

# Quaternions and Quantum Theory

by

Matthew A. Graydon

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## Abstract

The orthodox formulation of quantum theory invokes the mathematical apparatus of complex Hilbert space. In this thesis, we consider a quaternionic quantum formalism for the description of quantum states, quantum channels, and quantum measurements. We prove that probabilities for outcomes of quaternionic quantum measurements arise from canonical inner products of the corresponding quaternionic quantum effects and a unique quaternionic quantum state. We embed quaternionic quantum theory into the framework of usual complex quantum information theory. We prove that quaternionic quantum measurements can be simulated by usual complex positive operator valued measures. Furthermore, we prove that quaternionic quantum channels can be simulated by completely positive trace preserving maps on complex quantum states. We also derive a lower bound on an orthonormality measure for sets of positive semi-definite quaternionic linear operators. We prove that sets of operators saturating the aforementioned lower bound facilitate a reconciliation of quaternionic quantum theory with a generalized Quantum Bayesian framework for reconstructing quantum state spaces.

This thesis is an extension of work found in [42].

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## Dedication

*I dedicate this thesis to the loving memory of my grandparents  
Roy Graydon and Phyllis Graydon.*

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# Chapter 1

## Introduction

The theory of relativity and quantum theory are the great triumphs of 20th century physics. The theory of relativity revolutionized humankind's conceptions of matter and spacetime, and quantum theory inspired paradigm shifts in atomic physics and information science. The foundations for the orthodox formulations of the aforementioned theories are, however, quite dissimilar. The theory of relativity is founded on physical principles. In particular, the principle of relativity and the principle of constant velocity of light in vacuo form the physical basis for the theory of relativity. The former principle, in the case of general relativity, corresponds to the idea that all Gaussian coordinate systems are physically equivalent [18]; whereas, in special relativity the principle of relativity is restricted to the physical equivalence of inertial reference frames [19]. Einstein emphasized that these physical principles are not hypothetically constructed, but rather empirically discovered characteristics of natural processes that give rise to the mathematical framework of the theory of relativity [20].

In stark contrast to the theory of relativity, the orthodox formulation of quantum theory is founded on abstract mathematics, not physical principles of the type described by Einstein in [20]. To this day, with over one century now past since Planck's discovery of the quantization of radiation [62], the search for deep physical principles underpinning the orthodox formulation of quantum theory remains ongoing. The discovery of such physical principles could contribute to a complete unification of general relativity and quantum theory into a physical theory applicable on all physical scales and to all known physical forces, which is one of the great problems facing 21st century physics.

The mathematical formalism of quantum theory presently serves with unrivaled success to predict and control nongravitational physical phenomena on atomic and subatomic scales<sup>1</sup>. Research in the field of quantum foundations is particularly concerned with the identification of physical characteristics of the natural world that may be inferred in light of the great empirical success of quantum theory. One of the most significant open questions in quantum foundations is: 'What is the fundamental ontology underlying quantum theory?' In [21], Einstein, Podolsky, and Rosen (EPR) expound a sufficient condition for elements of physical reality:

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<sup>1</sup>In principle, quantum theory can be applied on all physical scales.

“If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of reality corresponding to that quantity.”

– A. Einstein, B. Podolsky, and N. Rosen, 1935 [21].

EPR also define a necessary condition for the completeness of a physical theory, stating that “every element of the physical reality must have a counterpart in the physical theory” [21], and EPR argue that the description of reality as given by quantum theory is incomplete in that sense. The EPR argument is tacitly based on the principle of *locality*, i.e. that the elements of physical reality attributed to spatially separated physical systems are independent [73], or put otherwise causally disconnected. EPR believed in the existence of a complete physical theory that respects the principle of locality *and* reproduces the predictions of quantum theory [21]. However, in [9], Bell proved that a complete physical theory that reproduces the predictions of quantum theory *must* be nonlocal. Thus, in light of the great empirical success of quantum theory, one must reject either the premise of locality or the EPR criterion of reality, or both.

Rejecting the principle of locality permits the construction of hidden-variable quantum models, such as Bohmian mechanics [11], wherein the fundamental ontology consists of quantum states together with additional elements of physical reality that suffice for the retention of EPR realism. The Kochen-Specker theorem [55] dictates that such hidden-variable quantum models must be *contextual*<sup>2</sup>, i.e. the elements of physical reality in such models cannot be independent of the contexts in which they are measured. Adopting an epistemic view of quantum states, one can reject the EPR criterion of reality and retain both the principle of locality and noncontextuality of probability assignments for measurement outcomes, as in Quantum Bayesianism (see [32] and references therein.) Indeed, the theorems of Bell [9] and Kochen-Specker [55] significantly constrain coherent interpretations of quantum theory; consequently, different paths are established for seeking out physical principles underpinning the orthodox quantum formalism.

In the past decade, the emerging field of quantum information science has motivated reconstructions of finite-dimensional quantum theory based on information-theoretic physical principles, including those found in [47][48][36][35][16][74][57]. Hardy points out in [48] that this program “was very much inspired by Fuchs’s suggestion that we need to find information-theoretic reasons for the quantum axioms (presented in a number of talks and written up in [30]).” Moreover, in [30], Fuchs emphasizes that quantum states are epistemic states, not states of nature. On that view, quantum theory prescribes a probability calculus for computing measurement outcome expectations based on states of knowledge, or degrees of belief regarding physical systems and experimental apparatus.

A common technique applied in recent quantum foundational reconstruction programs is to first *reformulate* the orthodox quantum formalism into new language using the tools, techniques, and ideas of quantum information theory, and then to attempt to *reconstruct* that reformulation based on physical principles. It is important to point out that, in all of the above cited cases, the goal is to derive a formalism equivalent to the orthodox quantum formalism invoking *complex Hilbert space*; wherein, quantum states for physical systems are defined by unit-trace positive semi-definite linear operators on complex Hilbert spaces; physical transformations are associated with completely

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<sup>2</sup>The Kochen-Specker theorem does not hold for 2-dimensional quantum systems.

positive maps on quantum states; and physical measurements are associated with positive operator valued measures comprised of complex quantum effects. But an important question remains:

### Why complex Hilbert space?

The probability calculi prescribed by classical physical theories are fundamentally different from the probability calculus prescribed by the orthodox formulation of quantum theory over complex Hilbert spaces. For instance, Birkhoff and von Neumann pointed out that classical experimental propositions regarding physical systems form Boolean algebras; whereas, quantum experimental propositions – *e.g.* projections onto subspaces of a complex Hilbert space – comprise nondistributive orthomodular lattices [10]. Moreover, Feynman emphasized that the classical Markovian law of probability composition fails to hold in the description of quantum phenomena [26]. Instead, quantum *probability amplitudes* superpose. These essential features are not, however, unique to the calculus prescribed by usual complex quantum theory – they are enjoyed in quantum theories formulated over any of the associative normed division algebras  $\mathbb{R}$ ,  $\mathbb{C}$ , or  $\mathbb{H}$  [10][1].

What then, if anything, *does* distinguish quantum theories formulated over  $\mathbb{R}$  or  $\mathbb{H}$  from usual complex quantum theory? In the case of real quantum theory [70], multipartite systems are endowed with some rather unusual properties. For example, in real quantum theory, there exist states associated with  $n$ -partite systems for which every subsystem is maximally entangled with each of the other subsystems, where  $n$  can be arbitrarily large [75]. Furthermore, real quantum theory is not a locally tomographic theory (wherein states for composite systems can be determined by the probabilities for outcomes of local measurements of the component systems) – it is instead a bilocally tomographic theory (wherein states for composite systems can be determined by the probabilities for outcomes of joint measurements of pairs of component systems) [49]. These observations point to aspects of real quantum theory that are not realized within the usual complex quantum theoretical framework. However, real quantum theory is equivalent to complex quantum theory equipped with a superselection rule [70], and conversely, the evolution and measurement of a multipartite complex quantum state under discrete or continuous evolution in complex quantum theory can be simulated using states and operators in real quantum theory [58].

In the case of quaternionic quantum theory [29][27][28], the very notion of ‘independent subsystems’ is a subject of debate. In fact, quaternion-linear tensor products of quaternionic modules do not exist [63]. This has constituted a significant obstacle for the development of a consistent definition of local quaternionic operations, and it has been argued that one is actually prevented from speaking of absolutely independent systems in quaternionic quantum theory [27]. Nevertheless, the experimental propositions in quaternionic quantum theory that commute with a fixed anti-Hermitian unitary operator are isomorphic to the experimental propositions of complex quantum theory, where the isomorphism preserves the logical operations of intersection, span, and orthocomplement [27]. Furthermore, in the context of quantum information processing involving unitary transformations and projective measurements, it has been shown that circuits acting on  $n$  2-dimensional quaternionic systems can be simulated by circuits acting on  $n + 1$  qubits [25].

In this thesis, we consider a *generalized* formulation of quaternionic quantum theory, rather than only considering the restricted class of quantum processes treated in [25][29][27][28][1]. We treat generalized quantum measurements as quaternionic positive operator valued measures, and we treat quantum channels as completely positive trace preserving quaternionic maps. Our primary purpose

is to demonstrate that *any* quaternionic quantum process can be embedded into the formalism of usual complex quantum information theory, as established in [42]. We also aim to investigate the status of quaternionic quantum theory within a generalized Quantum Bayesian framework for the reconstruction of quantum state spaces.

The remainder of this thesis is structured as follows. In Chapter 2, we formally introduce the quaternions, and we review prerequisite quaternionic linear algebraic theory for the chapters that follow. In Chapter 3, we consider a finite-dimensional quaternionic quantum formalism for the description of quantum states, quantum channels, and quantum measurements; wherein, quaternionic quantum states are defined by unit-trace positive semi-definite linear operators on finite dimensional right quaternionic modules; quaternionic quantum channels are defined by completely positive trace preserving quaternionic maps; and quaternionic quantum measurements are defined by quaternionic positive operator valued measures. We also prove a theorem dictating the quaternionic quantum probability rule. In Chapter 4, we prove that any physical process described via quaternionic quantum formalism has an equivalent description in usual complex quantum information theory. In Chapter 5, we derive a lower bound on an orthonormality measure for sets of positive semi-definite quaternionic matrices. We consider the expansion of quaternionic quantum states in bases that saturate the aforementioned lower bound, and we prove that such expansions permit a reconciliation of quaternionic quantum theory with a generalized Quantum Bayesian framework for the reconstruction of quantum state spaces. In Chapter 6, we conclude and outline directions for future research. In Appendix A, we present elements of symplectic group theory referred to throughout the thesis. In Appendix B, we describe the connection between unit-norm quaternions and rotations on  $\mathbb{R}^3$ . Finally, in Appendix C, we explicitly compute the universal  $C^*$ -algebras enveloping universally reversible self-adjoint quaternionic matrix algebras.

## Chapter 2

# Quaternionic Linear Algebra

Unlike the complex number field and its subfield of real numbers, the division ring of quaternions admits a *noncommutative* multiplication operation. The noncommutativity of quaternionic multiplication results in significant distinctions between complex and quaternionic linear algebraic theories. To state just one remarkable example at the outset, note that there exist finite-dimensional quaternionic matrices admitting an *infinite* spectrum of eigenvalues. In this chapter, we introduce the quaternions and review the linear algebraic theory of quaternionic modules and matrices required for the formulation of a quaternionic quantum formalism in Chapter 3.

### 2.1 Historical Prelude

The quaternions were discovered by Sir William Rowan Hamilton, whose important contributions to classical mechanics and geometrical optics are well known. In 1835, Hamilton demonstrated that the complex numbers could be regarded as an algebra of ordered pairs in  $\mathbb{R}^2$ . Hamilton referred to such points as *couples*, and he showed that couples could be added and multiplied together according to the rules of complex arithmetic [43]. For many years thereafter, Hamilton's attention was occupied with attempts to develop a similar algebra of ordered triples, or *triplets*, that could be represented by points in  $\mathbb{R}^3$  [44]. Shortly before his death in 1865, Hamilton would write the following in a letter to his son Archibald:

“Every morning in the early part of the above-cited month [author's note: October 1843], on my coming down to breakfast, your (then) little brother William Edwin, and yourself, used to ask me: ‘Well, Papa, can you multiply triplets?’ Whereto I was always obliged to reply, with a sad shake of the head: ‘No, I can only add and subtract them’.”

– W. R. Hamilton, August 5, 1865 [41].

Alas, Hamilton's attempts were in vain, for, as Hurwitz's theorem of 1898 would later establish, the three-dimensional algebra Hamilton sought does not exist [52]. On the 16<sup>th</sup> day of October

1843, while walking with his wife along the Royal Canal, Hamilton realized the key was to consider a *fourth* dimension [44], and he proceeded to carve the following equation into the stonework of the Brougham Bridge in Dublin:

$$i^2 = j^2 = k^2 = ijk = -1, \quad (2.1)$$

and so, in a flash of genius, the quaternions were discovered.

## 2.2 Quaternions

The set of quaternions is denoted by  $\mathbb{H}$  in honour of Hamilton, and is defined as follows:

$$\mathbb{H} = \{1a_0 + ia_1 + ja_2 + ka_3 \mid (\forall r : a_r \in \mathbb{R}) \wedge (i^2 = j^2 = k^2 = ijk = -1)\}, \quad (2.2)$$

where the real numbers  $a_r$  are referred to as *constituents*, and where the *quaternion basis elements* defined in (2.1) also obey the following anticommutation relations:

$$ij = k \quad \text{and} \quad ji = -k, \quad (2.3)$$

$$jk = i \quad \text{and} \quad kj = -i, \quad (2.4)$$

$$ki = j \quad \text{and} \quad ik = -j. \quad (2.5)$$

These anticommutation relations are trivial consequences of (2.1), for example

$$-1 = ijk \iff -k = ijk^2 \iff ij = k. \quad (2.6)$$

We denote the *real* and *quaternionic imaginary* parts of  $1a_0 + ia_1 + ja_2 + ka_3 \in \mathbb{H}$  by  $\Re(a) = a_0$  and  $\Im(a) = ia_1 + ja_2 + ka_3$  respectively. Before going any further, it will be useful to recall the definition of an abelian group:

**Definition 2.2.1** *Abelian Groups:*

*A group is a set  $\mathcal{S}$  equipped with a binary operation  $\star$  such that the following properties hold:*

- *Closure:*  $\forall a, b \in \mathcal{S}, a \star b \in \mathcal{S}$ ,
- *Associativity:*  $\forall a, b, c \in \mathcal{S}, (a \star b) \star c = a \star (b \star c)$ ,
- *Identity:*  $\exists \mathbf{I} \in \mathcal{S}$  such that  $\forall a \in \mathcal{S} : a \star \mathbf{I} = \mathbf{I} \star a = a$ ,
- *Invertibility:*  $\forall a \in \mathcal{S} \exists a^{-1} \in \mathcal{S}$  such that  $a \star a^{-1} = a^{-1} \star a = \mathbf{I}$ .

*An abelian group is a group equipped with a commutative binary operation such that*

- *Commutativity:*  $\forall a, b \in \mathcal{S}, a \star b = b \star a$ .

In fact,  $\mathbb{H}$  is an abelian group with respect to an associative commutative addition operation defined via

$$a + b = 1(a_0 + b_0) + i(a_1 + b_1) + j(a_2 + b_2) + k(a_3 + b_3), \quad (2.7)$$

for it is clear that  $\mathbb{H}$  is closed under addition, that 0 is neutral with respect to addition, and that  $\forall a \in \mathbb{H} \exists (-a) \in \mathbb{H} : a + (-a) = 0$ , namely  $-a = -1a_0 - ia_1 - ja_2 - ka_3$ .

Recall that a *monoid* enjoys all the properties of a group except for invertibility. The quaternions are in fact a monoid with respect to an associative noncommutative multiplication defined via

$$\begin{aligned} ab = & 1(a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3) + \\ & i(a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2) + \\ & j(a_0b_2 + a_2b_0 - a_1b_3 + a_3b_1) + \\ & k(a_0b_3 + a_3b_0 + a_1b_2 - a_2b_1), \end{aligned} \quad (2.8)$$

for, again, it is clear that  $\mathbb{H}$  is closed under multiplication, and that 1 is neutral with respect to multiplication. Furthermore,  $\forall a \neq 0 \in \mathbb{H} \exists a^{-1} \in \mathbb{H} : aa^{-1} = a^{-1}a = 1$ , where

$$a^{-1} = \frac{1a_0 - ia_1 - ja_2 - ka_3}{a_0^2 + a_1^2 + a_2^2 + a_3^2}. \quad (2.9)$$

Quaternionic addition and multiplication are distributive in the sense that  $\forall a, b, c \in \mathbb{H}$ :

$$a(b + c) = ab + ac, \quad (2.10)$$

and

$$(a + b)c = ac + bc. \quad (2.11)$$

Therefore, by virtue of all the aforementioned properties,  $\mathbb{H}$  is a *division ring*, but not a field.

The quaternions admit an involutory antiautomorphic conjugation operation taking  $a \rightarrow \bar{a}$  defined  $\forall a = 1a_0 + ia_1 + ja_2 + ka_3 \in \mathbb{H}$  via

$$\bar{a} = 1a_0 - ia_1 - ja_2 - ka_3, \quad (2.12)$$

where we say that conjugation is *involutory* because  $\overline{\bar{a}} = a$ . Furthermore, observing that

$$\begin{aligned} \overline{ab} &= 1(a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3) - i(a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2) \\ &\quad - j(a_0b_2 + a_2b_0 - a_1b_3 + a_3b_1) - k(a_0b_3 + a_3b_0 + a_1b_2 - a_2b_1) \\ &= (b_0 - ib_1 - jb_2 - kb_3)(a_0 - ia_1 - ja_2 - ka_3) \\ &= \bar{b}\bar{a}, \end{aligned} \quad (2.13)$$

one sees that quaternionic conjugation is a bijection reversing multiplicative structure, *i.e.* an anti-automorphism from the quaternions to themselves.

Quaternionic conjugation induces a multiplicative norm  $|\cdot| : \mathbb{H} \rightarrow \mathbb{R}$  via

$$|a| = \sqrt{a\bar{a}} = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}, \quad (2.14)$$

which is to say that the norm obeys the following properties  $\forall a, b \in \mathbb{H}$ :



- Multiplicativity:  $|ab| = |a||b|$ ,
- Nonnegativity:  $|a| \geq 0$  with equality *iff*  $a = 0$ ,
- Triangle Inequality:  $|a + b| \leq |a| + |b|$ ,

all of which follow directly from (2.14). Evidently,  $\forall a \in \mathbb{H}$ :

$$a^{-1} = (|a|)^{-2}\bar{a}. \quad (2.15)$$

The unit norm quaternions,

$$\text{Sp}(1) = \{a \in \mathbb{H} \mid |a| = 1\}, \quad (2.16)$$

form a group that is isomorphic to  $\text{SU}(2)$ . For the proof, see Remark A.2.5 in Appendix A. It is well known that  $\text{SU}(2)$  is the double cover of  $\text{SO}(3)$ , meaning that there is a two-to-one homomorphic surjection from  $\text{SU}(2)$  onto  $\text{SO}(3)$  preserving group-multiplicative structure. In Appendix B, we detail the connection between the group of unit-norm quaternions and the group of rotations on  $\mathbb{R}^3$ .

Equipped with a multiplicative norm, the quaternions obviously enjoy  $\mathbb{R}$ -homogeneity, which is to say that  $\forall \alpha \in \mathbb{R}, \forall a \in \mathbb{H} : |\alpha a| = |\alpha||a|$ . Therefore  $\mathbb{H}$  is in fact an *associative normed division algebra* over the field of real numbers. By a *normed algebra*, we mean a real vector space that is equipped with an associative  $\mathbb{R}$ -bilinear multiplication operation and a multiplicative norm. The aforementioned theorem of Hurwitz [52] dictates that  $\mathbb{R}$ ,  $\mathbb{C}$ , and  $\mathbb{H}$  are in fact the *only* associative normed division algebras up to isomorphism<sup>1</sup>. However, since  $\mathbb{H}$  is not a field, one must take care when extending the notion of a *vector space* over  $\mathbb{R}$  or  $\mathbb{C}$  to that of a *module* over  $\mathbb{H}$ .

## 2.3 Quaternionic Modules

In the chapters to follow, we are interested in a *finite-dimensional* quaternionic quantum formalism. Therefore, we shall restrict our attention in this chapter to the linear algebraic theory of finite-dimensional quaternionic modules and matrices. In the linear algebraic theory of complex vector spaces formulated over the *commutative* associative normed division algebra  $\mathbb{C}$ , it is inconsequential whether scalar multiplication is carried out from the *right* or from the *left*. In the case of quaternionic linear algebra, the same statement does not hold. For example, if  $A$  is a quaternionic matrix,  $\phi$  is a quaternionic column vector, and  $\lambda$  is a quaternion, then  $A\phi = \phi\lambda$  does not imply that  $A\phi = \lambda\phi$ . Following Finkelstein et al. [29], we adopt the convention wherein finite-dimensional quaternionic modules are taken as *right* modules. This choice of convention allows us to adopt the familiar *bra-ket* notation from complex quantum theory to formulate aspects of the theory of quaternionic linear operators detailed in Section 2.4. We now proceed with a formal definition of right quaternionic modules:

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<sup>1</sup>The only other normed division algebra (up to isomorphism) is the nonassociative algebra  $\mathbb{O}$  – the octonions [52].

**Definition 2.3.1** *Right Quaternionic Modules:*

A right quaternionic module is a set  $\mathcal{V}$  that is an abelian group under an addition operation  $+$  :  $\mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$ , and that is equipped and closed under right-multiplication by quaternions where  $\forall \phi, \xi \in \mathcal{V}$  and  $\forall a, b \in \mathbb{H}$  one has that  $(\phi + \xi)a = \phi a + \xi a$ ,  $(\phi)ab = (\phi a)b$ , and  $\phi(a + b) = \phi a + \phi b$ . The elements of  $\mathcal{V}$  are referred to as rays.

The familiar concepts of a *linear independence*, *spanning sets*, and *bases* from complex vector space theory all naturally carry over to the theory of right quaternionic modules. In particular, we say that a set of rays in a module  $\mathcal{V}$  is *linearly independent* if no ray in the set can be realized as a right-quaternion-linear combination of the others. Also, if any ray in  $\mathcal{V}$  can be realized as a right-quaternion-linear combination of rays in a set  $\mathcal{S} \subseteq \mathcal{V}$ , then we say that  $\mathcal{S}$  is a *spanning set*. A linearly independent spanning set is referred to as a *basis*, and the cardinality of a basis is referred to as the *dimension* of the right quaternionic module.

If  $\mathcal{B}_d = \{\phi_1, \dots, \phi_d\}$  is a basis for a finite  $d$ -dimensional right quaternionic module, then for any element  $\xi \in \mathcal{V}$ ,  $\exists \{a_1, \dots, a_d\} \subset \mathbb{H}$  such that  $\xi = \phi_1 a_1 + \dots + \phi_d a_d$ , and we can represent  $\xi$  via the  $d$ -tuple  $(a_1, \dots, a_d)$  with respect to  $\mathcal{B}_d$ . Whenever we make such a reference to an element in a finite  $d$ -dimensional quaternionic module, the basis shall be taken as implied. Moreover, the  $d$ -fold Cartesian product  $\mathbb{H}^d$  shall be assumed to be equipped with the standard addition operation, defined  $\forall \phi = (\phi_1, \dots, \phi_d)$ ,  $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{H}^d$  via

$$\phi + \xi = (\phi_1 + \xi_1, \dots, \phi_d + \xi_d), \quad (2.17)$$

and the standard right quaternionic multiplication operation, defined  $\forall \phi \in \mathbb{H}^d$  and  $\forall a \in \mathbb{H}$  via

$$\phi a = (\phi_1 a, \dots, \phi_d a), \quad (2.18)$$

so that  $\mathbb{H}^d$  is a  $d$ -dimensional right quaternionic module. We also equip  $\mathbb{H}^d$  with the *standard symplectic inner product*, which naturally induces the *canonical real-valued norm* on  $\mathbb{H}^d$  (see Definition A.2.2). Furthermore, we shall often adopt the familiar *bra-ket* notation from complex quantum theory, identifying  $\xi \in \mathbb{H}^d$  with the *ket* column ray

$$|\xi\rangle = \begin{pmatrix} \xi_1 \\ \cdot \\ \cdot \\ \cdot \\ \xi_d \end{pmatrix}, \quad (2.19)$$

and identifying *bra* conjugated row rays, comprising the dual right-module<sup>2</sup> to  $\mathbb{H}^d$ , as

$$\langle \phi | = (\overline{\phi_1} \dots \overline{\phi_d}), \quad (2.20)$$

so that

$$\langle \phi | \xi \rangle = \sum_{r=1}^d \overline{\phi_r} \xi_r \quad (2.21)$$

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<sup>2</sup>The set of quaternionic linear operators (see Section 2.4)  $T : \mathbb{H}^d \rightarrow \mathbb{H}$  is referred to as the *dual right-module* of  $\mathbb{H}^d$ , and is in one-to-one correspondence with  $\mathbb{H}^d$ . Therefore,  $\mathbb{H}^d$  is in fact *self-dual*.

defines the standard symplectic inner product, and so that the canonical real-valued norm on  $\mathbb{H}^d$  is defined via  $\|\phi\|^2 = \langle\phi|\phi\rangle$ . The group of automorphisms on  $\mathbb{H}^d$  preserving the standard symplectic inner product is defined and described in Definition A.2.3 and Remark A.2.4. Referring to Definition A.2.2, we note that the standard symplectic inner product admits  $\forall\phi, \xi \in \mathbb{H}^d$  and  $\forall a, b \in \mathbb{H}$  that  $\langle\phi a|\xi b\rangle = \bar{a}\langle\phi|\xi\rangle b$ . It is important to point out that if we had equipped  $\mathbb{H}^d$  to be a *left* quaternionic module, instead of a right quaternionic module, then it would follow that  $\langle a\phi|b\xi\rangle = \sum_{r=1}^d \bar{\phi}_r \bar{a} b \xi_r$ , which is *not* equal to  $\bar{a}\langle\phi|\xi\rangle b$  in general, and so the *bra-ket* would *not* define a symplectic inner product on  $\mathbb{H}^d$ . This would have marked *physical* consequences for quaternionic quantum theory. For instance, one could always find  $\phi \in \mathbb{H}^d$  such that the set of pure states associated with the set of rays  $\mathcal{S} = \{|a\phi\rangle \mid a \in \mathbb{H}\}$  would be inequivalent in the sense that one could also always find  $\xi \in \mathbb{H}^d$  and  $a \in \mathbb{H}$  such that

$$\langle\xi|\phi\rangle = 0, \quad (2.22)$$

and

$$\langle\xi|a\phi\rangle \neq 0, \quad (2.23)$$

both hold. Put otherwise, quaternionic phases would play a key role in such a theory. Of course, one could argue that it is simply not appropriate to adopt the *bra-ket* formalism in a left-modular formulation of a quaternionic quantum theory. And in fact one *can* do just that and develop a left-modular formulation of quaternionic quantum theory that is *equivalent* to a right-modular formulation of quaternionic quantum theory [66]. One major disadvantage of such an approach is that linear operators in the left-modular theory must act unusually – from the *right*. We adopt the right-modular convention so that linear operators act on our modules according to ordinary matrix multiplication from the left, which can be defined in terms of *bras* and *kets* as described in the following section.

## 2.4 Quaternionic Matrices

In this thesis, we define a *quaternionic linear operator* as a function  $T : \mathcal{V} \rightarrow \mathcal{W}$ , where  $\mathcal{V}$  and  $\mathcal{W}$  are right quaternionic modules, such that  $\forall\phi, \xi \in \mathcal{V}$  and  $\forall a, b \in \mathbb{H}$ :

$$T(\phi a + \xi b) = T(\phi)a + T(\xi)b, \quad (2.24)$$

and we shall restrict our attention to finite dimensions. On that view, quaternionic linear operators from finite  $d$ -dimensional  $\mathcal{V}$  to finite  $p$ -dimensional  $\mathcal{W}$  correspond to elements in the set of  $p \times d$  quaternionic matrices, which we denote by  $\mathcal{M}_{p,d}(\mathbb{H})$ . In particular, if

$$\mathcal{B}_d = \{\phi_1, \dots, \phi_d\}, \quad (2.25)$$

is a basis for  $\mathcal{V}$ , and if

$$\mathcal{B}_p = \{\xi_1, \dots, \xi_p\}, \quad (2.26)$$

is a basis for  $\mathcal{W}$ , then  $\forall s \in \{1, \dots, d\}$ ,  $\exists\{T_{1s}, \dots, T_{ps}\} \subset \mathbb{H}$  such that

$$T(\phi_s) = \sum_{r=1}^p \xi_r T_{rs}, \quad (2.27)$$

and the matrix  $A \in \mathcal{M}_{p,d}(\mathbb{H})$  with entries  $A_{rs} = T_{rs}$  is said to be the *matrix representation* of  $T$  with respect to  $\mathcal{B}_d$  and  $\mathcal{B}_p$ . Indeed, we can identify the action of a quaternionic linear operator  $T : \mathbb{H}^d \rightarrow \mathbb{H}^p$  on  $\chi \in \mathbb{H}^d$ ,  $T(\chi)$ , with the left action of a quaternionic matrix  $A \in \mathcal{M}_{p,d}(\mathbb{H})$  with entries  $A_{rs} = T_{rs}$  as follows:

$$T(\phi) = A\phi = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1d} \\ A_{21} & A_{22} & \cdots & A_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ A_{p1} & A_{p2} & \cdots & A_{pd} \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \\ \vdots \\ \chi_d \end{pmatrix} = \begin{pmatrix} A_{11}\chi_1 + A_{12}\chi_2 + \cdots + A_{1d}\chi_d \\ A_{21}\chi_1 + A_{22}\chi_2 + \cdots + A_{2d}\chi_d \\ \vdots \\ A_{p1}\chi_1 + A_{p2}\chi_2 + \cdots + A_{pd}\chi_d \end{pmatrix}, \quad (2.28)$$

where the underlying bases  $\mathcal{B}_d$  and  $\mathcal{B}_p$  for  $\mathcal{V}$  and  $\mathcal{W}$  are respectively assumed. On that view, there is biunique correspondence between  $\mathcal{M}_{p,d}(\mathbb{H})$  and the set of all quaternionic linear operators  $T : \mathcal{V} \rightarrow \mathcal{W}$ . Therefore, we use the terms ‘linear operator’ and ‘matrix’ interchangeably. For completeness, observe that that  $\forall A \in \mathcal{M}_{p,d}$ ,  $\forall B \in \mathcal{M}_{d,q}(\mathbb{H})$ ,  $\forall \phi, \xi \in \mathbb{H}^d$ ,  $\forall \eta \in \mathbb{H}^q$ , and  $\forall a, b \in \mathbb{H}$  that

$$A(\phi a + \xi b) = A(\phi a) + A(\xi b) = (A\phi)a + (A\xi)b, \quad (2.29)$$

$$A(B\eta) = AB(\eta). \quad (2.30)$$

Evidently,  $\mathcal{M}_{p,d}(\mathbb{H})$  is a  $pd$ -dimensional right quaternionic module. If  $\mathcal{B}_d$  and  $\mathcal{B}_p$  are both orthonormal in the usual sense with respect to the standard symplectic inner product, then matrix multiplication can be carried out explicitly as follows

$$A\phi = \left( \sum_{r=1}^p \sum_{s=1}^d |\xi_r\rangle A_{rs} \langle \phi_s| \right) \left( \sum_{t=1}^d |\phi_t\rangle \chi_t \right) = \sum_{r=1}^p \sum_{s=1}^d |\xi_r\rangle A_{rs} \chi_s, \quad (2.31)$$

where we have employed *bra-ket* notation.

The set of quaternionic matrices enjoys several involutions, including an involutory *conjugation* operation taking  $A \rightarrow \bar{A}$  defined via

$$\bar{A}_{rs} = \overline{(A_{rs})}. \quad (2.32)$$

$\mathcal{M}_{p,d}(\mathbb{H})$  also admits an involutory *transposition* operation taking  $A \rightarrow A^T$  defined via

$$A_{rs}^T = A_{sr}. \quad (2.33)$$

The conjugation and transposition operations commute on  $\mathcal{M}_{p,d}(\mathbb{H})$ ; although,  $\exists A \in \mathcal{M}_{p,d}(\mathbb{H})$  and  $B \in \mathcal{M}_{d,q}(\mathbb{H})$  such that

$$(AB)^T \neq B^T A^T, \quad (2.34)$$

and

$$\overline{AB} \neq \bar{A} \bar{B}. \quad (2.35)$$

$\mathcal{M}_{p,d}(\mathbb{H})$  does, however, admit an involutory star operation taking  $A \rightarrow A^*$  defined via

$$A^* = \bar{A}^T = \overline{A^T}, \quad (2.36)$$

so that  $\forall A \in \mathcal{M}_{p,d}(\mathbb{H})$  and  $\forall B \in \mathcal{M}_{d,q}(\mathbb{H})$ :

$$(AB)^* = B^* A^*. \quad (2.37)$$

If  $p = d$  and  $U \in \mathcal{M}_{d,d}(\mathbb{H})$  is such that  $UU^* = \mathbb{1}_{\mathbb{H}^d}$ <sup>3</sup> then as we say that  $U$  is a *unitary quaternionic matrix*. The set of unitary quaternionic matrices form a subgroup of the automorphism group on  $\mathbb{H}^d$  that preserves the standard symplectic inner product (see Remark A.2.4). Furthermore, when  $p = d$ , it follows  $\forall A \in \mathcal{M}_{d,d}(\mathbb{H})$  and  $\forall \phi, \xi \in \mathbb{H}^d$  that

$$\begin{aligned}
\langle \phi | A \xi \rangle &= \sum_{r=1}^d \overline{\phi}_r (A \xi)_r \\
&= \sum_{r=1}^d \sum_{s=1}^d \overline{\phi}_r A_{rs} \xi_s \\
&= \sum_{r=1}^d \sum_{s=1}^d \overline{A_{rs} \phi_r \xi_s} \\
&= \sum_{r=1}^d \sum_{s=1}^d \overline{A_{sr} \phi_s \xi_r} \\
&= \langle A^* \phi | \xi \rangle,
\end{aligned} \tag{2.38}$$

and we refer to the set  $\mathcal{M}_{d,d}(\mathbb{H})_{sa} = \{A \in \mathcal{M}_{d,d}(\mathbb{H}) \mid A = A^*\}$  as the *self-adjoint* quaternionic matrices. The aforementioned biunique correspondence between quaternionic linear operators and quaternionic matrices dictates that  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$  is in one-to-one correspondence with the set of *self-adjoint quaternionic linear operators*  $T = T^* : \mathcal{V} \rightarrow \mathcal{V}$  on a finite  $d$ -dimensional right quaternionic module  $\mathcal{V}$ , where  $T^*$  is defined via

$$\langle T(\phi) | \xi \rangle = \langle \phi | T^*(\xi) \rangle. \tag{2.39}$$

The set of self-adjoint quaternionic matrices (linear operators) is a *real vector space* of dimension  $d(2d - 1)$ , and we shall view it as such for the remainder of this thesis. As usual, we say that  $A \in \mathcal{M}_{d,d}(\mathbb{H})_{sa}$  is *positive semi-definite* if  $\forall \phi \in \mathbb{H}^d$  the following holds:

$$\langle \phi | A \phi \rangle \in \mathbb{R}_+, \tag{2.40}$$

and we will often write  $\langle \phi | A \phi \rangle \geq 0$ , where it is implied that the LHS of (2.40) is real. We denote the set of positive semi-definite linear operators on  $\mathbb{H}^d$  by  $\mathcal{L}_+(\mathbb{H}^d)$ , and we refer to  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$  as the *ambient space* for  $\mathcal{L}_+(\mathbb{H}^d)$ . In fact, positive semi-definiteness *implies* self-adjointness for quaternionic matrices:

**Lemma 2.4.1** *Positive semi-definiteness implies self-adjointness:*

*If  $A \in \mathcal{M}_{d,d}(\mathbb{H})$  is such that  $\forall \phi \in \mathbb{H}^d : \langle \phi | A \phi \rangle \geq 0$ , then  $A = A^*$ .*

*Proof:*

Applying  $\mathbb{R}$ -linearity of quaternionic conjugation we have that

$$\overline{\langle \phi | A \phi \rangle} = \sum_{r=1}^d \sum_{s=1}^d \overline{\overline{\phi}_r A_{rs} \phi_s} = \sum_{r=1}^d \sum_{s=1}^d \overline{\phi}_s \overline{A_{rs}} \phi_r = \sum_{r=1}^d \sum_{s=1}^d \overline{\phi}_r A_{rs}^* \phi_s = \langle \phi | A^* \phi \rangle. \tag{2.41}$$

---

<sup>3</sup>The condition  $UU^* = \mathbb{1}_{\mathbb{H}^d}$  is equivalent to the condition  $U^*U = \mathbb{1}_{\mathbb{H}^d}$ . For the proof, see Lemma A.2.6.

