Application of Complex Vectors and Complex Transformations in Solving Maxwell's Equations

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Application and implication of using complex vectors and complex transformations in solutions of Maxwell's equations is investigated. Complex vectors are used in complex plane waves and help to represent this type of waves geometrically. It is shown that they are also useful in representing inhomogeneous plane waves in plasma, single-negative and double-negative metamaterials. In specific I will investigate the Otto configuration and Kretschmann configuration and I will show that in order to observe the minimum in reflection coefficient it is necessary for the metal to be lossy. We will compare this to the case of plasmon-like resonance when a PEC periodic structure is illuminated by a plane wave.

Complex transformations are crucial in deriving Gaussian beam solutions of paraxial Helmholtz equation from spherical wave solution of Helmholtz equation. Vector Gaussian beams also will be discussed shortly.

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List of Abbreviations

LP: Linear Polarization- Linearly Polarized

NLP: Nonlinear Polarization- Nonlinearly Polarized

CP: Circular Polarization- Circularly Polarized

TE: Transverse Electric

TM: Transverse Magnetic

SNG: Single Negative (Metamaterial)

DNG: Double Negative (Metamaterial)

PHE: Paraxial Helmholtz Equation

GB: Gaussian Beam

SAGB: Simple Astigmatic Gaussian Beam

GAGB: General Astigmatic Gaussian Beam

Chapter 1

Complex Vectors

1.1 Review of Complex Linear Algebra

In this thesis vectors with real components are called “real vectors” and vectors with complex components are called “complex vectors”. Although phasors that are complex vectors are vastly used in time-harmonic Electromagnetic, there has not been enough consideration regarding differences between real vectors and complex vectors.

There are two types of representation for vectors with real components: Component-wise representation and geometrical representation.

It is possible to define operations such as inner product, cross product, length etc. between vectors using one representation and derive it in another representation. For example we can define inner product of two vectors using component representation of vectors and derive its geometrical formulation and vice-versa.

Linear algebra techniques are used to deal with the vectors component-wise.

For example the geometrical operations such as rotation and translation can be represented by matrixes with real components which we will call them “real matrixes” henceforward.

The situation is different for complex vectors because there is no geometrical representation for a complex vector unless we decompose it to its real part and imaginary part that are real vectors.

To define the operations such as inner product, cross product, length etc. between complex vectors we have to use the corresponding component-wise definition for real vectors with some modification.

For inner product we have to modify the definition by making the components of the second vector conjugates.

$$\begin{aligned}
\mathbf{V} &= \sum v_k \hat{x}_k \\
\mathbf{W} &= \sum w_k \hat{x}_k \\
\mathbf{V} \bullet \mathbf{W} &= \sum v_k w_k \\
\left(\mathbf{V} \bullet \mathbf{W}\right)^* &= \mathbf{V}^* \bullet \mathbf{W}^* \\
\left|\mathbf{V}\right|^2 &= \mathbf{V} \bullet \mathbf{V}^* = \sum \left|v_k\right|^2
\end{aligned} \qquad (1.1)$$

The definition of inner product guarantees that the length is a non-negative real number.

A complex vector can be written in terms of its real and imaginary parts:

$$\mathbf{V} = \mathbf{V}_r + j\,\mathbf{V}_i \qquad (1.2)$$

in which $\mathbf{V}_r$ and $\mathbf{V}_i$ are real vectors.

$$\left|\mathbf{V}\right|^2 = \mathbf{V} \bullet \mathbf{V}^* = \left|\mathbf{V}_r\right|^2 + \left|\mathbf{V}_i\right|^2 = \sum \left|v_k\right|^2 \qquad (1.3)$$

but

$$\mathbf{V} \bullet \mathbf{V} = \left|\mathbf{V}_r\right|^2 - \left|\mathbf{V}_i\right|^2 + 2j\mathbf{V}_r \bullet \mathbf{V}_i = \sum v_k^2 \qquad (1.4)$$

$$\mathbf{V} \times \mathbf{V} = \mathbf{0}$$

$$\mathbf{V} \times \mathbf{V}^* = -2\,j\,\mathbf{V_r} \times \mathbf{V_i} \qquad (1.5)$$

From (1.4) we observe that it is possible to have can have $\mathbf{V} \bullet \mathbf{V} = \mathbf{0}$ with $\mathbf{V} \neq \mathbf{0}$.

The best geometrical representation that we can have for a complex vector is by representing its real part ($\mathbf{V}_r$) and imaginary part ($\mathbf{V}_i$).

1.2 Vector Identities for Complex Vectors

Most of the identities that are valid for real vectors are also valid for complex vectors with some exceptions. For example the implication $\mathbf{V}\bullet\mathbf{V}=\mathbf{0}\Rightarrow\mathbf{V}=\mathbf{0}$ is not valid for complex vectors. Rather this statement is true:

$\mathbf{V}\bullet\mathbf{V}^*=\mathbf{0}\Rightarrow\mathbf{V}=\mathbf{0}$

A multilinear function F of vector arguments $\mathbf{V}_k$ (k=1,2,...,n) is a function that is linear in every argument:

$$F\left(\mathbf{V}_1,\mathbf{V}_2,\ldots,\alpha\mathbf{V}_k'+\beta\mathbf{V}_k'',\ldots,\mathbf{V}_n\right)=\alpha F\left(\mathbf{V}_1,\mathbf{V}_2,\ldots,\mathbf{V}_k',\ldots,\mathbf{V}_n\right)+\beta F\left(\mathbf{V}_1,\mathbf{V}_2,\ldots,\mathbf{V}_k'',\ldots,\mathbf{V}_n\right)$$
$$\forall\alpha,\beta\in\mathbb{C} \qquad (1.6)$$

A multilinear identity is of the form $F\left(\mathbf{V}_1,\mathbf{V}_2,\ldots,\mathbf{V}_k,\ldots,\mathbf{V}_n\right)=\mathbf{0}$

Theorem:

All multilinear identities that are valid for real vectors are also valid for complex vectors.

For example the identity:

$$\mathbf{A}\times\left(\mathbf{B}\times\mathbf{C}\right)=\mathbf{B}\left(\mathbf{A}\bullet\mathbf{C}\right)-\mathbf{C}\left(\mathbf{A}\bullet\mathbf{B}\right) \qquad (1.7)$$

is also valid for complex vectors.

Having $\mathbf{A}\bullet\mathbf{X}$ and $\left(\mathbf{A}\times\mathbf{X}\right)$ in which $\mathbf{A}$ is a known vector and $\mathbf{X}$ is unknown, using (1.7) we have

$$\mathbf{X} = \frac{(\mathbf{A}\bullet\mathbf{X})\mathbf{A} + (\mathbf{A}\times\mathbf{X})\times\mathbf{A}}{\mathbf{A}\bullet\mathbf{A}} \qquad (1.8)$$

which is valid for both real and complex vectors, but since $\mathbf{A}\bullet\mathbf{A}$ can be zero without $\mathbf{A}$ being zero we derive an alternative formulation for complex vectors:

$$\mathbf{X} = \frac{(\mathbf{A}^*\bullet\mathbf{X})\mathbf{A} + (\mathbf{A}\times\mathbf{X})\times\mathbf{A}^*}{\mathbf{A}\bullet\mathbf{A}^*} \qquad (1.9)$$

Equation (1.8) implies that $\mathbf{A}\bullet\mathbf{X} = 0 \quad \textit{and} \quad \mathbf{A}\times\mathbf{X} = \mathbf{0} \not\Rightarrow \mathbf{A} = \mathbf{0} \quad \textit{or} \quad \mathbf{X} = \mathbf{0}$

Equation (1.9) implies that $\mathbf{A}^*\bullet\mathbf{X} = 0 \quad \textit{and} \quad \mathbf{A}\times\mathbf{X} = \mathbf{0} \Rightarrow \mathbf{A} = \mathbf{0} \quad \textit{or} \quad \mathbf{X} = \mathbf{0}$

1.3 Parallel and Perpendicular Vectors

We know that for parallel real vectors $\mathbf{A}\times\mathbf{B} = \mathbf{0}$ and for perpendicular real vectors $\mathbf{A}\bullet\mathbf{B} = 0$. The same definitions are used for complex vectors but obviously the geometrical interpretation doesn't exist anymore.

$$\mathbf{A}\times\mathbf{B} = \mathbf{0} \quad \textit{and} \quad \mathbf{A} \neq \mathbf{0} \quad \Rightarrow \quad \exists\alpha \in \mathbb{C}, \quad \mathbf{B} = \alpha\mathbf{A} \qquad (1.10)$$

$$\mathbf{A}\bullet\mathbf{B} = 0 \quad \textit{and} \quad \mathbf{A} \neq \mathbf{0} \quad \Rightarrow \quad \exists\mathbf{C} \in \mathbb{C}^3, \quad \mathbf{B} = \mathbf{C}\times\mathbf{A} \qquad (1.11)$$

We can calculate α and $\mathbf{C}$ in equations (1.10) and (1.11) using equation (1.7).

$$\mathbf{A}^*\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\left(\mathbf{A}^*\bullet\mathbf{B}\right) - \mathbf{B}\left(\mathbf{A}\bullet\mathbf{A}^*\right), \ \mathbf{A}\times\mathbf{B} = \mathbf{0} \quad \Rightarrow \quad \alpha = \frac{\mathbf{A}^*\bullet\mathbf{B}}{\mathbf{A}\bullet\mathbf{A}^*} \qquad (1.12)$$

$$\mathbf{A}\times\left(\mathbf{A}^*\times\mathbf{B}\right) = \mathbf{A}^*(\mathbf{A}\bullet\mathbf{B}) - \mathbf{B}\left(\mathbf{A}^*\bullet\mathbf{A}\right) \quad \Rightarrow \quad \mathbf{C} = \frac{\mathbf{A}^*\times\mathbf{B}}{\mathbf{A}\bullet\mathbf{A}^*} \qquad (1.13)$$

1.4 Polarization of Complex Vectors

It is very important to notice that the definition given here for the term "polarization" is different from the classical definition of polarization which assumes the complex vector is a phasor that represents a real vector field with harmonic time dependency. However this general definition coincides with the standard definition of polarization of a phasor when the complex vector is a phasor.

Definition:

A complex vector given as $\mathbf{V} = \mathbf{V}_\mathbf{r} + j\,\mathbf{V}_\mathbf{i}$ is linearly polarized (LP) if and only if $\mathbf{V}_\mathbf{r} \times \mathbf{V}_\mathbf{i} = \mathbf{0}$, which means $\mathbf{V}_\mathbf{r}$ and $\mathbf{V}_\mathbf{i}$ are parallel.

It can be shown that a complex vector is LP if and only if it can be written as multiplication of a complex scalar and a real vector.

$$\mathbf{V} = c\mathbf{A},\ c \in \mathbb{C},\ \mathbf{A} \in \mathbb{R}^3 \qquad (1.14)$$

Definition:

A complex vector given as $\mathbf{V} = \mathbf{V}_r + j\,\mathbf{V}_i$ is nonlinearly polarized (NLP) if $\mathbf{V}_\mathbf{r} \times \mathbf{V}_\mathbf{i} \neq \mathbf{0}$, which means $\mathbf{V}_\mathbf{r}$ and $\mathbf{V}_\mathbf{i}$ are not parallel and form a plane.

We define a curve using parameter $\varphi \in [0, 2\pi]$, and real vector $\mathbf{v}(\varphi)$ as

$$\mathbf{v}(\varphi) = \mathrm{Re}\left(\mathbf{V}e^{j\varphi}\right) = \mathbf{V}_\mathbf{r}\cos(\varphi) - \mathbf{V}_\mathbf{i}\sin(\varphi) \qquad (1.15)$$

This curve is in the plane formed by $\mathbf{V}_\mathbf{r}$ and $\mathbf{V}_\mathbf{i}$ (polarization plane) and to find equation of this curve we define to auxiliary vectors $\mathbf{a}$ and $\mathbf{b}$ which are also in polarization plane:

(1.16)

$\mathbf{a} = \mathbf{V_r} \times (\mathbf{V_r} \times \mathbf{V_i})$ $\qquad$ $\mathbf{b} = \mathbf{V_i} \times (\mathbf{V_r} \times \mathbf{V_i})$

With some straightforward vector operations we can find the equation of the curve as:

$$(\mathbf{a} \cdot \mathbf{v}(\varphi))^2 + (\mathbf{b} \cdot \mathbf{v}(\varphi))^2 = |\mathbf{V_r} \times \mathbf{V_i}|^4 \qquad (1.17)$$

Equation (1-17) is equation of an ellipse in the polarization plane.

If $\mathbf{V}$ is multiplied by a complex scalar the polarization ellipse will remain the same.

If $\mathbf{V_r} \cdot \mathbf{V_i} = 0$ the curve will be a circle and vector $\mathbf{V}$ is called circularly polarized (CP)

The following properties are straightforward to prove:

$\mathbf{V}$ is LP $\Leftrightarrow$ $\mathbf{V} \times \mathbf{V}^* = \mathbf{0}$

$\mathbf{V}$ is CP $\Leftrightarrow$ $\mathbf{V} \cdot \mathbf{V} = 0$

1.5 Dot Product and Cross Product for Complex Vectors

Real part and imaginary part of the complex vectors are similar to real and imaginary part of the complex vectors:

$$\mathrm{Re}(\mathbf{A}) = \frac{1}{2}(\mathbf{A} + \mathbf{A}^*) \qquad \mathrm{Im}(\mathbf{A}) = \frac{1}{2j}(\mathbf{A} - \mathbf{A}^*) \qquad (1.18)$$

There is no corresponding quantity to absolute value and argument (phase) of complex numbers unless we consider the components.

Conjugation distributes over the standard vector operations:

$$(\mathbf{A} \times \mathbf{B})^* = \mathbf{A}^* \times \mathbf{B}^*$$
$$(\mathbf{A} \bullet \mathbf{B})^* = \mathbf{A}^* \bullet \mathbf{B}^* \qquad (1.19)$$

Using (1.18) and (1.19) it is easy to prove that if $\mathbf{A} \in \mathbb{C}^3$ and $\mathbf{B} \in \mathbb{R}^3$ then we have:

$$\mathrm{Re}(\mathbf{A} \bullet \mathbf{B}) = \mathrm{Re}(\mathbf{A}) \bullet \mathbf{B} \quad , \quad \mathrm{Im}(\mathbf{A} \bullet \mathbf{B}) = \mathrm{Im}(\mathbf{A}) \bullet \mathbf{B} \qquad (1.20)$$

$$\mathrm{Re}(\mathbf{A} \times \mathbf{B}) = \mathrm{Re}(\mathbf{A}) \times \mathbf{B} \quad , \quad \mathrm{Im}(\mathbf{A} \times \mathbf{B}) = \mathrm{Im}(\mathbf{A}) \times \mathbf{B} \qquad (1.21)$$

If $\mathbf{V}$ and $\mathbf{W}$ are two LP vectors with different polarizations then we have:

$$\mathbf{V} + j\mathbf{W} = \mathbf{0} \Rightarrow \mathbf{V} = \mathbf{0} \wedge \mathbf{W} = \mathbf{0} \qquad (1.22)$$

If $\mathbf{V}$ is LP and $\mathbf{W} = \mathbf{W_r} + j\mathbf{W_i}$ is NLP then we have:

$$\mathbf{V} \times \mathbf{W} = \mathbf{0} \Rightarrow \mathbf{V} \times \mathbf{W_r} = \mathbf{0} \quad \wedge \quad \mathbf{V} \times \mathbf{W_i} = \mathbf{0} \qquad (1.23)$$

$$\mathbf{V} \bullet \mathbf{W} = 0 \Rightarrow \mathbf{V} \bullet \mathbf{W_r} = \mathbf{0} \quad \wedge \quad \mathbf{V} \bullet \mathbf{W_i} = \mathbf{0} \qquad (1.24)$$

Having real unit vector $\hat{u}$ we can write any given complex vector as summation of LP vector parallel to $\hat{u}$ and another vector that is normal to $\hat{u}$. In (1-8) we put $\mathbf{A} = \hat{u}$:

$$\mathbf{X} = (\hat{u} \bullet \mathbf{X})\hat{u} + (\hat{u} \times \mathbf{X}) \times \hat{u} \qquad (1.25)$$

The first term is a LP vector parallel to $\hat{u}$ and the second term is normal to $\hat{u}$.

Let's define $\mathbf{C} = \mathbf{A} \times \mathbf{B}$

If $\mathbf{A}$ and $\mathbf{B}$ are LP then $\mathbf{C}$ is also LP.

If $\mathbf{A}$ is LP but $\mathbf{B}$ is NLP then $\mathbf{C}$ is also NLP with the same plane of polarization as $\mathbf{B}$ and with $\mathbf{C_r}$ perpendicular to $\mathbf{B_r}$ and $\mathbf{C_i}$ perpendicular to $\mathbf{B_i}$.

Chapter 2

Plane Waves with Complex Wave Vectors

2.1 Introduction

In this chapter I would like to use the identities and properties of the complex vectors that were introduced in chapter 1. When total reflection happens at the interface between two dielectrics the refracted wave would be an evanescent wave and usually is studied by defining a fictitious angle that its sine or cosine is greater than one. Using complex vectors we show that propagating plane waves and evanescent plane waves are different cases of a plane wave with complex wave vector.

Using the concept of complex wave vector I will provide a unified approach for analysis of multilayered media consisting layers of dielectric, meta-material (single-negative and double negative) and highly conductive medium and active medium.

If a complex vector field has this form:

$$\mathbf{E}(\mathbf{r}) = \mathbf{E_0}\, f(\mathbf{r}) \qquad (2\text{-}1)$$

in which $\mathbf{E_0}$ is a constant complex vector and $f(\mathbf{r})$ is a complex scalar field. Then the polarization properties of complex vector field $\mathbf{E}(\mathbf{r})$ are determined by $\mathbf{E_0}$ and they don't change with position. Properties of scalar field $f(\mathbf{r})$ such as surfaces of constant phase and surfaces of constant amplitude are also attributed to the complex field $\mathbf{E}(\mathbf{r})$. We must keep in mind that all complex vector fields cannot be written the form of equation (2-1) necessarily and so properties such as surfaces of constant phase and amplitude are not defined in general for complex vector fields.

2.2 Complex Plane Wave Solution of Maxwell’s Equations

We start with source-free phasor-domain (frequency-domain) Maxwell Equations in a simple (linear, homogenous, isotropic) medium with complex permittivity ε and permeability μ regardless of realisability of them in real world.
The time dependence $\exp(j\omega t)$ is assumed henceforward.

$$\begin{cases} \nabla\times\mathbf{E} = -j\omega\mathbf{B} \\ \nabla\times\mathbf{H} = j\omega\mathbf{D} \\ \nabla\cdot\mathbf{D} = 0 \\ \nabla\cdot\mathbf{B} = 0 \end{cases} \qquad \begin{cases} \mathbf{B} = \mu\mathbf{H} \\ \mathbf{D} = \varepsilon\mathbf{E} \end{cases} \qquad \text{(2-2)}$$

and if we assume that the scalar function in equation (2-1) has this form:

$$f(\mathbf{r}) = \exp(-j\mathbf{k}\cdot\mathbf{r}) \qquad \text{(2-3)}$$

in which $\mathbf{k}$ is a complex vector.

Substituting $\mathbf{E}(\mathbf{r}) = \mathbf{E_0} \exp(-j\mathbf{k} \cdot \mathbf{r})$ in Maxwell equations after some straightforward simplification we obtain:

$$\begin{cases} \mathbf{k} \times \mathbf{E} = \omega\mu\mathbf{H} \\ \mathbf{k} \times \mathbf{H} = -\omega\varepsilon\mathbf{E} \\ \mathbf{k} \cdot \mathbf{E} = 0 \\ \mathbf{k} \cdot \mathbf{H} = 0 \end{cases} \qquad (2\text{-}4)$$

And with more simplification we arrive at:

$$\mathbf{k} \cdot \mathbf{k} = \omega^2 \mu\varepsilon \hat{=} k_1^2 \qquad (2\text{-}5)$$

Equation (2-5) shows that the wave vector $\mathbf{k}$ only depends on the product $\mu\varepsilon$ rather than ε and μ. The wave number is also defined as: $\omega^2 \mu\varepsilon \hat{=} k_1^2$, which in general is a complex number. It is very important to notice that $\mu\varepsilon$ can be real or pure imaginary regardless of ε and μ being complex, real or pure imaginary.

By writing $\mathbf{k}$ with respect to its real and imaginary components we have

$$\mathbf{k} = \mathbf{k_r} + j\mathbf{k_i} \qquad (2\text{-}6)$$

From (2-5) and (2-6) we obtain:

$$\begin{cases} |\mathbf{k_r}|^2 - |\mathbf{k_i}|^2 = \omega^2 \operatorname{Re}(\mu\varepsilon) \\ \mathbf{k_r} \cdot \mathbf{k_i} = \frac{1}{2}\omega^2 \operatorname{Im}(\mu\varepsilon) \end{cases} \qquad (2\text{-}7)$$

Equation (2-7) shows that it is useful to consider the special cases that $\mu\varepsilon$ is real or pure imaginary ($\operatorname{Re}(\mu\varepsilon) = 0$ or $\operatorname{Im}(\mu\varepsilon) = 0$).

Based on this we can recognize four types of material:

Type I	Type II	Type III	Type IV
$\begin{cases} \operatorname{Re}(\mu\varepsilon) > 0 \\ \operatorname{Im}(\mu\varepsilon) = 0 \end{cases}$	$\begin{cases} \operatorname{Re}(\mu\varepsilon) < 0 \\ \operatorname{Im}(\mu\varepsilon) = 0 \end{cases}$	$\begin{cases} \operatorname{Re}(\mu\varepsilon) = 0 \\ \operatorname{Im}(\mu\varepsilon) < 0 \end{cases}$	$\begin{cases} \operatorname{Re}(\mu\varepsilon) = 0 \\ \operatorname{Im}(\mu\varepsilon) > 0 \end{cases}$

In types I and II $\mu\varepsilon$ is real and in types III and IV $\mu\varepsilon$ is pure imaginary.

Lossless dielectrics and double negative meta-materials fall into the type I.

Plasma and single-negative meta-materials are of type II.

Highly conductive (lossy) media belong to type III.

Highly active media are in type IV.

But these categories are not limited to above-mentioned examples nonetheless.

