Topology in Fundamental Physics

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Authorship Statement

This thesis is based on the following articles:

- **J. Hackett**, "Locality and translations in braided ribbon networks," Class. Quant. Grav. **24** (2007) 5757-5766. [hep-th/0702198 [HEP-TH]]
- S. Bilson-Thompson, **J. Hackett** and L. H. Kauffman, "Particle Topology, Braids, and Braided Belts," J. Math. Phys. **50**, 113505 (2009) [arXiv:0903.1376 [math.AT]].
- S. Bilson-Thompson, **J. Hackett**, L. Kauffman, L. Smolin, "Particle Identifications from Symmetries of Braided Ribbon Network Invariants," [arXiv:0804.0037 [hep-th]].
- **J. Hackett** and Y. Wan, "Infinite Degeneracy of States in Quantum Gravity," J. Phys. Conf. Ser. **306**, 012053 (2011) arXiv:0811.2161 [hep-th].
- **J. Hackett**, "Invariants of Braided Ribbon Networks," arXiv: 1106.5096 [math-ph].
- **J. Hackett**, "Invariants of Spin Networks from Braided Ribbon Networks," arXiv: 1106.5095 [math-ph].

All the articles that include collaborators are a result of even collaboration on the part of all authors. Of these papers I prepared the original drafts of all manuscripts save for the second. Passages and figures from these papers have been adapted for inclusion here with the consent of the other authors.

Abstract

In this thesis I present a mathematical tool for understanding the spin networks that arise from the study of the loop states of quantum gravity. The spin networks that arise in quantum gravity possess more information than the original spin networks of Penrose: they are embedded within a manifold and thus possess topological information. There are limited tools available for the study of this information. To remedy this I introduce a slightly modified mathematical object - Braided Ribbon Networks - and demonstrate that they can be related to spin networks in a consistent manner which preserves the diffeomorphism invariant character of the loop states of quantum gravity.

Given a consistent definition of Braided Ribbon Networks I then relate them back to previous trinion based versions of Braided Ribbon Networks. Next, I introduce a consistent evolution for these networks based upon the duality of these networks to simplicial complexes. From here I demonstrate that there exists an invariant of this evolution and smooth deformations of the networks, which captures some of the topological information of the networks.

The principle result of this program is presented next: that the invariants of the Braided Ribbon Networks can be transferred over to the original spin network states of loop quantum gravity.

From here we represent other advances in the study of braided ribbon networks, accompanied by comments of their context given the consistent framework developed earlier including: the meaning of isolatable substructures, the particular structure of the capped three braids in trivalent braided ribbon networks and their application towards emergent particle physics, and the implications of the existence of microlocal topological structures in spin networks.

Lastly we describe the current state of research in braided ribbon networks, the implications of this study on quantum gravity as a whole and future directions of research in the area.

Acknowledgements

I would like to thank my supervisor and mentor Lee Smolin for all of his help, guidance, patronage, support and most of all for his friendship over the course of my studies. Additionally I would like to thank him for the excitement that he has for physics: were he to not have this, I may never have had the chance for him to tell me (during a happenstance encounter on a bus) of exciting new research from an Australian named Sundance presenting particle physics in terms of braided ribbons.

My greatest thanks are to my wife, Sonia Markes, without whom I would have forgotten that the role of a physicist is not simply to solve problems, but also to ask all questions that we can of a theory. Her aspiration to ask exceptional questions rather than to answer solvable ones humbles me, as it should humble all researchers who - like me - have taken pride in the ease with which one can prove an incorrect idea false without seeing that even a bridge with one end can lead to a better perspective.

I would like to thank my brother Benjamin for helping me both to edit my work and to put my thoughts in order. He is a credit to his profession, and proof that though your career path might lead you away from the sciences, you can certainly still be a scientist. I would also like to thank my mother Judith for her offers to edit my work, her understanding when I refused her help and my Step-father Richard for his not offering - all of these were appreciated in equal measure.

I would like to thank my collaborators on the subject - Louis Kauffman, Sundance Bilson-Thompson and Yidun Wan - along with the other researchers I've had the chance to work with: Steve Weinstein Fotini Markopoulou-Kalamara, Simone Speziale, Isabeau Premont-Schwarz, Francesco Caravelli, Yuxiang Gu, and Jorge Escobedo.

My work has been supported by a generous grant from the National Sciences and Engineering Research Council of Canada (NSERC), support from the Perimeter Institute for Theoretical Physics, and the University of Waterloo.

I was fortunate enough to have been hosted for some time at the Centre de Physique Thorique de Luminy while holding a Michael Smith Foreign Studies Supplement (from NSERC), and it was there that some of the central ideas of this thesis were formed. This would not have happened were it not for those at the institute, particularly: Carlo Rovelli, Simone Speziale, Alejandro Perez, You Ding, Francesca Vidotto, Mingyi Zhang, and Wolfgang Wieland. I am indebted to them for both their hospitality and their openly defiant stance to my ideas (for which they should be lauded).

I would be remiss were I not to thank all of my friends made at the Physics Undergraduate Society (PhUS) at the University of Waterloo including: Maximilien Schirm, Matthew

Peters, Matthew Dimitrov, Peter MacDonald, Daniel Iwasa-Madge, Adrian Popescu, and Alex Parent. Were it not for the ability to walk into the lounge and ask that people help me to hold ribbons and perform physical operations with them, I may never have gained the necessary intuition to carry out the calculations that follow. I must also particularly thank Adam Ramer for his willingness to aid an artistically challenged physicist in producing the best of the figures that follow.

Lastly I would like to thank the members of my advisory and examination committees for the time they took to examine my research critically and help me to present my work in its best light.

Dedication

Above any other, this work is dedicated to my wife Sonia. Though I acknowledge her intellectual help, it is because of her love and support that this work is for her.

My grandfather Henry Pollit (of blessed memory) must be honoured for all of his teachings, and so I also dedicate this work to his memory. I know it will stand as but a single point in the legacy of his intellect, but it is with great pride that I add it to that list.

Lastly, I dedicate this work to all those of my friends and relations that may in the future pursue a similar path. That you may reach your goals and maintain a tradition of inquiry and discovery is my most sincere wish.

Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

In [11] Dirac laid out his vision of the future of mathematical physics. Despite the early date of the paper, Dirac's prediction was eerily accurate: mathematical physics has transitioned from its traditional role (of finding solutions to more and more complicated differential equations) to a field which pushes into a large variety of pure mathematical disciplines. Though he never explicitly mentioned topology in it, his paper actually laid out the components that would one day lead to the foundations of this thesis: the introduction of a gauge freedom in the phase of the wave function leading to flux quantization and - though it was only in the introduction - the introduction of the concept of anti-matter.

Later, in [12], Dirac laid out what would become the canonical procedure for quantization. This procedure provided the basis for the study of quantum gravity.[28] Through [30] this procedure bore fruit - using the same notion of fluxes as [11] - giving a loop space representation for quantum gravity. Later in [31] these states were related to spin networks, leading to the picture of loop quantum gravity that is used today.[29]

These spin networks differed from the previous spin networks due to their connection to the Wilson loops that gave Loop Quantum Gravity its name. The states of Loop Quantum Gravity were found on the basis of trying to solve the three constraints of General relativity: the Gauss constraint, the Diffeomorphism constraint and the Hamiltonian constraint. The loop states satisfy the Gauss constraint directly, but it is by identifying all diffeomorphisms of the loops with the same state that we solve the diffeomorphism constraint. Moving over to the spin network picture from the loops it is then imperative that the embedding information be taken over with it: otherwise we lose having solved the diffeomorphism constraint and lose our ability to connect to the knotted Wilson loops. This gives the modification of spin networks that we will examine in this thesis: embedded spin networks, which are graphs embedded into a manifold together with some labelings on each edge and vertex. We will make two simplifications in how we look at these states: we will ignore the labelings and we will also fix the valence of a given graph so that each vertex has the same number of edges. The last thing we will keep in mind is that the states of loop quantum gravity are the identification of these networks up to diffeomorphism.

It is useful when considering the spin network states to have some understanding of the basic physical interpretation associated with these embedded spin networks. The physical picture comes from the action of the volume and area operators on these networks.[28] The nodes of a spin network in this picture become associated with chunks of space - they are the parts of the network that give rise to the volume spectrum. The edges which connect these volumes then become associated with areas, giving us information about how these volumes are connected. This picture is what gives rise to the understanding of a discrete space-time.

In this thesis I will present braided ribbon networks, a deviation from the spin networks of loop quantum gravity, which incorporate more structure. Braided ribbon networks were informed from previous modifications of spin networks coming from studying quantum gravity with a positive cosmological constant [32] and attempting to institute local causality [25]. These approaches combined with an attempt to construct particle states from topological structures in a preon model[5], leading to the introduction of braided ribbon networks as an attempt to have matter emerge from quantum gravity.[6] The braided ribbon network approach from here split into two programs: one studying 3-valent networks [16, 3, 4] and one studying 4-valent networks [34, 33, 20, 19, 35, 17]. Though there was some initial success in attempting to use this framework to construct emergent matter in quantum gravity, more recent work has focused on the realization that results in braided ribbon networks could have more direct applications in the spin network states of loop quantum gravity.[18] This recent work has led to a restructuring of the framework in [14] to allow both the three and four valent formalisms to be addressed together, and to apply the results more universally to the states of loop quantum gravity [15].

In chapter 2 I will outline the unified framework of braided ribbon networks from [14]. This will include defining the dual of nodes as simplices, introducing the local evolution moves and demonstrating how this unified framework connects to the original form of braided ribbon networks from [6].

In chapter 3 I will present the result from [15] that there is a map from embedded spin networks to a subset of the braided ribbon networks. I will also present a new - though commonly assumed - result that there is a map from the braided ribbon networks to the embedded spin networks.

In chapter 4 I will present the reduced link as defined in [14] as a topological property of the Braided Ribbon Networks that is invariant under deformations and the evolution moves. I will then relate this to the original concept of a reduced link from [6]. Lastly I will present from [15] what could be considered the largest result towards the more recent goals of braided ribbon networks: that the reduced link can be made into an invariant of the embedded spin network states that emerge from loop quantum gravity through the map of chapter 3.

In chapter 5 I will present the ideas from [16] of isolated substructures. These structures represent parts of the topological structure of the braided ribbon networks that take a local form with respect to the underling graph of the braided ribbon networks. I'll also show how the evolution moves can generate translations of these objects with respect to the natural distance on the graph. Lastly I'll compare this notion to the notion of a subsystem introduced in [6].

In chapter 6 I will present the work from [3] and [4], which examines the properties of a specific subclass of isolated substructures called capped 3-braids. Here I'll present a classification of the capped 3-braids and examine the implications of it on the original goals of Braided Ribbon Networks as laid out in [6]. The work also goes further, creating a proof-of-concept mapping from these structures to the quantum numbers of the standard model of particle physics. Along with this I'll give a critical appraisal of the implications of this work for future attempts at looking for emergent particle physics in quantum gravity.

In chapter 7 I will present the results from [18] where Braided Ribbon Networks were first used to connect back to the spin network states of loop quantum gravity. Here I will demonstrate the existence of ultra-local structures (topological features of the embedding that can be localized to a single edge of the graph) and discuss the implication of the presence of these kinds of structures on the study of loop quantum gravity.

Finally in chapter 8 I will present an appraisal of the state of research and a vision for the future of research in Braided Ribbon Networks as the subject continues.

Chapter 2

Braided Ribbon Networks

In the past there have been two different approaches used under the title of Braided Ribbon Networks (see for example [33] and [16]). In the following I will introduce a single formalism which incorporates both of these results and therefore unifies them so that results from one can be applied to the other. I originally presented these results in [14] and the content that follows is mostly from that paper.

2.1 The Formalism

Here I introduce a new unified definition of Braided Ribbon Networks of valence n (with $n \geq 3$) as follows:

> We begin by considering an n-valent graph embedded in a compact 3 dimensional manifold. We construct a 2-surface from this by replacing each node by

a 2-sphere with n punctures (1-sphere boundaries on the 2-sphere), and each edge by a tube which is then attached to each of the nodes that it connects to by connecting the tube to one of the punctures on the 2-sphere corresponding to the node.

Lastly we add to each tube $n-1$ curves from one puncture to the other and then continue these curves across the sphere in such a way that each of the n tubes connected to a node shares a curve with each of the other tubes.

We will freely call the tubes between spheres *edges*, the spheres *nodes* and the curves on the tubes *racing stripes* (terminology from [32, 10, 9]) or less formally *stripes*.

We will call a Braided Ribbon Network the equivalence class of smooth deformations of such an embedding that do not involve intersections of the edges or the racing stripes.

We immediately face the following consequence: under this definition there are only braided ribbon networks of valence 2,3 or 4 (with valence 2 being a collection of framed loops). To see this fact we consider a 5-valent node - a 2-sphere with 5 punctures, with each puncture connected to each other puncture by a non-intersecting curve. Taking each puncture as a node, and the curves as edges, we then get that these objects would constitute the complete graph on 5 nodes and as they lie in the surface of a 2-sphere, such a graph would have to be planar. This is impossible by Kuratowski's theorem[22]: the complete graph on 5 nodes is non-planar. Likewise, we have for any higher valence n that the graph that would be constructed would have the complete graph on 5 nodes as a subgraph, and so they too can not be planar. If the reader desires an intuition for this, it may be instructive to recall that these statements follow from the four colour theorem - the existence of such a node would imply the existence of a map requiring five (or more) colours.

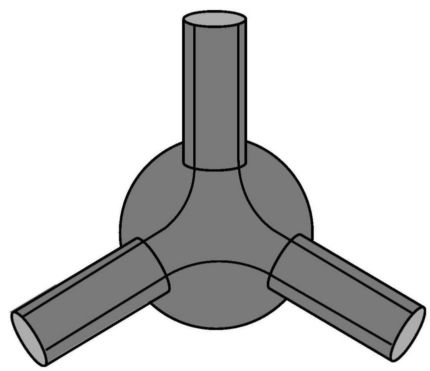


Figure 2.1: Three Valent Node

We will demonstrate later that this restriction is natural, and that it could be lifted if we instead allowed the dimension of the manifold to increase, along with the dimension of the surface which we extend the graph into. We can also introduce a modification to the framework that allows for higher valence vertices. To do this we first make a few definitions.

Definition 1. *We define the **natural valence** of a braided ribbon network to be the number of racing stripes on each edge.*

Definition 2. *We say that a node is **natural** if each of the tubes which intersect share a racing stripe with each of the other tubes. Otherwise we will say that a node is **composite**.*

We can then define a n-valent BRN with natural valence m (here n can take values of $n = km - 2(k-1)$ for any integer k) as a braided ribbon network where each of the

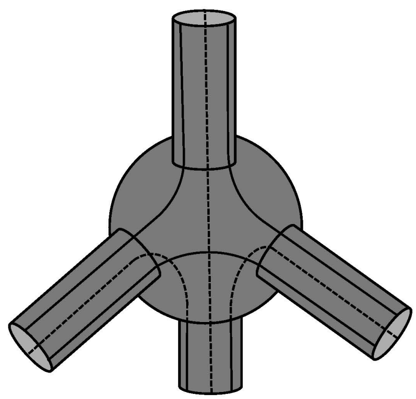


Figure 2.2: Four Valent Node

nodes has n tubes which intersect it but where each of the tubes has $m-1$ racing stripes. Likewise we can define a multi-valent BRN with natural valence m in a similar manner but without fixing the value of k for all nodes. We then construct composite nodes by connecting natural nodes in series by simple edges and shortening the edges which connect them internally until all of these nodes combine into a single sphere with the appropriate number of punctures (see figure 2.3). As these combined nodes are simply glued they are then dual to gluings of simplices which when grouped together would be equivalent to a polygon (for a natural valence of 3) or a polyhedron with triangular faces(for natural valence of 4).

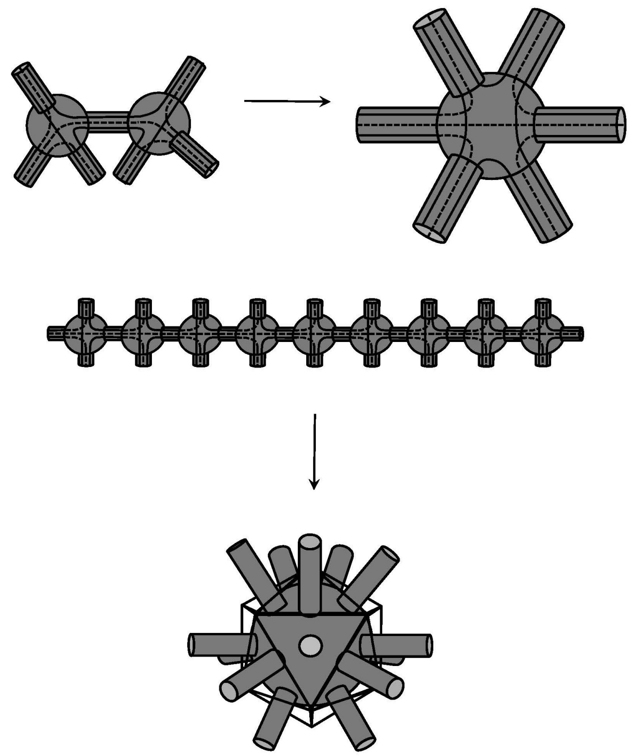


Figure 2.3: Forming Composite nodes

2.1.1 Duality of Nodes to Simplices

The nodes of an n-valent braided ribbon network can each individually be considered dual to an $(n-1)$-simplex. The edges which intersect the node are identified with the $(n-2)$-faces of the simplex, and the curves on the edges are identified with the $(n-3)$-faces of the simplex. We require that the map from the BRN node to the simplex be consistent in the following way: when a curve is common to two edges that intersect a node, we require that the $(n-3)$-face that it is identified with is the $(n-3)$-face common to the $(n-2)$-faces that the two edges are identified with.

