path drawn on a straight line. Since the outer face is convex, there is no vertex in  $G$  with two outer angles. Therefore the boundary of the outer face of  $G$  is a cycle  $S$  of length  $k$ . Consider the graph  $G[S]$  formed by the edges of  $S$ , in the induced drawing. By property (c) the sum of the AC label of the inner angles of  $G[S]$  is  $2k - 4$ . By property (a) the sum of the AC label on all angles of  $G[S]$  is  $4k$ . Therefore the sum of the AC labels on the outer angles of  $G[S]$  which is the same as the sum of the AC labels on the outer angles of  $G$  would be  $4k - (2k - 4) = 2k + 4$ .

□

### 3.1 AC labelings of Orthogonal Drawings

As mentioned earlier, our AC labeling extends a labeling for orthogonal drawings defined as follows:

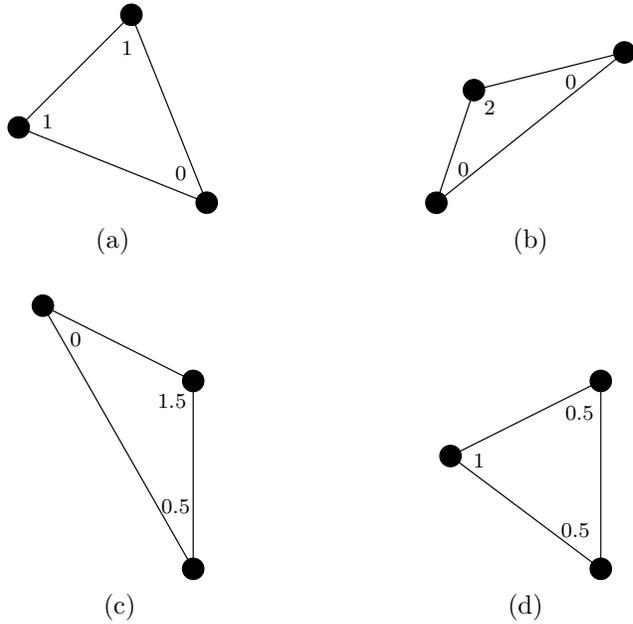


Figure 3.4: Some possible AC labelings of a triangle.

In an orthogonal drawing of a graph  $G$ , a (graph-theoretic) angle has label  $i \in \{1, 2, 3, 4\}$  if it is drawn with (geometric) angle  $i\pi/2$ . The following result was proved by Miura *et al.* [22] and also follows from a prior result on orthogonal-shape drawings by Tamassia [31].

**Lemma 3.2** [22] *A labeling of a plane connected graph  $G$  with labels in  $\{1, 2, 3, 4\}$  can be realized as an orthogonal drawing such that each (graph-theoretic) angle of label  $i$  has (geometric) angle of  $i\pi/2$ , if the labeling satisfies*

- (a) *For each vertex, the labels of incident angles sum to 4.*
- (b) *The sum of the labels on an inner face  $f$  is  $2k - 4$ , where  $k$  is the number of angles on the face.*
- (c) *The sum of the labels on the outer-face is  $2k + 4$ , where  $k$  is the number of angles on the outer-face.*

We call such a labeling *OD-admissible*. Note that by Lemma 3.1 an AC labeling satisfies the above conditions. Furthermore, Lemma 3.2 implies that

any OD-admissible labeling of a graph  $G$  is an AC labeling of some drawing of  $G$ . Therefore, an AC labeling corresponds to some orthogonal drawing, if and only if all its labels are in  $\{1, 2, 3, 4\}$ .

### 3.2 AC labeling of RI-drawings

In this section, we give some necessary conditions of AC labels in RI-drawings; some of these will be crucial for our arguments in the next chapters.

Most of our RI-drawings are oblique (no axis-aligned edges). In this case  $\ell(\alpha) \in \{0, 1, 2, 3, 4\}$  counts the number of coordinate axes contained in angle  $\alpha$ . This type of labeling for an oblique RI-drawing was introduced and used extensively in [23].

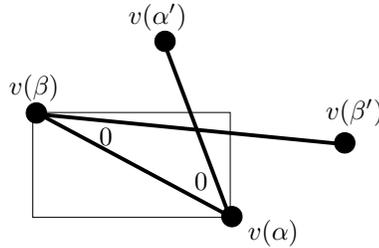


Figure 3.5: Two adjacent angles having label 0 in an RI drawing.

The rest of this section is concerned with proving properties of labelings of RI-drawings that will be needed later. Unfortunately, some of them (Lemma 3.12 and the results leading to Corollary 3.1 and 3.2) require a lengthy proof.

**Lemma 3.3** *In any planar RI-drawing, the sequence of AC labels along a face  $f$  does not contain 00.*

**Proof.** Assume two angles  $\alpha$  and  $\beta$  are adjacent in face  $f$ . Let  $\alpha'$  and  $\beta'$  be the two other angles of  $f$  that are adjacent with  $\alpha$  and  $\beta$ , respectively (possibly  $\alpha' = \beta'$ .) The vertices  $v(\alpha')$  and  $v(\beta')$  must be outside the axis-aligned rectangle induced by  $(v(\alpha), v(\beta))$ . If both angles  $\alpha$  and  $\beta$  have label 0, then the segments  $(v(\alpha), v(\alpha'))$  and  $(v(\beta), v(\beta'))$  intersect, contradicting planarity (see Figure 3.5).  $\square$

**Lemma 3.4** *In an oblique RI-drawing of a graph, the AC labels of any triangular inner face consists of two 1s and one 0.*

**Proof.** This was already proved in [23], using a case analysis, but an alternative proof is as follows: By Property (c) of Lemma 3.1 the sum of the labels at inner angles of a triangle is 2. Since we have an oblique drawing, all labels are integers. Also by Lemma 3.3 there is at most one angle of label 0 in the inner face. Therefore there must be exactly two angles of label 1 and one angle of label 0 in the inner face of any triangle.  $\square$

### 3.2.1 Relative Closeness

The following definition will be useful: Let  $C = (\alpha_0, \alpha_1, \alpha_2)$  be a chain of three angles in a drawing of a graph such that  $(v(\alpha_0), v(\alpha_1))$  and  $(v(\alpha_1), v(\alpha_2))$  are oblique and  $\ell(\alpha_1) = 1$ . We say that  $v(\alpha_2)$  is *strictly closer* to  $v(\alpha_1)$  than  $v(\alpha_0)$  if the order of coordinates with respect to the free ray of  $\alpha_1$  is  $v(\alpha_1), v(\alpha_2), v(\alpha_0)$  (see Figure 3.6(a)). We say that  $v(\alpha_2)$  is *closer* to  $v(\alpha_1)$  than  $v(\alpha_0)$  if  $v(\alpha_2)$  is strictly closer to  $v(\alpha_1)$  than  $v(\alpha_0)$ , or  $v(\alpha_2)$  and  $v(\alpha_0)$  tie with respect to the free ray (see Figure 3.6(b)). For example, if the free ray from  $\alpha_1$  goes upward, then  $v(\alpha_2)$  is closer to  $v(\alpha_1)$  than  $v(\alpha_0)$  if the  $y$ -coordinates satisfy  $y(v(\alpha_1)) < y(v(\alpha_2)) \leq y(v(\alpha_0))$ ; and if the  $y$ -coordinates satisfy  $y(v(\alpha_1)) < y(v(\alpha_2)) < y(v(\alpha_0))$ , then  $v(\alpha_2)$  is strictly closer to  $v(\alpha_1)$  than  $v(\alpha_0)$ .

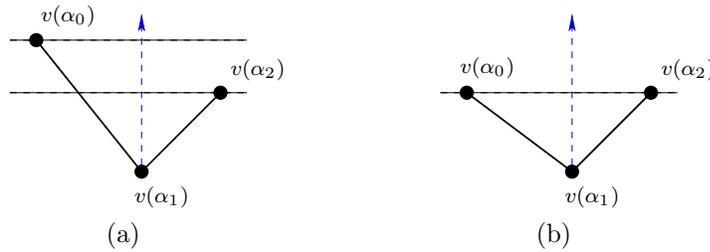


Figure 3.6: A chain  $(\alpha_0, \alpha_1, \alpha_2)$  such that  $v(\alpha_2)$  is *strictly closer* to  $v(\alpha_1)$  than  $v(\alpha_0)$  (a) and the same vertices in a drawing where  $v(\alpha_2)$  and  $v(\alpha_0)$  tie with respect to the free ray (b).

### 3.2.2 Hooks and Cages

We need two definitions that will make our work easier later on. Each of these two gadgets is a short path with some constraints on the placement of each vertex. Later we will prove that any chain of angles that has labels of the form  $01^+0$  has one of these two gadgets at one of its ends.

**Definition 3.1** *Let  $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$  be a chain of four angles on four distinct vertices drawn in the plane. We say the sequence  $(v(\alpha_0), v(\alpha_1), v(\alpha_2), v(\alpha_3))$  is a hook if the following holds (see Figure 3.7):*

- $\ell(\alpha_1) = 0$  and  $\ell(\alpha_2) = 1$ , and
- $v(\alpha_1)$  is closer to  $v(\alpha_2)$  than  $v(\alpha_3)$

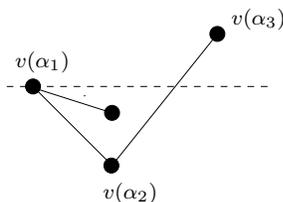


Figure 3.7: A hook  $(v(\alpha_0), v(\alpha_1), v(\alpha_2), v(\alpha_3))$  rotated so that  $y(v(\alpha_2)) < y(v(\alpha_0))$ .

**Lemma 3.5** *In any planar open RI-drawing, if  $(v(\alpha_0), v(\alpha_1), v(\alpha_2), v(\alpha_3))$  is a hook, then  $v(\alpha_0)$  is located on the free ray of  $\alpha_2$ .*

**Proof.** Presume after possible rotation and mirroring of the drawing that the free ray of  $\alpha_2$  goes upward as in Figure 3.7, with  $v(\alpha_1)$  left of the free ray. If  $v(\alpha_0)$  were left of the free ray, it would be in the rectangle of  $(v(\alpha_1), v(\alpha_2))$ . If it were right of the free ray, it would be in the rectangle of  $(v(\alpha_2), v(\alpha_3))$ , or  $(v(\alpha_1), v(\alpha_0))$  would intersect  $(v(\alpha_2), v(\alpha_3))$ . So it must be on the free ray.  $\square$

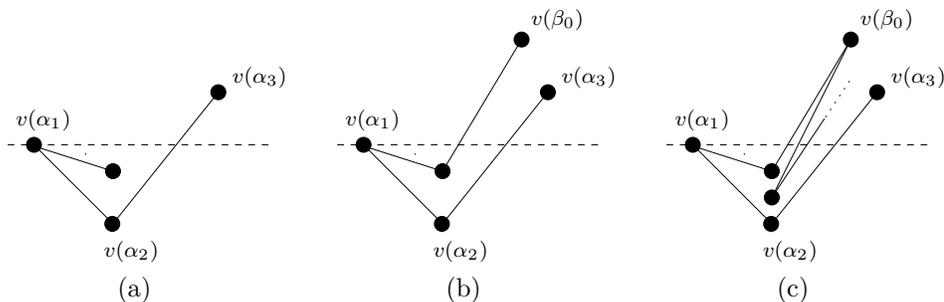


Figure 3.8: For the proof of Lemma 3.6.

**Lemma 3.6** *Let  $C = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$  be a chain in a face with more than 5 angles such that  $(v(\alpha_0), v(\alpha_1), v(\alpha_2), v(\alpha_3))$  is a hook in a planar oblique RI-drawing. Let  $\beta_0$  and  $\beta_1$  be the angles before  $\alpha_0$  on the same face such that  $v(\beta_0)$  is adjacent with  $v(\alpha_0)$ . Then one of the following holds:*

- $\ell(\alpha_0) = 4$ ;
- $\ell(\alpha_0) = 3$  and  $\ell(\beta_0) > 0$ ;
- $\ell(\alpha_0) = 3$ ,  $\ell(\beta_0) = 0$ , and  $\ell(\beta_1) = 4$ .

**Proof.** By Lemma 3.3,  $\alpha_0$  cannot have label 0 since  $\alpha_1$  does. If  $\alpha_0$  had label 1, then  $v(\beta_0)$  would either be in the rectangle defined by  $(v(\alpha_1)v(\alpha_2))$  or  $(v(\alpha_0)v(\beta_0))$  would intersect  $(v(\alpha_1)v(\alpha_2))$ . If  $\alpha_0$  had label 2, then  $v(\beta_0)$  would either be in the rectangle defined by  $(v(\alpha_2)v(\alpha_3))$  or  $(v(\alpha_0)v(\beta_0))$  would intersect  $(v(\alpha_2)v(\alpha_3))$ . If  $\alpha_0$  has label 4 then the lemma holds.

Now assume that  $\alpha_0$  has label 3 (see Figure 3.8(b)). If  $\beta_0$  has a label in  $\{1, 2, 3, 4\}$ , the second condition of the lemma holds. Otherwise, the only valid placement for  $v(\beta_1)$  would be on the line segment between  $v(\alpha_0)$  and  $v(\alpha_2)$ . In that case, by similar arguments to what we already showed,  $\beta_1$  can only have label 4 (see Figure 3.8(c)). Such sequence satisfies the last condition of the Lemma.  $\square$

**Definition 3.2** *Let  $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$  be a chain of five angles on five distinct vertices drawn in the plane. We say  $(v(\alpha_0), v(\alpha_1), v(\alpha_2), v(\alpha_3), v(\alpha_4))$  is a cage if the following holds (see Figure 3.9):*

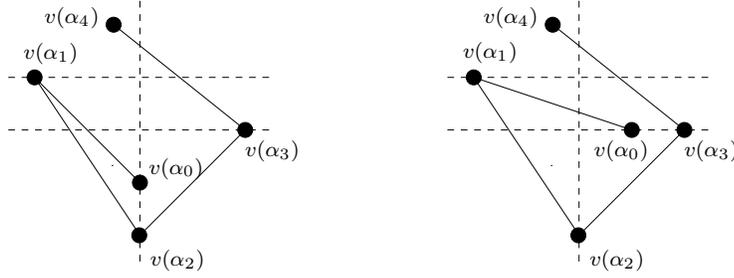


Figure 3.9: A cage  $(v(\alpha_0), v(\alpha_1), v(\alpha_2), v(\alpha_3), v(\alpha_4))$  with  $v(\alpha_0)$  on the free ray of  $\alpha_2$  (a) and on the free ray of  $\alpha_3$  (b).

- $\ell(\alpha_1) = 0$  and  $\ell(\alpha_2) = \ell(\alpha_3) = 1$ ;
- $v(\alpha_2)$  is closer to  $v(\alpha_3)$  than  $v(\alpha_4)$ ;
- $v(\alpha_3)$  is closer to  $v(\alpha_2)$  than  $v(\alpha_1)$ .

**Lemma 3.7** *If  $(v(\alpha_0), v(\alpha_1), v(\alpha_2), v(\alpha_3), v(\alpha_4))$  is a cage, then in any planar open RI-drawing  $v(\alpha_0)$  is located on the free ray of  $\alpha_2$  or  $\alpha_3$ .*

**Proof.** The arguments are similar to that of Lemma 3.5 and we do not repeat them here.  $\square$

Lemmas 3.5 and 3.7 imply that:

**Corollary 3.1** *A non-aligned RI-drawing of a graph does not contain a cage or a hook.*

**Lemma 3.8** *Let  $C = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$  be a chain in a face with more than 5 angles in an oblique RI-drawing of a 2-connected graph such that  $(v(\alpha_0), v(\alpha_1), v(\alpha_2), v(\alpha_3), v(\alpha_4))$  is a cage. Let  $\beta_0$  and  $\beta_1$  be the angles before  $\alpha_0$  on the same face such that  $v(\beta_0)$  is adjacent with  $v(\alpha_0)$ . Then one of the following holds:*

- $\ell(\alpha_0) = 4$ ;
- $\ell(\alpha_0) = 3$  and  $\ell(\beta_0) > 0$ ;

- $\ell(\alpha_0) = 3$ ,  $\ell(\beta_0) = 0$ , and  $\ell(\beta_1) = 4$ ;
- $\ell(\alpha_0) = 1$  and  $\ell(\beta_0) = 4$ ;

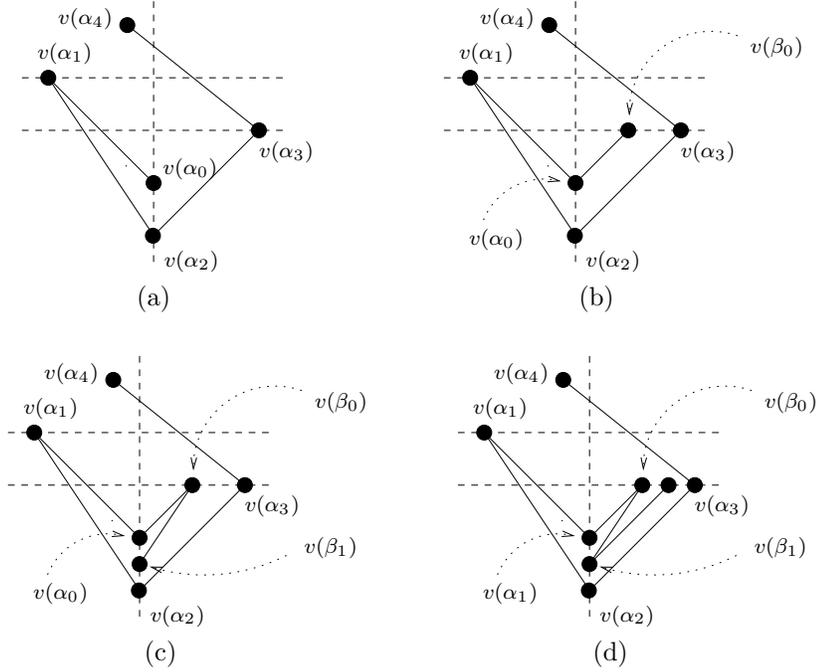


Figure 3.10: For the proof of Lemma 3.8, Case 1.