Here are four cases of ε and μ that results in medium of type I:

$$\begin{cases}\varepsilon=\varepsilon' \\ \mu=\mu'\end{cases} \qquad \begin{cases}\varepsilon=-\varepsilon' \\ \mu=-\mu'\end{cases} \qquad \begin{cases}\varepsilon=j\varepsilon'' \\ \mu=-j\mu''\end{cases} \qquad \begin{cases}\varepsilon=-j\varepsilon'' \\ \mu=j\mu''\end{cases}$$

(ε', μ', ε'' and μ'' are positive real numbers)

Another equivalent form of expressing the conditions for different types of media is as follows:

$$\varepsilon=|\varepsilon|\exp(-j\delta_\varepsilon) \qquad\qquad \mu=|\mu|\exp(-j\delta_\mu)$$

$$\begin{cases}\delta_\varepsilon+\delta_\mu=2m\pi & type\ I \\ \delta_\varepsilon+\delta_\mu=(2m+1)\pi & type\ II \\ \delta_\varepsilon+\delta_\mu=2m\pi+\dfrac{\pi}{2} & type\ III \\ \delta_\varepsilon+\delta_\mu=2m\pi+\dfrac{3\pi}{2} & type\ IV\end{cases}$$

2.3 Complex Poynting Vector

In this section first the most general formulation for complex Poynting vector:

$$\mathbf{E}(\mathbf{r})=\mathbf{E_0}\exp(-j\mathbf{k}\cdot\mathbf{r}) \quad \text{and} \quad \mathbf{H}=\frac{1}{\omega\mu}\mathbf{k}\times\mathbf{E}=\mathbf{H}_0\exp(-j\mathbf{k}\cdot\mathbf{r}) \qquad (2\text{-}8)$$

We obtain:

$$\mathbf{S}(\mathbf{r}) = \frac{1}{\omega\mu^*}\exp(2\mathbf{k_i}\cdot\mathbf{r})\,\mathbf{E_0}\times\left(\mathbf{k}^*\times\mathbf{E_0}^*\right) \qquad (2\text{-}9)$$

And using identity (1.7) it can be written as:

$$\boxed{\mathbf{S}(\mathbf{r}) = \frac{1}{\omega\mu^*}\exp(2\mathbf{k_i}\cdot\mathbf{r})\left[\mathbf{k}^*\left|\mathbf{E_0}\right|^2 - \mathbf{E_0^*}\left(\mathbf{E_0}\cdot\mathbf{k}^*\right)\right]} \qquad (2\text{-}10)$$

Similarly we can obtain an alternative formulation with respect to magnetic fields:

$$\mathbf{S}(\mathbf{r}) = \frac{1}{\omega\varepsilon}\exp(2\mathbf{k_i}\cdot\mathbf{r})\,\mathbf{H_0^*}\times\left(\mathbf{k}\times\mathbf{H_0}\right) \qquad (2\text{-}11)$$

$$\boxed{\mathbf{S}(\mathbf{r}) = \frac{1}{\omega\varepsilon}\exp(2\mathbf{k_i}\cdot\mathbf{r})\left[\mathbf{k}\left|\mathbf{H_0}\right|^2 - \mathbf{H_0}\left(\mathbf{H_0^*}\cdot\mathbf{k}\right)\right]} \qquad (2\text{-}12)$$

2.3.1 LP Wave Vector

If $\mathbf{k}$ is linearly polarized, which means $\mathbf{k_r}$ and $\mathbf{k_i}$ are parallel:

$$\mathbf{k_r}\times\mathbf{k_i} = \mathbf{0} \qquad (2\text{-}13)$$

Since $\mathbf{k}$ is LP, $\mathbf{E_0}\cdot\mathbf{k} = 0$ results in $\mathbf{E_0}\cdot\mathbf{k}^* = 0$ regardless of polarization of $\mathbf{E_0}$.

Similarly we have $\mathbf{H_0}\cdot\mathbf{k}^* = 0$. Therefore equations (2-10) and (2-12) become:

$$\mathbf{S} = \frac{1}{\omega\mu^*}\exp(2\mathbf{k_i}\cdot\mathbf{r})\left|\mathbf{E_0}\right|^2\mathbf{k}^* = \frac{1}{\omega\varepsilon}\exp(2\mathbf{k_i}\cdot\mathbf{r})\left|\mathbf{H_0}\right|^2\mathbf{k} \qquad (2\text{-}14)$$

Therefore $\mathbf{S}$ is linearly polarized with same polarization as $\mathbf{k}$

Assuming $\varepsilon = \varepsilon' - j\varepsilon''$ and $\mu = \mu' - j\mu''$, the average of time-domain Poynting vector can be calculated as:

$$\boxed{\begin{aligned}\left\langle \vec{\mathscr{S}}(\mathbf{r},t)\right\rangle &= \frac{1}{2}\mathrm{Re}\left[\mathbf{S}(\mathbf{r})\right] = \frac{1}{2\omega|\varepsilon|^2}\exp(2\mathbf{k_i}\cdot\mathbf{r})|\mathbf{H_0}|^2(\varepsilon'\mathbf{k_r} - \varepsilon''\mathbf{k_i}) \\ &= \frac{1}{2\omega|\mu|^2}\exp(2\mathbf{k_i}\cdot\mathbf{r})|\mathbf{E_0}|^2(\mu'\mathbf{k_r} - \mu''\mathbf{k_i})\end{aligned}} \qquad (2\text{-}15)$$

We have the following observations:

1- When both $\mathbf{k_r}$ and $\mathbf{k_i}$ are non-zero, direction of flow of power ($\mathbf{k_r}$) and direction of changing its amplitude ($\mathbf{k_i}$) are the same. In other words the planes of constant amplitude and planes of constant phase are parallel.

2-For media of type I and II we have $\mathbf{k_r}\bullet\mathbf{k_i} = 0$ which together with equation (2-13) implies that $\mathbf{k_r} = \mathbf{0}$ or $\mathbf{k_i} = \mathbf{0}$.

3- For type I media using equation (2-7) only $\mathbf{k_i} = \mathbf{0}$ is acceptable. This implies that there is no change in amplitude. In other words linearly polarized wave vector in media of type I is always a real vector. The complex Poynting vector using equation (2-14) is:

$$\mathbf{S} = \frac{1}{\omega\mu^*}|\mathbf{E_0}|^2\mathbf{k_r} = \frac{1}{\omega\varepsilon}|\mathbf{H_0}|^2\mathbf{k_r} \qquad (2\text{-}16)$$

which is a constant LP vector in the direction of $\mathbf{k} = \mathbf{k_i}$

The average time-domain Poynting vector simplifies to the following:

$$\left\langle \vec{\mathscr{S}}(\mathbf{r},t)\right\rangle = \frac{\varepsilon'}{2\omega|\varepsilon|^2}|\mathbf{H_0}|^2\mathbf{k_r} = \frac{\mu'}{2\omega|\mu|^2}|\mathbf{E_0}|^2\mathbf{k_r} \qquad (2\text{-}17)$$

Depending on the sign of ε', the time average Poynting vector $\left\langle \vec{\mathscr{S}}(\mathbf{r},t)\right\rangle$ and $\mathbf{k_r}$ are parallel or anti-parallel.

4- For type II based on equation (2-7) only $\mathbf{k_r} = \mathbf{0}$ is acceptable.

In other words linearly polarized wave vector for type II media is pure imaginary. The complex Poynting vector using equation (2-14) is:

$$\mathbf{S} = \frac{-j}{\omega\mu^*}\exp(2\mathbf{k_i}\cdot\mathbf{r})|\mathbf{E_0}|^2\,\mathbf{k_i} = \frac{j}{\omega\varepsilon}\exp(2\mathbf{k_i}\cdot\mathbf{r})|\mathbf{H_0}|^2\,\mathbf{k_i} \qquad (2\text{-}18)$$

The complex Poynting vector is a LP vector with same polarization as $\mathbf{k_i}$.

The average time-domain Poynting vector becomes:

$$\langle\vec{\mathscr{S}}(\mathbf{r},t)\rangle = \frac{1}{2}\mathrm{Re}[\mathbf{S}(\mathbf{r})] = \frac{-\varepsilon''}{2\omega|\varepsilon|^2}\exp(2\mathbf{k_i}\cdot\mathbf{r})|\mathbf{H_0}|^2\,\mathbf{k_i}$$
$$= \frac{\mu''}{2\omega|\mu|^2}\exp(2\mathbf{k_i}\cdot\mathbf{r})|\mathbf{E_0}|^2\,\mathbf{k_i} \qquad (2\text{-}19)$$

So if $\varepsilon''=0$ there will be no flow of power in media of type II.

5- For type III and IV media, both $\mathbf{k_r}$ and $\mathbf{k_i}$ are non-zero. They are anti-parallel for type III and they are parallel for type IV.

If $\mathbf{k}$ is LP, $\mathbf{E_0}$ and $\mathbf{H}_0$ are both LP (and normal to each other) or both are NLP with the same plane of polarization which is normal to $\mathbf{k}$.

2.3.2 NLP Wave Vector

Both $\mathbf{k_r}$ and $\mathbf{k_i}$ are nonzero because otherwise $\mathbf{k}$ would be LP.

If $\mathbf{k}$ is NLP, $\mathbf{E_0}$ and $\mathbf{H}_0$ cannot be LP at the same time, so we distinguish the following three cases

1- $\mathbf{E_0}$ is LP:

$$\mathbf{E_0}\cdot\mathbf{k} = 0 \Rightarrow \mathbf{E_0}\cdot\mathbf{k}^* = 0 \quad \wedge \quad \mathbf{E_0}\cdot\mathbf{k_r} = 0 \quad \wedge \quad \mathbf{E_0}\cdot\mathbf{k_i} = 0 \qquad (2\text{-}20)$$

This implies that $\mathbf{E_0}$ is perpendicular to plane of polarization of $\mathbf{k}$. $\mathbf{E_0}$ is also normal to plane of polarization of $\mathbf{H}_0$. Therefore $\mathbf{H}_0$ and $\mathbf{k}$ have the same plane of polarization. Using equation (2-20) we can simplify the general equation for complex Poynting vector (equation (2-10)) as follows:

$$\mathbf{S}(\mathbf{r}) = \frac{1}{\omega\mu^*}\exp(2\mathbf{k_i}\cdot\mathbf{r})|\mathbf{E_0}|^2\,\mathbf{k}^* \qquad (2\text{-}21)$$

and the average time-domain Poynting vector is obtained as:

$$\boxed{\left\langle \vec{\mathcal{S}}(\mathbf{r},t)\right\rangle = \frac{1}{2}\mathrm{Re}\left[\mathbf{S}(\mathbf{r})\right] = \frac{1}{2\omega|\mu|^2}\exp(2\mathbf{k_i}\cdot\mathbf{r})|\mathbf{E_0}|^2(\mu'\mathbf{k_r} - \mu''\mathbf{k_i})} \qquad (2\text{-}22)$$

We observe that $\left\langle \vec{\mathcal{S}}(\mathbf{r},t)\right\rangle$ is a function of position and it is parallel to $\mathbf{k_r}$ if $\mu''=0$. In this case for media of type I and II in which $\mathbf{k_r}\cdot\mathbf{k_i}=0$ the direction that $\left\langle \vec{\mathcal{S}}(\mathbf{r},t)\right\rangle$ decays is $-\mathbf{k_i}$ and the direction of the flow of power is $\mathbf{k_r}$. Since the direction of $\left\langle \vec{\mathcal{S}}(\mathbf{r},t)\right\rangle$ is parallel to the plane of constant $\left|\left\langle \vec{\mathcal{S}}(\mathbf{r},t)\right\rangle\right|$ this type of wave is called surface wave, which is the characteristic of the wave in infinite section of dielectric waveguide structures.

The criteria for the propagation to be attenuating, lossless or amplifying, is determined by sign of $\mathbf{k_i}\cdot\left\langle \vec{\mathcal{S}}(\mathbf{r},t)\right\rangle$. If $\mathbf{k_i}\cdot\left\langle \vec{\mathcal{S}}(\mathbf{r},t)\right\rangle$ is negative the propagation is lossy because $\left|\left\langle \vec{\mathcal{S}}(\mathbf{r},t)\right\rangle\right|$ decreases in the direction of flow of power. If it is positive the propagation is amplifying because $\left|\left\langle \vec{\mathcal{S}}(\mathbf{r},t)\right\rangle\right|$ increases in the direction of flow of energy. If $\mathbf{k_i}\cdot\left\langle \vec{\mathcal{S}}(\mathbf{r},t)\right\rangle$ is zero the $\left|\left\langle \vec{\mathcal{S}}(\mathbf{r},t)\right\rangle\right|$ is constant in the direction of propagation.

$$\mathbf{k_i}\cdot\left\langle \vec{\mathscr{S}}(\mathbf{r},t)\right\rangle > 0 \Rightarrow \quad \text{amplifying propagation}$$
$$\mathbf{k_i}\cdot\left\langle \vec{\mathscr{S}}(\mathbf{r},t)\right\rangle < 0 \Rightarrow \quad \text{attenuating propagation}$$
$$\mathbf{k_i}\cdot\left\langle \vec{\mathscr{S}}(\mathbf{r},t)\right\rangle = 0 \Rightarrow \quad \text{lossless propagation}$$

Using equation (2-22) sign of $\mathbf{k_i}\cdot\left\langle \vec{\mathscr{S}}(\mathbf{r},t)\right\rangle$ is determined by

$$\mathbf{k_i}.\left(\mu'\mathbf{k_r} - \mu''\mathbf{k_i}\right) = \mu'\mathbf{k_i}.\mathbf{k_r} - \mu''\left|\mathbf{k_i}\right|^2$$

So the criteria for type of propagation (amplifying, attenuating, and lossless) in general depend on both medium and wave propagation vector:

$$\frac{\omega^2}{2}\mu'\operatorname{Im}(\mu\varepsilon) - \mu''\left|\mathbf{k_i}\right|^2 > 0 \Rightarrow \quad \text{amplifying propagation}$$
$$\frac{\omega^2}{2}\mu'\operatorname{Im}(\mu\varepsilon) - \mu''\left|\mathbf{k_i}\right|^2 < 0 \Rightarrow \quad \text{attenuating propagation}$$
$$\frac{\omega^2}{2}\mu'\operatorname{Im}(\mu\varepsilon) - \mu''\left|\mathbf{k_i}\right|^2 = 0 \Rightarrow \quad \text{lossless propagation}$$

Hence for media of types I and II that we have $\mathbf{k_i}.\mathbf{k_r} = 0$ the sign is independent of wave vector and only depends on medium. Medium is lossy if $\mu'' > 0$ it is lossless if μ'' =0 and it is active if $\mu'' < 0$.

2-$\mathbf{H}_0$ is LP:

$$\mathbf{H_0}\cdot\mathbf{k} = 0 \Rightarrow \mathbf{H_0}\cdot\mathbf{k}^* = 0 \quad \wedge \quad \mathbf{H_0}\cdot\mathbf{k_r} = 0 \quad \wedge \quad \mathbf{H_0}\cdot\mathbf{k_i} = 0 \qquad \text{(2-23)}$$

Since $\mathbf{H_0}$ is normal to plane of polarization of both $\mathbf{k}$ and $\mathbf{E_0}$, they have the same plane of polarization. Using equation (2-23) we can simplify the general equation for complex Poynting vector (equation (2-12)) to the following form:

$$\mathbf{S}(\mathbf{r}) = \frac{1}{\omega\varepsilon}\exp\left(2\mathbf{k_i}\cdot\mathbf{r}\right)\left|\mathbf{H_0}\right|^2\mathbf{k} \qquad \text{(2-24)}$$

and the average time-domain Poynting vector is obtained as:

$$\boxed{\left\langle \vec{\mathcal{S}}(\mathbf{r},t)\right\rangle = \frac{1}{2}\mathrm{Re}\left[\mathbf{S}(\mathbf{r})\right] = \frac{1}{2\omega|\varepsilon|^2}\exp\left(2\mathbf{k_i}\cdot\mathbf{r}\right)|\mathbf{H_0}|^2\left(\varepsilon'\mathbf{k_r} - \varepsilon''\mathbf{k_i}\right)} \qquad (2\text{-}25)$$

$\left\langle \vec{\mathcal{S}}(\mathbf{r},t)\right\rangle$ is a function of position and it is parallel to $\mathbf{k_r}$ if $\varepsilon'' = 0$.

Similar to the previous case we can determine that medium is lossy, lossless or active based on sign of $\mathbf{k_i}\cdot\left\langle \vec{\mathcal{S}}(\mathbf{r},t)\right\rangle$.

Using equation (2-22) the sign of $\mathbf{k_i}\cdot\left\langle \vec{\mathcal{S}}(\mathbf{r},t)\right\rangle$ is determined by

$$\mathbf{k_i}.\left(\varepsilon'\mathbf{k_r} - \varepsilon''\mathbf{k_i}\right) = \varepsilon'\mathbf{k_i}.\mathbf{k_r} - \varepsilon''|\mathbf{k_i}|^2$$

So the criteria for type of propagation (amplifying, attenuating, and lossless) in general depend on both medium and wave propagation vector:

$$\frac{\omega^2}{2}\varepsilon'\,\mathrm{Im}(\mu\varepsilon) - \varepsilon''|\mathbf{k_i}|^2 > 0 \Rightarrow \quad \text{amplifying propagation}$$
$$\frac{\omega^2}{2}\varepsilon'\,\mathrm{Im}(\mu\varepsilon) - \varepsilon''|\mathbf{k_i}|^2 < 0 \Rightarrow \quad \text{attenuating propagation}$$
$$\frac{\omega^2}{2}\varepsilon'\,\mathrm{Im}(\mu\varepsilon) - \varepsilon''|\mathbf{k_i}|^2 = 0 \Rightarrow \quad \text{lossless propagation}$$

3- Both $\mathbf{E_0}$ and $\mathbf{H}_0$ are NLP. This is the most general case and there is no simplification possible. Also there is no geometrical interpretation for two normal vectors that both are nonlinear polarized.

By comparing equations (2-22) and (2-25) we conclude that for waves with NLP wave vector, $\left\langle \vec{\mathcal{S}}(\mathbf{r},t)\right\rangle$ depends on polarization of $\mathbf{E_0}$ and $\mathbf{H}_0$ but according to equation (2-15), $\left\langle \vec{\mathcal{S}}(\mathbf{r},t)\right\rangle$ is independent of $\mathbf{E_0}$ and $\mathbf{H}_0$ polarization for waves with LP wave vector.

To investigate case 3, we need to use the TE/TM decomposition theorem.

In next section we will review this theorem and derive a formulation for $\left\langle \vec{\mathcal{S}}(\mathbf{r},t)\right\rangle$ and study its implications and applications.

2.4 TE/TM Decomposition

According to TE/TM decomposition theorem if $(\mathbf{E},\mathbf{H})$ are a solution to source free Maxwell equation in a simple medium, and $\hat{u}$ is an arbitrary real unit vector then we can express $(\mathbf{E},\mathbf{H})$ as superposition of two set of TE fields $(\mathbf{E}_{TE},\mathbf{H}_{TE})$ and TM fields $(\mathbf{E}_{TM},\mathbf{H}_{TM})$ that each one is a solution to Maxwell equations in that medium and they have these properties:

$$\begin{cases} \mathbf{E} = \mathbf{E}_{TE} + \mathbf{E}_{TM} \quad , \quad \mathbf{E}_{TE} \cdot \hat{u} = 0 \\ \mathbf{H} = \mathbf{H}_{TE} + \mathbf{H}_{TM} \quad , \quad \mathbf{H}_{TM} \cdot \hat{u} = 0 \end{cases}$$

Moreover all components of TE fields $(\mathbf{E}_{TE},\mathbf{H}_{TE})$ can be expressed with respect to a scalar function (field) $\psi_{TE}(\mathbf{r})$ and all components of TM fields $(\mathbf{E}_{TM},\mathbf{H}_{TM})$ can be expressed with respect to another scalar function $\psi_{TM}(\mathbf{r})$.

Both $\psi_{TE}(\mathbf{r})$ and $\psi_{TM}(\mathbf{r})$ satisfy Helmholtz equation.

In the special case that there is a direction $\hat{u}$ such that the initial field $(\mathbf{E},\mathbf{H})$ is independent of the projection of position vector in $\hat{u}$ direction (i.e. $(\mathbf{r}\cdot\hat{u})\hat{u}$) (for example if $\hat{u}=\hat{y}$ then the field are independent of *y* and are only function of *x* and *z*)

Then the scalar functions $\psi_{TE}(\mathbf{r})$ and $\psi_{TM}(\mathbf{r})$ will be the components of $(\mathbf{E},\mathbf{H})$ in $\hat{u}$ direction:

$\psi_{TM}(\mathbf{r}) = \mathbf{E}\cdot\hat{u}$ and $\psi_{TE}(\mathbf{r}) = \mathbf{H}\cdot\hat{u}$

$\mathbf{E}_{TM}$ and $\mathbf{H}_{TE}$ will be LP vectors parallel to $\hat{u}$ and given as:

$\mathbf{E}_{TM} = (\mathbf{E}\cdot\hat{u})\hat{u}$ and $\mathbf{H}_{TE} = (\mathbf{H}\cdot\hat{u})\hat{u}$

And the total field can be written as:

$$\begin{cases}\mathbf{E} = (\mathbf{E}\cdot\hat{u})\hat{u} + \mathbf{E}_{TE}\\ \mathbf{H} = (\mathbf{H}\cdot\hat{u})\hat{u} + \mathbf{H}_{TM}\end{cases}$$

Now given the plane wave with fields $\mathbf{E}(\mathbf{r}) = \mathbf{E_0}\exp(-j\mathbf{k}\cdot\mathbf{r})$ and $\mathbf{H}(\mathbf{r}) = \mathbf{H_0}\exp(-j\mathbf{k}\cdot\mathbf{r})$ and NLP wave vector $\mathbf{k}$, if we choose $\hat{u}$ normal to $\mathbf{k}$ we have:

$$\mathbf{r} = \hat{u}\times(\mathbf{r}\times\hat{u}) + \hat{u}(\mathbf{r}\cdot\hat{u}) \Rightarrow \mathbf{k}\cdot\mathbf{r} = \mathbf{k}\cdot\hat{u}\times(\mathbf{r}\times\hat{u})$$

which shows $\mathbf{k}\cdot\mathbf{r}$ is independent of $\mathbf{r}\cdot\hat{u}$ and we can decompose the field to TE/TM modes as follows:

$$\mathbf{E}_{TM}(\mathbf{r}) = \mathbf{E}_{\mathbf{0}TM}\exp(-j\mathbf{k}\cdot\mathbf{r}) \qquad \mathbf{H}_{TE}(\mathbf{r}) = \mathbf{H}_{\mathbf{0}TE}\exp(-j\mathbf{k}\cdot\mathbf{r})$$

$$\begin{cases}\mathbf{E}_{\mathbf{0}TM} = (\mathbf{E_0}\cdot\hat{u})\hat{u}\\ \mathbf{H}_{\mathbf{0}TM} = \dfrac{\mathbf{E_0}\cdot\hat{u}}{\omega\mu}(\mathbf{k}\times\hat{u})\end{cases} \qquad \begin{cases}\mathbf{H}_{\mathbf{0}TE} = (\mathbf{H_0}\cdot\hat{u})\hat{u}\\ \mathbf{E}_{\mathbf{0}TE} = \dfrac{-\mathbf{H_0}\cdot\hat{u}}{\omega\varepsilon}(\mathbf{k}\times\hat{u})\end{cases}$$

Now we find the relationship between total complex Poynting vector $(\mathbf{S})$ and complex Poynting vector of each mode ($\mathbf{S}_{TE}$ and $\mathbf{S}_{TM}$)

$$\mathbf{S} = \mathbf{E}\times\mathbf{H}^* = (\mathbf{E}_{TE} + \mathbf{E}_{TM})\times(\mathbf{H}^*_{TE} + \mathbf{H}^*_{TM})$$

This can be simplified to

$$\mathbf{S} = \mathbf{S}_{TE} + \mathbf{S}_{TM} + \mathbf{S}_{cross}$$

$$\mathbf{S}_{cross} = \mathbf{E}_{TE}\times\mathbf{H}^*_{TM} = \frac{E^*_{0u}H_{0u}}{\omega^2\mu^*\varepsilon}\exp(2\mathbf{k_i}\cdot\mathbf{r})\,\hat{u}\cdot(\mathbf{k}^*\times\mathbf{k})\hat{u}$$

$$E_{0u} = \mathbf{E_0}\cdot\hat{u},\ H_{0u} = \mathbf{H_0}\cdot\hat{u}$$

Therefore $\mathbf{S}_{cross}$ is a LP vector parallel to $\hat{u}$.

