Facility location with economies of scale and congestion

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Most literature on facility location assumes a fixed set-up cost and a linear variable cost. However, as production volume increases, cost savings are achieved through economies of scale, and then when production exceeds a certain capacity level, congestion occurs and costs start to increase significantly. This leads to an S-shaped cost function that makes the location-allocation decisions challenging. This thesis presents a nonlinear mixed integer programming formulation for the facility location problem with economies of scale and congestion and proposes a Lagrangian solution approach. Testing on a variety of functions and cost settings reveals the efficiency of the proposed approach in finding solutions that are within an average gap of 3.79% from optimal.

Acknowledgments

I would like to thank my supervisors Professor Elhedhli and Professor Gzara for their help on this research.

Dedication

This is dedicated to my parents.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

Facility location problems deal with the location of facilities to serve demand from a set of customer regions. The problem is of strategic importance and may affect competitiveness if not addressed in the proper way. The best location of facilities and the allocation of demand are based on the cost of locating and operating a facility, and on the cost of allocating demand.

The literature is rich with models that consider different cost structures. Generally, most models can be categorized into four classes according to the form of the production cost function. The first class uses a linear cost function subject to either uncapacitated or capacitated production capability. The uncapacitated version is much easier to solve [20], [22], [25]. However, for the capacitated version, the difficulty increases with the tightness of the capacity [15]. Solution approaches are based on either tightening the feasible region of linear programming (LP) relaxation by cutting planes or using Lagrangian relaxation to get tighter bounds.

For tightening the feasible region of LP relaxation, a number of facets and valid inequalities are provided in [1]. For Lagrangian relaxation, [4] carries an exhaustive comparison between different Lagrangian bounds with heuristics.

The second class considers concave cost functions to model economies of scale resulting from a decrease in unit costs as output increases. The most common technique to tackle the concavity, is to use a set of linear piecewise functions as approximation [11], [8], [30], [27], [20]. [7] provides a branch-and-bound algorithm for the uncapacitated case, where all costs, except fixed cost, are concave functions.

The third class uses a convex cost function where the marginal cost increases with the output (e.g. [15], [6], [9], [10], [18]), mainly due to congestion resulting from delays, overtime costs and higher maintenance cost. As in the second class, the convexity is approximated using linear piecewise functions [15] [31] [29]. [6] provides a column generation using Dantzig-Wolf decomposition embedded in a branch-and-bound scheme.

Instead of focusing on each case independently, [31] and [29] consider an S-shaped function to model costs. When the production exceeds a certain level, congestion begins to dominate and causes costs to rise. This point of view has been used by economists especially with respect to short-run production. A number of functions can represent this feature, such as the cubic function shown in Figure 1.1 which is a typical choice in microeconomics [28].

As shown in Figure 1.1, the average cost (AC) and marginal cost (MC) first decline as output increases. After a certain point, which we refer to as the economic point, the marginal and average costs start to increase with average cost lagging

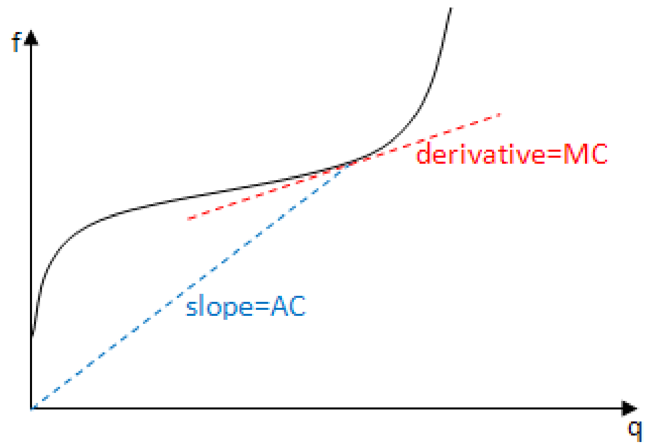


Figure 1.1: An example of S-shaped production cost based on $f(q) = aq^3 - bq^2 + cq + d, where\ a, b, c, d \geq 0$.

behind. (Note that: AC is the slope of the segment connecting the origin and the point on the curve; whereas MC is the derivative of the function.)

To the best of our knowledge, the only work that looked at an S-shaped production function is by Schutz et al. [29] and Van den Broek et al. [31]. The solution methodology is based on Lagrangian relaxation and linear piecewise approximation of the nonlinearity. Even though the shape of the production cost curve in these models is similar to the one considered here, some fundamental differences do exist and will be discussed in section 3.1. Moreover, if breakpoints are used to approximate the nonlinear function, especially for concave part, an extra set of binary variables is needed to model the selection of segments.

This thesis uses an S-shaped production cost function to take account of economies of scale and congestion as in [31] and [29]. The thesis introduces a new model where the production cost function is divided into two parts that capture concave and convex cost, respectively. The advantage of such a separation is that differentiability at the economic point can be relaxed in the model. This gives the practitioners more freedom in selecting the appropriate cost function. For example, economists often assume the total production cost function is a cubic function, then use regression analysis to get the best parameters to fit the function to real data. But if they could use a concave function and a convex function to fit different parts of the data without the worrying about differentiability at the point connecting the two functions, then the accuracy of the regression may improve. This is supported by the model in this thesis. Moreover, such a model possesses nice structures which are suitable for its decomposition using Lagrangian relaxation. In addition, the model does not approximate the nonlinear curve by linear piecewise functions, making its results independent of the accuracy of the approximation.

To solve the model we propose an efficient algorithm that exploits the decomposability of the objective function and the constraints. A Lagrangian relaxation approach is proposed to generate a lower bound and a feasible solution based on the solution of a nonlinear knapsack problem. The heuristic uses the dual variables of Lagrangian master problem as weights to find a combination of the solutions from subproblem. An enhance search algorithm is proposed. It creates a new problem from the original problem by closing one facility operating under the economic point. Numerical testing for different function forms on problems with up to 150 customers and 30 facilities are within an average gap of 3.79% from optimal cost.

The thesis is organized as follows: Chapter two presents literature review for various versions of facility location problem in terms of modeling approaches and solution methodologies. Chapter three describes the assumptions of the problem, proposes a model, and provides a Lagrangian relaxation based solution methodology. Chapter four is devoted to the numerical testing. Chapter five concludes the thesis.

Chapter 2

Literature Review

Facility location problems deal with the location of facilities to serve demand from a set of customer regions. Depending on the production capacity restrictions, the problems fall into two categories. Those problems with capacity constraints are capacitated facility location problem (CFLP) while those without capacity constraints are uncapacitated facility location problem (UFLP). In this chapter, the literature on the capacitated facility location problem (CFLP) is reviewed mainly. Also, a few UFLPs are also included in each section of this chapter. First, we introduce the mathematical model that is typically used. Let's consider a problem with m customers, indexed by i, and n candidate facility locations, indexed by j. The demand from customer i ($i = 1, ..., m$) is defined as d_i. Also we define u_{ij} as the variable cost of satisfying demand d_i from facility j ($j = 1, ..., n$), f_j as the fixed cost of facility j, K_j as the capacity of facility j, variable x_{ij} as the volume supplied to customer i by facility j, y_j as the binary variable which is 1 if facility j is open and 0 otherwise. Then the model is:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{m}\sum_{j=1}^{n} u_{ij}x_{ij} + \sum_{j=1}^{n} f_j y_j \\
\text{s.t.} \quad & \sum_{j=1}^{n} x_{ij} \geq d_i && \forall i = 1, ..., m \\
& \sum_{i=1}^{m} x_{ij} \leq K_j && \forall j = 1, ..., n \\
& x_{ij} \leq d_i y_j && \forall i = 1, ..., m, \ \forall j = 1, ..., n \\
& x_{ij} \geq 0 && \forall i = 1, ..., m, \ \forall j = 1, ..., n \\
& y_j \text{ binary} && \forall j = 1, ..., n
\end{aligned}
$$

The objective function minimizes the setup costs and the variable costs. The first set of constraints makes sure each customer's demand is satisfied. The second set of constraints makes sure that the capacity of each facility is not violated. The third set of constraints is a set of cuts that links y_j and x_{ij} for all i and j. Depending on the cost functions, we focus on three classes: CFLP with a concave cost function, CFLP with a convex cost function and CFLP with an S-shaped cost function.

2.1 Capacitated Facility Location Problem with Concave Cost Functions

In this family of problems, the unit production cost decreases as the output increases, resulting in a concave production cost function. The main reason for concavity is economies of scale. The common approach of handling concavity is to approximate the concave curve by a set of piecewise linear functions. Techniques include using dummy facilities and breakpoints, which will be explained in this section and section 2.2 respectively. Although both techniques are effective, using dummy facilities technique does not require introducing extra binary variables in concave case. Thus, it is much easier to apply.

Figure 2.1 is an example showing the basic idea of using dummy facilities. For each facility j, we associate k dummy facilities with fixed cost f_{jk} and unit cost u_{jk}. t_{ij} represents the transportation cost. Moreover, $f_{jk} < f_{jk+1}$ and $u_{jk} > u_{jk+1}$ must hold to insure the concavity of the approximating curve. The resulting approximation should resemble the lower envelope of the lines in Figure 2.1. It is obvious that there is no need to restrict either one or no facility being selected. This is because the minimization automatically chooses the segment on the lower envelope where the cost is smaller than any convex combination of the other dummy facilities at a given location.

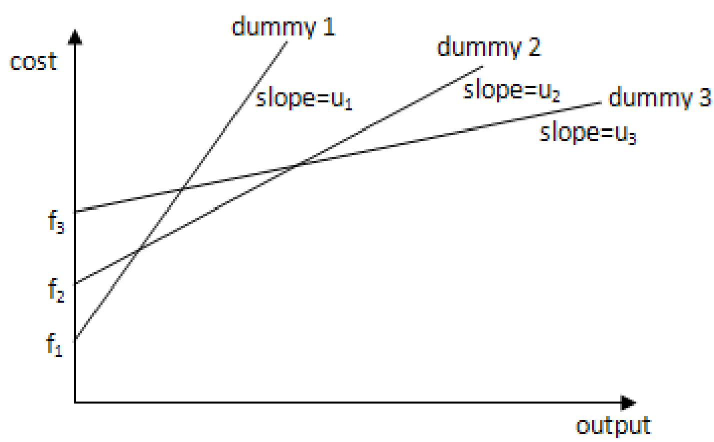


Figure 2.1: Linear piecewise approximation of facility j cost function using 3 dummy facilities.

