Testing the Equivalence Principle in the Quantum Domain

by

Catalina Alvarez

A thesis
presented to the University of Waterloo
in fulfilment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Physics

Waterloo, Ontario, Canada, 1997

0-612-21329-3

The University of Waterloo requires the signatures of all persons using or photocopying this thesis. Please sign below, and give address and date.

Abstract

Metric theories of gravity offer the singular beauty of endowing spacetime with a symmetric, second-rank tensor field $g_{\mu\nu}$ that couples universally to all non-gravitational fields. This unique operational geometry is embodied in the validity of the Einstein Equivalence Principle (EEP).

Although the empirical evidence in support of EEP has reached an impressive level of precision, it has only probed effects that are sensitive to nuclear electromagnetic interactions (*i.e.*, the baryon/photon sector of the standard model). In this thesis we provide the theoretical framework to confront EEP with the interaction realm of quantum electrodynamics (QED).

We reformulate QED within the context of non-metric theories of gravity and calculate the main radiative corrections affecting the atomic energy levels (Lamb shift) and the gyromagnetic ratio of fermions (anomalous magnetic moment).

We find that a non-metric spacetime structure induces qualitatively new effects in the behavior of radiative corrections that leave distinctive physical signatures. Such effects allow the possibility of setting new bounds on the validity of the EEP. In fact from present experiments, we obtain the most stringent bound yet noted for the non-metric parameters related to leptonic matter.

Acknowledgements

I would like to thank Prof. R. B. Mann for his valuable supervision and support which made this work possible. I also thank the Natural Sciences and Engineering Research Council of Canada, the Government of Ontario, and the Department of Physics at Waterloo for their financial support.

Me gustaría agradecer a mis padres, Sonia y Sergio, que a la distancia, me han apoyado incondicionalmente durante mis tantos años universitarios. Gracias por su paciencia y comprensión, gracias por aceptar las sinrazones que me llevaron a tan larga y misteriosa carrera academica.

A Marcus, mi precioso tesoro

Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 The Einstein Equivalence Principle (EEP)

The postulate that the equivalence between uniform acceleration and a uniform gravitational field applies to all physical phenomena allowed Einstein to construct a theory of gravitation, general relativity, which revolutionized our conceptual understanding of the universe. It allowed a description of physics in which the effects of gravitation are manifest as the dynamics of the geometry of a curved spacetime. That this geometry is unique for all forms of mass-energy is a consequence of Einstein's equivalence postulate.

Only decades later was it realized that this postulate is the foundation for a rather broad class of theories of gravitation (which includes general relativity) known as metric theories. Any theory of gravity that describes spacetime via a symmetric, second-rank tensor field $g_{\mu\nu}$ that couples universally to all non-gravitational fields respects the aforementioned equivalence between uniform acceleration and uniform gravitational fields, and is by definition a member of this class.

Metric theories therefore describe the interaction of matter (and any other non-gravitating field) with respect to an external gravitational field via an action

$$S_{NG} = S_{NG}(\psi_{NG}, g_{\mu\nu}) \tag{1.1}$$

where S_{NG} represents the non-gravitational action (*i.e.*, it excludes self-gravitational systems) as given for the current standard model of particle physics (where ψ_{NG} denotes the bosonic and fermionic fields). The different fields ψ_{NG} feel the gravitational influence of the external world only through their coupling to one and the same metric tensor $g_{\mu\nu}$. Non-metric theories of gravity break this universality by adding extra gravitational fields (scalar, tensor, *etc.*), which couple differently to different forms of matter.

A number of physically distinct principles can be derived from the condition (1.1) [1]. The most basic of these is the Weak Equivalence Principle, or WEP, which states that all freely falling bodies (*i.e.* bodies which are not acted upon by non-gravitational forces such as electromagnetism and which are small enough so that tidal effects are negligible in a given gravitational field) move independently of their internal structure or composition, and has as its one of this implications that bodies of differing internal composition (or mass-energy) fall with the same acceleration in a gravitational field. A natural extension of this to include all non-gravitational phenomena states that, in addition to WEP, the outcomes of non-gravitational test experiments (such as the measurement of an electromagnetic current in a wire) performed within a local, freely falling frame are independent of the frame's location (local position invariance, LPI) and velocity (local Lorentz invariance, LLI) in a background gravitational field. The combination of WEP, LLI and LPI embody what is now known as the Einstein Equivalence Principle, or EEP. Note that EEP does not refer to the structure and dynamics of the gravitational field itself, but states only the universality of the gravitational coupling with respect to matter. The further extension of this principle to include self-gravitating systems is known as the Strong Equivalence Principle, or SEP.

The assumption (1.1) of a universal gravitational coupling is a very strong condition, which further implies that the outcome of local non-gravitational experiments should be independent of the effects of an external (slowly varying) gravitational field. In this respect, direct tests of EEP may be carried out as follows. Consider an Earth-based laboratory in which local non-gravitational experiments are performed. External gravitational potentials generated by the Earth, the Sun, the planets, the Galaxy, *etc.* pervade this laboratory, and any non-metric couplings of these potentials to matter can cause the outcomes of experiments to depend on the laboratory's position, orientation or velocity relative to these sources. This is a direct violation of (respectively) LPI and LLI. The character of a violation reflects the form of the specific non-metric coupling responsible for it. It is only when LPI and LLI are valid that local non-gravitational dynamics is indistinguishable from special relativistic dynamics as predicted by metric theories of gravity.

In summary, the idea that gravity can be understood as a manifestation of spacetime curvature is rooted in the validity of the EEP. Metric theories (such as general relativity and Brans-Dicke Theory) endow spacetime with a symmetric, second-rank tensor field that couples universally to all non-gravitational fields [2], so that in a local freely falling frame the three postulates of EEP are satisfied. By definition, non-metric theories do not have this feature; by coupling auxiliary gravitational fields directly to matter they violate universality and so permit observers performing local experiments to detect effects due to their position and/or velocity in an external gravitational field.

1.2 Enough Empirical Support for EEP?

Specific empirical consequences of the postulate (1.1) are given by the universality of free-fall (or UFF, a necessary consequence of WEP), the universality of the gravitational redshift (LPI), the constancy of the constants (another manifestation of LPI), and the isotropy of space (LLI).

The UFF has been probed via torsion balance or Eõtvos type experiments, which search for quantitative differences between the passive gravitational mass and the inertial mass of a given body. The former is a dynamical quantity that determines the gravitational force acting on a body (*i.e.* its weight), whereas the latter is a kinematical quantity that determines the response of a body to any applied force. There is no logically necessary reason why these quantities must be equal (in appropriate units), and so we therefore expect

$$m_p = m_I + \sum_A \eta^A E^A/c^2 \tag{1.2}$$

where E^A is the internal energy generated by interaction A, and η^A is a dimensionless parameter that measures the strength of the WEP violation for body A.

For two different bodies we can write the acceleration as

$$a_1 = (1 + \sum_A \eta^A \frac{E_1^A}{m_1 c^2})g \qquad a_2 = (1 + \sum_A \eta^A \frac{E_2^A}{m_2 c^2})g \quad . \tag{1.3}$$

A measurement on the relative difference in acceleration yields the so called "Eõtvos ratio" given by

$$\eta \equiv 2|\frac{a_1 - a_2}{a_1 + a_2}| = \sum_A \eta^A \left(\frac{E_1^A}{m_1 c^2} - \frac{E_2^A}{m_2 c^2}\right) \tag{1.4}$$

In the gravitational field of the Sun, this ratio was constrained to be

$$|\eta| < \begin{cases} 10^{-11} & [3] \\ 10^{-12} & [4] \end{cases} \tag{1.5}$$

where Dicke and collaborators [3] tested for differences in acceleration between aluminum and gold, and Braginskii and Panov [4] for aluminum and platinum. More recent experiments, which were sensitive to the gravitational field of the Earth achieved similar bounds for beryllium and copper, and aluminum and beryllium [5].

The previous limit in turns constrains the violating parameter η^A related to each A-type interaction. This is possible provided the various interactions do not conspire towards special types of cancellations so that independent bounds can be gathered in each case (see Ref.[6] for quotations of those limits when referred to interactions stemming from the atomic nucleus: strong, electrostatic, magnetostatic, hyperfine, *etc.*).

In a redshift experiment the local energies at emission w_{em} and at reception w_{rec} of a photon transmitted between observers at different points in an external gravitational field are compared in terms of

$$Z = \frac{w_{em} - w_{rec}}{w_{em}} \equiv \Delta U \Big(1 - \Xi\Big) \tag{1.6}$$

The anomalous redshift parameter (Ξ) measures the degree of LPI violation. It signals the breakdown of the universality of gravity, and so depends on the nature of the transition involved in the experiment (e.g., fine, hyperfine, *etc.*).

The most accurate test for the gravitational redshift corresponds to the gravity probe A experiment [7], which was able to constrain $|\Xi^{Hf}| < 2 \times 10^{-4}$. This experiment employed hydrogen maser clocks, where the governing energy transition is given by the hyperfine splitting due to the interaction between the magnetic moment (spin) of the nucleus (proton) and electron.

One class of experiments probing variation of the fundamental constants corresponds to those searching for a temporal variation of the fine structure constant α. These tests can be divided into two categories: cosmological and laboratory measurements. The first ones look for variations within cosmological time scales and the others are based on clock comparisons over time durations of months or years.

Laboratory measurements rely on the ultra-high stability of the atomic standard clocks and set limits a few orders of magnitude less stringent than the cosmological measurements. One of the most sensitive tests for α–variation comes from the clock comparison between Hg^+ and H hyperfine transitions [8]. This experiment set an upper bound of $\dot{\alpha}/\alpha \leq 3.7 \times 10^{-14}/yr$ after a 140 day observation period. Note that any variation of α, whether a cosmological time variation or a spatial variation via a dependence of α on the gravitational potential, will force a variation in the relative clock rates between any such pair of clocks.

Time dilation experiments look for violation of isotropy, or similarly for a preferred direction in space. If LLI were violated the energy levels of a bound system such as a nucleus could be shifted in a way that correlates the motion of the bound particles in each state with the preferred direction, leading to an orientation–dependent binding energy. The most precise experiments of this sort [9, 10] search for a time dependent quadrupole splitting of Zeeman levels. They compare the nuclear-spin- precession frequencies between two gases with nuclear spin $I = 3/2$ and $I = 1/2$, the latter being insensitive to a quadrupole splitting. These results place the constraint $(1 - c_*^2/c_B^2) < 6 \times 10^{-21}$ on the relative gravitational coupling between electromagnetism and baryonic matter, given by the discrepancy between the speed of light (c_*) and the limiting speed for baryons (c_B).

We see then that tests of the validity of the various facets of EEP have been carried out to impressive levels of precision. Why, then, ought one to resist the temptation to conclude that future experiments should ignore non-metric theories and focus only on winnowing out the correct metric theory of gravity? There are four basic reasons. One is the anticipated improvements in precision of upcoming experiments by as much as six orders of magnitude [11]. If such experiments yield improved limits on EEP-violation, this will afford us a much greater degree of confidence in our physical theories under the extreme conditions present in many astrophysical and cosmological situations. Another is historical: attempts to unify gravity with the other forces of nature have yielded a number of logically possible, physically well-motivated, alternatives to general relativity which do not naturally respect the EEP [12]. A third reason is that tests of the EEP can provide us with a unique way (perhaps the only way) of testing modern physical theories that unify gravity with the other forces of nature insofar as such theories typically generate new interactions which violate the equivalence principle [13]. Finally, EEP experiments to date have probed effects that are predominantly sensitive to nuclear electrostatic energy. Although violations of WEP/EEP due to other forms of energy (virtually all of which are associated with baryonic matter) have also been estimated [14], the bulk of our empirical knowledge about the validity of the equivalence principle is in the baryon/photon sector of the standard model.

Comparatively little is known about the empirical validity of the EEP for systems dominated by other forms of mass-energy [15]. Such systems include photons of differing polarization [16], antimatter systems [17], neutrinos [18], mesons [19], massive leptons [20], hypothesized dark matter [21], second and third generation matter, and quantum vacuum energies [22]. There is no logically necessary reason why such systems should respect any or all of WEP, EEP or SEP.

In order to establish the universal behavior of gravity, we are therefore compelled to consider the validity of the EEP over as diverse a range of non-gravitational interactions as is possible. It is the aim of this thesis to extend this regime to the quantum field domain of radiative correction, namely vacuum fluctuations of leptonic fields in the presence of an electromagnetic source. Also, in this way, we will be gathering information about the less explored non-baryonic sector of the standard model.

1.3 Quantum Field Domain

Potential violations of the EEP due to vacuum energy shifts, which are peculiarly quantum-mechanical in origin (*i.e.* do not have a classical or semi-classical description) provide an interesting empirical regime for gravitation and quantum mechanics. Effects of this type include Lamb-shift transition energies in Hydrogenic atoms and anomalous magnetic moments of massive leptons. Tests of the EEP in this sector will provide us with qualitatively new empirical windows on the foundations of gravitational theory.

Quantum electrodynamics (QED) is the theory of charged leptons with photons, in which all observable effects can be expressed in terms of measured charge and mass. It offers more than a mere marriage of quantum mechanics and relativity. The wave-particle duality of quantum mechanics is fully incorporated into the theory, and charged particles and photons are treated as quantized fields. The tremendous success in predicting experimental facts ranges from very refined details of the properties of electrons and muons and atomic spectra to interactions in the multi-GeV range. It was the experimental discoveries of the Lamb shift in hydrogen and the anomalous moment of the electron in the 1940's, which stimulated the evolution of QED to its present precise form. These two low energy phenomena represent the most precise tests of QED and are the ones relevant in this thesis.

The Lamb shift is the shift in energy levels of a Hydrogenic atom due to radiative corrections. Such energy shifts break the degeneracy between states with the same principal quantum number and total angular momentum, but differing orbital and spin angular momenta. The best known example is the energy shift between the $2S_{1/2}$ and $2P_{1/2}$ states in a Hydrogen-like atom, which arises due to interactions of the electron with the quantum-field-theoretic fluctuations of the electromagnetic field. For metric theories, the lowest order contribution for the Lamb shift is 1052 MHz for hydrogen atoms. There is a 5 MHz discrepancy with the experimental value of 1057.845(9) MHz [23] or 1057.851(2) MHz [24], that can be improved with the inclusion of higher order terms and corrections coming from the structure and recoil of the nucleus. The main difficulty in comparing QED theory and experiment is the lack of an agreed upon value for the radius of the proton. However there is no conflict up to the relative level of 10^{-5}.

The anomalous magnetic moment of an electron in a weak magnetic field, $a(e)$ (one-half the deviation of the g factor from the value of 2, as predicted by the Dirac theory) is the simplest quantity that can be calculated from quantum electrodynamics. It accounts for the radiative corrections coming from the free scattering of an electron by a weak , slowly varying magnetic field. The most recent experimental value of the magnetic moment anomaly of the electron is [25]; $a(e) = 1159652188.4(4.3) \times 10^{-12}$. Agreement between theory and measurement of $a(e)$ is at the relative level of 10^{-7} [26].

The success of QED as a quantized field theory sets the proper grounds to test gravity in this domain. Non metric effects will show distinctly in a modified QED, and so will be tightly constrained by the present narrow gap between the empirical and theoretical (metric) values.

1.4 Non-metric Framework

Any breakdown of LPI/LLI is determined entirely by the form of the couplings of the gravitational field to matter since local, non-gravitational test experiments simply respond to their external gravitational environment. To explore such effects it is necessary to develop a formalism capable of representing such couplings for as wide a class of gravitational theories as possible. We consider in this thesis Lagrangian-based theories in which the dynamical equations governing the evolution of the gravitational and matter fields can be derived from the action principle

$$\delta \int d^4x \, \mathcal{L} \equiv \delta \int d^4x (\mathcal{L}_G + \mathcal{L}_{NG}) = 0 \quad . \tag{1.7}$$

The gravitational part $\mathcal{L}_G$ of the Lagrangian density contains only gravitational fields; it determines the dynamics of the free gravitational field. The non-gravitational part $\mathcal{L}_{NG}$ contains both gravitational and matter fields and defines the couplings between them. The dynamics of matter in an external gravitational field follow from the action principle

$$\delta \int d^4x \, \mathcal{L}_{NG} = 0 \tag{1.8}$$

by varying all matter fields in an external gravitational environment.

We work in the context of a wide class of non-metric theories of gravity as described by the $TH\epsilon\mu$ formalism [27]. Phenomenological models of $\mathcal{L}_{NG}$ provide a general framework for exploring the range of possible couplings of the gravitational field to matter and, thus, the range of mechanisms that might conceivably break LPI or LLI. The $TH\epsilon\mu$ formalism is one such model. It deals with the dynamics of charged particles and electromagnetic fields in a static, spherically symmetric gravitational field. In addition to all metric theories of gravitation, the $TH\epsilon\mu$ formalism encompasses a wide class of non-metric theories.

A quantum-mechanical extension of the original classical $TH\epsilon\mu$ formalism was developed by Will [28] to calculate the energy shifts (due to *e.g.* hyperfine effects) in Hydrogenic atoms at rest in a $TH\epsilon\mu$ gravitational field. Since the ticking rate of a hydrogen-maser clock is governed by the transition between a pair of these atomic states, this extension can be used to determine the effect of the gravitational field on the ticking rate of such clocks. This provides a basis for a quantitative interpretation of gravitational redshift experiments which employ hydrogen-maser clocks, for example, the gravity probe A rocket-redshift experiment [7]. Such experiments are a direct test of LPI.

This formalism was further extended by Gabriel and Haugan [29] who calculated the effects the motion of an atomic system through a gravitational field would have on the ticking rate of hydrogen-maser and other atomic clocks. Their extension can be used to compute energies of hyperfine and other energy shifts of hydrogen atoms in motion through a $TH\epsilon\mu$ field. Here the physical effect under consideration is time dilation rather than the gravitational redshift. When LLI is broken, the rates of clocks of different types that move together through the gravitational field are slowed by different time-dilation factors. This non-universal behavior is a characteristic symptom of the breakdown of LLI [30], just as non-universal gravitational redshift is the hallmark of LPI violation [28].

In this thesis, we will be concerned with the study of effects that could violate LPI or LLI. Radiative corrections are too small to be relevant for torsion balance experiments, where the leading binding energies stem from the atomic nucleus. In these experiments the different atomic binding energies are attenuated by the total mass of the atom (see Eq. (1.4)), whereas LPI/LLI violating experiments are clean experiments that are sensitive to energy transition itself.

We begin by considering a general idealized composite body made up of structureless test particles that interact by some non-gravitational force to form a bound system. The conserved energy function of the body E is assumed to have the quasi-Newtonian form [30]

$$E = M_R c_0^2 - M_R U(\vec{X}) + \frac{1}{2} M_R |\vec{V}|^2 + .. \tag{1.9}$$

where $\vec{X}$ and $\vec{V}$ are respectively the quasi-Newtonian coordinates and velocity of the center of mass of the body, M_R is the rest energy of the body and U is the external gravitational potential. Potential violations of the EEP arise when the rest energy M_R has the form

$$M_R c_0^2 = M_0 c_0^2 - E_B(\vec{X}, \vec{V}) \tag{1.10}$$

where M_0 is the sum of the rest masses of the structureless constituent particles and E_B is the binding energy of the body. It is the position and velocity dependence of E_B which signals the breakdown of the EEP. Expanding E_B in powers of U and V^2 to an order consistent with (1.9) we have

$$E_B(\vec{X}, \vec{V}) = E_B^0 + \delta m_P^{ij} U^{ij} - \frac{1}{2} \delta m_I^{ij} V^i V^j \tag{1.11}$$

where U^{ij} is the external gravitational potential tensor, satisfying $U^{ii} = U$. The quantities δm_P^{ij} and δm_I^{ij} are respectively called the anomalous passive gravitational and inertial mass tensors. They depend upon the detailed internal structure of the composite body. In an atomic system they can be expected to consist of terms proportional to the electrostatic, hyperfine, Lamb shift, and other contributions to the binding energy of an atomic state.

In a gravitational redshift experiment one compares the local energies at emission E_{em} and at reception E_{rec} of a photon transmitted between observers at different points in an external gravitational field. The measured redshift is defined as

$$Z = \frac{E_{em} - E_{rec}}{E_{em}}$$

Using (1.9) (with $\vec{V} = 0$) to relate the transition energies at the two different points, this parameter can be expressed as [30]

$$Z = \Delta U\Big(1 - \Xi\Big), \qquad \Xi = \frac{\delta m_P^{ij}}{\Delta E_B^0}\frac{\Delta U^{ij}}{\Delta U} \quad . \tag{1.12}$$

Clearly Z depends (through δm_P^{ij}) upon the specific test system used in the experiment. An absence of LPI violations will mean $\Xi = 0$, and so Z will be independent of the detailed physics underlying the energy transition .

The LLI violations may be empirically probed through time dilation experiments. These experiments compare atomic energy transitions as measured by the moving frame (ΔE_B) and preferred frame (ΔE_B^0), which can be related via [29]

$$\Delta E_B = \Delta E_B^0\Big(1 - [A-1]\frac{\vec{V}^2}{2}\Big) \tag{1.13}$$

with the time dilation coefficient A defined by

$$A = 1 - \frac{\delta m_I^{ik}}{\Delta E_B^0}\frac{V^i V^k}{V^2} \quad . \tag{1.14}$$

Here δm_I^{ik} represents the difference between the anomalous inertial tensors related to the atomic states involved in the transition. The coefficient A represents the dilation of the rate of a moving atomic clock whose frequency is governed by the transition. Since the anomalous mass tensor is not isotropic, A depends upon the orientation of the atom's quantization axis relative to its velocity through the preferred frame. Note that if LLI is valid the anomalous inertial mass tensor associated with every atomic state vanishes, so that $A = 1$.

In the following we shall calculate the radiative corrections of interest in a context of non-metric theories of gravity, such that we can derive the expressions analogous to (1.11) for each case (Lamb shift and anomalous magnetic moment), and therefore be able to make the empirical connection.

1.5 Overview

Although the empirical evidence in support of EEP has reached an impressive level of precision, it has only probed effects that are sensitive to nuclear electromagnetic interactions (*i.e.*, the baryon/photon sector of the standard model). To this end, the empirical validity of the EEP in physical regimes where radiative corrections cannot be neglected remains an open question. In this thesis we provide the theoretical framework to confront EEP with the interaction realm of quantum electrodynamics, which is the most successful quantum field theory describing the vacuum field interactions between fermions and photons.

In the next chapter, we reformulate QED within the context of non-metric theories of gravity as described by the $TH\epsilon\mu$ formalism. The main radiative corrections affecting the atomic energy levels (Lamb shift) and the gyromagnetic ratio of fermions (anomalous moment) are calculated in chapters 3 and 4 respectively. The analysis of the non-metric results is presented at the end of each chapter, along with their possible implications for present data and future experiments. Details of the computation so as further clarification in certain matters, are given in two main appendices, which complements chapters 3 and 4. We conclude this thesis with chapter 5, which presents a general overview and summary of this work.

Chapter 2

Gravitationally Modified Action

2.1 $TH\epsilon\mu$ Action

The $TH\epsilon\mu$ formalism was constructed to study electromagnetically interacting charged structureless test particles in an external, static, spherically symmetric (SSS) gravitational field, encompassing a wide class of non-metric (and all metric) gravitational theories. Originally employed as a computational framework designed to test Schiff's conjecture [6], it permits one to extract quantitative information about the implications of EEP-violation that can be compared to experiment. It assumes that the non-gravitational laws of physics can be derived from an action:

$$\begin{aligned} S_{NG} &= -\sum_a m_a \int dt\, (T - H v_a^2)^{1/2} + \sum_a e_a \int dt\, v_a^\mu A_\mu(x_a^\nu) \\ &\quad + \frac{1}{2}\int d^4x\, (\epsilon E^2 - B^2/\mu), \end{aligned} \tag{2.1}$$

where m_a, e_a, and $x_a^\mu(t)$ are the rest mass, charge, and world line of particle a, $x^0 \equiv t$, $v_a^\mu \equiv dx_a^\mu/dt$, $\vec{E} \equiv -\vec{\nabla} A_0 - \partial \vec{A}/\partial t$, $\vec{B} \equiv \vec{\nabla} \times \vec{A}$. The parameters T, H, ϵ, and μ are arbitrary functions of the Newtonian gravitational potential $U = GM/r$, which approach unity as $U \to 0$. For an arbitrary non-metric theory, these functions will depend upon the type of matter, *i.e.* the species of particle or field coupling to gravity. The functions ϵ and μ parameterize the 'photon metric', whereas T and H parameterize the 'particle metric' in the static, spherically symmetric case. Although we shall generically employ the notation T and H throughout this paper, it should be kept in mind that these functions shall in general have one set of values for electrons, another set for muons, another for protons, *etc.*. Universality of gravitational coupling in the particle sector implies that the T and H functions are species independent. It is an empirical question as to whether or not such universality holds for all particle species. The stringent limits on universality violation set by previous experiments [9] have only been with regards to the relative gravitational couplings in the baryon/photon sector of the standard model. For the leptonic sector relevant to our considerations, relatively little is known [15].

