A Simplified Method for Hedging Jump Diffusions

by

Wenjie Xiao

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Geometric Brownian Motion (GBM) and has been widely used in the Black Scholes option-pricing framework to model the return of assets. However, many empirical investigations show that market returns have higher peaks and fatter tails than GBM. Contrary to the Black Scholes model, an option-pricing model which contains jumps reflects the evolution of stock prices more accurately. Therefore, hedging a model under jump diffusion would be desirable. This thesis develops a simplified method for hedging jump diffusions.

In order to hedge the jump risk, other instruments besides the underlying asset must be used in the hedging procedure. We start with a the Partial Integro Differential Equation (PIDE) that models contingent claims with jumps and consider a dynamic hedging strategy that uses a hedging portfolio with the underlying asset and liquidly traded options. We introduce a simple hedging method, where, at each rebalance time, we minimize the instantaneous jump risk by finding proper weights for the underlying asset and instruments.

We use a simulation method to test our approach using a Truncated SVD method to solve the linear system of equations resulting from our minimization procedure. Our results indicate that the proposed dynamic hedging strategy provides sufficient protection against diffusion and jump risk.

The method also provides a firm theoretical basis for a method which is used in practice.

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Chapter 1

Introduction

1.1 Overview

Minimizing risk by using hedging instruments is a well-known problem. In a complete market, where no jumps occur, the standard Black-Scholes analysis assumes that risk can be eliminated by using a continuously re-balanced delta hedge. However, if the underlying process is a jump diffusion, then a market consisting of the underlying asset and a bond is no longer complete. Consequently, the standard delta hedge will not result in an instantaneously risk-free portfolio. A random gain or loss will be produced if we simply use the delta hedging strategy.

1.2 Previous Work

1.2.1 Model

The Black-Scholes model with constant volatility has been commonly used because it is easy to implement. However, this model does not reproduce the volatility smile typically seen in real option prices. In order to improve the Black-Scholes model, a volatility surface [2] is often used. However, a volatility surface cannot account for sudden jumps in asset prices. For example, recent stock market crashes are obviously not consistent with a

Geometric Brownian Motion (GBM) model. A more powerful model which allows jumps is needed.

The most general form of a model which allows jumps is a Lévy process [18, 19]. In this work, we restrict our attention to a finite activity Lévy process. The finite activity Lévy model is also known as jump diffusion model [18, 19]. Anderson and Andreasen [1] showed that a jump diffusion model with a local volatility surface is able to produce a good fit to S&P 500 option prices. In spite of this, jump diffusion models are not widely used. The pricing equations are more complex and, in theory, we need an infinite number of hedging instruments to complete the market.

1.2.2 Pricing the Portfolio

Efficient ways to estimate the price of derivative securities (e.g,, options), where the underlying assets follow a jump diffusion process have been discussed in several papers. A tractable option pricing model, which is valid even when jump risk is systematic, has been developed by Bates in 1988 [8]. Andersen and Andreasen [1] developed the idea of combining a deterministic local volatility approach with lognormally distributed Poisson jumps and constant parameters. They pointed out the following advantage in their paper

> " ... by letting the jump-part of the process dynamics explain a significant part of the volatility smile/skew, we generally obtain a 'reasonable', stable [deterministic volatility] function, without the extreme short-term variation typical of the pure diffusion approach" [1].

However, if we remove the assumption of constant local volatilities, there is no existing analytic method that can be applied directly to price the options under a jump diffusion model, even for simple European options. In order to solve this problem, numerical techniques are required [1]. Unfortunately, even if we assume constant volatilities in the model, most existing methods can only price vanilla European options when jump processes are included.

Generally, solving a partial integro-differential equation (PIDE) problem is required when we want to value an option under a jump diffusion model. One method suggested by Amin is based on multinomial trees [14]. However, this explicit type method has time step limitation problems because of stability considerations. Even worse, this approach also has accuracy issues (this method is only first order accurate in general). Zhang [22] introduced a method which divided the PIDE into two parts and solved them at different time levels. This method treats the jump integral term explicitly, and the rest of the PIDE implicitly. Although this method works well on pricing American options, there are severe restrictions due to stability conditions. In [23], pricing American options with Poisson distributed jumps was solved with the method of lines. More general/complex models which use a Lévy process can be solved by either a method which combines a finite difference method and a fast Fourier transform (FFT) [1] or a finite element method based on wavelets [24]. An implicit method has been introduced in [13], in which the pricing error can be reduced to second order in many cases.

1.2.3 Hedging the Portfolio

It is very challenging to develop a hedging strategy when the underlying asset follows a jump diffusion process. It is impossible to have a perfect hedge with a finite number of instruments when the underlying asset follows a jump diffusion process, no matter how small the hedging interval. There have been relatively few studies of hedging jump diffusion. Carr and Wu suggested using a semi-static approach for hedging in [29]. They used the exact relationship between the value of a target option in terms of its payoff and the risk-adjusted density function. Mathematically, we can transform this relationship into the idea of hedging the primary option by use of several short term options with certain weights. The weights of the spanning options are given by the gamma (second derivative of the option price) of the primary option at the expiry date of the short term options. In practice, the dynamics of the market is unpredictable, so a fitted model is commonly used to compute the portfolio weights. Of course, a finite number of hedging options is used.

Another approach uses a dynamic self financing portfolio. This was first suggested by Bates in 1988 [8]. In 2000, Andersen and Andreasen put forward this idea again [1]. They suggested using a dynamic self financing portfolio which consists of the underlying asset and a finite number of options to minimize jump risk. However, there are two important issues which need to be addressed. It is not clear how many options are needed in the hedging portfolio in order to obtain a satisfactory risk reduction. In addition, if the portfolio is re-balanced too frequently, then large transaction costs may accumulate.

1.3 Main Results

In this thesis, we assume the underlying asset follows a Merton jump diffusion [4]. We use dynamic hedging to minimize the instantaneous risk. A jump-diffusion model is subject to two sources of risk: the diffusion risk from the Brownian motion component and the jump risk. It is well-known that the diffusion risk can be removed by applying delta hedging. However, the jump risk can be eliminated only by adding an infinite number of hedging instruments, which is impossible in practice. Even so, we can address instantaneous jump risk by using a finite and practical number of hedging instruments. This kind of dynamic hedging can be used for both European and American-style exercise rights. As suggested by Andersen and Andreasen in [1], we use a finite number of options as part of a dynamic hedging strategy to minimize jump risk. More specifically, we will follow the idea of He et al [26]: minimizing the jump risk in some sense, subject to the delta-neutral constraint that eliminates the instantaneous jump diffusion risk. Mathematically, we transform this optimization problem into a linear equation problem. This linear equation problem can be solved by using a TSVD (truncated singular value decomposition). We show that the overall jump risk will be controlled if the instantaneous jump risk is minimized.

The main difference between the approach in this thesis and that in He el al [26] is that we minimize the jump risk by forcing the jump risk to be identically zero at a finite set of jump sizes. This contrasts to the weighted integral approach in [12]. We believe that the method used in this thesis is simple and intuitive and therefore may find use by practitioners. In fact, a simple form of this approach is used in the energy industry [33].

1.4 Outline

The outline of this thesis is as follows. In Chapter 2, we introduce the model and the derivation of instantaneous jump risk. In Chapter 3, we present a method that minimizes the risk and its implementation. Numerical results are provided in Chapter 4. Finally, in Chapter 5, we provide conclusions.

Chapter 2

Jump Diffusion

2.1 Hedging Model and Theoretical Hedging Risk

In this section, we will first briefly introduce the Black-Scholes under geometric Brownian motion (GBM) model and the Merton jump-diffusion model. Then, we will derive the expression for instantaneous jump risk.

2.1.1 The Black-Scholes Model

The Black-Scholes model was first introduced by Black and Scholes in [31]. It is widely used as a tool for pricing equity options. There are several assumptions underlying the Black-Scholes model. The most well-known assumptions are:

- the volatility (i.e. the standard deviation of the continuously compounded returns of a financial instrument in a specific time period) is constant;
- there is no transaction costs and taxes;
- the underlying asset follows Geometric Brownian Motion (GBM) with constant drift;
- there are no arbitrage opportunities;
- there are no restrictions on short selling.

Assuming the underlying asset follows GBM, the price can be written as the solution of the stochastic differential equation (SDE):

$$dS_t = \alpha S_t dt + \sigma S_t dZ_t, \tag{2.1.1}$$

where S_t is underlying asset, α is the drift term, σ is the volatility and dZ_t is the increment of a standard Wiener processes. Under the GBM assumptions, underlying asset paths are positive and continuous; asset returns are independent and uncorrelated over non-overlapping time periods [25].

By combining all of these assumptions together with the idea that there is no immediate gain for selling or buying, the Black-Scholes price can be obtained by solving the following partial differential equations (PDE):

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{2.1.2}$$

where V is the no-arbitrage value of a European option. When $t = T$ and T is the expiry time, the resulting value V is the option payoff.

Stock prices are continuous in the Black-Scholes model, but in reality stock prices can jump suddenly. In the next section, we discuss the Merton jump-diffusion model which better reflects reality.

2.1.2 Merton Jump Diffusion

In the Merton Jump Diffusion model, the change in the asset price can be divided into two parts. One part comes from the continuous diffusion of the model which is modeled by a Geometric Brownian Motion, while the other part is generated from discontinuous jumps, and is modeled by a compound Poisson Process. Based on the assumption that the underlying process is a jump diffusion with constant volatility, the underlying asset

S is given from the solution to the SDE

$$\frac{dS_t}{S_{t^-}} = (\alpha - \kappa\lambda)dt + \sigma dZ_t + (J_t - 1)d\pi_t \tag{2.1.3}$$

where

- t^- is the time instant immediately before time t,
- α is the instantaneous expected rate of return on the asset,
- $\lambda > 0$ is the intensity of the jump process which is independent of time t,
- κ is the mean of $(J_t - 1)$,
- $J_t - 1$ is a jump amplitude function which determines a jump from S to J_tS where J_t is an independent and identically distributed random variable which represents the jump amplitude and is nonnegative,
- σ denotes the diffusive volatility of the asset return when a jump does not occur,
- Z_t is a standard Brownian motion process, dZ_t is the increment of a Wiener process,
- $d\pi_t$ follows a Poisson distribution, where

$$d\pi_t = \begin{cases} 0 & \text{with probability } 1 - \lambda dt, \\ 1 & \text{with probability } \lambda dt. \end{cases}$$

For simplicity, J_t is assumed to be lognormally distributed. We also assume that Z_t, π_t and J_t are independent processes. The process π_t determines the possibility of getting a jump at a particular point in time. More specifically, $d\pi_t$ is the probability that an asset price jumps during a small time interval dt.