Cohen and Moon [3] propose a plant loading model incorporating economies of scale and scope. The model resembles a facility location problem since it involves a decision of allocating production lines among candidate plants. Each production line has a setup cost and a linear piecewise concave production cost function with respect to production volume. Benders decomposition is used to solve the resulting model.

The linear piecewise approximation using dummy facility technique can be improved by repeatedly adding dummy facilities when necessary. Dasci and Verter [5] consider a single-echelon production-distribution system with multiple products system which determines the facility locations, capacity acquisition and technology selection. The objective function is nonlinear and concave because the technology acquisition cost is assumed to capture economies of scale with respect to the quantity produced with that technology. This concavity is approximated by a set of linear piecewise functions. Each segment of the linear approximation represents the cost of what the authors refer to as pseudo-facilities. The later are similar to dummy facilities. The approximation is strengthened when necessary by Progressive Piecewise Linear Underestimation technique [32]. The resulting approximated problem is solved repeatedly by a branch-and-bound algorithm. The authors also extend the same model to a flexible technology acquisition case where some technologies can be applied to produce multiple product, compared to the case in [5] where a technology is only able to produce a single product. In both cases, the technology acquisition cost is assumed to be concave. Thus, the solution methodology is the same.

Hajiaghayi et al. [14] consider a generalized facility location problem where the facility operating cost is a function of the number of assigned clients. The authors focus on a concave cost function and propose a greedy heuristic which is proven to have a 1.861 approximation factor. Moreover, convex case is proven to be solvable exactly in polynomial time. Romeijn et al. [26] consider an uncapacitated facility location problem with single-sourcing, in which the production cost is a concave function of the quantity produced. A greedy heuristic which generalizes the algorithm of Jain et al. [19] is devised. The approximation factor for this heuristic is 1.52.

Lin et al. [21] analyze the strategic design of a distribution system with 4 echelons including plants, consolidation centers, distribution centers and retailers. One characteristic of the study is that the transportation cost is assumed to be a concave function of the volume shipped on each link except those connecting distribution centers and retailers, to take account of economies of scale. A greedy heuristic is proposed, which iterates between locating distribution centers and consolidation centers.

Dupont [7] considers a UFLP where the production cost as well as shipment cost are concave functions of the output at each facility. He show that the following two properties hold:

- There exists an optimal solution in which any customer is supplied by a single facility.
- For two customers a and b, if each customer can indifferently be delivered by facility C and D, the solution in which a and b are delivered by the same facility dominates the solution in which customer a is delivered by C and customer b is delivered by D.

2.2 Capacitated Facility Location Problem with Convex Cost Functions

In this class of problems, the unit production cost increases with the output, resulting in a convex cost function. The convexity is suitable to model the congestion at facilities. As in the concave case, the most common approach is to use linear piecewise approximation. Consequently, the techniques also duplicate those used for concave cost functions. Based on the technique of dummy facility, Harkness and Revelle [15] propose four formulations with appropriate constraints to restrict the output at a facility within the right range associated with the corresponding dummy facility.

Besides this, breakpoints are also applicable in this case. For each facility j, we can define l breakpoints indexed by $k (k = 1, ..., l)$. q_{jk} is the quantity supplied by facility j at breakpoint k. f_{jk} is the production cost of facility j at breakpoint k. The formulation using this technique is as follows :

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{m}\sum_{j=1}^{n} u_{ij}x_{ij} + \sum_{j=1}^{n}\sum_{k=1}^{l} f_{jk}y_{jk} & \\
\text{s.t.} \quad & \sum_{j=1}^{n} x_{ij} \geq d_i & \forall i = 1, ..., m \\
& \sum_{i=1}^{m} x_{ij} \leq K_j & \forall j = 1, ..., n \\
& \sum_{i=1}^{m} x_{ij} = \sum_{k=1}^{l} q_{jk}y_{jk} & \forall j = 1, ..., n \\
& \sum_{k=1}^{l} y_{jk} = 1 & \forall j = 1, ..., n \\
& x_{ij} \geq 0 & \forall i = 1, ..., m, \forall j = 1, ..., n \\
& 0 \leq y_{jk} \leq 1 & \forall j = 1, ..., n, \forall k = 1, ..., l
\end{aligned}
$$

In the above formulation, t_{ij} accounts for transportation cost only; points (f_{jk}, q_{jk})'s are those breakpoints on the convex curve of cost function at facility j, with $(f_{j0}, q_{j0}) = (0, 0)$ and f_{j1} being reasonably large while q_{j1} being reasonably small to represent the fixed cost. Given the last two constraint sets in the above formulation, at facility j, the output $\sum_{i=1}^{m} x_{ij}$ is a convex combination of the breakpoint volumes q_{jk}'s and the production cost $\sum_{k=1}^{l} f_{jk}y_{jk}$ is a convex combination of the breakpoint values f_{jk}'s. Figure 2.2 shows the idea.

This technique can also be applied in concave case, but it requires the use of a special ordered set of type 2 (SOS2) [33], because in the concave case the solution is not necessarily a combination of two neighboring minimization points. Thus, another two sets of variables and constraints are needed. For example, given l breakpoints for the production cost function of facility j, the following set of constraints are used :

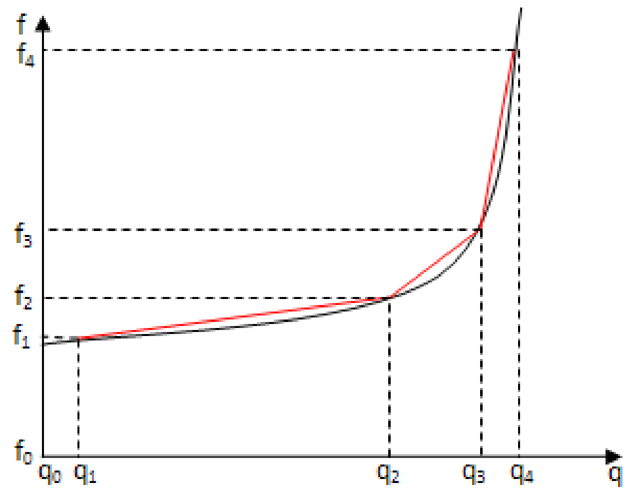


Figure 2.2: Linear piece-wise approximation by breakpoints.

$$\sum_{k=2}^{l} s_{jk} = 1$$

$$0 \leq \varepsilon_{jk} \leq s_{jk} \qquad \forall k = 2, ..., l$$

$$y_{j1} = s_{j2} - \varepsilon_{j2}$$

$$y_{j2} = s_{j3} - \varepsilon_{j3} + \varepsilon_{j2}$$

$$...$$

$$y_{jk} = s_{j,k+1} - \varepsilon_{j,k+1} + \varepsilon_{jk}$$

$$...$$

$$y_{jl-1} = s_{jl} - \varepsilon_{jl} + \varepsilon_{j,l-1}$$

$$y_{jl} = s_{jl}$$

where s_{jk} is a binary variable, and $s_{j,k} = 1(\forall k = 2, ...l)$ implies that only $y_{j,k-1}$ and y_{jk} are nonzero.

Another type of linear approximation uses tangent lines as shown in Figure 2.3. This approximation can be tightened. For example, Elhedhli [9] considers a service

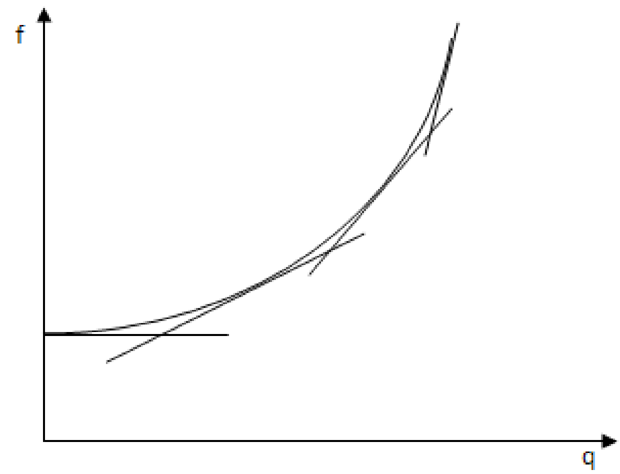


Figure 2.3: Linear piecewise approximation by tangent lines.

system design problem which involves locating a set of service facilities and captures congestion by a penalty cost on waiting time. As a result, the objective function is convex. The author eliminates the nonlinearity in the objective function by introducing two continuous variables, one of which is a concave function of the other. This concave function is approximated by a set of tangent linear piecewise functions. During the procedure, the approximation is tightened by adding more tangent points, resulting in adding more cuts in the reformulated model. Consequently, the lower bound of the relaxed problem improves monotonically.

Desrochers et al. [6] consider a convex CFLP that models congestion by a delay function. To solve the resulting model, they devise a column generation algorithm using Dantzig-Wolf decomposition and embed it into a branch-and-bound scheme.

Holmberg [18] presents an exact solution method for uncapacitated facility location problem with a convex transportation cost function. The transportation cost is assumed to be a convex function of distance due to the consideration of customers' preferences. The decision variables capture the flow from facilities to customers, and are defined to be integer. As a result, a nonlinear integer model is presented. The author linearizes the convexity exactly by a set of linear piecewise functions defined between each pair of neighboring integers on horizontal axis. The resulting model becomes a pure zero-one model, and is solved by a dual ascent method embedded in a branch-and-bound method.