A quantum mechanical extension of the action (2.1) which incorporates the Dirac Lagrangian was used by Will [28] to study the energy levels of hydrogen atoms. In that case a local approximation to the action is employed. The spacetime scale of atomic systems allows one to ignore the spatial variations of T, H, ϵ, and μ, and evaluate them at the center of mass position of the system, $\vec{X} = 0$. This work was further extended by Gabriel and Haugan [29] who showed that after rescaling coordinates, charges, and electromagnetic potentials, the field theoretic extension of the action (2.1) can be written in the form

$$S = \int d^4x\, \overline{\psi}(i \not{\partial} + e \not{A} - m)\psi + \frac{1}{2}\int d^4x\, (E^2 - c^2 B^2), \tag{2.2}$$

where local natural units are used, $\not{A} = \gamma_\mu A^\mu$, and $c^2 = H_0/T_0\epsilon_0\mu_0$ with the subindex "0" denoting the functions evaluated at $\vec{X} = 0$. The parameter c is the ratio of the local speed of light to the limiting speed of the species of massive particle under consideration. This action emerges upon replacing the point-particle part of the action in (2.1) with the Dirac Lagrangian, expanding the $TH\epsilon\mu$ parameters about the origin, neglecting their spatial variation over atomic distance scales, and rescaling coordinates and fields.

The action (2.1) (or (2.2)) has been widely used in the study of LPI/LLI violating effects such as the effect of non-metric gravitational fields on the differential ticking rates of different types of atomic clocks, a violation of LPI [28]. An analysis of the electrostatic structure of atoms and nuclei in motion through a $TH\epsilon\mu$ gravitational field using (2.1) shows that the non-metric couplings encompassed by the $TH\epsilon\mu$ formalism can also break LLI [30]. This symmetry is broken when the local speed of light $c_* \equiv (\mu_0\epsilon_0)^{-1/2}$ differs from the limiting speed of a given species of massive particle $c_0 \equiv (T_0/H_0)^{1/2}$, the latter being normalized to unity in (2.2). Further implications of the breakdown of LLI on various aspects of atomic and nuclear structure have also been investigated. Shifts in energy levels (including the hyperfine splitting) of hydrogenic atoms in motion through a $TH\epsilon\mu$ gravitational field have been calculated [29] by transforming the representation of the action (2.2) to a local coordinate system in which the atom is initially at rest and then analyzing the atom's structure in that frame. The local coordinate system in which the $TH\epsilon\mu$ action is represented by Eq. (2.2), is called the preferred frame; moving frames are those systems of local coordinates that move relative to the preferred frame (or to the rest frame of the external gravitational field U).

In this thesis we generalize this analysis by using the Gravitationally Modified (GM) action (2.2) to study the radiative correction contributions to the bound state energy levels in hydrogenic atoms and to the elastic scattering of free leptons by a magnetic field. To deal with the non-metric effects, we follow the scheme given in Ref. [29], and analyze the radiative corrections in the given rest frame of the system (moving frames).

Consider an atom (or particle) that moves with velocity $\vec{u}$ relative to the preferred frame. The moving frame in which this atom (or particle) is at rest is defined by means of a standard Lorentz transformation. A convenient representation [29] of the $TH\epsilon\mu$ action in this new coordinate system, if the non-gravitational fields ψ, $\vec{A}$, $\vec{E}$, and $\vec{B}$ transform via the corresponding Lorentz transformations laws for Dirac, vector, and electromagnetic fields, is

$$\begin{aligned} S &= \int d^4x\,\overline{\psi}(i\,\partial\!\!\!/ + e\,A\!\!\!/ - m)\psi + \int d^4x\,J_\mu A^\mu \\ &+ \frac{1}{2}\int d^4x\,\Big[(E^2 - B^2) \\ &+ \xi\gamma^2\left(\vec{u}^2E^2 - (\vec{u}\cdot\vec{E})^2 + B^2 - (\vec{u}\cdot\vec{B})^2 + 2\vec{u}\cdot(\vec{E}\times\vec{B})\right)\Big]. \end{aligned} \tag{2.3}$$

where J^μ is the electromagnetic 4-current associated with some external source (taken to be a pointlike spinless nucleus in the case of the Lamb shift) and $\gamma^2 = (1-\vec{u}^2)^{-1}$. In our formulation, all non-metric effects arise from the inequality between c_0 and c_* in the electromagnetic sector of the action. The dimensionless parameter $\xi \equiv 1-(c_*/c_0)^2 = 1-c^2$ measures the degree to which LPI/LLI is broken for a given species of particle. Comparatively little is known about such empirical limits on EEP-violation relative to the baryonic sector [15], for which previous experiments have set the limit [9] $|\xi_B| \equiv |1-c_B^2| < 6\times10^{-21}$ where c_B is the ratio of the limiting speed of baryonic matter to the speed of light. We can therefore safely neglect any putative effects of ξ_B in our analysis. The natural scale for ξ in theories that break local Lorentz invariance is set by the magnitude of the dimensionless Newtonian potential, which empirically is much smaller than unity in places we can imagine performing experiments [6]. We are therefore able to compute effects of the terms in Eq. (2.3) that break local Lorentz invariance via a perturbative analysis about the familiar and well-behaved $c\to1$ or $\xi\to0$ limit.

2.2 (GM) Field Equations

The fermion sector of the action (2.3) implies that the equation of motion for the ψ field is simply the Dirac equation coupled in the usual fashion to the potential A_μ, that is:

$$(i \not\partial + e \not A - m)\psi = 0 \tag{2.4}$$

On the other hand, the pure electromagnetic part of the action is modified with an extra term proportional to the small (species-dependent) parameter ξ. This will affect the electromagnetic field equations, and the photon propagator. In both cases we can calculate effects of the additional terms perturbatively.

The field equations coming from the action (2.3) are up to $O(\vec{u}^2)$[29]

$$\begin{aligned}\vec{\nabla}\cdot\vec{E} &= \rho+\xi\left[\vec{u}\cdot\vec{\nabla}(\vec{u}\cdot\vec{E})-\vec{u}\cdot\vec{\nabla}\times\vec{B}-\vec{u}^2\vec{\nabla}\cdot\vec{E}\right], \qquad (2.5)\\ \vec{\nabla}\times\vec{B}-\dot{\vec{E}} &= \vec{j}+\xi\left[\vec{\nabla}\times(\vec{u}\times\vec{E})+\vec{u}\times\vec{\nabla}(\vec{u}\cdot\vec{B})+(1+\vec{u}^2)\vec{\nabla}\times\vec{B}\right.\\ &+ \left.\vec{u}^2\dot{\vec{E}}-\vec{u}(\vec{u}\cdot\dot{\vec{E}})-\vec{u}\times\dot{\vec{B}}\right]\end{aligned}$$

where ρ and $\vec{j}$ are the charge density and current associated with the fermion field plus an external source (such as a nucleus.) Perturbatively solving these equations for electromagnetic potentials produced by a pointlike nucleus of charge Ze at rest in the moving frame yields

$$\begin{aligned}A_0 &= [1-\frac{\xi}{2}(\vec{u}^2+(\vec{u}\cdot\hat{n})^2)]\phi\equiv\phi+\xi\phi'\\ \vec{A} &= \frac{\xi}{2}[\vec{u}+\hat{n}(\vec{u}\cdot\hat{n})]\phi\equiv\xi\vec{A}'\end{aligned}\tag{2.6}$$

where $\hat{n}=\vec{x}/|\vec{x}|$, $\phi=Ze/4\pi|\vec{x}|$, and $\vec{\nabla}\cdot\vec{A}=0$. Note that Eq. (2.6) agrees with the corresponding result from Ref. [29].

The primed fields in Eq. (2.6) signal a breakdown of LLI. Consequently we expect that this electromagnetic potential will modify the energy states of hydrogenic atoms prior to the inclusion of radiative corrections.

Note that Eqs. (2.3) and (2.6) are the (GM) analogous of the Maxwell equations and the "Coulomb" potential respectively.

2.3 (GM) QED

Radiative corrections arise from the vacuum fluctuations of the interaction between the fermion (Dirac Spinor) and the electromagnetic field. In the case of atomic energy levels (Lamb shift), the fermion field is bounded by the electromagnetic field of the nucleus, and therefore needs to be considered accordingly. This complication is not present for the anomalous magnetic moment of fermions, where we basically study the free scaterring of fermions by a slowly varying magnetic field. Both situations are particular cases in the general framework of Quantum Electrodynamics (QED).

In the sequel, we proceed to reformulate QED within a non-metric context as introduced by the action (2.3). We expect the reader to be familiar with standard concepts in Quantum Field Theory, or to refer to *e.g.*, [31] for further clarification on basic matters.

We need to generate an approach that leads to a consistent, regularized, and renormalized quantum field theory. The procedure reduces to that of finding the corresponding (GM) fermion and photon propagators, along with the vertex rule describing the interaction between the fields. Despite the absence of LPI/LLI symmetries, the theory is still gauge invariant and therefore its consistency can be checked via the validity of the Ward identities.

We use the path integral approach to find the propagators. That is, we look for the inverse of the operator appearing in the quadratic term (either for photon or fermion field) of the Lagrangian [31]. Given our model (action (2.3)), we do not introduce changes into the fermion and interaction sector of the action, and so the fermion propagator and vertex rule remain unchanged with respect to the original (metric) situation. The reformulation of QED up to this level reduces just to finding the (GM) photon propagator.

To find the photon propagator, we go back to the action (2.2) and add a gauge fixing term of the form

$$S_{GF} = -\frac{1}{2}\int d^4x \left[(1-\xi)(\partial\cdot A)^2 + 2\xi\partial^0 A_0 \partial\cdot A\right], \tag{2.7}$$

after which the resulting electromagnetic part can be written as

$$S_{EM} = \int d^4x \left[\frac{1}{2}A_\mu\partial^\nu\partial_\nu A^\mu + \frac{\xi}{2}(A_\mu\partial_0\partial^0 A^\mu + A_0\partial^\mu\partial_\mu A^0 - A_\mu\partial^\nu\partial_\nu A^\mu)\right] \tag{2.8}$$

where we have integrated by parts and neglected surface terms.

This action is still given in preferred frame coordinates. We can go to the moving frame by performing the Lorentz transformations

$$\begin{aligned} A_0 \to A_0' &= \gamma(A_0 - \vec{u}\cdot\vec{A}) \equiv \gamma\beta\cdot A \\ \partial_0 \to \partial_0' &= \gamma(\partial_0 - \vec{u}\cdot\vec{\nabla}) \equiv \gamma\beta\cdot\partial \end{aligned} \tag{2.9}$$

where $\gamma^2 \equiv 1/(1-\vec{u}^2)$ and $\beta^\mu \equiv (1,\vec{u})$; henceforth $\beta^2 \equiv 1-\vec{u}^2$. Transforming Eq. (2.8) by using Eq.(2.9) gives

$$S_{EM} = \frac{1}{2}\int d^4k\, A^\mu \mathcal{K}_{\mu\nu} A^\nu \tag{2.10}$$

where (in momentum space)

$$\mathcal{K}_{\mu\nu} = -\eta_{\mu\nu}k^2(1-\xi) - \xi\gamma^2\left[\eta_{\mu\nu}(\beta\cdot k)^2 + \beta_\mu\beta_\nu k^2\right] \tag{2.11}$$

where $\eta_{\mu\nu}$ is the Minkowski tensor with a signature $(+---)$ and $\mathcal{K}_{\mu\nu}$ is the inverse of the photon propagator $G_{\mu\nu}$. Therefore after solving

$$\mathcal{K}_{\mu\delta}G^{\delta\nu} = \delta_{\mu}^{\nu}, \tag{2.12}$$

we find up to first order in ξ

$$G_{\mu\nu} = -(1+\xi)\frac{\eta_{\mu\nu}}{k^2} + \xi\frac{\gamma^2}{k^2}\left[\eta_{\mu\nu}\frac{(\beta\cdot k)^2}{k^2} + \beta_\mu\beta_\nu\right]. \tag{2.13}$$

where actually k^2 stands for $k^2 + i\eta$, with η being a small positive number which is set to zero after the relevant integrations are performed.

The terms proportional to ξ in Eq. (2.13) signal the breakdown of both LPI and LLI, since those terms are still present even if $\vec{u} = 0$. Concerning the Feynman rules, Eq. (2.13) is the only change needed to obtain the analogous ones. Note that the computation of radiative corrections involves the calculation of loop integrals as given by the Feynman rules up to a given order.

2.4 Renormalization

As with the metric case, we expect to find divergences, which after an adequate regularization process are removed via a parameter redefinition. In our case, the addition of more parameters to the theory also entails new renormalizations beyond those of the wavefunctions, charge and mass of the fermion. The $TH\epsilon\mu$ parameters appear as functions of $c_0^2 \equiv T_0/H_0$ and $c_*^2 \equiv 1/\mu_0\epsilon_0$, and must then be correspondly redefined. In the following, we just describe in general terms the type of countert-erms needed to achieve this procedure. We leave for the next chapters more specific details about the renormalization procedure, which are better understood within the appropriate context (*e. g.*, bound system or free scattering).

In units where $c_0 \equiv 1$ ($c_* = c$), EEP-violating corrections only appear in the electromagnetic sector of the action (as terms proportional to ξ). However we could choose more generally $c_0 \neq 1$, for which the particle sector of the Lagrangian density is of the form

$$\mathcal{L}_D = \overline{\psi}(\not{p} - \not{V} - m)\psi + \xi_0 \overline{\psi}(p_0 - A_0)\gamma^0 \psi \tag{2.14}$$

with $\xi_0 \equiv 1 - c_0^{-1}$; or in the moving frame (after using (2.9)) is

$$\begin{aligned} \mathcal{L}'_D &= \overline{\psi}(\not{p} - \not{V} - m)\psi \\ &+ \xi_0 \gamma^2 \overline{\psi}(\beta \cdot p - \beta \cdot V)\ \not{\beta}\psi \end{aligned} \tag{2.15}$$

up to a constant.

From (2.15) we see that quantum corrections of the form

$$\delta\mathcal{L}_D = \overline{\psi}(\delta\xi_0^{(1)}\beta \cdot p - \delta\xi_0^{(2)}\beta \cdot V)\ \not{\beta}\psi \tag{2.16}$$

can still be expected. Note that gauge invariance will guarantee $\delta\xi_0^{(1)} = \delta\xi_0^{(2)} = \delta\xi_0$. Hence, in order to renormalize the $TH\epsilon\mu$ parameters, we have to include counterterms of the form

$$\delta\xi_0\ \not{\beta}(\beta \cdot p - \beta \cdot V) \tag{2.17}$$

where $\delta\xi_0$ is chosen such that there are no radiative correction contributions as the source is turned off. Finally, given the form of the electromagnetic action (see Eq. (2.10)), we expect also quantum fluctuations of the form

$$\delta\mathcal{L}_{EM} = \delta\xi A^\mu\{(k^2 - (\beta \cdot k)^2)\eta_{\mu\nu} - \beta_\mu\beta_\nu k^2\}A^\nu \tag{2.18}$$

to occur, and so similar counterterms need to be considered.

In the next chapters, we will see how to use Eqs. (2.17) and (2.18) to get rid of the unwanted divergences.

Chapter 3

(GM) Lamb Shift

3.1 Bound System

Since the first accurate measurement by Lamb and Retherford of the shift between the the $2S_{1/2}$ and $2P_{1/2}$ states in Hydrogen atoms [32] (sometimes known as the *classical* Lamb shift), several Lamb shifts related to Hydrogen ($1S$ [33], $2S - 2P_{3/2}$ [34]) and Helium [35] have been measured.

However, in most of this chapter, we will refer to the classical Lamb shift only. In this case, the Dirac equation for a Coulomb potential predicts those states to be degenerate, the difference between them in metric theories comes only from radiative corrections. For non-metric theories which can be described by the $TH\epsilon\mu$ formalism, these energy levels will be modified by the EEP-violating terms introduced in the source (Eq. (2.6)), removing this degeneracy before introducing radiative corrections. Note that the fermion sector of the $TH\epsilon\mu$ action does not change and therefore neither does the Dirac equation. The preferred frame effects appear only in the expression for the electromagnetic source produced by the nucleus.

The calculation of radiative corrections to the atomic energy levels involves a bound state formalism for QED, which deals with a bound electron propagator. This makes the computation substantially more complicated than in the free case. There are several approaches for coping with the boundness of the propagator (see [36] and references therein), and we shall closely follow one of them.

3.2 (GM) Dirac States

The Dirac equation in the presence of an external electromagnetic field still reads like the metric case:

$$H|n\rangle = (\vec{\alpha}\cdot\vec{p} + \beta m - eA^0 + e\vec{\alpha}\cdot\vec{A})|n\rangle = E_n|n\rangle \tag{3.1}$$

where the various symbols have their usual meaning.

The (GM) energy levels of hydrogenic atoms are found by solving (3.1) in the presence of the electromagnetic field (2.6) produced by the nucleus which entirely accounts for the preferred frame effects. If we replace Eq. (2.6) in (3.1), the Hamiltonian can be written as

$$H = H_0 + \xi H', \qquad H' = -e\phi' + e\vec{\alpha}\cdot\vec{A}' \tag{3.2}$$

where H_0 corresponds to the standard Hamiltonian (with Coulomb potential only), and the primed fields are defined as in Eq. (2.6). In terms of the known solutions for $H_0|n\rangle^0 = E_n^0|n\rangle^0$, we can perturbatively solve Eq. (3.1) by writing

$$E_n = E_n^0 + \xi E_n' \qquad |n\rangle = |n\rangle^0 + \xi|n\rangle' \tag{3.3}$$

with

$$E'_n = {}^0\langle n|H'|n\rangle^0 \equiv E'^{(E)}_n + E'^{(M)}_n \tag{3.4}$$

$$|n\rangle' = \sum_{r\neq n} \frac{{}^0\langle r|H'|n\rangle^0}{E^0_n - E^0_r}|r\rangle^0 \tag{3.5}$$

where $E'^{(E)}_n$ and $E'^{(M)}_n$ account for the contributions coming from the respective electric and magnetic potentials.

We now proceed to calculate the energy levels related to the Lamb shift states. To obtain these, we find it convenient to use the exact solution for the Dirac spinor $|n\rangle^0$, expanding the final answer in powers of $Z\alpha$ to $O((Z\alpha)^4)$. The relationship between this approach and an alternate one in which the Hamiltonian is first expanded in powers of $Z\alpha$ using a Foldy-Wouthuysen transformation is discussed in appendix A.1.

The unperturbed Dirac state $|n\rangle^0$ can be expressed as:

$$|n\rangle^0 = \begin{pmatrix} G_{lj}(r)\,|l;jm\rangle \\ -iF_{lj}(r)\,\vec{\sigma}\cdot\hat{n}\,|l;jm\rangle \end{pmatrix} \tag{3.6}$$

where $|l;jm\rangle$ is the spinor harmonic eigenstate of J^2,L^2 and J_z, with respective quantum numbers j, l and m. The functions F and G can be written in terms of confluent hypergeometric functions that depend in a non-trivial way on $Z\alpha$ for a given l and j. In the case of the Lamb states, they can be expressed by [37]

$$\begin{aligned} F &= -N(1-W)^{1/2}r^{\gamma-1}e^{-\lambda r}(a_0 + a_1 r) \\ G &= N(1+W)^{1/2}r^{\gamma-1}e^{-\lambda r}(c_0 + c_1 r), \end{aligned} \tag{3.7}$$

where the various parameters for the $2S$ level take the form (taking $m = 1$)

$$\begin{array}{ll} \gamma = (1-(Z\alpha)^2)^{1/2} & W = \left(\frac{1+\gamma}{2}\right)^{1/2} \\ \lambda = \frac{Z\alpha}{2W} & N = \frac{(2Z\alpha)^{\gamma+1/2}}{2(2W)^{\gamma+1}}\left[\frac{2\gamma+1}{\Gamma(2\gamma+1)(2W+1)}\right]^{1/2} \\ a_0 = 2(W+1) & a_1 = -\frac{Z\alpha}{W}\frac{2W+1}{2\gamma+1} \\ c_0 = 2W & c_1 = a_1 \end{array} \tag{3.8}$$

Inserting the fields from (2.6) and (3.6) in E'_n, we write

$$E'^{(E)}_n = (R_{GG} + R_{FF})\langle jm; l|u^2 + (\vec{u}\cdot\hat{n})^2|l; jm\rangle \tag{3.9}$$

$$E'^{(M)}_n = -iR_{GF}\langle jm; l|(\vec{\sigma}\cdot\hat{n})(\sigma\cdot\vec{u}) + \vec{u}\cdot\hat{n}|l; jm\rangle + \text{h.c.} \tag{3.10}$$

where "h.c." means Hermitian conjugate and where

$$R_{GG} = \frac{1}{2}\int G\frac{Z\alpha}{r}Gr^2dr \tag{3.11}$$

with R_{FF} and R_{GF} defined in an analogous manner.

We now evaluate this energy for the $2S_{1/2}$ and $2P_{1/2}$ states in this semiclassical approximation, prior to the inclusion of any radiative corrections. Since the angular operator in (3.10) has odd parity (as given by $\hat{n}$), it is straightforward to show that the magnetic contribution $E'^{(M)}_n = 0$, so $E'_n = E'^{(E)}_n$ for any state.

If we now substitute Eq. (3.7) in (3.11), we obtain

$$R_{GG} = \frac{1}{16}(2\gamma+1)\frac{(1+W)}{2W+1}\left[\frac{c_0^2}{2\gamma}\left(\frac{Z\alpha}{W}\right)^2 + 2c_0c_1\frac{Z\alpha}{W} + (2\gamma+1)c_1^2\right] \tag{3.12}$$

$$R_{FF} = \frac{1}{16}(2\gamma+1)\frac{(1-W)}{2W+1}\left[\frac{a_0^2}{2\gamma}\left(\frac{Z\alpha}{W}\right)^2 + 2a_0a_1\frac{Z\alpha}{W} + (2\gamma+1)a_1^2\right] \tag{3.13}$$

where we have used

$$\begin{aligned}\Gamma(z) &\equiv \int_0^\infty e^{-x}x^{z-1}dx, \qquad Re(z)>0 \\ \Gamma(z+1) &= z\Gamma(z)\end{aligned} \tag{3.14}$$

to simplify the integrand in (3.11).

Using the corresponding expressions for the harmonic spinors and the appropriate parameters in (3.12) for each Lamb state [37], we eventually find

$$E'_{2S_{1/2}} = \frac{1}{6}u^2 m(Z\alpha)^2\left[1+(\frac{7}{16}+\frac{19}{16})(Z\alpha)^2\right]+\cdots \tag{3.15}$$

$$E'_{2P_{1/2}} = \frac{1}{6}u^2 m(Z\alpha)^2\left[1+(\frac{7}{16}+\frac{3}{16})(Z\alpha)^2\right]+\cdots \tag{3.16}$$

where we have expanded the exact solutions for R_{GG} and R_{FF} in powers of $(Z\alpha)^2$, and kept the first relativistic correction only. The angular integration and the R_{GG} term are the same for both states, and so the non-relativistic limit is still degenerate for them. However the first relativistic correction coming from the R_{FF} factor (term proportional to 19/16 for the $2S$ state) breaks the degeneracy, yielding

$$\Delta E_L^{(D)} = E_{2S_{1/2}} - E_{2P_{1/2}} = \xi\frac{u^2}{6}m(Z\alpha)^4 + O\left((Z\alpha)^6\right) \tag{3.17}$$

We obtain the result that the $2S_{1/2}$–$2P_{1/2}$ degeneracy is lifted before radiative corrections are introduced. This 'semiclassical' non-metric contribution to the Lamb shift is isotropic in the 3-velocity $\vec{u}$ of the moving frame and vanishes when $\vec{u}=0$. Hence it violates LLI but not LPI.

In order to proceed to a computation of the relevant radiative corrections, we need to find the perturbative corrections for the energies and spinor states given by (3.4) and (3.5) respectively. The radiative correction δE_n to the Dirac energy E_n can be formally expressed as

$$\delta E_n = \langle n|\delta H|n\rangle \tag{3.18}$$

where δH accounts for the loop contributions as given by the gravitationally modified QED. Since EEP violating effects appear in both the photon propagator and the classical electromagnetic field, we expect

$$\delta H = \delta H^0 + \xi \delta H' \tag{3.19}$$

In addition, the state $|n\rangle$ may be analogously expanded. Up to first order in ξ, we can therefore write (3.18) in the form

$$\delta E_n =^0 \langle n|\delta H^0|n\rangle^0 + \xi \left[{}^0\langle n|\delta H'|n\rangle^0 + \{ {}^0\langle n|\delta H^0|n\rangle' + \text{h.c.}\} \right] \tag{3.20}$$

The contributions from the $|n\rangle'$ states are of the same order of magnitude (in terms of powers of $Z\alpha$) as the $\delta H'$ terms and so cannot be neglected. This may be seen by noting that, apart from the $\vec{u}$ dependence, $\phi' \sim \phi$ and so ${}^0\langle n|H'|r\rangle^0 \sim E_n^0 - E_r^0$. Inserting this in (3.5) proves the statement. Note that the effect of the $|n\rangle'$ states was overlooked in Ref. [29]. If we identify $\delta H \to H_{(hf)}$,where $H_{(hf)}$ represents the perturbation to the Dirac Hamiltonian due to the spin of the nucleus, then by the same arguments as before we can show that the term $\{ {}^0\langle n|H_{(hf)}{}^0|n\rangle' + \text{h.c.}\}$ was omitted in the corresponding expression for the hyperfine energy.