The compound Poisson Process in model (2.1.3) includes two pieces of information. First, π_t determines if the jump occurs in the current time interval dt. Second, J_t gives the jump amplitude if a jump occurs. For example, suppose a jump occurs in the current time interval dt, (i.e. $d\pi_t = 1$) then the price of the asset jumps from S_{t^-} to $J_t S_{t^-}$. So the relative price jump size is

$$\frac{dS_t}{S_{t^-}} = \frac{J_t S_{t^-} - S_{t^-}}{S_{t^-}} = J_t - 1.$$

As mentioned in model (2.1.3), J_t is a nonnegative independent identically random number which is generated from lognomal distribution, $\ln(J_t) \sim$ i.i.d. $N(\mu, \gamma)$, that is:

$$\begin{aligned} E(J_t) &= e^{\mu + \frac{\gamma}{2}} \\ Var(J_t) &= e^{2\mu+\gamma}(e^{\gamma} - 1). \end{aligned}$$

This implies,

$$\begin{aligned} \kappa = E(J_t) - 1 = E(J_t - 1) &= e^{\mu + \frac{\gamma}{2}} - 1 \\ Var(J_t - 1) &= e^{2\mu+\gamma}(e^{\gamma} - 1). \end{aligned}$$

2.1.3 Option Pricing under the Jump-Diffusion Process

Following standard arguments [25, 1], the value of a European option under the process (2.1.3) can be found by solving a partial integro-differential equation (PIDE).
Define $\mathcal{L}$ as:

$$\begin{aligned} \mathcal{L}V \equiv\ & \frac{\partial V}{\partial \tau} \\ & - \left(\frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + (r - \kappa^{\mathbb{Q}} \lambda^{\mathbb{Q}}) S \frac{\partial V}{\partial S} - rV + \lambda^{\mathbb{Q}} \left[\int_0^{\infty} V(SJ, \tau) g^{\mathbb{Q}}(J) dJ - V(S, \tau) \right] \right), \end{aligned}$$

where $\mathbb{Q}$ represents the $\mathbb{Q}$ measure, V is the value of the option, T represents the expiry time of the option, t represents the current time, $\tau = T - t$, r is risk-free interest rate, κ, λ and J are defined as in the SDE (2.1.3), $g^{\mathbb{Q}}(J)$ is the risk-adjusted distribution of jumps. The price of a European option, which may only be exercised at its expiration data, is given by

$$\mathcal{L}V = 0. \tag{2.1.4}$$

In order to price the American style contract, which can be exercised at any time before the maturity, the PIDE Variational Inequality

$$\min(\mathcal{L}V, V - V_e) = 0,$$

needs to be solved [32], where V_e denotes the payoff of the claim.

By using the jump-diffusion process, the model is clearly improved in several ways. First, the probability of a large change in the underlying asset is larger than under the GBM model. This difference gives us a heavier tail in the distribution of returns. Second, the implied volatility generated under the jump-diffusion model produces a volatility smile, which is consistent with observed option prices.

2.1.4 Hedging Risk under a Jump-Diffusion Process

In this section, the mathematical representation of jump risk will be derived. Under the $\mathbb{P}$ measure, an asset follows

$$dS_t = (\alpha^{\mathbb{P}} - \kappa^{\mathbb{P}}\lambda^{\mathbb{P}})S_{t^-}dt + \sigma S_{t^-}dZ^{\mathbb{P}} + (J_t - 1)S_{t^-}d\pi^{\mathbb{P}} \tag{2.1.5}$$

where $\mathbb{P}$ is the real world measure, and a jump occurs with intensity $\lambda^{\mathbb{P}}$. The jump size J_t is distributed according to $g^{\mathbb{P}}(J_t)$ and has a mean of $\kappa^{\mathbb{P}} + 1$.

Assume we are short a derivative (primary option) V, then our position is $-V$ in the contract. In order to hedge a target option V, we will start with the standard hedging portfolio. Holding a long position in e units of the underlying asset, and a long position in N additional hedging options with prices $\vec{I} = [I_1, I_2, ..., I_N]$, with corresponding weight

$\vec{\phi} = [\phi_1, \phi_2, ..., \phi_N]$ and an amount B in cash, the overall hedged position has value

$$\Pi = -V + eS + \vec{\phi} \cdot \vec{I} + B \tag{2.1.6}$$

where $\vec{I}$ is a vector which includes all possible hedging instruments we will use for the entire hedging time period. If I_i is not used in the current time period, ϕ_i is set to zero. For simplicity, all the explicit dependence on time t and asset price S have been dropped. In order to represent changes of the components of Π when a jump occurs with size J, we define

$$\begin{aligned} \Delta V &= V(JS) - V(S), \\ \Delta S &= JS - S, \\ \Delta \vec{I} &= \vec{I}(JS) - \vec{I}(S). \end{aligned}$$

If a change in the short position $-V$ is always explicitly equal to the change in the hedge portfolio $eS + \vec{\phi} \cdot \vec{I} + B$, then we will say the hedge is perfect and then $d\Pi$ equals zero (i.e. no variation) over $t \to t + dt$. Therefore, we must consider the infinitesimal change in the overall hedged position value Π. Because we are considering the real-world evolution of this portfolio, the underlying jump-diffusion process of interest is governed by the $\mathbb{P}$ measure, and is given in (2.1.5). So we have (using Ito's formula):

$$\begin{aligned} dS &= \xi^{\mathbb{P}} S dt + \sigma S dZ^{\mathbb{P}} + \Delta S d\pi^{\mathbb{P}} \\ dV &= \left[\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + \xi^{\mathbb{P}} S \frac{\partial V}{\partial S} \right] dt + \sigma S \frac{\partial V}{\partial S} dZ^{\mathbb{P}} + \Delta V d\pi^{\mathbb{P}} \\ d\vec{I} &= \left[\frac{\partial \vec{I}}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 \vec{I}}{\partial S^2} + \xi^{\mathbb{P}} S \frac{\partial \vec{I}}{\partial S} \right] dt + \sigma S \frac{\partial \vec{I}}{\partial S} dZ^{\mathbb{P}} + \Delta \vec{I} d\pi^{\mathbb{P}} \\ dB &= rBdt, \end{aligned}$$

where $\xi = \alpha^{\mathbb{P}} - \kappa^{\mathbb{P}} \lambda^{\mathbb{P}}$. Therefore, the immediate change in the value of the overall hedged position can be written as

$$
\begin{aligned}
d\Pi &= -dV + e\ dS + \vec{\phi} \cdot d\vec{I} + dB \\
&= -\left[\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2}\right] dt + \vec{\phi} \cdot \left[\frac{\partial \vec{I}}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 \vec{I}}{\partial S^2}\right] dt \\
&\quad + rBdt + \left[-\Delta V + (e\Delta S + \vec{\phi} \cdot \Delta \vec{I})\right] d\pi^{\mathbb{P}} \\
&\quad + \xi^{\mathbb{P}} S \left[-\frac{\partial V}{\partial S} + e + \vec{\phi} \cdot \frac{\partial \vec{I}}{\partial S}\right] dt + \sigma S \left[-\frac{\partial V}{\partial S} + e + \vec{\phi} \cdot \frac{\partial \vec{I}}{\partial S}\right] dZ^{\mathbb{P}}, \quad (2.1.7)
\end{aligned}
$$

where e and $\vec{\phi}$ are constant over dt, as they are specified at the beginning of this time interval. If the portfolio is delta neutral, then we have

$$
-\frac{\partial V}{\partial S} + e + \vec{\phi} \cdot \frac{\partial \vec{I}}{\partial S} = 0. \qquad (2.1.8)
$$

Substituting (2.1.8) into equation (2.1.7), we will have

$$
\begin{aligned}
d\Pi &= -\left[\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2}\right] dt + \vec{\phi} \cdot \left[\frac{\partial \vec{I}}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 \vec{I}}{\partial S^2}\right] dt \\
&\quad + rBdt + \left[-\Delta V + (e\Delta S + \vec{\phi} \cdot \Delta \vec{I})\right] d\pi^{\mathbb{P}}, \qquad (2.1.9)
\end{aligned}
$$

indicating that $d\Pi$ is now a pure jump process with drift. Rewriting PIDE (2.1.4) as

$$
\begin{aligned}
\frac{\partial V}{\partial \tau} &= \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} \\
&+ (r - \kappa^{\mathbb{Q}}\lambda^{\mathbb{Q}})S\frac{\partial V}{\partial S} - rV + \lambda^{\mathbb{Q}}\left(\int_0^\infty V(SJ,\tau)g^{\mathbb{Q}}(J)dJ - V(S,\tau)\right),
\end{aligned}
$$

and then by using elementary rearrangement, and we obtain:

$$
\begin{aligned}
\frac{\partial V}{\partial t} &+ \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} = rV + \lambda^{\mathbb{Q}}\mathbb{E}^{\mathbb{Q}}(\Delta S) - rS\frac{\partial V}{\partial S} - \lambda^{\mathbb{Q}}\mathbb{E}^{\mathbb{Q}}(\Delta V) \\
\frac{\partial \vec{I}}{\partial t} &+ \frac{\sigma^2 S^2}{2}\frac{\partial^2 \vec{I}}{\partial S^2} = r\vec{I} + \lambda^{\mathbb{Q}}\mathbb{E}^{\mathbb{Q}}(\Delta S) - rS\frac{\partial \vec{I}}{\partial S} - \lambda^{\mathbb{Q}}\mathbb{E}^{\mathbb{Q}}(\Delta \vec{I}), \qquad (2.1.10)
\end{aligned}
$$

where $\mathbb{E}^{\mathbb{Q}}(\Delta S) = \mathbb{E}^{\mathbb{Q}}(S[J-1]) = S\mathbb{E}^{\mathbb{Q}}(J-1) = S\kappa^{\mathbb{Q}}$. Substituting (2.1.10) into (2.1.9) gives

$$
\begin{aligned}
d\Pi = & - \left[rV + \lambda^{\mathbb{Q}}\mathbb{E}^{\mathbb{Q}}(\Delta S) - rs\frac{\partial V}{\partial S} - \lambda^{\mathbb{Q}}\mathbb{E}^{\mathbb{Q}}(\Delta V) \right] dt \\
& + \vec{\phi} \cdot \left[r\vec{I} + \lambda^{\mathbb{Q}}\mathbb{E}^{\mathbb{Q}}(\Delta S) - rs\frac{\partial \vec{I}}{\partial S} - \lambda^{\mathbb{Q}}\mathbb{E}^{\mathbb{Q}}(\Delta \vec{I}) \right] dt \\
& + rBdt + \left[-\Delta V + (e\Delta S + \vec{\phi} \cdot \Delta \vec{I}) \right] d\pi^{\mathbb{P}} \\
= & \quad r \left[-V + \left(\frac{\partial V}{\partial S} - \vec{\phi} \cdot \frac{\partial \vec{I}}{\partial S} \right) S + \vec{\phi} \cdot \vec{I} + B \right] dt \\
& + \lambda^{\mathbb{Q}} \left[\mathbb{E}^{\mathbb{Q}}(\Delta V) - \left(\frac{\partial V}{\partial S} - \vec{\phi} \cdot \frac{\partial \vec{I}}{\partial S} \right) \mathbb{E}^{\mathbb{Q}}(\Delta S) - \vec{\phi} \cdot \mathbb{E}^{\mathbb{Q}}(\Delta \vec{I}) \right] dt \\
& + \left[-\Delta V + (e\Delta S + \vec{\phi} \cdot \Delta \vec{I}) \right] d\pi^{\mathbb{P}}. \qquad (2.1.11)
\end{aligned}
$$

Since we assume the delta neutral constraint holds, then substituting (2.1.8) into (2.1.11) gives,

$$
\begin{aligned}
d\Pi = & \quad r \left[-V + eS + \vec{\phi} \cdot \vec{I} + B \right] dt \\
& + \lambda^{\mathbb{Q}} \left[\mathbb{E}^{\mathbb{Q}}(\Delta V) - e\mathbb{E}^{\mathbb{Q}}(\Delta S) - \vec{\phi} \cdot \mathbb{E}^{\mathbb{Q}}(\Delta \vec{I}) \right] dt \\
& + \left[-\Delta V + (e\Delta S + \vec{\phi} \cdot \Delta \vec{I}) \right] d\pi^{\mathbb{P}} \\
= & \quad r\Pi dt \\
& + \lambda^{\mathbb{Q}} dt \mathbb{E}^{\mathbb{Q}} \left[(\Delta V) - (e\Delta S + \vec{\phi} \cdot \Delta \vec{I}) \right] \\
& + d\pi^{\mathbb{P}} \left[-\Delta V + (e\Delta S + \vec{\phi} \cdot \Delta \vec{I}) \right]. \qquad (2.1.12)
\end{aligned}
$$

Therefore, equation (2.1.12) shows that the value change of the overall hedged position depends on two components: the risk free gain and the instantaneous jump risk, denoted by

$$
\underbrace{\lambda^{\mathbb{Q}} dt \mathbb{E}^{\mathbb{Q}} \left[(\Delta V) - (e\Delta S + \vec{\phi} \cdot \Delta \vec{I}) \right] + d\pi^{\mathbb{P}} \left[-\Delta V + (e\Delta S + \vec{\phi} \cdot \Delta \vec{I}) \right]}_{\text{instantaneous jump risk}}.
$$

Note that if the jump process under measure $\mathbb{Q}$ and $\mathbb{P}$ are the same, then the real-world expected value of instantaneous jump risk is zero.