If $\mathbf{k}$ is LP then $\mathbf{k}^* \times \mathbf{k} = \mathbf{0}$ and therefore $\mathbf{S}_{cross} = \mathbf{0}$ and superposition is valid for any direction $\hat{u}$ (which is normal to $\mathbf{k}$). This can also be verified using equation (2-14).

If $\mathbf{k}$ is NLP then $\mathbf{k}^* \times \mathbf{k} = 2j\mathbf{k}_r \times \mathbf{k}_i$ is a LP vector parallel to $\hat{u}$ and $\hat{u} \cdot \left(\mathbf{k}^* \times \mathbf{k}\right)$ is never zero. So superposition is only valid for the directions defined by real unit vector $\hat{v}$ that $\hat{v} \cdot \hat{u} = 0$ because for these directions we have $\mathbf{S}_{cross} \cdot \hat{v} = 0$ and

$$\mathbf{S} \cdot \hat{v} = \mathbf{S}_{TE} \cdot \hat{v} + \mathbf{S}_{TM} \cdot \hat{v}$$

$$\left\langle \vec{\mathscr{S}}\left(\mathbf{r},t\right)\right\rangle \cdot \hat{v} = \left\langle \vec{\mathscr{S}}_{TE}\left(\mathbf{r},t\right)\right\rangle \cdot \hat{v} + \left\langle \vec{\mathscr{S}}_{TM}\left(\mathbf{r},t\right)\right\rangle \cdot \hat{v}$$

And since magnetic field of TE mode is LP, from equations (2-24) and (2-25) we obtain:

$$\mathbf{S}_{TE} = \frac{1}{\omega\varepsilon} \exp\left(2\mathbf{k_i} \cdot \mathbf{r}\right)\left|H_{0u}\right|^2 \mathbf{k}$$

$$\left\langle \vec{\mathscr{S}}_{TE}\left(\mathbf{r},t\right)\right\rangle = \frac{1}{2\omega\left|\varepsilon\right|^2} \exp\left(2\mathbf{k_i} \cdot \mathbf{r}\right)\left|H_{0u}\right|^2 \left(\varepsilon'\mathbf{k_r} - \varepsilon''\mathbf{k_i}\right)$$

And since electric field of TM mode is LP, using equation (2-21) and (2-22) we have:

$$\mathbf{S}_{TM} = \frac{1}{\omega\mu^*} \exp\left(2\mathbf{k_i} \cdot \mathbf{r}\right)\left|E_{0u}\right|^2 \mathbf{k}^*$$

$$\left\langle \vec{\mathscr{S}}_{TM}\left(\mathbf{r},t\right)\right\rangle = \frac{1}{2\omega\left|\mu\right|^2} \exp\left(2\mathbf{k_i} \cdot \mathbf{r}\right)\left|E_{0u}\right|^2 \left(\mu'\mathbf{k_r} - \mu''\mathbf{k_i}\right)$$

TE mode has a LP magnetic field $\mathbf{H}$ and the TM mode has a LP electric field $\mathbf{E}$.

$$\left\langle \vec{\mathscr{S}}\left(\mathbf{r},t\right)\right\rangle = \left\langle \vec{\mathscr{S}}_{TE}\left(\mathbf{r},t\right)\right\rangle + \left\langle \vec{\mathscr{S}}_{TM}\left(\mathbf{r},t\right)\right\rangle + \frac{1}{2}\mathrm{Re}\left(\mathbf{S}_{cross}\right)$$

$$\mathbf{k_i} \cdot \left\langle \vec{\mathscr{S}}\left(\mathbf{r},t\right)\right\rangle = \mathbf{k_i} \cdot \left\langle \vec{\mathscr{S}}_{TE}\left(\mathbf{r},t\right)\right\rangle + \mathbf{k_i} \cdot \left\langle \vec{\mathscr{S}}_{TM}\left(\mathbf{r},t\right)\right\rangle + \frac{1}{2}\mathrm{Re}\left(\mathbf{k_i} \cdot \mathbf{S}_{cross}\right)$$

$$\mathbf{k} \cdot \hat{u} = 0 \Rightarrow \mathbf{k}_i \cdot \hat{u} = 0 \Rightarrow \mathbf{k_i} \cdot \mathbf{S}_{cross} = 0$$

Therefore we have:

$$\mathbf{k_i}\cdot\left\langle \vec{\mathscr{S}}(\mathbf{r},t)\right\rangle = \mathbf{k_i}\cdot\left\langle \vec{\mathscr{S}}_{TE}(\mathbf{r},t)\right\rangle + \mathbf{k_i}\cdot\left\langle \vec{\mathscr{S}}_{TM}(\mathbf{r},t)\right\rangle$$

If $\mathbf{k_i}\cdot\left\langle \vec{\mathscr{S}}_{TE}(\mathbf{r},t)\right\rangle < 0$ and $\mathbf{k_i}\cdot\left\langle \vec{\mathscr{S}}_{TM}(\mathbf{r},t)\right\rangle < 0$ then $\mathbf{k_i}\cdot\left\langle \vec{\mathscr{S}}(\mathbf{r},t)\right\rangle < 0$. In other words if a wave is attenuating in both TE and TM modes then it is also attenuating when both $\mathbf{E_0}$ and $\mathbf{H_0}$ are NLP.

In next section we will investigate the problem of plane boundary between two simple medium.

We will define an incident plane that contains the normal to the boundary.

If $\mathbf{k}$ is LP then the incidence plane is defined by normal to the boundary and $\mathbf{k}$. If $\mathbf{k}$ is NLP we assume that normal to the boundary is parallel to polarization plane of $\mathbf{k}$ and call it incidence plane.

We choose the normal to the boundary as *z* axis and the incidence plane as *xz* plane (see figures 2.1 and 2.5). So the problem is 2D along y axis and the unit vector $\hat{u} = \hat{y}$ would satisfy the condition for 2D TE/TM decomposition theorem as stated above.

One of the important results that is useful for multilayer media is:

$$\hat{z}\cdot\left\langle \vec{\mathscr{S}}(\mathbf{r},t)\right\rangle = \hat{z}\cdot\left\langle \vec{\mathscr{S}}_{TE}(\mathbf{r},t)\right\rangle + \hat{z}\cdot\left\langle \vec{\mathscr{S}}_{TM}(\mathbf{r},t)\right\rangle$$

All possible cases of polarization of $\mathbf{k}$, $\mathbf{E_0}$ and $\mathbf{H}_0$ are shown in table 2-1.

Table 2.1: All possible polarizations for parameters of a plane wave

$\mathbf{k}$	$\mathbf{E_0}$	$\mathbf{H}_0$	TE_y/TM_y decomposition
LP	LP	LP	TE_y or TM_y or TE_y+TM_y
LP	NLP	NLP	TE_y+TM_y
NLP	LP	NLP	TM_y
NLP	NLP	LP	TE_y
NLP	NLP	NLP	TE_y+TM_y

2.5 Geometrical Representation of Complex Wave Vector for Different Types of Media

Using equation (2-7) we can draw different configurations of complex wave vector in a given coordinate system. We will use these representations later in study of multilayer media.

Since a NLP (nonlinear polarized) complex vector defines a plane we can choose this plane to be *xz* plane without loss of generality (WLOG)

For a medium of type I different possibilities are shown in Figure 2.1.

The black vectors are real part and the red vectors are imaginary part of complex vectors.

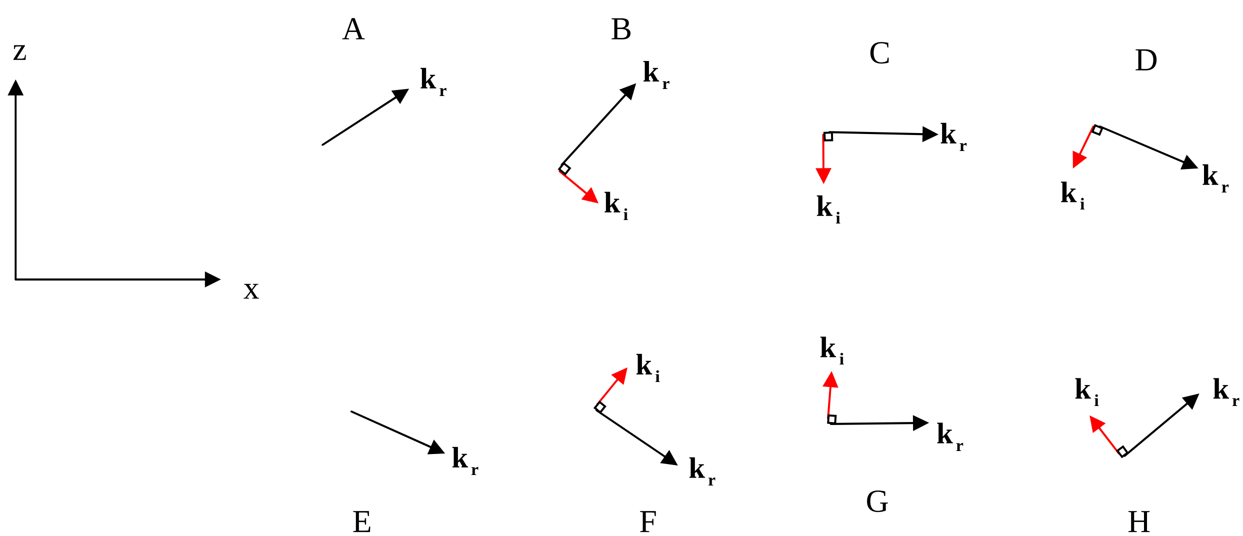

Figure 2.1: Different configurations of complex wave vector in type I medium

Since later we will consider multilayer media that interfaces are normal to z-axis, we don't need to consider the cases that are symmetric to z-axis in Figure 2.1

Cases A and E are LP and other cases are NLP.

The possible cases for type II media are very similar to type I except that $|\mathbf{k_i}| > |\mathbf{k_r}|$ and $\mathbf{k_r} = \mathbf{0}$ for linear polarization. These are shown in Figure (2.2):

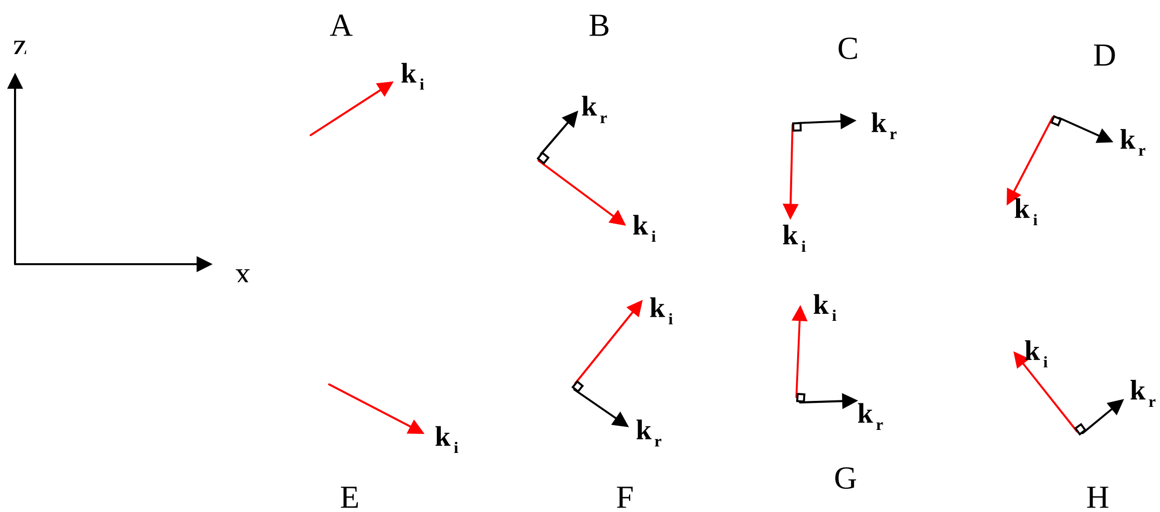

Figure 2.2: Different configurations of complex wave vector in type II medium

For type III medium we have $|\mathbf{k_i}| = |\mathbf{k_r}|$ and $\mathbf{k_r} \cdot \mathbf{k_i} < 0$. Different cases are depicted in figure (2.3)

In this figure cases A and E are LP and other cases are NLP.

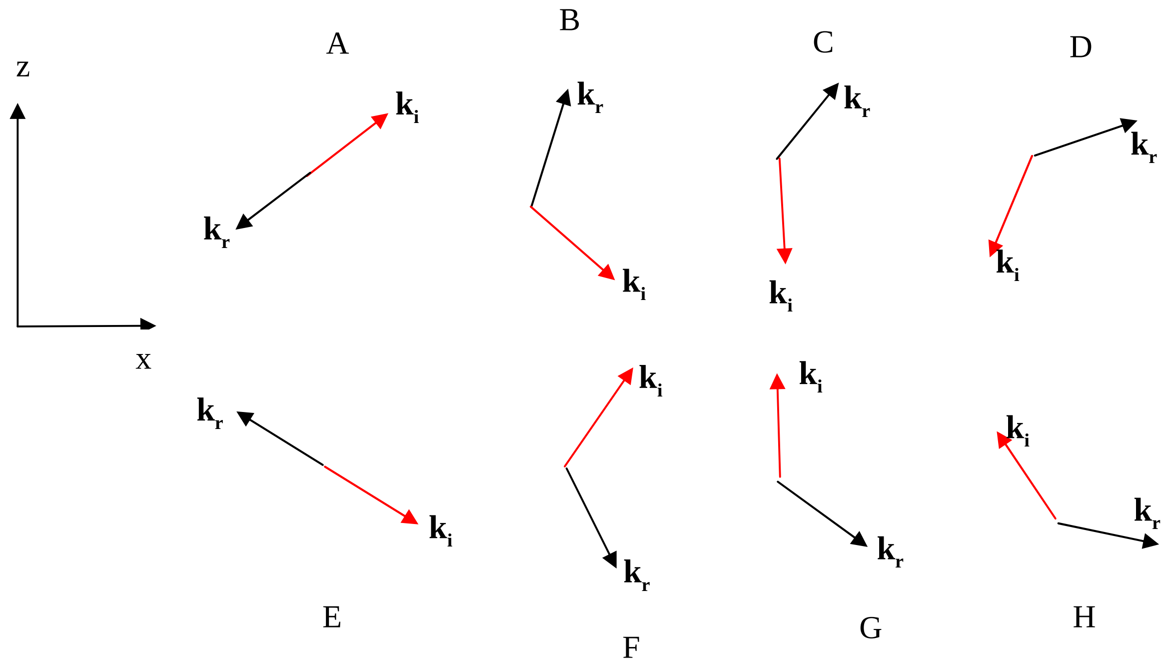

Figure 2.3: Different configurations of complex wave vector in type III medium

For type IV medium we have $|\mathbf{k_i}| = |\mathbf{k_r}|$ and $\mathbf{k_r} \cdot \mathbf{k_i} > 0$. Different cases are illustrated in figure (2.4). In cases A and E the complex wave vector is LP and in other cases it is NLP.

This type media is active because for example when the complex vector is LP according to equation (2-15) the average time-domain Poynting vector flows in the same direction ($\mathbf{k_r}$) that its amplitude increases ($\mathbf{k_i}$) which is characteristic of an active medium.

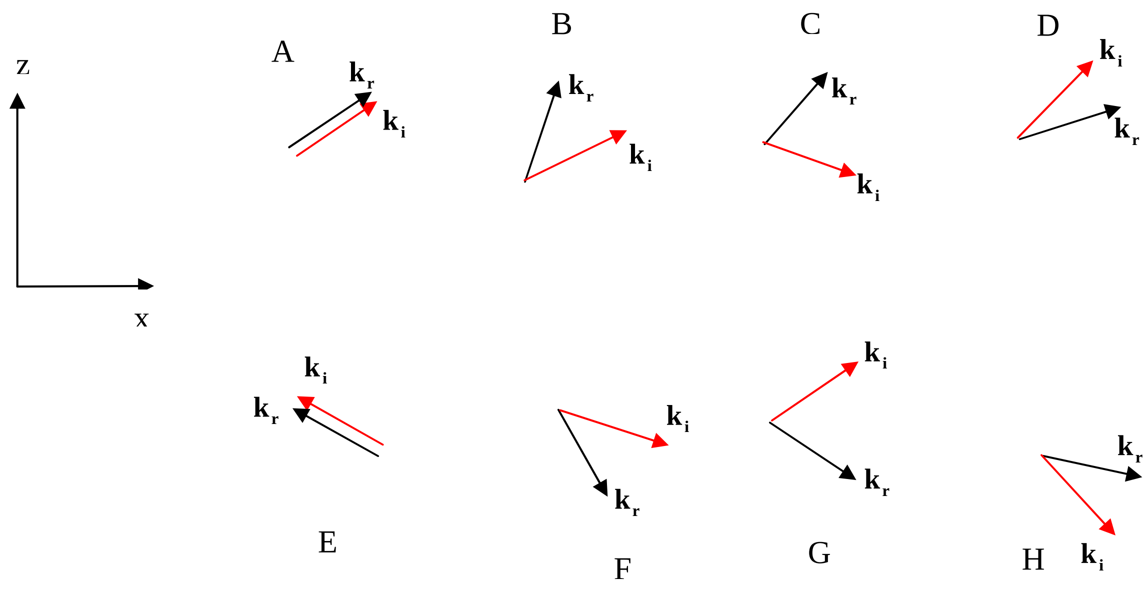

Figure 2.4: Different configurations of complex wave vector in type IV medium

Chapter 3

Complex Plane Wave Solution of Maxwell Equations in Presence of Plane Boundaries

3.1 Complex Wave Vector Matching

In this chapter first we investigate the plane wave solution of Maxwell equations in a plane boundary between two simple medium. A plane wave solution in medium 1 is considered as incident wave or excitation.

The *z* axis is the normal to the plane boundary between two simple medium with given permittivity and permeability.

The incident wave (which is solution of Maxwell equations for a simple medium with ε_1, μ_1) is represented as $\mathbf{E}_1(\mathbf{r}) = \mathbf{E}_{\mathbf{01}} \exp(-j\mathbf{k}_1 \cdot \mathbf{r})$. The complex wave vector $\mathbf{k}_1$ can be LP or NLP. If it is LP we use its polarization to define the incidence plane (*xz*). If it is NLP we assume that z axis is parallel to its polarization plane and so the *xz* will be the polarization plane of $\mathbf{k}_1$, so it will not have any *Y*-component.

We define two (unknown) plane waves that together with the incident wave are going to satisfy both Maxwell equations and boundary conditions.

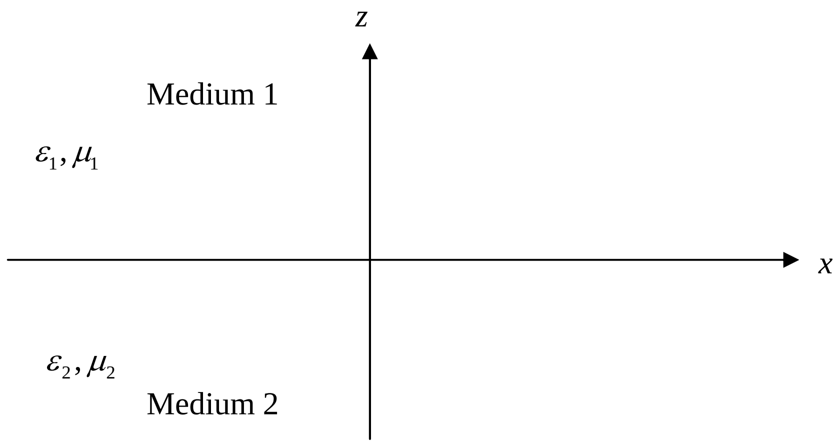

Figure 3.1: Two simple medium with plane boundary between them

The unknown plane wave in medium 1 and is called "reflected wave" and is also a solution of Maxwell equations in this medium:

$$\mathbf{E}_3(\mathbf{r}) = \mathbf{E}_{\mathbf{0}3} \exp(-j\mathbf{k}_3 \cdot \mathbf{r})$$

in which $\mathbf{E}_{\mathbf{0}3}$ and $\mathbf{k}_3$ are unknown.

Another unknown plane wave (transmitted or refracted wave) is defined in medium 2 and is represented as $\mathbf{E}_2(\mathbf{r}) = \mathbf{E}_{\mathbf{0}2} \exp(-j\mathbf{k}_2 \cdot \mathbf{r})$ in which $\mathbf{E}_{\mathbf{0}2}$ and $\mathbf{k}_2$ are unknown. This is a solution of Maxwell equations in medium 2.