For 3-valent Braided Ribbon Networks this identification gives us that each node is dual to a triangle (see figure 2.4), whereas for 4-valent Braided Ribbon Networks this identification give us a tetrahedron for each node (see figure 2.5). This gives some insight into the restriction on the natural valence of Braided Ribbon Networks: a 5-valent node would be dual to a 4-simplex, which given that the graph is embedded in 3 dimensions would be a significant feat, whereas if we extended to a higher dimensional surface embedded in a higher dimensional manifold, this should be possible.

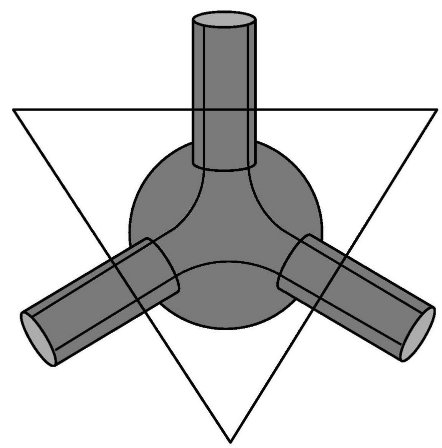


Figure 2.4: Three valent node dual to a triangle

Definition 3. *An edge is knotted if there exists a closed compact* 2*-surface which intersects the edge only at its boundary (the node) such that the contraction of the edge is a knotted arc not equivalent to the unknotted arc.*

Definition 4. *An edge is twisted if there does not exist a smooth deformation of a plane on which the racing stripes can be projected down onto without them intersecting.*

We extend this notion of duality to multiple nodes in a significantly restricted manner.

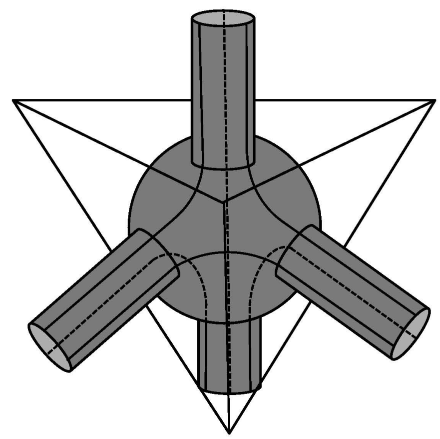


Figure 2.5: Four valent node dual to a tetrahedron

If an edge connecting two nodes has neither knotting nor twisting we will call the edge *simple* - we extend the dual picture to gluings of simplices to allow the $(n-1)$-simplex of each node to be glued together along the $(n-2)$-face corresponding to the shared edge. We continue to require the consistency of all of the shared subsimplices in this scenario (see figure 2.6). We will refer to a collection of nodes such that any edges which connect them are simple as being *simply glued*.

2.1.2 The Evolution Moves

With the duality established in section 2.1.1 we define evolution moves on the graph by making reference to the Pachner moves on the dual gluings of $n-1$-simplices. The oper-

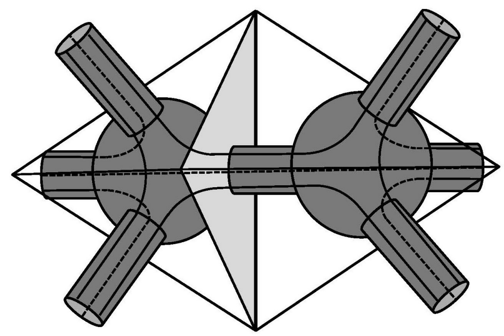


Figure 2.6: Gluing of two simplices dual to nodes

ations are defined as preserving the external identifications of sub simplices, and we will go through these explicitly for both 3 and 4-valent BRNs. These operations are referred to as evolution moves due to the fact that the corresponding operations in spin-networks are the result of the action of the Hamiltonian on the graphs - which would give rise to a sense of evolution.

Pachner moves generate transformations between triangulations of piecewise linear manifolds. They take the form of moves between triangulations which have the same boundary. We label them by the number of glued simplices which the move has as its origin and the number that it has as its target. We will then construct from these operations corresponding operations on collections of simply glued nodes.

For 3-valent networks, the corresponding simplices are triangles. The Pachner moves on triangles are the $1-3$ move (figure 2.7), the $3-1$ move (figure 2.8) and the $2-2$ move (figure 2.9). These correspond to similarly named evolution moves for the braided ribbon networks (figures 2.10 and 2.11).

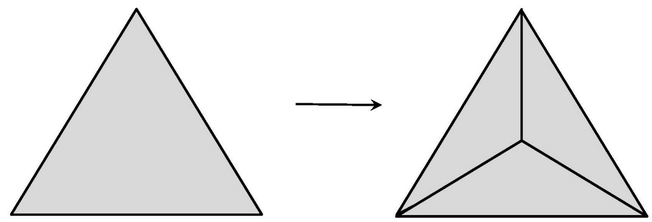


Figure 2.7: The 1-3 Pachner Move

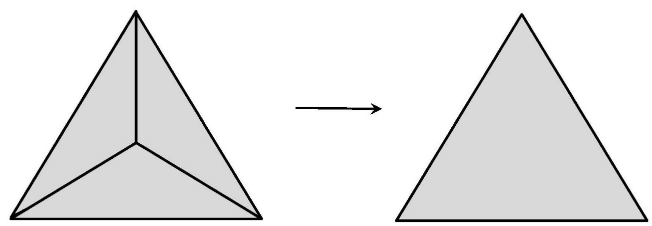


Figure 2.8: The 3-1 Pachner Move

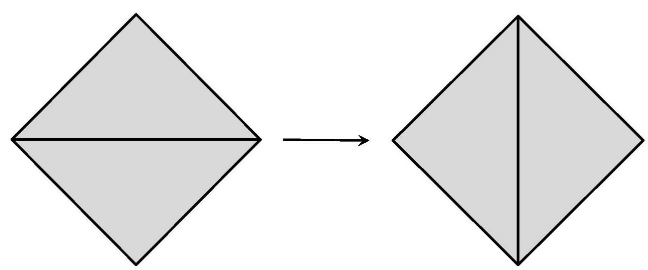


Figure 2.9: The 2-2 Pachner Move

For 4-valent networks, the corresponding simplices are tetrahedra. Here the Pachner moves are the $2-3$ and $3-2$ moves (figure 2.12), and the $1-4$ and $4-1$ moves (figure 2.13). The corresponding evolution moves on braided ribbon networks are then figures

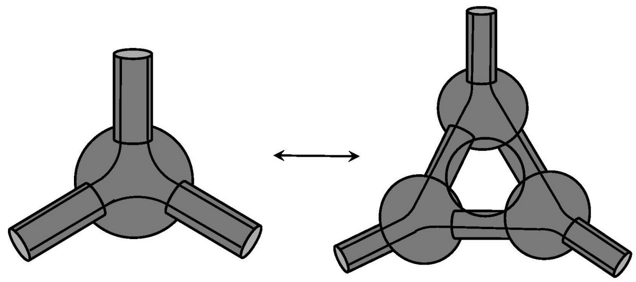


Figure 2.10: The 1-3 and 3-1 Evolution Moves

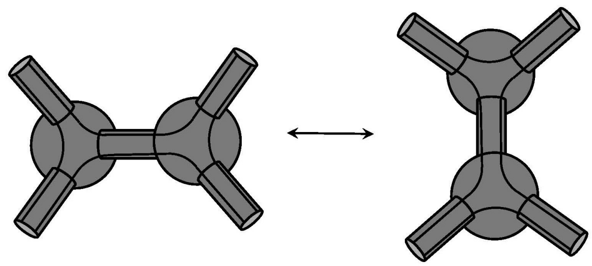


Figure 2.11: The 2-2 Evolution Move

2.14, and 2.15.

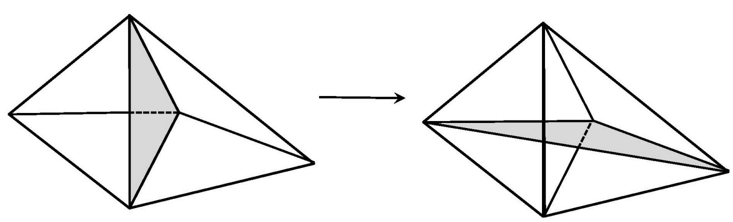


Figure 2.12: The 2-3 and 3-2 Pachner Moves

In all of these cases we require that the moves are only allowed if they can be performed smoothly, that is to say that the operation can be performed through smooth deformations

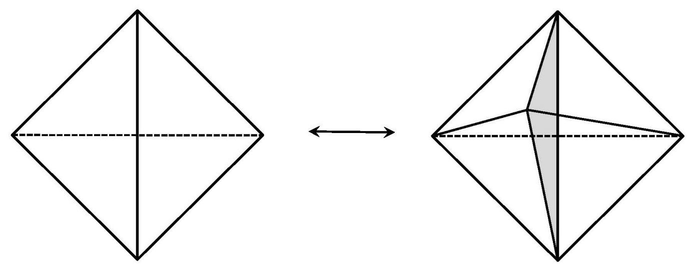


Figure 2.13: The 1-4 and 4-1 Pachner Moves

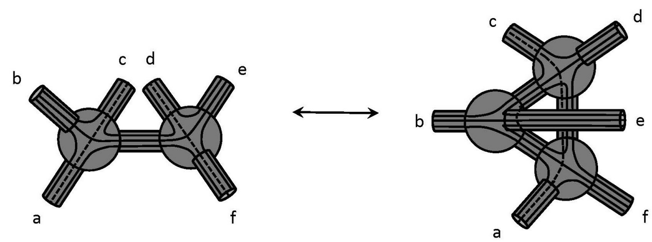


Figure 2.14: The 2-3 and 3-2 Evolution Moves

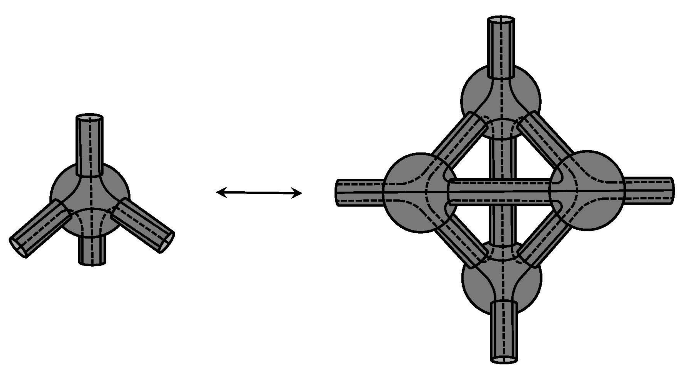


Figure 2.15: The 1-4 and 4-1 Evolution Moves

of the 2-surface together with point-like changes in genus of the surface which occur at points which are in the interior of the 3-manifold. This restricts us from performing something like the $1-3$ move in such a manner that the three nodes now encircle some other part of the 2-surface or a feature of the topology of the 3-manifold (such as would occur if the space were a 3 dimensional torus).

2.2 Ribbons vs. Tubes: an Equivalence

We will take this opportunity to explain why these graphs made from tubes and spheres are referred to as braided ribbon graphs by demonstrating the connection to the graphs of [6].

In [6] trivalent braided ribbon networks are defined as two dimensional surfaces embedded in a 3-manifold constructed by taking the union of trinions (2-surfaces with three distinct 'legs' along which they can be connected to one another) as shown in figure 2.16. Each trinion defines a node, and each of the legs of the trinions is an edge. Like the edges of braided ribbon networks the legs which connect the trinions to one another can be braided, twisted or knotted. These ribbon graphs are considered up to equivalence class defined by smooth deformations of the resulting surfaces.

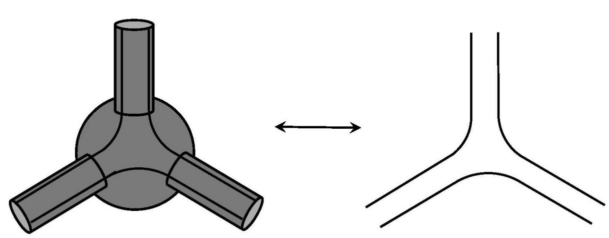


Figure 2.16: From BRN to trinions

We construct from one of these ribbon graphs a braided ribbon network as we've defined in chapter 2 as follows: for each node of the network we consider a closed ball in the embedding space which has the node on its surface but which has an empty intersection with the rest of the ribbon graph. These spheres then define the nodes of the braided ribbon network. The edges of the braided ribbon networks are then defined by similarly constructing tubes between these spheres so that the boundaries of the edges of the ribbon graph coincide with the boundary of the tubes. The boundaries of the surface of the ribbon graph then become the racing stripes of the braided ribbon network.

Likewise we can construct a ribbon graph from a 3-valent braided ribbon network by making the following observation: at each node the racing stripes divide the sphere into two parts, likewise along each edge the tube is divided in two by the racing stripes. We can consistently choose one side or the other and identify this as the surface of a ribbon graph (alternatively we can think of 'squishing' the two halves together into a single surface, in a sort defining one side to be the 'front' and the other the 'back').

It is often simpler to understand work done in the 3-valent case in terms of these trinion BRNs, and so where the original work was done in the formalism of these trinion BRNs we will remain in this formalism. This will not impact the validity of any of the results, and the reader can easily translate the results between these pictures as needed.

Chapter 3

The Relation of Braided Ribbon Networks to Spin Networks

In order to make use of Braided Ribbon Networks, it is helpful to make contact between them and the spin-network states of loop quantum gravity. We will make this contact incrementally: in section 3.1 we will move from BRNs to spin networks (this step is the easiest), in section 3.2 we will introduce a small mathematical formalism that will allow us to establish a map from spin networks to BRNs in section 3.2. Through these maps between BRNs and spin networks we will have a full picture of how we can use BRNs as a tool study spin networks. In particular, this shall allow us to import the results that we will develop in chapter 4 into our understanding of spin networks. For our purposes we will take a definition of a spin network that corresponds to the origins of Loop Quantum Gravity along with a further assumption of fixed valence.[31] To do this we will use the term to refer to a graph of some fixed valence embedded in a 3-manifold, the edges and vertices of this graph could be labeled but we will ignore those labelings throughout as they can be associated with the combinatorial structure of the graph rather than the topological.

3.1 Mapping from Braided Ribbon Networks to Spin Networks

The construction of the map from braided ribbon networks to spin networks is a rather straightforward process inherent to the definition of the BRNs. As the basic object underlying each BRN is a graph embedded in a 3-manifold, we can simply consider a map which replaces each BRN with this underlying graph.

Though this map is easily defined, it is helpful to elaborate upon it. Though we can define this map, it is illustrative to examine the ways in which one can construct it given an arbitrary BRN. The first map is constructed from an arbitrary BRN to a spin network through three steps. Firstly, for each edge and node of the BRN we define a radius r (with the radii of the edges much smaller than the radii of the spheres) and deform these tubes and spheres to have their respective radius with respect to an arbitrary metric on the embedding space. Secondly, we smoothly contract each tube by taking the radii to 0 one by one while maintaining contact between each component. This will leave us with a spin network embedded in the 3-manifold.

We observe two important details about this map. The first is that if we consider the volume interior to the 2-surface of the BRN, this map is a retraction of this volume down to its underlying graph. In fact, excluding the racing stripes this map is a deformation retract. The second detail that is important is that the map from BRN to spin network exists regardless of twists in the BRN (the rotations of the racing stripes with respect to the surface). As we are contracting the surface down to an infinitely thin edge, any behaviour of the stripes disappear.

3.2 Mapping from Spin Networks to Braided Ribbon Networks

We mention here - for future use - the idea of a blackboard embedding of a graph: an embedding which places the nodes of a graph into a plane and keeps the edges of the graph in the plane except for those locations where they cross one another. Such an embedding is named for the fact that it is how graphs are drawn in practice. Such an embedding introduces ambiguities for a four-valent (or higher) graph, and so we require that a blackboard embedding of such a graph to also include a labeling of each node as shown in figure 3.1 with a slash on one of the incident edges to move it up out of the plane of embedding - giving an orientation to the node. Blackboard embeddings of higher valence graphs can similarly be marked in order to make such an embedding unambiguous. With this ambiguity removed we can define each node in such a graph to be locally dual to a simplex (a triangle for a 3-valent node, or a tetrahedron for a 4-valent node) in the same way as we did for BRNs in [14]. We do so in the manner prescribed by figures 3.2 and 3.3. We also note that instead of this disambiguation we could instead not allow planar embeddings of 4-valent nodes, but as this thesis is constrained to diagrams in two dimensions we will use this notation. The need to remove these planar embeddings comes from the fact that there are two possible orientations of a tetrahedron: given a choice of a face there are two possible orderings of the other three faces. Removing the planar embeddings of the spin-networks allows us to have a well defined dual by fixing the orientation.