**Proof.** By Lemma 3.7 there are two possible placements for  $v(\alpha_0)$ , one where  $v(\alpha_0)$  lies on the free ray of  $\alpha_2$ , and one where  $v(\alpha_0)$  lies on the free ray of  $\alpha_3$ .

**Case 1** Assume  $v(\alpha_0)$  lies on the free ray of  $\alpha_2$  (see Figure 3.10(a)): The proof is similar to the proof of Lemma 3.6. By Lemma 3.3,  $\alpha_0$  cannot have label 0. If  $\alpha_0$  had label 1, then  $v(\beta_0)$  would either be in the rectangle defined by  $(v(\alpha_1)v(\alpha_2))$  or  $(v(\alpha_0)v(\beta_0))$  would intersect  $(v(\alpha_1)v(\alpha_2))$ . If  $\alpha_0$  had label 2, then  $v(\beta_0)$  would either be in the rectangle defined by  $(v(\alpha_2)v(\alpha_3))$  or  $(v(\alpha_0)v(\beta_0))$  would intersect  $(v(\alpha_2)v(\alpha_3))$ . If  $\alpha_0$  has label 4 then the lemma holds. Now assume that  $\alpha_0$  has label 3 (see Figure 3.10(b)). If  $\beta_0$  has a label in  $\{1, 2, 3, 4\}$ , the second condition of the lemma holds. Otherwise, the only valid placement for  $v(\beta_1)$  is on the line segment between  $v(\alpha_0)$  and  $v(\alpha_2)$

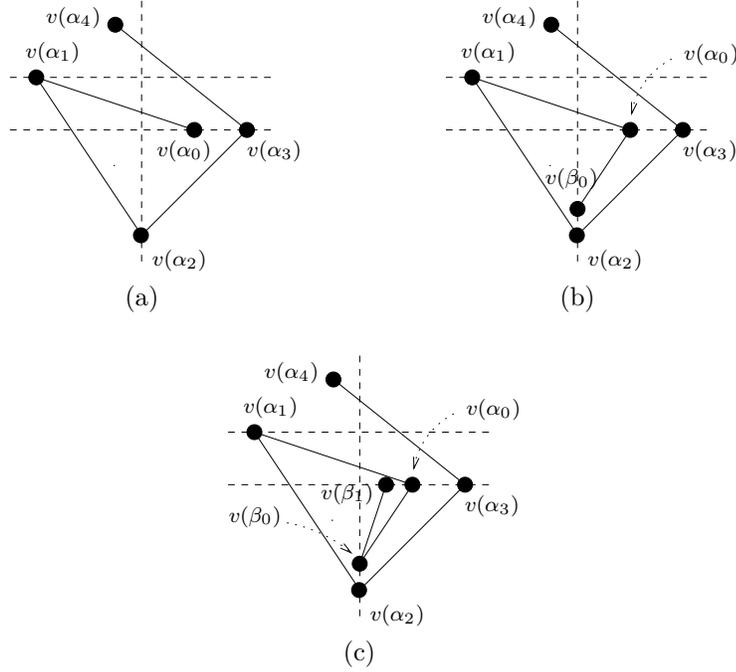


Figure 3.11: For the proof of Lemma 3.8, Case 2.

(see Figure 3.10(c)). In that case, by similar arguments to what we already showed,  $\beta_1$  can only have label 4 (see Figure 3.10(d)). Such a sequence satisfies the last condition of the lemma.

**Case 2** Assume  $v(\alpha_0)$  lies on the free ray of  $\alpha_3$  (see Figure 3.11(a)): By Lemma 3.3,  $\alpha_0$  cannot have label 0. If  $\alpha_0$  had label 2, then  $v(\beta_0)$  would either be in the rectangle defined by  $(v(\alpha_2)v(\alpha_3))$  or  $(v(\alpha_0)v(\beta_0))$  would intersect  $(v(\alpha_2)v(\alpha_3))$ . If  $\alpha_0$  had label 3, then  $v(\beta_0)$  would either be in the rectangle defined by  $(v(\alpha_3)v(\alpha_4))$  or  $(v(\alpha_0)v(\beta_0))$  would intersect  $(v(\alpha_3)v(\alpha_4))$ . If  $\alpha_0$  had label 4 then the lemma holds. Now, assume that  $\alpha_0$  has label 1 (see Figure 3.11(b)). Then  $\beta_0$  can only have a label in  $\{0, 4\}$ . If  $\beta_0$  has label 4 the lemma holds. Assume that  $\beta_0$  has label 0 (see Figure 3.11(c)). In that case there can be no path connecting  $v(\beta_1)$  to  $v(\alpha_0)$  other than the path  $(v(\beta_1), v(\beta_0), v(\alpha_0))$ , for all vertices on such a path would have to be on the free rays of  $\alpha_2$  and  $\alpha_3$  and hence the path can never leave that quadrant. But there are two such paths by 2-connectivity, a contradiction.  $\square$

### 3.2.3 Upper Bound on the Sum of the Labels of a Chain

Next we show that in a non-aligned drawing there can be no chain with labels of the form  $01^+0$ . This will be crucial for our recognition algorithm later.

**Lemma 3.9** *Let  $C = (\alpha_0, \alpha_1, \dots, \alpha_{k+1})$  be a chain of angles in a planar oblique RI-drawing. Assume that the sequence of labels of the angles in  $C' = (\alpha_1, \alpha_2, \dots, \alpha_k)$  is of the form  $01^+0$ . Then at least one of the following holds:*

- $(v(\alpha_0), v(\alpha_1), v(\alpha_2), v(\alpha_3))$  is a hook.
- $(v(\alpha_{k+1}), v(\alpha_k), v(\alpha_{k-1}), v(\alpha_{k-2}))$  is a hook.
- $(v(\alpha_0), v(\alpha_1), v(\alpha_2), v(\alpha_3), v(\alpha_4))$  is a cage.
- $(v(\alpha_{k+1}), v(\alpha_k), v(\alpha_{k-1}), v(\alpha_{k-2}), v(\alpha_{k-3}))$  is a cage.

**Proof.** Assume neither  $(v(\alpha_0), v(\alpha_1), v(\alpha_2), v(\alpha_3))$  nor  $(v(\alpha_{k+1}), v(\alpha_k), v(\alpha_{k-1}), v(\alpha_{k-2}))$  is a hook. We prove that in such a case,  $(v(\alpha_0), v(\alpha_1), v(\alpha_2), v(\alpha_3), v(\alpha_4))$  or  $(v(\alpha_{k+1}), v(\alpha_k), v(\alpha_{k-1}), v(\alpha_{k-2}), v(\alpha_{k-3}))$  is a cage. Observe that  $k \geq 3$  since  $C'$  has the form  $01^+0$ .

For any  $1 < i < k$ ,  $\ell(\alpha_i) = 1$ , and hence  $\alpha_i$  has exactly one free ray, which is an axis-aligned ray that extends from vertex  $v(\alpha_i)$  of the angle  $\alpha_i$  into the face  $f(\alpha_i)$ .

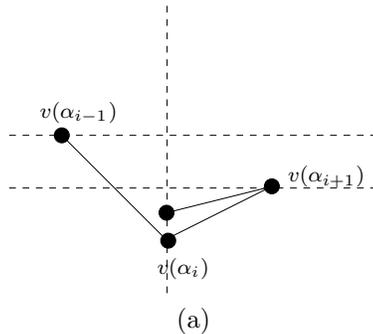


Figure 3.12:  $\ell(\alpha_{i+1}) = 0$  implies a hook.

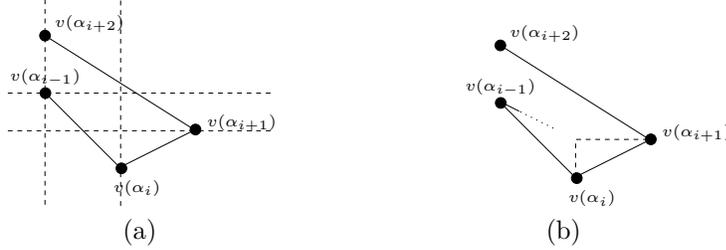


Figure 3.13: For the proof of Lemma 3.9, Case 1. The drawing in Case 1 (a) and the valid placements of  $v(\alpha_{i-2})$  where it can only lie on the dashed segments (b).

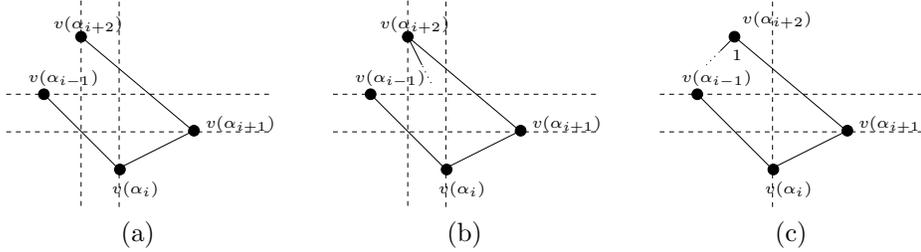


Figure 3.14: For the proof of Lemma 3.9, Case 2.

Since  $\alpha_1$  has label 0 and  $(v(\alpha_0), v(\alpha_1), v(\alpha_2), v(\alpha_3))$  is not a hook, therefore  $v(\alpha_3)$  is closer to  $v(\alpha_2)$  than  $v(\alpha_1)$ . Now let  $1 < i < k$  be maximal such that  $\alpha_i$  has label 1 and  $v(\alpha_{i+1})$  is closer to  $v(\alpha_i)$  than  $v(\alpha_{i-1})$ . After possible rotation and flipping we may assume that the free ray at  $\alpha_i$  goes upward and  $v(\alpha_{i-1})$  is left of  $v(\alpha_i)$ . If  $\alpha_{i+1}$  had label 0 then  $i + 1 = k$  and  $(v(\alpha_{k+1}), v(\alpha_k), v(\alpha_{k-1}), v(\alpha_{k-2}))$  would be a hook (see Figure 3.12) contradicting the assumption. So  $\ell(\alpha_{i+1}) = 1$ . By choice of  $i$ ,  $v(\alpha_{i+2})$  is not closer to  $v(\alpha_{i+1})$  than  $v(\alpha_i)$ , so  $x(v(\alpha_{i+2})) < x(v(\alpha_i))$ . Also,  $x(v(\alpha_{i+2})) \geq x(v(\alpha_{i-1}))$ , otherwise either  $v(\alpha_{i-1})$  would be in the rectangle defined by  $(v(\alpha_{i+1}), v(\alpha_{i+2}))$  or  $(v(\alpha_{i+1}), v(\alpha_{i+2}))$  and  $(v(\alpha_{i-1}), v(\alpha_i))$  cross. We have two cases:

**Case 1**  $x(v(\alpha_{i+2})) = x(v(\alpha_{i-1}))$  (see Figure 3.13(a)): In this case, we prove that  $\ell(\alpha_{i-1}) = 0$ . Since  $i > 1$ ,  $\ell(\alpha_{i-1}) \leq 2$ , so  $x(v(\alpha_{i-1})) < x(v(\alpha_{i-2}))$ . Because of the three segments  $(v(\alpha_{i+1}), v(\alpha_{i+2}))$ ,  $(v(\alpha_{i+1}), v(\alpha_{i+2}))$ , and  $(v(\alpha_{i+1}), v(\alpha_{i+2}))$  the only valid placement of  $v(\alpha_{i-2})$  is on the free ray of  $\alpha_i$  or  $\alpha_{i+1}$ ,

and below  $v(\alpha_{i+1})$  (see Figure 3.13(b)). Hence  $\ell(\alpha_{i-1}) = 0$  and therefore,  $i = 2$ . This means that  $(v(\alpha_0), v(\alpha_1), v(\alpha_2), v(\alpha_3), v(\alpha_4))$  is a cage.

**Case 2**  $v(\alpha_{i+2})$  is to the right of  $v(\alpha_{i-1})$  (see Figure 3.14(a)): We can assume that  $\alpha_{i+2}$  does not have label 0, for if it did, then we could argue that  $i = k - 2$  and by similar arguments to what we had in the last case  $(v(\alpha_{k+1}), v(\alpha_k), v(\alpha_{k-1}), v(\alpha_{k-2}), v(\alpha_{k-3}))$  would be a cage (see Figure 3.14(b)). So  $\alpha_{i+2}$  has label 1, and by choice of  $i$ ,  $v(\alpha_{i+1})$  is strictly closer to  $v(\alpha_{i+2})$  than  $v(\alpha_{i+3})$  (see Figure 3.14(c)). So  $y(v(\alpha_{i+3})) < y(v(\alpha_{i+1})) \leq y(v(\alpha_{i-1}))$ . But now regardless of the placement of  $v(\alpha_{i+3})$ , either edge  $(v(\alpha_{i+2}), v(\alpha_{i+3}))$  intersects  $(v(\alpha_{i-1}), v(\alpha_i))$ , or  $v(\alpha_{i+3})$  is in the rectangle defined by  $(v(\alpha_{i-1}), v(\alpha_i))$ , or  $v(\alpha_{i-1})$  is in the rectangle defined by  $(v(\alpha_{i+2}), v(\alpha_{i+3}))$ . Either way, we do not have a planar oblique RI-drawing, a contradiction.  $\square$

Combining Lemma 3.9 with Corollary 3.1 yields:

**Corollary 3.2** *Let  $C$  be a chain of angles in a planar non-aligned RI-drawing. Then the sequence of the labels of the angles in  $C$  cannot be of the form  $01^+0$ .*

If an RI-drawing is oblique, then it may have hooks or cages, but only if nearby angles have special labels. More precisely:

**Lemma 3.10** *In an oblique RI-drawing of a 2-connected graph let  $C = (\alpha_1, \alpha_2, \dots, \alpha_k)$  be a chain appearing in a face with more than 5 angles such that  $C$  has labels of the form  $01^+0$ . Then there is a chain  $C'$  containing  $C$  such that the sequence of the labels of the elements of  $C'$  has one of the following forms or the reverse thereof:*

- $01^+04$ , or
- $01^+03x$  where  $x > 0$ , or
- $01^+0304$ , or
- $01^+014$ .

**Proof.** Let  $\beta_1, \beta_0$ , and  $\alpha_0$  be the three angles preceding  $\alpha_1$ . By Lemma 3.9 there is a cage or a hook at one end of the chain  $C$ .

Suppose after possible reversal of  $C$  that  $(v(\alpha_0), v(\alpha_1), v(\alpha_2), v(\alpha_3))$  is a hook. By Lemma 3.6 one of the following holds:

- $\ell(\alpha_0) = 4$ , in which case  $C' = (\alpha_0, \alpha_1, \dots, \alpha_k)$  satisfies the first condition of the lemma;
- $\ell(\alpha_0) = 3$  and  $\ell(\beta_0) > 0$ , in which case  $C' = (\beta_0, \alpha_0, \dots, \alpha_k)$  satisfies the second condition of the lemma;
- $\ell(\alpha_0) = 3$ ,  $\ell(\beta_0) = 0$ , and  $\ell(\beta_1) = 4$ , in which case  $C' = (\beta_1, \beta_0, \alpha_0, \dots, \alpha_k)$  satisfies the third condition of the lemma.