Continuity of tangential electric field at the boundary gives us this equation:

$$\hat{z} \times \mathbf{E}_{\mathbf{0}1} \exp(-j\mathbf{k}_1 \cdot \mathbf{r}) + \hat{z} \times \mathbf{E}_{\mathbf{0}3} \exp(-j\mathbf{k}_3 \cdot \mathbf{r}) = \hat{z} \times \mathbf{E}_{\mathbf{0}2} \exp(-j\mathbf{k}_2 \cdot \mathbf{r}) \quad @ \quad z = 0 \quad \text{(3-1)}$$

The necessary condition for equation (3-1) to hold true for all $\mathbf{r}\ @\ z = 0$ is:

$$\mathbf{k}_1 \cdot \mathbf{r} = \mathbf{k}_2 \cdot \mathbf{r} = \mathbf{k}_3 \cdot \mathbf{r} \quad @ \quad z = 0 \qquad \text{(3-2)}$$

Equation (3-2) is valid regardless of the polarization of vectors $\mathbf{E}_{\mathbf{0}m}$, $m = 1, 2, 3$.

For (3-2) to be true it is necessary that tangential components of the three complex wave vectors:

$$\mathbf{k}_1 \cdot \hat{x} = \mathbf{k}_2 \cdot \hat{x} = \mathbf{k}_3 \cdot \hat{x} \qquad (3\text{-}3)$$

There is no restriction to the polarization of wave vectors.

Assuming $\mathbf{k}_m = k_{mx}\hat{x} + k_{mz}\hat{z} \quad , \; m = 1,2,3$

It is easy to find the wave vector of reflected wave:

$$\mathbf{k_1} \cdot \mathbf{k_1} = \mathbf{k}_3 \cdot \mathbf{k}_3 = \omega^2 \mu_1 \varepsilon_1 \Rightarrow k_{1x}^2 + k_{1z}^2 = k_{3x}^2 + k_{3z}^2 \qquad (3\text{-}4)$$

From equations (3-3) and (3-4), we conclude that:

$$k_{3z} = -k_{1z} \qquad (3\text{-}5)$$

Geometrical interpretation of equation (3-5) is that $\mathbf{k}_3$ is symmetric of $\mathbf{k_1}$ with respect to *x*-axis. Therefore $\mathbf{k}_3$ will have the same polarization as $\mathbf{k}_1$ (LP or NLP).

For finding $\mathbf{k}_2$ we have:

$$\mathbf{k}_2 \cdot \mathbf{k}_2 = \omega^2 \mu_2 \varepsilon_2 \quad \wedge \quad k_{2x} = k_{1x} \Rightarrow \; k_{2z} = \pm\sqrt{\omega^2 \mu_2 \varepsilon_2 - k_{1x}^2} \qquad (3\text{-}6)$$

The correct branch of k_{2z} can be selected using direction of flow of power. Since incident plane is in medium 1 direction of flow of power is from media 1 to media 2 and this means $\hat{z} \cdot \left\langle \vec{\mathscr{S}}_2(\mathbf{r},t) \right\rangle$ is negative.

It is very important to notice that there is no restriction to polarization of $\mathbf{k}_2$. $\mathbf{k}_2$ can be LP or NLP regardless of polarization of $\mathbf{k_1}$ as long as the continuity of tangential component of complex wave vector across the boundary ($k_{2x} = k_{1x}$) is satisfied.

3.1.1 Criteria for Designating Waves as Incident, Reflected and Transmitted (Refracted)

When we are dealing with lossless dielectrics it is straightforward to designate incident, reflected and transmitted waves because the direction of flow of energy is parallel with the wave vector. As it was shown in equation (2-17) the vectors $\left\langle \vec{\mathscr{S}}(\mathbf{r},t)\right\rangle$ and $\mathbf{k}_\mathbf{r}$ are parallel.

That is not the case when we are dealing with media that have negative or complex permeability and permittivity.

In figure (3.1) we assume that flow of power is from media 1 to media 2. So we will have incident and reflected waves in medium 1 and transmitted wave in medium 2.

Normal components of complex Poynting vector and time average Poynting vector are continuous across the boundary. Therefore it is consistent that we use $\hat{z}\cdot\left\langle \vec{\mathscr{S}}(\mathbf{r},t)\right\rangle$ as criteria.

For incident wave must be towards $\hat{z}\cdot\left\langle \vec{\mathscr{S}}(\mathbf{r},t)\right\rangle$ boundary, so in figure (3.1) $\hat{z}\cdot\left\langle \vec{\mathscr{S}}(\mathbf{r},t)\right\rangle$ is negative for incident wave. Similarly $\hat{z}\cdot\left\langle \vec{\mathscr{S}}(\mathbf{r},t)\right\rangle>0$ for reflected wave we use the condition $\hat{z}\cdot\left\langle \vec{\mathscr{S}}(\mathbf{r},t)\right\rangle>0$ and for transmitted wave $\hat{z}\cdot\left\langle \vec{\mathscr{S}}(\mathbf{r},t)\right\rangle<0$.

For example if medium 1 is a double negative metamaterial and medium 2 is a lossless dielectric, incident, reflected and transmitted waves are shown in figure (3.2).

Using equation (2-15) we have $\hat{z}.\left\langle \vec{\mathcal{S}}(\mathbf{r},t)\right\rangle = A\varepsilon'\hat{z}.\mathbf{k_r}$ in which $A>0$. Using this relationship we obtain the wave designation based on the above-mentioned procedure.

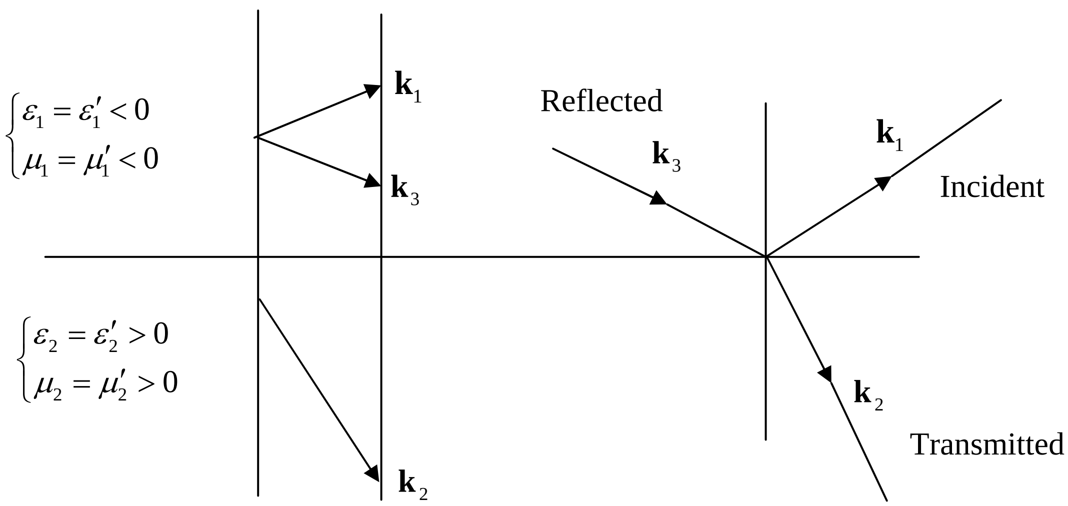

Figure 3.2: Determining incident reflected and transmitted waves using power flow criterion

In figure (3.2) $\mathbf{k}_1$ is incident wave, $\mathbf{k}_3$ is reflected wave and $\mathbf{k}_2$ is the transmitted wave.

3.2 Geometrical representation of Complex Wave Vectors in Plane Boundary Problem

In this section we use the results of previous section to different media types introduced in section (2.2). Since there are four types of media, it results in 16 different plane boundary problems.

We will consider several cases that are of more practical importance, in the following subsections.

3.2.1 Plane Boundary between Two Medium of Type I

This is the most familiar case of a plane boundary between two lossless dielectrics. It is important to notice that knowing the type of medium doesn’t give us enough information to determine the appropriate sign in equation (3-6). For that purpose we need to know some information about permittivity and/or permeability.

Figure (3.3) shows all possible cases of different wave vectors that can be matched when we have a plane boundary between two medium of type I.

Real vectors are shown in black and imaginary vectors are shown in red.

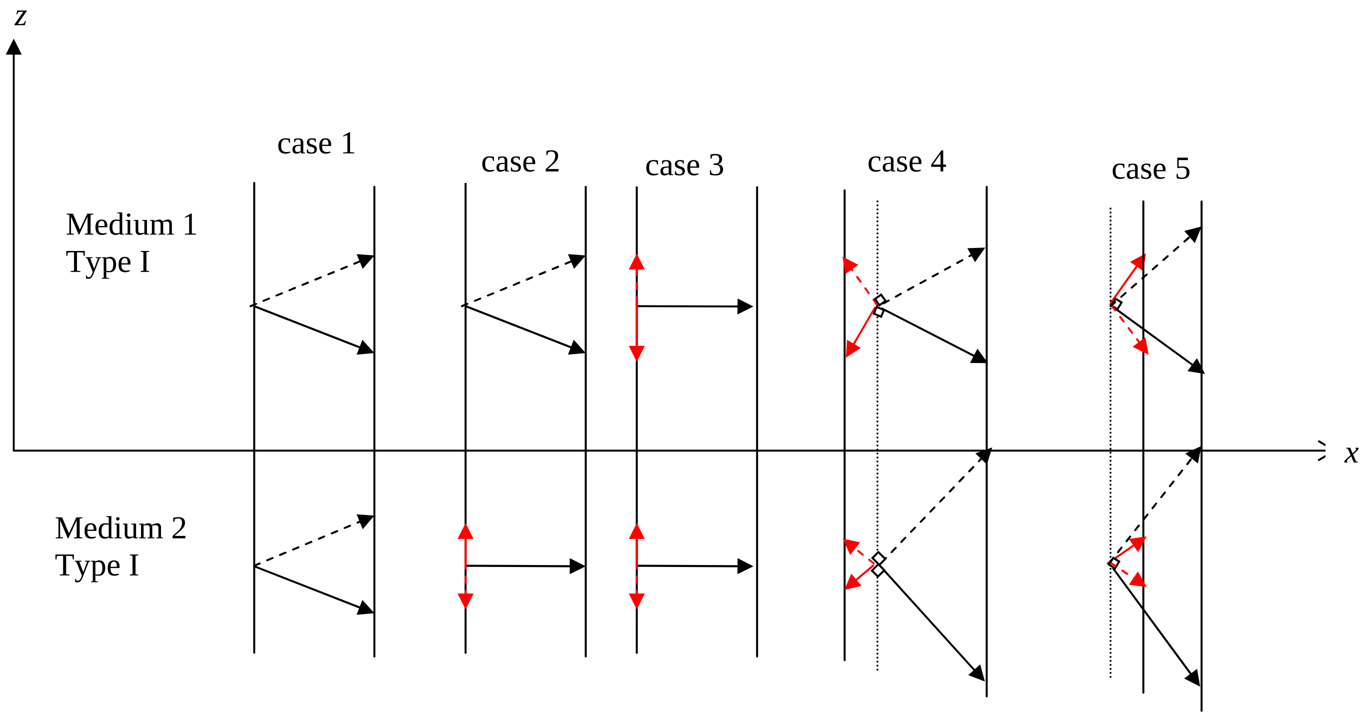

Figure 3.3: Two simple medium of type I with plane boundary between them

In figure (3.3) each of medium 1 or medium 2 can be chosen as the medium that contains the incident wave and any wave vector in a symmetric pair can be selected as incident wave, then the other wave vector will be the incident wave.

Any pair of wave vectors in medium 1 or 2 can be selected as incident/reflected waves. The wave with $\hat{z}\cdot\left\langle \vec{\mathscr{S}}(\mathbf{r},t)\right\rangle$ toward boundary is incident wave and the wave with $\hat{z}\cdot\left\langle \vec{\mathscr{S}}(\mathbf{r},t)\right\rangle$ away from boundary is reflected wave. If $\hat{z}\cdot\left\langle \vec{\mathscr{S}}(\mathbf{r},t)\right\rangle=0$ then we use $\left|\left\langle \vec{\mathscr{S}}(\mathbf{r},t)\right\rangle\right|$ as criteria. If $\left|\left\langle \vec{\mathscr{S}}(\mathbf{r},t)\right\rangle\right|$ decreases towards boundary the wave is incident and the wave is reflected if $\left|\left\langle \vec{\mathscr{S}}(\mathbf{r},t)\right\rangle\right|$ decreases away from boundary.

After determining incident and reflected wave in one of the mediums, we must choose the correct branch among two possibilities. If $\hat{z}\cdot\left\langle \vec{\mathscr{S}}(\mathbf{r},t)\right\rangle$ is away from boundary we choose that wave as transmitted wave. If $\hat{z}\cdot\left\langle\vec{\mathscr{S}}(\mathbf{r},t)\right\rangle$=0, we use $\left|\left\langle\vec{\mathscr{S}}(\mathbf{r},t)\right\rangle\right|$ as criteria and the wave with decreasing $\left|\left\langle\vec{\mathscr{S}}(\mathbf{r},t)\right\rangle\right|$ away from boundary will be the transmitted wave.

We apply the above mentioned criteria to several cases.

Assuming that medium 1 is a media of type I with $\varepsilon_1 = \varepsilon_1'$ such that $\varepsilon_1' > 0$ and media 2 is also of type I with $\varepsilon_2 = \varepsilon_2'$ such that $\varepsilon_2' < 0$. We want to determine the designation of waves in case 1 shown in figure (2-6). Using equation (2-17) we have :

$$\left\langle\vec{\mathscr{S}}_1(\mathbf{r},t)\right\rangle = \frac{\varepsilon_1'}{2\omega|\varepsilon_1|^2}|\mathbf{H}_{01}|^2\,\mathbf{k}_{\mathbf{r}1} \qquad (3\text{-}7)$$

which index 1 is used for incident wave with criteria $\hat{z}\cdot\left\langle\vec{\mathscr{S}}_1(\mathbf{r},t)\right\rangle < 0$ which is equivalent to $\varepsilon_1'\,\hat{z}\cdot\mathbf{k}_{\mathbf{r}1} > 0$ and since $\varepsilon_1' > 0$ it is equivalent to $\hat{z}\cdot\mathbf{k}_{\mathbf{r}1} < 0$.

Similarly the reflected wave with index 3 is

$$\left\langle\vec{\mathscr{S}}_3(\mathbf{r},t)\right\rangle = \frac{\varepsilon_1'}{2\omega|\varepsilon_1|^2}|\mathbf{H}_{03}|^2\,\mathbf{k}_{\mathbf{r}3} \qquad (3\text{-}8)$$

and the condition $\hat{z}\cdot\left\langle\vec{\mathscr{S}}_3(\mathbf{r},t)\right\rangle > 0$ is equivalent to $\hat{z}\cdot\mathbf{k}_{\mathbf{r}3} > 0$.

For the transmitted wave with index 2 in media 2 we have

$$\left\langle\vec{\mathscr{S}}_2(\mathbf{r},t)\right\rangle = \frac{\varepsilon_2'}{2\omega|\varepsilon_2|^2}|\mathbf{H}_{02}|^2\,\mathbf{k}_{\mathbf{r}2} \qquad (3\text{-}9)$$

and $\hat{z}\cdot\left\langle\vec{\mathscr{S}}_2(\mathbf{r},t)\right\rangle < 0$ is equivalent to $\varepsilon_2'\,\hat{z}\cdot\mathbf{k}_{\mathbf{r}2} < 0$ which means $\hat{z}\cdot\mathbf{k}_{\mathbf{r}2} > 0$

The result is shown in figure (3.4) both in wave vector representation and ray representation.

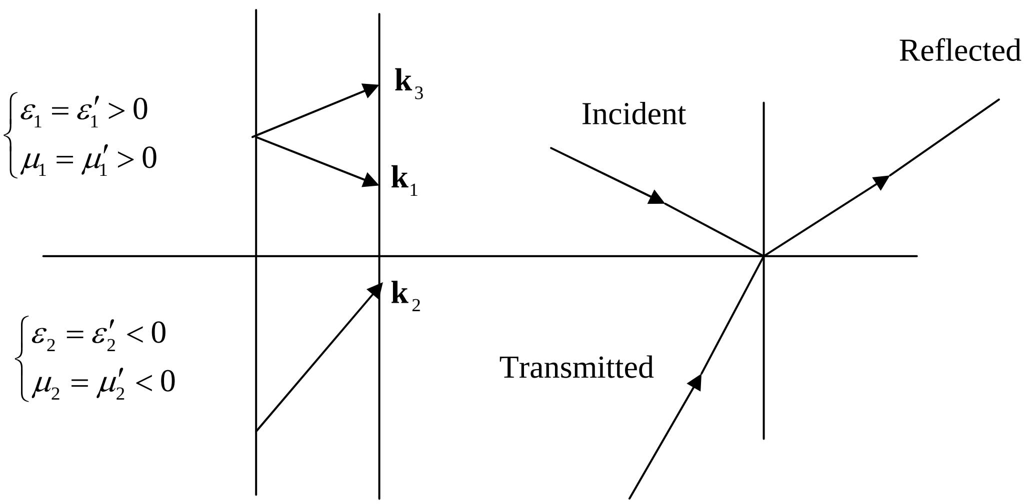

Figure 3.4: incident, reflected and transmitted wave at interface between a lossless dielectric and lossless DNG metamaterial (both media are type I)

Another observation in figure 3.4 is that the same incident wave in one medium can excite different transmitted waves in the other medium. For example in cases 1 and 2 the incident wave is a wave with LP (real) wave vector in medium 1 that can excite a LP wave vector in (case 1) or NLP wave vector (case 2) in medium 2. Case 2 is the familiar phenomenon of total reflection and the LP (real) wave vector in medium 1 excites the NLP wave vector in medium 2.

For a lossless dielectric equation (2-22) or (2-25) shows that direction of flow of power is $\hat{x}$ and it exponentially decays in $-\hat{z}$ direction, which is the well known evanescent or surface wave.

Due to reciprocity if in case 2 we consider the evanescent waves in medium 2 as incident wave, the transmitted wave will be a LP wave with real wave vector.

3.2.2 Plane Boundary between Medium of Type I and II

All different possibilities are illustrated in figure 3.5. The black vectors are real part of complex wave vectors and the red vectors are imaginary part of complex wave vectors.

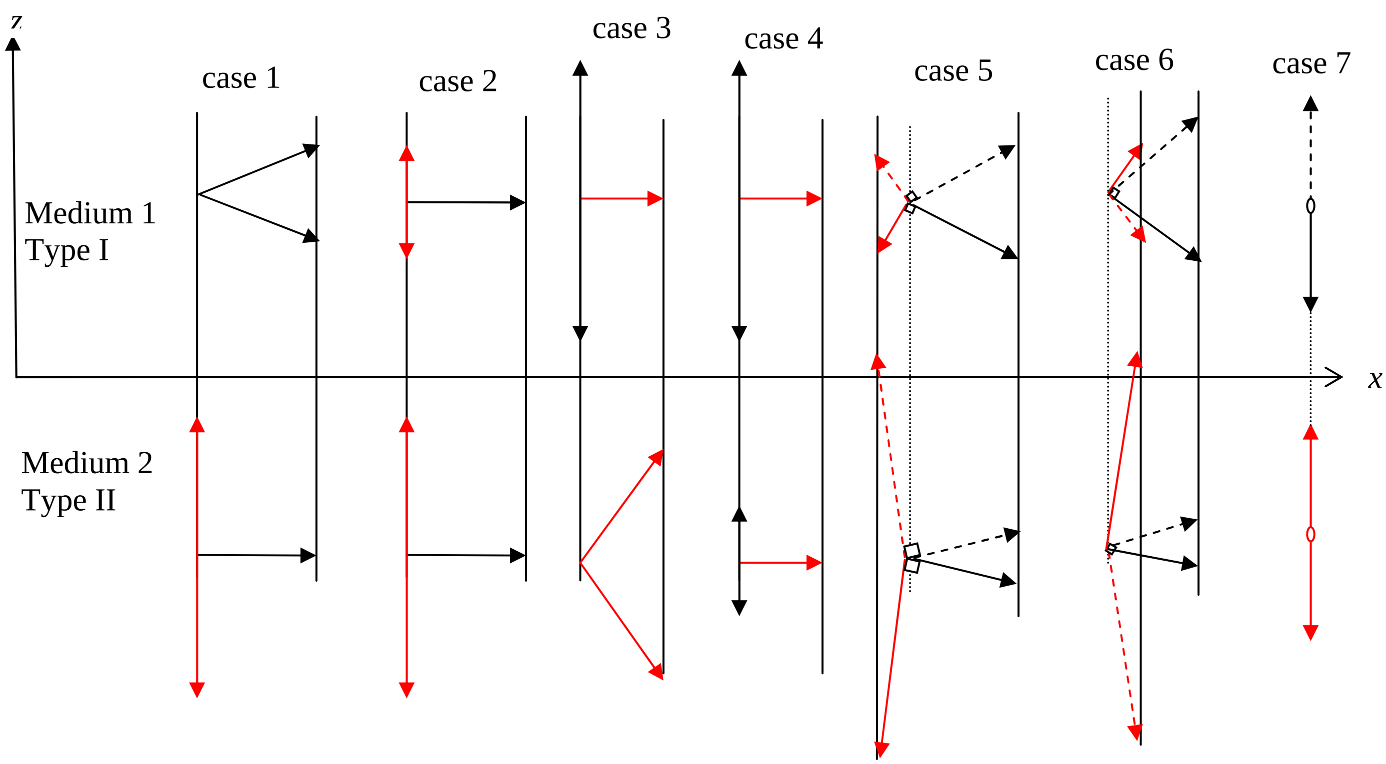

Figure 3.5: Two simple medium of type I and II with plane boundary between them

We observe that in cases 1 and 2 the same incident wave in medium 2 could have different transmitted waves depending on the actual value of parameters. The same thing happens in cases 3 and 4 when the incident waves is in medium 1 are the same but the they can excite different transmitted waves in medium 2.

