Variations on the Erdős Discrepancy Problem

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The Erdős discrepancy problem asks, "Does there exist a sequence $t = \{t_i\}_{i=1}^{\infty}$ with each $t_i \in \{-1, 1\}$ and a constant c such that $|\sum_{i=1}^{n} t_{id}| \leq c$ for all $n, d \in \mathbb{N} = \{1, 2, 3, \ldots\}$?" The discrepancy of t equals $\sup_{n \geq 1} |\sum_{i=1}^{n} t_{id}|$. Erdős conjectured in 1957 that no such sequence exists [4].

We examine versions of this problem with fixed values for c and where the values of d are restricted to particular subsets of $\mathbb{N}$. By examining a wide variety of different subsets, we hope to learn more about the original problem.

When the values of d are restricted to the set $\{1, 2, 4, 8, \ldots\}$, we show that there are exactly two infinite $\{-1, 1\}$ sequences with discrepancy bounded by 1 and an uncountable number of infinite $\{-1, 1\}$ sequences with discrepancy bounded by 2. We also show that the number of $\{-1, 1\}$ sequences of length n with discrepancy bounded by 1 is $2^{s_2(n)}$ where $s_2(n)$ is the number of 1s in the binary representation of n.

When the values of d are restricted to the set $\{1, b, b^2, b^3, \ldots\}$ for $b > 2$, we show there are an uncountable number of infinite sequences with discrepancy bounded by 1. We also give a recurrence for the number of sequences of length n with discrepancy bounded by 1. When the values of d are restricted to the set $\{1, 3, 5, 7, \ldots\}$ we conjecture that there are exactly 4 infinite sequences with discrepancy bounded by 1 and give some experimental evidence for this conjecture.

We give descriptions of the lexicographically least sequences with D-discrepancy c for certain values of D and c as fixed points of morphisms followed by codings. These descriptions demonstrate that these automatic sequences.

We introduce the notion of discrepancy-1 maximality and prove that $\{1, 2, 4, 8, \ldots\}$ and $\{1, 3, 5, 7, \ldots\}$ are discrepancy-1 maximal while $\{1, b, b^2, \ldots\}$ is not for $b > 2$.

We conclude with some open questions and directions for future work.

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Chapter 1

The Erdős Discrepancy Problem and Background

The Erdős discrepancy problem asks, "Does there exist a sequence $\{t_i\}_{i=1}^{\infty}$ with each $t_i \in \{-1, 1\}$ and a constant c such that $|\sum_{i=1}^{n} t_{id}| \leq c$ for all $n, d \in \mathbb{N} = \{1, 2, 3, \ldots\}$?" [4]

In January, 2010, the PolyMath group began investigating the Erdős discrepancy problem as their PolyMath5 project [6]. While the question remains open, the PolyMath group has discovered many significant experimental results, including a sequence t of length 1124 with $|\sum_{i=1}^{n} t_{id}| \leq 2$ for all $d \in \mathbb{N}$ and $n \leq 1124/d$. They have also discovered an infinite sequence t such that $|\sum_{i=1}^{n} t_{id}| \leq \log_9(nd)$ for all $d \in \mathbb{N}$ and sufficiently large n [6].

In order to discuss variations on this problem, we will introduce some notation and use this notation to restate the problem.

1.1 Sequences

A sequence is a finite or infinite ordered list of symbols from an set Σ. The set Σ is called the alphabet. The Erdős discrepancy problem is a question about infinite sequences over the alphabet $\Sigma = \{-1, 1\}$. In this thesis we will consider both finite and infinite length sequences and all sequences will be over the alphabet $\Sigma = \{-1, 1\}$

unless otherwise stated. All sequences are indexed over $\mathbb{N} = \{1, 2, 3, \dots\}$ which means that if t is a sequence then t_1 is the first term, t_2 is the second term, and so on.

If t is a finite sequence then t^ω is the infinite sequence resulting from repeating t over and over. For example if $t = (-1, -1, 1, 1)$ then $t^\omega = (-1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, \dots)$. Writing t^α with $\alpha \in \mathbb{Q}$ means the longest prefix of t^ω of length at most $\alpha|t|$ where $|t|$ denotes the length of t. For example if $t = (-1, 1)$ then $t^{3/2} = (-1, 1, -1)$.

1.2 Morphisms

A map on sequences ϕ is a morphism if it has the property that for any sequence $t = (t_1, t_2, t_3, \dots)$

$$\phi(t_1, t_2, t_3, \dots) = (\phi(t_1), \phi(t_2), \phi(t_3), \dots).$$

The action of a morphism can be fully described by describing its action on each member of the alphabet. For example, we can define a morphism as follows:

$$\begin{aligned} -1 &\mapsto -1, -1, 1 \\ 1 &\mapsto 1, 1 \end{aligned}$$

Let $a \in \Sigma$. If

1. $\phi(a) = ax$ for some x and
2. $\phi^n(x) \neq \epsilon$ for all $n \geq 0$

then the limit $\lim_{n\to\infty} \phi^n(a)$ exists and is equal to $ax\phi(x)\phi^2(x)\phi^3(x)\cdots$ [2]. We denote this limit by $\phi^\omega(a)$ and call it the fixed point of ϕ on a.

A morphism with $|\phi(a)| = 1$ for all $a \in \Sigma$ is called a coding.

1.3 Discrepancy

We will now define the discrepancy of a sequence and restate the Erdős discrepancy problem using this definition.

Definition 1. a) Let $D \subseteq \mathbb{N} = \{1, 2, 3, \dots\}$ and $c \in \mathbb{N}$. A finite integer sequence $\{t_i\}_{i=1}^{N}$ has D-discrepancy c if

$$\left| \sum_{i=1}^{k} t_{di} \right| \leq c$$

for all $d \in D$ and for all k with $0 < k \leq N/d$.

b) Let $D \subseteq \mathbb{N}$ and $c \in \mathbb{N}$. An infinite integer sequence $\{t_i\}_{i=1}^{\infty}$ has D-discrepancy c if

$$\left| \sum_{i=1}^{k} t_{di} \right| \leq c$$

for all $d \in D$ and for all $k > 0$.

Note that if a sequence has D-discrepancy c then it also has D-discrepancy c' for all $c' > c$.

The Erdős discrepancy question asks if there exists an infinite sequence with finite $\mathbb{N}$-discrepancy. We will examine variations on this question where instead of requiring the sequence to have finite $\mathbb{N}$-discrepancy, we will require the sequence to have D-discrepancy bounded by c for some $D \subseteq \mathbb{N}$ and some $c \in \mathbb{N}$.

We will count the number of infinite sequences with $\{1, b, b^2, \dots\}$-discrepancy c for $b > 1$ and $c \leq 2$. We will also count the number of finite sequences of length n with $\{1, b, b^2, \dots\}$-discrepancy 1 for $b > 1$. These results are summarized in Table 8.1. We will describe the lexicographically least infinite sequences with $\{1, b, b^2, \dots\}$-discrepancy c for $b > 1$ and $c \leq 2$ as well as the lexicographically least infinite sequence with $\{1, 2, 4, 8, \dots\}$-discrepancy 3. Discrepancy-1 graphs will be used to prove that $\{1, 3, 5, 7, \dots\}$ is a discrepancy-1 maximal set and that $\{1, b, b^2, \dots\}$ is discrepancy-1 maximal only when $b = 2$. Discrepancy-1 graphs will also be used to state a conjecture about the number of infinite sequences with $\{1,3,5,7,\dots\}$-discrepancy 1.

In order to simplify the proofs to follow, we now define homogeneous arithmetic progression subsequences and absolute running sums. The d-homogeneous arithmetic progression subsequence is the subsequence obtained by considering at every dth term. The absolute running sum is the absolute value of the sum of the terms in a sequence. Intuitively, the discrepancy of a sequence is calculated by taking the absolute running sum of the homogeneous arithmetic progression subsequence. We formalize this as follows.

Definition 2. Let $\{t_i\}_{i=1}^{\infty}$ be an integer sequence and $d \in \mathbb{N}$. The d-homogeneous arithmetic progression subsequence (d-HAPS) of t is

$$\{t_{id}\}_{i=1}^{\infty}$$

Definition 3. Let $\{t_i\}_{i=1}^{\infty}$ be an integer sequence and $n \in \mathbb{N}$. The absolute running n-sum of t is

$$\left|\sum_{i=1}^{n} t_i\right|$$

Remark 4. Let $D \subseteq \mathbb{N}$, t be an integer sequence, and $c \in \mathbb{N}$. The sequence t has D-discrepancy c iff for all $d \in D$ and for all $n \in \mathbb{N}$, the absolute running n-sum of the d-HAPS of t is at most c.

For example, consider the sequence $(-1,1,1,1,1,-1,-1)$. The 2-HAPS of this sequence is $(1,1,-1)$ and the absolute running $1,2,3$-sums of the 2-HAPS are 1, 2, and 1 respectively. The 3-HAPS of the sequence is $(1,-1)$ and the absolute running $1,2$-sums of the 3-HAPS are 1 and 0 respectively. The largest absolute running sum is 2, therefore the sequence has $\{2,3\}$-discrepancy 2 but not $\{2,3\}$-discrepancy 1.

Chapter 2

Powers of 2 Discrepancy

In this chapter, we will focus on the D-discrepancy of sequences when $D = \{1, 2, 4, 8, \dots\}$. We will show that there are exactly 2 infinite sequences over $\{-1, 1\}$ with D-discrepancy 1 and an uncountable number of such sequences with D-discrepancy 2. We go on to count the number of sequences of length n with D-discrepancy 1. Finally, we will describe the lexicographically least infinite sequences with D-discrepancy 3.

Lemma 5. *An infinite sequence $\{t_i\}_{i=1}^{\infty}$ over $\{-1, 1\}$ has D-discrepancy 1 iff for all $d \in D$ and odd n,*

$$t_{nd} = -t_{(n+1)d}.$$

Proof. ($\Rightarrow$)

Suppose that t has D-discrepancy 1. Let $d \in D$ and n be odd. Since t has D-discrepancy 1,

$$\left| \sum_{i=1}^{n-1} t_{id} \right| \leq 1$$

and since n is odd,

$$\sum_{i=1}^{n-1} t_{id} \equiv 0 \pmod{2}.$$

Therefore, $\sum_{i=1}^{n-1} t_{id} = 0$. To get a contradiction, suppose that $t_{nd} = t_{(n+1)d}$. Then

$$\begin{aligned}\left|\sum_{i=1}^{n+1} t_{id}\right| &= \left|\sum_{i=1}^{n-1} t_{id} + t_{nd} + t_{(n+1)d}\right| \\ &= |0 + t_{nd} + t_{nd}| \\ &= |2t_{nd}| \\ &= 2.\end{aligned}$$

This is a contradiction because t has D-discrepancy 1. Therefore, $t_{nd} = -t_{(n+1)d}$.

($\Leftarrow$)

Consider the absolute running n-sum of the d-HAPS of t for an arbitrary $d \in D$ and $n \in \mathbb{N}$. If n is even then

$$\begin{aligned}\left|\sum_{i=1}^{n} t_{id}\right| &= \left|\sum_{i=1}^{n/2} (t_{(2i-1)d} + t_{2id})\right| \\ &= 0.\end{aligned}$$

If n is odd then

$$\begin{aligned}\left|\sum_{i=1}^{n} t_{id}\right| &= \left|\sum_{i=1}^{(n-1)/2} (t_{(2i-1)d} + t_{2id}) + t_{nd}\right| \\ &= |t_{nd}| \\ &= 1.\end{aligned}$$

Therefore the absolute running sums of the d-HAPS of t are bounded by 1 for all $d \in D$ so the D-discrepancy of t is 1. □

2.1 Discrepancy of Infinite Sequences

In this section we will describe the two infinite sequences with $\{1, 2, 4, 8, \dots\}$-discrepancy 1. We then go on to prove that there are an uncountable number of infinite sequences with $\{1, 2, 4, 8, \dots\}$-discrepancy 2. This demonstrates that while having discrepancy 1 is a very restrictive condition that only a small number of sequences satisfy, a very large number of sequences satisfy the slightly relaxed condition of having discrepancy 2.

Definition 6. The Thue-Morse sequence, denoted $\mathbf{t}$, is the fixed point of the morphism μ defined by

$$\begin{aligned} -1 &\mapsto -1, 1 \\ 1 &\mapsto 1, -1 \end{aligned}$$

iterated on 1. That is,

$$\mathbf{t} = \mu^{\omega}(1) = 1, -1, -1, 1, -1, 1, 1, -1, \dots$$

The Thue-Morse sequence was originally discovered by Axel Thue in 1906. Since the discovery of the Thue-Morse sequence, it has had applications in many seemingly unrelated branches of mathematics. For a summary, see [1, 3]. We will show that the Thue-Morse sequence and its complement are the only infinite sequences with $\{1, 2, 4, 8, \dots\}$-discrepancy 1.

Theorem 7. *Let $D = \{1, 2, 4, 8, \dots\}$. The only infinite sequences over the set $\{-1, 1\}$ with D-discrepancy 1 are $\mathbf{t} = \mu^{\omega}(1)$ and $-\mathbf{t} = \mu^{\omega}(-1)$.*

Proof. Let $\mathbf{s}$ be an infinite sequence over $\{-1, 1\}$ with D-discrepancy 1. By Lemma 5, we have that $\mathbf{s}$ can be split into blocks of $(-1, 1)$ and $(1, -1)$.

Now we claim that if a sequence s has D-discrepancy 1, then $\mu^{-1}(s)$ also has D-discrepancy 1. Note that μ^{-1} is well-defined on s because s can be split as above. To get a contradiction, let $s' = \mu^{-1}(s)$ and suppose that there exists $d \in D$ and $n \in \mathbb{N}$

such that $|\sum_{i=1}^{n} s'_{di}| > 1$. By the construction of μ, we have $s'_i = -s_{2i}$. Therefore,

$$\begin{aligned}\left|\sum_{i=1}^{N} s'_{di}\right| &> 1\\ \left|\sum_{i=1}^{N} -s_{2di}\right| &> 1\\ \left|\sum_{i=1}^{N} s_{2di}\right| &> 1.\end{aligned}$$

Because $2d \in D$, this shows that s does not have D-discrepancy 1, a contradiction. Thus, μ^{-1} preserves the property of having D-discrepancy 1.

We now show that $\mathbf{s} = \pm\mathbf{t}$ by showing that every prefix of $\mathbf{s}$ of length 2^i is $\mu^i(1)$ or $\mu^i(-1)$. Consider repeated applications of μ^{-1} to $\{s_j\}_{j=1}^{2^i}$. Before each application, the sequence has D-discrepancy 1 and so it can be split into blocks of $(-1, 1)$ and $(1, -1)$ by Lemma 5. Thus, μ^{-1} is well-defined on the sequence. Hence, $\mu^{-1}(\{s_j\}_{j=1}^{2^i})$ will also have D-discrepancy 1, and it can be split again. This can be repeated, with each application halving the length of the sequence until, after i applications, we must be left with one of the two sequences of length 1. Hence,

$$\mu^{-i}(\{s_j\}_{j=1}^{2^i}) = \pm 1$$

and

$$\{s_j\}_{j=1}^{2^i} = \mu^i(\pm 1),$$

as required. Thus, by letting $i \to \infty$, we see any sequence with D-discrepancy 1 must have a prefix that is $\mu^i(\pm 1)$ for all i. Therefore, the only infinite sequences over $\{-1, 1\}$ with D-discrepancy 1 are $\pm\mathbf{t}$. □

Theorem 8. *Let $D = \{1, 2, 4, 8, \dots\}$. There are an uncountable number of infinite sequences over $\{-1, 1\}$ with D-discrepancy 2.*

This proof is based on an idea due to Kevin Hare.

Proof. We will give an injective mapping from the power set of the naturals to the set of infinite sequences over $\{-1,1\}$ with D-discrepancy 2, thereby showing that there are uncountably many such sequences. Define the morphism σ as follows:

$$\begin{aligned} -1 &\mapsto -1,-1 \\ 1 &\mapsto 1,1. \end{aligned}$$

Let $K \subseteq \mathbb{N}$ be a subset of the naturals. Then for each $i \geq 1$ we define the morphism f_i as follows:

$$\begin{aligned} f_1 &= \mu \\ f_{i+1} &= \begin{cases} f_i \circ \mu \circ f_i, & \text{if } i+1 \in K; \\ f_i \circ \sigma \circ f_i, & \text{if } i+1 \notin K. \end{cases} \end{aligned}$$

Let K map to $\lim_{i\to\infty} f_i(1)$. We claim that:

1. For each $K \subseteq \mathbb{N}$, the limit $\lim_{i\to\infty} f_i(1)$ exists and is an infinite sequence.
2. For each $K \subseteq \mathbb{N}$, the infinite sequence $\lim_{i\to\infty} f_i(1)$ has D-discrepancy 2.
3. No two distinct subsets of the naturals map to the same infinite sequence.

This suffices to show that the set of infinite sequences with D-discrepancy 2 is uncountable.