If there is no hook at either end of the chain, then by Lemma 3.9 we can assume that after possible reversal of  $C$  that  $(v(\alpha_0), v(\alpha_1), v(\alpha_2), v(\alpha_3), v(\alpha_4))$  is a cage. Since  $\ell(\alpha_1) = 0$  by Lemma 3.8, we have four cases of labels for  $\alpha_0, \beta_0, \beta_1$ . The first three cases are handled as in the last paragraph. For the last case, we have that:

- $\ell(\alpha_0) = 1$  and  $\ell(\beta_0) = 4$ , in which case  $C' = (\beta_0, \alpha_0, \dots, \alpha_k)$  satisfies the fourth condition of the lemma.

□

We need these observations to lower-bound the sum of labels in a chain.

**Lemma 3.11** *Let  $C$  be a chain of angles in a planar oblique RI-drawing of a 2-connected graph. Then the sum of the labels of the angles of  $C$  is at least  $|C| - 3$ .*

**Proof.** We prove this using induction. Since the labels are not negative and by Lemma 3.3 there are no two adjacent angles of label 0 in  $C$ , the claim holds if  $C$  has 5 or fewer angles. Let  $S$  be the sequence of labels of the angles of  $C$ ; we aim to show  $\sum_{s \in S} s \geq |S| - 3$ . We have several cases:

**Case 1**  $S$  begins with a label  $x > 0$ : Let  $S_2$  be the sequence of labels such that  $S = (x, S_2)$ . Then by induction  $\sum_{s \in S_2} s \geq |S_2| - 3$  and therefore

$$\sum_{s \in S} s = \sum_{s \in S_2} s + x \geq |S_2| - 3 + 1 = |S| - 3.$$

**Case 2**  $S$  ends with a label  $x > 0$ : This is similar to Case 1.

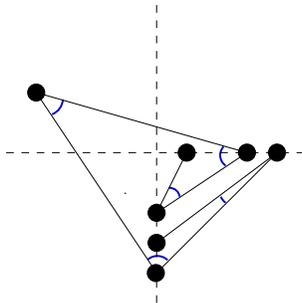


Figure 3.15: An RI-drawing with a chain with labels of the form 01010.

For the next cases we can assume  $S = (0, S_1, 0, S_2, \dots, S_k, 0)$  where  $S_i$  contains no 0.  $S_i$  is non-empty by Lemma 3.3. We call each such a  $S_i$  a *block*. A block that is of the form  $1^+$  is called a *bad block*. By Lemma 3.9 each bad block has a cage or a hook on one of its ends. We say a block is *left leaning* if it has a cage or a hook on its beginning, *right leaning* otherwise. Note that each bad block satisfies the conditions of Lemma 3.9, therefore, if  $S_i$ ,  $i < k$ , is right-leaning, then  $S_{i+1}$  begins with 4 or 3 or 14, and similarly for left-leaning.

**Case 3** There is a block  $S_i$  that has label-sum at least  $|S_i| + 3$ : Assume  $S = (S', S_i, S'')$ . By induction  $\sum_{s \in S'} s \geq |S'| - 3$  and  $\sum_{s \in S''} s \geq |S''| - 3$ . Therefore

$$\begin{aligned} \sum_{s \in S} s &= \sum_{s \in S'} s + \sum_{s \in S_i} s + \sum_{s \in S''} s \\ &\geq |S'| - 3 + |S_i| + 3 + |S''| - 3 \\ &= |S| - 3 \end{aligned}$$

as desired.

**Case 4** There are two bad blocks  $S_{i-1}$  and  $S_{i+1}$  leaning towards the same block  $S_i$ : We show that this case is included in Case 3. In this case, each of these bad blocks fit into the description of Lemma 3.9. Note that the third item of Lemma 3.9 cannot apply, otherwise we would have a 4 in one of the bad blocks which contradicts the definition. Therefore, by Lemma 3.10  $S_i$  contains a 4, or  $S_i$  contains a  $3x$  and a  $x'3$  with  $x, x' \geq 1$  as prefix and suffix. In either case, the label-sum of  $S_i$  would be greater or equal to  $|S_i| + 3$ , and hence we are in Case 3.

**Case 5** None of the previous cases applies:

Note that by Lemma 3.10 no two bad blocks can be leaning towards each other. Since we are not in Case 4, each bad block leans towards a unique block in  $S$ , except perhaps the leftmost and rightmost block (which might be bad blocks that are leaning towards the two ends of  $S$ ). By Lemma 3.10 the label-sum of a block  $S_i$  that is being leaned upon by some bad block is at least  $|S_i| + 2$ . Any block  $S_j$  that is not a bad block has label-sum at least  $|S_j| + 1$ . This means that excluding the leftmost and rightmost bad blocks, on average the label-sum of each block  $S_l$  is at least  $|S_l| + 1$ . There are  $k$  blocks and  $k + 1$  0s in  $S$ , therefore the label-sum of all labels in all blocks is at least  $|S| - 3$ .  $\square$

**Lemma 3.12** *In an oblique RI-drawing of a 2-connected graph the sum of the labels of a chain  $C$  of angles is in the range  $[|C| - 3, 3|C| + 3]$ .*

**Proof.** The lower bound is implied by Lemma 3.11 directly. For the upper bound, let  $S$  be the sequence of labels of angles of  $C$ .



Figure 3.16: A chain of angles on distinct vertices (a) and the drawing induced by the chain as discussed in Lemma 3.12 (b).

By 2-connectivity we can assume that no vertex has two angles in  $C$ , otherwise it would appear twice on a face and be a cut-vertex. Consider Figure 3.16(a). In that case the vertices of  $C$  induce a path. Let us look at the drawing of this path along with the edges of the first and last elements of  $C$  (Figure 3.16(b)). Then there is a unique chain  $C'$  with  $|C'| = |C|$  in such drawing, such that  $C'$  lies on the other side of the path and on the same vertices as  $C$ . Let  $S$  and  $S'$  be the sequence of labels of angles of  $C$  and  $C'$ , respectively. Then  $\sum_{s \in S} s + \sum_{s \in S'} s = 4|S|$ . Also, by Lemma 3.11 we have  $\sum_{s \in S'} s \geq |S'| - 3 = |S| - 3$ . Therefore

$$\sum_{s \in S} s = 4|S| - \sum_{s \in S'} s \leq 4|S| - (|S| - 3) = 3|S| + 3$$

as desired. □

### 3.3 Graphs with a Triangular Outer Face

We will eventually use AC labels to characterize exactly which graphs have an RI-drawing with a non-aligned frame. For this we use the frame of a graph, which is the graph obtained by removing the inside of filled triangles. In this section, we need to discuss how to draw these filled triangles, i.e. how to draw a planar graph with a triangular outer face.

This investigation was started by Miura *et al.* [23]. They showed that if  $T$  is a maximal filled triangle of an inner triangulated graph  $G$ , and if  $T$  has an RI-drawing then the inside of  $T$  has some easily verifiable structural properties. Also, the structure of the graph inside  $T$  imposes restrictions as to which labels of the angles of  $T$  must or must not be 0 or 1 in an oblique RI-drawing of the frame-graph of  $G$  (and vice versa, any oblique RI-drawing of the frame-graph can be expanded to an RI-drawing of all of  $G$ .)

Now we will refine these restrictions so that they still hold even if the graph  $G$  is not inner triangulated.

**Lemma 3.13** *Given a plane graph  $G$  with triangular outer face, one can compute in linear time all possible labelings of the outer face that could occur in an open RI-drawing of  $G$  with oblique outer face. Furthermore, any drawing of the outer face with such an AC labeling can be completed to an RI-drawing of  $G$  in linear time.*

**Proof.** First, we observe some properties of such an RI-drawing. Let  $\alpha, \beta$  and  $\gamma$  be the outer angles of  $G$ . By Lemma 3.4 the inner angles of a triangle are  $\{0, 1, 1\}$ , so by Lemma 3.1 the outer angles have labels  $\{4, 3, 3\}$ . Therefore, without loss of generality, assume  $\ell(\alpha) = 4$  and  $\ell(\beta) = \ell(\gamma) = 3$ . Then all vertices must be on the boundary of the rectangle defined by  $(v(\beta), v(\gamma))$  (see Figure 3.17(a)). Some of these vertices lie on one of the horizontal edges and some on one of the vertical edges of the border of this rectangle. Traversing these vertices along the border of this rectangle therefore gives an arrangement  $(v_1, v_2, \dots, v_k)$  of the vertices in  $V(G) \setminus \{v(\alpha)\}$  with  $v_1 = v(\beta)$

and  $v_k = v(\gamma)$ , and an index  $l$ , such that for every edge  $e$  of  $G$  one of the following holds:

- $e$  connects  $v(\alpha)$  and a vertex  $v_i$ , with  $1 \leq i \leq k$ ,
- $e$  connects  $v_i$  and  $v_j$ , with  $1 \leq i \leq l < j \leq k$ ,
- $e$  connects  $v_i$  and  $v_{i+1}$ , with  $1 \leq i < k$ .

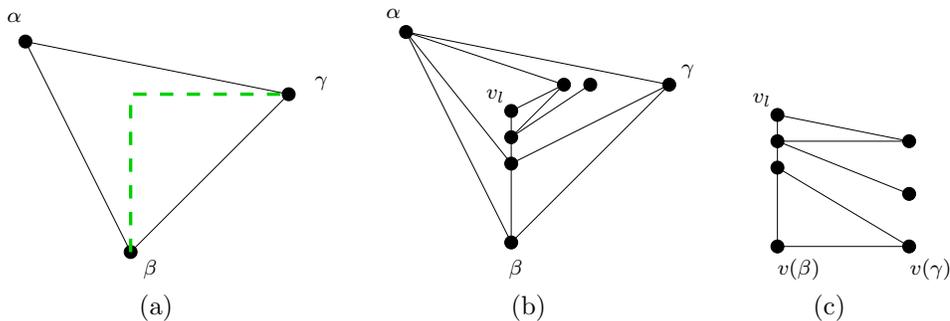


Figure 3.17: A triangle and the locus (dashed) of possible placement of vertices inside the triangle in an open RI-drawing (a). An open RI-drawing of a filled triangle  $G$  (b), and the corresponding drawing of vertices of  $V(G) \setminus \{v(\alpha)\}$  on two lines (c).

In fact, vertices  $v_1, v_2, \dots, v_l$  are on the same axis aligned line as  $v(\beta)$  while vertices  $v_{l+1}, v_{l+2}, \dots, v_k$  are on the same axis aligned line as  $v(\gamma)$  (see Figure 3.17(b)). Note that if there is a vertex on the common end of the two lines, we can move it by some small distance so that it is only on one of the lines and the drawing is still valid. Therefore, we can restate the problem as follows: To draw  $G$ , we need to find a drawing of vertices  $V(G) \setminus \{v(\alpha)\}$  on two lines such that  $v(\gamma)$  and  $v(\beta)$  are the first vertices of the two lines (see Figure 3.17(c)). Cornelsen *et al.* [10] give a linear algorithm that finds a drawing of a graph on two lines, if one exists. Their algorithm works even if some edges are constrained to have their endpoints on different lines. It is easy to modify their algorithm to work for graphs with a fixed embedding. We need to make sure that  $v(\gamma)$  and  $v(\beta)$  to be the last vertices. For this, we only need to care about the connected component of  $V(G) \setminus \{v(\alpha)\}$  that contains  $v(\gamma)$  and  $v(\beta)$ . To force  $v(\gamma)$  and  $v(\beta)$  are the last vertices in the

drawing of such a component, we can add two dummy vertices  $v'_\gamma$  and  $v'_\beta$  that are adjacent to  $v(\gamma)$  and  $v(\beta)$  respectively. We also add dummy edges  $(v'_\gamma, v'_\beta)$  and  $(v(\gamma), v(\beta))$  that are restricted to have their ends on different lines. The only way to draw the cycle  $(v(\gamma), v(\beta), v'_\beta, v'_\gamma)$  without crossing is to place  $v'_\beta$  and  $v'_\gamma$  on the same side of  $v(\beta)$  and  $v(\gamma)$ . Since there can be no edge from that side to  $v(\beta)$  and  $v(\gamma)$ ,  $v'_\beta$  and  $v'_\gamma$  are on the ends of the two lines. Removing them will result in a drawing of the component containing  $v(\beta)$  and  $v(\gamma)$  such that  $v(\beta)$  and  $v(\gamma)$  are on the two ends of the two lines. Hence, finding the drawing of this modified graph will give us the required structure, and we can add  $v(\alpha)$  to it to obtain an open RI-drawing of  $G$ .

So to find all possible labelings, try all possible assignments of  $\{3, 3, 4\}$  to  $\{\alpha, \beta, \gamma\}$ , and for each of them, try whether graph  $G$  can be drawn on two lines as explained above. Since there are  $O(1)$  labellings and the algorithm in [10] takes linear time, this can be done in linear time.  $\square$

# Chapter 4

## Testing RI-Drawability of Plane Graphs

No necessary and sufficient conditions or testing algorithms are known for the existence of weak open RI-drawings, even for inner triangulated graphs. Miura *et al.* [23] aimed to develop necessary and sufficient condition for all inner triangulated graphs, but did not succeed. As discussed in Section 3.3 such a drawing imposes conditions on how filled triangles are drawn; a natural first step is hence to remove the interior of all filled triangles and try to draw the resulting graph while satisfying these conditions.

So, let  $G$  be a plane graph. Let  $\mathcal{F}$  be the *frame graph* of  $G$ , that is, the graph obtained by removing the inside of filled triangles of  $G$ . In this chapter we give a constructive algorithm to decide whether  $G$  admits an open RI drawing such that  $\mathcal{F}$  is drawn non-aligned. Here, we assume the outer face is not a triangle, as we already handled that case in Lemma 3.13. First we review the results of Miura *et al.* [23] who also used the frame graph.

### 4.1 Review of Miura *et al.* [23]

Miura *et al.* first test for every filled triangle  $T$  whether the graph inside  $T$  has an RI-drawing. If this fails for any  $T$  then clearly  $G$  has no RI-drawing either. So in the following we always assume that all interiors of all filled

triangles of  $T$  have an RI-drawing, at least under some restrictions on the drawing of  $T$ .

Next, Miura *et al.* compute the restrictions made by a filled triangle  $T$ . Note that Miura *et al.* study only oblique drawings of the frame graph, and for such drawings their labeling is the same as the AC labeling. Recall that  $T$  must have labels  $\{0, 1, 1\}$  in any non-aligned RI-drawing  $\Gamma_{\mathcal{F}}$  of the frame graph. In a corresponding RI-drawing of  $G$ , the only place to put the vertices inside  $T$  is on the two coordinate axes from the two vertices labeled 1 in  $T$  as discussed in Section 3.3. In an inner triangulated graph, the vertex labeled 0 on  $T$  hence must be adjacent to all vertices inside  $T$ . So they obtain:

**Lemma 4.1** [23] *Let  $G$  be a plane inner triangulated graph. If  $T = \{a, b, c\}$  is a triangle of the frame graph that is a filled triangle in  $G$ , and if  $a$  is not adjacent to all vertices inside  $T$ , then in any open RI-drawing of  $G$  with oblique frame, the induced oblique RI-drawing of the frame has AC label 1 at  $a$ .*

Note that the generalization of this lemma to general plane graphs would be Lemma 3.13 as we give an algorithm that decides what angles of the outer triangle need to have label 1 so that we can complete the RI-drawing. So there is a set  $A$  of inner angles of the frame graph  $\mathcal{F}$  that must be labeled 1 in any non-aligned (hence oblique) RI-drawing of  $\mathcal{F}$  induced by an RI-drawing of  $G$ . Moreover, if we can find a non-aligned RI-drawing of  $\mathcal{F}$  that has these AC labels, then it can be expanded into an open RI-drawing of graph  $G$ .

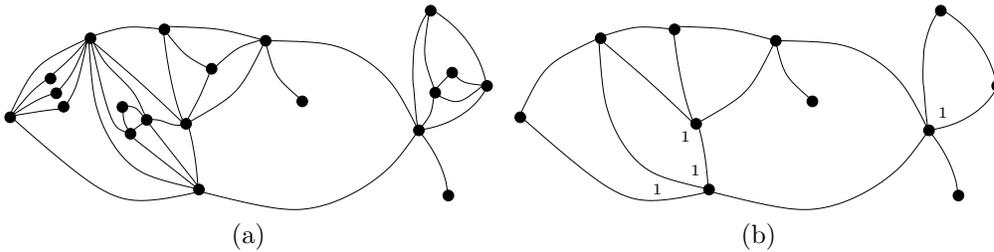


Figure 4.1: Graph  $G$  (a) and its frame graph  $\mathcal{F}$  with forced AC labels (b).

