Free semigroup algebras and the structure of an isometric tuple

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

An n-tuple of operators $V = (V_1, \dots, V_n)$ acting on a Hilbert space H is said to be isometric if the operator $[V_1 \ \cdots \ V_n] : H^n \to H$ is an isometry. A free semigroup algebra is the weakly closed algebra $\mathrm{W}(V_1, \dots, V_n)$ generated by an isometric n-tuple V. The structure of the free semigroup algebra generated by V contains a great deal of information about V. Thus it is natural to study this algebra in order to study V.

A free semigroup algebra is said to be analytic if it is isomorphic to the noncommutative analytic Toeplitz algebra, which is a higher-dimensional generalization of the classical algebra H^∞ of bounded analytic functions on the complex unit disk. This notion of analyticity is of central importance in the general theory of free semigroup algebras. A vector x in H is said to be wandering for an isometric n-tuple V if the set

$$\{x\} \cup \{V_{i_1} \cdots V_{i_k} x \mid 1 \leq i_1, \dots, i_k \leq n \text{ and } k \geq 1\}$$

is orthonormal. As in the classical case of H^∞, the analytic structure of the noncommutative analytic Toeplitz algebra is determined by the existence of wandering vectors for the generators of the algebra.

In the first part of this thesis, we prove the following dichotomy: either an isometric n-tuple V has a wandering vector, or the free semigroup algebra it generates is a von Neumann algebra. This implies the existence of wandering vectors for every analytic free semigroup algebra. As a consequence, it follows that every free semigroup algebra is reflexive, in the sense that it is completely determined by its invariant subspace lattice.

In the second part of this thesis we prove a decomposition for an isometric tuple of operators which generalizes the classical Lebesgue-von Neumann-Wold decomposition of an isometry into the direct sum of a unilateral shift, an absolutely continuous unitary and a singular unitary. The key result is an operator-algebraic characterization of an absolutely continuous isometric tuple in terms of analyticity. We show that, as in the classical case, this decomposition determines the weakly closed algebra and the von Neumann algebra generated by the tuple.

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To L.Z.

Table of Contents

1 Introduction **1**

2 Wandering vectors and the reflexivity of free semigroup algebras **4**

2.1 Background and preliminaries 7

2.2 The noncommutative Toeplitz operators 10

2.3 Wandering vectors 16

2.4 The hyperreflexivity of free semigroup algebras 31

3 The structure of an isometric tuple **38**

3.1 Motivation 40

3.2 Background and preliminaries 42

3.3 Absolutely continuous isometric tuples 48

3.4 Singular isometric tuples 67

3.5 The Lebesgue-von Neumann-Wold decomposition 68

Bibliography **72**

Chapter 1

Introduction

This thesis concerns the structure of an isometric tuple of operators, an object that appears frequently in mathematics and mathematical physics. From the perspective of an operator theorist, the notion of an isometric tuple is a natural higher-dimensional generalization of the notion of an isometry.

An n-tuple of operators $(V_1, \ldots, V_n)$ acting on a Hilbert space H is said to be isometric if the row operator $[V_1 \ \cdots \ V_n] : H^n \to H$ is an isometry. This is equivalent to requiring that the operators $V_1, \ldots, V_n$ satisfy the algebraic relations

$$V_i^* V_j = \begin{cases} I & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

These relations are often referred to as the Cuntz relations.

The weakly closed (non-self-adjoint) algebra $\mathrm{W}(V_1, \ldots, V_n)$ generated by V is called the free semigroup algebra generated by V. It turns out that the structure of this algebra contains a great deal of information about V. Thus it is natural to study this algebra in order to study V. This idea, along with the definition of a free semigroup algebra, was introduced by Davidson and Pitts in [DP99]. Free semigroup algebras, and various generalizations, have subsequently been studied by a number of authors, and many applications have been found (see for example [Dav01]).

The work of Davidson, Katsoulis and Pitts in [DKP01] revealed that the notion of analyticity is of central importance in the general theory of free semigroup algebras. A free semigroup algebra is said to be analytic if it is isomorphic to the noncommutative analytic Toeplitz algebra. This free semigroup algebra, introduced by Davidson and Pitts in [DP98], is a higher-dimensional generalization of the algebra H^∞ of bounded analytic functions on the complex unit disk. We will also say that an isometric tuple is analytic if the free semigroup algebra it generates is analytic. The general structure theorem for free semigroup algebras obtained in [DKP01] implies that every free semigroup algebra can be decomposed as the sum of a slice of a von Neumann algebra and an analytic free semigroup algebra.

The analytic structure of an operator algebra often reveals itself in the form of wandering vectors. A vector x is said to be wandering for the isometric n-tuple V if the set of vectors

$$\{x\} \cup \{V_{i_1} \cdots V_{i_k} x \mid 1 \leq i_1, \ldots, i_k \leq n \text{ and } k \geq 1\}$$

is orthonormal. In this case, we will also say that x is wandering for the free semigroup algebra generated by V.

The main result in the first part of this thesis is a proof of the existence of wandering vectors for an analytic free semigroup algebra. In fact, we prove the following stronger dichotomy: either a free semigroup algebra has a wandering vector, or it is a von Neumann algebra. This result implies that every isometric tuple is reflexive, which means that the free semigroup algebra it generates is completely determined by its invariant subspaces. As an application of this result, we show that every analytic free semigroup algebra satisfies a very strong factorization property. This implies that every anaytic free semigroup algebra is actually hyperreflexive, which is a stronger quantitative form of reflexivity.

The existence of wandering vectors for an analytic free semigroup algebra was conjectured by Davidson, Katsoulis and Pitts in [DKP01]. They observed that it was equivalent to the question of the reflexivity of an arbitrary free semigroup algebra, and more generally, to the invariant subspace problem for an isometric tuple.

The main result in the second part of this thesis is a decomposition of an isometric tuple that generalizes the classical Lebesgue-von Neumann-Wold decomposition of an isometry into the direct sum of a unilateral shift, an absolutely continuous unitary and a singular unitary. We show that, as in the classical case, this decomposition determines the structure of the weakly closed algebra and the von Neumann algebra generated by the tuple.

The existence of a higher-dimensional Lebesgue-von Neumann-Wold decomposition was conjectured by Davidson, Li and Pitts in [DLP05]. They observed that the measure-theoretic definition of an absolutely continuous operator was equivalent to an operator-theoretic property of the functional calculus for that operator. Since this property naturally extends to the higher-dimensional setting, this allowed them to define the notion of an absolutely continuous isometric tuple.

To develop the technical portion of this thesis, we extend ideas from the commutative theory of dual algebras to the present noncommutative setting. The commutative theory, based on Brown's proof of the existence of invariant subspaces for subnormal operators [Bro78], was developed and applied with great success by Bercovici, Brown, Foias, Pearcy and many others (see for example [BFP85]). We were inspired to use this approach by Bercovici's results in [Ber98].

In Chapter 2 we prove the existence of wandering vectors for an analytic free semigroup algebra, and obtain as a consequence the reflexivity of an arbitrary free semigroup algebra and the hyperreflexivity of an analytic free semigroup algebra. In Chapter 3 we prove the Lebesgue-von Neumann-Wold decomposition of an an isometric tuple, and determine the structure of the free semigroup algebra and the von Neumann algebra generated by an isometric tuple.

The content comprising Chapter 2 and Chapter 3 of this thesis was taken from two different papers. While we have attempted to eliminate any inconsistencies in the material, the reader may notice a small amount of overlap in the preliminary sections of these chapters.

Chapter 2

Wandering vectors and the reflexivity of free semigroup algebras

A **free semigroup algebra** $\mathcal{S}$ is the weak-operator-closed (non-self-adjoint) algebra generated by n isometries $S_1, ..., S_n$ on a Hilbert space H which have pairwise orthogonal ranges, or equivalently, which satisfy

$$S_i^* S_j = \begin{cases} I & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Although n can be finite or infinite, for notational convenience we treat n as finite and make note of any issues that arise. We say that the n-tuple $S = (S_1 \ \dots \ S_n)$ is **isometric**, since the row operator $[V_1 \ \cdots \ V_n] : H^n \to H$ is an isometry.

Isometric tuples appear throughout operator theory. A theorem of Frazho, Bunce, and Popescu shows that n operators $A_1, ..., A_n$ which satisfy $\sum A_k A_k^* \leq I$ can be dilated to an isometric n-tuple $S = (S_1, \dots, S_n)$ such that each S_k is of the form

$$S_k = \begin{pmatrix} A_k & 0 \\ * & * \end{pmatrix}.$$

This is a noncommutative multivariable analogue of the Sz.-Nagy dilation theorem.

Popescu [Pop96] showed that the norm-closed algebra generated by any row isometry of size n is completely isometrically isomorphic to the noncommutative disk algebra $\mathcal{A}_n$, and it is well known that the C*-algebra generated by a row isometry of size n is isomorphic to the Cuntz algebra $\mathcal{O}_n$ if $\sum A_k A_k^* = I$, and otherwise is isomorphic to the Cuntz-Toeplitz algebra $\mathcal{E}_n$. By contrast, the weak-operator-closed algebras generated by distinct isometric tuples can be dramatically different (see for example [DKP01]).

In some sense then, it is natural to study an isometric tuple by looking at the free semigroup algebra it generates. This idea, and with it the definition of a free semigroup algebra, was introduced by Davidson and Pitts [DP99]. They observed that free semigroup algebras often contain interesting information about the unitary invariants of their generators.

The prototypical example of a free semigroup algebra is the noncommutative analytic Toeplitz algebra generated by the left regular representation of the free semigroup on n letters. This algebra, which we denote by $\mathcal{L}_n$, was first studied by Popescu [Pop91] in the context of noncommutative multivariable dilation theory.

For $n = 1$, $\mathcal{L}_n$ is the familiar algebra of analytic Toeplitz operators, which is singly generated by the unilateral shift. For $n \geq 2$, $\mathcal{L}_n$ is no longer commutative, but it turns out that a number of classical results about the analytic Toeplitz operators have straightforward generalizations to this setting. This is a large part of the motivation for the name "noncommutative analytic Toeplitz algebra."

The role of $\mathcal{L}_n$ is of central importance in the general theory of free semigroup algebras, and it turns out to be desirable to isolate "$\mathcal{L}_n$-like" behavior. A free semigroup algebra is said to be analytic if it is algebraically isomorphic to $\mathcal{L}_n$. It is important to emphasize the word "algebraically" here. Examples have been constructed (see for example [DKP01]) of free semigroup algebras which are analytic, and so behave algebraically like $\mathcal{L}_n$, but which have a very different spatial structure.

The general structure theorem for free semigroup algebras [DKP01] shows that every free semigroup algebra can be decomposed into 2×2 block-lower-triangular form, where the left column is a slice of a von Neumann algebra, and the bottomright entry is an analytic free semigroup algebra. It is well known (see for example [Wer52]) that the weak-operator-closed algebra generated by a single isometry can be self-adjoint. Davidson, Katsoulis, and Pitts [DKP01] asked whether it was possible for a free semigroup algebra on 2 or more generators to be self-adjoint, and some time later Read [Read05] (see also [Dav06]) answered in the affirmative by showing that $\mathcal{B}(H)$ was a free semigroup algebra.

A notion of fundamental importance is that of a wandering vector. A unit vector x is said to be wandering for the free semigroup algebra generated by an isometric n-tuple $(S_1, \ldots, S_n)$ if the set of vectors

$$\{x\} \cup \{S_{i_1} \cdots S_{i_k} x \mid 1 \leq i_1, \ldots, i_k \leq n \text{ and } k \geq 1\}$$

is orthonormal. It is known (see for example [DP99]) that the spatial structure of $\mathcal{L}_n$ is completely determined by the existence of a large number of wandering vectors.

It is easy to see that the restriction of any free semigroup algebra to the cyclic subspace generated by a wandering vector is unitarily equivalent to $\mathcal{L}_n$, and so in particular is analytic. It has been an open question for some time, however, whether every analytic free semigroup algebra necessarily has a wandering vector. It turns out that this question is equivalent to the question of whether every free semigroup algebra is reflexive. This can be shown using the general structure theorem for free semigroup algebras: since every von Neumann algebra is reflexive, the reflexivity of a free semigroup algebra depends on the reflexivity of its analytic part.

The purpose of this chapter is to prove that every analytic free semigroup algebra has wandering vectors, and hence to prove that every free semigroup algebra is reflexive.

Our approach is very much in the spirit of the "dual algebra arguments" which have been used with great success by Bercovici, Foias, Pearcy and many others (see for example [BFP85]), and which are based on Brown's proof of the existence of invariant subspaces for subnormal operators [Bro78]. The fundamental idea at the heart of these arguments is that it is often possible to prove the existence of invariant subspaces for a weak*-closed operator algebra by showing that, in an appropriate sense, the predual of the algebra is small.

Typically, these arguments are employed in a commutative setting, where certain spectral and function-theoretic tools are available. In the present noncommutative context, we rely instead on various operator-theoretic techniques.

A clue that it might be possible to attack the present problem using dual algebra techniques came from a recent paper of Bercovici [Ber98], who used them to establish the hyperreflexivity of a class of algebras which includes the noncommutative analytic Toeplitz algebra on two or more generators. The hyperreflexivity of this algebra had already been shown by Davidson and Pitts [DP99], with an upper bound of 51 on the hyperreflexivity constant, but Bercovici's approach yielded a surprisingly low upper bound of 3.

Motivated by Bercovici's result, once we have shown that every analytic free semigroup algebra has a wandering vector, we go further and show that every analytic free semigroup algebra on two or more generators is hyperreflexive with hyperreflexivity constant at most 3.

2.1 Background and preliminaries

Let $\mathbb{F}_n^+$ denote the free semigroup in n noncommuting letters $\{1, ..., n\}$, including the empty word $\varnothing$. For a word w in $\mathbb{F}_n^+$, let $|w|$ denote its length, and let $\mathbb{F}_n^k$ denote the set of all words in $\mathbb{F}_n^+$ of length at most k.

Let F_n^2 denote the "Fock" space $F_n^2 = \ell^2(\mathbb{F}_n^+)$ with orthonormal basis $\{\xi_w : w \in \mathbb{F}_n^+\}$ consisting of words in $\mathbb{F}_n^+$. For each v in $\mathbb{F}_n^+$, define an isometry L_v by

$$L_v \xi_w = \xi_{vw}, \quad w \in \mathbb{F}_n^+.$$

The map $v \to L_v$ gives a representation of $\mathbb{F}_n^+$, called the **left regular representation**.

The isometries $L_1, ..., L_n$ have pairwise orthogonal ranges. The free semigroup algebra they generate, denoted by $\mathcal{L}_n$, is called the **noncommutative analytic Toeplitz algebra**. For $n = 1$, $\mathcal{L}_n$ is the classical analytic Toeplitz algebra, but for $n \geq 2$, $\mathcal{L}_n$ is no longer commutative.

We require a result for $\mathcal{L}_n$ which generalizes a classical result about the analytic Toeplitz operators. An element in $\mathcal{L}_n$ is said to be **inner** if it is an isometry, and **outer** if it has dense range. It was shown in [DP99] that a nonzero element A in $\mathcal{L}_n$ can be written as $A = BC$, where B is inner and C is outer. This generalizes the classical inner-outer factorization for elements in the analytic Toeplitz algebra.

Every element A in $\mathcal{L}_n$ is completely determined by its Fourier series

$$A \sim \sum_{w \in \mathbb{F}_n^+} a_w L_w,$$

which is a formal power series with coefficients in $\mathcal{L}_n$, where

$$A\xi_\varnothing = \sum_{w \in \mathbb{F}_n^+} a_w \xi_w.$$

For $k \geq 1$, define the k-th Cesàro sum of the Fourier series of A by

$$\Gamma_k(A) = \sum_{|w|<k} (1 - \frac{|w|}{k}) a_w L_w.$$

Then the sequence $\Gamma_k(A)$ is strongly convergent to A.

By symmetry, for each v in $\mathbb{F}_n^+$ we can define an isometry R_v by

$$R_v \xi_w = \xi_{wv}, \quad w \in \mathbb{F}_n^+,$$

and the map $v \to R_v$ gives an anti-representation (i.e. a multiplication-reversing representation) of $\mathbb{F}_n^+$, called the **right regular representation**. The isometries $R_1, \ldots, R_n$ also have orthogonal ranges, and the free semigroup algebra they generate, denoted by $\mathcal{R}_n$, is unitarily equivalent to $\mathcal{L}_n$. It was shown in [DP99] that $\mathcal{R}_n$ is the commutant of $\mathcal{L}_n$.

A free semigroup algebra $\mathcal{S}$ is said to be **analytic** if it is algebraically isomorphic to $\mathcal{L}_n$. It was shown in [DKP01] that if $\mathcal{S}$ is analytic, then there is a completely isometric isomorphism Φ from $\mathcal{L}_n$ to $\mathcal{S}$ which takes the generators of $\mathcal{L}_n$ to the generators of

$\mathcal{S}$. Moreover, Φ is a weak*-to-weak* homeomorphism, and the inverse map Φ^{-1} is the dual of an isometric isomorphism ϕ from the predual of $\mathcal{L}_n$ to the predual of $\mathcal{S}$.

For a free semigroup algebra $\mathcal{S}$, let $\mathcal{S}_0$ denote the weak-operator-closed ideal generated by $S_1, ..., S_n$. Then either $\mathcal{S}_0 = \mathcal{S}$, or $\mathcal{S}/\mathcal{S}_0 \cong \mathbb{C}$. In the latter case, the general structure theorem for free semigroup algebras [DKP01] implies that $\mathcal{S}$ has an analytic part. If $\mathcal{S}_0 = \mathcal{S}$, then $\mathcal{S}$ is a von Neumann algebra.

The set of weak*-continuous linear functionals on $\mathcal{B}(H)$, i.e. the predual, can be identified with the set of trace class operators $\mathcal{C}^1(H)$, where K in $\mathcal{C}^1(H)$ corresponds to the linear functional

$$T \to \mathrm{tr}(TK), \quad T \in \mathcal{B}(H).$$

With this identification, the set of weak-operator-continuous linear functionals on $\mathcal{B}(H)$ corresponds to the set of finite rank operators. The predual of a weak*-closed subspace $\mathcal{S}$ of $\mathcal{B}(H)$ can be identified with the quotient space $\mathcal{C}^1(H) \diagup {}^{\perp}\mathcal{S}$, where ${}^{\perp}\mathcal{S}$ denotes the set of elements in $\mathcal{C}^1(H)$ which annihilate $\mathcal{S}$, i.e. the preannihilator.

It was shown in [DP99] that the weak* topology and the weak operator topology coincide on $\mathcal{L}_n$. This means that the equivalence class of every weak*-continuous linear functional on $\mathcal{L}_n$ contains an element of finite rank.

Let $\mathcal{S}$ be a free semigroup algebra on a Hilbert space H. A unit vector x in H is said to be **wandering** for $\mathcal{S}$ if the set $\{S_w x : w \in \mathbb{F}_n^+\}$ is orthonormal. The following theorem from [DKP01] will be important for our results.

Theorem 2.1.1. *Let $\mathcal{S}$ be an analytic free semigroup algebra. Then for some $m \geq 1$, the ampliation $\mathcal{S}^{(m)}$ has a wandering vector.*

Suppose that $\mathcal{S}$ is an analytic free semigroup algebra, and let π_0 be the weak-operator-continuous linear functional on $\mathcal{S}$ such that π_0 annihilates $\mathcal{S}_0$ and $\pi_0(I) = 1$. Then the equivalence class in $\mathcal{C}^1(H)$ corresponding to π_0 contains an operator of finite rank, say $m \geq 1$. This m corresponds to the m in the statement of Theorem 2.1.1. Since the restriction of $\mathcal{S}^{(m)}$ to the cyclic subspace generated by a wandering vector is unitarily equivalent to $\mathcal{L}_n$, it follows that the weak* topology and the weak operator topology agree on $\mathcal{S}$.

A subspace $\mathcal{S}$ of $\mathcal{B}(H)$ is said to be *reflexive* if $\mathcal{S}$ contains every operator T in $\mathcal{B}(H)$ with the property that Tx belongs to $\mathcal{S}[x]$ for every x in H. This definition of reflexivity was introduced by Loginov and Shulman [LS75].

The notion of hyperreflexivity, which was introduced by Arveson [Arv75], is a quantitative analogue of reflexivity. Let $d_{\mathcal{S}}$ denote the distance seminorm

$$d_{\mathcal{S}}(T) = \inf\{\|T - A\| : A \in \mathcal{S}\}, \quad T \in \mathcal{B}(H),$$

and define another seminorm $r_{\mathcal{S}}$ by

$$r_{\mathcal{S}}(T) = \sup\{|(Tx, y)| : \|x\|, \|y\| \leq 1 \text{ and } (Ax, y) = 0 \text{ for all } A \in \mathcal{S}\}$$

for T in $\mathcal{B}(H)$. Then the reflexivity of $\mathcal{S}$ is equivalent to the condition that $d_{\mathcal{S}}(T) = 0$ if and only if $r_{\mathcal{S}}(T) = 0$.

The equality $r_{\mathcal{S}}(T) \leq d_{\mathcal{S}}(T)$ always holds. We say that $\mathcal{S}$ is **hyperreflexive** if there is a constant $C > 0$ such that $d_{\mathcal{S}}(T) \leq Cr_{\mathcal{S}}(T)$ for all T in $\mathcal{B}(H)$. The smallest such C is called the **hyperreflexivity constant** of $\mathcal{S}$. Of course, hyperreflexivity implies reflexivity.