Proof of Claim 1. We will show that $f_i(1)$ is a proper prefix of $f_{i+1}(1)$ for each $i > 1$, thereby showing that $f_i(1)$ converges to an infinite sequence as $i \to \infty$.

We prove this by induction on i.

Base case: $i = 1$.

We have that

$$f_2(1) = \mu \circ \mu \circ \mu(1) = (1,-1,-1,1,-1,1,1-1)$$

or

$$f_2(1) = \mu \circ \sigma \circ \mu(1) = (1,-1,1,-1,-1.1,-1,1).$$

In both cases, $f_2(1)$ begins with $(1,-1) = f_1(1)$.

Inductive case: $i \geq 2$.

By the induction hypothesis, $f_{i-1}(1)$ has $f_{i-2}(1)$ as a prefix. Repeatedly applying the induction hypothesis, we get that $f_{i-1}(1)$ has $f_1(1) = (1,-1)$ as a prefix. Thus, $f_{i-1}(1)$ has 1 as its first term. Therefore, $\mu \circ f_{i-1}(1)$ has $(1,-1)$ as a prefix and $\sigma \circ f_{i-1}(1)$ has $(1,1)$ as a prefix. Recalling that

$$f_i(1) = f_{i-1} \circ \mu \circ f_{i-1}(1)$$

or

$$f_i(1) = f_{i-1} \circ \sigma \circ f_{i-1}(1),$$

we have that $f_i(1)$ starts with $(f_{i-1}(1), f_{i-1}(-1))$ or $(f_{i-1}(1), f_{i-1}(1))$. In either case, $f_i(1)$ has $f_{i-1}(1)$ as a proper prefix as required.

Therefore, since the $f_i(1)$ agree on longer and longer prefixes as i increases, the limit $\lim_{i\to\infty} f_i(1)$ exists and is equal to the infinite sequence that has each $f_i(1)$ as a prefix.

$\square$

Proof of Claim 2. We will first state and prove some facts about discrepancies of images of particular morphisms. We will go on to prove that the $\{1\}$-discrepancy achieved by applying certain sequences of morphisms is bounded by 2. Together with the facts, this will show that each $f_i(1)$ has D-discrepancy 2 and therefore $\lim_{i\to\infty} f_i(1)$ has D-discrepancy 2.

Subclaim 9. *Let $\alpha \in \{\mu, \sigma\}$ and let s be a sequence over $\{1,-1\}$. The $\{2^j\}$-discrepancy of $\alpha(s)$ is equal to the $\{2^{j-1}\}$-discrepancy of s for all $j > 0$.*

Proof of Subclaim 9. Let $s' = \alpha(s)$. If $\alpha = \mu$ then by the construction of μ we have that $s'_{2t} = -s_t$ for $1 \leq t \leq |s|$. For arbitrary k, consider the absolute running k-sum of the 2^j-HAPS of s',

$$\begin{aligned}\left|\sum_{i=1}^{k} s'_{i2^j}\right| &= \left|\sum_{i=1}^{k} -s_{i2^{j-1}}\right| \\ &= \left|\sum_{i=1}^{k} s_{i2^{j-1}}\right|.\end{aligned}$$

Thus, the absolute running sums of the 2^j-HAPS of s' are exactly the absolute running sums of the 2^{j-1}-HAPS of s and so the $\{2^j\}$-discrepancy of s' is equal to the $\{2^{j-1}\}$-discrepancy of s.

Now suppose that $\alpha = \sigma$. By the construction of σ we have that $s'_{2t} = s_t$ for $1 \leq t \leq |s|$. Again, we can consider any absolute running sum of the 2^j-HAPS of s',

$$\left|\sum_{i=1}^{k} s'_{i2^j}\right| = \left|\sum_{i=1}^{k} s_{i2^{j-1}}\right|.$$

Therefore, the $\{2^j\}$-discrepancy of s' is equal to the $\{2^{j-1}\}$-discrepancy of s. □

Subclaim 10. *Let s be a $\{-1,1\}$ sequence. The sequence $\mu(s)$ has $\{1\}$-discrepancy 1.*

Proof of Subclaim 10. By the construction of μ, the sequence $\mu(s)$ can be split into blocks of either $(-1,1)$ and $(1,-1)$. Therefore, the absolute running sums of the 1-HAPS of s are bounded by 1. □

Subclaim 11. *Let s be a $\{-1,1\}$ sequence with $\{1\}$-discrepancy c. The $\{1\}$-discrepancy of $\sigma(s)$ is $2c$.*

Proof of subclaim 11. Since σ replaces each 1 with $(1,1)$ and each -1 with $(-1,-1)$ and the absolute running sums of the 1-HAPS of s are bounded by c, the absolute running sums of the 1-HAPS of $\sigma(s)$ are bounded by $2c$. □

Let us write $f_i(1)$ as $\alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_{2^i-2} \circ \alpha_{2^i-1}(1)$ with each $\alpha_j \in \{\mu, \sigma\}$ for $1 \leq j < 2^i$.

Let $\mathbf{s} = \lim_{i\to\infty} f_i(1)$ and let $j \geq 0$. Consider any absolute running sum of the 2^j-HAPS,

$$\left|\sum_{t=1}^{k} \mathbf{s}_{t2^j}\right|.$$

There exists i such that $f_i(1)$ contains all the terms of this sum. Since $f_i(1)$ can be written as $\alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_{2^i-2} \circ \alpha_{2^i-1}(1)$, the $\{2^j\}$-discrepancy of $f_i(1)$ is equal to the $\{2^{j-1}\}$-discrepancy of $\alpha_2 \circ \cdots \circ \alpha_{2^i-2} \circ \alpha_{2^i-1}(1)$ by Subclaim 9 for any $0 < j \leq 2^i - 1$. Continuing in this way we get that the $\{2^j\}$-discrepancy of $f_i(1)$ is equal to the $\{1\}$-discrepancy of $\alpha_j \circ \cdots \circ \alpha_{2^i-1}(1)$. Since each application of μ resets the $\{1\}$-discrepancy to 1 (Subclaim 10) and each application of σ doubles the $\{1\}$-discrepancy (Subclaim 11), the $\{1\}$-discrepancy of $\alpha_j \circ \cdots \circ \alpha_{2^i-1}(1)$ can only be greater than 2 if both $\alpha_j = \sigma$ and $\alpha_{j+1} = \sigma$.

We note that the first and last morphisms in f_i are μ for all $i > 0$. This can be easily seen by induction on i. By the definition of f_i, the morphism σ can only appear between applications of f_{i-1} for some i. Therefore, it is never the case that there are two applications of σ in a row in f_i.

Since α_j and α_{j+1} cannot both be σ, the $\{1\}$-discrepancy of $\alpha_j \circ \cdots \circ \alpha_{2^i-1}(1)$ is 2 and, by Subclaim 9, the $\{2^j\}$-discrepancy of $f_i(1)$ is 2. Our choice of j was arbitrary, so the D-discrepancy of $f_i(1)$ is 2 and thus the D-discrepancy of $\mathbf{s}$ is 2.

□

Proof of Claim 3. Let K and K' be distinct subsets of $\mathbb{N}$ which map to $\lim_{i\to\infty} f_i(1)$ and $\lim_{i\to\infty} f_i'(1)$ respectively. Let t be the smallest natural number at which K and K' differ. Without loss of generality, we have $t \in K$ and $t \notin K'$.

Then $f_{t-1} = f'_{t-1}$ and so

$$f_t(1) = f_{t-1} \circ \phi \circ f_{t-1}(1)$$

and

$$f'_t(1) = f'_{t-1} \circ \sigma \circ f'_{t-1}(1) = f_{t-1} \circ \sigma \circ f_{t-1}(1).$$

By Claim 1, $f_{t-1}(1)$ has $\mu(1)$ as a prefix so its first term is 1. Therefore, $\mu \circ f_{t-1}(1)$ starts with $(1, -1)$ and $f_{t-1} \circ \phi \circ f_{t-1}(1)$ starts with $(f_{t-1}(1), f_{t-1}(-1))$.

On the other hand, $\sigma \circ f_{t-1}(1)$ starts with $(1, 1)$ and so $f_{t-1} \circ \sigma \circ f_{t-1}(1)$ starts with $(f_{t-1}(1), f_{t-1}(1))$. Thus, $f_t(1) \neq f'_t(1)$. Since these sequences are prefixes of $\lim_{i\to\infty} f_i(1)$ and $\lim_{i\to\infty} f'_i(1)$ respectively, we have $\lim_{i\to\infty} f_i(1) \neq \lim_{i\to\infty} f'_i(1)$. $\square$

Since we can map each subset of the natural numbers to an infinite sequence over $\{-1, 1\}$ (Claim 1), each sequence is distinct (Claim 3), and each sequence over has D-discrepancy 2 (Claim 2), the number of such sequences is uncountable.

$\square$

2.2 Lexicographically Least Sequence with D-discrepancy 3

In the previous section, we showed that there are an uncountable number of infinite sequences with $\{1, 2, 4, 8, \dots\}$-discrepancy 2. This means that there are also an uncountable number of infinite sequences with $\{1, 2, 4, 8, \dots\}$-discrepancy 3. We will now describe the unique lexicographically least infinite sequence with $\{1, 2, 4, 8, \dots\}$-discrepancy 3. We say that a sequence t is lexicographically less than another sequence t' if $t_i = -1$ where i is the least index where $t_i \neq t'_i$.

We will describe the lexicographically least infinite sequence with $\{1, 2, 4, 8, \dots\}$-discrepancy 3 as the fixed point of a morphism followed by a coding. The lexicographically least infinite sequence with $\{1, 2, 4, 8, \dots\}$-discrepancy 2 can also be described in this way and is a special case of Theorem 19. Sequences that can be described as the fixed point of a morphism followed by a coding are called automatic sequences [2].

In order to describe the lexicographically least infinite sequence with $\{1, 2, 4, 8, \dots\}$-discrepancy 3, we will need a generalization of discrepancy.

Definition 12. a) Let $D \subseteq \mathbb{N}$ and $a, b \in \mathbb{Z}$. A finite integer sequence $\{t_i\}_{i=1}^{N}$ has asymmetric D-discrepancy (a, b) if

$$a \leq \sum_{i=1}^{k} t_{id} \leq b$$

for all $d \in D$ and for all k with $0 < k \leq N/d$.

b) Let $D \subseteq \mathbb{N}$ and $a, b \in \mathbb{Z}$. An infinite integer sequence $\{t_i\}_{i=1}^{\infty}$ has asymmetric D-discrepancy (a, b) if

$$a \leq \sum_{i=1}^{k} t_{id} \leq b$$

for all $d \in D$ and for all $k > 0$.

We can now describe the lexicographically least infinite sequence with $\{1, 2, 4, 8, \ldots\}$-discrepancy 3 as the fixed point of a morphism followed by a coding.

Let ϕ be a morphism over the alphabet $\{A, B, C, D, E, F, G, H\}$ defined as follows:

$$\begin{aligned}
A &\mapsto AB \\
B &\mapsto CD \\
C &\mapsto EF \\
D &\mapsto EG \\
E &\mapsto EG \\
F &\mapsto DH \\
G &\mapsto GE \\
H &\mapsto GE.
\end{aligned}$$

Let ψ be a coding from $\{A, B, C, D, E, F, G, H\}$ to $\{-1, 1\}$ defined as follows:

$$\begin{aligned}
A &\mapsto -1 \\
B &\mapsto -1 \\
C &\mapsto -1 \\
D &\mapsto 1 \\
E &\mapsto -1 \\
F &\mapsto 1 \\
G &\mapsto -1 \\
H &\mapsto 1.
\end{aligned}$$

We will show that $\psi(\phi^\omega(A))$ is the lexicographically least infinite sequence with $\{1, 2, 4, 8, \ldots\}$-discrepancy 3. To do so, we require a few lemmas.

Lemma 13. *The sequence $\psi(\phi^\omega(A))$ can be written as*

$\psi(\phi^4(A)), \mu^2(-1,-1,-1,1), \mu^3(-1,-1,-1,1), \mu^4(-1,-1,-1,1), \mu^5(-1,-1,-1,1)\ldots.$

Proof. We will prove by induction on n that

$$\phi^n(A) = \phi^4(A), \phi^2(E,E,E,G), \ldots, \phi^{n-3}(E,E,E,G)$$

for all $n \geq 5$.

Base case: $n = 5$.

We have,

$$\begin{aligned}
\phi^5(A) &= A,B,C,D,E,F,E,G,E,G,D,H,E,G,G,E,E, \\
&\quad E,G,G,E,E,G,G,E,E,G,G,E,G,E,E,G \\
&= \phi^4(A), E,G,G,E,E,G,G,E,E,G,G,E,G,E,E,G \\
&= \phi^4(A), \phi^2(E), \phi^2(E), \phi^2(E), \phi^2(G) \\
&= \phi^4(A), \phi^2(E,E,E,G).
\end{aligned}$$

Inductive case: $n > 5$.

We have,

$$\phi^n(A) = \phi(\phi^{n-1}(A)).$$

By the induction hypothesis,

$$\begin{aligned}
\phi^n(A) &= \phi(\phi^4(A), \phi^2(-1,-1,-1,1), \ldots, \phi^{n-4}(-1,-1,-1,1)) \\
&= \phi^5(A), \phi^3(-1,-1,-1,1), \ldots, \phi^{n-3}(-1,-1,-1,1)) \\
&= \phi^4(A), \phi^2(-1,-1,-1,1), \phi^3(-1,-1,-1,1), \ldots, \phi^{n-3}(-1,-1,-1,1))
\end{aligned}$$

as required.

Therefore,

$$\phi^{\omega}(A) = \phi^4(A), \phi^2(E,E,E,G), \phi^3(E,E,E,G), \phi^4(E,E,E,G), \ldots.$$

Notice that since $\phi(E) = (E,G)$ and $\phi(G) = (G,E)$, the morphism ϕ behaves like the Thue-Morse morphism μ on E and G. Therefore, $\psi(\phi^k(E)) = \mu^k(-1)$ and $\psi(\phi^k(G)) = \mu^k(1)$. Thus,

$$\begin{aligned}
\psi(\phi^{\omega}(A)) &= \psi(\phi^4(A), \phi^2(E,E,E,G), \phi^3(E,E,E,G), \phi^4(E,E,E,G), \ldots) \\
&= \psi(\phi^4(A)), \psi(\phi^2(E,E,E,G), \psi(\phi^3(E,E,E,G), \psi(\phi^4(E,E,E,G), \ldots \\
&= \psi(\phi^4(A)), \mu^2(-1,-1,-1,1), \mu^3(-1,-1,-1,1), \mu^4(-1,-1,-1,1), \ldots.
\end{aligned}$$

□

Lemma 14. *The sequence $\mu^k(-1)$ is the lexicographically least sequence of length 2^k with asymmetric $\{1,4,16,\ldots\}$-discrepancy $(-1,5)$ and asymmetric $\{2,8,32,\ldots\}$-discrepancy $(-5,1)$ for all $k \geq 0$.*

Proof. We will prove the following claims for $k \geq 0$ simultaneously by induction on k.

1. The sequence $\mu^k(-1)$ is the lexicographically least sequence of length 2^k with asymmetric $\{1,4,16,\ldots\}$-discrepancy $(-1,5)$ and asymmetric $\{2,8,32,\ldots\}$-discrepancy $(-5,1)$.

2. The sequence $\mu^k(1)$ is the lexicographically least sequence of length 2^k that ends with $(-1)^k$ with asymmetric $\{1,4,16,\ldots\}$-discrepancy $(-1,5)$ and asymmetric $\{2,8,32,\ldots\}$-discrepancy $(-5,1)$.

Base case: $k = 0$.

1. The sequence $\mu^0(-1) = -1$ is the lexicographically least sequence of length 1 with asymmetric $\{1,4,16,\ldots\}$-discrepancy $(-1,5)$ and asymmetric $\{2,8,32,\ldots\}$-discrepancy $(-5,1)$.

2. The sequence $\mu^0(1) = 1$ is the lexicographically least sequence of length 1 that ends with 1 with asymmetric $\{1, 4, 16, \ldots\}$-discrepancy $(-1, 5)$ and asymmetric $\{2, 8, 32, \ldots\}$-discrepancy $(-5, 1)$.

Inductive case: $k > 0$.

1. Let t be the lexicographically least sequence of length 2^k with asymmetric $\{1, 4, 16, \ldots\}$-discrepancy $(-1, 5)$ and asymmetric $\{2, 8, 32, \ldots\}$-discrepancy $(-5, 1)$. By the induction hypothesis (1), the sequence $\mu^{k-1}(-1)$ is the lexicographically least sequence of length 2^{k-1} with asymmetric $\{1, 4, 16, \ldots\}$-discrepancy $(-1, 5)$ and asymmetric $\{2, 8, 32, \ldots\}$-discrepancy $(-5, 1)$. Therefore, if there exists a sequence of length 2^k with asymmetric $\{1, 4, 16, \ldots\}$-discrepancy $(-1, 5)$ and asymmetric $\{2, 8, 32, \ldots\}$-discrepancy $(-5, 1)$ that starts with $\mu^{k-1}(-1)$ then t starts with $\mu^{k-1}(-1)$.