**Definition 4.1** (based on [23]) *A labeling of the angles of the frame graph  $\mathcal{F}$  with labels in  $\{0, 1, 2, 3, 4\}$  is a decent labeling if*

- (a) the labels at every vertex sum to 4, and
- (b) every inner triangle has labels  $\{0, 1, 1\}$ , and every angle in  $A$  is labeled 1, where  $A$  is the set of restrictions implied by the filled triangles. The labeling is called good if additionally
- (c) the outer angles have labels  $\{2, 3, 4\}$ .

Note that (a) and (b) are also properties of AC labelings, hence the necessity of them is justified. In other words (and as also shown by Miura *et al.*) if  $G$  is an inner triangulated graph and has an open RI-drawing with oblique frame, then  $\mathcal{F}$  has a decent labeling. However, Miura *et al.* also showed a graph where this is not sufficient. Hence they added condition (c) which forces the outer face to consist of four chains that are monotone in  $x$  and  $y$ . This condition is not necessary, but they show that adding it gives sufficient conditions: any graph that has a good labeling has an oblique RI-drawing.

## 4.2 Overview of the Algorithm

We show here that modifying restrictions (b) and (c) of Definition 4.1 gives conditions that are both necessary and sufficient, for RI-drawings with non-aligned frame. We switched the drawing model from “oblique frame” to “non-aligned frame” precisely so that we could find necessary and sufficient conditions. Characterizing the existence of drawings with oblique frame remains an open problem.

We also realized that with minor changes our results (first presented in [1]) work for all plane graphs, not just inner triangulated graphs as in [23] and [1].

**Definition 4.2** *A labeling of the angles of the frame graph  $\mathcal{F}$  with labels in  $\{0, 1, 2, 3, 4\}$  is RI-admissible if it satisfies*

- (a) the labels at every vertex sum to 4,
- (b') the labels at each inner face with  $k$  angles sums to  $2k - 4$ , and every angle in  $A$  is labeled 1, where  $A$  is the set of restrictions implied by the filled triangles, and
- (c') there is no chain of angles with labels of the form  $01^*0$ .

Note that condition (b') implies condition (b). Replacing (b) with (b') was not necessary if we only cared about inner triangulated graphs.

The main result of this chapter is:

**Theorem 4.1** *A plane graph  $G$  has a planar weak open RI-drawing with non-aligned frame if and only if the frame graph  $\mathcal{F}$  has an RI-admissible labeling.*

The proof of this theorem will require multiple steps.

The necessity of conditions (a) and (b') follows directly from Lemma 3.1 and condition (c') is necessary by Corollary 3.2.

We do not prove sufficiency directly; instead we give an algorithm that tests whether a graph  $G$  has a planar weak open RI-drawing with non-aligned frame, and the steps of the algorithm will imply sufficiency of the conditions on the labeling. Here is an outline of our algorithm:

- (i) Compute the frame graph  $\mathcal{F}$  (see Figure 4.1).
- (ii) For every triangle  $T$  of  $\mathcal{F}$  that was filled in  $G$ , compute whether the interior of  $T$  is realizable in an open RI-drawing as in Lemma 3.13. If this fails for any triangle, then  $G$  has no open RI-drawing. Else, let  $A$  be the set of angles of  $\mathcal{F}$  that must have label 1 (see Figure 4.1).
- (iii) Construct  $D$  (see Figure 4.2), which is a subdivision of the surround-dual. More precisely, it is the dual graph of  $\mathcal{F}$  after adding a degree  $3k$  vertex in each non-triangle face with  $k$  angles.
- (iv) Find an OD-admissible labeling of  $D$  that corresponds to a rectangle-contact dual drawing of  $\mathcal{F}$  and respects  $A$  in some sense. See Figure 4.7. If there is none, stop:  $G$  does not have an RI-drawing with  $\mathcal{F}$  drawn non-aligned. Otherwise, use the OD-admissible labeling to create a rectangle-contact dual drawing  $\Gamma_D$  of  $\mathcal{F}$  that respects  $A$  in some sense.
- (v) Expand the orthogonal drawing  $\Gamma_D$  into a rectangular drawing  $\Gamma_{D'}$  of a super-graph  $D'$  of  $D$ . Do this by adding edges and vertices to the graph, without changing the rectangular faces, so that  $\Gamma_{D'}$  also respects  $A$  (see Figure 4.7).

- (vi) Construct the dual graph  $\mathcal{D}(D')$  and then remove the outer face vertex. The resulting inner triangulated graph  $\mathcal{F}'$  is a super-graph of the frame-graph  $\mathcal{F}$  (see Figure 4.8).
- (vii) From the OD-admissible labeling of  $D'$ , extract an RI-admissible labeling of  $\mathcal{F}'$ . This labeling is decent, and in fact, it is good. See Figure 4.8.
- (viii) Using this good labeling and the fact that  $\mathcal{F}'$  is inner triangulated, create a non-aligned RI-drawing of  $\mathcal{F}'$  using a variant of the algorithm presented in [23]. See Figure 4.9.
- (ix) Then insert the filled triangles (which is possible by choice of  $A$ ) to obtain an open RI-drawing with non-aligned frame of a super-graph  $G'$  of  $G$ .
- (x) Remove the vertices of  $V_{G'} \setminus V_G$  from the drawing (see Figure 4.9).

Steps (i), (iii) and (x) are straightforward. Also, steps (ii) and (ix) are doable by Lemma 3.13. We give definitions and details for the other steps below.

### 4.2.1 Definition of $D$

$D$  is a subdivision of the surrounding dual of  $\mathcal{F}$ ,  $\mathcal{D}^*(\mathcal{F})$ . Here we show how graph  $D$  is defined. Let  $\mathcal{F}$  be the frame-graph, i.e.,  $\mathcal{F}$  is a plane graph without any filled triangle. Let  $\mathcal{F}^+$  be the graph obtained from  $\mathcal{F}$  by adding one vertex  $v_f$  in each non-triangular face  $f$ . For every angle  $\alpha$  at a vertex  $u$  on a non-triangular face  $f$ , we add three edges from  $u$  to  $v_f$  in  $\mathcal{F}^+$  at the place (in the cyclic order around  $u$ ) where  $\alpha$  was. Thus, a vertex that appears twice on a non-triangular face  $f$  of  $\mathcal{F}$  would have 6 edges to  $v_f$ , though not all of them would be consecutive. Now let  $D$  be the dual graph of  $\mathcal{F}^+$ , i.e.,  $\mathcal{D}(\mathcal{F}^+)$ . See Figure 4.2.

A *vertex-face* of  $D$  is a face  $f$  of  $D$  that corresponds to a vertex  $v$  of  $\mathcal{F}$  (see Figure 4.3). The rest of the faces of  $D$  are called *facial-cycle-faces*. Note that each facial-cycle-face  $f$  corresponds to some non-triangular face  $g$  of  $F$ . Also, any vertex  $v$  of  $D$  that corresponds to a triangle  $T$  of  $\mathcal{F}^+$  (and consequently  $\mathcal{F}$ ), is called a *triangle-vertex*. Any vertex of  $D$  that is not a

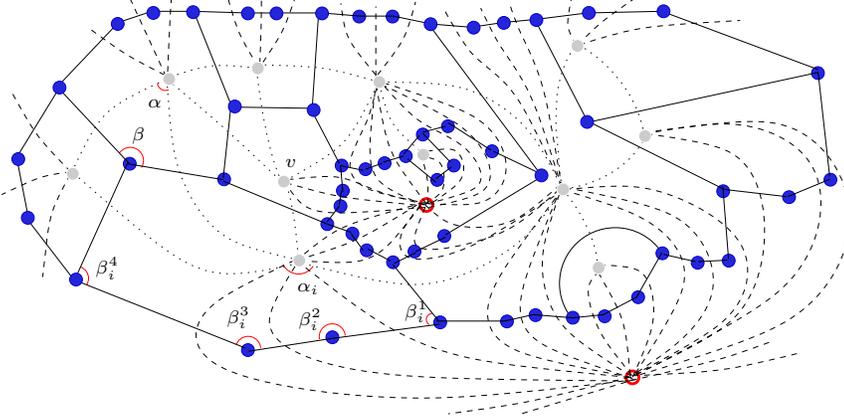


Figure 4.2: The graph  $\mathcal{F}$  from Figure 4.1 (gray vertices and dotted edges), the added vertices to each non-triangular face of  $\mathcal{F}$  (circles as vertices and dashed edges), and the graph  $D$  (solid).

triangle-vertex is a *facial-cycle-vertex*. Figure 4.3 highlights the vertex-face of  $D$  corresponding to the vertex  $v$  of  $\mathcal{F}$  as drawn in Figure 4.2.

There is a correspondence between angles of  $D$  and  $\mathcal{F}^+$ . For every angle  $\alpha$  at a triangular face of  $\mathcal{F}$  there is a corresponding inner angle  $\beta$  of  $D$  at a triangle-vertex. For every angle  $\alpha_i$  at a non-triangular face  $f$  of  $\mathcal{F}$ , there are four corresponding inner angles  $\beta_i^1, \beta_i^2, \beta_i^3, \beta_i^4$  of  $D$  at four facial-cycle-vertices (see Figure 4.2).

## 4.2.2 Non-aligned RI-Drawings and Rectangle-contact Dual Drawings

Recall that in step (ii) we determined a set of angles  $A$  of the frame graph  $\mathcal{F}$ . We know that any open RI-drawing of  $G$ , the induced drawing of  $F$  has AC labeling 1 at all angles in  $A$ . We use the same set  $A$  to restrict orthogonal drawings of  $D$ . Note that all such angles are on triangular inner faces. More precisely, we say that an orthogonal drawing  $\Gamma_D$  of  $D$  *respects*  $A$  if for every angle  $\alpha \in A$ , the corresponding angle in  $\Gamma_D$  has AC label 1, i.e. it has geometric angle  $\pi/2$ . Since  $\alpha$  belonged to a triangular face this corresponding angle is well-defined.

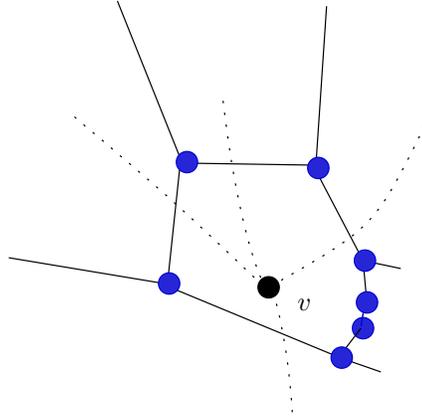


Figure 4.3: A vertex-face of the graph  $D$  (solid edges) depicted in Figure 4.2 corresponding to the vertex  $v$  of  $\mathcal{F}$  (dotted edges).

In this part, we aim to show that step (iv) is correct: If  $D$  cannot be drawn as a rectangle-contact dual drawing of  $\mathcal{F}$  that respects  $A$ , then  $\mathcal{F}$  does not have a non-aligned open RI-drawing. We prove this by showing that from any non-aligned open RI-drawing of  $\mathcal{F}$ , a drawing of  $D$  that is a rectangle-contact dual drawing of  $\mathcal{F}$  with corresponding angles can be constructed.

**Definition 4.3** *We say that a non-aligned RI-drawing  $\Gamma_{\mathcal{F}}$  of  $\mathcal{F}$  and a drawing  $\Gamma_D$  of  $D$  that is a rectangle-contact dual drawing of  $\mathcal{F}$  have the same inner structure if for any angle  $\alpha$  at a triangular face of  $\mathcal{F}$  and its corresponding angle  $\beta$  in  $D$ , angle  $\alpha$  has AC label 1 if and only if the angle  $\beta$  has AC label 1.*

We will not be countering drawings directly, but instead go via labels; by Lemma 3.2 we know exactly when a set of labels can be realized as an orthogonal drawing.

**Theorem 4.2** *For any RI-admissible labeling of  $\mathcal{F}$ , there exists an OD-admissible labeling of  $D$  that corresponds to a rectangle-contact drawing of  $F$  and has the same inner structure.*

**Proof.** Assume we have a non-aligned RI-drawing of  $\mathcal{F}$ , and let  $\ell_{\mathcal{F}}(\cdot)$  be the corresponding AC labeling of angles of  $\mathcal{F}$ . We show how to convert  $\ell_{\mathcal{F}}(\cdot)$

into an OD-admissible labeling  $\ell_D(\cdot)$  of the angles of  $D$ . To avoid confusion, we use *RI-label* to refer to the labels of angles of  $F$  and *OD-labels* to refer to the labels of angles of  $D$ . We define  $\ell_D(\cdot)$  as follows:

If  $\alpha$  is an angle of  $\mathcal{F}$  at a triangular face with corresponding angle  $\beta$  of  $D$ , then set  $\ell_D(\beta) = 2 - \ell_{\mathcal{F}}(\alpha)$ . Since  $\alpha$  has RI-label 0 or 1, hence  $\beta$  has OD-label 1 or 2, and it has OD-label 1 if and only if  $\alpha$  has RI-label 1, so the two sets of labels have the same inner structure (see Figure 4.4).

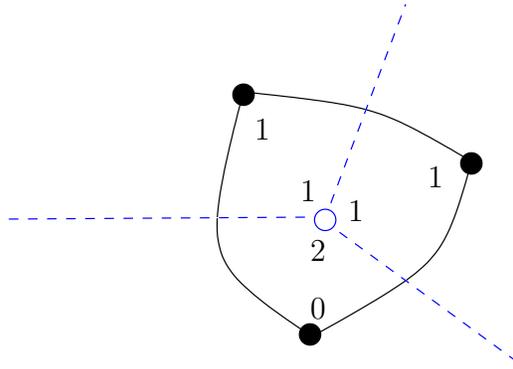


Figure 4.4: Translation of the labels of angles of triangular inner faces of  $\mathcal{F}$  (solid edges) to labels of the angles of  $D$  (dashed edges).

If  $\alpha$  is an angle of  $\mathcal{F}$  at a non-triangular face, then assigning OD-labels to its corresponding 4 angles of  $D$  is more complicated (and in particular, not always a local operation.) Let  $\alpha_0, \dots, \alpha_{k-1}$  be angles in clockwise order around some non-triangular face of  $\mathcal{F}$ ; addition in the following is modulo  $k$ . For each  $\alpha_i$ , let  $\beta_i^1, \dots, \beta_i^4$  be the four corresponding inner angles of  $D$ , in clockwise order around the face. Now for each  $i$  (see also Figure 4.5):

- If  $\ell_{\mathcal{F}}(\alpha_i) = 0$ , then assign OD-labels 2, 2, 2, 2 to  $\beta_i^1, \beta_i^2, \beta_i^3, \beta_i^4$ .
- If  $\ell_{\mathcal{F}}(\alpha_i) = 2$ , then assign OD-labels 1, 2, 2, 1 to  $\beta_i^1, \beta_i^2, \beta_i^3, \beta_i^4$ .
- If  $\ell_{\mathcal{F}}(\alpha_i) = 3$ , then assign OD-labels 1, 1, 2, 1 to  $\beta_i^1, \beta_i^2, \beta_i^3, \beta_i^4$ .
- If  $\ell_{\mathcal{F}}(\alpha_i) = 4$ , then assign OD-labels 1, 1, 1, 1 to  $\beta_i^1, \beta_i^2, \beta_i^3, \beta_i^4$ .
- The most complicated case is  $\ell_{\mathcal{F}}(\alpha_i) = 1$ . We assign either OD-labels 1, 2, 2, 2 or OD-labels 2, 2, 2, 1 to  $\beta_i^1, \beta_i^2, \beta_i^3, \beta_i^4$ , but the choice between these depends on the neighborhood.

Explore from angle  $\alpha_i$  both clockwise and counter-clockwise along the face until we obtain a maximal subsequence where all RI-labels are 1. Say this sequence is  $\alpha_j, \dots, \alpha_l$ . By Corollary 3.2, the sequence  $\alpha_{j-1}, \alpha_j, \dots, \alpha_l, \alpha_{l+1}$  does *not* have the form  $01^+0$ , so one of  $\alpha_{j-1}$  and  $\alpha_{l+1}$  has RI-label  $\geq 2$ . If  $\ell_{RI}(\alpha_{j-1}) \geq 2$ , then assign OD-labels 1, 2, 2, 2 to  $\beta_i^1, \beta_i^2, \beta_i^3, \beta_i^4$  (and also to all other corresponding angles in that subsequence), else assign OD-labels 2, 2, 2, 1 to  $\beta_i^1, \beta_i^2, \beta_i^3, \beta_i^4$ .