We choose case 1 and apply the procedure explained in previous section to designate incident, reflected and transmitted waves (Figure 3.6)

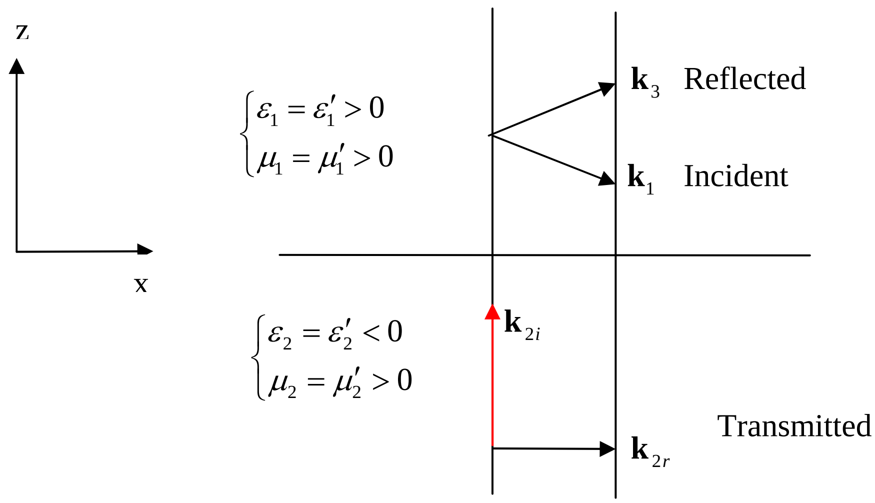

Figure 3.6: Plane boundary between a lossless dielectric and a lossless SNG metamaterial (Plasmons)

There is no flow of power in z direction to medium 2 and the transmitted wave is the so-called surface Plasmon. The Poynting vector of the transmitted wave is parallel to x axis but its direction depends on polarization of the incident wave.

For TM_y mode using equation (2-22) we have

$$\left\langle \vec{\mathcal{S}}_2(\mathbf{r},t)\right\rangle = \frac{1}{2\omega|\mu_2|^2}\exp\left(2\mathbf{k}_{\mathbf{i}2}\cdot\mathbf{r}\right)\left|\mathbf{E}_{\mathbf{0}2}\right|^2\mu_2'\mathbf{k}_{\mathbf{r}2} \qquad (3\text{-}10)$$

and for TE_y using equation (2-25) we have

$$\left\langle \vec{\mathcal{S}}_2(\mathbf{r},t)\right\rangle = \frac{1}{2\omega|\varepsilon_2|^2}\exp\left(2\mathbf{k}_{\mathbf{i}2}\cdot\mathbf{r}\right)\left|\mathbf{H}_{\mathbf{0}2}\right|^2\varepsilon_2'\mathbf{k}_{\mathbf{r}2} \qquad (3\text{-}11)$$

By comparing (3-10) and (3-11) we observe that in TM_y mode power flows in $\hat{x}$ direction but for TE_y mode power flows in $-\hat{x}$ direction. This is the difference between surface Plasmon and the surface wave (evanescent wave) that is formed between two dielectrics, although they have the same wave vectors.

3.2.3 Plane Boundary between Medium of Type I and III

All different possibilities are illustrated in figure 3.7. The black vectors are real part of complex wave vectors and the red vectors are imaginary part of complex wave vectors.

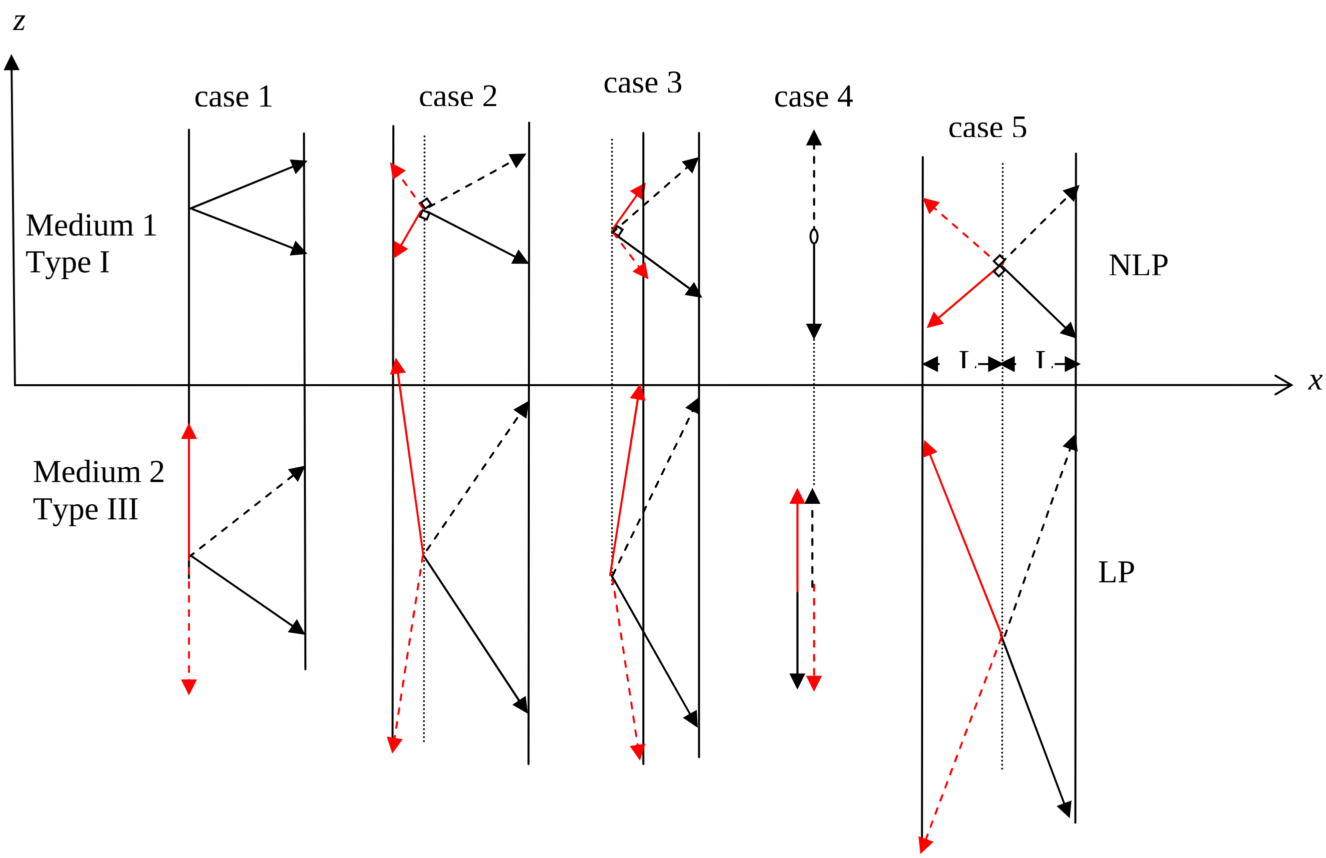

Figure 3.7: Two simple medium of type I and III with plane boundary between them

We consider case 2 when media 1 is a lossless dielectric with $\varepsilon_1 = \varepsilon_1'$ and $\mu_1 = \mu_1'$ (for example free space) and media 2 is a dielectric with high loss with $\varepsilon_2 = -j\varepsilon_2''$ and $\mu_2 = \mu_2'$. (Figure 3.8)

The incident wave is in medium 1 so we need to determine the correct branch of transmitted wave in medium2. For that we need t calculate $\left\langle \vec{\mathscr{S}}_2(\mathbf{r},t)\right\rangle$ which depends on TE/TM mode (polarization).

For TM$_y$ mode using equation (2-22) we obtain:

$$\left\langle \vec{\mathscr{S}}_2(\mathbf{r},t)\right\rangle = \frac{1}{2\omega\left|\mu_2\right|^2}\exp\left(2\mathbf{k}_{\mathbf{i}2}\cdot\mathbf{r}\right)\left|\mathbf{E}_{\mathbf{0}2}\right|^2 \mu_2'\mathbf{k}_{\mathbf{r}2} \qquad (3\text{-}11)$$

For TM$_y$ mode we must choose $\mathbf{k}_2$ such that $\hat{z}\cdot\mathbf{k}_{\mathbf{r}2} < 0$.

For TE$_y$ mode using equation (2-25) we have:

$$\left\langle \vec{\mathscr{S}}_2(\mathbf{r},t)\right\rangle = \frac{1}{2\omega\left|\varepsilon_2\right|^2}\exp\left(2\mathbf{k}_{\mathbf{i}2}\cdot\mathbf{r}\right)\left|\mathbf{H}_{\mathbf{0}2}\right|^2 \left(-\varepsilon_2''\mathbf{k}_{\mathbf{i}2}\right) \qquad (3\text{-}12)$$

For TE$_y$ mode we have to select $\mathbf{k}_2$ so that $\hat{z}\cdot\mathbf{k}_{\mathbf{i2}} > 0$.

By investigating case 2 in figure (3.5) we see that for both modes the same branch can be selected for transmitted wave.

For a highly lossy dielectric $\varepsilon_2'' >> 1$ we have $\left|\mathbf{k}_{\mathbf{i2}}\right| >> \left|\mathbf{k}_{\mathbf{i}1}\right|$ because according to

$$\begin{cases} \left|\mathbf{k}_{\mathbf{r}2}\right| = \left|\mathbf{k}_{\mathbf{i}2}\right| \\ \mathbf{k}_{\mathbf{r}2}\cdot\mathbf{k}_{\mathbf{i}2} = \dfrac{-1}{2}\omega^2\left(\mu_2'\varepsilon_2''\right) \end{cases} \qquad (3\text{-}13)$$

So the vector $\mathbf{k}_{\mathbf{i2}}$ will be very close to normal to the boundary and since in the TE$_y$ mode the propagation is in $-\mathbf{k}_{\mathbf{i2}}$ direction the flow of power in media 2 is negligible and power flow density remains constant along the boundary.

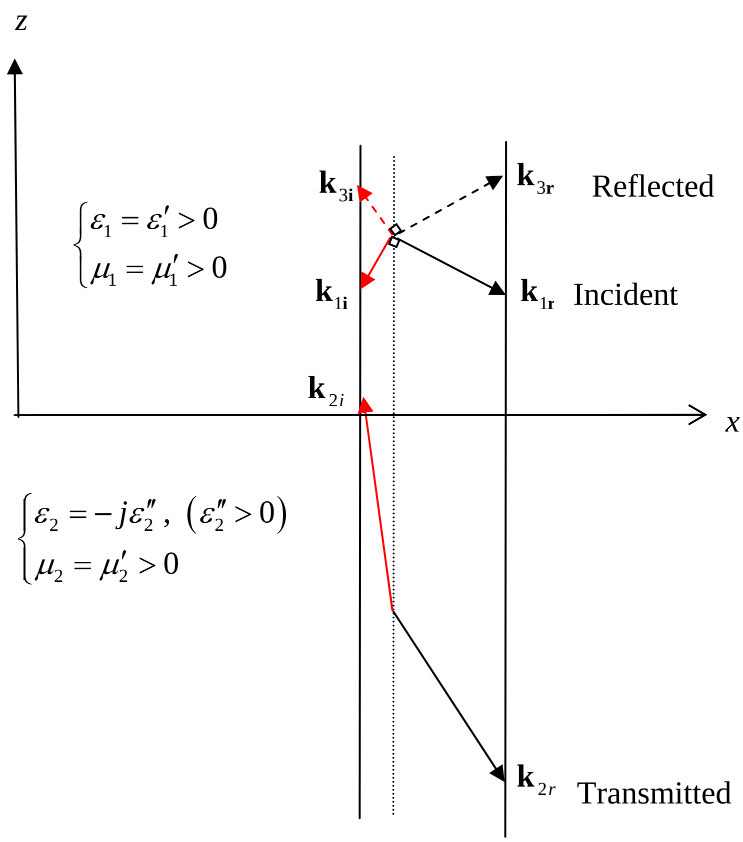

Figure 3.8: Plane boundary between a lossless dielectric and a highly lossy dielectric (Zenneck Waves)

For TM_y polarization we have

$$R_{TM_y} = \frac{E_{y3}}{E_{y1}} = \frac{\dfrac{k_{z1}}{\mu_1} - \dfrac{k_{z2}}{\mu_2}}{\dfrac{k_{z1}}{\mu_1} + \dfrac{k_{z2}}{\mu_2}} \qquad \text{(3-14)}$$

Assuming both media are non-magnetic materials we have $\mu_1 = \mu_2$ and

$$R_{TM_y} = \frac{k_{z1} - k_{z2}}{k_{z1} + k_{z2}} \quad , \quad |k_{z2}| >> |k_{z1}| \quad \Rightarrow R_{TM} \simeq -1 \qquad \text{(3-15)}$$

So for TM_y polarization (which is usually referred as TE_z in literature) total field close to boundary is small.

For TE_y polarization we have

$$R_{TE_y} = \frac{H_{y3}}{H_{y1}} = \frac{\frac{k_{z1}}{\varepsilon_1} - \frac{k_{z2}}{\varepsilon_2}}{\frac{k_{z1}}{\varepsilon_1} + \frac{k_{z2}}{\varepsilon_2}} \qquad (3\text{-}16)$$

In this case we can show that $\left|\frac{k_{z2}}{\varepsilon_2}\right| << \left|\frac{k_{z1}}{\varepsilon_1}\right|$ then $R_{TE_y} \simeq 1$ and the total field is maximum close to surface.

This property is used for low frequency broadcasting and signal reaches to points beyond the line of sight, because at those frequencies ant antenna over earth can be considered as to be a point close to boundary. And the fact that $\mathbf{k}_{\mathbf{i2}}$ is close to normal to the boundary shows that $\left\langle \vec{\mathscr{S}}_2(\mathbf{r},t) \right\rangle$ decreases slowly along the boundary.

3.3 Poynting vector of superposition of incident and reflected waves

To investigate the flow of power in multi-layer media it is essential to calculate the Poynting vector of the total field resulted as superposition of incident and reflected field.

For this purpose we consider the TM_y and TE_y modes separately.

3.3.1 Superposition of incident and reflected wave in TE_y mode

To be consistent with the notations that we used in reflection and transmission from a plane boundary, we use index 1 for incident wave and index 3 for reflected wave. Our goal is to calculate the total Poynting vector with respect to parameters of incident and reflected plane waves. Assuming that *xz* is the incidence plane, we have:

$$\mathbf{k}_1 = k_x \hat{x} + k_z \hat{z} \quad \text{and} \quad \mathbf{k}_3 = k_x \hat{x} - k_z \hat{z}$$

$$\mathbf{H}_{01} = H_{01}\hat{y} \quad \text{and} \quad \mathbf{H}_{03} = H_{03}\hat{y} \quad , \quad H_{01} = \left|H_{01}\right| e^{j\varphi_1} \quad , \quad H_{03} = \left|H_{03}\right| e^{j\varphi_3}$$

$$\mathbf{S} = \left(\mathbf{E}_1 + \mathbf{E}_3\right) \times \left(\mathbf{H}_1^* + \mathbf{H}_3^*\right) = \mathbf{S}_1 + \mathbf{S}_3 + \underbrace{\mathbf{E}_1 \times \mathbf{H}_3^* + \mathbf{E}_3 \times \mathbf{H}_1^*}_{\mathbf{S}_{13}} \qquad (3\text{-}17)$$

$$\mathbf{S}_{13} = \frac{1}{\omega\varepsilon}\left[H_{01}H_{03}^* \exp 2\left(k_{xi}x - jk_{zr}z\right)\mathbf{k}_1 + H_{01}^* H_{03} \exp 2\left(k_{xi}x + jk_{zr}z\right)\mathbf{k}_3\right]$$

In which $\mathbf{S}_1$ and $\mathbf{S}_3$ are complex Poynting vector of incident and reflected wave respectively and were investigated in section (2-3). The term $\mathbf{S}_{13}$ is result of the cross-term that results from cross product of incident electric field ($\mathbf{E}_1$) with reflected magnetic field ($\mathbf{H}_3$) and cross product of incident magnetic field ($\mathbf{H}_1$) with reflected electric field ($\mathbf{E}_3$). We call $\mathbf{S}_{13}$ "cross-term". So we have established a correspondence between decomposition of total field to incident and reflected components and decomposition of total Poynting vector.

Using equation (2-24) we have $\hat{y}\cdot\mathbf{S}_1=0$, $\hat{y}\cdot\mathbf{S}_3=0$ and from (3-17) we have $\hat{y}\cdot\mathbf{S}_{13}=0$. So the total Poynting vector also doesn't have any component normal to plane of incidence: $\hat{y}\cdot\mathbf{S}=0$

We are interested in normal component of time-average Poynting vector which can be obtained using equation (2-25):

$$\hat{z}.\left\langle \vec{S}_1(\mathbf{r},t)\right\rangle=\frac{|H_{01}|^2}{2\omega|\varepsilon_1|^2}\exp\left[2\left(k_{xi}x+k_{zi}z\right)\right]\left(\varepsilon_1' k_{zr}-\varepsilon_1'' k_{zi}\right)\quad,\quad k_{zr}=\mathrm{Re}\left(k_z\right)\quad,\quad k_{zi}=\mathrm{Im}\left(k_z\right)$$

$$\hat{z}.\left\langle \vec{S}_3(\mathbf{r},t)\right\rangle=\frac{|H_{03}|^2}{2\omega|\varepsilon_1|^2}\exp\left[2\left(k_{xi}x-k_{zi}z\right)\right]\left(-\varepsilon_1' k_{zr}+\varepsilon_1'' k_{zi}\right)\qquad(3\text{-}18)$$

And the mutual interaction term $\mathbf{S}_{13}$ after some simplification is obtained as:

$$\boxed{\hat{z}\cdot\mathrm{Re}\left(\mathbf{S}_{13}\right)=\frac{2|H_{01}H_{03}|}{\omega|\varepsilon_1|^2}\exp\left(2k_{xi}x\right)\sin\left(2k_{zr}z+\varphi_3-\varphi_1\right)\left(\varepsilon_1' k_{zi}-\varepsilon_1'' k_{zr}\right)}\qquad(3\text{-}19)$$

So we can write the normal component of total field as sum of the normal components of incident and reflected wave and a third term which is result of interaction of electric field of incident wave with magnetic field of reflected wave and magnetic field of incident wave with electric field of reflected wave.

$$\hat{z}.\left\langle \vec{S}(\mathbf{r},t)\right\rangle=\hat{z}.\left\langle \vec{S}_1(\mathbf{r},t)\right\rangle+\hat{z}.\left\langle \vec{S}_3(\mathbf{r},t)\right\rangle+\frac{1}{2}\hat{z}\cdot\mathrm{Re}\left(\mathbf{S}_{13}\right)\qquad(3\text{-}20)$$

There are several special cases that are of practical importance.

If we have incident and reflected waves with NLP wave vector in a lossless dielectric as shown in case 3 in figure 2-7, then we have

(3-21)

$$\mathbf{k}_1=k_{xr}\hat{x}+jk_{zi}\hat{z}\quad\text{and}\quad\mathbf{k}_3=k_{xr}\hat{x}-jk_{zi}\hat{z}=\mathbf{k}_1^*\quad\text{and}\quad\varepsilon_1\in\mathbb{R}\,,\,\mu_1\in\mathbb{R}$$

$$\left\langle \vec{S}_1(\mathbf{r},t)\right\rangle = \frac{|H_{01}|^2}{2\omega\varepsilon_1}\exp(2k_{zi}z)k_{xr}\,\hat{x} \qquad \text{and} \qquad \left\langle \vec{S}_3(\mathbf{r},t)\right\rangle = \frac{|H_{03}|^2}{2\omega\varepsilon_1}\exp(-2k_{zi}z)k_{xr}\,\hat{x}$$

$$\mathrm{Re}(\mathbf{S}_{13}) = \frac{2|H_{01}H_{03}|}{\omega\varepsilon_1}\left\{k_{xr}\cos(\varphi_3-\varphi_1)\,\hat{x} + k_{zi}\sin(\varphi_3-\varphi_1)\,\hat{z}\right\} \qquad \text{(3-22)}$$

The total Poynting vector is obtained as:

$$\left\langle \vec{S}(\mathbf{r},t)\right\rangle = \frac{k_{xr}}{2\omega\varepsilon_1}\left[|H_{01}|^2\exp(2k_{zi}z) + |H_{03}|^2\exp(-2k_{zi}z) + 2|H_{01}H_{03}|\cos(\varphi_3-\varphi_1)\right]\hat{x}$$
$$+\frac{|H_{01}H_{03}|}{\omega\varepsilon_1}k_{zi}\sin(\varphi_3-\varphi_1)\,\hat{z} \qquad \text{(3-23)}$$

So the total time-average Poynting vector contains an *x* component which is a function of *z* and a *z* component which is constant:

$$\hat{z}.\left\langle \vec{S}_1(\mathbf{r},t)\right\rangle = 0 \quad \text{and} \quad \hat{z}.\left\langle \vec{S}_3(\mathbf{r},t)\right\rangle = 0$$
$$\hat{z}\cdot\left\langle \vec{S}(\mathbf{r},t)\right\rangle = \frac{|H_{01}H_{03}|}{\omega\varepsilon_1}k_{zi}\sin(\varphi_3-\varphi_1) \qquad \text{(3-24)}$$

So we have shown although incident wave and reflected wave individually don't transfer any power in *z* direction, their superposition transfers power in the *z* direction.

We can use these results to show that how power is coupled between two dielectrics with an air gap between them as shown in figure 3.9. It is assumed that the incidence angle of incident wave in medium 2 is greater than critical angle so that the wave in the air gap is evanescent.

$\left\langle \vec{S}(\mathbf{r},t) \right\rangle$ is a constant vector in mediums 1 and 3. In medium 2 $\hat{z} \cdot \left\langle \vec{S}(\mathbf{r},t) \right\rangle$ is constant but $\hat{x} \cdot \left\langle \vec{S}(\mathbf{r},t) \right\rangle$ is a function of *z*, as shown by equation (3-23).

Since $\hat{z} \cdot \left\langle \vec{S}(\mathbf{r},t) \right\rangle$ is continuous across the boundary we conclude that it is constant and equal in all three media and power transfers in *z* direction from layer 2 to layer 3. (Figure 3.9)

Although the polarization of wave vector and polarization of magnetic field change from linear to nonlinear when the wave enters into air gap, the transmitted wave to medium 3 is a plane wave with LP wave vector and LP magnetic field. This property is used in optics to sample the optical signal from a dielectric waveguide by a prism.