Benjaafar et al. [2] consider a demand allocation problem which has multiple inventory locations and multiple demand sources. The convexity is due to the fact that supply lead time depends on load and the production capacity at the facility. The solution methodology resembles the one in [9], where the linear piecewise approximation gets tighter at every iteration.

2.3 Capacitated Facility Location with S-shaped Cost Functions

The previous CFLP models have either a concave or a convex production cost function. However, none of them take into account both economies of scale and congestion. It is worth mentioning that for many production activities, the unit cost first declines to a certain level and rises afterwards. There are a number of explanations for this behavior. For example, at the early stage of the production, as output increases, the fixed cost per unit decreases; maintenance cost reduces as machines reach their most efficient production levels; raw material price drops as discounts from suppliers become available with bulk purchases; managerial cost increases in a rate less than that of output growth. All of these factors lower both average cost and marginal cost, resulting in a concave curve. After the production fully explores the benefit of economies of scale, the marginal cost begins to increase, with average cost lagging behind. Similarly, some reasonable explanations could be extra cost due to overtime labor; higher maintenance cost or depreciation cost due to exhausting the machines; or penalties such as delay cost [6] for sacrificing customer service.

There are two models tackling this characteristic in [31] and [29]. The model in [31] considers only economies of scale even though the production cost curve is S-shaped. This is because average cost decreases monotonically whereas marginal cost converges to average cost from below after declining to its lowest, with both costs asymptotic to a certain value, as illustrated in Figure 2.4.

However, the production activity is often separated into two stages, i.e. the long-term and the short-term. In the long-term, the average cost is monotonically decreasing to represent economies of scale, and the production cost curve resembles the characteristics in Figure 2.4. However, in the short-term, the production cost does not change along the long-term cost curve, but along a convex short-term curve. The short-term cost curve is tangent to the long-term cost curve and keeps above on both sides of the tangent point. Under perfect conditions, the long-term cost curve is the lower envelope tangent to a set of short-run curves at every production level, as presented in Figure 2.5.

Given this, the model in [29] divides the long-term cost curve into several stages

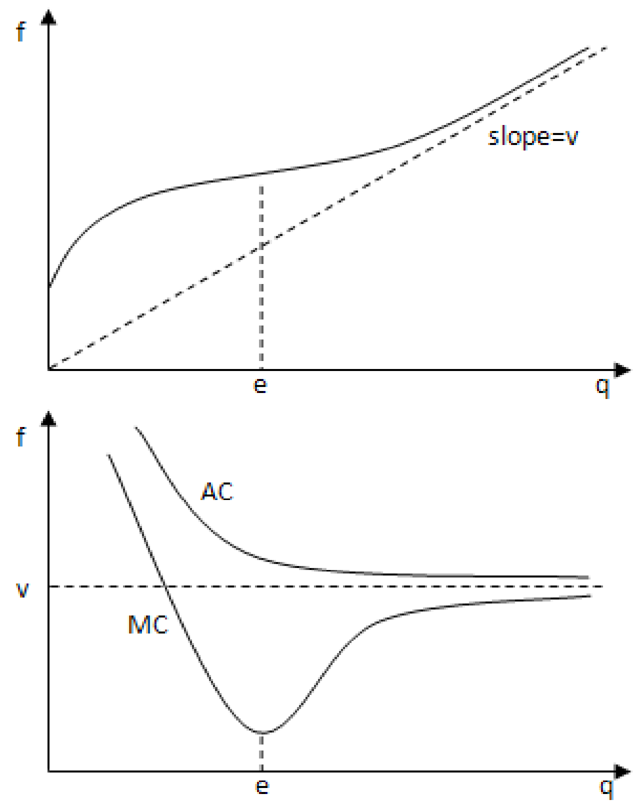


Figure 2.4: Features of production cost in Van den Broek (2006).

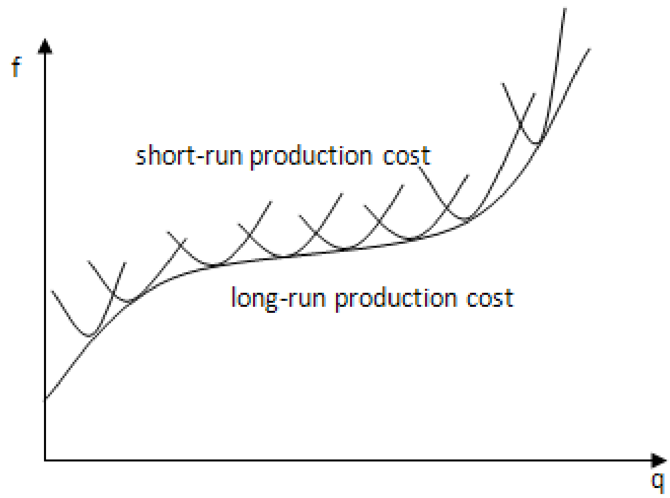


Figure 2.5: Short-run and long-run production cost in Schuz (2008).

or scenarios with a probability and a short-term cost curve assigned for each. Then the model minimizes the expected total cost.

Despite these differences, both models use the same solution methodology, which first approximates the nonlinear cost curve using breakpoints discussed in previous section and then solves the model by Lagrangian relaxation together with a Lagrangian heuristic.

In [24], a two-echelon supply chain design problem is considered. The main difference from other supply chain design problem is that the retailers are allowed be supplied from multiple warehouses. The problem considered is similar to our problem since it includes both the assignment decision with multi-sourcing and the location decision. Also, the limitation on capacity is captured in the model by capacity constraints. Another similarity is that the objective function is S-shaped. This is because the replenishment and holding costs show economies of scale as each order quantity increases, then exhibits diseconomies of scale because of capacity constraints. Lagrangian relaxation is used to solve the problem with a subgradient method. The author applies an algorithm developed by Ozsen [23] to solve the subproblem.

Holmberg [16] considers a facility location problem where the objective function is neither concave, nor convex, nor S-shaped. The fixed costs and the linear production cost coefficients vary at different levels of production. As a result, the production cost is staircase and neither convex nor concave. To solve this problem, a convex piecewise linearization is proposed with Benders' decomposition as the solution methodology. The convex piecewise linearization can be tightened by "branching" a part of previous linearization into two convex piecewise linearization and then apply Benders' decomposition to the new problem. Since the linearization gets closer to the original staircase curve, the bound improves after each iteration until the optimum is reached or an acceptable gap is reached. Holmberg et al. [18] study a production-transportation problem characterized by a linear transportation cost function, a concave production cost function and a convex shortage penalty cost function. Although facility location is not included, it is argued by the authors that the model can be directly extended to incorporate the location decision. The basic idea of solving this nonconcave and nonconvex model is to partition the feasible region into rectangles in terms of the production variables. This reduces the problem to stochastic transportation subproblems with a convex objective function. The subproblems can be efficiently solved by a method proposed by Holmberg [17].

You and Grossmann [34] propose a mixed-integer nonlinear (MINP) model and algorithms for large scale supply chain design with stochastic demand and three-echelons. The selection of distribution centers is actually a facility location problem. The objective function is nonconvex since the stochastic demand and the economic order quantity (EOQ) result in the safety stock cost and the replenishment cost with square root. The authors reformulate the problem into a MINP by replacing the square root terms in the objective function with continuous variables and moving the nonlinearity into constraints. Then, they present a heuristic based on convexity assumption on which MINP solver relies. Moreover, a Lagrangian relaxation algorithm is proposed with a subgradient method to update Lagrangian multipliers and a Lagrangian heuristic to give a feasible solution.

A summary of the literature discussed in this chapter is given in Table 2.1.

Reference	Cost function	Solution Methodology
Romeijn et al. (2010)	concave	greedy heuristic
Ozsen et al. (2009)	S-shaped	Lagrangian relaxation
Dupon (2008)	concave	branch-and-bound
Schuz et al. (2008)	S-shaped	Lagrangian relaxation and piecewise linearization
You and Grossmann (2008)	neither concave nor convex	heuristic and Lagrangian relaxation
Elhedhli (2006)	convex	piecewise linearization with automatic improvement
Benjaafar et al. (2006)	convex	piecewise linearization with automatic improvement
Lin et al. (2006)	concave	greedy heuristic
Van den Broek et al. (2006)	S-shaped	Lagrangian relaxation andr piecewise linearization
Harkness et al. (2003)	convex	branch-and-bound and piecewise linearization
Hajiaghayi et al. (2003)	concave	greedy heuristic
Dasci et al. (2001)	concave	branch-and-bound and piecewise linearization
Holmberg (1999)	convex	dual ascent method embedded in branch-and-bound and piecewise linearization
Holmberg et al. (1999)	neither concave nor convex	partition feasible region
Desrochers et al. (1995)	convex	column generation within branch-and-bound
Holmberg (1994)	neither concave nor convex	Benders' decomposition and piecewise linearization
Cohen et al. (1991)	concave	Benders' decomposition and piecewise linearization

Table 2.1: A classification of the literature based on objective function type and solution method

Chapter 3

Formulation and Solution Methodology

3.1 Formulation

The focus of this thesis is the facility location problem where production costs follow an S-shaped function to model economies of scale and congestion. To the best of our knowledge, the work that is closest to the one treated here is that of [31] and [29]. The main difference is due to the slope of the production cost function. The convex part of the production cost curve in [31] and [29] is asymptotically linear as in Figure 2.4, whereas in this work it is convex as in Figure 3.1.

The second difference concerns the short term production cost. Unlike [31] and [29], where the short-term production cost is convex, we assume in this thesis that it is concave due to economies of scale.