Finally, for all angles of  $D$  that have not been labeled yet, we set the OD-label such that the sum of labels around the vertex is 4. (This happens only at facial-cycle-vertices at the angles that do not have a corresponding angle in  $\mathcal{F}$ .) We verify that the labeling is OD-admissible and corresponds to a drawing of  $D$  that is a rectangle-contact dual drawing of  $\mathcal{F}$ :

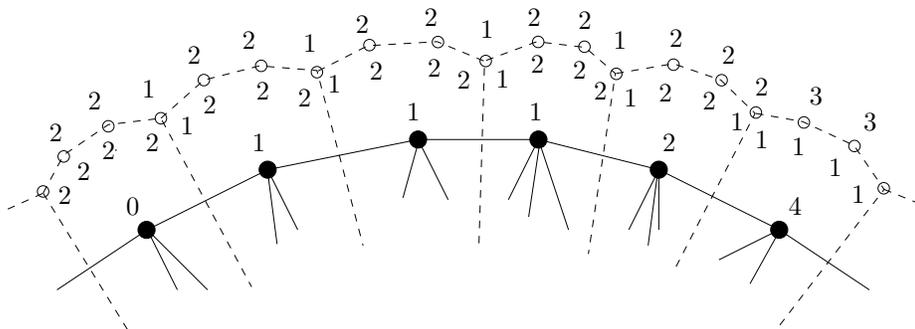


Figure 4.5: Conversion of RI-labels of  $\mathcal{F}$  (solid) to OD-admissible labels of  $D$  (dashed).

- Every vertex-face  $f$  of  $D$  with  $k$  angles has exactly 4 angles that have OD-label 1 and  $k - 4$  angles with OD-label 2, since the RI-labels at the corresponding vertex  $v$  sum to 4. By construction, an RI-label  $i$  at vertex  $v$  gives rise to  $i$  angles with OD-label 1 at  $f$  (this holds even if  $v$  is on a non-triangular face of  $\mathcal{F}$ .) Also by construction, the rest of the angles of  $f$  are assigned OD-label 2. Therefore, in any realization of this OD-labeling as an orthogonal drawing, vertex-faces are drawn as rectangles.
- The OD-labels at every vertex  $v$  of  $D$  sum to 4. For if  $v$  is a triangle-vertex corresponding to a triangular face  $T$  of  $\mathcal{F}$ ,  $T$  has RI-labels

$\{0, 1, 1\}$ , which correspond to OD-labels  $\{2, 1, 1\}$  in  $D$ . Otherwise, if  $v$  is a facial-cycle-vertex, then by construction of the OD-labels the total is 4 (see Figure 4.5).

- We claim that every angle  $\alpha$  of  $D$  that is on a facial-cycle-face of  $\mathcal{F}$  has OD-label in  $\{1, 2, 3\}$  (see Figure 4.6).

Recall that by construction of  $D$  the boundary of  $f(\alpha)$  is a cycle. Also  $\ell_D(\alpha)$  is defined as 4 minus the sum of other OD-labels at the vertex  $v$  that supports  $\alpha$ . Since there is at least one other OD-label at  $v$ , and it is 1 or 2, hence  $\ell_D(\alpha) \leq 3$ .

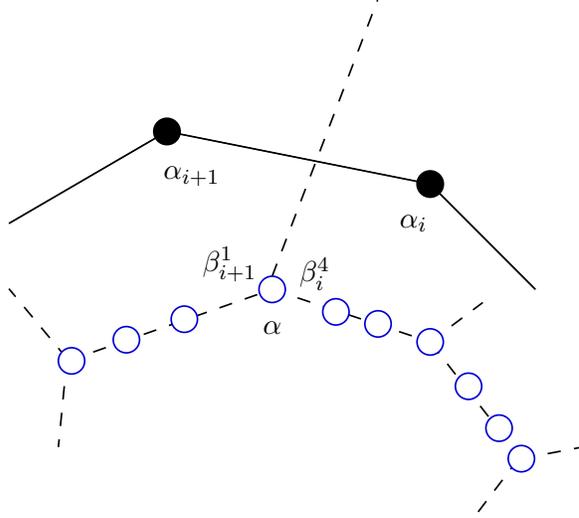


Figure 4.6: The angle  $\alpha$  on a facial-cycle-face.

It is harder to show that  $\ell_D(\alpha) > 0$ . Assume to the contrary that  $\ell_D(\alpha) \leq 0$ . Since, by the construction of  $D$ , there are at most two other angles at  $v$ , hence there must be exactly two (say  $\beta_i^4$  and  $\beta_{i+1}^1$ ) and they must both have OD-label 2. From the construction of the OD-labels of  $D$  and the way we handled the case  $\ell_{\mathcal{F}}(\alpha_i) = 1$ , this can happen only if  $\ell_{\mathcal{F}}(\alpha_i) = 0 = \ell_{\mathcal{F}}(\alpha_{i+1})$ . By Lemma 3.3 no two consecutive angles in a non-aligned RI-drawing have AC label 0, so this cannot happen.

- Now we show that the OD-labels on each inner face of  $D$  sum to  $2k - 4$ , where  $k$  is the number of angles on  $f$ . This is easy to observe for

vertex-faces of  $D$ , since by what we showed in the first bullet they are labeled with  $k - 4$  labels of 2 and 4 labels of 1. So we assume  $f$  to be a facial-cycle-face of  $D$  that corresponds to an inner face  $g$  of  $\mathcal{F}$ . Let  $t = k/3$  be the number of the angles on  $g$ . By Lemma 3.1, the sum of the RI-labels of the angles  $\alpha_1, \dots, \alpha_t$  of  $g$  is  $2t - 4$ . The angles  $B = \{\beta_1^1, \dots, \beta_1^4, \beta_2^1, \dots, \beta_2^4, \dots, \beta_t^1, \dots, \beta_t^4\}$  cover all angles at the vertices of  $f$  except the angles that belong to  $g$ . Observe that any RI-label  $i$  on  $g$  creates total OD-label of  $8 - i$  on its four corresponding angles. Therefore, the sum of the OD-labels in  $B$  is  $8t - (2t - 4) = 6t + 4$ . There are  $k$  vertices on  $f$ , hence the sum of the OD-labels of the vertices on  $f$  is  $4k = 12t$ ; therefore the sum of the OD-labels at  $f$  is  $12t - (6t + 4) = 6t - 4 = 2k - 4$  as desired.

- Finally we need to show that the OD-labels on the outer angles of  $D$  sum to  $2k + 4$ , where  $k$  is the number of angles on the outer face of  $D$ .

The proof is identical to the one for the last statement. The only difference is that for the outer face of  $\mathcal{F}$ , the RI-labels of  $\mathcal{F}$  sum to  $4k + 4$ . This, combined with the logic of the last statement, yields the desired equation.

Hence the labeling of  $D$  is OD-admissible. Since any vertex-face of  $D$  has exactly four labels 1 and all other labels are 2, any orthogonal drawing that realizes the labeling (which exists by Lemma 3.2) is a rectangle-contact dual drawing of  $\mathcal{F}$ .  $\square$

**Remark 4.1** *Note that Theorem 3.2 implies a correspondence between rectangle-contact dual drawings and non-aligned RI-drawings: For any non-aligned RI-drawing of a graph  $\mathcal{F}$ , there exists a rectangle-contact dual drawing of  $\mathcal{F}$  that has the same inner structure.*

We note here that the proof did not use anything about the RI-drawing except that the corresponding AC labeling is a decent labeling for which the sequence of labels at non-triangular faces does not contain 01\*0. This will be crucial for the sufficiency in Theorem 4.1 later.

The contrapositive of Theorem 4.2 proves correctness of step (iv). If  $D$  does not admit a rectangle-contact dual drawing of  $\mathcal{F}$  that respects  $A$ , then

$\mathcal{F}$  cannot have a non-aligned RI-drawing with all angles in  $A$  having AC label 1.

**Corollary 4.1** *Step (iv) is correct.*

Figure 4.7 shows an OD-admissible labeling that respects the restrictions of Figure 4.2, and the corresponding rectangle-contact dual drawing. We have not yet discussed how to implement step (iv); this will be done in Section 4.3.

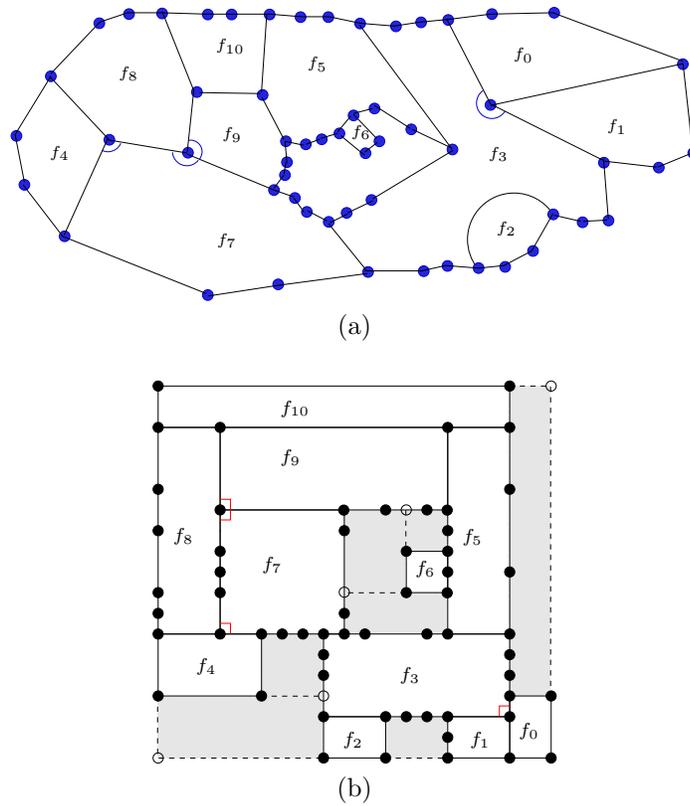


Figure 4.7: Graph  $D$  with the restrictions on AC labels (a), and an rectangle-contact dual drawing expanded to a rectangular drawing by adding dashed edges and shaded rectangles (b).

### 4.2.3 From Rectangle-contact Dual Drawing to Non-aligned RI-drawing

**Lemma 4.2** *Any drawing  $\Gamma_D$  of  $D$  that is a rectangle-contact dual drawing of  $\mathcal{F}$  and respects  $A$  can be expanded into a rectangular drawing  $\Gamma_{D'}$  of a plane graph  $D'$  such that:*

- $\Gamma_{D'}$  is a rectangular dual drawing of some super graph  $\mathcal{F}'$  of  $\mathcal{F}$ ,
- $D'$  has size  $O(|D|)$ ,
- No edge of is drawn inside faces of  $D$  that correspond to vertices of  $\mathcal{F}$ , though subdivision vertices might be added on their boundary, and
- $\Gamma_{D'}$  respects  $A$ .

**Proof.** As part of his orthogonal-shape approach to orthogonal graph drawing, Tamassia ([31], see also [3]) provided an algorithm to add a linear number of vertices and edges to an orthogonal drawing to turn it into a rectangular drawing without changing directions of edges. The algorithm does not create any vertex of degree 4. Applying this algorithm to the drawing  $\Gamma_D$  gives a rectangular drawing  $\Gamma_{D'}$  of a graph  $D'$  and only adds vertices and edges in non-rectangular faces. Hence all angles at faces of  $D$  that correspond to vertices of  $\mathcal{F}$  are preserved. All inner faces of  $\Gamma_{D'}$  are rectangles, hence it is a rectangle-contact dual drawing of some graph  $\mathcal{F}'$  that is a super graph of  $\mathcal{F}$ . All AC labels at rectangular faces of  $\gamma_D$  are unchanged in  $\gamma_{D'}$ , so  $\gamma_{D'}$  respects  $A$ .  $\square$

**Lemma 4.3** *If  $D'$  has a rectangular drawing  $\Gamma_{D'}$  that respects  $A$ , then there is a super graph  $\mathcal{F}'$  of  $\mathcal{F}$  that has a good labeling.*

**Proof.** We prove this by converting the OD-admissible labeling of  $\Gamma_{D'}$  into an RI-admissible labeling of  $\mathcal{F}'$ , hence more or less the reverse of the proof of Theorem 4.2. We refer to the labels of angles of  $D'$  and  $\mathcal{F}'$  by OD-labels and RI-labels, respectively. Note that any two axis aligned rectangles can touch at most once. Let  $\mathcal{F}'$  be the touching graph of the faces of  $\Gamma_{D'}$ , that is,  $\mathcal{F}'$  is the dual of  $D'$  after removing all degree 2 vertices of  $D'$ . For every



algorithm only promises an oblique drawing of  $\mathcal{F}'$ ; it need not be non-aligned. But we can modify their algorithm to make the drawing non-aligned.

They construct two directed acyclic graphs (DAGs)  $X$  and  $Y$  of linear size on the vertices of the given graph (here  $\mathcal{F}'$ ), where a  $(u, v)$  edge in  $X$  implies  $x(u) < x(v)$  (respectively  $(u, v)$  in  $Y$  implies  $y(u) < y(v)$ .) They show that any placement of the vertices in the plane that respects the restrictions forced by these two graphs can be completed into an open RI-drawing that induces the given labeling as its AC labeling. Then, they construct a drawing using the fact that  $x(v)$  (resp.  $y(v)$ ) can be the length of the longest path ending in  $v$  in  $X$  (resp. in  $Y$ ). Here we modify their algorithm by using a topological order in place of longest paths; then all coordinates are distinct. This gives us a non-aligned drawing of  $\mathcal{F}'$  in a  $|V(\mathcal{F}')| \times |V(\mathcal{F}')|$  grid that has the given good labeling as its AC labeling.  $\square$

### 4.3 Putting it All Together

If a plane graph  $G$  has an open RI-drawing with non-aligned frame  $\mathcal{F}$ , then  $F$  has a rectangle-contact dual drawing with the same inner structure (Theorem 4.2). Hence we can find a rectangle-contact dual drawing of  $\mathcal{F}$  that respects  $A$ , expand it to a rectangular drawing that is a rectangular dual drawing of some super graph  $\mathcal{F}'$  of  $\mathcal{F}$  (Lemma 4.2), extract a good labeling from it (Lemma 4.3), and create a non-aligned RI-drawing from it (Lemma 4.4). See also Figure 4.9. Insert the filled triangles which is possible since  $A$  has been respected (Lemma 3.13.) Now delete the added vertices and edges; this results in the desired open RI-drawing with non-aligned frame of  $G$ . This proves correctness of the algorithm.

Our proof was constructive and gives rise to an algorithm to test whether  $G$  has an open RI-drawing with non-aligned frame. It remains to analyze the run-time of this algorithm. Most steps are clearly doable in linear time. The bottleneck is the time to test whether  $D$  can be drawn as a rectangle-contact drawing of  $\mathcal{F}$  that respects  $A$ .

There are multiple approaches to related problems, and none of the faster ones unfortunately seems applicable. Rahman *et al.* [28] presented a linear algorithm to find an orthogonal drawing of a plane graph that has no bends,

if one exists. But it is not clear how to impose the restriction of respecting  $A$  or having rectangular faces.

Miura *et al.* [22] reduced testing the existence of OD-admissible labels to a matching problem, and it is a simple exercise to additionally impose that angles corresponding to  $A$  obtain AC label 1. They show that this matching can be found  $O(n^{1.5}/\log n)$  time. But their algorithm requires knowing which of the angles on the outer face are convex and reflex (otherwise the time complexity increases much more.)

We use here a flow-approach inspired by Tamassia's work [31] on orthogonal drawing with minimum number of bends on edges. Tamassia created a flow network of a plane graph that encodes the shapes (i.e., abstract descriptions via bends and angles) of all possible plane orthogonal drawings. Given a plane graph  $G$ , his algorithm creates a graph  $H$  that has a vertex  $\phi_f$  for every face  $f$  and a vertex  $\phi_v$  for every vertex of  $G$ . Here, we simplify the construction as we do not allow bends. For each vertex  $v$  of  $G$  we set  $s(\phi_v) = -4$ . For each face  $f$  of  $G$  with  $k$  angles we set  $s(\phi_f) = 2k - 4$  if  $f$  is an inner face, otherwise  $s(\phi_f) = 2k + 4$ . Then for angle  $\alpha$  in  $G$ , there is an edge  $e$  from vertex  $\phi_{f(\alpha)}$  of  $H$  to vertex  $\phi_{v(\alpha)}$  of  $H$  with capacity set  $s(e) = \{1, 2, 3, 4\}$ . Then, in the solution of the corresponding network flow, the label of  $\alpha$  would be  $f(e)$ . It is easy to verify such a labeling has the properties of Lemma 3.2 and is OD-admissible, and hence realizable as an orthogonal drawing.