Davidson [Dav87] showed that the analytic Toeplitz algebra is hyperreflexive with hyperreflexivity constant at most 19. Davidson and Pitts [DP99] showed that for $n \geq 2$, $\mathcal{L}_n$ is hyperreflexive with hyperreflexivity constant at most 51. This was later improved by Bercovici [Ber98], who showed that this hyperreflexivity constant is at most 3.

2.2 The noncommutative Toeplitz operators

The Toeplitz operators are precisely the operators T in $\mathcal{B}(\ell^2(\mathbb{N}))$ which satisfy $S^*TS = T$, where S is the unilateral shift. This motivates the following definition, which was introduced by Popescu [Pop89b].

Definition 2.2.1. Let S=$(S_1, \ldots, S_n)$ be an isometric tuple. We say that T is an

S-**Toeplitz operator** if

$$S_i^* T S_j = \begin{cases} T & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

and we let $\mathcal{T}_S$ denote the set of all S-Toeplitz operators.

If an S-Toeplitz operator T is strictly positive, then by Theorem 4.3 of [Pop89b], it can be factored as $T = A^* A$, for some A in the commutant of the free semigroup algebra generated by $\mathcal{S}$.

Define isometric tuples L and R by $L = (L_1, \ldots, L_n)$ and $R = (R_1, \ldots, R_n)$. The size, n, will always be clear from the context. In this section we will establish some properties of the set $\mathcal{T}_R$ of R-Toeplitz operators which we will need later. Note that since $\mathcal{L}_n$ is unitarily equivalent to $\mathcal{R}_n$, the set $\mathcal{T}_R$ of R-Toeplitz operators is unitarily equivalent to the set $\mathcal{T}_L$ of L-Toeplitz operators. This means that any properties of $\mathcal{T}_R$ will correspond in an obvious way to properties of $\mathcal{T}_L$.

The following Lemma is implied by Corollary 1.3 of [Pop09]. Here we give a short direct proof.

Lemma 2.2.2. *The set $\mathcal{T}_R$ of R-Toeplitz operators is precisely the weak* closure of the operator system $\mathcal{L}_n^* + \mathcal{L}_n$.*

Proof. It is clear that the weak* closure of $\mathcal{L}_n^* + \mathcal{L}_n$ is contained in $\mathcal{T}_R$, since

$$R_i^* L_w R_j = R_i^* R_j L_w = \begin{cases} L_w & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose then that T belongs to $\mathcal{T}_R$. It's clear that T^* also belongs to $\mathcal{T}_R$, and hence that the real and imaginary parts of T belong to $\mathcal{T}_R$. Since the scalar operators also belong to $\mathcal{T}_R$, it follows that we can write T as a finite linear combination of strictly positive operators in $\mathcal{T}_R$. Hence we may suppose that T is strictly positive.

By Theorem 4.3 of [Pop89b], we can write $T = A^* A$ for some A in $\mathcal{L}_n$. Note that $A^* \Gamma_k(A)$ belongs to $\mathcal{L}_n + \mathcal{L}_n^*$ for $k \geq 1$, where $\Gamma_k(A)$ denotes the k-th Cesaro sum of the Fourier series for A. The sequence $\Gamma_k(A)$ is weak*-convergent to A, so it follows that $A^*\Gamma_k(A)$ is weak*-convergent to $A^*A = T$, and hence that T belongs to the weak* closure of $\mathcal{L}_n + \mathcal{L}_n^*$. $\square$

Note that based on the definition of the set $\mathcal{T}_R$ of R-Toeplitz operators, Lemma 2.2.2 implies that the weak* closure of $\mathcal{L}_n^* + \mathcal{L}_n$ is closed in the weak operator topology.

Lemma 2.2.3. *For $n \geq 2$, every R-Toeplitz operator T can be factored as $T = B^*C$ for some B and C in $\mathcal{L}_n$. Moreover, B and C can be taken to be bounded below.*

Proof. As in the proof of Lemma 2.2.2, we can write T as a finite linear combination of strictly positive R-Toeplitz operators, say $T = \sum_{i=1}^m c_i T_i$ for some $c_1, ..., c_m$ in $\mathbb{C}$ and strictly positive $T_1, ..., T_m$ in $\mathcal{T}_R$. By Theorem 4.3 of [Pop89b], we can factor each T_i as $T_i = A_i^* A_i$ for some A_i in $\mathcal{L}_n$. Set $B = \sum_{i=1}^m L_{1^i 2} A_i$ and $C = \sum_{i=1}^m c_i L_{1^i 2} A_i$. Then B and C both belong to $\mathcal{L}_n$ and $T = B^*C$.

To see that B and C can be taken to be bounded below, take $B' = B + L_{1^{m+1}2}$ and $C' = C + L_{1^{m+2}2}$, where m is as above. Then B' and C' both belong to $\mathcal{L}_n$. Since the isometries $L_{12}, ..., L_{1^{m+2}2}$ have pairwise orthogonal ranges, B' and C' are bounded below, and $T = (B')^*C'$. $\square$

Lemma 2.2.3 provides another characterization of the R-Toeplitz operators for $n \geq 2$.

Corollary 2.2.4. *For $n \geq 2$, the set $\mathcal{T}_R$ of R-Toeplitz operators is precisely $\mathcal{L}_n^*\mathcal{L}_n = \{B^*C : B, C \in \mathcal{L}_n\}$.*

Popescu [Pop09] showed that every R-Toeplitz operator T has a Fourier series

$$T \quad \sim \quad \sum_{w \in \mathbb{F}_n^+} a_w L_w + \sum_{w \in \mathbb{F}_n^+ \setminus \{\varnothing\}} \overline{b_w} L_w^*,$$

which is a formal power series with coefficients in $\mathcal{L}_n$ and $\mathcal{L}_n^*$. This completely determines T in the sense that for every word u in $\mathbb{F}_n^+$,

$$T\xi_u = \sum_{w \in \mathbb{F}_n^+} a_w L_w \xi_u + \sum_{w \in \mathbb{F}_n^+ \setminus \{\varnothing\}} \overline{b_w} L_w^* \xi_u.$$

Let $\mathcal{S}$ be an analytic free semigroup algebra. By Theorem 1.1 of [DKP01], the canonical map from $\mathcal{L}_n$ to $\mathcal{S}$ is a complete isometry and a weak*-to-weak* homeomorphism. Our goal for the remainder of this section is to show that this map extends in a natural way to a map from the weak* closure of $\mathcal{L}_n + \mathcal{L}_n^*$ (i.e. from the set $\mathcal{T}_R$ of R-Toeplitz operators) to the weak* closure of $\mathcal{S} + \mathcal{S}^*$, and that this extension is also a complete isometry and a weak*-to-weak* homeomorphism.

Lemma 2.2.5. *Let $\mathcal{S}$ be an analytic free semigroup algebra with $n \geq 2$ generators , and let Φ be the canonical map from $\mathcal{L}_n$ to $\mathcal{S}$. Then Φ^{-1} maps isometries in $\mathcal{S}$ to isometries in $\mathcal{L}_n$.*

Proof. By Theorem 2.1.1, $\mathcal{S}^{(m)}$ has a wandering vector w for some m, and the restriction of $\mathcal{S}^{(m)}$ to $\mathcal{S}^{(m)}[w]$ is unitarily equivalent to $\mathcal{L}_n$. The map Φ^{-1} from $\mathcal{S}$ to $\mathcal{L}_n$ is given by taking $\mathcal{S}$ to $\mathcal{S}^{(m)}$, restricting to $\mathcal{S}^{(m)}[w]$, and applying this equivalence. If G is an isometry in $\mathcal{S}$, then $G^{(m)}$ is an isometry in $\mathcal{S}^{(m)}$, and so clearly the restriction of $G^{(m)}$ to $\mathcal{S}^{(m)}[w]$ is an isometry. $\square$

Theorem 2.2.6. *Let $\mathcal{S}$ be an analytic free semigroup algebra with $n \geq 2$ generators on a Hilbert space H. Then the canonical map Φ from $\mathcal{L}_n$ to $\mathcal{S}$ extends to a completely isometric weak*-to-weak* homeomorphism from the weak* closure of $\mathcal{L}_n + \mathcal{L}_n^*$ to the weak* closure of $\mathcal{S} + \mathcal{S}^*$.*

Proof. Applying Arveson's extension theorem [Arv69] gives a completely positive map Ψ from $C^*(\mathcal{L}_n)$ to $\mathcal{B}(H)$ which extends Φ. Since Ψ extends Φ, we have $\|\Psi\| = \|\Psi(I)\| = \|\Phi(I)\| = 1$. Let $\mathcal{Z} = \{A \in C^*(\mathcal{L}_n) : \Psi(A)^*\Psi(A) = \Psi(A^*A)\}$. By [Cho74], we have

$$\mathcal{Z} = \{A \in C^*(\mathcal{L}_n) : \Psi(B)\Psi(A) = \Psi(BA) \text{ for all } B \text{ in } C^*(\mathcal{L}_n)\}.$$

By Theorem 4.1 of [DKP01], Φ maps isometries in $\mathcal{L}_n$ to isometries in $\mathcal{S}$, so every isometry in $\mathcal{L}_n$ belongs to $\mathcal{Z}$. Since, by Theorem 4.5 of [DKP01], every element in $\mathcal{L}_n$ can be written as a finite linear combination of isometries in $\mathcal{L}_n$, this implies that $\mathcal{Z}$ contains all of $\mathcal{L}_n$. Hence for A in $\mathcal{L}_n$, $\Psi(TA) = \Psi(T)\Psi(A)$ for all T in $C^*(\mathcal{L}_n)$.

Note that by Corollary 2.2.4, $C^*(\mathcal{L}_n)$ contains $\mathcal{T}_R$. For the remainder of the proof, we restrict Ψ to $\mathcal{T}_R$.

Let T be a self-adjoint element in $\mathcal{T}_R$ such that $\Psi(T) = 0$. For sufficiently large $\lambda > 0$, $T + \lambda I$ is strictly positive, so by Theorem 4.3 of [Pop89b], we can write $T + \lambda I = B^* B$ for some B in $\mathcal{L}_n$. Let $V = \lambda^{-1/2} B$. Then

$$\begin{aligned}
\Phi(V)^*\Phi(V) - I &= \Psi(V^*V - I) \\
&= \Psi(\lambda^{-1}B^*B - I) \\
&= \Psi(\lambda^{-1}(T + \lambda I) - I) \\
&= \lambda^{-1}\Psi(T) \\
&= 0,
\end{aligned}$$

which shows that $\Phi(V)$ is an isometry in $\mathcal{S}$. By Lemma 2.2.5, this implies that V is an isometry in $\mathcal{L}_n$. Hence

$$T = \lambda(V^*V - I) = 0.$$

Since, for arbitrary T in $\mathcal{T}_R$, $\mathrm{re}(T)$ and $\mathrm{im}(T)$ are self-adjoint, and since

$$\Psi(T) = \Psi(\mathrm{re}(T) + \mathrm{im}(T)) = \mathrm{re}(\Psi(T)) + \mathrm{im}(\Psi(T)) = 0$$

if and only if $\psi(\mathrm{re}(T)) = 0$ and $\psi(\mathrm{im}(T)) = 0$, it follows that Ψ is injective.

Arguing exactly as above, the canonical map Φ^{-1} from $\mathcal{S}$ to $\mathcal{L}_n$ also has a completely positive extension Ω from $C^*(\mathcal{S})$ to $\mathcal{B}(F_n^2)$, and for G in $\mathcal{S}$, $\Omega(HG) = \Omega(H)\Omega(G)$ for all H in $C^*(\mathcal{S})$. Since Ω extends Φ^{-1}, we have $\|\Omega\| = \|\Omega(I)\| = \|\Phi^{-1}(I)\| = 1$. For the remainder of the proof we restrict Ω to the intersection of $C^*(\mathcal{S})$ and the range of Ψ.

Note that the range of Ψ is contained in the weak* closure of $\mathcal{S} + \mathcal{S}^*$. Indeed, by Lemma 2.2.4, every element in the range of Ψ can be written as $\Psi(B^*C) = \Psi(B^*)\Psi(C) = \Phi(B)^*\Phi(C)$ for some B and C in $\mathcal{L}_n$. The sequence $\Gamma_k(C)$ is weak operator convergent to C, so by the weak operator continuity of Φ, the sequence $\Phi(\Gamma_k(C))$ is weak operator convergent to $\Phi(C)$, and hence the sequence $\Phi(B)^*\Phi(\Gamma_k(C))$ is weak operator convergent to $\Phi(B)^*\Phi(C)$, which implies that $\Phi(B)^*\Phi(C)$ is contained in the weak* closure of $\mathcal{S} + \mathcal{S}^*$.

We claim that $\Omega(\Psi(T)) = T$ for all T in $\mathcal{T}_R$. Indeed, apply Lemma 2.2.4 to write $T = B^*C$ for some B and C in $\mathcal{L}_n$, and let $G = \Phi(B)$ and $H = \Phi(C)$. Then we have

$$\begin{aligned} \Psi(T) &= \Psi(B^*C) \\ &= \Phi(B)^*\Phi(C) \\ &= G^*H, \end{aligned}$$

which gives

$$\begin{aligned} \Omega(\Phi(T)) &= \Omega(G^*H) \\ &= (\Phi^{-1}(G))^*\Phi^{-1}(H) \\ &= B^*C \\ &= T. \end{aligned}$$

Then

$$\frac{\|T\|}{\|\Psi(T)\|} = \frac{\|\Omega(\Psi(T))\|}{\|\Psi(T)\|} \leq 1,$$

which gives

$$\|T\| \leq \|\Psi(T)\| \leq \|T\|,$$

and shows that Ψ maps $\mathcal{T}_R$ isometrically onto its range.

We now show that Ψ is weak*-to-weak* continuous. Since the predual of $\mathcal{T}_R$ is separable, by an application of the Krein-Smulian theorem it suffices to show that if T_n is a sequence in $\mathcal{T}_R$ which is weak*-convergent to zero, then $\Psi(T_n)$ is weak* convergent to zero.

Let $\mathcal{A} = \{A \oplus \Phi(A) : A \in \mathcal{L}_n\}$, and note that $\mathcal{A}$ is the free semigroup algebra generated by the isometries $L_1 \oplus S_1, ..., L_n \oplus S_n$. Fix u in H. By Theorem 1.6 of

[DKP01], there exists a vector x in F_n^2 such that the restriction of $\mathcal{A}$ to $\mathcal{W} = \mathcal{A}[x \oplus u]$ is unitarily equivalent to $\mathcal{L}_n$. Letting P denote the projection of $\mathcal{F} \oplus H$ onto $\mathcal{W}$, and letting $\mathcal{K}$ denote the weak* closure of the restriction of $P(\mathcal{A} + \mathcal{A}^*)P$ to $\mathcal{W}$, it follows that $\mathcal{K}$ is unitarily equivalent to $\mathcal{T}_R$. By Lemma 2.2.4, every element of $\mathcal{K}$ can be written as the restriction to $\mathcal{W}$ of an element of the form

$$P(B^* \oplus \Phi(B)^*)(C \oplus \Phi(C))P = P(B^*C \oplus \Psi(B^*C))P.$$

Hence $\mathcal{K}$ is the restriction to $\mathcal{W}$ of $\{T \oplus \psi(T) : T \in \mathcal{T}_R\}$.

If T_n is weak* convergent to zero in $\mathcal{T}_R$, the unitary equivalence between $\mathcal{T}_R$ and $\mathcal{K}$ implies the restriction of the sequence $T_n \oplus \Psi(T_n)$ to $\mathcal{W}$ is weak*-convergent to zero in $\mathcal{K}$. Hence

$$((T_n \oplus \Psi(T_n))(x \oplus u), x \oplus u) = (T_n x, x) + (\Psi(T_n)u, u) \to 0,$$

and since $(T_n x, x) \to 0$, this implies that $(\Psi(T_n)u, u) \to 0$. Since u was chosen arbitrarily, we deduce that $(\Psi(T_n)u, u) \to 0$ for all u in H. By the polarization identity, we get that $\Psi(T_n)$ is weak operator convergent to zero. By the uniform boundedness principle, the sequence $\Psi(T_n)$ is bounded. It follows that $\Psi(T_n)$ is weak* convergent to zero. We therefore conclude that Ψ is weak* continuous.

It now follows by another application of the Krein-Smulian theorem that Ψ has weak* closed range, and that Ψ is a weak*-to-weak* homeomorphism onto its range. But it's clear that the range of Ψ is weak* dense in the weak* closure of $\mathcal{S} + \mathcal{S}^*$, so Ψ maps $\mathcal{T}_R$ weak*-to-weak* homeomorphically onto the weak* closure of $\mathcal{S} + \mathcal{S}^*$. From above, Ψ is a completely positive isometry, with completely positive inverse Ω. Hence Ψ is completely isometric. $\square$

2.3 Wandering vectors

Let $\mathcal{S}$ be a weak*-closed subspace of $\mathcal{B}(H)$, and let x and y be vectors in H. Then $[x \otimes y]_{\mathcal{S}}$ denotes the weak-operator-continuous linear functional on $\mathcal{S}$ which is given by the equivalence class of the rank one tensor $x \otimes y$. In other words,

$$(A, [x \otimes y]_{\mathcal{S}}) = (Ax, y), \quad A \in \mathcal{S}.$$

Definition 2.3.1. A weak*-closed subspace $\mathcal{S}$ of $\mathcal{B}(H)$ is said to have property $\mathbb{A}_1(1)$ if, for every weak*-continuous linear functional π on $\mathcal{S}$ and every $\varepsilon > 0$, there are vectors x and y in H with $\|x\|\|y\| < (1 + \epsilon)\|\pi\|$ such that $\pi(A) = (Ax, y)$ for all A in $\mathcal{S}$.

It was shown in [DP99] that $\mathcal{L}_n$ has property $\mathbb{A}_1(1)$, and it is well known (see for example Proposition B of [Dav87]) that a singly generated analytic free semigroup algebra has property $\mathbb{A}_1(1)$. In this section, we will use dual algebra techniques to show that every analytic free semigroup algebra with $n \geq 2$ generators has property $\mathbb{A}_1(1)$. From this result, it will follow easily that every analytic free semigroup algebra has a wandering vector.

For the remainder of this section we fix an analytic free semigroup algebra $\mathcal{S}$ with $n \geq 2$ generators acting on a Hilbert space H. The general outline of our approach is as follows. Let π be a weak*-continuous linear functional on $\mathcal{S}$. We will show that we can construct convergent sequences (x_k) and (y_k) such that

$$\lim_{k \to \infty} \|\pi - [x_k \otimes y_k]_{\mathcal{S}}\| = 0.$$

This will then give $\pi = [x \otimes y]_{\mathcal{S}}$, where $x = \lim_k x_k$ and $y = \lim_k y_k$.

The following idea will allow us to iteratively construct the sequences (x_k) and (y_k). Fix x_k and y_k. Suppose we can find vectors x' and y' such that

1. $[x' \otimes y']_{\mathcal{S}}$ approximates the error $\pi - [x_k \otimes y_k]_{\mathcal{S}}$ arbitrarily closely,
2. $\|[x_k \otimes y']_{\mathcal{S}}\|$ and $\|[x' \otimes y_k]_{\mathcal{S}}\|$ are arbitrarily small,
3. $\|x'\|$ and $\|y'\|$ are arbitrarily close to $\|\pi - [x_k \otimes y_k]_{\mathcal{S}}\|$.

Set $x_{k+1} = x_k + x'$ and $y_{k+1} = y_k + y'$. Then

$$\|\pi - [x_{k+1} \otimes y_{k+1}]_{\mathcal{S}}\| \leq \|\pi - [x_k \otimes y_k]_{\mathcal{S}} - [x' \otimes y']_{\mathcal{S}}\| + \|[x' \otimes y_k]_{\mathcal{S}}\| + \|[x_k \otimes y']_{\mathcal{S}}\|,$$

so $[x_{k+1} \otimes y_{k+1}]_{\mathcal{S}}$ is an arbitrarily good approximation to π, and the sequences (x_k) and (y_k) can be made Cauchy. Of course, the main difficulty will be in showing that it is possible to find x' and y' as above.

Definition 2.3.2. An operator $X : F_n^2 \to H$ is said to **intertwine** $\mathcal{L}_n$ and $\mathcal{S}$ if $XL_i = S_i X$ for $1 \leq i \leq n$.

Let $\overline{x} = (x_1, ..., x_m)$ be a wandering vector for $\mathcal{S}^{(m)}$. By Theorem 1.3 of [Pop89a], we know that the restriction of $\mathcal{S}^{(m)}$ to $\mathcal{S}^{(m)}[\overline{x}]$ is unitarily equivalent to $\mathcal{L}_n$. Let $X : F_n^2 \to H$ denote the map which follows this equivalence with the projection onto the first coordinate. Then X intertwines $\mathcal{L}_n$ and $\mathcal{S}$. It was shown in [DLP05] that every vector in H is in the range of some intertwining operator of this form.

The following result shows that every intertwining operator gives rise to an L-Toeplitz operator. This allows us to use the results of section 2.2 to work with intertwining operators.

Lemma 2.3.3. *Suppose $X : F_n^2 \to H$ intertwines $\mathcal{L}_n$ and $\mathcal{S}$. Then X^*X is an L-Toeplitz operator.*

Proof. This follows immediately from the identity

$$\begin{aligned} L_i^* X^* X L_j &= X^* S_i^* S_j X \\ &= \begin{cases} X^*X & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

□

We require several technical results about L-Toeplitz operators.