   Suppose that t begins with $\mu^{k-1}(-1)$. The sequence $\mu^{k-1}(-1)$ has $\{1, 2, 4, \ldots, 2^{k-2}\}$-discrepancy 1 and the running sums of the $\{1, 2, 4, \ldots, 2^{k-2}\}$-HAPS of $\mu^{k-1}(-1)$ are 0.

   The 2^{k-1}th term of $\mu^{k-1}(-1)$ is $(-1)^k$. If $k-1$ is odd then the 2^{k-1}th term is -1 and t must have asymmetric $\{2^{k-1}\}$-discrepancy $(-1, 5)$. Thus the 2^kth term of t must be 1. By the induction hypothesis (2), the lexicographically sequence of length 2^{k-1} that ends with 1 with asymmetric $\{1, 4, 16, \ldots\}$-discrepancy $(-1, 5)$ and asymmetric $\{2, 8, 32, \ldots\}$-discrepancy $(-5, 1)$ is $\mu^{k-1}(1)$. Otherwise, if $k-1$ is even then the 2^{k-1}th term is 1 and t must have asymmetric $\{2^{k-1}\}$-discrepancy $(-5, 1)$. Thus the 2^kth term of t must be -1. By the induction hypothesis (2), the lexicographically least sequence of length 2^{k-1} that ends with -1 with asymmetric $\{1, 4, 16, \ldots\}$-discrepancy $(-1, 5)$ and asymmetric $\{2, 8, 32, \ldots\}$-discrepancy $(-5, 1)$ is $\mu^{k-1}(1)$. Thus, in either case we have

$$\begin{aligned} t &= \mu^{k-1}(-1), \mu^{k-1}(1) \\ &= \mu^{k-1}(-1, 1) \\ &= \mu^k(-1). \end{aligned}$$

2. Let t be the lexicographically least sequence of length 2^k that ends with $(-1)^k$ with asymmetric $\{1, 4, 16, \dots\}$-discrepancy $(-1, 5)$ and asymmetric $\{2, 8, 32, \dots\}$-discrepancy $(-5, 1)$.

   If $k-1$ is odd then t ends with 1 and t has asymmetric $\{2^{k-1}\}$-discrepancy $(-5, 1)$. Therefore, the 2^{k-1}th term of t is -1. By the induction hypothesis (2), the lexicographically least sequence of length 2^{k-1} that ends with -1 with asymmetric $\{1, 4, 16, \dots\}$-discrepancy $(-1, 5)$ and asymmetric $\{2, 8, 32, \dots\}$-discrepancy $(-5, 1)$ is $\mu^{k-1}(1)$. Otherwise, if $k-1$ is even then t ends with -1 and t has asymmetric $\{2^{k-1}\}$-discrepancy $(-1, 5)$. Therefore the 2^{k-1}th term of t is 1. By the induction hypothesis (2), the lexicographically least sequence of length 2^{k-1} that ends with 1 with asymmetric $\{1, 4, 16, \dots\}$-discrepancy $(-1, 5)$ and asymmetric $\{2, 8, 32, \dots\}$-discrepancy $(-5, 1)$ is $\mu^{k-1}(1)$.

   In either case, if there exists a sequence of length 2^k that ends with $(-1)^k$ with asymmetric $\{1, 4, 16, \dots\}$-discrepancy $(-1, 5)$ and asymmetric $\{2, 8, 32, \dots\}$-discrepancy $(-5, 1)$ that starts with $\mu^{k-1}(1)$ then t starts with $\mu^{k-1}(1)$. Suppose that t begins with $\mu^{k-1}(1)$. The sequence $\mu^{k-1}(1)$ has $\{1, 2, 4, \dots, 2^{k-2}\}$-discrepancy 1 and the running sums of the $\{1, 2, 4, \dots, 2^{k-2}\}$-HAPS of $\mu^{k-1}(1)$ are 0.

   By the induction hypothesis (1), the lexicographically least sequence of length 2^{k-1} with asymmetric $\{1, 4, 16, \dots\}$-discrepancy $(-1, 5)$ and asymmetric $\{2, 8, 32, \dots\}$-discrepancy $(-5, 1)$ is $\mu^{k-1}(-1)$ and $\mu^{k-1}(-1)$ ends with $(-1)^{k-1}$. Therefore,

$$\begin{aligned} t &= \mu^{k-1}(1), \mu^{k-1}(-1) \\ &= \mu^{k-1}(1, -1) \\ &= \mu^k(1). \end{aligned}$$

□

Theorem 15. *Let $D = \{1, 2, 4, 8, \dots\}$. The sequence $\psi(\phi^\omega(A))$ is the lexicographically least infinite sequence with D-discrepancy* 3.

Proof. Let t be the lexicographically least infinite sequence with D-discrepancy 3.

For all $k \geq 5$, define t_k as

$$\psi(\phi^4(A)), \mu^2(-1,-1,-1,1), \ldots, \mu^{k-3}(-1,-1,-1,1).$$

We will first show that the running sum of the 2^{k-2}-HAPS of t_k is 0 for all $k \geq 5$.

When $k = 5$, we have $t_5 = \psi(\phi^4(A)), \mu^2(-1,-1,-1,1)$ and the running sum of the 8-HAPS of t_5 is

$$1 - 1 - 1 + 1 = 0.$$

When $k = 6$, we have $t_6 = \psi(\phi^4(A)), \mu^2(-1,-1,-1,1), \mu^3(-1,-1,-1,1)$ and the running sum of the 16-HAPS of t_6 is

$$-1 + 1 + 1 - 1 = 0.$$

When $k = 7$,

$$t_7 = \psi(\phi^4(A)), \mu^2(-1,-1,-1,1), \mu^3(-1,-1,-1,1), \mu^4(-1,-1,-1,1)$$

and the running sum of the 32-HAPS of t_7 is

$$1 - 1 - 1 + 1 = 0.$$

When $k \geq 8$,

$$t_k = t_{k-2}, \mu_{k-5}(-1,-1,-1,1), \mu_{k-4}(-1,-1,-1,1), \mu_{k-3}(-1,-1,-1,1)$$

and the running sum of the 2^{k-2}-HAPS of t_k is

$$(-1)^{k-4} + (-1)^{k-3} + (-1)^{k-2} + (-1)^{k-3} = 0.$$

Now, we will prove that the following claims for all $k \geq 5$ simultaneously by induction on k.

1. The lexicographically least sequence of length 2^k with D-discrepancy 3 is t_k.

2. The running sum of the 2^i-HAPS of t_k is -2 for all even i with $0 \leq i \leq k-3$.

3. The running sum of the 2^i-HAPS of t_k is 2 for all odd i with $0 \le i \le k-3$.

Base case: $k = 5$.

1. The lexicographically least sequence of length 32 with D-discrepancy 3 is

$$-1,-1,-1,1,-1,1,-1,1,-1,1,1,-1,-1,1,1,-1,$$
$$-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,1,-1,-1,1$$

which is equal to $\psi(\phi(A))\mu^2(-1,-1,-1,1) = t_5$.

2. The running sum of the 1-HAPS of t_5 is -2.

3. The running sum of the 2-HAPS of t_5 is 2.

Inductive case: $k > 5$.

1. Let t be the lexicographically least sequence of length 2^k with D-discrepancy 3. By the induction hypothesis (1), the lexicographically least sequence of length 2^{k-1} with D-discrepancy 3 is t_{k-1}. Therefore, if there exists a sequence of length 2^k with D-discrepancy 3 that starts with t_{k-1} then t starts with t_{k-1}.

   Suppose that t starts with t_{k-1}. By induction hypotheses $(2,3)$, the running sums of the 2^i-HAPS of t_{k-1} are -2 for all even i with $0 \le i \le k-4$ and the running sums of the 2^i-HAPS of t_{k-1} are 2 for all odd i with $0 \le i \le k-4$. This means that the next 2^{k-3} terms of t must have asymmetric $\{2^i : i \text{ is even and } i \le k-4\}$-discrepancy $(-1,5)$ and asymmetric $\{2^i : i \text{ is odd and } i \le k-4\}$-discrepancy $(-5,1)$. The lexicographically least such sequence is $\mu^{k-3}(-1)$ by Lemma 14. The sequence $\mu^{k-3}(-1)$ leaves the running sums of the 2^i-HAPS unchanged for $0 \le i \le k-4$ so the sequence $t_{k-1}, \mu^{k-3}(-1)$ can be followed by $\mu^{k-3}(-1)$ and then $\mu^{k-3}(-1)$ again.

   However, the running sum of the 2^{k-2}-HAPS of t_{k-1} is 0 and so the running sum of the 2^{k-2}-HAPS of

$$t_{k-1}, \mu^{k-3}(-1), \mu^{k-3}(-1), \mu^{k-3}(-1)$$

is $3(-1)^{k-2}$. Therefore, the last 2^{k-3} terms of t must end with $(-1)^{k-3}$ and have have asymmetric $\{2^i : i$ is even and $i \leq k-4\}$-discrepancy $(-1, 5)$ and asymmetric $\{2^i : i$ is odd and $i \leq k-4\}$-discrepancy $(-5, 1)$. The lexicographically least such sequence is $\mu^{k-3}(1)$ by Lemma 14. Therefore,

$$\begin{aligned} t &= t_{k-1}, \mu^{k-3}(-1), \mu^{k-3}(-1), \mu^{k-3}(-1), \mu^{k-3}(1) \\ &= t_{k-1}, \mu^{k-3}(-1, -1, -1, 1) \\ &= t_k. \end{aligned}$$

2. By the induction hypothesis (2), the running sum of the 2^i-HAPS of t_{k-1} is -2 for all even i with $0 \leq i \leq k-4$. The running sum of the 2^i-HAPS of $\mu^{k-3}(-1, -1, -1, 1)$ is 0 for all i with $0 \leq i \leq k-4$. Therefore, the running sum of the 2^i-HAPS of t_k is -2 for all even i with $0 \leq i \leq k-4$.

   If $k-3$ is even then the running sum of the 2^{k-3}-HAPS of $\mu^{k-3}(-1, -1, -1, 1)$ is -2. The running sum of the 2^{k-3}-HAPS of t_{k-1} is 0. Therefore, the running sum of the 2^{k-3}-HAPS of t_k is -2.

3. By the induction hypothesis (3), the running sum of the 2^i-HAPS of t_{k-1} is 2 for all odd i with $0 \leq i \leq k-4$. The running sum of the 2^i-HAPS of $\mu^{k-3}(-1, -1, -1, 1)$ is 0 for all i with $0 \leq i \leq k-4$. Therefore, the running sum of the 2^i-HAPS of t_k is 2 for all even i with $0 \leq i \leq k-4$.

   If $k-3$ is odd then the running sum of the 2^{k-3}-HAPS of $\mu^{k-3}(-1, -1, -1, 1)$ is 2. The running sum of the 2^{k-3}-HAPS of t_{k-1} is 0. Therefore, the running sum of the 2^{k-3}-HAPS of t_k is -2.

Since t_k is the lexicographically least sequence of length 2^k with D-discrepancy 3, it follows that the lexicographically least infinite sequence with D-discrepancy 3 is

$$\psi(\phi^4(A)), \mu^2(-1, -1, -1, 1), \mu^3(-1, -1, -1, 1), \ldots.$$

By Lemma 13, this is equal to $\psi(\phi^\omega(A))$.

$\square$

2.3 Discrepancy of Finite Sequences

We give an explicit formula for the number of sequences of length n with $\{1, 2, 4, 8, \dots\}$-discrepancy 1.

Theorem 16. *Let $D = \{1, 2, 4, 8, \dots\}$. Let $s_2(n)$ equal the sum of the digits of n in base 2. The number of sequences of length n over the alphabet $\{-1, 1\}$ with D-discrepancy 1 is $2^{s_2(n)}$.*

For example, 13 is 1101 in base 2. Therefore, the number of sequences of length 13 with D-discrepancy 1 is

$$2^{s_2(13)} = 2^{1+1+0+1} = 8.$$

Proof. Let us write n as $n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_j}$ where $n_1 > n_2 > \cdots > n_j$.

We claim that the sequences of length n over $\{-1, 1\}$ with D-discrepancy 1 are exactly the sequences

$$\mu^{n_1}(\pm 1) \cdot \mu^{n_2}(\pm 1) \cdot \cdots \cdot \mu^{n_j}(\pm 1).$$

($\Rightarrow$) We will show that any sequence t of length n over $\{-1, 1\}$ with D-discrepancy 1 can be written as above. We shall proceed by induction on n.

Base case:

The only sequences of length 1 are ± 1 and can be written as $\mu^0(\pm 1)$.

Inductive case:

Consider the length 2^{n_1} prefix of t. This prefix must have D-discrepancy 1 and we showed in the Theorem 1 that we can apply μ^{-1} n_1 times to obtain ± 1. Thus, the first 2^{n_1} terms of t are $\mu^{n_1}(\pm 1)$. We would like to show that the rest of t has D-discrepancy 1. Let $d \in D$. There are two cases:

- $d \geq 2^{n_1}$.

  Because 2^{n_1} is the largest power of 2 less than or equal to n, 2^{n_1} is greater than the length of the rest of t. Therefore, $d > n - 2^{n_1}$ and the d-HAPS of the rest of t is the empty sequence.

- $d < 2^{n_1}$.

  Let k be such that $2^{n_1}/d < k \leq n/d$. Then

$$\sum_{i=1}^{k} t_{id} = \sum_{i=1}^{2^{n_1}} t_{id} + \sum_{i=2^{n_1}+1}^{k} t_{id}$$

  Because the first sum has an even number of terms and $\mu^{n_1}(\pm 1)$ has D-discrepancy 1, the first sum must be 0. Thus, the second sum cannot exceed 1 in absolute value.

In both cases, $|\sum_{i=2^{n_1}+1}^{n/d} t_{id}| \leq 1$ so the rest of sequence has D-discrepancy 1. By the inductive hypothesis, the rest of the sequence can be written

$$\mu^{n_2}(\pm 1) \cdot \cdots \cdot \mu^{n_j}(\pm 1)$$

and so t can be written

$$\mu^{n_1}(\pm 1) \cdot \mu^{n_2}(\pm 1) \cdot \cdots \cdot \mu^{n_j}(\pm 1).$$

($\Leftarrow$) Now we show that $t = \mu^{n_1}(\pm 1) \cdot \cdots \cdot \mu^{n_2}(\pm 1) \cdot \cdots \cdot \mu^{n_j}(\pm 1)$ has D-discrepancy 1. Again, let us proceed by induction on n, the length of t.

Base case:

The length 1 sequences $\mu^0(\pm 1) = \pm 1$ both have D-discrepancy 1.

Inductive case:

Suppose that all sequences of this form and of length $< n$ have D-discrepancy 1. Let $d \in D$. There are three cases:

- $d < 2^{n_1}$.

  Let $1 \leq k \leq n/d$ and consider the sum $\sum_{i=1}^{k} t_{id}$. If $k \leq 2^{n_1}/d$ then this sum does not exceed 1 in absolute value because $\mu^{n_1}(\pm 1)$ has D-discrepancy 1. Otherwise we can split the sum as

$$\sum_{i=1}^{2^{n_1}/d} t_{id} + \sum_{i=2^{n_1}/d+1}^{k} t_{id}.$$

Since the left sum has an even number of terms and since $\mu^{n_1}(\pm 1)$ has D-discrepancy 1, the left sum is 0. By the induction hypothesis, the right sum does not exceed 1 in absolute value. Thus, the whole sum does not exceed 1 in absolute value.

- $d = 2^{n_1}$.

  We know that $2 \cdot 2^{n_1} > n$ so for $k \geq 1$, we have $\sum_{i=1}^{k} t_{id} = t_d$. Thus, the sum cannot exceed 1 in absolute value.

- $d > 2^{n_1}$.

  We know that $2 \cdot 2^{n_1} > n$ so therefore for $k \geq 1$, we have $\sum_{i=1}^{k} t_{id} = 0$. Thus, the sum cannot exceed 1 in absolute value.

Thus, t has D-discrepancy 1.