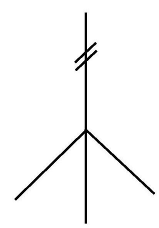


Figure 3.1: Unambiguous Blackboard Embedding of a 4 Valent Node

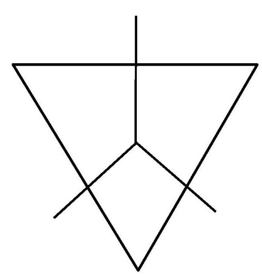


Figure 3.2: 3 Valent Spin Network Node Dual to a Triangle

As the content of the states of loop quantum gravity we are concerned with are not dependent on the labelings of the edges or vertices of the graphs, we will ignore these labelings and consider instead only the equivalence classes of graphs of fixed valence embedded in a 3-manifold under the restriction of the diffeomorphism group. The restriction of the group is to omit a class of transformations which change the orientation of the nodes of the graph - that is to say the transformation which takes a graph between two blackboard em-

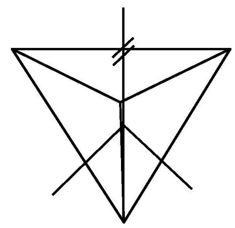


Figure 3.3: 4 Valent Spin Network Node Dual to a Tetrahedron (the Edge Opposite the Slash Corresponds to the Rear Face)

beddings which have different cyclic orderings of the edges around each node (for a 3-valent node) or the cyclic orderings of the edges around each node with one edge removed (for a 4-valent node) - these are demonstrated in figure 3.4 and 3.5. We will call such equivalence classes *diffnets* and the different elements of the equivalence class will be called embeddings of a diffnet. We extend this notion to those graphs that don't possess blackboard embeddings by making the requirement that they not change the orientation of the dual simplices of their nodes - this is equivalent to considering the blackboard embedding to be a local property rather than a global one.

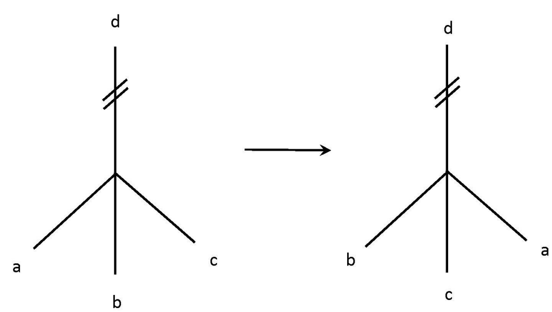


Figure 3.4: Allowed Ordering Changes

We now set about to demonstrate that there exists a map from 3 or 4-valent diffnets

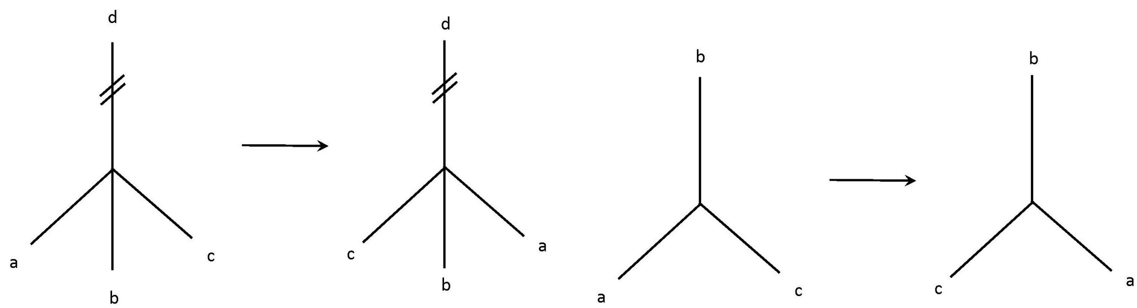


Figure 3.5: Disallowed Ordering Changes

in compact manifolds to BRNs. To do so we must first construct a map from a given embedding of a fixed valence graph to an embedding of a BRN. To do this we first observe that since our graph is embedded in a compact manifold that any open cover has a finite subcover. We will construct an open cover as follows: around each node of the graph we take an open ball, around each edge we take an open region which intersects the open balls of the corresponding nodes but does not contain the nodes themselves (we also require that the open region not contain any portion of any other edges), we then take some open cover of the rest of the manifold of which none of the open sets contain any of the nodes or portions of the edges. As the manifold is compact there exists some finite subcover of this open cover, and by construction this subcover will contain each of the balls around nodes and regions around edges. We now define on each part of the open cover (labeled by i) a map F_i which will eventually be assembled into a map F on the entire space. On each of the open balls the map F_i takes each node and replaces it with a BRN node which is dual to the same simplex (demonstrated in figure 3.6. For each region containing an edge the map F_i takes the edge and replaces it with a tube with racing stripes. We then require that the choice of racing stripes be one that does not introduce any twisting. This is insufficient to fix the F_i for the edges, however by requiring these F_i to be compatible with those on the open balls around the nodes we can fix them sufficiently. The maps F_i on the open sets which don't contain any portion of the graph are trivial. We thus can construct the map F from this collection of maps F_i as a map from an embedding of a diffnet to an embedding of a BRN.

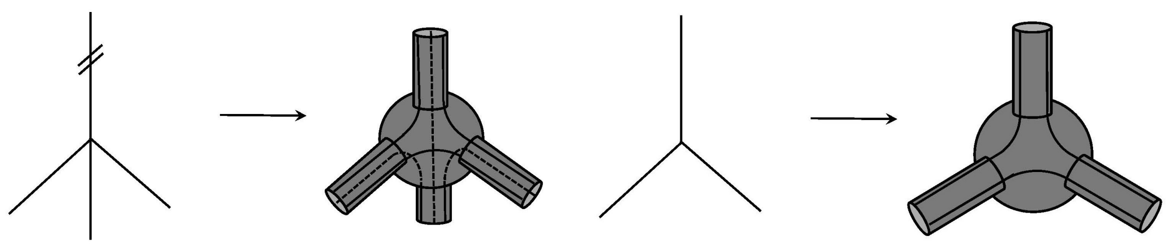


Figure 3.6: Diffnet to BRN Transformation

Our next step is to show that the diffeomorphisms commute with this F, by which we mean: that for any diffeomorphism (which preserves node orientation), ϕ, which takes one embedding of a diffnet to embedding there exists a deformation, χ, between the corresponding elements of the BRN. To demonstrate this we need only realize that any such diffeomorphism can be mapped consistently into a deformation of the corresponding embedding of a BRN. We construct this map as follows: diffeomorphisms are continuous and thus are continuous across the edges (or nodes) of the graph. We can 'inflate' this continuity as we 'inflate' the edges and nodes - leaving the deformation constant across the surface itself - but giving an identical transformation on the overall structure of the graph. This deformation then exactly mimics the diffeomorphism, barring changing the orientation of a node (which is already forbidden). There is one other form of diffeomorphism which might cause concern - diffeomorphisms which add twists to the edges - and so we will address it specifically.

What we mean by a diffeomorphism introducing twists to edges should be clarified: the edges here are simply curves in space and so cannot have twist in the same way that a tube or any other 3-dimensional object would. Instead here we mean that the diffeomorphism replicates the process described by the first Reidemeister move, shown in figure 3.7, or its inverse. It is tempting to think that it would not be possible to map such a diffeomorphism into a deformation of a BRN or that worse still should we do so we would be left with an operation that introduces twist. This is not the case. As we show in figures 3.8 and 3.9 (for a pair of racing stripes) the corresponding deformation to this diffeomorphism preserves twist (the same follows in four valent networks with three racing stripes). The source of the confusion here is most likely due to the fact that the ribbon in figure 3.8 is the corresponding inflation of the edge in figure 3.6, not the ribbon in figure 3.9 as might be people's first instinct.

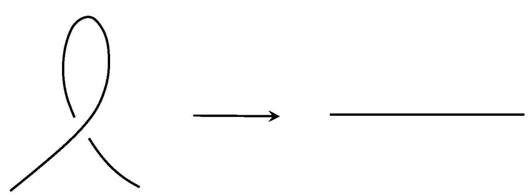


Figure 3.7: The First Reidemeister Move

As F maps every embedding in the equivalence class of a diffnet to an embedding in a single equivalence class of the BRNs, we can promote it to a map on the equivalence classes.

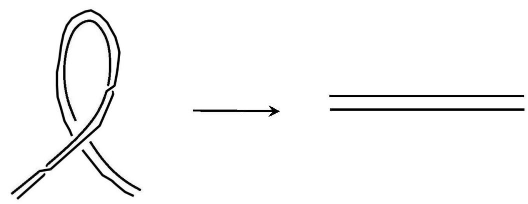


Figure 3.8: An Untwisted Ribbon

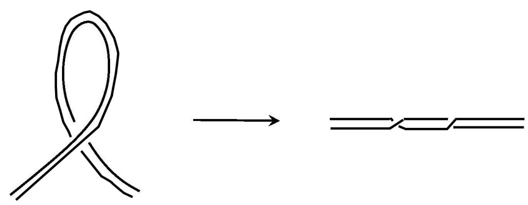


Figure 3.9: A Twisted Ribbon

That is to say that F maps diffnets to BRNs in the same way that it maps members of these equivalence classes. This gives us an important result: we have that the equivalence classes of spin-networks embedded up to orientation preserving diffeomorphisms can be mapped consistently into the braided ribbon networks. For clarity we point out that this map is not onto for the space of Braided Ribbon Networks: by our construction we will not map into any BRN with twisted edges.

Chapter 4

Topological Objects in Braided Ribbon Networks

With the relation between BRNs and spin networks as laid out in chapter 3, the next step is to demonstrate the advantage of braided ribbon networks: their affinity for the study of the embedding of the graphs. To do this we need to examine the kinds of structures that can result from the embedding of these graphs, and then determine if there are properties within the BRNs that can be used to study these structures.

In considering an embedded graph we are confronted with two topological structures: linking of edges (including knotting when considering linking of an edge with itself) and non-contractible cycles in the graph (for a manifold with a non-trivial fundamental group). When one considers the additional structure of a BRN we also introduce the linking of racing stripes, which is commonly referred to as twist of the edges.

In this chapter we will introduce a property which captures some of this information - the reduced link - and prove that it is invariant under the evolution moves.

4.1 The Reduced Link

We move now to the goal of this chapter: to demonstrate the existence of invariants of the braided ribbon networks under these evolution moves. To do this we first observe the fact that the evolution moves are defined in such a way that the external edges (and stripes) are unchanged. We will first study trivalent networks and then demonstrate that the invariant for these networks likewise works for 4-valent BRNs. We consider the set of curves defined by the racing stripes in the manifold which the braided ribbon network is embedded in, which we call the *stripe diagram* of the BRN.

Theorem 1. *For a BRN generated from a finite graph, the curves which constitute the stripe diagram is made up of loops.*

Proof. Curves can either take the form of loops - those curves with no boundary points - or terminating curves (which do have boundary elements. We prove by contradiction by assuming that there is a curve which is not a loop. This curve can be parameterized and - as the graph is finite - has an origin and a terminus. The racing stripes however are formed by composing paths along the edges and the nodes - with the stripe along each edge connecting two nodes and the stripe along each node connecting two edges. We thus find that there are no points which could act as an origin or a terminus, and we thus have a contradiction. □

The stripe diagram is thus a collection of loops embedded in a manifold. We now make an observation: examining the $1-3$ move we see that the operation changes the corresponding stripe diagram by the introduction of a single loop. Also due to the restriction we imposed in section 2.1.2, we see that this loop cannot be linked with any other loop, or be a non-trivial element of the fundamental group. From this insight we ask the following: what if we considered the stripe diagram, removing any such loops?

Definition 5. *We construct the **Reduced Link** of the three valent BRN by removing from the stripe diagram all unlinked and unknotted loops which are isotopic to the identity of the fundamental group of the 3-manifold the BRN is embedded in.*

The structure of the reduced link illustrates its promise of capturing the topological information: being composed of loops from the stripe diagram it is capable of capturing all three types of topological information we have identified. As the stripes on the edges necessarily follow their path they can capture the linking of edges and non-trivial cycles in the graph, and as they are in fact the stripes it should also capture any linking of the stripes.

Now we demonstrate that the reduced link of the three valent BRN is in fact an invariant of the evolution moves (the existence of this proof was originally mentioned in [16] for the original picture of trivalent BRNs).

Theorem 2. *The reduced link of a three valent BRNs is invariant under the* $1-3$*,* $2-2$ *and* $3-1$ *evolution moves dual to the Pachner moves.*

Proof. To prove this, we consider the reduced link of a BRN both before and then after the application of each of the evolution moves. The contributions from the involved nodes is unchanged as is demonstrated by the stripe diagrams in figure 4.1, and thus the reduced link of a network is preserved under the evolution moves. □

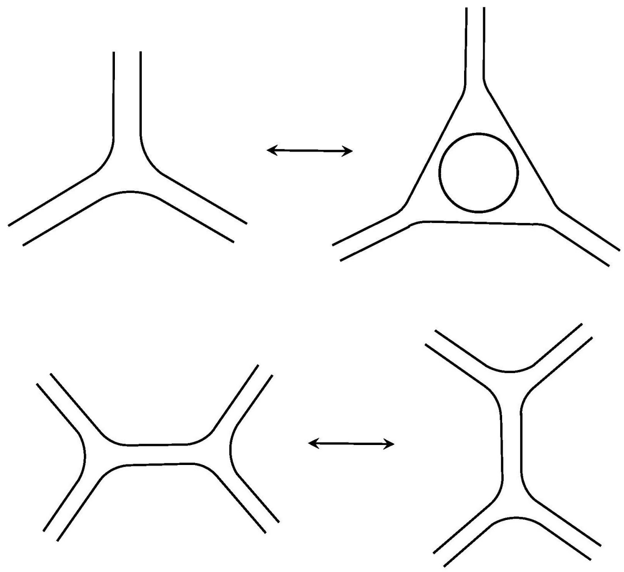


Figure 4.1: Stripe Diagrams of 3 valent moves

We now extend the definition of the reduced link to the four valent BRNs. This extension is obvious: there is nothing in the definition of the reduced link that would prevent us from taking it - as is - for 4-valent BRNs as well. Thus all that remains is to demonstrate that the reduced link is also invariant under the 4-valent evolution moves.

Theorem 3. *The reduced link of a four valent BRN is invariant under the evolution moves.*

Proof. We do this - as in the proof of theorem 2 - through looking at the stripe diagrams in figure 4.2.

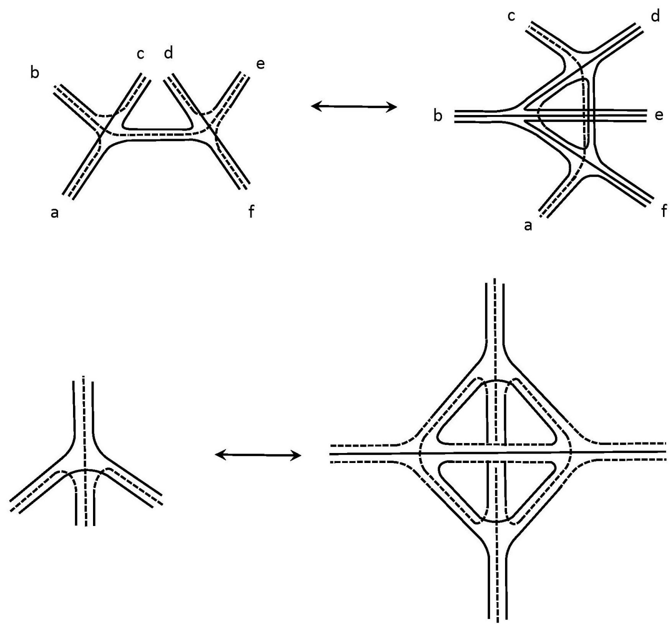


Figure 4.2: Stripe Diagrams of 4 valent moves

Looking at these figures, we see that all the stripes connect the same exterior edges as before (this corresponds to the exterior faces and edges of the glued tetrahedra being unchanged by the Pachner move). It only remains to examine whether any linking, knotting or non-trivially contractible curves have been introduced. We demonstrate this by going step by step through the deformations (shown here again in figures 4.3 and 4.4)that give rise to the evolution moves (we will only proceed in a single direction, as the processes are invertible).

The $1-4$ move is achieved, beginning with a single node, by introducing four closed loops on the surface of the sphere one in each of the four regions bounded by the racing stripes - these loops each correspond to the introduction of an edge in the dual Pachner move. From here we separate the sphere into four parts (this process is allowed under the caveat of point-like changes in the genus in the interior of the manifold), leaving the only aesthetic deformations remaining (straightening out the connections between the sphere into tubes etc). In this process, the only changes to the stripe space were the introduction of the four closed loops on the surface of the sphere. By definition these cannot have any knotting or linking, and must be in the trivial element of the fundamental group. These therefore are removed in the map from the stripe space to the reduced link, and so leave the reduced link unchanged.