3.3 (GM) Radiative Corrections

To lowest order in QED there are two types of radiative corrections to the energy levels of an electron bound in an external electromagnetic potential: the vacuum polarization (Π) and self-energy (Σ), along with a counterterm (δC) that subtracts the analogous processes for a free electron. These contributions are illustrated in Fig. 3.1

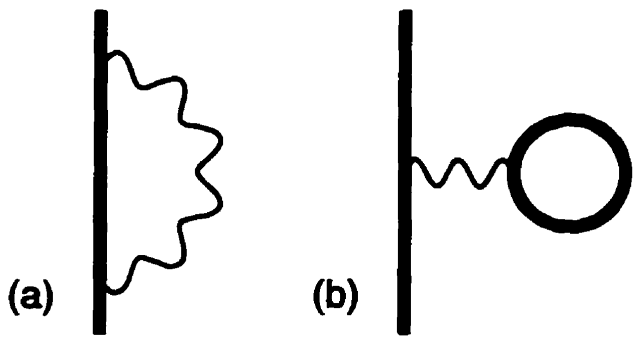

Figure 3.1: Radiative corrections of order α : (a) self-energy and (b) vacuum polarization.

The energy shift due to these contributions for the state $|n\rangle$ can then be written as

$$\delta E_n = \delta E_S + \delta E_P \tag{3.21}$$

where

$$\delta E_S = \langle n|\Sigma - \delta C|n\rangle, \tag{3.22}$$

which corresponds to the self-energy contribution in Fig. 3.1(a) minus the corresponding counterterm, and

$$\delta E_P = \langle n|\Pi|n\rangle, \tag{3.23}$$

which is the vacuum polarization contribution illustrated in Fig. 3.1(b).

In Fig. 3.1 the bold line represents the bound electron propagator. This propagator can be written in operator form as $(\not{p} - \not{V} - m)^{-1}$, with

$$V^\mu(\vec{x}) \equiv -eA^\mu(\vec{x}) \quad \text{and} \quad p^\mu \equiv (E_n, \vec{p})$$

where A^μ is the external electromagnetic potential. Here E_n is the total energy of the state $|n\rangle$, which satisfies the Dirac equation $(\not{p} - \not{V} - m)|n\rangle = 0$

Eq. (3.21) represents the one loop correction (one power of α) to the atomic energy levels as given by E_n. We are interested in obtaining the "lowest order" Lamb shift, which is the $\alpha(Z\alpha)^4$ contribution. (There are still more approximations that come after expanding the bound propagator, which introduce additional nonanalytic terms in the expression for the Lamb shift that behave like $\alpha(Z\alpha)^4 \ln(Z\alpha)$).

The GM radiative corrections are found by evaluating (3.21) where the external electromagnetic potential and the photon propagator are respectively given by Eqs. (2.6) and (2.13). All expressions will be expanded in terms of the LPI/LLI violating parameter ξ, and the velocity of the moving frame $\vec{u}$ up to $O(\xi)$ and $O(\vec{u}^2)$ as implied by (2.6) and (2.13). EEP-violating effects are all contained in the terms proportional to these quantities.

A variety of methods are available for evaluating the corrections in (3.21), each differing primarily in the manner in which the bound electron propagator is treated. We shall follow the method of Baranger, Bethe and Feynman [38] (hereafter referred to as BBF), in which the corrections in (3.22) are separated into a term in which the external potential acts only once, and another term in which it acts at least twice. This latter 'many-potential' term can be further separated into a nonrelativistic part, and a relativistic part which can be calculated by considering the intermediate states as free. This approach is sufficient for the lowest order calculation we consider here. We now proceed to outline the main steps of this method.

The self-energy term in Eq. (3.21) can be written as

$$\delta E_S = \frac{\alpha}{4\pi^3} \int d^4k \, iG^{\mu\nu}(k) \langle n|\gamma_\mu \frac{1}{\not{p} - \not{V} - \not{k} - m} \gamma_\nu |n\rangle - \langle n|\delta C|n\rangle \quad . \quad (3.24)$$

This expression gives a complex result for the level shift, since the denominators in the integral each have a small positive imaginary part. The resulting imaginary part of δE_S represents the decay rate of the state $|n\rangle$ through photon emission. The

Lamb shift refers to the real part of the shift, and only that part will be retained in the computation of Eq. (3.24).

The difficulty in evaluating Eq. (3.24) arises entirely from choosing a convenient expression for the bound propagator. The integrand in (3.24) is rearranged in order to obtain one part which is of first order in the potential (δE_1), and another part (δE_2) which contains the potential at least twice. Using the identity[38]

$$\hat{O} \equiv (\not{p}_b - m)\frac{\not{p}_b\hat{O} + \hat{O}\not{p}_a}{p_b^2 - p_a^2} - \frac{\not{p}_b\hat{O} + \hat{O}\not{p}_a}{p_b^2 - p_a^2}(\not{p}_a - m), \tag{3.25}$$

to re-express γ_μ and γ_ν in (3.24) and respectively identifying $p_b = p$, $p_a = p - k$, and $p_b = p - k$, $p_a = p$ yields after some manipulation

$$\delta E_S = \delta E_1 + \delta E_2, \tag{3.26}$$

where

$$\delta E_1 = \frac{\alpha}{\pi}\int d^3p\, d^3p' \overline{\psi}_n(\vec{p}\,')\{I_1 + I_2 + I_3\}\psi_n(\vec{p}), \tag{3.27}$$

with

$$\begin{aligned}
I_1 &= \frac{i}{4\pi^2}\int \frac{2p'_\mu - \gamma_\mu \not{k}}{k^2 - 2p'\cdot k} \not{V} \frac{2p_\nu - \not{k}\gamma_\nu}{k^2 - 2p\cdot k} G^{\mu\nu}(k) d^4k \\
I_2 &= \frac{i}{4\pi^2}\not{V}\int \frac{2p_\mu - \gamma_\mu \not{k}}{k^2 - 2p\cdot k}\frac{2p_\nu - \not{k}\gamma_\nu}{k^2 - 2p\cdot k} G^{\mu\nu}(k) d^4k \\
I_3 &= \frac{i}{4\pi^2}\int \frac{2p_\mu - \gamma_\mu \not{k}}{k^2 - 2p\cdot k}\gamma_\nu G^{\mu\nu}(k) d^4k - \delta C
\end{aligned} \tag{3.28}$$

and where

$$\begin{aligned}
\delta E_2 &= \frac{\alpha}{4\pi^3}\int \overline{\psi}_n(\vec{p}\,')M_\mu(p', p' - s' - k) \\
&\times K_+^V(E_0 - k_0; \vec{p}\,' - \vec{s}\,' - \vec{k}, \vec{p} + \vec{s} - \vec{k}) \\
&\times M_\nu^\dagger(p + s - k, p)\psi_n(\vec{p})G^{\mu\nu}(k) d^4k\, d^3p\, d^3p' d^3s\, d^3s' \\
&\equiv \langle M K_+^V M \rangle
\end{aligned} \tag{3.29}$$

with

$$\begin{aligned} M_\mu(p', p-k) &= \not{V}(\vec{p}\,'-\vec{p})\frac{2p_\mu - \gamma_\mu \not{k}}{2p\cdot k - k^2} - \frac{2p'_\mu - \gamma_\mu \not{k}}{2p'\cdot k - k^2}\not{V}(\vec{p}\,'-\vec{p}) \\ M^\dagger_\nu(p'-k, p) &= \not{V}(\vec{p}\,'-\vec{p})\frac{2p_\nu - \not{k}\gamma_\nu}{k^2 - 2p\cdot k} - \frac{2p'_\nu - \not{k}\gamma_\nu}{k^2 - 2p'\cdot k}\not{V}(\vec{p}\,'-\vec{p}) \end{aligned}$$

The quantity K^V_+ is defined as $-iK^V_+ \equiv (\not{p} - \not{V} - m)^{-1}$,where in momentum space $K^V_+ = \delta(E'-E)K^V_+(E;\vec{p}\,',\vec{p})$.

In Eqs.(3.27) and (3.29) the p's have time component E_n and the s's have time component 0. Note that the above derivations are independent of the specific form of the photon propagator $G_{\mu\nu}$.

Further evaluation entails a lengthy computation which in principle is analogous to that of BBF. In practice though, the calculation is substantially more complicated than in the metric case due to the additional non-metric terms present in the photon propagator and the electromagnetic source related to a charged point particle. Regularization and renormalization procedures have to be modified accordingly. Details involving the subsequent computation of the self energy (and vacuum polarization) term are given in appendix A.2.

The final result for the loop corrections related to the Lamb shift is of the form

$$\Delta E_L^{(Q)} = \delta E_{2S_{1/2}} - \delta E_{2P_{1/2}} \tag{3.30}$$

where each term is obtained from Eq. (A.55) (and its relevant subsidiary equations) as calculated for the corresponding atomic state. By adding the "semiclasical" correction coming from the Dirac level (labeled by (D) in Sec. III), the total Lamb shift reads

$$\begin{aligned} \Delta E_L &= \Delta E_L^{(D)} + \Delta E_L^{(Q)} \\ &= \frac{m}{6\pi}(Z\alpha)^4\alpha\Big\{-2.084 + \ln\frac{1}{\alpha^2} + \xi\Big[-4.534 + \frac{3}{2}\ln\frac{1}{\alpha^2} \end{aligned} \tag{3.31}$$

$$+ \quad \vec{u}^2 \left[\frac{\pi}{\alpha} - 3.486 + \frac{2}{3}\ln\frac{1}{\alpha^2} - 0.011\cos^2\theta\right] + u_i u_j \Delta\hat{\epsilon}_{ij}\Big]\Big\}$$

where we have introduced the dimensionless parameter $\Delta\hat{\epsilon}_{ij} \equiv 2\Delta\hat{E}_{ij}/((Z\alpha)^4 m^3)$ (see (A.56)), and used Eqs. (A.59) and (A.60) in the evaluation of (3.30) through Eq. (A.55).

The former result is the energy shift associated with the particular states in (3.30). However in Eq. (A.55) we have derived a general expression for the one-loop radiative corrections related to any atomic state. These are

$$\delta E_{n0} = \frac{4}{3\pi}\frac{(Z\alpha)^4\alpha}{n^3} m \left[\frac{19}{30} - \frac{\xi}{30} + (1+\frac{3}{2}\xi)\ln(\frac{m}{2E_*^{n0}}) + O(u^2)\right] \tag{3.32}$$

for $l = 0$, and

$$\delta E_{nl} = \frac{4}{3\pi}\frac{(Z\alpha)^4\alpha}{n^3} m \left[(1+\frac{3}{2}\xi)\ln(\frac{Z^2 Ryd}{E_*^{nl}}) + \frac{3}{8}\frac{C_{lj}}{2l+1}(1+\frac{\xi}{2}) + O(u^2)\right] \tag{3.33}$$

for $l \neq 0$; where we have not explicitly written the terms proportional to the moving frame velocity. Here

$$C_{lj} = \begin{cases} 1/(l+1) & \text{for } j = l + 1/2 \\ -1/l & \text{for } j = l - 1/2 \end{cases} \tag{3.34}$$

and E_* is defined by (A.57). Values for this reference energy can be obtained from Ref.[39] up to states with $n = 4$.

Note that in addition to the explicit dependence on the frame velocity in Eq. (3.31), there exists a position dependence hidden by the rescaling of the original action (Eq. (2.3)), which was considered locally constant throughout the computation. The full $TH\epsilon\mu$ parameter dependence in Eq. (3.31) can be recovered by replacing

$$\alpha \to \alpha\frac{1}{\epsilon}\sqrt{\frac{H}{T}}, \qquad m \to m\sqrt{H}, \qquad \Delta E_L \to \sqrt{\frac{H}{T}}\Delta E_L \tag{3.35}$$

in the preceding equations.

Note that ξ in Eq. (3.31) accounts for any EEP violation coming from a non-universal gravitational coupling between photons and leptons. A further distinction can still be made between leptons and antileptons. In principle a matter/antimatter violation of the EEP could be measured in a Lamb shift transition, through the appearance of virtual positron/electron pairs in the vacuum polarization loop contribution [40]. This will add a non-metric term to Eq. (3.31), of the form (see appendix A.3 for more details):

$$\Delta E_L^{(+)} = -\xi_{e+} \frac{m}{120\pi} (Z\alpha)^4 \alpha (1 + 2|\vec{u}|^2) \tag{3.36}$$

where $\xi_{e+} = 1 - c_{e_-}/c_{e_+}$ accounts for the difference between the limiting speed of electrons ($c_{e_-} = c_0$) and positrons (c_{e_+}).

We turn next to the question of relating the Lamb shift to observable quantities in order to parameterize possible violations of the EEP.

3.4 Test for LPI/LLI Violations

Here we consider the possibility of employing the Lamb shift as the atomic transition governing the appropriate experiment. To do so we must compute the relevant Ξ and A coefficients respectively.

In order to calculate the corresponding δm_p^{ij} related to the Lamb shift, we must find the manner in which ΔE_L varies as the location of the atom is changed. Setting $\vec{u} = 0$ in (3.31) and performing the rescaling given in (3.35), we obtain

$$\Delta E_L = \mathcal{E}_L \frac{\sqrt{T}}{\epsilon^5} \left(\frac{H}{T}\right)^{5/2} \left\{1 + a\xi + b(1 + \frac{3}{2}\xi) \ln\left(\epsilon^2 \frac{T}{H}\right)\right\} \tag{3.37}$$

with $\mathcal{E}_L = \frac{m}{6\pi}(Z\alpha)^4\alpha/b$, and

$$a = b(-4.534 + \frac{3}{2}\ln\frac{1}{\alpha^2}) \qquad b = 1/(-2.084 + \ln\frac{1}{\alpha^2})$$

where $\mathcal{E}_L$ represents the metric value (within the given approximations) for the Lamb shift. Note that there is still a position dependence in (3.37) through the definition of

$$\xi \equiv 1 - \frac{H}{T}\frac{1}{\mu\epsilon} \quad . \tag{3.38}$$

We recall that the total energy of the system can be expressed in terms of

$$E = m\sqrt{T} + \Delta E_L + \cdots \tag{3.39}$$

where the ellipsis represents other contributions for the binding energy of the system.

The functions T, H, ϵ and μ, considered to be functions of U and evaluated at the instantaneous center of mass location $\vec{X} = 0$ for purposes of the calculation of ΔE_L, are now expanded in the form

$$T(U) = T_0 + T_0'\vec{g}_0 \cdot \vec{X} + O(\vec{g}_0 \cdot \vec{X})^2 \tag{3.40}$$

where $\vec{g}_0 = \vec{\nabla}U|_{\vec{X}=0}$, $T_0 = T|_{\vec{X}=0}$, and $T_0' = dT/dU|_{\vec{X}=0}$. It is useful to redefine the gravitational potential U by

$$U \to -\frac{1}{2}\frac{T_0'}{H_0}\vec{g}_0 \cdot \vec{X} \tag{3.41}$$

whose gradient yields the test-body acceleration $\vec{g}$.

If the above is used to expand (3.39), we get

$$E = (m + \mathcal{E}_L)(1 - U) + \mathcal{E}_L U\Big\{(5 - a - 2b)\Gamma_0 - a\Lambda_0\Big\} \tag{3.42}$$

where we have used (3.35); and neglected terms proportional to ξ, since the main position dependence parameterization is given in terms of:

$$\Gamma_0 = \frac{2T_0}{T_0'}(\frac{\epsilon_0'}{\epsilon_0} + \frac{T_0'}{2T_0} - \frac{H_0'}{2H_0}), \qquad \Lambda_0 = \frac{2T_0}{T_0'}(\frac{\mu_0'}{\mu_0} + \frac{T_0'}{2T_0} - \frac{H_0'}{2H_0}) \tag{3.43}$$

If we now identify (3.42) with Eqs. (1.9) and (1.11), we can obtain the corresponding Lamb shift contributions to the binding energy and anomalous passive mass tensor as

$$\begin{aligned} \Delta E_B^{0(L)} &= -\mathcal{E}_L \\ \delta m_P^{ij(L)} &= \Delta E_B^{0(L)}\Big\{(5-a-2b)\Gamma_0 - a\Lambda_0\Big\} \quad . \end{aligned} \tag{3.44}$$

This result was first presented in Ref. [41], where in (3.44) we have corrected the latter for a sign error in the coefficient multiplying Λ_0 and a missing factor b in the Γ_0 term.

Inserting (3.44) in (1.12), we obtain

$$\Xi^L = 3.424\,\Gamma_0 - 1.318\,\Lambda_0 \tag{3.45}$$

as the LPI violating parameter associated with the Lamb shift transition. Note that if LPI is valid then $\Gamma_0 = \Lambda_0 = 0$.

In comparing the result (3.45) to anomalous redshift parameters computed for other systems, it is important to note that we are working with units that are species dependent. Recall that the choice of $c_0 = 1$, and the redefinition of the gravitational potential (3.41) involves the T and H functions associated with electrons (or more generally a given species of lepton). Note that we are working within a context where the universality of gravity among all species of particles does not hold. That is, the T and H functions are species dependent.

Consider, for example, hyperfine transitions (maser clocks). In this case the leptonic and baryonic gravitational parameters appear simultaneously. This atomic splitting comes from the interaction between the magnetic moments of the electron and proton (nucleus). The proton metric appears only in the latter, and so it does not affect the principal and fine structure atomic energy levels. It is simple to check that the hyperfine splitting scales as

$$\Delta E_{hf} = \mathcal{E}_{hf} \frac{{T_B}^{1/2}}{H_B} \frac{H_0^2}{T_0} \frac{\mu_0}{\epsilon_0^3} \tag{3.46}$$

where the label B is added to distinguish baryonic related functions from leptonic ones; and $\mathcal{E}_{hf}$ depends only on atomic parameters.

In expanding (3.46) according to (3.40), we obtain

$$\Delta E_{hf} = \mathcal{E}_{hf}(1 - U_B) + \mathcal{E}_{hf} U_B \Xi^{hf} \tag{3.47}$$

with

$$\Xi^{hf} = 3\Gamma_B - \Lambda_B + \Delta \tag{3.48}$$

where U_B, Γ_B and Λ_B are the baryonic analogues of (3.41), and (3.43) respectively. In (3.47) we rescaled the atomic parameters to absorb the $TH\epsilon\mu$ functions and chose units such that $c_B = 1$. The quantity Δ is given by

$$\Delta = 2\frac{T_B}{T_B'} \left[2(\frac{H_B'}{H_B} - \frac{H_0'}{H_0}) - \frac{T_B'}{T_B} + \frac{T_0'}{T_0} \right] \tag{3.49}$$

and would vanish under the assumption that the leptonic and baryonic $TH\epsilon\mu$ parameters were the same.

Turning next to experiments which test LLI, we need to obtain the tensor δm_I^{ij} appropriate to the Lamb shift. This tensor is obtained after taking partial derivatives of ΔE_L with respect to u_i and u_j (note $\vec{V} \equiv \vec{u}$). Substituting the result into

(1.14) yields

$$1 - A^L = \frac{\xi}{7.757}\left\{\frac{\pi}{\alpha} + 3.074 - 0.011\cos^2\theta + \frac{V_i V_j}{\vec{V}^2}\Delta\hat{\epsilon}_{ij}\right\} \tag{3.50}$$

for the Lamb shift time dilation coefficient, where θ is the angle between the atom's quantization axis and its velocity with respect to the preferred frame.

Note that the coefficient A_L depends upon $\Delta\hat{\epsilon}_{ij}$, the evaluation of which involves the computation of an infinite sum as given by (A.56). The dominant contribution in Eq. (3.50) comes from the Dirac part of the energy (proportional to $\frac{1}{\alpha}$), which produces an overall shift only. Non-isotropic effects arise solely from radiative corrections.

In general, an experimental test of LLI involves a search for the effects of motion relative to a preferred frame such as the rest frame of the cosmic microwave background. A detailed analysis about the interpretation of LLI violating experiments is presented in Ref. [29], which analyzed experiments concerned with hyperfine transitions, obtaining an expression for the time dilation parameter corresponding to that kind of transition[1]. This parameter is negligible in comparison with other sources of energy, such as nuclear electrostatic energy in the case of the $^9Be^+$ clock experiment [10].

In summary, we have been able to parameterize EEP violations arising from Lamb shift transitions associated with redshift and time dilation experiments. In these types of EEP violating experiments one typically looks for variations of the energy shift due to changes in either the gravitational potential or the direction of the preferred frame velocity. The feasibility of such experiments is hindered by the present level of precision of Lamb shift transitions (one part in 10^6) in comparison

[1]Note that the expression given there for A^{Hf} is incomplete according to discussion presented in Sec. III

$$+ \quad \frac{\alpha(Z\alpha)^2}{6\pi}(10.434 + O(u^2))\Big]$$

where the first term comes from the Dirac contributions (here + and - label the transition coming from the $2P_{3/2}$ state with $|M| = 3/2$ and $|M| = 1/2$ respectively) and the second one from radiative corrections. Note that the leading anisotropic effects stem from the nonrelativistic contributions, and so their ratio with the metric value, $O(m(Z\alpha)^4)$, is $O(\xi u^2/(Z\alpha)^2)$, instead of $O(\xi u^2)$ as for the *classical* Lamb shift. Time dilation experiments will look for changes on the $E_{2S_{1/2}} - E_{2P_{3/2}}$ splitting as the Earth rotates, which would single out only the preferred frame contributions. Current experiments [34] measure a value of 9911.200(12) MHz for that transition, which gives a nominal bound (coming from the experimental error) of $\frac{3}{2}\xi\cos^2\theta < 1\times10^{-4}$ for the preferred frame part. This bound should improve once appropriate experiments are carried out, since these will look for periodic behavior which can be isolated and measured with high precision.

Note that an empirical value for the Lamb shift is obtained from Ref.[34] by subtracting the theoretical result of the fine splitting $2P_{1/2} - 2P_{3/2}$. Now by following the previous formalism we can parameterize the LPI violation in the former experimental result through:

$$E^{exp}_{2S_{1/2}-2P_{3/2}} = (\mathcal{E}_f + \mathcal{E}_L)(1-U) + U(\mathcal{E}_f\Xi^f + \mathcal{E}_L\Xi^L) \qquad (3.52)$$

where we have added the corresponding parameters related to the fine transition [6]: $\mathcal{E}_f$ and Ξ^f. Constraining the ratio of this quantity to a direct measurement of the Lamb shift [23] to lie within experimental/theoretical error, we obtain the bound $|U(\Xi^L - \Xi^f)| = |U(0.576\Gamma_0 + 1.318\Lambda_0)| < 10^{-5}$. This result is sensitive to the absolute value of the total local gravitational potential [15, 43], whose magnitude has recently been estimated to be as large as 3×10^{-5} due to the local supercluster [19]. Hence measurements of this type can provide us with empirical information to the magnitudes of such changes. In the first case, any Earth based experiments will be limited by the small size of the Earth's gravitational potential ($\approx 10^{-9}$), which is well beyond any foreseeable improvement in Lamb shift precision. Similar problems appear in the second case, where the known upper bound $|\vec{u}| < 10^{-3}$ [6] for the preferred frame velocity leaves no room for any improvement on the EEP violating parameter ξ, since anisotropic effects go as $\xi|\vec{u}|^2$.

However useful information can still be extracted from Eq. (3.31) if we use the current level of discrepancy between the experimental result [23] and the theoretical (metric) value [42] to bound the non-metric contributions for the Lamb shift. This constrains $\xi < 1(1) \times 10^{-5}$. Similar bounds can be obtained by considering empirical information about other atomic states. In this context, the indirect measurement of the $1S$ Lamb shift [33] gives a limit $\xi < 1.4(1) \times 10^{-5}$, and the measurement of the $2S_{1/2} - 2P_{3/2}$ fine structure interval [34]: $\xi < 0.7(1.4) \times 10^{-5}$. If we drop the assumption that positrons and electrons have equivalent couplings to the gravitational field [40], we find that there is an additional contribution to (3.31) due to $\xi_{e^+} \neq \xi_{e^-}$. This contribution arises entirely from radiative corrections and is given by Eq. (3.36). Making the same comparisons as above, we find the most stringent bound on this quantity to be $|\xi_{e^+}| < 10^{-3}$.

The previous bounds were obtained by using (3.9) and (3.32) or (3.33) to calculate the corresponding non-metric Dirac and radiative corrections contributions respectively. The $1S$ Lamb shift experiment, actually measures the transition: $(E_{4S} - E_{2S}) - \frac{1}{4}(E_{2S} - E_{1S})$, and so we use this one to make the comparison, where experimental and theoretical values are given in Ref.[33]. In the other experiment we need to use the non-metric part of $E_{2S_{1/2}} - E_{2P_{3/2}}$ ($\equiv \Delta_\xi$), namely:

$$\Delta_\xi = \xi(Z\alpha)^2 m \left[\pm\frac{u^2}{60}(\frac{3}{2}\cos^2\theta - 1) + O((Z\alpha)^2 u^2)\right. \tag{3.51}$$

sensitive to radiative corrections that constrains the allowed regions of (Γ_0, Λ_0) parameter space. Unfortunately the present level of precision in measuring the Lamb shift allows only a rather weak constraint.