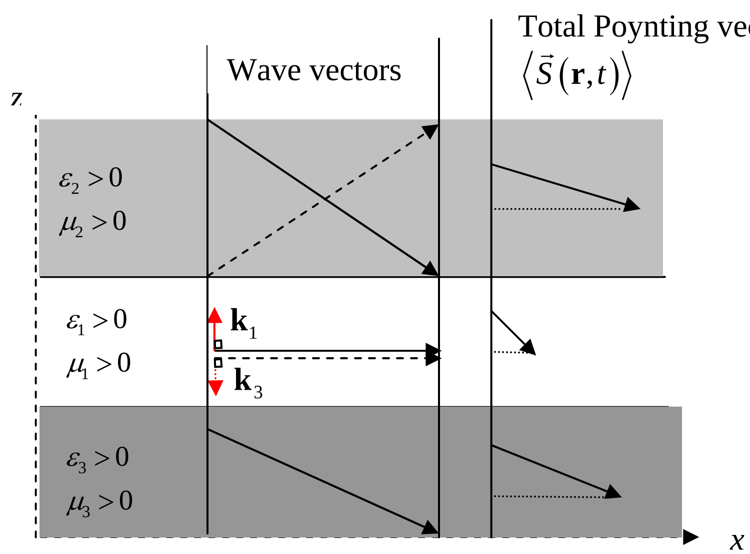

Figure 3.9: The air gap between two dielectrics with wave vectors and average time Poynting vectors. The z component of Poynting vectors in all three media is constant and equal.

We have exactly the same situation as shown in Figure 3.9 when medium 1 is a slab of single negative metamaterial in air (or surrounded by other dielectrics i.e. medium 2 and 3 are air or any other dielectric). The only difference is that we have NLP wave vector in the medium1 regardless of the incident angle of the plane wave in medium 2. The case for normal incident wave to a slab of single negative metamaterial has been shown in figure 3.10.

The total Poynting vector in the slab is:

$$\left\langle \vec{S}(\mathbf{r},t) \right\rangle = \frac{|H_{01}H_{03}|}{\omega\varepsilon_1} k_{zi} \sin(\varphi_3 - \varphi_1)\hat{z} \qquad (3\text{-}25)$$

which is constant in the slab and equal to Poynting vector in mediums 2 and 3.

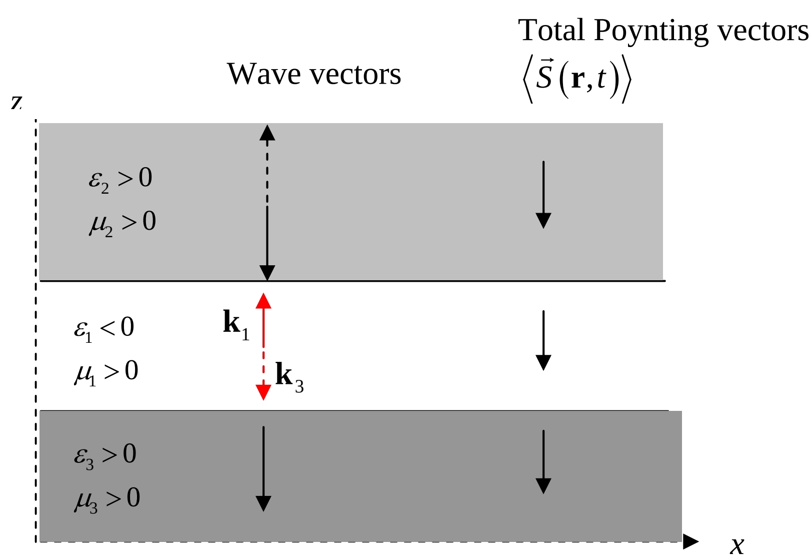

Figure 3.10: Normal incidence on a single negative metamaterial slab. All Poynting vectors are constant and equal.

Another special case is superposition of incident and reflected fields with LP wave vector in a lossless dielectric. In this case we have

$$\begin{aligned}\mathbf{k}_1 &= k_{xr}\hat{x} + k_{zr}\hat{z} \\ \mathbf{k}_3 &= k_{xr}\hat{x} - k_{zr}\hat{z}\end{aligned} \quad \text{and} \quad \varepsilon_1 \in \mathbb{R}\,,\ \mu_1 \in \mathbb{R}$$

$$\mathrm{Re}\left(\mathbf{S}_{13}\right) = \frac{2\left|H_{01}H_{03}\right|}{\omega\varepsilon_1}\cos\left(2k_{zr}z + \varphi_3 - \varphi_1\right)k_{xr}\hat{x}$$

Using (3-17) we have:

$$\left\langle \vec{S}\left(\mathbf{r},t\right)\right\rangle = \frac{k_{xr}}{2\omega\varepsilon_1}\left[\left|H_{01}\right|^2 + \left|H_{03}\right|^2 + 2\left|H_{01}H_{03}\right|\cos\left(2k_{zr}z + \varphi_3 - \varphi_1\right)\right]\hat{x} + \frac{k_{zr}}{2\omega\varepsilon_1}\left(\left|H_{01}\right|^2 - \left|H_{03}\right|^2\right)\hat{z} \tag{3-26}$$

This shows that the mutual interaction term $\hat{z}\cdot\mathrm{Re}\left(\mathbf{S}_{13}\right)$ is equal to zero and also $\hat{z}.\left\langle \vec{S}\left(\mathbf{r},t\right)\right\rangle$is a constant but $\hat{x}.\left\langle \vec{S}\left(\mathbf{r},t\right)\right\rangle$ is a function of *z*.

$$\hat{z}.\left\langle \vec{S}\left(\mathbf{r},t\right)\right\rangle = \hat{z}.\left\langle \vec{S}_1\left(\mathbf{r},t\right)\right\rangle + \hat{z}.\left\langle \vec{S}_3\left(\mathbf{r},t\right)\right\rangle = \frac{k_{zr}}{2\omega\varepsilon_1}\left(\left|H_{01}\right|^2 - \left|H_{03}\right|^2\right)$$

In the case of total reflection which absolute value of reflection coefficient is equal to one, we have $\left|H_{01}\right| = \left|H_{03}\right|$ and $\hat{z}.\left\langle \vec{S}\left(\mathbf{r},t\right)\right\rangle = 0$ which reconfirms the continuity of normal component of Poynting vector, considering the fact that the z-component of Poynting vector of surface wave is also zero.
This explains why a slab of dielectric in air guides the power in x-direction as shown in figure 3.11.

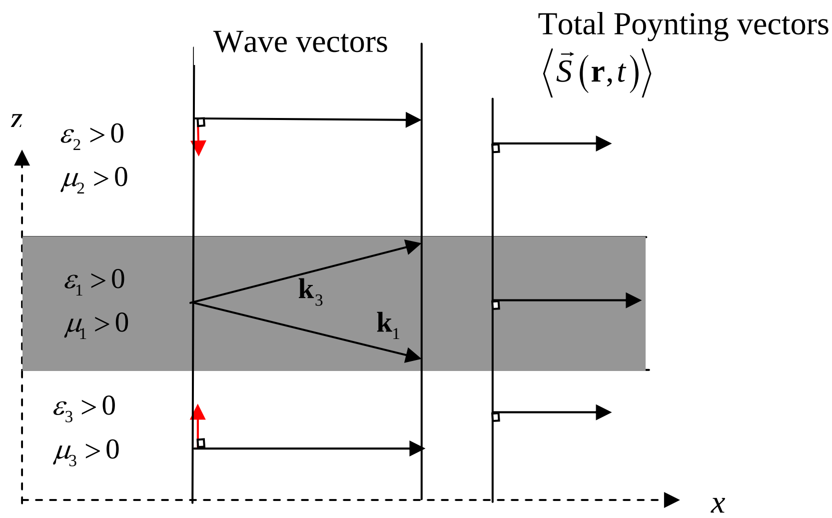

Figure 3.11: A dielectric slab in the air, with wave vectors and total time-average Poynting vectors. In this case there is no flow of power normal to the boundaries.

It is also worth mentioning that, the z component of total Poynting vector (normal to the boundaries) is constant in both (3-23) and (3-26). This guarantees the lossless flow of power from first semi-infinite medium to all middle layers to the most bottom semi-infinite medium which is also consistent with continuity of normal component of Poynting vector across the boundary.

3.3.2 Superposition of incident and reflected wave in TM_y mode

Similar to TE_y mode we can calculate the total Poynting vector of the superposition of the incident and reflected wave.

$$\mathbf{k}_1 = k_x\hat{x} + k_z\hat{z} \quad \text{and} \quad \mathbf{k}_3 = k_x\hat{x} - k_z\hat{z} \tag{3-27}$$

$$\mathbf{E}_{01} = E_{01}\hat{y} \quad \text{and} \quad \mathbf{E}_{03} = E_{03}\hat{y} \quad , \quad E_{01} = |E_{01}|e^{j\varphi_1} \quad , \quad E_{03} = |E_{03}|e^{j\varphi_3}$$

$$\mathbf{S} = (\mathbf{E}_1 + \mathbf{E}_3)\times(\mathbf{H}_1^* + \mathbf{H}_3^*) = \mathbf{S}_1 + \mathbf{S}_3 + \underbrace{\mathbf{E}_1 \times \mathbf{H}_3^* + \mathbf{E}_3 \times \mathbf{H}_1^*}_{\mathbf{S}_{13}} \qquad (3\text{-}28)$$

In which $\mathbf{S}_1$ and $\mathbf{S}_3$ are complex Poynting vector of incident and reflected wave respectively. The term $\mathbf{S}_{13}$ is result of interaction of incident electric field ($\mathbf{E}_1$) and reflected magnetic field ($\mathbf{H}_3$) and interaction of incident magnetic field ($\mathbf{H}_1$) with reflected electric field ($\mathbf{E}_3$). We call $\mathbf{S}_{13}$ "mutual interaction term".

Normal components of $\mathbf{S}_1$ and $\mathbf{S}_3$ can be calculated using equation (2-22)

$$\hat{z}.\left\langle \vec{S}_1(\mathbf{r},t)\right\rangle = \frac{|E_{01}|^2}{2\omega|\mu_1|^2}\exp\left[2(k_{xi}x + k_{zi}z)\right](\mu_1' k_{zr} - \mu_1'' k_{zi}) \quad , \quad k_{zr} = \mathrm{Re}(k_z) \quad , \quad k_{zi} = \mathrm{Im}(k_z)$$

$$\hat{z}.\left\langle \vec{S}_3(\mathbf{r},t)\right\rangle = \frac{|E_{03}|^2}{2\omega|\mu_1|^2}\exp\left[2(k_{xi}x - k_{zi}z)\right](-\mu_1' k_{zr} + \mu_1'' k_{zi}) \qquad (3\text{-}29)$$

And the mutual interaction term $\mathbf{S}_{13}$ after some simplification is obtained as:

$$\boxed{\hat{z}\cdot\mathrm{Re}(\mathbf{S}_{13}) = \frac{2|E_{01}E_{03}|}{\omega|\mu_1|^2}\exp(2k_{xi}x)\sin(2k_{zr}z + \varphi_3 - \varphi_1)(\mu_1' k_{zi} + \mu_1'' k_{zr})} \qquad (3\text{-}30)$$

And the normal component of the total Poynting vector can be expressed as

$$\hat{z}.\left\langle \vec{S}(\mathbf{r},t)\right\rangle = \hat{z}.\left\langle \vec{S}_1(\mathbf{r},t)\right\rangle + \hat{z}.\left\langle \vec{S}_3(\mathbf{r},t)\right\rangle + \frac{1}{2}\hat{z}\cdot\mathrm{Re}(\mathbf{S}_{13}) \qquad (3\text{-}31)$$

The same arguments that we made for special structures in TE_y mode are valid also for TM_y mode.

3.3.3 Superposition of incident and reflected wave in mixed mode

We can decompose the incident, reflected and total field to TE/TM components. Assuming that *xz* is incident plane we can decompose the total field to TE/TM components as discussed in section 2.4:

$$\begin{cases} \mathbf{E} = E_y \hat{y} + \mathbf{E}_{TE} \\ \mathbf{H} = H_y \hat{y} + \mathbf{H}_{TM} \end{cases} \quad (3\text{-}32)$$

The total Poynting vector is obtained as:

$$\mathbf{S} = \mathbf{S}_{TM} + \mathbf{S}_{TE} + \mathbf{E}_{TE} \times \mathbf{H}^{*}_{TM} \quad (3\text{-}33)$$

$$\mathbf{S}_{TM} = E_y \hat{y} \times \mathbf{H}^{*}_{TM} \qquad \text{and} \qquad \mathbf{S}_{TE} = \mathbf{E}_{TE} \times H^{*}_{y} \hat{y}$$

since the term $\mathbf{E}_{TE} \times \mathbf{H}^{*}_{TM}$ is parallel to $\hat{y}$ we can write:

$$\hat{z} \cdot \mathbf{S} = \hat{z} \cdot \mathbf{S}_{TM} + \hat{z} \cdot \mathbf{S}_{TE} \quad (3\text{-}34)$$

$$\hat{z} \cdot \left\langle \vec{S}\left(\mathbf{r},t\right) \right\rangle = \hat{z} \cdot \left\langle \vec{S}_{TM}\left(\mathbf{r},t\right) \right\rangle + \hat{z} \cdot \left\langle \vec{S}_{TE}\left(\mathbf{r},t\right) \right\rangle$$

using the results that we obtained in 3.1.1 and 3.1.2 we can express $\mathbf{S}_{TM}$ and $\mathbf{S}_{TE}$ with respect to reflected and incident waves:

$$\begin{cases} \mathbf{S}_{TM} = \mathbf{S}_{1TM} + \mathbf{S}_{3TM} + \mathbf{S}_{13TM} \\ \mathbf{S}_{TE} = \mathbf{S}_{1TE} + \mathbf{S}_{3TE} + \mathbf{S}_{13TE} \end{cases} \quad (3\text{-}35)$$

$$\begin{cases} \hat{z} \cdot \left\langle \vec{S}_{TM}\left(\mathbf{r},t\right) \right\rangle = \hat{z} \cdot \left\langle \vec{S}_{1TM}\left(\mathbf{r},t\right) \right\rangle + \hat{z} \cdot \left\langle \vec{S}_{3TM}\left(\mathbf{r},t\right) \right\rangle + \frac{1}{2}\hat{z} \cdot \mathrm{Re}\left(\mathbf{S}_{13TM}\right) \\ \hat{z} \cdot \left\langle \vec{S}_{TE}\left(\mathbf{r},t\right) \right\rangle = \hat{z} \cdot \left\langle \vec{S}_{1TE}\left(\mathbf{r},t\right) \right\rangle + \hat{z} \cdot \left\langle \vec{S}_{3TE}\left(\mathbf{r},t\right) \right\rangle + \frac{1}{2}\hat{z} \cdot \mathrm{Re}\left(\mathbf{S}_{13TE}\right) \end{cases} \quad (3\text{-}36)$$

Adding the equations together we have:

$$\hat{z} \cdot \left\langle \vec{S}\left(\mathbf{r},t\right) \right\rangle = \hat{z} \cdot \left\langle \vec{S}_{1}\left(\mathbf{r},t\right) \right\rangle + \hat{z} \cdot \left\langle \vec{S}_{3}\left(\mathbf{r},t\right) \right\rangle + \frac{1}{2}\hat{z} \cdot \mathrm{Re}\left(\mathbf{S}_{13TM} + \mathbf{S}_{13TE}\right) \quad (3\text{-}37)$$

Assuming $\mathbf{H}_1 = H_{y1}\hat{y}$ and $\mathbf{H}_3 = H_3\hat{y}$ are magnetic fields of incident and reflected waves in TE_y mode and $\mathbf{E}_1 = E_{y1}\hat{y}$ and $\mathbf{E}_3 = E_{y3}\hat{y}$ are electric fields of incident and reflected waves in TM_y mode equation (3-33) can be further simplified to:

$$\mathbf{E}_{TE} \times \mathbf{H}_{TM}^{*} = \hat{y}\left[2j\,\mathrm{Im}\left(k_x^{*}k_z\right)\left(H_{y1}E_{y1}^{*} - H_{y3}E_{y3}^{*}\right) + 2\,\mathrm{Re}\left(k_x^{*}k_z\right)\left(H_{y1}E_{y3}^{*} - H_{y3}E_{y1}^{*}\right)\right] \qquad (3\text{-}38)$$

The term $\mathbf{E}_{TE} \times \mathbf{H}_{TM}^{*}$ has $\hat{y}$ component, and this is contrast to total complex Poynting vector of pure TE/TM modes ($\mathbf{S}_{TM}$ and $\mathbf{S}_{TE}$) that don't have any $\hat{y}$ component. This is because both $\mathbf{E}_{TE}$ and $\mathbf{H}_{TM}$ are perpendicular to $\hat{y}$ and so their cross product will be in $\hat{y}$ direction.
We use the results that we have obtained to this point to compare wave vectors, time-average Poynting vector and magnetic fields (in $\mathrm{TM_y}$ mode) for plane wave incident on interface of two lossless dielectrics for two cases of partial and total reflection.

It is very important to notice that complex Poynting vector of pure modes $\mathbf{S}_{TM}$ and $\mathbf{S}_{TE}$ are in *xz* plane, in other words from (3-17) and (3-28) we have: $\hat{y}\cdot\mathbf{S}_{TE} = 0$ and $\hat{y}\cdot\mathbf{S}_{TM} = 0$, while the mixed mode Poynting vector given in equation (3-33) the cross term $\mathbf{E}_{TE} \times \mathbf{H}_{TM}^{*}$ is in $\hat{y}$ direction.

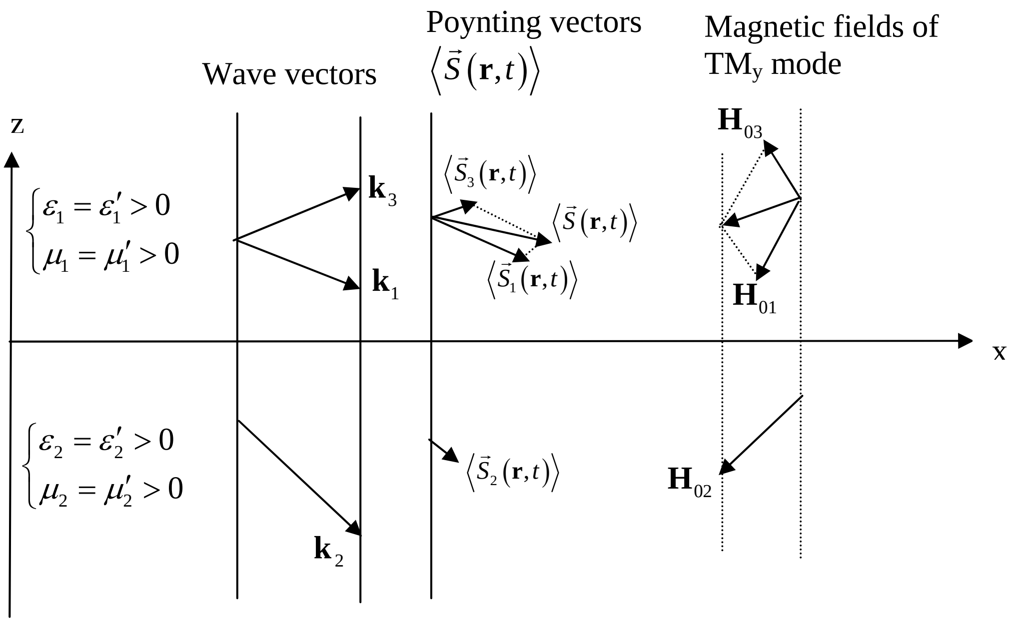

Figure 3.12: wave vectors, time average Poynting vector and magnetic field in partial reflection from interface of two lossless dielectrics

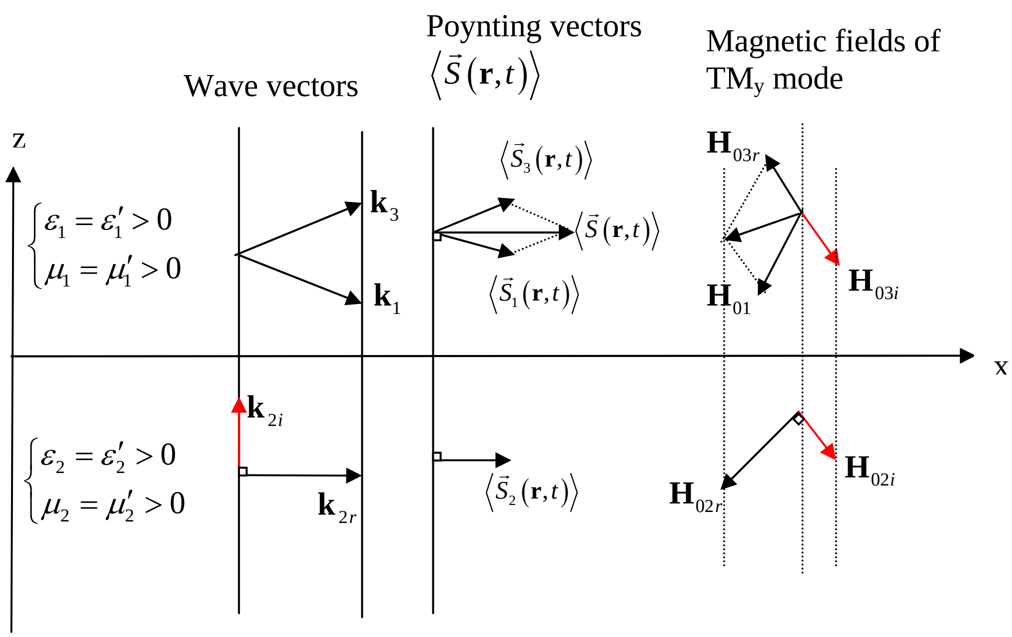

Figure 3.13: wave vectors, time average Poynting vector and magnetic field for total reflection, from interface of two lossless dielectrics

Assuming $\mathbf{E}_1 = E_{01}\exp\left(-j\mathbf{k}_1\cdot\mathbf{r}\right)\hat{y}$ and E_{01} is real (without loss of generality) and considering the reflection coefficient:

$$R_{TM_y} = \frac{E_{03}}{E_{01}} = \frac{\dfrac{k_{z1}}{\mu_1} - \dfrac{k_{z2}}{\mu_2}}{\dfrac{k_{z1}}{\mu_1} + \dfrac{k_{z2}}{\mu_2}} \qquad \text{(3-39)}$$

We notice that R_{TM_y} is real when partial reflection happens and is complex with modulus equal to 1. There are several remarks that we can make by comparing total reflection and partial reflection as shown in figures 3.12 and 3.13.

1- $\mathbf{H}_{03}$ is LP in both cases but it is real (co-phase with incident wave) in partial reflection and it is complex in total reflection. The total magnetic field in medium 1 for total reflection is NLP. So the difference between total reflection and partial reflection is not just in refracted wave.