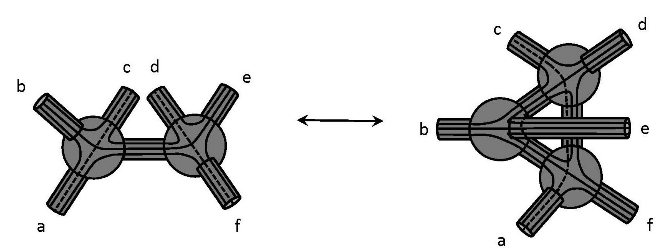


Figure 4.3: The 2-3 and 3-2 Evolution Moves

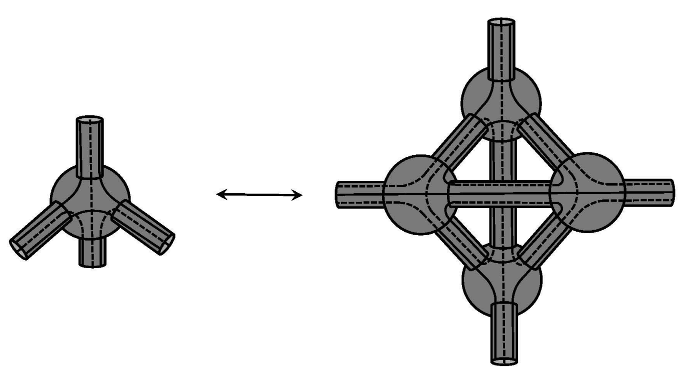


Figure 4.4: The 1-4 and 4-1 Evolution Moves

The $2-3$ move begins with 2 nodes connected by a single edge. To make the steps clear we will label the edges as in the diagram, and though the steps we take will specify particular edges they aren't unique and so can be made general through relabeling. This interior edge is expanded to combine it and the two spheres into a single sphere. We now introduce a single closed loop corresponding to the introduction of the single edge in the dual Pachner move. This loop is placed in the region interior to the stripes which connect f to d, d to e, and e to f (or equivalently a to c, c to b, and b to a). Splitting the sphere into a torus through the center of this loop, we then regroup the edges into pairs c with d, b with e and a with f. For aesthetics we constrict the region of the sphere between the pairs of edges into tubes, making our three nodes. We can see here that the only change to the stripe space was the introduction of a single trivial loop. This change does not alter the reduced link, completing the proof of the invariance of the reduced link. □

From the identification in section 2.2 we can see the correspondence between the reduced link defined on these two equivalent types of graph. In [6] the reduced link is defined by considering the boundary of the 2-surface independently and removing unlinked and unknotted curves. Here we've seen that the boundary of the 2-surface is equivalent to the racing stripes of the BRN, and thus that this operation is equivalent to the reduced link we've defined. We will review this in section 4.1.2.

Just as we have the reduced link for BRNs, we can define the reduced link for a spin network through the map F between diffnets and BRNs. The reduced link of a spin network is computed by first applying a diffeomorphism that puts the spin network into a blackboard embedding, applying the map F and then taking the reduced link of the resulting BRN. As F commutes with the evolution moves on the graphs this reduced link is also an invariant of the networks.

4.1.1 The Reduced Link for Spin Networks

The point of finding such a relation between spin networks and braided ribbon networks in section 3.2 is to be able to bring just this type of object over to spin-networks. To do this we need to show one further thing: that the evolution moves dual to the Pachner moves also commute with the map F of section 3.2. This follows immediately if we impose the same requirement on the evolution moves for embedded spin-networks that we used to construct the evolution moves for BRNs: that the orientation of the simplices dual to the nodes is reflected in the orientation of the nodes and that just as that orientation of the external faces is preserved by the Pachner moves, it is preserved by the evolution moves.

Stated more formally: in section 3.2 we have shown that there exists a functor from the category of diffnets with morphisms equal to the applications of the evolution moves to the category of BRNs with morphisms equal to the applications of the evolution moves. The category of BRNs can be considered as the sum of its connected subcategories (each corresponding to the set of BRNs which are connected by a series of evolution moves). The reduced link is a functor from the category of BRNs to the category of links embedded in the manifold up to diffeomorphism (with all morphisms being identity morphisms). Due to the lack of non-identity morphisms in this category of links, it assigns the same link to any two connected elements of the BRN category. With these two functors we then construct the composite functor - the reduced link of diffnets (or more generally, spin networks).

It is worth noting that though the reduced link is an invariant which captures information of how spin networks or BRNs are embedded in a manifold, it is not proven to capture all of the topological information. Whether this is actually the case, or whether there exists some other invariant which would capture all of the information remains an open question.

4.1.2 The Reduced Link in the Trinion BRNs

As a large number of the results which occurred in the trinion BRNs relied heavily on the reduced link, it is helpful to review how the reduced link was formulated in that context. In the trinion picture the stripe space is instead the boundary of the 2-surface which defines our graph.

Here the reduced link is taken by considering the boundary of the 2-surface, and then removing the unlinked, unknotted curves which are trivial elements of the fundamental group of the manifold. We can see this process in figure 4.5 which demonstrates the process of taking the reduced link for a portion of a trinion BRN.

The invariance of the reduced links of the trinion BRNs follows directly from the proofs of the reduced links of the standard BRNs through the fact that the stripe diagrams are identical to the boundaries of the trinions.

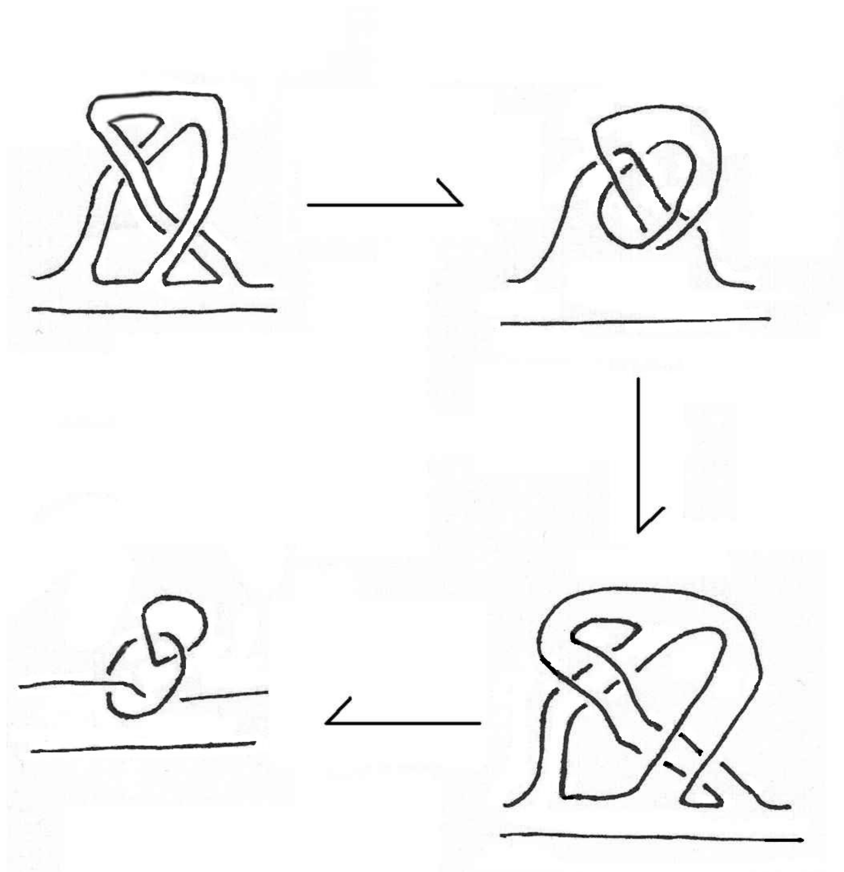


Figure 4.5: Taking the reduced link in Trinion BRNs

Chapter 5

Subsystems and Isolated Substructures: Localization of Topological Information

The dual nature (here we use the term non-mathematically) of braided ribbon networks is that they possess both the combinatorial information and topological information. This dual nature makes understanding the properties of the networks difficult, especially as the topological information is not somehow constrained to exist on particular parts of the graph. With the reduced link from chapter 4 we can attempt to understand the relationship between these two types of information, but to do so we must first lie down some mathematical definitions. We do this in section 5.1, then in section 5.2 we will define isolated substructures - a construction which illustrates the idea of a localized component of the topological structure - and introduce the concept of how they can translate with respect to the concept of locality in the graph through the evolution moves. Then in section 5.3 we discuss the implication of these results on the attempts to have particle physics emerge from braided ribbon networks, and how progress could be made in the matter. As all of this work was originally presented in the trinion BRN formalism it is reproduced here in the same. The trinion formalism also serves to help illustrate the concepts within. These sections were originally presented by myself in [16] and we have only provided minor alterations to them here.

Lastly in section 5.4 we present the concept of subsystem - originally from [6] - and generalize it to work in the context of 4-valent BRNs.

Throughout this chapter and also chapter 6 we will refer to capped braids (or equivalently encapsulated braids). This term will refer to a braid which exists in a BRN in a particular format where all the strands involved in the braid have one end that connects into a set of nodes disconnected from the rest of the graph. For an example of a capped braid see the leftmost image in figure 5.1 below.

5.1 Mathematical Definitions

In order to properly discuss the idea of a translation we must first discuss the topology with respect to which the translations shall occur.

The Microlocal Metric Space

Consider a ribbon graph Γ consisting of N nodes and M ribbons having some braiding and twisting content. We construct a new metrical space $\tilde{\Gamma}$ as follows: let X be the set of trinions within the ribbon graph. We shall take each trinion x within X as a node in a pseudograph, and construct edges for this pseudograph in the natural way: by making an edge between two nodes if their respective trinions share a ribbon. This is the reverse of the framing process that can be used to construct a ribbon network.

> ***The Microlocal Distance Function***
>
> Considering the set of all possible paths between two nodes on the pseudograph, the distance between the nodes is the minimum number of edges in any such path. This satisfies the four requirements for a distance function: that it is non-negative for any choice of two nodes, that it is strictly positive for any two non-identical nodes, that it is reflexive and that it satisfies the triangle inequality. This metric is equivalent to the standard metric of graph theory.

Thus, the set X of nodes, along with the microlocal distance function, create a metric space and, therefore, have a standard topology defined by microlocal distance function on X.

5.2 Isolated Substructures and Microlocal Translations

We shall now demonstrate that there are indeed translations of braided-local structures with respect to the microlocal distance function. Also, we shall demonstrate that even when using a more general notion of microlocal distance we can nonetheless demonstrate situations where braided-local structures have undergone a translation. These translations are generated by the evolution moves.

We shall first introduce a series of definitions and then prove a result using them.

Definition 6. *Two nodes a and b are* ***Ribbon Connected*** *if there exists a sequence of $N+1$ nodes x_n such that $x_1 = a$, $x_{N+1} = b$ and for each n the trinion with node x_n and the trinion with node x_{n+1} share a ribbon. This is equivalent to the nodes being connected in the graph $\tilde{\Gamma}$.*

Definition 7. *A* ***Connected Ribbon Network*** *is a set of nodes X, such that all nodes in X are ribbon connected to all other nodes in X.*

Definition 8. *The* ***edge space*** *of a ribbon graph is a space made up by the boundary of the ribbon graph. An* ***edge segment*** *is a connected subset of the edge space.*

Definition 9. *Two edge segments a and b are Edge Connected if there exists an edge segment c such that both a and b are subsets of c.*

5.2.1 Isolated Substructures

In order to demonstrate translations within ribbon networks we must first define a special class of elements within ribbon networks.

Definition 10. *An* ***Isolated Substructure*** *is a ribbon connected set of nodes where a closed surface can be placed around it with exactly one ribbon intersecting the surface. We call this ribbon the Isolated Substructure's "tether".*

It should be understood that isolated substructures are not the same as 'subsystems' as defined by [6] or in section 5.4. This is readily apparent by considering the form of the reduced link of an isolated substructure.

It is interesting to note that, though the definition of an isolated substructure appears to be restrictive at first glance, there are a significant number of structures that can be 'packed up' into the form of an isolated substructure. For instance, all of the example definitions of particles from [6] can be changed into isolated substructures through the use of the $2-2$ evolution move as shown in figure 5.1.

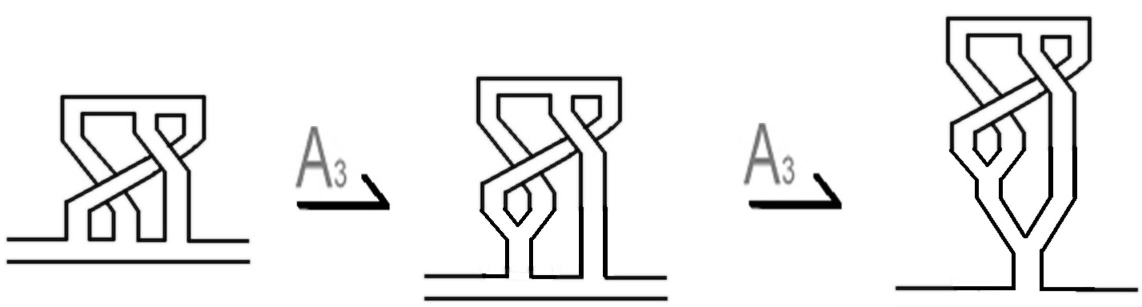


Figure 5.1: Transforming a capped braid into an isolated substructure

Definition 11. *An edge segment is* ***Replaceable*** *if it can be replaced by an isolated substructure's tether without producing a node which is four valent. Specifically, a Replaceable*

Edge Segment cannot be an edge of a node (for instance the edge segment labeled A in figure 5.2).

The purpose of defining replaceable edge segments is purely to keep a consistent formalism: we've defined ribbon networks only in terms of trinions. Though a four valent node could be resolved into two different versions of two three valent nodes (and these versions would be related by the $2-2$ move), such objects aren't defined in our formalism and so we exclude them.

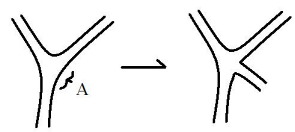


Figure 5.2: Edge segment A is not replaceable without introducing a four valent node.

These definitions together allow us to consider the dynamics of isolated substructures under the evolution moves. We can consider a graph Γ to be composed of a set of isolated substructures attached to replaceable edge segments of a second graph Λ. As we do not require that all such isolated substructures be so removed, this procedure can be done without ambiguity.

Theorem 4. *Given a finite closed network* Γ *with two edge connected replaceable edge segments* a *and* b*, there exists a sequence of evolution moves such that a graph* Γ_a *- composed of* Γ *with an isolated substructure* A *tethered to* a *- evolves to* Γ_b*, where* Γ_b *is composed of the same graph* Γ *but with* A *now tethered to* b*.*

Proof. We shall proceed by induction on the number of nodes between a and b, say N. As

a and b are edge connected, the node created by A being tethered to a is ribbon connected to the two nodes that are at either side of b. We shall label these nodes x_0 (for the node created by A at a) through x_{N+1} in such a way that each x_j shares a single ribbon with x_{j+1}. The nodes on either side of b are then labeled x_N and x_{N+1}.

Before we proceed we need to show the ability to move an isolated substructure through intermediate topological structures that are not composed of nodes. These are comprised of three categories: knots, twists and braidings. Examples of each of these is shown in figure 5.3. As isolated substructures only have a single connection to the outside network, we can move it past this 'terrain' through the following procedures.

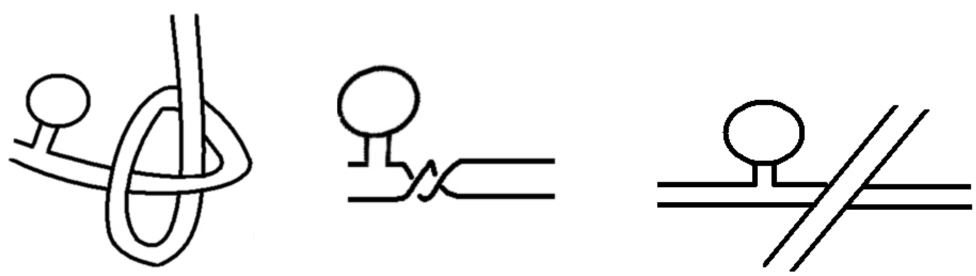


Figure 5.3: Examples of Terrain

For each knot the isolated substructure is pulled through the knot by stretching out the knot until the substructure can pass through it. As the substructure is unconnected except through its tether, this leaves the network unchanged other than the reversal of the position of the knot and the substructure.

For each twist (consisting of a rotation by π), the isolated substructure can run along the edge of the twist. Alternatively, one can view the procedure as deforming the network itself by twisting the segment with the isolated substructure in a manner that undoes the twist on the one side and create a twist on the other.

For each braiding, we deform the ribbon of the braid by sliding it over the isolated substructure to the other side. This is reminiscent of the Reidemeister move of the second kind.

The above demonstrates that we can move an isolated substructure tethered to a node R so that it is tethered with no intermediate 'terrain' between it and some edge segment t (which is not a component of a piece of 'terrain') that connects the node R to its nearest neighbours and is edge connected to the edge which the isolated substructure is tethered to. This is proven by induction on the number of elements of 'terrain' between R and t and the use of the above prescriptions. A consequence of this is that the same method can be used for a node S which has microlocal distance 1 to R. This ability shall be used heavily in our proof.

Now, returning to the proof, we shall first prove the case of $N = 1$. We apply the above lemma to move the isolated substructure through any intermediate terrain between x_0 and x_1, giving us A tethered to a new node x_0' (we shall use primes to denote nodes that have undergone some change) that is immediately adjacent to x_1 with no intermediate terrain. We then perform an exchange move from $\mathcal{A}_{evol}$ on the node x_0' and x_1 to move x_0' onto the edge on the other side of x_1'. We can then again use the above lemma (in its more general case) to move x_0' to its final resting place at b.