Lemma 2.3.4. *Let T be an L-Toeplitz operator with Fourier series*

$$T \sim \sum_{w \in \mathbb{F}_n^+} a_w R_w + \sum_{w \in \mathbb{F}_n^+ \setminus \{\varnothing\}} \overline{b_w} R_w^*.$$

Then for any word u in $\mathbb{F}_n^+$,

$$\left\| \sum_{w \in \mathbb{F}_n^+ \setminus \{\varnothing\}} \overline{b_w} R_w^* \xi_u \right\| \le \left\| \sum_{w \in \mathbb{F}_n^+ \setminus \{\varnothing\}} b_w R_w \xi_u \right\|.$$

Proof. We have

$$\begin{aligned} \left\| \sum_{w \in \mathbb{F}_n^+ \setminus \{\varnothing\}} \overline{b_w} R_w^* \xi_u \right\|^2 &= \sum_{\substack{w \in \mathbb{F}_n^+ \setminus \{\varnothing\} \\ w = w'u}} |b_w|^2 \\ &\le \sum_{w \in \mathbb{F}_n^+ \setminus \{\varnothing\}} |b_w|^2 \\ &= \left\| \sum_{w \in \mathbb{F}_n^+ \setminus \{\varnothing\}} b_w R_w \xi_u \right\|^2. \end{aligned}$$

□

Lemma 2.3.5. *Let T be an L-Toeplitz operator with Fourier series*

$$T \sim \sum_{w \in \mathbb{F}_n^+} a_w R_w + \sum_{w \in \mathbb{F}_n^+ \setminus \{\varnothing\}} \overline{b_w} R_w^*.$$

Then given $p \ge 1$ and $\epsilon > 0$, there is a word v in $\mathbb{F}_n^+$ such that

$$\|R_v^* T R_v \xi_u - a_\varnothing \xi_u\| < \epsilon$$

for any word $u \in \mathbb{F}_n^p$.

Proof. For $k \ge 1$, let v_k be the word $v_k = 12^k$. Then for any word w in $\mathbb{F}_n^+$,

$$R_{v_k}^* R_w R_{v_k} = \begin{cases} I & \text{if } w = \varnothing, \\ R_{v_k w'} & \text{if } w = w' v_k \text{ for } w' \in \mathbb{F}_n^+, \\ 0 & \text{otherwise.} \end{cases}$$

This implies the Fourier series for $R_{v_k}^* T R_{v_k}$ is given by

$$R_{v_k}^* T R_{v_k} \sim a_\varnothing I + \sum_{\substack{w \in \mathbb{F}_n^+ \setminus \{\varnothing\} \\ w = w' v_k}} a_w R_{v_k w'} + \sum_{\substack{w \in \mathbb{F}_n^+ \setminus \{\varnothing\} \\ w = w' v_k}} \overline{b_w} R_{v_k w'}^*.$$

Hence for u in $\mathbb{F}_n^+$,

$$R_{v_k}^* T R_{v_k} \xi_u \sim a_\varnothing \xi_u + \sum_{\substack{w \in \mathbb{F}_n^+ \setminus \{\varnothing\} \\ w = w' v_k}} a_w R_{v_k w'} \xi_u + \sum_{\substack{w \in \mathbb{F}_n^+ \setminus \{\varnothing\} \\ w = w' v_k}} \overline{b_w} R_{v_k w'}^* \xi_u.$$

This gives

$$\begin{aligned} \|R_{v_k}^* T R_{v_k} \xi_u - a_\varnothing \xi_u\| &= \left\| \sum_{\substack{w \in \mathbb{F}_n^+ \setminus \{\varnothing\} \\ w = w' v_k}} a_w R_{v_k w'} \xi_u + \sum_{\substack{w \in \mathbb{F}_n^+ \setminus \{\varnothing\} \\ w = w' v_k}} \overline{b_w} R_{v_k w'}^* \xi_u \right\| \\ &\leq \left\| \sum_{\substack{w \in \mathbb{F}_n^+ \setminus \{\varnothing\} \\ w = w' v_k}} a_w R_{v_k w'} \xi_u \right\| + \left\| \sum_{\substack{w \in \mathbb{F}_n^+ \setminus \{\varnothing\} \\ w = w' v_k}} \overline{b_w} R_{v_k w'}^* \xi_u \right\| \\ &\leq \left\| \sum_{\substack{w \in \mathbb{F}_n^+ \setminus \{\varnothing\} \\ w = w' v_k}} a_w R_{v_k w'} \xi_u \right\| + \left\| \sum_{\substack{w \in \mathbb{F}_n^+ \setminus \{\varnothing\} \\ w = w' v_k}} b_w R_{v_k w'} \xi_u \right\|, \end{aligned}$$

where the last inequality follows from Lemma 2.3.4. Now

$$\left\| \sum_{\substack{w \in \mathbb{F}_n^+ \setminus \{\varnothing\} \\ w = w' v_k}} a_w R_{v_k w'} \xi_u \right\|^2 = \sum_{\substack{w \in \mathbb{F}_n^+ \setminus \{\varnothing\} \\ w = w' v_k}} |a_w|^2 = \|R_{v_k}^* T \xi_\varnothing\|^2,$$

and similarly,

$$\left\| \sum_{\substack{w \in \mathbb{F}_n^+ \setminus \{\varnothing\} \\ w = w' v_k}} b_w R_{v_k w'} \xi_u \right\|^2 = \sum_{\substack{w \in \mathbb{F}_n^+ \setminus \{\varnothing\} \\ w = w' v_k}} |b_w|^2 = \|R_{v_k}^* T^* \xi_\varnothing\|^2,$$

The result follows from the fact that for all ξ in F_n^2, $\|R_v^* \xi\| \to 0$ as $|v| \to \infty$. $\square$

Recall that $\phi : (\mathcal{L}_n)_* \to \mathcal{S}_*$ is the predual of the map $\Phi^{-1} : \mathcal{S} \to \mathcal{L}_n$.

Lemma 2.3.6. *Let $X : F_n^2 \to H$ be an intertwining operator, and let $x = X\xi_\varnothing$. Then given $p \geq 1$ and $\epsilon > 0$, there exists a word v in $\mathbb{F}_n^+$ such that*

$$\left\| [S_{u_1 v} x \otimes S_{u_2 v} x]_{\mathcal{S}} - \|x\|^2 \phi([\xi_{u_1} \otimes \xi_{u_2}]_{\mathcal{L}_n}) \right\| < \epsilon$$

for all words u_1 and u_2 in $\mathbb{F}_n^p$.

Proof. By scaling X if necessary, we can suppose that $\|x\| = 1$. Let $T = X^* X$. Then T is an L-Toeplitz operator by Lemma 2.3.3. Writing the Fourier series for T as

$$T \sim \sum_{w \in \mathbb{F}_n^+} a_w R_w + \sum_{w \in \mathbb{F}_n^+ \setminus \{0\}} \overline{b_w} R_w^*,$$

it follows that $a_\varnothing = \|x\|^2 = 1$. Hence by Lemma 2.3.5, there exists a word v in $\mathbb{F}_n^+$ such that $\|R_v^* T R_v \xi_{u_2} - \xi_{u_2}\| < \epsilon$ for any word u_2 in $\mathbb{F}_n^p$. Then for A in $\mathcal{S}$,

$$\begin{aligned}
(A, [S_{u_1 v} x \otimes S_{u_2 v} x]_{\mathcal{S}}) &= (A S_{u_1 v} x, S_{u_2 v} x) \\
&= (A S_{u_1 v} X \xi_\varnothing, S_{u_2 v} X \xi_\varnothing) \\
&= (X \Phi^{-1}(A) L_{u_1 v} \xi_\varnothing, X L_{u_2 v} \xi_\varnothing) \\
&= (X \Phi^{-1}(A) R_v \xi_{u_1}, X R_v \xi_{u_2}) \\
&= (X R_v \Phi^{-1}(A) \xi_{u_1}, X R_v \xi_{u_2}) \\
&= (\Phi^{-1}(A) \xi_{u_1}, R_v^* T R_v \xi_{u_2})
\end{aligned}$$

for all words u_1 and u_2 in $\mathbb{F}_n^p$. This gives

$$\begin{aligned} |(A, [S_{u_1 v}x \otimes S_{u_2 v}x]_{\mathcal{S}} - [\xi_{u_1} \otimes \xi_{u_2}]_{\mathcal{L}_n})| &= \left|(\Phi^{-1}(A)\xi_{u_1}, R_v^* T R_v \xi_{u_2} - \xi_{u_2})\right| \\ &\leq \|\Phi^{-1}(A)\xi_{u_1}\| \|R_v^* T R_v \xi_{u_2} - \xi_{u_2}\| \\ &< \epsilon\|A\|. \end{aligned}$$

Therefore,

$$\|[S_{u_1 v}x \otimes S_{u_2 v}x]_{\mathcal{S}} - \phi([\xi_{u_1} \otimes \xi_{u_2}]_{\mathcal{L}_n})\| < \epsilon. \qquad \square$$

Lemma 2.3.7. *Let $M \geq 2$ be minimal such that $\mathcal{S}^{(M)}$ has a wandering vector $\overline{w} = (w_1, ..., w_M)$. Then given $\epsilon \in (0,1)$ there exists a unit vector $\overline{x} = (x_1, ..., x_M)$ in $\mathcal{S}^{(M)}[\overline{w}]$ such that $x_1 = X\xi_\varnothing$ for some intertwining operator $X : F_n^2 \to H$, and $\|x_1\| > 1 - \epsilon$.*

Proof. Let P denote the projection map from $\mathcal{S}^{(M)}[\overline{w}]$ to $H^{(M-1)}$ which takes $\overline{x} = (x_1, ..., x_M)$ to $(x_2, ..., x_M)$. Then P intertwines the restriction of $\mathcal{S}^{(M)}$ to $\mathcal{S}^{(M)}[\overline{w}]$ and $\mathcal{S}^{(M-1)}$. The restriction of $\mathcal{S}^{(M)}$ to $\mathcal{S}^{(M)}[\overline{w}]$ is unitarily equivalent to $\mathcal{L}_n$. Let U be a unitary implementing this equivalence. Then setting $Y = PU$, Y intertwines $\mathcal{L}_n$ and $S^{(M-1)}$. Suppose that for all $\overline{x}$ in $\mathcal{S}^{(M)}[\overline{w}]$, $\|x_1\| \leq (1-\epsilon)\|\overline{x}\|$. Then

$$\|\overline{x}\|^2 = \sum_{i=1}^{M} \|x_i\|^2 \leq (1-\epsilon)^2 \|\overline{x}\|^2 + \sum_{i=2}^{M} \|x_i\|^2,$$

which gives

$$\sum_{i=2}^{mM} \|x_i\|^2 \geq (1 - (1-\epsilon)^2)\|\overline{x}\|^2,$$

implying that P is bounded below, and hence that Y is bounded below. By Theorem 2.8 of [DLP05], this implies that the range of Y is a wandering subspace for $\mathcal{S}^{(M-1)}$, contradicting the minimality of M. Hence there must be some unit vector $\overline{x}$ in $\mathcal{S}^{(M)}[\overline{w}]$ such that $\|x_1\| > 1 - \epsilon$.

Let Q denote the projection map from $\mathcal{S}^{(M)}[\overline{w}]$ to H which takes $\overline{y} = (y_1, ..., y_M)$ to $\overline{y}_1$, and let $Z = QU$. Note that x_1 is contained in the range of Z. For every R

in $\mathcal{R}_n$, the operator ZR intertwines $\mathcal{L}_n$ and $\mathcal{S}$. Moreover, since the set of vectors $\{R\xi_\varnothing : R \in \mathcal{R}_n\}$ is dense in F_n^2, the set $\{ZR\xi_\varnothing : R \in \mathcal{R}_n\}$ is dense in the closure of the range of Z. It follows that we can choose the vector $\overline{x}$ as above such that $x_1 = X\xi_\varnothing$ for some intertwining operator $X : F_n^2 \to H$. $\square$

Let $M \geq 1$ be minimal such that the ampliation $\mathcal{S}^{(M)}$ has a wandering vector $\overline{x}$. Such M exists by Theorem 2.1.1. Then $H^{(M)}$ contains an infinite family of pairwise orthogonal subspaces $\mathcal{W}_k$, for $k \geq 1$, which are wandering for $\mathcal{S}^{(M)}$. For example, we can take $\mathcal{W}_k = \mathcal{S}^{(m)}[S_{v_k}^{(m)}\overline{x}]$, where $v_k = 12^k$. For $k \geq 1$, let $\mathcal{M}_k$ denote the linear manifold in H given by

$$\mathcal{M}_k = \{z \in H : z = \overline{z}_1 \text{ for some } \overline{z} = (\overline{z}_1, ..., \overline{z}_M) \text{ in } \mathcal{W}_k\}.$$

Let $\mathcal{W}$ denote the algebraic span of the $\mathcal{W}_k$, and let $\mathcal{M}$ denote the algebraic span of the $\mathcal{M}_k$.

Lemma 2.3.8. *Given $h_1, ..., h_q$ in $\mathcal{M}$ and $\epsilon > 0$, there exists a unit vector y in $\mathcal{M}$ such that $y = Y\xi_\varnothing$ for some intertwining operator $Y : F_n^2 \to H$, and such that $\|[S_u y \otimes h_j]_{\mathcal{S}}\| < \epsilon$ and $\|[h_j \otimes S_u y]_{\mathcal{S}}\| < \epsilon$ for any word $u \in \mathbb{F}_n^+$ and $1 \leq j \leq q$.*

Proof. For each j, there exists $\overline{h}^{(j)} = (h_1^{(j)}, ..., h_M^{(j)})$ in $\mathcal{W}$ such that $h_j = h_1^{(j)}$. Choose $\epsilon_0 \in (0,1)$ such that $\epsilon_0/(1-\epsilon_0) < \epsilon$ and $\epsilon_0/\|h^{(j)}\| < 1$ for $1 \leq j \leq q$, and choose r sufficiently large that $\overline{h}^{(j)}$ is orthogonal to $\mathcal{W}_r$ for $1 \leq j \leq q$.

By Lemma 2.3.7, there exists a unit vector $\overline{x} = (x_1, ..., x_M)$ in $\mathcal{M}_r$ such that $x_1 = X\xi_\varnothing$ for some intertwining operator $X : F_n^2 \to H$, and such that

$$\|x_1\| > \max\left\{1 - \epsilon_0, \left(1 - \frac{\epsilon_0^2}{\|\overline{h}^{(j)}\|^2}\right)^{1/2} : 1 \leq j \leq q\right\}.$$

This gives $1/\|x_1\| < 1/(1-\epsilon_0)$ and

$$\sum_{i=2}^{M} \|x_i\|^2 = 1 - \|x_1\|^2 < \frac{\epsilon_0^2}{\|\overline{h}^{(j)}\|^2}, \quad 1 \leq j \leq q.$$

For any word u in $\mathbb{F}_n^+$,

$$\begin{aligned}
\|[S_u x_1 \otimes h_1^{(j)}]_{\mathcal{S}}\| &\leq \left\|\sum_{i=1}^{M}[S_u x_i \otimes h_i^{(j)}]_{\mathcal{S}}\right\| + \left\|\sum_{i=2}^{M}[S_u x_i \otimes h_i^{(j)}]_{\mathcal{S}}\right\| \\
&= \left\|[S_u^{(M)}\overline{x} \otimes \overline{h}^{(j)}]_{\mathcal{S}}\right\| + \left\|\sum_{i=2}^{M}[S_u x_i \otimes h_i^{(j)}]_{\mathcal{S}}\right\| \\
&= \left\|\sum_{i=2}^{M}[S_u x_i \otimes h_i^{(j)}]_{\mathcal{S}}\right\| \\
&\leq \left(\sum_{i=2}^{M}\|S_u x_i\|^2\right)^{1/2}\left(\sum_{i=2}^{M}\|h_i^{(j)}\|^2\right)^{1/2} \\
&= \left(\sum_{i=2}^{M}\|x_i\|^2\right)^{1/2}\left(\sum_{i=2}^{M}\|h_i^{(j)}\|^2\right)^{1/2} \\
&< \frac{\epsilon_0}{\|\overline{h}^{(j)}\|}\left(\sum_{i=2}^{M}\|h_i^{(j)}\|^2\right)^{1/2} \\
&\leq \epsilon_0,
\end{aligned}$$

where we have used the fact that $\overline{x}$ and $\overline{h}^{(j)}$ belong to orthogonal $\mathcal{S}^{(M)}$-invariant subspaces, which implies that $\|[S_u^{(M)}\overline{x} \otimes \overline{h}^{(j)}]_{\mathcal{S}}\| = 0$. Multiplying this inequality by $1/\|x_1\| = 1/\|S_u x_1\|$ then gives

$$\|[S_u(x_1/\|x_1\|) \otimes h_j]\| < \epsilon_0/(1-\epsilon_0)$$

for $1 \leq j \leq q$. In the same way we get

$$\|h_j \otimes S_u(x_1/\|x_1\|)]\| < \epsilon_0/(1-\epsilon_0)$$

for $1 \leq j \leq q$. Hence we can take $y = x_1/\|x_1\|$ and $Y = X/\|x_1\|$. $\square$

Lemma 2.3.9. *Given vectors $h_1, ..., h_q$ in $\mathcal{M}$, $p \geq 1$, and $\epsilon > 0$, there exists a unit* *vector z in $\mathcal{M}$ such that*

$$\|[S_{u_1}z \otimes S_{u_2}z]_{\mathcal{S}} - \phi([\xi_{u_1} \otimes \xi_{u_2}]_{\mathcal{L}_n})\| < \epsilon$$

for all u_1 and u_2 in $\mathbb{F}_n^p$, and such that $\|[S_w z \otimes h_j]_{\mathcal{S}}\| < \epsilon$ and $\|[h_j \otimes S_w z]_{\mathcal{S}}\| < \epsilon$ for all $w \in \mathbb{F}_n^+$ and $1 \leq j \leq q$.

Proof. By Lemma 2.3.8, there exists a unit vector y in $\mathcal{M}$ such that $y = Y\xi_\varnothing$ for some intertwining operator $Y : F_n^2 \to H$, and such that for any word w in $\mathbb{F}_n^+$, $\|[S_w y \otimes h_j]_{\mathcal{S}}\| < \epsilon$ and $\|[h_j \otimes S_w y]_{\mathcal{S}}\| < \epsilon$ for $1 \leq j \leq q$. By Lemma 2.3.6, there exists a word v in $\mathbb{F}_n^+$ such that $\|[S_{u_1}S_v y \otimes S_{u_2}S_v y]_{\mathcal{S}} - \phi([\xi_{u_1} \otimes \xi_{u_2}]_{\mathcal{L}_n})\| < \epsilon$ for any words u_1 and u_2 in $\mathbb{F}_n^p$. Then $\|[S_w S_v y \otimes h_j]_{\mathcal{S}}\| = \|[S_{wv} y \otimes h_j]_{\mathcal{S}}\| < \epsilon$ and $\|[h_j \otimes S_w S_v y]_{\mathcal{S}}\| = \|[h_j \otimes S_{wv} y]_{\mathcal{S}}\| < \epsilon$, so we can take $z = S_v y$. □

Lemma 2.3.10. *Given a weak*-continuous linear functional π on $\mathcal{S}$, $h_1, ..., h_q$ in $\mathcal{M}$, and $\epsilon > 0$, there are vectors x and y in $\mathcal{M}$ such that*

1. *$\|\pi - [x \otimes y]_{\mathcal{S}}\| < \epsilon$,*
2. *$\|x\| < (1+\epsilon)\|\pi\|^{1/2}$ and $\|y\| < (1+\epsilon)\|\pi\|^{1/2}$,*
3. *$\|[x \otimes h_j]_{\mathcal{S}}\| < \epsilon$ and $\|[h_j \otimes y]_{\mathcal{S}}\| < \epsilon$ for $1 \leq j \leq q$.*

Proof. By scaling π and ϵ if necessary, we can assume that $\|\pi\| = 1$. Choose $\epsilon_0 > 0$ such that $2\epsilon_0 + 3\epsilon_0^2 < \epsilon/2$ and $4\epsilon_0 + 4\epsilon_0^2 < \epsilon + \epsilon^2/2$. Since $\mathcal{L}_n$ has property $\mathbb{A}_1(1)$, there are vectors ξ and η in F_n^2 such that $[\xi \otimes \eta]_{\mathcal{L}_n} = \phi^{-1}(\pi)$, with $\|\xi\| < 1 + \epsilon_0$ and $\|\eta\| < 1 + \epsilon_0$.