Thus, we have from the condition  $\overline{\langle \phi | A \phi \rangle} = \langle \phi | A \phi \rangle$  that  $\langle \phi | A^* \phi \rangle = \langle \phi | A \phi \rangle$ . Put otherwise,

$$\sum_{r=1}^d \sum_{s=1}^d \phi_r^* A_{sr}^* \phi_s = \sum_{r=1}^d \sum_{s=1}^d \phi_r^* A_{rs} \phi_s. \quad (2.42)$$

Given that (2.42) holds for all  $\phi \in \mathbb{H}^d$ , we may choose and substitute  $\phi_r = \delta_{rn}$  into (2.42) to get  $A_{nn}^* = A_{nn}$ , where  $\delta_{rn}$  is the usual Kronecker delta function for arbitrary  $n \in \{1, \dots, d\}$ . Thus, the diagonal elements of  $A$  are purely real. Now, we may also choose and substitute  $\phi_r = \delta_{rn} + i\delta_{rm}$  into (2.42) to get

$$\sum_{r=1}^d \sum_{s=1}^d (\delta_{rn} - i\delta_{rm}) A_{sr}^* (\delta_{sn} + i\delta_{sm}) = \sum_{r=1}^d \sum_{s=1}^d (\delta_{rn} - i\delta_{rm}) A_{rs} (\delta_{sn} + i\delta_{sm}), \quad (2.43)$$

which yields

$$-iA_{nm}^* + A_{mn}^* i = -iA_{mn} + A_{nm} i. \quad (2.44)$$

Let  $A_{nm} = a_0 + ia_1 + ja_2 + ka_3$ ,  $A_{mn} = b_0 + ib_1 + jb_2 + kb_3$ . We have from (2.44) that

$$-i(a_0 - ia_1 - ja_2 - ka_3) + (b_0 - ib_1 - jb_2 - kb_3)i = -i(b_0 + ib_1 + jb_2 + kb_3) + (a_0 + ia_1 + ja_2 + ka_3)i, \quad (2.45)$$

and distributivity of quaternionic multiplication yields

$$-ia_0 - a_1 + ija_2 + ika_3 + ib_0 + b_1 - jib_2 - kib_3 = -ib_0 + b_1 - jib_2 - kib_3 + ia_0 - a_1 + jia_2 + kia_3. \quad (2.46)$$

Simplifying (2.46) we get

$$i(b_0 - a_0) - j(a_3 + b_3) + k(a_2 + b_2) = i(a_0 - b_0) + j(a_3 + b_3) - k(a_2 + b_2). \quad (2.47)$$

Both the LHS and the RHS of (2.47) are pure quaternion imaginary, and are thus identically zero implying that  $a_0 = b_0$ ,  $a_3 = -b_3$ , and  $a_2 = -b_2$ . It remains only to show that  $a_1 = -b_1$ , which can be done easily by choosing and substituting, for example,  $\phi_r = \delta_{rn} + j\delta_{rm}$  into (2.42) and following through an entirely similar calculation. Therefore  $A = A^*$ , finishing the proof. In Section 2.5, we prove the spectral theorem for self-adjoint quaternionic matrices, which implies that the eigenvalues of a positive semi-definite quaternionic matrix are elements of  $\mathbb{R}_+$ .  $\square$

We shall equip the real vector space  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$  with the *canonical inner product*

$$(\cdot, \cdot) : \mathcal{M}_{d,d}(\mathbb{H})_{sa} \times \mathcal{M}_{d,d}(\mathbb{H})_{sa} \rightarrow \mathbb{R}, \quad (2.48)$$

defined  $\forall A, B \in \mathcal{M}_{d,d}(\mathbb{H})_{sa}$  via

$$(A, B) = \text{tr}(AB), \quad (2.49)$$

where the *quaternionic trace* is defined for  $A \in \mathcal{M}_{d,d}(\mathbb{H})$  with entries  $A_{rs}$  via

$$\text{tr}(A) = \Re\left(\sum_{t=1}^d A_{tt}\right), \quad (2.50)$$

which is manifestly  $\mathbb{R}$ -linear:  $\forall \alpha, \beta \in \mathbb{R}$  and  $\forall A, B \in \mathcal{M}_{d,d}(\mathbb{H})$

$$\text{tr}(A\alpha + B\beta) = \text{tr}(A)\alpha + \text{tr}(B)\beta. \quad (2.51)$$

In fact, the value of the quaternionic trace is independent of the choice of the underlying basis for matrix representation [29], and on that view it is easy to see that the quaternionic trace enjoys the *cyclic* property  $\forall A, B \in \mathcal{M}_{d,d}(\mathbb{H})$ :

$$\mathrm{tr}(AB) = \mathrm{tr}(BA). \quad (2.52)$$

To see that (2.52) holds, we represent  $A$  and  $B$  with respect to an orthonormal basis for  $\mathbb{H}^d$  so that

$$\begin{aligned} \mathrm{tr}(AB) &= \Re\left(\sum_{t=1}^d \sum_{r=1}^d \sum_{s=1}^d \sum_{u=1}^d \sum_{v=1}^d \langle t|r\rangle A_{rs} \langle s|u\rangle B_{uv} \langle v|t\rangle\right) \\ &= \sum_{r=1}^d \sum_{s=1}^d \Re\left(A_{rs} B_{sr}\right) \\ &= \Re\left(\sum_{r=1}^d \sum_{s=1}^d B_{rs} A_{sr}\right) \\ &= \mathrm{tr}(BA), \end{aligned} \quad (2.53)$$

where we have used the facts that  $\forall a, b \in \mathbb{H}$ :

$$\Re(a + b) = \Re(a) + \Re(b), \quad (2.54)$$

$$\Re(ab) = \Re(ba). \quad (2.55)$$

The canonical inner product on  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$  enjoys the following properties  $\forall A, B, C \in \mathcal{M}_{d,d}(\mathbb{H})_{sa}$  and  $\forall \alpha, \beta \in \mathbb{R}$ :

- $\mathbb{R}$ -bilinearity:  $(A\alpha, B\beta) = \alpha(A, B)\beta$  and  $(A + B, C) = (A, C) + (B, C)$
- Symmetry:  $(A, B) = (B, A)$ ,
- Nonnegativity:  $(A, A) \geq 0$  with equality *iff*  $A = 0$ ,

all of which follow from the definition of the quaternionic trace. The canonical inner product on  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$  obeys the Cauchy–Schwarz inequality:

$$|(A, B)|^2 \leq (A, A)(B, B), \quad (2.56)$$

which can be proven by virtue of Theorem 2.5.4. The canonical inner product on  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$  induces a real-valued norm  $\|\cdot\| : \mathcal{M}_{d,d}(\mathbb{H})_{sa} \rightarrow \mathbb{R}$  defined  $\forall A \in \mathcal{M}_{d,d}(\mathbb{H})_{sa}$  via

$$\|A\| = \sqrt{(A, A)} = \sqrt{\mathrm{tr}(A^2)}. \quad (2.57)$$

The norm obeys  $\forall A, B \in \mathcal{M}_{d,d}(\mathbb{H})_{sa}$  and  $\forall \alpha \in \mathbb{R}$ :

- $\mathbb{R}$ -homogeneity:  $\|\alpha A\| = |\alpha| \|A\|$ ,
- Nonnegativity:  $\|A\| \geq 0$  with equality *iff*  $A = 0$ , and

- Triangle Inequality:  $\|A + B\| \leq \|A\| + \|B\|$ .

Multiplicativity is trivial to establish, for by definition:

$$\|aA\| = \sqrt{\operatorname{tr}((aA)^2)} = |a|\sqrt{\operatorname{tr}(A^2)}, \quad (2.58)$$

and nonnegativity follows just as easily since

$$\|A\| = \sqrt{\operatorname{tr}(A^2)} = \sqrt{\sum_{r,s} |A_{rs}|^2}. \quad (2.59)$$

The triangle inequality is established in the usual way by observing that

$$\begin{aligned} \|A + B\|^2 &= (A + B, A + B) \\ &\leq \|A\|^2 + \|B\|^2 + 2|(A, B)| \\ &\leq \|A\|^2 + \|B\|^2 + 2\|A\|\|B\| \\ &= (\|A\| + \|B\|)^2. \end{aligned} \quad (2.60)$$

It also turns out that  $\|A\|$  is equal to the square root of the sum of the squares of the eigenvalues of  $A \in \mathcal{M}_{d,d}(\mathbb{H})_{sa}$ . For the proof, see Corollary 2.5.5.

## 2.5 The Spectral Theorem for $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$

This section is devoted to proving the spectral theorem for self-adjoint quaternionic linear operators. We begin with a definition of the right-modular quaternionic eigenproblem:

**Definition 2.5.1** *The Right-Modular Quaternionic Eigenproblem:*

Let  $T : \mathcal{V} \rightarrow \mathcal{V}$  be a quaternionic linear operator on a finite  $d$ -dimensional right quaternionic module  $\mathcal{V}$ . The right-modular quaternionic eigenproblem for  $T$  is

$$T(\phi) = \phi a, \quad (2.61)$$

where  $\phi \in \mathbb{H}^d$  is nonzero and  $a \in \mathbb{H}$ . Any solution  $(\phi, a)$  to (2.61) is called an *eigenpair*, where  $\phi$  is referred to as an *eigenvector*, and where  $a$  is referred to as an *eigenvalue*. The set of eigenvalues of all eigenpairs is referred to as the *spectrum* of  $T$ .

Every quaternionic linear operator admits at least one eigenpair [12], and it is interesting to note that if  $\Im(a) \neq 0$ , then the spectrum of  $T$  is infinite. In particular, if  $a \neq \Re(a)$  is an eigenvalue of  $T$  with corresponding eigenvector  $\phi$ , then every element of the set  $\{b^{-1}ab \mid b \in \mathbb{H}\}$  is also an eigenvalue of  $T$  corresponding to the eigenvector  $\phi b$ , since

$$T(\phi b) = (T(\phi))b = (\phi a)b = (\phi b)b^{-1}ab. \quad (2.62)$$

However, the following lemma dictates the reality of eigenvalues of self-adjoint quaternionic linear operators:



**Lemma 2.5.2** *Reality of eigenvalues of self-adjoint quaternionic linear operators:*

Let  $T : \mathcal{V} \rightarrow \mathcal{V}$  be a self-adjoint quaternionic linear operator on a finite  $d$ -dimensional right quaternionic module  $\mathcal{V}$ . If  $T$  is such that

$$T(\phi) = \phi a, \quad (2.63)$$

where  $(\phi, a) \in \mathcal{V} \times \mathbb{H}$ , then  $\Im(a) = 0$ .

*Proof:*

$T(\phi) = \phi a$  implies that

$$a = \frac{\langle \phi | T(\phi) \rangle}{\langle \phi | \phi \rangle} = \frac{\langle \phi | T^*(\phi) \rangle}{\langle \phi | \phi \rangle} = \bar{a}, \quad (2.64)$$

where we have used (2.41).  $\square$

In fact, every self-adjoint quaternionic linear operator admits exactly  $d$  (possibly degenerate) real eigenvalues [23], and the corresponding eigenvectors can be chosen to constitute an orthonormal basis for  $\mathbb{H}^d$  according to Lemma (2.5.3).

**Lemma 2.5.3** *Eigenvectors and orthonormal bases:*

If  $T : \mathcal{V} \rightarrow \mathcal{V}$  is a self-adjoint quaternionic linear operator on a finite  $d$ -dimensional right quaternionic module  $\mathcal{V}$ , then  $\mathcal{V}$  admits an orthonormal basis consisting of eigenvectors of  $T$ .

*Proof:*

We prove the lemma by induction on  $d$ .

- *Base case:*

With  $d = 1$ , the orthonormal basis can be chosen as any unit norm quaternion  $\phi \in \text{Sp}(1)$ .

- *Inductive step:*

Suppose that the result holds for  $d \in \mathbb{Z}_+$ . We will show that the result holds for  $p = d + 1$ .

Let  $(\phi, a) \in \mathcal{V} \times \mathbb{R}$  such that

$$T(\phi) = \phi a. \quad (2.65)$$

By Lemma 2.5.2,  $a \in \mathbb{R}$ . Furthermore,  $\phi \in \mathcal{V}$  can be chosen such that  $\langle \phi | \phi \rangle = 1$ . Next, let

$$\mathcal{W} = \{\xi \in \mathcal{V} \mid \langle \xi | \phi \rangle = 0\} \subset \mathcal{V}. \quad (2.66)$$

$\mathcal{W}$  is a right quaternionic module. We recall a result due to Farenick and Pidkowich [23]: If  $\phi \in \mathcal{V}$  admits  $\langle \phi | \phi \rangle = 1$ , then there exists an orthonormal basis  $\mathcal{B}_p = \{\phi_1, \dots, \phi_p\}$  for  $\mathcal{V}$  with  $\phi = \phi_1$ . Therefore, if  $\xi \in \mathcal{W} \subset \mathcal{V}$ , then  $\exists \{b_1, \dots, b_p\} \subset \mathbb{H}$  such that

$$\xi = \phi b_1 + \dots + \phi_p b_p, \quad (2.67)$$

with  $b_1 = 0$  for any choice of  $\xi$ . Thus,  $\mathcal{W}$  is a  $d$ -dimensional right quaternionic module. Next, observe that  $\mathcal{W}$  is an invariant submodule under  $T$ ,  $\forall \xi \in \mathcal{V}$ :

$$\langle T(\xi)|\phi \rangle = \langle \xi|T^*(\phi) \rangle = \langle \xi|T(\phi) \rangle = \langle \xi|\phi \rangle a = 0, \quad (2.68)$$

which establishes that the restriction of  $T$  to  $\mathcal{W}$  is a self-adjoint quaternionic linear operator. Therefore, by the inductive hypothesis,  $\mathcal{W}$  admits an orthonormal basis  $\mathcal{B}$  consisting of eigenvectors of  $T$ . Adjoining  $\phi$  to  $\mathcal{B}_p$  completes the proof.  $\square$

Equipped with Lemma 2.5.2 and 2.5.3, we now prove the spectral theorem for self-adjoint quaternionic linear operators, which is proven in [29].

**Theorem 2.5.4** *The Spectral Theorem:*

*If  $T$  is a self-adjoint quaternionic linear operator on a finite  $d$ -dimensional right quaternionic module  $\mathcal{V}$ , then  $\mathcal{V}$  admits an orthonormal basis  $\mathcal{B}_d = \{\phi_1, \dots, \phi_d\}$  consisting of the eigenvectors of  $T$  such that the matrix representation of  $T$  with respect  $\mathcal{B}_d$  is real and diagonal.*

*Proof:*

By Lemma 2.5.3 we have that  $\mathcal{V}$  admits an orthonormal basis  $\mathcal{B}_d = \{\phi_1, \dots, \phi_d\}$  consisting of the eigenvectors of  $T$ . Therefore, the action of  $T$  on  $\mathcal{V}$  can be defined via  $A \in \mathcal{M}_{d,d}(\mathbb{H})_{sa}$ , where  $A$  acts according to

$$A = \sum_{r=1}^d \sum_{s=1}^d |\phi_r\rangle A_{rs} \langle \phi_s|. \quad (2.69)$$

It follows that

$$A_{rs} = \langle \phi_r|A\phi_s \rangle = \langle \phi_r|\phi_s a_s \rangle = \delta_{rs} a_s, \quad (2.70)$$

where the eigenvalue  $a_s$  is an element of  $\mathbb{R}$  by Lemma 2.5.2. Put otherwise,

$$A = \sum_{r=1}^d |\phi_r\rangle a_r \langle \phi_r|, \quad (2.71)$$

finishing the proof.  $\square$

Theorem 2.5.4 is used to characterize the norm of  $A \in \mathcal{M}_{d,d}(\mathbb{H})_{sa}$  in terms of its eigenvalues in Corollary 2.5.5.

**Corollary 2.5.5** *On Eigenvalues and the Norm:*

*If  $A \in \mathcal{M}_{d,d}(\mathbb{H})_{sa}$  admits the spectrum  $\{\lambda_1, \dots, \lambda_d\}$  with corresponding orthonormalized eigenvectors  $\{\xi_1, \dots, \xi_d\}$ , then*

$$\|A\| = \sqrt{\sum_{r=1}^d \lambda_r^2}. \quad (2.72)$$

*Proof:*

By explicit computation:

$$\begin{aligned}
\|A\|^2 &= \operatorname{tr}(A^2) \\
&= \operatorname{tr}\left(\sum_{r=1}^d \sum_{r'=1}^d |\xi_r\rangle \lambda_r \langle \xi_r | \xi_{r'}\rangle \lambda_{r'} \langle \xi_{r'} | \right) \\
&= \operatorname{tr}\left(\sum_{r=1}^d |\xi_r\rangle \lambda_r^2 \langle \xi_r | \right) \\
&= \sum_{r=1}^d \lambda_r^2 \operatorname{tr}(|\xi_r\rangle \langle \xi_r |) \\
&= \sum_{r=1}^d \lambda_r^2.
\end{aligned} \tag{2.73}$$

In deriving (2.73), we have used Lemma 2.5.2 and Lemma 2.5.3 , and the fact that  $\forall \xi, \phi \in \mathbb{H}^d$  one has that:

$$\begin{aligned}
\operatorname{tr}(|\xi\rangle \langle \phi |) &= \Re\left(\sum_{t \in \mathcal{B}_d} \langle t | \xi \rangle \langle \phi | t \rangle\right) \\
&= \sum_{t \in \mathcal{B}_d} \Re\left(\langle t | \xi \rangle \langle \phi | t \rangle\right) \\
&= \sum_{t \in \mathcal{B}_d} \Re\left(\langle \phi | t \rangle \langle t | \xi \rangle\right) \\
&= \Re\left(\sum_{t \in \mathcal{B}_d} \langle \phi | t \rangle \langle t | \xi \rangle\right) \\
&= \Re\left(\langle \phi | \xi \rangle\right),
\end{aligned} \tag{2.74}$$

where  $\mathcal{B}_d$  is taken as an orthonormal basis for  $\mathbb{H}^d$ , which turns out to be a useful identity in the chapters that follow.  $\square$

## Chapter 3

# Quaternionic Quantum Formalism

In this chapter, we consider a quaternionic quantum formalism for the description of quantum states, quantum channels, and quantum measurements. Our formulation will take place in a *finite-dimensional* setting, and shall be restricted to the description of experiments of the type depicted in Figure 3.1, which is the scenario considered by Hardy in [47].

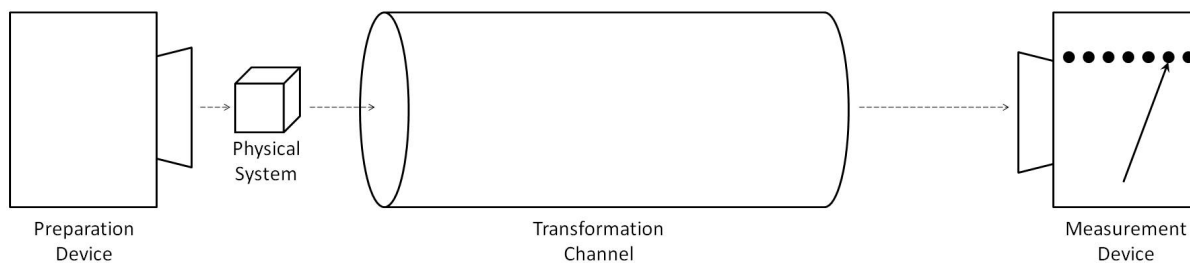


Figure 3.1: A generic physical scenario

We consider experiments wherein the experimentalist has access to three types of equipment: preparation devices, transformation channels, and measurement devices. In particular, we consider the type of experiment depicted in Figure 3.1. A preparation device emits a physical system with an initial associated state. The initial associated state is defined by the settings of the preparation device. The system then enters a transformation channel that can transform the initial associated state. The transformation of the initial associated state is defined by the settings of the transformation channel. Finally, a physical system exits the channel and enters a measurement device that registers one classical outcome. The number of classical outcomes depends on the settings of the measurement device. The probabilities for the classical outcomes depend on the settings of the measurement device and the transformed initial associated state.<sup>1</sup> Hardy pointed out that this type of experiment covers a very wide range of physical phenomena:

---

<sup>1</sup>In the preceding sentences, we have provided an operational definition for such an experiment. In practice, the *description* of the preparation, transformation, and measurement processes are, at bottom, governed by the experimentalist.

The situation described here [author’s comment: *i.e.* in Figure 3.1 ] is quite generic. Although we have described the set up as if the system were moving along one dimension, in fact the system could equally well be regarded as remaining stationary whilst being subjected to transformations and measurements. Furthermore, the system need not be localized but could be in several locations. The transformations could be due to controlling fields or simply due to the natural evolution of the system. Any physical experiment, quantum, classical or other, can be viewed as an experiment of the type described here.

– L. Hardy, 2001 [47].

In the following sections, we formulate a quaternionic quantum description of states, transformation channels, and measurement devices. There is a rich history of thought on the subject of formulating quaternionic quantum theories [10][29][27][28][51][59][1]. In this chapter, we develop a quaternionic quantum formalism mirroring the *full apparatus* of usual complex quantum information theory. Our definition of a quaternionic quantum state is the standard one [51]. We define quaternionic quantum channels by completely positive trace preserving maps, which have been also been considered in [6]. We define quaternionic quantum measurements via quaternionic positive operator valued measures, which appear naturally in convex operational approaches to quantum theory [8]. We prove that probabilities for quaternionic quantum measurement outcomes arise from canonical inner products between the corresponding quaternionic quantum effects and a unique quaternionic quantum state.

### 3.1 Quaternionic Quantum States

We define a quaternionic quantum state for a physical system  $\mathfrak{S}$  by a unit-trace positive semi-definite matrix  $\rho \in \mathcal{M}_{d,d}(\mathbb{H})_{sa}$ , which is to say that

$$\text{tr}(\rho) = 1, \tag{3.1}$$

and

$$\forall \phi \in \mathbb{H}^d : \langle \phi | \rho \phi \rangle \geq 0. \tag{3.2}$$

We denote the convex set of all  $d$ -dimensional quaternionic quantum states by  $\mathcal{L}_+^1(\mathbb{H}^d)$ . It is clear that  $\mathcal{L}_+^1(\mathbb{H}^d)$  is convex, since given  $\rho_1, \rho_2 \in \mathcal{L}_+^1(\mathbb{H}^d)$  and  $0 < \lambda < 1$  then

$$\rho = ((1 - \lambda)\rho_1 + \lambda\rho_2) \in \mathcal{L}_+^1(\mathbb{H}^d), \tag{3.3}$$

for by linearity of the quaternionic trace we have that

$$\text{tr}(\rho) = \text{tr}((1 - \lambda)\rho_1 + \lambda\rho_2) = (1 - \lambda)\text{tr}(\rho_1) + \lambda\text{tr}(\rho_2) = 1, \tag{3.4}$$

and by linearity of the standard symplectic inner product we have that

$$\forall \phi \in \mathbb{H}^d : \langle \phi | \rho \phi \rangle = (1 - \lambda)\langle \phi | \rho_1 \phi \rangle + \lambda\langle \phi | \rho_2 \phi \rangle \in \mathbb{R}_+. \tag{3.5}$$

We refer to the extreme points of  $\mathcal{L}_+^1(\mathbb{H}^d)$  as *pure* quaternionic quantum states. All other points are referred to as *mixed* states, which can be thought of as probabilistic mixtures of pure states. Lemma 3.1.1 characterizes the set of pure quaternionic quantum states.

**Lemma 3.1.1** *Extreme Quaternionic Quantum States:*

The extreme points of  $\mathcal{L}_+^1(\mathbb{H}^d)$  are rank-1 projection operators.

*Proof:*

Let

$$\pi^2 = \pi = |\phi\rangle\langle\phi| \in \mathcal{L}_+^1(\mathbb{H}^d) \quad (3.6)$$

be a rank-1 projection operator onto  $\phi \in \mathbb{H}^d$ . We will show that  $\pi$  is an extreme point of  $\mathcal{L}_+^1(\mathbb{H}^d)$  via *reductio ad absurdum*. Let  $a, b \in \mathcal{L}_+^1(\mathbb{H}^d)$  with  $a \neq b$ , and let  $t \in (0, 1)$  and suppose  $\pi = ta + (1-t)b$ , so that  $\pi^2 = t^2a^2 + (1-t)^2b^2 + t(1-t)(ab + ba)$ . It follows that

$$1 = t^2\text{tr}(a^2) + (1-t)^2\text{tr}(b^2) + 2t(1-t)\text{tr}(ab). \quad (3.7)$$

It will be useful to show that  $\forall a, b \in \mathcal{L}_+^1(\mathbb{H}^d)$ :

$$\text{tr}(ab) \leq \text{tr}(a)\text{tr}(b), \quad (3.8)$$

with equality iff  $a = b$ . Expand  $a$  and  $b$  in their respective eigenbases

$$a = \sum_{r=1}^d |r\rangle\lambda_r\langle r| \quad (3.9)$$

$$b = \sum_{s=1}^d |s\rangle\mu_s\langle s| \quad (3.10)$$

and then expand  $b$  in terms of the eigenbasis for  $a$  as

$$b = \sum_{s,r,r'} |r\rangle s_r \mu_s \bar{s}_{r'} \langle r'|. \quad (3.11)$$

Next, compute  $\text{tr}(ab)$  with respect to the eigenbasis for  $a$ :

$$\text{tr}(ab) = \sum_{r=1}^d \sum_{r'=1}^d \sum_{r''=1}^d \sum_{r'''=1}^d \sum_{s=1}^d \langle r|r'\rangle \lambda_{r'} \langle r'|r''\rangle s_{r''} \mu_s \bar{s}_{r'''} \langle r'''|r\rangle = \sum_{r=1}^d \sum_{s=1}^d |\langle r|s\rangle|^2 \lambda_r \mu_s, \quad (3.12)$$

whereas

$$\text{tr}(a)\text{tr}(b) = \sum_{r,s} \lambda_r \mu_s. \quad (3.13)$$

With  $\lambda_r \mu_s \geq 0$ , one has that  $|\langle r|s\rangle|^2 \leq 1$  with equality iff  $a = b$ , proving (3.8). Applying (3.8) to (3.7) with  $a \neq b$  we have that  $1 < 1$ , a contradiction. Therefore our supposition is false, and we conclude that rank-1 projection operators are extreme points of  $\mathcal{L}_+^1(\mathbb{H}^d)$ . Since every element of  $\mathcal{L}_+^1(\mathbb{H}^d)$  can be expanded as a convex combination of rank-1 projection operators – as dictated by Theorem 2.5.4 – only rank-1 projection operators can be extreme points.  $\square$

The ambient space for  $\mathcal{L}_+^1(\mathbb{H}^d)$  is  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$ , which means that  $(d(2d-1)-1)$  real parameters are required for the definition of an arbitrary quaternionic quantum state. Therefore, the surface

of  $\mathcal{L}_+^1(\mathbb{H}^d)$  is  $(d(2d - 1) - 2)$ -dimensional. Extreme points require  $4(d - 1)$  real parameters for their specification, which is strictly less than the dimensionality of the surface of the convex set of quaternionic quantum states, except in dimension 2. Indeed, quaternionic quantum theory shares this property with usual complex quantum theory. Let us consider  $\mathcal{L}_+^1(\mathbb{H}^2)$  – the set of quaternionic quantum states for a 2-dimensional quaternionic quantum system, or *quabit* – in some detail.

Define the following matrices:

$$\mathbb{1}_{\mathbb{H}^2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \sigma_4 = \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}, \sigma_5 = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}. \quad (3.14)$$

Indeed,  $\{\sigma_1, \dots, \sigma_5\}$  are a *spin system*<sup>2</sup> thereby admitting that

$$\sigma_r \sigma_s + \sigma_s \sigma_r = 2\delta_{rs} \mathbb{1}_{\mathbb{H}^d}, \quad (3.15)$$

and

$$\forall r \in \{1, 2, 3, 4, 5\} : \sigma_r^2 = \mathbb{1}_{\mathbb{H}^d}. \quad (3.16)$$

The set of  $2 \times 2$  self-adjoint quaternionic matrices is obtained from all  $\mathbb{R}$ -linear combinations of the matrices defined in (3.14), *i.e.*

$$\mathcal{M}_{2,2}(\mathbb{H})_{sa} = \text{lin}_{\mathbb{R}} \left\{ \mathbb{1}_{\mathbb{H}^2}, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5 \right\}, \quad (3.17)$$

and the set of 2-dimensional quaternionic quantum states is

$$\mathcal{L}_+^1(\mathbb{H}^2) = \left\{ \rho \in \mathcal{M}_{2,2}(\mathbb{H})_{sa} \mid \rho = \frac{1}{2} \left( \mathbb{1}_{\mathbb{H}^2} + \vec{\mathfrak{r}} \cdot \vec{\sigma} \right) \right\}, \quad (3.18)$$

where

$$\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5), \quad (3.19)$$

and  $\vec{\mathfrak{r}} \in \mathbb{R}^5$  is such that

$$|\vec{\mathfrak{r}}| \leq 1, \quad (3.20)$$

which is a necessary and sufficient condition for  $\rho$  to be positive semi-definite. The value of the upper bound on the norm of  $\vec{\mathfrak{r}}$  in (3.20) is computed using the fact that  $\rho$  can be diagonalized as

$$\rho = |\xi_+\rangle \lambda_+ \langle \xi_+| + |\xi_-\rangle \lambda_- \langle \xi_-| \quad (3.21)$$

where from [24] we have that

$$\lambda_+ = \frac{1}{2}(1 + |\mathfrak{r}|), \quad (3.22)$$

and

$$\lambda_- = \frac{1}{2}(1 - |\mathfrak{r}|), \quad (3.23)$$

so that, in particular,

$$\langle \xi_- | \rho | \xi_- \rangle \geq 0 \implies |\vec{\mathfrak{r}}| \leq 1. \quad (3.24)$$

Given that any element  $\phi$  in  $\mathbb{H}^d$  can be expressed as

$$|\phi\rangle = |\xi_+\rangle c_+ + |\xi_-\rangle c_- \quad (3.25)$$

---

<sup>2</sup>A spin system is a collection of anticommuting unitaries not equal to  $\pm 1$  in a real unital Jordan algebra that generates a Jordan-Banach algebra known as a *spin factor*, *e.g.*  $V_5 = \mathcal{M}_{2,2}(\mathbb{H})_{sa}$  [46].

one has that

$$\langle \phi | \rho \phi \rangle = \bar{c}_+ \lambda_+ c_+ + \bar{c}_- \lambda_- c_- = |c_+|^2 \lambda_+ + |c_-|^2 \lambda_-, \quad (3.26)$$

and given the reality and positivity of  $\lambda_+$ , and of the squares of  $|c_+|$  and  $|c_-|$ , we conclude that (3.20) is indeed a necessary and sufficient condition for  $\rho$  to be positive semi-definite. Pure qubit states comprise the surface of a sphere with  $|\vec{r}| = 1$ . For arbitrary dimension  $d \in \mathbb{Z}_+$ , we can make use of the following lemma to characterize the set of pure quaternionic quantum states corresponding to  $d$ -dimensional quaternionic systems, which we refer to as *quadits*.

**Lemma 3.1.2** *Characterization of Extreme Quaternionic Quantum States:*

$A \in \mathcal{M}_{d,d}(\mathbb{H})_{sa}$  is such that  $A = |\phi\rangle\langle\phi|$  if and only if

$$\text{tr}(A^2) = \text{tr}(A^3) = 1. \quad (3.27)$$

*Proof:*

We follow the complex-case proof given by Fuchs and Schack [36]. Let  $A \in \mathcal{M}_{d,d}(\mathbb{H})_{sa}$  with real eigenvalues  $\lambda_r$  and orthogonal eigenprojectors  $|\xi_r\rangle\langle\xi_r|$ . Recall from Chapter 2 that

$$\text{tr}(A^2) = \sum_{r=1}^d \lambda_r^2, \quad (3.28)$$

and one also has that

$$\begin{aligned} \text{tr}(A^3) &= \text{tr} \left( \sum_{r=1}^d \sum_{r'=1}^d \sum_{r''=1}^d |\xi_r\rangle \lambda_r \langle \xi_r | \xi_{r'} \rangle \lambda_{r'} \langle \xi_{r'} | \xi_{r''} \rangle \lambda_{r''} \langle \xi_{r''} | \right) \\ &= \text{tr} \left( \sum_{r=1}^d |\xi_r\rangle \lambda_r^3 \langle \xi_r | \right) \\ &= \sum_{r=1}^d \lambda_r^3 \text{tr}(|\xi_r\rangle\langle\xi_r|) \\ &= \sum_{r=1}^d \lambda_r^3. \end{aligned} \quad (3.29)$$

Now, (3.27) implies that

$$\forall r \in \{1, \dots, d\} : |\lambda_r| \leq 1 \implies \forall r \in \{1, \dots, d\} : 1 - \lambda_r \geq 0. \quad (3.30)$$

Next, observe that (3.29)-(3.28) yields

$$\sum_{r=1}^d \lambda_r^2 (\lambda_r - 1) = 0, \quad (3.31)$$

which implies that there is one and only one non-zero  $\lambda_r = 1$ . The converse holds trivially, and so the lemma is proven.  $\square$



Given that  $\mathcal{L}_+^1(\mathbb{H}^d)$  is the convex hull of all pure states, the conditions (3.27) fully characterize the outer shape of quaternionic quantum state space. We shall analyze such spaces in further detail in Chapter 5.