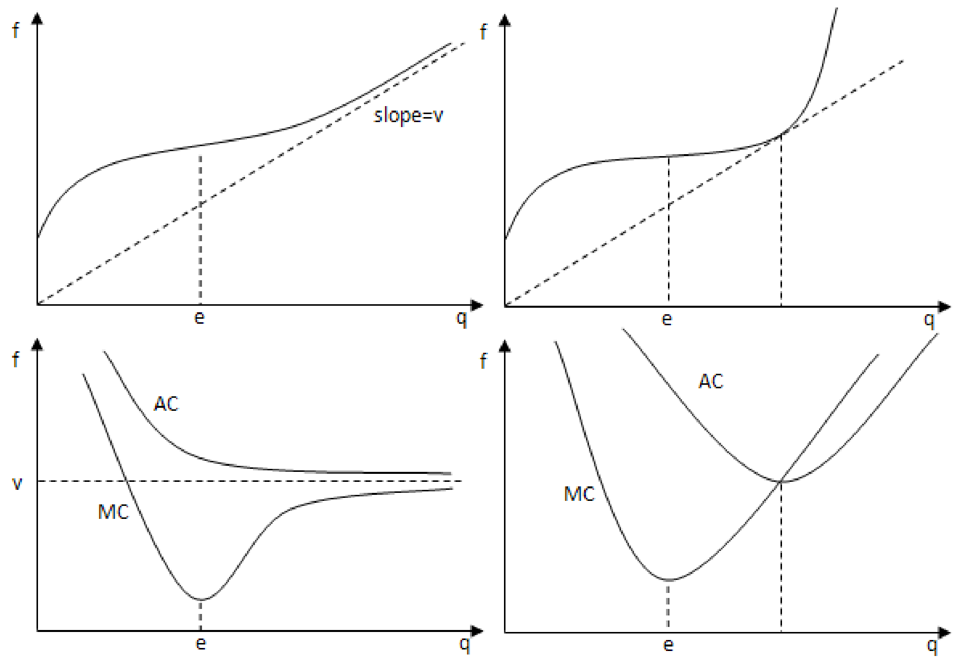


Figure 3.1: Difference in terms of MC and AC compared to Van den Broek (2006).

Before presenting the formulation, the following notation is introduced:

Indices:

$i = 1, ..., m$: index for customers.

$j = 1, ..., n$: index for facilities.

Parameters:

F_j: The fixed cost of opening and operating at location j.

d_i: The demand for customer i.

e_j: The economic point at facility j.

where cost shifts from concave to convex.

K_j: The capacity of facility j.

c_{ij}: The variable cost of satisfying demand i from facility j.

Decision variables:

x_{ij}: The quantity produced by facility j for customer i.

$$y_j^e = \begin{cases} 1, & \text{if facility } j \text{ is producing at or under economic point.} \\ 0, & \text{otherwise.} \end{cases}$$

$$y_j^e = \begin{cases} 1, & \text{if facility } j \text{ is producing above the economic point.} \\ 0, & \text{otherwise.} \end{cases}$$

The resulting facility location model with economies of scale and congestion is:

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} F_j(y_j^e + y_j^c) + \sum_{j=1}^{n} y_j^e g_j(\sum_{i=1}^{m} x_{ij}) \\ & + \sum_{j=1}^{n} y_j^c f_j(\sum_{i=1}^{m} x_{ij}) + \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} && (1) \\ \text{s.t.} \quad & \sum_{j=1}^{n} x_{ij} \geq d_i && \forall i = 1, ..., m \quad (2) \\ & e_j y_j^c < \sum_{i=1}^{m} x_{ij} \leq K_j y_j^c + e_j y_j^e && \forall j = 1, ..., n \quad (3) \\ & y_j^e + y_j^c \leq 1 && \forall j = 1, ..., n \quad (4) \\ & y_j^e, y_j^c \in \{0, 1\} && \forall j = 1, ..., n \quad (5) \\ & x_{ij} \geq 0 && \forall i = 1, ..., m, \forall j = 1, ..., n \quad (6) \end{aligned}$$

where g_j is the concave function to model economies of scale and f_j is the convex function to model congestion.

The objective function (1) minimizes the sum of fixed costs, production costs and variable costs (transportation costs) across all facilities. Constraints (2) insure all customer demands are satisfied. Constraints (3) specify the interval for $\sum_{i=1}^{m} x_{ij}$

under each combination of (y_j^e, y_j^c). Constraints (4) ensure that a facility is either closed, operating under economies of scale, or operating under congestion.

3.2 Lagrangian Relaxation

Lagrangian relaxation has been a popular choice for a number of large-scale optimization [13]. Algorithms for a variety of important problems in areas of routing, location, scheduling, assignment and set covering [12] are improved dramatically by Lagrangian relaxation. It works by moving the hard constraints into the objective so as to add a penalty on the objective if they are not satisfied. The penalty variables are referred to as Lagrangian multipliers. The relaxed problem is usually much easier to solve. For a minimization problem, it provides a lower bound on the original problem. Thus, it can be embedded into a branch-and-bound algorithm to find global optimum. The best lower bound depends on the Lagrangian multipliers and is at least as good as that of LP relaxation. Subgradient method is the most common choice to search for the best Lagrangian multipliers. In the remaining section, we will apply Lagrangian relaxation to the model proposed earlier. Section 3.3 provides the algorithm for solving the relaxed problem. Section 3.4 creates a Lagrangian master problem to update the Lagrangian multipliers and the last section proposes a Lagrangian heuristic.

By applying Lagrangian relaxation to constraint set (2) with $\mu_i \geq 0$ for all i's as the Lagrangian multipliers, the corresponding Lagrangian subproblem $LR(\mu)$ is:

$$
\begin{aligned}
LR(\mu) = \min \sum_{j=1}^{n} F_j(y_j^e + y_j^c) + \sum_{j=1}^{n} y_j^e g_j(\sum_{i=1}^{m} x_{ij}) + \sum_{j=1}^{n} y_j^c f_j(\sum_{i=1}^{m} x_{ij}) \\
+ \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + \sum_{i=1}^{m} \mu_i (d_i - \sum_{j=1}^{n} x_{ij}) \\
= \min \sum_{j=1}^{n} F_j(y_j^e + y_j^c) + \sum_{j=1}^{n} y_j^e g_j(\sum_{i=1}^{m} x_{ij}) + \sum_{j=1}^{n} y_j^c f_j(\sum_{i=1}^{m} x_{ij}) \\
+ \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} - \mu_i) x_{ij} + \sum_{i=1}^{m} \mu_i d_i
\end{aligned}
$$

subject to (3), (4), (5), (6).

$LR(\mu)$ decomposes into j subproblems :

$$
\begin{aligned}
SP_j(\mu) = \quad & \min \quad F_j(y_j^e + y_j^c) + y_j^e g_j(\sum_{i=1}^{m} x_{ij}) + y_j^c f_j(\sum_{i=1}^{m} x_{ij}) + \sum_{i=1}^{m} (c_{ij} - \mu_i) x_{ij} \\
& \text{s.t.} \quad e_j y_j^c \le \sum_{i=1}^{m} x_{ij} \le K_j y_j^c + e_j y_j^e \\
& \qquad y_j^e + y_j^c \le 1 \\
& \qquad y_j^e, y_j^c \in \{0, 1\} \\
& \qquad x_{ij} \ge 0, \forall i = 1, ..., m
\end{aligned}
$$

For each $SP_j(\mu)$, the possible feasible solutions for y_j^e and y_j^c fall into three cases, i.e. $(y_j^e, y_j^c) \in \{(0,0), (1,0), (0,1)\}$. For each case, $SP_j(\mu)$ is simplified as follows :

Case 1, $(y_j^e, y_j^c) = (0, 0)$:

$$\begin{aligned} SP_j(\mu) = \quad & \min \quad \sum_{i=1}^{m}(c_{ij} - \mu_i)x_{ij} \\ & \text{s.t.} \quad 0 \leq \sum_{i=1}^{m} x_{ij} \leq 0 \\ & \qquad\quad x_{ij} \geq 0, \, \forall i = 1, ..., m \end{aligned}$$

Obviously, the solution is $x_{ij} = 0$, $\forall i$, and $SP_j(\mu) = 0$.

Case 2, $(y_j^e, y_j^c) = (1, 0)$:

$$\begin{aligned} SP_j(\mu) = \quad & \min \quad F_j + g_j(\sum_{i=1}^{m} x_{ij}) + \sum_{i=1}^{m}(c_{ij} - \mu_i)x_{ij} \\ & \text{s.t.} \quad 0 \leq \sum_{i=1}^{m} x_{ij} \leq e_j \\ & \qquad\quad x_{ij} \geq 0, \, \forall i = 1, ..., m \end{aligned}$$

is a concave optimization problem.

Case 3, $(y_j^e, y_j^c) = (0, 1)$:

$$\begin{aligned} SP_j(\mu) = \quad & \min \quad F_j + f_j(\sum_{i=1}^{m} x_{ij}) + \sum_{i=1}^{m}(c_{ij} - \mu_i)x_{ij} \\ & \text{s.t.} \quad e_j \leq \sum_{i=1}^{m} x_{ij} \leq K_j \\ & \qquad\quad x_{ij} \geq 0, \, \forall i = 1, ..., m \end{aligned}$$

is a convex optimization problem.

The solution to $SP_j(\mu)$ is the minimum solution achieved in case 1, 2 and 3.

3.3 Solving the Subproblems

Before discussing the solution of subproblems, we devise a set of valid cuts to strengthen the solution of subproblems and the Lagrangian lower bound. It is obvious that the original model forces $\sum_{j=1}^{n} x_{ij}$ to be exactly equal d_i because the problem is a minimization problem, so

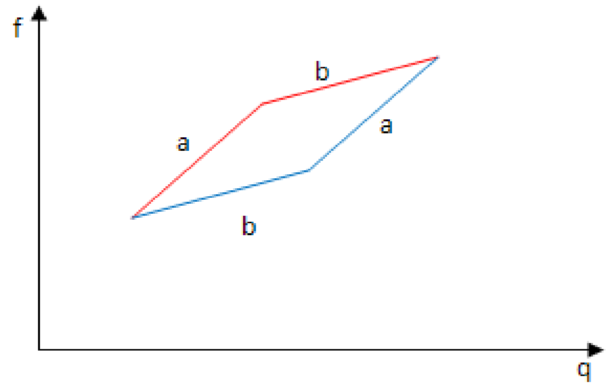
$$x_{ij} \leq d_i, \quad \forall i = 1, ..., m, \forall j = 1, ..., n \quad (7)$$

These constraints are redundant in the original problem but not in the subproblems as (2) are relaxed.