3.5 Discussion

We have computed for the first time radiative corrections to a physical process, namely the energy shift between two hydrogenic energy levels that are semi-classically degenerate, within the context of the $TH\epsilon\mu$ formalism. The corresponding (GM) QED was derived, and the (GM) expressions for the propagators were obtained. The non-metric aspects of a theory describable by the $TH\epsilon\mu$ formalism can be all included in the photon propagator, given an appropriate choice of coordinates, leaving the fermion propagator unchanged. The addition of more parameters to the theory (by the $TH\epsilon\mu$ functions) entail new renormalizations, where not only charge and mass need to be redefined but also the $TH\epsilon\mu$ parameters.

The approach we took to solve for the semi-classical Dirac energies (Sec. III) differs from the one given in Ref. [29], in which the Dirac Hamiltonian was expanded using Foldy-Wouthuysen transformations yielding the first relativistic correction to the Schrödinger Hamiltonian (as introduced for example, for the Darwin and spin-orbit terms), and subsequently the energies. Instead we began from the fully relativistic expression, where the perturbations come only from the preferred frame terms of the electromagnetic potential. Our approach involved evaluating expectation values with respect to the relativistic spinors instead of their nonrelativistic extensions (or Pauli states). The effects of relativistic corrections such as spin-orbit coupling are therefore included exactly in this approach. Once this is done, the final result is expanded to keep it within the desired order. The semi-relativistic approach is not suitable when preferred frame effects are studied.

Qualitatively new information on the validity of the EEP will be obtained by setting new empirical bounds on the parameters ξ, A_L and Ξ_L which are associated with purely *leptonic* matter. Relatively little is known about empirical limits on EEP-violation in this sector [15]. Previous experiments have set the limits [9] $|\xi_B| \equiv |1 - c_B^2| < 6 \times 10^{-21}$ where c_B is the ratio of the limiting speed of baryonic matter to the speed of light. In our case we obtain an analogous bound on ξ for electrons from the difference between current experimental and theoretical values, giving $|\xi| < 10^{-5}$. Although much weaker than the bounds on ξ_B, it is comparable to that noted in a different context by Greene *et. al.* [44]. They considered a similar formalism ($TH\epsilon\mu$ with $\vec{u} = 0$) for analyzing the measurement of the photon wavelength emitted in a transition where a mass Δm is converted into electromagnetic radiation, thereby providing an empirical relationship between the limiting speed of massive particles (electrons) and light.

The breakdown of LPI for the Lamb shift in the context of a non-metric theory of gravity describable by the $TH\epsilon\mu$ formalism is embodied in the the anomalous gravitational redshift parameter (3.45). Recall that Ξ depends on the nature of the atomic transition through the evaluation of the anomalous passive tensor. This tensor will have differing expressions for differing types of atomic transitions [6]. An atomic clock based on the Lamb shift transition will, in a non-metric theory, exhibit a ticking rate that is dependent upon the location of the spacetime frame of reference and that differs from frequencies of clocks of differing composition. For example, the gravity probe A experiment [7] employed hydrogen-maser clocks, and was able to constrain the corresponding LPI violating parameter related to hyperfine transitions (c.f. (3.48)):

$$|\Xi^{Hf}| = |3\Gamma_B - \Lambda_B + \Delta| < 2 \times 10^{-4} \tag{3.53}$$

where Δ measures the relative gravitational coupling between leptons and baryons (c.f. (3.49)). This experiment involves interactions between nuclei and electrons and so does not (at least to the leading order to which we work) probe the leptonic sector in the manner that Lamb-shift experiments would. In general Eq. (1.12) will describe the gravitational redshift of a photon emitted due to a given transition in a hydrogenic atom; for a hyperfine transition the redshift parameter is (3.53), whereas it is (3.45) for the Lamb shift transition.

An analogous experiment to test for LPI violations based on Lamb shift transition energies poses a formidable experimental challenge because of the intrinsic uncertainties of excited states of Hydrogenic atoms. Setting empirical bounds on Ξ_L by precisely comparing two identical Lamb shift transitions at different points in a gravitational potential would appear unfeasible since the anticipated redshift in the background potential of the earth ($\approx 10^{-9}$) is much smaller than any foreseeable improvement in the precision of Lamb-shift transition measurements [42]. One would at least need to perform the experiment in a stronger gravitational field (such as on a satellite in close solar orbit) with 1-2 orders-of-magnitude improvement in precision. A 'clock-comparison' type of experiment between a 'Lamb-shift clock' and some other atomic frequency standard [6] is, in principle, sensitive to the absolute value of the total local gravitational potential [15, 43], as noted earlier. With this interpretation, comparative transition measurements of the type discussed in the previous section can more effectively constrain the allowed regions of (Γ_0, Λ_0) parameter space than can measurements which depend upon changes in the gravitational potential. Of course exploiting anticipated improvements in precision of measurements of atomic vacuum energy shifts [42] will yield better bounds on ξ_{e^-} and ξ_{e^+} via (3.31).

Violations of LLI single out a preferred frame of reference. In fact, the search for a preferred direction motivated the most precise tests of LLI performed so far [10, 9]. We have extended the analysis of the effects of motion relative to a preferred frame to account for the radiative correction for the atomic energies associated with the Lamb shift, as embodied in the expression (3.50). This non-universality reflects the breakdown of spatial isotropy for quantum-mechanical vacuum energies. The coefficient A_L depends upon $\Delta\hat{\epsilon}_{ij}$, the evaluation of which involves the numerical computation of the sum in (A.56). Unfortunately, the intrinsic linewidths of the relevant states render direct measurement of such effects unfeasible. More precise empirical information on the value of ξ can be obtained by precisely measuring changes in the $E_{2S_{1/2}} - E_{2P_{3/2}}$ splitting as functions of terrestrial or solar motions. However these effects are insensitive to radiative corrections, depending instead upon the semi-classical non-metric effects discussed in section III.

Finally, we note that our formalism could be applied to muonic atoms. For a muon-proton bound system, we will obtain an expression similar to that of (A.55), but where all parameters refer to muons. For an anti-muon electron bound system (a muonic atom) a similar analysis would apply. However in both cases the mass and spin of the muon could not be neglected.

Chapter 4

(GM) Anomalous Magnetic Moment

4.1 (GM) Free Scattering

We shall consider the lowest order radiative correction to the elastic scattering of electrons by a static external field A^μ. These one loop contributions can be summarized in terms of the Feynman diagrams illustrated in Fig. 4.1.

The Feynman amplitudes for the diagrams follow from the Feynman rules giving the result [45]:

$$\Lambda^\mu(p',p) = \bar{u}(\vec{p}\,')\left\{\Gamma^\mu + P^\mu + L^\mu\right\}u(\vec{p}) \tag{4.1}$$

where

$$\Gamma^\mu(p',p) = \frac{(ie)^2}{(2\pi)^4}\int d^4k\gamma^\alpha iS_F(p'-k)\gamma^\mu iS_F(p-k)\gamma^\beta iG_{\alpha\beta}(k) \tag{4.2}$$

$$P^\mu(p',p) = \gamma^\alpha iG_{\alpha\beta}(q)i\Pi^{\beta\mu}(q) \tag{4.3}$$

$$L^\mu(p',p) = i\Sigma(p')iS_F(p')\gamma^\mu + \gamma^\mu iS_F(p)i\Sigma(p) \tag{4.4}$$

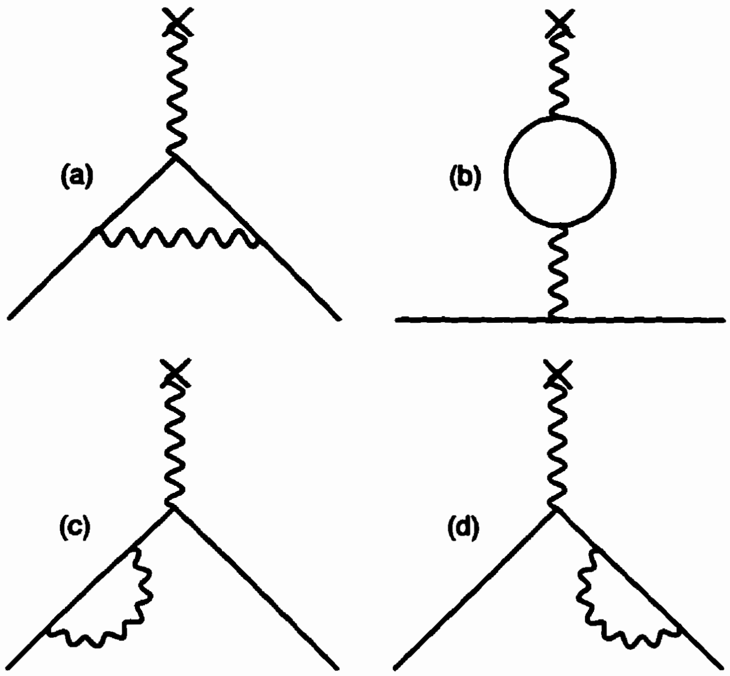

Figure 4.1: One loop corrections to the elastic scattering of an electron by an external electromagnetic source

with

$$i\Sigma(p) = \frac{(ie)^2}{(2\pi)^4}\int d^4k iG_{\alpha\beta}(k)\gamma^\alpha iS_F(p-k)\gamma^\beta \tag{4.5}$$

$$i\Pi^{\beta\mu}(q) = \frac{(ie)^2}{(2\pi)^4}(-)\mathrm{Tr}\int d^4k\gamma^\beta iS_F(k+q)\gamma^\mu iS_F(k) \tag{4.6}$$

and $q \equiv p' - p$.

We refer to Eqs. (4.2), (4.3), and (4.4) as the Vertex, Polarization, and Leg contributions, which respectively correspond to diagrams (a), (b) and (c) plus (d). We also note that expressions (4.2), (4.5), (4.6) represent the one loop corrections to the vertex, fermion and photon self energy parts respectively.

Given the form of the photon propagator it is convenient to introduce:

$$\Lambda^\mu = (1+\xi)\Lambda_0^\mu + \gamma^2\xi\Lambda_\xi^\mu \tag{4.7}$$

where the subscript "0" denotes the (known) result coming from the standard part of the photon propagator, and "ξ" for the part proportional to γ^2 in (2.13)

$$G^\xi_{\mu\nu} = \frac{\beta_\mu\beta_\nu}{k^2} + \eta_{\mu\nu}\frac{(\beta\cdot k)^2}{k^4} \tag{4.8}$$

In the remainder of this section we consider this part of the propagator only, omitting the "ξ" label in the corresponding expressions.

The procedure for evaluating the loop integrals is equivalent to that of standard (or metric) QED. We need to regularize them first and then renormalize the parameters, which include the $TH\epsilon\mu$ parameters along with the fermion charge and mass. The regularization of the photon propagator is carried out using

$$\frac{1}{k^2} \to -\int_{\mu^2}^{\Lambda^2}\frac{dL}{(k^2-L)^2}, \qquad \frac{1}{k^4} \to -2\int_{\mu^2}^{\Lambda^2}\frac{dL}{(k^2-L)^3}, \tag{4.9}$$

with the assumed limits $\mu \to 0$ and $\Lambda \to \infty$, and the parameter renormalization by the inclusion of the corresponding counterterms to each loop integral. Details about this procedure and the corresponding calculations are given in the appendix B.1. We quote the final result for the loop integrals:

$$\begin{aligned}
\Sigma(p) &= \frac{\alpha}{\pi}(\not p - m)\left\{\frac{(\beta\cdot p)^2}{m^2}\frac{2}{3} - \beta^2\left[\frac{5}{24}\ln(\frac{\Lambda}{m})^2 + \frac{133}{144} - \frac{1}{4}\ln(\frac{m}{\mu})^2\right]\right\} \\
&- \frac{\alpha}{\pi}\frac{\beta\cdot p}{6m}\{(\not p - m)\not\beta + \not\beta(\not p - m)\} + O\big((\not p - m)^2\big) \\
\Gamma^\mu(p',p) &= \frac{\alpha}{\pi}\Big\{\gamma^\mu\left[\frac{2}{3}\frac{(\beta\cdot p)^2}{m^2} - \beta^2\left(\frac{5}{24}\ln(\frac{\Lambda}{m})^2 + \frac{133}{144} - \frac{1}{4}\ln(\frac{m}{\mu})^2\right)\right. \\
&+ \left[\frac{q^2}{m^2}\left(\frac{17}{144}\beta^2 + \frac{1}{12} + \ln(\frac{m}{\mu})^2(\frac{1}{8} - \frac{\beta^2}{24})\right)\right. \\
&+ \left.\frac{3}{2}\frac{\beta\cdot p}{m}\frac{\beta\cdot q}{m} + (\frac{\beta\cdot q}{m})^2\left(\frac{19}{36} - \frac{1}{6}\ln(\frac{m}{\mu})^2\right)\right]
\end{aligned} \tag{4.10}$$

$$
\begin{aligned}
&- \frac{\not{q}}{m}\gamma^\mu \not\beta\left(\frac{1}{6}\frac{\beta\cdot p}{m}+\frac{1}{12}\frac{\beta\cdot q}{m}\right)+\left(\frac{1}{9}\frac{q^2}{m^2}-\frac{1}{3}\right)\not\beta\beta^\mu \qquad (4.11)\\
&- \frac{1}{6}\frac{\not{q}}{m}\not\beta\beta^\mu-\frac{1}{2}\frac{\beta\cdot q}{m}\not\beta\gamma^\mu+\frac{2}{3}\frac{\beta\cdot q}{m}\beta^\mu+\left(\frac{\beta^2}{24}+\frac{1}{6}+\frac{1}{6}\frac{\beta\cdot p}{m}\frac{\beta\cdot q}{m}\right)\frac{\not{q}}{m}\gamma^\mu\\
&+ \left(\frac{1}{6}\frac{\beta\cdot p}{m}-\frac{5}{36}\frac{\beta\cdot q}{m}\right)\frac{q^\mu}{m}\not\beta-\left(\frac{\beta^2}{24}+\frac{1}{6}\right)\frac{q^\mu}{m}\Big\}+O(q^3)
\end{aligned}
$$

$$
\Pi^{\alpha\beta}(q) = -\frac{\alpha}{\pi}\left(q^2\eta^{\alpha\beta}-q^\alpha q^\beta\right)\frac{q^2}{15m^2}+O(q^6) \qquad (4.12)
$$

where we have implicitly assumed that (4.11) is acting on free spinors.

The Ward identity

$$
\frac{\partial\Sigma(p)}{\partial p_\mu}=\Gamma^\mu(p,p). \qquad (4.13)
$$

is a consequence of gauge invariance, and therefore it holds even in the absence of Lorentz invariance. It is straightforward to check that (4.10) and (4.11) satisfy (4.13).

The evaluation of (4.1) is also straightforward once the loop integrals have been calculated. We just comment on the computation of the Leg correction, which is ambiguous since it contains terms like "0/0", which are indeterminate. To obtain an unambiguous result, we must explicitly introduce a damping factor, which is necessary for the correct definition of the initial and final "bare" states. Details of this adiabatic approach are presented in appendix B.2. The final result for the Leg correction is

$$
\begin{aligned}
L^\mu &= \frac{\alpha}{\pi}\Big\{\gamma^\mu\left[\frac{2}{3}\left(\frac{(\beta\cdot p)^2}{m^2}+\frac{\beta\cdot p}{m}\frac{\beta\cdot q}{m}+\frac{1}{2}\frac{(\beta\cdot q)^2}{m^2}\right)\right.\\
&+ \left.\beta^2\left(\frac{5}{24}\ln(\frac{\Lambda}{m})^2+\frac{133}{144}-\frac{1}{4}\ln(\frac{m}{\mu})^2\right)\right] \qquad (4.14)\\
&+ \frac{1}{3}\not\beta\beta^\mu-\frac{1}{6}\frac{\not{q}}{m}\not\beta\beta^\mu+\frac{1}{6}\frac{\beta\cdot q}{m}\not\beta\gamma^\mu\Big\}
\end{aligned}
$$

Note that this part gives a contribution to the total amplitude that *cannot* be removed after renormalization. Furthermore, the gauge invariance of the Feynman amplitude which is manifest as

$$q \cdot \Lambda = 0 \tag{4.15}$$

requires the presence of such terms, a condition that is not satisfied by the vertex contribution only.

The final result for the scattering amplitude is

$$\Lambda^\mu = F^\mu + G^\mu + I^\mu \tag{4.16}$$

with

$$\begin{aligned} F^\mu &= \frac{\alpha}{\pi}\Big\{\gamma^\mu\left[\frac{q^2}{m^2}\left(\frac{17}{144}\beta^2 + \frac{1}{12} + \ln(\frac{m}{\mu})^2(\frac{1}{8} - \frac{\beta^2}{24})\right) + \frac{5}{6}\frac{\beta\cdot p}{m}\frac{\beta\cdot q}{m}\right. \\ &+ \left.(\frac{\beta\cdot q}{m})^2\left(\frac{47}{180} - \frac{1}{6}\ln(\frac{m}{\mu})^2\right)\right] - \frac{\not q}{m}\gamma^\mu\,\not\beta\frac{1}{12}\frac{\beta\cdot q}{m} \\ &+ \frac{8}{45}\frac{q^2}{m^2}\,\not\beta\beta^\mu + \frac{1}{6}\frac{\beta\cdot p}{m}\frac{\beta\cdot q}{m}\frac{\not q}{m}\gamma^\mu\Big\} \end{aligned} \tag{4.17}$$

$$G^\mu = \frac{\alpha}{\pi}\Big\{-\frac{\not q}{m}\gamma^\mu\,\not\beta\frac{1}{6}\frac{\beta\cdot p}{m} - \frac{1}{3}\frac{\not q}{m}\,\not\beta\beta^\mu - \frac{1}{3}\frac{\beta\cdot q}{m}\,\not\beta\gamma^\mu + (\frac{\beta^2}{24} + \frac{1}{6})\frac{\not q}{m}\gamma^\mu\Big\} \tag{4.18}$$

$$I^\mu = \frac{\alpha}{\pi}\Big\{\frac{2}{3}\frac{\beta\cdot q}{m}\beta^\mu + \left(\frac{1}{6}\frac{\beta\cdot p}{m} - \frac{37}{180}\frac{\beta\cdot q}{m}\right)\frac{q^\mu}{m}\,\not\beta - \left(\frac{\beta^2}{24} + \frac{1}{6}\right)\frac{q^\mu}{m}\Big\} \tag{4.19}$$

The various terms in (4.16) distinguish the different contributions to the scattering amplitude. In (4.17) we group terms of order q^2 or higher. G^μ accounts for terms of order q at least, and I^μ for the gauge terms or those who give no contribution to the amplitude. Note that the remaining infrared divergence in F^μ can be understood in terms of soft photon radiation, analogous to the metric case.

In the next section we will use the above results to compute the $g-2$ anomaly.

4.2 (GM) g-2 Experiments

To lowest order the Feynman amplitude associated with the elastic scattering of an electron by a static external field is

$$ie\bar{u}(p')\ \not{A}(q)u(p) \quad . \tag{4.20}$$

The radiative correction of order α to this process is given by

$$ie\bar{u}(p')\{(1+\xi)\Lambda_0 \cdot A + \gamma^2\xi\Lambda_\xi \cdot A\}u(p) \tag{4.21}$$

where Λ_0 represents the (known) metric result and Λ_ξ represents the contribution from (4.16).

In the nonrelativistic limit of slowly moving particles ($|\vec{q}| \to 0$) and a static magnetic field , it is straightforward to show that

$$e\ \not{A}(q) \ \to\ -\frac{e}{2m}\vec{B}\cdot\vec{\sigma} \tag{4.22}$$

$$e\Lambda_0 \cdot A \ \to\ -\frac{e}{2m}(\frac{\alpha}{2\pi})\vec{B}\cdot\vec{\sigma} \tag{4.23}$$

$$e\Lambda_\xi \cdot A \ \to\ e\,G\cdot A \tag{4.24}$$

with G^μ given by (4.18), which is the dominant term as $q \to 0$.

In order to simplify this contribution, we consider a constant magnetic field $\vec{B}$, that is $\vec{A} = \frac{1}{2}\vec{r}\times\vec{B}$, in which case

$$\not{q}\ \not{\beta}\beta\cdot A \to -\frac{1}{2}(\vec{B}\cdot\vec{u}\,\vec{\sigma}\cdot\vec{u} - \vec{B}\cdot\vec{\sigma}\,\vec{u}^2) \tag{4.25}$$

where we have neglected the terms that mix the large and small spinor components. Similarly, we can show

$$\beta\cdot q\ \not{\beta}\ \not{A} \to -\frac{1}{2}(\vec{B}\cdot\vec{u}\,\vec{\sigma}\cdot\vec{u} - \vec{B}\cdot\vec{\sigma}\,\vec{u}^2) \tag{4.26}$$

and in the nonrelativistic limit

$$\not{u}\,\not{A}\,\not{\beta}\frac{\beta\cdot p}{m}\simeq\not{u}\,\not{A}\,\not{\beta}\rightarrow-\vec{B}\cdot\vec{\sigma} \tag{4.27}$$

and

$$\not{u}\,\not{A}\rightarrow-\vec{B}\cdot\vec{\sigma} \quad . \tag{4.28}$$

If we put everything together in (4.18):

$$e\,G\cdot A\rightarrow-\frac{e}{2m}\frac{\alpha}{\pi}\{\vec{B}\cdot\vec{\sigma}(\frac{1}{12}+\frac{7}{12}\vec{u}^2)-\frac{2}{3}\vec{B}\cdot\vec{u}\,\vec{\sigma}\cdot\vec{u}\} \tag{4.29}$$

As a cross-check on the above result, we take the limit $u_iu_j \rightarrow -\delta_{ij}$ obtaining $G\cdot A\rightarrow -2\Lambda_0\cdot A$, which is the required limit consistent with the structure of Eq. (4.8) in that case. The previous result is the contribution of (4.24) to (4.21), which added to (4.20), give us the relevant part of the Hamiltonian as

$$H_\sigma=-\{\Gamma\,\vec{S}\cdot\vec{B}+\Gamma_*\vec{S}\cdot\vec{u}\,\vec{B}\cdot\vec{u}\}+O(\xi^2)O(\alpha^2)\equiv-\Gamma^{ij}S_iB_j \tag{4.30}$$

with

$$\Gamma\equiv\frac{e}{2m}g \quad\equiv\quad \frac{e}{2m}\{2+\frac{\alpha}{\pi}[1+\xi(1+\frac{\gamma^2}{6}(1+7\vec{u}^2))]\} \tag{4.31}$$

$$\Gamma_*\equiv\frac{e}{2m}g_* \quad\equiv\quad -\frac{e}{2m}\frac{\alpha}{\pi}\xi\frac{4}{3}\gamma^2 \tag{4.32}$$

where we have identified $\vec{S}\equiv\frac{\vec{\sigma}}{2}$, and $\hat{u}=\vec{u}/|\vec{u}|$. The Γ parameters account for the coupling strength between the magnetic field and spin. We see that Γ_{ij} generalizes the gyromagnetic ratio of a fermion analogous to the manner in which the anomalous mass tensor generalizes the mass of a particle [30]. We therefore identify the parameters $\Gamma^{ij}\equiv\Gamma\delta^{ij}+\Gamma_*u^iu^j$ with the components of the *anomalous gyromagnetic ratio tensor* of the fermion in the class of $TH\epsilon\mu$ theories.

Note that the presence of preferred frame effects induces a qualitatively new form of interaction between the spin and magnetic field which is quantified by Γ_*.

Here, instead of coupling with each other, they both couple independently to the fermion velocity relative to the preferred frame. This interaction stems purely from radiative corrections, and would be absent in any tree-level analysis of (GM)QED.

Hence, Eq. (4.30) describes the interaction (as seen from the particle rest frame) between the particle spin and an external homogeneous magnetic field. From this we can extract the energy difference between electrons with opposite spin projection in the direction of the magnetic field as:

$$\Delta E_\sigma = -\frac{eB}{2m}\left[g + g_* u^2 \cos^2\Theta\right] \tag{4.33}$$

where Θ is the angle between the magnetic field and the preferred frame velocity. The influence of the radiative corrections (coming from $g-2$ and g_*) in (4.33) is negligible in comparison to the dominant factor of 2 in g. Since we want to single out the effects of the non-metric corrections, it is more interesting to study the precession of the spin or, more specifically, the oscillation of the longitudinal spin polarization. In the metric case, this frequency is proportional to the factor $g-2$, and so it is a distinctive signature of radiative corrections.

The observable quantity in the $g-2$ experiments is actually the electron polarization, which is proportional to the quantum mechanical expectation value of $\vec{S}$, that is, $\langle\vec{S}\rangle$. Using Ehrenfest's theorem, a quantum mechanical solution for the motion of $\langle\vec{S}\rangle$ is obtained from the equation

$$\frac{d\vec{S}}{dt'}|_{R.F.} = -i[\vec{S}, H_\sigma] = \vec{S} \times \left[\Gamma\vec{B}' + \Gamma_*(\vec{B}'\cdot\vec{u})\vec{u}\right] \tag{4.34}$$

where the primed variables are referred explicitly to the particle rest frame ($R.F.$). Note that the preferred frame effect will show distinctly as a temporal variation of the spin component parallel to the magnetic field.