Since $\xi_\varnothing$ is cyclic for $\mathcal{L}_n$, there is $p \geq 1$ and C and D in the span of $\{L_u : u \in \mathbb{F}_n^p\}$ such that $\|C\xi_\varnothing - \xi\| < \epsilon_0$ and $\|D\xi_\varnothing - \eta\| < \epsilon_0$. Then

$$\|C\xi_\varnothing\| \leq \|C\xi_\varnothing - \xi\| + \|\xi\| < 1 + 2\epsilon_0,$$

so $\|C\xi_\varnothing\|^2 < 1+\epsilon+\epsilon^2/2$, and similarly, $\|D\xi_\varnothing\|^2 < 1+\epsilon+\epsilon^2/2$. Also,

$$\begin{aligned}
\|\pi - \phi([C\xi_\varnothing \otimes D\xi_\varnothing]_{\mathcal{L}_n})\| &= \|[\xi\otimes\eta]_{\mathcal{L}_n} - [C\xi_\varnothing \otimes D\xi_\varnothing]_{\mathcal{L}_n}\| \\
&\leq \|[(\xi - C\xi_\varnothing)\otimes\eta]_{\mathcal{L}_n}\| + \|[(C\xi_\varnothing - \xi)\otimes(\eta - D\xi_\varnothing)]_{\mathcal{L}_n}\| + \\
&\quad \|[\xi\otimes(\eta - D\xi_\varnothing)]_{\mathcal{L}_n}\| \\
&\leq \|\xi - C\xi_\varnothing\|\|\eta\| + \|\xi - C\xi_\varnothing\|\|\eta - D\xi_\varnothing\| + \\
&\quad \|\xi\|\|\eta - D\xi_\varnothing\| \\
&< 2\epsilon_0 + 3\epsilon_0^2 \\
&< \epsilon/2.
\end{aligned}$$

Set $A = \Phi(C)$ and $B = \Phi(D)$. If we expand C and D as

$$C = \sum_{u\in\mathbb{F}_n^p} c_u L_u \quad \text{and} \quad D = \sum_{u\in\mathbb{F}_n^p} d_u L_u,$$

then

$$A = \sum_{u\in\mathbb{F}_n^p} c_u S_u \quad \text{and} \quad B = \sum_{u\in\mathbb{F}_n^p} d_u S_u.$$

Choose $\epsilon_1 > 0$ such that

$$\epsilon_1 \sum_{u\in\mathbb{F}_n^p} |c_u| < \epsilon, \quad \epsilon_1 \sum_{u\in\mathbb{F}_n^p} \left|\overline{d_u}\right| < \epsilon, \quad \epsilon_1 \sum_{u\in\mathbb{F}_n^p}\sum_{v\in\mathbb{F}_n^p} \left|c_u\overline{d_v}\right| < \epsilon/2,$$

$$\epsilon_1 \sum_{u\in\mathbb{F}_n^N}\sum_{v\in\mathbb{F}_n^N} |c_u\overline{c_v}| < \epsilon+\epsilon^2/2, \quad \epsilon_1 \sum_{u\in\mathbb{F}_n^N}\sum_{v\in\mathbb{F}_n^N} \left|d_u\overline{d_v}\right| < \epsilon+\epsilon^2/2.$$

By Lemma 2.3.9, there exists a unit vector z in $\mathcal{M}$ such that

$$\|[S_u z\otimes S_v z]_{\mathcal{S}} - \phi([\xi_u\otimes\xi_v]_{\mathcal{L}_n})\| < \epsilon_1$$

for any words u and v in $\mathbb{F}_n^p$, and such that $\|[S_u z\otimes h_j]_{\mathcal{S}}\| < \epsilon_1$ and $\|[h_j\otimes S_u z]_{\mathcal{S}}\| < \epsilon_1$ for any word u in $\mathbb{F}_n^+$ and $1\leq j\leq q$. Then

$$
\begin{aligned}
&\|[Az \otimes Bz]_{\mathcal{S}} - \phi([C\xi_\varnothing \otimes D\xi_\varnothing]_{\mathcal{L}_n})\| \\
&\qquad = \left\| \sum_{u \in \mathbb{F}_n^p} \sum_{v \in \mathbb{F}_n^p} c_u \overline{d_v} ([S_u z \otimes S_v z]_{\mathcal{S}} - \phi([L_u \xi_\varnothing \otimes L_v \xi_\varnothing]_{\mathcal{L}_n})) \right\| \\
&\qquad \le \sum_{u \in \mathbb{F}_n^p} \sum_{v \in \mathbb{F}_n^p} \left| c_u \overline{d_v} \right| \|[S_u z \otimes S_v z]_{\mathcal{S}} - \phi([L_u \xi_\varnothing \otimes L_v \xi_\varnothing]_{\mathcal{L}_n})\| \\
&\qquad < \epsilon_1 \sum_{u \in \mathbb{F}_n^p} \sum_{v \in \mathbb{F}_n^p} \left| c_u \overline{d_v} \right| \\
&\qquad < \epsilon/2.
\end{aligned}
$$

Hence from above,

$$\|\pi - [Az \otimes Bz]_{\mathcal{S}}\| \le \|\pi - \phi([C\xi_\varnothing \otimes D\xi_\varnothing]_{\mathcal{L}_n})\| + \|\phi([C\xi_\varnothing \otimes D\xi_\varnothing]_{\mathcal{L}_n}) - [Az \otimes Bz]_{\mathcal{S}}\| < \epsilon.$$

By a similar estimation,

$$\|[Az \otimes Az]_{\mathcal{S}} - \phi([C\xi_\varnothing \otimes C\xi_\varnothing]_{\mathcal{L}_n})\| < \epsilon_1 \sum_{u \in \mathbb{F}_n^p} \sum_{v \in \mathbb{F}_n^p} |c_u \overline{c_v}| < \epsilon + \epsilon^2/2.$$

Evaluation of these functionals at the identity then implies

$$\epsilon + \epsilon^2/2 > \|Az\|^2 - \|C\xi_\varnothing\|^2 \ge \|Az\|^2 - (1 + \epsilon + \epsilon^2/2),$$

and hence that $\|Az\| < 1 + \epsilon$. In the same way we get $\|Bz\| < 1 + \epsilon$.

Finally,

$$\begin{aligned}
\|[Az \otimes h_j]_{\mathcal{S}}\| &= \|\sum_{u \in \mathbb{F}_n^p} c_u [S_u z \otimes h_j]_{\mathcal{S}}\| \\
&\leq \sum_{u \in \mathbb{F}_n^p} |c_u| \, \|[S_u z \otimes h_j]_{\mathcal{S}}\| \\
&< \epsilon_1 \sum_{u \in \mathbb{F}_n^p} |c_u| \\
&< \epsilon,
\end{aligned}$$

and in the same way we get

$$\|[h_j \otimes Bz]_{\mathcal{S}}\| < \epsilon_1 \sum_{u \in \mathbb{F}_n^p} \left|\overline{d}_u\right| < \epsilon.$$

Hence we can take $x = Az$ and $y = Bz$. $\square$

The next result follows from Lemma 2.3.10 by a standard iterative argument from the theory of dual algebras. We include the details for the convenience of the reader.

Theorem 2.3.11. *Given a weak*-continuous linear functional π on $\mathcal{S}$ and $\epsilon > 0$, there are vectors x and y in H such that $\pi = [x \otimes y]_{\mathcal{S}}$, $\|x\| < (1+\epsilon)\|\pi\|^{1/2}$, and $\|y\| < (1+\epsilon)\|\pi\|^{1/2}$. In other words, $\mathcal{S}$ has property $\mathbb{A}_1(1)$.*

Proof. By scaling π if necessary, we can assume that $\|\pi\| = 1$. Choose $\alpha > 0$ such that $(1+\alpha)/(1-\alpha) < 1+\epsilon$. Note that $\alpha^k \to 0$ as $k \to \infty$. We claim that for $k \geq 1$, we can find x_k and y_k in $\mathcal{M}$ such that

1. $\|\pi - [x_k \otimes y_k]_{\mathcal{S}}\| < \alpha^{2k}$,
2. $\|x_k\| < (1+\alpha)(1+\alpha+\ldots+\alpha^{k-1})$ and $\|y_k\| < (1+\alpha)(1+\alpha+\ldots+\alpha^{k-1})$,
3. $\|x_k - x_{k-1}\| < (1+\alpha)\alpha^{k-1}$ and $\|y_k - y_{k-1}\| < (1+\alpha)\alpha^{k-1}$ for $k \geq 2$.

Setting $x_0 = 0$ and $y_0 = 0$, Lemma 2.3.10 easily implies this is true for $k = 1$. Proceeding by induction, suppose that we have found x_k and y_k satisfying these conditions. Choose $\epsilon_0 > 0$ such that $\epsilon_0 < \alpha$ and $\epsilon_0 < \alpha^{2(k+1)}/3$. By Lemma 2.3.10, there are x' and y' in $\mathcal{M}$ such that

1. $\|\pi - [x_k \otimes y_k]_{\mathcal{S}} - [x' \otimes y']_{\mathcal{S}}\| < \epsilon_0$,
2. $\|x'\| < (1 + \epsilon_0)\|\pi - [x_k \otimes y_k]_{\mathcal{S}}\|^{1/2}$ and
   $\|y'\| < (1 + \epsilon_0)\|\pi - [x_k \otimes y_k]_{\mathcal{S}}\|^{1/2}$,
3. $\|[x' \otimes y_k]_{\mathcal{S}}\| < \epsilon_0$ and $\|[x_k \otimes y']_{\mathcal{S}}\| < \epsilon_0$.

Set $x_{k+1} = x_k + x'$, and $y_{k+1} = y_k + y'$. Then

$$\begin{aligned}
\|\pi - [x_{k+1} \otimes y_{k+1}]_{\mathcal{S}}\| &= \|\pi - [(x_k + x') \otimes (y_k + y')]_{\mathcal{S}}\| \\
&\leq \|\pi - [x_k \otimes y_k]_{\mathcal{S}} - [x' \otimes y']_{\mathcal{S}}\| + \|[x_k \otimes y']_{\mathcal{S}}\| + \\
&\quad \|[x' \otimes y_k]_{\mathcal{S}}\| \\
&< 3\epsilon_0 \\
&< \alpha^{2(k+1)}.
\end{aligned}$$

Also,

$$\|x'\| < (1 + \epsilon_0)\|\pi - [x_k \otimes y_k]_{\mathcal{S}}\|^{1/2} < (1 + \alpha)\alpha^k,$$

which gives

$$\|x_{k+1}\| = \|x_k + x'\| \leq \|x_k\| + \|x'\| < (1 + \alpha)(1 + \alpha + ... + \alpha^k)$$

and

$$\|x_{k+1} - x_k\| = \|x'\| < (1 + \alpha)\alpha^k.$$

Symmetrically, $\|y_{k+1}\| < (1 + \alpha)(1 + \alpha + ... + \alpha^k)$ and $\|y_{k+1} - y_k\| < (1 + \alpha)\alpha^k$, which establishes the claim.

Now for $l > k$,

$$\begin{aligned}\|x_l - x_k\| &\leq \|x_l - x_{l-1}\| + ... + \|x_{k+1} - x_k\| \\ &< (1+\alpha)(\alpha^{l-1} + ... + \alpha^k) \\ &\leq \alpha^{l-1}(1+\alpha)/(1-\alpha),\end{aligned}$$

so the sequence (x_k) is Cauchy. Let $x = \lim_k x_k$. Then

$$\|x\| = \lim_k \|x_k\| \leq \lim_k (1+\alpha)(1+\alpha+...+\alpha^{k-1}) = (1+\alpha)/(1-\alpha) < 1+\epsilon.$$

Similarly, the sequence $\{y_k\}$ is Cauchy. Letting $y = \lim_k y_k$ be its limit, $\|y\| < 1+\epsilon$. Finally, we have

$$\|\pi - [x \otimes y]_{\mathcal{S}}\| = \lim \|\pi - [x_k \otimes y_k]_{\mathcal{S}}\| \leq \lim \alpha^{2k} = 0,$$

so $\pi = [x \otimes y]_{\mathcal{S}}$. □

Theorem 2.3.12. *Every analytic free semigroup algebra has a wandering vector.*

Proof. Let $\mathcal{S}$ be an analytic free semigroup algebra, and let $\mathcal{S}_0$ denote the weak-operator-closed ideal generated by $S_1, ..., S_n$. Since $\mathcal{S}$ is analytic, $\mathcal{S}_0$ is proper, and in particular doesn't contain the identity. Let π_0 denote the weak-operator continuous linear functional which annihilates $\mathcal{S}_0$ and satisfies $\pi(I) = 1$.

Since $\mathcal{S}$ has property $\mathbb{A}_1(1)$, there are vectors x and y in H such that $\pi_0(A) = (Ax, y)$ for all A in $\mathcal{S}$. This implies $(S_w x, y) = 0$ for all $w \in \mathbb{F}_n^+ \backslash \{\varnothing\}$, so y is orthogonal to the subspace $\mathcal{S}_0[x]$. However, $(x, y) = \pi(I) = 1$, so y is not orthogonal to the subspace $\mathcal{S}[x]$. Hence $\mathcal{S}[x] \ominus \mathcal{S}_0[x]$ is nonempty.

Let z be a unit vector in $\mathcal{S}[x] \ominus \mathcal{S}_0[x]$. Then the subspace $\mathcal{S}_0[z]$ is contained in the subspace $\mathcal{S}_0[x]$, and in particular, is orthogonal to z. Hence $(S_w z, z) = 0$ for all $w \in \mathbb{F}_n^+ \backslash \{\varnothing\}$. Let u and v be distinct words in $\mathbb{F}_n^+$ such that $|u| \leq |v|$. Then $S_u^* S_v$ is in $\mathcal{S}_0$, so $(S_u z, S_v z) = (z, S_u^* S_v z) = 0$. By symmetry, it follows that $(S_u z, S_v z) = 0$ for every pair of distinct words u and v in $\mathbb{F}_n^+$. Thus z is a wandering vector for $\mathcal{S}$. □

Corollary 2.3.13. *A free semigroup algebra is either a von Neumann algebra, or it has a wandering vector.*

Proof. Let $\mathcal{S}$ be a free semigroup algebra. By the general structure theorem for free semigroup algebras [DKP01], $\mathcal{S}$ is either a von Neumann algebra, or it has an analytic part. In the latter case, by Theorem 2.3.12, $\mathcal{S}$ has a wandering vector. $\square$

By Theorem 4.1 of [DLP05], every free semigroup algebra which has a wandering vector is reflexive. Thus we have established the following result.

Corollary 2.3.14. *Every free semigroup algebra is reflexive.*

Theorem 4.2 of [DLP05] shows that every analytic free semigroup algebra which has a wandering vector is hyperreflexive with hyperreflexivity constant at most 55. This gives the following result, which we will refine in section 2.4.

Corollary 2.3.15. *Every analytic free semigroup algebra is hyperreflexive with hyperreflexivity constant at most* 55.

2.4 The hyperreflexivity of free semigroup algebras

In Section 2.3, we established that every analytic free semigroup algebra has a wandering vector. In this section, we will build on this result to show that the predual of every analytic free semigroup algebra with $n \geq 2$ generators satisfies a very strong factorization property. By a result of Bercovici [Ber98], we will obtain as a consequence that every such algebra is hyperreflexive with hyperreflexivity constant at most 3.

Definition 2.4.1. A weak*-closed subspace $\mathcal{S}$ of $\mathcal{B}(H)$ is said to have property $\mathcal{X}_{0,1}$ if given a weak*-continuous linear functional π on $\mathcal{S}$ with $\|\pi\| \leq 1$, $h_1, ..., h_q$ in H, and $\epsilon > 0$, there are vectors x and y in H such that

1. $\|\pi - [x \otimes y]_{\mathcal{S}}\| < \epsilon$,
2. $\|x\| \leq 1$ and $\|y\| \leq 1$,

3. $\|[x \otimes h_j]_{\mathcal{S}}\| < \epsilon$ and $\|[h_j \otimes y]_{\mathcal{S}}\| < \epsilon$ for $1 \leq j \leq q$.

Bercovici [Ber98] showed that any weak*-closed algebra whose commutant contains two isometries with pairwise orthogonal ranges has property $\mathcal{X}_{0,1}$, and showed that any weak*-closed algebra with propety $\mathcal{X}_{0,1}$ is hyperreflexive with hyperreflexivity constant at most 3. For $n \geq 2$, this includes $\mathcal{L}_n$. We will show that every analytic free semigroup algebra with $n \geq 2$ generators has property $\mathcal{X}_{0,1}$.

We require the following result which is implied by Lemma 1.2 in [Kri01].

Lemma 2.4.2. *Given a proper isometry V in $\mathcal{R}_n$, vectors $\nu_1, ..., \nu_q$ in F_n^2, and $\epsilon > 0$, there exists m such that $\|(V^*)^m \nu_j\| < \epsilon$ for $1 \leq j \leq q$.*

For the remainder of this section we fix an analytic free semigroup algebra $\mathcal{S}$ with $n \geq 2$ generators acting on a Hilbert space H, and we let $\mathcal{Z}$ denote the weak* closure of $\mathcal{S} + \mathcal{S}^*$. Let Φ denote the canonical map from $\mathcal{L}_n$ to $\mathcal{S}$. By Theorem 2.2.6, we can extend Φ to a map from the set $\mathcal{T}_R$ of R-Toeplitz operators to $\mathcal{Z}$, and this extension is a complete isometry and a weak*-to-weak* homeomorphism.

For x and y in H, we will need to take care to distinguish between the weak-operator-continuous vector functional $[x \otimes y]_{\mathcal{S}}$ defined on $\mathcal{S}$, and the weak-operator-continuous vector functional $[x \otimes y]_{\mathcal{Z}}$ defined on $\mathcal{Z}$.

The following lemma is a variation of an argument of Bercovici [Ber98]. It was kindly provided by Ken Davidson.

Lemma 2.4.3. *Given isometries U and V in $\mathcal{R}_n$ with orthogonal ranges, vectors ξ and ν in F_n^2 with ν in the kernel of U^*, and $\epsilon > 0$, define*

$$\eta_k = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} U^i V \xi.$$

Then $\lim_k \|[\nu \otimes \eta_k]_{\mathcal{T}_R}\| = 0$.

Proof. Let H^2 denote the Hardy-Hilbert space with orthonormal basis $\{e_k : k \geq 0\}$. For $k \geq 0$, define $Y : H^2 \to F_n^2$ by $Y e_k = U^k V \xi$ and $Z : H^2 \to F_n^2$ by $Z e_k = U^k \nu$

for $k \geq 0$. Note that Y and Z are isometries. For T in $\mathcal{T}_R$, by Lemma 2.2.3 we can factor T as $T = A^*B$, for A and B in $\mathcal{L}_n$. Then

$$\begin{aligned}
(Y^*TZe_j, e_i) &= (A^*BU^j\nu, U^iV\xi) \\
&= (A^*V^*(U^*)^iU^jB\nu, \xi) \\
&= \begin{cases} 0 & \text{if } i < j \\ c_{i-j} & \text{if } i \geq j \end{cases},
\end{aligned}$$

where $c_{i-j} = (A^*B\nu, U^{i-j}V\xi) = (T\nu, U^{i-j}V\xi)$. This implies that Y^*TZ is an analytic Toeplitz operator with symbol f, for some f in H^∞. Note that $\|f\|_\infty = \|Y^*TZ\| \leq \|T\|$. Hence

$$\begin{aligned}
|(T, [\nu \otimes \eta_k]_{\mathcal{T}_R})| &= \left|(T\nu, \frac{1}{\sqrt{k}}\sum_{i=1}^{k} U^iV\xi)\right| \\
&= \frac{1}{\sqrt{k}}\left|\sum_{i=1}^{k}(T\nu, U^iV\xi)\right| \\
&= \frac{1}{\sqrt{k}}\left|\sum_{i=1}^{k} c_i\right| \\
&\leq \frac{1}{\sqrt{k}}\|D_k\|_1\|f\|_\infty \\
&\leq \frac{1}{\sqrt{k}}\|D_k\|_1\|T\|,
\end{aligned}$$

where $\|D_k\|_1$ denotes the L^1-norm of the Dirichlet kernel. Using the well-known fact that $\|D_k\|_1$ grows logarithmically as $k \to \infty$ gives $\lim_k \|[\nu \otimes \eta_k]_{\mathcal{T}_R}\| = 0$. $\square$

Lemma 2.4.4. *Given vectors $h_1, ..., h_q$ in H and $\epsilon > 0$, there exists an intertwining operator $Y : F_n^2 \to H$ such that $\|Y\xi_\varnothing\| = 1$ and $\|[Y\xi_\varnothing \otimes h_i]_{\mathcal{Z}}\| < \epsilon$* for $1 \leq i \leq q$.

Proof. For $1 \leq i \leq q$, let $H_i : F_n^2 \to H$ be an intertwining operator such that $\|H_i\xi_\varnothing - h_i\| < \epsilon/2$. Since $\mathcal{S}$ is analytic, by Theorem 2.3.12 there is an isometric intertwining operator $X : F_n^2 \to H$. Then each H_i^*X is an L-Toeplitz operator, so by

Lemma 2.2.3, we can write $H_i^* X = A_i^* B_i$, for some A_i and B_i in $\mathcal{R}_n$ such that A_i and B_i are bounded below. Let $C_i = R_{1^i 2} B_i$, and let $D = \sum_{i=1}^k R_{1^i 2} A_i$. Then D is bounded below and $H_i^* X = C_i^* D$. Using inner-outer factorization, write $D = UF$ for U and F in $\mathcal{R}_n$, where U is inner and F is outer. Then F is bounded below since D is, and hence is invertible.

By Lemma 2.4.2, there exists m such that $\|(U^*)^m C_i \xi_\varnothing\| < \epsilon/(8\|F\|)$ for $1 \leq i \leq q$. Write $C_i \xi_\varnothing = \nu_i + \omega_i$, where $\|\omega_i\| < \epsilon/(8\|F\|)$, and ν_i is in the kernel of $(U^*)^m$. Set $V = U^m R_1$ and $W = U^m R_2$. Then V and W are isometries in $\mathcal{R}_n$ with pairwise orthogonal ranges. Note that ν_i is in the kernel of V^*. For $k \geq 1$, define intertwining operators $Y_k : F_n^2 \to H$ by

$$Y_k = XF^{-1} \frac{1}{\sqrt{k+1}} \sum_{j=0}^{k} U^{m-1} R_1 V^j W,$$

and define

$$\eta_k = \frac{1}{\sqrt{k}} \sum_{j=1}^{k} V^j W \xi_\varnothing.$$

Note that η_k is a unit vector.