The following proposition is needed in the solution methodology.

Proposition 1 *Given a set of slopes and a starting point, the linear piecewise function formed by picking the slopes in descending order is above any other linear piecewise function formed by any other order.*

Proof. Consider two segments a and b with slope h_a and h_b, where $h_a \geq h_b$, starting from an initial point, the linear piecewise function formed of a then b is above that of b then a as shown in Figure 3.3. Since this holds for any two neighboring segments of a linear piecewise function, it holds for any arbitrary number of segments. ■



Proof of proposition 1.

In our model, both g_j and f_j are functions of $\sum_{i=1}^{m} x_{ij}$, which means that as long as the total production quantity does not change at facility j, the change in individual values of x_{ij} does not change the production cost. So we can define $q_j = \sum_{i=1}^{n} x_{ij}$. But choosing which x_{ij} to be positive affects $\sum_{i=1}^{m}(c_{ij} - \mu_i)x_{ij}$, the linear part in the objective function. Due to this fact, it is more preferable to assign value to x_{ij} with negative $c_{ij} - \mu_i$, while keeping x_{ij} with nonnegative $c_{ij} - \mu_i$ to 0. Thus, only x_{ij}'s with negative $c_{ij} - \mu_i$'s are considered. Since Case 1 has 0 as its optimum, Case 2 and Case 3 can only dominate when their objective values are nonpositive. Thus, at facility j, if $c_{ij} - \mu_i$'s are all nonnegative, then Case 1 dominates the solution at this facility. If not, we first sort negative $c_{ij} - \mu_i$'s in ascending order and define an ordered set $\Pi_j = \{\pi_j^t | c_{\pi_j^t,j} - \mu_{\pi_j^t} \le c_{\pi_j^{t+1},j} - \mu_{\pi_j^{t+1}} \le 0, t = 1, 2, ..., T\}$. The potential demand $\sum_{t=1}^{T} d_{\pi_j^t}$ faced by facility j can fall into one of the three intervals: $(0, e_j]$, $(e_j, K_j]$ and $(K_j, \sum_i d_i]$. If $\sum_{\pi_j^t \in \Pi_j} d_{\pi_j^t} \in (0, e_j]$, only Case 2 is solved and compared with Case 1. Otherwise, both Case 2 and Case 3 are considered.

To show the idea of solving the subproblem, let's denote $q_j^t = \sum_{t=1}^{t} d_{\pi_j^t}$, plot

$$h_j(q_j) = \begin{cases} F_j + g_j(q_j) & ,0 \leq q_j \leq e_j \\ F_j + f_j(q_j) & ,e_j \leq q_j \leq K_j \end{cases}$$

in Figure 3.2, and connect those ordered points $(0,0)$, $(q_j^1, h_j(q_j^1))$, $(q_j^2, h_j(q_j^2))$, $(q_j^3, h_j(q_j^3))$, ... , $(q_j^T, h_j(q_j^T))$. Given proposition 1, the resulting linear piecewise function

$$p_j(q_j) = \begin{cases} (\mu_{\pi_j^1} - c_{\pi_j^1,j})x_{\pi_j^1,j}, & \begin{array}{l} 0 \leq x_{\pi_j^1,j} \leq d_{\pi_j^1}, \\ q_j = x_{\pi_j^1,j}; \end{array} \\ (\mu_{\pi_j^1} - c_{\pi_j^1,,j})d_{\pi_j^1} + (\mu_{\pi_j^2} - c_{\pi_j^2,j})x_{\pi_j^2,j}, & \begin{array}{l} 0 \leq x_{\pi_j^2,j} \leq d_{\pi_j^2}, \\ q_j = q_j^1 + x_{\pi_j^2,j}; \end{array} \\ \sum_{t=1}^{2}(\mu_{\pi_j^t} - c_{\pi_j^t,,j})d_{\pi_j^t} + (\mu_{\pi_j^3} - c_{\pi_j^3,j})x_{\pi_j^3,j}, & \begin{array}{l} 0 \leq x_{\pi_j^3,j} \leq d_{\pi_j^3}, \\ q_j = q_j^2 + x_{\pi_j^3,j}; \end{array} \\ \ldots & \\ \sum_{t=1}^{t-1}(\mu_{\pi_j^t} - c_{\pi_j^t,,j})d_{\pi_j^t} + (\mu_{\pi_j^t} - c_{\pi_j^t,j})x_{\pi_j^t,j}, & \begin{array}{l} 0 \leq x_{\pi_j^t,j} \leq d_{\pi_j^t}, \\ q_j = q_j^{t-1} + x_{\pi_j^t,j}; \end{array} \\ \ldots & \\ \sum_{t=1}^{T-1}(\mu_{\pi_j^t} - c_{\pi_j^t,,j})d_{\pi_j^t} + (\mu_{\pi_j^T} - c_{\pi_j^T,j})x_{\pi_j^T,j}, & \begin{array}{l} 0 \leq x_{\pi_j^T,j} \leq d_{\pi_j^T}, \\ q_j = q_j^{T-1} + x_{\pi_j^T,j}. \end{array} \end{cases}$$

in Figure 3.2, is above any other linear piecewise function formed in any other order. As a result, $\min h_j(\sum_{i=1}^{m} x_{ij}) + \sum_{i=1}^{m}(c_{ij} - \mu_i)x_{ij} = \max \sum_{i=1}^{m}(\mu_i - c_{ij})x_{ij} -$

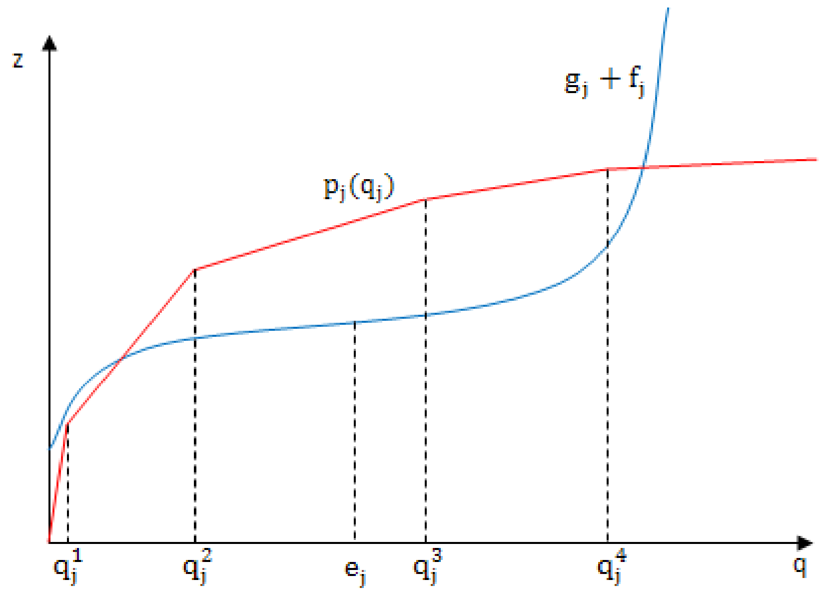


Figure 3.2: A graphically example of the basic idea for solving subproblem.

$h_j(\sum_i^m x_{ij}) = \max \sum_{i\in\Pi_j}(\mu_i - c_{ij})x_{ij} - h_j(\sum_{i\in\Pi_j} x_{ij}) = \max p_j(q_j) - h_j(q_j)$. The first equality holds because minimizing the objective function on the right is equivalent to maximizing its negative. The second equality holds because only x_{ij}'s with negative $c_{ij} - \mu_i$'s are considered. The third equality holds because of proposition 1.

Algorithm for Case 2:

Define $Z^*_{left} = \max\ p_j(q_j) - h_j(q_j)$ for $0 \le q_j \le e_j$; l such that $q_j^{l-1} < e_j$ and $q_j^l \ge e_j$. Then, Figure 3.3 describes the steps of solving Case 2. The idea is similar to the one used to solve the subproblem in [23].

To prove that the solution found by the algorithm in Figure 3.3 is optimal for Case 2, let's consider a piece of $p_j(q_j)$:

Case 2 algorithm
1. If $\Pi_j \neq \emptyset$, go to step 2;
otherwise, $Z^*_{left} = -\infty$, stop.
2. If l exists, go to setp 3;
otherwise, go to step 8.
3. Calculate q_t for $t \leq l$.
4. $Z^*_{left} = \max\{p_j(q^t_j) - h_j(q^t_j), e_j | t < l\}$.
5. If Z^*_{left} is reached at $q^t_j < e_j$, go to step 6;
otherwise, go to step 7.
6. Denote the optimal t as t^*,
set $x_{\pi^t_j j}$ to $d_{\pi^t_j}$, $\forall t \leq t^*$, stop.
7. $Z^*_{left} = p_j(e_j) - h_j(e_j)$,
set $x_{\pi^t_j j}$ to $d_{\pi^t_j}$, $\forall t < t^*$, and $x_{\pi^{t^*}_j j}$ to $e_j - q^{t^*-1}_j$,
stop.
8. $Z^*_{left} = \max\{p_j(q^t_j) - h_j(q^t_j)\}$,
denote the optimal t as t^*,
set $x_{\pi^t_j j}$ to $d_{\pi^t_j}$, $\forall t \leq t^*$,
stop.