We have shown that the sequences of length n over $\{-1, 1\}$ with D-discrepancy 1 are exactly the sequences

$$\mu^{n_1}(\pm 1) \cdot \mu^{n_2}(\pm 1) \cdot \ldots \mu^{n_j}(\pm 1).$$

The number of such sequences is $2^{s_2(n)}$. □

For example, we can write 13 as $2^3 + 2^2 + 2^0$. Therefore, the sequences of length 13 with D-discrepancy 1 are $\mu^3(\pm 1) \cdot \mu^2(\pm 1) \cdot \mu^0(\pm 1)$. That is,

$$\begin{aligned}
\mu^3(-1)\mu^2(-1)\mu^0(-1) &= -1, 1, 1, -1, 1, -1, -1, 1, -1, 1, 1, -1, -1 \\
\mu^3(-1)\mu^2(-1)\mu^0(1) &= -1, 1, 1, -1, 1, -1, -1, 1, -1, 1, 1, -1, 1 \\
\mu^3(-1)\mu^2(1)\mu^0(-1) &= -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, -1 \\
\mu^3(-1)\mu^2(1)\mu^0(1) &= -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1 \\
\mu^3(1)\mu^2(-1)\mu^0(-1) &= 1, -1, -1, 1, -1, 1, 1, -1, -1, 1, 1, -1, -1 \\
\mu^3(1)\mu^2(-1)\mu^0(1) &= 1, -1, -1, 1, -1, 1, 1, -1, -1, 1, 1, -1, 1 \\
\mu^3(1)\mu^2(1)\mu^0(-1) &= 1, -1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, -1 \\
\mu^3(1)\mu^2(1)\mu^0(1) &= 1, -1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, 1.
\end{aligned}$$

Chapter 3

Powers of b Discrepancy

In this chapter, we will focus on the D-discrepancy of sequences when $D = \{1, b, b^2, \dots\}$ for $b \geq 2$. We will show that there are an uncountable number of infinite sequences with D-discrepancy 1 when $b > 2$. We will describe the lexicographically least infinite sequences with D-discrepancy 1 and 2. Finally, we will count the number of finite sequences of length n with D-discrepancy 1.

3.1 Discrepancy of Infinite Sequences

While there are a finite number of infinite sequences with $\{1, 2, 4, 8, \dots\}$-discrepancy 1, we will show that there are an uncountable number of infinite sequences with $\{1, b, b^2, \dots\}$-discrepancy 1 for $b > 2$.

Theorem 17. *Let b be a natural number greater than 2 and let $D = \{1, b, b^2, \dots\}$. The number of infinite sequences over $\{-1, 1\}$ with D-discrepancy 1 is uncountable.*

Proof. Define the morphism ϕ_d as follows:

$$\begin{aligned} 1 &\mapsto (1, -1)^{d/2} \\ -1 &\mapsto (-1, 1)^{d/2}. \end{aligned}$$

For example, ϕ_3 is

$$\begin{aligned} 1 &\mapsto 1, -1, 1 \\ -1 &\mapsto -1, 1, -1 \end{aligned}$$

and ϕ_6 is

$$\begin{aligned} 1 &\mapsto 1, -1, 1, -1, 1, -1 \\ -1 &\mapsto -1, 1, -1, 1, -1, 1 \end{aligned}$$

We claim that the sequence $\phi_b^\omega(1)$ has D-discrepancy 1. We will prove by induction on i that $\phi_b^\omega(1)$ has $\{b^i\}$-discrepancy 1.

Base case: $i = 0$.

If b is odd then $\phi_b(1)$ is an alternating sequence of 1 and -1 that starts and ends with 1 and $\phi_b(-1)$ is an alternating sequence of -1 and 1 that starts and ends with -1. Thus, $\phi_b \circ \phi_b(1)$ is also such an alternating sequence that starts and ends with 1. Continuing in this way, we see that $\phi_b^\omega(1)$ is an alternating sequence of 1 and -1 that starts with 1. Thus, the $\{1\}$-discrepancy of $\phi_b^\omega(1)$ is 1.

If b is even, then $\phi_b(1)$ and $\phi_b(-1)$ can be split into blocks of $(1, -1)$ and $(-1, 1)$. Thus, $\phi_b^\omega(1)$ can be split into blocks of $(1, -1)$ and $(-1, 1)$. So the 1-discrepancy of $\phi_b^\omega(1)$ is 1.

Inductive case.

Consider an arbitrary absolute running sum of the b^i-HAPS for $i > 0$.

$$\left| \sum_{j=1}^{k} \phi_b^\omega(1)_{jb^i} \right|$$

By the construction of ϕ_b we have that $t_j = \phi_b(t)_{bj}$ if b is odd and $t_j = -\phi_b(t)_{bj}$ if b

is even for all sequences t. Therefore,

$$\begin{aligned}\left|\sum_{j=1}^{k}\phi_b^\omega(1)_{jb^i}\right| &= \left|\sum_{j=1}^{k}\pm\phi_b^{-1}(\phi_b^\omega(1))_{jb^{i-1}}\right| \\ &= \left|\sum_{j=1}^{k}\phi_b^\omega(1)_{jb^{i-1}}\right|\end{aligned}$$

By the induction hypothesis, the $\{b^{i-1}\}$-discrepancy of $\phi_b^\omega(1)$ is 1 and so this sum is bounded by 1. Therefore, the $\{b^i\}$-discrepancy of $\phi_b^\omega(1)$ is 1.

By induction, the sequence $\phi_b^\omega(1)$ has D-discrepancy 1. Now we will show that this sequence can be modified to get an uncountable number of sequences with D-discrepancy 1.

Since $\phi_b^\omega(1)$ has $\{1\}$-discrepancy 1, it can be split into blocks of $(1, -1)$ and $(-1, 1)$.

Suppose b is odd. For any $i > 0$, if we change the block at position $b^i - 2$ from $(-1, 1)$ to $(1, -1)$ or from $(1, -1)$ to $(-1, 1)$, the D-discrepancy of the resultant sequence is still 1. Since the running sum of the $\{1\}$-discrepancy is 0 at position $b^i - 3$, this change does not affect the $\{1\}$-discrepancy. Since $b^i - 2$ and $b^i - 1$ are not divisible by b, this change does not affect the $\{b^k\}$-discrepancy for any $k > 0$.

Suppose b is even and thus $b > 3$. Similarly, for any $i > 0$ we can change the block at position $b^i - 3$ from $(-1, 1)$ to $(1, -1)$ or from $(1, -1)$ to $(-1, 1)$ and not change the D-discrepancy of the resultant sequence.

At each $j > 0$, we can choose to apply this change or not. This gives us an infinite number of binary choices and in all cases we get infinite sequences with D-discrepancy 1. This shows that the number of such sequences is uncountable. □

3.2 Lexicographically Least Sequence with Discrepancy 1

In this section we describe the lexicographically least infinite sequence with $\{1, b, b^2, \ldots\}$-discrepancy 1 for each even $b \geq 2$. These sequences are the fixed points of morphisms followed by codings. Note that if b is odd then the lexicographically least infinite sequence with $\{1, b, b^2, \ldots\}$-discrepancy 1 is $(-1, 1)^\omega$.

Let $D = \{1, b, b^2, \ldots\}$ with b even and $b > 2$. Let ϕ be the morphism defined as follows:

$$\begin{aligned} -1 &\mapsto (-1, 1)^{b/2} \\ 1 &\mapsto (-1, 1)^{b/2-2}, (1, -1). \end{aligned}$$

Theorem 18. *The lexicographically least infinite sequence with D-discrepancy 1 is $\phi^\omega(-1)$.*

For example, if $b = 4$ then ϕ is

$$\begin{aligned} -1 &\mapsto -1, 1, -1, 1 \\ 1 &\mapsto -1, 1, 1, -1 \end{aligned}$$

and the lexicographically least infinite sequence with D-discrepancy 1 is

$$\phi^\omega(-1) = -1, 1, -1, 1, -1, 1, 1, -1, -1, 1, -1, 1, -1, 1, 1, -1, \ldots.$$

Proof. We will begin by proving the following claims simultaneously by induction on k.

1. The sequence $\phi^k(-1)$ is the lexicographically least sequence of length b^k with D-discrepancy 1.

2. The sequence $\phi^k(-1)$ ends with $(-1)^{k-1}$.

3. The sequence $\phi^k(1)$ is the lexicographically least sequence of length b^k with D-discrepancy 1 that ends with $(-1)^k$.

Base case: $k = 0$.

1. The sequence (-1) is the lexicographically least sequence of length 1 with D-discrepancy 1.

2. The sequence (-1) ends with -1.

3. The sequence (1) is the lexicographically least sequence of length 1 with D-discrepancy 1 that ends with 1.

Inductive case: $k > 0$.

1. Let t be the lexicographically least sequence of length b^k with D-discrepancy 1. By the induction hypothesis (1), the lexicographically least sequence of length b^{k-1} with D-discrepancy 1 is $\phi^{k-1}(-1)$. If there exists a sequence of length b^k with D-discrepancy 1 that starts with $\phi^{k-1}(-1)$ then t starts with $\phi^{k-1}(-1)$.

   Suppose that t starts with $\phi^{k-1}(-1)$. By the construction of ϕ, the running sums of the b^i-HAPS of $\phi^{k-1}(-1)$ are 0 for $i < k-1$. Therefore, the next b^{k-1} terms of t must have $\{1, b, b^2, \ldots, b^{k-2}\}$-discrepancy 1. In order for t to have $\{b^{k-1}\}$-discrepancy 1, the $2b^{k-1}$th term cannot be the same as the b^{k-1}th term. By the induction hypothesis (2), the b^{k-1}th term is $(-1)^{k-1}$ and therefore the $2b^{k-1}$th term must be $(-1)^k$. By the induction hypothesis (3), the lexicographically least possibility for the next b^{k-1} terms of t is $\phi^{k-1}(1)$. After these $2b^{k-1}$ terms, the running sums of the 2^i-HAPS are 0 for $i < k$ so we can continue to alternate $\phi^{k-1}(-1)$ and $\phi^{k-1}(1)$. Therefore,

$$\begin{aligned}
t &= \phi^{k-1}(-1), \phi^{k-1}(1), \ldots, \phi^{k-1}(-1), \phi^{k-1}(1) \\
&= \phi^{k-1}(-1, 1, \ldots, -1, 1) \\
&= \phi^k(-1).
\end{aligned}$$

2. The sequence $\phi^k(-1)$ ends with $\phi^{k-1}(1)$. By the induction hypothesis, (3), the sequence $\phi^{k-1}(1)$ ends with $(-1)^{k-1}$.

3. Let t be the lexicographically least sequence of length b^k with D-discrepancy 1 that ends with $(-1)^k$. As above, the lexicographically least choice for the first $b^{k-1}(b-2)$ symbols of t is $(\phi^{k-1}(-1), \phi^{k-1}(1))^{b/2-1}$.

   Suppose that t starts with $(\phi^{k-1}(-1), \phi^{k-1}(1))^{k/2-1}$. Since t must end with $(-1)^k$, the $(b-1)b^{k-1}$th term must be $(-1)^{k-1}$. Since $\phi^{k-1}(1)$ is the lexicographically least sequence of length b^{k-1} with D-discrepancy that ends with

$(-1)^{k-1}$ by the induction hypothesis (3), suppose that the next b^{k-1} terms of t are $\phi^{k-1}(1)$. By the induction hypothesis (1), the lexicographically least sequence of length b^{k-1} with D-discrepancy 1 is $\phi^{k-1}(-1)$ and this sequence ends with $(-1)^k$ by the induction hypothesis (2). Therefore,

$$\begin{aligned} t &= \phi^{k-1}(-1), \phi^{k-1}(1), \ldots, \phi^{k-1}(-1), \phi^{k-1}(1), \phi^{k-1}(1), \phi^{k-1}(-1) \\ &= \phi^{k-1}(-1, 1, \ldots, -1, 1, 1, -1) \\ &= \phi^k(1). \end{aligned}$$

Since $\phi^k(-1)$ is the lexicographically least sequence of length b^k with D-discrepancy 1, it follows that the lexicographically least infinite sequence with D-discrepancy 1 is $\phi^\omega(-1)$.

□

3.3 Lexicographically Least Sequence with Discrepancy 2

In this section, we describe the lexicographically least infinite sequence with $\{1, b, b^2, \ldots\}$-discrepancy 2 for each even $b \geq 2$. As in the previous section, these sequences are the fixed points of morphisms followed by codings. Note that if b is odd then the lexicographically least infinite sequence with $\{1, b, b^2, \ldots\}$-discrepancy 2 is $(-1, (-1, 1)^\omega)$ and, more generally, the lexicographically least infinite sequence with $\{1, b, b^2, \ldots\}$-discrepancy c is $((-1)^{c-1}, (-1, 1)^\omega)$.

Let $D = \{1, b, b^2, \ldots\}$ and let ϕ be a morphism over the alphabet $\{A, B, C, D\}$ defined as follows.

If $b = 2$ then

$$\begin{aligned} A &\mapsto A, B \\ B &\mapsto C, A \\ C &\mapsto C, D \\ D &\mapsto A, C. \end{aligned}$$

If $b > 2$ then

$$\begin{aligned}
A &\mapsto A, B, (C, A)^{b/2-1} \\
B &\mapsto (C, A)^{b/2} \\
C &\mapsto (C, A)^{b/2-1}, C, D \\
D &\mapsto A, B, (C, A)^{b/2-2}, C, D.
\end{aligned}$$

For example, if $b = 8$ then ϕ is

$$\begin{aligned}
A &\mapsto A, B, C, A, C, A, C, A \\
B &\mapsto C, A, C, A, C, A, C, A \\
C &\mapsto C, A, C, A, C, A, C, D \\
D &\mapsto A, B, C, A, C, A, C, D.
\end{aligned}$$

Let ψ be a coding from $\{A, B, C, D\}$ to $\{-1, 1\}$ defined as follows:

$$\begin{aligned}
A &\mapsto -1 \\
B &\mapsto -1 \\
C &\mapsto 1 \\
D &\mapsto 1.
\end{aligned}$$

We will show that $\psi(\phi^\omega(A))$ is the lexicographically least infinite sequence with D-discrepancy 2.

Theorem 19. *The lexicographically least infinite sequence with D-discrepancy 2 is $\psi(\phi^\omega(A))$.*

Proof. We will prove the following claims for all $k > 0$ simultaneously by induction on k.

1. The sequence $\psi(\phi^k(A))$ is the lexicographically least sequence of length b^k with $\{1, b, b^2, \ldots, b^k\}$-discrepancy 2.

2. For all $0 \leq i < k$, the running b^{k-i}-sum of the b^i-HAPS of $\psi(\phi^k(A))$ is -2. That is,
$$\sum_{j=1}^{b^{k-i}} \psi(\phi^k(A))_{jb^i} = -2.$$

3. The sequence $\psi(\phi^k(B))$ is the lexicographically least sequence of length b^k with asymmetric $\{1, b, b^2, \ldots, b^{k-1}\}$-discrepancy $(0, 4)$.

4. For all $0 \leq i < k$, the running b^{k-i}-sum of the b^i-HAPS of $\psi(\phi^k(B))$ is 0. That is,
$$\sum_{j=1}^{b^{k-i}} \psi(\phi^k(B))_{jb^i} = 0.$$

5. The sequence $\psi(\phi^k(C))$ is the lexicographically least sequence of length b^k with asymmetric $\{1, b, b^2, \ldots, b^k\}$-discrepancy $(0, 4)$.

6. For all $0 \leq i < k$, the running b^{k-i}-sum of the b^i-HAPS of $\psi(\phi^k(C))$ is 2. That is,
$$\sum_{j=1}^{b^{k-i}} \psi(\phi^k(C))_{jb^i} = 2.$$

7. The sequence $\psi(\phi^k(D))$ is the lexicographically least sequence of length b^k with $\{1, b, b^2, \ldots, b^{k-1}\}$-discrepancy 2 and with b^kth term having the value 1.

8. For all $0 \leq i < k$, the running b^{k-i}-sum of the b^i-HAPS of $\psi(\phi^k(D))$ is 0. That is,
$$\sum_{j=1}^{b^{k-1}} \psi(\phi^k(\Delta))_{jb^i} = 0.$$

Base case: $k = 1$.

1. The lexicographically least sequence of length b with $\{1, b\}$-discrepancy 2 is

$(-1,-1,(1,-1)^{b/2-1})$.

$$\begin{aligned}\psi(\phi(A)) &= \psi(A,B,(C,A)^{b/2-1}) \\ &= (-1,-1,(1,-1)^{b/2-1}).\end{aligned}$$

2. When $i=0$, the running b-sum of the 1-HAPS of $(-1,-1,(1,-1)^{b/2-1})$ is -2.

3. The lexicographically least sequence of length b with asymmetric $\{1\}$-discrepancy $(0,4)$ is $(1,-1)^{b/2}$.

$$\begin{aligned}\psi(\phi(B)) &= \psi((C,A)^{b/2}) \\ &= (1,-1)^{b/2}.\end{aligned}$$

4. When $i=0$, the running b-sum of the 1-HAPS of $(1,-1)^{b/2}$ is 0.

5. The lexicographically least sequence of length b with asymmetric $\{1,b\}$-discrepancy $(0,4)$ is $((1,-1)^{b/2-1},1,1)$.

$$\begin{aligned}\psi(\phi(C)) &= \psi((C,A)^{b/2-1},C,D) \\ &= ((1,-1)^{b/2-1},1,1).\end{aligned}$$

6. When $i=0$, the running b-sum of the 1-HAPS of $((1,-1)^{b/2-1},1,1)$ is 2.

7. The lexicographically least sequence of length b with $\{1,b\}$-discrepancy 2 and bth term having the value 1 is $(-1,1)$ if $b=2$ and $(-1,-1,(1,-1)^{b/2-2},1,1)$ if $b>2$. If $b=2$ then

$$\begin{aligned}\psi(\phi(D)) &= \psi(A,C) \\ &= (-1,1).\end{aligned}$$

If $b > 2$ then

$$\begin{aligned}\psi(\phi(D)) &= \psi(A, B, (C, A)^{b/2-2}, C, D) \\ &= (-1, -1, (1, -1)^{b/2-2}, 1, 1).\end{aligned}$$

8. When $i = 0$, the running b-sum of the 1-HAPS of $(-1, 1)$ is 0 and the running b-sum of the 1-HAPS of $(-1, -1, (1, -1)^{b/2-2}, 1, 1)$ is 0.