2- $\mathbf{H}_{02}$ is LP in partial reflection but it is NLP in total reflection

3- There is flow of power in z direction in partial reflection but there is no flow of power in z direction when we have total reflection.

4- $\left\langle \vec{S}_2\left(\mathbf{r},t\right)\right\rangle$ is exponentially decaying when there is total reflection in contrast to the partial reflection which is a constant vector.

3.4 Plane waves in plasma and surface plasmon and their properties

The formulation that we have developed to this point will be used to study the properties of interface between a dielectric and plasma. The analysis here is limited to classical Maxwell's equation and modeling of materials in each layer as simple medium (linear, homogeneous, and anisotropic). We will not consider quantum effects and other physical phenomena beyond this model. For example the fact that Plasmon is only created in the surface is modeled by a very thin homogeneous layer with negative permittivity.
Plasma is defined as a dielectric with permittivity $\varepsilon = \varepsilon' - j\varepsilon''$ with $\varepsilon' < 0$. The physical implementation and frequency dependence of this permittivity is beyond the scopes of this model.

3.4.1 Surface plasmonic resonance

Surface plasmon as far as our model is concerned, is a thin layer of plasma in the metal close to its interface with a dielectric. Although we will analyze interface of a bulk dielectric and bulk plasma, but the fact that field decays exponentially as we move far from interface, justifies this assumption. Figure 3.14 shows interface of a lossless dielectric with lossless plasma. This is case 2 shown in figure 3.5. Total time-average Poynting vector in media 1 can be obtained from equation (3-23) in which since reflection coefficient

(R) is real for both TE$_y$ and TM$_y$ modes we have $\varphi_3 - \varphi_1 = \arg(R) = k\pi$ and so $\sin(\varphi_3 - \varphi_1) = 0$, thus the z component of total Poynting vector becomes zero. And it can be shown that its x-component is positive.

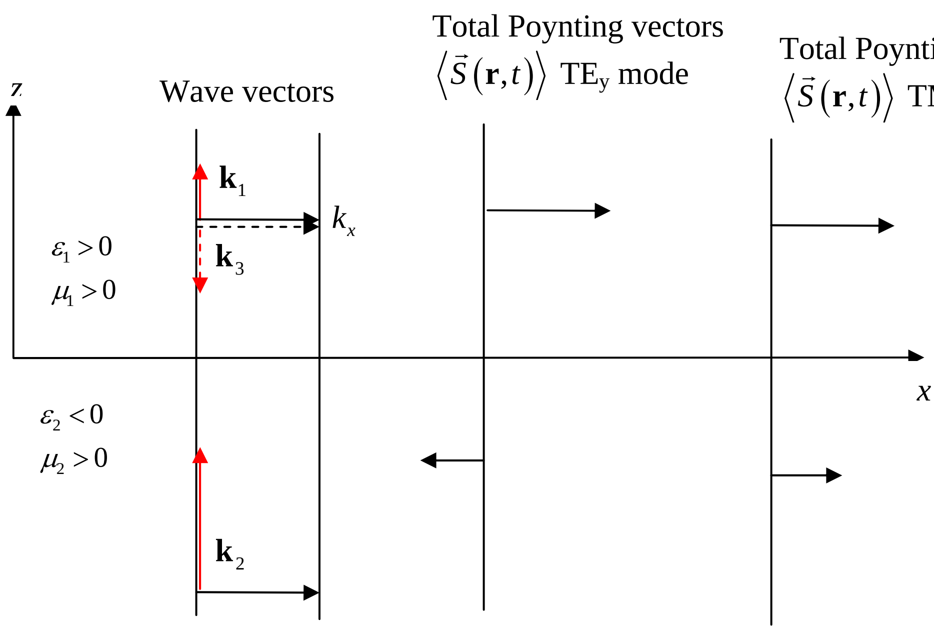

Figure 3.14- Surface Plasmon Polaritons and Surface Plasmon Resonance

The reflection coefficients are

$$R_{TM_y} = \frac{\dfrac{k_{z1}}{\mu_1} - \dfrac{k_{z2}}{\mu_2}}{\dfrac{k_{z1}}{\mu_1} + \dfrac{k_{z2}}{\mu_2}} \quad \text{and} \quad R_{TE_y} = \frac{\dfrac{k_{z1}}{\varepsilon_1} - \dfrac{k_{z2}}{\varepsilon_2}}{\dfrac{k_{z1}}{\varepsilon_1} + \dfrac{k_{z2}}{\varepsilon_2}} \qquad (3\text{-}40)$$

We have $k_{z1} = jk_{z1i}$ and $k_{z2} = jk_{z2i}$ with k_{z1i} and k_{z2i} positive real numbers given by $k_{z1i} = \sqrt{k_x^2 - \omega^2 \mu_1 \varepsilon_1}$ and $k_{z2i} = \sqrt{k_x^2 - \omega^2 \mu_2 \varepsilon_2}$. Since $\varepsilon_1 \varepsilon_2 < 0$ the equation

$$\frac{k_{z1i}}{\varepsilon_1} + \frac{k_{z2i}}{\varepsilon_2} = 0 \qquad (3\text{-}41)$$

has a solution for k_x. For this particular k_x we have $R_{TE_y} = \infty$ and this implies that incident wave is zero and is called *surface plasmon resonance*. But since $\frac{k_{z1i}}{\mu_1} + \frac{k_{z2i}}{\mu_2}$ is always positive there is no k_x for which $R_{TM_y} = \infty$.

Equation (3-40) can be obtained by assuming the plane wave solution for Maxwell's equations in both media 1 and 2 and apply the boundary condition.

$\mathbf{H}_1 = H_{01}\hat{y}\exp(-j\mathbf{k}_3 \cdot \mathbf{r})$ $\qquad$ $\mathbf{H}_3 = H_{03}\hat{y}\exp(-j\mathbf{k}_2 \cdot \mathbf{r})$

After applying the continuity of H_y and E_x we have:

$\exp(-j\mathbf{k}_3 \cdot \mathbf{r}) = \exp(-j\mathbf{k}_2 \cdot \mathbf{r}) \quad @\, z = 0$ and $H_{01} = H_{02}$

and

$$\frac{k_{z3}}{\varepsilon_1} = \frac{k_{z2}}{\varepsilon_2} \qquad (3\text{-}42)$$

considering the fact that $k_{z3} = -k_{z1}$, equations (3-41) and (3-42) are equivalent. This is similar to the property of LTI circuits that natural frequencies and poles of the transfer function are the same.

Assuming that $\mu_1 = \mu_2 = \mu$ the k_x for surface plasmon resonance is obtained as

$$k_x^{SP} = \omega\sqrt{\frac{\mu\varepsilon_1\varepsilon_2}{\varepsilon_1 + \varepsilon_2}} \qquad (3\text{-}43)$$

Since $\varepsilon_1\varepsilon_2 < 0$ for equation (3-43) to give a real number we should have $\varepsilon_1 + \varepsilon_2 < 0$.

As we can see in Figure 3.14 the direction of $\langle \vec{S}(\mathbf{r},t) \rangle$ for TE_y and TM_y polarizations are opposite. These waves are called *polaritons*.

Also for the case of metals at optical frequencies since the plasma only forms at the boundary of metal with dielectric the medium is called *surface plasmon*. Although we assumed bulk plasma medium 2 with $\varepsilon_2 < 0$ and $\mu_2 > 0$ since the field exponentially decays very fast in medium 2, its permittivity “far” from surface is insignificant. So these waves are called *surface plasmon polaritons* due to the fact that in different polarizations power flows in opposite directions and we can separate them. In TE_y mode we can have the so-called *surface plasmon resonance* as it was shown by equations (3-42) and (3-43).

3.4.2 Otto and Kretschmann configurations for exciting surface plasmons

The Kretschmann’s configuration is shown in Figure (3-15) assuming that the incidence angle is greater than the critical angle between prism and dielectric. In this configuration the thickness of metal (*d*) should be very small so that our assumption about negative permittivity in the whole layer is correct.

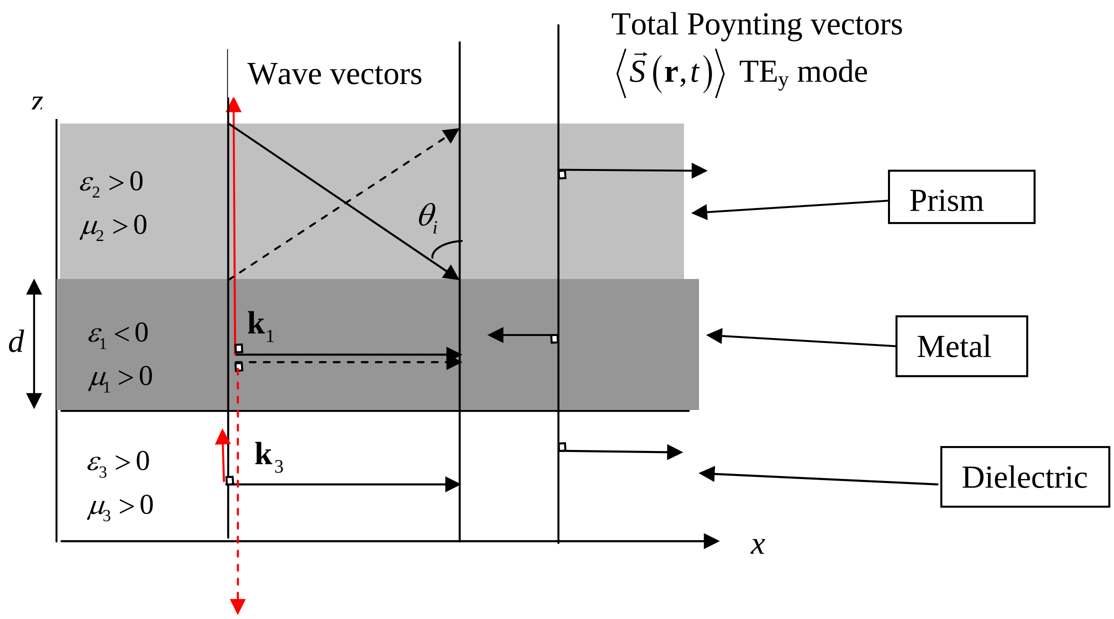

Figure 3.15- Kretschmann's configuration for exciting surface plasmons Polaritons with lossless metal.

The total reflection coefficient at the boundary between medium 2 and 1 with respect to partial reflection coefficients given above can be obtained as:

$$\Gamma = \frac{R_1 + R_2 \exp(-2k_{1zi}d)}{1 + R_1 R_2 \exp(-2k_{1zi}d)}, \quad k_{1z} = jk_{1zi} \qquad (3\text{-}44)$$

in which $k_{1zi} = \mathrm{Im}(k_{1z})$, R_1 is the reflection coefficient between medium 2 and 1 and R_2 is the reflection coefficient between 1 and 3 given by equations (3-40)

Since $|R_1| = 1$ and R_2 is real, it can be shown that $|\Gamma| = 1$, which means we have total reflection at boundary between 2 and 1 and also justifies the Poynting vectors shown in figure (3.15).

All time-average Poynting vectors are function of z. The time-average Poynting vector in medium 1 can be obtained using equation (3-23) and since R_2 is real, the *z* component of total Poynting vector is zero.

The time-average Poynting vector in medium 2 is calculated using equation (3-26), and since $|\Gamma| = 1$ the z component is zero. The time-average Poynting vector in medium 3 is obtained by equation (2-25).

As it is shown in figure (3.16) if the metal is lossless we don't observe any significant change in $|\Gamma|$ by changing the incident angle, but if the metal is lossy there is a sharp drop in $|\Gamma|$ as shown in figure (3.17).

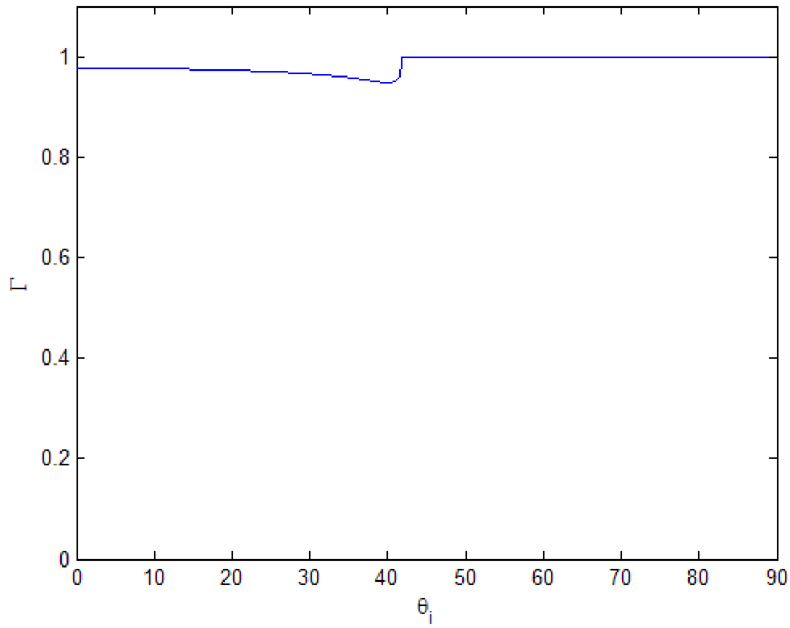

Figure 3.16- Magnitude of total reflection coefficient as a function of incident angle in Kretschmann's configuration, when the metal is lossless

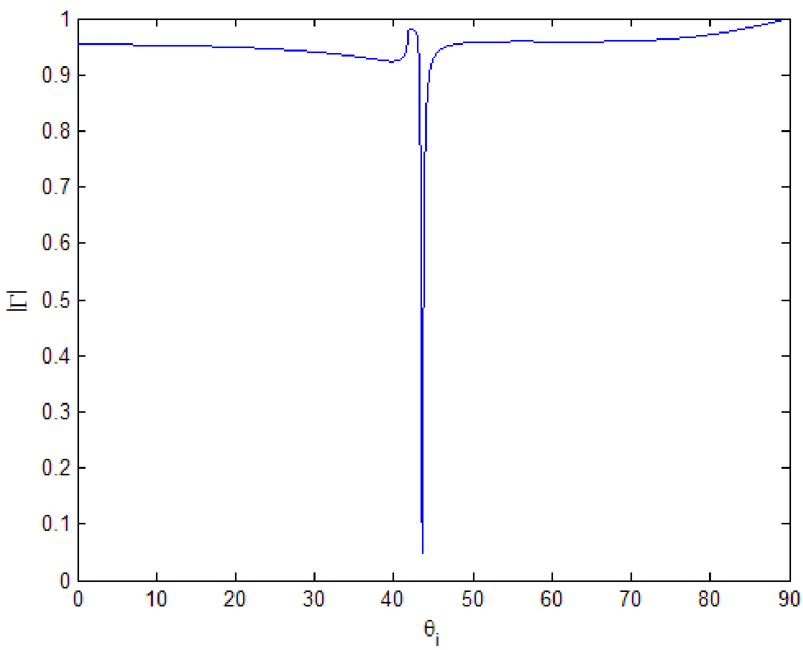

Figure 3.17- Magnitude of total reflection coefficient as a function of incidence angle in Kretschmann's configuration, when the metal is lossy

The wave vectors and Poynting vectors when the metal is lossy are shown in figure (3.18)

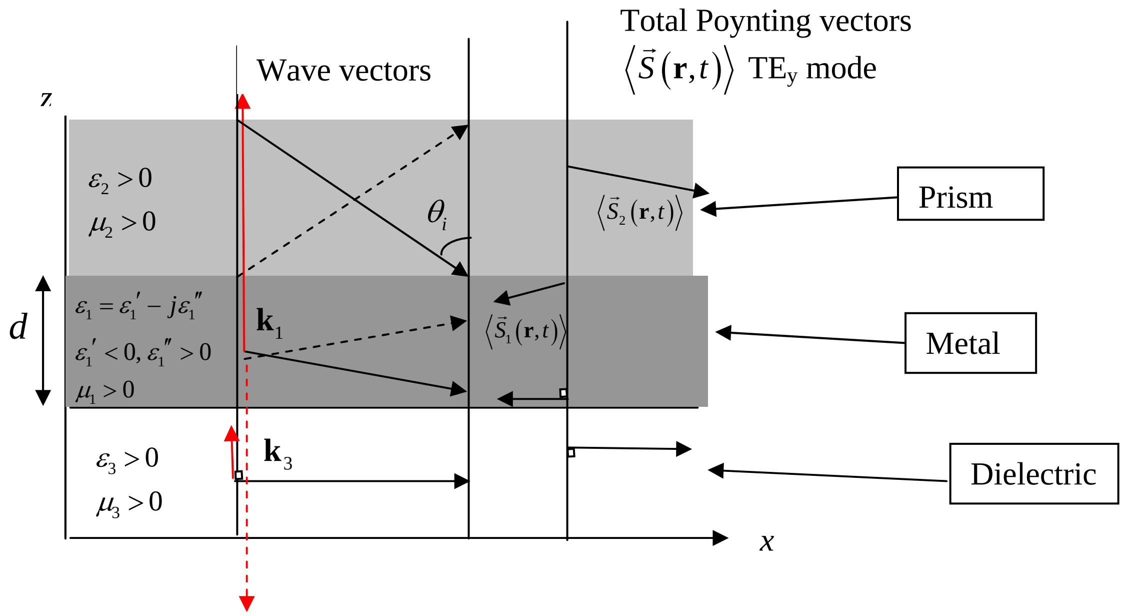

Figure 3.18- Wave vectors and Poynting vectors in Kretschmann's configuration when the metal is lossy.

The Poynting vectors in media 2 and 3 still follow the equations (3-26) and (2-25) respectively, so the *z* component of $\langle \vec{S}_2(\mathbf{r},t) \rangle$ is constant. But the Poynting vector in medium 1 can not be calculated by equation (3-23) because its permittivity is complex. Both *x* and *z* components of $\langle \vec{S}_1(\mathbf{r},t) \rangle$ are function of *z*. Due to continuity of normal Poynting vector, $\langle \vec{S}_1(\mathbf{r},t) \rangle$ has non-zero *z* component at boundary with2 but its *z* component is zero at boundary with 3.

In Otto's configuration (figure 3.19) inhomogeneous waves (with complex wave vectors) are excited by prism in dielectric, and they excite surface plasmons in dielectric-metal interface.

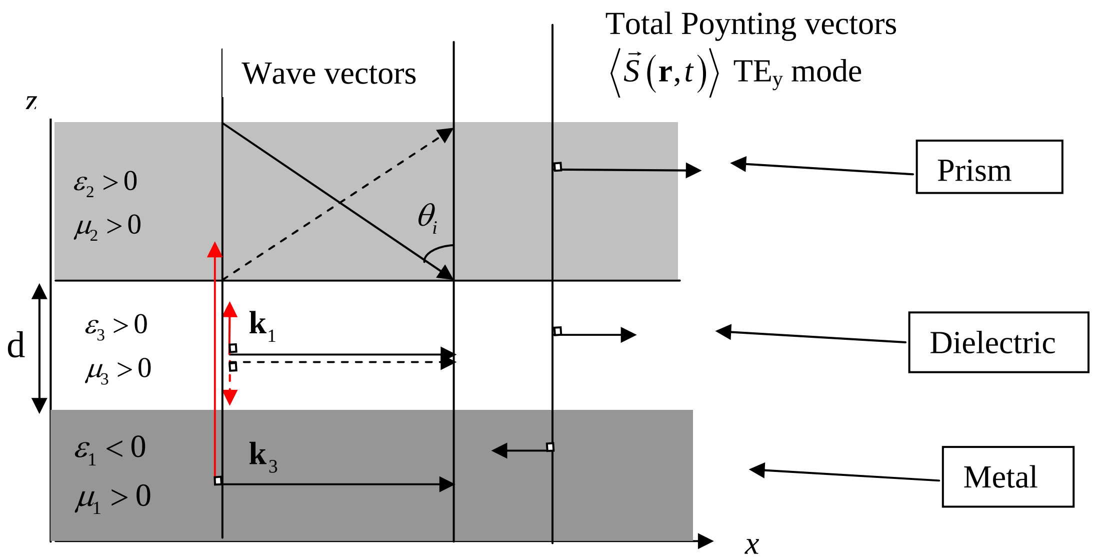

Figure 3.19- Otto's configuration for exciting surface plasmon Polaritons with lossless metal

For Otto's configuration in equation (3-44) if $\theta_i > \theta_c$ we have $|R_1| = 1$ and $R_2 \in \mathbb{R}$ which results in $|\Gamma| = 1$. If $\theta < \theta_c$ we have

$$\Gamma = \frac{R_1 + R_2 \exp(-j2k_{1z}d)}{1 + R_1 R_2 \exp(-j2k_{1z}d)}, \quad k_{1z} = k_{1zr} = \mathrm{Re}(k_{1z}) \qquad (3\text{-}45)$$

with $k_{1z} \in \mathbb{R}$, $R_1 \in \mathbb{R}$ and $|R_2| = 1$ which results in $|\Gamma| = 1$.

So for a lossless metal we have $|\Gamma| = 1$ for any angle of incidence θ_i regardless of the air gap d . This is confirmed by calculation as shown in figure 3.20.