The first component of the instantaneous jump risk is deterministic and the second component is stochastic since it depends on whether or not a jump occurs over the instant dt and the size of the jump. In order to minimize the jump risk, we only need to consider the stochastic part. We define a random variable for this stochastic part:

$$\Delta H(J) = -\Delta V + e\Delta S + \vec{\phi} \cdot \Delta \vec{I}. \tag{2.1.13}$$

When $\Delta H(J)$ is small (i.e. the second component of the instantaneous jump risk is small), the deterministic component of the jump risk also becomes small (i.e. the first component of the instantaneous jump risk is small), so we have small overall instantaneous jump risk. Therefore, the change in the overall hedged position due to a jump is small. If we can find the optimal weight $\{e, \phi\}$ that minimizes the jump risk over the instant dt, then the hedging risk can be controlled.

2.2 SVD and TSVD

Our method for determination of the hedging portfolio weights will use a Singular Value Decomposition (SVD). In this section, we give a review of the Singular Value Decomposition (SVD) and Truncated Singular Value Decomposition (TSVD).

2.2.1 Singular Value Decomposition: Definition

A singular value decomposition (SVD) is a very useful tool for calculating the pseudoinverse, least squares fitting of data, matrix approximation and determining the rank, range and null space of a matrix. The SVD can be applied to any kind of matrix. However, for simplicity, we only consider square matrices in this thesis. Let A be a $n \times n$ matrix. Then we can decompose A as follows:

$$A = U\Lambda V^T, \tag{2.2.1}$$

where U and V are orthonormal matrices which have the property $UU^T = I_n$ and $VV^T = I_n$. I_n is an $n \times n$ identity matrix. U has size $n \times n$. V is a $n \times n$ matrix. Λ is a $n \times n$ diagonal matrix with form:

$$\begin{bmatrix} w_1 & 0 & \cdots & 0 \\ 0 & w_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & w_n \end{bmatrix}$$

The diagonal entries are called the singular values of matrix A (including zero). Each singular value has its corresponding singular vectors. The i^{th} column in matrix U (i.e. u_i) and i^{th} column of matrix V (i.e. v_i) are called left and right singular vectors for the i^{th} singular value. Normally, all w_i are in decreasing order, i.e. $w_1 \geq w_2 \geq \cdots \geq w_n$.

2.2.2 SVD and Matrix Norms

Given the matrix equation $Ax = b$, we define x to be a minimum norm solution of this equation when x is a solution of this equation and has minimum norm amongst all possible solutions of the equation. In this thesis, we only consider square matrices. Therefore, if A is a $n \times n$ matrix, there are two possible cases for solving $Ax = b$:

- If the rank of A (i.e. rank(A)) is n. There exists a unique solution for the equation. This solution is the minimum norm solution.
- if the rank of $A < n$. We say A is rank deficient. Then the solution is found which minimizes both $\| x \|_2$ and $\| Ax - b \|_2$, known as the minimum norm solution.

We define a pseudoinverse matrix of Λ, say $\tilde{\Lambda}$, as the diagonal matrix with:

$$\tilde{w}_{i,i} = \begin{cases} \frac{1}{w_{i,i}} & \text{if } w_{i,i} > 0 \\ 0 & \text{if } w_{i,i} = 0 \end{cases}$$

Then the pseudoinverse of A, denoted by $\tilde{A}$, is given by

$$\tilde{A} = V\tilde{\Lambda}U^T$$

If A is a square matrix, we have that $\tilde{x} = V\tilde{W}U^T b$ solves the linear equation $Ax = b$ in the following sense:

- If A is non-singular, then $\tilde{x}$ is the unique solution to the equation.
- If A is singular and $b \in R_A$, then $\tilde{x}$ is minimum norm solution.
- If A is singular and $b \in N_A$, then $\tilde{x} = \arg\min_x |Ax - b|$,

where

- N_A is the null space of A such that $N_A = \{x \in R^n : Ax = 0\}$.
- R_A is the range of A such that $R_A = \{x \in R^n : Ax \neq 0\}$.

If A is a nonsingular matrix, then all the singular values $w_i > 0$. We then have $\Lambda^{-1} = \tilde{\Lambda}$, and

$$\begin{aligned} A^{-1} &= (U\Lambda V^T)^{-1} \\ &= V\Lambda^{-1}U^T \\ &= V\tilde{\Lambda}U^T \\ &= \tilde{A} \end{aligned}$$

Therefore, $\tilde{A} = A^{-1}$. The pseudoinverse solution $\tilde{x}$ can be determined by using an SVD approach. $\tilde{x}$ is a unique solution and it is a minimum norm solution. If A is a singular matrix, the minimum-norm solution to $Ax \approx b$ is given by $\tilde{A}b$.

2.2.3 Truncated Singular Value Decomposition

In real applications, due to efficiency considerations or other benefits, the SVD is usually not chosen. Instead, truncated SVDs are used for the computation.

A truncated SVD (TSVD) is a reduced rank approximation to A obtained by setting all but first k largest singular values equal to zero and using only the first k column vectors of U and k row vectors of V in the calculation. The rest of matrices of U and V are discarded. If we have an SVD decomposition $A = U\Lambda V^T$ and U, V and Λ are defined as in section (2.2.1), the TSVD can be written as:

$$A_k = U_k \Lambda_k V_k^T. \tag{2.2.2}$$

If a matrix A has rank r, a full rank decomposition of A is usually denoted by

$$A_r = U_r \Lambda_r V_r^T. \tag{2.2.3}$$

A theorem proven by Eckart and Young [30] shows that the error in approximating a matrix A (with rank r) by A_r can be written:

$$\| A - A_r \|_F \leq \| A - B \|_F \tag{2.2.4}$$

where B is any matrix with rank r, $\| \, . \, \|_F$ means Frobenius norm. This formula (2.2.4) tells us the difference between A and A_r is smaller than the difference between A and any other rank r matrix B. Therefore, there does not exist a matrix that has rank r and is closer to matrix A than A_r.

If A is a singular matrix (a matrix that is not invertible) with rank r, we can find the best approximate minimum-norm solution to $Ax \approx b$ by solving $A_k^{-1}b$ where $k = r$ by using a TSVD decomposition (2.2.3).

Furthermore, k can be any number $< n$, even for $k < r$. Although (2.2.3) is no longer a full rank decomposition of A when $k \neq r$, it is still the closest approximation for matrix A with rank k from Eckart's theorem (2.2.4).

We can decompose any singular or nonsingular matrix using an SVD or a TSVD. In this thesis, we will use a TSVD, since this will tend to produce more stable weights in the hedging portfolio.

2.3 Hedging Strategy

As we discussed in Section (2.1.4), if we can find a proper weight $\{e, \vec{\phi}\}$ to lower the value of the stochastic term in the instantaneous jump risk (2.1.13), we can lower the overall jump risk over the instant dt. Kennedy [12] points out that the weights can be found by solving the optimization problem:

$$\begin{aligned} \underset{\{e,\vec{\phi}\}}{\arg\min} \quad & \int_0^\infty \left[-\Delta V + (e\Delta S + \vec{\phi}\cdot\Delta\vec{I})\right]^2 W(J)dJ && (2.3.1) \\ \text{subject to} \quad & e + \vec{\phi}\cdot\frac{\partial\vec{I}}{\partial S} = \frac{\partial V}{\partial S}, && (2.3.2) \end{aligned}$$

where $e+\vec{\phi}\cdot\frac{\partial\vec{I}}{\partial S} = \frac{\partial V}{\partial S}$ is the delta neutral condition. In equation (2.3.1), $W(J)$ is a proper weighting function (see [12]).

In this thesis, we will use a simpler method to find the weights $\{e, \vec{\phi}\}$ which makes the jump risk small. Recall the expression for the jump risk of given jump size J:

$$\Delta H(J) = -\Delta V + e\Delta S + \vec{\phi}\cdot\Delta\vec{I} \tag{2.3.3}$$

If $dim(\vec{I}) = N$, then we have $N+1$ hedging weights, $\{e, \phi_1, \cdots, \phi_N\}$. Selecting N distinct values of $J_i, i = 1, \cdots, N$, let

$$\Delta H(J_i) = 0, i = 1, \cdots, N. \tag{2.3.4}$$

This gives us a set of N equations.

The delta neutral condition (2.3.2) adds to the above equations. This gives a total of $N+1$ equations and $N+1$ unknowns. If, for example, we select J_i to be equally spaced in $[0, J_{max}]$ and we assume $W(J) \approx 0$ for $J \geq J_{max}$, then

$$\begin{aligned} \sum\left[\Delta H(J_i)\right]^2 W(J_i)\delta J_i &\approx \int_0^{J_{max}} \left[-\Delta V + (e\Delta S + \vec{\phi}\cdot\Delta\vec{I})\right]^2 W(J)dJ \\ &\approx \int_0^\infty \left[-\Delta V + (e\Delta S + \vec{\phi}\cdot\Delta\vec{I})\right]^2 W(J)dJ && (2.3.5) \end{aligned}$$

where $\delta J_i = J_{i+1} - J_i$ and hence we can view this hedging method as a quadrature rule for approximating equation (2.3.1). Since $\Delta H(J_i) = 0$, then in the limit as $N \to \infty$, equations (2.3.4) and (2.3.5) approximates equation (2.3.1), for any nonnegative weighting function $W(J)$.

Chapter 3

Testing Procedure

In this chapter, we will discuss the method used to test our proposed algorithm for hedging jump risk. The algorithm consists of generating a stochastic price path, and along each path, constructing the hedging portfolio described in Chapter 2. At each re-balancing date, we determine the hedging weight, using the TSVD described in Chapter 2. We then repeat this for many stochastic paths. This will generate summary statistics for our proposed hedging strategy.

3.1 Data Generation

In this section, we will describe the method used to generate a stochastic price path. Since we are working with a jump diffusion process, the model that we will use is the Merton jump diffusion model. Recall the Merton model:

$$\frac{dS_t}{S_{t^-}} = (\alpha - \kappa\lambda)dt + \sigma dZ_t + (J_t - 1)d\pi_t.$$

To implement this model, we start with a given initial value S_0. The path of a underlying asset can be generated using the following steps:

$$
\begin{aligned}
&1. \quad \text{Let } Y(t_i) = \log(S(t_i)) && (3.1.1)\\
&2. \quad \text{set } Y(t_{i+1}) = Y(t_i) + (\alpha - \kappa\lambda - \frac{\sigma^2}{2})\Delta t + \sigma\Phi_1(t_i)\sqrt{\Delta t} && (3.1.2)\\
&3. \quad \text{if } \Phi_2(t_i) \leq \lambda dt, \text{ then} \\
&\quad Y(t_{i+1}) := Y(t_{i+1}) + \mu + \gamma\Phi_3(t_i) && (3.1.3)
\end{aligned}
$$

where Φ_1, Φ_3 are random numbers which are generated from a normal distribution and Φ_2 are random numbers with a uniform distribution on the interval $(0, 1)$, Δt is the unit of the time interval given by $\Delta t = \frac{T}{nh}$, T is the expiry time of the target option, and nh is the number of timesteps we use to approximate the solution to the SDE.