Inductive case.
Suppose the claims hold for all $k < \ell$. We now show that they hold for ℓ.

1. Let t be the lexicographically least sequence of length b^ℓ with $\{1, b, b^2, \ldots b^\ell\}$-discrepancy 2. By the induction hypothesis (1), the sequence $\psi(\phi^{\ell-1}(A))$ is the lexicographically least sequence of length $b^{\ell-1}$ with $\{1, b, b^2, \ldots, b^{\ell-1}\}$-discrepancy 2. Therefore, if there exists a sequence of length b^ℓ with $\{1, b, b^2, \ldots, b^\ell\}$-discrepancy 2 that starts with $\psi(\phi^{\ell-1}(A))$ then t starts with $\psi(\phi^{\ell-1}(A))$.

   The running sums of the $\{1, b, \ldots, b^{\ell-2}\}$-HAPS of $\psi(\phi^{\ell-1}(A)$ are -2 by the induction hypothesis (2). Therefore, for a sequence that starts with $\psi(\phi^{\ell-1}(A))$ to have $\{1, b, b^2, \ldots, b^\ell\}$-discrepancy 2, its next $b^{\ell-1}$ terms must have asymmetric $\{1, b, b^2, \ldots, b^{\ell-2}\}$-discrepancy $(0, 4)$. The sequence $\psi(\phi^{\ell-1}(B))$ has this property, by the induction hypothesis (3). Furthermore, the induction hypothesis (3) says that $\psi(\phi^{\ell-1}(B))$ is the lexicographically least such sequence and so let us suppose that t starts with $\psi(\phi^{\ell-1}(A, B))$.

   Since, by the induction hypothesis (4), the running sums of the $\{1, b, \ldots, b^{\ell-2}\}$-HAPS of $\psi(\phi^{\ell-1}(B))$ are 0, the running sums of the $\{1, b, \ldots, b^{\ell-2}\}$-HAPS of $\psi(\phi^{\ell-1}(A, B))$ are -2. Furthermore, since both $\psi(\phi^{\ell-1}(A))$ and $\psi(\phi^{\ell-1}(B))$ end with -1, the running sum of the $\{b^{\ell-1}\}$-HAPS of $\psi(\phi^{\ell-1}(A, B))$ is -2 as well. Therefore, the next $b^{\ell-1}$ terms of t must have asymmetric $\{1, b, b^2, \ldots, b^{\ell-1}\}$-discrepancy $(0, 4)$. By the induction hypothesis (5), the lexicographically least such sequence is $\psi(\phi^{\ell-1}(C))$. Let us suppose that t starts with $\psi(\phi^{\ell-1}(A, B, C))$.

   Since, by the induction hypothesis (6), the running sums of the $\{1, b, b^2, \ldots, b^{\ell-1}\}$-HAPS of $\psi(\phi^{\ell-1}(C))$ are 2, the running sums of the $\{1, b, b^2, \ldots, b^{\ell-1}\}$-HAPS of $\psi(\phi^{\ell-1}(A,B,C))$ are 0. Therefore, the next $b^{\ell-1}$ terms of t must have $\{1,b,b^2,\ldots,b^{\ell-1}\}$-discrepancy 2. By the induction hypothesis (1), the lexicographically least such sequence is $\psi(\phi^{\ell-1}(A))$. Let us suppose that t starts with $\psi(\phi^{\ell-1}(A,B,C,A))$.

By using the induction hypothesis (2) again, we see that the running sums of the $\{1,b,\ldots,b^{\ell-2}\}$-HAPS of $\psi(\phi^{\ell-1}(A,B,C,A))$ are, again, -2. So t can continue to alternate between $\psi(\phi^{\ell-1}(C))$ and $\psi(\phi^{\ell-1}(A))$ as above.

Since we made the lexicographically least choice at each step and the final construction of t has the desired property,

$$\begin{aligned} t &= \psi(\phi^{\ell-1}(A,B,(C,A)^{b/2-1}) \\ &= \psi(\phi^{\ell}(A)). \end{aligned}$$

2. Recall that

$$\psi(\phi^{\ell}(A)) = \psi(\phi^{\ell-1}(A,B(C,A)^{b/2-1})).$$

Let i be such that $0 \leq i < \ell-1$. By the induction hypothesis (2) we know that the running sum of the b^i-HAPS of $\psi(\phi^{\ell-1}(A))$ is -2. By the induction hypothesis (4) we know that the running sum of the b^i-HAPS of $\psi(\phi^{\ell-1}(B))$ is 0. Therefore, the running sum of the b^i-HAPS of $\psi(\phi^{\ell-1}(A,B))$ is -2. Similarly, by induction hypotheses (6) and (2), the running sum of the b^i-HAPS of $\psi(\phi^{\ell-1}(C,A))$ is 0. Therefore, the running sum of the b^i-HAPS of $\psi(\phi^{\ell}(A))$ is -2.

Now let $i=\ell-1$. The running sum of the $b^{\ell-1}$-HAPS of $\psi(\phi^{\ell}(A))$ is

$$\psi(\phi^{\ell-1}(A))_{b^{\ell-1}} + \psi(\phi^{\ell-1}(B))_{b^{\ell-1}} + (b/2-1)\left(\psi(\phi^{\ell-1}(C))_{b^{\ell-1}} + \psi(\phi^{\ell-1}(A))_{b^{\ell-1}}\right).$$

We can see that $\psi(\phi^{\ell-1}(A))_{b^{\ell-1}} = -1$ since $\psi(\phi^{\ell-1}(A))$ is the lexicographically least sequence of length $b^{\ell-1}$ with $\{1,b,b^2,\ldots,b^{\ell-1}\}$-discrepancy 2 and the term at index $b^{\ell-1}$ can always be -1 without violating the discrepancy. We can also see that $\psi(\phi^{\ell-1}(B))_{b^{\ell-1}} = -1$ since $\psi(\phi^{\ell-1}(B))$ is the lexicographically least sequence of length $b^{\ell-1}$ with $\{1, b, b^2, \ldots, b^{\ell-2}\}$-discrepancy 2 and the term at index $b^{\ell-1}$ does not participate in any discrepancy sum. Finally, we know that $\psi(\phi^{\ell-1}(C))_{b^{\ell-1}} = 1$ since the asymmetric $\{b^{\ell-1}\}$-discrepancy of $\psi(\phi^{\ell-1}(C))$ is $(0, 4)$. Therefore, the running sum of the $b^{\ell-1}$-HAPS of $\psi(\phi^{\ell}(A))$ is

$$-1 - 1 + (b/2 - 1)(1 - 1) \;\; = \;\; -2.$$

3. Let t be the lexicographically least sequence of length b^ℓ with asymmetric $\{1, b, b^2, \ldots, b^{\ell-1}\}$-discrepancy $(0, 4)$. The first $b^{\ell-1}$ terms of t must have asymmetric $\{1, b, b^2, \ldots, b^{\ell-1}\}$-discrepancy $(0, 4)$ and by the induction hypothesis (5), the sequence $\psi(\phi^{\ell-1}(C))$ is the lexicographically least such sequence.

   Let i be such that $0 \leq i < \ell - 1$. The induction hypothesis (6) tells us that the running sum of the b^i-HAPS of $\psi(\phi^{\ell-1}(C))$ is 2. In order for a sequence that starts with $\psi(\phi^{\ell-1}(C))$ to have asymmetric $\{1, b, b^2, \ldots, b^{\ell-1}\}$-discrepancy $(0, 4)$, its next $b^{\ell-1}$ terms must have $\{1, b, b^2, \ldots, b^{\ell-2}\}$-discrepancy 2. The sequence $\psi(\phi^{\ell-1}(A))$ has this property and is the lexicographically least such sequence.

   Since the running sum of the b^i-HAPS of $\psi(\phi^{\ell-1}(A))$ is -2 by the induction hypothesis (2), the running sum of the b^i-HAPS of $\psi(\phi^{\ell-1}(C, A))$ is 0. So we can continue to alternate between $\psi(\phi^{\ell-1}(C))$ and $\psi(\phi^{\ell-1}(A))$. Therefore,

$$\begin{aligned} t &= \psi(\phi^{\ell-1}((C, A)^{b/2}) \\ &= \psi(\phi^{\ell}(B)). \end{aligned}$$

4. Recall that

$$\psi(\phi^{\ell}(B)) \;\; = \;\; \psi(\phi^{\ell-1}((C, A)^{b/2}).$$

   Let i be such that $0 \leq i < \ell - 1$. By the induction hypothesis (6), we know that the running sum of the b^i-HAPS of $\psi(\phi^{\ell-1}(C))$ is 2. By the induction hypothesis (2), we know that the running sum of the b^i-HAPS of $\psi(\phi^{\ell-1}(A))$ is -2. Therefore, the running sum of the b^i-HAPS of $\psi(\phi^{\ell}(B))$ is 0.

Now let $i = \ell - 1$. The running sum of the $b^{\ell-1}$-HAPS of $\psi(\phi^{\ell}(B))$ is

$$\psi(\phi^{\ell-1}(C))_{b^{\ell-1}} + \psi(\phi^{\ell-1}(A))_{b^{\ell-1}}.$$

Since $\psi(\phi^{\ell-1}(C))$ has asymmetric $\{b^{\ell-1}\}$-discrepancy $(0, 4)$, it must be that $\psi(\phi^{\ell-1}(\Gamma))_{b^{\ell-1}} = 1$. We know from above that $\psi(\phi^{\ell-1}(A))_{b^{\ell-1}} = -1$. Therefore, the running sum of the $b^{\ell-1}$-HAPS of $\psi(\phi^{\ell}(B))$ is 0.

5. Let t be the lexicographically least sequence of length b^{ℓ} with asymmetric $\{1, b, b^2, \ldots, b^{\ell}\}$-discrepancy $(0, 4)$.

   Notice that t is the lexicographically least sequence of length b^{ℓ} with asymmetric $\{1, b, b^2, \ldots, b^{\ell-1}\}$-discrepancy $(0, 4)$ that ends with 1. Thus, as in part 3, the sequence t starts by alternating $\psi(\phi^{\ell-1}(C))$ and $\psi(\phi^{\ell-1}(A))$. However, since t must end with 1, the last $b^{\ell-1}$ terms of t must have D-discrepancy 1 and end with 1. By the induction hypothesis (7), the lexicographically least such sequence is $\psi(\phi^{\ell-1}(D))$. Therefore,

$$\begin{aligned} t &= \psi(\phi^{\ell-1}((C, A)^{b/2-1}, C, D)) \\ &= \psi(\phi^{\ell}(D)). \end{aligned}$$

6. Recall that

$$\psi(\phi^{\ell}(C)) = \psi(\phi^{\ell-1}((C, A)^{b/2-1}, C, D)).$$

   Let i be such that $0 \leq i < \ell - 1$. By the induction hypothesis (6), we know that the running sum of the b^i-HAPS of $\psi(\phi^{\ell-1}(C))$ is 2. By the induction hypothesis (2), we know that the running sum of the b^i-HAPS of $\psi(\phi^{\ell-1}(A))$ is -2. By the induction hypothesis (8), we know that the running sum of the b^i-HAPS of $\psi(\phi^{\ell-1}(D)$ is 0. Therefore, the running sum of the b^i-HAPS of $\psi(\phi^{\ell}(C))$ is 2.

Now let $i = \ell - 1$. The running sum of the $b^{\ell-1}$-HAPS of $\psi(\phi^{\ell}(C))$ is

$$(b/2-1)\left(\psi(\phi^{\ell-1}(C))_{b^{\ell-1}} + \psi(\phi^{\ell-1}(A))_{b^{\ell-1}}\right) + \psi(\phi^{\ell-1}(C))_{b^{\ell-1}} + \psi(\phi^{\ell-1}(D))_{b^{\ell-1}}.$$

Since $\psi(\phi^{\ell-1}(D))_{b^{\ell-1}} = 1$ by the induction hypothesis (7) and we know that $\psi(\phi^{\ell-1}(C))_{b^{\ell-1}} = 1$ and $\psi(\phi^{\ell-1}(A))_{b^{\ell-1}} = -1$ from above, the running sum of the $b^{\ell-1}$-HAPS of $\psi(\phi^{\ell}(C))$ is 2.

7. Let t be the lexicographically least sequence of length b^{ℓ} with $\{1, b, b^2, \ldots, b^{\ell-1}\}$-discrepancy 2 and b^{ℓ}th term having the value 1. The first $b^{\ell-1}$ terms of t must have $\{1, b, b^2, \ldots, b^{\ell-1}\}$-discrepancy 2 and by the induction hypothesis (1), the sequence $\psi(\phi^{\ell-1}(A))$ is the lexicographically least such sequence.

   Let i be such that $0 \leq i < \ell - 1$. The induction hypothesis (2) tells us that the running sum of the b^i-HAPS of $\psi(\phi^{\ell-1}(A))$ is -2. In order for a sequence that starts with $\psi(\phi^{\ell-1}(A))$ to have $\{1, b, b^2, \ldots, b^{\ell-1}\}$-discrepancy 2, its next $b^{\ell-1}$ terms must have asymmetric $\{1, b, b^2, \ldots, b^{\ell-2}\}$-discrepancy $(0, 4)$. The lexicographically least such sequence is $\psi(\phi^{\ell-1}(B))$ by the induction hypothesis (3).

   As in the previous claims, the running sums of the b^i-HAPS of $\psi(\phi^{\ell-1}(A, B))$ is -2 for i such that $0 \leq i < \ell$. Thus, t alternates between $\psi(\phi^{\ell-1}(C))$ and $\psi(\phi^{\ell-1}(A))$. However, the sequence t must end with 1 and so the last $b^{\ell-1}$ terms of t must have $\{1, b, b^2, \ldots, b^{\ell-2}$-discrepancy 2 and end with 1. By the induction hypothesis (7), the lexicographically least such sequence is $\psi(\phi^{\ell-1}(D))$. Therefore,

$$\begin{aligned} t &= \psi(\phi^{\ell-1}(A, B, (C, A)^{b/2-2}, C, D)) \\ &= \psi(\phi^{\ell}(D)). \end{aligned}$$

   When $b = 2$, the $b^{\ell-1}$ terms that follow $\psi(\phi^{\ell-1}(A))$ must end with 1 and so the lexicographically least such sequence is $\psi(\phi^{\ell-1}(C))$ instead of $\psi(\phi^{\ell-1}(B))$.

Therefore, when $b = 2$

$$\begin{aligned} t &= \psi(\phi^{\ell-1}(A, C)) \\ &= \psi(\phi^{\ell}(D)). \end{aligned}$$

8. When $b = 2$, recall that

$$\psi(\phi^{\ell}(D)) = \psi(\phi^{\ell-1}(A, C)).$$

Let i be such that $0 \leq i < \ell - 1$. By the induction hypothesis (2), we know that the running sum of the b^i-HAPS of $\psi(\phi^{\ell-1}(A))$ is -2. By the induction hypothesis (6), we know that the running sum of the b^i-HAPS of $\psi(\phi^{\ell-1}(C))$ is 2. Therefore, the running sum of the b^i-HAPS of $\psi(\phi^{\ell}(D))$ is 0.

Now let $i = \ell - 1$. The running sum of the $b^{\ell-1}$-HAPS of $\psi(\phi^{\ell}(D))$ is

$$\psi(\phi^{\ell-1}(A))_{b^{\ell-1}} + \psi(\phi^{\ell-1}(C))_{b^{\ell-1}}.$$

Since we know from above that $\psi(\phi^{\ell-1}(A))_{b^{\ell-1}} = -1$ and $\psi(\phi^{\ell-1}(C))_{b^{\ell-1}} = 1$, the running sum of the $b^{\ell-1}$-HAPS of $\psi(\phi^{\ell}(D))$ is 0.

When $b > 2$, recall that

$$\psi(\phi^{\ell}(D)) = \psi(\phi^{\ell-1}(A, B, (C, A)^{b/2-2}, C, D)).$$

Let i be such that $0 \leq i < \ell - 1$. By the induction hypothesis (2), we know that the running sum of the b^i-HAPS of $\psi(\phi^{\ell-1}(A))$ is -2. By the induction hypothesis (4), we know that the running sum of the b^i-HAPS of $\psi(\phi^{\ell-1}(B))$ is 0. By the induction hypothesis (6), we know that the running sum of the b^i-HAPS of $\psi(\phi^{\ell-1}(C))$ is 2. By the induction hypothesis (6), we know that the running sum of the b^i-HAPS of $\psi(\phi^{\ell-1}(D))$ is 0. Therefore, the running sum of the b^i-HAPS of $\psi(\phi^{\ell}(D))$ is 0.