## 3.2 Quaternionic Quantum Channels

We define the evolution of a quaternionic quantum state  $\rho$  associated with a physical system  $\mathfrak{S}$  via a *quaternionic quantum channel*  $\Phi : \mathcal{M}_{d,d}(\mathbb{H}) \longrightarrow \mathcal{M}_{p,p}(\mathbb{H})$  whose action on  $\rho \in \mathcal{M}_{d,d}(\mathbb{H})$  is defined in terms of  $\{A_1, \dots, A_n\} \subset \mathcal{M}_{p,d}(\mathbb{H})$  via

$$\Phi(\rho) = \sum_{r=1}^n A_r \rho A_r^*, \quad (3.32)$$

where

$$\sum_{r=1}^n A_r^* A_r = \mathbf{1}_{\mathbb{H}^d}. \quad (3.33)$$

In order to prove that quaternionic quantum channels transform quaternionic quantum states into quaternionic quantum states, we require Lemma 3.2.1.

**Lemma 3.2.1** *On the Quaternionic Trace:*

*If  $A \in \mathcal{M}_{p,d}(\mathbb{H})$  and  $\rho \in \mathcal{M}_{d,d}(\mathbb{H})_{sa}$ , then*

$$\mathrm{tr}(A\rho A^*) = \mathrm{tr}(A^*A\rho). \quad (3.34)$$

*Proof:*

Observing that  $A\rho A^* \in \mathcal{M}_{p,p}(\mathbb{H})_{sa}$ , while  $A^*A\rho \in \mathcal{M}_{d,d}(\mathbb{H})_{sa}$ , we recognize that the LHS and RHS of (3.34) are actually being computed with respect to bases for right quaternionic modules of generally different dimensions  $p$  and  $q$ , and therefore the cyclic property of the trace does not suffice for a proof of (3.34) when  $p \neq d$ . We will use the Greek letters  $\alpha, \beta, \gamma, \delta$ , and  $\tau$  to keep track of elements in an arbitrary orthonormal basis for  $\mathbb{H}^d$ , and we will use the Latin letters  $a, b$  and  $t$  to keep track of elements in an arbitrary orthonormal basis for  $\mathbb{H}^p$ . Explicitly computing the LHS of (3.34) we have that

$$\begin{aligned} \mathrm{tr}(A\rho A^*) &= \Re \left( \sum_{t=1}^p \sum_{a=1}^p \sum_{\alpha=1}^d \sum_{\beta=1}^d \sum_{b=1}^p \sum_{\gamma=1}^d \sum_{\delta=1}^d \langle t|a \rangle A_{a\alpha} \langle \alpha|\gamma \rangle \rho_{\gamma\delta} \langle \delta|\beta \rangle A_{\beta b}^* \langle b|t \rangle \right) \\ &= \Re \left( \sum_{t=1}^p \sum_{\alpha=1}^d \sum_{\beta=1}^d A_{t\alpha} \rho_{\alpha\beta} A_{\beta t}^* \right). \end{aligned} \quad (3.35)$$

Explicitly computing the RHS of (3.34) we have that

$$\begin{aligned}
\mathrm{tr}(A^*A\rho) &= \Re\left(\sum_{\tau=1}^d \sum_{\beta=1}^d \sum_{b=1}^p \sum_{a=1}^p \sum_{\alpha=1}^d \sum_{\gamma=1}^d \sum_{\delta=1}^d \langle \tau | \beta \rangle A_{\beta b}^* \langle b | a \rangle A_{a\alpha} \langle \alpha | \gamma \rangle \rho_{\gamma\delta} \langle \delta | \tau \rangle\right) \\
&= \Re\left(\sum_{\tau=1}^d \sum_{\alpha=1}^d \sum_{a=1}^p A_{\tau a}^* A_{a\alpha} \rho_{\alpha\tau}\right) \\
&= \sum_{\tau=1}^d \sum_{\alpha=1}^d \sum_{a=1}^p \Re\left(A_{\tau a}^* A_{a\alpha} \rho_{\alpha\tau}\right) \\
&= \sum_{\tau=1}^d \sum_{\alpha=1}^d \sum_{a=1}^p \Re\left(A_{a\alpha} \rho_{\alpha\tau} A_{\tau a}^*\right) \\
&= \Re\left(\sum_{\tau=1}^d \sum_{\alpha=1}^d \sum_{a=1}^p A_{a\alpha} \rho_{\alpha\tau} A_{\tau a}^*\right) \\
&= \Re\left(\sum_{t=1}^p \sum_{\alpha=1}^d \sum_{\beta=1}^d A_{t\alpha} \rho_{\alpha\beta} A_{\beta t}^*\right)
\end{aligned} \tag{3.36}$$

where we have exchanged summation indices in arriving at the last equality, finishing the proof.  $\square$

Equipped with Lemma 3.2.1, we now prove Theorem 3.2.2, which states that

$$\Phi(\mathcal{L}_+^1(\mathbb{H}^d)) \subseteq \mathcal{L}_+^1(\mathbb{H}^p) \tag{3.37}$$

for any quaternionic quantum channel.

**Lemma 3.2.2** *On Quaternionic Quantum Channels:*

Let  $\rho \in \mathcal{L}_+^1(\mathbb{H}^d)$  and let  $\Phi : \mathcal{M}_{d,d} \rightarrow \mathcal{M}_{p,p}$  be a quaternionic quantum channel defined via

$$\Phi(\rho) = \sum_{r=1}^n A_r \rho A_r^*, \tag{3.38}$$

where  $\{A_1, \dots, A_n\} \subset \mathcal{M}_{p,d}(\mathbb{H})$  are such that

$$\sum_r A_r^* A_r = \mathbf{1}_{\mathbb{H}^d}. \tag{3.39}$$

Then

$$\mathrm{tr}(\Phi(\rho)) = 1, \tag{3.40}$$

and

$$\forall \chi \in \mathbb{H}^p : \langle \chi | \Phi(\rho) \chi \rangle \geq 0. \tag{3.41}$$

*Proof:*

To prove (3.40) we apply properties of the quaternionic trace and Lemma 3.2.1:

$$\begin{aligned}
\text{tr}(\Phi(\rho)) &= \text{tr}\left(\sum_{r=1}^n A_r \rho A_r^*\right) \\
&= \sum_{r=1}^n \text{tr}(A_r \rho A_r^*) \\
&= \sum_{r=1}^n \text{tr}(\rho A_r^* A_r) \\
&= \text{tr}\left(\rho\left(\sum_{r=1}^n A_r^* A_r\right)\right) \\
&= \text{tr}(\rho \mathbb{1}_{\mathbb{H}^d}) \\
&= \text{tr}(\rho) \\
&= 1.
\end{aligned} \tag{3.42}$$

To prove (3.41), let

$$\mathcal{B}_p = \{r \in \mathbb{H}^p \mid \langle r | r' \rangle = \delta_{rr'}\}_{r=1}^p, \tag{3.43}$$

and

$$\mathcal{B}_d = \{\alpha \in \mathbb{H}^d \mid \langle \alpha | \alpha' \rangle = \delta_{\alpha\alpha'}\}_{\alpha=1}^d, \tag{3.44}$$

be orthonormal bases for  $\mathbb{H}^p$  and  $\mathbb{H}^d$  respectively.

Let  $\chi = \sum_{r=1}^p |r\rangle \chi_r \in \mathbb{H}^p$  with components

$$\chi_r = \langle r | \chi \rangle, \tag{3.45}$$

and  $A = \sum_{r=1}^p \sum_{\alpha=1}^d |r\rangle A_{r\alpha} \langle \alpha| \in \mathcal{M}_{p,d}$  with components

$$A_{r\alpha} = \langle r | A \alpha \rangle. \tag{3.46}$$

Also, let  $\rho = \sum_{\alpha=1}^d \sum_{\beta=1}^d |\alpha\rangle \rho_{\alpha\beta} \langle \beta| \in \mathcal{M}_{d,d}(\mathbb{H})_{sa}$

$$\rho_{\alpha\beta} = \langle \alpha | \rho \beta \rangle. \tag{3.47}$$

Observe that

$$\begin{aligned}
\langle \chi | A \rho A^* \chi \rangle &= \sum_{r=1}^p \bar{\chi}_r (A \rho A^* \chi)_r \\
&= \sum_{r=1}^p \sum_{s=1}^p \bar{\chi}_r (A \rho A^*)_{rs} \chi_s \\
&= \sum_{r=1}^p \sum_{s=1}^p \sum_{\alpha=1}^d \bar{\chi}_r A_{r\alpha} (\rho A^*)_{\alpha s} \chi_s \\
&= \sum_{r=1}^p \sum_{s=1}^p \sum_{\alpha=1}^d \sum_{\beta=1}^d \bar{\chi}_r A_{r\alpha} \rho_{\alpha\beta} A_{\beta s}^* \chi_s.
\end{aligned} \tag{3.48}$$

Next, observe that

$$\begin{aligned}
\langle A^* \chi | \rho A^* \chi \rangle &= \sum_{\alpha=1}^d \overline{(A^* \chi)_\alpha} (\rho A^* \chi)_\alpha \\
&= \sum_{\alpha=1}^d \sum_{r=1}^p \overline{A_{\alpha r}^* \chi_r} (\rho A^* \chi)_\alpha \\
&= \sum_{\alpha=1}^d \sum_{r=1}^p \sum_{s=1}^p \overline{A_{\alpha r}^* \chi_r} (\rho A^*)_{\alpha s} \chi_s \\
&= \sum_{\alpha=1}^d \sum_{r=1}^p \sum_{s=1}^p \sum_{\beta=1}^d \overline{A_{\alpha r}^* \chi_r} \rho_{\alpha \beta} A_{\beta s}^* \chi_s \\
&= \sum_{r=1}^p \sum_{s=1}^p \sum_{\alpha=1}^d \sum_{\beta=1}^d \overline{\chi_r} A_{r \alpha} \rho_{\alpha \beta} A_{\beta s}^* \chi_s \\
&= \langle \chi | A \rho A^* \chi \rangle.
\end{aligned} \tag{3.49}$$

Therefore,

$$\begin{aligned}
\langle \chi | \Phi(\rho) \chi \rangle &= \langle \chi | \sum_{r=1}^n A_r \rho A_r^* \chi \rangle \\
&= \sum_{r=1}^n \langle \chi | A_r \rho A_r^* \chi \rangle \\
&= \sum_{r=1}^n \langle A_r^* \chi | \rho A_r^* \chi \rangle \\
&= \sum_{r=1}^n \langle \phi_r | \rho \phi_r \rangle,
\end{aligned} \tag{3.50}$$

where  $\phi_r = A_r^* \chi$  is an arbitrary element of  $\mathbb{H}^d$ . Given that  $\rho \in \mathcal{L}_+^1(\mathbb{H}^d)$ , have we reach the desired conclusion.  $\square$

We have established that quaternionic quantum channels transform quaternionic quantum states into quaternionic quantum states. In the context of usual complex quantum theory, such a transformation is referred to as a *positive map*. The usual complex quantum formalism demands that physical transformations be defined by *completely positive* maps. A completely positive map is positive map,  $\Phi : \mathcal{M}_{d,d}(\mathbb{C}) \rightarrow \mathcal{M}_{p,p}(\mathbb{C})$ , such that for all positive semi-definite complex matrices  $A \in \mathcal{M}_{qd,qd}(\mathbb{C})_{sa}$  one has that  $(\mathbb{1}_{\mathbb{C}^q} \otimes \Phi)(A)$  is positive semi-definite  $\forall q \in \mathbb{Z}_+$  [60]. Choi's theorem for completely positive maps [17] dictates that any completely positive map acts according to an operator-sum decomposition analogous to (3.32). As we shall see in the following chapter, any quaternionic quantum channel can be simulated by a complex quantum channel (see Theorem 4.2.5). On that view, we conclude that our definition of quaternionic quantum channels is consistent with the usual definition of physical transformations. We refer to such channels as *completely positive trace preserving quaternionic maps*. Such terminology has previously been adopted in the  $p = q$  case by Kossakowski [56]. We say more on the subject of tensor products in Section 6.2.

### 3.3 Quaternionic Quantum Measurements

We define a measurement device to be a physical device that receives a physical system  $\mathfrak{S}$  as input and outputs one classical outcome  $r \in \{1, \dots, n\}$  with  $n \in \mathbb{Z}_+$ . We associate each classical outcome with a *quaternionic quantum effect*. We define the set of quaternionic quantum effects for physical systems with an associated state  $\rho \in \mathcal{L}_+^1(\mathbb{H}^d)$  as

$$\mathcal{E}(\mathbb{H}^d) = \{E \in \mathcal{L}_+(\mathbb{H}^d) \subset \mathcal{M}_{d,d}(\mathbb{H})_{sa} \mid \forall \phi \in \mathbb{H}^d : \langle \phi | E \phi \rangle \leq 1\}, \quad (3.51)$$

from which it follows that  $\forall E \in \mathcal{E}(\mathbb{H}^d)$ :

$$\|E\| = \sqrt{\text{tr}(E^2)} \leq \sqrt{d}. \quad (3.52)$$

A *quaternionic quantum measurement* corresponds to a measurement device with classical outcomes  $r \in \{1, \dots, n\}$  to which there is an associated set of quaternionic quantum effects  $\{E_1, \dots, E_n\}$  admitting the following normalization condition:

$$\sum_{r=1}^n E_r = \mathbf{1}_{\mathbb{H}^d}, \quad (3.53)$$

and we refer to such a set as a *quaternionic positive operator valued measure* (quaternionic POVM).

The set  $\mathcal{E}(\mathbb{H}^d)$  is convex, for if  $E_1, E_2 \in \mathcal{E}(\mathbb{H}^d)$  and  $0 < \lambda < 1$  then

$$E = (1 - \lambda)E_1 + \lambda E_2 \in \mathcal{E}(\mathbb{H}^d), \quad (3.54)$$

since one has that  $\forall \phi \in \mathbb{H}^d$

$$0 \leq \langle \phi | E \phi \rangle = (1 - \lambda)\langle \phi | E_1 \phi \rangle + \lambda\langle \phi | E_2 \phi \rangle \leq 1. \quad (3.55)$$

A distinguished class of quaternionic quantum POVMs are *quaternionic projection valued measures* (quaternionic PVMs), which are defined by sets of  $d$  rank-1 projection operators that are orthonormal with respect to the canonical inner product on  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$ , thereby resolving the identity on  $\mathbb{H}^d$  as in (3.53). Given Theorem 2.5.4, we have that the elements of a quaternionic PVM correspond to the eigenvectors of a self-adjoint quaternionic matrix.

The formulations of quaternionic quantum theory given by Finkelstein et al. [27] and Adler [1] restrict their attention to PVMs by identifying observables with self-adjoint matrices and physical propositions with the corresponding eigenprojectors. The full class of quaternionic POVMs offers a wider range of possibilities for the description of measurement devices. For instance, a quaternionic POVM in  $\mathcal{E}(\mathbb{H}^d)$  may admit a cardinality larger than  $d$ , thereby describing a measurement device with any number of classical outcomes. Moreover, the elements of a quaternionic POVM need not be orthogonal. In the context of complex quantum theory, the POVM formalism has two major advantages over the traditional PVM formalism. Firstly, complex POVMs provide a compact formalism for the description of physical scenarios wherein  $\mathfrak{S}$  interacts with ancillary systems [38][60]. Secondly, complex POVMs provide optimal solutions to important problems in complex quantum information theory, such as distinguishing a set complex quantum states [38][60]. These

lessons from complex quantum theory motivate us to consider a quaternionic POVM formalism for the description of quaternionic quantum measurements.

In usual complex quantum theory, Neumark's Theorem dictates that any complex POVM in  $\mathcal{E}(\mathbb{C}^d)$  can be realized as a complex PVM acting on a higher-dimensional complex Hilbert space. In Theorem 4.2.5, we prove that every quaternionic quantum measurement process corresponds to a measurement process described by usual complex quantum theory. Therefore, quaternionic POVMs can also be described by PVMs on higher-dimensional *complex* Hilbert spaces.

### 3.4 The Quaternionic Quantum Probability Rule

In this section we prove that the probability for an outcome of a quaternionic quantum measurement is equal the canonical inner product between the corresponding quaternionic quantum effect and a unique quaternionic quantum state. Our proof is based on the proof given by Caves et al. in the complex case [13]. In particular, we assume that probabilities for outcomes of quaternionic quantum measurements are *noncontextual*, so that they are given by *quaternionic frame functions* defined in Definition 3.4.1. The Kochen-Specker theorem [55] thereby requires us to denounce EPR realism [21]. Consequently, we deny the existence of the results of *unperformed* quaternionic quantum measurements [61]. Our assumption of noncontextuality is consistent with a Quantum Bayesian interpretation of quaternionic quantum states, wherein quaternionic quantum states reflect personalist Bayesian degrees of belief regarding the outcomes of quaternionic quantum measurements, and wherein measurements *generate* physical reality (see [32] and references therein, as well as Section 5.1). Our assumption of noncontextuality is not consistent with hidden-variable quantum models. We now proceed with the definition of quaternionic frame functions, and then we present and prove Theorem 3.4.2 dictating the quaternionic probability rule. In the special cases of PVMs, Theorem 3.4.2 reduces to a quaternionic version of Gleason's theorem [40].

**Definition 3.4.1** *Quaternionic Frame Functions*

Let

$$f : \mathcal{E}(\mathbb{H}^d) \rightarrow [0, 1]. \quad (3.56)$$

If  $\forall X = \{E_r \in \mathcal{E}(\mathbb{H}^d) \mid \sum_r E_r = \mathbf{1}_{\mathbb{H}^d}\}$ :

$$\sum_{E_r \in X} f(E_r) = 1, \quad (3.57)$$

then we say that  $f$  is a quaternionic frame function.

**Theorem 3.4.2** *The Quaternionic Quantum Probability Rule:*

For every quaternionic frame function there exists a unique unit-trace positive semi-definite matrix  $\rho \in \mathcal{M}_{d,d}(\mathbb{H})_{sa}$  such that  $\forall E \in \mathcal{E}(\mathbb{H}^d)$ :

$$f(E) = (E, \rho) = \text{tr}(E\rho). \quad (3.58)$$

*Proof:*

For the proof, we closely follow the steps taken by Caves et al. in the complex case [13]. First, we demonstrate that every quaternionic frame function is linear with respect to the nonnegative rational numbers  $\mathbb{Q}_+$ . Next, we prove that every quaternionic frame function is continuous. These two results together imply that every quaternionic frame function is linear on all of  $\mathcal{E}(\mathbb{H}^d)$ . We then extend  $f$  to a linear function on all of  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$ , and we explicitly show that it arises from the canonical inner product on  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$ .

Step 1: Linearity with respect to the nonnegative rationals

Let

$$X_1 = \{E_1, E_2, E_3\} \subset \mathcal{E}(\mathbb{H}^d) \quad (3.59)$$

define a quaternionic POVM, so that

$$X_2 = \{E_1 + E_2, E_3\} \subset \mathcal{E}(\mathbb{H}^d) \quad (3.60)$$

also defines a quaternionic POVM. If  $f$  is a quaternionic frame function, then

$$f(E_1) + f(E_2) + f(E_3) = f(E_1 + E_2) + f(E_3), \quad (3.61)$$

which implies that

$$f(E_1) + f(E_2) = f(E_1 + E_2), \quad (3.62)$$

trivially establishing that  $f$  is additive on  $\mathcal{E}(\mathbb{H}^d)$ .

Next, consider the effect

$$nE \in \mathcal{E}(\mathbb{H}^d), \quad (3.63)$$

where  $n \in \mathbb{Z}_+$  and  $E \in \mathcal{E}(\mathbb{H}^d)$  with  $\|E\| \leq \frac{\sqrt{d}}{n}$ . If  $m \in \mathbb{Z}_+$ , it follows that

$$\frac{n}{m}E \in \mathcal{E}(\mathbb{H}^d). \quad (3.64)$$

We have from additivity of quaternionic frame functions on  $\mathcal{E}(\mathbb{H}^d)$  that

$$mf\left(\frac{n}{m}E\right) = f(nE) = nf(E) \implies f\left(\frac{n}{m}E\right) = \frac{n}{m}f(E), \quad (3.65)$$

establishing that every quaternionic frame function is linear with respect to  $\mathbb{Q}_+$  on  $\mathcal{E}(\mathbb{H}^d)$ .

Step 2: Continuity

In this step, we prove that every quaternionic frame function is continuous. Recall the definition of continuity of functions between metric spaces [13]:

*Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces and let  $f : (X, d_X) \rightarrow (Y, d_Y)$ . We say that  $f$  is continuous at  $x_0 \in (X, d_X)$  if  $\forall \epsilon > 0 \exists \delta > 0$  such that  $d_Y(f(x), f(x_0)) < \epsilon \forall x$  satisfying  $d_X(x, x_0) < \delta$ .*

In our case, we have  $X = \mathcal{M}_{d,d}(\mathbb{H})_{sa}$  and the metric  $d_X(x_1, x_2)$  is given in terms of the norm induced by the canonical inner product on  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$ :

$$d_X(x, y) = \|x - y\| = \sqrt{\text{tr}((x - y)^2)}. \quad (3.66)$$

Indeed, the following properties hold for all quaternionic effects  $x, y, z \in \mathcal{E}(\mathbb{H}^d)$  by virtue of  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$  being a real vector space equipped with a norm induced by the canonical inner product (2.49):

- Nonnegativity:  $d_X(x, y) \geq 0$  with equality *iff*  $x = y$ ,
- Symmetry:  $d_X(x, y) = d_X(y, x)$ ,
- Triangle inequality:  $d_X(x, z) \leq d_X(x, y) + d_X(y, z)$ .

Nonnegativity of the metric follows directly from nonnegativity of the norm. Symmetry of the metric is a trivial consequence  $\mathbb{R}$ -homogeneity of the norm, in particular

$$d_X(x, y) = \|x - y\| = \|(-1)(y - x)\| = |-1|\|y - x\| = \|y - x\| = d_X(y, x). \quad (3.67)$$

The triangle inequality for the metric simply follows from the triangle inequality for the norm:

$$d_X(x, z) = \|x - z\| = \|(x - y) + (y - z)\| \leq \|x - y\| + \|y - z\| = d_X(x, y) + d_X(y, z). \quad (3.68)$$

We have thus established that  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$  equipped with the aforementioned metric is indeed a metric space. It is obvious that  $Y = [0, 1]$  equipped with the standard metric is a metric space.

We now proceed to show that any quaternionic frame function is continuous. Following Caves et al. [13], we shall proceed via *reductio ad absurdum*. First, suppose that a quaternionic frame function  $f$  is *discontinuous* – that is, not continuous – at the zero matrix  $\mathbf{0}$ . Additivity of quaternionic frame functions on  $\mathcal{E}(\mathbb{H}^d)$  implies that  $\forall E \in \mathcal{E}(\mathbb{H}^d)$ :

$$f(E) = f(E + \mathbf{0}) = f(E) + f(\mathbf{0}) \implies f(\mathbf{0}) = 0. \quad (3.69)$$

Now, if  $f$  is discontinuous at  $\mathbf{0}$ , then there exists  $\epsilon > 0$  such that  $\forall \delta > 0$  there exists some  $E \in \mathcal{E}(\mathbb{H}^d)$  satisfying  $\|E\| < \delta$  and  $|f(E)| \geq \epsilon$ . Consider the choice  $\delta = \frac{1}{n} < \epsilon$  where  $n \in \mathbb{Z}_+$ , and let  $E \in \mathcal{E}(\mathbb{H}^d)$  be such that  $\|E\| < \frac{1}{n}$  and  $|f(E)| \geq \epsilon$ . Now, consider the matrix  $F = nE$ . By multiplicativity of the norm induced by the canonical inner product on  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$ , one has that  $\|F\| = |n|\|E\| = n\|E\| < 1 < \sqrt{d}$ , for all dimensions of interest  $d \geq 2$ . Therefore, we have that  $F \in \mathcal{E}(\mathbb{H}^d)$ ; however, from additivity of quaternionic frame functions on  $\mathcal{E}(\mathbb{H}^d)$ , we also have that  $f(F) = nf(E) = n|f(E)| \geq n\epsilon > 1$ , which is impossible since  $f$  is a frame function. We have arrived at a contradiction, therefore our supposition that  $f$  is discontinuous at the zero matrix is false. We conclude that  $f$  is continuous at the zero matrix.

With the continuity of quaternionic frame functions established at the zero matrix, we can now easily establish continuity of quaternionic frame functions at any arbitrary effect  $E_0 \in \mathcal{E}(\mathbb{H}^d)$ . Consider a neighbouring  $E \in \mathcal{E}(\mathbb{H}^d)$  to  $E_0$  such that  $\|E - E_0\| < 1$ . It is clear that  $G = E - E_0 \in$



$\mathcal{M}_{d,d}(\mathbb{H})_{sa}$ , and applying Theorem 2.5.5 we may write  $G = A - B$  where  $A$  and  $B$  are respectively the positive and negative parts of the spectral resolution of  $G$ . Phrased explicitly, Theorem 2.5.4 theorem implies that we can write

$$\begin{aligned} G &= \sum_{r=1}^d |\xi_r\rangle \lambda_r \langle \xi_r| \\ &= \underbrace{\left( |\xi_1\rangle \lambda_1 \langle \xi_1| + |\xi_2\rangle \lambda_2 \langle \xi_2| + \cdots + |\xi_n\rangle \lambda_n \langle \xi_n| \right)}_A \\ &\quad - \underbrace{\left( (|\xi_{n+1}\rangle \mu_{n+1} \langle \xi_{n+1}| + |\xi_{n+2}\rangle \mu_{n+2} \langle \xi_{n+2}| + \cdots + |\xi_d\rangle \mu_d \langle \xi_d|) \right)}_B, \end{aligned} \quad (3.70)$$

where we have grouped the eigenvalues such that  $\forall r$  satisfying  $1 \leq r \leq n$  one has that  $\lambda_r \geq 0$ , and  $\forall s$  satisfying  $n+1 \leq s \leq d$  one has that  $\lambda_s < 0$ , with  $\mu_s = -\lambda_s$ . One has that  $A, B \in \mathcal{L}_+(\mathbb{H}^d)$ , since  $\forall \phi \in \mathbb{H}^d$ :

$$\langle \phi | A \phi \rangle = \sum_{r=1}^n \langle \phi | \xi_r \rangle \lambda_r \langle \xi_r | \phi \rangle = \sum_{r=1}^n |\langle \phi | \xi_r \rangle|^2 \lambda_r \geq 0, \quad (3.71)$$

where we have used the fact that  $\lambda_r \in \mathbb{R}_+$  commutes with the quaternion  $\langle \phi | \xi_r \rangle$  and its conjugate  $\langle \xi_r | \phi \rangle$ . Similarly, one has that  $\forall \phi \in \mathbb{H}^d$ :

$$\langle \phi | B \phi \rangle = \sum_{s=n+1}^d \langle \phi | \xi_s \rangle \mu_s \langle \xi_s | \phi \rangle = \sum_{s=n+1}^d |\langle \phi | \xi_s \rangle|^2 \mu_s \geq 0. \quad (3.72)$$

We have chosen  $E$  in the neighbourhood of  $E_0$  such that  $\|E - E_0\| < 1$ , and so, in fact, it follows that  $A, B \in \mathcal{E}(\mathbb{H}^d)$ . To see that  $A$  and  $B$  are indeed effects, observe that

$$\begin{aligned} \|A - B\|^2 &= \text{tr}((A - B)(A - B)) \\ &= \text{tr}(A^2) + \text{tr}(B^2) - 2\text{tr}(AB) \\ &= \|A\|^2 + \|B\|^2, \end{aligned} \quad (3.73)$$

where we have used the cyclic property of the canonical inner product on  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$ , and where we have used the fact that eigenvectors of self-adjoint  $G = A - B$  may be taken as orthogonal, hence

$$\text{tr}(AB) = \text{tr}\left( \sum_{r=1}^n \sum_{s=n+1}^d |\xi_r\rangle \lambda_r \langle \xi_r | \xi_s \rangle \mu_s \langle \xi_s | \right) = 0. \quad (3.74)$$

Summarizing the above, we have that

$$\|A\|^2 = \|A - B\|^2 - \|B\|^2 < 1 - \|B\|^2 < 1 \implies \|A\| < 1, \quad (3.75)$$

and similarly

$$\|B\|^2 = \|A - B\|^2 - \|A\|^2 < 1 - \|A\|^2 < 1 \implies \|B\| < 1, \quad (3.76)$$

hence  $A, B \in \mathcal{E}(\mathbb{H}^d)$ . Next, consider the matrix  $E + B$ . By definition,  $E \in \mathcal{E}(\mathbb{H}^d) \subset \mathcal{L}_+(\mathbb{H}^d)$ , and we have already established that  $B \in \mathcal{L}_+(\mathbb{H}^d)$ . Letting  $\alpha = \|E + B\| \in \mathbb{R}_+$  and choosing  $q \in \mathbb{Q}_+ > \alpha$  we have that

$$\left\| \frac{E + B}{q} \right\| = \left\| \frac{A + B}{q} \right\| < 1, \quad (3.77)$$

and applying the frame function to the subnormalization of  $E + B = E_0 + A$  by  $q$  we get

$$f\left(\frac{E+B}{q}\right) = f\left(\frac{E_0+A}{q}\right) \implies f\left(\frac{E}{q}\right) + f\left(\frac{B}{q}\right) = f\left(\frac{E_0}{q}\right) + f\left(\frac{A}{q}\right), \quad (3.78)$$

and therefore

$$f(E) - f(E_0) = f(A) - f(B). \quad (3.79)$$

In deriving (3.79) we have applied linearity of quaternionic frame functions with respect to the nonnegative rationals. Now, continuity of  $f$  at the zero matrix says that  $\forall \epsilon = \frac{\epsilon'}{2} \exists \delta > 0$  such that  $\|A\|, \|B\| < \delta$  implies that  $|f(A)|, |f(B)| < \epsilon'$ . So, if  $\|E - E_0\| = \|A - B\| < \delta$ , then  $|f(E) - f(E_0)| = |f(A) - f(B)| \leq |f(A)| + |f(B)| < 2\epsilon' = \epsilon$ . Therefore, we have established that any quaternionic frame function is continuous. Taken together with the result from Step 1 in the proof – that is, any quaternionic frame function is linear on  $\mathcal{E}(\mathbb{H}^d)$  with respect to  $\mathbb{Q}_+$  – we conclude that any quaternionic frame function is in fact linear on  $\mathcal{E}(\mathbb{H}^d)$  with respect to  $\mathbb{R}$ . Indeed, the rational numbers are a dense subset of the real numbers, meaning that for any real number  $\alpha \in \mathbb{R}$  there exists a rational number  $q \in \mathbb{Q}$  that can be chosen such that  $|\alpha - q|$  is arbitrarily small, and so by continuity the difference between  $f(E\alpha)$  and  $f(E)q$  can be made arbitrarily small.

### Step 3: Linearity and the canonical inner product

We have established that any quaternionic frame function  $f$  is  $\mathbb{R}$ -linear on  $\mathcal{E}(\mathbb{H}^d)$ . We now show that any such  $f$  has a unique  $\mathbb{R}$ -linear extension to cover the domain of all self-adjoint quaternionic matrices  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$ . Let  $H \in \mathcal{M}_{d,d}(\mathbb{H})_{sa}$ , and let  $G_1, G_2 \in \mathcal{L}_+(\mathbb{H}^d)$  be the positive and negative parts of the spectral resolution of  $H$  respectively, so that  $H = G_1 - G_2$  in analogy with (3.70). Now, if  $G \in \mathcal{L}_+(\mathbb{H}^d)$ , then one can always find  $E \in \mathcal{E}(\mathbb{H}^d)$  such that  $G = \alpha E$ , where  $\alpha \in \mathbb{R}_+$ . We define the extension of  $f$  acting on arbitrary  $H \in \mathcal{M}_{d,d}(\mathbb{H})_{sa}$  as follows:

$$f(H) = f(G_1) - f(G_2) = \alpha_1 f(E_1) - \alpha_2 f(E_2). \quad (3.80)$$

The extension of  $f$  is  $\mathbb{R}$ -linear. Indeed, if  $a \in \mathbb{R}$  and  $H \in \mathcal{M}_{d,d}(\mathbb{H})_{sa}$  then

$$\begin{aligned} f(aH) &= f(aG_1 - aG_2) \\ &= f(a\alpha_1 E_1 - a\alpha_2 E_2) \\ &= a\alpha_1 f(E_1) - a\alpha_2 f(E_2) \\ &= af(H). \end{aligned} \quad (3.81)$$

Furthermore, if  $H^{(1)}, H^{(2)} \in \mathcal{M}_{d,d}(\mathbb{H})_{sa}$  with

$$H^{(1)} = \alpha_1^{(1)} E_1^{(1)} - \alpha_2^{(1)} E_2^{(1)}, \quad (3.82)$$

and

$$H^{(2)} = \alpha_1^{(2)} E_1^{(2)} - \alpha_2^{(2)} E_2^{(2)}, \quad (3.83)$$

then

$$\begin{aligned}
f(H^{(1)} + H^{(2)}) &= f\left(\underbrace{\alpha_1^{(1)} E_1^{(1)} + \alpha_1^{(2)} E_1^{(2)}}_{\alpha_1 E_1} - \underbrace{(\alpha_2^{(1)} E_2^{(1)} + \alpha_2^{(2)} E_2^{(2)})}_{\alpha_2 E_2}\right) \\
&= f(\alpha_1 E_1 - \alpha_2 E_2) \\
&= f(\alpha_1 E_1) - f(\alpha_2 E_2) \\
&= f\left(\alpha_1^{(1)} E_1^{(1)} + \alpha_1^{(2)} E_1^{(2)}\right) - f\left(\alpha_2^{(1)} E_2^{(1)} - \alpha_2^{(2)} E_2^{(2)}\right) \\
&= \alpha_1^{(1)} f\left(E_1^{(1)}\right) - \alpha_2^{(1)} f\left(E_2^{(1)}\right) + \alpha_1^{(2)} f\left(E_1^{(2)}\right) - \alpha_2^{(2)} f\left(E_2^{(2)}\right) \\
&= f\left(H^{(1)}\right) + f\left(H^{(2)}\right), \tag{3.84}
\end{aligned}$$

where we have repeatedly used  $\mathbb{R}$ -linearity of  $f$  on  $\mathcal{E}(\mathbb{H}^d)$ . Of course, there are infinitely many ways to decompose a positive semi-definite quaternionic matrix as the product of a positive real number and an effect. In particular, one can always find distinct  $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}_+$  and distinct  $E_1, E_2, E_3, E_4 \in \mathcal{E}(\mathbb{H}^d)$  such that

$$H = \alpha_1 E_1 - \alpha_2 E_2 = \alpha_3 E_3 - \alpha_4 E_4, \tag{3.85}$$

which implies that

$$\alpha_1 E_1 + \alpha_4 E_4 = \alpha_2 E_2 + \alpha_3 E_3. \tag{3.86}$$

Despite the existence of such distinct unravellings of  $H$ , the extension of  $f$  acting on  $H$  actually is unique. To see this, choose  $\beta = \max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$  and divide (3.86) by  $\beta$ :

$$\frac{\alpha_1}{\beta} E_1 + \frac{\alpha_4}{\beta} E_4 = \frac{\alpha_2}{\beta} E_2 + \frac{\alpha_3}{\beta} E_3. \tag{3.87}$$

It is obvious that  $\forall r \in \{1, 2, 3, 4\} : \frac{\alpha_r}{\beta} E_r \in \mathcal{E}(\mathbb{H}^d)$  – the original domain of  $f$ . Acting on (3.87) with  $f$ , and applying  $\mathbb{R}$ -linearity of  $f$  on  $\mathcal{E}(\mathbb{H}^d)$ , we conclude that

$$f(\alpha_1 E_1 - \alpha_2 E_2) = f(\alpha_3 E_3 - \alpha_4 E_4), \tag{3.88}$$

establishing a unique  $\mathbb{R}$ -linear extension of  $f$  to the domain  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$ .