Now we shall assume that the case of $N-1$ nodes is correct and prove the case of N nodes. The prescription for this is analogous to the $N = 1$ case. Given that there is a method for moving past $N-1$ nodes (by inductive hypothesis), we shall use that method to change the situation to a single intermediate node, and then invoke the method of the $N = 1$ case to bypass the final node.

The preceding gives the inductive argument and completes the proof. □

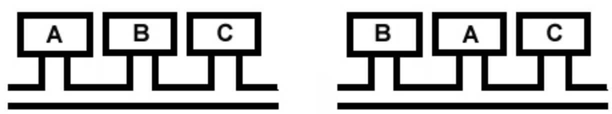
To demonstrate translations we will need a further tool. We therefore consider also the following lemma:

Lemma 1. *Translations Through an Isolated Substructures*
Given an isolated substructure A that has been moved to the edge of the tether of another substructure B, it is possible to translate A to the opposite edge of the tether of B.

Proof. Due to the above theorem, it only remains to show that the two edges of the tether of an isolated substructure are edge connected. Proceeding by contradiction, we assume that they are not edge connected. As we see that an edge of the network enters the isolated substructure and does not exit, there must be some terminus of the edge within the isolated substructure. However, such a situation is impossible, as the edges of a ribbon network must form closed links or terminate at some boundary (which we have not introduced into the theory of ribbon networks). We therefore have a contradiction. Thusly we see that if an isolated substructure can be moved to the edge of a tether, by the above theorem, it can be moved to the other edge. □

5.2.2 Microlocal Translations

The application of theorem 4 is straightforward and results in the ability to demonstrate translations under the microlocal distance function. For instance, it is possible to construct a sequence of evolution moves such that figure 5.4a evolves to figure 5.4b. Under the microlocal distance function, the isolated substructure A is now less distant from the isolated substructure C (measuring the distance between substructures from the node at which they are tethered). It should be understood that there is no guarantee of a correspondence between these microlocal translations and translations of objects within a macroscopic space-time.



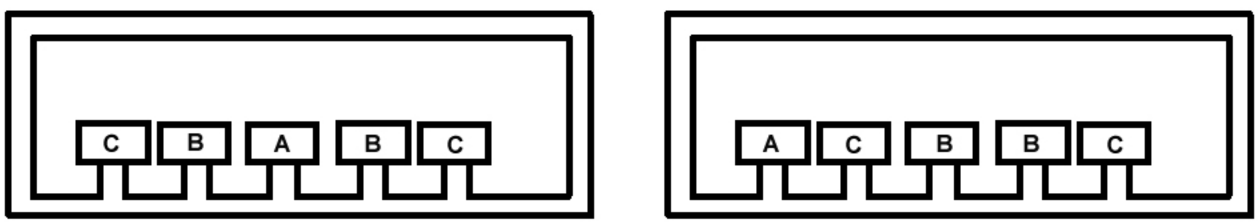
(a) (b)

Figure 5.4: Microlocal Translations

Even applying a more restrictive definition of distances, we can demonstrate translations in some form. Consider the following definition of closeness:

Definition 12. *An isolated substructure A is said to be* α***-closer*** *to an isolated substructure B than it is to another substructure C, if for all paths along the ribbons of the network, leaving the node at which A is tethered and intersecting the node at which C is tethered the path intersects the node at which B is tethered.*

By expanding our definitions slightly to allow us to consider isolated substructures with identical structure to be treated equally, we can show that it is possible to evoke a translation. Specifically, it is possible to take a situation where a substructure A is α-closer to substructures of type B than to those of type C and to apply a series of evolution moves such that the reverse is true afterwards. For instance, consider figure 5.5a and figure 5.5b. Thus we see that we have translations even under stringent requirements, thereby concluding our result.



(a) (b)

Figure 5.5: α-closer Translations

5.3 Macro, Micro and Braided Locality

The braided-local structure of the braid network can be characterized by the reduced link of the structure. The reduced link of a braided ribbon network has been shown to be invariant under the evolution moves. As a result, it is clear that the braided-local content of a braid network is invariant under the evolution moves. This invariance is a double edged sword. On one hand, it allows us to assign some meaning to these invariant structures, as was done by the authors of [6]. On the other hand, it means that there is no way that these microlocal moves can provide any form of dynamics in this context. Any two networks related by the evolution moves must necessarily have the same reduced link, and hence would correspond to the same particle content. As a result, I suggest that, to construct a theory of quantum gravity containing particle physics from a ribbon network, it is necessary to consider the existence of a second evolution algebra, which I shall call $\mathcal{A}_{braid}$.

In [26] and [23], the concept of macrolocality in networks is put forward as the locality derived from the classical metric that would arise for a network with a space-time as its classical limit. In [26], the authors then remind us that there is no need for macrolocality to be coincidental to microlocality. It seems to be a consequence of the ideas of [6] to suggest that, though microlocality and macrolocality are not necessarily coincidental, the braided-local content - the invariants that we associate with particles - should be part of the bridge of the gap between the two. I therefore suggest that $\mathcal{A}_{braid}$ could be the bridge between microlocality and macrolocality.

For future consideration, I outline some general possibilities of $\mathcal{A}_{braid}$. Regardless, it should be noted that any such algebra that could provide macrolocal dynamics, particle interactions included, would need to alter the reduced link of the network if the identifications in [6] are to be considered seriously.

> A candidate based upon the assumption that any move within the second evolution algebra should be as close to being microlocal as possible is called a **Nearly Microlocal Algebra**. This could be completed by introducing moves involving next to nearest neighbor nodes. This suggestion corresponds to the

idea that there is a degree of coincidence between microlocality and macrolocality (again, we should remember that such a coincidence is not needed).[26]

An algebra based upon moves that alter the braiding content of the network in ways that are roughly equivalent to elements of the standard braid group is referred to as a **Braid Algebra**. Also, it can contain moves that allow the composition of multiple braided isolated substructures.

An algebra premised upon the idea that microlocality should be dual or completely unrelated to macrolocality is called an **Anti-Microlocal Algebra**. Such an algebra can be constructed from a set of moves that act upon the reduced links of a graph. Such moves could be realized through the following algorithm:

> Take the reduced link of the graph Γ and apply a move that composes or interacts parts of the reduced link (whether through cutting and repairing links, or through allowing links that correspond in some manner to annihilate each other). Then take the new reduced link and equate it with a superposition of all graphs $\Gamma_{x'}$ which produce the same reduced link. In this situation we use the term superposition to mean a sum of weighted probabilities of each result. That any such graph $\Gamma_{x'}$ should exist could be provable by a generalization of the theorem that allows the construction of a closed braid that corresponds to any link, though this is supposition. [1]

It is also possible that a candidate could draw upon multiple such programs.

5.4 Subsystems

In [6] the concept of a subsystem of a BRN was introduced to refer to a portion of the graph which has a reduced link composed of a closed link which is unconnected to the reduced link of the rest of the BRN. Though this definition is correct, the lack of an effective technique for testing if a portion of a BRN can be considered a subsystem causes confusion. Not the least of these confusions is that the first example of a subsystem given in [6] of encapsulated braids (which we call capped braids) as subsystems is in fact not universally correct.

We present here a definition of a subsystem which should eliminate confusion through it having a criteria which can be easily tested. It additionally works for both 3 and 4-valent BRNs.

Definition 13. *A portion of a BRN is a* ***subsystem*** *if there exists a compact 2-surface which it is interior to, but which no racing stripe intersects.*

This definition is of course in some ways constructive: we desire the reduced link of a subsystem to separate from the reduced link of the rest of the graph, and this is guaranteed through the requirement that the racing stripes are divided between the interior and exterior of the surface. We can also see from this definition why capped braids (or encapsulated braids) cannot be subsystems in general: they would need to have a reduced link which separates from the exterior racing stripe of the first and last strands of the braid.

Chapter 6

Twist Numbers: an Example of a Localized Topological Structure

> The content of this chapter is a result of a collaboration with Lee Smolin, Sundance Bilson-Thompson, and Louis Kauffman previously published in [4] and additionally in [3]. It was done in the context of the trinion BRNs, and is reproduced here in the same. Some modifications have been made - primarily to the motivations and conclusions - but nothing which alters the substance of the results. The content presented here acts as an example of how we can use the topological information within the reduced link to build structures that could have physical interpretations.

Throughout what follows we will make reference to 'Braided Belts'. This term was not originally intended to be a mathematical one - rather it referred to a specific construction of a leather belt by way of cutting slits along the length of a belt and feeding the top of the belt through the spaces created by these slits. If one desires to understand this more formally it could be taken to be a compact two surface formed by taking a disk and adding two holes to it. We refer to the belt as if it were a ribbon network by talking about the top of the belt (above the slits) as the top node and the other end of the belt as the bottom node.

The work that follows in this chapter was part of a program to demonstrate how matter could emerge in the context of a theory of quantum gravity. The intent was to utilize the fact that the topological structures of Braided Ribbon Networks could be studied more easily than those of the spin networks in Loop Quantum Gravity to demonstrate that such a result was possible. This approach was broken down into two parts. The first step was in developing the tools with which we can understand the braids in BRNs, this allowed us to study previous attempts to have matter encoded into braids [5] and to see if they could be incorporated into the theory. The second part came from understanding that within the context of BRNs the preon model of [5] breaks down and so a new construction is needed. This second step is then focused on demonstrating that despite that setback there still exists another map between the braids in BRNs and the observed states of matter.

6.1 Twists, Braids and Belts

A trinion may be converted into a structure with both crossings and twists, by keeping the ends fixed and flipping over the node in the middle, as illustrated in Fig. 6.1 (we shall refer to this process of flipping over a node while keeping the ends of the legs fixed as a “trinion

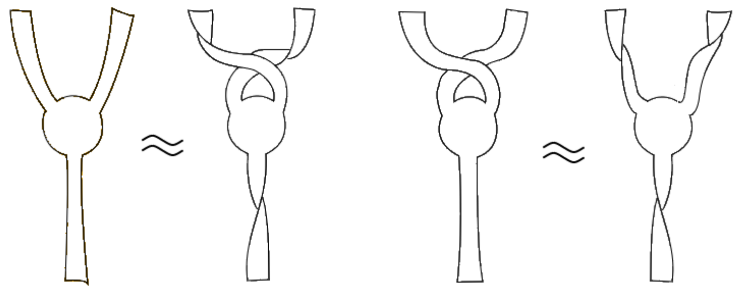


Figure 6.1: Turning over a node induces crossings and twists in the "legs" of that node (left). A trinion flip may be used to eliminate crossing while creating twists (right).

flip", or "trip" for short). Conversely, on the right of Fig. 6.1 we show how a trinion with untwisted ribbons, but whose upper ribbons are crossed, can be converted into a trinion with uncrossed ribbons and oppositely-directed half-twists in the upper and lower ribbons by performing an appropriate trinion flip (in the illustration, a negative half-twist in the lower ribbon of the trinion and positive half-twists in the upper ribbons). In Fig. 6.2, we show the same process performed on a trinion whose (crossed) upper ribbons have been bent downwards to lie besides and to the left of the (initially) lower ribbon. This configuration is nothing other than a framed 3-braid corresponding to the generator σ_1 (with the extra detail that the tops of all three strands are joined at a node). Keeping the ends of the ribbons fixed as before and flipping over the node so as to remove the crossings now results in three unbraided (i.e. trivially braided) strands, with a positive half-twist on the leftmost strand, a positive half-twist on the middle strand, and a negative half-twist on the rightmost strand. Hence the associated twist-word is $[\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}]$. This illustrates that in the case of braids on three strands, each of the crossing generators can be isotoped to uncrossed strands bearing half-integer twists. By variously bending the top two ribbons

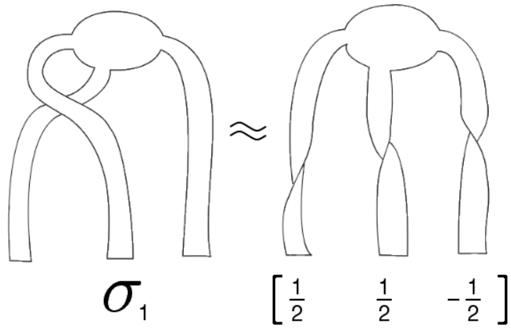


Figure 6.2: Trinion bent to form a generator of the braid group

down to the right of the bottom ribbon, and/or taking mirror images, and performing the appropriate trinion flips we can determine that the generators may be exchanged for twists according to the pattern:

$$\begin{aligned}
\sigma_1 &\rightarrow \left[\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}\right] \\
\sigma_1^{-1} &\rightarrow \left[-\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}\right] \\
\sigma_2 &\rightarrow \left[-\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right] \\
\sigma_2^{-1} &\rightarrow \left[\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}\right]
\end{aligned} \tag{6.1}$$

All braids on three strands can be built up as products of these generators.

6.1.1 Framed braid multiplication

As mentioned above, a braid with several crossings is specified by its braid word, which corresponds with several generators multiplied together. Unframed braids are multiplied together by joining the tops of the strands of the second braid to the bottoms of the strands of the first braid. Framed braids are multiplied in an analogous way, however there are extra complications introduced by the presence of twists. Therefore when forming the product of two (or more) framed braids, we shall firstly join the bottom of the ribbons in the first braid to the tops of the ribbons in the second braid, and so on (in the case of braided belts, we must eliminate the intervening nodes first, while retaining the nodes at the "top" of the first belt in the product, and the "bottom" of the last belt). Secondly we shall isotop the twists from each of the component braids upwards to render the product into standard form (as described above). They will get permuted in the process, according to the permutation $P_{\mathcal{B}}$ associated with the braid $\mathcal{B}$ that they pass through. Given a permutation $P_{\mathcal{B}}$ and a twist-word $[x, y, z]$, we shall write the permuted twist-word as $P_{\mathcal{B}}([x, y, z])$. Thus given two braids $[r, s, t]\mathcal{B}_1$ and $[u, v, w]\mathcal{B}_2$, we can form their product by joining strands to form $[r, s, t]\mathcal{B}_1[u, v, w]\mathcal{B}_2$, and then move the twists $[u, v, w]$ upward along the strands of the braid $\mathcal{B}_1$. Thus

$$[r, s, t]\mathcal{B}_1[u, v, w]\mathcal{B}_2 = [r, s, t]P_{\mathcal{B}_1}([u, v, w])\,\mathcal{B}_1\mathcal{B}_2 \tag{6.2}$$

where $[r, s, t][x, y, z] = [r + x, s + y, t + z]$ and $\mathcal{B}_1\mathcal{B}_2$ denotes the usual product of braid words. In general the twists will be permuted by all the braids they pass through. For example, remembering that $P_{\sigma_i} = P_{i,(i+1)}$,

$$\begin{aligned} [r, s, t]\sigma_1\sigma_2[x, y, z] &= [r, s, t]\sigma_1[x, z, y]\sigma_2 \\ &= [r, s, t][z, x, y]\sigma_1\sigma_2 \\ &= [r + z, s + x, t + y]\sigma_1\sigma_2. \end{aligned} \tag{6.3}$$

It should also be clear to the reader that since crossings can be exchanged for twists, in the case of braided belts - which we shall be discussing exclusively from this point onwards, we may go a step further and entirely eliminate the crossings from a 3-braid. When we do so we uncross the strands (hence permuting them by the permutation associated with the crossing being eliminated) and introduce the twists indicated in eqn. (6.1). In general, this means that we iterate the process

$$\begin{aligned}[a_1, a_2, a_3][b_1, b_2, b_3]\sigma_i\sigma_j \ldots \sigma_m &\to [a_1 + b_1, a_2 + b_2, a_3 + b_3]\sigma_i\sigma_j \ldots \sigma_m \\ &\to P_{\sigma_i}([a_1 + b_1, a_2 + b_2, a_3 + b_3])[x, y, z]\sigma_j \ldots \sigma_m \end{aligned} \tag{6.4}$$

where $[x, y, z]$ is the twist-word associated to σ_i (as listed in Eqn. 6.1, when i is specified). We iterate this procedure until the braid word becomes the identity. Hence continuing the example above, from Eqn. (6.3),

$$\begin{aligned}[r+z, s+x, t+y]\sigma_1\sigma_2 \quad &\to \quad P_{\sigma_1}([r+z, s+x, t+y])\left[\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right]\sigma_2 \\ &\to \quad [s+x, r+z, t+y]\left[\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right]\sigma_2 \\ &\to \quad \left[s+x+\frac{1}{2}, r+z+\frac{1}{2}, t+y-\frac{1}{2}\right]\sigma_2 \\ &\to \quad P_{\sigma_2}\left(\left[s+x+\frac{1}{2}, r+z+\frac{1}{2}, t+y-\frac{1}{2}\right]\right) \\ &\qquad\qquad \times\left[-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right] \\ &\to \quad \left[s+x+\frac{1}{2}, t+y-\frac{1}{2}, r+z+\frac{1}{2}\right] \\ &\qquad\qquad \times\left[-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right] \\ &\to \quad [s+x, t+y, r+z+1]. \end{aligned} \tag{6.5}$$

We shall refer to the form of a braid in which all the crossings have been exchanged for twists as the *pure twist form*. The list of three numbers which characterise the twists on the strands in the pure twist form will be referred to as the *pure twist-word*. The pure twist-word is of interest because it is a topological invariant (since it is obtained when a braid is reduced to a particularly simple form i.e. all crossings removed).