3.2 Weight Function

In this section, we briefly review the method for finding the hedging weights, and provide a matrix form for the equations.

As we discussed in section (2.1.4), if we can find hedging weights $\{e, \vec{\phi}\}$ such that $\Delta H(J)$ is small, then the change in the overall hedged position due to a jump is small. Furthermore, in section (2.3), we provided a simplified method that can be used to determine the hedging portfolio weights. Recall the linear system:

$$
\begin{cases}
\Delta H(J_i) = 0, i = 1, \cdots, N. \\
e + \vec{\phi} \cdot \frac{\partial \vec{I}}{\partial S} = \frac{\partial V}{\partial S},
\end{cases}
\tag{3.2.1}
$$

where

$$
\Delta H(J_i) = -[V(J_iS, t) - V(S, t)] + e[J_iS - S] + \vec{\phi} \cdot [\vec{I}(J_iS, t) - \vec{I}(S, t)]. \tag{3.2.2}
$$

Suppose $dim(\vec{I}) = N$, then we can write equation 3.2.1 as:

$$\begin{pmatrix} 1 & (I_1)_S & (I_2)_S & \cdots & (I_N)_S \\ J_1S - S & I_1(J_1S,t) - I_1(S,t) & I_2(J_1S,t) - I_2(S,t) & \cdots & I_N(J_1S,t) - I_N(S,t) \\ J_2S - S & I_1(J_2S,t) - I_1(S,t) & I_2(J_2S,t) - I_2(S,t) & \cdots & I_N(J_2S,t) - I_N(S,t) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ J_NS - S & I_1(J_NS,t) - I_1(S,t) & I_2(J_NS,t) - I_2(S,t) & \cdots & I_N(J_NS,t) - I_N(S,t) \end{pmatrix} \begin{pmatrix} e \\ \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix}$$

$$= \begin{pmatrix} V_S \\ V(J_1S,t) - V(S,t) \\ V(J_2S,t) - V(S,t) \\ \vdots \\ V(J_NS,t) - V(S,t) \end{pmatrix}. \qquad (3.2.3)$$

As we discussed in previous chapter, the weights will be determined using a TSVD.

3.3 Solution of the Linear Equation

In this section, first we will discuss the relationship between the linear system equation (3.2.1) and the least square problem. Then we will discuss singular problems we may encounter during the hedging simulations. Lastly, we will provide the pseudocode that generates the hedging weights.

3.3.1 Relationship to Least Square Problem

Suppose we have linear equations $Ax = b$, where A is a square matrix of size $n \times n$ and x and b are vectors with size $n \times 1$. Let r be the residual vector form x, defined as:

$$r = Ax - b.$$

Then we call a vector x^* the least squares solution if

$$\| r \|_2 = \| Ax^* - b \|_2 \leq \| Ax - b \|_2 \text{ for all } x \in R^n,$$

where R^n is set of all real vectors with size $n \times 1$ and $\| \, . \, \|_2$ means 2-norm.

Although a least squares solution might not be unique, the least squares solution x with the smallest norm, say $\| \, x \, \|_2$, is unique. The minimum norm solution x can be solved using the normal equations

$$x = (A^T A)^{-1} A^T b,$$

if $(A^T A)$ is nonsingular.

There always exists a least squares solution x, even when matrix $A^T A$ is ill-conditioned or singular. In these cases (singular or ill-conditioned), we can find pseudoinverse solution $\tilde{x}$ by using TSVD which discussed in section (2.2.2).

Let us write a linear equations $Ax = b$ for matrix (3.2.3), where

$$A = \begin{pmatrix} 1 & (I_1)_s & (I_2)_s & \cdots & (I_N)_s \\ J_1S - S & I_1(J_1S,t) - I_1(S,t) & I_2(J_1S,t) - I_2(S,t) & \cdots & I_N(J_1S,t) - I_N(S,t) \\ J_2S - S & I_1(J_2S,t) - I_1(S,t) & I_2(J_2S,t) - I_2(S,t) & \cdots & I_N(J_2S,t) - I_N(S,t) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ J_NS - S & I_1(J_NS,t) - I_1(S,t) & I_2(J_NS,t) - I_2(S,t) & \cdots & I_N(J_NS,t) - I_N(S,t) \end{pmatrix} \quad (3.3.1)$$

$$b = \begin{pmatrix} V_s \\ V(J_1S,t) - V(S,t) \\ V(J_2S,t) - V(S,t) \\ \vdots \\ V(J_NS,t) - V(S,t) \end{pmatrix} ; x = \begin{pmatrix} e \\ \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix}, \quad (3.3.2)$$

then the residual norm is $\| \, r \, \|_2 = \| \, Ax - b \, \|_2$.

Obviously, the smallest value of $\| \, r \, \|_2$ is zero, in which case x is also the solution to $Ax = b$.

3.3.2 Put-call Parity Problem

We need be careful when selecting the additional instruments for $\vec{I}$, since some combination of options will make the matrix (3.3.1) singular or nearly singular.

For example, suppose we have two hedging options and they have the same strike prices and expiration dates. One is a European put option, say $P(S,t)$ and the other is a European call option, say $C(S,t)$. Then the corresponding columns of matrix A are

$$\begin{pmatrix} 1 & C_S & P_S \\ J_1S - S & C(J_1S,t) - C(S,t) & P(J_1S,t) - P(S,t) \\ J_2S - S & C(J_2S,t) - C(S,t) & P(J_2S,t) - P(S,t) \\ \dots & & \\ J_NS - S & C(J_NS,t) - C(S,t) & P(J_NS,t) - P(S,t) \end{pmatrix}. \tag{3.3.3}$$

Since C and P have same strike price and expiry time, then we know that put-call parity holds. Recall the formula for put-call parity

$$C(S,t) + Ke^{-r(T-t)} = P(S,t) + S, \tag{3.3.4}$$

where K is the positive strike price and r is the risk free rate. If a jump occurs, then the corresponding put-call parity rule is

$$C(JS,t) + Ke^{-r(T-t)} = P(JS,t) + JS. \tag{3.3.5}$$

Subtract (3.3.4) from (3.3.5), which gives

$$C(JS,t) - C(S,t) = P(JS,t) - P(S,t) + (J-1)S. \tag{3.3.6}$$

If we take derivative of (3.3.4) w.r.t S, we obtain

$$P_S = C_S - 1. \tag{3.3.7}$$

substitute (3.3.7) and (3.3.6) into (3.3.3), we get the following columns in A

$$\begin{pmatrix} 1 & C_S & C_S - 1 \\ J_1S - S & C(J_1S,t) - C(S,t) & C(J_1S,t) - C(S,t) - (J_1 - 1)S \\ J_2S - S & C(J_2S,t) - C(S,t) & C(J_2S,t) - C(S,t) - (J_2 - 1)S \\ \cdots & & \\ J_NS - S & C(J_NS,t) - C(S,t) & C(J_NS,t) - C(S,t) - (J_N - 1)S \end{pmatrix}. \tag{3.3.8}$$

Clearly, in this matrix, we have the following relationship

$$\text{the third column} = \text{the second column - the first column},$$

so the columns in matrix (3.3.8) are linearly dependent. Therefore, the matrix (3.3.3) is singular.

In any real hedging situation, we would normally not include puts and calls with the same strike and maturity, due to the put-call parity problem above. However, suppose we have a put at strike K_1 and a call at strike K_2, and that $J_1S \gg K_1$, $J_2S \gg K_2$, then put-call parity will hold approximately, and the matrix (3.3.3) can be almost singular. In this case, solution of (3.2.3) using a TSVD should give reasonable weights, and produce a weighting vector with a small norm.

3.3.3 Nearly Singular System

Suppose we have a target option near maturity $t = T$ and we have two call options (hedging options) expiring at $t = T$, with different strikes K_1, K_2. The value of these two call options are (near $t = T$)

$$I_1 \sim S - K_1, \text{ and}$$
$$I_2 \sim S - K_2$$

when $S \gg K_1$ and $S \gg K_2$. Clearly,

$$
\begin{aligned}
(I_1)_S &\sim (I_2)_S, \text{ and} \\
I_1(J_i S) - I_1(S) &\sim I_2(J_i S) - I_2(S).
\end{aligned}
$$

Hence the two of columns of A corresponding to the weights ϕ_1, ϕ_2 are almost linearly dependent. In this case, if we use a standard method for solving the equation (3.2.3), we will find that ϕ_1, ϕ_2 fluctuate wildly as we approach maturity. This will lead to large transaction costs. Hence, it is again desirable to use a TSVD so that we obtain reasonable (i.e. small norm) hedging weights.

3.4 Hedging Simulations

In this section, we will describe the algorithm we use to determine the effectiveness of our hedging strategy using Monte Carlo simulations.

In order to speed up the computation, we precompute the tables of prices and deltas of the target option and the hedging option at each discrete time t_i, for a range of discrete prices.

We then simulate a random path using equation (3.1.1), (3.1.2) and (3.1.3). At each discrete time t_i, we use a table look-up to construct the matrix (3.2.3). This system can be solved using a TSVD.

3.4.1 Using the TSVD

To avoid unstable hedge weights, we will use a cutoff parameter to set all singular values to zero if they are too small. More specifically, any singular value

$$
|w_i| < \text{tol} \max_j |w_j|
$$

is set to zero. Typically, we use tol $= 10^{-6}$.

Consider the writer of an option who wants to use the hedge portfolio

$$\Pi = -V + eS + \vec{\phi} \cdot \vec{I} + B$$

to isolate their position from the hedging risk (i.e. jump diffusion risk) over each time step before the target option matures. With a given initial stock price S_0 (i.e. $S_{t_0} = S_0$), we use the algorithm (3.4.1) to generate statistics for our proposed hedging strategy.

Instead of computing the price and delta at each particular time step, we will create precomputed tables which list the values (prices or delta) we will need to use. In the simulation process, we either read the value directly from table or interpolate from the values given in the precomputed table.

The precomputed tables have a grid size of $ns \times nh$ where ns is the number of grid points for the underlying S and nh is the number of time steps. In this thesis, we pick the range of times in the table to be $[0, 1]$. The maximum and minimum value of the underlying S in the table are chosen as

$$\begin{aligned} S_{max} &= \exp\{\log(S_0) + (\alpha_{\mathbb{P}} - 0.5\sigma^2) * T + \sigma_{max} * \sigma_{eff} * \sqrt{T}\}, \text{ and} \\ S_{min} &= \exp\{\log(S_0) + (\alpha_{\mathbb{P}} - 0.5\sigma^2) * T - \sigma_{max} * \sigma_{eff} * \sqrt{T}\} \end{aligned}$$

where we use $\sigma_{max} = 3$ in this thesis, T is maturity time, $\alpha_{\mathbb{P}}$ is drift value under P measure, σ is the volatility and σ_{eff} ([36]) is defined as

$$\sigma_{eff} = \sqrt{\sigma^2 + \lambda_{\mathbb{P}}(\mu_{\mathbb{P}}^2 + \gamma_{\mathbb{P}}^2)}$$

where

- $\lambda_{\mathbb{P}}$ is the intensity of the jump process under P measure,
- $\mu_{\mathbb{P}}$ is P measure jump mean
- $\gamma_{\mathbb{P}}$ is P measure jump standard deviation

The jump size is assumed to be log normally distributed.