Using the fact that $V = DF^{-1} U^{m-1} R_1$, we compute

$$\begin{aligned} H_i^* Y_k &= H_i^* X F^{-1} \frac{1}{\sqrt{k+1}} \sum_{j=0}^{k} U^{m-1} R_1 V^j W \\ &= C_i^* D F^{-1} \frac{1}{\sqrt{k+1}} \sum_{j=0}^{k} U^{m-1} R_1 V^j W \\ &= C_i^* \frac{1}{\sqrt{k+1}} \sum_{j=1}^{k+1} V^j W. \end{aligned}$$

Then for T in $\mathcal{T}_R$,

$$\begin{aligned}(TH_i\xi_\varnothing, Y_k\xi_\varnothing) &= (\Phi^{-1}(T)C_i\xi_\varnothing, \frac{1}{\sqrt{k+1}}\sum_{j=1}^{k+1} V^jW\xi_\varnothing)\\ &= (\Phi^{-1}(T)C_i\xi_\varnothing, \eta_{k+1}),\end{aligned}$$

Hence $\|[H_i\xi_\varnothing \otimes Y_k\xi_\varnothing]_{\mathcal{Z}}\| = \|[C_i\xi_\varnothing \otimes \eta_{k+1}]_{\mathcal{T}_R}\|$. By Lemma 2.4.3, we can choose r sufficiently large that $\|[\nu_i \otimes \eta_{r+1}]_{\mathcal{T}_R}\| < \epsilon/(8\|F\|)$. This gives

$$\begin{aligned}\|[C_i\xi_\varnothing \otimes \eta_{r+1}]_{\mathcal{T}_R}\| &\leq \|[\nu_i \otimes \eta_{r+1}]_{\mathcal{T}_R}\| + \|[\omega_i \otimes \eta_{r+1}]_{\mathcal{T}_R}\|\\ &\leq \|[\nu_i \otimes \eta_{r+1}]_{\mathcal{T}_R}]\| + \|\omega_i\|\|\eta_{r+1}\|\\ &< \epsilon/(4\|F\|).\end{aligned}$$

Thus $\|[H_i\xi_\varnothing \otimes Y_k\xi_\varnothing]_{\mathcal{Z}}\| < \epsilon/(4\|F\|)$ for $1 \leq i \leq q$.

Now,

$$\begin{aligned}\|Y_r\xi_\varnothing\|^2 &= \|XF^{-1}\frac{1}{\sqrt{r+1}}\sum_{j=0}^{r} U^{m-1}R_1V^jW\xi_\varnothing\|^2\\ &\geq \frac{1}{(r+1)\|F\|^2}\|\sum_{j=0}^{r} U^{m-1}R_1V^jW\xi_\varnothing\|^2\\ &= \frac{r}{(r+1)\|F\|^2},\end{aligned}$$

which implies

$$\|Y_r\xi_\varnothing\| \geq \frac{1}{2\|F\|}.$$

Setting $Y = Y_r/\|Y_r\xi_\varnothing\|$, it follows that

$$\begin{aligned}\|[H_i\xi_\varnothing \otimes Y\xi_\varnothing]_{\mathcal{Z}}\| &= \frac{1}{\|Y_p\xi_\varnothing\|}\|[H_i\xi_\varnothing \otimes Y_p\xi_\varnothing]_Z\|\\ &\leq 2\|F\|\|[H_i\xi_\varnothing \otimes Y_k\xi_\varnothing]_{\mathcal{Z}}\|\\ &< \epsilon/2.\end{aligned}$$

Thus

$$\begin{aligned}
\|[h_j\xi_\varnothing \otimes Y\xi_\varnothing]_Z\| &\leq \|[(h_j - H_j\xi_\varnothing) \otimes Y\xi_\varnothing]_\mathcal{Z}\| + \|[H_j\xi_\varnothing \otimes Y\xi_\varnothing]_\mathcal{Z}\| \\
&\leq \|h_j - H_j\xi_\varnothing\|\|Y\xi_\varnothing\| + \|[H_j\xi_\varnothing \otimes Y\xi_\varnothing]_\mathcal{Z}\| \\
&< \epsilon.
\end{aligned}$$

□

Lemma 2.4.5. *Given vectors $h_1, ..., h_q$ in H, $p \geq 1$, and $\epsilon > 0$, there exists a unit vector z in H such that*

$$\|[S_{u_1}z \otimes S_{u_2}z]_\mathcal{S} - \phi([\xi_{u_1} \otimes \xi_{u_2}]_{\mathcal{L}_n})\| < \epsilon$$

for all u_1 and u_2 in $\mathbb{F}_n^p$, and such that $\|[S_w z \otimes h_i]_\mathcal{S}\| < \epsilon$ and $\|[h_i \otimes S_w z]_\mathcal{S}\| < \epsilon$ for all $w \in \mathbb{F}_n^+$ and $1 \leq i \leq q$.

Proof. By Lemma 2.4.4, there is an intertwining operator $Y : F_n^2 \to H$ such that $\|Y\xi_\varnothing\| = 1$ and $\|[Y\xi_\varnothing \otimes h_i]_\mathcal{Z}\| < \epsilon$ for $1 \leq i \leq q$. By Lemma 2.3.6, there is a word v in $\mathbb{F}_n^+$ such that $\|[S_{u_1 v}Y\xi_\varnothing \otimes S_{u_2 v}Y\xi_\varnothing]_\mathcal{S} - \phi([\xi_{u_1} \otimes \xi_{u_2}]_{\mathcal{L}_n})\| < \epsilon$ for all words $u_1$and u_2 in $\mathbb{F}_n^p$. Set $z = S_v Y\xi_\varnothing$.

For T in $\mathcal{Z}$ and $w \in \mathbb{F}_n^+$,

$$\begin{aligned}
|(T, [S_w z \otimes h_j]_Z)| &= |(TS_w, [z \otimes h_j]_\mathcal{Z})| \\
&\leq \|TS_w\|\|[z \otimes h_j]_\mathcal{Z}\| \\
&\leq \|T\|\|[z \otimes h_J]_\mathcal{Z}\|.
\end{aligned}$$

Hence $\|[S_w z \otimes h_i]_\mathcal{Z}\| \leq \|[z \otimes h_i]_\mathcal{Z}\| < \epsilon$ and similarly, $\|[h_i \otimes S_w z]_Z\| \leq \|[h_i \otimes z]_\mathcal{Z}\| < \epsilon$. In particular, restricting to $\mathcal{S}$ gives $\|[S_w z \otimes h_i]_\mathcal{S}\| < \epsilon$ and $\|[h_i \otimes S_w z]_\mathcal{S}\| < \epsilon$. □

Lemma 2.4.5 is essentially a strengthened version of Lemma 2.3.9, in the sense that the h_i's in the hypothesis can be completely arbitrary.

Lemma 2.4.6. *Given a weak*-continuous linear functional π on $\mathcal{S}$, $h_1, ..., h_q$ in H, and $\epsilon > 0$, there are vectors x and y in H such that*

1. $\|\pi - [x \otimes y]_{\mathcal{S}}\| < \epsilon$,
2. $\|x\| < (1+\epsilon)\|\pi\|^{1/2}$ *and* $\|y\| < (1+\epsilon)\|\pi\|^{1/2}$,
3. $\|[x \otimes h_j]_{\mathcal{S}}\| < \epsilon$ *and* $\|[h_j \otimes y]_{\mathcal{S}}\| < \epsilon$ *for* $1 \leq j \leq q$.

Proof. The proof follows exactly as in the proof of Lemma 2.3.10, using Lemma 2.4.5 in place of Lemma 2.3.9. □

Lemma 2.4.6 clearly implies the desired result.

Theorem 2.4.7. *Every analytic free semigroup algebra with $n \geq 2$ generators has property $\mathcal{X}_{0,1}$.*

Corollary 2.4.8. *Every analytic free semigroup algebra with $n \geq 2$ generators is hyperreflexive with hyperreflexivity constant at most 3.*

Chapter 3

The structure of an isometric tuple

This chapter concerns the structure of an isometric tuple of operators, an object that appears frequently in mathematics and mathematical physics. From the perspective of an operator theorist, the notion of an isometric tuple is a natural higher-dimensional generalization of the notion of an isometry.

An n-tuple of operators $(V_1, \ldots, V_n)$ acting on a Hilbert space H is said to be **isometric** if the row operator $[V_1 \ \cdots \ V_n] : H^n \to H$ is an isometry. This is equivalent to requiring that the operators $V_1, \ldots, V_n$ satisfy the algebraic relations

$$V_i^* V_j = \begin{cases} I & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

These relations are often referred to as the Cuntz relations.

The main result in this chapter is a decomposition of an isometric tuple that generalizes the classical Lebesgue-von Neumann-Wold decomposition of an isometry into the direct sum of a unilateral shift, an absolutely continuous unitary and a singular unitary. We show that, as in the classical case, this decomposition determines the structure of the weakly closed algebra and the von Neumann algebra generated by the tuple.

The existence of a higher-dimensional Lebesgue-von Neumann-Wold decomposition was conjectured by Davidson, Li and Pitts in [DLP05]. They observed that the measure-theoretic definition of an absolutely continuous operator was equivalent to an operator-theoretic property of the functional calculus for that operator. Since this property naturally extends to the higher-dimensional setting, this allowed them to define the notion an absolutely continuous isometric tuple.

The key technical result in this chapter is a more effective operator-algebraic characterization of an absolutely continuous isometric tuple. The lack of such a characterization had been identified as the biggest obstruction to establishing the conjecture in [DLP05] (see also [DY08]). As we will see, the difficulty here can be attributed to the lack of a higher-dimensional analogue of the spectral theorem.

In this chapter, we overcome this difficulty by extending ideas from the commutative theory of dual algebras to the noncommutative setting. A similar approach was used in Chapter 2 to prove that certain isometric tuples are hyperreflexive. In the present chapter, our assumptions on the isometric tuples we consider are much weaker, and the problem is substantially more difficult. The idea to use this approach was inspired by results of Bercovici in [Ber98].

In Section 3.1, we review the Lebesgue-von Neumann-Wold decomposition of a single isometry that is the motivation for our results. In Section 3.2, we provide a brief review of the requisite background material on higher-dimensional operator theory, and we introduce the notions of absolute continuity and singularity. In Section 3.3, we prove an operator-algebraic characterization of an absolutely continuous isometric tuple. In Section 3.4, we prove an operator-algebraic characterization of a singular isometric tuple. In Section 3.5, we prove the Lebesgue-von Neumann-Wold decomposition of an isometric tuple, and we obtain some consequences of this result.

3.1 Motivation

The structure of a single isometry V is well understood. By the Wold decomposition of an isometry, V can be decomposed as

$$V = V_u \oplus U,$$

where V_u is a unilateral shift of some multiplicity, and U is a unitary. By the Lebesgue decomposition of a measure applied to the spectral measure of U, we can decompose U as

$$U = V_a \oplus V_s,$$

where V_a is an absolutely continuous unitary and V_s is a singular unitary, in the sense that their spectral measures are absolutely continuous and singular respectively with respect to Lebesgue measure. This allows us to further decompose V as

$$V = V_u \oplus V_a \oplus V_s.$$

We will refer to this as the **Lebesgue-von Neumann-Wold decomposition** of an isometry.

It will be convenient to consider the above notions of absolute continuity and singularity from a different perspective. Let $A(\mathbb{D})$ denote the classical disk algebra of analytic functions on the complex unit disk $\mathbb{D}$ with continuous extension to the boundary. An isometry V induces a contractive representation of $A(\mathbb{D})$, namely the $A(\mathbb{D})$ functional calculus for V, given by

$$f \to f(V), \quad f \in A(\mathbb{D}).$$

Recall that the algebra $A(\mathbb{D})$ is a weak-* dense subalgebra of the algebra H^∞ of bounded analytic functions on the complex unit disk. In certain cases, the representation of $A(\mathbb{D})$ induced by V is actually the restriction to $A(\mathbb{D})$ of a weak-* continuous representation of H^∞, namely the H^∞ functional calculus for V, given by

$$f \to f(V), \quad f \in H^\infty.$$

It follows from Theorem III.2.1 and Theorem III.2.3 of [SF70] that this occurs if and only if $V_s = 0$ in the Lebesgue-von Neumann-Wold decomposition of V. This motivates the following definitions.

Definition 3.1.1. Let V be an isometry. We will say that V is **absolutely continuous** if the representation of $A(\mathbb{D})$ induced by V extends to a weak-* continuous representation of H^∞. If V has no absolutely continuous restriction to an invariant subspace, then we will say that V is **singular**.

The importance of the Lebesgue-von Neumann-Wold decomposition of an isometry V is that it determines the structure of the weakly closed algebra $\mathrm{W}(V)$ and the von Neumann algebra $\mathrm{W}^*(V)$ generated by V. Recall that $\mathrm{W}(V)$ is the weak closure of the polynomials in V, and $\mathrm{W}^*(V)$ is the weak closure of the polynomials in V and V^*.

Let α denote the multiplicity of V_u as a unilateral shift, and let μ_a and μ_s be scalar measures equivalent to the spectral measures of V_a and V_s respectively. Since a unilateral shift of multiplicity one is irreducible, $\mathrm{W}^*(V)$ is given by

$$\mathrm{W}^*(V) \simeq B(\ell^2)^\alpha \oplus L^\infty(V_a) \oplus L^\infty(\mu_s)(V_s).$$

It was established by Wermer in [Wer52] that $\mathrm{W}(V)$ can be self-adjoint, depending on α and μ_a. If $\alpha \neq 0$ or if Lebesgue measure is absolutely continuous with respect to μ_a, then $\mathrm{W}(V)$ is given by

$$\mathrm{W}(V) \simeq H^\infty(V_u \oplus V_a) \oplus L^\infty(\mu_s)(V_s).$$

Otherwise, if neither of these conditions holds, then $\mathrm{W}(V) = \mathrm{W}^*(V)$.

The following example shows that it is possible for the weakly closed algebra generated by an absolutely continuous isometry to be self-adjoint. We will see later that there is no higher-dimensional analogue of this phenomenon.

Example 3.1.2. Let U denote the operator of multiplication by the coordinate function on $L^2(\mathbb{T}, m)$, where m denotes Lebesgue measure. Let m_1 and m_2 denote Lebesgue measure on the upper and lower half of the unit circle respectively, and let $U_1$and U_2 denote the operator of multiplication by the coordinate function on $L^2(\mathbb{T}, m_1)$ and $L^2(\mathbb{T}, m_2)$ respectively.

Since the spectral measure of $U \simeq U_1 \oplus U_2$ is equivalent to Lebesgue measure, U is absolutely continuous. Thus U_1 and U_2 are also absolutely continuous. From above,

$$\mathrm{W}^*(U) \simeq L^\infty(U), \qquad \mathrm{W}(U) \simeq H^\infty(U).$$

However, since Lebesgue measure is not absolutely continuous with respect to m_1 or m_2,

$$\mathrm{W}(U_i) = \mathrm{W}^*(U_i) = L^\infty(U_i), \quad i = 1, 2.$$

In particular, the weakly closed algebras $\mathrm{W}(U_1)$ and $\mathrm{W}(U_2)$ generated by U_1 and U_2 respectively are self-adjoint.

3.2 Background and preliminaries

3.2.1 The noncommutative function algebras

The noncommutative Hardy space F_n^2 is defined to be the full Fock-Hilbert space over $\mathbb{C}^n$, i.e.

$$F_n^2 = \oplus_{k=0}^\infty (\mathbb{C}^n)^{\otimes k},$$

where we will write $\xi_\varnothing$ to denote the vacuum vector, so that $(\mathbb{C}^n)^{\otimes 0} = \mathbb{C}\xi_\varnothing$. Let $\xi_1, \ldots, \xi_n$ be an orthonormal basis of $\mathbb{C}^n$ and let $\mathbb{F}_n^*$ denote the unital free semigroup on n generators $\{1, \ldots, n\}$ with unit $\varnothing$. For a word $w = w_1 \cdots w_k$ in $\mathbb{F}_n^*$, it will be convenient to write $\xi_w = \xi_{w_1} \otimes \cdots \otimes \xi_{w_k}$. We can identify F_n^2 with the set of power series in n noncommuting variables $\xi_1, \ldots, \xi_n$ with square-summable coefficients, i.e.

$$F_n^2 = \left\{ \sum_{w \in \mathbb{F}_n^*} a_w \xi_w : \sum_{w \in \mathbb{F}_n^*} |a_w|^2 < \infty \right\}.$$

In particular, we can identify the noncommutative Hardy space F_1^2 with the classical Hardy space H^2 of analytic functions having power series expansions with square-summable coefficients.

The left multiplication operators $L_1, \ldots, L_n$ are defined on F_n^2 by

$$L_i \xi_w = \xi_i \otimes \xi_w = \xi_{iw}, \quad w \in \mathbb{F}_n^*.$$

It is clear that the n-tuple $L = (L_1, \ldots, L_n)$ is isometric. We will call it the **unilateral n-shift** since, for $n = 1$, L_1 can be identified with the unilateral shift on H^2. For a word $w = w_1 \cdots w_k$ in $\mathbb{F}_n^*$, it will be convenient to write $L_w = L_{w_1} \cdots L_{w_k}$.

The **noncommutative disk algebra** $\mathcal{A}_n$ is the norm closed unital algebra generated by $L_1, \ldots, L_n$ and the **noncommutative analytic Toeplitz algebra** $\mathcal{L}_n$ is the weakly closed unital algebra generated by $L_1, \ldots, L_n$. These algebras were introduced by Popescu in [Pop96], and have subsequently been studied by a number of authors (see for example [DP98] and [DP99]).

The noncommutative disk algebra $\mathcal{A}_n$ and the noncommutative analytic Toeplitz algebra $\mathcal{L}_n$ are higher-dimensional analogues of the classical disk algebra $A(\mathbb{D})$ and the classical algebra H^∞ of bounded analytic functions. In particular, the algebra $\mathcal{A}_n$ is a proper weak-* dense subalgebra of the algebra $\mathcal{L}_n$. If we agree to identify functions in H^∞ with the corresponding multiplication operators on H^2, then we can identify $A(\mathbb{D})$ with $\mathcal{A}_1$ and H^∞ with $\mathcal{L}_1$.

As in the classical case, an element A in $\mathcal{L}_n$ is uniquely determined by its Fourier series

$$A \sim \sum_{w \in \mathbb{F}_n^*} a_w L_w,$$

where $a_w = (A\xi_\varnothing, \xi_w)$ for w in $\mathbb{F}_n^*$. The Cesaro sums of this series converge strongly to A, and it is often useful heuristically to work directly with this representation.

We will also need to work with the right multiplication operators $R_1, \ldots, R_n$ defined on F_n^2 by

$$R_i \xi_w = \xi_w \otimes \xi_i = \xi_{wi}, \quad w \in \mathbb{F}_n^*.$$

The n-tuple $R = (R_1, \ldots, R_n)$ is unitarily equivalent to $L = (L_1, \ldots, L_n)$. The unitary equivalence is implemented by the "unitary flip" on F_n^2 that, for a word $w_1 \cdots w_k$ in $\mathbb{F}_n^*$, takes $\xi_{w_1 \cdots w_k}$ to $\xi_{w_k \cdots w_1}$. We will let $\mathcal{R}_n$ denote the weakly closed algebra generated by $R_1, \ldots, R_n$.

3.2.2 Free semigroup algebras

Let $V = (V_1, \ldots, V_n)$ be an isometric n-tuple. The weakly closed unital algebra $\mathrm{W}(V)$ generated by $V_1, \ldots, V_n$ is called the **free semigroup algebra** generated by V. As in Section 3.2.1, for a word $w = w_1 \cdots w_k$ in the unital free semigroup $\mathbb{F}_n^*$, it will be convenient to write $V_w = V_{w_1} \cdots V_{w_k}$.

Example 3.2.1. The noncommutative analytic Toeplitz algebra $\mathcal{L}_n$ introduced in Section 3.2.1 is a fundamental example of a free semigroup algebra. We will see that it plays an important role in the general theory of free semigroup algebras.

The study of free semigroup algebras was initiated by Davidson and Pitts in [DP99]. They observed that information about the unitary invariants of an isometric tuple can be detected in the algebraic structure of the free semigroup algebra it generates, and used this fact to classify a large family of representations of the Cuntz algebra. Free semigroup algebras have subsequently received a great deal of interest (see for example [Dav01]).

It was shown in [DP98] that $\mathcal{L}_n$ has a great deal of structure that is analogous to the analytic structure of H^∞. This motivates the following definition.

Definition 3.2.2. An isometric n-tuple $V = (V_1, \ldots, V_n)$ is said to be **analytic** if the free semigroup algebra generated by V is isomorphic to the noncommutative analytic Toeplitz algebra $\mathcal{L}_n$.

The notion of analyticity is of central importance in the theory of free semigroup algebras. This is apparent from the work of Davidson, Katsoulis and Pitts in [DKP01]. They proved the following general structure theorem.

Theorem 3.2.3 (Structure theorem for free semigroup algebras)**.** *Let $\mathcal{V} = \mathrm{W}(V)$ be a free semigroup algebra. Then there is a projection P in $\mathcal{V}$ with range invariant under $\mathcal{V}$ such that*

1. *if $P \neq 0$, then the restriction of $\mathcal{V}$ to the range of P is an analytic free semigroup algebra,*
2. *the compression of $\mathcal{V}$ to the range of $P^\perp$ is a von Neumann algebra,*
3. *$\mathcal{V} = P\mathcal{V}P + (\mathrm{W}^*(V))P^\perp$.*

The analytic structure of a free semigroup algebra reveals itself in the form of wandering vectors. Let $V = (V_1, \ldots, V_n)$ be an isometric n-tuple acting on a Hilbert space H. A vector x in H is said to be **wandering** for V if the set of vectors $\{V_w x : w \in \mathbb{F}_n^*\}$ is orthonormal. In this case we will also say that x is wandering for the free semigroup algebra generated by V.