6.1.2 Making 3-belts

Consider a braided belt (or framed braid) on three strands. In the particular (trivial) case where the strands do not cross each other, the associated braid word is clearly the identity, I. In the case where the strands are untwisted, the associated twist-word is also the identity. Such an untwisted trivial braid - the identity braid on three strands - can be made by cutting two parallel slits in a strip of leather, as shown on the left of Fig. 6.3. The resulting surface is topologically equivalent to three parallel strips capped at the top and bottom by an attached disk. We will refer to three strands (not necessarily unbraided) attached to disks in this manner as a *3-belt*. In Fig. 6.3, we show the consequence of trinion flips in the making of a braided leather belt.

Recall that $\sigma_1 \equiv [\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}]$, and so we may write $I = [-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}]\sigma_1$. The right-hand side of Fig. 6.3 illustrates that the strands are now crossed and twisted, but still the 3-belt we have obtained is isotopic to the trivial 3-belt. Figure 6.4 shows the result of six consecutive repetitions of this process (alternately to the first two strands and the last two strands) of a 3-belt. The reader will note by direct observation that along each of the three

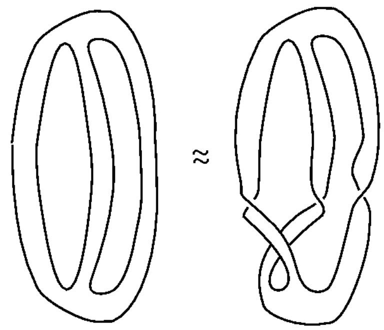


Figure 6.3: A Trip performed on a closed 3-Belt

ribbon strands, all the twists cancel. Thus when we isotop all the twists to the top of the braid we obtain a flat (untwisted) braided belt, with braid-word $(\sigma_2^{-1}\sigma_1)^3$. Iteration of the procedure that yields Fig. 6.4 is actually used by belt-makers.

Since the same physical structure can be isotoped to have the braid word I or $(\sigma_2^{-1}\sigma_1)^3$, it is clear that the braid word is not a topological invariant. However, as noted above, the pure twist-word *is* a topological invariant. If any two braids $[a, b, c]\mathcal{B}_1$ and $[x, y, z]\mathcal{B}_2$ have the same pure twist-word, then they are isotopic. For the remainder of this section we shall be mostly interested in classifying braided, twisted 3-belts by their pure twist numbers, rather than inducing twists and crossings on an initially trivial 3-belt. We are therefore primarily interested in the procedure illustrated in Eqn. (6.4). We now apply this procedure to the braid word $(\sigma_2^{-1}\sigma_1)^3$, and confirm that its pure twist-word is $I = [0, 0, 0]$. This corresponds with reversing the procedure (used by belt-makers) described in the previous paragraph.

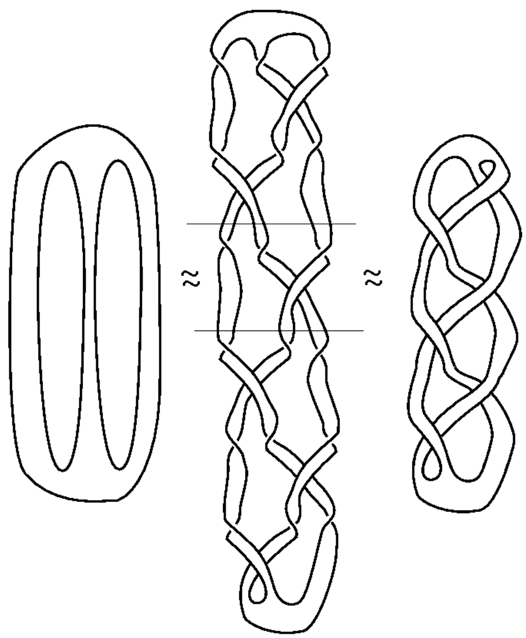


Figure 6.4: Braiding a Belt

Firstly we consider the braid word $\sigma_2^{-1}\sigma_1$

$$\begin{aligned}
\sigma_2^{-1}\sigma_1 &\rightarrow \left[\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right]\sigma_1 \\
&\rightarrow P_{1,2}\left(\left[\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right]\right)\left[\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right] \\
&\rightarrow \left[-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right]\left[\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right] \\
&\rightarrow [0,1,-1]
\end{aligned} \tag{6.6}$$

To consider the full braid with six crossings we need to multiply this result with itself three times, but also keep track of the permutations induced by the braid word $\sigma_2^{-1}\sigma_1$. This permutation will be $P_{1,2}(P_{2,3}) = P_{(123)}$, that is, the cyclic permutation which sends $[a,b,c] \rightarrow [c,a,b]$. Hence,

$$\begin{aligned}
(\sigma_2^{-1}\sigma_1)^3 &\rightarrow [0,1,-1](\sigma_2^{-1}\sigma_1)^2 \\
&\rightarrow P_{(123)}([0,1,-1])[0,1,-1]\sigma_2^{-1}\sigma_1 \\
&\rightarrow [-1,0,1][0,1,-1]\sigma_2^{-1}\sigma_1 \\
&\rightarrow [-1,1,0]\sigma_2^{-1}\sigma_1 \\
&\rightarrow P_{(123)}([-1,1,0])[0,1,-1] \\
&\rightarrow [0,-1,1][0,1,-1] \\
&\rightarrow [0,0,0]
\end{aligned} \tag{6.7}$$

as expected.

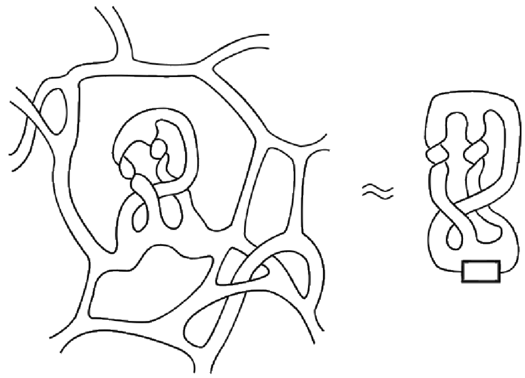


Figure 6.5: A braided belt can be embedded within a larger network, represented by the box.

The reader will notice that it is unnecessary to keep track of the permutation induced by the first generator in a braid word (or the first braid word in a product, when the corresponding pure twist-word is known), as there is nothing for this permutation to act upon.

The braided belts we consider in this section are of interest not only from a purely topological basis, but also due to a possible connection with theoretical physics. In [6] it was shown that braided belts attached at one end to a larger network of ribbons could be used to represent the elementary quarks and leptons. In order to keep the discussion here relevant to the work in [6], we shall henceforth treat the ordering of generators in a braid word as indicative of a "top end" which is free to be trinion-flipped, and a "bottom end" which is attached to a larger network (which for all practical purposes is fixed and static). The left-most generator in a product is equated with the top end, and this is why we shall always work from left to right when we resolve a braid word to find the associated

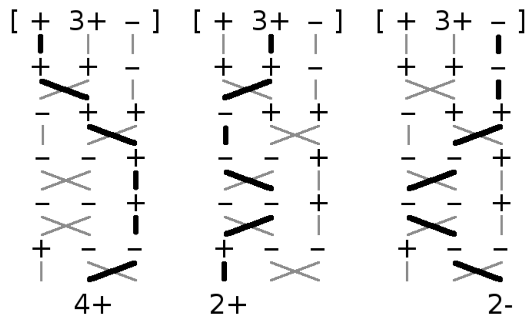


Figure 6.6: Finding the pure twist-word for a braid $[\frac{1}{2}, \frac{3}{2}, -\frac{1}{2}]\sigma_1\sigma_2\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}$

pure twist-word. In diagrams of braided belts we shall henceforth include a box on the boundary of the belt at the bottom end, to represent the presence of a larger network to which the braided belt is attached, as shown in Fig. 6.5

There is a simple schematic technique for finding the pure twist-word associated with a braid word, as follows:

- Replace each generator in the braid word by one of the following triples of symbols;

$$\begin{aligned} \sigma_1 &\rightarrow + + - \\ \sigma_1^{-1} &\rightarrow - - + \\ \sigma_2 &\rightarrow - + + \\ \sigma_2^{-1} &\rightarrow + - - \end{aligned} \tag{6.8}$$

  such that they form a vertical stack, leftmost generator at the top.

- Beneath each pair of similar symbols, place a cross, $\times$, connecting each symbol to the diagonally opposite symbol in the pair below. Beneath each dissimilar symbol,

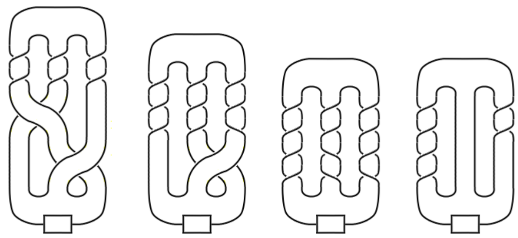


Figure 6.7: The Positron, in fully-braided form (left), resolved in stages to its pure twist form (right).

place a vertical bar, |, connecting it directly to the symbol below it. The lines below the bottom row of symbols will end on three blank spaces.

- Sum up the symbols along each vertical path, starting at the top of the symbol stack, and ending on the blank spaces at the bottom. Each + stands in for $\frac{1}{2}$. Each − stands in for $-\frac{1}{2}$. The resulting triplet of numbers is the pure twist-word.

In the case where the braid is written in standard form $[a, b, c]\mathcal{B}$ we construct the symbol stack for the braid word as described above, and then place the twist word at the top, with three vertical lines descending to the top of the symbol stack.

In Fig. 6.6 we give an example of using this process to find the pure twist-word corresponding to the braid $[\frac{1}{2}, \frac{3}{2}, -\frac{1}{2}]\sigma_1\sigma_2\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}$. The three diagrams in the figure correspond with the addition of twists along each of the three strands. The resulting pure twist-word is found to be $[1, 2, -1]$.

A further example is given by the braid $e_{\mathrm{L}}^{+} = [1,1,1]\sigma_1\sigma_2^{-1}$. This is the braid structure assigned to the left-handed positron in [5]. We find that

$$\begin{aligned}
e_{\mathrm{L}}^{+} &\rightarrow [1,1,1]\sigma_1\sigma_2^{-1} \\
&\rightarrow [1,1,1]\left[\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right]\sigma_2^{-1} \\
&\rightarrow \left[\frac{3}{2},\frac{3}{2},\frac{1}{2}\right]\sigma_2^{-1} \\
&\rightarrow P_{\sigma_2^{-1}}\left(\left[\frac{3}{2},\frac{3}{2},\frac{1}{2}\right]\right)\left[\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right] \\
&\rightarrow \left[\frac{3}{2},\frac{1}{2},\frac{3}{2}\right]\left[\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right] \\
&\rightarrow [2,0,1]. \qquad (6.9)
\end{aligned}$$

Thus $[2,0,1]$ gives the framings on the equivalent parallel flat strip belt. See Fig. 6.7 for a graphical version of this calculation, and a depiction of the boundary of the surface that corresponds to e_{L}^{+}. We denote the boundary of this surface by $\partial e_{\mathrm{L}}^{+}$. Note that $\partial e_{\mathrm{L}}^{+}$ is independent (topologically) of the deformation that we have applied to straighten out the braiding from the original definition of e_{L}^{+}. Thus our algebraic reduction gives us an algorithm for finding the boundary link for each particle in Bilson-Thompson's tables.

In Fig. 6.9 - Fig. 6.12 we illustrate the correspondence between the braids proposed in [5] to match the first-generation fermions of the Standard Model, and their pure twist form. Notice that the pure twist forms are distinct in each illustrated case, except for the neutrino and anti-neutrino. In this case, the left-handed "negative" neutrino is isomorphic to the right-handed "positive" anti-neutrino. Likewise the left-handed "positive" anti-neutrino is

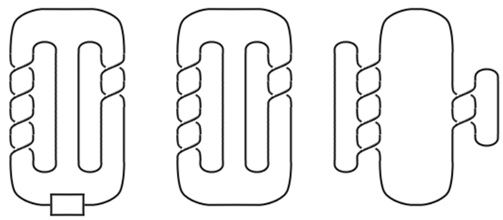


Figure 6.8: By assuming that the outer edge of the braided belt is closed (when we trace it through the rest of the network), we can equate a link (right) to the pure twist form (left) of any braid, in this case the left-handed positron as illustrated in Fig 6.7

isomorphic to the right-handed "negative" neutrino. In other words, there are half as many topologically distinct states for neutral particles as one would expect for charged particles. This is in agreement with the Standard Model, where neutrinos are purely left-handed, and anti-neutrinos are purely right-handed. The pure twist numbers in each case are listed with the corresponding particles in the table below:

6.2 Algebra

In this section we give a representation of the three-strand braid group in terms of *permutation matrices*. A permutation matrix P is a matrix whose columns are a permutation of the columns of the identity matrix. Such a matrix acts as a permutation on the standard basis (column vectors that have a single unit entry). We shall sometimes refer to P as a

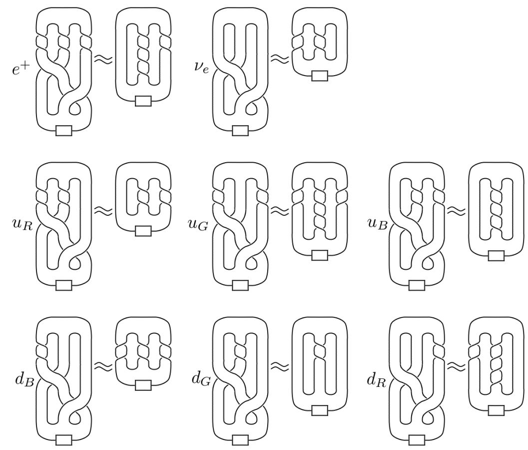


Figure 6.9: Left-handed negatively-charged fermions, as per the structure proposed by Bilson-Thompson, and their associated pure twist form

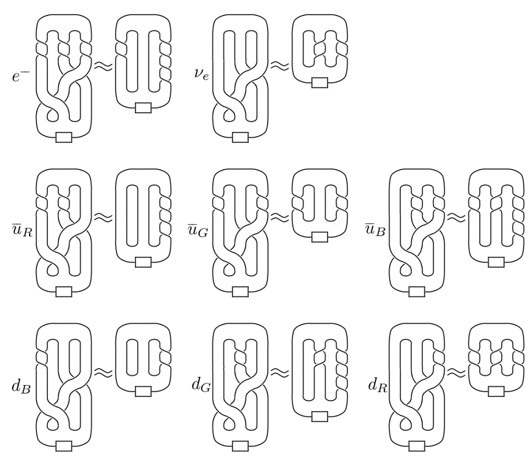


Figure 6.10: Right-handed negatively-charged fermions, as per the structure proposed by Bilson-Thompson, and their associated pure twist form

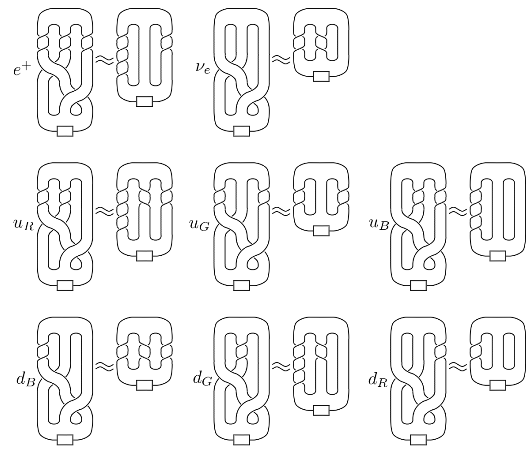


Figure 6.11: Left-handed positively-charged fermions, as per the structure proposed by Bilson-Thompson, and their associated pure twist form

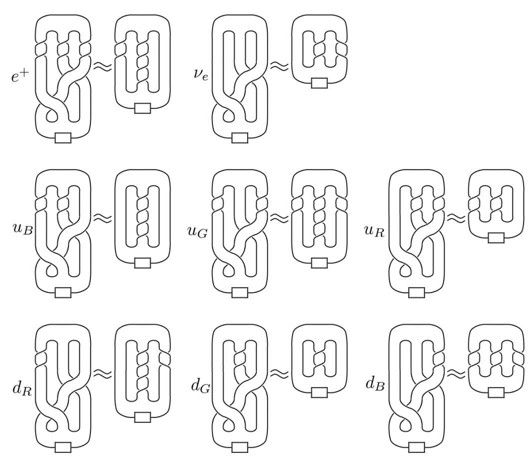


Figure 6.12: Right-handed positively-charged fermions, as per the structure proposed by Bilson-Thompson, and their associated pure twist form

left-handed		right-handed	
e^-	[0,-2,-1]	e^-	[-1,0,-2]
$\bar{u}_B$	[0,-1,-1]	$\bar{u}_B$	[-1,1,-2]
$\bar{u}_G$	[1,-2,-1]	$\bar{u}_G$	[-1,0,-1]
$\bar{u}_R$	[0,-2,0]	$\bar{u}_R$	[0,0,-2]
d_B	[1,-2,0]	d_B	[0,0,-1]
d_G	[0,-1,0]	d_G	[0,1,-2]
d_R	[1,-1,-1]	d_R	[-1,1,-1]
ν_L	[1,-1,0]	ν_R	[0,1,-1]
$\bar{d}_B$	[1,0,0]	$\bar{d}_B$	[0,2,-1]
$\bar{d}_G$	[2,-1,0]	$\bar{d}_G$	[0,1,0]
$\bar{d}_R$	[1,-1,1]	$\bar{d}_R$	[1,1,-1]
u_B	[2,-1,1]	u_B	[1,1,0]
u_G	[1,0,1]	u_G	[1,2,-1]
u_R	[2,0,0]	u_R	[0,2,0]
e^+	[2,0,1]	e^+	[1,2,0]

Table 6.1: Twist number identification of the Bilson-Thompson Model

permutation, rather than a permutation matrix, for brevity. For example,

$$P_{1,2} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.10}$$

and

$$P_{2,3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{6.11}$$

Obviously

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

and so on.