Simplified Hedging Simulation

Compute table of prices and deltas of V and $\vec{I}$

For each simulation $j, 0 < j < M$

Solve the linear system as shown in (3.2.3) using a TSVD

$x(t_0, j) = [e(t_0, j), \vec{\phi}(S(t_0, j))]^T$

$B(t_0, j) = V(S(t_0, j)) - e(t_0, j)S(t_0, j) - \vec{I}(S(t_i, j)) \cdot \vec{\phi}(S(t_0, j))$

For each time step $t_i, 0 < i < nh - 2$,

Generate $Y(t_{i+1}, j)$ by using steps $(3.1.1), (3.1.2)$ and $(3.1.3)$

Set $S(t_{i+1}, j) = \exp(Y(t_{i+1}, j))$

If this is a rebalancing time, then

Form a matrix A as shown in (3.3.1)

Solve the linear system as shown in (3.2.3) using a TSVD

$x(t_{i+1}, j) = [e(t_{i+1}, j), \vec{\phi}(S(t_{i+1}, j))]^T$

$B(t_{i+1}, j) = \exp(r\Delta t)B(t_i, j) - [e(t_{i+1}, j) - e(t_i, j)]S(t_i)$

$- [\vec{\phi}(S(t_{i+1}, j)) - \vec{\phi}(S(t_i, j))] \cdot \vec{I}(S(t_i, j)) \cdot \vec{\phi}(S(t_i, j))$

EndIf

EndFor

Generate $Y(t_{N_R}, j)$ by using steps $(3.1.1), (3.1.2)$ and $(3.1.3)$

Set $S(t_{N_R}, j) = \exp(Y(t_{N_R}, j))$

$\Pi(t_{N_R}, j) = -V(S(t_{N_R}, j)) + e(t_{N_R - 1}, j)S(t_{N_R}, j)$

$+ \vec{\phi}(S(t_{N_R - 1}, j)) \cdot \vec{I}(S(t_{N_R}, j)) + B(t_{N_R - 1}, j) * \exp(r\Delta t)$

EndFor

(3.4.1)

Note that we are using the real measure $\mathbb{P}$ in the data generation step, and the $\mathbb{Q}$ measure for the prices of the hedging instruments.

The relative profit and loss is normally used as the measurement for the hedging error. We can find relative profit and loss by calculating the hedge portfolio through the above algorithm (3.4.1) and substituting into the formula

$$\text{Relative } P\&L = \frac{\exp(-rT)\Pi(T)}{V(S(0), t = 0)} \tag{3.4.2}$$

to get the value of the discounted relative $P\&L$ (also called discounted hedging error).

The relative profit and loss along the j^{th} stochastic path is given by

$$(P\&L)_j = \frac{\exp(-rT)\Pi(t_N, j)}{V(S(0), t = 0)}.$$

We can then compute mean, standard deviation and VAR from these results for $j = 1, \cdots, M$.

Chapter 4

Numerical Examples

In this Chapter, we will present numerical results for the hedging strategies discussed in the previous Chapters. In particular, we discuss

- The cutoff parameter used in the TSVD [see section (4.7)].
- The maximum jump size considered in the approximation of the hedging error integral [see section (4.8)].
- Selection of the basic method used to approximate the hedging error integral [see section (4.9, 4.4)]

4.1 Computational Parameters

Unless otherwise specified, we will use the parameters listed in Table (4.1). Later in this Chapter, we will verify that the above choice of parameters gives accurate results.

4.2 Approximation of the Hedging Error Integral

In the rest of the Chapter, we will compare the results obtained using two methods to approximate the hedging error integral (2.3.5): equally spaced jump sizes and Gaussian quadrature jump sizes. Their definitions are stated in following subsections.

parameter	value
ns (table size of S grid)	400
nh (table size of number of time points)	800
rebalance times	400
TSVD cutoff	10^{-6}
the range of jump size	[0,2]
number of simulations	100000
number of time steps	800

TABLE 4.1: *The pre-computed table has size $ns \times nh$, rebalancing times indicate the number of time we will rebalance our portfolio in a one year period. The number of simulations is the sample size we used in our Monte Carlo Simulations.*

4.2.1 Equally Spaced Jump Sizes

Equally spaced jump sizes are easy to understand. The best way to define this approximation is to give an example. Suppose in the hedging process, we choose the range of probable jump size to be in $J \in [0, 2]$ and we decide to use four hedging instruments to hedge our target option. We need to pick four J_i from $[0, 2]$ and we require that $\Delta H(J_i) = 0$, and that the distance between each J_i which gives $\Delta H(J_i) = 0$ to be constant. Recall the expression for jump risk $\Delta H(J)$ from Chapter 2.

$$\Delta H(J) = -[V(JS) - V(S)] + e(JS - S) + \vec{\phi} \cdot [\vec{I}(JS) - \vec{I}(S)].$$

Note that when $J = 1$, $\Delta H(J) = 0$. Therefore, it is not necessary to specify $\Delta H(J) = 0$ when $J = 1$. Thus the equally spaced jump sizes when we specify $\Delta H(J) = 0$ for four hedging options, are $0, 0.5, 1.5, 2$.

4.2.2 Gaussian Quadrature Jump Size

Normally, in order to evaluate the integral of a given function $h(x)$, we seek to obtain the best numerical estimate of the integral by selecting the optimal values x_i to evaluate $h(x_i)$. Generally, numerical integration methods are developed based on a rather simple choice of evaluation points for x_i. However if we carefully choose the points to evaluate $h(x)$, this may lead to higher accuracy in evaluating the integral. In numerical analysis, a quadrature rule is an approximation of the integral function and we usually calculate the weighted sum of the function value at specified points in the domain of integration.

The fundamental theorem of Gaussian quadrature states that the optimal x_i of the n-point Gaussian quadrature formulas are precisely the roots of the orthogonal polynomial for the same interval and weighting function w_i.

We will discuss two types of Gaussian quadrature in this thesis: the basic Gaussian quadrature with Legendre polynomials as its orthogonal polynomials (also called Gauss-Legendre quadrature), as well as Gauss-Laguerre quadrature which uses Laguerre polynomials as its orthogonal polynomials.

The standard form of basic Gaussian quadrature uses a finite interval and is defined by:

$$\int_{-1}^{1} h(x)dx \approx \sum_{i=1}^{n} w_i h(x_i), \tag{4.2.1}$$

where x_i is the i^{th} optimal value for evaluating $h(x)$ and weights w_i can be determined by

$$w_i = \frac{2}{(1-x_i^2)[\frac{dp_n(x_i)}{dx}]^2},$$

where $p_n(x)$ is a Legendre polynomial of degree n.

The conventional domain of integration for a Gaussian quadrature formulas is $[-1, 1]$. The basic Gaussian quadrature can be used for any finite domain of integration simply by changing the integration over the standard interval $[-1, 1]$ before applying the Gaussian quadrature formulas. Let's say we have interval $[a, b]$ and the integral we wish to approximate is given by

$$\int_{a}^{b} h(x)dx,$$

then carrying out a change of variables, gives us

$$\int_a^b h(x)dx \approx \frac{b-a}{2}\sum_{i=1}^{n} w_i h(\frac{b-a}{2}x_i + \frac{a+b}{2}).$$

Gauss-Laguerre quadrature is used when the upper bound of the integral over x is infinite and is defined as

$$\int_0^\infty e^{-x}e^{x}h(x)dx \approx \sum_{i=1}^{n} w_i[e^{x_i}h(x_i)], \tag{4.2.2}$$

where the weights w_i are given by

$$w_i = \frac{x_i}{(n+1)^2[L_{n+1}(x_i)]^2},$$

where $L_n(x)$ is a Laguerre polynomial with degree n.

We will refer to the optimal x_i for the Gaussian quadrature formula (either under finite interval or infinite interval) as a Gaussian quadrature jump size.

4.3 Example

To provide a simplified illustration of the hedging strategy, we introduce a specific rebalancing example to study the behavior of the jump size which we will choose in two different ways. Before showing the results, let us first provide some necessary data, under both $\mathbb{P}$ and $\mathbb{Q}$ measure. Recall that the $\mathbb{P}$ measure is the real world probability measure. We will use $\mathbb{P}$ measure parameters when simulating the stochastic paths in our hedging simulations. The $\mathbb{Q}$ measure parameters are those used in pricing options.

Jumps are assumed to occur with a Possion distribution. We assume that if a jump occurs, then the log of the size of jump, defined as J_{gen}, follows a normal distribution with $\log(J_{gen}) \sim N(\mu, \gamma)$ and the values that characterize the jump diffusion model are reported in Table (4.2), the values listed in the table are taken from [12].

Probability measure	λ	μ	γ	σ	α
$\mathbb{Q}$	0.1	-0.92	0.425	0.2	0.05
$\mathbb{P}$	0.0228	-0.5588	0.425	0.2	0.1779

TABLE 4.2: *The dividend yield $q = 0$, interest rate $r = \alpha^{\mathbb{Q}}$ and drift rate in Brownian motion is $\alpha^{\mathbb{P}}$.*

Recall that, in finance, the option that allows the holder to profit based on the change of the price of the underlying asset, regardless of the direction of price movement, is called a straddle option. A straddle option is equivalent to the investor holding a position in both a put and a call with the same strike price, and expiration date. The purchase of the option derivatives is known as a long straddle, while a short straddle indicates the sale of the option. The payoff of stradle option is given by

$$\text{payoff} = \max(K - S, 0) + \max(S - K, 0),$$

where K is strike price and S is underlying price.

Now, suppose a financial institution has sold an at-the-money one year straddle option and the initial stock price is $S_0 = 100$. To hedge this option, we take an underlying asset and some additional hedging options. We start with two hedging options, then four options and then eight. The corresponding options are shown in Table (4.3), (4.4) and (4.5) respectively. We will reuse the same options to hedge the portfolio when the current hedging options have expired. The computational parameters are given in Table (4.1). To keep things simple, we will use these examples throughout this thesis.

Instrument	Maturity(year)	Strike
Straddle	1	100
call	0.5	70
put	0.5	140

TABLE 4.3: *The first option is the target option, the rest are hedging options. All options are European options.*

In the case of equally spaced jump sizes, the points where $\Delta H(J_i) = 0$ are given in Tables (4.6), and we use these values throughout the thesis.

Table (4.7) shows the values for the Gaussian-Legendre quadrature rule and Table

Instrument	Maturity(year)	Strike
Straddle	1	100
call	0.5	70
call	0.5	90
put	0.5	120
put	0.5	140

TABLE 4.4: *The first option is the target option, the rest are hedging options. All options are European options.*

Instrument	Maturity(year)	Strike
Straddle	1	100
call	0.5	70
put	0.5	80
call	0.5	90
put	0.5	105
call	0.5	110
put	0.5	120
call	0.5	130
put	0.5	140

TABLE 4.5: *The first option is the target option, the rest are hedging options. All options are European options.*

(4.8) shows the values for the Gaussian-Laguerre quadrature rule.

Number of hedging options	Value of J: equally spaced
2	$0, 2$
4	$0, 0.5, 1.5, 2$
8	$0, 0.25, 0.5, 0.75, 1.25, 1.5, 1.75, 2$

TABLE 4.6: *The value $J = 1$ is automatically hedged due to the delta neutrality condition(2.1.8). The rest of the numbers are equally spaced.*

Number of hedging options	Value of J under Gauss-Legendre Quadrature rule
2	$0.2254, 1.7746$
4	$0.0938, 0.4615, 1.5385, 1.9062$
8	$0.0318, 0.1640, 0.3866, 0.6758, 1.3242, 1.6134, 1.8360, 1.9682$

TABLE 4.7: *The value $J = 1$ is automatically hedged due to the delta neutrality condition(2.1.8), Legendre polynomial quadrature.*

Number of hedging options	Value of J under Gauss-Laguerre Quadrature rule
2	$0.585786, 3.41421$
4	$0.322548, 1.74576, 4.53662, 9.39507$
8	$0.17027, 0.9037, 2.25108, 4.2667, 7.0459, 10.75851, 14.74067, 22.8632$

TABLE 4.8: *The value $J = 1$ is automatically hedged due to the delta neutrality condition(2.1.8), Laguerre polynomial quadrature.*

4.4 Hedging Simulations

In this section, we provide the results of our numerical studies of various of hedging strategies.