The existence of wandering vectors for an analytic free semigroup algebra was established in Chapter 2, settling a conjecture first made in [DKP01] (see also [DLP05] and [DY08]). Examples show that the structure of an analytic free semigroup algebra can be quite complicated, making this result far from obvious.

3.2.3 Dilation theory

Recall that an operator T is said to be contractive if $\|T\| \leq 1$. An n-tuple of operators $T = (T_1, \ldots, T_n)$ acting on a Hilbert space H is said to be **contractive** if the row operator $[T_1 \ \cdots \ T_n] : H^n \to H$ is contractive.

Sz.-Nagy showed that every contractive operator T acting on a Hilbert space H has a unique minimal dilation to an isometry V, acting on a bigger Hilbert space K

(see for example [SF70]). This means that $H \subseteq K$, H is cyclic for V and

$$T^k = P_H V^k \mid_H, \quad k \geq 1.$$

Sz.-Nagy's dilation theorem was generalized in the work of Bunce, Frazho and Popescu in [Bun84], [Fra82] and [Pop89a] respectively. They showed that every contractive n-tuple of operators $T = (T_1, \dots, T_n)$ acting on a Hilbert space H has a unique minimal dilation to an isometric n-tuple $V = (V_1, \dots, V_n)$, acting on a bigger Hilbert space K. This means that $H \subseteq K$, H is cyclic for $V_1, \dots, V_n$ and

$$P_H V_{i_1} \cdots V_{i_k} \mid_H = T_{i_1} \cdots T_{i_k}, \quad i_1, \dots, i_k \in \{1, \dots, n\} \text{ and } k \geq 1.$$

3.2.4 The Wold decomposition

The classical Wold decomposition decomposes a single isometry into the direct sum of a unilateral shift of some multiplicity and a unitary. In order to state the Wold decomposition of an isometric tuple, we need to generalize these notions.

In Section 3.2.1, we introduced the unilateral n-shift $L = (L_1, \dots, L_n)$, and we saw that it is the natural higher-dimensional generalization of the classical unilateral shift. An isometric n-tuple is said to be a **unilateral shift of multiplicity** α if it is unitarily equivalent to the ampliation $L^{(\alpha)} = (L_1^{(\alpha)}, \dots, L_n^{(\alpha)})$, for some positive integer α.

The higher-dimensional generalization of a unitary is based on the fact that a unitary is the same thing as a surjective isometry. An n-tuple of operators $U = (U_1, \dots, U_n)$ is said to be **unitary** if the operator $[U_1 \ \cdots \ U_n] : H^n \to H$ is a surjective isometry. This is equivalent to requiring that the operators $U_1, \dots, U_n$ satisfy

$$\sum_{i=1}^{n} U_i U_i^* = I.$$

Note that a unilateral shift is not unitary. This is because the "vacuum" vector $\xi_\varnothing$ in F_n^2 is not contained in the range of the unilateral n-shift $L = (L_1, \dots, L_n)$.

In [DP99], Davidson and Pitts studied a family of "atomic" isometric tuples that arise from certain infinite directed trees. As the following example shows, this family contains a large number of unitary tuples.

Example 3.2.4. Fix an infinite directed n-ary tree B with vertex set V such that every vertex has a parent. For a vertex v in V, let $c_i(v)$ denote the i-th child of v. Let $H = \ell^2(V)$, so that the set $\{e_v : v \in V\}$ is an orthonormal basis for H. Define operators $S_1, \ldots, S_n$ on H by

$$S_i e_v = e_{c_i(v)}, \quad 1 \leq i \leq n.$$

It's clear that $S_1, \ldots, S_n$ are isometries, and the fact that B is an infinite directed n-ary tree implies that the range of S_i and the range of S_j are orthogonal for $i \neq j$. Thus $S = (S_1, \ldots, S_n)$ is an isometric n-tuple. The fact that every vertex has a parent implies that every basis vector is in the range of some S_i. Thus S is a unitary n-tuple.

Let $V = (V_1, \ldots, V_n)$ be an arbitrary isometric n-tuple. If V is unitary, then the C*-algebra $\mathrm{C}^*(V_1, \ldots, V_n)$ generated by V is isomorphic to the Cuntz algebra $\mathcal{O}_n$. Otherwise, it is isomorphic to the extended Cuntz algebra $\mathcal{E}_n$, the extension of the compacts by $\mathcal{O}_n$. Since the only irreducible *-representation of the compacts is the identity representation, and since $\mathcal{O}_n$ is simple, a *-representation of $\mathcal{E}_n$ can be decomposed into a multiple of the identity representation and a representation of $\mathcal{O}_n$. The Wold decomposition of an isometric n-tuple, which was proved by Popescu in [Pop89a], can be obtained as a consequence of these C*-algebraic facts, based on the observation that the C*-algebra generated by a unilateral n-shift is isomorphic to $\mathcal{E}_n$.

Proposition 3.2.5 (The Wold decomposition)**.** *Let $V = (V_1, \ldots, V_n)$ be an isometric n-tuple. Then we can decompose V as*

$$V = V_u \oplus U,$$

where V_u is a unilateral n-shift and U is a unitary n-tuple.

3.2.5 Absolutely continuous and singular isometric tuples

As in the classical case, an isometric n-tuple $V = (V_1, \ldots, V_n)$ induces a contractive representation of the noncommutative disk algebra $\mathcal{A}_n$, called the $\mathcal{A}_n$ functional calculus for V, determined by

$$L_{i_1} \cdots L_{i_k} \to V_{i_1} \cdots V_{i_k}, \quad i_1, \ldots, i_k \in \{1, \ldots, n\} \text{ and } k \geq 1.$$

This is a consequence of Popescu's generalization of von Neumann's inequality in [Pop91].

Recall from Section 3.2.1 that $\mathcal{A}_n$ is a proper weak-* dense subalgebra of the noncommutative analytic Toeplitz algebra $\mathcal{L}_n$. The following definition is the natural generalization of Definition 3.1.1.

Definition 3.2.6. Let $V = (V_1, \ldots, V_n)$ be an isometric n-tuple. We will say that V is **absolutely continuous** if the representation of $\mathcal{A}_n$ induced by V is the restriction to $\mathcal{A}_n$ of a weak-* continuous representation of $\mathcal{L}_n$. We will say that V is **singular** if V has no absolutely continuous restriction to an invariant subspace.

It is clear from Definition 3.2.2 and Definition 3.2.6 that an analytic isometric tuple is absolutely continuous. In order to obtain the Lebesgue-von Neumann-Wold decomposition of an isometric tuple, we will prove the converse result that an absolutely continuous isometric tuple is analytic.

3.3 Absolutely continuous isometric tuples

The main result in this section is an operator-algebraic characterization of an absolutely continuous isometric tuple. Specifically, we will show that for $n \geq 2$, every absolutely continuous isometric n-tuple is analytic.

For $n \geq 2$, fix an absolutely continuous isometric n-tuple $S = (S_1, \ldots, S_n)$ acting on a Hilbert space H. Let Φ denote the corresponding representation of the noncommutative disk algebra $\mathcal{A}_n$, given by

$$\Phi(L_w) = S_w, \quad w \in \mathbb{F}_n^*.$$

Since S is absolutely continuous, Φ extends to a representation of $\mathcal{L}_n$ that is weak-* continuous.

It was shown in Corollary 1.2 of [DY08] that Φ is actually a completely isometric isomorphism and a weak-* homeomorphism from $\mathcal{L}_n$ to the weak-* closed algebra generated by $S_1, \ldots, S_n$. This is equivalent to the fact that an infinite ampliation of S is an analytic isometric tuple. Evidently, it is much more difficult to show that S is analytic. As an explanation, we offer the aphorism that things are generally much nicer in the presence of infinite multiplicity.

Showing that S is analytic amounts to showing that the free semigroup algebra (i.e. the weakly closed algebra) $\mathrm{W}(S)$ generated by $S_1, \ldots, S_n$ is isomorphic to the noncommutative analytic Toeplitz algebra $\mathcal{L}_n$. Since we know from above that the weak-* closed algebra generated by $S_1, \ldots, S_n$ is isomorphic to $\mathcal{L}_n$, our strategy will be to show that this algebra is actually equal to $\mathrm{W}(S)$.

3.3.1 The noncommutative Toeplitz operators

Let $\mathcal{S}$ denote the weak-* closed algebra generated by $S_1, \ldots, S_n$. The map Φ introduced at the beginning of this section is a completely isometric isomorphism and a weak-* homeomorphism from $\mathcal{L}_n$ to $\mathcal{S}$. It will be useful for what follows to extend Φ even further. Let $\mathcal{M}_n$ denote the weak-* closure of the operator system $\mathcal{L}_n + \mathcal{L}_n^*$. We will call the elements of $\mathcal{M}_n$ the **noncommutative Toeplitz operators**, because they are a natural higher-dimensional generalization of the classical Toeplitz operators.

The noncommutative Toeplitz operators were introduced by Popescu in [Pop89b]. It was shown in Corollary 1.3 of [Pop09] that A belongs to $\mathcal{M}_n$ if and only if

$$R_i^* A R_j = \begin{cases} A & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

where $R_1, \ldots, R_n$ are the right multiplication operators introduced in Section 3.2.1. A short proof of this fact was also given in Lemma 2.2.2 of Chapter 2. It follows from this characterization that $\mathcal{M}_n$ is weakly closed.

Let $\mathcal{T}$ denote the weak-* closure of the operator system $\mathcal{S} + \mathcal{S}^*$. The proof of the following proposition is nearly identical to the proof of Theorem 2.2.6 of Chapter 2.

Proposition 3.3.1. *Let $S = (S_1, \ldots, S_n)$ be an absolutely continuous isometric n-tuple. The representation Φ of $\mathcal{L}_n$ induced by S extends to a completely isometric and weak-* homeomorphic *-map from $\mathcal{M}_n$ to $\mathcal{T}$.*

We will need to exploit the fact that $\mathcal{M}_n$ and $\mathcal{T}$ are dual spaces. Let $\mathcal{T}_*$ denote the predual of $\mathcal{T}$, i.e. the set of weak-* continuous linear functionals on $\mathcal{T}$. Similarly, let $\mathcal{M}_{n*}$ denote the predual of $\mathcal{M}_n$. Basic functional analysis implies that the inverse map Φ^{-1} is the dual of an isometric isomorphism ϕ from $\mathcal{M}_{n*}$ to $\mathcal{T}_*$. Moreover, since Φ^{-1} is isometric, so is ϕ.

We can identify the predual of $B(F_n^2)$, i.e. the set of weak-* continuous linear functionals on $B(F_n^2)$, with the set of trace class operators $C^1(F_n^2)$ on F_n^2, where K in $C^1(F_n^2)$ corresponds to the linear functional

$$(T, K) = \mathrm{tr}(TK), \quad T \in B(F_n^2).$$

If we let $(\mathcal{M}_n)_\perp$ denote the preannihilator of $\mathcal{M}_n$, i.e.

$$(\mathcal{M}_n)_\perp = \{K \in C^1(F_n^2) : \mathrm{tr}(AK) = 0, \quad \forall A \in \mathcal{M}_n\},$$

then we can identify the predual $(\mathcal{M}_n)_*$ with the quotient space $C^1(F_n^2)/(\mathcal{M}_n)_\perp$. Similarly, we can identify the predual $\mathcal{T}_*$ with the quotient space $C^1(H)/\mathcal{T}_\perp$.

For ξ and η in F_n^2, it will be convenient to let $[\xi \otimes \eta]_{\mathcal{M}_n}$ denote the weak-* continuous linear functional on $\mathcal{M}_n$ given by

$$(A, [\xi \otimes \eta]_{\mathcal{M}_n}) = (A\xi, \eta), \quad A \in \mathcal{M}_n.$$

In other words, $[\xi \otimes \eta]_{\mathcal{M}_n}$ denotes the equivalence class of the rank one tensor $x \otimes y$

in $(\mathcal{M}_n)_*$. Similarly, for x and y in H, let $[x \otimes y]_{\mathcal{T}}$ denote the weak-* continuous linear functional on $\mathcal{T}$ given by

$$(T, [x \otimes y]_{\mathcal{T}}) = (Tx, y), \quad T \in \mathcal{T}.$$

3.3.2 Intertwining operators

An operator $X : F_n^2 \to H$ is said to **intertwine** the isometric n-tuple $S = (S_1, \ldots, S_n)$ and the unilateral n-shift $L = (L_1, \ldots, L_n)$ if it satisfies

$$XL_i = S_i X, \quad 1 \leq i \leq n.$$

Observe that if X intertwines S and L, then the operator JX^*XJ is a noncommutative Toeplitz operator, where J is the unitary flip introduced in Section 3.2.1. Indeed, using the fact that $JR_i = L_i J$ for $1 \leq i \leq n$, we compute

$$\begin{aligned} R_i^* JX^*XJR_j &= JL_i^* X^* X L_j J \\ &= JX^* S_i^* S_j XJ \\ &= \begin{cases} JX^*XJ & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Since S is absolutely continuous, it follows from Theorem 2.7 of [DLP05] that every vector x in H is in the range of an operator that intertwines S and L.

3.3.3 Dual algebra theory

Recall that to prove the isometric n-tuple $S = (S_1, \ldots, S_n)$ is analytic, our strategy is to show that the weak-* closed algebra $\mathcal{S} = \mathrm{W}^*(S_1, \ldots, S_n)$ is actually equal to the weakly closed algebra $\mathrm{W}(S_1, \ldots, S_n)$. This amounts to showing that $\mathcal{S}$ is already weakly closed. However, instead of working directly with $\mathcal{S}$, it will be necessary to work with the operator system $\mathcal{T}$. In fact, we will need to consider the general structure of the predual of $\mathcal{T}$.

In Section 3.3.1, we saw that an element in the predual $\mathcal{T}_*$ of the operator system $\mathcal{T}$ can be identified with an equivalence class of trace class operators. We will show that $\mathcal{T}$ satisfies a very powerful predual "factorization" property, in the sense that the equivalence class of an element in the predual $\mathcal{T}_*$ always contains "nice" representatives. We will see that $\mathcal{S}$ inherits this property from $\mathcal{T}$, and that this will imply the desired result.

The idea of studying factorization in the predual of an operator algebra is the central idea in dual algebra theory, which has been applied with great success to a number of problems in the commutative setting (see for example [BFP85]). As we will see, many of the factorization properties that were introduced in the commutative setting make sense even in the present noncommutative setting.

Definition 3.3.2. A weak-* closed subspace $\mathcal{A}$ of operators acting on a Hilbert space H is said to have property $\mathbb{A}_1(1)$ if, given a weak-* continuous linear functional τ on $\mathcal{A}$ with $\|\tau\| \leq 1$ and $\epsilon > 0$, there are vectors x and y in H such that $\|x\| \leq (1+\epsilon)^{1/2}$, $\|y\| \leq (1+\epsilon)^{1/2}$ and $\tau = [x \otimes y]_{\mathcal{A}}$.

If a weak-* closed subspace of $\mathcal{B}(H)$ has property $\mathbb{A}_1(1)$, then the equivalence class of any weak-* continuous linear functional on the subspace contains an operator of rank one. Note that in this case, every weak-* continuous linear functional on the subspace is actually weakly continuous. It was shown in [DP99] that $\mathcal{L}_n$ has property $\mathbb{A}_1(1)$, and the same proof also shows that $\mathcal{M}_n$ has property $\mathbb{A}_1(1)$.

Of course, the main difficulty with a predual factorization property like property $\mathbb{A}_1(1)$ is that it is often extremely difficult to show that it holds. The next factorization property turns out to be much stronger than property $\mathbb{A}_1(1)$, but it is sometimes easier to show that it holds due to its approximate nature.

Definition 3.3.3. A weak-* closed subspace $\mathcal{A}$ of operators acting on a Hilbert space H is said to have property $\mathcal{X}_{0,1}$ if, given a weak-* continuous linear functional τ on $\mathcal{A}$ with $\|\tau\| \leq 1$, $z_1, ..., z_q$ in H and $\epsilon > 0$, there are vectors x and y in H such that

1. $\|x\| \leq 1$ and $\|y\| \leq 1$,

2. $\|[x \otimes z_j]_{\mathcal{A}}\| < \epsilon$ and $\|[z_j \otimes y]_{\mathcal{A}}\| < \epsilon$ for $1 \le j \le q$,

3. $\|\tau - [x \otimes y]_{\mathcal{A}}\| < \epsilon$.

It's easy to see that the infinite ampliation of a weak-* closed subspace of $\mathcal{B}(H)$ has property $\mathcal{X}_{0,1}$. Thus, intuitively, a weak-* closed subspace of $\mathcal{B}(H)$ that has property $\mathcal{X}_{0,1}$ can be thought of as having "approximately infinite" multiplicity. It was shown in [BFP85] that property $\mathcal{X}_{0,1}$ implies property $\mathbb{A}_1(1)$.

We will show that $\mathcal{T}$ has property $\mathcal{X}_{0,1}$. Since this property is inherited by weak-* closed subspaces, it will follow that $\mathcal{S}$ has property $\mathcal{X}_{0,1}$, and hence that $\mathcal{S}$ has property $\mathbb{A}_1(1)$. It is easy to show that any weak-* closed subspace of operators with property $\mathbb{A}_1(1)$ is weakly closed (see for example Proposition 59.2 of [Con00]). Thus this will imply the desired result that $\mathcal{S}$ is weakly closed.

3.3.4 Approximate factorization

Lemma 3.3.4. *Given unit vectors $x, z_1, ..., z_q$ in H and $\epsilon > 0$, there are vectors $\xi, \zeta_1, ..., \zeta_q$ in F_n^2 such that*

1. $\|\xi\| < \sqrt{q}(1+\epsilon)^{1/2}$,

2. $\|\zeta_i\| < (1+\epsilon)^{1/2}$ *for* $1 \le i \le q$,

3. $[x \otimes z_i]_{\mathcal{T}} = \phi([\xi \otimes \zeta_i]_{\mathcal{M}_n})$ *for* $1 \le i \le q$.

Proof. Since $\mathcal{M}_n$ has property $\mathbb{A}_1(1)$, there are vectors $\upsilon_1', ..., \upsilon_q', \zeta_1', ..., \zeta_q'$ in F_n^2 such that $\|\upsilon_i'\| < (1+\epsilon)^{1/2}$, $\|\zeta_i'\| < (1+\epsilon)^{1/2}$ and $[x \otimes z_i]_{\mathcal{T}} = \phi([\upsilon_i' \otimes \zeta_i']_{\mathcal{M}_n})$ for $1 \le i \le q$.

Let $V_i = R_{12^k}$ for $1 \le i \le q$, so that $V_1, ..., V_q$ are isometries in $\mathcal{R}_n$ with pairwise orthogonal ranges. Set $\xi = \sum_{i=1}^q V_i \upsilon_i'$ and $\zeta_i = V_i \zeta_i'$ for $1 \le i \le q$. Then

$\|\xi\| < \sqrt{q}(1+\epsilon)^{1/2}$, $\|\zeta_i\| < (1+\epsilon)^{1/2}$ and for T in $\mathcal{T}$,

$$\begin{aligned}
(\phi([\xi \otimes \zeta_i]_{\mathcal{M}_n}), T) &= (\Phi^{-1}(T)\xi, \zeta_i) \\
&= (\Phi^{-1}(T)\sum_{j=1}^{q} V_j \upsilon_j', V_i \zeta_i') \\
&= (\Phi^{-1}(T)\upsilon_i', \zeta_i') \\
&= (\phi([\upsilon_i' \otimes \zeta_i']_{\mathcal{M}_n}), T) \\
&= ([x \otimes z_i]_{\mathcal{T}}, T).
\end{aligned}$$

Hence $[x \otimes z_i]_{\mathcal{T}} = \phi([\xi \otimes \zeta_i]_{\mathcal{M}_n})$. $\square$

Lemma 3.3.5. *Let η be a unit vector contained in the algebraic span of $\{\xi_w : w \in \mathbb{F}_n^*\}$. Then there are words u and v in $\mathbb{F}_n^*$ such that*

$$L_u R_v \eta = L\xi_\varnothing = R\xi_\varnothing,$$

where L is an isometry in $\mathcal{L}_n$, and R is an isometry in $\mathcal{R}_n$ with range orthogonal to the range of R_1.

Proof. Expand η as $\eta = \sum_{|w|\leq m} a_w \xi_w$ for some $m \geq 0$. Let $u = 12^m$ and let $v = 1^m 2$. Then $L_u R_v \eta = \sum_{|w|\leq m} a_w \xi_{uwv}$. Set $L = \sum_{|w|\leq m} a_w L_{uwv}$ and $R = \sum_{|w|\leq m} a_w R_{uwv}$. Then $L_u R_v \eta = L\xi_\varnothing = R\xi_\varnothing$, and it's clear that the range of R is orthogonal to the range of R_1.