Now let $i = \ell - 1$. The running sum of the $b^{\ell-1}$-HAPS of $\psi(\phi^{\ell}(D))$ is

$$\begin{aligned} & \psi(\phi^{\ell-1}(A))_{b^{\ell-1}} + \psi(\phi^{\ell-1}(B))_{b^{\ell-1}} \\ + & \ (b/2-2)\left(\psi(\phi^{\ell-1}(C))_{b^{\ell-1}} + \psi(\phi^{\ell-1}(A))_{b^{\ell-1}}\right) \\ + & \ \psi(\phi^{\ell-1}(C))_{b^{\ell-1}} + \psi(\phi^{\ell-1}(D))_{b^{\ell-1}}. \end{aligned}$$

Since we know from above that $\psi(\phi^{\ell-1}(A))_{b^{\ell-1}} = -1$ and $\psi(\phi^{\ell-1}(B))_{b^{\ell-1}} = -1$ and $\psi(\phi^{\ell-1}(C))_{b^{\ell-1}} = 1$ and $\psi(\phi^{\ell-1}(D))_{b^{\ell-1}} = 1$, the running sum of the $b^{\ell-1}$-HAPS of $\psi(\phi^{\ell}(D))$ is 0.

Since $\psi(\phi^k(A))$ is the lexicographically least sequence of length b^k with $\{1, b, b^2, \dots, b^k\}$-discrepancy 2 for all $k > 0$, it follows that $\psi(\phi^{\omega}(A))$ is the lexicographically least infinite sequence with D-discrepancy 2.

$\square$

3.4 Discrepancy of Finite Sequences

As in the case of $\{1, 2, 4, 8, \dots\}$-discrepancy, we can find an explicit recurrence for the number of sequences of length n with $\{1, b, b^2, \dots\}$-discrepancy 1 when $b > 2$.

Let $\nu_b(n)$ be the exponent of the largest power of b that divides n. For example, $\nu_3(18) = 2$ since 3^2 is the highest power of 3 that divides 18.

Theorem 20. *Let $b > 2$ be a positive integer and let $D = \{1, b, b^2, b^3, \dots\}$. Let $A(n)$ be defined as follows:*

$$A(n) = \begin{cases} 1, & \text{if } n = 0; \\ 2A(n-1), & \text{if } n \text{ is odd;} \\ 2^{-\nu_b(n/2)} A(n-1), & \text{if } n \text{ is even and positive.} \end{cases}$$

Then the number of sequences of length n over the alphabet $\{-1, 1\}$ with D-discrepancy 1 is $A(n)$.

Proof. Let $t = \{t_1, t_2, \dots, t_n\}$ be a sequence of length n over the alphabet $\{-1, 1\}$. By Lemma 5, the sequence t has D-discrepancy 1 if and only if $t_{(2k-1)b^m} + t_{2kb^m} = 0$

for each d and m such that $1 < 2kb^m \leq n$. Let t' be the sequence in which

$$t'_i = \begin{cases} 1, & \text{if } t_i = 1; \\ 0, & \text{if } t_i = -1. \end{cases}$$

Now we can see that t has D-discrepancy 1 if and only if $t'_{(2k-1)b^m} \neq t'_{2kb^m}$ for all k and m such that $1 < 2kb^m \leq n$. We translate this inequality to $t'_{(2k-1)b^m} + t'_{2kb^m} = 1$ over GF(2).

This gives us the infinite set of equations $\{t'_{(2k-1)b^m} + t'_{2kb^m} = 1 : k > 0, m \geq 0\}$. Only finitely many of these have indices which are at most n, and we let $K(n)$ be the number of such equations with indices which are at most n. The sequences of length n with D-discrepancy 1 are the solutions to a linear system over GF(2) with n variables and $K(n)$ equations. We represent the linear system as a matrix with a row for each equation. For example, if $b = 3$ and $n = 6$ then $K(n) = 4$ and we get the system

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t'_1 \\ t'_2 \\ t'_3 \\ t'_4 \\ t'_5 \\ t'_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

We notice that if n is odd then t'_n does not participate in any equations. This is because, in each equation, the index of the larger-indexed variable is even and n is the largest index. Thus, if n is odd then $K(n) = K(n-1)$. If n is even then t'_n participates in an equation for each power of b that divides $n/2$. Thus, if n is even then $K(n) = K(n-1) + \nu_b(n/2) + 1$. We will now show that the rank of this $K(n) \times n$ matrix is $K(n)$.

We will perform row operations on the matrix until it is in row echelon form, that is, each row has a leading non-zero entry in a distinct column. Let $c_{k,m}$ be the equation $t'_{(2k-1)b^m} + t'_{2kb^m} = 1$.

Case 1: b is odd.

For each $m > 0$ we perform a row addition operation on $c_{k,m}$ by adding $c_{kb-(b-1)/2,m-1}$

to it. Call the resulting equation $c'_{k,m}$.

$$\begin{aligned}
c'_{k,m} &= c_{k,m} + c_{kb-(b-1)/2,m-1} \\
&= (t'_{(2k-1)b^m} + t'_{2kb^m} = 1) + (t'_{(2(kb-(b-1)/2)-1)b^{m-1}} + t'_{2(kb-(b-1)/2)b^{m-1}} = 1) \\
&= (t'_{(2k-1)b^m} + t'_{2kb^m} + t'_{2kb^m-b^m+b^{m-1}-b^{m-1}} + t'_{2kb^m-b^m+b^{m-1}} = 0) \\
&= (t'_{(2k-1)b^m} + t'_{2kb^m} + t'_{(2k-1)b^m} + t'_{(2k-1)b^m+b^{m-1}} = 0) \\
&= (t'_{(2k-1)b^m+b^{m-1}} + t'_{2kb^m} = 0).
\end{aligned}$$

Continuing our example, we would add the equation $c_{2,0} : t'_3 + t'_4 = 1$ to the equation $c_{1,1} : t'_3 + t'_6 = 1$ to get $c'_{1,1} : t'_4 + t'_6 = 0$. The corresponding row transformation results in the system,

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} t'_1 \\ t'_2 \\ t'_3 \\ t'_4 \\ t'_5 \\ t'_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

To see that each row has its leading non-zero entry in a distinct column, first note that each of the $c_{k,0}$ equations has a leading non-zero entry in column $2k-1$. These are distinct odd-indexed columns. Every other equation $c'_{k,m}$ with $m > 0$ has a leading non-zero entry in column $(2k-1)b^m + b^{m-1}$. We claim that each distinct equation has a leading non-zero entry in a distinct even-indexed column: if $(k, m) \neq (k', m')$ then $(2k-1)b^m + b^{m-1} \neq (2k'-1)b^{m'} + b^{m'-1}$.

Suppose otherwise. The highest power of b that divides $(2k-1)b^m + b^{m-1}$ is $m-1$, while the highest power of b that divides $(2k'-1)b^{m'} + b^{m'-1}$ is $m'-1$. Thus, if $m \neq m'$ then the equations have leading non-zero entries in different columns as claimed. Otherwise if $m = m'$ then

$$(2k-1)b^m + b^{m-1} = (2k'-1)b^m + b^{m-1}$$

and so $k = k'$, a contradiction. Thus, each $c'_{k,m}$ has a leading non-zero entry in a distinct column.

Therefore, the matrix has full row rank $K(n)$.

Case 2: b is even.

We will show that if b is even then the matrix is already in row echelon form and no row operations are required. We claim that the equations have leading non-zero entries in distinct columns. Suppose otherwise. Then $c_{k,m}$ has a leading non-zero entry in the same column as $c_{k',m'}$ for $k' \neq k$ and $m' \neq m$, and

$$(2k-1)b^m = (2k'-1)b^{m'}.$$

Suppose without loss of generality that $m' < m$ and

$$(2k-1)b^{m-m'} = (2k'-1).$$

The left hand side $(2k-1)b^{m-m'}$ is even but $2k'-1$ is odd. This is a contradiction. Therefore, the $c_{k,m}$ have leading non-zero entries in distinct columns and the matrix has full row rank $K(n)$.

Thus, the dimension of the solution space of the linear system is $n - K(n)$. Over GF(2) this means that the number of solutions $A(n)$ is $2^{n-K(n)}$. If n is odd then

$$\begin{aligned} A(n) &= 2^{n-K(n)} \\ &= 2^{1+n-1-K(n-1)} \\ &= 2 \cdot 2^{n-1-K(n-1)} \\ &= 2A(n-1). \end{aligned}$$

If n is even then

$$\begin{aligned} A(n) &= 2^{n-K(n)} \\ &= 2^{1+n-1-K(n-1)-\nu_b(n/2)-1} \\ &= 2^{-\nu_b(n/2)} \cdot 2^{n-1-K(n-1)} \\ &= 2^{-\nu_b(n/2)} A(n-1) \end{aligned}$$

as required. $\square$

Some example values of $A(n)$ for small values of b are given in Table 3.1.

b	3	4	5
$A(0)$	1	1	1
$A(1)$	2	2	2
$A(2)$	2	2	2
$A(3)$	4	4	4
$A(4)$	4	4	4
$A(5)$	8	8	8
$A(6)$	4	8	8
$A(7)$	8	16	16
$A(8)$	8	8	16
$A(9)$	16	16	32
$A(10)$	16	16	16
$A(11)$	32	32	32
$A(12)$	16	32	32
$A(13)$	32	64	64
$A(14)$	32	64	64
$A(15)$	64	128	128
$A(16)$	64	64	128
$A(17)$	128	128	256
$A(18)$	32	128	256

Table 3.1: Example values of $A(n)$

Chapter 4

Discrepancy-1 Graphs

In order to further characterize the infinite sequences over $\{-1, 1\}$ with D-discrepancy 1 for various sets D, we introduce the notion of discrepancy-1 graphs. The conditions of Lemma 5 can represented by a graph on the natural numbers.

Definition 21. The discrepancy-1 graph of a set $D \subseteq \mathbb{N}$ is the undirected graph $G = (V, E)$ where

$$\begin{aligned} V &= \mathbb{N}, \\ E &= \{((2k-1)d, 2kd) : k \in \mathbb{N}, d \in D\}. \end{aligned}$$

The subgraph of the discrepancy-1 graph of $\{1, 2, 4, 8, \ldots\}$ containing vertices 1 to 16 is shown in Figure 4.1.

Lemma 5 tells us that an infinite sequence t has D-discrepancy 1 iff $t_i = -t_j$ for all edges (i, j) in the discrepancy-1 graph of D.

A graph is bipartite if the vertices can be partitioned into two sets, A and B, such that there is no edge from a vertex in A to another vertex in A and there is no edge from a vertex in B to another vertex in B.

Lemma 22. *There exists an infinite sequence with D-discrepancy 1 iff the discrepancy-1 graph of D is bipartite.*

Proof. $(\Rightarrow)$

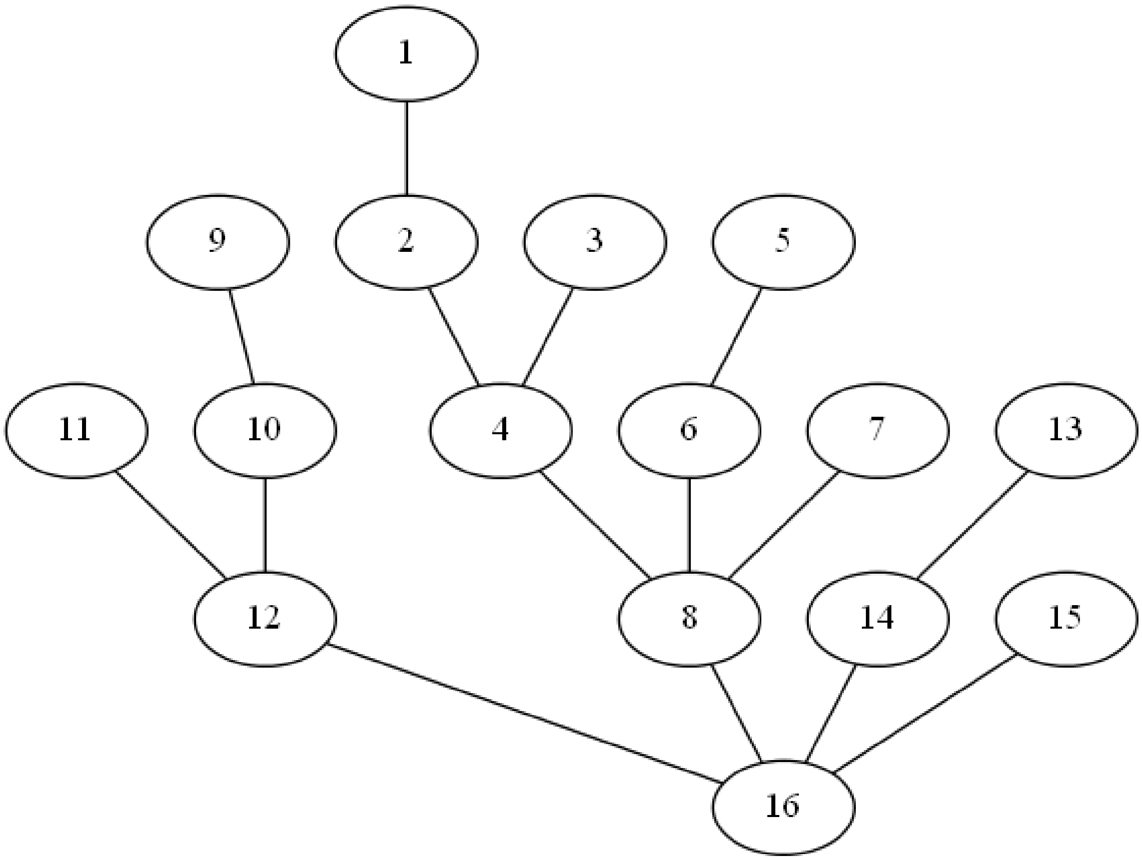

Figure 4.1: Vertices 1 to 16 of the discrepancy-1 graph of $\{1, 2, 4, 8, \dots\}$.

Let t be an infinite sequence with D-discrepancy 1. Let $A = \{i \in \mathbb{N} : t_i = 1\}$ and $B = \{i \in \mathbb{N} : t_i = -1\}$. Since $t_i = -t_j$ for all edges (i, j) in the discrepancy-1 graph of D, we cannot have that both i and j are in A or that both i and j are in B. Therefore, the graph is bipartite.

($\Leftarrow$)

Let A and B be disjoint sets such that $A \cup B = \mathbb{N}$ and for each edge (i, j) in the discrepancy-1 graph of D, we have $i \in A$ and $j \in B$ or vice versa. Define the infinite sequence t as follows:

$$t_i = \begin{cases} -1, & \text{if } i \in A; \\ 1, & \text{if } i \in B. \end{cases}$$

We have that $t_i = -t_j$ for all edges (i, j) in the discrepancy-1 graph of D so by Lemma 5, the sequence t has D-discrepancy 1. $\square$

Furthermore, this gives us a bijection between infinite sequences with D-discrepancy 1 and bipartitions of the discrepancy-1 graph of D. Since each connected component of a bipartite graph has two bipartitions (which can be obtained from each other by exchanging A and B), the number of bipartitions of the graph is 2^k where k is the number of connected components. Therefore, by the bijection to infinite sequences with D-discrepancy 1 we get the following corollary.

Corollary 23. *If the discrepancy-*1 *graph of a set* $D \subseteq \mathbb{N}$ *is bipartite and has* k *connected components then the number of infinite sequences with* D*-discrepancy* 1 *is* 2^k.

Lemma 22 lets us easily identify sets D such that there are no infinite sequences with D-discrepancy 1. We use the fact that a graph is bipartite iff it contains no odd-length cycles [8, Thm 1.2.17, p. 21]. For example, if $D = \{1, 2, 3\}$ then the discrepancy-1 graph of D contains the odd-length cycle $(9, 12, 10, 9)$ and thus is not bipartite. Therefore, by Lemma 22 there are no infinite sequences with D-discrepancy 1. Similarly, the discrepancy-1 graph of $\{2, 4, 6\}$ contains the odd-length cycle $(18, 20, 24, 18)$ and the discrepancy-1 graph of $\{2, 3, 5\}$ contains the odd-length cycle $(15, 18, 20, 15)$.

Chapter 5

Odd Discrepancy

When $D = \{1, 3, 5, 7, 9, \dots\}$, we conjecture that there are exactly 4 infinite sequences with D-discrepancy 1. This will follow from a conjecture about the discrepancy-1 graph of D. We will also present some experimental evidence for the conjecture.

Conjecture 24. Let $D = \{1, 3, 5, 7, \dots\}$. There are exactly 4 infinite sequences with D-discrepancy 1. They are:

$$(-1, 1), (-1, 1)^{\omega}$$
$$(-1, 1), (1, -1)^{\omega}$$
$$(1, -1), (-1, 1)^{\omega}$$
$$(1, -1), (1, -1)^{\omega}$$

Conjecture 25. Let G be the discrepancy-1 graph of D. For all $n \geq 4$, there exists a path in G from n to a smaller number.