4.4.1 Equally Spaced Jump Size

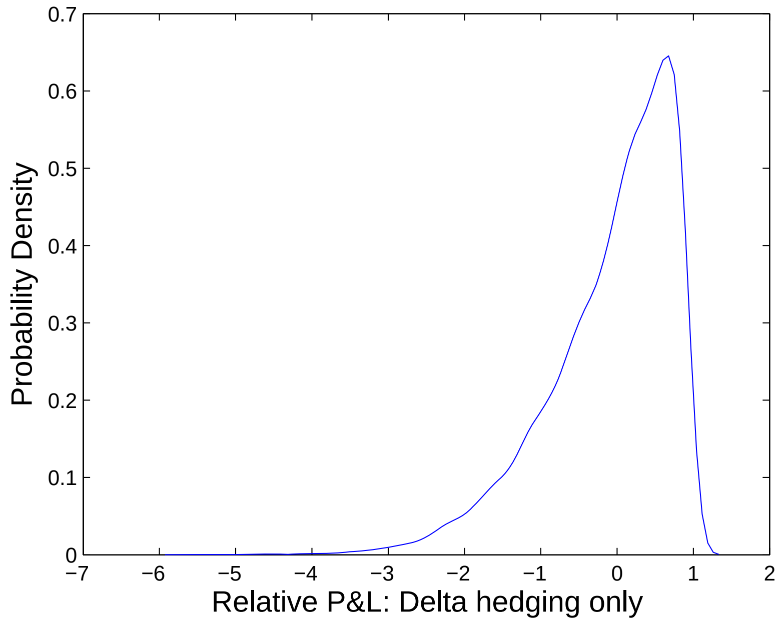
In this section, we plot the distributions of relative $P\&L$ (3.4.1) with four hedging options and compare the results with the plot of probability of relative $P\&L$ with delta hedging only. We use the interval $[0, 2]$ as our range of jump size. Therefore, the values of J are $0, 0.5, 1.5, 2$. The corresponding plot is given in Figure (4.1).

As shown in Figure (4.1), when we increase the number of hedging options, the plot becomes more peaked. In (a) of Figure (4.1), the values of the relative $P\&L$ are in the range $[-3, 1]$. When we include four options as our hedging instruments, we find that most relative $P\&L$ values are in the range $[-0.1, 0.1]$, which is closer to zero.

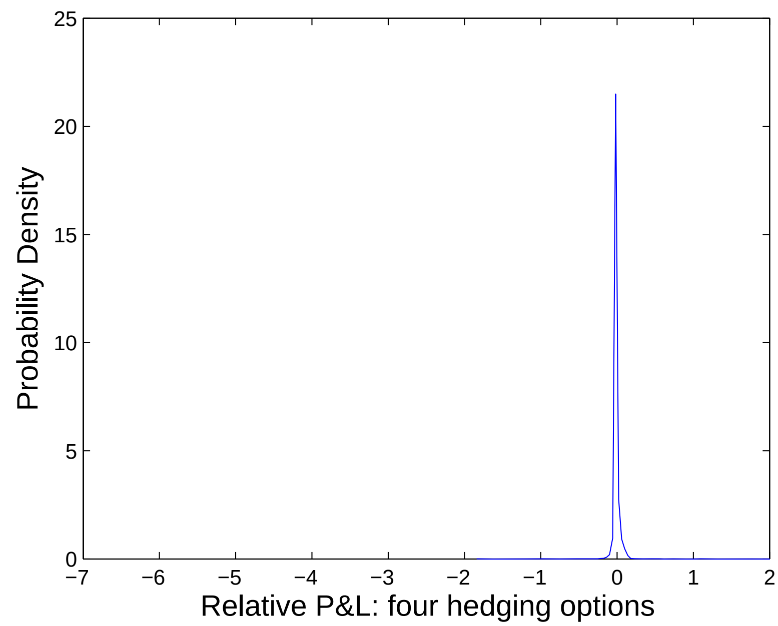
4.4.2 Gaussian Quadrature Jump Size

Now, we use the jump size from the Gaussian Quadrature formula to calculate the relative $P\&L$ using four hedging options. The corresponding jump sizes where we set $\Delta H(J_i) = 0$ are $0.0938, 0.4615, 1.5385, 1.9062$. The density of the $P\&L$ is given in Figure (4.2).

Comparing (a) and (b) in Figure (4.2), shows the same pattern again: the distribution becomes narrower when we include more hedging instruments. In addition, we see a normal-like distribution with roughly zero mean.

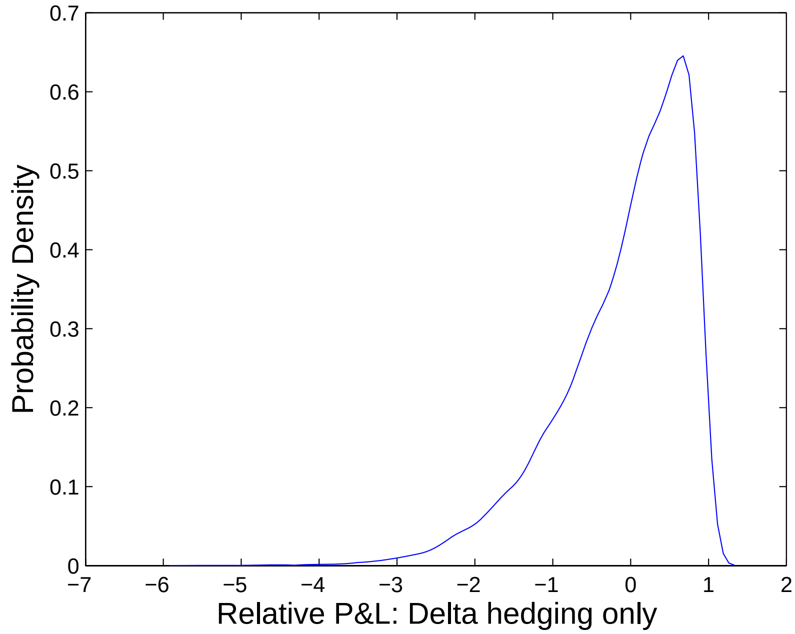


(a)

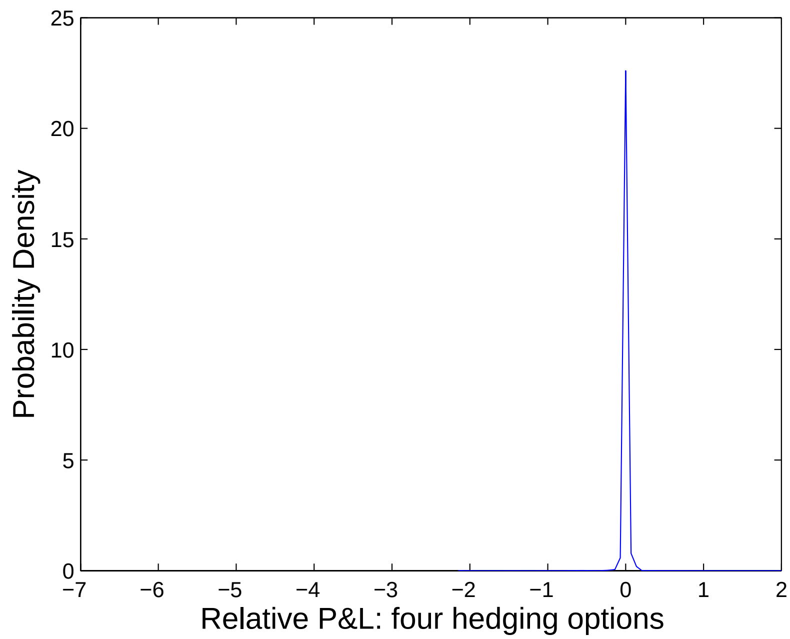


(b)

FIGURE 4.1: *Distributions of relative P&L for hedging the one-year European straddle. Parameters are given in Table 4.2. We rebalance our option 400 times a year. (a) Distributions of relative P&L with delta hedging; (b) Distributions of relative P&L with 4 hedging options using equally space jump size.*



(a)



(b)

FIGURE 4.2: *Distributions of relative P&L for hedging the one-year European straddle. Parameters are given in Table 4.2. The number rebalancing times is 400. (a) Distributions of relative P&L with delta hedging; (b) Distributions of relative P&L with 4 hedging options where we use Gaussian Quadrature jump size.*

4.4.3 Comparison of Gaussian Quadrature and Equally Spaced jump size

We will now examine the performance of the hedging strategies discussed in section 4.3 in more detail. In particular, we will look at the value of standard deviation of relative $P\&L$ and the percentiles of the $P\&L$. The resulting data are listed in Table (4.9).

number of hedging options	std	Percentiles(%)		
		0.02	0.2	99.8
2	0.156693	-3.1522	-1.9638	0.2651
4	0.040933	-0.6027	-0.1696	0.1436
8	0.024017	-0.2548	-0.0693	0.1174

TABLE 4.9: *Parameters used in Table (4.2). Equally spaced jump size. Rebalancing 400 times a year. "std" refers to standard deviation.*

number of hedging options	std	Percentiles(%)		
		0.02	0.2	99.8
2	0.169368	-2.3776	-2.0207	0.2793
4	0.041128	-0.5064	-0.1639	0.1443
8	0.018814	-0.2019	-0.0478	0.1023

TABLE 4.10: *Parameters used in Table (4.2). Gaussian quadrature jump size. Rebalancing 400 times a year. "std" refers to standard deviation.*

We see that the standard deviation becomes smaller when we include more hedging instruments, and the tail losses become smaller.

We recalculate all the data using the same example and parameters, but now we use the Gaussian Quadrature jump sizes. The resulting standard deviation and percentiles for relative $P\&L$ are shown in Table (4.10)

The performance of these two strategies is quite similar. However, for a large number of hedging options, it appears that the Gaussian quadrature method is slightly better than the equally spaced technique.

4.5 Convergence with Respect to Rebalancing Frequency

In this section, we show that for a fixed number of hedging options, the standard deviation converges to a finite value as the number of rebalances becomes large. In this section and the next section, we hedge with the underlying asset and 4 additional options.

4.5.1 Effect of Rebalancing Frequency

Now we increase the number of rebalancing times in one year, and observe the changes in the standard deviation. These results shown in Table (4.11).

Rebalancing frequency	standard deviation	Delta hedge only standard deviation
Rebalancing every half year	0.163359	0.487705
Rebalancing monthly	0.093157	0.381079
Rebalancing 50 times a year	0.055472	0.377799
Rebalancing 100 times a year	0.044935	0.375368
Rebalancing 200 times a year	0.041093	0.375229
Rebalancing 400 times a year	0.040933	0.375166

TABLE 4.11: *Parameters are given in Table 4.2. We use 4 hedging instruments to hedge our target European straddle option and using equally spaced jump sizes.*

Rebalancing frequency	standard deviation	Delta hedge only standard deviation
Rebalancing every half year	0.170389	0.487705
Rebalancing monthly	0.120692	0.381079
Rebalancing 50 times a year	0.060978	0.377799
Rebalancing 100 times a year	0.046232	0.375368
Rebalancing 200 times a year	0.041793	0.375229
Rebalancing 400 times a year	0.041128	0.375166

TABLE 4.12: *Parameters are given in Table 4.2. We using 4 hedging instruments to hedge our target European straddle option and using Gaussian quadrature jump sizes.*

In Table (4.11), the standard deviation of relative $P\&L$ becomes smaller when rebalancing the hedging portfolio more frequently, but converges to a finite limit. This is, of course, because the residual risk cannot be eliminated with a finite number of hedging instruments.