It remains to show that L and R are isometries. For w and w' in $\mathbb{F}_n^+$ with $|w| \leq m$ and $|w'| \leq m$,

$$L_v^* L_w^* L_{w'} L_v = \begin{cases} I & \text{if } w = w', \\ 0 & \text{otherwise.} \end{cases}$$

This gives

$$\begin{aligned}
L^*L &= \sum_{|w|\le m}\sum_{|w'|\le m} \overline{a_w} a_{w'} L^*_{uwv} L_{uw'v} \\
&= \sum_{|w|\le m}\sum_{|w'|\le m} \overline{a_w} a_{w'} L^*_v L^*_w L_{w'} L_v \\
&= \sum_{|w|\le m} |a_w|^2 I \\
&= I,
\end{aligned}$$

where the last equality follows from the fact that η is a unit vector. Thus L is an isometry, and it follows from a similar computation that R is an isometry. □

Lemma 3.3.6. *Given unit vectors $z_1, ..., z_q$ in H and $\epsilon > 0$, there exists a unit vector x in H and vectors $\xi, \zeta_1, ..., \zeta_q$ in F^2_n such that*

1. $\|\xi\| < \sqrt{q}(1+\epsilon)^{1/2}$,
2. $\|\zeta_i\| < (1+\epsilon)^{1/2}$ *for* $1 \le i \le q$,
3. $\xi = \|\xi\| L\xi_\varnothing = \|\xi\| R\xi_\varnothing$, *where L is an isometry in $\mathcal{L}_n$, and R is an isometry in $\mathcal{R}_n$ with range orthogonal to the range of R_1,*
4. $\|[x \otimes z_i]_{\mathcal{T}} - \phi([\xi \otimes \zeta_i]_{\mathcal{M}_n})\| < \epsilon$ *for* $1 \le i \le q$.

Proof. Let x' be any unit vector in H. By Lemma 3.3.4, there are vectors $\xi', \zeta'_1, ..., \zeta'_q$ in F^2_n such that

1. $\|\xi'\| < \sqrt{q}(1+\epsilon)^{1/2}$,
2. $\|\zeta'_i\| < (1+\epsilon)^{1/2}$ for $1 \le i \le q$,
3. $[x' \otimes z_i]_{\mathcal{T}} = \phi([\xi' \otimes \zeta'_i]_{\mathcal{M}_n})$ for $1 \le i \le q$.

Let η be a vector contained in the algebraic span of $\{\xi_w : w \in \mathbb{F}_n^*\}$ such that $\|\eta\| < \sqrt{q}(1+\epsilon)^{1/2}$ and $\|\xi' - \eta\| < \epsilon/(1+\epsilon)^{1/2}$. Then

$$\begin{aligned}
\|[x' \otimes z_i]_{\mathcal{T}} - \phi([\eta \otimes \zeta_i']_{\mathcal{M}_n})\| &\leq \|[x' \otimes z_i]_{\mathcal{T}} - \phi([\xi' \otimes \zeta_i']_{\mathcal{M}_n})\| \\
&\quad + \|[(\xi' - \eta) \otimes \zeta_i']_{\mathcal{M}_n}\| \\
&\leq \|\xi' - \eta\|\|\zeta_i'\| \\
&< \epsilon
\end{aligned}$$

for $1 \leq i \leq q$.

By Lemma 3.3.5, there are words u and v in $\mathbb{F}_n^+$ such that

$$L_u R_v \eta = \|\eta\| L \xi_\varnothing = \|\eta\| R \xi_\emptyset,$$

where L is an isometry in $\mathcal{L}_n$, and R is an isometry in $\mathcal{R}_n$ with range orthogonal to the range of R_1. Set $x = S_u x'$, $\xi = L_u R_v \eta$ and $\zeta_i = R_v \zeta_i'$ for $1 \leq i \leq q$. Then for T in $\mathcal{T}$,

$$\begin{aligned}
|([x \otimes z_i]_{\mathcal{T}} - \phi([\xi \otimes \zeta_i]_{\mathcal{M}_n}), T)| &= |([S_u x' \otimes z_i]_{\mathcal{T}} - \phi([\eta \otimes \zeta_i']_{\mathcal{M}_n}), T)| \\
&= |([x' \otimes z_i]_{\mathcal{T}} - \phi([\eta \otimes \zeta_i']_{\mathcal{M}_n}), T S_u)| \\
&\leq \|[x' \otimes z_i]_{\mathcal{T}} - \phi([\eta \otimes \zeta_i']_{\mathcal{M}_n})\| \|T S_u\| \\
&< \epsilon \|T\|.
\end{aligned}$$

Hence $\|[x \otimes z_i]_{\mathcal{T}} - \phi([\xi \otimes \zeta_i]_{\mathcal{M}_n})\| < \epsilon$. $\square$

The following result is implied by Lemma 1.2 in [Kri01].

Lemma 3.3.7. *Given a proper isometry R in $\mathcal{R}_n$, vectors $\zeta_1, ..., \zeta_q$ in F_n^2 and $\epsilon > 0$, there exists $k \geq 1$ such that $\|(R^*)^k \zeta_i\| < \epsilon$ for $1 \leq i \leq q$.*

Lemma 3.3.8. *Given a proper isometry S in $\mathcal{S}$, vectors u and v in H and $\epsilon > 0$, there exists $k \geq 1$ such that $\|[u \otimes (S^*)^k v]_{\mathcal{S}}\| < \epsilon$.*

Proof. Since $\mathcal{L}_n$ has property $\mathbb{A}_1$, there are vectors μ and ν in F_n^2 such that $[u \otimes v]_{\mathcal{S}} = \phi([\mu \otimes \nu]_{\mathcal{L}_n})$. Thus for A in $\mathcal{S}$,

$$\begin{aligned} |([u \otimes (S^*)^k v]_{\mathcal{S}}, A)| &= |([\mu \otimes (\Phi^{-1}(S)^*)^k \nu]_{\mathcal{L}_n}, \Phi^{-1}(A))| \\ &= |(\Phi^{-1}(A)\mu, (\Phi^{-1}(S)^*)^k \nu)| \\ &\leq \|A\|\|\mu\|\|(\Phi^{-1}(S)^*)^k \nu\|, \end{aligned}$$

which gives $\|[u \otimes (S^*)^k v]_{\mathcal{S}}\| \leq \|\mu\|\|(\Phi^{-1}(S)^*)^k \nu\|$. Since $\Phi^{-1}(S)$ is a a proper isometry in $\mathcal{L}_n$, and since $\mathcal{L}_n$ and $\mathcal{R}_n$ are unitarily equivalent, the result now follows by Lemma 3.3.7. $\square$

Lemma 3.3.9. *Given unit vectors $z_1, ..., z_q$ in H and $\epsilon > 0$, there exists a unit vector x in H and vectors $\xi, \zeta_1, ..., \zeta_q$ in F_n^2 such that*

1. $\|\xi\| < \sqrt{q}(1+\epsilon)^{1/2}$,
2. $\|\zeta_i\| < (1+\epsilon)^{1/2}$ *for* $1 \leq i \leq q$,
3. $\xi = \|\xi\| L\xi_\varnothing = \|\xi\| R\xi_\varnothing$, *where L is an isometry in $\mathcal{L}_n$, and R is an isometry in $\mathcal{R}_n$ with range orthogonal to the range of R_1,*
4. $\|R^*\zeta_i\| < \epsilon$ *for* $1 \leq i \leq q$,
5. $|(\Phi(L)^k x, x)| < \epsilon$ *for* $k \geq 1$,
6. $\|[x \otimes z_i]_{\mathcal{T}} - \phi([\xi \otimes \zeta_i]_{\mathcal{M}_n})\| < \epsilon$ *for* $1 \leq i \leq q$.

Proof. By Lemma 3.3.6, there exists a unit vector x' in H and vectors $\xi', \zeta_1, ..., \zeta_q$ in F_n^2 such that

1. $\|\xi'\| < \sqrt{q}(1+\epsilon)^{1/2}$,
2. $\|\zeta_i\| < (1+\epsilon)^{1/2}$ for $1 \leq i \leq q$,
3. $\xi' = \|\xi'\| L'\xi_\varnothing = \|\xi'\| R'\xi_\varnothing$, where L' is an isometry in $\mathcal{L}_n$ and R' is an isometry in $\mathcal{R}_n$with the range of R' orthogonal to the range of R_1,

4. $\|[x' \otimes z_i]_{\mathcal{T}} - \phi([\xi' \otimes \zeta_i]_{\mathcal{M}_n})\| < \epsilon$ for $1 \leq i \leq q$.

By Lemma 3.3.7 and Lemma 3.3.8, there exists $m \geq 1$ such that $\|(R_1^*)^m (R')^* \zeta_i\| < \epsilon$ for $1 \leq i \leq q$ and $\|[x' \otimes (S_1^*)^m \Phi(L')^* x']_{\mathcal{S}}\| < \epsilon$. Set $\xi = L_1^m \xi'$, $L = L_1^m L'$ and $R = R' R_1^m$. Then $\xi = \|\xi\| L \xi_\varnothing = \|\xi\| R \xi_\varnothing$, L is an isometry in $\mathcal{L}_n$, and R is an isometry in $\mathcal{R}_n$ with range orthogonal to the range of R_1. For $1 \leq i \leq q$, this gives $\|R^* \zeta_i\| = \|(R_1^*)^m (R')^* \zeta_i\| < \epsilon$.

Let $x = S_1^m x'$. Then for $k \geq 1$, we compute

$$\begin{aligned}
|(\Phi(L)^k x, x)| &= |(\Phi(L_1^m L')^k S_1^m x', S_1^m x')| \\
&= |(S_1^m \Phi(L' L_1^m)^k x', S_1^m x')| \\
&= |(\Phi(L' L_1^m)^k x', x')| \\
&= |(\Phi(L' L_1^m)^{k-1} x', (S_1^*)^m \Phi(L')^* x')| \\
&= |([x' \otimes (S_1^*)^m \Phi(L')^* x']_{\mathcal{S}}, \Phi(L' L_1^m)^{k-1})| \\
&\leq \|[x' \otimes (S_1^*)^m \Phi(L')^* x']_{\mathcal{S}}\| \|(L' L_1^m)^{k-1}\| \\
&< \epsilon.
\end{aligned}$$

Finally, for T in $\mathcal{T}$ we have

$$\begin{aligned}
|([x \otimes z_i]_{\mathcal{T}} - \phi([\xi \otimes \zeta_i]_{\mathcal{M}_n}), T)| &= |([x' \otimes z_i]_{\mathcal{T}} - \phi([\xi' \otimes \zeta_i]_{\mathcal{M}_n}), T S_1^m)| \\
&\leq \|[x' \otimes z_i]_{\mathcal{T}} - \phi([\xi' \otimes \zeta_i]_{\mathcal{M}_n}\| \|T S_1^m\| \\
&< \epsilon \|T\|.
\end{aligned}$$

Thus $\|[x \otimes z_i]_{\mathcal{T}} - \phi([\xi \otimes \zeta_i]_{\mathcal{M}_n})\| < \epsilon$. $\square$

3.3.5 Approximately orthogonal vectors

The following lemma is extracted from the proof of Theorem 4.3 in [Ber98].

Lemma 3.3.10. *Given two isometries R and R' in $\mathcal{R}_n$ with orthogonal ranges and* *vectors ξ and μ in F_n^2 with μ in the kernel of R^*, define*

$$\mu_k = \frac{1}{\sqrt{k}} \sum_{j=1}^{k} R^j R' \mu.$$

Then

$$\|[\xi \otimes \mu_k]_{\mathcal{M}_n}\| \leq \frac{1}{\sqrt{k}} \|\mu\| \|D_k - 1\|_1,$$

where D_k denotes the k-th Dirichlet kernel and $\|\cdot\|_1$ denotes the L^1 norm.

Lemma 3.3.11. *Given unit vectors $z_1, \ldots, z_q$ in H and $\epsilon > 0$, there exists a unit vector x in H such that $\|[x \otimes z_i]_{\mathcal{T}}\| < \epsilon$ for $1 \leq i \leq q$.*

Proof. We may suppose that $\epsilon < 1$. Using the fact that $\lim k^{-1/2} \|D_k\|_1 = 0$, where D_k denotes the k-th Dirichlet kernel and $\|\cdot\|_1$ denotes the L^1 norm, choose $k \geq 1$ such that $2(q/k)^{-1/2} \|D_k - 1\|_1 < \epsilon/(3(1+\epsilon))$. Next choose $\epsilon' > 0$ such that

$$\epsilon' < \min\left\{1, \frac{\epsilon(1-\epsilon)}{3\sqrt{k}}, \frac{\epsilon(1-\epsilon)}{6\sqrt{q}}, \frac{k\epsilon}{k^2 - k}\right\}.$$

By Lemma 3.3.9, there exists a unit vector x' in H and vectors $\xi', \zeta_1, \ldots, \zeta_q$ in F_n^2 such that

1. $\|\xi'\| < \sqrt{q}(1+\epsilon')^{1/2}$,
2. $\|\zeta_i\| < (1+\epsilon')^{1/2}$ for $1 \leq i \leq q$,
3. $\xi' = \|\xi'\| L \xi_\varnothing = \|\xi'\| R \xi_\varnothing$, where L is an isometry in $\mathcal{L}_n$, and R is an isometry in $\mathcal{R}_n$ with range orthogonal to the range of R_1,
4. $\|R^* \zeta_i\| < \epsilon'$ for $1 \leq i \leq q$,
5. $|(\Phi(L)^k x', x')| < \epsilon'$ for $k \geq 1$,
6. $\|[x' \otimes z_i]_{\mathcal{T}} - \phi([\xi' \otimes \zeta_i]_{\mathcal{M}_n})\| < \epsilon'$ for $1 \leq i \leq q$.

By (4) we can write $\zeta_i = \mu_i + \nu_i$, where μ_i is in the kernel of R^* and $\|\nu_i\| < \epsilon'$.

Let $\xi = k^{-1/2} \sum_{j=0}^{k-1} L_1 L^j \xi'$. Then by (3) we can write ξ as

$$\begin{aligned} \xi &= \frac{1}{\sqrt{k}} \sum_{j=0}^{k-1} L_1 L^j \xi' \\ &= \frac{\|\xi'\|}{\sqrt{k}} \sum_{j=0}^{k-1} L_1 L^{j+1} \xi_{\varnothing} \\ &= \frac{\|\xi'\|}{\sqrt{k}} \sum_{j=1}^{k} L_1 L^j \xi_{\varnothing}, \end{aligned}$$

which implies $\|\xi\| = \|\xi'\|$. Applying (3) again, we can also write ξ as

$$\begin{aligned} \xi &= \frac{1}{\sqrt{k}} \sum_{j=0}^{k-1} L_1 L^j \xi' \\ &= \frac{\|\xi'\|}{\sqrt{k}} \sum_{j=0}^{k-1} L_1 L^j R \xi_{\varnothing} \\ &= \frac{\|\xi'\|}{\sqrt{k}} \sum_{j=0}^{k-1} R^{j+1} R_1 \xi_{\varnothing} \\ &= \frac{\|\xi'\|}{\sqrt{k}} \sum_{j=1}^{k} R^j R_1 \xi_{\varnothing}. \end{aligned}$$

By Lemma 3.3.10 and the choice of k, this gives

$$\begin{aligned} \|[\xi \otimes \mu_i]_{\mathcal{M}_n}\| &\leq \frac{1}{\sqrt{k}} \|\xi\| \|\mu_i\| \|D_k - 1\|_1 \\ &\leq \sqrt{\frac{q}{k}} (1+\epsilon')^{1/2} \|\mu_i\| \|D_k - 1\|_1 \\ &< \epsilon(1-\epsilon)/3. \end{aligned}$$

Let $y = Sx'$, where $S = k^{-1/2} \sum_{j=0}^{k-1} \Phi(L_1 L^j)$. Then $\|S\| \leq \sqrt{k}$, so for T in

$\mathcal{T}$,

$$\begin{aligned}
|([y\otimes z_i]_{\mathcal{T}}-\phi([\xi\otimes\zeta_i]_{\mathcal{M}_n}),T)| &= |([x'\otimes z_i]_{\mathcal{T}}-\phi([\xi'\otimes\zeta_i]_{\mathcal{M}_n}),TS)| \\
&\leq \|[x'\otimes z_i]_{\mathcal{T}}-\phi([\xi'\otimes\zeta_i]_{\mathcal{M}_n})\|\|TS\| \\
&< \epsilon'\sqrt{k}\|T\| \\
&< (\epsilon(1-\epsilon)/3)\|T\|,
\end{aligned}$$

which gives $\|[y\otimes z_i]_{\mathcal{T}}-\phi([\xi\otimes\zeta_i]_{\mathcal{M}_n})\|<\epsilon(1-\epsilon)/3$. Since

$$\|[\xi\otimes\nu_i]_{\mathcal{M}_n}\|\leq\|\xi\|\|\nu_i\|<\sqrt{q}(1+\epsilon')^{1/2}\epsilon'<\epsilon(1-\epsilon)/3,$$

this gives

$$\begin{aligned}
\|[y\otimes z_i]_{\mathcal{T}}\| &\leq \|[y\otimes z_i]_{\mathcal{T}}-\phi([\xi\otimes\zeta_i]_{\mathcal{M}_n})\|+\|[\xi\otimes\mu_i]_{\mathcal{M}_n}\|+\|[\xi\otimes\nu_i]_{\mathcal{M}_n}\| \\
&< \epsilon(1-\epsilon).
\end{aligned}$$

Finally, we compute

$$\begin{aligned}
\|y\|^2 &= \|Sx'\|^2 \\
&= \left\|\frac{1}{\sqrt{k}}\sum_{j=0}^{k-1}\Phi(L_1L^j)x'\right\|^2 \\
&= \|x'\|^2+\frac{1}{k}\sum_{0\leq i<j\leq k-1}(x',\Phi(L)^{j-i}x')+\frac{1}{k}\sum_{0\leq j<i\leq k-1}(\Phi(L)^{i-j}x',x') \\
&\geq 1-\frac{1}{k}\sum_{0\leq i<j\leq k-1}|(x',\Phi(L)^{j-i}x')|-\frac{1}{k}\sum_{0\leq j<i\leq k-1}|(\Phi(L)^{i-j}x',x')| \\
&\geq 1-\frac{k^2-k}{k}\epsilon'. \\
&> 1-\epsilon.
\end{aligned}$$

Hence taking $x=(1-\epsilon)^{-1}y$, $\|x\|\geq 1$ and $\|[x\otimes z_i]_{\mathcal{T}}\|<\epsilon$ for $1\leq i\leq q$. $\square$

Lemma 3.3.12. *Given unit vectors $z_1, \ldots, z_q$ in H and $\epsilon > 0$, there exists an intertwining operator $X : F_n^2 \to H$ such that $\|X\xi_\varnothing\| = 1$ and $\|[X\xi_\varnothing \otimes z_i]_\mathcal{T}\| < \epsilon$ for $1 \leq i \leq q$.*

Proof. By Lemma 3.3.11, there exists a unit vector x in H such that $\|[x \otimes z_i]_\mathcal{T}\| < \epsilon$ for $1 \leq i \leq q$. By Theorem 2.7 of [DLP05], x is in the range of an intertwining operator $X' : F_n^2 \to H$. Hence there is a vector ξ in F_n^2 such that $X'\xi = x$. The result now follows from the fact that the set of vectors $\{R\xi_\varnothing : R \in \mathcal{R}_n\}$ is dense in F_n^2, and the fact that for R in $\mathcal{R}_n$, the operator $X'R$ is intertwining. $\square$

Lemma 3.3.13. *Let $X : F_n^2 \to H$ be an intertwining operator with $\|X\xi_\varnothing\| = 1$. Then given $\epsilon > 0$, there is a word v in $\mathbb{F}_n^*$ such that*

$$\|[XR_v\xi_\varnothing \otimes XR_v\xi_\varnothing]_\mathcal{T} - \phi([\xi_\varnothing \otimes \xi_\varnothing]_{\mathcal{M}_n})\| < \epsilon.$$

Proof. Since X^*X is an L-Toeplitz operator, by Lemma 2.4.5 of Chapter 2, there is a word v in $\mathbb{F}_n^*$ such that $\|R_v^*X^*XR_v\xi_\varnothing - \xi_\varnothing\| < \epsilon/2$. Note that $R_v^*X^*XR_v$ is also an L-Toeplitz operator. Let $\xi = (R_v^*X^*XR_v - I)\xi_\varnothing$, so that $\|\xi\| < \epsilon/2$. For w in $\mathbb{F}_n^*$, since $(L_w\xi, \xi_\varnothing) = 0$ we can write

$$\begin{aligned}
(S_wXR_v\xi_\varnothing, XR_v\xi_\varnothing) &= (L_w\xi_\varnothing, R_v^*X^*XR_v\xi_\varnothing) \\
&= (L_w\xi_\varnothing, \xi_\varnothing) + (L_w\xi_\varnothing, \xi) + (L_w\xi, \xi_\varnothing).
\end{aligned}$$

Similarly,

$$\begin{aligned}
(S_w^*XR_v\xi_\varnothing, XR_v\xi_\varnothing) &= (L_w^*R_v^*X^*XR_v\xi_\varnothing, \xi_\varnothing) \\
&= (L_w^*\xi_\varnothing, \xi_\varnothing) + (L_w^*\xi_\varnothing, \xi) + (L_w^*\xi, \xi_\varnothing).
\end{aligned}$$