Figure 3.3: Algorithm for solving subproblem of Case 2.

$$p_j(q_j) = \sum_{t=1}^{t-1}(\mu_{\pi_j^t} - c_{\pi_j^t,,j})d_{\pi_j^t} + (\mu_{\pi_j^t} - c_{\pi_j^t,j})x_{\pi_j^t,j} \quad \begin{matrix} 0 \le x_{\pi_j^t,j} \le d_{\pi_j^t} \\ q_j = q_j^{t-1} + x_{\pi_j^t,j} \end{matrix} \quad ,$$

which is a linear function of the form $p(q_j) = aq_j + b$, where $a = \mu_{\pi_j^t} - c_{\pi_j^t,j}$, $b = \sum_{t=1}^{t-1}(\mu_{\pi_j^t} - c_{\pi_j^t,j})d_{\pi_j^t}$. Since $g_j(q_j)$ is concave, $p_j(q_j) - g_j(q_j)$ is convex. Thus, $\max p_j(q_j) - g_j(q_j)$, where $q_j \in [q_j^{t-1}, q_j^t] \subseteq [0, e_j]$, is reached at either q_j^{t-1} or q_j^t. Consequently, for $p_j(q_j)$ with limited number of linear functions, $\max p_j(q_j) - g_j(q_j)$ is reached at one of q_j^t's, i.e. $Z^*_{left} = \max\{p_j(q_j^t) - h_j(q_j^t)\}$.

Algorithm for Case 3:

Define $Z^*_{right} = \max\ p_j(q_j) - h_j(q_j)$ for $q_j \in [e_j, K_j]$; r such that $\mu_{\pi_j^r} - c_{\pi_{j'}^r j} = f_j'(q_j')$, where $q_j' \in [q_j^{r-1}, q_j^r] \subseteq [e_j, K_j]$; w such that $q_j^w \le K_j$ and $q_j^{w+1} > K_j$; s such that $\mu_{\pi_j^s} - c_{\pi_j^s j} \ge f_j'(q_j^s) \ge \mu_{\pi_j^{s+1}} - c_{\pi_j^{s+1} j}$ for $q_j^s \in [e_j, K_j]$. The steps of the algorithm are illustrated in Figure 3.4.

To prove that the solution found by the algorithm in Figure 3.4 is optimal for Case 3, let's consider the objective function $\max p_j(q_j) - f_j(q_j)$, where $q_j \in [e_j, q_j^T] \cap [e_j, K_j]$. Because $f_j(q_j)$ is convex and $p_j(q_j)$ on $[e_j, q_j^T]$ is concave, $p_j(q_j) - f_j(q_j)$ is concave on $[e_j, q_j^T] \cap [e_j, K_j]$. If $p_j'(e_j) \le f_j'(e_j)$, the optimal value is reached at e_j, since $p_j(q_j)$ increases at a rate less than $f_j(q_j)$ after e_j. Moreover, since Case 2 and Case 3 share the same objective value at e_j, Case 3 can be ignored in this case. If $p_j'(e_j) > f_j'(e_j)$, $p_j(q_j) - f_j(q_j)$ reaches its optimal at q_j such that $p_j'(q_j) - f_j'(q_j) = 0$. However, such q_j may not exist because of the nondifferentiability of $p_j(q_j)$ at discrete points q_j^t's. But the number of the nondifferentiable points are limited, thus,

Case 3 algorithm
1. If $\Pi_j \neq \emptyset$, go to step 2;
   otherwise, $Z^*_{right} = -\infty$, stop.
2. If $q_j^T > e_j$, go to step 3;
   otherwise, $Z^*_{right} = -\infty$, stop.
3. If $p_j(e_j) > f'_j(e_j)$ go to step 4;
   otherwise, $Z^*_{right} = -\infty$, stop.
4. If r exists, go to step 5;
   otherwise, go to step 6.
5. Set $x_{\pi_j^t j}$ to $d_{\pi_j^t}$ for $t < r$, and $x_{\pi_j^r j}$ with $q'_j - q_j^{r-1}$,
   $Z^*_{right} = p_j(q'_j) - h_j(q'_j)$, stop.
6. If s exists, go to step 7; otherwise, go to step 8.
7. Set $x_{\pi_j^s j}$ to $d_{\pi_j^s}$ for $t \leq s$
   $Z^*_{right} = p_j(q_j^s) - h_j(q_j^s)$, stop.
8. If $q_j^T \leq K_j$, set $x_{\pi_j^t}$ to $d_{\pi_j^t}$ for $t \leq T$, and $Z^*_{right} = p_j(q_j^T) - h_j(q_j^T)$;
   otherwise, set $x_{\pi_j^t j}$ to $d_{\pi_j^t}$ for $t \leq w$, and $x_{\pi_j^w j} = K_j - \sum_{t=1}^{w-1} d_{\pi_j^t}$,
   $Z^*_{right} = p_j(K_j) - h_j(K_j)$, stop.

Figure 3.4: Algorithm for solving subproblem of Case 3.

we can analyze on each interval $[q_j^{t-1}, q_j^t] \subseteq [e_j, q_j^T] \cap [e_j, K_j]$. Since $p'_j(q_j) = \mu_{\pi_j^t} - c_{\pi_j^t, j}$ on $[q_j^{t-1}, q_j^t]$, then $f'_j(q_j^{t-1})$ and $f'_j(q_j^t)$ can tell us whether $p'_j(q_j)$ can equal $f'_j(q_j)$ on $[q_j^{t-1}, q_j^t] \subseteq [e_j, q_j^T] \cap [e_j, K_j]$. If $f'_j(q_j^{t-1}) \leq \mu_{\pi_j^t} - c_{\pi_j^t, j} \leq f'_j(q_j^t)$, there is a $q_j \in [q_j^{t-1}, q_j^t] \subseteq [e_j, q_j^T] \cap [e_j, K_j]$ such that $p'_j(q_j) = f'_j(q_j)$. If $\mu_{\pi_j^t} - c_{\pi_j^t, j} \geq f'_j(q_j^t)$, then the optimal point is on $[q_j^t, K_j]$. If $\mu_{\pi_j^t} - c_{\pi_j^t, j} \leq f'_j(q_j^{t-1})$, then the optimal point is on $[e_j, q_j^{t-1}]$. Obviously, if $\mu_{\pi_j^t} - c_{\pi_j^t, j} \geq f'_j(q_j^t) \geq \mu_{\pi_j^{t+1}} - c_{\pi_j^{t+1}, j}$, then $t = s$ and the optimal point is q_j^s. However, such q_j^s may not exist in two cases. The first case is when $q_j^T \leq K_j$ and $\mu_{\pi_j^T} - c_{\pi_j^T, j} \geq f'_j(q_j^T)$. Clearly, the optimum is reached at q_j^T. The second case is when $q_j^T > K_j$ and $\mu_{\pi_j^w} - c_{\pi_j^w, j} \geq f'_j(K_j)$. Obviouly, the optimum reaches at K_j. Consequently, the algorithm in Figure 3.4 captures all possibilities and gives the optimal solution of Case 3.

In the end, after getting the optimal solution for each case, the optimal value of subproblem j is :

$$SP_j(\mu) = \min\{0, -Z^*_{left}, -Z^*_{right}\}.$$

3.4 Updating the Lagrangian Multipliers

To update μ, we create a master problem and add cuts using the solution of the subproblems. First, let's define the feasible set V as $\{(x, y) : (3) - (7)\}$, which is

$$\left\{\begin{array}{ll} (x_{ij}, y^e_j, y^c_j) : & \\ e_j y^c_j < \sum_{i=1}^{m} x_{ij} \leq K_j y^c_j + e_j y^e_j & \forall j, \\ y^e_j + y^c_j \leq 1 & \forall j, \\ y^e_j, y^c_j \in \{0, 1\} & \forall j, \\ x_{ij} \geq 0 & \forall i, j, \\ x_{ij} \leq d_i & \forall i, j. \end{array}\right\}$$

As we can see, V can be separated into j subsets, denoted by V_j. Let the corresponding feasible solution be denoted by $(x^h_{ij}, y^{e^h}_j, y^{c^h}_j)$. Then the subproblem can be written as :

$$\theta_j = \min_{(x^h_{ij}, y^{e^h}_j, y^{c^h}_j) \in V_j} \{F_j(y^{e^h}_j + y^{c^h}_j) + y^{e^h}_j g_j(\sum_{i=1}^{m} x^h_{ij}) + y^{c^h}_{ij} f_j(\sum_{i=1}^{m} x^h_{ij}) + \sum_{i=1}^{m} (c_{ij} - \mu_i) x^h_{ij}\}$$

The master problem is then:

[$MP - SP$]
Initiate $LRUB = \infty, LRLB = -\infty$
1. Start with a set of $\mu \geq 0$ (usually μ is set to be larger than the maximum of c_{ij}'s)
While $LRUB \neq LRLB$
2. Solve $SP_j(\mu)$ using the algorithms in Figure 3.3 and 3.4 for each j
get a solution of $(x_{ij}^h, y_j^{e^h}, y_j^{c^h})$ and a lower bound $LRLB^h = \sum_j SP_j(\mu)$
3. Update the lower bound $LRLB = \max(LRLB, LRLB^h)$
4. Use $(x_{ij}^h, y_j^{e^h}, y_j^{c^h})$ to add j cuts in the form of constraint (8) to MP
5. Solve MP to get a new set of μ and an upper bound $LRUB$
End while

Figure 3.5: Algorithm for updating Lagrangian multipliers.

$$
\begin{aligned}
[MP]: \quad \max \quad & \sum_{i=1}^{m} \mu_i d_i + \sum_{j=1}^{n} \theta_j \\
\text{s.t.} \quad & \sum_{i=1}^{m} x_{ij}^h \mu_i + \theta_j \leq F_j(y_j^{e^h} + y_j^{c^h}) + y_j^{e^h} g_j(\sum_{i=1}^{m} x_{ij}^h) && \forall j = 1, ..., n \quad (8) \\
& \qquad + y_{ij}^{c^h} f_j(\sum_{i=1}^{m} x_{ij}^h) + \sum_{i=1}^{m} c_{ij} x_{ij}^h \\
& (x_{ij}^h, y_j^{e^h}, y_j^{c^h}) \in V_j && \forall j = 1, ..., n \quad (9) \\
& \mu_i \geq 0, \ \theta_j \text{ unrestricted} && \forall i = 1, ..., m \quad (10)
\end{aligned}
$$

The master problem gives a Lagrangian upper bound $LRUB$. The subproblems give a Lagrangian lower bound $LRLB = LR(\mu) = \sum_{j=1}^{n} SP_j(\mu) + \sum_{i=1}^{m} \mu_i d_i$. Then the algorithm proceeds as explained in Figure 3.5.