It is interesting to observe that even if we rebalance only twice a year, with the underlying asset and four options, then the standard deviation is smaller compared with delta hedging, rebalanced 400 times.

We repeat the same process as in the previous section with Gaussian quadrature jump sizes and again test the convergence. The corresponding standard deviation of relative $P\&L$ is shown in Table (4.12)

We see that when we increase the rebalancing time from 200 to 400, the standard deviation does not change significantly. Therefore, we will use 400 as our default rebalancing frequency.

4.5.2 Convergence with Respect to Number of Simulation

In this section, we will test the Monte Carlo sampling error of our simulations. In this section and the next section, we hedge with the underlying asset and 4 additional options.

4.5.3 Effect of Number of Simulations

We will start with simulation size $M = 12500$, and double the size in each test. We start with an equally spaced jump size. The results are shown in Table (4.13).

number of simulation	standard deviation	Delta hedge only standard deviation
M=12500	0.048211	0.392437
M=25000	0.042792	0.386737
M=50000	0.041073	0.377551
M=100000	0.040933	0.375166

TABLE 4.13: *Parameters are given in Table 4.2. We use 4 hedging instruments to hedge our target options where the target option is a European straddle and using equally spaced jump size.*

We recompute the standard deviation of relative $P\&L$ again, but now we will use the Gaussian quadrature jump size. The results are shown in Table (4.14)

By examining these tables, we see that the change in the standard deviation is quite small, going from 50,000 to 100,000 samples. Consequently, we will use $M = 100,000$ as

number of simulation	standard deviation	Delta hedge only standard deviation
M=12500	0.047921	0.392437
M=25000	0.043685	0.386737
M=50000	0.041572	0.377551
M=100000	0.041128	0.375166

TABLE 4.14: *Parameters are given in Table 4.2. We use 4 hedging instruments to hedge our target options where the target option is a European straddle and using Gaussian quadrature jump size.*

our default sample size.

4.6 Minimizing the Hedging Error

If the number of discrete jump sizes J_i used to approximate the hedging error (equation (3.4.2)) is large enough, then the standard deviation of the change in the hedging portfolio should become small. We will verify this in the following section.

4.6.1 Change in Hedging Portfolio Value with Increasing Number of Hedging Instruments

In order to illustrate the issues involved, we first compute the change in the total hedging portfolio for a fixed stock price S at a fixed time t due to different values of the jump size J.

Consider the following scenario: suppose an option writer sells a one-year straddle option with strike price \$100, and the option writer wants to form a portfolio to hedge the possible risk (including the jump risk). Recall that the hedging portfolio is

$$\Pi = -V + eS + \vec{\phi} \cdot \vec{I} + B.$$

Figures (4.3) and (4.4) show the value of the portfolio as a function of the jump size J (for fixed S, t). Note that a good hedging portfolio will be close to zero for all values of J.

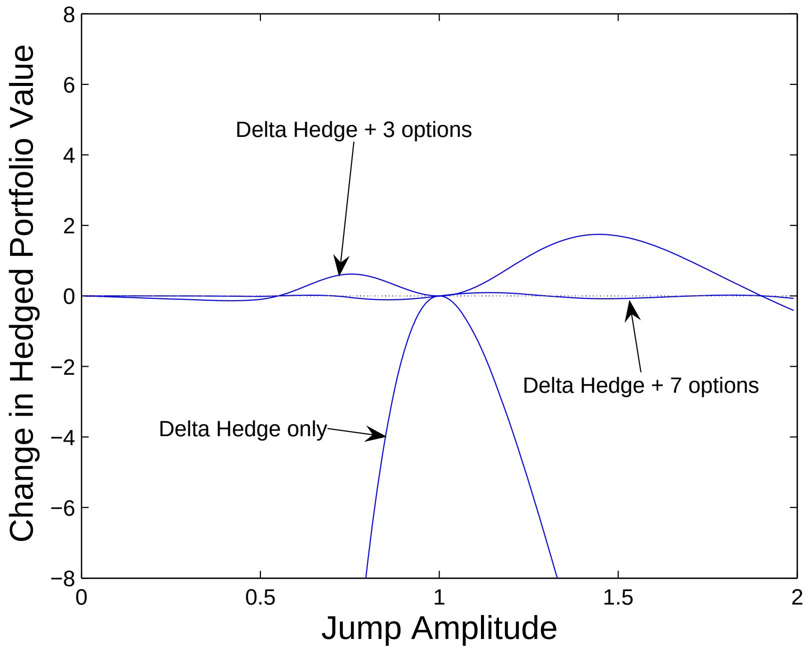


FIGURE 4.3: *Change in the value of the overall hedged position resulting from the jump, parameters given in Table 4.2 with fixed stock price* $S = 106.5$ *at time* $t = 0.05$. *The target option is a one-year straddle option with strike* \$100. *Hedging options are all European calls with strike: K =[110,140,150] (Delta hedge + 3 options), K = [80,90,110,120,130,140,150] (Delta hedge + 7 options). Options we used are listed in Table (4.15) and (4.16).*

We can clearly see from Figure (4.3) that the variance of the hedging portfolio becomes smaller as we increase the number of hedging options. In the example we used for Figures (4.3) and (4.4), we are only hedging once at one specific time. The change in the portfolio is given as

$$\Delta H(J) = -\Delta V + e\Delta S + \vec{\phi} \cdot \Delta \vec{I}.$$

Assuming that J is lognormally distributed, the expectation of $\Delta H(J)$ is defined as

$$E(\Delta H(J)) = \int_0^\infty \Delta H(J) f(J) dJ,$$

where $f(J)$ is the lognormal density function and the standard deviation of $\Delta H(J)$ is

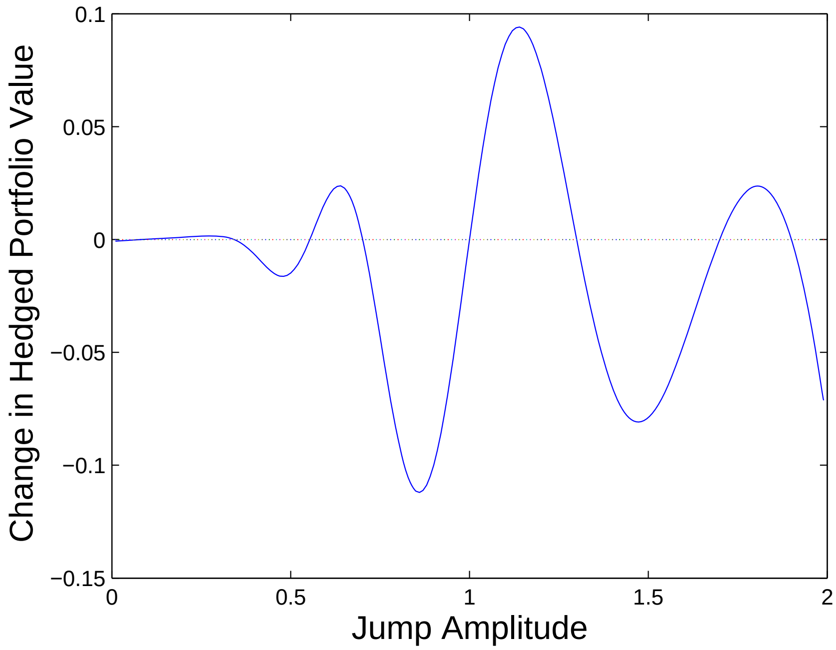


FIGURE 4.4: *Enlargement of the curve when 7 options are used, Parameters are given in Table 4.2 with fixed stock price* $S = 106.5$ *at time* $t = 0.05$. *The target option is a one-year straddle option with strike* \$100. *Hedging options are all European calls with strike:* $K =$ *[80,90,110,120,130,140,150]. Options we used are listed in Table (4.15) and (4.16).*

defined as

$$std(\Delta H(J)) = \left(E(\Delta H(J)^2) - E(\Delta H(J))^2\right)^{\frac{1}{2}}.$$

Instrument	Maturity(year)	Strike
Straddle	1	100
call	1.5	110
put	2	140
call	1.2	150

TABLE 4.15: *The first option is the target option, the rest are hedging options. All options are European options.*

Table (4.17) shows the sample mean and standard deviation of $\Delta H(J)$. The standard deviation is reduced as we increase the number of hedging options.

Instrument	Maturity(year)	Strike
Straddle	1	100
call	1.5	110
put	2	140
call	1.2	150
call	1.1	90
call	1.7	80
call	1.4	120
call	1.6	130

TABLE 4.16: *The first option is the target option, the rest are hedging options. All options are European options.*

Hedging Strategy	mean	standard deviation
Delta hedging only	-17.0613	36.7962
Delta hedging +3 options	-0.392836	0.62987
Delta hedging +7 options	-0.0087528	0.0470654

TABLE 4.17: *Mean and standard deviation of $\Delta H(J)$. Parameters are given in Table (4.2) with fixed stock price $S = 106.5$ at time $t = 0.05$. The options we used are list in Table (4.15) and (4.16)*

4.6.2 Comparison of Gaussian Quadrature Jump Sizes

We have introduced two different Gaussian quadrature approaches in section (4.2.2). Now, let us compute the mean and standard deviation of hedging error using both Gaussian quadrature strategies. We compute the mean and standard deviation of relative $P\&L$ with Gauss-Legendre quadrature jump size using the simulation methods which were discussed in section (3.4). The results are given in Table (4.18). Table (4.19) lists the results with the Gauss-Laguerre jump sizes using the same simulation method.

Comparing the results in the two tables, we also find that the standard deviation in Table (4.19) is much larger than the value in Table (4.18) when we are hedging with the same number of additional instruments. This is because we waste our resources hedging large, unlikely jump amplitudes. Therefore, in the rest of this thesis, we will only consider the Gauss-Legendre jump sizes.

Hedging strategy	mean	standard deviation	Percentiles(%)		
		0.02	0.2	99.8	
Delta hedging only	0.24982	0.375166	-2.3776	-2.0207	0.2793
Delta hedging +4 options	-0.00783517	0.041128	-0.5064	-0.1639	0.1443
Delta hedging +8 options	-0.00027752	0.018814	-0.2019	-0.0478	0.1023

TABLE 4.18: *Parameters are given in Table (4.2) and additional hedging options are given in Table (4.4) and (4.5). We generate the jump size for each options with the Gauss-Legendre Gaussian quadrature rule.*

Hedging strategy	mean	standard deviation	Percentiles(%)		
		0.02	0.2	99.8	
Delta hedging only	0.24982	0.375166	-3.1545	-1.7045	3.8847
Delta hedging +4 options	-0.0280768	0.225342	-2.5506	-1.4335	0.8139
Delta hedging +8 options	-0.00284226	0.0601099	-1.7887	-0.5291	0.1792

TABLE 4.19: *Parameters are given in Table (4.2) and additional hedging options are given in Table (4.4) and (4.5). We generate the jump size for each options with Gauss-Laguerre Quadrature rule.*

4.7 Transaction Cost Considerations

In [34], the transaction costs are modelled explicitly. Here, we take a simpler approach. We would like to generate a hedging strategy which does not result in large amounts of buying/selling of assets. If the hedging portfolio is

$$\Pi = -V + eS + \vec{\phi} \cdot \vec{I} + B,$$

then, a simple measure of an efficient strategy, in terms of transaction costs is to examine

$$\Pi_{abs} = |V| + |eS| + |\vec{\phi} \cdot \vec{I}| + |B|. \tag{4.7.1}$$

Recall that a good hedging strategy has $\Pi \sim 0$. If $\Pi_{abs} \sim 0$, then the portfolio does not have large long and short positions, which would lead to large transaction costs. Now, recall the TSVD cutoff parameter described in section (3.4.1). If we increase the value of the cutoff parameter, then we keep fewer non-zero singular values, and we are effectively, regularizing the solution of the matrix problem.