Having established  $\mathbb{R}$ -linearity of any quaternionic frame function  $f$  on the real vector space  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$ , we now proceed to explicitly show that  $f$  arises from the canonical inner product on  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$ .

Let  $\{\Upsilon_1, \dots, \Upsilon_{d(2d-1)}\}$  be an orthonormal basis for  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$  such that  $(\Upsilon_r, \Upsilon_{rs}) = \delta_{rs}$ . We can expand any effect  $E \in \mathcal{E}(\mathbb{H}^d)$  as a real-linear combination of the  $\Upsilon_r$  in terms of coefficients  $(\Upsilon_r, E)$ :

$$E = \sum_{r=1}^{d(2d-1)} \Upsilon_r (\Upsilon_r, E). \tag{3.89}$$

Also, there exists a *unique* matrix  $\rho \in \mathcal{M}_{d,d}(\mathbb{H})_{sa}$  that we can expand as a real-linear combination of the  $\Upsilon_r$  in terms of coefficients  $f(\Upsilon_r)$ :

$$\rho = \sum_{r=1}^{d(2d-1)} \Upsilon_r f(\Upsilon_r), \quad (3.90)$$

so that  $(\rho, \Upsilon_r) = f(\Upsilon_r)$ . We stress that  $\rho$  is indeed unique, for

$$\sum_{r=1}^{d(2d-1)} \Upsilon_r \alpha_r = \sum_{r=1}^{d(2d-1)} \Upsilon_r \beta_r \iff 0 = \sum_{r=1}^{d(2d-1)} (\alpha_r - \beta_r) \Upsilon_r \iff \forall r : \alpha_r = \beta_r, \quad (3.91)$$

since  $\Upsilon_r$  are linearly independent. It follows that

$$\begin{aligned} f(E) &= \sum_{r=1}^{d(2d-1)} f(\Upsilon_r)(\Upsilon_r, E) \\ &= \sum_{r=1}^{d(2d-1)} (\rho, \Upsilon_r)(\Upsilon_r, E) \\ &= \left( \sum_{r=1}^{d(2d-1)} \Upsilon_r (\rho, \Upsilon_r), E \right) \\ &= (E, \rho). \end{aligned} \quad (3.92)$$

The matrix  $\rho$  is positive semi-definite, which is verified by letting  $E = |\phi\rangle\langle\phi|$  for arbitrary  $\phi \in \mathbb{H}^d$  and observing from (2.74) that

$$0 \leq f(|\phi\rangle\langle\phi|) = \text{tr}(\rho|\phi\rangle\langle\phi|) = \langle\phi|\rho\phi\rangle. \quad (3.93)$$

We also have that  $\rho$  is unit-trace, which follows from the observation that

$$\text{tr}(\rho) = (\rho, \mathbb{1}_{\mathbb{H}^d}) = \left( \rho, \sum_{E_r \in X} E_r \right) = \sum_{E_r \in X} f(E_r) = 1, \quad (3.94)$$

where  $X$  is an arbitrary quaternionic POVM, finishing the proof.  $\square$

## Chapter 4

# Quaternionic Quantum Dynamics in Complex Quantum Theory

In this chapter, we embed quaternionic quantum theory into the framework of complex quantum theory. We prove that every physical process described via the quaternionic quantum formalism considered in Chapter 3 corresponds to a physical process described via the formalism of usual complex quantum information theory. Moreover, we show that any quaternionic quantum description of a physical preparation  $\rightarrow$  transformation  $\rightarrow$  measurement process of the type depicted in Figure 3.1 is *equivalent* to a complex quantum description in the sense that both descriptions admit identical probabilities for the outcomes of the final measurement process. Our results generalize previously discovered equivalences between quaternionic and complex quantum theories highlighted in Chapter 1. In particular, we prove that quaternionic quantum theory and complex quantum theory are consistent with respect to the *full apparatus* of quantum information theory, rather than only considering the restricted class of quantum processes treated in [25]. The results given by Fernandez and Schneeberger in [25] establish the equivalence of quaternionic and complex quantum theories with respect unitary evolution and PVM quantum measurements. Specifically, Fernandez and Schneeberger show that any quaternionic preparation  $\rightarrow$  unitary evolution  $\rightarrow$  PVM measurement process on  $n$  qubits corresponds to an equivalent process involving  $n + 1$  qubits. Their proof is based on the group isomorphism

$$\mathrm{Sp}(d) \cong \mathrm{U}(2d, \mathbb{C}) \cap \mathrm{Sp}(2d, \mathbb{C}) \tag{4.1}$$

(see Lemma A.2.6 for the proof of this symplectic group isomorphism). We also appeal to the symplectic embedding of quaternionic matrices into complex matrices to establish our main results. In Section 4.1, we recall the symplectic coordinate representation of quaternions, and we apply the symplectic coordinate representation to establish injective mappings  $\psi_{p,d}$  from  $\mathcal{M}_{p,d}(\mathbb{H})$  into  $\mathcal{M}_{2p,2d}(\mathbb{C})$ , which reduce to the injective  $*$ -homomorphisms pointed out by Farenick and Pidkovich in [23] when  $p = d$ . We proceed to prove several key properties of the maps  $\psi_{p,d}$ , including a lemma relating canonical inner products on  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$  to Hilbert-Schmidt inner products on  $\mathcal{M}_{2d,2d}(\mathbb{C})_{sa}$ . Equipped with the lemmas of Section 4.1, we prove our main results in Section 4.2.

## 4.1 Symplectic Embeddings of $\mathcal{M}_{p,d}(\mathbb{H})$ into $\mathcal{M}_{2p,2d}(\mathbb{C})$

Finkelstein et al. [27] pointed out that every quaternion can be represented by a unique pair of complex numbers in accordance with Definition 4.1.1:

**Definition 4.1.1** *Symplectic coordinate representations of quaternions:*

Let  $a \in \mathbb{H}$  with  $a = a_0 + ia_1 + ja_2 + ka_3 = \gamma_1 + \gamma_2 j$  where  $\gamma_1, \gamma_2 \in \mathbb{C}$  are defined in terms of the constituents of  $a$  as follows:

$$\gamma_1 = a_0 + ia_1 \in \mathbb{C}, \quad (4.2)$$

and

$$\gamma_2 = a_2 + ia_3 \in \mathbb{C}. \quad (4.3)$$

We say that  $\gamma_1 + \gamma_2 j$  is the symplectic coordinate representation of  $a$ .

The correspondence between symplectic coordinate representations of quaternions and the associated  $\mathbb{C}$ -linear endomorphisms on  $\mathbb{C}^2$  are described in Section A.3. Symplectic coordinate representations of quaternions are used to define symplectic coordinate representations of quaternionic matrices in accordance with Definition 4.1.2:

**Definition 4.1.2** *Symplectic coordinate representations of quaternionic matrices:*

Let  $A \in \mathcal{M}_{p,d}(\mathbb{H})$  with  $A = \Gamma_1 + \Gamma_2 j$  where  $\Gamma_1, \Gamma_2 \in M_{p,d}(\mathbb{C})$  are obtained from the symplectic coordinate representations of matrix elements according to (4.2) and (4.3). We say that  $\Gamma_1 + \Gamma_2 j$  is the symplectic coordinate representation of  $A$ .

Evidently, the symplectic coordinate representation of a *self-adjoint* quaternionic matrix  $A \in \mathcal{M}_{d,d}(\mathbb{H})_{sa}$  is given in terms of complex self-adjoint  $\Gamma_1 = \Gamma_1^*$  and complex antisymmetric  $\Gamma_2 = -\Gamma_2^T$ , for  $A = A^*$  implies that

$$\Gamma_1 + \Gamma_2 j = (\Gamma_1 + \Gamma_2 j)^* = \Gamma_1^* - j\Gamma_2^* = \Gamma_1^* - \Gamma_2^T j. \quad (4.4)$$

Symplectic coordinate representations of quaternionic matrices are used to define symplectic embeddings of quaternionic matrices into complex matrices according to Definition 4.1.3.

**Definition 4.1.3** *Symplectic Embeddings:*

Let  $A = \Gamma_1 + \Gamma_2 j \in \mathcal{M}_{p,d}(\mathbb{H})$ . Define

$$\psi_{p,d} : \mathcal{M}_{p,d}(\mathbb{H}) \rightarrow \mathcal{M}_{2p,2d}(\mathbb{C}) \quad (4.5)$$

via:

$$\psi_{p,d}(A) = \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ -\overline{\Gamma_2} & \overline{\Gamma_1} \end{pmatrix}. \quad (4.6)$$

We say (4.6) is the symplectic embedding of  $A$  into  $\mathcal{M}_{2p,2d}(\mathbb{C})$ .

The symplectic embeddings enjoy the properties summarized in Lemma 4.1.4.

**Lemma 4.1.4** *Properties of Symplectic Embeddings:*

The following holds  $\forall A, A' \in \mathcal{M}_{p,d}(\mathbb{H})$ ,  $\forall B \in \mathcal{M}_{d,q}(\mathbb{H})$  and  $\forall \alpha, \alpha' \in \mathbb{R}$ :

$$\psi_{p,d}(A\alpha + A'\alpha') = \psi_{p,d}(A)\alpha + \psi_{p,d}(A')\alpha', \quad (4.7)$$

$$\psi_{d,d}(\mathbf{1}_{\mathbb{H}^d}) = \mathbf{1}_{\mathbb{C}^{2d}}, \quad (4.8)$$

$$\psi_{p,d}(A)\psi_{d,q}(B) = \psi_{p,q}(AB), \quad (4.9)$$

$$\psi_{p,d}(A)^* = \psi_{d,p}(A^*), \quad (4.10)$$

$$\psi(A) = \psi(A') \iff A = A'. \quad (4.11)$$

*Proof:*

Let

$$A = \Gamma_1 + \Gamma_2 j, \quad (4.12)$$

$$A' = \Gamma'_1 + \Gamma'_2 j, \quad (4.13)$$

and

$$B = \Lambda_1 + \Lambda_2 j, \quad (4.14)$$

where  $\Gamma_1, \Gamma'_1, \Gamma_2, \Gamma'_2 \in \mathcal{M}_{p,d}(\mathbb{C})$  and where  $\Lambda_1, \Lambda_2 \in \mathcal{M}_{d,q}(\mathbb{C})$ . Linearity (4.7) follows directly from the definition of the symplectic embeddings:

$$\begin{aligned} \psi_{p,d}\left((\Gamma_1\alpha + \Gamma_2 j\alpha) + (\Gamma'_1\alpha' + \Gamma'_2 j\alpha')\right) &= \psi_{p,d}\left((\Gamma_1\alpha + \Gamma'_1\alpha') + (\Gamma_2\alpha + \Gamma'_2\alpha')j\right) \\ &= \begin{pmatrix} \Gamma_1\alpha + \Gamma'_1\alpha' & \Gamma_2\alpha + \Gamma'_2\alpha' \\ -\overline{\Gamma_2\alpha + \Gamma'_2\alpha'} & \overline{\Gamma_1\alpha + \Gamma'_1\alpha'} \end{pmatrix} \\ &= \psi_{p,d}(A)\alpha + \psi_{p,d}(A')\alpha'. \end{aligned} \quad (4.15)$$

If  $A = \mathbf{1}_{\mathbb{H}^d}$ , then  $\Gamma_1 = \mathbf{1}_{\mathbb{H}^d}$  and  $\Gamma_2 = \mathbf{0}$ , and so  $\psi_{d,d}(\mathbf{1}_{\mathbb{H}^d})$  is the identity on  $\mathbf{1}_{\mathbb{C}^{2d}}$  as in (4.8). To prove multiplicativity (4.9) of the symplectic embeddings, we observe that

$$AB = (\Gamma_1\Lambda_1 - \Gamma_2\overline{\Lambda_2}) + (\Gamma_1\Lambda_2 + \Gamma_2\overline{\Lambda_1})j \implies \psi_{p,q}(AB) = \begin{pmatrix} \Gamma_1\Lambda_1 - \Gamma_2\overline{\Lambda_2} & \Gamma_1\Lambda_2 + \Gamma_2\overline{\Lambda_1} \\ -\overline{\Gamma_1\Lambda_2} - \overline{\Gamma_2\Lambda_1} & \overline{\Gamma_1\Lambda_1} - \overline{\Gamma_2\Lambda_2} \end{pmatrix}, \quad (4.16)$$

whereas

$$\psi_{p,d}(A)\psi_{d,p}(B) = \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ -\overline{\Gamma_2} & \overline{\Gamma_1} \end{pmatrix} \begin{pmatrix} \Lambda_1 & \Lambda_2 \\ -\overline{\Lambda_2} & \overline{\Lambda_1} \end{pmatrix} = \begin{pmatrix} \Gamma_1\Lambda_1 - \Gamma_2\overline{\Lambda_2} & \Gamma_1\Lambda_2 + \Gamma_2\overline{\Lambda_1} \\ -\overline{\Gamma_2\Lambda_1} - \overline{\Gamma_1\Lambda_2} & \overline{\Gamma_1\Lambda_1} - \overline{\Gamma_2\Lambda_2} \end{pmatrix} = \psi_{p,q}(AB). \quad (4.17)$$

To prove that symplectic embeddings commute with the composition of conjugation and transposition (4.10), we observe that  $A^* = \Gamma_1^* - \Gamma_2^T j$  so

$$\psi_{d,p}(A^*) = \begin{pmatrix} \Gamma_1^* & (-\overline{\Gamma_2})^* \\ \Gamma_2^* & (\overline{\Gamma_1})^* \end{pmatrix} = \begin{pmatrix} \Gamma_1^* & -\Gamma_2^T \\ \Gamma_2^* & \Gamma_1^T \end{pmatrix} = \psi_{p,d}(A)^*. \quad (4.18)$$

Finally, symplectic embeddings are injective (4.11), for if  $\psi_{p,d}(A) = \psi_{p,d}(A')$ , then  $\Gamma_1 = \Gamma_1'$  and  $\Gamma_2 = \Gamma_2'$ , finishing the proof.  $\square$

In addition to Lemma 4.1.4, we require Lemma 4.1.5 in order to establish our results in Section 4.2.

**Lemma 4.1.5** *Inner Product Correspondence:*

*The following holds  $\forall A, B \in \mathcal{M}_{d,d}(\mathbb{H})_{sa}$ :*

$$\text{tr}(AB) = \frac{1}{2} \text{tr}(\psi_{d,d}(A)\psi_{d,d}(B)), \quad (4.19)$$

*where by virtue of  $A = A^*$  and  $B = B^*$  we have that the RHS of (4.19) is the standard Hilbert-Schmidt inner product [60] on  $\mathcal{M}_{2d,2d}(\mathbb{C})_{sa}$ .*

*Proof:* Given (4.4), the symplectic coordinate representations of quaternionic self-adjoint  $A = \Gamma_1 + \Gamma_2 j$  and  $B = \Lambda_1 + \Lambda_2 j$  are in terms of complex self-adjoint  $\Gamma_1 = \Gamma_1^*$  and  $\Lambda_1 = \Lambda_1^*$  and complex antisymmetric  $\Gamma_2 = -\Gamma_2^T$  and  $\Lambda_2 = -\Lambda_2^T$ . Expanding the LHS of (4.19) we get

$$\begin{aligned} \text{tr}(AB) &= \underbrace{\frac{1}{2} \text{tr}(\Gamma_1 \Lambda_1 + \Lambda_1 \Gamma_1)}_{\mathbf{a}} + \underbrace{\frac{1}{2} \text{tr}(\Gamma_1 \Lambda_2 j + \Lambda_1 \Gamma_2 j)}_{\mathbf{b}} \\ &+ \underbrace{\frac{1}{2} \text{tr}(\Gamma_2 j \Lambda_1 + \Lambda_2 j \Gamma_1)}_{\mathbf{c}} + \underbrace{\frac{1}{2} \text{tr}(\Gamma_2 j \Lambda_2 j + \Lambda_2 j \Gamma_2 j)}_{\mathbf{d}}. \end{aligned} \quad (4.20)$$

Expanding the RHS of (4.19) we get

$$\frac{1}{2} \text{tr}(\psi_{d,d}(A)\psi_{d,d}(B)) = \underbrace{\frac{1}{2} \text{tr}(\Gamma_1 \Lambda_1 + \overline{\Gamma_1} \overline{\Lambda_1})}_{\mathbf{a}'} + \underbrace{\frac{1}{2} \text{tr}(-\Gamma_2 \overline{\Lambda_2} - \overline{\Gamma_2} \Lambda_2)}_{\mathbf{d}'}. \quad (4.21)$$

Our method of proof will be to show that  $\mathbf{a} = \mathbf{a}'$ ,  $\mathbf{d} = \mathbf{d}'$ , and  $\mathbf{b} + \mathbf{c} = 0$ . We begin by showing that  $\mathbf{a} = \mathbf{a}'$ . Since  $\Gamma_1, \Lambda_1 \in M_n(\mathbb{C})_{sa}$ , we have that

$$\text{tr}(\overline{\Gamma_1} \overline{\Lambda_1}) = \text{tr}(\Gamma_1^T \Lambda_1^T) = \text{tr}((\Lambda_1 \Gamma_1)^T) = \text{tr}(\Lambda_1 \Gamma_1) = \text{tr}(\Gamma_1 \Lambda_1), \quad (4.22)$$

and therefore by the cyclic property of the trace we have that  $\mathbf{a} = \mathbf{a}'$ . We now proceed to show that  $\mathbf{b} + \mathbf{c} = 0$ . It will be useful to express the components of the symplectic coordinate representations as follows:

$$\Gamma_1 = \sum_{r=1}^n \sum_{s=1}^n |r\rangle (a_{rs} + ib_{rs}) \langle s|, \quad \text{with } a_{rs} + ib_{rs} = a_{sr} - ib_{sr}, \quad (4.23)$$



$$\Gamma_2 = \sum_{r=1}^n \sum_{s=1}^n |r\rangle(\alpha_{rs} + i\beta_{rs})\langle s|, \quad \text{with } \alpha_{rs} + i\beta_{rs} = -\alpha_{sr} - i\beta_{sr}, \quad (4.24)$$

$$\Lambda_1 = \sum_{u=1}^n \sum_{v=1}^n |u\rangle(d_{uv} + ig_{uv})\langle v|, \quad \text{with } d_{uv} + ig_{uv} = d_{vu} - ig_{vu}, \quad (4.25)$$

$$\Lambda_2 = \sum_{u=1}^n \sum_{v=1}^n |u\rangle(\delta_{uv} + i\gamma_{uv})\langle v|, \quad \text{with } \delta_{uv} + i\gamma_{uv} = -\delta_{vu} - i\gamma_{vu}, \quad (4.26)$$

where  $|r\rangle, |s\rangle, |u\rangle, |v\rangle$  are elements of an arbitrary orthonormal basis for  $\mathbb{H}^d$ . We can see from (4.23) that  $j\Gamma_1 = \overline{\Gamma_1}j$ , and similarly  $j\Lambda_1 = \overline{\Lambda_1}j$ . It follows that we may express  $2(\mathbf{b} + \mathbf{c})$  as

$$\sum_{t=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{u=1}^n \sum_{v=1}^n \langle t|r\rangle(a_{rs} + ib_{rs})\langle s|u\rangle(\delta_{uv} + i\gamma_{uv})\langle v|t\rangle \quad (4.27)$$

$$+ \sum_{t=1}^n \sum_{u=1}^n \sum_{v=1}^n \sum_{r=1}^n \sum_{s=1}^n \langle t|u\rangle(d_{uv} + ig_{uv})\langle v|r\rangle(\alpha_{rs} + i\beta_{rs})\langle r|t\rangle \quad (4.28)$$

$$+ \sum_{t=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{u=1}^n \sum_{v=1}^n \langle t|r\rangle(\alpha_{rs} + i\beta_{rs})\langle s|u\rangle(d_{uv} - ig_{uv})\langle v|t\rangle \quad (4.29)$$

$$+ \sum_{t=1}^n \sum_{u=1}^n \sum_{v=1}^n \sum_{r=1}^n \sum_{s=1}^n \langle t|u\rangle(\delta_{uv} + i\gamma_{uv})\langle v|r\rangle(a_{rs} - ib_{rs})\langle r|t\rangle. \quad (4.30)$$

Switching  $u$  to  $v$ ,  $v$  to  $u$ ,  $r$  to  $s$ , and  $s$  to  $r$  in (4.30), we then condense the summation notation and consider (4.27) + (4.30)

$$\begin{aligned} &= \sum_{t,r,s,v} \langle t|r\rangle(a_{rs} + ib_{rs})(\delta_{sv} + i\gamma_{sv})\langle v|t\rangle + \sum_{t,r,s,v} \langle t|v\rangle(\delta_{vs} + i\gamma_{vs})(a_{sr} - ib_{sr})\langle r|t\rangle \\ &= \sum_{t,r,s,v} \langle t|r\rangle(a_{rs} + ib_{rs})(\delta_{sv} + i\gamma_{sv})\langle v|t\rangle + \sum_{t,r,s,v} \langle t|v\rangle(-\delta_{sv} - i\gamma_{sv})(a_{rs} + ib_{rs})\langle r|t\rangle \\ &= \sum_{t,r,s,v} (a_{rs}\delta_{sv} - b_{rs}\gamma_{sv}) \underbrace{(\langle t|r\rangle\langle v|t\rangle - \langle t|v\rangle\langle r|t\rangle)}_{\Delta_{rv} - \Delta_{vr}} \\ &\quad + \sum_{t,r,s,v} (a_{rs}\gamma_{sv} + b_{rs}\delta_{sv}) \underbrace{(\langle t|r\rangle i\langle v|t\rangle - \langle t|v\rangle i\langle r|t\rangle)}_{i(\Delta_{rv} - \Delta_{vr})} \\ &= 0 \end{aligned} \quad (4.31)$$

where we have used the unusual notation  $\Delta_{rv}$  for the Kronecker delta function so as not to confuse it with the entries of  $\Gamma_2$ . An entirely similar calculation shows that (4.28)+(4.29)=0. Therefore  $\mathbf{b} + \mathbf{c} = 0$ . We now proceed to show that  $\mathfrak{d} = \mathfrak{d}'$ . Observe that

$$j\Gamma_2j = \sum_{r=1}^n \sum_{s=1}^n |r\rangle j(\alpha_{rs} + i\beta_{rs})j\langle s| = \sum_{r=1}^n \sum_{s=1}^n |r\rangle(-\alpha_{rs} + i\beta_{rs})\langle s|, \quad (4.32)$$

and similarly,

$$j\Lambda_2j = \sum_{u=1}^n \sum_{v=1}^n |r\rangle j(\delta_{uv} + i\gamma_{uv})j\langle s| = \sum_{u=1}^n \sum_{v=1}^n |r\rangle(-\delta_{uv} + i\gamma_{uv})\langle s|. \quad (4.33)$$

We therefore compute  $2\mathfrak{d}$  as follows:

$$\mathrm{tr}\left(\Gamma_{2j}\Lambda_{2j} + \Lambda_{2j}\Gamma_{2j}\right) = -\mathrm{tr}\left(\Gamma_2\overline{\Lambda_2} + \Lambda_2\overline{\Gamma_2}\right), \quad (4.34)$$

establishing that  $\mathfrak{d} = \mathfrak{d}'$  and finishing the proof.  $\square$

Equipped with Lemma 4.1.4 and Lemma 4.1.5, we proceed in Section 4.2 to prove the equivalence of quaternionic and complex quantum theories.

## 4.2 Complex Realizations of Quaternionic Quantum Dynamics

In this section, we prove that every physical preparation  $\rightarrow$  transformation  $\rightarrow$  measurement process described via quaternionic quantum formalism has an equivalent description given via the formalism of usual complex quantum information theory. We begin by proving that a subnormalization of the symplectic embedding of any quaternionic quantum state is a complex quantum state.

**Lemma 4.2.1** *Symplectic Embeddings of Quaternionic Quantum States:*

If  $\rho \in \mathcal{L}_+^1(\mathbb{H}^d)$ , then

$$\sigma_{d,d}(\rho) = \frac{1}{2}\psi_{d,d}(\rho) \in \mathcal{L}_+^1(\mathbb{C}^{2d}), \quad (4.35)$$

where  $\psi_{d,d}$  is the symplectic embedding defined in (4.6). Moreover,  $\sigma_{d,d}(\mathcal{L}_+^1(\mathbb{H}^d))$  is a convex subset of  $\mathcal{L}_+^1(\mathbb{C}^{2d})$ .

*Proof:*

Applying Theorem 2.5.4:

$$\rho = \sum_{r=1}^d |\xi_r\rangle \lambda_r \langle \xi_r|, \quad (4.36)$$

where the  $\{\xi_1, \dots, \xi_d\}$  form an orthonormal basis for  $\mathbb{H}^d$  and where  $\lambda \in \mathbb{R}_+$ . Applying the symplectic embedding:

$$\psi_{d,d}(\rho) = \psi_{d,d}\left(\sum_{r=1}^d |\xi_r\rangle \lambda_r \langle \xi_r|\right) = \sum_{r=1}^d \psi_{d,d}(|\xi_r\rangle \langle \xi_r|) \lambda_r, \quad (4.37)$$

where we have used (4.7) and the reality of  $\lambda_r$ . Next, let  $\Pi_{\xi_r}^2 = \Pi_{\xi_r} = |\xi_r\rangle \langle \xi_r|$  and observe that

$$\psi_{d,d}(\Pi_{\xi_r}) = \psi_{d,d}(\Pi_{\xi_r}\Pi_{\xi_r}) = \psi_{d,d}(\Pi_{\xi_r})^2 \quad (4.38)$$

where we have used (4.9), establishing that the symplectic embedding of a projection operator is a projection operator. Projection operators are manifestly positive semi-definite, and given positivity of  $\lambda_r$  we have that  $\psi_{d,d}(\rho) \in \mathcal{L}_+(\mathbb{C}^{2d})$ . Now, from (4.19) we have that

$$1 = \mathrm{tr}(\rho) = \mathrm{tr}(\rho\mathbf{1}_{\mathbb{H}^d}) = \frac{1}{2}\mathrm{tr}(\psi_{d,d}(\rho)\psi_{d,d}(\mathbf{1}_{\mathbb{H}^d})) = \frac{1}{2}\mathrm{tr}(\psi_{d,d}(\rho)), \quad (4.39)$$

hence

$$\sigma(\rho) = \frac{1}{2}\psi_{d,d}(\rho) \in \mathcal{L}_+^1(\mathbb{C}^{2d}). \quad (4.40)$$

It remains to show that  $\sigma_{d,d}(\mathcal{L}_+^1(\mathbb{H}^d))$  is convex. If  $\sigma_{d,d}(\rho_1), \sigma_{d,d}(\rho_2) \in \sigma_{d,d}(\mathcal{L}_+^1(\mathbb{H}^d))$  and  $t \in (0, 1)$ , then

$$t\sigma_{d,d}(\rho_1) + (1-t)\sigma_{d,d}(\rho_2) = \sigma_{d,d}(\underbrace{t\rho_1 + (1-t)\rho_2}_{\rho \in \mathcal{L}_+^1(\mathbb{H}^d)}) = \sigma_{d,d}(\rho) \in \sigma_{d,d}(\mathcal{L}_+^1(\mathbb{H}^d)), \quad (4.41)$$

where we have used (4.7) and the convexity of  $\mathcal{L}_+^1(\mathbb{H}^d)$ . The proof is complete.  $\square$

Lemma 4.2.2 characterizes the extreme points of the image of quaternionic quantum state space under subnormalized symplectic embedding into complex quantum state space.

**Lemma 4.2.2** *Extreme points of  $\sigma_{d,d}(\mathcal{L}_+^1(\mathbb{H}^d)) \subset \mathcal{L}_+^1(\mathbb{C}^{2d})$ :*

*The extreme points of  $\sigma_{d,d}(\mathcal{L}_+^1(\mathbb{H}^d))$  are subnormalized rank-2 projectors that are the images of the extreme points of  $\mathcal{L}_+^1(\mathbb{H}^d)$ .*

*Proof:*

For the proof, we employ a technique used in another context in [13]. With  $\rho \in \mathcal{L}_+^1(\mathbb{H})$ , Theorem 2.5.4 dictates that  $\rho$  can be decomposed as in (4.36). Consider the unique complex quantum state

$$\sigma_{d,d}(\rho) = \frac{1}{2}\psi_{d,d}(\rho) = \frac{1}{2}\psi\left(\sum_{r=1}^d |\xi_r\rangle\lambda_r\langle\xi_r|\right) = \sum_{r=1}^d \frac{\lambda_r}{2}\psi_{d,d}(|\xi_r\rangle\langle\xi_r|) = \sum_{r=1}^d \lambda_r\sigma_{d,d}(|\xi_r\rangle\langle\xi_r|). \quad (4.42)$$

With (4.42), we see that the image of a quaternionic quantum state under  $\sigma_{d,d}$  can be expanded as a convex combination of subnormalized projectors – rank-2 projectors to be specific, since

$$\psi_{d,d}(|\xi_r\rangle\langle\xi_r|) = ||r\rangle\rangle\langle\langle r|| + ||r+d\rangle\rangle\langle\langle r+d||, \quad (4.43)$$

where  $||r\rangle\rangle$  are used to denote the standard basis vectors of  $\mathbb{C}^{2d}$ . Therefore, only subnormalized rank-2 projectors can be extreme points of  $\sigma_{d,d}(\mathcal{L}_+^1(\mathbb{H}))$ . It remains to show that, in fact, *all*  $\sigma_{d,d}(\pi) \in \sigma_{d,d}(\mathcal{L}_+^1(\mathbb{H}))$  with  $\pi^2 = \pi$  are extreme points. We proceed via *reductio ad absurdum*. Suppose  $\sigma_{d,d}(\pi)$  can be written as a *proper* convex combination of other subnormalized rank-2 projectors:

$$\sigma_{d,d}(\pi) = tA + (1-t)B, \quad (4.44)$$

where  $\exists a^2 = a, b^2 = b \in \mathcal{L}_+^1(\mathbb{H}^d)$ ,  $a \neq b$ , such that  $A = \sigma_{d,d}(a)$  and  $B = \sigma_{d,d}(b)$ , and where  $t \in (0, 1)$ . For any normalized vector  $|\eta\rangle \in \mathbb{C}^{2d}$  we have

$$\langle\eta|\sigma_{d,d}(\pi)\eta\rangle = t\langle\eta|A\eta\rangle + (1-t)\langle\eta|B\eta\rangle. \quad (4.45)$$

If  $\langle\eta|\sigma_{d,d}(\pi)\eta\rangle = 0$ , then  $\langle\eta|A\eta\rangle = 0$  and  $\langle\eta|B\eta\rangle = 0$ ; this shows that the supports of  $A$  and  $B$  are contained in the support of  $\sigma_{d,d}(\pi)$ . Now, pick  $\eta$  in the support of  $\sigma_{d,d}(\pi)$  so that  $\langle\eta|\sigma_{d,d}(\pi)\eta\rangle = 1$ ,

which implies that  $\langle \eta|A\eta \rangle = \langle \eta|B\eta \rangle = 1$ . Therefore,  $A = B = \sigma_{d,d}(\pi)$ , contradicting our assumption of a proper convex combination for  $\sigma_{d,d}(\pi)$ . We conclude that the extreme points of  $\sigma_{d,d}(\mathcal{L}_+^1(\mathbb{H}^d))$  are subnormalized rank-2 projectors that are the images of the extreme points of  $\mathcal{L}_+^1(\mathbb{H}^d)$ .  $\square$

Lemma 4.2.1 dictates that a subnormalization of the symplectic embedding of a quaternionic quantum state space is a convex subset of a complex quantum state space, and Lemma 4.2.2 characterizes the extreme points of that convex set. The next step towards proving that any quaternionic quantum description of a physical process has an equivalent complex quantum description is to show that quaternionic quantum channels correspond to completely positive trace preserving maps on complex matrices.

**Lemma 4.2.3** *Symplectic embeddings of quaternionic quantum channels:*

Let  $\Phi : \mathcal{M}_{d,d}(\mathbb{H}) \longrightarrow \mathcal{M}_{p,p}(\mathbb{H})$  be a completely positive trace preserving quaternionic map whose action on  $\rho \in \mathcal{M}_{d,d}(\mathbb{H})_{sa}$  is defined in terms of  $\{A_1, \dots, A_n\} \subset \mathcal{M}_{p,d}(\mathbb{H})$  via

$$\Phi(\rho) = \sum_{r=1}^n A_r \rho A_r^*, \quad (4.46)$$

where

$$\sum_{r=1}^n A_r^* A_r = \mathbf{1}_{\mathbb{H}^d}. \quad (4.47)$$

Then the map  $\Theta : \mathcal{M}_{2d,2d}(\mathbb{C}) \longrightarrow \mathcal{M}_{2p,2p}(\mathbb{H})$  whose action on  $x \in \mathcal{M}_{2d,2d}(\mathbb{C})$  is defined in terms of  $\{\psi_{p,d}(A_1), \dots, \psi_{p,d}(A_n)\} \subset \mathcal{M}_{2p,2d}(\mathbb{C})$  – where  $\psi_{d,d}$  is the symplectic embedding defined in (4.6) – via

$$\Theta(x) = \sum_{r=1}^n \psi_{p,d}(A_r) x (\psi_{p,d}(A_r))^*, \quad (4.48)$$

is a completely positive trace preserving complex map.