This gives

$$[XR_v\xi_\varnothing \otimes XR_v\xi_\varnothing]_\mathcal{T} = \phi([\xi_\varnothing \otimes \xi_\varnothing]_{\mathcal{M}_n} + [\xi \otimes \xi_\varnothing]_{\mathcal{M}_n} + [\xi_\varnothing \otimes \xi]_{\mathcal{M}_n}),$$

so we conclude that

$$\begin{aligned}\|[XR_v\xi_\varnothing \otimes XR_v\xi_\varnothing]_{\mathcal{T}} - \phi([\xi_\varnothing \otimes \xi_\varnothing]_{\mathcal{M}_n})\| &\leq \|[\xi \otimes \xi_\varnothing]_{\mathcal{M}_n} + [\xi_\varnothing \otimes \xi]_{\mathcal{M}_n}\| \\ &\leq 2\|\xi\|\|\xi_\varnothing\| \\ &< \epsilon,\end{aligned}$$

as required. □

Lemma 3.3.14. *Given unit vectors $z_1, \ldots, z_q$ in H and $\epsilon > 0$, there exists an intertwining operator $X : F_n^2 \to H$ such that $\|X\xi_\varnothing\| = 1$, $\|[X\xi_\varnothing \otimes z_i]_{\mathcal{T}}\| < \epsilon$ for $1 \leq i \leq q$ and*

$$\|[X\xi_\varnothing \otimes X\xi_\varnothing]_{\mathcal{T}} - \phi([\xi_\varnothing \otimes \xi_\varnothing]_{\mathcal{M}_n})\| < \epsilon.$$

Proof. By Lemma 3.3.12, there exists an intertwining operator $X' : F_n^2 \to H$ such that $\|X'\xi_\varnothing\| = 1$ and $\|[X'\xi_\varnothing \otimes z_i]_{\mathcal{T}}\| < \epsilon$ for $1 \leq i \leq q$. By Lemma 3.3.13, there is a word v in $\mathbb{F}_n^*$ such that

$$\|[X'R_v\xi_\varnothing \otimes X'R_v\xi_\varnothing]_{\mathcal{T}} - \phi([\xi_\varnothing \otimes \xi_\varnothing]_{\mathcal{M}_n})\| < \epsilon.$$

Let $X = X'R_v$. Then

$$\|X\xi_\varnothing\| = \|X'R_v\xi_\varnothing\| = \|X'L_v\xi_\varnothing\| = \|S_vX'\xi_\varnothing\| = \|X'\xi_\varnothing\| = 1.$$

For T in $\mathcal{T}$,

$$\begin{aligned}|([X\xi_\varnothing \otimes z_i]_{\mathcal{T}}, T)| &= |([X'R_v\xi_\varnothing \otimes z_i]_{\mathcal{T}}, T)| \\ &= |([X'L_v\xi_\varnothing \otimes z_i]_{\mathcal{T}}, T)| \\ &= |([S_vX'\xi_\varnothing \otimes z_i]_{\mathcal{T}}, T)| \\ &= |([X'\xi_\varnothing \otimes z_i]_{\mathcal{T}}, TS_v)| \\ &\leq \|[X'\xi_\varnothing \otimes z_i]_{\mathcal{T}}\|\|T\|.\end{aligned}$$

Hence $\|[X\xi_\varnothing \otimes z_i]_{\mathcal{T}}\| < \epsilon$ for $1 \leq i \leq q$. □

3.3.6 The strong factorization property

Theorem 3.3.15. *Given a weak-* continuous linear functional τ on $\mathcal{T}$ with $\|\tau\| \leq 1$, unit vectors $z_1, ..., z_q$ in H and $\epsilon > 0$, there are vectors x and y in H such that*

1. $\|x\| \leq 1$ *and* $\|y\| \leq 1$,

2. $\|\tau - [x \otimes y]_{\mathcal{T}}\| < \epsilon$,

3. $\|[x \otimes z_i]_{\mathcal{T}}\| < \epsilon$ *and* $\|[z_i \otimes y]_{\mathcal{T}}\| < \epsilon$ *for* $1 \leq i \leq q$.

In other words, $\mathcal{T}$ has property $\mathcal{X}_{0,1}$.

Proof. Choose $\epsilon' > 0$ such that $\epsilon' < \epsilon$ and $1 - (1 + 2\epsilon')^{-2}(1 - \epsilon') < \epsilon$. Since $\mathcal{M}_n$ has property $\mathbb{A}_1(1)$, there are vectors ξ and υ in F_n^2 with $\|\xi\| \leq 1 + \epsilon'/2$ and $\|\upsilon\| \leq 1 + \epsilon'/2$ such that $\tau = \phi([\xi \otimes \upsilon]_{\mathcal{M}_n})$. Since $\xi_\varnothing$ is cyclic for $\mathcal{L}_n$, there are A and B in $\mathcal{L}_n$ such that $\|A\xi_\varnothing - \xi\| < \epsilon'/(4(1+\epsilon'))$ and $\|B\xi_\varnothing - \upsilon\| < \epsilon'/(4(1+\epsilon'))$. Then

$$\|A\xi_\varnothing\| \leq \|A\xi_\varnothing - \xi\| + \|\xi\| < 1 + \epsilon',$$

and similarly $\|B\xi_\varnothing\| < 1 + \epsilon'$. This gives

$$\begin{aligned}
\|[A\xi_\varnothing \otimes B\xi_\varnothing]_{\mathcal{M}_n} - [\xi \otimes \upsilon]_{\mathcal{M}_n}\| &\leq \|[(A\xi_\varnothing - \xi) \otimes B\xi_\varnothing]\| \\
&\quad + \|[\xi \otimes (B\xi_\varnothing - \upsilon)]_{\mathcal{M}_n}\| \\
&\leq \|A\xi_\varnothing - \xi\|\|B\xi_\varnothing\| + \|\xi\|\|B\xi_\varnothing - \upsilon\| \\
&< \epsilon'/2.
\end{aligned}$$

By Lemma 3.3.14, there is an intertwining operator $X : F_n^2 \to H$ such that $\|X\xi_\varnothing\| = 1$, $\|[X\xi_\varnothing \otimes z_i]_{\mathcal{T}}\| < \epsilon'/(\|A\| + \|B\|)$ for $1 \leq i \leq q$ and $\|[X\xi_\varnothing \otimes X\xi_\varnothing]_{\mathcal{T}} - \phi([\xi_\varnothing \otimes \xi_\varnothing]_{\mathcal{M}_n})\| < \epsilon'/(2(\|A\| + \|B\|)^2)$. Note that since $\mathcal{T}$ is self-adjoint, we also have $\|[z_i \otimes X\xi_\varnothing]_{\mathcal{T}}\| < \epsilon'/(\|A\| + \|B\|)$ for $1 \leq i \leq q$.

Define vectors x' and y' in H by $x' = \Phi(A)X\xi_\varnothing$ and $y' = \Phi(B)X\xi_\varnothing$. Then

$$\begin{aligned}
\|x'\|^2 &= \|\Phi(A)X\xi_\varnothing\|^2 \\
&= \|\Phi(A)X\xi_\varnothing\|^2 - \|A\xi_\varnothing\|^2 + \|A\xi_\varnothing\|^2 \\
&= |([X\xi_\varnothing \otimes X\xi_\varnothing]_{\mathcal{T}} - \phi([\xi_\varnothing \otimes \xi_\varnothing]_{\mathcal{M}_n}), \Phi(A^*A))| + \|A\xi_\varnothing\|^2 \\
&\leq \|[X\xi_\varnothing \otimes X\xi_\varnothing]_{\mathcal{T}} - \phi([\xi_\varnothing \otimes \xi_\varnothing]_{\mathcal{M}_n})\|\|A\|^2 + \|A\xi_\varnothing\|^2 \\
&< 1 + 2\epsilon',
\end{aligned}$$

and similarly, $\|y'\|^2 < 1 + 2\epsilon'$. For T in $\mathcal{T}$,

$$\begin{aligned}
&|([x' \otimes y']_{\mathcal{T}} - \phi([A\xi_\varnothing \otimes B\xi_\varnothing]_{\mathcal{M}_n}), T)| \\
&\qquad = |([\Phi(A)X\xi_\varnothing \otimes \Phi(B)X\xi_\varnothing]_{\mathcal{T}} - \phi([A\xi_\varnothing \otimes B\xi_\varnothing]_{\mathcal{M}_n}), T)| \\
&\qquad = |([X\xi_\varnothing \otimes X\xi_\varnothing]_{\mathcal{T}} - \phi([\xi_\varnothing \otimes \xi_\varnothing]_{\mathcal{M}_n}), \Phi(A)^*T\Phi(B))| \\
&\qquad \leq \|[X\xi_\varnothing \otimes X\xi_\varnothing]_{\mathcal{T}} - \phi([\xi_\varnothing \otimes \xi_\varnothing]_{\mathcal{M}_n})\|\|A\|\|B\|\|T\| \\
&\qquad < \frac{\epsilon'}{2}\|T\|,
\end{aligned}$$

which implies $\|[x' \otimes y']_{\mathcal{T}} - \phi([A\xi_\varnothing \otimes B\xi_\varnothing]_{\mathcal{M}_n})\| < \epsilon'/2$. Thus

$$\begin{aligned}
\|[x' \otimes y']_{\mathcal{T}} - \tau\| &= \|[x' \otimes y']_{\mathcal{T}} - \phi([\xi \otimes \upsilon]_{\mathcal{M}_n})\| \\
&\leq \|[x' \otimes y']_{\mathcal{T}} - \phi([A\xi_\varnothing \otimes B\xi_\varnothing]_{\mathcal{M}_n})\| \\
&\quad + \|[A\xi_\varnothing \otimes B\xi_\varnothing]_{\mathcal{M}_n} - [\xi \otimes \upsilon]_{\mathcal{M}_n}\| \\
&< \epsilon'.
\end{aligned}$$

For $1 \leq i \leq q$,

$$\|[x' \otimes z_i]_{\mathcal{T}}\| = \|[AX\xi_\varnothing \otimes z_i]_{\mathcal{T}}\| \leq \|A\|\|[X\xi_\varnothing \otimes z_i]_{\mathcal{T}}\| < \epsilon',$$

and similarly, $\|[z_i \otimes y']_{\mathcal{T}}\| < \epsilon'$.

Now take $x = (1 + 2\epsilon')^{-1}x'$ and $y = (1 + 2\epsilon')^{-1}y'$. Then by choice of ϵ' we get $\|x\| \leq 1$ and $\|y\| \leq 1$. Similarly, $\|[x \otimes z_i]_{\mathcal{T}}\| < \epsilon$ and $\|[z_i \otimes y]_{\mathcal{T}}\| < \epsilon$ for $1 \leq i \leq q$. Finally, we have

$$\begin{aligned}
\|[x \otimes y]_{\mathcal{T}} - \tau\| &\leq (1+2\epsilon')^{-2}\|[x' \otimes y']_{\mathcal{T}} - \tau\| + (1-(1+2\epsilon')^{-2})\|\tau\| \\
&< 1-(1+2\epsilon')^{-2}(1-\epsilon') \\
&< \epsilon,
\end{aligned}$$

as required. □

3.3.7 Absolute continuity and analyticity

Theorem 3.3.16. *For $n \geq 2$, every absolutely continuous isometric n-tuple is analytic.*

Proof. For $n \geq 2$, let $S = (S_1, \ldots, S_n)$ be an absolutely continuous isometric n-tuple, and let $\mathcal{S}$ denote the weak-* closed unital algebra generated by $S_1, \ldots, S_n$. By Corollary 1.2 of [DY08], $\mathcal{S}$ is isomorphic to the noncommutative analytic Toeplitz algebra $\mathcal{L}_n$. By Theorem 3.3.15, $\mathcal{S}$ has property $\mathcal{X}_{0,1}$, and hence has property $\mathbb{A}_1(1)$. Therefore, by the discussion in Section 3.3.3, $\mathcal{S}$ is weakly closed, and hence $\mathcal{S}$ is actually the free semigroup algebra (i.e. the weakly closed algebra) generated by $S_1, \ldots, S_n$. Since $\mathcal{S}$ is isomorphic to $\mathcal{L}_n$, this implies that S is analytic. □

The next result follow from Theorem 2.3.12 of Chapter 2.

Corollary 3.3.17. *For $n \geq 2$, let $S = (S_1, \ldots, S_n)$ be an absolutely continuous isometric n-tuple acting on a Hilbert space H. Then the wandering vectors for S span H.*

It was shown in Corollary 2.4.8 of Chapter 2 that every analytic isometric tuple is hyperreflexive with hyperreflexivity constant at most 3, but the next result can also be proved directly using Theorem 3.3.15 of the present paper and Theorem 3.1 of [Ber98].

Corollary 3.3.18. *Absolutely continuous row isometries are hyperreflexive with hyperreflexivity constant at most* 3.

3.4 Singular isometric tuples

In Theorem 3.3.16, we showed that for $n \geq 2$, an isometric n-tuple is absolutely continuous if and only if it is analytic. With this operator-algebraic characterization of an absolutely continuous isometric tuple, we are now able to give an operator-algebraic characterization of a singular isometric tuple.

Theorem 3.4.1. *For $n \geq 2$, an isometric n-tuple is singular if and only if the free semigroup algebra it generates is a von Neumann algebra.*

Proof. Let $V = (V_1, \ldots, V_n)$ be an isometric n-tuple, and let $\mathcal{V}$ denote the free semigroup algebra (i.e. the weakly closed algebra) generated by V. If $\mathcal{V}$ is a von Neumann algebra, then V has no absolutely continuous part since, by Theorem 3.3.16, an absolutely continuous isometric tuple is analytic, and the noncommutative analytic Toeplitz algebra $\mathcal{L}_n$ is not self-adjoint by Corollary 1.5 of [DP99].

Conversely, if V is singular then it has no analytic restriction to an invariant subspace since, by Theorem 3.3.16, an absolutely continuous isometric tuple is analytic. Thus by Theorem 3.2.3, $\mathcal{V}$ is a von Neumann algebra. □

Example 3.1.2 showed that it is possible for an absolutely continuous unitary to generate a von Neumann algebra. Theorem 3.4.1 implies that there is no higher-dimensional analogue of this phenomenon.

Recall that a family of operators is said to be reductive if every subspace invariant for the family is also coinvariant.

Corollary 3.4.2. *For $n \geq 2$, every reductive unitary n-tuple is singular.*

Proof. Let $V = (V_1, \ldots, V_n)$ be a reductive isometric n-tuple, and let $\mathcal{V}$ denote the free semigroup algebra generated by V. By the dichotomy for free semigroup algebras, Corollary 2.3.12 of Chapter 2, if $\mathcal{V}$ is not a von Neumann algebra, then there is a vector x that is wandering for V. Let $\mathcal{V}[x]$ denote the cyclic invariant subspace generated by x. Then the subspace $\sum_{i=1}^{n} V_i \mathcal{V}[x]$ is invariant for V but not coinvariant, which would contradict that V is reductive. Thus $\mathcal{V}$ is a von Neumann algebra and V is singular by Theorem 3.4.1. □

Example 3.4.3. By Theorem 3.4.1, for $n \geq 2$ an isometric n-tuple is singular if and only if the free semigroup algebra it generates is a von Neumann algebra. The existence of a self-adjoint free semigroup algebra on two or more generators was conjectured in [DKP01], but it took some time for the first example to be constructed. In [Read05], Read showed that $B(\ell^2)$ is generated as a free semigroup algebra on two generators. In [Dav06], Davidson gave an exposition of Read's construction and showed that it could be generalized to show that $B(\ell^2)$ is generated as a free semigroup algebra on n generators for every $n \geq 2$. By our characterization of singularity, this gives an example of a singular isometric n-tuple for every $n \geq 2$.

3.5 The Lebesgue-von Neumann-Wold decomposition

In Theorem 3.3.16, we showed that for $n \geq 2$, an isometric n-tuple is absolutely continuous if and only if it is analytic. In Theorem 3.4.1, we showed that for $n \geq 2$, an isometric n-tuple is singular if and only if the free semigroup algebra (i.e. the weakly closed algebra) it generates is a von Neumann algebra. With these operator-algebraic characterizations of absolute continuity and singularity, we will be able to prove the Lebesgue-von Neumann-Wold decomposition of an isometric tuple.

In the classical case, the Lebesgue decomposition of a measure guarantees that every unitary splits into absolutely continuous and singular parts. For $n \geq 2$, it turns out that it is possible for a unitary n-tuple to be irreducible and neither absolutely continuous nor singular.

Definition 3.5.1. An isometric n-tuple $V = (V_1, \ldots, V_n)$ is said to be of **dilation type** if it has no summand that is absolutely continuous or singular.

Note that by the Wold decomposition of an isometric tuple, Proposition 3.2.5, an isometric n-tuple of dilation type is necessarily unitary. The next result provides a characterization of an isometric tuple of dilation type as a minimal dilation, in the sense of Section 3.2.3.

Proposition 3.5.2. *Let $V = (V_1, \ldots, V_n)$ be an isometric n-tuple of dilation type. Then there is a subspace H coinvariant under V such that H is cyclic for V and the compression* *of V to $H^\perp$ is a unilateral n-shift. In other words, V is the minimal isometric dilation of its compression to H.*

Proof. Note that since V has no summand that is absolutely continuous, by Proposition 3.2.5 V is necessarily a unitary n-tuple. Let $\mathcal{V}$ denote the free semigroup algebra generated by V, and let P be the projection from Theorem 3.2.3 applied to $\mathcal{V}$. Let H be the range of P, so that H is coinvariant under V.

Let $K = (H + \sum_{i=1}^n V_i H) \ominus H$. Then K is wandering for the compression of V to $H^\perp$. If $K = 0$, then by Theorem 3.2.3, $\mathcal{V}$ can be decomposed into the direct sum of a self-adjoint free semigroup algebra and an analytic free semigroup algebra. By the characterization of singular isometric tuples, Corollary 3.4.1, this would contradict that V is of dilation type. Thus $K \neq 0$. The fact that K is cyclic follows from the fact that H is cyclic. □

Example 3.5.3 (Irreducible isometric tuple of dilation type)**.** It was shown in Corollary 6.6 of [DKS01] that the minimal isometric dilation of a contractive n-tuple $A = (A_1, \ldots, A_n)$ acting on a finite-dimensional space is an irreducible unitary n-tuple if and only if both $\sum_{i=1}^n A_i A_i^* = I$ and $\mathrm{C}^*(A)$ has a minimal coinvariant subspace that is cyclic for $\mathrm{C}^*(A)$. These conditions are satisfied by the contractive tuple $A = (A_1, A_2)$, where

$$A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Thus the minimal isometric dilation of A is an example of an irreducible isometric tuple of dilation type. A similar construction can be carried out for all $n \geq 2$.

Theorem 3.5.4 (Lebesgue-von Neumann-Wold Decomposition)**.** *Let $V = (V_1, \ldots, V_n)$ be an isometric n-tuple. Then V decomposes as*

$$V = V_u \oplus V_a \oplus V_s \oplus V_d,$$

where V_u is a unilateral n-shift, V_a is an absolutely continuous unitary n-tuple, V_s is a singular unitary n-tuple, and V_d is a unitary n-tuple of dilation type.

Proof. The case for $n = 1$ follows by the discussion in Section 3.1. Thus we can suppose that $n \geq 2$. By the Wold decomposition of an isometric tuple, Proposition 3.2.5, we can decompose V as

$$V = V_u \oplus U,$$

where V_u is a unilateral n-shift and U is a unitary n-tuple.

By the characterization of an absolutely continuous isometric n-tuple as analytic, Theorem 3.3.16, and the characterization of a singular isometric n-tuple, Corollary 3.4.1, an isometric n-tuple cannot be both absolutely continuous and singular. Therefore, we can decompose U as

$$U = V_a \oplus V_s \oplus V_d,$$

where V_a is an absolutely continuous isometric n-tuple, V_s is a singular isometric n-tuple, and V_d is of dilation type. Thus we can further decompose V as

$$V = V_u \oplus V_a \oplus V_s \oplus V_d,$$

as required. $\square$

The next result follows from combining Proposition 3.5.2 and Theorem 3.2.3.

Proposition 3.5.5. *Let $V = (V_1, \ldots, V_n)$ be an isometric n-tuple of dilation type acting on a Hilbert space H. Then there is a projection P and $\alpha \geq 1$ and such that the weakly closed algebra $\mathrm{W}(V)$ generated by V is of the form*

$$\mathrm{W}(V_1, \ldots, V_n) = \mathrm{W}^*(V)P + P^\perp \mathrm{W}(V) P^\perp,$$

where $P^\perp \mathrm{W}(V_1, \ldots, V_n) \mid_{P^\perp H} \simeq \mathcal{L}_n^{(\alpha)}$.

The next result follows from the Lebesgue-von Neumann-Wold decomposition of an isometric tuple, Proposition 3.2.5, and Proposition 3.5.5.

Theorem 3.5.6. *Let $V = (V_1, \ldots, V_n)$ be an isometric n-tuple acting on a Hilbert space H, and let $V = V_u \oplus V_a \oplus V_s \oplus V_d$ be the Lebesgue-von Neumann-Wold decomposition of V as in Theorem 3.5.4. Then there is a projection P and $\alpha, \beta \geq 0$ such that the weakly closed algebra $\mathbb{W}(V)$ generated by V is*

$$\mathbb{W}(V) \simeq (\mathcal{L}_n(V_u \oplus V_a))^{(\alpha)} \oplus \mathbb{W}^*(V_s) \oplus \left(\mathbb{W}^*(V_d)P + P^\perp \mathbb{W}(V_d)P^\perp\right),$$

where $P^\perp \mathbb{W}(V_1, \ldots, V_n) \mid_{P^\perp H} \simeq \mathcal{L}_n^{(\beta)}$. The von Neumann algebra $\mathbb{W}^*(V_1, \ldots, V_n)$ generated by V is

$$\mathbb{W}^*(V) \simeq (B(\ell^2))^{(\alpha)} \oplus \mathbb{W}^*(V_s) \oplus \mathbb{W}^*(V_d).$$

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