If D is a diagonal matrix and P is a permutation matrix, then

$$PD = D^P P \tag{6.12}$$

where $D^P = P(D)$ denotes the result of permuting the elements of D along the diagonal according to the permutation P. This is directly analogous to eqn. (6.2)

The permutation matrices $P_{1,2}$ and $P_{2,3}$ generate (by taking products) all permutations on three letters. These permutations can be used to represent permutations induced by a braid, and hence they can stand in for the braid word. A general twist word $[a, b, c]$ may be represented by the matrix

$$[a, b, c] = \begin{pmatrix} t^a & 0 & 0 \\ 0 & t^b & 0 \\ 0 & 0 & t^c \end{pmatrix}.$$

The reader should be aware that the permutation matrices do not give a complete image of the braid group, because they do not define the direction of crossing. Hence the Ps contain less information than the σ_is. However we can find the pure twist form of a braid using this matrix representation in a manner we will now describe.

Consider the identity 3-belt, as illustrated in fig. 6.3. If we perform trinion flips on the top of the belt, we induce both crossings and twistings. When the crossings created in this process correspond to a given generator, σ_i, the twists induced are the *negative* of the twists corresponding to σ_i, as listed in eqns (6.1). The relative minus sign occurs because eqns (6.1) list the twist words created when crossings are eliminated, rather than created. In the interests of notational clarity, we shall therefore write ρ_i to denote the twists *and* permutations induced on an initially trivial 3-belt by performing a trinion flip to cross strand i over strand $i+1$. The corresponding twists and permutations are then;

$$\begin{aligned} \rho_1 &\quad\rightarrow\quad \left[-\tfrac{1}{2},\ -\tfrac{1}{2},\ \tfrac{1}{2}\right] P_{1,2} \\ \rho_1^{-1} &\quad\rightarrow\quad \left[\tfrac{1}{2},\ \tfrac{1}{2},\ -\tfrac{1}{2}\right] P_{1,2} \\ \rho_2 &\quad\rightarrow\quad \left[\tfrac{1}{2},\ -\tfrac{1}{2},\ -\tfrac{1}{2}\right] P_{2,3} \\ \rho_2^{-1} &\quad\rightarrow\quad \left[-\tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2}\right] P_{2,3} \end{aligned} \tag{6.13}$$

compare this with eqns (6.1). Applying eqn. (6.12) we see that

$$P_{1,2}[a,b,c] = [b,a,c]P_{1,2}$$

and

$$P_{2,3}[a,b,c] = [a,c,b]P_{2,3}.$$

To find the pure twist-word corresponding to a given braid word, we first replace the

σ_i with the equivalent ρ_i e.g.

$$\sigma_1\sigma_2\sigma_1^{-1} \to \rho_1\rho_2\rho_1^{-1}.$$

We next substitute in the twist words and permutations from eqns (6.13), and apply eqn. (6.12). Once all the twist words have been shifted to the far left of the resulting expression and summed, we read off the negative of this twist word to find the pure twist-word corresponding to our initial braid word.

Example: Let us find the pure twist word corresponding to the braid $\sigma_1\sigma_2\sigma_1^{-1}$. We proceed as follows;

$$\begin{aligned}
\rho_1\rho_2\rho_1^{-1} &= \left[-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right] P_{1,2} \left[\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right] P_{2,3} \left[\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right] P_{1,2} \\
&= \left[-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right] \left[-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right] P_{1,2}P_{2,3} \left[\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right] P_{1,2} \\
&= \left[-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right] \left[-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right] \left[-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right] P_{1,2}P_{2,3}P_{1,2} \\
&= \left[-\frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right] P_{1,2}P_{2,3}P_{1,2}
\end{aligned}$$

We then take the negative of the computed twist word, to obtain our result, $[\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}]$.

These assignments of framed permutations to braids gives a representation of the framed braid group into framed permutations. To see this we can directly verify that $\sigma_1\sigma_2\sigma_1 =$

$\sigma_2\sigma_1\sigma_2$ at the level of the framed permutations as follows;

$$\begin{aligned}
\rho_1\rho_2\rho_1 &= \left[-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right] P_{1,2} \left[\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right] P_{2,3} \left[-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right] P_{1,2} \\
&= \left[-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right] \left[-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right] P_{1,2}P_{2,3} \left[-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right] P_{1,2} \\
&= [-1, 0, 0]\, P_{1,2}P_{2,3} \left[-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right] P_{1,2} \\
&= [-1, 0, 0]\, P_{1,2} \left[-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right] P_{2,3}P_{1,2} \\
&= [-1, 0, 0] \left[\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right] P_{1,2}P_{2,3}P_{1,2} \\
&= \left[-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right] P_{1,2}P_{2,3}P_{1,2}. \qquad (6.14)
\end{aligned}$$

Similarly, we find that $\rho_2\rho_1\rho_2 = \left[-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right] P_{2,3}P_{1,2}P_{2,3}$. Since $\rho_1\rho_2\rho_1$ and $\rho_2\rho_1\rho_2$ yield the same twist word we conclude that $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$. It is noteworthy that $P_{1,2}P_{2,3}P_{1,2} = P_{2,3}P_{1,2}P_{2,3}$, however this has no bearing on the result because the twist words alone are sufficient to define isomorphism.

6.2.1 Relation to the Reduced Link

It is worth noting that we can examine these results in the context of chapter 4. Looking to the reduced link we can understand the twist numbers in this context: for a general capped 3-braid the twist numbers correspond to the number of windings that the loops of the reduced link have about one another. This description breaks down in certain

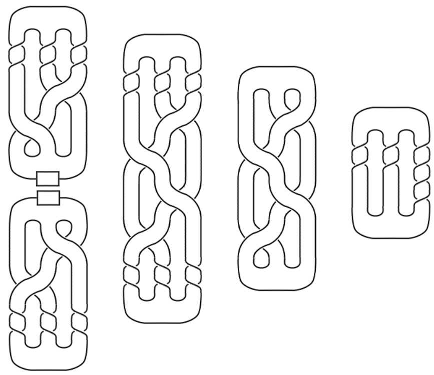


Figure 6.13: Uniquely forming the product of two braids (left) by joining them box-to-box, to yield a link (right).

situations: particularly those where some of the twist numbers are 0 and so one of the loops may drop out of the reduced link.

On seeing this relationship between the reduced link and the twist numbers it is tempting to suggest that the twist numbers are therefore conserved. In making such a statement though we must be careful to specify that what we mean is that the twist number is the information within a capped braid that is conserved, not that such a capped braid will always look like a capped braid after applications of the evolution moves.

6.3 Trying to Reach a Map to Particles

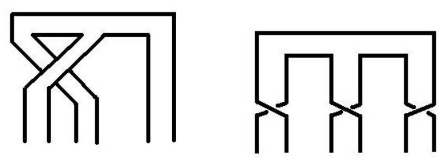
It now behooves us, equipped with a better understanding of the invariant content of the capped three braids, to attempt to construct an embedding of the states of particle physics into them which is compatible with this invariants. Though from table **??** we can see that the first generation of the standard model from [5] is compatible with the twist numbers, it fails upon expanding to multiple generations. To see this fact we present the following observation: the addition of a single pair of crossings as is done in [5] will only introduce a single full twist difference to each of the twist numbers, whereas when we introduce positive twisting on one generation and negative twisting on the other we more than make-up for this difference. This leaves us with overlap between the assigned particles and the twist numbers.

We thus set out to produce an example of how such an embedding can be found. We shall proceed in as systematic of a matter as we can to avoid the construction becoming artificial - an infinite number of such maps are easily constructible given that there are a finite variety of particles in the standard model and a countable infinity of twist numbers. In section 6.3.2 we will give a construction of the C, P and T transformations as operations on the twist numbers and then in section 6.3.3 we will use this construction as a guiding principle to construct our map. The guiding principle of this process will be that each twist number should only correspond to a single particle variety, and thus that the generations should form a sort of segmentation of the space of twist numbers.

6.3.1 A Caveat to the Twist Number

As was shown earlier in chapter 6 each capped braid is equivalent to one with trivial braiding but various twists. This is illustrated in figure 6.14. Those three twists (ordered clockwise) are an invariant (a,b,c) by which we label states. This invariant is generated by the fact that fig.6.14a can be continuously deformed into fig.6.14b making the two surfaces equivalent. This same identification can be made for the other generators of the group B_3

and thus we can iteratively remove all braiding from a capped three braid. The twists on each of the three strands when all braiding has been removed define an invariant triplet of half-integers, (a, b, c). Thus any capped three braid is deformable into any other capped three braid that shares the same invariant triplet. Additionally if we restrict our attention to braids which are orientable surfaces we find that our vectors will not mix integers and half-integers.



(a) (b)

Figure 6.14: Equivalence move by which braids are equivalent to twists

There is a single exception to the uniqueness of this invariant: situations where the ordering of the ribbons is not unique. This exception is demonstrated by a situation where we can use the evolution algebra to isolate the braid in multiple ways each of which give a distinct ordering to the invariant (fig.6.15), equivalent to choosing making different choices for the left-most strand in a triplet. This exception will not be considered in the rest of our discussion.

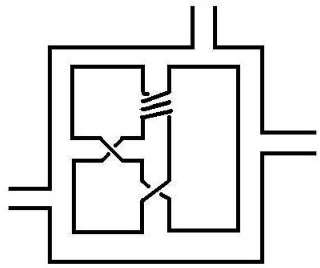


Figure 6.15: Invariant special case

6.3.2 Discrete Symmetries of Capped Three Braids and C, P and T.

In studying the invariants of capped three braids it is useful to determine the group of symmetries which govern them. By considering the most general group we are initially inclined to consider S_3. However, if we consider the fact that left-handed and right-handed twists are differentiated only by a minus sign within the invariant, we can also consider the two element group as an additional symmetry, giving $G = S_3 \times Z_2$.

We will write the group as follows:

> Let $a \in S_3$ and let $+1$ and -1 be the two elements of Z_2 (with -1 being the element of order 2). Then a_{+1} and a_{-1} $\forall a \in S_3$ will be the elements of $S_3 \times Z_2$.

The group would act on the invariants as follows:

$$(12)_{-1} \triangleleft [a, b, c] = [-b, -a, -c] \tag{6.15}$$

$$(123)_{+1} \triangleleft [a, b, c] = [c, a, b] \tag{6.16}$$

The task then becomes to determine whether there is a subgroup of G which yields appropriate elements corresponding to C, P and T. The choice of these elements will give us a foundation upon which to build a physical interpretation of these capped three braids. Considering the idea that magnitude of charge should be invariant under these operations, we find that charge must take on the form of:

$$\text{Charge} = \chi(a+b+c) \tag{6.17}$$

where χ is some function. To respect convention we are led to suggest that χ be related to a factor of $\frac{1}{3}$. As charge is additive, and so is twist, we are led to expect that χ involves only $(a+b+c)$, and no other powers this sum. The simplest choice is then that charge is $\frac{1}{3}(a+b+c)$. C must then be an element of S_3 combined with a factor of -1. Charge conjugation, parity and time reversal each have order 2, and should, for the case of states of massive particles at rest satisfy,

$$CPT = \mathbf{1} \tag{6.18}$$

Taking this into account, our only option is for (C, P, T) to be the set $(a_{-1}, a_1, \mathbf{1}_{-1})$ for some element $a \in S_3$, with order two.

Considering the three options of (12), (13) and (23), (13) becomes the preferred option as it corresponds to a mathematical operation that we can perform upon a Braided Ribbon Network as a whole. A (13) permutation can be generated across the entire network by looking at the network from 'the other side' - this will reverse the ordering of the invariant of all braids. Similarly a $(13)_{-1}$ can be achieved through a reflection of the network in its entirety, and a $\mathbf{1}_{-1}$ by performing the two operations in succession.

Armed with this, we are left with our complete set of options: C can only be $(13)_{-1}$ or $\mathbf{1}_{-1}$, with P and T taking the other roles, as listed below.

We shall choose for the rest of the article the following definitions:

$$C = \mathbf{1}_{-1} \tag{6.19}$$

$$P = (13) \tag{6.20}$$

$$T = (13)_{-1} \tag{6.21}$$

additionally the charge of a braid defined by the invariant $[a, b, c]$ will be $\frac{1}{3}(a + b + c)$.

6.3.3 Classification of Particle States

We are now ready to present the main result of [3], which is the complete scheme for the identification of the standard model fermions and weak vector bosons with excitations of quantum geometry.

Invariant Classes

By considering the action of the operators C, P and T upon classes of braids we can divide the braids into four categories. The simplest category is that of braids which are invariant under both C and P (and hence also invariant under T). These braids would then be equivalent to charge-less particles that do not have left- or right-handed versions (either scalars, or integer spin objects with secondary spin quantum number of zero). A second class is objects which are invariant under T. These objects necessarily must have the form $(a, 0, -a)$. We can then identify the two parity states in this situation with being left- and right-handed versions of the same object.

Class	Invariant	Number of Corresponding Objects	Invariant under
I	$[0, 0, 0]$	One	C,P,T
II	$[a, 0, -a]$	Two	T
III	$[a, b, a]$	Two	P
IV	$[a, b, c]$	Four	None

Table 6.2: Classes of Braids

Table 6.2 demonstrates these categories. We can see that the categories discussed above fall exactly into class I and II. This leaves the other two categories to explain. Class III braids are then in correspondence with particles that are invariant under parity - charged scalars and objects with secondary spin quantum number of zero (i.e. the $m_s = 0$ states of integer spin objects). It is interesting to note that Class II braids respond to C and P in the same manner, implying that their parity transforms are also their charge conjugates. Class IV objects are then sets of four braids that map to one another under C,P and T corresponding with fermions.

Interaction Assumptions

We will assume that there is some mechanism that allows interactions between braids in the three forms of:

$$[a,b,c] + [d,e,f] = [a+d, b+e, c+f] \tag{6.22}$$

$$[a,b,c] = [d,e,f] + [g,h,i]\,, \tag{6.23}$$

$$d+g=a, \quad e+h=b, \quad f+i=c$$

$$[a,b,c] + [d,e,f] = [g,h,i] + [j,k,l]\,, \tag{6.24}$$

$$a+d=g+j, \quad b+e=h+k, \quad c+f=i+l$$

The Z

From this, we make a choice for the Z-boson. As the Z is a spin-1 particle without a charge conjugation pair, it falls into the category of class I & II braids. Adding in a further requirement that we desire our Z-boson to be representable by a braid which does not contain any twists, we find that our first successful candidate for it is the $[2,0,-2]$ state (the $m_s = 0$ state necessarily becomes the $[0,0,0]$ state). The W^+ and W^- states should then take on the form of the Z states with an extra integer twist placed on each strand, giving a set of eight objects (these are explicitly shown in table 6.3).

The Neutrino

We shall require that a neutrino be a particle, represented by a braid which does not contain any twists, with a neutral charge. Additionally it should satisfy that its parity conjugate is equal to its interaction with one of the Z bosons (i.e. given the evidence that the neutrinos of at least two generations are massive, we expect that right-handed neutrinos exist, and will interact with left-handed Z bosons to produce left-handed neutrinos). We find a countable number of such particles satisfying:

$$[a, 2-2a, a-2]\,, \quad a \in 2Z \tag{6.25}$$

The first such example is then $[2, -2, 0]]$ which corresponds to a left-handed neutrino. It follows that all such objects are class IV braids, and therefore come in sets of four objects. This implies that there are both left-handed and right-handed neutrinos in all generations and that there are an infinite series of such neutrinos.

The Rest of the Scheme

Further particles are found by adding twists to the individual strands of the neutrinos (without double twisting a single strand), the colour of a quark being determined by the 'odd strand out' (the untwisted strand when two strands are twisted, or the only twisted strand when a single one is). This twisting gives them their expected charges, and the colouring gives the expected results for interactions with the W's. Table 6.3 shows the first two generations of particles, along with the bosons.

This pattern then continues for infinitely many higher generations, each made from successive neutrino states.

6.3.4 Conclusions

Though these results could be considered a positive step towards the emergence of matter from quantum gravity, I hesitate to trumpet them as such. In reality what has been demonstrated here is the existence of a consistent map from the set of particles from the standard model to the infinite space of invariants of braided ribbon networks. That we can do so is necessary, but nowhere near sufficient to demonstrate that particle physics can emerge from quantum gravity.

The reality is that until there exists a means to introduce dynamics to such states - even heuristically - all that can be done in such studies is to match quantum numbers to countable invariants. This should be understood not as a failure, but instead as a clear message of where future efforts should be directed.