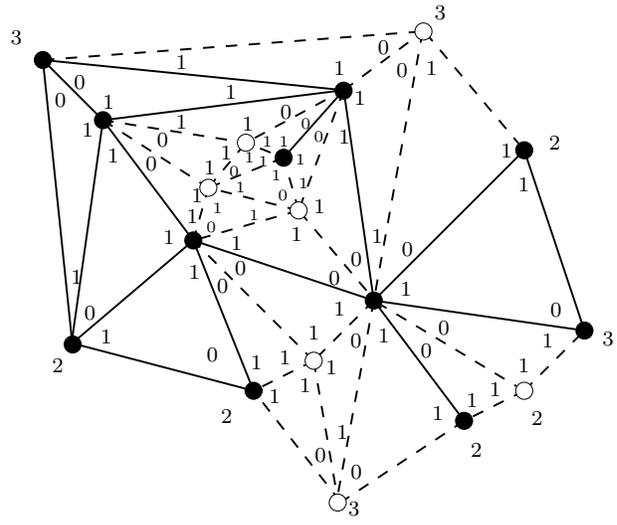
To apply this algorithm to the graph  $D$  for constructing a rectangle-contact dual drawing of  $\mathcal{F}$  that respects  $A$ , we modify capacity sets of edges in two steps. First, any edge of  $H$  that corresponds to an angle  $\alpha$  on a vertex-face of  $D$  has its capacity restricted to  $\{1, 2\}$ . This guarantees that vertex-faces will be drawn as rectangles. Second, for any edge of  $H$  that corresponds to an angle  $\alpha \in A$  the only element in the capacity set is 1. This ensures that the OD-admissible labeling respects  $A$ .

Tamassia's result required finding a minimum-cost flow as his construction has some extra edges that correspond to the bends on the final drawing. Since we forbid bends on edges, we only need to find a feasible flow, which can be done in  $O(n^{1.5} \log n)$  time [16]. Recently, Cornelsen and Karrenbauer [9] introduced ways to use the planarity of  $H$  to find a solution in  $O(n^{1.5})$ .

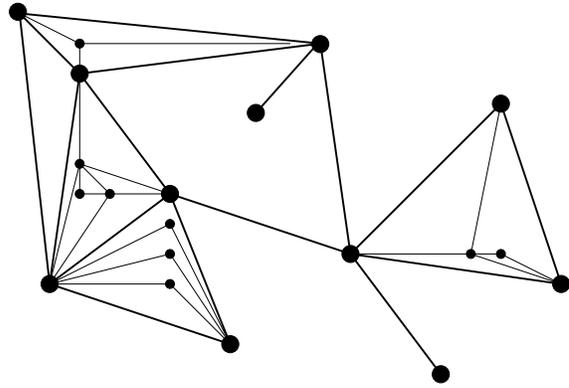
**Theorem 4.3** *Let  $G$  be a plane graph. In  $O(n^{1.5})$  time, we can test whether*

*G has a planar weak open RI-drawing with non-aligned frame, and if so, construct it.*

We briefly return to the sufficiency for Theorem 4.1. If  $\mathcal{F}$  has a decent labeling with no chain of angles having labels of the form  $01^*0$ , then as mentioned after Theorem 4.2,  $D$  can be drawn as a rectangle-contact dual drawing of  $\mathcal{F}$  that respects  $A$ . Steps (iv-x) of the algorithm then construct a planar weak open RI-drawing of  $G$  with non-aligned frame, proving Theorem 4.1.



(a)



(b)

Figure 4.9: The RI-drawing of  $\mathcal{F}'$  obtained by the labeling (unspecified labels are 1) (a) and the RI-drawing of  $G$  obtained by putting the inside of filled triangles back in and removing the dummy vertices (b).

## Chapter 5

# Hardness of Testing RI-Drawability of Planar Graphs

All the results established so far study only plane graphs. For the case of planar graphs, no specific algorithm has been proposed. Of course, in the case of triconnected graphs, the two problems are quite similar in the sense that by fixing the outer face of a triconnected graph, a unique planar embedding is implied. Therefore the positive results regarding triconnected plane graphs, imply some positive results about some classes of planar graphs. Here, we prove that deciding whether a planar graph admits an open RI drawing or not is NP-hard. This is in contrast with closed RI drawings, where given a planar graph, we only need to find an embedding without filled triangles, if one exists. Biedl *et al.* [6] present a polynomial time algorithm to find such an embedding. Therefore, the same problem is polynomially solvable, for closed RI drawings.

The reduction presented here is similar to, and in fact based on, the one by Garg and Tamassia [15] used for proving NP-hardness of finding orthogonal drawings of a given planar graph. Their proof uses an intermediate reduction from NOT-ALL-EQUAL-3-SAT to the integer switch-flow problem. Given an undirected graph  $G$  with a set of integer capacities  $s(e)$  assigned to each edge  $e$  of  $G$ , the *integer switch-flow* problem asks for a valid orientation of the edges of  $G$ ,  $\vec{G}$ , such that there is a valid solution for the corresponding

circulation instance on  $\vec{G}$ . We study this problem in a special case which is still NP-hard.

Given an instance  $\mathcal{I}$  of NOT-ALL-EQUAL-3-SAT, our reduction are as follows:

- (i) Then  $\mathcal{I}$  is reduced to an instance  $\mathcal{D}(\mathcal{I})$  of the integer switch-flow problem, consisting of a graph  $\Phi$  with special properties and a function  $s$ , associating a set of integers to each edge of  $\Phi$ .
- (ii) A graph  $G_{OD}$  is defined in such a way that that  $G_{OD}$  has an orthogonal drawing if and only if  $\mathcal{D}(\mathcal{I})$  is satisfiable.
- (iii) Based on  $G_{OD}$ , a graph  $G_{RI}$  is defined such that  $G_{OD}$  is a subgraph of  $G_{RI}$ .
- (iv) Then, we prove that if  $G_{OD}$  admits an orthogonal drawing then  $G_{RI}$  admits an open RI drawing.
- (v) Finally, we show that if  $G_{RI}$  admits an open RI drawing, then  $\mathcal{I}$  is satisfiable.

Steps (i) and (ii) are exactly as published by Garg and Tamassia [15] (except for some change in constants); we briefly review them. Steps (iii), (iv) and (v) define our contribution. All the steps are explained in following sections.

Here we show that the problem is in NP. Note that any graph that admits an RI-drawing, admits one with all vertices on an  $n \times n$  grid. The reason of this is that so long as the order of vertices on both coordinates is preserved, we can move vertices around and no vertex would enter the interior of a rectangle-of-influence. This means that our problem is in NP. Assuming correctness of these steps gives us the following result:

**Theorem 5.1** *The problem of deciding if a planar graph is open RI drawable is NP-complete.*

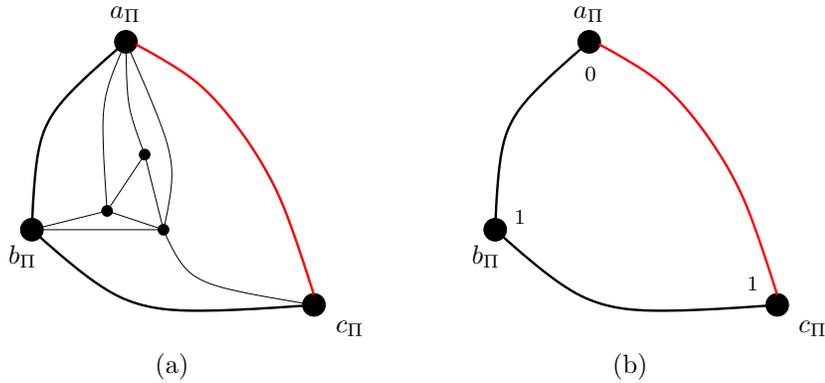


Figure 5.1: The graph  $\Pi$  (a) and the only labeling of its frame that admits an RI-drawing (b).

## 5.1 Tendrils and Wiggles

To define the graphs  $G_{RI}$  and  $G_{OD}$ , we require some definitions and intermediate gadgets first.

A *rectilinear  $i$ -wiggle*  $M$ , for a positive integer  $i$ , is a path of length  $8i + 1$ . The two ends of a rectilinear  $i$ -wiggle  $M$  are its *designated vertices*  $s_M$  and  $t_M$ . The 2-wiggle is depicted in Figure 5.2. The *RI  $i$ -wiggle*  $W$  is constructed from the rectilinear  $i$ -wiggle  $M$  by replacing each edge  $(u, v)$  of the rectilinear  $i$ -wiggle where  $u, v \notin \{s_M, t_M\}$  with an instance of the graph  $\Pi$  depicted in Figure 5.1 such that  $u$  and  $v$  correspond to  $b_\Pi$  and  $a_\Pi$ , respectively, where  $u$  is closer to  $s_M$  in  $M$  than  $v$ . Then the designated vertices of  $W$ ,  $s_W$  and  $t_W$ , are the designated vertices of  $M$ . An RI 1-wiggle is depicted in Figure 5.3. We refer to the initial rectilinear  $i$ -wiggle that  $W$  was built upon as the *backbone* of  $W$ .

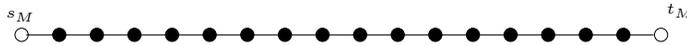


Figure 5.2: A rectilinear 2-wiggle.

The *partial  $i$ -tendril*  $P$  is an undirected graph, with four designated vertices,  $a_P$ ,  $b_P$ ,  $c_P$  and  $d_P$ . The partial 1-tendril  $P$  is the graph shown in Figure 5.4(a). The partial  $i$ -tendril  $P$  is constructed from the partial  $(i - 1)$ -tendril  $P_{i-1}$ , by attaching a partial 1-tendril  $P_1$  to  $P_{i-1}$ . To be exact, this

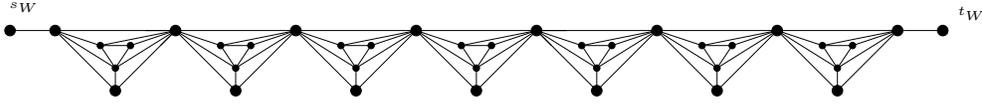


Figure 5.3: An RI 1-wiggle.

attachment is done by adding an edge connecting  $d_{P_1}$  with  $a_{P_{i-1}}$  and another edge connecting  $c_{P_1}$  with  $b_{P_{i-1}}$ . Then,  $a_P, b_P, c_P$  and  $d_P$  are  $a_{P_1}, b_{P_1}, c_{P_{i-1}}$  and  $d_{P_{i-1}}$  (see Figure 5.4(b)).

A *rectilinear  $i$ -tendrill*  $R$  is an undirected graph with two designated vertices  $s_R$  and  $t_R$ . The rectilinear  $i$ -tendrill  $R$  is constructed from the partial  $i$ -tendrill  $P$ , as follows:

- There are four vertices  $a, b, c$  and  $d$  in  $R$ , with edges  $(a, b)$  and  $(c, d)$ .
- There is an edge between  $a_P$  and  $a, b_P$  and  $b, c_P$  and  $c, d_P$  and  $d$ .
- There are two designated vertices  $s_R$  and  $t_R$  in  $R$ .
- There is an edge connecting  $s_R$  to  $a$  and an edge connecting  $t_R$  to  $d$ .

(See Figures 5.4(c) and 5.6.)

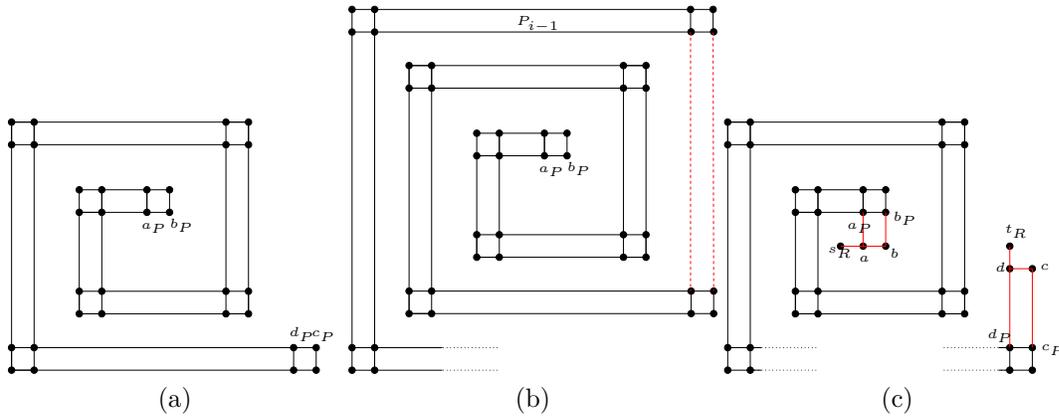


Figure 5.4: A partial 1-tendrill  $P$  (a), construction of a partial  $i$ -tendrill  $P$  from a partial 1-tendrill  $P_1$  and a partial  $(i - 1)$ -tendrill  $P_{i-1}$  (b) and the construction of a rectilinear  $i$ -tendrill from a partial  $i$ -tendrill (c).

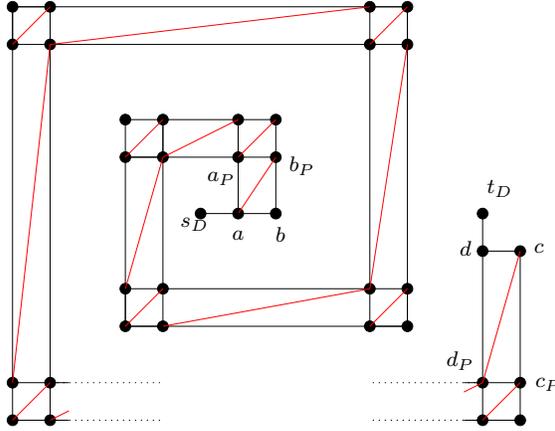


Figure 5.5: The construction of a diagonal  $i$ -tendrils from a rectilinear  $i$ -tendrils.

**Lemma 5.1** [15] *Any rectilinear  $i$ -tendrils  $R$  has exactly one planar embedding that places both  $s_R$  and  $t_R$  in the outer face.*

The *diagonal  $i$ -tendrils* is constructed from the rectilinear  $i$ -tendrils by adding a diagonal edge in each four-cycle such that:

- For any two four-cycles that share an edge, the diagonal edges inside them cover both ends of the common edge.
- A diagonal edge meets the vertex adjacent to the designated vertex  $s_R$  of the  $i$ -tendrils.

(See Figure 5.5.)

The *RI  $i$ -tendrils* is constructed from the diagonal  $i$ -tendrils by replacing each triangle by the graph  $\Pi$  depicted in Figure 5.1, such that:

- The edge  $(a_\Pi, c_\Pi)$  corresponds to the diagonal edge,
- The planar embedding of  $\Pi$  is maintained, i.e., the counter clockwise order is  $\{a_\Pi, b_\Pi, c_\Pi\}$  along the triangle.

(See Figure 5.6(b))

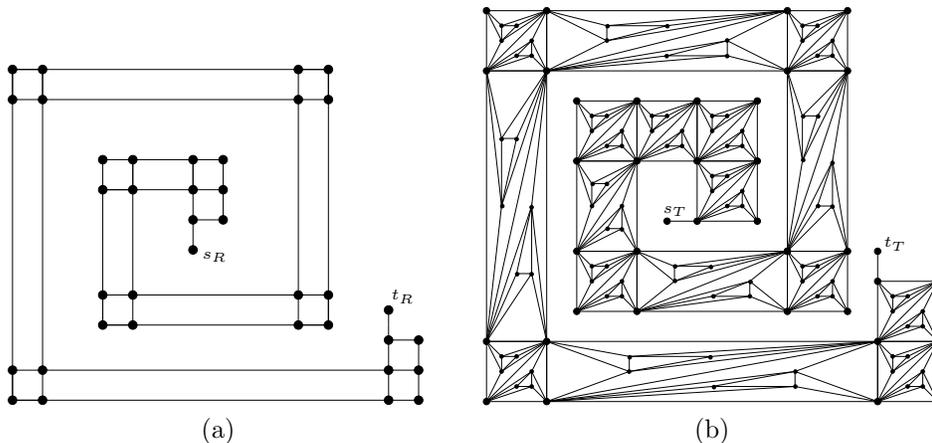


Figure 5.6: A rectilinear 1-tendrill  $R$  (a) and an RI 1-tendrill  $T$  (b).