*Proof:*

By Choi's theorem on completely positive maps [17], we have that any map of the form (4.48) is completely positive. It remains to show that  $\Theta$  is trace preserving. Observe that (4.47) implies that

$$\begin{aligned} \sum_{r=1}^n (\psi_{p,d}(A_r))^* \psi_{p,d}(A_r) &= \sum_r \psi_{d,p}(A_r^*) \psi_{p,d}(A_r) \\ &= \psi_{d,d} \left( \sum_{r=1}^n A_r^* A_r \right) \\ &= \psi_{d,d}(\mathbf{1}_{\mathbb{H}^d}) \\ &= \mathbf{1}_{\mathbb{C}^{2d}}, \end{aligned} \quad (4.49)$$

where we have used (4.7) and (4.8). Next, let  $x \in \mathcal{M}_{2d,2d}(\mathbb{C})$  and compute

$$\begin{aligned}
\mathrm{tr}(\Theta(x)) &= \mathrm{tr}\left(\sum_{r=1}^n \psi_{p,d}(A_r)x(\psi_{p,d}(A_r))^*\right) \\
&= \sum_{r=1}^n \mathrm{tr}\left(\psi_{p,d}(A_r)x(\psi_{p,d}(A_r))^*\right) \\
&= \sum_{r=1}^n \mathrm{tr}\left((\psi_{p,d}(A_r))^* \psi_{p,d}(A_r)x\right) \\
&= \mathrm{tr}\left(\left(\sum_{r=1}^n (\psi_{p,d}(A_r))^* \psi_{p,d}(A_r)\right)x\right) \\
&= \mathrm{tr}(x),
\end{aligned} \tag{4.50}$$

where we have applied linearity of the trace, Lemma 3.2.1, and (4.49), finishing the proof.  $\square$

With Lemma 4.2.3, we have established that the symplectic embedding of a completely positive trace preserving quaternionic map corresponds to a quantum channel in usual complex quantum information theory. In 2000, Kossakowski proved a statement of a different variety equating the action of every decomposable complex map<sup>1</sup> on a complex matrix  $x$  to the *complex projection* of the action of a completely positive quaternionic map on  $x$  [56]. Specifically, Kossakowski showed that the action of a decomposable complex map  $\alpha : \mathcal{M}_{d,d}(\mathbb{C}) \rightarrow \mathcal{M}_{d,d}(\mathbb{C})$  on  $x \in \mathcal{M}_{d,d}(\mathbb{C})$  is equivalent to  $\frac{1}{2}(\phi(x) - i\phi(x)i)$  for some completely positive quaternionic map  $\phi : \mathcal{M}_{d,d}(\mathbb{H}) \rightarrow \mathcal{M}_{d,d}(\mathbb{H})$ . Asorey et al. went on to show that the complex projection of any completely positive quaternionic map is positive, but not necessarily completely positive [6].

We now turn our attention to quaternionic quantum measurements and prove that quaternionic POVMs correspond to complex quantum measurements.

**Lemma 4.2.4** *Symplectic embeddings of quaternionic quantum measurements:*

*If  $X = \{E_1, \dots, E_n\} \subset \mathcal{E}(\mathbb{H}^d)$  is a quaternionic POVM quantum measurement, then*

$$\{\psi_{d,d}(E_1), \dots, \psi_{d,d}(E_n)\} \subset \mathcal{E}(\mathbb{C}^{2d}) \tag{4.51}$$

*where  $\psi_{d,d}$  is the symplectic embedding defined in (4.6). Moreover,  $\psi_{d,d}(X)$  is a complex POVM quantum measurement.*

*Proof:*

Applying the Theorem 2.5.4 to  $E_r \in \mathcal{E}(\mathbb{H}^d)$  we have that

$$E_r = \sum_{r=1}^d |\phi_r\rangle \mu_r \langle \phi_r| \tag{4.52}$$

---

<sup>1</sup>The general form of a decomposable complex map is given by the sum of completely positive and completely copositive complex maps [69].

with  $0 \leq \mu_r \leq 1$ . Letting  $\Pi_{\phi_r}^2 = \Pi_{\phi_r} = \psi_{d,d}(|\phi_r\rangle\langle\phi_r|)$  it follows  $\forall \eta \in \mathbb{C}^{2d}$  that

$$0 \leq \langle \eta | \psi_{d,d}(E_r) \eta \rangle = \left\langle \eta \left| \sum_{r=1}^d \mu_r \Pi_{\phi_r} \eta \right. \right\rangle = \sum_{r=1}^d \langle \eta | \Pi_{\phi_r} \eta \rangle \mu_r \leq 1, \quad (4.53)$$

establishing that  $\psi_{d,d}(E_r) \in \mathcal{E}(\mathbb{C}^{2d})$ . It remains to show that the elements of  $\psi_{d,d}(X)$  resolve the identity:

$$\sum_{r=1}^n \psi_{d,d}(E_r) = \psi_{d,d}\left(\sum_{r=1}^d E_r\right) = \psi_{d,d}(\mathbb{1}_{\mathbb{H}^d}) = \mathbb{1}_{\mathbb{C}^{2d}}, \quad (4.54)$$

as required, where we have applied Lemma 4.1.3.  $\square$

Equipped with Lemma 4.2.1, Lemma 4.2.3, and Lemma 4.2.4, we are now in the position to prove our main result in Theorem 4.2.5; wherein  $\mathfrak{P}$ ,  $\mathfrak{S}$ ,  $\mathfrak{C}$ , and  $\mathfrak{M}$  refer to, respectively, a physical preparation device, a physical system, a physical transformation channel, and a physical measurement device as depicted in Figure 4.1.

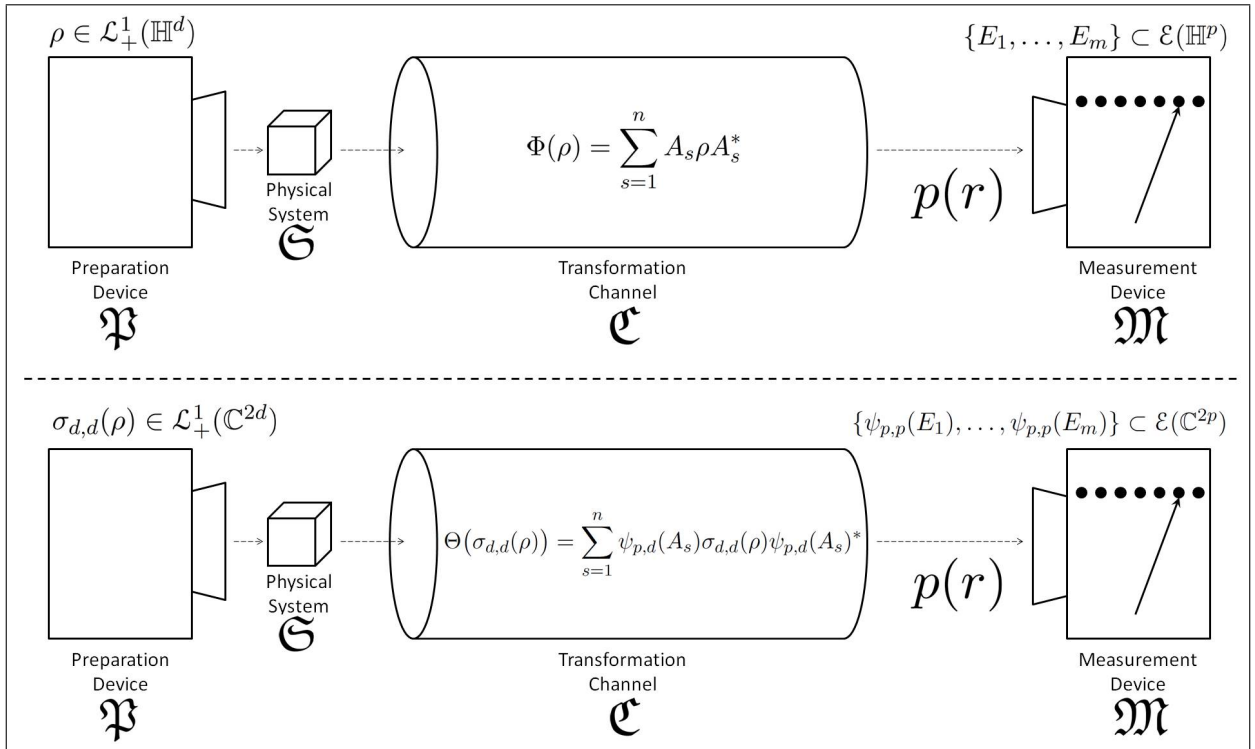


Figure 4.1: One experiment, two equivalent descriptions

**Theorem 4.2.5** *Quaternionic Quantum Theory is consistent with Complex Quantum Theory:*

Let  $\mathfrak{P}$  be a physical preparation device that outputs a physical system  $\mathfrak{S}$ . Let  $\mathfrak{C}$  be a physical transformation channel that acts on  $\mathfrak{S}$  and outputs a physical system  $\mathfrak{C}(\mathfrak{S})$ . Let  $\mathfrak{M}$  be a physical measurement device that acts on  $\mathfrak{C}(\mathfrak{S})$  and registers one classical outcome  $r \in \{1, \dots, m\}$ . Let  $\mathcal{P}$  represent the aforementioned generic physical scenario. Let the triple  $(\rho, \Phi, \mathbf{X})$  represent a quaternionic quantum description of  $\mathcal{P}$ , where  $\rho \in \mathcal{L}_+^1(\mathbb{H}^d)$  is an initial associated quaternionic quantum state for  $\mathfrak{S}$ ; where

$$\Phi : \mathcal{M}_{d,d}(\mathbb{H}) \rightarrow \mathcal{M}_{p,p}(\mathbb{H}) \quad (4.55)$$

is a completely positive trace preserving quaternionic map whose whose action on the initial associated state  $\rho$  is defined via

$$\Phi(\rho) = \sum_{s=1}^n A_s \rho A_s^*, \quad (4.56)$$

to yield a quaternionic quantum state  $\Phi(\rho) \in \mathcal{L}_+^1(\mathbb{H}^p)$  associated with  $\mathfrak{C}(\mathfrak{S})$ , such that  $\{A_1, \dots, A_n\} \subset \mathcal{M}_{p,d}(\mathbb{H})$  admit

$$\sum_{s=1}^n A_s^* A_s = \mathbf{1}_{\mathbb{H}^d}; \quad (4.57)$$

and where  $\mathbf{X} = \{E_1, \dots, E_m\} \subset \mathcal{E}(\mathbb{H}^p)$  is a quaternionic POVM comprised of quaternionic quantum effects corresponding to the outcomes of  $\mathfrak{M}$ , where the probabilities for the outcomes of  $\mathfrak{M}$  are given  $\forall r \in \{1, \dots, m\}$  by

$$p(r) = \text{tr}\left(E_r \Phi(\rho)\right). \quad (4.58)$$

Then there exists a triple  $(\sigma_{d,d}(\rho), \Theta, \psi_{p,p}(\mathbf{X}))$  representing a complex quantum description of  $\mathcal{P}$ , where  $\sigma_{d,d}(\rho) \in \mathcal{L}_+^1(\mathbb{C}^{2d})$  is an initial associated complex quantum state for  $\mathfrak{S}$ ; where

$$\Theta : \mathcal{M}_{2d,2d}(\mathbb{C}) \rightarrow \mathcal{M}_{2p,2p}(\mathbb{C}) \quad (4.59)$$

is a completely positive trace preserving complex map whose whose action on the initial associated state  $\sigma_{d,d}(\rho)$  is defined via

$$\Theta(\sigma_{d,d}(\rho)) = \sum_{s=1}^n \psi_{p,d}(A_s) \sigma_{d,d}(\rho) \psi_{p,d}(A_s)^*, \quad (4.60)$$

to yield a complex quantum state  $\Theta(\sigma_{d,d}(\rho)) \in \mathcal{L}_+^1(\mathbb{C}^{2p})$  associated with  $\mathfrak{C}(\mathfrak{S})$ , such that  $\{\psi_{p,d}(A_1), \dots, \psi_{p,d}(A_n)\} \subset \mathcal{M}_{2p,2d}(\mathbb{C})$  admit

$$\sum_{s=1}^n \psi_{p,d}(A_s)^* \psi_{p,d}(A_s) = \mathbf{1}_{\mathbb{C}^{2d}}; \quad (4.61)$$

and where  $\psi_{p,p}(\mathbf{X}) = \{\psi_{p,p}(E_1), \dots, \psi_{p,p}(E_m)\} \subset \mathcal{E}(\mathbb{C}^{2p})$  is a complex POVM comprised of complex quantum effects corresponding to the outcomes of  $\mathfrak{M}$ , where the probabilities for the outcomes of  $\mathfrak{M}$  are given  $\forall r \in \{1, \dots, m\}$  by

$$q(r) = \text{tr}\left(\psi_{p,p}(E_r) \Theta(\sigma_{d,d}(\rho))\right), \quad (4.62)$$

such that  $\forall r \in \{1, \dots, m\}$

$$q(r) = p(r). \quad (4.63)$$

*Proof:*

We divide the proof into two parts. First, we prove that the triple  $(\sigma_{d,d}(\rho), \Theta, \psi_{p,p}(\mathbf{X}))$  is a valid complex quantum description of  $\mathcal{P}$ . Next, we prove that  $(\sigma_{d,d}(\rho), \Theta, \psi_{p,p}(\mathbf{X}))$  gives rise to the same outcome probabilities for  $\mathfrak{M}$  as the quaternionic triple  $(\rho, \Phi, \mathbf{X})$  according to (4.63).

Lemma 4.2.1 implies that  $\sigma_{d,d}(\rho)$  is a valid complex quantum state:  $\sigma_{d,d}(\rho) \in \mathcal{L}_+^1(\mathbb{C}^{2d})$ . Lemma 4.2.3 implies that  $\Theta$ , whose action is defined according to (4.59), (4.60), and (4.61), is a completely positive trace preserving complex map; thus,  $\Theta$  is a valid complex quantum channel and  $\Theta(\sigma_{d,d}(\rho))$  is a valid complex quantum state:  $\Theta(\sigma_{d,d}(\rho)) \in \mathcal{L}_+^1(\mathbb{C}^{2p})$ . Lemma 4.2.4 implies that  $\psi_{p,p}(\mathbf{X})$  is a valid complex POVM quantum measurement. Therefore, the triple  $(\sigma_{d,d}(\rho), \Theta, \psi_{p,p}(\mathbf{X}))$  is a valid complex quantum description of  $\mathcal{P}$ , completing the first part of the proof.

The complex triple  $(\sigma_{d,d}(\rho), \Theta, \psi_{p,p}(\mathbf{X}))$  gives rise to outcome probabilities  $q(r)$  for  $\mathfrak{M}$  according to the usual Born rule in complex quantum theory according to (4.62). Applying Lemma 4.1.4 and Lemma 4.1.5, as well as 2.51, we calculate  $\forall r \in \{1, \dots, m\}$ :

$$\begin{aligned}
q(r) &= \text{tr}\left(\psi_{p,p}(E_r)\Theta(\sigma_{d,d}(\rho))\right) \\
&= \text{tr}\left(\psi_{p,p}(E_r)\left(\sum_{s=1}^n \psi_{p,d}(A_s)\sigma_{d,d}(\rho)\psi_{p,d}(A_s)^*\right)\right) \\
&= \sum_{s=1}^n \text{tr}\left(\psi_{p,p}(E_r)\psi_{p,d}(A_s)\sigma_{d,d}(\rho)\psi_{p,d}(A_s)^*\right) \\
&= \sum_{s=1}^n \text{tr}\left(\psi_{p,p}(E_r)\psi_{p,d}(A_s)\sigma_{d,d}(\rho)\psi_{d,p}(A_s^*)\right) \\
&= \frac{1}{2} \sum_{s=1}^n \text{tr}\left(\psi_{p,p}(E_r)\psi_{p,d}(A_s)\psi_{d,d}(\rho)\psi_{d,p}(A_s^*)\right) \\
&= \frac{1}{2} \sum_{s=1}^n \text{tr}\left(\psi_{p,p}(E_r)\psi_{p,d}(A_s)\psi_{d,p}(\rho A_s^*)\right) \\
&= \frac{1}{2} \sum_{s=1}^n \text{tr}\left(\psi_{p,p}(E_r)\psi_{p,p}(A_s\rho A_s^*)\right) \\
&= \sum_{s=1}^n \text{tr}\left(E_r(A_s\rho A_s^*)\right) \\
&= \text{tr}\left(E_r\left(\sum_{s=1}^n A_s\rho A_s^*\right)\right) \\
&= \text{tr}\left(E_r\Phi(\rho)\right) \\
&= p(r),
\end{aligned} \tag{4.64}$$

establishing that the outcome probabilities for the final measurement process in  $\mathcal{P}$  given by the quaternionic triple and the complex triple are identical. The proof is complete.  $\square$



With the proof of Theorem 4.2.5 now complete, we note that if  $\Phi$  is restricted to be unitary, *i.e.* if

$$\Phi : \mathcal{M}_{d,d}(\mathbb{H}) \rightarrow \mathcal{M}_{d,d}(\mathbb{H}) \quad (4.65)$$

acts on  $\rho \in \mathcal{L}_+^1(\mathbb{H}^d)$  according to  $U \in \text{Sp}(d)$  via

$$\Phi(\rho) = U\rho U^*, \quad (4.66)$$

and if  $X$  is restricted to be a PVM quaternionic quantum measurement, *i.e.* if

$$X = \{\pi_1^2 = \pi_1, \dots, \pi_d^2 = \pi_d\} \subset \mathcal{E}(\mathbb{H}^d), \quad (4.67)$$

such that  $\forall r, s \in \{1, \dots, d\}$ :

$$\text{tr}(\pi_r \pi_s) = \delta_{rs}, \quad (4.68)$$

then Theorem 4.2.5 reproduces the simulation given by Fernandez and Schneeberger in [25].

Theorem 4.2.5 establishes that any description of a generic physical process  $\mathcal{P}$  given in terms of quaternionic quantum formalism is equivalent to a description of  $\mathcal{P}$  given in terms of the language of usual complex quantum information theory. Put otherwise, any process described by quaternionic quantum theory can be simulated using states and operations in usual complex quantum information theory. Furthermore, given a quaternionic quantum triple  $(\rho, \Phi, X)$  describing  $\mathcal{P}$ , Theorem 4.2.5 prescribes an *explicit construction* for a complex quantum triple  $(\sigma_{d,d}(\rho), \Theta, \psi_{p,p}(X))$  that yields identical probabilities for the outcomes of  $\mathfrak{M}$ . We conclude that the formalism of usual complex quantum information theory is sufficient for an information-theoretic description – phrased in terms probabilities for measurement outcomes – of all physical processes that can be described via the quaternionic quantum formalism presented in Chapter 3. In closing this chapter, we recall that John Archibald Wheeler, one of the founding fathers of complex quantum information theory, emphasized that such descriptions are fundamental:

“The thesis *it from bit*: every *it*, every particle, every field of force, even the spacetime continuum itself, derives its way of action and its very existence entirely, even if in some contexts indirectly, from the detector-elicited answers to yes or no questions, binary choices, *bits*. Otherwise stated, all things physical, all *its*, must in the end submit to an information-theoretic description.”

– J. A. Wheeler, 1992 [72].

## Chapter 5

# Quaternionic Quantum Bayesian State Spaces

In this chapter, we consider quaternionic quantum state spaces taken as convex sets of unit-trace positive semi-definite linear operators on finite  $d$ -dimensional right quaternionic modules. These state spaces correspond to quads in the quaternionic quantum formalism considered in Chapter 3 and Chapter 4. The usual Quantum Bayesian, or QBist reformulation of complex quantum theory expresses quantum states as points on a probability simplex over  $d^2$  outcomes for some fixed reference symmetric informationally complete complex quantum measurement. We investigate the status of quaternionic quantum theory within a generalized QBist framework for the reconstruction of quantum state spaces suggested by Fuchs and Schack [36][37]. We explore the possibility of expanding quaternionic quantum states in terms of maximally symmetric bases for self-adjoint quaternionic matrices, and we chart the geometry of quaternionic quantum state spaces on the corresponding probability simplexes. We also analyze the geometry of symplectic embeddings of quaternionic quantum state spaces on probability simplexes for symmetric informationally complete complex quantum measurements.

### 5.1 Introduction to the Quantum Bayesian Program

The quantum information revolution has brought with it a new approach to understanding the foundations of quantum theory. In the early dawn of the quantum information age, John A. Wheeler pointed out several interconnections between quantum physics, information theory, and the nature of existence [77]. Wheeler envisioned existence as an information-theoretic concept: the reality of a physical system being *generated* via quantum measurements, not everlasting physical law. Quantum theory, on that view, prescribes a probability calculus for the use of agents in Wheeler’s ‘participatory Universe’. Wheeler himself submitted that one day all of physics would be expressed in the language of information. Some thirty years have now past, and though it is not yet Wheeler’s tomorrow, much progress has been made in the reformulation of quantum theory solely in terms of probabilities.

Among those considering quantum theory as a probability calculus, and in the light of quantum information [30], are those espousing the Quantum Bayesian interpretation of quantum states originally developed by C. M. Caves, C. A. Fuchs, and R. Schack [31][14][30][34][15][35][36][5] [37][33][32]. From the Quantum Bayesian, or QBist point of view, the quantum state assigned to an objectively existent physical system represents an agent’s personalist degrees of belief with regard to it – particularly in light of the participatory character of the Universe that Wheeler highlighted. The Quantum Bayesian approach towards understanding the foundations of quantum theory, dubbed QBism [33], adopts the Born rule as an addition to the classical probability rules required by Dutch-book coherence [36]. This additional structure in fact causes a restriction on the class of priors offered by Bayesian probability theory.

The character of the probability calculus prescribed by quantum theory is manifest in the geometry of its state space. The mathematical formalism of QBism replaces the usual space of finite  $d$ -dimensional complex quantum states – that is, the unit-trace positive semi-definite linear operators on a  $d$ -dimensional complex Hilbert space – with a QBist state space. A QBist state space is a convex subset of a probability simplex over  $d^2$  outcomes for some fixed reference symmetric informationally complete quantum measurement [76][64]. Appleby et al. have recently explored the structure of QBist state spaces [5]. Their approach is motivated by a QBist framework for quantum state spaces suggested by Fuchs and Schack [36][37]. There is a parameter,  $q$ , in the QBist framework, which must be set  $q = 2$  in order to recover usual quantum state space. It has been suggested that the case  $q = 4$  may correspond to state spaces in quaternionic quantum theory [37].

In the sections to follow, we strive to answer the following question: does quaternionic quantum theory fit into the QBist framework? We shall consider the general case of a quantum state space given by a convex set of unit-trace positive semi-definite linear operators on a finite dimensional right quaternionic module, as defined in the preceding chapters and denoted by  $\mathcal{L}_+^1(\mathbb{H}^d)$ .

## 5.2 The QBist Framework

In the usual formulation of finite-dimensional quantum theory, the space of valid quantum states is given by all unit-trace  $\rho \in \mathcal{L}_+^1(\mathbb{C}^d)$ . If<sup>1</sup> there exists a symmetric informationally complete (SIC) positive operator valued measure [64] in dimension  $d$ ,

$$\text{SIC} = \left\{ \frac{1}{d} \Pi_r = \frac{1}{d} |\phi_r\rangle\langle\phi_r| \mid |\phi_r\rangle \in \mathbb{C}^d \wedge \|\langle\phi_r|\phi_s\rangle\|^2 = \frac{d\delta_{rs} + 1}{d + 1} \right\}_{r=1}^{d^2}, \quad (5.1)$$

then one can establish an injective map[36]

$$\omega : \mathcal{L}_+^1(\mathbb{C}^d) \rightarrow \Delta_{d^2} = \left\{ \vec{p} \in \mathbb{R}^{d^2} \mid p(r) \geq 0 \wedge \sum_{r=1}^{d^2} p(r) = 1 \right\}, \quad (5.2)$$

---

<sup>1</sup>It has been conjectured that SICs exist in all finite dimensions [76]. There is numerical evidence of SIC existence for  $d = 2 - 67$  given by Scott and Grassl in [65], where analytic constructions are also found for  $d = 2 - 15, 19, 24, 35$ , and 48.

defined via

$$\omega(\rho) = \vec{p}, \quad (5.3)$$

with components

$$p(r) = \text{tr}\left(\frac{1}{d}\Pi_r\rho\right). \quad (5.4)$$

In this way, one can identify any quantum state  $\rho$  with a unique point on the probability simplex  $\Delta_{d^2}$  over  $d^2$  outcomes for a SIC quantum measurement. Moreover, one can expand any quantum state as [36]:

$$\rho = \sum_{r=1}^{d^2} \left( (d+1)p(r) - \frac{1}{d} \right) \Pi_r, \quad (5.5)$$

from which it follows that the probabilities  $f(s)$  for the outcomes of any quantum measurement associated with a POVM  $\mathcal{F} = \{F_s \in \mathcal{L}_+(\mathbb{C}^d) \mid \sum_s F_s = \mathbb{1}\}$  are given by [36]:

$$f(s) = \sum_{r=1}^{d^2} \left( (d+1)p(r) - \frac{1}{d} \right) t(s|r), \quad (5.6)$$

with

$$t(s|r) = \text{tr}(F_s\Pi_r). \quad (5.7)$$

Fuchs and Schack refer to (5.6) as the *Urgleichung* – the primal equation for the quantum probability calculus. The *Urgleichung* expresses the Born rule in the language of probabilities. From the QBist viewpoint, it is the Born rule alone that restricts classical Bayesian probability theory to yield quantum theory, so (5.6) is an equation of great importance. We have pointed out that the SIC representation of quantum states can be used to arrive at (5.6). Fuchs and Schack have sought to *derive* (5.6), without any reference to complex Hilbert spaces, starting from the *generalized* *Urgleichung* [36][37]

$$f(s) = \sum_{r=1}^n (\alpha p(r) - \beta) t(s|r). \quad (5.8)$$

In the generalized *Urgleichung* (5.8),  $f(s)$  represent prior probabilities for a factual measurement with  $d$  outcomes,  $p(r)$  represent prior probabilities for a counterfactual reference measurement with  $n$  outcomes, and  $t(s|r)$  represents the conditional probability for factual outcome  $s$  given counterfactual outcome  $r$ . The parameters  $\alpha, \beta \in \mathbb{R}_+$  in (5.8) set the form of the corresponding theory. In the derivation given by Fuchs and Schack [36][37], there are initially no relationships imposed on  $d, n, \alpha$  and  $\beta$ , but it follows immediately from (5.8) that

$$\alpha = n\beta + 1. \quad (5.9)$$

Referring the reader to [36] and [37], we note that a set of assumptions motivated by the Bayesian interpretation of probability lead to the relation

$$\frac{d}{n}\alpha - \beta = 1, \quad (5.10)$$

where  $n$  is given by

$$n(q, d) = \frac{1}{2}qd(d-1) + d, \quad (5.11)$$

with  $q \in \mathbb{Z}_+$  parameterizing the character of the calculus. The choice  $q = 2$  yields the *Urgleichung*. Incidentally, we observe that if  $q = 4$ , then  $n$  corresponds to the dimension of the ambient space for the set of  $d$ -dimensional quaternionic quantum states. Also, observing that the real vector space of real symmetric matrices –  $\mathcal{M}_{d,d}(\mathbb{R})_s$  – has dimension  $\frac{1}{2}d(d+1)$ , the case  $q = 1$  could correspond to a quantum formalism where states for *redits* are taken as unit-trace positive semi-definite matrices in  $\mathcal{M}_{d,d}(\mathbb{R})_s$ . The case  $q = 0$  corresponds to the classical law of total probability; however, the correspondence is strictly formal. Indeed, classical probability theory does *not* constrain the prior probabilities for a factual measurement in terms of probabilities involving a counterfactual process via the law of total probability.

One can easily solve for  $\alpha$  and  $\beta$  to yield

$$f(s) = \sum_{r=1}^n \left( \frac{n-1}{d-1} p(r) - \frac{n-d}{n(d-1)} \right) t(s|r), \quad (5.12)$$

which reduces to

$$f(s) = \sum_{r=1}^{d(2d-1)} \left( (2d+1)p(r) - \frac{2}{2d-1} \right) t(s|r), \quad (5.13)$$

in the quaternionic case. (5.13) is the *Quaternionic Urgleichung*. In the Section 5.4, we show that one may formally recover (5.13) *if* maximally symmetric bases (analogues of SICs) exist for  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$ . We derive the structure maximally symmetric bases in Section 5.3, where, for the sake of completeness, we carry out our computations so that they apply to any of the associative normed division algebras. As such, we introduce the notation  $\mathcal{L}_+(\mathbf{R}^d)$  to denote the set of positive semi-definite matrices with  $\mathbf{R} = \mathbb{R}, \mathbb{C}$ , or  $\mathbb{H}$ .<sup>2</sup> Also, noting that  $\mathcal{L}_+(\mathbb{R}^d) \subset \mathcal{L}_+(\mathbb{C}^d) \subset \mathcal{L}_+(\mathbb{H}^d) \subset \mathcal{M}_{d,d}(\mathbb{H})_{sa}$ , we adopt the quaternionic trace and associated canonical inner product on  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$  defined in the preceding chapters.

### 5.3 Maximally Symmetric Bases

The *Urgleichung* (5.6) can be derived from a SIC representation of quantum states. It stands to reason that it may be possible to derive the generalized *Urgleichung* by expanding general (re-, qu-, or qua-)dit quantum states in terms of a *maximally symmetric basis*. How should one quantify such symmetry? In the complex case, Appleby, Dang, and Fuchs proposed a measure to quantify the symmetry of a set of  $d^2$  positive semi-definite complex linear operators [4]. Their measure has a lower bound saturated only by SICs. We now proceed to generalize their result. Let

$$\mathcal{A} = \{A_r \in \mathcal{L}_+(\mathbf{R}^d) \mid \text{tr}(A_r^2) = 1\}_{r=1}^n \quad (5.14)$$

be a set of  $n$ , as in (5.11), positive semi-definite linear operators normalized with respect to the norm induced by the canonical inner product. We say that  $\mathcal{A}$  is *maximally symmetric* when its elements minimize the measure:

$$K_t = \sum_{r=1}^n \sum_{s \neq r}^n |(A_r, A_s)|^t = \sum_{r=1}^n \sum_{s \neq r}^n (\text{tr}(A_r A_s))^t \quad (5.15)$$

---

<sup>2</sup>By a positive semi-definite real matrix we shall mean a *symmetric* positive semi-definite real matrix.

for any real number  $t \geq 1$ . Our measure  $K_t$  is of exactly the same form as the one originally proposed by Appleby, Dang, and Fuchs in the complex case, and is in some sense a measure of how close  $\mathcal{A}$  comes to being an orthonormal set [4]. We now prove the following theorem:

**Theorem 5.3.1** *Maximally Symmetric Bases:*

If  $\{A_r \in \mathcal{L}_+(\mathbf{R}^d) \mid \text{tr}(A_r^2) = 1\}_{r=1}^n$ , then

$$K_t \geq \frac{n(n-d)^t}{d^t(n-1)^{t-1}}. \quad (5.16)$$

Moreover, the lower bound in (5.16) is saturated iff

$$\mathcal{A} = \left\{ \Pi_r = |\phi_r\rangle\langle\phi_r| \in \mathcal{L}_+(\mathbf{R}^d) \mid \text{tr}(\Pi_r \Pi_s) = \frac{n-d}{d(n-1)} \left( 1 + \delta_{rs} \frac{n(d-1)}{n-d} \right) \right\}_{r=1}^n. \quad (5.17)$$

*Proof:*

We will prove that (5.16) holds using an argument in the same spirit as the one given in [4]. We consider first the case  $t = 1$ :

$$K_1 = \sum_{r=1}^n \sum_{s \neq r}^n \text{tr}(A_r A_s). \quad (5.18)$$

Now we define

$$G = \sum_{r=1}^n A_r, \quad (5.19)$$

and we apply the spectral theorem to get

$$G = \sum_{m=1}^d |m\rangle \lambda_m \langle m|, \quad (5.20)$$

where  $\lambda_m \in \mathbb{R}_+$  and  $\langle m|m'\rangle = \delta_{mm'}$ , from which it follows that

$$G^2 = \sum_{m=1}^d |m\rangle \lambda_m^2 \langle m|. \quad (5.21)$$

It also follows that

$$\text{tr}(G) = \sum_{m=1}^d \lambda_m, \quad (5.22)$$

and

$$\text{tr}(G^2) = \sum_{m=1}^d \lambda_m^2. \quad (5.23)$$

Equivalently,

$$\text{tr}(G^2) = \sum_{r=1}^n \sum_{s=1}^n \text{tr}(A_r A_s). \quad (5.24)$$

Therefore,

$$K_1 = \text{tr}(G^2) - n. \quad (5.25)$$

Applying the Cauchy-Schwarz inequality to the (real and positive) eigenvalues of  $G$  we get

$$\text{tr}(G^2) \geq \frac{1}{d} \left( \text{tr}(G) \right)^2. \quad (5.26)$$

By our demand that  $\text{tr}(A_r^2) = 1$ , it follows that  $\text{tr}(A_r) \geq 1$ <sup>3</sup> and so

$$\text{tr}(G^2) \geq \frac{n^2}{d}. \quad (5.27)$$

Therefore

$$K_1 \geq \frac{n^2}{d} - n. \quad (5.28)$$

We now show that equality obtains if and only if  $\forall r : \text{tr}(A_r) = 1$  and  $G = \frac{n}{d} \mathbb{1}$ . To prove sufficiency we note that

$$\text{tr}(A_r) = 1 \implies \text{tr}(G) = n, \quad (5.29)$$

and we note that

$$G = \frac{n}{d} \mathbb{1} \implies \text{tr}(G^2) = \frac{n^2}{d}. \quad (5.30)$$

To prove necessity, we observe that saturation of the bound implies that

$$\text{tr}(G^2) = \frac{1}{d} (\text{tr}(G))^2 = \frac{n^2}{d}, \quad (5.31)$$

so  $\text{tr}(G) = n$ , and together with  $\text{tr}(A_r) \geq 1$  we have that  $\text{tr}(A_r) = 1$ . It remains to show that the eigenvalues  $\lambda_m$  of  $G$  are all equal to  $\frac{n}{d}$ . We have that

$$\sum_{m=1}^d \lambda_m^2 = \frac{1}{d} \left( \sum_{m=1}^d \lambda_m \right)^2, \quad (5.32)$$

and  $\lambda_m \in \mathbb{R}_+$ , and so by the Cauchy-Schwarz inequality the  $\lambda_m$  are equal to the same value  $\lambda = \frac{n}{d}$ . Therefore,  $A_r$  are rank-1 projectors and the sum of their subnormalization by  $\frac{d}{n}$  resolves the identity on  $\mathbf{R}^d$ .