In general we want to know the spin precession relative to some specific laboratory system, with respect to which the particle is moving with some velocity

$\vec{\beta}$. This frame need not *a-priori* be the previously defined preferred frame, and so $\vec{\beta} \neq \vec{u}$.

Since the $TH\epsilon\mu$ formalism does not change (locally) the fermion electromagnetic field interaction, we expect that a charged particle in the presence of an homogeneous magnetic field will satisfy the equation

$$\frac{d\vec{\beta}}{dt} = \vec{\beta} \times \vec{\Omega}_c \tag{4.35}$$

with the cyclotron frequency $\vec{\Omega}_c = \frac{e}{m\gamma}\vec{B}$ and $\gamma = (1 - \vec{\beta}^2)^{-1/2}$. Relating (4.34) to the laboratory system yields

$$\frac{d\vec{S}}{dt}|_{Lab} = \frac{d\vec{S}}{dt}|_{R.F.} + \vec{\Omega}_T \times \vec{S} \tag{4.36}$$

due to Thomas precession, with $\vec{\Omega}_T = \frac{\gamma^2}{\gamma+1}(\frac{d\vec{\beta}}{dt} \times \vec{\beta})$. This frequency is kinematic in origin and it is a consequence of the non-commutativity of the Lorentz transformations.

Relating the primed variables in (4.34) to the laboratory ones by a Lorentz transformation gives

$$\frac{d\vec{S}}{dt}|_{Lab} = \vec{S} \times \vec{\Omega}_s \tag{4.37}$$

with

$$\vec{\Omega}_s = \Gamma\vec{B} + (1-\gamma)\vec{\Omega}_c + \Gamma_*(\vec{B}\cdot\vec{u})\vec{u} \tag{4.38}$$

where we have set $\vec{E} = 0$ and considered (for simplicity) the case of orbital motion perpendicular to the magnetic field ($\vec{\beta}\cdot\vec{B} = 0$) in the above. Note that the spin precession about $\vec{\Omega}_s$ is no longer parallel to the magnetic field (axial direction), but has a component parallel to $\vec{u}$ that comes from radiative and non-metric effects.

At this point it becomes necessary to define the preferred coordinate system. There are several candidates (such as the rest frame of the cosmic microwave background) for this frame [6]. To study this issue it is sufficient to assume that the laboratory system (Earth) moves with a non-relativistic velocity ($\vec{V}$) with respect to the preferred frame, and so we can identify

$$\vec{u} = \vec{V} + \vec{\beta} \quad .$$

In order to single out the effects of radiative corrections, we study the spin precession relative to the rotational motion of the electron, that is:

$$\frac{d\vec{S}}{dt}|_{rot} = \vec{S} \times \vec{\Omega}_D \tag{4.39}$$

with $\vec{\Omega}_D = \vec{\Omega}_s - \vec{\Omega}_c$ and $\vec{S} = (S_\perp^{\|}, S_\perp^{\perp}, S_{\|})$, where the first two components are perpendicular to $\vec{B}$ (lower index) but parallel and perpendicular to $\vec{\beta}$ (upper index), and the last one parallel to $\vec{B}$. In the following we refer to the difference frequency (Ω_D) as the anomalous frequency (given its connection with the anomalous magnetic moment in the metric case). It is convenient to rewrite:

$$\vec{\Omega}_D = \vec{\Omega}_a + \Omega_a^* \cos\Theta(\vec{V}_\perp + \vec{\beta}) \tag{4.40}$$

with

$$\vec{\Omega}_a = \frac{e}{2m}\left(g + g_* V^2 \cos^2\Theta - 2\right)\vec{B} \tag{4.41}$$

and $\Omega_a^* = \frac{e}{2m} g_* BV$; where Θ represents the angle between V and the magnetic field, and $V_\perp$ the component of the velocity perpendicular to B. In Ω_a we group all the terms parallel to the magnetic field that contribute to the anomalous frequency (including non-metric effects). The remaining terms perpendicular to B arise from non-metric effects only, and produce a temporal variation of the spin component parallel to the magnetic field. This effect is absent in the metric case, and so represents a qualitatively new manifestion of possible EEP violation.

In general we are interested in solving (4.39) for the cases $\beta >> V$ or $\beta << V$ so that $\gamma(u) \simeq \gamma(\beta)$ or $\gamma(V)$, but is otherwise constant. Since Ω_a^* is proportional to

ξ, we can perturbatively solve for each component in (4.39). Taking, for example, the initial condition $\vec{S}(0) = S\hat{\beta}$ we find

$$\begin{aligned} S_{\perp}^{\parallel} &= S\cos\Omega_a t \qquad S_{\perp}^{\perp} = S\sin\Omega_a t \\ S_{\parallel} &= S\frac{\Omega_a^*}{\Omega_a}\beta\cos\Theta(1-\cos\Omega_a t) + S\frac{\Omega_a^*}{\Omega_a+\Omega_c}V\frac{\sin 2\Theta}{2}\left[\cos(\Omega_a+\Omega_c)t - 1\right] \end{aligned} \tag{4.42}$$

where we have chosen a coordinate system where $\hat{B} = \hat{z}$ so that

$$\hat{V} = \hat{B}\cos\Theta + \hat{x}\sin\Theta, \quad \hat{\beta} = \hat{y}\cos\Omega_c t - \hat{x}\sin\Omega_c t \tag{4.43}$$

and assumed that any rotation related to Θ is negligible in comparison to other frequencies involved in the problem (Ω_a or Ω_c).

The fact that Ω_a was (in the metric case) proportional to $g-2$, motived the very precise $g-2$ experiments which were designed to specifically measure that anomalous frequency. We see that this frequency is modified from its metric value by the additional terms present in (4.41). If we assume that the EEP-violating contributions to Ω_a are bounded by the current level of precision for anomalous magnetic moments [26], then the discrepancy between the best empirical and theoretical values for the electron yields the bounds

$$|\xi_{e^-}| < 3.5\times 10^{-8} \quad \text{and} \quad |\xi_{e^-} - \xi_{e^+}| < 10^{-9} \tag{4.44}$$

the latter following from a comparison of positron and electron magnetic moments. For muons, a similar analysis yields

$$|\xi_{\mu^-}| < 10^{-8} \quad \text{and} \quad |\xi_{\mu^-} - \xi_{\mu^+}| < 10^{-8} \quad . \tag{4.45}$$

Even though the accuracy of the muon anomaly is lower than the electron one, the slightly stronger bound in (4.45) arises because the experiments are carried out for high-velocity muons [46]. To our knowledge these bounds on violation of gravitational universality are the most stringent yet noted for leptonic matter. Torsion balance experiments and laser experiments yield the weaker bound $\xi_{e^-} < 10^{-7}$ when these tests are analyzed in a similar context [47].

Newman *et. al.* analyzed the $g-2$ experiments [48] in order to find new bounds for the validity of special relativity. They assumed that the parameter γ involved in the electron motion had a different value ($\tilde{\gamma}$) from that which arises kinematically (in Thomas precession and Lorentz transformations). The equivalent equation for (4.41) is in that case

$$\Omega_a^{NFRS} = \frac{eB}{m}\left(\frac{g}{2} - \frac{\gamma}{\tilde{\gamma}}\right) \tag{4.46}$$

and by comparing with two electron $g-2$ experiments, one at electron relativistic energy ($\beta = 0.57$) and the other nearly at rest ($\beta = 5 \times 10^{-5}$), they obtained the constraint $\delta\gamma/\tilde{\gamma} < 5.3 \times 10^{-9}$. Our approach is qualitatively different from theirs, in that we assume $\gamma = \tilde{\gamma}$ but include preferred frame effects in the evaluation of the anomalous magnetic moment. A similar analysis in our case yields the weaker bounds of $|\xi_e| < 7 \times 10^{-6}$ for electrons, and $|\xi_\mu| < 2 \times 10^{-7}$ for muons. In the latter we used the $g-2$ muon experiments carried at $\beta = 0.9994$ ($\gamma = 29$)[46], and $\beta = 0.92$ ($\gamma = 12$)[49].

Preferred effects not only modify the anomalous frequency according to (4.41), but also induce oscillations in the spin component parallel to B. As stated above, this is a qualitatively new consequence of EEP violations due solely to radiative corrections in (GM)QED. Searching for such oscillations therefore provides a new null test of the EEP. We can estimate the magnitude of such effects by taking the temporal average of $S_{||}$ over the main oscillation given by Ω_a, which gives

$$\delta = \frac{\langle S_{||}\rangle}{S} \sim \xi V \beta \cos\Theta \gamma^2 \tag{4.47}$$

This effect is enhanced in highly relativistic situations, and can be estimated by considering a typical experiment with $V \sim 10^{-3}$. For electrons $\beta \sim 0.5$, and so $\delta_e \sim 10^{-11}$; for muons $\beta = 0.9994$, yielding $\delta_\mu \sim 10^{-8}$. In both cases we used the corresponding present constraints for ξ given above.

The novelty of the $S_{||}$ oscillation suggests the possibility of putting tighter constraints on the non-metric parameter, once appropriate experiments are carried out. The same goes for the analysis of Ω_a at different values of Θ (the angle between the magnetic field and the velocity of the laboratory system with respect to the preferred frame). The rotation of the Earth will turn this orientation dependence into a time-dependence of the anomalous magnetic moment, with a period related to that of the sidereal day.

The previous analysis was concerned with effects related to spatial anisotropy. We turn now to a consideration of possible violations of local position invariance. The position dependence in the former section was implicit in the redefinitions of charge, mass and fields. These quantities were rescaled in terms of the $TH\epsilon\mu$ functions, which were considered constant throughout the computation. LPI violating experiments are of two types. One of these entails the measurement of a given frequency at two different points in a gravitational field (where differences in the gravitational potential could be significant) within the same reference system. The other type involves a comparison of frequencies arising from two different forms of energy (*i.e.* two different clocks) at the same point in a gravitational potential. We parameterize the gravitational dependence on a given frequency as:

$$\Omega = \Omega^0\left[1 - U + \Xi^{ij} U_{ij}\right] + \cdots \tag{4.48}$$

where U_{ij} represents the external gravitational tensor, satisfying $U_{ii} = U$, and the ellipsis represents higher order terms (going as either U^2 or velocity times U) in the gravitational potential or terms independent of it.

The measured redshift parameter related to this frequency may be written as

$$Z = \Delta U\Big(1 - \Xi\Big), \qquad \Xi = \Xi^{ij}\frac{\Delta U_{ij}}{\Delta U} \tag{4.49}$$

where Ξ^{ij} will depend upon the specific frequency measured in the experiments. Note that this tensor is equivalent to the anomalous passive gravitational mass tensor introduced for the study of atomic transitions.

In $g-2$ experiments the relevant frequency is Ω_a, which describes the precession of the longitudinal polarization in the presence of a constant magnetic field. Using the $TH\epsilon\mu$ formalism (see Eq.(4.41)) we obtain

$$\Omega_a = \frac{eB}{2m}[g-2] + \cdots = \frac{eB}{2m}\frac{\alpha}{\pi}\left[1 + \xi\frac{7}{6}\right] + \cdots \tag{4.50}$$

where we have omitted terms proportional to velocities, which eventually will contribute as $O(v^2U)$ terms at most.

In order to carry out the loop calculation, the $TH\epsilon\mu$ dependence was absorbed into the definition of the parameters under the rescaling

$$\Omega \to \Omega/c_0 \qquad m \to m\sqrt{T_0}/c_0 \qquad \alpha \to \alpha/\epsilon_0 c_0 \tag{4.51}$$

with $c_0 = (T_0/H_0)^{1/2}$ as the limiting speed of the massive particles, the subscript '0' denoting the $TH\epsilon\mu$ functions evaluated locally at $\vec{X} = 0$. Although the product eB remains invariant under this rescaling, the expression for the constant magnetic field still depends on the $TH\epsilon\mu$ parameters once it is written solely in terms of atomic parameters. This can be seen clearly by considering the magnetic field produced by a long solenoid of length L, with N turns and carrying a current I. The gravitationally modified Maxwell equation to solve is:

$$\vec{\nabla} \times (\mu^{-1}\vec{B}) = 4\pi\vec{J} \tag{4.52}$$

and so we find the non-vanishing magnetic field inside the solenoid to be $B = 4\pi\mu_0 IN/L$. Again we assume that the $TH\epsilon\mu$ functions are constant throughout the size of the experimental device. In terms of fundamental atomic parameters, L is proportional to an integer times the Bohr radius (the interatomic spacing), which is known to rescale as $a_0 \to a_0\epsilon_0 c_0^2/\sqrt{T_0}$ [6]. If we now write $I = \int \vec{J}\cdot d\vec{S}$, where J can be expressed in terms of a density charge ρ in motion (v) through a volume V, and then relate the Bohr radius to each spatial dimension along with the limiting particle velocity c_0 to the velocity distribution v, we can show $I \to I\sqrt{T_0}/\epsilon_0 c_0$, and so $B \to B\mu_0 T_0/\epsilon_0^2 c_0^3$. Along with (4.51), this gives the position dependence of (4.50) to be

$$\Omega_a = \Omega_a^0 \sqrt{T_0}\frac{\mu_0 c_0}{(\epsilon_0 c_0)^3}(1 + \xi\frac{7}{6}) \tag{4.53}$$

with $\Omega_a^0 = eB\alpha/2m\pi$ (recall $\xi = 1 - 1/\mu_0\epsilon_0 c_0^2$).

Note that the $TH\epsilon\mu$ functions are evaluated at some representative point of the system, which we have chosen to be the origin $\vec{X} = 0$. In order to determine how Ω_a changes as the position of the system varies, we expand the $TH\epsilon\mu$ functions in (4.53) according to (3.40), which in turns yields

$$\Omega_a = \Omega_a^0\big[1 - U + (\frac{11}{6}\Gamma_0 - \frac{13}{6}\Lambda_0)U\big] \tag{4.54}$$

where we have rescaled again according to (4.51), and omitted terms proportional to ξ, since the main position dependence parameterization is given in terms of the LPI-violating parameters Γ_0 and Λ_0, introduced by Eq. (3.43).

By comparing Eq. (4.54) with (4.48), we can identify

$$\Xi^{g-2} = \frac{11}{6}\Gamma_0 - \frac{13}{6}\Lambda_0 \tag{4.55}$$

as the LPI-violating parameter. Note that this depends on the anomalous frequency related to the longitudinal polarization of the beam. It is also species-dependent, with the value of Γ_0 and Λ_0 for the electron differing from that of the muon. A search for possible position dependence of anomalous spin precession frequencies provides another qualitatively new test of LPI sensitive to radiative corrections.

Actually the most precise $g-2$ experiments for electron measure the ratio $a = \Omega_a/\Omega_c$ at nonrelativistic electron energies ($\beta \sim 10^{-5}$), and so $\Omega_c \simeq eB/m$. This is interesting because by following the former parameterization we can write:

$$\Omega_c = \Omega_c^0\left[1 - U + (2\Gamma_0 - \Lambda_0)U\right] \tag{4.56}$$

or by taking the ratio of (4.54) to (4.56), we obtain the anomalous magnetic moment:

$$a = a^0(1 + U\Xi^a), \qquad \Xi^a = \frac{1}{6}\Gamma_0 + \frac{7}{6}\Lambda_0 \tag{4.57}$$

and then by identifying a with the most precise experimental value [25] and a^0 with the theoretical one [26], we can constrain through the resulting theoretical/experimental errors $|U\Xi^a| < 3 \times 10^{-8}$. This result is sensitive to the absolute value of the total local gravitational potential [15], whose magnitude has recently been estimated to be as large as 3×10^{-5} due to the local supercluster [19]. Hence measurements of this type can provide us with empirical information sensitive to radiative corrections that constrains the allowed regions of (Γ_0, Λ_0) parameter space, giving in this case:

$$|\frac{1}{6}\Gamma_0 + \frac{7}{6}\Lambda_0| < 10^{-3} \tag{4.58}$$

For muons the analogous constraint is $|U\Xi_\mu^a| < 10^{-5}$, and so a much weaker bound is obtained.

We note that a similar experiment to that employing hydrogen-maser clocks could be carried out for the energy shift defined in (4.33), which can be used as a frequency test to look for position or frame dependence. This can be done by following the same procedure as for atomic energy shifts, where the anomalous passive and inertial gravitational tensor are introduced in order to relate non-metric effects to redshift and time dilation parameters. Since radiative corrections are irrelevant in that energy shift, we omit that procedure here.

4.3 Discussion

Refined measurements of anomalous magnetic moments can provide an interesting new arena for investigating the validity of the EEP in physical systems where radiative corrections are important. We have considered this possibility explicitly for the class of non-metric theories described by the $TH\epsilon\mu$ formalism. The non-universal character of the gravitational couplings in such theories affects the one loop corrections to the scattering amplitude of a free fermion in an external electromagnetic field in a rather complicated way, giving rise to several novel effects.

An evaluation of the one-loop diagrams reveals that the leg corrections, which in the metric case give no contribution to the total amplitude after a proper renormalization of mass and spinor field, provide contributions which cannot be removed after renormalization. Moreover they are essential in ensuring the gauge invariance of the scattering amplitude, which is not fulfilled by the vertex correction alone. The consistency of the calculation is verified explicitly through the Ward identity, which furnishes a cross-check between the fermion self energy and the vertex correction. The non-metric corrections to the scattering amplitude also have an infrared divergence, which could be understood in terms of inelastic soft photon radiation, as in the metric case. This does not affect the term associated with the anomalous magnetic moment.

The presence of preferred frame effects induces a new type of coupling between the magnetic field and the spin as described by (4.30). This interaction stems purely from radiative corrections, and generalizes the gyromagnetic ratio of a fermion to a tensorial coupling described by Γ_{ij}. We emphasize that qualitatively new information on the validity of the EEP will be obtained by setting new empirical bounds on this coupling, as it is associated with purely *leptonic* matter.

Consequently, discussion of a $g-2$ contribution to the magnetic moment no longer makes sense, and we instead refer to the anomalous frequency as the main connection with experiment. Note that this frequency, defined as the relative electron spin precession with respect to its velocity, comes from radiative corrections and it becomes proportional to $g-2$ in the metric case. This frequency shows an explicit dependence on both the preferred frame velocity and its relative direction with respect to the external magnetic field. There is also a dependence on the electron velocity, which makes the other contributions negligible at relativistic electron energies. Two $g-2$ experiments on the electron (one at relativistic energies and the other almost at rest) may then be used to limit the preferred frame parameter to be no larger than 10^{-5}, analogous to the work of Newman *et al.*. Constraining any possible EEP violation to be no larger than the present discrepancy between theory and experiment we found the most stringent bounds for ξ yet obtained for leptonic matter, as given in (4.44) and (4.45).

We expect that new experiments which probe the anisotropic character (or angular dependence) of the frequency could be used to impose stronger limits in different physical regimes. For example, as the Earth rotates, the spatial orientation of the magnetic field changes – this should in turn diminish the experimental errors involved in the comparison between two energetically different $g-2$ experiments.

The relativistic generalization of the spin polarization equation (4.37), followed the same procedure as for the metric case, where non-metric effects where included in the interaction only (Eq. (4.30)). This yields an equation of motion for the spin (as seen from the rest frame) which is qualitatively different from that expected from its classical counterpart, where the angular momentum rate is related to the torque applied on the system. This approach for dealing with violations of Lorentz invariance is dynamical; from a kinematical viewpoint we assume that standard Lorentz transformations relate coordinates and fields from one system to another.

Perhaps the most remarkable feature of the non-metric effects is that of the oscillations of the component of spin polarization parallel to the magnetic field. Since this component remains constant in the metric case, an experiment which searches for such oscillations is a new null test of the equivalence principle that is uniquely sensitive to radiative corrections in the leptonic sector. Hence an empirical investigation of its behavior will provide qualitatively new information about the validity of EEP, and could constrain even further the limits on the preferred frame parameters.

Finally, we analyzed the behavior of the anomalous frequency in the context of redshift experiments, which can put constraints on the LPI-violating parameters (Γ_0, Λ_0) once the corresponding experiments are carried out. This region of parameter space is qualitatively different from that probed by either Lamb-shift or hyperfine effects. In the electron sector a bound on the magnitude of $U\Xi^a$ can be obtained by demanding that it be no larger than the error bounds in the discrepancy between the experimental and theoretical values of the ratio $a = \Omega_a/\Omega_c$. Assuming the local potential to be as large as that estimated from the local supercluster, we obtain a bound on $|\Xi^a|$ that is comparable to the limit on an analogous quantity in the baryonic sector obtained from redshift experiments [7]. However this latter experiment is proportional to changes in the local potential, which are $\sim 10^{-10}$. More direct limits on $|\Xi^a|$ must be set by performing a similar sort of redshift experiment on anomalous magnetic moments [50]. The logistics and higher precision demanded by such an experiment will be a major challenge to undertake.

Chapter 5

Concluding Remarks

In summary, we studied for the first time the validity of the EEP in the realm of quantum field fluctuations. We reformulated Quantum Electrodynamics in the context of non-metric theories of gravity, which involved the development of an approach that led to a consistent, regularized and renormalized quantum field theory. We used perturbation methods (loop counting) to calculate the relevant radiative corrections, and derived the corresponding Feynman rules for bound systems and free scattering. Finally we made the empirical connections via the interpretation of present data and the design and assessment of future experiments.

We find that a non-metric spacetime structure induces qualitatively new effects in the behavior of radiative corrections that leave distinctive physical signatures. Such effects allow the possibility of setting new bounds on the validity of the EEP. In fact from present experiments, we obtain the most stringent bound yet noted for the non-metric parameters related to leptonic matter. A summary of those constraints is presented in table 5.1. Recall that the relative gravitational coupling between massive particles and photons is measured by $\xi = 1 - c_*^2/c_0^2$, where c_0 and

c_* are the limiting speed and speed of light respectively. The remaining parameters (c.f. (3.43) and (3.49)) account for the differences between the local variations of the metric between nucleons and/or electrons, and photons. Note that the stringent limits on universality violation set by previous experiments have only been with regards to the relative gravitational coupling in the baryon/photon sector of the standard model. For the leptonic sector relevant to our consideration, relatively little was known.

In addition, we set the proper grounds to perform future experiments which could greatly improve our empirical knowledge of EEP or else refute its validity. In this regard, it is important to note that almost all the attempts at unifying gravity with the other interactions predict the existence of a new long-range, macroscopically coupled interactions appearing as auxiliary fields of gravitation [12]. Indeed this is the case in string theory where gravity always appears accompanied by a scalar field (the dilaton) [13].

Up to now, the main observable consequences of the EEP have been verified with high precision by all existing experiments. However, as stated already by Damour [12], the fact that present tests are at the 10^{-12} level does not diminish the possibility of small violations of the equivalence principle because there exist string-inspired models in which one gets, in a non fine-tuned way, violations of the universality of the free fall at the level of $\eta \sim 10^{-18}\epsilon$, where ϵ is a dimensionless quantity which could be of order unity [51]

The condition of "metricity", or "universality of the gravitational coupling" is an *ad hoc* assumption of the theory, and *not a natural consequence of an extended formalism*. In fact, nearly all the new interactions that naturally appear in extensions of the present framework of physics violate the equivalence principle [12]. In view of this, it is important to continue improving the precision of the experiments probing the EEP. Indeed the project of a Satellite Test of the Equivalence Principle (STEP) aims at probing the universality of the free fall of pairs of test masses orbiting the Earth at the impressive level of precision $\eta \sim 10^{-17}$, and there are plans for flying very stable clocks near the Sun to improve the testing of the gravitational redshift down to the 10^{-6} fractional level (see [52] and references therein). Following that direction, we expect that the intrinsically quantum-mechanical character of the radiative corrections will motivate the development of new LPI/LLI experiments based on the Lamb shift transition and anomalous magnetic moments. In so doing we will extend our understanding of the validity of the equivalence principle into the regime of quantum-field theory.

Non-metric	Baryons	Leptons			
parameters	Nucleons	e^-	$e^- - e^+$	μ^-	$\mu^- - \mu^+$
ξ	–	3.5×10^{-8}	10^{-9}	10^{-8}	10^{-8}
	6×10^{-21} [9]	10^{-7} [47]	–	–	

	Nucleons	Electrons
Γ	2×10^{-10} [6]	10^{-3} [47]
Λ	3×10^{-6} [6]	–
$\lvert 3\Gamma_B - \Lambda_B + \Delta \rvert < 2 \times 10^{-4}$ [47]		$\lvert \Gamma_e + 7\Lambda_e \rvert < 6 \times 10^{-3}$

Table 5.1: Comparison of the constraints (upper limits) for the $TH\epsilon\mu$ parameters obtained from this thesis (radiative corrections) and from other experiments as indicated.