**Lemma 5.2** *Any RI  $i$ -tendrill  $T$  has exactly one plane embedding that places both  $s_T$  and  $t_T$  in the outer face.*

**Proof.** We know that there exists a planar embedding of  $T$ , as our construction can produce one. Let  $T'$  be the rectilinear  $i$ -tendrill that was the base of construction of  $T$ . Adding the edge connecting  $s_{T'}$  and  $t_{T'}$  to  $T'$  will result in a subdivision of a triconnected graph. Since  $\Pi$  is triconnected, after adding the diagonal edges and copies of  $\Pi$ , the resulting graph would also be a subdivision of a triconnected graph, and hence will have a unique plane embedding.  $\square$

In following sections we construct a graph and substitute some of its edges with tendrills and wiggles. Then the truth assignment to each pair of literals in the NOT-ALL-EQUAL-3-SAT instance is done based on whether some tendrill is flipped or not.

Now we define a measure for the amount that a subgraph can “turn” in some sense. Presume a straight-line drawing is fixed. The *contribution* of a set of angles is the sum of the labels of the angles in the set minus twice of the size of the set. Let the *contribution* of a subgraph to a face  $f$  be the contribution of the set of angles of  $f$  that have both edges in the subgraph. The contribution roughly corresponds to the turn-angle, i.e., by how much

we deviate from the straight path at the angle. Note that in the drawing of an RI  $i$ -tendrils, the two paths on the outer face “spiral”; we prove this formally in the following lemma.

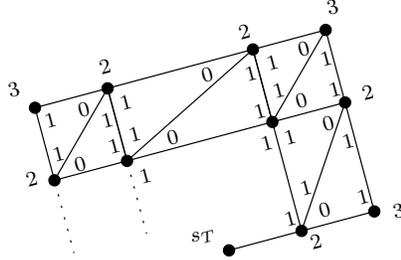


Figure 5.7: An RI-drawing of the frame graph of an RI tendril along with its AC labeling that is an RI-admissible labeling.

**Lemma 5.3** *Let  $A$  and  $B$  be the two maximal chains on the outer face of an RI  $i$ -tendrils that have no angle on  $s_R$  and  $t_R$  such that  $|A| < |B|$ . Then in any open RI-drawing, the contribution of  $A$  and  $B$  is in  $[8i - 3, 8i + 4]$  and  $[-8i - 4, -8i + 2]$ , respectively.*

**Proof.** Fix an arbitrary RI-drawing of the RI  $i$ -tendrils, and consider the induced RI-drawing of its frame graph (which is a diagonal  $i$ -tendrils) and graph  $\Pi$ . Let  $F_\Pi$  be the frame graph of the graph  $\Pi$ ;  $F_\Pi$  is a triangle. Any open RI drawing of  $\Pi$  induces the same angular labeling of  $F_\Pi$ , which is the one shown in the Figure 5.1(b). This holds because only one vertex of  $F_\Pi$ , is adjacent with all the vertices inside  $\Pi$ .

Therefore, by Lemmas 3.4 and 4.1, we have the unique AC labeling for  $F_\Pi$  and hence a unique AC labelling of the inner angles of the diagonal  $i$ -tendrils (see Figure 5.7). Therefore, we know that the labels assigned to the outside angles of the RI tendril are uniquely defined, except at the vertices  $s'_T$  and  $t'_T$  that are adjacent to  $s_T$  and  $t_T$ , respectively. Since the inner angles at  $t'_T$  and  $s'_T$  already have total label 1 because of the restrictions of filled triangles, therefore, each of the outer angles at  $t'_T$  and  $s'_T$  has label in  $[0, 3]$ ,

so can contribute anything between  $-2$  to  $1$  to the total contribution of  $A$  and  $B$ . As for the rest of the angles, by the fixed label of inner angles there are  $8i + 2$  angles of label 3 in  $B$  and  $8i$  angles of label 1 in  $A$ . The rest have label 2 which do not change the contribution. Therefore, the contribution of  $B$  is in  $[8i - 2, 8i + 4]$  and the contribution of  $A$  is in  $[-8i - 4, -8i + 2]$ , and the lemma holds.  $\square$

By *substituting* an edge  $(u, v)$  by a graph  $H$  with two designated vertices  $s_H$  and  $t_H$ , we mean identifying each of  $u$  and  $v$  with  $s_H$  and  $t_H$ , respectively, and then removing  $(u, v)$ . We later substitute some of the edges of a special 2-connected graph with wiggles and tendrils, such that the contribution of these wiggles and tendrils to the faces of that graph help us solve NOT-ALL-EQUAL-3-SAT.

**Lemma 5.4** *Let  $\Gamma_G$  be an RI-drawing of a 2-connected graph that has one of its edges substituted by an  $i$ -wiggly  $W$ . Then the contribution of  $W$  to each of its adjacent faces is in  $[-8i - 4, 8i + 4]$ .*

**Proof.** Let us look at the contribution of any (of the two) maximal chain of outer angles that does not have an element on  $s_W$  and  $t_W$ , in the drawing induced by the vertices of  $W$ . It is easy to see that the backbone of  $W$  (which is a path with  $8i + 1$  edges) is drawn oblique, since each triangle in  $W$  contains  $\Pi$  and therefore a clique of size 3 inside, and hence cannot have an axis aligned edge. By Lemma 3.12 on page 34 the sum of the labels of this backbone on either side (excluding the angles at  $s_W$  and  $t_W$ ) must be in  $[8i - 3, 3(8i) + 3]$ . This implies that the contribution of the backbone is in  $[-8i - 3, 8i + 3]$ . Now we consider the triangles attached to the backbone. Let us start by looking at the drawing induced by the vertices of the backbone. Attaching such triangle  $\{x, y, z\}$  to edge  $(x, y)$  of the backbone does not change the contribution to the outer face of this partial drawing (since it is always 4 by Lemma 3.1). Since the outer face consists of the two paths from  $s_W$  to  $t_W$ , and one of them was not changed when attaching  $\{x, y, z\}$ , both paths retain the same contribution after adding to one of them. Repeating the argument and attaching triangles one by one shows the result.  $\square$

## 5.2 The Intermediate Reduction

The reduction in this section is taken from [15].

Let  $\mathcal{I}$  be an instance of NOT-ALL-EQUAL-3-SAT with  $n$  pairs of literals  $\{x_i, \bar{x}_i\}$  and  $m$  clauses  $\{c_1, c_2, \dots, c_m\}$ . For  $1 \leq i \leq n$ , let  $\alpha_i$  and  $\beta_i$  denote the number of occurrences of  $x_i$  and  $\bar{x}_i$ , respectively. Let  $\theta = 74mn + 9$ . Also, we define  $\gamma_i = (2i - 1)\theta$  and  $\delta_i = 2i\theta$ . Let the graph  $\Phi$  be an undirected graph with a set of integer capacities  $s(e)$  associated to each of its edges  $e \in E(\Phi)$ , constructed as follows:

- For any literal  $x_i$  (resp.  $\bar{x}_i$ ), there is a *literal vertex*  $v_{x_i}$  (resp.  $v_{\bar{x}_i}$ ) in  $\Phi$ ,
- For any clause  $c_i$ , there is a *clause vertex*  $v_{c_i}$  in  $\Phi$ ,
- There is a dummy vertex  $z$  in  $\Phi$ ,
- For any pair of literals  $\{x_i, \bar{x}_i\}$  there is a literal-literal edge  $e = (v_{x_i}, v_{\bar{x}_i})$  in  $\Phi$  with  $s(e) = \{\alpha_i\gamma_i + \beta_i\delta_i\}$ .
- For any literal  $x_i$  and any clause  $c$ , there is a clause-literal edge  $e = (v_{x_i}, v_c)$  in  $\Phi$ . If  $x_i \in c$ ,  $s(e) = \{\gamma_i\}$ , otherwise,  $s(e) = \{0\}$ .
- For any literal  $\bar{x}_i$  and any clause  $c$ , there exists a clause-literal edge  $e = (v_{\bar{x}_i}, v_c)$  in  $\Phi$ . If  $\bar{x}_i \in c$ ,  $s(e) = \{\delta_i\}$ , otherwise,  $s(e) = \{0\}$ .
- For any clause  $c_j$ , there exists a clause-dummy edge  $e = (v_{c_j}, z)$  in  $\Phi$ . Also,  $s(e) = \{0, 1, \dots, \eta_j - 2\theta\}$ , where  $\eta_j$  is the sum of the members of the capacity sets of the clause-literal edges incident with  $v_{c_j}$ . Note that clause-literal edges have a single valid capacity, so  $\eta_j$  is well-defined.
- For any literal  $x_i$ , there is a literal-dummy edge  $e = (v_{x_i}, z)$  in  $\Phi$ , with  $s(e) = \{\beta_i\delta_i\}$ .
- For any literal  $\bar{x}_i$ , there is a literal-dummy edge  $e = (v_{\bar{x}_i}, z)$  in  $\Phi$ , with  $s(e) = \{\alpha_i\gamma_i\}$ .

**Theorem 5.2** [15] *Let  $\mathcal{D}(\mathcal{I})$  be the switch-flow instance corresponding to the graph  $\Phi$  and the capacity function  $s$ . Then  $\mathcal{D}(\mathcal{I})$  is satisfiable if and only if  $\mathcal{I}$  is satisfiable. Also, a satisfying solution for  $\mathcal{I}$  can be constructed from a satisfying solution of  $\mathcal{D}(\mathcal{I})$  in polynomial time.*

### 5.3 The Graphs $G_{RI}$ and $G_{OD}$

In this section we review the construction of  $G_{OD}$  which was used in [15] to prove NP-hardness of orthogonal drawability of planar graphs. In the end of the section we will introduce a new graph  $G_{RI}$ .

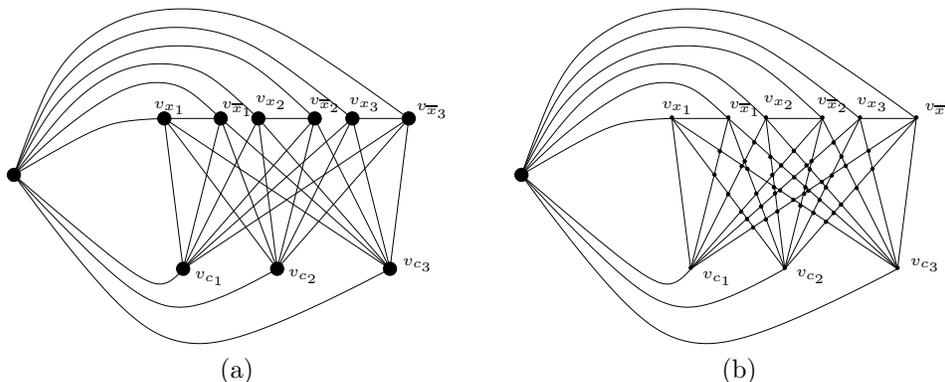


Figure 5.8: An example of the graph  $\Phi$  corresponding to a NOT-ALL-EQUAL-3-SAT with three variables and three clauses (a) and the corresponding graph  $P$  (b)

Let us construct a drawing  $\Gamma_\Phi$  of  $\Phi$  (see Figure 5.8(a)). Place all vertices associated with literals on a horizontal line, so that pairs of literals corresponding to the same boolean variable are adjacent on the line. Place all vertices associated with clauses on another horizontal line below the first one, such that no three of the clause-literal edges, when drawn as straight line segments, cross at the same point. Place  $z$  between the two lines, and to the left of all vertices. Draw all edges that are not incident to  $z$  as straight lines. Draw edges incident to  $z$  as curves that do not intersect any other edge. The crossings only occur on edges between clauses and literals. Also, only two edges pass through each crossing.

Now, construct a plane graph  $P$  from the drawing  $\Gamma_\Phi$ , by simply replacing each crossing with a vertex of degree 4, called a *crossing vertex* (see Figure 5.8(b)). Doing this, each clause-literal edge is replaced by a series of edges. We call these edges the *fragments* of the original edge. The capacity set of each fragment is inherited from the original edge. It is easy to verify that there is a solution on the integer switch-flow instance on  $P$ , if and only if

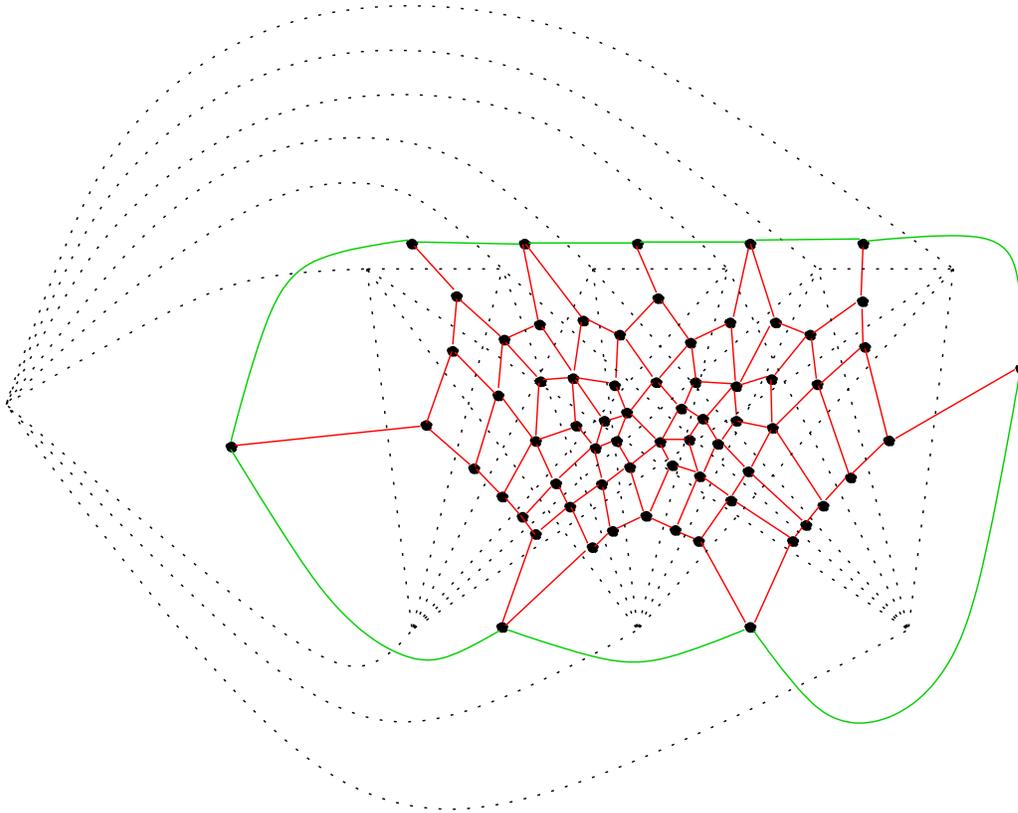


Figure 5.9: The dual-like  $D$  corresponding to the graph  $\Phi$  of Figure 5.8.

there is a valid solution on the integer switch-flow instance on  $\Phi$ . The reason is that by construction no two clause-literal edges share a common value in their capacities. Therefore, in any valid solution of the integer switch-flow instance, at each crossing vertex, the two fragments originating from the same edge will have the same direction.

**Lemma 5.5** [15]  $P$  is *triconnected*.

Note that this means that there is a unique dual  $\mathcal{D}(P)$  of  $P$ . Let  $D$  be the graph  $\mathcal{D}(P)$  (see Figure 5.9). Construct the graph  $G_{OD}$  from the graph  $P$  as follows:

First, replace each vertex of  $D$  by a binary tree with  $d$  leaves so that each of its neighbors in  $D$  is adjacent with one of the leafs of the binary tree. Let  $D'$  be the result of this modification. Then substitute some edges of  $D'$  with tendrils and wiggles as follows. Let  $e$  be an edge of  $D'$  that also is an edge of  $D$  (i.e., not part of the binary trees that were added.) Let  $e'$  be its dual in  $P$ .

- If  $e'$  is a clause-dummy edge then we know  $s(e') = \{1, \dots, c\}$  for some  $c$ . Substitute  $e$  with a rectilinear  $c$ -wiggle  $W$ .
- Otherwise  $s(e')$  has only one entry, say  $s(e') = \{c\}$  for some  $c$ . Substitute  $e$  with a rectilinear  $c$ -tendril  $T_c$ .

We refer to the vertices and faces of  $G_{OD}$  that correspond to vertices and faces of  $D'$  as the *primary* vertices and faces of  $G_{OD}$ , respectively.

$G_{RI}$  is constructed similarly, with the difference that we use RI tendrils and RI wiggles instead of rectilinear tendrils and rectilinear wiggles, respectively. Note that each RI tendril (resp. wiggle) contains a rectilinear tendril (resp. wiggle) as a subgraph, so  $G_{OD}$  is a subgraph of  $G_{RI}$ .