Particle				Invariant			
Z_L	Z_R	Z_0		$[2,0,-2]$	$[-2,0,2]$	$[0,0,0]$	
W_L^+	W_R^+	W_0^+		$[3,1,-1]$	$[-1,1,3]$	$[1,1,1]$	
W_L^-	W_R^-	W_0^-		$[1,-1,-3]$	$[-3,-1,1]$	$[-1,-1,-1]$	
$\nu_L^{(e)}$	$\nu_R^{(e)}$	$\bar{\nu}_L^{(e)}$	$\bar{\nu}_R^{(e)}$	$[2,-2,0]$	$[0,-2,2]$	$[-2,2,0]$	$[0,2,-2]$
e_L^-	e_R^-	e_L^+	e_R^+	$[1,-3,-1]$	$[-1,-3,1]$	$[-1,3,1]$	$[1,3,-1]$
$d_{L,r}$	$d_{L,g}$	$d_{L,b}$		$[1,-2,0]$	$[2,-3,0]$	$[2,-2,-1]$	
$d_{R,r}$	$d_{R,g}$	$d_{R,b}$		$[-1,-2,2]$	$[0,-3,2]$	$[0,-2,1]$	
$\bar{d}_{L,r}$	$\bar{d}_{L,g}$	$\bar{d}_{L,b}$		$[-1,2,0]$	$[-2,3,0]$	$[-2,2,1]$	
$\bar{d}_{R,r}$	$\bar{d}_{R,g}$	$\bar{d}_{R,b}$		$[1,2,-2]$	$[0,3,-2]$	$[0,2,-1]$	
$u_{L,r}$	$u_{L,g}$	$u_{L,b}$		$[2,-1,1]$	$[3,-2,1]$	$[3,-1,0]$	
$u_{R,r}$	$u_{R,g}$	$u_{R,b}$		$[0,-1,3]$	$[1,-2,3]$	$[1,-1,2]$	
$\bar{u}_{L,r}$	$\bar{u}_{L,g}$	$\bar{u}_{L,b}$		$[-2,1,-1]$	$[-3,2,-1]$	$[-3,1,0]$	
$\bar{u}_{R,r}$	$\bar{u}_{R,g}$	$\bar{u}_{R,b}$		$[0,1,-3]$	$[-1,2,-3]$	$[-1,1,-2]$	
$\nu_L^{(\mu)}$	$\nu_R^{(\mu)}$	$\bar{\nu}_L^{(\mu)}$	$\bar{\nu}_R^{(\mu)}$	$[4,-6,2]$	$[2,-6,4]$	$[-4,6,-2]$	$[-2,6,-4]$
μ_L	μ_R	$\bar{\mu}_L$	$\bar{\mu}_R$	$[3,-7,1]$	$[1,-7,3]$	$[-3,7,-1]$	$[-1,7,-3]$
$c_{L,r}$	$c_{L,g}$	$c_{L,b}$		$[4,-5,3]$	$[5,-6,3]$	$[5,-5,2]$	
$c_{R,r}$	$c_{R,g}$	$c_{R,b}$		$[2,-5,5]$	$[3,-6,5]$	$[3,-5,4]$	
$\bar{c}_{L,r}$	$\bar{c}_{L,g}$	$\bar{c}_{L,b}$		$[-4,5,-3]$	$[-5,6,-3]$	$[-5,5,-2]$	
$\bar{c}_{R,r}$	$\bar{c}_{R,g}$	$\bar{c}_{R,b}$		$[-2,5,-5]$	$[-3,6,-5]$	$[-3,5,-4]$	
$s_{L,r}$	$s_{L,g}$	$s_{L,b}$		$[5,-6,2]$	$[4,-7,2]$	$[4,-6,1]$	
$s_{R,r}$	$s_{R,g}$	$s_{R,b}$		$[1,-6,4]$	$[2,-7,4]$	$[2,-6,3]$	
$\bar{s}_{L,r}$	$\bar{s}_{L,g}$	$\bar{s}_{L,b}$		$[-3,6,-2]$	$[-4,7,-2]$	$[-4,6,-1]$	
$\bar{s}_{R,r}$	$\bar{s}_{R,g}$	$\bar{s}_{R,b}$		$[-1,6,-4]$	$[-2,7,-4]$	$[-2,6,-3]$	

Table 6.3: Braid assignments for particles

Chapter 7

Implications for Loop Quantum Gravity

> The content of this chapter is a result of a collaboration with Yidun Wan and was originally published in [18]. It was originally presented in the context of trinion BRNs and spin networks, and will be left as such here.

The intent of this chapter is to take a different tack from chapter 6 and to focus on using the understanding derived from studying BRNs to point out implications for Loop Quantum Gravity. The ability to do this comes from the intuition about the properties of embedded spin networks derived from looking at BRNs where these properties can be studied directly. In this way this chapter is more related to chapter 3 and should be understood in that context.

7.1 Conserved Structures in Embedded Networks

Given the existence of conserved structures in braided ribbon networks demonstrated in chapter 4 we shall consider specifically the more local structures (in the sense of micro-locality from section 5.1). This specifically shall ignore structures where ribbons are knotted or braided with ribbons that are distant under the standard distance function of graph theory. We can consider the local structures in order of reducing locality: the most local are those that involve only a single ribbon (ultra local), then those that involve ribbons sharing a node (1st degree local), then those that involve the ribbons of two adjacent nodes (2nd degree local), and so on. In particular, we know from [24] that a structure such as that in Fig. 7.2 is conserved as we are unable to perform the $2-2$ move without first deforming the knot away from these two nodes, and is therefore an ultra local conserved structure. The only exception to this is if the ribbon we are concerned with connects two 'halves' of the network (i.e. that the ribbon divides the network between two parts that are not connected to one another through anything other than that single ribbon - in the language of the next section, it corresponds to both b and b' being tethers of isolated substructures). In this situation we are able to remove the knotting and twisting by isotopy. As this case is artificial in nature and uninteresting, we shall ignore it for our investigation.

7.1.1 Isolating the Conserved Structures

In section 5.2 the concept of an isolated substructure was introduced as a means of understanding the ability to translate features through a braided ribbon graph. An isolated substructure is a subset of a graph which connects to the rest of the graph only through

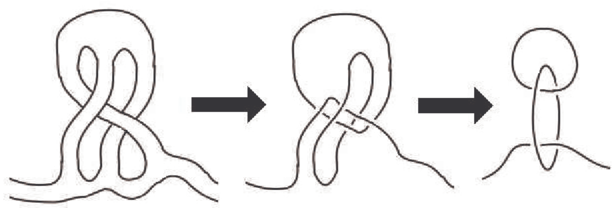


Figure 7.1: The Reduced Link

a single ribbon (called its tether) and is not part of any larger topological features. If a substructure can be evolved into an isolated substructure by a sequence of applications of elements of the evolution algebra, we will call them isolatable. Isolatable substructures are essentially propagating locally conserved quantities[16] able to move via the evolution algebra to any point edge-connected to its tether, and having conserved structure inside of it. We can also see via the form of the reduced link of an isolated substructure - and the invariance of the reduced link under the evolution moves - that an isolatable substructure corresponds to a 'piece' of the reduced link that is essentially cut and paste into the link of the edge its tether is on (see for example fig.7.1). That a general reduced link can be considered a direct product of these 'pieces' means that a structure being isolatable does not require the evolution to acquire its meaning as a part of an invariant of the network, and that this meaning is invariant under interpretation of the meaning of the evolution. We shall demonstrate that a specific class of ultra-local conserved quantities are isolatable. To do this we shall review some definitions and then introduce a few more.

Definition 14. *Two edge segments are said to be* ***edge connected*** *if they are connected in the space consisting of the edge of the network. Equivalently two edge segments are said to be edge connected if they are part of the same link in the reduced link of the network.*

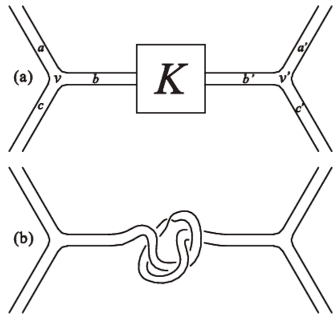


Figure 7.2: An example of an ultra-local structure

The path in the space of edges between the two edge segments is called the ***edge path****.*

Definition 15. *Two ribbons or two nodes in a network have a* ***path*** *between them if there exists a sequence of ribbons and nodes that can be traversed between them. The sequence of ribbons and nodes taken is called the path.*

Definition 16. *A* ***free path*** *is a path which does not have any twists, knots or links along it. Specifically, each ribbon connecting the nodes of the path does not have any knotting or twisting on it, and there is no ribbon that crosses a ribbon in the path in such a way that cannot be undone by the Reidemeister moves applied to the ribbons.*

Definition 17. *A* ***free edge path*** *is an edge path which does not have any knots or links along it. This corresponds similarly to requiring that each ribbon the edges of the path belong to does not have any knotting or twisting on it, and there is no ribbon that crosses one of these ribbons in such a way that cannot be undone by the Reidemeister moves applied to the ribbons.*

We shall now prove that a general class of ultra-local structures (see Fig. 7.2) can be made isolated.

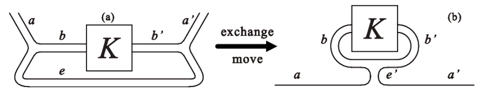


Figure 7.3: 0 Node Case

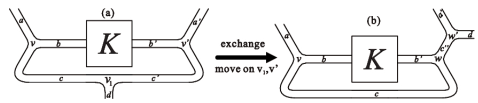


Figure 7.4: 1 Node Case

Theorem 5. *An ultra-local structure in a BRN which possesses a free edge path between one of the edges of each of the external ribbons can be isolated.*

Proof. We shall prove this using induction on the number of intervening nodes. First we shall prove for one intervening nodes, then assume true for $n-1$ nodes and prove true for n nodes. Consider the situation depicted in Fig. 7.4 (where the apparent orientation of the nodes is for simplicity, and is in fact general), the application of the exchange move between nodes v_1 and v' reduces the situation to that of Fig. 7.3 which can then be isolated by using the exchange move on the two nodes involved. Now, we examine the situation in Fig. 7.5 to demonstrate that the n node situation can be reduced to $n-1$ nodes by applying the exchange move on v' and v_n. We can then use the assumption of truth on the $n-1$ case to isolate the knot. □

This result lets us examine a peculiar situation: that where a knot on a ribbon is *not* a conserved quantity. Examining Fig. 7.6 (where the unattached ribbons connect to a larger network) we can see that it is possible in certain situations to reduce the number of ultra

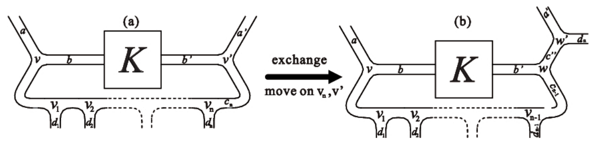


Figure 7.5: n Node Case

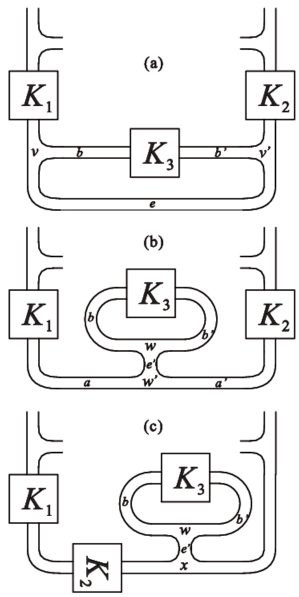


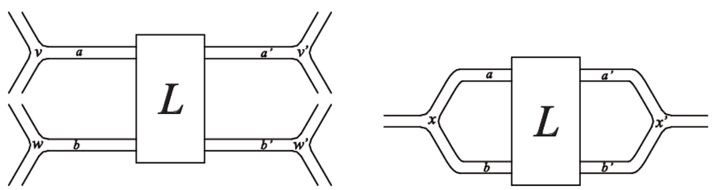
Figure 7.6: Reduction of the number of ultra local structures

local structures, in the sense that two knots, e.g. the K_1 and K_2 in the figure, merge with each other. From this we see that we can always obtain a graph with only conserved local structures remaining.

Theorem 6. *A knot in an embedded graph which possesses a free path between its two external edges can be isolated.*

Proof. The proof of this follows inherently from the above proof and the fact that in an embedded graph - instead of a BRN - one can rotate an edge without introducing a twist. □

We can apply the above theorems to reduce less local structures to more local situations. Consider for example the situation in Fig. 7.7a, we can apply the results of the above theorems to transform it to Fig. 7.7b if ribbons a and b, and a' and b' are connected by a free edge path.



(a) (b)

Figure 7.7: Less Local knottings

We can also see from these results that if there is no (edge) free path between the two edges of any of the knots in a network, there is no means to combine the knots onto single edges. This leads us to conclude that if one isolates all isolatable knots on a graph the remaining knots are completely invariant.

7.1.2 Immediate Results of Ultra-Local Structures

Considering the idea of ultra local structures, we find that there is only a single type of conserved structure. A structure formed by the topological deformation of a single ribbon can only possess two features: knots and twists. As twists can be passed through the knotting of a ribbon by an isotopy, we can consider any ultra local structure to be exactly characterized by a half integer (corresponding to the number of rotations) and a knot or a connected sum of knots.

This leads us to the following results:

The existence of countably infinite many species of local conserved structures

There exist infinitely many species of local conserved structures.

Any edge can be replaced by an edge with an isolated edge with some half-integer twist and a knot or a connected sum of knots. As there are infinitely many half-integers and knots, there are therefore infinitely many such species of structures.

Maximal number of local conserved quanities

For a closed 3-valent BRN with N nodes, the maximum number of ultra-local conserved quantities is $\frac{3N}{2}$.

We can immediately lift these results to the scenario of un-framed spin networks: excepting the twists, all the results follow immediately. Additionally the first result does not depend on the valence of the spin-network involved in any way and is therefore a general result for embedded spin-networks.

7.2 Conclusions and Discussion

We have demonstrated the existence of a countable infinity of species of local conserved structures within Braided Ribbon Networks (and embedded spin-networks in general). We have also provided several results of use for isolating these structures and understanding when they are actually preserved.

In a theory of Quantum Gravity where the states are given by spin-networks embedded in a 3-manifold all of these states will be part of the Hilbert space. The difficulty this poses comes from the fact that these locally conserved structures correspond to an infinite number of conserved quantities that don't correspond with anything that commutes with the constraints of general relativity. This poses a significant problem in any attempt to recover the classical limit from a generic embedded spin-network - there is no reason to believe that these conserved quantities will simply cease to exist in the classical limit. This leaves a significant dilemma: we must change something in the theory for general relativity to be the classical limit.

There are two immediately obvious alternatives for resolving this, the first being to modify the Hamiltonian constraint in such a way that we introduce new generators or the evolution algebra. The alternative to this is that we should reduce the physical Hilbert space of a theory of quantum gravity to require that there do not exist any knots or linking. Our ability to consider this super-selection rule and still do certain things (including considering embedded spin networks) is questionable and requires investigation before this can be adopted as an 'easy' solution.

Chapter 8

Conclusions and Future Directions

The study of Braided Ribbon networks has come a significant way from their origins as an attempt to have matter emerge from loop quantum gravity. We have outlined several results which develop this transformation. In chapter 1 I presented a unified framework for 3 and 4-valent BRNs and how they can be expanded to general valences. In chapter 3 I demonstrated the relationship between BRNs and the embedded spin network states of loop quantum gravity. In chapter 4 I presented the concept of the reduced link and demonstrated it as an invariant for both the 3 and 4-valent BRNs. In chapter 5 I presented the concepts of subsystems and isolated substructures which allow us to understand way in which the parts of the BRN contribute to the reduced link. With these tools I reproduced the previous results which used the invariants from BRNs: the twist invariants of chapter 6 and the microlocal invariants of chapter 7.

Though the hopes of encoding particle physics in these invariants has made little headway through this program, we have instead developed tools for studying embedded spin networks. With this perspective we will discuss implications and possible avenues of further research from it.

The first implication of having a tool with which we can study the embedding information of BRNs or spin networks is that we now have a simpler object which we can use to attempt to find physical interpretations of the embedding information. In analogy with $2+1$ quantum gravity, those links which are non-trivial elements in the fundamental group of the embedding space can already be assigned some meaning: they correspond to parts of the network that grant cross-sectional area to different handles of the manifold. It is possible that we can extend from this starting point and find physical meaning for all of the embedding information.

The second implication comes from examining the objects which make up the reduced link: the racing stripes. In the 3-valent case these are dual to nodes, and in the four valent case they are edges of tetrahedra. Our ability to take the dual of a grouping of nodes is reliant upon the triviality of the stripes of these nodes, and so we cannot construct the dual of nodes which have a reduced link. With this understanding it is possible that we could construct a dual which somehow combines glued tetrahedra and a modification from the reduced link. Such work is indeed already underway.

Lastly we present one further possible application of these results, one which takes a longer view. Some recent work, [13], has attempted to make contact between loop quantum gravity and spin foams. This contact was made at the expense of working with unembedded spin networks. It is possible that the development of the reduced link could allow for this work to be reproduced without this sacrifice. One particular approach would be to consider cobordisms between embedded spin-networks, which could allow (depending upon the genus of the four dimensional surface) the reduced link to change. This work is only in its infancy, but should it succeed it would allow a connection between spin foams and the actual states of loop quantum gravity. Additionally, allowing for an evolution of this form could resolve the concerns of chapter 7 through the properties of cobordant knots.

Having many paths forward and a significant body of results behind, the braided ribbon network research program has a bright future. With the connection to the states of loop quantum gravity demonstrated in chapter 3 this future can now be shared and I hope the two research programs can both inform one another as they move forward.

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