Consider now the case  $t > 1$ . Let  $x = \text{tr}(A_r A_s) \in \mathbb{R}_+$ . The function  $f(x) = x^t$  is strictly convex for  $t > 1$ . Recall the Jensen inequality [53] for a convex function  $f$

$$\sum_{r=1}^N a_r f(x_r) \geq \left( \sum_{r=1}^N a_r \right) f \left( \frac{\sum_{r=1}^N a_r x_r}{\sum_{r=1}^N a_r} \right) \quad (5.33)$$

Identifying  $a_r = 1$  and  $N = n(n-1)$ , and after some algebra we get

$$K_t \geq \frac{K_1^t}{(n(n-1))^{t-1}} \geq \frac{n(n-d)^t}{d^t (n-1)^{t-1}}. \quad (5.34)$$

---

<sup>3</sup>To see this, denote the eigenvalues of  $A_r$  by  $\lambda_r \in \mathbb{R}_+$ . By our demand  $\sum_r \lambda_r^2 = 1$  this implies  $\lambda_r \leq 1$ , for suppose  $\lambda_r > 1$  in which case our demand fails. It follows that  $\sum_r \lambda_r \geq 1$ , since  $\lambda_r \leq 1 \implies \lambda_r^2 \leq \lambda_r$  and thus  $1 = \sum_r \lambda_r^2 \leq \sum_r \lambda_r$ .

The Jensen inequality [53] also implies that the lower bound of (5.16) is saturated if and only if  $\text{tr}(A_r A_s)$  is equal to a unique constant  $\mu^2$  for all  $r \neq s$ . We can compute the value of  $\mu^2$  given that

$$\mathbb{1}_{\mathbf{R}^d} = \frac{d}{n} \sum_{r=1}^n A_r, \quad (5.35)$$

and so by operating on (5.35) with  $A_s$  and taking the trace of both sides it follows that

$$\frac{n}{d} \text{tr}(A_s) = \frac{n}{d} = \sum_{r=1}^n \text{tr}(A_r A_s) = \mu^2(n-1) + 1 \implies \mu^2 = \frac{n-d}{d(n-1)}. \quad (5.36)$$

Therefore, maximally symmetric  $\mathcal{A}$  take the form

$$\mathcal{A} = \left\{ \Pi_r = |\phi_r\rangle\langle\phi_r| \in \mathcal{L}_+(\mathbb{R}^d) \mid \text{tr}(A_r A_s) = \frac{n-d}{d(n-1)} \left( 1 + \delta_{rs} \frac{n(d-1)}{n-d} \right) \right\}_{r=1}^n, \quad (5.37)$$

where we have denoted the rank-1 projectors  $A_r$  by  $\Pi_r$ , finishing the proof.  $\square$

**Corollary 5.3.2** *The elements of a maximally symmetric set are linearly independent:*

*Proof:* For the proof, we follow [36] by taking  $\alpha_r \in \mathbb{R}$  and setting

$$0 = \sum_{r=1}^n \Pi_r \alpha_r. \quad (5.38)$$

Taking the trace of both sides of (5.38) we get

$$0 = \sum_{r=1}^n \alpha_r. \quad (5.39)$$

Now, operating on (5.38) with  $\Pi_s$ , and taking the trace of both sides, we use (5.39) to get  $\alpha_s = 0$ .  $\square$

It follows immediately from Corollary 5.3.2 that maximally symmetric sets with  $n$  as in (5.11) form *bases* for the corresponding real vector spaces  $\mathcal{M}_{d,d}(\mathbf{R})_{sa}$ , with forms given in Table 5.1. An obvious question at this stage is: do maximally symmetric bases *exist*? For the case of  $\mathbf{R} = \mathbb{R}$ , they exist for  $d = 2$  (the regular 2-simplex) and for  $d = 3$  (with projections  $\Pi_r$  onto vectors defining the vertices of an regular icosahedron). For cases where  $d > 3$ , it can be shown that a necessary condition for the existence of a maximally symmetric basis for  $\mathcal{M}_{d,d}(\mathbb{R})_{sa}$  is that  $d+2$  must be the square of an odd integer [54]. For the case of  $\mathbf{R} = \mathbb{C}$ , maximally symmetric bases are SICs. In the quaternionic case of  $\mathbf{R} = \mathbb{H}$ , much less is known. However, it is at least known that maximally symmetric bases for  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$  *do* exist for  $d = 2$  and  $d = 3$  [54]. In the sections to follow, we will *assume the existence* of maximally symmetric bases for  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$ .



$\mathbf{R}$	Maximally symmetric bases
$\mathbb{R}$	$\left\{ \Pi_r =  \phi_r\rangle\langle\phi_r  \in \mathcal{L}_+(\mathbb{R}^d) \mid \text{tr}(\Pi_r\Pi_s) = \frac{1+\delta_{rs}(1+d)}{d+2} \right\}_{r=1}^{\frac{d(d+1)}{2}}$
$\mathbb{C}$	$\left\{ \Pi_r =  \phi_r\rangle\langle\phi_r  \in \mathcal{L}_+(\mathbb{C}^d) \mid \text{tr}(\Pi_r\Pi_s) = \frac{1+\delta_{rs}d}{d+1} \right\}_{r=1}^{d^2}$
$\mathbb{H}$	$\left\{ \Pi_r =  \phi_r\rangle\langle\phi_r  \in \mathcal{L}_+(\mathbb{H}^d) \mid \text{tr}(\Pi_r\Pi_s) = \frac{2+\delta_{rs}(2d-1)}{2d+1} \right\}_{r=1}^{d(2d-1)}$

Table 5.1: Forms of maximally symmetric bases for  $\mathcal{M}_{d,d}(\mathbf{R})_{sa}$  for  $\mathbf{R} = \mathbb{R}, \mathbb{C}$ , and  $\mathbb{H}$ .

## 5.4 Recovering the Quaternionic Urgleichung

In this section, we show that the existence of a maximally symmetric basis for  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$  allows one to recover the quaternionic Urgleichung (5.13). Let  $\mathcal{A}$  be a maximally symmetric basis for  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$  as in Table 5.1. From the proof of Theorem 5.3.1, we have that

$$\sum_{r=1}^d \frac{1}{(2d-1)} \Pi_r = \mathbb{1}_{\mathbb{H}^d}. \quad (5.40)$$

Also, given that  $\Pi_r^2 = \Pi_r = |\phi_r\rangle\langle\phi_r|$ , the following holds  $\forall r \in \{1, \dots, d(2d-1)\}$ :

$$\forall \xi \in \mathbb{H}^d : \langle \xi | \Pi_r \xi \rangle = |\langle \xi | \phi_r \rangle|^2 \geq 0. \quad (5.41)$$

Therefore

$$\text{HSIC} = \left\{ E_r = \frac{1}{2d-1} \Pi_r \right\}_{r=1}^{d(2d-1)} \quad (5.42)$$

is a quaternionic POVM. The quaternionic quantum effects  $E_r \in \text{HSIC}$  inherit linearly independence from the  $\Pi_r$ , and so for all quaternionic quantum states  $\rho \in \mathcal{L}_+(\mathbb{H}^d) \subset \mathcal{M}_{d,d}(\mathbb{H})_{sa}$   $\exists \{\alpha_1, \dots, \alpha_{d(2d-1)}\} \subset \mathbb{R}$  such that

$$\rho = \sum_{r=1}^{d(2d-1)} E_r \alpha_r. \quad (5.43)$$

Right-multiplying (5.43) by  $E_s$  for some arbitrary  $s \in \{1, \dots, d(2d-1)\}$  and taking the quaternionic trace of both sides we get, by virtue of the reality of  $\alpha_r$ , that

$$\text{tr}(\rho E_s) = p(s) = \sum_{r=1}^{d(2d-1)} \text{tr}(E_r E_s) \alpha_r, \quad (5.44)$$

where  $p(s)$  is the  $s^{\text{th}}$  component of the vector of probabilities  $\vec{p}$  for the outcomes of HSIC given the quaternionic quantum state  $\rho$ . Now, we write (5.44) in matrix notation

$$\vec{p} = M\vec{\alpha}, \quad (5.45)$$

where the matrix  $M \in \mathcal{M}_{d(2d-1), d(2d-1)}(\mathbb{R})_s$  has entries

$$M_{rs} = \text{tr}(E_r E_s) = \frac{1}{(2d-1)^2} \frac{2}{2d+1} \left( 1 + \delta_{rs} \frac{2d-1}{2} \right). \quad (5.46)$$

The symmetries of  $M$  prompt the ansatz  $M_{rs}^{-1} = \delta_{rs}a + b$ . The matrix inverse is unique, and solving  $MM^{-1} = \mathbb{1}_{\mathbb{R}^{d(2d-1)}}$  we calculate:

$$a = 4d^2 - 1, \quad (5.47)$$

$$b = -2. \quad (5.48)$$

Inverting (5.45) we have that

$$\vec{\alpha} = M^{-1}\vec{p}, \quad (5.49)$$

therefore

$$\rho = \sum_{r=1}^{d(2d-1)} \left( (2d+1)p(r) - \frac{2}{2d-1} \right) \Pi_r. \quad (5.50)$$

With (5.50), one sees that any quaternionic quantum state can be expanded in terms of the probabilities for the outcomes of the elements of (5.42). Moreover, the elements of (5.42) are symmetric in the sense that they admit a constant pairwise canonical inner product. Hence we refer to (5.42) as a *symmetric informationally complete quaternionic quantum measurement*.

Now, consider any quaternionic POVM  $\mathcal{F}$  with elements  $F_s$ . Right-multiplying (5.50) by  $F_s$  and taking the quaternionic trace of both sides, and identifying  $t(s|r) = \text{tr}(\Pi_r F_s)$  we recover the quaternionic Urgleichung

$$f(s) = \sum_{r=1}^{d(2d-1)} \left( (2d+1)p(r) - \frac{2}{2d-1} \right) t(s|r). \quad (5.51)$$

Therefore, under the assumption that maximally symmetric bases exist for  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$ , quaternionic quantum theory is consistent with the generalized QBist framework proposed by Fuchs and Schack with the choice  $q = 4$ . We characterize the geometry of the corresponding quaternionic QBist state spaces in Section 5.5.

## 5.5 Quaternionic Quantum Bayesian State Space Geometry

In this section, we assume the existence of an HSIC in dimension  $d$  and chart the geometry of *quaternionic Quantum Bayesian state spaces*. A quaternionic Quantum Bayesian state space is a convex subset of a probability simplex  $\Delta_{d(2d-1)}$  over  $d(2d-1)$  outcomes for an HSIC quaternionic

quantum measurement. The usual space of quaternionic quantum states  $\mathcal{L}_+^1(\mathbb{H}^d)$  can be injected into  $\Delta_{d(2d-1)}$  via the map  $\theta : \mathcal{L}_+^1(\mathbb{H}^d) \rightarrow \Delta_{d(2d-1)}$  defined  $\forall \rho \in \mathcal{L}_+^1(\mathbb{H}^d)$  via

$$\vec{p} = \theta(\rho) = \begin{pmatrix} \text{tr}(E_1\rho) \\ \vdots \\ \text{tr}(E_{d(2d-1)}\rho) \end{pmatrix}, \quad (5.52)$$

where  $E_r$  comprise an HSIC as in (5.42). In Chapter 3, we proved that the extreme points of the space of quaternionic quantum states in  $\mathcal{L}_+^1(\mathbb{H}^d)$  are completely determined by the conditions  $\text{tr}(\rho^2) = \text{tr}(\rho^3) = 1$ . These conditions translate into constraints on coherent probability assignments on HSIC probability simplexes. The unit-trace condition for quaternionic quantum states implies normalization of probabilities:

$$\text{tr}(\rho) = 1 \implies \text{tr} \left( \sum_{r=1}^{d(2d-1)} E_r \rho \right) = 1 \implies \sum_r p(r) = 1, \quad (5.53)$$

where  $p(r)$  denote the components of the image of a quaternionic quantum state under the injection defined in (5.52). The condition  $\text{tr}(\rho^2) = 1$  dictates that pure quaternionic quantum states constitute points on the surface of a sphere according to Lemma 5.5.1.

**Lemma 5.5.1** *The Quaternionic Quadratic Condition:*

*If  $\theta$  is an injection of quaternionic quantum state space into an HSIC probability simplex  $\Delta_{d(2d-1)}$  as defined in (5.52), then*

$$\forall \vec{p} \in \theta \left( \{ |\xi\rangle\langle\xi| \in \mathcal{L}_+^1(\mathbb{H}^d) \mid \xi \in \mathbb{H}^d \} \right) \subset \Delta_{d(2d-1)} \quad (5.54)$$

*the following holds:*

$$\sum_{r=1}^{2d^2-d} p(r)^2 = \frac{3}{4d^2-1}. \quad (5.55)$$

*Proof:*

We expand an arbitrary quaternionic quantum state  $\rho$  according to (5.50) and explicitly compute  $\text{tr}(\rho^2)$ :

$$1 = \frac{1}{(d-1)^2} \left( (n-1)^2 \sum_{r=1}^n \sum_{s=1}^n p(r)p(s) \text{tr}(\Pi_r \Pi_s) - \frac{2(n-1)(n-d)}{d} \sum_{r=1}^n p(r) + \frac{(n-d)^2}{d} \right), \quad (5.56)$$

where we have let  $n = d(2d-1)$ . Algebraic simplification yields (5.55), finishing the proof.  $\square$

The condition  $\text{tr}(\rho^3) = 1$  dictates that not all points on the surface of the sphere correspond to pure quaternionic quantum states according to Lemma 5.5.2 <sup>4</sup>.

<sup>4</sup>For the case  $d = 2$ , the cubic condition is in fact redundant and pure quaternionic quantum states comprise the entire surface of the sphere, as they do in usual  $\mathcal{L}_+^1(\mathbb{H}^2)$  state space (see Section 3.1).

**Lemma 5.5.2** *The Quaternionic Cubic Condition:*

If  $\theta$  is an injection of quaternionic quantum state space into an HSIC probability simplex  $\Delta_{d(2d-1)}$  as defined in (5.52), then

$$\forall \vec{p} \in \theta\left(\left\{|\xi\rangle\langle\xi| \in \mathcal{L}_+^1(\mathbb{H}^d) \mid \xi \in \mathbb{H}^d\right\}\right) \subset \Delta_{d(2d-1)} \quad (5.57)$$

the following holds:

$$\sum_{r=1}^{2d^2-d} \sum_{s=1}^{2d^2-d} \sum_{t=1}^{2d^2-d} c_{rst} p(r)p(s)p(t) = \frac{8d+19}{(2d+1)^3}, \quad (5.58)$$

where  $c_{rst} = \text{tr}(\Pi_r \Pi_s \Pi_t)$ .

*Proof:*

We expand an arbitrary quaternionic quantum state  $\rho$  according to (5.50) and explicitly compute  $\text{tr}(\rho^3)$  as

$$\left(\frac{n-1}{d-1}\right)^3 \sum_{r,s,t} c_{rst} p(r)p(s)p(t) - \frac{2(n-1)^2(n-d)}{d(d-1)^3} \sum_{r,s} p(r)p(s) \text{tr}(\Pi_r \Pi_s) + \frac{(n-d)^2(n-1)}{d^2(d-1)^3} - \frac{n-d}{d(d-1)}, \quad (5.59)$$

where  $c_{rst} = \text{tr}(\Pi_r \Pi_s \Pi_t)$  and where we have let  $n = d(2d-1)$ . We apply Lemma 5.5.1 to simplify the double sum in (5.59) and then apply routine algebraic simplification to yield (5.58), finishing the proof.  $\square$

Lemma 5.5.1 and Lemma 5.5.2 are necessary and sufficient conditions for  $\vec{p} \in \Delta_{d(2d-1)}$  to represent a pure quaternionic quantum state. All quaternionic quantum states arise as convex combinations of the extreme points satisfying (5.55) and (5.58). Therefore, Lemma 5.5.1 and Lemma 5.5.2 characterize the surface of quaternionic quantum state spaces on HSIC probability simplexes.

## 5.6 QBist Quadits inside QBist Qudits

In Chapter 4, we proved that all  $d$ -dimensional quaternionic quantum states for a physical system  $\mathfrak{S}$  can be viewed  $2d$ -dimensional complex quantum states for  $\mathfrak{S}$ . The converse does not hold. There are  $2d$ -dimensional complex quantum states that do not correspond to  $d$ -dimensional quaternionic states. In this section, we will define the boundary of quaternionic quantum state spaces inside SIC probability simplexes. Furthermore, we analyze relations between representations of quaternionic quantum states on HSIC and SIC simplexes.

Let  $\{\Pi_\alpha \in \mathcal{M}_{2d,2d}(\mathbb{C})_{sa}\}_{\alpha=1}^{4d^2}$  be a SIC so that

$$\text{tr}(\Pi_\alpha \Pi_\beta) = \frac{1}{2d+1} \quad \forall \alpha \neq \beta. \quad (5.60)$$

$\forall \rho \in \mathcal{L}_+^1(\mathbb{H})$ , we can expand  $\sigma_{d,d}(\rho)$  according to (5.5)

$$\sigma_{d,d}(\rho) = (D+1) \sum_{\alpha=1}^{D^2} q(\alpha) \Pi_\alpha - \mathbf{1}_{\mathbb{C}^D}, \quad (5.61)$$

where we have let  $D = 2d$  and  $q(\alpha) = \text{tr}(\sigma_{d,d}(\rho) \Pi_\alpha)$ , and where we have used the fact that

$$\sum_{r=1}^{D^2} \frac{1}{D} \Pi_r = \mathbf{1}_{\mathbb{C}^D}. \quad (5.62)$$

In Chapter 4, we established that the images of the extreme points of  $\mathcal{L}_+^1(\mathbb{H}^d)$  under  $\sigma_{d,d}$  are the extreme points of  $\sigma_{d,d}(\mathcal{L}_+^1(\mathbb{H}^d)) \subset \mathcal{L}_+^1(\mathbb{C}^{2d})$ . Therefore, the following conditions

$$\text{tr}(\sigma_{d,d}(\rho)) = 1, \quad (5.63)$$

and

$$\text{tr}(\sigma_{d,d}(\rho)^2) \leq \frac{1}{2}, \quad (5.64)$$

and

$$\text{tr}(\sigma_{d,d}(\rho)^3) \leq \frac{1}{4}, \quad (5.65)$$

completely define the boundary of quaternionic quantum state space inside complex quantum state space. The upper bounds in (5.64) and (5.65) are saturated only by the images of pure quaternionic quantum states<sup>5</sup>. The conditions (5.63), (5.64) and (5.65) translate to boundary conditions on the injection of  $\sigma_{d,d}(\mathcal{L}_+^1(\mathbb{H}^d))$  into a SIC probability simplex. Aside from the normalization condition (5.63), the condition (5.64) defines the boundary of a sphere – the quaternionic sphere – whose radius is given in Lemma 5.6.1. The condition (5.65) defines a cubic constraint for quaternionic quantum states on SIC probability simplexes. The cubic constraint is given in Lemma 5.6.2.

**Lemma 5.6.1** *The Quaternionic Sphere:*

*If  $\omega$  is an injection of complex quantum state space into a SIC probability simplex  $\Delta_{D^2}$  as defined in (5.3), then  $\forall \vec{q} \in \omega(\sigma_{d,d}(\mathcal{L}_+^1(\mathbb{H}^d))) \subset \Delta_{D^2}$  the following holds:*

$$\sum_{\alpha=1}^{D^2} q(\alpha)^2 \leq \frac{3}{2D(D+1)}. \quad (5.66)$$

*Proof:*

---

<sup>5</sup>There may be complex mixed states in  $\mathcal{L}_+^1(\mathbb{C}^d)$  that saturate these bounds, but which do not correspond to the image of a quaternionic quantum state under  $\sigma_{d,d}$ .

By explicit computation of  $\text{tr}(\sigma_{d,d}(\rho)^2)$ :

$$\begin{aligned} \text{tr}\left((D+1)^2 \sum_{\alpha,\beta} q(\alpha)q(\beta)\Pi_\alpha\Pi_\beta - 2(D+1) \sum_{\alpha} q(\alpha)\Pi_\alpha + \mathbf{1}_{\mathbb{C}^D}\right) &= \\ (D+1)^2 \sum_{\alpha,\beta} q(\alpha)q(\beta)\text{tr}(\Pi_\alpha\Pi_\beta) - 2(D+1) + D &= \\ \left((D+1)^2 - (D+1)\right) \sum_{\alpha} q_\alpha^2 + (D+1) - 2D - 2 + D. & \end{aligned} \quad (5.67)$$

Therefore, given (5.64):

$$\sum_{\alpha=1}^{D^2} q(\alpha)^2 \leq \mathfrak{r}_{\mathbb{H}}, \quad (5.68)$$

where the radius of the quaternionic sphere inside the SIC probability simplex is

$$\mathfrak{r}_{\mathbb{H}} = \frac{3}{2D(D+1)}, \quad (5.69)$$

finishing the proof.  $\square$

In [36], it is derived that the full set of complex quantum states in  $\Delta_{D^2}$  is bounded by the surface of a sphere of radius

$$\mathfrak{r}_{\mathbb{C}} = \frac{2}{D(D+1)}. \quad (5.70)$$

Evidently,

$$\mathfrak{r}_{\mathbb{H}} < \mathfrak{r}_{\mathbb{C}}. \quad (5.71)$$

**Lemma 5.6.2** *The Quaternionic Cubic Constraint:*

*If  $\omega$  is an injection of complex quantum state space into a SIC probability simplex  $\Delta_{D^2}$  as defined in (5.3), then  $\forall \vec{q} \in \omega\left(\sigma_{d,d}(\mathcal{L}_+^1(\mathbb{H}^d))\right) \subset \Delta_{D^2}$  the following holds:*

$$\sum_{\alpha=1}^{D^2} \sum_{\beta=1}^{D^2} \sum_{\gamma=1}^{D^2} c_{\alpha,\beta,\gamma} q(\alpha)q(\beta)q(\gamma) \leq \frac{D + \frac{23}{4}}{(D+1)^3}, \quad (5.72)$$

where

$$c_{\alpha,\beta,\gamma} = \text{tr}(\Pi_\alpha\Pi_\beta\Pi_\gamma). \quad (5.73)$$

*Proof:*

By explicit computation of  $\text{tr}(\sigma_{d,d}(\rho)^3)$ :

$$\text{tr}\left((D+1)^3 \sum_{\alpha,\beta,\gamma} q(\alpha)q(\beta)q(\gamma)\Pi_\alpha\Pi_\beta\Pi_\gamma - 2(D+1)^2 \sum_{\alpha,\beta} q(\alpha)q(\beta)\Pi_\alpha\Pi_\beta + (D+1) \sum_{\alpha} q(\alpha)\Pi_\alpha - \sigma(\rho)^2\right). \quad (5.74)$$

Denoting the triple products  $\text{tr}(\Pi_\alpha \Pi_\beta \Pi_\gamma)$  by  $c_{\alpha,\beta,\gamma}$  we have that

$$\sum_{\alpha,\beta,\gamma} c_{\alpha,\beta,\gamma} q(\alpha)q(\beta)q(\gamma) \leq \frac{1}{(D+1)^3} \left( \frac{1}{4} + \underbrace{\text{tr}(\sigma(\rho)^2)}_{\leq \frac{1}{2}} - (D+1) + 2(D+1)^2 \sum_{\alpha,\beta} q(\alpha)q(\beta) \text{tr}(\Pi_\alpha \Pi_\beta) \right). \quad (5.75)$$

Now, given Lemma 5.6.1 we have that

$$\sum_{\alpha,\beta} q(\alpha)q(\beta) \text{tr}(\Pi_\alpha \Pi_\beta) = \frac{1}{D+1} + \left(1 - \frac{1}{D+1}\right) \sum_{\alpha} q(\alpha)^2 \leq \frac{D+3}{(D+1)^2}, \quad (5.76)$$

and it follows that

$$\sum_{\alpha,\beta,\gamma} c_{\alpha,\beta,\gamma} q(\alpha)q(\beta)q(\gamma) \leq \frac{1}{(D+1)^3} \left( \frac{1}{4} + \frac{1}{2} - D - 1 + 2D + 6 \right). \quad (5.77)$$

Therefore, given (5.65):

$$\sum_{\alpha=1}^{D^2} \sum_{\beta=1}^{D^2} \sum_{\gamma=1}^{D^2} c_{\alpha,\beta,\gamma} q(\alpha)q(\beta)q(\gamma) \leq \mathfrak{t}_{\mathbb{H}}, \quad (5.78)$$

where

$$\mathfrak{t}_{\mathbb{H}} = \frac{D + \frac{23}{4}}{(D+1)^3}, \quad (5.79)$$

finishing the proof.  $\square$

In [36], it is derived that the full set of complex quantum states in  $\Delta_{D^2}$  is subject to the cubic constraint

$$\sum_{\alpha=1}^{D^2} \sum_{\beta=1}^{D^2} \sum_{\gamma=1}^{D^2} c_{\alpha,\beta,\gamma} q(\alpha)q(\beta)q(\gamma) \leq \mathfrak{t}_{\mathbb{C}}, \quad (5.80)$$

with

$$\mathfrak{t}_{\mathbb{C}} = \frac{D+7}{(D+1)^3}. \quad (5.81)$$

Evidently,

$$\mathfrak{t}_{\mathbb{H}} < \mathfrak{t}_{\mathbb{C}}. \quad (5.82)$$

The equations (5.66) and (5.72), together with (5.63) completely specify the boundary of the convex set of quaternionic quantum states under the affine injection of their images under  $\sigma_{d,d}$  induced via the Born rule using complex SICs. All quaternionic quantum states arise as convex combinations of the extreme points saturating the bounds in (5.66) and (5.72), thus these equations indicate the outer shape of quaternionic quantum state spaces on SIC probability simplexes.

We now turn our attention to relations between representations of quaternionic quantum states on HSIC and SIC simplexes. Let

$$\{\pi_r \in \mathcal{M}_{d,d}(\mathbb{H})_{sa}\}_{r=1}^{2d^2-d} \quad (5.83)$$

be an HSIC so that

$$\text{tr}(\pi_r \pi_s) = \frac{2}{2d+1} \quad \forall r \neq s, \quad (5.84)$$

and let  $\{\Pi_\alpha \in \mathcal{M}_{2d,2d}(\mathbb{C})_{sa}\}_{\alpha=1}^{4d^2}$  be a SIC as in (5.60). We expand

$$\psi_{d,d}(\pi_r) = \sum_{\alpha=1}^{4d^2} \Pi_\alpha c_\alpha^r, \quad (5.85)$$

where  $c_\alpha^r \in \mathbb{R}$ . Given a quaternionic quantum state  $\rho$  and its complex image  $\sigma_{d,d}(\rho) = \frac{1}{2}\psi_{d,d}(\rho)$ , we relate the probabilities for HSIC measurement outcomes –  $p(r)$  – to the probabilities for SIC measurement outcomes –  $q(\alpha)$  – as follows:

$$p(r) = \text{tr}(\pi_r \rho) = \text{tr}(\psi_{d,d}(\pi_r) \sigma_{d,d}(\rho)) = \text{tr} \left( \sum_{\alpha=1}^{4d^2} \Pi_\alpha c_\alpha^r \sigma(\rho) \right) = \sum_{\alpha=1}^{4d^2} c_\alpha^r q(\alpha). \quad (5.86)$$

The probabilities  $p(r)$  are constrained by (5.53), (5.55), and (5.58). In terms of probabilities on the complex SIC simplex, these conditions become:

$$\sum_{r=1}^{2d^2-d} \sum_{\alpha=1}^{4d^2} c_\alpha^r q(\alpha) = 1, \quad (5.87)$$

$$\sum_{r=1}^{2d^2-d} \sum_{\alpha=1}^{4d^2} \sum_{\beta=1}^{4d^2} c_\alpha^r c_\beta^r q(\alpha) q(\beta) = \frac{3}{4d^2-1}, \quad (5.88)$$

$$\sum_{r=1}^{2d^2-d} \sum_{s=1}^{2d^2-d} \sum_{t=1}^{2d^2-d} \sum_{\alpha=1}^{4d^2} \sum_{\beta=1}^{4d^2} \sum_{\gamma=1}^{4d^2} c_{rst} c_\alpha^r c_\beta^s c_\gamma^t q(\alpha) q(\beta) q(\gamma) = \frac{8d+19}{(2d+1)^3}, \quad (5.89)$$

where

$$\sum_{\alpha=1}^{4d^2} c_\alpha^r = 2, \quad (5.90)$$

because  $\psi_{d,d}(\pi_r)$  is a rank-2 projector. These relations relate constraints on coherent probability assignments for SIC measurement outcomes to the constraints on coherent probability assignments for HSIC measurement outcomes. Together with Lemma 5.6.1 and Lemma 5.6.2, we observe that the images of quaternionic quantum state spaces on SIC probability simplexes are characterized by constraints on the images of pure quaternionic quantum states that involve the explicit forms of the SICs and HSICs involved. It remains an open question as to whether maximally symmetric bases for quaternionic and complex quantum states exist in all finite dimensions.



## Chapter 6

# Conclusion and Future Directions

### 6.1 Conclusion

In this thesis, we considered a generalized quaternionic quantum formalism for the description of quantum states, quantum channels, and quantum measurements. We proved Theorem 3.4.2, which prescribes a rule for computing noncontextual probability assignments for outcomes of physical measurements via quaternionic quantum formalism. We applied symplectic embeddings from  $\mathcal{M}_{p,d}(\mathbb{H}) \rightarrow \mathcal{M}_{2p,2d}(\mathbb{C})$  to establish correspondences between quaternionic and complex quantum states, completely positive trace preserving maps, and positive operator valued measures. In Theorem 4.2.5 – the main result of this thesis – we proved that quaternionic quantum theory and complex quantum theory are consistent with respect to the *full apparatus* of quantum information theory. In particular, we proved that every quaternionic quantum description of a generic physical preparation  $\rightarrow$  transformation  $\rightarrow$  measurement process is equivalent to a description given within the framework of complex quantum information theory. Furthermore, given a quaternionic quantum description of a physical process, Theorem 4.2.5 defines an *explicit construction* of an equivalent complex quantum description. We also considered the possibility of reconciling quaternionic quantum theory with a generalized Quantum Bayesian framework for the reconstruction of quantum state spaces. In Theorem 5.3.1, we derived a lower bound on an orthonormality measure for sets of positive semi-definite self-adjoint linear operators acting on real, complex, and quaternionic modules. In the quaternionic case, we proved that if maximally symmetric bases saturating the aforementioned lower bound exist, then a quaternionic version of the Quantum Bayesian generalized Urgleichung can be derived from the formalism of quaternionic quantum theory.

### 6.2 Future Directions

Our treatment of quaternionic quantum states, channels, and measurements in the preceding chapters is quite general. In particular, we have considered a quaternionic quantum formalism for the description of the generic physical scenario depicted in Figure 3.1, which is not restricted to the

dynamics of localized systems [47]. However, the development of a quaternionic quantum formalism for the *explicit* description of local operations and product states for composite systems remains an open problem. For instance, observe that in all finite-dimensions  $d$  and  $p$ :

$$\dim\left(\mathcal{M}_{d,d}(\mathbb{H})_{sa} \otimes_{\mathbb{R}} \mathcal{M}_{p,p}(\mathbb{H})_{sa}\right) = dp(2d-1)(2p-1), \quad (6.1)$$

as a real-vector space, where  $\otimes_{\mathbb{R}}$  denotes the real-algebraic tensor product equipped with component wise multiplication. However, one also has that

$$\dim\left(\mathcal{M}_{dp,dp}(\mathbb{H})_{sa}\right) = dp(2dp-1), \quad (6.2)$$

as a real-vector space, which is *smaller* than (6.1). Therefore, one cannot form general product states and local operations for composite systems in quaternionic quantum theory using the usual Kronecker matrix product method [60] from complex quantum theory. Indeed, taking  $\otimes$  to denote the Kronecker matrix product,  $\exists \rho_1, \rho_2 \in \mathcal{L}_+^1(\mathbb{H}^d)^1$  such that  $\rho_1 \otimes \rho_2$  is *not* self-adjoint, and therefore not a quaternionic quantum state. This observation rules out the Kronecker matrix product as a viable candidate for the constructing composite quaternionic states and operations; however, it does not rule out altogether the possibility of developing a tensor product formalism for the explicit description of composite systems in quaternionic quantum theory.