Appendix A

Lamb Shift Energy

A.1 Semi-Relativistic Calculation of Hydrogenic Energy Levels

Consider a hydrogenic atom immersed in an external gravitational field, moving with velocity $\vec{u}$ relative to the preferred frame. In Sec. III we follow a fully relativistic approach to solve for the atomic energy levels. That is we perturbatively solve the Dirac equation in the presence of the electromagnetic field of the nucleus, where the unperturbed states correspond to the Dirac solution in the presence of a Coulomb potential only (the metric case).

We consider here the use of the Foldy-Wouthuysen transformation in solving (3.1). In this approach, we write

$$H = H_c + H_{mag} + H_{mv} + H_{SO} + H_D \tag{A.1}$$

with

$$H_c = m + \frac{\vec{p}^2}{2m} - eA_0$$

$$\begin{aligned} H_{mag} &= \frac{e}{2m}(\vec{p}\cdot\vec{A}+\vec{A}\cdot\vec{p})+\frac{e}{2m}\vec{\sigma}\cdot\vec{B} \\ H_{mv} &= -\frac{\vec{p}^4}{8m^3} \\ H_{SO} &= \frac{ie}{8m^2}\vec{\sigma}\cdot\vec{\nabla}\times\vec{E}+\frac{e}{4m^2}\vec{\sigma}\cdot\vec{E}\times\vec{p} \\ H_D &= \frac{e}{8m^2}\vec{\nabla}\cdot\vec{E} \end{aligned} \tag{A.2}$$

where A_μ is given by Eq. (2.6).

As shown in section III, we can take $H_{mag} \to 0$, since the magnetic field does not contribute to the atomic energy levels. We can then group the terms in the Hamiltonian as

$$\begin{aligned} H &= H_c + H_f \\ H_f &= H_{mv} + H_{SO} + H_D \end{aligned} \tag{A.3}$$

where we have defined the fine contribution to the Hamiltonian (H_f), in order to account for the first relativistic correction $O((Z\alpha)^4)$ to the atomic energy levels.

We start writing a formal solution for $H|n\rangle = E_n|n\rangle$, in terms of its non-relativistic limit:

$$H_c|n\rangle_c = E_n^c|n\rangle_c, \tag{A.4}$$

as

$$|n\rangle = |n\rangle_c + |n\rangle_f, \qquad E_n = E_n^c + {}_c\langle n|H_f|n\rangle_c \tag{A.5}$$

where the index "*f*" accounts for the first relativistic correction to the states and energies.

Since $A_0 = \phi + \xi\phi'$, and so $H_c = H_c^0 + \xi H_c'$, we do not know the exact solution for (A.4), but only the perturbative expansion:

$$|n\rangle_c = |n\rangle_c{}^0 + \xi|n\rangle_c' \qquad E_n{}^c = E_n{}^{c(0)} +_c^0 \langle n|H_c'|n\rangle_c^0 \tag{A.6}$$

where

$$H_c^0|n\rangle_c^0 = (m + \frac{\vec{p}^2}{2m} - e\phi)|n\rangle_c^0 = E_n^{c(0)}|n\rangle_c^0 \tag{A.7}$$

If we use (A.6) along with $H_f = H_f^0 + \xi H_f'$ in (A.5), we can finally write up to $O(\xi)$,

$$\begin{aligned} E_n &= E_n^0 + \xi E_n' = {}_c^0\langle n| \left(H_c^0 + H_f^0\right) |n\rangle_c^0 \\ &+ \xi \left[{}_c^0\langle n| \left(H_c' + H_f'\right) |n\rangle_c^0 + \left\{ {}_c^0\langle n|H_f^0|n\rangle_c' + \text{h.c.}\right\}\right] + O((Z\alpha)^6) \end{aligned} \tag{A.8}$$

We see then that under this semi-relativistic approach, we must address the problem of finding the states $|n\rangle_c'$, whose contribution to (A.8) is between the brace brackets. This is equivalent to including the first relativistic correction which comes after solving

$$H^0|n\rangle^0 = (H_c^0 + H_f^0 + \cdots)|n\rangle^0 \tag{A.9}$$

as

$$|n\rangle^0 = |n\rangle_c^0 + |n\rangle_f^0 + \cdots, \tag{A.10}$$

since, we can show

$$\left\{ {}_c^0\langle n|H_f^0|n\rangle_c' + \text{h.c.}\right\} = \left\{ {}_f^0\langle n|H_c'|n\rangle_c^0 + \text{h.c.}\right\} \tag{A.11}$$

This relation allows us to rewrite part of (A.8) as

$$\begin{aligned} E_n' &= \left({}_c^0\langle n| + {}_f^0\langle n| + \cdots\right) \left(H_c' + H_f' + \cdots\right) \left(|n\rangle_c^0 + |n\rangle_f^0 + \cdots\right) \\ &= {}^0\langle n|H'|n\rangle^0. \end{aligned} \tag{A.12}$$

It is clear then that if we start with the exact solution for the Dirac equation in the presence of a Coulomb potential, we can avoid working with the states $|n\rangle_c'$. Note that since we are interested only in the first relativistic correction, the result (A.12) must be expanded to $O((Z\alpha)^4)$.

Unfortunately for hyperfine or Lamb shift energies, the effect of the primed states cannot be removed, since they both come from perturbations to the (known) relativistic solution of the Dirac equation in the presence of a Coulomb potential only.

A semi-relativistic expression for the Hamiltonian of a hydrogenic system was worked out in Ref. [29], where the effects of nuclear spin (hyperfine effect) were also included within the context of LLI violations. The result presented there for the atomic energy levels is incomplete though, since the contribution of the prime states was overlooked, as discussed at the end of Sec. III.

A.2 Loop calculations

Given the form of the photon propagator (2.13), it is convenient to divide the calculation into two parts

$$\delta E_S = \delta E_S^{(A)} + \delta E_S^{(B)} \tag{A.13}$$

where $\delta E_S^{(A)}$ groups the contributions of the terms proportional to $\eta_{\mu\nu}$ in $G_{\mu\nu}$, whereas $\delta E_S^{(B)}$ contains those proportional to $\gamma^2 = 1/(1-\vec{u}^2)$ and ξ. We are interested in solving for the shift in energy levels up to first order in ξ, so it is enough to consider a Coulomb potential as the source for part B, while for part A the full source as defined in Eq. (2.6) needs to be included.

We mention again that we are interested in calculating the GM Lamb shift to lowest nontrivial order in α, *i.e.* up to $O(\alpha(Z\alpha)^4)$. To this order, we can use the nonrelativistic expressions for both the large and small component of the electron spinor ψ. So for example, if we make the substitution

$$\psi(\vec{p}) = (Z\alpha m)^{-3/2} w(\vec{t}), \tag{A.14}$$

where $w(\vec{t})$ is a dimensionless spinor whose first two components are of order unity, and the last two are of order $Z\alpha$, we can assign orders to the various terms according to

$$\begin{aligned} p_i &\sim Z\alpha m, \quad E_0 - m \sim (Z\alpha)^2 m \\ eA_0\, d^3p' &\sim eA_i\, d^3p' \sim (Z\alpha)^2 m \\ \overline{\psi}\gamma_i \psi_n d^3p &\sim Z\alpha m. \end{aligned} \tag{A.15}$$

These approximations will be used in the sequel to simplify the expressions we obtain.

A.2.1 Type A Contributions to the Self-energy

Here we will consider

$$G^{(A)}_{\mu\nu} = -\frac{\eta_{\mu\nu}}{k^2}(1+\xi) \tag{A.16}$$

and $\not{V} = -eA_\mu\gamma^\mu$, with A_μ given by Eq.(2.6). This part of the calculation is almost identical to that of BBF [38]; the only difference is that now we have to consider a source that contains a magnetic part in addition to the electric one.

We begin by computing δE_1. Relating the counterterm δC to the renormalization of the electron mass and regularizing the photon propagator via

$$\frac{1}{k^2} \to -\int_{\mu^2}^{\Lambda^2} \frac{dL}{(k^2-L)^2}. \tag{A.17}$$

we find that I_2 and I_3 in (3.27) become

$$\begin{aligned} I_2 &= (1+\xi) \quad \not{V}\{\frac{1}{2}\ln(p^2/\mu^2) - \ln(\Lambda^2/p^2)\} \\ I_3 &= (1+\xi) \quad \frac{3}{4}\{\not{V}(\ln(\Lambda^2/p^2) + \frac{1}{2}) + m\ln(m^2/p^2)\} \quad . \end{aligned} \tag{A.18}$$

On the other hand, we obtain for I_1

$$\begin{aligned}
I_1 &= (1+\xi)\Big\{-\frac{3}{8}\,\not{V} - \frac{1}{2}p'\cdot p\,\not{V}\int_0^1 \frac{dx}{p_x^2}\ln(p_x^2/\mu^2) + \frac{1}{4}\,\not{V}\int_0^1 dx\,\ln(\Lambda^2/p_x^2) \\
&+ \frac{1}{2}\int_0^1 \frac{dx}{p_x^2}\{(1-x)p^2 + xp'^2 + 2p\cdot'\,\not{V} + \not{p}'\,\not{V}\,\not{p} \\
&- 2V\cdot p'(1-x)\,\not{p} - 2V\cdot px\,\not{p}' + V\cdot p_x\,\not{p}_x\}\Big\},
\end{aligned} \tag{A.19}$$

where $p_x = xp' + (1-x)p$.

We can simplify this expression by letting the momentum operators $\not{p}'$ and $\not{p}$ respectively act on the spinors $\overline{\psi}(\vec{p}\,')$ and $\psi(\vec{p})$, using the Dirac equation and (A.15) to keep terms up to the desired order.

Adding together I_1, I_2, and I_3 we obtain a result correct to order $\alpha(Z\alpha)^4$:

$$\begin{aligned}
\delta E_1^{(A)} &= \frac{\alpha}{\pi}(1+\xi)\int \overline{\psi}(\vec{p}\,')\left[\not{V}\frac{q^2}{m^2}\left[\frac{1}{3}\ln(\frac{m}{\mu}) - \frac{1}{8}\right] + \frac{i}{4m}q_\nu\sigma^{\mu\nu}V_\mu\right]\psi(\vec{p})d^3p'd^3p \\
&- \frac{\alpha}{\pi}(1+\xi)\langle n|\frac{-3V_0^2 + 5\vec{V}^2}{4m}|n\rangle,
\end{aligned} \tag{A.20}$$

with $q = p' - p$, and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. Note that the term proportional to q^2 in Eq. (A.20) needs to be evaluated with only the large component of ψ and $\not{V} \simeq V_0$ ($\gamma_0 \sim 1$).

We point out that the initial ultraviolet divergence in (A.18) is cancelled after the addition of the I's in (A.20). The remaining infrared divergence will be cancelled by a similar term which comes from the many-potential part of the level shift. A similar cancelation occurs in the non-gauge invariant term present in Eq. (A.20). These cancelations are non-trivial, and provide useful cross checks to our calculation.

Consider next the evaluation of δE_2. Since the operator M_μ satisfies the transversality condition

$$k\cdot M = k\cdot M^\dagger = 0 \tag{A.21}$$

we can write $M_0 = \vec{k}\cdot\vec{M}/k_0$.

Using

$$\not{V}\not{k}\gamma_\mu = 2V\cdot k\gamma_\mu - 2V_\mu\not{k} + \not{k}\gamma_\mu\not{V}, \tag{A.22}$$

in the first term of Eq.(3.30) the operator $M_\mu^\dagger$can be decomposed into

$$M_\mu{}^\dagger = M_\mu{}^{\dagger I} + M_\mu{}^{\dagger II} \tag{A.23}$$

with

$$\begin{aligned} M_\mu^{\dagger I} &= \{\frac{2p_\mu}{k^2 - 2p\cdot k} - \frac{2p'_\mu}{k^2 - 2p'\cdot k} \qquad \text{(A.24)} \\ &+ \not{k}\gamma_\mu(\frac{1}{k^2 - 2p'\cdot k} - \frac{1}{k^2 - 2p\cdot k})\}\not{V} \\ M_\mu^{\dagger II} &= 2(V_\mu\not{k} - V\cdot k\gamma_\mu)/(k^2 - 2p\cdot k), \qquad \text{(A.25)} \end{aligned}$$

each of which still satisfies

$$M^{\dagger I}\cdot k = M^{\dagger II}\cdot k = 0. \tag{A.26}$$

In terms of these operators we now have

$$\delta E_2 = \langle M^I K_+^V M^I\rangle + \langle M^{II} K_+^V M^{II}\rangle + \langle M^I K_+^V M^{II}\rangle + \langle M^{II} K_+^V M^I\rangle, \tag{A.27}$$

where each term represents a contribution to Eq. (3.29) involving the products of only M^I or M^{II} or cross terms operators. The simplification of these terms is quite analogous to that shown in BBF [38]. The decomposition of the M operator in (A.23) allows one to use simpler expressions for the bound propagator K_+^V. In appendix A.2.5 it is shown that only in the part $\langle M^I K_+^V M^I\rangle$ will it be necessary to use the bound electron propagator; in all other contributions it is sufficient to replace K_+^V by the propagator for free electrons, K_+^0. Moreover the main contribution to $\langle M^I K_+^V M^I\rangle$ arises from intermediate states of the electron with nonrelativistic energy so that both K_+^V and M^I can be replaced by their simpler nonrelativistic approximations. It is also shown that the cross term in Eq. (A.27) gives a contribution of order $\alpha(Z\alpha)^5$ and is therefore not relevant in our calculation. According to the above considerations we can then approximate Eq. (A.27) by

$$\delta E_2 \simeq \langle M_{NR}^I K_{NR}^V M_{NR}^I \rangle + \langle M^{II} K_+^0 M^{II} \rangle \equiv \langle M^I \rangle + \langle M^{II} \rangle. \tag{A.28}$$

We start evaluating the first term of Eq. (A.28). The nonrelativistic prescription for K_+^V is given by

$$K_{NR}^V(x',x) = \begin{cases} \sum_r \varphi_r(\vec{x}')\varphi_r^*(\vec{x}) \exp(-iE_r(t'-t)) & \text{for } (t'-t) > 0 \\ 0 & \text{for } (t'-t) < 0 \end{cases} \tag{A.29}$$

or in momentum space

$$K_{NR}^V(E_n - k_0; \vec{p}', \vec{p}) = -i \sum_r \varphi_r(\vec{p}')\varphi_r^*(\vec{p})(E_r - E_n + k_0)^{-1} \tag{A.30}$$

where φ_r represents the large component of the Dirac spinor.

In the same nonrelativistic approach M_μ^I reduces to

$$M_\mu^{I(NR)} \simeq (p'_\mu - p_\mu)\frac{V_0(\vec{p}' - \vec{p})}{mk_0} \equiv R_\mu, \tag{A.31}$$

where we have approximated $\not{V} \simeq V_0$, because although the magnetic and electric potential have the same order of magnitude (as powers of $Z\alpha$), the $\vec{\gamma}$ matrix mixes large components of the intermediate states with small ones and therefore introduces corrections one order higher in $Z\alpha$.

Therefore, after replacing Eq. (A.30) and (A.31) in Eq. (3.29) we obtain

$$\langle M^I \rangle = \frac{\alpha}{4\pi^3 i} \int d^4k\, G^{\mu\nu}(k) \times \sum_r \frac{\langle n|R_\mu|r\rangle\langle r|R_\nu|n\rangle}{k_0 - E_n - E_r} \tag{A.32}$$

where we have neglected the contribution of the photon momentum k to the momentum of the intermediate electron states. This is equivalent to leaving out the factor $\exp(i\vec{k}\cdot\vec{x})$ in the spatial integration. This can be done because $k \sim E_n - E_r \sim m(Z\alpha)^2$, which is small compared with the electron momentum $\vec{p} \sim mZ\alpha$ for nonrelativistic states.

Inserting (A.16) into (A.32), and using Eq. (A.26) to relate the temporal component of R with its spatial components, which satisfy

$$\langle n|\vec{R}|r\rangle = \frac{-1}{mk_0}(E_n - E_r)\langle n|\vec{p}|r\rangle, \tag{A.33}$$

we find, after integration

$$\langle M^I\rangle = \frac{2\alpha}{3\pi m^2}(1+\xi)\sum_r |\langle n|\vec{p}|r\rangle|^2(E_r - E_n)\left[\ln(\frac{\mu}{2|E_n - E_r|}) + \frac{5}{6}\right] \tag{A.34}$$

where all the states and energies represent the non relativistic limit of the Dirac solution.

Eq.(A.34) can be simplified by using

$$\sum_r |\langle n|\vec{p}|r\rangle|^2(E_r - E_n) = \frac{1}{2}\langle n|\nabla^2 V_0|n\rangle, \tag{A.35}$$

which finally gives

$$\langle M^I\rangle = \frac{\alpha}{3\pi m^2}(1+\xi)\left[\left(\ln(\frac{\mu}{2E_*}) + \frac{5}{6}\right)\langle n|\nabla^2 V_0|n\rangle + \hat{C}\right] \tag{A.36}$$

with

$$\hat{C} \equiv \hat{C}_{ii}, \qquad \hat{C}_{ij} = 2\sum_r \langle r|p_i|n\rangle\langle n|p_j|r\rangle(E_r - E_n)\ln|\frac{E_*}{E_n - E_r}| \tag{A.37}$$

where E_* is a reference energy to be defined, and $\hat{C}_{ij}$ has been introduced for later convenience. To obtain this result we have neglected the imaginary part of $\langle M^I\rangle$ retaining only the leading terms of $\langle M^I\rangle$ in the limit $\mu \to 0$.

In computing $\langle M^{II}\rangle$, we can take K_+^V to be the free electron propagator, which is

$$K_+^V(E_n - k_0; \vec{p}\,' - \vec{s}\,' - \vec{k}, \vec{p} + \vec{s} - \vec{k}) = \frac{i\delta^3(\vec{s}\,' - \vec{p}\,' + \vec{p} + \vec{s})}{\not r - \not k - m}, \qquad \text{(A.38)}$$

where

$$r^\mu = (m, \vec{s}_*), \qquad \vec{s}_* = \vec{p}\,' - \vec{s}\,' = \vec{p} + \vec{s} \qquad \text{(A.39)}$$

upon which $\langle M^{II}\rangle$ becomes

$$\langle M^{II}\rangle = \frac{\alpha}{\pi}\int d^3p' d^3p\, d^3s_* \overline{\psi}(\vec{p}\,')V_\alpha(\vec{p}\,' - \vec{s}_*)N^\alpha_\beta(p_*, s_*)V^\beta(\vec{s}_* - \vec{p})\psi_n(\vec{p}), \qquad \text{(A.40)}$$

with

$$N^\alpha_\beta(p_*, s_*) = -\frac{4}{i}\int \frac{(\eta^{\alpha\mu}\not k - k^\alpha\gamma^\mu)(\not r - \not k + m)(\eta^\nu_\beta \not k - k_\beta\gamma^\nu)}{(k^2 - 2p_*\cdot k)^2(k^2 - 2r\cdot k - \vec{s}_*^{\,2})}G_{\mu\nu}(k)d^4k. \qquad \text{(A.41)}$$

In the nonrelativistic domain $\int d^3pV_\alpha \approx (Z\alpha)^2m$ and so the constant value of N^β_α (independent of the momentum and energy of the intermediate states) will already yield an overall contribution to Eq. (A.40) of the desired order $\alpha(Z\alpha)^4$. Note that N^β_α can be expanded in powers of the momentum $\vec{p}\,'$, $\vec{p}$ or $\vec{s}_*$, which are of order $mZ\alpha$, and therefore any contribution beyond the constant, $Z\alpha$-independent term will be of higher order. The same argument can be used to neglect the binding energy of the intermediate states. We can therefore evaluate (A.41) by approximating $p \sim p_*$ and $p' \sim p_*$ in the denominator of $M^{I\dagger}$ and M^I respectively, so that $p_* \approx (m, 0)$ and $s_* \approx 0$.

Evaluating N as in reference [38] we find that (A.40) becomes

$$\langle M^{II}\rangle = \frac{\alpha}{\pi}(1+\xi)\langle n|\frac{-3V_0^2 + 5\vec{V}^2}{4m}|n\rangle. \qquad \text{(A.42)}$$

Note that this term will exactly cancel the non-gauge invariant term present in Eq. (A.20).

Finally we add Eq.(A.36) to Eq.(A.42) to obtain $\delta E_2^{(A)}$, and then add it to Eq. (A.20) to give the final result for the type-A contribution to the self-energy:

$$\begin{aligned} \delta E_S^{(A)} &= \frac{\alpha}{3\pi m^2}(1+\xi)\Big[\hat{C} + \Big(\ln(\frac{m}{2E_*}) + \frac{11}{24}\Big)\langle n|\nabla^2 V_0|n\rangle \\ &+ \frac{3}{4}m\int \overline{\psi}(\vec{p}')i\sigma_{\mu\nu}V^\mu q^\nu \psi_n(\vec{p})d^3p'd^3p\Big]. \end{aligned} \quad \text{(A.43)}$$

Apart from the constant $(1+\xi)$ factor, there is no formal difference between the result (A.43) for this contribution to the level shift and the standard one [38]. However there are implicit differences which appear in the expression for V^μ and the solution for the Dirac states $|n\rangle$ (in the non-relativistic approach here) in the presence of that source.

A.2.2 Type B Contributions to the Self-energy

To solve the type-B contributions we have to consider the photon propagator

$$G_{\mu\nu}^{(B)} = \xi\frac{\gamma^2}{k^2}\left[\beta_\mu\beta_\nu + \eta_{\mu\nu}\frac{(\beta\cdot k)^2}{k^2}\right] \quad \text{(A.44)}$$

and a source $A_\mu \simeq \eta_{\mu 0}\phi$.

The evaluation of $\delta E_S^{(B)}$ is achieved by the same procedure as for part A, where now we use Eq. (A.44) in (3.27) and (3.29) to solve for $\delta E_1^{(B)}$ and $\delta E_2^{(B)}$ respectively. This computation is somewhat more laborious than that in part A, due to the $\beta_\mu\beta_\nu$ tensorial dependence and the factor $\frac{(\beta\cdot k)^2}{k^2}$ present in this part of the (GM) photon propagator.

To evaluate I_1, I_2, and I_3 we need to modify the BBF technique by using (A.17) along with

$$\frac{1}{k^4} \to -2\int_{\mu^2}^{\Lambda^2}\frac{dL}{(k^2-L)^3}. \quad \text{(A.45)}$$

to regulate (A.44). The expressions for the I's are somewhat more complicated than those for $\delta E_S^{(A)}$ (as expected); but their manipulation and further algebra follow from BBF [38]. The relevant details are in appendix A.2.6; the result for the one potential part is

$$\begin{aligned}
\delta E_1^{(B)} &= \frac{\alpha}{3\pi m^2}\gamma^2\xi\int\overline{\psi}(\vec{p}')\Big\{\ \not{V}q^2\left[\frac{17}{48}\beta^2-\frac{5}{4}+(\frac{\beta^2}{2}-1)\ln(\frac{\mu}{m})\right] \\
&+ \not{V}(\beta\cdot q)^2\left[\frac{5}{6}+\ln(\frac{\mu}{m})\right] \qquad\qquad (A.46)\\
&+ (\frac{\beta\cdot p}{2}\not{V}-\beta\cdot Vm)i\sigma^{ij}u_iq_j-m\beta\cdot qi\sigma^{\mu\nu}V_\mu\beta_\nu \\
&+ m(\frac{\beta^2}{8}-\frac{1}{2})i\sigma^{\mu\nu}V_\mu q_\nu\Big\}\psi(\vec{p})d^3p'd^3p \\
&- \frac{\alpha}{\pi}\gamma^2\xi(1+\frac{7}{8}\beta^2)\langle n|\frac{V_0^2}{3m}|n\rangle
\end{aligned}$$

which is good up to order $\alpha(Z\alpha)^4$, and we have retained only the leading terms as $\mu\to 0$.