Note that one way to force a small norm for Π_{abs} is to solve the following problem for the hedging weights

$$x_\rho = \arg\min \left[\| AX - b \|_2^2 + \rho \| x \|_2^2 \right], \tag{4.7.2}$$

for a given value of regularization parameter ρ.

Suppose we have a value $k \le r$, where r is the rank of matrix A. In [28], it is pointed out that a TSVD can be considered to be an SVD with a filter factor g_i for the singular value. Given a cutoff σ_k, the filter is defined by,

$$g_i = \begin{cases} 0 & \text{if } \sigma_i < \sigma_k, \\ 1 & \text{if } \sigma_i \ge \sigma_k. \end{cases}$$

In [28], it is also noted that solving (4.7.2) can be viewed as using a TSVD with a smooth filter function. The TSVD can be seen as a "sharp" filter, applied to (4.7.2), with $\rho = \sigma_k$. Hence, using a larger value of σ_k, is roughly equivalent to using a larger value of ρ in equation (4.7.2). As a result, increasing the size of σ_k should make Π_{abs} smaller.

Next we consider the change of Π_{abs} with respect to the cutoff parameter σ_k/σ_1. We simulate 100000 random paths and consider hedging 100 times in one year. At each hedging time, we compute the average of the Π_{abs} with the simulated data at this particular time. We plot the resulting average value in Figure (4.5). For this example, we include four additional hedging options.

In Figure (4.5), we can see that increasing the size of the cutoff parameter σ_k/σ_1 decreases the size of Π_{abs}, as we would expect from our previous discussion. However, we would also expect that increasing the size of the cutoff increases the hedging error, since we have given up some of the reduction in hedging error in return for keeping the size of the portfolio positions small.

Now, let us examine the effect of the value of the TSVD cutoff on the relative $P\&L$. We use three different values of the cutoff $[10^{-4}, 10^{-6}, 10^{-8}]$ to see the change in the value of relative $P\&L$. The results are listed in Table (4.20).

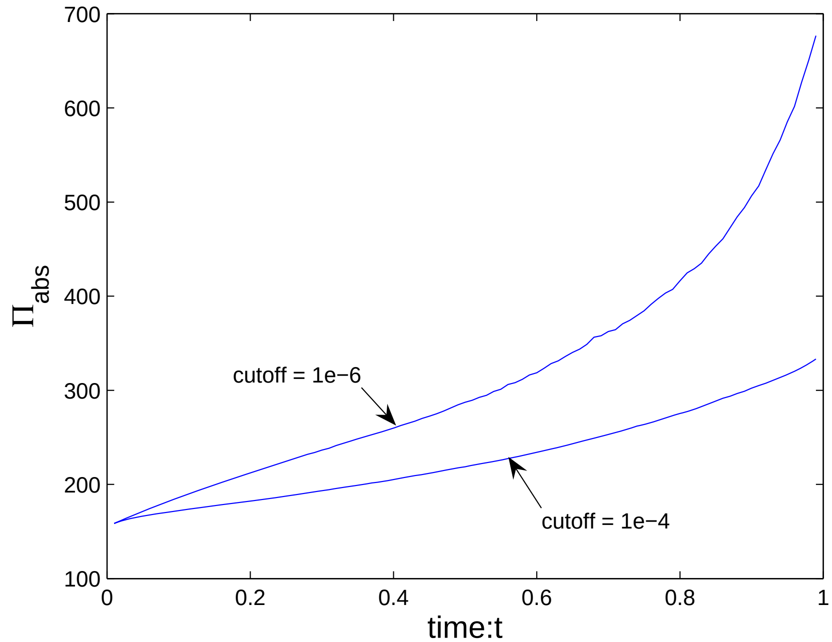


Figure 4.5: *Expected value of* Π_{abs}. *Stability w.r.t cutoff. Parameters: initial stock price* $S_0 = 100$, *Strike* $k = 100$, *the* $\mathbb{P}$ *and* $\mathbb{Q}$ *measure data are listed in Table 4.2.*

Value of cutoff	standard deviation
10^{-4}	0.046856
10^{-6}	0.041128
10^{-8}	0.090047

Table 4.20: *Parameters are given in Table 4.2. Using four hedging instruments to hedge our target option which is European straddle option. Hedging options are given in Table (4.4). We use Gauss-Legendre jump size.*

The standard deviation of the relative $P\&L$ becomes smaller while decreasing the cutoff from 10^{-4} to 10^{-6}, but the standard deviation suddenly goes up when we use 10^{-8} as our cutoff. We still keep some very small singular values when using 10^{-8}, and these singular values are close to zero. We suspect that the increase in the standard deviation for very small cutoff is due to ill-conditioning.

Note that if the value of the cutoff is too small, we can expect large transaction costs. On the other hand, if we pick the cutoff too large, the hedge error is larger. Therefore, the tradeoff between the reduction of transaction costs and accuracy of the hedging process is controlled by the value of cutoff. In this thesis, we use 10^{-6} as our default TSVD cutoff value.

4.8 The Range of Jump Size

Given the range of jump size $[J_{min}, J_{max}]$, if we have no idea which jump sizes are more likely than others under the $\mathbb{P}$ measure, we normally assume the range of jump size is $[0, 2]$ [12]. Setting $J_{min} = 0$ appears to be an obvious choice. In paper [26], it is shown that the range $[0, 2]$ performs well in general, and it has much better performance than the range for an educated guess of the $\mathbb{P}$ measure jump size distribution.

In order to protect upward jumps, we need to select J_{max}. However finding the right value of J_{max} is a bit complex. If J_{max} is too large, we will waste our resources to protect highly unlikely jump amplitudes. If we pick J_{max} too small, we will miss a probable jump event. Table (4.21) shows the results for various values of J_{max}.

Interval of $[0, J_{max}]$	standard deviation
$[0, 3]$	0.112812
$[0, 2]$	0.041128
$[0, 1.5]$	0.051364

TABLE 4.21: *Parameters are given in Table 4.2. We using four hedging instruments to hedge our target options where target options is European straddle. Hedging options are given in Table (4.4). We used the Gauss-Legendre jump sizes.*

From the result in Table (4.21), we see that when we use a larger value for the maximum jump size J, we get larger standard deviations. When we use the small interval $[0, 1.5]$, we found the standard deviation is larger than for $J_i \in [0, 2]$. This may be caused by ignoring too many probable jump sizes when hedging. Therefore, in this thesis, the range of jump size is chosen to be in $[0, 2]$.

4.9 Analysis of Errors due to Pre-computed Table Size

Recall that in Chapter 3, we described the algorithm used to simulate the hedging strategies. We use a pre-computed table of hedging option values, and deltas. During the simulation, we interpolate the necessary values from these tables. In this section, we examine the interpolation errors from interpolating these tables.

We use the same example as before to show the changes of standard deviation with respect to changing table size. The pre-computed table dimensions are $ns \times nh$, where ns is the number of entries in the asset price axis and nh is the number of entries in time grid axis of the table. We will test the errors due to changing ns in the next subsections.

4.9.1 Interpolation Error

In this section, we use the case of hedging with the underlying asset and four additional options to test the convergence. We study the convergence with respect to ns (price grid). We use the Gaussian quadrature jump size method. The convergence table with respect to ns with nh held fixed is shown in Table (4.22). Since we select nh to be the number of time steps in pre-computed table, there is no interpolation error in the time direction.

ns	standard deviation
25	0.229439
50	0.135256
100	0.061084
200	0.041365
400	0.041128

TABLE 4.22: *Parameters are given in Table (4.2). We use four hedging instruments to hedge our target European straddle option. The hedging options are given in Table (4.4). We used Gaussian quadrature jump sizes. We fixed nh as $nh = 800$. We rebalance our portfolio 400 times a year.*

We can see from Table (4.22), the standard deviation does not change much as we change $ns = 200$ to $ns = 400$. We use $ns = 400$ in this thesis.

4.10 Convergence with Respect to Time steps

In contrast to the number of rebalancing times, the number of time steps is used when we generate the random path in the MC simulation (see section 3.4). The change in the standard derivation corresponding to different numbers of time steps is listed in Table (4.23).

time steps	standard deviation
400	0.042401
800	0.041128
1600	0.041109

TABLE 4.23: *Parameters are given in Table (4.2). We use 4 hedging instruments to hedge our target European straddle option. The hedging options are given in Table (4.4). We used Gaussian quadrature jump sizes. We fixed* $ns = 400$*. We are rebalancing our portfolio 400 times a year.*

From Table (4.23), we find that the standard deviation of relative $P\&L$ become smaller when we increase the number of time steps. However, when the number of time steps changes from 800 to 1600, the difference in the standard deviation is small. Therefore, we use 800 as the default number of time steps.

4.11 Summary of Numerical Experiments

In this Chapter, we gave the experimental results for different input parameters and various hedging strategies. We conclude with a brief summary.

- The hedging error decreases as the number of hedging options used increases.
- With a fixed number of hedging options, increasing the rebalancing frequency, eventually results in the standard deviation converging to a finite value. This is due to the residual jump risk.
- In order to construct the hedging portfolio, we need to specify the jump sizes where we force the hedging error to be zero. We have tested three choices: equally

spaced, Gauss-Legendre points and Gauss-Laguerre points. Equally spaced points and Gauss-Legendre points are very close, with Gauss-Legendre points being superior with a larger number of hedging instruments.

- We can produce a hedging strategy which keeps the norm of the hedge portfolio weights small by adjusting the TSVD cutoff parameters. This may be useful to avoid large transaction costs.

- Consistent with previous results, we find that restricting the range of jumps sizes (where we require the hedging error to be zero) to $[0, 2]$ seems to work well for reasonable parameters.

Chapter 5

Conclusion

5.1 Summary

The Black-Scholes model assuming the underlying asset follows a geometric Brownian motion and delta hedging are widely used in practice. However, if we consider a market with jumps, the delta hedging strategy is no longer effective. It is by now well established that jump diffusions are more realistic models of real world asset price processes.

In the energy world, it is common place to hedge jumps by estimating a small number of possible jump sizes, and to construct a hedging portfolio which protects against these events. This approach is simple to explain to practitioners, and is easy to implement.

In this thesis, we have shown that this simple idea can be viewed as a quadrature rule for evaluating the jump risk integral. Since the error in the evaluation of this integral will be small if a large number of quadrature points are used, then it follows that the error in the hedging strategy will be small if a large number of instruments are used in the hedging portfolio.

We have experimented with selection of quadrature points based on an equally spaced midpoint rule, and two forms of Gaussian Quadrature. Our experiments indicate that the Gaussian Quadrature method is slightly better than equally spaced points.

In order to determine the hedging portfolio weights at each rebalancing time, a linear system of equations must be solved. In some cases, this linear system is nearly singular. We used a Truncated SVD(TSVD) method to solve this system. As a by-product of using the TSVD, we have shown that if a suitable cut-off value is used in the TSVD, then the norm of the hedging portfolio weights is small. Hence this will tend to reduce transaction costs.

Overall, we find that using a hedging portfolio consisting of the underlying asset and four additional liquid options is very effective at reducing the standard deviation and the tail risk of the hedging strategy.

5.2 Future Work

Some directions for the future research are:

- In our hedging strategy, we control the jump risk by minimizing the instantaneous risk. A global hedging strategy could be devised which would presumably lead to better results.
- In this thesis, the maximum jump size in the range of jump sizes used in the quadrature rule is constant. However, this maximum jump size was determined by experiment. It would be desirable to develop a more mathematically sound approach for determining this parameter.
- The TSVD cutoff value is another issue which may be considered for future investigation. Transaction costs and hedge error move in opposite directions as we change the value of the cutoff. Finding an optimal cutoff which gives us the smallest possible hedge error, and with the least possible transaction cost, would be an interesting avenue for future research.

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