One possible route towards the construction of quaternionic tensor products, which was pointed out by Barnum [7], could appeal to the fact that if  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$  is equipped with the standard symmetric product  $\bullet : \mathcal{M}_{d,d}(\mathbb{H})_{sa} \times \mathcal{M}_{d,d}(\mathbb{H})_{sa} \rightarrow \mathcal{M}_{d,d}(\mathbb{H})_{sa}$  defined  $\forall A, B \in \mathcal{M}_{d,d}(\mathbb{H})_{sa}$  via

$$A \bullet B = \frac{1}{2}(AB + BA), \quad (6.3)$$

then  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$  is  $J\mathcal{C}$ -algebra [46], *i.e.* a norm closed Jordan subalgebra of the self-adjoint subspace of a set of bounded linear operators acting on a complex Hilbert space, where the isomorphism preserves the norm (see Section C.1 for supplementary definitions). In [45], Hanche-Olsen defines a *universal tensor product* for  $J\mathcal{C}$ -algebras, which could serve as a useful mathematical apparatus for the construction of tensor products of quaternionic quantum states and operations [7]. The universal tensor product of  $J\mathcal{C}$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$  is denoted by  $\mathcal{A} \tilde{\otimes} \mathcal{B}$ , and is defined as the  $J\mathcal{C}$ -subalgebra of  $(C_u^*(\mathcal{A}) \otimes_{max} C_u^*(\mathcal{B}))$  generated by the subspace  $\psi_{\mathcal{A}}(\mathcal{A}) \otimes_{\mathbb{R}} \psi_{\mathcal{B}}(\mathcal{B})$ , where  $C_u^*(\mathcal{A})$  is the *universal  $C^*$ -algebra* of  $\mathcal{A}$ , and where  $\psi_{\mathcal{A}}$  is the associated injection of  $\mathcal{A}$  into  $C_u^*(\mathcal{A})$  (see Section C.2 for definitions). It turns out, for  $d \geq 3$ , that the universal  $C^*$ -algebra of  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$  is  $\mathcal{M}_{2d,2d}(\mathbb{C})$ , and the associated injection of  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$  into  $C_u^*(\mathcal{M}_{d,d}(\mathbb{H})_{sa})$  is the symplectic embedding defined in Definition 4.1.3 (see Example C.2.3 for the proof). For  $d = 2$ ,  $C_u^*(\mathcal{M}_{2,2}(\mathbb{H})_{sa})$  is given by the direct sum of two copies of  $\mathcal{M}_{4,4}(\mathbb{C})$  [46]. The universal tensor product is referred to as *universal* due to Theorem 6.2.1:

**Theorem 6.2.1** ([45], p 1071) *Universality of  $\mathcal{A} \tilde{\otimes} \mathcal{B}$ :*

*If  $\mathcal{A}, \mathcal{B}$  and  $\mathcal{C}$  are unital  $J\mathcal{C}$ -algebras, and  $\phi : \mathcal{A} \rightarrow \mathcal{C}$  and  $\varphi : \mathcal{B} \rightarrow \mathcal{C}$  are unital homomorphisms such that  $\forall a \in \mathcal{A}, \forall b \in \mathcal{B},$  and  $\forall x \in \mathcal{C}$*

$$\phi(a) \bullet (\varphi(b) \bullet x) = \varphi(b) \bullet (\phi(a) \bullet x), \quad (6.4)$$

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<sup>1</sup>*e.g.* The Kronecker matrix product of  $\rho_1 = \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$  with  $\rho_2 = \frac{1}{2} \begin{pmatrix} 1 & j \\ -j & 1 \end{pmatrix}$  is not self-adjoint.

then there is a unique homomorphism  $\chi$  of  $\mathcal{A}\tilde{\otimes}\mathcal{B}$  onto  $\mathcal{C}$  such that

$$\chi(a \otimes b) = \phi(a) \bullet \varphi(b). \quad (6.5)$$

Moreover, the universal  $C^*$ -algebra of  $\mathcal{A}\tilde{\otimes}\mathcal{B}$  is  $C^*(\mathcal{A}) \otimes_{\max} C^*(\mathcal{B})$ .

The universal tensor product  $\tilde{\otimes}$  is a potentially viable candidate for use in the development of an explicit description of composite systems in quaternionic quantum theory. For suppose that  $\mathcal{A} = \mathcal{M}_{d,d}(\mathbb{H})_{sa}$  and  $\mathcal{B} = \mathcal{M}_{p,p}(\mathbb{H})_{sa}$  are ambient spaces for quaternionic quantum states associated with two physical systems  $\mathfrak{S}_{\mathcal{A}}$  and  $\mathfrak{S}_{\mathcal{B}}$  respectively. In that case, the sets  $\mathcal{E}(\mathbb{H}^d)$  and  $\mathcal{E}(\mathbb{H}^p)$  of local quaternionic quantum effects defining local quaternionic quantum measurements on  $\mathfrak{S}_{\mathcal{A}}$  and  $\mathfrak{S}_{\mathcal{B}}$  are such that  $\mathcal{E}^d \subset \mathcal{A}$  and  $\mathcal{E}^d \subset \mathcal{B}$ . By its definition alone,  $\mathcal{A}\tilde{\otimes}\mathcal{B}$  therefore contains a copy of the local state spaces and the local effect sets associated with  $\mathfrak{S}_{\mathcal{A}}$  and  $\mathfrak{S}_{\mathcal{B}}$ . Put otherwise,  $\mathcal{A}\tilde{\otimes}\mathcal{B}$  contains all product states and all product effects. Furthermore, suppose that  $\mathcal{C}$  is a  $J\mathcal{C}$ -algebra – not necessarily  $\mathcal{A}\tilde{\otimes}\mathcal{B}$  – that is an ambient space for quantum states associated with the composite system  $\mathfrak{S}_{\mathcal{A}\mathcal{B}}^2$ , where the local state spaces are mapped to  $\mathcal{C}$  by  $\phi$  and  $\varphi$  respectively. Further still, suppose that  $\mathcal{C}$  is such that  $\forall x \in \mathcal{C}$  the action of  $\phi(a)$  and  $\varphi(b)$  operator commute in the sense of (6.4). Then, according to Theorem 6.2.1, there is a unique homomorphism  $\chi$  of the universal tensor product  $\mathcal{A}\tilde{\otimes}\mathcal{B}$  onto  $\mathcal{C}$  such that  $\chi(a\tilde{\otimes}b) = \phi(a) \bullet \varphi(b) = \varphi(b) \bullet \phi(a) = \chi(b\tilde{\otimes}a)$ . So, in particular, if  $a$  and  $b$  are local quaternionic quantum effects whose images operator commute on states in  $\mathcal{C}$ , then there is a unique homomorphism  $\chi$  taking the universal tensor product of  $a\tilde{\otimes}b$  onto  $\mathcal{C}$  such that the action of  $\chi(a\tilde{\otimes}b)$  on states in  $\mathcal{C}$  is symmetric under the interchange of  $a$  and  $b$ . In light of these observations, it would interesting to see if one could develop a complete model for the explicit description of composite systems in quaternionic quantum theory via the mathematical apparatus of the universal tensor product.

Another interesting direction for future research would be to investigate the existence of symmetric informationally complete quaternionic quantum measurements (HSICs) for dimensions  $d > 3$ . In Section 5.4, we proved that quaternionic quantum theory is consistent with a generalized Quantum Bayesian framework for reconstructing quantum state spaces under the assumption that HSICs exist. Furthermore, in Section 5.5 and Section 5.6, we assumed the existence of HSICs and considered the consequences of that assumption for the geometry of quaternionic quantum state spaces. But do HSICs exist in all dimensions? At present, all that can be said for sure is that they do exist for  $d = 2$  and  $d = 3$  [54]. In complex quantum theory, the existence of SICs remains an open problem as well, although *much* more effort has been put towards the SIC existence problem than the HSIC existence problem. It has been shown numerically that there exists a fiducial vector  $\phi \in \mathbb{C}^d$  whose orbit under the action of the Weyl-Heisenberg group forms a set of vectors whose corresponding subnormalized projection operators form a SIC  $\forall d \in \{2, \dots, 67\}$  [65]. The Weyl-Heisenberg group is generated by 2 order  $d$  elements and a complex phase, and the quotient group of the Weyl-Heisenberg modulo its center is isomorphic to  $\mathbb{Z}_d^2$  [3]. Hence, the projective unitary representation of the Weyl-Heisenberg group on  $U(d, \mathbb{C})$  is order  $d^2$  – just the right number to possibly yield the  $d^2$  symmetric states comprising a SIC via the action of products of powers of the generators on a fiducial vector. In the quaternionic case, the subject of group covariant HSICs has yet to be explored. The cardinality

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<sup>2</sup>In finite dimensions, it follows that the only possible choices for  $\mathcal{C}$  are self-adjoint matrix algebras over the associative normed division algebras, as well as spin factors (see footnote 2) [46].

of an HSIC is equal to  $d(2d - 1)$ . Thus, a search for group covariant HSICs might explore groups with projective quaternionic unitary representations on  $\text{Sp}(d)$  of order at least  $d(2d - 1)$ , if they exist. Another possible approach might consider a translation of the HSIC existence problem into the language of complex quantum theory using symplectic embeddings.

In complex quantum theory, Neumark's Theorem dictates that POVM quantum measurements are equivalent to PVM quantum measurements involving higher-dimensional complex Hilbert spaces. Nevertheless, the POVM formalism has proven to be a powerful tool in complex quantum information theory, with distinct advantages over the PVM formalism. The POVM formalism provides a compact formalism for the description of physical scenarios involving ancillary systems, and it provides optimal solutions to important problems such as distinguishing a set complex quantum states [38][60]. In this thesis, we proved that quaternionic quantum descriptions of physical processes are equivalent to complex quantum descriptions involving higher-dimensional complex Hilbert spaces. Thus, in analogy with the power of POVMs, a quaternionic quantum formalism for the description of physical phenomena may illuminate new insights into quantum information science and the foundations of quantum theory, which is another interesting direction for future research.

# APPENDICES

# Appendix A

## Symplectic Group Theory

We refer to

$$q = \gamma_1 + \gamma_2 j \tag{A.1}$$

as the “*symplectic coordinate representation*” of  $q = q_0 + iq_1 + jq_2 + kq_3 \in \mathbb{H}$ , with  $\gamma_1 = q_0 + iq_1$ ,  $\gamma_2 = q_2 + iq_3 \in \mathbb{C}$ . Horwitz and Biedenharn refer to (A.1) as the symplectic representation of a component of a vector in a *right* quaternionic module [51]. Horwitz and Biedenharn provide an implicit justification for adopting such terminology via their citation of seminal papers on quaternionic quantum theory written by Finkelstein et al. in the early 1960s [27][28]. Looking back on [27][28], however, one finds that Finkelstein et al. actually refer to an *embedding* of  $q$  into  $\mathcal{M}_{2,2}(\mathbb{C})$  – i.e.  $\psi_{1,1}(q)$  – as the symplectic representation of an endomorphism on  $\mathbb{H}$  accomplishing *left*-multiplication by  $q$ . In this appendix, we explore the origins of such terminology and we consider relations between the symplectic groups.

### A.1 The Symplectic Group

**Definition A.1.1** ([71] p. 159) *Symplectic Forms:*

A *symplectic form* on a complex vector space  $\mathcal{V}$  is a  $\mathbb{C}$ -bilinear skew-symmetric non-degenerate form

$$\omega : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{C}. \tag{A.2}$$

Put otherwise, if  $u, u_1, u_2, v \in \mathcal{V}$  and  $\lambda, \lambda_1, \lambda_2, \mu \in \mathbb{C}$ , then the following holds for a symplectic form:

- *$\mathbb{C}$ -bilinearity:*  $\omega(u_1\lambda_1 + u_2\lambda_2, v) = \lambda_1\omega(u_1, v) + \lambda_2\omega(u_2, v)$ ,
- *Skew-symmetry:*  $\omega(u, v) = -\omega(v, u)$  and  $\omega(u, u) = 0$ ,
- *Nondegeneracy:*  $\omega(u, v) = 0 \forall v \implies u = 0$ .

With  $u = (u_1, u'_1, u_2, u'_2, \dots, u_\nu, u'_\nu)$  and  $v = (v_1, v'_1, v_2, v'_2, \dots, v_\nu, v'_\nu)$ , we can define a symplectic form  $\omega(u, v)$  via

$$\omega(u, v) = (u_1 v'_1 - u'_1 v_1) + \dots + (u_\nu v'_\nu - u'_\nu v_\nu), \quad (\text{A.3})$$

such that the  $n = 2\nu$  basis vectors  $e_r$  satisfy  $\omega(e_r, e_s) = \omega(e'_r, e'_s) = 0$  and  $\omega(e_r, e'_s) = -\omega(e'_r, e_s) = \delta_{rs}$ . Indeed, the existence of a symplectic form requires an even number of dimensions. We say that  $\{e_r\}$  is a symplectic coordinate system.

**Example A.1.2** *Symplectic Form on  $\mathbb{C}^2$ :*

The standard basis vectors  $e_1 = (1, 0)$  and  $e_2 = (0, 1)$  are suitable symplectic coordinate system. If  $u = (u_x, u_y)$ ,  $v = (v_x, v_y) \in \mathbb{C}^2$ , then  $\omega(u, v) = u_x v_y - u_y v_x$  is a symplectic form on  $\mathbb{C}^2$ .

**Definition A.1.3** ([71] p. 159) *Symplectic Groups:*

The group of transformations on a complex vector space preserving the symplectic form (A.3) is called the symplectic group, and is denoted by  $Sp(2d, \mathbb{C})$ .

**Remark A.1.4** *Equivalence of Symplectic Groups:*

Given  $x, y \in \mathcal{V}$ , another symplectic form  $\omega'$  may be defined as  $\omega'(x, y) = {}^t x \Omega y$ , where  $\Omega = -\Omega^T = -\Omega^{-1}$  is the real skew-symmetric matrix defined as

$$\Omega = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}, \quad (\text{A.4})$$

where  $\mathbf{0}$  and  $\mathbf{1}$  are the zero and unit matrices in  $M_{d \times d}(\mathbb{C})$  respectively. If  $B \in GL(2d, \mathbb{C})$  preserves  $\omega'$ , then  $\omega'(Bx, By) = \omega'(x, y) \implies B^T \Omega B = \Omega$ . That  $\mathbf{1}_{\mathbb{C}^{2d}}$  preserves  $\omega'$  is clear. If  $B$  preserves  $\omega'$ , then  $B^{-1} = \Omega^T B^T \Omega$ , which we can see from

$$B^T \Omega B = \Omega \implies \Omega^T B^T \Omega B = \mathbf{1}_{\mathbb{C}^{2d}} \implies B^{-1} = \Omega^T B^T \Omega. \quad (\text{A.5})$$

If  $B$  and  $C$  both preserve  $\omega'$ , then  $BC$  preserves  $\omega'$  as well:

$$(BC)^T \Omega (BC) = C^T B^T \Omega BC = C^T \Omega C = \Omega. \quad (\text{A.6})$$

Given the associativity of matrix multiplication, we conclude that the set of matrices preserving  $\omega'$  is a group, namely

$$G = \left\{ B \in GL(2d, \mathbb{C}) \mid B^T \Omega B = \Omega \right\}. \quad (\text{A.7})$$

We should point out that (A.7) is actually equivalent to the compact symplectic group  $Sp(2d, \mathbb{C})$  defined in Definition B.2.3. Indeed, if we define

$$J_{2d} = \begin{pmatrix} J & 0 & 0 & \dots & 0 \\ 0 & J & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & J \end{pmatrix}, \quad (\text{A.8})$$

with

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \text{ and } 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad (\text{A.9})$$

then symplectic form  $\omega : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{C}$  defined via  $\omega(x, y) = {}^t x J_{2d} y$  is exactly the one considered by [71] and pointed out in Definition A.2.1. Notice that  $J_{2d} = -J_{2d}^T = -J_{2d}^{-1}$ , and so by following through our preceding arguments, one can show that the set of matrices preserving  $\omega$  form a group, namely  $\text{Sp}(2d, \mathbb{C})$ . Let us show that  $G \cong \text{Sp}(2d, \mathbb{C})$ . There exists [22] a real orthogonal matrix  $P$  with  $PP^T = \mathbb{1}_{\mathbb{C}^{2d}}$  such that  $\Omega = PJ_{2d}P^T$ . Therefore

$$B^T \Omega B = \Omega \implies B^T P J_{2d} P^T B = P J_{2d} P^T \implies P^T B^T P J_{2d} P^T B P = J_{2d} \implies \tilde{B}^T J_{2d} \tilde{B} = J_{2d}, \quad (\text{A.10})$$

where  $\tilde{B}$  is the unique matrix  $P^T B P$ . Hence  $G \cong \text{Sp}(2d, \mathbb{C})$ . Moreover, we can repeat these arguments for *any* skew-symmetric  $J$  defining a symplectic form on  $\mathcal{V}$ . So, in particular, we can define the symplectic group with respect to  $\Omega$ , which turns out to be a convenient choice for proving Theorem A.3.5.

**Example A.1.5**  $Sp(2, \mathbb{C}) = SL(2, \mathbb{C})$ :

Let  $u = (u_x, u_y), v = (v_x, v_y) \in \mathbb{C}^2$  and  $A = \begin{bmatrix} \alpha & \beta \\ \delta & \gamma \end{bmatrix} \in \text{Sp}(2, \mathbb{C})$ , so that

$$\omega(Au, Av) = (\alpha u_x + \beta u_y)(\delta v_x + \gamma v_y) - (\delta u_x + \gamma u_y)(\alpha v_x + \beta v_y) = (\alpha\gamma - \delta\beta)\omega(u, v) \implies \det(A) = 1, \quad (\text{A.11})$$

establishing that  $\text{Sp}(2, \mathbb{C}) = \text{SL}(2, \mathbb{C})$ .

**Remark A.1.6** *Etymology of Symplectic*:

Weyl originally referred to symplectic groups as ‘complex groups’, as he recalled in [71]:

“The name ‘complex group’ formerly advocated by me in allusion to line complexes, as these are defined by the vanishing of antisymmetric bilinear forms, has become more and more embarrassing through collision with the word ‘complex’ in the connotation of complex number. I therefore propose to replace it by the corresponding Greek adjective ‘symplectic.’ Dickson calls the group ‘Abelian linear group’ in homage to Abel who first studied it”

– H. Weyl, 1939 [71].



## A.2 The Compact Symplectic Group

**Definition A.2.1** ([39] p. 99) *Symplectic Inner Product :*

A symplectic inner product on a right quaternionic module  $\mathcal{V}$  is an  $\mathbb{R}$ -bilinear form

$$K : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{H} \tag{A.12}$$

that is conjugate symmetric, conjugate  $\mathbb{H}$ -linear in the first factor, and  $\mathbb{H}$ -linear in the second. Put otherwise, if  $u, u_1, u_2, v \in \mathcal{V}$ ,  $\lambda_1, \lambda_2 \in \mathbb{R}$ , and  $\lambda, \mu \in \mathbb{H}$ , then a symplectic inner product satisfies

- $\mathbb{R}$ -linearity:  $K(u_1\lambda_1 + u_2\lambda_2, v) = \lambda_1K(u_1, v) + \lambda_2K(u_2, v)$ ,
- conjugate symmetry:  $K(u, v) = \overline{K(v, u)}$ ,
- $\mathbb{H}$ -(conjugate)linearity:  $K(u\lambda, v\mu) = \overline{\lambda}K(u, v)\mu$ .

It is called nonnegative if  $K(u, u) \geq 0$  with equality iff  $u = 0$ .

**Definition A.2.2** *Standard Symplectic Inner Product on  $\mathbb{H}^d$  and the Induced Norm:*

Let  $u, v \in \mathbb{H}^d$ . Then,

$$\langle u|v \rangle = \sum_{r=1}^d \overline{u_r}v_r, \tag{A.13}$$

is a nonnegative symplectic inner product. We call (A.13) the standard symplectic inner product. The standard standard symplectic inner product induces the canonical real-valued norm on  $\mathbb{H}^d$   $\|\cdot\| : \mathbb{H}^d \rightarrow \mathbb{R}$  defined  $\forall \phi \in \mathbb{H}^d$  via  $\|\phi\|^2 = \langle \phi|\phi \rangle$  and satisfying  $\forall \phi, \xi \in \mathbb{H}^d$  and  $\forall a \in \mathbb{H}$ :

- Positive Homogeneity:  $\|\phi a\|^2 = |a|\|\phi\|^2$ ,
- Nonnegativity:  $\|\phi\| \geq 0$  with equality iff  $\phi = 0$ , and
- Triangle Inequality:  $\|\phi + \xi\| \leq \|\phi\| + \|\xi\|$ .

**Definition A.2.3** ([39] p. 99) *Compact Symplectic Group:*

The group of automorphisms on a  $d$ -dimensional right quaternionic module preserving the standard symplectic inner product (A.13) is called the compact symplectic group, and is denoted by  $Sp(d)$ .

**Remark A.2.4**  $Sp(d) = U(d, \mathbb{H})$ :

The compact symplectic group is just the group of quaternionic unitary matrices. Let  $\varphi \in Sp(d) \subset GL(d, \mathbb{H})$  be defined with respect to an arbitrary orthonormal basis as follows:  $\varphi = \sum_{rs} |r\rangle \varphi_{rs} \langle s|$ , so that

$$\langle \varphi(u) | \varphi(v) \rangle = \sum_{s,s',t} \overline{u_s} \overline{\varphi_{ts}} \varphi_{ts'} v_{s'} = \langle u | v \rangle \iff \sum_t \overline{\varphi_{ts}} \varphi_{ts'} = \delta_{ss'}, \quad (\text{A.14})$$

establishing that  $\varphi \in U(d, \mathbb{H})$ .

**Remark A.2.5**  $Sp(1) \cong SU(2)$ :

Let  $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SU(2)$ .  $UU^* = \mathbf{1}_{\mathbb{C}^2}$  and  $\det(U) = 1$  imply that  $|a|^2 + |b|^2 = 1$ ,  $c = -\bar{b}$ , and  $a = \bar{d}$ .

We will set up a group isomorphism between  $Sp(1)$  and  $SU(2)$ . Let  $\varphi \in Sp(1)$  be decomposed according to (A.1):  $\varphi = \gamma_1 + \gamma_2 j$ , so that  $\varphi \bar{\varphi} = 1 \implies |\gamma_1|^2 + |\gamma_2|^2 = 1$ . Let  $\psi : Sp(1) \rightarrow SU(2)$  be defined via

$$\psi(\varphi) = \begin{bmatrix} \gamma_1 & \gamma_2 \\ -\gamma_2 & \gamma_1 \end{bmatrix}. \quad (\text{A.15})$$

Indeed, it is obvious that  $\psi$  is a bijection, and Lemma 4.1.4 tells that  $\psi(\varphi_1 \varphi_2) = \psi(\varphi_1) \psi(\varphi_2)$ , establishing that  $\psi$  is indeed a group isomorphism.

**Lemma A.2.6**  $Sp(d) \cong U(2d) \cap Sp(2d, \mathbb{C})$ :

*Proof:*

This is a well-known result [39]. We have already proved the case  $n = 1$  (consult Example A.1.5 and Remark A.2.5). We recall that

$$U(2d, \mathbb{C}) = \left\{ B \in GL(2d, \mathbb{C}) \mid BB^* = \mathbf{1}_{\mathbb{C}^{2d}} \right\}. \quad (\text{A.16})$$

As mentioned in Remark A.2.4, the compact symplectic group  $Sp(d)$  is just the group of quaternionic unitary matrices, that is

$$Sp(d) = \left\{ A \in GL(d, \mathbb{H}) \mid AA^* = \mathbf{1}_{\mathbb{H}^d} \right\}. \quad (\text{A.17})$$

The condition  $AA^* = \mathbf{1}_{\mathbb{H}^d}$  is equivalent to the condition  $A^*A = \mathbf{1}_{\mathbb{H}^d}$ , as is the case with complex unitaries. To see this, we appeal to properties of the \*-homomorphic symplectic embedding  $\psi_{d,d} : \mathcal{M}_{d,d}(\mathbb{H}) \rightarrow \mathcal{M}_{2d,2d}(\mathbb{C})$  defined in Chapter 4, Definition 4.1.3 to see that

$$AA^* = \mathbf{1}_{\mathbb{H}^d} \implies \psi_{d,d}(A) \psi_{d,d}(A^*) = \mathbf{1}_{\mathbb{C}^{2d}} \implies \psi_{d,d}(A^*) \psi_{d,d}(A) = \mathbf{1}_{\mathbb{C}^{2d}} \implies A^*A = \mathbf{1}_{\mathbb{H}^d}. \quad (\text{A.18})$$

One also has that  $A \in Sp(d) \iff \psi_{d,d}(A) \in U(2d, \mathbb{C})$ , for indeed:

$$\mathbf{1}_{\mathbb{C}^{2d}} = \psi_{d,d}(\mathbf{1}_{\mathbb{H}^d}) = \psi_{d,d}(AA^*) = \psi_{d,d}(A) \psi_{d,d}(A^*) = \psi_{d,d}(A) \psi_{d,d}(A)^*. \quad (\text{A.19})$$

The image of  $\text{Sp}(d)$  under  $\psi_{d,d}$  is a subgroup of  $\text{U}(2d, \mathbb{C})$ . To see this, we note that  $A, B \in \text{Sp}(d) \implies \psi_{d,d}(AB)\psi_{d,d}(AB)^* = \psi_{d,d}(A)\psi_{d,d}(B)\psi_{d,d}(B)^*\psi_{d,d}(A)^* = \mathbf{1}_{\mathbb{C}^{2d}} \implies \psi_{d,d}(AB) \in \text{U}(2d, \mathbb{C})$ . We also recall that  $\mathbf{1}_{\mathbb{C}^{2d}}$  is image of  $\mathbf{1}_{\mathbb{H}^d} \in \text{Sp}(d)$ , and  $\psi_{d,d}(A) \in \text{U}(2d, \mathbb{C})$  implies  $\psi_{d,d}(A)^{-1} = \psi_{d,d}(A)^* \in \text{U}(2d, \mathbb{C})$  – it is just the image of  $A^*$ . So,  $\psi_{d,d}(\text{Sp}(d))$  is a subgroup of the unitary group.

We also have that

$$\Omega\psi_{d,d}(A) = \overline{\psi_{d,d}(A)}\Omega \quad \forall A \in \text{Sp}(d). \quad (\text{A.20})$$

Indeed, calculation yields

$$\Omega\psi_{d,d}(A) = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ -\Gamma_2 & \Gamma_1 \end{pmatrix} = \begin{pmatrix} -\overline{\Gamma_2} & \overline{\Gamma_1} \\ -\Gamma_1 & -\Gamma_2 \end{pmatrix} = \begin{pmatrix} \overline{\Gamma_1} & \overline{\Gamma_2} \\ -\Gamma_2 & \Gamma_1 \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} = \overline{\psi_{d,d}(A)}\Omega. \quad (\text{A.21})$$

Now,  $A, B \in \text{Sp}(d)$  implies that

$$\Omega\psi_{d,d}(AB) = \Omega\psi_{d,d}(A)\psi_{d,d}(B) = \overline{\psi_{d,d}(A)}\Omega\psi_{d,d}(B) = \overline{\psi_{d,d}(A)}\overline{\psi_{d,d}(B)} = \overline{\psi_{d,d}(AB)}\Omega. \quad (\text{A.22})$$

Given that  $\psi_{d,d}(A)$  are unitary, i.e.  $\overline{\psi_{d,d}(A)}\psi_{d,d}(A)^T = \mathbf{1}_{\mathbb{C}^{2d}}$ , we have established that

$$A \in \text{Sp}(d) \implies \psi_{d,d}(A) \in \text{U}(2d, \mathbb{C}) \cap \text{Sp}(2d, \mathbb{C}). \quad (\text{A.23})$$

Now, let  $B \in \mathcal{M}_{2d,2d}(\mathbb{C})$  such that  $B \in \text{U}(2d, \mathbb{C}) \cap \text{Sp}(2d, \mathbb{C})$ . We will show that  $\exists! A \in \text{Sp}(d)$  such that  $B = \psi_{d,d}(A)$ . Let us write

$$B = \begin{pmatrix} c & d \\ e & f \end{pmatrix}, \quad (\text{A.24})$$

where  $c, d, e, f \in \mathcal{M}_{d,d}(\mathbb{C})$  are arbitrary  $d \times d$  blocks. If  $\Omega B = \overline{B}\Omega$ , then  $f = \bar{c}$  and  $e = -\bar{d}$ . Indeed,

$$\Omega B = \overline{B}\Omega \implies \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} \begin{pmatrix} c & d \\ e & f \end{pmatrix} = \begin{pmatrix} \bar{c} & \bar{d} \\ \bar{e} & \bar{f} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} \implies \begin{pmatrix} e & f \\ -c & -d \end{pmatrix} = \begin{pmatrix} -\bar{d} & \bar{c} \\ -\bar{f} & \bar{e} \end{pmatrix}. \quad (\text{A.25})$$

Put otherwise,  $B \in \text{U}(2d, \mathbb{C}) \cap \text{Sp}(2d, \mathbb{C}) \implies B = \psi_{d,d}(A)$  for a unique  $A \in \text{Sp}(d)$ , finishing the proof.  $\square$

### A.3 Quaternionic Endomorphisms on $\mathbb{C}^2$

“It is sometimes convenient to represent quaternions by pairs of complex numbers  $(c^0, c^1)$  according to  $q = c^0 + i_2 c^1$ , where  $c^0, c^1$  commute with  $i_3$ , and are therefore essentially complex numbers. Treating these pairs as vectors in a two-dimensional complex vector space  $\mathbb{C}^2$ , we find that every linear transformation of  $\mathcal{Q}$  is represented by a linear transformation of  $\mathbb{C}^2$ , that is by a  $2 \times 2$  complex matrix. In particular the *left multiplication*  $q \rightarrow aq$  by a fixed quaternion  $a$ , is represented by a matrix  $a_{ij}$ , the *symplectic* representation of  $a$ . The symplectic representations of left multiplication by  $i_1, i_2, i_3$  are just the Pauli spin operators (times  $i$ ).”

– Finkelstein et al., 1962 [27].

Viewing  $\mathbb{Q} = \mathbb{H} = \mathbb{C} \oplus j\mathbb{C} \cong \mathbb{C}^2$  one does see that left-multiplication by elements of  $\mathbb{H}$  correspond to  $\mathbb{C}$ -linear endomorphisms on  $\mathbb{C}^2$ . Let us adopt the Finkelstein notation with

$$i_1^2 = i_2^2 = i_3^2 = i_1 i_2 i_3 = -1 \quad (\text{A.26})$$

and write  $q = q_0 + i_1 q_1 + i_2 q_2 + i_3 q_3 = (q_0 + i_3 q_3) + i_2(q_2 + i_3 q_1) = c^0 + i_2 c^1$  and embed

$$q \hookrightarrow \begin{bmatrix} q_0 + i q_3 \\ q_2 + i q_1 \end{bmatrix} \in \mathbb{C}^2. \quad (\text{A.27})$$

Then

$$i_1 q = (-q_1 + i_3 q_2) + i_2(-q_3 + i_3 q_0) \hookrightarrow \begin{bmatrix} -q_1 + i q_2 \\ -q_3 + i q_0 \end{bmatrix} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \begin{bmatrix} q_0 + i q_3 \\ q_2 + i q_1 \end{bmatrix}, \quad (\text{A.28})$$

which shows that left-multiplication by  $i_1$  corresponds to  $i\sigma_x$ . Similarly,

$$i_2 q = (-q_2 - i_3 q_1) + i_2(q_0 + i_3 q_3) \hookrightarrow \begin{bmatrix} -q_2 - i q_1 \\ q_0 + i q_3 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} q_0 + i q_3 \\ q_2 + i q_1 \end{bmatrix}, \quad (\text{A.29})$$

$$i_3 q = (-q_3 + i_3 q_0) + i_2(q_1 - i_3 q_2) \hookrightarrow \begin{bmatrix} -q_3 + i q_0 \\ q_1 - i q_2 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} q_0 + i q_3 \\ q_2 + i q_1 \end{bmatrix}, \quad (\text{A.30})$$

showing that left-multiplication by  $i_2$  corresponds to  $i\sigma_y$ , and left-multiplication by  $i_3$  corresponds to  $i\sigma_z$ . Left-multiplication by the remaining unit, 1, corresponds to  $\mathbb{1}_{\mathbb{C}^2}$ . We have shown that  $\text{Sp}(1) \cong \text{SU}(2) \subset \text{Sp}(2, \mathbb{C})$ . One has that  $i\sigma_x, i\sigma_y, i\sigma_z, \mathbb{1}_{\mathbb{C}^2} \in \text{SU}(2)$ , and so we see that the group of unit quaternions has a symplectic representation on  $(\mathbb{C}^2, \omega)$ , which is to say that this representation preserves the symplectic form  $\omega$ . In this sense, Finkelstein et al. were on the right track when they used the words ‘‘symplectic representation’’, but strictly speaking the endomorphism on  $\mathbb{C}^2$  corresponding to left-multiplication by  $a \in \mathbb{H}$  only preserves the symplectic form when  $a\bar{a} = 1$ .

There are, however, linear-algebraic motivations for adopting a *right*-module formalism as we have pointed out in Chapter 2. Let us take  $n = 1$  and show that right-multiplications by unit quaternions in  $\mathbb{H}$  correspond to elements of  $\text{SU}(2)$ . With  $q \in \mathbb{H}$  decomposed according to (A.1) we embed

$$q = \gamma_1 + \gamma_2 j \hookrightarrow \begin{bmatrix} q_0 + i q_1 \\ q_2 + i q_3 \end{bmatrix} \in \mathbb{C}^2, \quad (\text{A.31})$$

and observe that

$$qi \hookrightarrow \begin{bmatrix} -q_1 + i q_0 \\ q_3 - i q_2 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} q_0 + i q_1 \\ q_2 + i q_3 \end{bmatrix}, \quad (\text{A.32})$$

$$qj \hookrightarrow \begin{bmatrix} -q_2 - q_3 \\ q_0 + i q_1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} q_0 + i q_1 \\ q_2 + i q_3 \end{bmatrix}, \quad (\text{A.33})$$

$$qk \hookrightarrow \begin{bmatrix} -q_3 + i q_2 \\ -q_1 + i q_0 \end{bmatrix} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \begin{bmatrix} q_0 + i q_1 \\ q_2 + i q_3 \end{bmatrix}, \quad (\text{A.34})$$

showing that right-multiplications by unit quaternions correspond to elements of  $\text{SU}(2)$ .

## A.4 Summary

We have shown that the standard basis for  $\mathbb{C}^2$  is a suitable symplectic coordinate system (see Example A.1.2). On that view, Horwitz and Biedenharn's reference to (A.1) as the *symplectic representation* of components of elements of  $\mathbb{H}$  seems almost acceptable. We must, however, conclude from this analysis that it is *the representation of the unit quaternions on  $\mathbb{C}^2$*  that is truly "symplectic" in the full sense of the word. On that view, we refer to (A.1) as the "symplectic coordinate representation" of  $q \in \mathbb{H}$ , and we refer to  $\psi_{p,d} : \mathcal{M}_{p,d}(\mathbb{H}) \rightarrow \mathcal{M}_{2p,2d}(\mathbb{C})$  as the "symplectic embedding".

# Appendix B

## Sp(1) and SO(3)

In this appendix, we detail the connection between the group of unit norm quaternions – Sp(1) – and the group of rotations on  $\mathbb{R}^3$  – SO(3). Our primary references are [67] and [50].