The evaluation of $\delta E_2^{(B)}$ is quite analogous to that for $\delta E_2^{(A)}$. The starting point is Eq. (A.28), where $\langle M^I\rangle$ and $\langle M^{II}\rangle$ are still defined by (A.32) and (A.40) respectively. We give calculational details in appendix A.2.6, and quote here only the final result:

$$\begin{aligned}
\delta E_2^{(B)} &= \frac{\alpha}{3\pi m^2}\gamma^2\xi\Big\{\left[\frac{5}{12}\vec{u}^2-\frac{1}{12}+(\frac{1}{2}+\frac{\vec{u}^2}{2})\ln(\frac{\mu}{2E_*})\right]\langle n|\nabla^2V_0|n\rangle \\
&+ \left[\frac{5}{6}+\ln(\frac{\mu}{2E_*})\right]\langle n|(\vec{u}\cdot\vec{\nabla})^2V_0|n\rangle+u_iu_j\hat{C}_{ij}+(\frac{1}{2}+\frac{\vec{u}^2}{2})\hat{C}\Big\} \\
&+ \frac{\alpha}{\pi}\gamma^2\xi(1+\frac{7}{8}\beta^2)\langle n|\frac{V_0^2}{3m}|n\rangle \qquad\qquad (A.47)
\end{aligned}$$

We now add (A.46) to (A.47) to obtain

$$\begin{aligned}
\delta E_S^{(B)} &= \frac{\alpha}{3\pi m^2}\xi\Big\{\left[-\frac{11}{12}\vec{u}^2-\frac{47}{48}+(\frac{1}{2}+\vec{u}^2)\ln(\frac{m}{2E_*})\right]\langle n|\nabla^2V_0|n\rangle \\
&+ \ln(\frac{m}{2E_*})\langle n|(\vec{u}\cdot\vec{\nabla})^2V_0|n\rangle+u_iu_j\hat{C}_{ij}+(\frac{1}{2}+\vec{u}^2)\hat{C} \qquad\qquad (A.48)
\end{aligned}$$

$$+ \quad \int \bar{\psi}(\vec{p}\,')\Big[\big(\frac{\beta\cdot p}{2}\not{V} - \beta\cdot Vm\big)i\sigma^{ij}u_i q_j + m\vec{u}\cdot q i\sigma^{\mu\nu}V_\mu\beta_\nu$$
$$- \quad m\big(\frac{\vec{u}^2}{2}+\frac{3}{8}\big)i\sigma^{\mu\nu}V_\mu q_\nu\Big]\psi(\vec{p})d^3p'd^3p\Big\}$$

where we approximated $\gamma^2 \simeq 1+\vec{u}^2$ in order to keep terms only up to order $\vec{u}^2$. As a cross-check on the above result we note that, before expanding γ^2, the limit $\beta_\mu\beta_\nu \to \eta_{\mu\nu}$, yields $\delta E_S^{(B)} \to -2\xi\gamma^2\delta E_S^0$. This is as expected since according to (A.44), $G_{\mu\nu}^{(B)} \to -2\xi\gamma^2 G_{\mu\nu}^0$, where $G_{\mu\nu}^0$ is the standard (metric) propagator.

We close this section with a comment on the renormalization procedure. For $\delta E_S^{(A)}$, the counterterm δC was related to mass renormalization. However in this part of the calculation we must also account for the renormalization of the $TH\epsilon\mu$ parameters, which show up as functions of the limiting speed for massive particles ($c_0^2 \equiv T_0/H_0$), and the photon velocity ($c_*^2 \equiv 1/\mu_0\epsilon_0$). Charge renormalization is not necessary here because the Ward Identity forces a cancelation between the divergences coming from the one potential part and many potential part of the self energy in the same manner as in the metric case.

A.2.3 Vacuum Polarization

We now need to obtain the vacuum polarization contribution. To the desired approximation, the electrons forming the loop in diagram 3.1(b) can be considered free. This is because Furry's theorem implies that the next-order correction to this is a diagram which contains a loop with 4 vertices, which is expected to be of order $\alpha(Z\alpha)^6$. In that case the result is known to be

$$\delta E_P = \int \bar{\psi}(\vec{p}\,')i\Pi^{\mu\nu}(q)iG_{\nu\sigma}(q)\gamma^\sigma V_\mu(\vec{q})\psi(\vec{p})d^3p'd^3p, \tag{A.49}$$

The evaluation of $\Pi^{\mu\nu}$ is identical to the standard (metric) case, since it only involves the product of fermion propagators, which are unchanged by the $TH\epsilon\mu$ action. The differences appear in the renormalization process, where both the charge and the $TH\epsilon\mu$ parameters must be renormalized. This procedure follows from condition (2.18), which introduces the appropriate counterterm needed to renormalize the $TH\epsilon\mu$ parameters. Subsequent analysis is similar to the metric case, and the renormalized solution for the vacuum polarization turns to be

$$\Pi^{\mu\nu}(q) \simeq -\frac{\alpha}{15\pi}\frac{q^2}{m^2}(q^2\eta^{\mu\nu} - q^\mu q^\nu) \tag{A.50}$$

If we substitute Eqs. (2.13) and (A.50) in (A.49), we obtain after some manipulation

$$\delta E_P = \frac{\alpha}{3\pi m^2}\Big\{\langle n|\nabla^2 V_0|n\rangle\Big(-\frac{1}{5}+\xi\frac{\vec{u}^2}{5}\Big) - \frac{\xi}{5}\langle n|(\vec{u}\cdot\vec{\nabla})^2 V_0)|n\rangle\Big\} \tag{A.51}$$

We next proceed to add together the self energy and vacuum polarization contributions to the level shift.

A.2.4 The total GM Radiative Correction

Up to this point we have been able to solve the level shift in terms of

$$\delta E_n = \delta E_S^{(A)} + \delta E_S^{(B)} + \delta E_P \tag{A.52}$$

where each term has been defined in Eqs. (A.43), (A.48) and (A.51).

We note that in δE_S there are terms proportional to $\vec{\gamma}$, which mix large (φ) and small component (χ) of ψ. Within the accuracy required we can relate them by $\chi = -i\frac{\vec{\sigma}\cdot\vec{\nabla}}{2m}\varphi$, and so write everything in terms of the large component only.

Replacing the expression for the external source (2.6) in (A.52), we obtain after some algebra

$$\delta E_n = \frac{\alpha}{3\pi m^2}\Big[\Big(1+\xi(\frac{3}{2}+\vec{u}^2)\Big)\hat{C} + \xi u_i u_j \hat{C}_{ij} + \langle n|\hat{E}|n\rangle\Big] \tag{A.53}$$

where $\hat{C}$ and $\hat{C}_{ij}$ are defined by Eq. (A.37), and

$$\begin{aligned}
\hat{E} &= 4\pi Z\alpha\delta(\vec{x})\left[\frac{19}{30}+\ln(\frac{m}{2E_*})+\xi\big[-\frac{1}{30}-\frac{58}{45}\vec{u}^2+(\frac{3}{2}+\frac{2}{3}\vec{u}^2)\ln(\frac{m}{2E_*})\big]\right] \\
&+ 3\frac{Z\alpha}{r^3}\left[\frac{1}{4}+\xi\big[\frac{1}{8}-\frac{\vec{u}^2}{2}-(\vec{u}\cdot\hat{n})^2\big]\right]\vec{\sigma}\cdot\vec{L} \\
&- \xi\frac{Z\alpha}{r^3}[3(\vec{u}\cdot\hat{n})^2-\vec{u}^2]\left[\frac{14}{15}+2\ln(\frac{m}{2E_*})\right] \\
&+ \frac{\xi}{2}\frac{Z\alpha}{r^2}\left[\frac{7}{2}\vec{u}\cdot\hat{n}\,\vec{\sigma}\cdot(\vec{u}\times\vec{p})-\vec{\sigma}\cdot(\vec{u}\times\hat{n})\vec{u}\cdot\vec{p}\right]
\end{aligned} \quad (A.54)$$

We have omitted operators with odd parity (such as $\vec{u}\times\hat{n}\cdot\vec{\sigma}$) in (A.54), since their expectation values vanish for states of definite parity.

There is still an implicit dependence on ξ and $\vec{u}$ in (A.53), which comes from the Dirac states (as seen at the end of Sec. III). Note that up to this order all atomic states and energies referred in Eqs. (A.53) and (A.37) are considered within a non relativistic approach.

In terms of the formal solution for the Dirac equation (3.3), we can single out the complete ξ dependence in (A.53), and write

$$\delta E_n = \frac{\alpha}{3\pi m^2}\left\{\left(1+\xi(\frac{3}{2}+\vec{u}^2)\right)\hat{C}^0+\xi u_i u_j\hat{E}_{ij}+{}^0\langle n|\hat{E}|n\rangle^0\right\} \quad (A.55)$$

with

$$u_i u_j\hat{E}_{ij} = u_i u_j\hat{C}_{ij}+\hat{C}'+({}^0\langle n|\hat{E}_{\xi=0}|n\rangle'+\text{h.c.}) \quad (A.56)$$

where $\hat{C}'$ groups all the terms in Eq. (A.37) depending on the perturbative states ($|n\rangle'$) or energies (E_n') as introduced in Eq. (3.3). These perturbative states are needed not only for the $|n\rangle$ state related to the level shift, but for all the intermediate states introduced by (A.37)as well. Eq. (A.55) is valid up to $O(\xi)O(\vec{u}^2)O(\alpha(Z\alpha)^4)$.

We can define the reference energy E_* as in the metric case by [53]

$$\begin{aligned} \ln(E_*^{n0}) &= \frac{\sum_r |\langle r|\vec{p}|n\rangle|^2 (E_r - E_n) \ln|E_r - E_n|}{\sum_r |\langle r|\vec{p}|n\rangle|^2 (E_r - E_n)} && \text{for } l = 0 \\ 2\frac{m^3}{n^3}(Z\alpha)^4 \ln(\frac{Z^2 Ryd}{E_*^{nl}}) &= \sum_r |\langle r|\vec{p}|n\rangle|^2 (E_r - E_n) \ln|\frac{1}{E_r - E_n}| && \text{for } l \neq 0 \end{aligned} \quad (A.57)$$

where the subscript 0 has been omitted in the energies and states. This definition reduces

$$\hat{C}^0 = \begin{cases} 0 & \text{for } l = 0 \\ 4\frac{m^3}{n^3}(Z\alpha)^4 \ln(\frac{Z^2 Ryd}{E_*^{nl}}) & \text{for } l \neq 0 \end{cases} \quad (A.58)$$

which provides an elegant way to write the "Bethe-sum". The presence of preferred frame effects will induce more "Bethe-sum"–like terms in $\hat{C}_{ij}$ which, along with the contribution from the perturbative states (both ones counted by $\delta\hat{E}_{ij}$) will have to be evaluated numerically for any particular state.

For the Lamb shift states we can use [53] :

$$E_*^{2S} = 16.640\,\text{Ryd} \qquad E_*^{2P} = 0.9704\,\text{Ryd} \quad (A.59)$$

and simplify the last term in Eq. (A.55) as

$$\begin{aligned} {}^0\langle\hat{E}\rangle^0_{2S_{1/2}} &= \frac{(Z\alpha)^4}{2} m^3 \left\{ \frac{19}{30} + \ln(\frac{m}{2E_*^{2S}}) \right. \\ &- \left. \xi\Big[\frac{1}{3} + \frac{58}{45}\vec{u}^2 - (\frac{3}{2} + \frac{2}{3}\vec{u}^2)\ln(\frac{m}{2E_*^{2S}})\Big] \right\} \\ {}^0\langle\hat{E}\rangle^0_{2P_{1/2}} &= \frac{(Z\alpha)^4}{2} m^3 \left\{ -\frac{3}{24} - \frac{\xi}{12}\left[\frac{3}{4} \right.\right. \\ &- \left.\left. \vec{u}^2 \left[\frac{107}{30} - \frac{1}{6\sqrt{10}} + \cos^2\theta \left(\frac{1}{12} + \frac{1}{6\sqrt{10}} \right) \right] \right] \right\} \end{aligned} \quad (A.60)$$

where θ represents the angle between the atom's quantization axis and the frame velocity $\vec{u}$.

A.2.5 Many potential part approximations

In this appendix we justify the following approximations:

$$\langle M^I K_+^V M^I \rangle \simeq \langle M_{NR}^I K_{NR}^V M_{NR}^I \rangle \tag{A.61}$$

$$\langle M^{II} K_+^V M^{II} \rangle \simeq \langle M^{II} K_+^0 M^{II} \rangle \tag{A.62}$$

$$\langle M^I K_+^V M^{II} \rangle \simeq O((Z\alpha)^5 \alpha) \tag{A.63}$$

following arguments similar to those presented by BBF [38].

We first note that, as powers of $Z\alpha$, the orders of magnitude of the different terms involved in the expressions in (A.61) are equivalent to those for the metric case. For example, if we look at the source, we see that each component of eA_μ is of the order of $\sim e\phi$, where A_μ is given by Eq. (2.6) and ϕ is the ordinary Coulomb potential, and so the relative order between the non-metric and metric case is the same. Furthermore, as discussed at the end of Sec. III, the states $|n\rangle$ and $|n\rangle^0$ also have the same order of magnitude, as do the quantities E_n and E_n^0. Discrepancies that could be expected from the photon propagator, particularly from the part proportional to $\beta^\mu \beta^\nu$ (in contrast to the $\eta_{\mu\nu}$ dependence for the standard case), are not important as long as the transversality condition is satisfied for the M operators, since this condition relates the differing components with the appropriate orders of magnitude. Finally, unlike the photon propagator, the bound propagator retains the same form as in the standard case, with differences arising only from the expression for the external source. As a consequence its further simplification is analogous to the metric (BBF) case.

Let us look at the many potential part. From (3.29) we get

$$\begin{aligned} \langle M K_+^V M \rangle &= \int \overline{\psi}_n(\vec{p}\,') M_\mu(p', p' - s' - k) \\ &\times K_+^V(E_n - k_0; \vec{p}\,' - \vec{s}\,' - \vec{k}, \vec{p} + \vec{s} - \vec{k}) \end{aligned} \tag{A.64}$$

$$\times \quad M^\dagger_\nu(p+s-k,p)\psi_n(\vec{p})G^{\mu\nu}(k)$$

for the generic structure of the terms on the left hand sides of (A.61)–(A.63), where the constant factors and integrations over p_i and s_i have been omitted. The nonrelativistic and relativistic regions are defined according to $|\vec{k}| \sim (Z\alpha)^2 m << m$ and $|\vec{k}| > m$, respectively. In considering the relevant orders of magnitude in each of the expressions (A.61)–(A.63) that follow from (A.64), we note that, to lowest order in $Z\alpha$, the relevant contribution from $G^{\mu\nu}$ comes when $k_0 \sim |\vec{k}|$, and that we can employ the nonrelativistic expressions for the ψ_n, making use of the approximations given by (A.15).

Turning now to the relation (A.61), we can prove it by showing that the contribution of relativistic states for M^I is of a higher order of magnitude than for M^{II}. We can see from (A.24) and (A.25) that M^I differs from M^{II} by a factor (leaving aside the temporal component) $(\vec{p}' - \vec{p})/k_0$, which in the relativistic region ($k_0 \sim m$) is of order $Z\alpha$. Therefore the contribution of M^I in that domain will be of at least one order higher than that of M^{II}. Since the latter is already of the desired order (assuming the validity of (A.62)) we can neglect the contribution of the relativistic states for M^I, and consider it, along with the bound propagator, in its nonrelativistic limit.

To prove the relation (A.62) we evaluate the error due to the neglect of the electromagnetic potential in the intermediate states. We imagine that one extra potential ($\not{V}$) acts between $M^{II\dagger}$ and M^{II}. This introduces an extra factor of order

$$\int d^3r' \frac{\not{V}(r'-r)}{\not{r}' - \not{k} - m} \sim \int d^3r' \frac{\not{V}(r'-r)}{2k_0 m}\not{k} \sim (Z\alpha)^2 \qquad \text{(A.65)}$$

which is negligible within the accuracy required. We have then shown that, in the evaluation of M^{II}, the intermediate states may indeed be regarded as free.

The relation (A.63) follows from arguments similar to those used to justify (A.61). Since in the relativistic region M^I is one order higher than M^{II}, the cross term in that region will also be one order higher than $\langle M^{II}\rangle$, and so is negligible. On the other hand in the nonrelativistic region M^I will be dominant (note the factor k_0 in its denominator) over M^{II}. That is

$$|\frac{M^{II}}{M^I}| \sim |\frac{k_0}{\vec{p}' - \vec{p}}| \sim Z\alpha \tag{A.66}$$

and so the product of these terms will be negligible in comparison with $\langle M^I\rangle$. Hence the cross terms yield results that are at least one order higher than the desired order, and so they do not need to be included.

A.2.6 Calculational Details of Type B Contributions

We present here further details underlying the computation leading to Eqs. (A.46) and (A.47), which are referred as the type-B contributions to the self energy. In this part the photon propagator to be considered is given by (A.44), where the first and second terms have respectively a tensor dependence like $\beta_\mu\beta_\nu$ and $\eta_{\mu\nu}$, and need to be regularized according to (A.17) and (A.45). We show the relevant details involving the first term of the propagator only, since the remainder can be computed in a similar way.

We begin then with the one potential part by simplifying I_1. After replacing (A.44) in (3.28), we get

$$I_1 = -\frac{i}{4\pi^2}\gamma^2\xi \int \frac{2p'\cdot\beta - \not\beta\not k}{k^2 - 2p'\cdot k} \not V \frac{2p\cdot\beta - \not k\not\beta}{k^2 - 2p\cdot k}\frac{d^4k}{(k^2 - L)^2} dL + \cdots \tag{A.67}$$

where from now on the ellipsis stands for the contributions coming from the second term of (A.44).

If we use

$$\frac{1}{abc^2} = 6\int_0^1 dx \int_0^1 \frac{z(1-z)dz}{[(ax+b(1-x))(1-z)+cz]} \tag{A.68}$$

we can rewrite Eq. (A.67) as

$$\begin{aligned} I_1 = & - 4p\cdot\beta p'\cdot\beta \not{V} J_0 + 2p\cdot\beta \not{\beta}\gamma^\mu \not{V} J_\mu \\ & + 2p'\cdot\beta \not{V}\gamma^\mu \not{\beta} J_\mu - \not{\beta}\gamma^\mu \not{V}\gamma^\nu \not{\beta} J_{\mu\nu} + \cdots \end{aligned}$$

where

$$J_{\{0;\mu;\mu\nu\}} = -\frac{3i}{2\pi^2}\gamma^2\xi \int_0^1 dx \int_0^1 z(1-z)dz \frac{dLd^4k}{[(k-p_x(1-z))^2-\Delta_L]^4}\{1;k_\mu;k_\mu k_\nu\} \tag{A.69}$$

with

$$p_x = xp' + (1-x)p \qquad \Delta_L = p_x^2(1-z)^2 + Lz \tag{A.70}$$

After evaluating (A.69), we can express

$$\begin{aligned} I_1 &= \gamma^2\xi \int \frac{dx}{p_x^2}\Big\{ \not{V}\Big[\frac{1}{2}\beta\cdot p\beta\cdot p'(\ln\frac{p_x^2}{\mu^2}-2) + \frac{\beta^2}{8}p_x^2(\frac{3}{2}-\ln\frac{\Lambda^2}{p_x^2})\Big] \\ &- x\frac{\not{p}'}{2}\Big(\beta\cdot p \not{V}\not{\beta} + \beta\cdot p' \not{\beta}\not{V}\Big) - \Big(\beta\cdot p \not{V}\not{\beta} + \beta\cdot p' \not{\beta}\not{V}\Big)(1-x)\frac{\not{p}}{2} \\ &+ \not{p}_x\Big(\beta\cdot V(\beta\cdot p + \beta\cdot p') - \frac{\beta^2}{4}p_x\cdot V\Big) \\ &+ \not{\beta}\Big(\frac{1}{2}p_x\cdot\beta p_x\cdot V - \frac{1}{4}V\cdot\beta p_x^2(1+\frac{1}{2}-\ln\frac{\Lambda^2}{p_x^2} \\ &- (1-x)\beta\cdot pV\cdot p - xp'\cdot\beta V\cdot p'\Big)\Big\} + \cdots \end{aligned} \tag{A.71}$$

The evaluation of the remaining I's is analogous, and so

$$I_2 = \not{V}\gamma^2\xi\Big\{\frac{\beta^2}{4}(\ln\frac{\Lambda^2}{p^2}-1) + \frac{(\beta\cdot p)^2}{p^2}(1+\frac{1}{2}\ln\frac{\mu^2}{p^2})\Big\} + \cdots \tag{A.72}$$

$$I_3 = -\frac{1}{4}\beta\cdot p \not{\beta}\gamma^2\xi(\ln\frac{\Lambda^2}{p^2}+\frac{5}{2}) - \frac{1}{8}\not{p}\beta^2\gamma^2\xi(\ln\frac{\Lambda^2}{p^2}-\frac{1}{2}) + \cdots + \delta C \tag{A.73}$$

The counterterm δC is chosen such that there is no radiative contribution when the source is turned off, or equivalently, it accounts for the free electron process.

That is, it forces I_3 equal to zero when acting on free spinors. From Eq. (2.17), we know $\delta C = \delta m + \delta\xi_0\ \not\beta(\beta\cdot p - \beta\cdot V)$, where in this case

$$\delta m = \frac{\beta^2}{8}\gamma^2\xi\Big(\ln\frac{\Lambda^2}{m^2} - \frac{1}{2}\Big) + \cdots \qquad \delta\xi_0 = \frac{1}{4}\gamma^2\xi\Big(\ln\frac{\Lambda^2}{m^2} + \frac{5}{2}\Big) + \cdots \tag{A.74}$$

Since here $V^\mu = \eta^{\mu 0}V_0$, we can rewrite after some manipulation

$$I_1 + I_2 + I_3 = \gamma^2\xi(K_1 + K_2 + K_3) + \cdots \tag{A.75}$$

where

$$\begin{aligned}
K_1 &= \not V \int \frac{dx}{p_x^2}\left[\Big(\ln(\frac{\mu}{E_0}) + 1\Big)\left(\frac{(\beta\cdot p)^2}{p^2} - \beta\cdot p\beta\cdot p'\right) + \frac{1}{2}\beta\cdot p\beta\cdot p'\ln(\frac{p_x^2}{E_0^2})\right] \\
K_2 &= \Big(m\frac{\beta^2}{2} + \beta\cdot p\ \not\beta\Big)\left[-\frac{1}{4}\ln(\frac{m^2}{p^2})\delta(p-p') - \frac{\not V}{2m} - \frac{1}{4}\beta\cdot V\ \not\beta\ln(\frac{p^2}{m^2})\right] \\
K_3 &= -\frac{1}{2}\int\frac{dx}{p_x^2}\Big\{x\beta\cdot p'\ \not p'\ \not\beta\ \not V + (1-x)\beta\cdot p\ \not V\ \not\beta\ \not p \\
&+\ V\cdot p\left(\frac{\beta^2}{2}\ \not p_x + \beta\cdot p_x\ \not\beta\right) - 2\beta\cdot V(\beta\cdot p + \beta\cdot p')\ \not p_x + x\beta\cdot p\ \not p'\ \not V\ \not\beta \\
&+\ (1-x)\beta\cdot p'\ \not\beta\ \not V\ \not p - p_x^2\left[\frac{\beta^2}{2}\ \not V - 2\beta\cdot V\ \not\beta + \not V\ \not\beta\frac{\beta\cdot p}{m}\right]\Big\}
\end{aligned}$$

We want a result good to $\alpha(Z\alpha)^4$, and so we can simplify the above expressions by using the assigned order given by (A.15), from which we can relate

$$\begin{aligned}
&q \equiv p' - p \sim Z\alpha m \\
&p'^2 - p^2 \sim (\beta\cdot p)^2 - p^2 \sim p_x^2 - m^2 \sim (Z\alpha)^2 m^2
\end{aligned} \tag{A.76}$$

and then reduce K_1 to

$$K_1 \simeq \frac{\not V}{2m^2}\left\{\left[(\beta\cdot q)^2 - \frac{q^2}{3}\right]\ln(\frac{\mu}{m}) - \frac{5}{6}q^2(\frac{1}{2} + \beta^2) + (\beta\cdot q)^2\right\} \tag{A.77}$$

where antisymmetric terms under $p' \leftrightarrow p$ vanish.

To simplify K_2 we follow BBF and use

$$\begin{aligned}(\not{p}-m)^2 \not{\beta} &= \not{V}(\not{p}-m)\not{\beta} = 2\beta\cdot p\not{V} - 2m\not{V}\not{\beta} - \not{V}\not{\beta}\not{V} \\ &\simeq 2(V\cdot p - m\not{V})\not{\beta} - \not{q}\not{\beta}\not{V} - V^2 \end{aligned} \quad (A.78)$$

where we have assumed the operator is acting on Dirac spinors of momentum p and omitted the integration coming from

$$\overline{\psi}(p)(\not{p}-m) = \delta(q_0)\int\overline{\psi}(p')\not{V}(q)d^3p' \quad (A.79)$$

Note that $\not{V}\not{\beta}\not{V} \simeq V^2$, since the square of the potential (after factoring out the spinors and integration variables) is already of the desired order $(Z\alpha)^4$ (see (A.15)) and so $\not{\beta} \simeq \gamma_0 \simeq 1$.

The final result is

$$K_2 \simeq -\frac{V^2}{4m}(\frac{\beta^2}{2}+5) + (V\cdot p - m\not{V})\frac{\not{\beta}}{2m} - \frac{\beta\cdot p}{4m^2}\not{q}\not{\beta}\not{V} \quad (A.80)$$

Following a similar approach we reduce

$$\begin{aligned}K_3 \simeq & -\frac{\not{\beta}}{2m}(V\cdot p - m\not{V}) + \frac{V^2}{m} + \beta\cdot V\frac{\beta\cdot q}{m} - \frac{\beta\cdot q}{4m^2}\not{\beta}V\cdot p \\ & - \frac{\beta\cdot q}{2m}\not{\beta}\not{V} - \frac{\beta\cdot V}{2m}\not{q}\not{\beta} + \frac{\beta^2}{8m}\not{q}\not{V} - \frac{q^2}{12m^2}(\frac{\beta^2}{2}-1)\not{V}\end{aligned} \quad (A.81)$$

We can make further simplifications by using

$$\int\overline{\psi}(p')B(p',p)\psi(p)d^3p'd^3p = 0, \quad (A.82)$$

provided $\gamma^0 B^\dagger(p',p)\gamma^0 = -B(p,p')$, where B represents any operator as a function of p' and p, as for example, $\beta\cdot q\not{V}$. Note that we are interested only in the real part of the level shift.