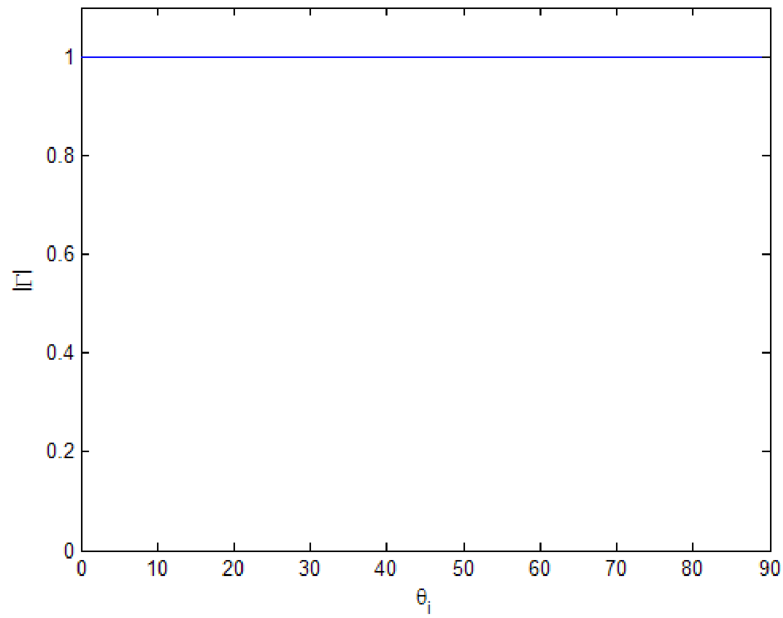

Figure 3.20- Reflection coefficient vs. incident angle in Otto's configuration, when the metal is lossless

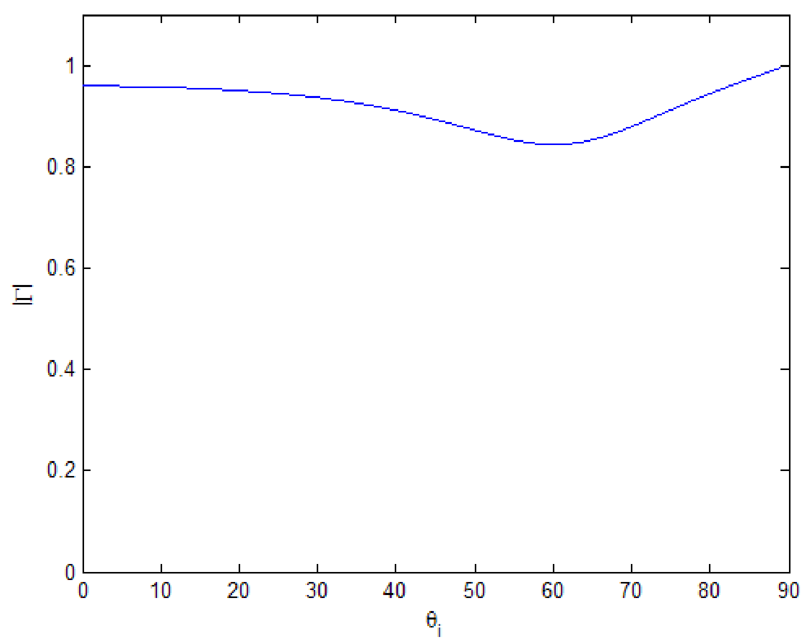

Figure 3.21- Reflection coefficient vs. incident angle in Otto's configuration, when the metal is lossy

But if the metal is lossy, $|\Gamma|$ has a minimum as a function of θ_i as shown in figure 3.21. $|\Gamma|_{min}$ and θ_i that $|\Gamma|_{min}$ happens are both function of air gap *d* and it is possible to get lower values for $|\Gamma|_{min}$ by changing *d*.

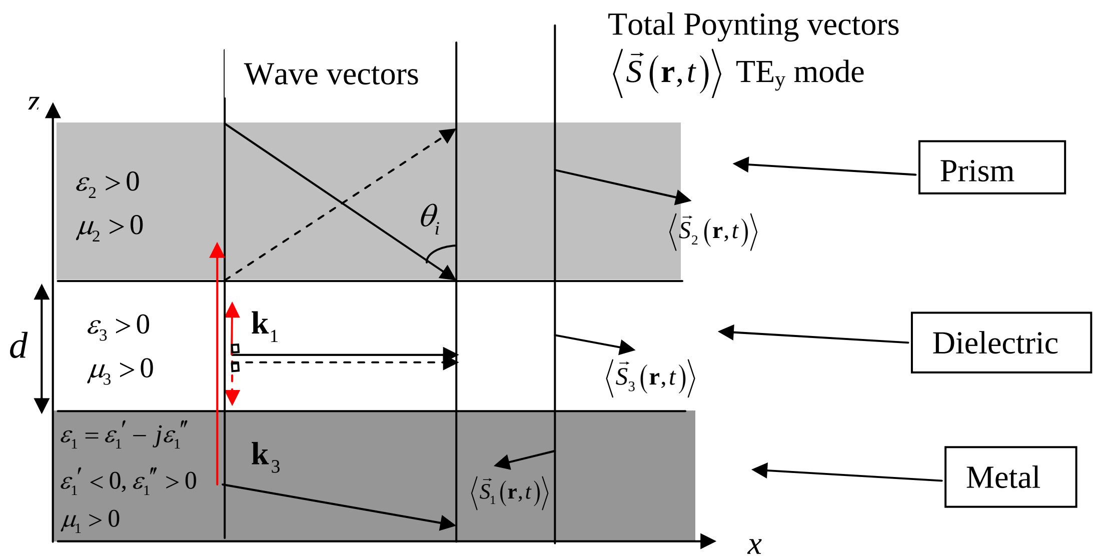

Figure 3.22- Wave vectors and Poynting vectors in Otto's configuration when the metal is lossy.

Wave vectors and Poynting vectors of Otto's configuration with lossy metal are shown in figure (3.22). $\left\langle \vec{S}_2(\mathbf{r},t) \right\rangle$ and $\left\langle \vec{S}_3(\mathbf{r},t) \right\rangle$ can be calculated using (3-26) and (3-23) respectively and both have constant *z* component. $\left\langle \vec{S}_1(\mathbf{r},t) \right\rangle$ is obtained by equation (2-25) and both *x* and *z* components are function of *z*.

Chapter 4

Complex Transformations and Gaussian Beams

4.1 Derivation and Properties of Scalar Gaussian Beam

In this section I lay the ground for next section which is complex transformations. By reformulating the well-known Gaussian beam formulation and expressing it as several theorems I will show why complex transformations are important. In next section we will investigate the complex transformations in detail.

Assuming that $U(\mathbf{r}) = \psi(\mathbf{r})\exp(-jk_0 z)$ satisfies scalar Helmholtz equation and $\left|\frac{\partial^2 \psi}{\partial z^2}\right| << \left|2k_0 \frac{\partial \psi}{\partial z}\right|$, then $\psi(\mathbf{r})$ satisfies the paraxial Helmholtz equation (PHE):

$$\nabla_T^2 \psi - 2jk_0 \frac{\partial \psi}{\partial z} = 0 \quad \text{and} \quad \nabla_T^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \qquad (4\text{-}1)$$

Theorem1. The ∇_T^2 operator is invariant under this transformation:

$$\boldsymbol{\rho}' = \overline{\mathbf{J}}_\phi \boldsymbol{\rho}$$

$$\boldsymbol{\rho} = [x, y], \quad \boldsymbol{\rho}' = [x', y'], \quad \overline{\mathbf{J}}_\phi = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \qquad (4\text{-}2)$$

in other words, we have:

$$\nabla_T'^2\psi = \frac{\partial^2\psi}{\partial x'^2} + \frac{\partial^2\psi}{\partial y'^2} = \nabla_T^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}$$

The parameter φ can be real or complex.

If φ is real, the transformation is a counterclockwise rotation with angle φ radian around the *z* axis.

Theorem 2. The function $\psi(\boldsymbol{\rho}, z) = \frac{1}{\sqrt{(z-z_1)(z-z_2)}} \exp\left[-j\frac{k}{2}\boldsymbol{\rho}\overline{\boldsymbol{\Lambda}}(z)\boldsymbol{\rho}^T\right]$ in which $\boldsymbol{\rho} = [x, y]$ and $\overline{\boldsymbol{\Lambda}}(z) = \begin{bmatrix} \frac{1}{z-z_1} & 0 \\ 0 & \frac{1}{z-z_2} \end{bmatrix}$, is a solution to PHE: $\nabla_T^2\psi - 2jk\frac{\partial\psi}{\partial z} = 0$. z_1 and z_2 can be real or complex.

$\boldsymbol{\rho}\overline{\boldsymbol{\Lambda}}(z)\boldsymbol{\rho}^T$ is a bivariable quadratic polynomial without any cross terms. For a constant *z* it defines conic sections with their axis be along *x* and *y*.

If z_1 and/or z_2 are complex we have

$$\overline{\boldsymbol{\Lambda}}(z) = \overline{\boldsymbol{\Lambda}}_r(z) + j\overline{\boldsymbol{\Lambda}}_i(z)$$

$$|\psi(\boldsymbol{\rho}, z)| = \frac{\exp\left(\frac{k}{2}\boldsymbol{\rho}\overline{\boldsymbol{\Lambda}}_i(z)\boldsymbol{\rho}^T\right)}{\sqrt{\left[(z-z_{1r})^2 + z_{1i}^2\right]\left[(z-z_{2r})^2 + z_{2i}^2\right]}} \qquad (4\text{-}3)$$

$$\angle\psi(\boldsymbol{\rho}, z) = \exp\left[-j\frac{k}{2}\boldsymbol{\rho}\overline{\boldsymbol{\Lambda}}_r(z)\boldsymbol{\rho}^T + j\eta(z)\right] \qquad (4\text{-}4)$$

$$\eta(z) = \frac{1}{2}\left[\tan^{-1}\left(\frac{z-z_{1r}}{z_{1i}}\right) + \tan^{-1}\left(\frac{z-z_{2r}}{z_{2i}}\right)\right]$$

$$\overline{\boldsymbol{\Lambda}}_r(z) = \mathrm{diag}\left(\frac{z-z_{1r}}{(z-z_{1r})^2 + z_{1i}^2}, \frac{z-z_{2r}}{(z-z_{2r})^2 + z_{2i}^2}\right)$$

$$\bar{\mathbf{\Lambda}}_i(z) = \mathrm{diag}\left(\frac{z_{1i}}{(z-z_{1r})^2 + z_{1i}^2}, \frac{z_{2i}}{(z-z_{2r})^2 + z_{2i}^2}\right)$$

Using theorems 1 and 2 we can state theorem 3.

Theorem 3. The function $\psi(\boldsymbol{\rho}, z) = \frac{1}{\sqrt{(z-z_1)(z-z_2)}} \exp\left[-j\frac{k}{2}\boldsymbol{\rho}\left(\bar{\mathbf{J}}_\phi \bar{\mathbf{\Lambda}}(z)\bar{\mathbf{J}}_{-\phi}\right)\boldsymbol{\rho}^T\right]$

is also a solution to PHE. The parameter φ can be real or complex.
$\bar{\mathbf{Q}}(z) = \bar{\mathbf{J}}_\phi \bar{\mathbf{\Lambda}}(z)\bar{\mathbf{J}}_{-\phi}$ is a symmetric matrix because $\bar{\mathbf{J}}_{-\phi} = \bar{\mathbf{J}}_\phi^T$.

Theorem 4. Any symmetric matrix $\bar{\mathbf{Q}}$ (real or complex) can be diagonalized in the form of $\bar{\mathbf{J}}_\phi \bar{\mathbf{\Lambda}} \bar{\mathbf{J}}_{-\phi}$ with $\bar{\mathbf{J}}_\phi$ (given in equation (4-2)) and $\bar{\mathbf{\Lambda}}$ is a diagonal matrix. Columns of $\bar{\mathbf{J}}_\phi$ are eigenvectors of $\bar{\mathbf{Q}}$ and diagonal entries of $\bar{\mathbf{\Lambda}}$ are corresponding eigenvalues.
If $\bar{\mathbf{Q}}$ is real, then φ is real and $\bar{\mathbf{\Lambda}}$ is also a real matrix.
If $\bar{\mathbf{Q}}$ is complex, then in general φ is complex and $\bar{\mathbf{\Lambda}}$ is a complex matrix.

Using theorems 3 and 4, theorem 5 is concluded:

Theorem 5. For any symmetric matrix $\bar{\mathbf{Q}}(z)$ (real or complex) , the function $\psi(\boldsymbol{\rho}, z)$ given as $\psi(\boldsymbol{\rho}, z) = \frac{1}{\sqrt{(z-z_1)(z-z_2)}} \exp\left[-j\frac{k}{2}\boldsymbol{\rho}\bar{\mathbf{Q}}(z)\boldsymbol{\rho}^T\right]$

is a solution to PHE. Based on theorem 4 the complex phase that has the form of bivariable quadratic polynomial $\frac{k}{2}\boldsymbol{\rho}\overline{\mathbf{Q}}(z)\boldsymbol{\rho}^T$ can always expressed in the form $\frac{k}{2}\boldsymbol{\rho}'\overline{\boldsymbol{\Lambda}}(z)\boldsymbol{\rho}'^T$ with $\boldsymbol{\rho}' = \overline{\mathbf{J}}_{\phi}\boldsymbol{\rho}$.

We have $\overline{\mathbf{Q}}(z) = \overline{\mathbf{Q}}_r(z) + j\overline{\mathbf{Q}}_i(z) = \overline{\mathbf{J}}_{\alpha}\overline{\boldsymbol{\Lambda}}_1\overline{\mathbf{J}}_{-\alpha} + j\overline{\mathbf{J}}_{\beta}\overline{\boldsymbol{\Lambda}}_2\overline{\mathbf{J}}_{-\beta} = \overline{\mathbf{J}}_{\phi}\overline{\boldsymbol{\Lambda}}\overline{\mathbf{J}}_{-\phi}$ (4-5)

Theorem 6. In equation (4-5) φ is real if and only if $\alpha = \beta$ and in this case we have $\alpha = \beta = \varphi$ and we can express both phase quadratic polynomial $\frac{k}{2}\boldsymbol{\rho}\overline{\mathbf{Q}}_r(z)\boldsymbol{\rho}^T$ and amplitude quadratic polynomial $\frac{k}{2}\boldsymbol{\rho}\overline{\mathbf{Q}}_i(z)\boldsymbol{\rho}^T$ in diagonal form using the same rotation transformation $\boldsymbol{\rho}' = \overline{\mathbf{J}}_{\phi}\boldsymbol{\rho}$.

In this case $\psi(\boldsymbol{\rho}, z)$ is called simple astigmatic Gaussian beam (SAGB). SAGB is created when a non-astigmatic beam passes through a non-astigmatic system.

Otherwise if φ is complex it is impossible to diagonalize $\overline{\mathbf{Q}}_r(z)$ and $\overline{\mathbf{Q}}_i(z)$ with the same transformation because $\alpha \neq \beta$. So either the phase quadratic polynomial or the amplitude quadratic polynomial will have the cross term. In this case $\psi(\boldsymbol{\rho}, z)$ is called general astigmatic Gaussian beam (GAGB). GAGB are created when we have reflection and transmission of simple non-astigmatic Gaussian beam from a curved boundary. [5]

The phase and amplitude of $\psi(\boldsymbol{\rho}, z)$ can be obtained similar to (4-3) and (4-4). When $\overline{\mathbf{Q}}_i(z) \neq \overline{\mathbf{0}}$ due to the term $\exp\left(\frac{k}{2}\boldsymbol{\rho}\overline{\mathbf{Q}}_i(z)\boldsymbol{\rho}^T\right)$ in amplitude, it is called Gaussian beam. If $z_1 \neq z_2$ it is called astigmatic Gaussian beam.

4.2 Complex linear transformations

Assuming that a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ is defines as following:

$$\mathbf{r}' = \bar{\mathbf{A}}\mathbf{r} + \mathbf{b} \qquad (4\text{-}6)$$

in which $\bar{\mathbf{A}}$ is a real matrix and $\mathbf{r}'$, $\mathbf{r}$ and $\mathbf{b}$ are real column vectors. We also assume that $\bar{\mathbf{A}}$ is an orthogonal matrix which means its columns are orthonormal vectors:

$\bar{\mathbf{A}}\bar{\mathbf{A}}^T = \bar{\mathbf{A}}^T\bar{\mathbf{A}} = \bar{\mathbf{I}}$.

$\bar{\mathbf{A}} = [\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3]$ and $\bar{\mathbf{I}} = [\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3]$

With $\bar{\mathbf{I}}$ being identity matrix , $\mathbf{e}_i$ are columns of transformation matrix $\bar{\mathbf{A}}$ and form an orthonormal basis for the new coordinate system.

Then we can define differential operators such as gradient, divergence and curl in the transformed coordinate system both in terms of transformed coordinates and original coordinate systems.

$$\nabla'\Phi = \sum \frac{\partial \Phi}{\partial x_i'}\mathbf{u}_i = \sum \frac{\partial \Phi}{\partial x_i}\mathbf{e}_i \qquad (4\text{-}7)$$

for example if we have

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (4\text{-}8)$$

we can verify that

$$\nabla'\Phi = \frac{\partial \Phi}{\partial x_1'}\mathbf{u}_1 + \frac{\partial \Phi}{\partial x_2'}\mathbf{u}_2 = \frac{\partial \Phi}{\partial x_1}\mathbf{e}_1 + \frac{\partial \Phi}{\partial x_2}\mathbf{e}_2 \qquad (4\text{-}9)$$

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \mathbf{e}_1 = \begin{bmatrix} \cos\varphi \\ -\sin\varphi \end{bmatrix} \quad \mathbf{e}_2 = \begin{bmatrix} \sin\varphi \\ \cos\varphi \end{bmatrix}$$

We can generalize these transformations to map $\mathbb{R}^3$ to $\mathbb{C}^3$. In this case $\bar{\mathbf{A}}$ can be a complex matrix and $\mathbf{b}$ can be a complex column vector.

Since the multivariable complex analysis is very involved we cannot define differential operators in the new complex space but we can use the results that we obtained in the real case to define these operators using the original real space. For example if the angle φ in equation (4-9) is complex we define the gradient operator in the complex space as

$$\nabla'\Phi \triangleq \frac{\partial\Phi}{\partial x_1}\mathbf{e}_1 + \frac{\partial\Phi}{\partial x_2}\mathbf{e}_2 \qquad (4\text{-}10)$$

so we don't need to involve multivariable complex analysis and partial complex derivatives.

Using this technique we can extend the geometrical transformations such as rotation and translation to complex domain. As we saw in theorem 1 the Laplace operator is invariant under the rotation transformation and so we can use the complex angle to obtain broader classes of solutions from existing solutions as was shown in section 4-1.

4.3 Vector Gaussian Beam

It is known that the far field of an antenna can be expressed as:

$$\mathbf{E}(\mathbf{r}) = \mathbf{F}(\theta,\varphi)U(r) \qquad (4\text{-}11)$$

$$U(r) = \frac{\exp(-jkr)}{r} \qquad (4\text{-}12)$$

$$\mathbf{F}(\theta,\varphi) = F_\theta(\theta,\varphi)\hat{\theta} + F_\varphi(\theta,\varphi)\hat{\varphi} \qquad (4\text{-}13)$$

We consider the y axis as the axis of beam so we write the Taylor expansion of all the components expansion of all the components around $\theta=\frac{\pi}{2},\varphi=\frac{\pi}{2}$ we obtain:

$$\mathbf{F}(x,y,z)=\left[-F_\varphi+\frac{\partial F_\varphi}{\partial\theta}\frac{z}{y}+\frac{\partial F_\varphi}{\partial\varphi}\frac{x}{y}\right]\hat{x}+\left[F_\theta\frac{z}{y}+F_\varphi\frac{x}{y}\right]\hat{y}+\left[-F_\theta+\frac{\partial F_\theta}{\partial\theta}\frac{z}{y}+\frac{\partial F_\theta}{\partial\varphi}\frac{x}{y}\right]\hat{z} \tag{4-14}$$

All the parameters $F_\theta,F_\varphi,\frac{\partial F_\varphi}{\partial\theta},\frac{\partial F_\varphi}{\partial\varphi},\frac{\partial F_\theta}{\partial\theta},\frac{\partial F_\theta}{\partial\varphi}$ are calculated at $\theta=\frac{\pi}{2},\varphi=\frac{\pi}{2}$

And for the scalar spherical wave we have:

$$\begin{aligned}U(r)&\approx\psi(\mathbf{r})\exp(-jky)\\ \psi(\mathbf{r})&=\frac{1}{y}\exp\left(-jk\frac{x^2+z^2}{y}\right)\end{aligned} \tag{4-15}$$

And the function $\psi(\mathbf{r})$ is a solution to paraxial Helmholtz equation:

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial z^2}-2jk\frac{\partial\psi}{\partial y}=0 \tag{4-16}$$

Using equations (4-11) and (4-15) we have

$$\begin{aligned}\mathbf{E}(\mathbf{r})&=\mathbf{A}(\mathbf{r})\exp(-jky)\\ \mathbf{A}(\mathbf{r})&=\mathbf{F}(x,y,z)\psi(x,y,z)\end{aligned} \tag{4-17}$$

$\mathbf{F}(x,y,z)$ is given in equation (4-14) and $\psi(x,y,z)$ is given in (4-15). As it was discussed in section (4-2) we can apply a complex transformation to the coordinate systems with $\psi(x, y, z)$ still satisfying equation (4-16). We choose the transformation to be a translation in y with complex value y_0

$$\mathbf{A}_1(\mathbf{r}) = \mathbf{F}(x, y + y_0, z)\psi(x, y + y_0, z)$$

$$\mathbf{E}_1(\mathbf{r}) = \mathbf{A}_1(\mathbf{r})\exp(-jky)$$

Chapter 5

Summary and Future Research

5.1 Summary

The concept of complex vectors and complex transformation were used to investigate the multilayer media. In addition to reflection and transmission coefficient we derived the polarization and Poynting vector that can be different from layer to layer. In most of the books and articles that address this problem, focus is only on reflection coefficient. In this thesis I have also studied the changes in polarization of electric and magnetic field and also the phenomenon of evanescent waves and surface waves is represented by different polarizations of complex wave vector as a subset of a more general problem. I have provided a more general definition of polarization that are applicable to complex vectors that are the superset of phasors. Poynting vector as the quantity that shows the direction of flow of power was investigated in detail.

A geometrical representation for complex waves such as surface (evanescent) waves and surface Plasmon polaritons was developed.

In the end we studied surface plasmon resonance with emphasis on behavior of Poynting vector and effect of different polarizations. Otto and Kretschmann configurations were investigated in detail and I offered an explanation for the fact that the metal must be lossy so that we observe minimum in total reflection coefficient.

In chapter 4 we investigated complex transformation and Gaussian beam with some background mathematical theorems.

5.2 Future work

In this study I applied the concept of complex plane waves (plane waves with complex wave vector) to some known problems and configurations such as plane boundary between two homogeneous media and multilayer media. Considering the fact that Gaussian beam and geometrical optics can be considered as generalization of real plane waves (plane wave with real wave vector or with LP complex wave vector) can be considered as generalization of a plane wave, we can apply them to complex waves too.

For example if we consider a plane as a first degree approximation of an arbitrary surface of constant phase, Gaussian beam can be interpreted as its second degree approximation. In the same way we can generalize the planes of constant amplitude of a complex plane wave as first degree approximation of an arbitrary surface of constant amplitude and investigate the implications of using second degree approximation to this surface.

One of the challenges in doing so is that the axis that we approximate these surfaces around them are not necessarily in the same direction (as $\mathbf{k_r}$ and $\mathbf{k_i}$ are not parallel when the wave vector is NLP)

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