3.5 A Lagrangian Heuristic

The above algorithm solves the Lagrangian dual problem and generates a lower bound (LB) to the original problem. However, the subproblem solution is rarely feasible to the original problem. In this section, we propose a heuristic based on the solutions from the subproblems. The basic idea is to find a combination of the proposed solutions from the subproblems that may be feasible to the original problem. To do this, we resort to the dual of the master problem, i.e. Dantzig-Wolfe master problem. Its solution gives a weight to each proposal from the subproblems. Let's define λ_j^h as the dual variable of constraint $\sum_{i=1}^m x_{ij}^h \mu_i + \theta_j \leq F_j(y_j^{e^h} + y_j^{c^h}) + y_j^{e^h} g_j(\sum_{i=1}^m x_{ij}^h) + y_{ij}^{c^h} f_j(\sum_{i=1}^m x_{ij}^h) + \sum_{i=1}^m c_{ij} x_{ij}^h$ generated by the solution h at facility j from SP_j and $b_j^h = F_j(y_j^{e^h} + y_j^{c^h}) + y_j^{e^h} g_j(\sum_{i=1}^m x_{ij}^h) + y_{ij}^{c^h} f_j(\sum_{i=1}^m x_{ij}^h) + \sum_{i=1}^m c_{ij} x_{ij}^h$. Suppose that when $LRLB = LRUB$, there are H solutions generated from subproblem. Then the dual problem of $[MP]$ is :

$$
\begin{array}{lll}
\min & \sum\limits_{h=1}^{H} \sum\limits_{j=1}^{n} b_j^h \lambda_j^h & \\
\text{s.t.} & \sum\limits_{h=1}^{H} \sum\limits_{j=1}^{n} x_{ij}^h \lambda_j^h \geq d_i & \forall i = 1, ..., m \\
& \sum\limits_{h=1}^{H} \lambda_j^h = 1 & \forall j = 1, ..., n \\
& \lambda_j^h \geq 0 & \forall j = 1, ..., n, \forall h = 1, ..., H
\end{array}
$$

It is clear that the the above minimization problem gives a combination of the subproblem solutions x_{ij}^h for each facility: $\sum_{h=1}^H \lambda_j^h x_{ij}^h$, $\forall j = 1, ..., n$.

After getting a feasible solution, we will apply an enhanced search based on the feasible solution. First, the closed facilities are removed from current problem. Then the facilities operating under their economic points e_j's are picked and the one with the minimal output is chosen and its y_j^e amd y_j^c are set to 0. The resulting new problem is solved again as done before, i.e. iterating between the master problem and subproblem and performing the heuristic. The Lagrangian lower bound for the new problem may become greater than the current best value, in that case the search stops as no feasible solution could improve the current one. During the process, whenever the feasible solution is better than the current best, update the best feasible solution. We stop the search when:

1. All facilities are operating above the economic points.

2. Closing any facility will result in an infeasible problem, i.e. total demand is greater than total capacity.

3. Each offspring problem generated by closing from its predecessor one facility working below the economic point, has been searched.

4. $LRLB \geq UB$

The algorithm steps are illustrated in Figure 3.6. Figure 3.7 gives an example. The original problem has 5 facilities. In each node, the upper set contains all facilities of the original problem and each offspring problems. The lower set contains those facilities operating under the economic points. Each arc with the number on it represents removing that facility and creating a new problem. $\{A\}\backslash\{B\}$ means remove set B from set A. After carrying out the initial Lagrangian search and heuristic, we get a lower bound and upper bound on the original problem. All lower bounds after the initial step are only used to compare with upper bound and are not valid lower bounds on the original problem. As shown in Figure 3.7, nodes 4, 5, 6, 7 are stopped because of criteria 3, 1, 4, 2 respectively.

It is obvious that the new problems with one facility operating under the economic point being closed are only a subset of all the possible problems that could

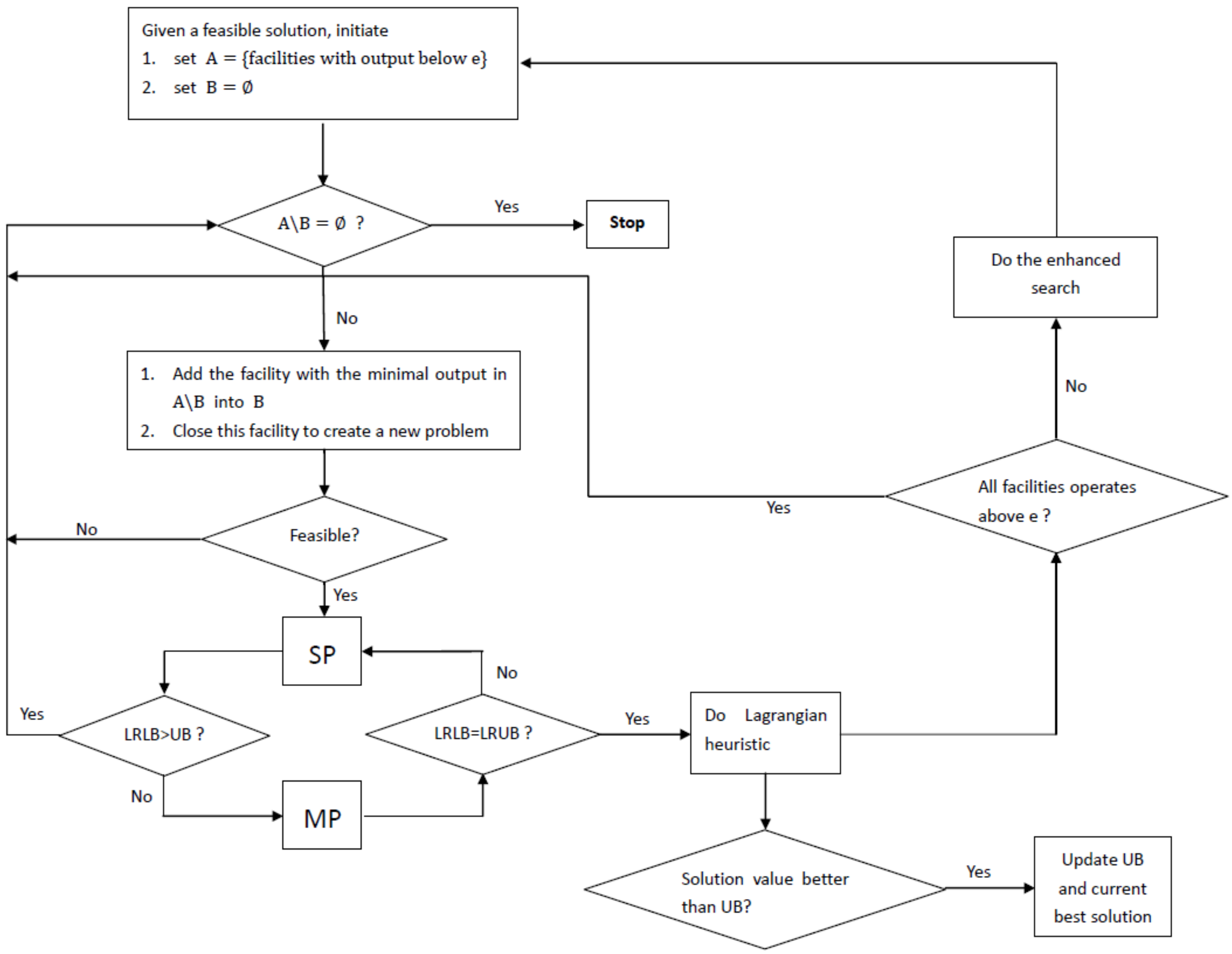


Figure 3.6: The enhanced search.

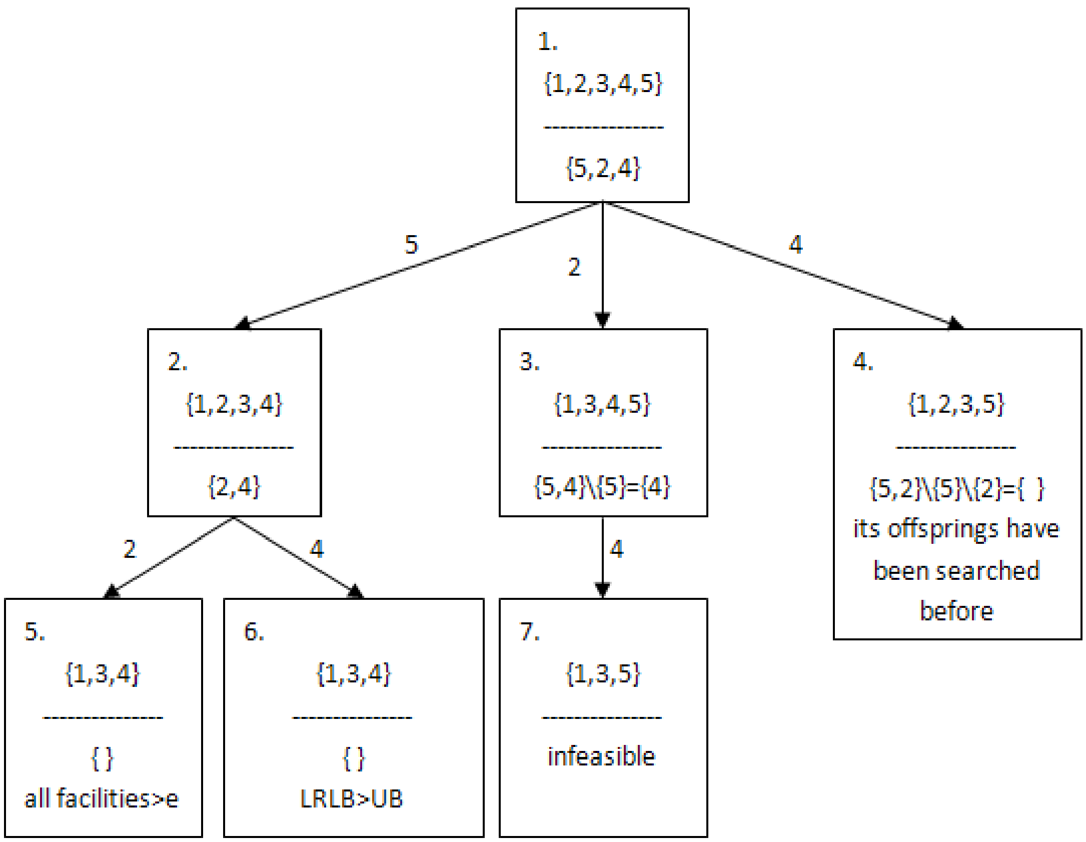


Figure 3.7: An example of the enhanced search.