Theorem 26. *Conjecture 25 $\Rightarrow$ Conjecture 24.*

Proof. Suppose Conjecture 25 holds. Since $1 \in D$, we have that $(1, 2)$ is an edge in G. There are no divisors of 1 or 2 in D other than 1, so these nodes have no other incident edges. Thus, the vertex set $\{1, 2\}$ is a connected component of G. If there exists a path in G from each $n \geq 4$ to a smaller number, we can easily see by induction on n that each $n \geq 4$ is connected to 3. Therefore, there are exactly

2 connected components in G and by Corollary 23 the number of infinite sequences with D-discrepancy 1 is 4. The 4 bipartitions of G give the sequences as required. $\square$

Theorem 27. *Conjecture 25 is true for all* $n < 2^{3217} - 1$.

Proof. We will proceed by cases.

- Case 1: n is even.

  Since n is even and 1 divides n, the edge $(n, n-1)$ is in G.

- Case 2: n is odd and $n+1$ is not a power of 2.

  Since n is odd and 1 divides n, the edge $(n, n+1)$ is in G. Because $n+1$ is not a power of 2, it has an odd divisor d with $1 < d < n$. Therefore, the edge $(n+1, n+1-d)$ is in G. We have that $n+1-d < n$ so there exists a path from n to a smaller number in G.

- Case 3: n is odd, $n+1$ is a power of 2, and n is composite.

  Let p be such that $n = 2^p - 1$. Since $n > 3$, we have that $p > 2$ and therefore $2^p - 1 \equiv 3 \pmod 4$. We claim that n has a divisor d such that $d \equiv 3 \pmod 4$. If n had no such divisor then each prime divisor of n would be congruent to $1 \pmod 4$ and consequently $n \equiv 1 \pmod 4$. But $n \equiv 3 \pmod 4$, so n has such a factor d as claimed. The edge $(n, n+d)$ is in G and $n + d \equiv 2 \pmod 4$. Therefore, $(n+d)/2$ is an odd number that divides $n+d$ and so the edge $(n+d, (n+d)/2)$ is in G. Since $d < n$, we have $(n+d)/2 < n$ and so this is a path from n to a smaller number in G.

- Case 4: n is odd, $n+1$ is a power of 2, and n is prime.

  Let p be such that $n = 2^p - 1$.

  - Subcase 1: $p \equiv 3 \pmod 4$.

    Clearly n divides n so the edge $(n, 2n) = (n, 2^{p+1} - 2)$ is in G. Since 1 divides $2^{p+1} - 2$, the edge $(2^{p+1} - 1, 2^{p+1} - 3)$ is in G. Continuing in this way we get that the edges $(2^{p+1} - 3, 2^{p+2} - 6)$ and $(2^{p+2} - 6, 2^{p+2} - 7)$ are in G. Since $p \equiv 3 \pmod 4$, we know that $p + 2 \equiv 1 \pmod 4$ and

Edge	Odd factor
$(2^p-1, 2^{p+1}-2)$	2^p-1
$(2^{p+1}-2, 2^{p+1}-3)$	1
$(2^{p+1}-3, 2^{p+2}-6)$	$2^{p+1}-3$
$(2^{p+2}-6, 2^{p+2}-7)$	1
$(2^{p+2}-7, 2^{p+2}-2)$	5
$(2^{p+2}-2, 2^{p+1}-1)$	$2^{p+1}-1$
$(2^{p+1}-1, 2^{p+1}+2)$	3
$(2^{p+1}+2, 2^p+1)$	2^p+1
$(2^p+1, 2^p+2)$	1
$(2^p+2, 2^{p-1}+1)$	$2^{p-1}+1$

Table 5.1: The path in G from 2^p-1 to a smaller number.

that $2^{p+2} \equiv 2 \pmod 5$. Thus, $2^{p+2}-7 \equiv 0 \pmod 5$ and 5 divides $2^{p+2}-7$. Therefore, the edge $(2^{p+2}-7, 2^{p+2}-7+5) = (2^{p+2}-7, 2^{p+2}-2)$ is in G. Now since $2^{p+1}-1$ is an odd number that divides $2^{p+2}-2$, the edge $(2^{p+2}-2, 2^{p+1}-1)$ is in G. Because $p \equiv 1 \pmod 2$, we have $p+1 \equiv 0 \pmod 2$ and thus $2^{p+1}-1 \equiv 0 \pmod 3$. Therefore, the edge $(2^{p+1}-1, 2^{p+1}+2)$ is in G. Since 2^p+1 is an odd number that divides $2^{p+1}+2$, we have that the edge $(2^{p+1}+2, 2^p+1)$ is in G. Because 1 divides 2^p+1, we have that the edge $(2^p+1, 2^p+2)$ is in G. Finally, the number $2^{p-1}+1$ is odd and divides 2^p+2, so the edge$(2^p+2, 2^{p-1}+1)$ is in G. Since $2^{p-1}+1$ is less than 2^p-1, this is a path from n to a smaller number in G. This path is summarized in Table 5.1.

– Subcase 2: $p \equiv 1 \bmod 4$ and $n < 2^{3217}-1$.

  For each such n, a path in G from n to a smaller number has been explicitly computed. Some of these paths are shown in Table 5.2 and the remainder can be found at (`http://www.student.cs.uwaterloo.ca/~aleong/mersenne_paths.txt`).

□

n	Path
$2^3 - 1$	$(7, 14, 13, 26, 25, 30, 15, 18, 9, 10, 5)$
$2^5 - 1$	$(31, 62, 61, 122, 121, 132, 99, 102, 51, 54, 27)$
$2^{13} - 1$	$(8191, 16382, 16381, 32762, 32761, 32942, 16471, 16472, 14413, 14414, 7207)$
$2^{17} - 1$	$(131071, 262142, 262141, 262152, 229383, 229386, 114693)$
$2^{61} - 1$	$(3058430092136939 51, 4611686018427387902,$ $4611686018427387901, 4611686018427387938,$ $4611369672836879817, 4611369672836879818,$ $2305684836418439909)$
$2^{89} - 1$	$(618970019642690137449562111, 1237940039285380274899124222,$ $1237940039285380274899124221, 1237940039285380274899126044,$ $928455029464035206174344533, 928455029464035206174344534,$ $464227514732017603087172267)$

Table 5.2: Paths from n to a smaller number in the discrepancy-1 graph of D.

Chapter 6

Discrepancy-1 Maximality

In this section we introduce the notion of discrepancy-1 maximality. A set D is discrepancy-1 maximal if there is an infinite sequence with D-discrepancy 1 but no element can be added to D and have this still be true. We will show that the sets $\{1, 2, 4, 8, \dots\}$ and $\{1, 3, 5, 7, \dots\}$ are discrepancy-1 maximal. We will also show the existence of a lexicographically least discrepancy-1 maximal set.

Definition 28. Let $D \subseteq \mathbb{N}$. We say D is discrepancy-1 maximal if there exists an infinite sequence with D-discrepancy 1 and for all $d \notin D$ there are no sequences with $(D \cup \{d\})$-discrepancy 1.

6.1 Maximality of the Odd Numbers

Theorem 29. *The set* $\{1, 3, 5, 7, \dots\}$ *is discrepancy-*1 *maximal.*

Proof. Let $D = \{1, 3, 5, 7, \dots\}$ and $d \notin D$. The sequence $(-1, 1)^\omega$ has D-discrepancy 1. Consider the discrepancy-1 graph of $(D \cup \{d\})$. This graph contains the odd-length cycle $(5d, 5d-5, 6d-6, 3d-3, 3d, 3d-1, 6d-2, 6d-3, 6d, 5d)$. Since the discrepancy-1 graph of $(D \cup \{d\})$ has an odd-length cycle, it is not bipartite and by Lemma 22 there are no infinite sequences with $(D \cup \{d\})$-discrepancy 1. Therefore, D is discrepancy-1 maximal. $\square$

Edge	Factor
$(5d, 5d-5)$	5
$(5d-5, 6d-6)$	$d-1$
$(6d-6, 3d-3)$	$3d-3$
$(3d-3, 3d)$	3
$(3d, 3d-1)$	1
$(3d-1, 6d-2)$	$3d-1$
$(6d-2, 6d-3)$	1
$(6d-3, 6d)$	3
$(6d, 5d)$	d

Table 6.1: An odd-length cycle in the discrepancy-1 graph of $(D \cup \{d\})$

6.2 Maximality of Powers of 2

We would like to prove that the set $\{1, 2, 4, 8, \ldots\}$ is discrepancy-1 maximal. By Theorem 7, the only infinite sequences with $\{1, 2, 4, 8, \ldots\}$-discrepancy 1 are the Thue-Morse sequence $\mathbf{t}$ and its complement $-\mathbf{t}$. We shall make use of the following property of these sequences [1]. Recall that $s_2(n)$ denotes the number of 1s in the binary representation of n.

Theorem 30. *Let $t \in \{\mathbf{t}, -\mathbf{t}\}$. For all $i, j \in \mathbb{N}$, it is the case that $t_i = t_j$ iff $s_2(i-1) \equiv s_2(j-1) \pmod 2$.*

In order to make use of this, we will prove a theorem about the Thue-Morse sequence.

Theorem 31. *Let $t \in \{\mathbf{t}, -\mathbf{t}\}$. For all integers $d > 1$ which are not powers of 2, there exists an odd integer n such that $s_2(nd-1) \equiv s_2((n+1)d-1) \pmod 2$. Furthermore, such an n always exists with $n \leq 2d-2$ and $s_2(n) \leq 2$.*

The existence of an n that satisfies the first part of this theorem follows from a theorem of Shevelev [7] as pointed out by Johannes Morgenbesser. The theorem states that for all $\epsilon > 0$ there exists $\alpha > 0$ such that

$$|\{0 \leq y < x : y \equiv d-1 \pmod d, s_2(y) \equiv 0 \pmod 2\}| = \frac{x}{2d} + \Omega(x^{\alpha-\epsilon}).$$

If no such n existed then $s_2(d-1) \not\equiv s_2(2d-1) \pmod 2$ and $s_2(3d-1) \not\equiv s_2(4d-1) \pmod 2$ and so on. Thus, half of the y such that $y \equiv d-1 \pmod d$ would have $s_2(y) \equiv 0 \pmod 2$ and the remainder term in the size of the above set would be $O(1)$, contradicting Shevelev's theorem. However, this does not say anything about the size of n or $s_2(n)$. We give a proof that demonstrates the required bounds.

Proof. Let ℓ be the number of trailing 1s in the binary representation of d.

Case 1: d is odd, $s_2(d)$ is odd, and ℓ is odd.

Let $m = 2^k$ be the smallest power of 2 greater than d and let $n = m + 1$.

Consider the binary representation of $nd = 2^k d + d$. Since 2^k is the smallest power of 2 greater than d, we have that $2^k d$ is a bitshift of d that does not overlap with d in any column. For example, if $d = 13$ then $m = 16$ and $n = 17$ and the sum $2^k d + d = 16 \cdot 13 + 13$ can be written as follows:

```
 11010000 (208)
+    1101 (13)
 --------
 11011101 (221)
```

Therefore, we have that $s_2(nd) = s_2(d) + s_2(d) \equiv 0 \pmod 2$. Since nd is odd, its binary representation ends with a 1. Thus $nd - 1$ will have 1 fewer 1s than nd, that is, $s_2(nd-1) = s_2(nd) - 1 \equiv 1 \pmod 2$.

Similarly, consider the binary representation of $(n+1)d = 2^k d + 2d$. Since 2^k is the smallest power of 2 greater than d, we have that $2^k d$ and $2d$ are bitshifts of d that overlap in exactly one column. For example, if $d = 13$ then $m = 16$ and $n = 17$ and the sum $2^k + 2d = 16 \cdot 13 + 2 \cdot 13$ can be written as follows:

```
 11010000 (208)
+   11010 (26)
---------
 11101010 (234)
```

Therefore, we have that $s_2((n+1)d) = s_2(d) + s_2(d) - \ell$. Since ℓ is odd, we get that $s_2(nd) \equiv 1 \pmod 2$. By writing $(n+1)d$ as $2(2^{k-1}d + d)$ and noting that $k > 1$ since $d > 1$, we see that 2 is the highest power of 2 that divides $(n+1)d$. Therefore, the binary representation of $(n+1)d$ ends with 10. Thus, the binary representation of $(n+1)d-1$ is the same as the binary representation of $(n+1)$ except that it ends with 01. Consequently, we have that $s_2((n+1)d-1) = s_2((n+1)d) \equiv 1 \equiv s_2(nd-1) \pmod 2$.

Case 2: d is odd, $s_2(d)$ is odd, and ℓ is even.

Let $m = 2^k$ be the smallest power of 2 greater than d and let $n = 2^{k-1} - 1$.

Consider the binary representation of $nd = 2^{k-1}d - d$. We have that d and $2^{k-1}d$ are bitshifts of d that overlap in exactly one column. For example, if $d = 463$ then $m = 512$ and $n = 255$ and the difference $2^{k-1}d - d = 256 \cdot 465 - 465$ can be written as follows:

```
 11100111100000000 (118528)
-        111001111 (463)
------------------
 11100110100110001 (118065)
```

Since d is odd and has an even number of trailing 1s, it has at least 2 trailing 1s. This means that the subtraction will set the 1s column to 1, set the next $k-2$ columns to their complement from d, set the next column to 1, set the next column to 0, and copy the remaining columns from $2^{k-1}d$. Therefore $s_2(nd) = s_2(d) + (k - s_2(d)) = k$. Since nd is odd, its binary representation ends with a 1. Thus $nd - 1$ will have 1 fewer 1s than nd, that is, $s_2(nd-1) = s_2(nd) - 1 = k - 1$.

Similarly, consider the binary representation of $(n+1)d = 2^{k-1}d$. Since $2^{k-1}d$ is a bitshift of d, we have that $s_2((n+1)d) = s_2(d)$. Since d is odd and $s_2(d)$ is odd, we have $s_2((n+1)d-1) = s_2(d) - 1 + k - 1 \equiv k - 1 \pmod 2$.

Thus, $s_2(nd-1) \equiv s_2((n+1)d-1) \pmod 2$.

Case 3: d is odd and $s_2(d)$ is even.

Let $m = 2^k$ be the smallest power of 2 greater than d and let $n = m - 1 = 2^k - 1$.

Consider the binary representation of $nd - 1 = 2^k d - d - 1 = (2^k d - 1) - d$. Since $2^k d$ is a bitshift of d and d ends with a 1, subtracting 1 from $2^k d$ removes a 1 and adds k 1s to the binary representation. Therefore, we have that $s_2(2^k d - 1) = s_2(d) - 1 + k$ and the binary representation of $2^k d - 1$ ends with k 1s. Since d has at most k digits, subtracting d from $2^k d - 1$ removes $s_2(d)$ 1s. Thus, we have $s_2(nd-1) =$

$s_2(2^k d - 1 - d) = s_2(d) - 1 + k - s_2(d) = k - 1$.

Similarly, consider the binary representation of $(n+1)d - 1 = 2^k d - 1$. As before, we have that $s_2(2^k d - 1) = s_2(d) - 1 + k$. Since $s_2(d) \equiv 0 \pmod 2$, we have that $s_2((n+1)d - 1) \equiv k - 1 \pmod 2$.

Thus, $s_2(nd - 1) \equiv s_2((n+1)d - 1) \pmod 2$.

Case 4: d is even.

Let us write $d = 2^k m$ with $m > 1$ odd and $k > 0$. Since $m > 1$ is odd, there exists an odd n such that $s_2(nm - 1) \equiv s_2((n+1)m - 1) \pmod 2$ by the previous cases. Consider the binary representation of $nd - 1$.

$$\begin{aligned} s_2(nd-1) &= s_2(nm2^k - 1) \\ &= s_2(nm) - 1 + k \\ &= s_2(nm - 1) + k \end{aligned}$$

Similarly, consider the binary representation of $(n+1)d - 1$. Let r be the number of trailing 0s in $(n+1)$.

$$\begin{aligned} s_2((n+1)d-1) &= s_2((n+1)m2^k - 1) \\ &= s_2((n+1)m) - 1 + r + k \\ &= s_2((n+1)m - 1) + 1 - r - 1 + r + k \\ &= s_2((n+1)m - 1) + k \\ &\equiv s_2(nd - 1) \pmod 2 \end{aligned}$$

□

Theorem 32. *The set $D = \{1, 2, 4, 8, \dots\}$ is discrepancy-1 maximal.*

Proof. Let $d \notin D$ and t be an infinite sequence with D-discrepancy 1. By Theorem 7, the sequence t must be either $\mathbf{t}$ or $-\mathbf{t}$. Since d is not a power of 2, Theorem 31 says that there exists an odd n such that $s_2(nd - 1) \equiv s_2((n+1)d - 1) \pmod 2$. By Theorem 30, we get that $t_{nd} = t_{(n+1)d}$. But this means that t does not have $\{d\}$-discrepancy 1 by Lemma 5. Since all the infinite sequences with D-discrepancy 1 do not have $\{d\}$-discrepancy 1, there are no infinite sequences with $(D \cup \{d\})$-discrepancy 1 and therefore, D is discrepancy-1 maximal. $\square$

6.3 Powers of b are Not Maximal

While we have seen several examples of discrepancy-1 maximal sets, the set $\{1, b, b^2, \dots\}$ is not discrepancy-1 maximal for $b > 2$. We will show that there exist infinite sequences with $(\{1, 3, 9, \dots\} \cup \{5\})$-discrepancy 1 and infinite sequences with $(\{1, b, b^2, \dots\} \cup \{2\})$-discrepancy 1 for all $b > 3$.