Putting everything together, we obtain after some manipulation

$$\begin{aligned}\delta E_1^{(B)} &= \frac{\alpha}{\pi m^2}\gamma^2\xi\int\overline{\psi}(\vec{p}\,')\Big\{\not{V}(\beta\cdot q)^2\left[\frac{5}{8}+\frac{1}{2}\ln(\frac{\mu}{m})\right]-\not{V}q^2\left[\frac{\beta^2}{16}+\frac{1}{8}+\frac{1}{6}\ln(\frac{\mu}{m})\right]\\ &- (\frac{\beta\cdot p}{4}\not{V}+\beta\cdot V\frac{m}{2})i\sigma^{ij}u_iq_j-\frac{m}{2}\beta\cdot qi\sigma^{\mu\nu}V_\mu\beta_\nu \qquad (A.83)\\ &+ m(\frac{\beta^2}{8}-\frac{1}{2})i\sigma^{\mu\nu}V_\mu q_\nu\Big\}\psi(\vec{p})d^3p'd^3p-\frac{\alpha}{\pi}\gamma^2\xi(1+\frac{\beta^2}{2})\langle n|\frac{V_0^2}{4m}|n\rangle+\cdots\end{aligned}$$

Note again that this represents the calculation involving only the first term of Eq. (A.44).

Now to evaluate the many potential part contribution we need to solve Eq. (A.28), with $\langle M^I\rangle$ and $\langle M^{II}\rangle$ given by Eqs. (A.32) and (A.40) respectively.

So, after substituting (A.44) in (A.32)

$$\langle M^I\rangle = \frac{\alpha}{4\pi^3 i}\gamma^2\xi\sum_r\int\frac{|\beta\cdot\mathcal{M}|^2}{k^2-\mu^2}\frac{d^4k}{k_0-E_n-E_r}+\cdots \qquad (A.84)$$

with

$$\mathcal{M}_\mu \equiv \langle n|R_\mu|r\rangle,$$

Using the transversality condition, we relate

$$\mathcal{M}_0 = \frac{\vec{k}\cdot\vec{\mathcal{M}}}{k_0} = \frac{|\vec{k}|}{k_0}|\vec{\mathcal{M}}|\cos\theta$$

which reduces the integral on the angles of $\vec{k}$ to

$$\int d\Omega|\beta\cdot\mathcal{M}|^2 = 4\pi\left(\frac{\vec{k}^2}{3k_0^2}|\vec{\mathcal{M}}|^2+|\vec{u}\cdot\vec{\mathcal{M}}|^2\right) \qquad (A.85)$$

We evaluate the remaining k_0 and $|\vec{k}|$ integrations in (A.84), by using (A.33), (A.35) along with the analogous relations

$$\begin{aligned}\vec{u}\cdot\vec{\mathcal{M}} &= \frac{-1}{mk_0}(E_r-E_n)\langle n|\vec{u}\cdot\vec{p}|r\rangle\\ \sum_r|\langle n|\vec{u}\cdot\vec{p}|r\rangle|^2(E_r-E_n) &= \frac{1}{2}\langle n|\vec{u}\cdot\vec{\nabla}V_0|n\rangle\end{aligned}$$

to finally obtain

$$\langle M^I \rangle = \frac{\alpha}{\pi m}\gamma^2\xi \quad \Big\{ \quad \frac{1}{6}\hat{C} + \frac{1}{2}u_i u_j \hat{C}_{ij} + \left[\frac{2}{9} + \frac{1}{6}\ln(\frac{\mu}{2E_*})\right] \langle n|\nabla^2 V_0|n\rangle \quad \text{(A.86)}$$
$$+ \quad \left[\frac{1}{2} + \frac{1}{2}\ln(\frac{\mu}{2E_*})\right] \langle n|(\vec{u}\cdot\vec{\nabla})^2 V_0|n\rangle \Big\} + \cdots$$

where we have kept only the leading terms as $\mu \to 0$ and neglected the imaginary part.

The computation of $\langle M^{II}\rangle$ is straightforward. Here we need to replace (A.44) in (A.40), and use $V_\alpha = \eta_{\alpha 0}V^0$. Further simplifications follow from BBF and the assigned order of magnitude given before. The final result is

$$\langle M^{II}\rangle = \frac{\alpha}{\pi}\gamma^2\xi(\frac{\beta^2}{2} + 1)\langle n|\frac{V_0^2}{4m}|n\rangle + \cdots \qquad \text{(A.87)}$$

Adding together (A.83), (A.86), and (A.87) will give us then the final expression for the self energy contribution for this part of the calculation. Note that the above results can be verified by taking the limit $\beta_\mu\beta_\nu \to \eta_{\mu\nu}$, which reduces

$$G^{(B)}_{\mu\nu} \to -\gamma^2\xi G^0_{\mu\nu} + \cdots,$$

and therefore the former expressions should reduce up to a constant, to the metric case.

A.3 Virtual non-metric anomaly

In the $TH\epsilon\mu$ formalism, gravity interacts with matter through the T and H functions, which are assumed locally constant within atomic scales. *A-priori* they do not need to be the same for different types of matter (like baryons and leptons), or furthermore for matter and antimatter. In this context for example, a non-metric anomaly related to electron/ positron difference will modified the Lagrangian density related to fermions by

$$\mathcal{L}_D = \overline{\psi}(\not p - \not V - m)\psi + \xi_+ \overline{\psi}^+(p_0 - A_0)\gamma^0\psi^+ \qquad (A.88)$$

where $\xi_+ \equiv 1 - c_-/c_+$ and $c_\mp = (T_\mp/H_\mp)^{1/2}$, with $-$ and $+$ labeling electrons and positrons respectively. After using (2.9), we can refer (A.88) to the moving frame as

$$\mathcal{L}'_D = \overline{\psi}(\not p - \not V - m)\psi + \xi_+\gamma^2\overline{\psi}^+(\beta\cdot p - \beta\cdot V)\ \not\beta\psi^+ \qquad (A.89)$$

The imposed broken symmetry between particle and antiparticle changes the fermion propagator (in the positron case) to (up to $O(\xi_+)$):

$$S_F^+ = (\not p - m)^{-1} + \xi_+(\not p - m)^{-1}\gamma^2\ \not\beta\beta\cdot p(\not p - m)^{-1} \qquad (A.90)$$

where the first term represents the unchanged electron propagator S_F^-.

The positron-electron pairs produced in the electric field of the atomic nucleus, are seen in the Lamb shift transition via the vacuum polarization contribution given by (A.49), where in this case:

$$i\Pi^{\mu\nu}(q) = \frac{(ie)^2}{(2\pi)^4}(-1)Tr\int d^4p\gamma^\mu iS_F^-(p+q)\gamma^\nu iS_F^+(p) \qquad (A.91)$$

After using Eq. (A.90) along with standard techniques [53], we obtain that the non-metric part of (A.91) is up to $O(q^2)$

$$i\Pi^{\mu\nu}(q)^+ = -\frac{\alpha}{2\pi}\gamma^2\eta^{\mu\nu}\frac{q^2}{m^2}\left\{\frac{1}{30}q^2\beta^2 - \frac{1}{5}(\beta\cdot q)^2\right\} + \cdots \qquad (A.92)$$

where the ellipsis accounts for the gauge dependent terms which give no contribution to (A.49). Eq (A.92) also comes after proper regularization and renormalization processes, which follow from previous sections.

In this EEP violating context, the radiative corrections related to atomic energy levels are modified by (up to $O(\alpha(Z\alpha)^4 O(u^2))$

$$\delta E_L^+ = \delta E_P^+ = -\xi_+ \frac{\alpha}{10\pi m^2}\Big\{\frac{1}{6}\langle n|\nabla^2 V_0|n\rangle + \langle n|(\vec{u}\cdot\vec{\nabla})^2 V_0)|n\rangle\Big\} \qquad \text{(A.93)}$$

where we have replaced (A.92) in (A.49) and simplified afterwards. By taking the Lamb atomic states, we finally obtain

$$\Delta E_L^+ = -\xi_+ \frac{m}{120\pi}(Z\alpha)^4 \alpha(1 + 2\vec{u}^2) \qquad \text{(A.94)}$$

Appendix B

(g-2) Anomaly

B.1 Loop integrations

We show the main steps leading to Eqs. (4.11), (4.10), and (4.12). Details are given throughout the computation by considering only the first term of the photon propagator (4.8), that is

$$G^{\xi}_{\mu\nu} = \frac{\beta_\mu \beta_\nu}{k^2} + \cdots \tag{B.1}$$

with the remaining term in (4.8) contributing in a similar manner.

We solve for the fermion self energy by replacing (B.1) in (4.5), and using (4.9) along with the Feynman parameters

$$\frac{1}{a^2 b} = 2\int_0^1 \frac{dz\, z}{[az + b(1-z)]^3}.$$

After integrating we obtain

$$\Sigma(p) = \frac{\alpha}{4\pi}\int \not\beta(\not p z + m)\not\beta \ln\left(\frac{m^2(1-z)^2 + z\Lambda^2 - \Delta z(1-z)}{m^2(1-z)^2 - \Delta z(1-z) + z\mu^2}\right) + \cdots \tag{B.2}$$

with $\Delta = p^2 - m^2$. We consider $\Delta/m^2 << 1$, and expand the above to obtain after some manipulation

$$\Sigma(p) = \frac{\alpha}{4\pi}\Big\{\beta\cdot p\ \not\beta\left(\frac{5}{2}+\ln(\frac{\Lambda}{m})^2\right) - (\not p - m)\frac{\beta^2}{2}\left(\ln(\frac{\Lambda}{m})^2+\frac{1}{2}\right) \tag{B.3}$$
$$-m\frac{\beta^2}{2}\left(\frac{1}{2}-\ln(\frac{\Lambda}{m})^2\right) + 2\frac{(\beta\cdot p)^2}{m^2}(\not p - m)\left(\ln(\frac{m}{\mu})^2 - 3\right)\Big\} + O\Big((\not p - m)^2\Big) + \cdots$$

where we have kept the leading terms as $\mu \to 0$ and $\Lambda \to \infty$, and $O\Big((\not p - m)^2\Big)$ stands for the terms satisfying

$$S_F(p)O\Big((\not p - m)^2\Big)u(\vec{p}) = 0.$$

We renormalize $\Sigma(p)$ by subtracting

$$\delta\Sigma = \delta m + \delta\xi_0\ \not\beta\beta\cdot p \tag{B.4}$$

where the counterterms respectively account for mass and $TH\epsilon\mu$ -parameter renormalization.

Choosing the counterterms so that

$$\overline{u}(\vec{p})\Sigma(p)u(\vec{p}) = 0,$$

and so

$$\delta m = -\frac{\alpha}{4\pi}m\frac{\beta^2}{2}\left(\frac{1}{2}-\ln(\frac{\Lambda}{m})^2\right) \tag{B.5}$$
$$\delta\xi_0 = \frac{\alpha}{4\pi}\left(\frac{5}{2}+\ln(\frac{\Lambda}{m})^2\right) \tag{B.6}$$

the regularized result is then

$$\Sigma(p) = \frac{\alpha}{\pi}(\not p - m)\left\{\frac{(\beta\cdot p)^2}{m^2}\left[\frac{1}{2}\ln(\frac{m}{\mu})^2 - \frac{3}{2}\right] - \beta^2\left[\frac{1}{8}\ln(\frac{\Lambda}{m})^2 + \frac{1}{16}\right]\right\}$$
$$+\ O\Big((\not p - m)^2\Big)\ . \tag{B.7}$$

Note that the remaining ultraviolet divergence related to this term could be removed after charge renormalization. We find it convenient to leave it in order to cross-check the calculation, since a similar term from the vertex part should cancel it, thereby removing the divergence from the resulting scattering amplitude.

The evaluation of the vertex function follows a similar procedure, giving the result

$$\begin{aligned}\Gamma^\mu &= \frac{\alpha}{\pi}\int\frac{dx}{p_x^2}\Big\{\gamma^\mu\Big[\frac{1}{2}\beta\cdot p\,\beta\cdot p'(\ln\frac{p_x^2}{\mu^2}-2)+\frac{\beta^2}{8}p_x^2(\frac{3}{2}-\ln\frac{\Lambda^2}{p_x^2})\Big]\\ &- x\frac{\not{p}'}{2}\Big(\beta\cdot p\gamma^\mu\,\not{\beta}+\beta\cdot p'\,\not{\beta}\gamma^\mu\Big)-\Big(\beta\cdot p\gamma^\mu\,\not{\beta}+\beta\cdot p'\,\not{\beta}\gamma^\mu\Big)(1-x)\frac{\not{p}}{2} \qquad \text{(B.8)}\\ &+ \not{\beta}\Big(\frac{1}{2}p_x\cdot\beta p_x^\mu-\frac{1}{4}\beta^\mu p_x^2(\frac{3}{2}-\ln\frac{\Lambda^2}{p_x^2})-(1-x)\beta\cdot p\,p^\mu-xp'\cdot\beta\,p'^{\mu}\Big)\\ &+ \not{p}_x\Big(\beta^\mu(\beta\cdot p+\beta\cdot p')-\frac{\beta^2}{4}p_x^\mu\Big)\Big\}+\cdots\end{aligned}$$

with $p_x = xp' + (1-x)p$. Since (B.8) is acting on a free spinor, we can use

$$p_x^2 = m^2 - x(1-x)q^2,$$

with $q = p' - p$, and so expand

$$\frac{m^2}{p_x^2} = 1 + x(1-x)\frac{q^2}{m^2} + O(q^4),$$

which after some algebra reduces (B.8) to

$$\begin{aligned}\Gamma^\mu(p',p) &= \frac{\alpha}{\pi}\Big\{\gamma^\mu\left[-\beta^2\left(\frac{1}{16}+\frac{1}{8}\ln(\frac{\Lambda}{m})^2\right)+\frac{(\beta\cdot p)^2}{m^2}\left(\frac{1}{2}\ln(\frac{m}{\mu})^2-\frac{3}{2}\right)\right.\\ &+ \left.\frac{q^2}{m^2}\left(\frac{1}{12}\ln(\frac{m}{\mu})^2-\frac{1}{3}-\frac{\beta^2}{16}\right)+\frac{\beta\cdot p}{m}\frac{\beta\cdot q}{m}\left(\frac{1}{2}\ln(\frac{m}{\mu})^2-\frac{5}{4}\right)\right]\\ &+ \frac{\not{q}}{m}\gamma^\mu\,\not{\beta}\left(\frac{1}{4}\frac{\beta\cdot p}{m}+\frac{1}{8}\frac{\beta\cdot q}{m}\right)+\frac{5}{24}\frac{q^2}{m^2}\,\not{\beta}\beta^\mu-\frac{1}{2}\frac{\not{q}}{m}\,\not{\beta}\beta^\mu \qquad \text{(B.9)}\\ &- \frac{1}{2}\frac{\beta\cdot q}{m}\,\not{\beta}\gamma^\mu+\frac{\beta\cdot q}{m}\beta^\mu+\frac{\beta^2}{8}\frac{\not{q}}{m}\gamma^\mu-\left(\frac{1}{4}\frac{\beta\cdot p}{m}+\frac{1}{3}\frac{\beta\cdot q}{m}\right)\frac{q^\mu}{m}\,\not{\beta}\\ &- \frac{\beta^2}{8}\frac{q^\mu}{m}\Big\}+O(q^3)\end{aligned}$$

where the vertex function has been renormalized by subtracting a term like

$$\delta\Gamma^{\mu} = \delta\xi_0 \, \not{\beta}\beta^{\mu} \tag{B.10}$$

with $\delta\xi_0$ is given by (B.6). We recall that gauge invariance forces this coefficient to be equal to the one participating in the renormalization of the fermion self energy.

B.2 Adiabatic hypothesis

In order to describe how self energy effects convert the incident electron from a bare particle to a physical one, it is convenient to introduce a damping function, $g(t)$, which adiabatically switches off the coupling between fields, such that the interaction lagrangian is replaced by

$$\mathcal{L}_I = eg(t)\overline{\psi}(x) \not{A}(x)\psi(x) \tag{B.11}$$

It is assumed that the time T over which $g(t)$ varies is very long compared to the duration of the scattering process. In momentum space

$$g(t) = \int G(\Omega_0)\exp(i\Omega \cdot x)d\Omega_0 \tag{B.12}$$

with $\Omega \equiv (\Omega_0, 0)$, and $g(0) = 1$. It is supposed that $G(\Omega_0)$ is almost a delta function, being large for Ω_0 in a range of about T^{-1}

In the presence of an external field A_{μ}, Eq. (4.4) will now read

$$L{\cdot}A \to \int G(\Omega_0)G(\Omega_0')d\Omega_0 d\Omega_0' \not{A}(p'-p-\Omega-\Omega')iS_F(p-\Omega-\Omega')i\Sigma(p-\Omega)+\cdots \tag{B.13}$$

where $\cdots$ represents the equivalent second term from (4.4).

As $T \to \infty$, and Ω_0, $\Omega_0' \to 0$, the fermion propagator reduces to

$$S_F = \frac{1}{\not{p} - \not{\Omega} - \not{\Omega}' - m} \simeq -\frac{\not{p} + m}{2p_0(\Omega_0 + \Omega_0')} \tag{B.14}$$

where we used $p^2 = m^2$. This implies that we can expand Σ up to order Ω only, since higher terms vanish after taking the previous limit. Here we employ the relation

$$\frac{1}{A-B} = \frac{1}{A} + \frac{1}{A}B\frac{1}{A} + \cdots \tag{B.15}$$

to expand

$$\Sigma(p-\Omega) \simeq \Sigma(p) - \frac{\partial\Sigma(p)}{\partial p^\mu}\Omega^\mu \quad . \tag{B.16}$$

After renormalization, Σ takes the form

$$\Sigma(p) = (\not{p}-m)(A + B(\beta\cdot p)^2) + C(\beta\cdot p)\left\{ \not{\beta}(\not{p}-m) + (\not{p}-m)\not{\beta}\right\} + \cdots \tag{B.17}$$

where the constants A, B, and C can be obtained from Eq. (4.10).

Let us introduce $\overline{\Omega} \equiv \Omega + \Omega'$, and symmetrize Ω by $\frac{1}{2}(\Omega+\Omega')$ in (B.16), to write

$$S_F(p-\Omega-\Omega')\Sigma(p-\Omega) = \frac{1}{\not{p}-\not{\overline{\Omega}}-m}(\Sigma(p) - \frac{1}{2}\frac{\partial\Sigma(p)}{\partial p^\mu}\overline{\Omega}^\mu) \tag{B.18}$$

which after using (B.17) can be written as

$$\frac{1}{2}(A + B(\beta\cdot p)^2) + C\beta\cdot p\ \not{\beta} \tag{B.19}$$

where we have used that $\Sigma(p-\Omega)$ is acting on a free spinor, and therefore terms of the form $(\not{p}-m)u(p)$ vanish. Now, the final evaluation of (4.4) follows directly from (B.19).

Bibliography

[1] R. H. Dicke in *Relativity, Groups and Topology, 1964*, Eds. C. DeWitt and B. DeWitt, p 165.

[2] K. S. Thorne, A. P. Lightman and D. L. Lee, Phys. Rev. D **7**, 3563 (1973).

[3] P. G. Roll, T. Krotkov, and R. H. Dicke, Ann. Phys. **26**, 442 (1964).

[4] V. B. Braginskii and V. I. Panov, Soviet Phys. JETP **34**, 463 (1972).

[5] E. G. Adelberger, *et al.*, Phys. Rev. D **42**, 3267 (1990).

[6] C. M. Will, *Theory and Experiment in Gravitational Physics*, 2nd edition (Cambridge University Press, Cambridge, 1992).

[7] R. F. C. Vessot and M.W. Levine, Gen. Relativ. Gravit. **10**, 181 (1979).

[8] J. D. Prestage, R. L. Tjoelker and L. Maleki, Phys. Rev. Lett. **74**, 3511 (1995).

[9] S. K. Lamoreaux *et al.*, Phys. Rev. Lett. **57**, 3125 (1986); T. E. Chupp et al., *ibid.* **63**, 1541 (1989).

[10] J. D. Prestage *et al.*, Phys. Rev. Lett. **54**, 2387 (1985).

[11] J.P. Blaser *et.al.*, *STEP Assessment Study Report*, ESA Document SCI (94)5, May (1994).

[12] T. Damour, Los Alamos preprint hep-ph/9606080 (June 1996).

[13] P. Fayet, Phys. Letter B **69** 489 (1986); *ibid.* **171** 261 (1986); *ibid.* **172** 363 (1986).

[14] M. P. Haugan and C. M. Will, Phys. Rev. Lett. **37**, 1 (1976); Phys. Rev D. **15**, 2711 (1977).

[15] R.J. Hughes, Contemporary Physics **34** 177 (1993).

[16] M. Gabriel, M. Haugan, R.B. Mann and J. Palmer, Phys. Rev. Lett. **67** (1991) 2123.

[17] M.H. Holzscheiter, T. Goldman and M.M. Nieto Los Alamos preprint LA-UR-95-2776, hep-ph/9509336 (Sept. 1995).

[18] R.B. Mann and U. Sarkar, Phys. Rev. Lett. **76** (1996) 865.

[19] I.R. Kenyon, Phys. Lett. B **237** 274 (1990).

[20] C. Alvarez and R. B. Mann, Gen. Relativ. Gravit., *in press.*

[21] G. Smith, E.G. Adelberger, B.R. Heckel and Y. Su Phys. Rev. Lett. **70** 123(1993).

[22] C. Alvarez and R. B. Mann, Mod. Phys. Lett. A **11**, 1757 (1996).

[23] S. R. Lundeen and F. M. Pipkin, Phys. Rev. Lett. **46**, 232 (1981).

[24] V. G. Pal'chickov, Yu. L. Sokolov, and V. P. Yakovlev, Pis'ma Zh. Eksp. Teor. Fiz. **38**, 347 (1983) [JETP Lett. 38, 418 (1983)].

[25] R. Van Dyck, P. Schwinberg, and H. Dehmelt, Phys. Rev. Lett. **59**, 26 (1987).

[26] T. Kinoshita and D. R. Yennie, *Quantum Electrodynamics*, Ed T. Kinoshita (World Scientific, Singapure, 1990), p 1.

[27] A. P. Lightman and D. L. Lee, Phys. Rev. D **8**, 364 (1973).

[28] C. M. Will, Phys. Rev. D **10**, 2330 (1974).

[29] M. D. Gabriel and M. P. Haugan, Phys. Rev. D **41**, 2943 (1990).

[30] M. P. Haugan, Ann. Phys. (N.Y.) **118**, 156 (1979).

[31] L. H. Ryder, *Quantum Field Theory* (Cambridge University Press, Cambridge, 1985).

[32] W. E. Lamb, Jr., and R. C. Retherford, Phys. Rev. **72**, 241 (1947).

[33] M. Weitz *et al.*, Phys. Rev. Lett. **72**, 328 (1994).

[34] E. W. Hagley and F. M. Pipkin, Phys. Rev. Lett. **72**, 1172 (1994).

[35] Ping Zhao *et al.*, Phys. Rev. Lett. **63**, 159 (1989).

[36] J. R. Sapirstein and D. R. Yennie, *Quantum Electrodynamics*, Ed T. Kinoshita (World Scientific, Singapure,1990), p 560.

[37] M. E. Rose *Relativistic Electron Theory* (Wiley, New York, 1961).

[38] M. Baranger, H. A. Bethe, and R. P. Feynman, Phys. Rev. **92**, 482 (1953).

[39] J. M. Harriman, Phys. Rev. **101**, 594 (1956).

[40] L. Schiff, Proc. Nat. Acad. Sci. **45**, 69 (1959).

[41] C. Alvarez and R. B. Mann, *in Proceedings of the* 5^{th} *Canadian Conference on General Relativity and Relativistic Astrophysics, 1993*, Eds. R. B Mann and R. G. McLenaghan, p 200.

[42] M. I. Eides and V. A. Shelyuto, Phys. Rev. A **52**, 954 (1995).

[43] M. L. Good, Phys. Rev. **121**, 311 (1961).

[44] G. Greene *et al.*, Phys. Rev. D **44**, R2216 (1991).

[45] F. Mandl, G. Shaw, *Quantum Field Theory* (Wiley, Toronto, 1984).

[46] J. Bailey *et.al.*, Phys. Lett. B **68**, 191 (1977).

[47] C. Alvarez and R. B. Mann, Los Alamos preprint gr-qc/9609039 (Sep. 1996).

[48] D. Newman, G. W. Ford, A. Rich, E. Sweetman, Phys. Rev. Lett. **40**, 1355 (1978).

[49] J. Bailey *et.al.*, Nuovo Cimento A **9**, 369 (1972).

[50] M. Gasperini, Mod. Phys. Lett. A **4**, 1681 (1989).

[51] T. Damour and A. M. Polyakov, Nucl. Phys. B **423**, 532 (1994); Gen. Rel. Grav. **26**, 1171 (1994).

[52] T. Damour, Los Alamos preprint gr-qc/9606079 (June 1996).

[53] C. Itzykson and J. B Zuber, *Quantum Field Theory* (McGraw-Hill, New York, International Editions, 1987).