**Lemma 5.6** [15] *All the embeddings of  $G_{OD}$  are obtained by choosing one of the two possible flips for each rectilinear tendril.*

**Corollary 5.1** *All the embeddings of  $G_{RI}$  are obtained by choosing one of the two possible flips for each RI tendril and fixing the embedding of each RI wiggle.*

So the choice of how to embed/flip each wiggle or tendril corresponds to choosing how to direct the corresponding edge in the switch-flow problem, and the difficulty will be to show that any RI-drawing yields a solution that maintains balances at vertices.

## 5.4 From $\Gamma_{OD}$ to $\Gamma_{RI}$

In this section, we prove that for every orthogonal drawing of  $G_{OD}$ , there is an open RI drawing of  $G_{RI}$ . In fact, we prove that for any OD-admissible

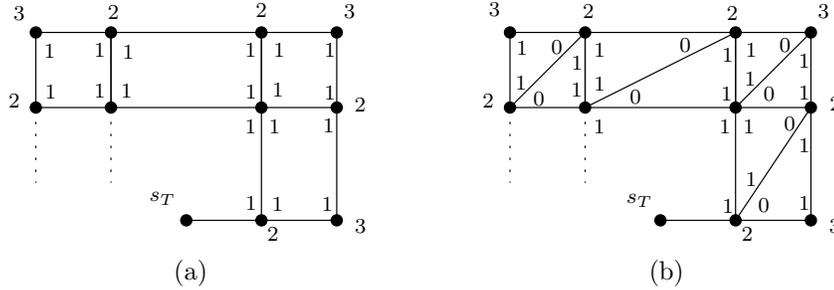


Figure 5.10: The OD-admissible labeling of a rectilinear tendril (a) and extracting RI-admissible labeling for the diagonal tendril (b).

labeling of  $G_{OD}$ , there is an open RI drawing of  $G_{RI}$  with non-aligned frame. This way, we can show that if  $\mathcal{I}$  is satisfiable, then  $G_{RI}$  is open RI drawable.

We do this by transforming an OD-admissible labeling  $\ell_{OD}(\cdot)$  of  $G_{OD}$  to an RI-admissible labeling  $\ell_{RI}(\cdot)$  of  $G_{RI}$ . Let  $H'$  be the subgraph of  $G_{RI}$  that is isomorphic to  $G_{OD}$  as described in previous sections. Fix the embedding of  $H'$  based on the embedding of  $G_{OD}$ . If an angle appears in both  $H'$  and  $G_{OD}$  assign the same label to it in  $H'$  as it has in the OD-admissible labeling of  $G_{OD}$ . Next, we add the diagonals of the RI tendrils. Each 4-cycle in  $G_{OD}$  must have 4 angles labelled 1 in the OD-admissible labeling. As shown in Figure 5.10, we can then assign labels 0 and 1 to the sides of the diagonals so that the constraints forced by the filled triangles are satisfied.

Next we must label the angles of the triangles of the frame of an RI-wiggle  $M$ . Let  $M'$  be the backbone of  $M$ , which is a wiggle in  $H'$  (see Figure 5.11(a)). Let  $u$  be the neighbor of  $s_T$  in  $M'$ . Let  $v$  be the other neighbor of  $u$  in  $M'$ . Let  $\alpha$  be an angle at  $u$  with label larger than 1. There exists such an  $\alpha$  since the sum of labels around  $u$  is 4. Then, we embed the triangle of  $M$  at  $(u, v)$  on the side of  $M'$  where  $\alpha$  is. Let the angles inside the triangle be labelled 1 at  $u$ , 0 at  $v$ , and 1 at the third vertex; this satisfies the condition for the filled triangle (see Figures 5.11(c)). Also, the remaining angle on this side of  $v$  maintains its label, so there is still a label of value greater than 1 at  $v$ . Therefore, we can iterate to the next edge on the path from  $s_T$  to  $t_T$  and replace it with a triangle (see Figure 5.11(d)). We do this iteratively for all edges of the back bone that have no end in  $\{s_T, t_T\}$ .

**Lemma 5.7** *The labeling of the frame graph of  $G_{RI}$  as constructed above is*

*RI-admissible.*

**Proof.** The construction preserves the total labels around each vertex. Also, for each vertex added to a face, the total label around that face is increased by 2. The labels of the angles of the binary trees with primary vertices has not been changed, and therefore they contain no label 0. Also, except for the inside of triangles, no label of 0 is placed on a face, meaning there is no chain with labels of the form  $01^*0$ . The only property that remains to be verified is that the labeling respects the inside of filled triangles, i.e., in each triangle the vertex that is not adjacent to all vertices inside that triangle has label 1. This property holds by the way labels 011 were assigned to angles of the frame of instances of the graph  $\Pi$ .  $\square$

## 5.5 From $\Gamma_{RI}$ to an Integer Switch-flow Solution

In this section, we will construct a valid solution for the integer switch-flow of  $P$ , and hence  $\Phi$ , from an open RI drawing  $\Gamma_{RI}$  of  $G_{RI}$ . The proof is similar to the ones proposed in [15] except that we need different constants since the contribution of RI  $i$ -tendrils fluctuates more. We will assign directions to edges of  $\Phi$  by looking at the tendrils of  $G_{RI}$  in  $\Gamma_{RI}$ . Recall that by Lemma 5.3 each side of an RI  $i$ -tendrils  $T$  has contribution either roughly  $8i$  or roughly  $-8i$ . We use this to model the direction and amount of flow in the edge of  $P$  that corresponds to  $T$  that has a single valid capacity value  $i$ . Also, based on the shape of the RI wiggles in  $\Gamma_{RI}$ , we will assign directions and flow to the clause-dummy edges of  $P$ . Recall that by Lemma 5.4 the contribution of either sides of an RI  $i$ -wiggles  $W$  is roughly between  $8i$  and  $-8i$ . This will help us assign direction and flow to the corresponding edge of  $P$  that has capacity  $[-i, i]$ .

Recall that by Lemma 5.3 the contribution of an RI  $i$ -tendrils of  $G_{RI}$  to a primary face is either between  $8i - 2$  and  $8i + 4$  or between  $-8i - 4$  and  $-8i + 2$ . Similarly, by Lemma 5.4 the contribution of an RI  $i$ -wiggles to a primary face of  $G_{RI}$  is between  $-8i - 3$  and  $8i + 3$ .

For each  $k$ -tendrils  $T$  in  $G_{RI}$ , let the *significant contribution* of  $T$  to  $f$  be  $8k$  if its contribution to  $f$  is between  $8k - 2$  and  $8k + 4$ , and  $-8k$  otherwise.

This means that by Lemma 5.3 the difference between the contribution and the significant contribution of a tendril is at most 4. Also, let the significant contribution of a wiggle  $W$  to a face  $f$  be its contribution to  $f$  rounded to the closest multiple of 8. Clearly the difference between the contribution and the significant contribution of a wiggle is also at most 4.

**Lemma 5.8** *If  $n \geq 3$  and  $m \geq 3$ , then the magnitude of the total contribution of primary vertices to a nondummy face is at most  $70nm$  and at least  $-70nm$ .*

**Proof.** Garg *et al.* [15] proved that the total number of primary vertices on a nondummy face is at most  $35nm$ . Since each primary vertex contributes at most 2 and at least  $-2$  to each face, the total contribution of primary vertices to a nondummy face is bounded by  $-70nm$  and  $70nm$ .  $\square$

**Lemma 5.9** *Let  $f$  be a primary face of  $G_{RI}$  that is not the outer face. Let  $\tau(f)$  and  $\omega(f)$  be the total significant contribution of tendrils and wiggles of  $f$  respectively. Then*

$$|\tau(f) + \omega(f)| < \theta$$

where  $\theta$  is  $74mn + 9$ .

**Proof.** Let  $\tau'(f)$  and  $\omega'(f)$  be the total contribution of tendrils and wiggles of  $f$  respectively. Let  $\nu(f)$  be the number of primary vertices on the boundary of  $f$ . Since  $G_{RI}$  is biconnected, each such a vertex has exactly one angle in  $f$ . Let  $\nu'(f)$  be the sum of the labels of these angles minus  $2\nu(f)$ . Then, by Lemma 3.1, we have  $\tau'(f) + \omega'(f) + \nu'(f) = -4$ . Since by construction of  $P$  and  $D$  each face has at most  $nm$  tendrils, and the difference of the contribution and significant contribution of a tendril is at most 4,  $|\tau'(f) - \tau(f)| \leq 4mn$ . Also, since any face except the outer face has at most 1 wiggle,  $|\omega'(f) - \omega(f)| \leq 4$ . By Lemma 5.8 we know  $|\nu'(f)| \leq 70nm$ .

Putting it all together, we have:

$$\begin{aligned} |\tau(f) + \omega(f)| &\leq |\tau'(f) + \omega'(f) + \nu'(f)| \\ &\quad + |\tau'(f) - \tau(f)| + |\omega'(f) - \omega(f)| + |-\nu'(f)| \\ &\leq 4 + 4mn + 4 + 70nm \end{aligned}$$

which concludes the lemma.  $\square$

**Lemma 5.10** *From any open RI drawing of  $G_{RI}$ , a valid solution to the integer switch-flow instance of  $\Phi$  can be constructed.*

**Proof.** Let the flow of each edge  $e$  to each vertex  $v$  in  $P$  be the same as the significant contribution of the dual edge  $e'$  of  $e$  to the dual face  $v'$  of  $v$  in  $G_{RI}$ , divided by 8. By Lemma 5.3 and definition of significant contribution of tendrils, the amount of the flow assigned to each edge corresponding to a tendril is a valid range of values assigned to that edge. By Lemma 5.4 and the definition of significant contribution of wiggles, the amount of the flow assigned to each edge corresponding to a wiggle is within the valid value assigned to that edge. Also, by Lemma 5.9 we know that the sum of the flow to all but one of the vertices is 0, and hence is 0 to all vertices. This means that the constructed switch-flow network is valid. Hence a valid solution for switch-flow network of  $\Phi$  can be constructed based on the flips of the tendrils and the labels of the angles in the open RI drawing of  $G_{RI}$ .  $\square$

This proves the other direction of the reduction and hence NP-completeness of testing whether a planar graph has a planar weak open RI-drawing.

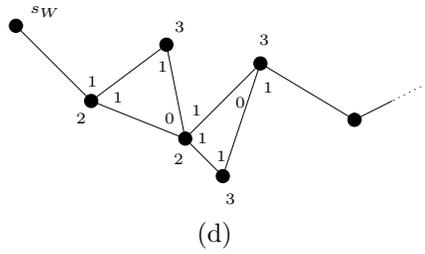
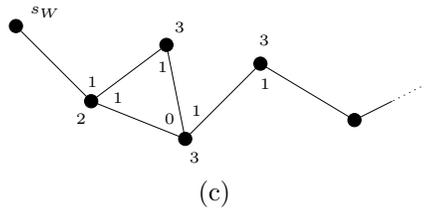
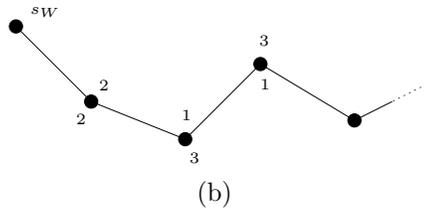
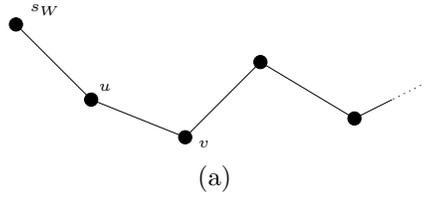


Figure 5.11: The backbone of an RI wiggle  $W$  (a) and its corresponding labeling (b) and the two first iterations of obtaining an RI-admissible labeling for  $W$  (c and d).

# Chapter 6

## Conclusion

In this chapter, we review and conclude the work done in this thesis. In Section 6.1 we will discuss different types of RI drawings that have been studied here, and look at some examples that distinguish between these types. Then, in Section 6.2 we will review some of the results presented in this document, and also propose some interesting open problems.

### 6.1 Three Models of RI-drawings

In this thesis, we have studied three types of planar weak open RI-drawings: those without any restrictions on coordinates; those where the frame is oblique; and those where the frame is non-aligned. Clearly every non-aligned RI-drawing is oblique, and any of them is an unrestricted RI-drawing. We now briefly discuss that these models are truly different in the sense that there are graphs that have a drawing in one model, but not the other.

**Lemma 6.1** *There exists an inner triangulated graph that has a planar weak open RI drawing, but admits no planar weak open RI drawing with oblique frame.*

**Proof.** Figure 6.1 shows an example of an open RI drawing of a graph that admits no open RI drawing in which the frame graph is oblique. For assume it did, and consider the induced AC labeling of the frame which then uses

only integer labels. Note that the label at angle  $\alpha$  must be 1, for the incident vertex must have label-sum 4, and every inner angle has AC label 0 or 1. Similarly the angle  $\beta$  must have AC label 1, forcing AC label 0 onto  $\gamma$ . But the vertex at  $\gamma$  is not adjacent to all vertices inside the separating triangle and so must not have AC label 0. Contradiction.  $\square$

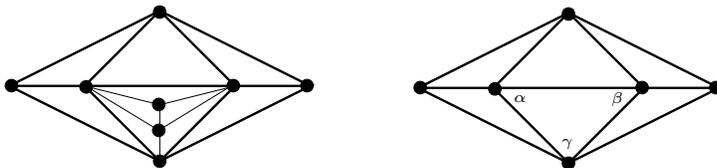


Figure 6.1: An RI drawable graph admitting no RI drawing with oblique frame.

**Lemma 6.2** *There exists an inner triangulated graph that has a planar weak open RI drawing with oblique frame, but does not admit a planar weak open RI drawing with non-aligned frame.*

**Proof.** Figure 6.2 shows a graph (and its frame) that has a planar weak open RI-drawing with oblique frame. Assume we could draw it with non-aligned frame. Observe that the structure of the separating triangles forces all the AC labels in the frame as indicated. Since the AC labels at a vertex sum to 4, therefore  $\alpha_0$  and  $\alpha_2$  have AC label 0, while  $\alpha_1$  has AC label 1. Hence the outer-face has a sub-sequence 010 among its AC labels, which is impossible in a non-aligned drawing (Corollary 3.2.)  $\square$

However, as was shown implicitly in Lemma 4.4, if a graph has an oblique RI-drawing such that all outer angles have AC label 2,3, or 4, then the drawing can be converted to a non-aligned RI-drawing with the same AC labels by extracting the good labeling. So the concepts of oblique and non-aligned coincide if the outer-face is “nice” in the sense of consisting of four chains that are  $x$ -monotone and  $y$ -monotone.

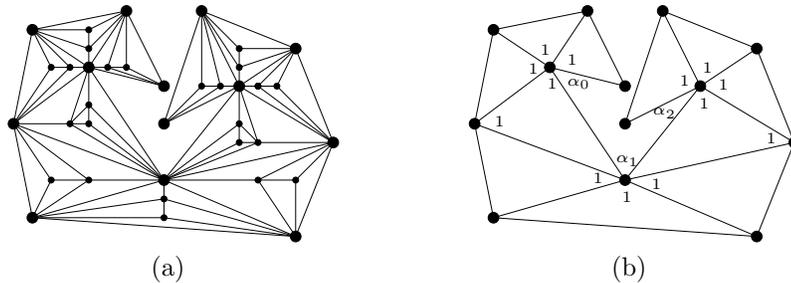


Figure 6.2: A graph that has an RI-drawing with an oblique frame (a), but the forced AC labels in the frame (b) force some vertices to be aligned.

## 6.2 Review and Open Problems

We have presented an algorithm to find a planar weak open RI drawing with non-aligned frame of a given planar graph  $G$  with fixed embedding, if there exists such a drawing. We also characterized existence of such drawings in terms of properties of AC labelings.

Also, we showed that if the embedding is not fixed, the problem of deciding if the given graph has a planar weak open RI drawing is NP-complete.

Our results also imply a correspondence between non-aligned planar weak RI-drawings and a class of orthogonal drawings. Theorem 4.2 shows that any non-aligned planar weak RI-drawing can be converted to a rectangular dual drawing with the same inner structure. Steps (v)-(x) of our algorithm (see Section 4.2) show that any rectangular dual drawing can be converted to a non-aligned planar weak RI-drawing, that preserves the inner structure. So apart from modifications near the outer-face (rectangles can “slide outward”), there is a one-to-one correspondence between non-aligned planar weak RI-drawings and rectangular dual drawings.

The most pressing open problem is what happens when we want to drop “with non-aligned frame” for planar weak open RI-drawings of plane graphs. Can we efficiently test whether a given plane graph has a weak open planar RI-drawing? It is quite easy to find necessary conditions for this problem, but are they sufficient? And if they are sufficient, how easy is it to test whether a graph has a labeling that satisfies these conditions? Neither of these questions appears straight forward to answer.

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