Define the set of pure imaginary quaternions  $\Im(\mathbb{H})$  as

$$\Im(\mathbb{H}) = i\mathbb{R} + j\mathbb{R} + k\mathbb{R} \subseteq \mathbb{H} = \left\{ ir_1 + jr_2 + kr_3 \mid (r_1, r_2, r_3 \in \mathbb{R}) \wedge (i^2 = j^2 = k^2 = ijk = -1) \right\}. \quad (\text{B.1})$$

There is, of course, a natural bijection  $f : \mathbb{R}^3 \rightarrow \Im(\mathbb{H})$  defined  $\forall \vec{r} = \hat{x}r_1 + \hat{y}r_2 + \hat{z}r_3 \in \mathbb{R}^3$  via

$$f(\vec{r}) = ir_1 + jr_2 + kr_3. \quad (\text{B.2})$$

The function  $f$  is clearly injective with  $f^{-1} : \Im(\mathbb{H}) \rightarrow \mathbb{R}^3$  defined  $\forall r = ir_1 + jr_2 + kr_3 \in \Im(\mathbb{H})$  via

$$f^{-1}(ir_1 + jr_2 + kr_3) = \hat{x}r_1 + \hat{y}r_2 + \hat{z}r_3. \quad (\text{B.3})$$

It is also obvious that  $f$  admits  $\forall \alpha, \beta \in \mathbb{R}$  and  $\forall \vec{r}, \vec{s} \in \mathbb{R}^3$  that

$$f(\alpha\vec{r} + \beta\vec{s}) = \alpha f(\vec{r}) + \beta f(\vec{s}), \quad (\text{B.4})$$

and that  $f^{-1}$  admits  $\forall \alpha, \beta \in \mathbb{R}$  and  $\forall p, q \in \Im(\mathbb{H})$  that

$$f^{-1}(\alpha p + \beta q) = \alpha f^{-1}(p) + \beta f^{-1}(q). \quad (\text{B.5})$$

Two familiar binary algebraic operations on  $\mathbb{R}^3$ , the standard scalar product and the standard vector product, correspond to quaternionic arithmetic operations through  $f$  as follows:

$$\vec{r} \cdot \vec{s} = r_1 s_1 + r_2 s_2 + r_3 s_3 = \frac{1}{2} \left( f(\vec{r}) \overline{f(\vec{s})} + f(\vec{s}) \overline{f(\vec{r})} \right), \quad (\text{B.6})$$

$$\vec{r} \times \vec{s} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \end{vmatrix} = f^{-1} \left( \frac{1}{2} (f(\vec{r}) f(\vec{s}) - f(\vec{s}) f(\vec{r})) \right). \quad (\text{B.7})$$

We will prove that rotations on  $\mathbb{R}^3$  also correspond to quaternionic arithmetic operations through  $f$ . To begin, let  $t \in \text{Sp}(1)$  admitting  $t\bar{t} = 1$  and define the transformation  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  via

$$T(\vec{r}) = f^{-1}(tf(\vec{r})\bar{t}). \quad (\text{B.8})$$

Indeed,

$$tf(\vec{r})\bar{t} + \overline{(tf(\vec{r})\bar{t})} = tf(\vec{r})\bar{t} + \overline{tf(\vec{r})\bar{t}} = t(f(\vec{r}) + \overline{f(\vec{r})})\bar{t} = 0 \implies tf(\vec{r})\bar{t} \in \Im(\mathbb{H}), \quad (\text{B.9})$$

so  $f^{-1}(tf(\vec{r})\bar{t}) \in \mathbb{R}^3$  is well defined and unique. Next, observe that

$$\begin{aligned} T(\vec{r}) \cdot T(\vec{s}) &= \frac{1}{2} \left( f(T(\vec{r}))\overline{f(T(\vec{s}))} + f(T(\vec{s}))\overline{f(T(\vec{r}))} \right) \\ &= \frac{1}{2} (tf(\vec{r})\bar{t}\overline{tf(\vec{s})\bar{t}} + tf(\vec{s})\bar{t}\overline{tf(\vec{r})\bar{t}}) \\ &= t \left( \frac{1}{2} (f(\vec{r})\overline{f(\vec{s})} + f(\vec{s})\overline{f(\vec{r})}) \right) \bar{t} \\ &= t(\vec{r} \cdot \vec{s})\bar{t} \\ &= \vec{r} \cdot \vec{s}, \end{aligned} \quad (\text{B.10})$$

establishing that  $T$  preserves the standard scalar product. Therefore,  $T$  preserves the standard Euclidean norm on  $\mathbb{R}^3$ . Put otherwise,  $T$  is a Euclidean isometry. The fixed points of  $T$  are all of those  $\vec{z} \in \mathbb{R}^3$  satisfying

$$\vec{z} = T(\vec{z}) \iff f(\vec{z}) = f\left(f^{-1}(tf(\vec{z})\bar{t})\right) = tf(\vec{z})\bar{t}. \quad (\text{B.11})$$

From (B.11) it follows that  $f(\vec{z})$  commutes with  $\bar{t}$ , therefore

$$i(z_1t_0 - z_2t_3 + z_3t_2) = i(z_1t_0 - z_3t_2 + z_2t_3), \quad (\text{B.12})$$

$$j(z_2t_0 - z_3t_1 + z_1t_3) = j(z_2t_0 - z_1t_3 + z_3t_1), \quad (\text{B.13})$$

$$k(z_3t_0 - z_1t_2 + z_2t_1) = k(z_3t_0 - z_2t_1 + z_1t_2). \quad (\text{B.14})$$

From (B.12), (B.13), and (B.14) it follows that

$$\frac{z_\alpha}{z_\beta} = \frac{t_\alpha}{t_\beta} \quad \forall \alpha, \beta \in \{1, 2, 3\}, \quad (\text{B.15})$$

establishing that  $\vec{z} = f^{-1}(iz_1 + jz_2 + kz_3)$  and  $-\vec{t} = f^{-1}(-it_1 - jt_2 - kt_3)$  are collinear. Therefore

$$\vec{z} = T(\vec{z}) \iff \vec{z} = \alpha\vec{t}, \quad \alpha \in \mathbb{R}. \quad (\text{B.16})$$

Next, observe that

$$\begin{aligned}
T(\vec{s}) \times T(\vec{v}) &= f^{-1} \left[ \frac{1}{2} \left( f(T(\vec{s}))f(T(\vec{v})) - f(T(\vec{v}))f(T(\vec{s}))) \right) \right] \\
&= f^{-1} \left[ \frac{1}{2} \left( tf(\vec{s})\bar{t}tf(\vec{v})\bar{t} - tf(\vec{v})\bar{t}tf(\vec{s})\bar{t} \right) \right] \\
&= f^{-1} \left[ \frac{1}{2} \left( t(f(\vec{s})f(\vec{v}) - f(\vec{v})f(\vec{s}))\bar{t} \right) \right] \\
&= f^{-1} \left[ tf \left( f^{-1} \left( \frac{1}{2} (f(\vec{s})f(\vec{v}) - f(\vec{v})f(\vec{s})) \right) \right) \bar{t} \right] \\
&= f^{-1} \left( tf(\vec{s} \times \vec{v})\bar{t} \right) \\
&= T(\vec{s} \times \vec{v}). \tag{B.17}
\end{aligned}$$

$\mathbb{R}$ -linearity of  $T$  follows from  $\mathbb{R}$ -linearity of  $f$  and  $\mathbb{R}$ -linearity of  $f^{-1}$ , that is  $\forall \alpha, \beta \in \mathbb{R}$  and  $\forall \vec{r}, \vec{s} \in \mathbb{R}^3$

$$\begin{aligned}
T(\alpha\vec{r} + \beta\vec{s}) &= f^{-1} \left( t(f(\alpha\vec{r} + \beta\vec{s}))\bar{t} \right) \\
&= f^{-1} \left( t(\alpha f(\vec{r}) + \beta f(\vec{s}))\bar{t} \right) \\
&= \alpha f^{-1} (tf(\vec{r})\bar{t}) + \beta f^{-1} (tf(\vec{s})\bar{t}) \\
&= \alpha T(\vec{r}) + \beta T(\vec{s}). \tag{B.18}
\end{aligned}$$

Therefore,  $T$  acts on  $\mathbb{R}^3$  via a matrix  $A$  defined with respect to an orthonormal basis  $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$  for  $\mathbb{R}^3$  as  $T(\vec{r}) = A\vec{r}$ . It will now be proven that  $\det(A) = 1$ . Expand the vector product  $T(\vec{u}) \times T(\vec{v}) = A\vec{u} \times A\vec{v}$  according to the orthonormal basis  $\{A\vec{e}_1, A\vec{e}_2, A\vec{e}_3\}$  as

$$A\vec{u} \times A\vec{v} = \sum_{r=1}^3 \left( (A\vec{u} \times A\vec{v}) \cdot A\vec{e}_r \right) A\vec{e}_r. \tag{B.19}$$

We require the following elementary result from linear algebra:

$$(A\vec{u} \times A\vec{v}) \cdot A\vec{e}_r = A\vec{e}_r \cdot (A\vec{u} \times A\vec{v}) = \begin{vmatrix} (A\vec{e}_r)_1 & (A\vec{e}_r)_2 & (A\vec{e}_r)_3 \\ (A\vec{u})_1 & (A\vec{u})_2 & (A\vec{u})_3 \\ (A\vec{v})_1 & (A\vec{v})_2 & (A\vec{v})_3 \end{vmatrix} = \det(X), \tag{B.20}$$

where  $(A\vec{e}_r)_1$  denotes the projection of  $A\vec{e}_r$  onto  $e_1$ , and so on. Now, observe that

$$A(\vec{e}_r | \vec{u} | \vec{v}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \vec{e}_{r1} & \vec{u}_1 & \vec{v}_1 \\ \vec{e}_{r2} & \vec{u}_2 & \vec{v}_2 \\ \vec{e}_{r3} & \vec{u}_3 & \vec{v}_3 \end{pmatrix} = \begin{pmatrix} (A\vec{e}_r)_1 & (A\vec{u})_1 & (A\vec{v})_1 \\ (A\vec{e}_r)_2 & (A\vec{u})_2 & (A\vec{v})_2 \\ (A\vec{e}_r)_3 & (A\vec{u})_3 & (A\vec{v})_3 \end{pmatrix} = X^T. \tag{B.21}$$



Returning to (B.19) we have that

$$\begin{aligned}
A\vec{u} \times A\vec{v} &= \sum_{r=1}^3 \det(X) A\vec{e}_r \\
&= \sum_{r=1}^3 \det(X^T) A\vec{e}_r \\
&= \sum_{r=1}^3 \det\left(A \left(\vec{e}_r | \vec{u} | \vec{v}\right)\right) A\vec{e}_r \\
&= \sum_{r=1}^3 \det(A) \det\left(\left(\vec{e}_r | \vec{u} | \vec{v}\right)^T\right) A\vec{e}_r \\
&= \det(A) \sum_{r=1}^3 \left(\left(\vec{u} \times \vec{v}\right) \cdot \vec{e}_r\right) A\vec{e}_r, \tag{B.22}
\end{aligned}$$

we we have used the elementary fact that  $\det(M) = \det(M^T) \forall M \in M_n(\mathbb{R})$ . Expanding  $\vec{u} \times \vec{v}$  in terms of the orthonormal basis  $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ , and using (B.17), we finally get that

$$A(\vec{u} \times \vec{v}) = A \sum_{r=1}^3 \left(\left(\vec{u} \times \vec{v}\right) \cdot \vec{e}_r\right) \vec{e}_r = \sum_{r=1}^3 \left(\left(\vec{u} \times \vec{v}\right) \cdot \vec{e}_r\right) A\vec{e}_r \implies \det(A) = 1. \tag{B.23}$$

We have already established that  $T$  is a linear Euclidean isometry, from which it immediately follows that  $AA^T = \mathbf{1}$ <sup>1</sup>. We conclude that  $T$  defines a rotation on  $\mathbb{R}^3$  about the axis defined by  $\vec{t}$  through an angle  $\theta$ .

Let us pause to introduce some notational conventions that will simplify the calculation of the rotation angle following [50]. Let  $p, q \in \mathbb{H}$  with  $p = 1p_0 + ip_1 + jp_2 + kp_3 = p_0 + f(\vec{p})$  and  $q = 1q_0 + iq_1 + jq_2 + kq_3 = q_0 + f(\vec{q})$  where  $\vec{p} = \hat{x}p_1 + \hat{y}p_2 + \hat{z}p_3$ . Furthermore, let us adopt the convention wherein we identify the basis elements  $\hat{x} \leftrightarrow i$ ,  $\hat{y} \leftrightarrow j$ , and  $\hat{z} \leftrightarrow k$ , so that we can write  $f(\vec{r})$  simply as  $\vec{r}$ . Further still, let us write  $p = (p_0, \vec{p})$  and  $q = (q_0, \vec{q})$  so that a separation of the real and imaginary parts is made even more explicit and so that quaternion multiplication becomes simply

$$pq = (p_0q_0 - \vec{p} \cdot \vec{q}, p_0\vec{q} + q_0\vec{p} + \vec{p} \times \vec{q}). \tag{B.24}$$

Now, let  $f(\vec{r}) = (u_0, \vec{u})$  and compute  $f(T(\vec{r})) = tf(\vec{r})\vec{t}$  as

$$\begin{aligned}
(t_0u_0 - \vec{t} \cdot \vec{u}, t_0\vec{u} + u_0\vec{t} + \vec{t} \times \vec{u})(t, -\vec{t}) &= \left((t_0u_0 - \vec{t} \cdot \vec{u})t_0 + (t_0\vec{u} + u_0\vec{t} + \vec{t} \times \vec{u}) \cdot \vec{q}, \right. \\
&\left. (t_0u_0 - \vec{t} \cdot \vec{u})(-\vec{t}) + t_0(t_0\vec{u} + u_0\vec{t} + \vec{t} \times \vec{u}) + (t_0\vec{u} + u_0\vec{t} + \vec{t} \times \vec{u}) \times (-\vec{t})\right) \tag{B.25}
\end{aligned}$$

We have already argued that the real part must vanish, which is readily verified given that  $u_0 = 0$  and  $(\vec{t} \times \vec{u}) \cdot \vec{t} = 0$ . Making use of vector identities (in particular  $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$  and  $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$ ) it follows that

$$T(\vec{r}) = \vec{u}' = (t_0^2 - \vec{t} \cdot \vec{t})\vec{u} + 2t_0(\vec{t} \times \vec{u}) + 2(\vec{t} \cdot \vec{u})\vec{t}. \tag{B.26}$$

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<sup>1</sup> $\delta_{rs} = \vec{e}_r \cdot \vec{e}_s = A\vec{e}_r \cdot A\vec{e}_s = \sum_{t=1}^3 A_{r,t}A_{t,s} \implies AA^T = \mathbf{1}$ .

Now, choosing  $\vec{u} = \vec{t} \times \vec{a}$  with  $|\vec{a}| = 1$  and  $\vec{a} \cdot \vec{t} \neq 1$  we have the angle of rotation  $\theta$  defined through

$$\vec{u}' \cdot \vec{u} = |\vec{u}'||\vec{u}|\cos\theta, \quad (\text{B.27})$$

and we also have that

$$\vec{u}' \cdot \vec{u} = \left( (t_0^2 - \vec{t} \cdot \vec{t})\vec{u} + 2t_0(\vec{t} \times \vec{u}) + 2(\vec{t} \cdot \vec{u})\vec{t} \right) \cdot \vec{u} = (t_0^2 - \vec{t} \cdot \vec{t})|\vec{u}|^2. \quad (\text{B.28})$$

Given that  $T$  is an isometry it follows that

$$\cos\theta = t_0^2 - \vec{t} \cdot \vec{t}, \quad (\text{B.29})$$

and given  $1 = |t| = t_0^2 + \vec{t} \cdot \vec{t}$  we have that

$$t_0^2 = \frac{1}{2}(\cos\theta + 1) \implies t_0 = \pm\cos\left(\frac{\theta}{2}\right), \quad (\text{B.30})$$

and we also have that

$$|\vec{t}|^2 = \frac{1}{2}(\cos\theta - 1) \implies \vec{t} = \pm\sin\left(\frac{\theta}{2}\right). \quad (\text{B.31})$$

Choosing the positive solution, we thus have

$$t = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\hat{\omega}, \quad (\text{B.32})$$

where the image of  $\hat{\omega} = \frac{f(\vec{t})}{|\vec{t}|}$  under  $f^{-1}$  is along the axis of rotation defined by  $\vec{t}$ . It is a trivial consequence of the definition of  $T$  that  $-t \in \text{Sp}(1)$  implements the same rotation as the one defined through  $t$ :

$$(-t)f(\vec{r})(\overline{-t}) = tf(\vec{r})\bar{t}, \quad (\text{B.33})$$

and it follows that  $\text{Sp}(1)/\{\pm 1\} \cong \text{SO}(3)$ . Indeed,  $T^{-1}$  is defined simply through  $\bar{t}$ , and  $T \circ Q$  is defined simply through  $h = tq \in \text{Sp}(1)$ .

# Appendix C

## Universal $C^*$ -algebras

In Section 6.2, we discuss one possible route that has been pointed out by Barnum [7] for the development of an explicit description of composite systems in quaternionic quantum theory via the universal tensor product introduced in [45] and defined in 6.2. The universal tensor product of  $JC$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$  is a  $JC$ -subalgebra of the maximal tensor product of their corresponding universal  $C^*$ -algebras. In this appendix, we explicitly calculate the universal  $C^*$ -algebras enveloping  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$  for  $d \geq 3$ . In Section C.1, we recall prerequisite algebraic definitions. In Section C.2, we recall theorems due to Hanche-Olsen and Størmer concerning universal  $C^*$ -algebras enveloping  $JC$ -algebras and we carry out the aforementioned explicit computation.

### C.1 Preliminaries

In this section, we recall some basic definitions following [2] and [46]. Recall that an *algebra* is a real or complex vector space equipped with a bilinear product. A *unital algebra* is an algebra equipped with a multiplicative identity. An algebra  $\mathcal{A}$  is said to be *normed* if it is equipped with a norm such that  $\forall a, b \in \mathcal{A}, \|ab\| \leq \|a\|\|b\|$ . An associative normed algebra which is complete is called a *Banach algebra*. A *Jordan algebra* is an algebra  $\mathcal{A}$  equipped with a commutative bilinear product denoted by  $a \bullet b$  satisfying  $a \bullet (b \bullet a^2) = (a \bullet b) \bullet a^2 \forall a, b \in \mathcal{A}$ . A *JB-algebra* is a Jordan algebra which is also a Banach algebra in which the norm satisfies the additional constraints  $\|a^2\| = \|a\|^2$  and  $\|a^2 + b^2\| \leq \|a^2\| \forall a, b \in \mathcal{A}$ . A *JC-algebra* is a *JB-algebra* that is isometrically isomorphic to a norm-closed Jordan subalgebra of the self-adjoint subspace of a set of bounded linear operators acting on a complex Hilbert space. In finite dimensions, *JB-algebras are JC-algebras*, except for  $M_3(\mathbb{O})_{sa}$ . A *\*-algebra* is an associative algebra  $\mathcal{A}$  equipped with an involution  $a \mapsto a^*$  satisfying  $(a+b)^* = a^* + b^*$ ,  $(\alpha a)^* = \bar{\alpha} a^*$ ,  $a^{**} = a$ , and  $(ab)^* = b^* a^* \forall a, b \in \mathcal{A}$  and  $\forall \alpha$  in the underlying field. A linear map  $\Phi$  from a \*-algebra  $\mathcal{A}_1$  to a \*-algebra  $\mathcal{A}_2$  is called a *\*-homomorphism* if  $\Phi(x^*) = \Phi(x)^*$  and  $\Phi(xy) = \Phi(x)\Phi(y) \forall x, y \in \mathcal{A}_1$ , and it is called a *\*-anti-homomorphism* if  $\Phi(x^*) = \Phi(x)^*$  and  $\Phi(xy) = \Phi(y)\Phi(x) \forall x, y \in \mathcal{A}_1$ . The terms *\*-isomorphism*, *\*-anti-isomorphism*, *\*-automorphism*, and *\*-anti-automorphism* are used accordingly. A  *$C^*$ -algebra* is a complex Banach \*-algebra  $\mathfrak{B}$  such that  $\|x^*x\| = \|x\|^2 \forall x \in \mathfrak{B}$ . If  $\mathfrak{B}$  is a  $C^*$ -algebra, then the set of all self-adjoint elements

$x \in \mathfrak{B}$  such that  $x = x^*$  is denoted  $\mathfrak{B}_{sa}$ .

## C.2 Theorems and Computation

In this section we compute the universal  $C^*$ -algebra enveloping  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$  for all  $d \geq 3$  in Example C.2.3. We begin by recalling a theorem due to Hanche-Olsen and Størmer [46]:

**Theorem C.2.1** ([46], p. 152) *Existence of Universal  $C^*$ -algebras:*

Let  $\mathcal{A}$  be a  $JB$ -algebra. Then there exists up to isomorphism a unique  $C^*$ -algebra  $C_u^*(\mathcal{A})$  and a homomorphism  $\psi_{\mathcal{A}} : \mathcal{A} \rightarrow C_u^*(\mathcal{A})_{sa}$  such that:

- $\psi_{\mathcal{A}}(\mathcal{A})$  generates  $C_u^*(\mathcal{A})$  as a  $C^*$ -algebra.
- If  $\mathfrak{B}$  is a  $C^*$ -algebra and  $\pi : \mathcal{A} \rightarrow \mathfrak{B}_{sa}$  is a homomorphism, then there exists a  $*$ -homomorphism  $\hat{\pi} : C_u^*(\mathcal{A}) \rightarrow \mathfrak{B}$  such that  $\pi = \hat{\pi} \circ \psi_{\mathcal{A}}$ .
- There is a involutive  $*$ -anti-automorphism  $\Phi$  of  $C_u^*(\mathcal{A})$  such that  $\Phi(\psi_{\mathcal{A}}(a)) = \psi_{\mathcal{A}}(a) \forall a \in \mathcal{A}$ .

*Proof:* See [46].

One refers to  $C_u^*(\mathcal{A})$  as the *universal  $C^*$ -algebra enveloping  $\mathcal{A}$* . The following morphism diagram is useful to keep in mind:

$$\begin{array}{ccc} C_u^*(\mathcal{A}) & \xrightarrow{\hat{\pi}} & \mathfrak{B} \\ \psi_{\mathcal{A}} \uparrow & & \uparrow \\ \mathcal{A} & \xrightarrow{\pi} & \mathfrak{B}_{sa} \end{array}$$

Recall that a  $JC$ -algebra  $\mathcal{A}$  is said to be *reversible* if

$$a_1, a_2, \dots, a_n \in \mathcal{A} \implies a_1 a_2 \cdots a_n + a_n a_{n-1} \cdots a_1 \in \mathcal{A}. \quad (\text{C.1})$$

$\mathcal{M}_{d,d}(\mathbb{H})_{sa}$  is clearly reversible, for if  $\{a_1, a_2, \dots, a_n\} \subset \mathcal{M}_{d,d}(\mathbb{H})_{sa}$  then

$$(a_1 a_2 \cdots a_n + a_n a_{n-1} \cdots a_1)^* = a_n^* a_{n-1}^* \cdots a_1^* + a_1^* a_2^* \cdots a_n^* = a_1 a_2 \cdots a_n + a_n a_{n-1} \cdots a_1. \quad (\text{C.2})$$

Furthermore, a  $JC$ -algebra  $\mathcal{A}$  is said to be *universally reversible* if  $\pi(\mathcal{A})$  is reversible for each concrete representation  $\pi$  of  $\mathcal{A}$  on an algebra of bounded linear operators on a complex Hilbert space.  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$  is universally reversible for all  $d \geq 3$  [68]. In [45] Hanche-Olsen proves the following theorem characterizing the universal  $C^*$ -algebra enveloping a universally reversible  $JC$ -algebra:

**Theorem C.2.2** ([45], p. 1069) *Characterization of  $C_u^*(\mathcal{A})$  for universally reversible JC-algebras:*

*Assume  $\mathcal{A}$  is a universally reversible JC-algebra, that  $\mathfrak{B}$  is a  $C^*$ -algebra, and that  $\theta : \mathcal{A} \rightarrow \mathfrak{B}_{sa}$  is an injective homomorphism such that  $\theta(\mathcal{A})$  generates  $\mathfrak{B}$ . If  $\mathfrak{B}$  admits an anti-automorphism  $\varphi$  such that  $\varphi \circ \theta = \theta$ , then  $\hat{\theta}$  is a  $*$ -isomorphism of  $C_u^*(\mathcal{A})$  onto  $\mathfrak{B}$ .*

*Proof:* See [45].

Concretely, Theorem C.2.2 tells us how to compute the universal  $C^*$ -algebra enveloping a universally reversible JC-algebra  $\mathcal{A}$ . In particular, if one can find an injective homomorphism  $\theta$  of  $\mathcal{A}$  into a  $C^*$ -algebra  $\mathfrak{B}$  that generates that  $\mathfrak{B}$ , and if one can find an  $*$ -anti-automorphism of  $\mathfrak{B}$  fixing the image of  $\mathcal{A}$  under injection by  $\theta$ , then  $\mathfrak{B}$  is the universal  $C^*$ -algebra enveloping  $\mathcal{A}$  up to isomorphism. In [45], it is stated that  $\mathcal{M}_{2d,2d}(\mathbb{C})$  is the universal  $C^*$ -algebra enveloping  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$  when  $d \geq 3$ , which we prove explicitly in Example C.2.3.

**Example C.2.3** *The Universal  $C^*$ -algebra enveloping  $\mathcal{M}_{d,d}(\mathbb{H})_{sa}$ :*

$$\forall d \geq 3 : C_u^*(\mathcal{M}_{d,d}(\mathbb{H})_{sa}) = \mathcal{M}_{2d,2d}(\mathbb{C}). \quad (\text{C.3})$$

*Proof:*

To begin the proof, recall the  $*$ -homomorphic injection  $\psi_{d,d} : \mathcal{M}_{d,d}(\mathbb{H})_{sa} \rightarrow \mathcal{M}_{2d,2d}(\mathbb{C})_{sa}$  defined for  $a = \Gamma_1 + \Gamma_2 j \in \mathcal{M}_{d,d}(\mathbb{H})_{sa}$  via

$$\psi_{d,d}(a) = \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ -\Gamma_2 & \Gamma_1 \end{pmatrix}. \quad (\text{C.4})$$

We will first show that  $\psi_{d,d}(\mathcal{M}_{d,d}(\mathbb{H})_{sa})$  generates  $\mathcal{M}_{2d,2d}(\mathbb{C})_{sa}$  as a  $C^*$ -algebra. Let us introduce some notation. The set of standard basis vectors for  $\mathbb{C}^d$  will be denoted by  $\{|r\rangle\}_{r=1}^d$ , and the set of standard basis vectors for  $\mathbb{C}^{2d}$  will be denoted by  $\{||r\rangle\rangle\}_{r=1}^{2d}$ , with the standard orthonormality relations  $\langle s|r\rangle = \delta_{rs}$  and  $\langle\langle s||r\rangle\rangle = \delta_{rs}$ .

From now on, let  $r \neq s$ . Choose  $Q_{rs} \in \mathcal{M}_{d,d}(\mathbb{H})_{sa}$  such that  $Q_{rs} = i|r\rangle\langle s| - i|s\rangle\langle r| \implies$

$$\psi_{d,d}(Q_{rs}) = i||r\rangle\rangle\langle\langle s|| - i||s\rangle\rangle\langle\langle r|| = i||r+n\rangle\rangle\langle\langle s+n|| + i||s+n\rangle\rangle\langle\langle r+n||. \quad (\text{C.5})$$

Choose  $P_{rs} \in \mathcal{M}_{d,d}(\mathbb{H})_{sa}$  such that  $P_{rs} = |r\rangle\langle s| + |s\rangle\langle r| \implies$

$$\psi_{d,d}(P_{rs}) = ||r\rangle\rangle\langle\langle s|| + ||s\rangle\rangle\langle\langle r|| = ||r+n\rangle\rangle\langle\langle s+n|| + ||s+n\rangle\rangle\langle\langle r+n||. \quad (\text{C.6})$$

Then

$$\frac{1}{2i} \left( \psi_{d,d}(Q_{rs}) + i\psi_{d,d}(P_{rs}) \right) = ||r\rangle\rangle\langle\langle s|| + ||s+n\rangle\rangle\langle\langle r+n||. \quad (\text{C.7})$$

Choose  $R_r \in \mathcal{M}_{d,d}(\mathbb{H})_{sa}$  such that  $R_r = |r\rangle\langle r| \implies$

$$\psi_{d,d}(R_r) = ||r\rangle\rangle\langle\langle r|| + ||r+n\rangle\rangle\langle\langle r+n||. \quad (\text{C.8})$$

Choose  $S_s \in \mathcal{M}_{d,d}(\mathbb{H})_{sa}$  such that  $S_s = |s\rangle\langle s| \implies$

$$\psi_{d,d}(S_s) = ||s\rangle\rangle\langle\langle s|| + ||s+n\rangle\rangle\langle\langle s+n||. \quad (\text{C.9})$$

Then,

$$\psi_{d,d}(R_r) \left( \frac{1}{2i} \left( \psi_{d,d}(Q_{rs}) + i\psi_{d,d}(P_{rs}) \right) \right) = ||r\rangle\rangle\langle\langle s||, \quad (\text{C.10})$$

$$\psi_{d,d}(S_s) \left( \frac{1}{2i} \left( \psi_{d,d}(Q_{rs}) + i\psi_{d,d}(P_{rs}) \right) \right) = ||s+n\rangle\rangle\langle\langle r+n||. \quad (\text{C.11})$$

Also, notice that

$$-\frac{1}{4} \left( i\psi_{d,d}(S_s) + \psi_{d,d}(P_{rs})\psi_{d,d}(R)\psi_{d,d}(Q_{rs}) \right)^2 = ||s\rangle\rangle\langle\langle s||, \quad (\text{C.12})$$

$$-\frac{1}{4} \left( i\psi_{d,d}(S_s) - \psi_{d,d}(P_{rs})\psi_{d,d}(R)\psi_{d,d}(Q_{rs}) \right)^2 = ||s+n\rangle\rangle\langle\langle s+n||. \quad (\text{C.13})$$

So, for arbitrary  $\xi \in \mathcal{M}_{d,d}(\mathbb{C})$ , we can generate the following elements of  $\mathcal{M}_{2d,2d}(\mathbb{C})$ :

$$\Xi_u = \begin{pmatrix} \xi & 0 \\ 0 & 0 \end{pmatrix}, \quad \Xi_b = \begin{pmatrix} 0 & 0 \\ 0 & \xi \end{pmatrix}. \quad (\text{C.14})$$

Choose  $W_{rs} \in \mathcal{M}_{d,d}(\mathbb{H})_{sa}$  such that  $W_{rs} = j|r\rangle\langle s| - j|s\rangle\langle r| \implies$

$$\psi_{d,d}(W_{rs}) = ||r\rangle\rangle\langle\langle s+n|| - ||s\rangle\rangle\langle\langle r+n|| - ||r+n\rangle\rangle\langle\langle s|| + ||s+n\rangle\rangle\langle\langle r||. \quad (\text{C.15})$$

It follows that

$$\psi_{d,d}(P_{rs})\psi_{d,d}(W_{rs}) = -||r\rangle\rangle\langle\langle r+n|| - ||s\rangle\rangle\langle\langle s+n|| + ||r+n\rangle\rangle\langle\langle r|| + ||s+n\rangle\rangle\langle\langle s||, \quad (\text{C.16})$$

and so

$$\sum_{r=1}^d \psi_{d,d}(R_r)\psi_{d,d}(P_{rs})\psi_{d,d}(W_{rs}) = \sum_{r=1}^d (-||r\rangle\rangle\langle\langle r+n|| + ||r+n\rangle\rangle\langle\langle r||) = X = \begin{pmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix}. \quad (\text{C.17})$$

Applying  $X$  to  $\Xi_u$  or  $\Xi_b$  we see that  $\psi_{d,d}(\mathcal{M}_{d,d}(\mathbb{H})_{sa})$  generates  $\mathcal{M}_{2d,2d}(\mathbb{C})$ .

It remains to show that there exists an involutive  $*$ -anti-automorphism of  $\mathcal{M}_{2d,2d}(\mathbb{C})$  that acts as the identity on  $\psi_{d,d}(\mathcal{M}_{d,d}(\mathbb{H})_{sa})$ . Let

$$J = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}, \quad (\text{C.18})$$

where  $\mathbf{0}$  and  $\mathbf{1}$  are the zero and unit matrices in  $\mathcal{M}_{d,d}(\mathbb{C})$  respectively. Notice that  $J^T = J^{-1} = -J$  and  $J^2 = -\mathbf{1}$ . Let

$$\Phi : \mathcal{M}_{2d,2d}(\mathbb{C}) \rightarrow \mathcal{M}_{2d,2d}(\mathbb{C}) \quad (\text{C.19})$$

be defined  $\forall X \in \mathcal{M}_{2d,2d}(\mathbb{C})$  via

$$\Phi(X) = (J^T X J)^T. \quad (\text{C.20})$$

$\Phi$  is  $\mathbb{C}$ -linear. Indeed,  $\forall \alpha, \beta \in \mathbb{C}$  and  $\forall X, Y \in \mathcal{M}_{2d,2d}(\mathbb{C})$  one has that

$$\Phi(\alpha X + \beta Y) = \alpha \Phi(X) + \beta \Phi(Y). \quad (\text{C.21})$$

Direct computation yields

$$\Phi^2(X) = \Phi(\Phi(X)) = (J^T (J^T X J)^T J)^T = (J^T J^T X^T J J)^T = X, \quad (\text{C.22})$$

$$\Phi(X)^* = ((J^T X J)^T)^* = \overline{(J^T X J)} = J^T \overline{X} J = (J^T X^* J)^T = \Phi(X^*), \quad (\text{C.23})$$

$$\Phi(XY) = (J^T X Y J)^T = J^T Y^T X^T J = J^T Y^T J J^T X^T J = (J^T Y J)^T (J^T X J)^T = \Phi(Y) \Phi(X). \quad (\text{C.24})$$

Let

$$g : \mathcal{M}_{2d,2d}(\mathbb{C}) \rightarrow \mathcal{M}_{2d,2d}(\mathbb{C}) \quad (\text{C.25})$$

be defined  $\forall Y \in \mathcal{M}_{2d,2d}(\mathbb{C})$  via

$$g(Y) = (J Y J^T)^T. \quad (\text{C.26})$$

It follows  $\forall X \in \mathcal{M}_{2d,2d}(\mathbb{C})$  that  $g(\Phi(X)) = (J \Phi(X) J^T)^T = (J J^T X^T J J^T)^T = X$ , so  $g = \Phi^{-1}$ . One also has that

$$\Phi(X) = \Phi(Y) \iff (J^T X J)^T = (J^T Y J)^T \iff J^T X J = J^T Y J \iff X = Y. \quad (\text{C.27})$$

All of the above demonstrates that  $\Phi$  is an involutive \*-anti-automorphism of  $\mathcal{M}_{2d,2d}(\mathbb{C})$ . The remarkable property of  $\Phi$  is that

$$\Phi(\psi_{d,d}(a)) = \psi_{d,d}(a) \quad \forall a \in \mathcal{M}_{d,d}(\mathbb{H})_{sa}. \quad (\text{C.28})$$

Therefore, by virtue of Theorem C.2.2, the proof is complete.  $\square$

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