Theorem 33. *The set $D = \{1, b, b^2, \dots\}$ is not discrepancy-1 maximal for $b > 2$.*

Proof. Let t be an infinite sequence with D-discrepancy 1. We know by Theorem 17 that there are an uncountable number of such sequences.

Case 1: $b = 3$.

We will show that the discrepancy-1 graph of $(D \cup \{5\})$ contains no odd-length cycles. Since t has D-discrepancy 1, we know that that the discrepancy-1 graph of D is bipartite by Lemma 22. The discrepancy graph of D can be converted to the discrepancy-1 graph of $(D \cup \{5\})$ by adding the edges $\{(5n, 5(n+1) : \ n \text{ odd}\}$. This means that if there is no even-length path between $5n$ and $5(n+1)$ in the discrepancy-1 graph of D for all odd n then adding the edges $\{5n, 5(n+1) : \ n \text{ odd}\}$ does not create any new odd-length cycles. Therefore, to prove that the discrepancy-1 graph of $(D \cup \{5\})$ is bipartite, it is sufficient to show that there does not exist an even-length path between $5n$ and $5(n+1)$ in the discrepancy-1 graph of D for all odd n.

Case 1a: $5n \equiv 0 \pmod 3$.

Since $5n + 5$ is not divisible by 3, the corresponding vertex's only incident edge is $(5n + 4, 5n + 5)$. But $5n + 4$ is not divisible by 3 either so $\{5n + 4, 5n + 5\}$ is an entire connected component. Therefore, there is no path between $5n$ and $5n + 5$ in the discrepancy-1 graph of D.

Case 1b: $5n \equiv 1 \pmod 3$.

The vertex $5n$ is adjacent $5n - 1$. Since $5n - 1$ is divisible by 3, the vertex $5n - 1$ is adjacent to $5n - 4$ which is adjacent to $5n - 3$. Similarly, the vertex $5n + 5$ is adjacent to $5n+4$ and $5n+2$ because 3 divides $5n+5$. The vertex $5n+2$ is adjacent to $5n+3$.

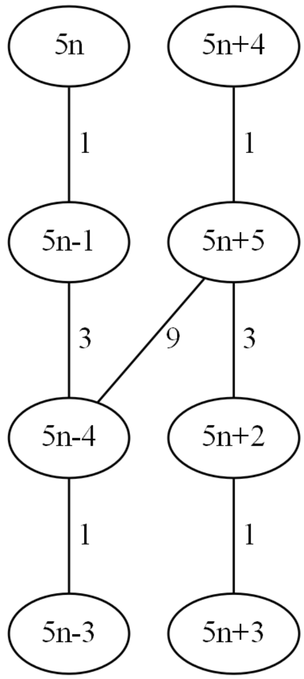

Figure 6.1: The discrepancy-1 graph of D in case $1b$ when 9 divides $5n-4$

If 9 does not divide $5n-1$ or $5n-4$ then $\{5n, 5n-1, 5n-4, 5n-3\}$ is an entire connected component, and there is no path between $5n$ and $5n+5$. If 9 divides $5n-1$ then 9 does not divide $5n+5$ or $5n+2$ and thus $\{5n+5, 5n+1, 5n+2, 5n+3\}$ is an entire connected component, and there is no path between $5n$ and $5n+5$. If 9 divides $5n-4$ then the vertex $5n-4$ is adjacent to $5n+5$ and there is the odd-length path $(5n, 5n-1, 5n-4, 5n+5)$. Only one of $5n-4$ or $5n+5$ can be divisible by 27, so this odd-length path is the only path between $5n$ and $5n+5$. Therefore, there is no even-length path between $5n$ and $5n+5$ in the discrepancy-1 graph of D for all odd n.

Case $1c$: $5n \equiv 2 \pmod 3$.

Since $5n$ is not divisible by 3, the corresponding vertex's only incident edge is $(5n, 5n-1)$. But $5n-1$ is not divisible by 3 either, so $\{5n, 5n-1\}$ is an entire connected component. Therefore, there is no path between $5n$ and $5n+5$ in the discrepancy-1 graph of D for all odd n.

Case 2: $b > 3$.

As in Case 1, we will show that the discrepancy-1 graph of $(D \cup \{2\})$ has no odd-length cycles by showing that the discrepancy-1 graph of D does not contain a path between $2n$ and $2(n+1)$ for all odd n.

The vertex $2n$ is adjacent to $2n-1$ and the vertex $2n+2$ is adjacent to $2n+1$. Since $b > 3$, at most one of these 4 vertices can be divisible by b. Therefore, either $\{2n, 2n-1\}$ or $\{2n+2, 2n+1\}$ is an entire connected component and thus, there is no path between $2n$ and $2n+2$ in the discrepancy-1 graph of D.

Since there exists an infinite sequence with D-discrepancy 1 and there exists $d \notin D$ such that there exists an infinite sequence with $(D \cup \{d\})$-discrepancy 1, the set D is not discrepancy-1 maximal.

□

6.4 Lexicographically Least Discrepancy-1 Maximal Set

In this section we will show that any finite set is not maximal. This implies the existence of a lexicographically least discrepancy-1 maximal set.

Theorem 34. *If $D \subsetneq \mathbb{N}$ is a finite set then D is not discrepancy-1 maximal.*

Proof. Let $\mathrm{lcm}(D)$ denote the lowest common multiple of the elements of D. Let $d' = 2\mathrm{lcm}(D)$. We claim that D is not discrepancy-1 maximal because there exists an infinite sequence with $(D \cup \{d'\})$-discrepancy 1. Consider the discrepancy-1 graph of D. We will show that nd' is not connected to $(n+1)d'$ for any odd n, thereby showing that adding d' to D will not create any new cycles in the discrepancy-1 graph.

For all $d \in D$ there exists m such that $nd' = 2n\mathrm{lcm}(D)$ can be written as $2md$. Thus the incident edges to nd' are $(nd', nd' - d)$ for all $d \in D$. We claim that no edge can connect a vertex with value less than or equal to nd' to a vertex with value greater than nd'. Suppose that $(k, k+d)$ is an edge with $k \leq nd'$ and $k+d > nd'$. This means that $nd' \geq k > nd' - d$. Since k must be a multiple of d, we have $k = nd'$. But k/d must be odd and nd'/k is even. This is a contradiction, therefore, nd' is the vertex with the greatest value in its connected component. Thus, the vertex nd' is not connected to $(n+1)d'$.

Therefore, adding d' to D does not create any new cycles in the discrepancy-1 graph and so the discrepancy-1 graph of $(D \cup \{d'\})$ is bipartite. By Theorem 22, there exists a sequence with $(D \cup \{d'\})$-discrepancy 1 and so D is not maximal. $\square$

We define the lexicographically least discrepancy-1 maximal set $\mathcal{D}$ as follows:

$$\begin{aligned}
D_0 &= \emptyset \\
D_i &= \begin{cases} D_{i-1} \cup \{i\}, & \text{if there exists an infinite sequence with } (D_{i-1} \cup \{i\})\text{-discrepancy 1;} \\ D_{i-1}, & \text{otherwise.} \end{cases} \\
\mathcal{D} &= \lim_{i \to \infty} D_i
\end{aligned}$$

We can see that $\mathcal{D}$ is maximal since if it were not, then there would exist $d \notin \mathcal{D}$ such that there exists an infinite sequence with $(\mathcal{D} \cup \{d\})$-discrepancy 1. But if this is the case, then $D_i = D_{i-1} \cup \{i\}$ and $i \in \mathcal{D}$, a contradiction.

In order to compute each D_i we must determine if there exists an infinite sequence with $(D_{i-1} \cup \{i\})$-discrepancy 1. Such a sequence exists iff the discrepancy-1 graph of $(D_{i-1} \cup \{i\})$ is bipartite. While it would be difficult to determine if an infinite graph is bipartite computationally, it suffices to examine a finite subgraph.

Theorem 35. *Let $D \subsetneq \mathbb{N}$ be a finite subset of the naturals and G be the discrepancy-1 graph of D. The graph G is bipartite iff the subgraph of G containing the vertices 1 to $2lcm(D)$ is bipartite.*

Proof. $(\Rightarrow)$

If G is bipartite then any subgraph of G is also bipartite by the same bipartition.

$(\Leftarrow)$

Suppose that G is not bipartite. Then G must contain an odd-length cycle $(n_1, n_2, \ldots, n_m)$. We saw in the previous theorem that $2\text{lcm}(D)$ is the vertex with the greatest value in its connected component. This means that either $n_i \leq 2\mathrm{lcm}(D)$ for $1 \leq i \leq m$ or $n_i > 2\mathrm{lcm}(D)$ for $1 \leq i \leq m$. If $n_i \leq 2\mathrm{lcm}(D)$ for $1 \leq i \leq m$ then the subgraph of G containing vertices 1 to $2\mathrm{lcm}(D)$ contains this odd-length cycle and is not bipartite. Otherwise, we can write

$$2\mathrm{lcm}(D)q + r_i = n_i$$

with $q \in \mathbb{N}$ and $1 \leq r_i \leq 2\mathrm{lcm}(D)$.

Consider the edge (n_i, n_{i+1}) and let us write it as $((2k-1)d, 2kd)$ for some $k \in \mathbb{N}$ and $d \in D$. If $n_i = (2k-1)d$ then $n_{i+1} = 2kd$ and

$$\begin{aligned} 2\mathrm{lcm}(D)q + r_i &= (2k-1)d, \\ \frac{2\mathrm{lcm}(D)q}{d} + \frac{r_i}{d} &= 2k - 1. \end{aligned}$$

Since $2\mathrm{lcm}(D)q/d$ is even and $2k-1$ is odd, we have $\frac{r_i}{d}$ is odd. Therefore,

$$\begin{aligned} \left(\frac{r_i}{d}d, \left(\frac{r_i}{d} + 1\right)d\right) &= (r_i, r_i + d) \\ &= (r_i, r_{i+1}) \end{aligned}$$

is an edge in G.

Otherwise, $n_i = 2kd$ and $n_{i+1} = (2k-1)d$. In this case,

$$\begin{aligned} 2\mathrm{lcm}(D)q + r_i &= 2kd, \\ \frac{2\mathrm{lcm}(D)q}{d} + \frac{r_i}{d} &= 2k. \end{aligned}$$

Since $2\mathrm{lcm}(D)q/d$ is even and $2k$ is even, we have $\frac{r_i}{d}$ is even. Therefore,

$$\begin{aligned} \left(\frac{r_i}{d}d, \left(\frac{r_i}{d} - 1\right)d\right) &= (r_i, r_i - d) \\ &= (r_i, r_{i+1}) \end{aligned}$$

is an edge in G.

In either case we see that (r_i, r_{i+1}) is an edge in G. Since $r_i \leq 2\mathrm{lcm}(D)$ for all

$1 \leq i \leq m$, the subgraph of G containing the vertices 1 to $2\text{lcm}(D)$ contains the odd-length cycle $(r_1, r_2, \ldots, r_m)$. Therefore, this subgraph is not bipartite. $\square$

Therefore, we can compute $\mathcal{D}$ as follows.

```
D := {}
for i = 1 to infinity
    D' := D union {i}
    G := subgraph of discrepancy-1 graph of D' containing vertices 1 to 2*lcm(D')
    if G is bipartite then
        add i to D
    end if
end for
```

This lexicographically least discrepancy-1 maximal set starts as follows:

$$\mathcal{D} = \{1, 2, 4, 7, 14, 25, 50, 100, 140, 350, 700, 1000, 2625, \ldots\}$$

Chapter 7

$\{1\}$-Discrepancy

So far we have been mostly concerned with counting the number of sequences with a fixed discrepancy in terms of the length of the sequence. We can also count the number of sequences in terms of both the discrepancy and the length. In particular we will count the number of sequences of length n with $\{1\}$-discrepancy c by relating the sequences to lattice paths.

Theorem 36. *The number of* $\{-1,1\}$ *sequences of length* n *with* $\{1\}$*-discrepancy* c *is*

$$\begin{aligned}\sum_{i+j=n;0\le i,j\le c}\Bigg[\binom{n}{i} &- \sum_{h=0}\binom{n}{i-(c+1)-2h(c+1)} \\ &- \sum_{h=0}\binom{n}{i+(c+1)+2h(c+1)} \\ &+ \sum_{h=1}\binom{n}{i+2h(c+1)} \\ &+ \sum_{h=1}\binom{n}{i-2h(c+1)}\Bigg].\end{aligned}$$

Proof. If we consider each 1 in the sequence to be a north step and each -1 in the sequence to be an east step, then each $\{-1,1\}$ sequence is a lattice path. If the length of the sequence is n, then a total of n steps are taken and the lattice path walks from $(0,0)$ to some (i,j) where $i+j=n$. The sequence has $\{1\}$-discrepancy c if the number of 1s does not exceed the number of -1s by c in any prefix of the sequence and vice versa. This corresponds to the conditions that the lattice path never crosses the lines $y=x+c$ or $y=x-c$. Note that both i and j must be at most c since otherwise (i,j) will be outside these lines. Fray and Roselle [5] discovered that the number of lattice paths from $(0,0)$ to (i,j) that do not touch these lines is

$$\begin{aligned} D(i,j;c,c) = \binom{i+j}{j} &- \sum_{h=0} \binom{i+j}{j-c-2hc} - \sum_{h=0} \binom{i+j}{j+c+2hc} \\ &+ \sum_{h=1} \binom{i+j}{j+2hc} + \sum_{h=1} \binom{i+j}{j-2hc} \end{aligned}$$

Therefore, the number of $\{-1,1\}$ sequences of length n with $\{1\}$-discrepancy c is

$$\sum_{i+j=n;0\leq i,j\leq c} D(i,j;c+1,c+1)$$

as required. $\square$

Chapter 8

Conclusions

We have been able to count the number of finite and infinite sequences with D-discrepancy 1 or 2 for various particular subsets of $\mathbb{N}$. These results are summarized in table 8.1.

D	Discrepancy	# of sequences of length n	# of infinite sequences
$\{1, 2, 4, 8, \dots\}$	1	$2^{s_2(n)}$	2
$\{1, b, b^2, \dots\}$ $(b > 2)$		$A(n)$	uncountable
$\{1, 3, 5, 7, \dots\}$		?	4 (conjectured)
$\{1, 2, 4, 8, \dots\}$	2	?	uncountable
$\{1, b, b^2, \dots\}$ $(b > 2)$		?	uncountable
$\{1, 3, 5, 7, \dots\}$		?	?

Table 8.1: Number of sequences with D-discrepancy c

We hope that these results can serve as building blocks and can be generalized to count the number of sequences with finite D-discrepancy. We have seen that there are no infinite sequences with $\mathbb{N}$-discrepancy 1, but it is not yet known if there exist infinite sequences with $\mathbb{N}$-discrepancy 2. Showing that such a sequence exists would solve the Erdős discrepancy problem. We hope that by studying the sequences with D-discrepancy 2 for particular sets D, we can learn more about the existence of sequences with $\mathbb{N}$-discrepancy 2.

We have also shown that certain sets (namely $\{1, 2, 4, 8, \dots\}$, $\{1, 3, 5, 7, \dots\}$, and $\mathcal{D}$) are discrepancy-1 maximal; if any element is added to one of these sets, there are no infinite sequences with D-discrepancy 1 where D is the resulting set. Showing that any finite set is not maximal allows us to demonstrate the existence of a lexicographically least maximal set $\mathcal{D}$, an interesting set in its own right.

Chapter 9

Open Problems

Many problems still remain open. We would like to complete Table 8.1 as well as proving Conjecture 24.

Open Problem 1. Complete Table 8.1.

We proved that the lexicographically least infinite sequences with $\{1, b, b^2, \dots\}$-discrepancy c are the fixed points of morphisms followed by codings when $c \leq 3$. Such sequences are called automatic sequences [2].

Open Problem 2. Is the lexicographically least infinite sequence with $\{1, b, b^2, \dots\}$-discrepancy c automatic for all $b > 1$ and $c > 0$?

The theory of discrepancy maximal sets leads to some interesting questions as well. In all of the cases we examined, the number of infinite sequences with D-discrepancy 1 was finite exactly when D was maximal. We would like to know if this is always the case.

Open Problem 3. Let $D \subseteq \mathbb{N}$. Prove that the number of infinite sequences with D-discrepancy 1 is finite iff D is discrepancy-1 maximal.

We can also generalize the concept of discrepancy-1 maximality to discrepancy-2 maximality. A set D is discrepancy-2 maximal if there exists an infinite sequence with D-discrepancy 2 and for all $d \notin D$, there do not exist any infinite sequences with $(D \cup \{d\})$-discrepancy 2.

Open Problem 4. Find a set that is discrepancy-2 maximal.

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