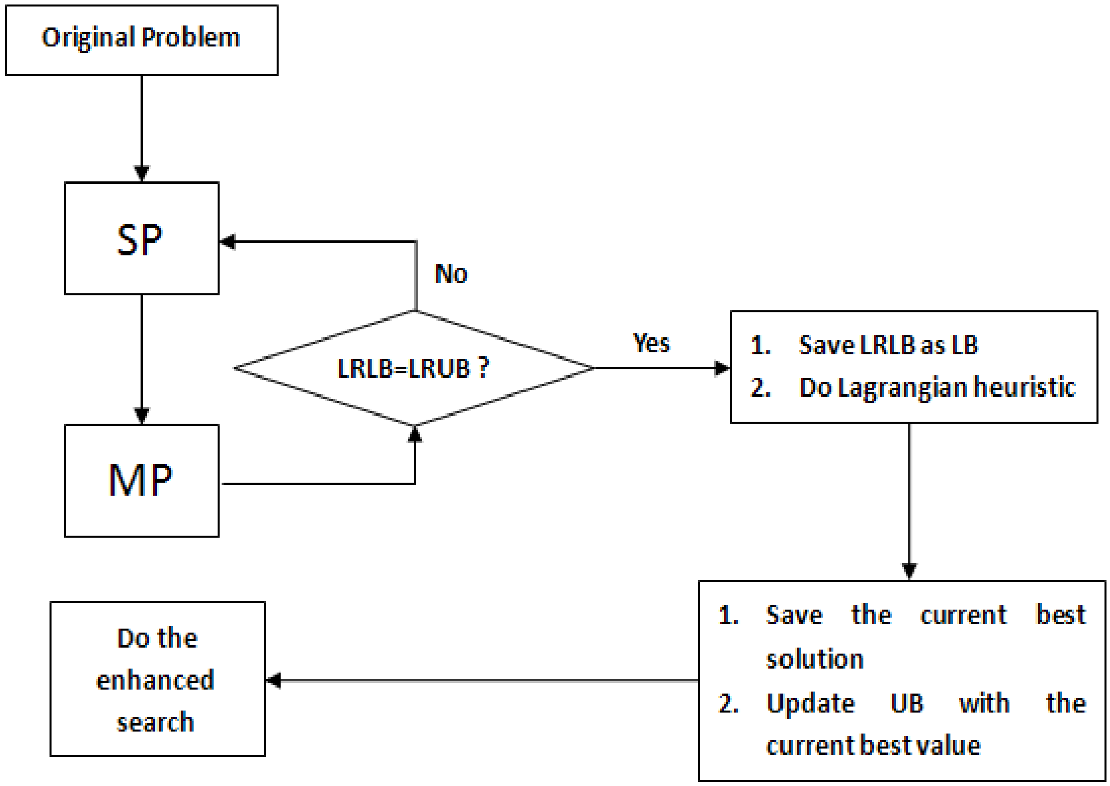


Figure 3.8: Overall Lagrangian algorithm.

be generated from their predecessor. Thus, the enhanced search does not promise to find a global optimum. However, exploring the entire set of all possible problems dramatically increases the computation time. Thus, the enhanced search is used when the computation time and the gap from optimum are acceptable.

The overall algorithm is depicted in Figure 3.8.

Chapter 4

Numerical Testing

In this Chapter, the computational performance of the proposed model and solution method is evaluated. All proposed algorithms and the heuristic are coded in Matlab 7.6. Gurobi 3.0.1 is used to solve problem $[MP]$. The tests are done on a PC Intel Core 2 2.40GHz with 3GB of RAM.

4.1 Test Instances

The test problems are based on the collection of facility location instances of Holmberg [18]. As [18] focuses on single sourcing, the transportation cost from facility j to customer i is the cost for the entire demand d_i. Thus, the transportation cost is divided by the corresponding demand to derive unit transportation costs c_{ij}. For the S-shaped production cost function, we use three types. The first type takes the form:

$$h(x) = \begin{cases} a_1 x^{b_1} & 0 < x \leq e,\ 0 < b_1 < 1 \\ a_2(x-e)^{b_2} + t(x-e) + a_1 e^{b_1} & e < x \leq K,\ 1 < b_1 \end{cases} \quad (11)$$

The second type takes the form:

$$h(x) = \begin{cases} a_1 x^{b_1} & 0 < x \leq e,\ 0 < b_1 < 1 \\ a_2(x-e)/(K-x) & e < x \leq K \end{cases} \quad (12)$$

Function type 1 and 2 are continuous but not necessarily differentiable at the economic point e for the above two functions. To achieve differentiability at e, t should be set to $a_1 b_1 e^{b_1 - 1}$ in function (11) and a_2 to $(1 - e/k)e^{b_1} a_1 b_1 k/e$ in function (12).

For the first two type of functions, we set K to the average capacity across the 55 instances (376), and $e = 0.5K = 188$. Figure 4.1 displays a plot of the above two types of functions of cost structure 1 in Table 4.2 and 4.7.

The third type is the cubic function:

$$f(q) = aq^3 - bq^2 + cq + d, where a, b, c, d \geq 0 \quad (13)$$

The cubic function (13) is a common choice in economics to represent production cost. Alternatively, it can be written as:

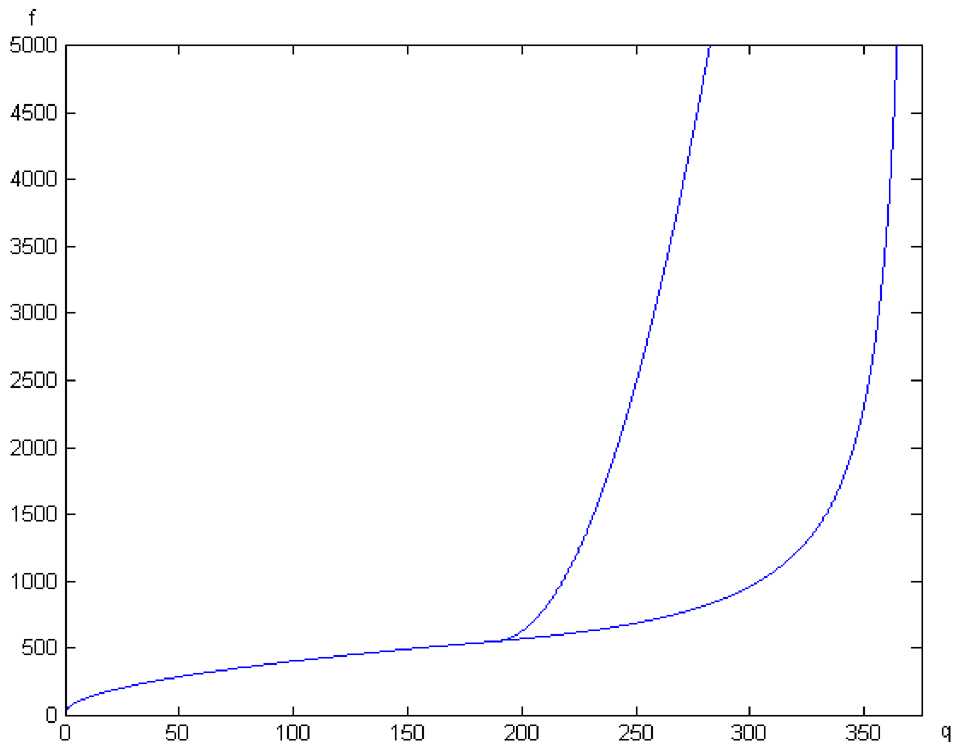


Figure 4.1: Shape difference between function type 1 and type 2.

$$f(q) = a(q-e)^3 + \epsilon(q-e) + h, a > 0, e > 0, \epsilon \geq 0 \qquad (14)$$

Since the function is required to be monotonically increasing, it is necessary that both $f'(q) = 3aq^2 - 2bq + c = 0$ and $f(q) = 3a(q-e)^2 + \epsilon = 0$ do not have two solutions, implying that $b^2 \leq 3ac$ and $\epsilon \geq 0$. If ϵ is positive, the slope at the economic point e is positive. Different cost structures can be obtained by varying a, e, d and ϵ. Figure 4.2 displays different forms of (14) for four combinations of a, e, d, ϵ.

To test the efficiency of the approach and explore the output of the model, we consider 4 cost structures depending on which cost component dominates. For each function type, numerical results are tabulated based on the following four structures in terms of fixed costs, production costs, and variable costs (transportation costs):

- Cost structure 1: The three cost components are around the same percentage of total cost.
- Cost structure 2: Fixed costs dominate other costs.
- Cost structure 3: Production costs dominate other costs.
- Cost structure 4: Variable costs dominate other costs.

The information in the tables is summarized in Table 4.1.

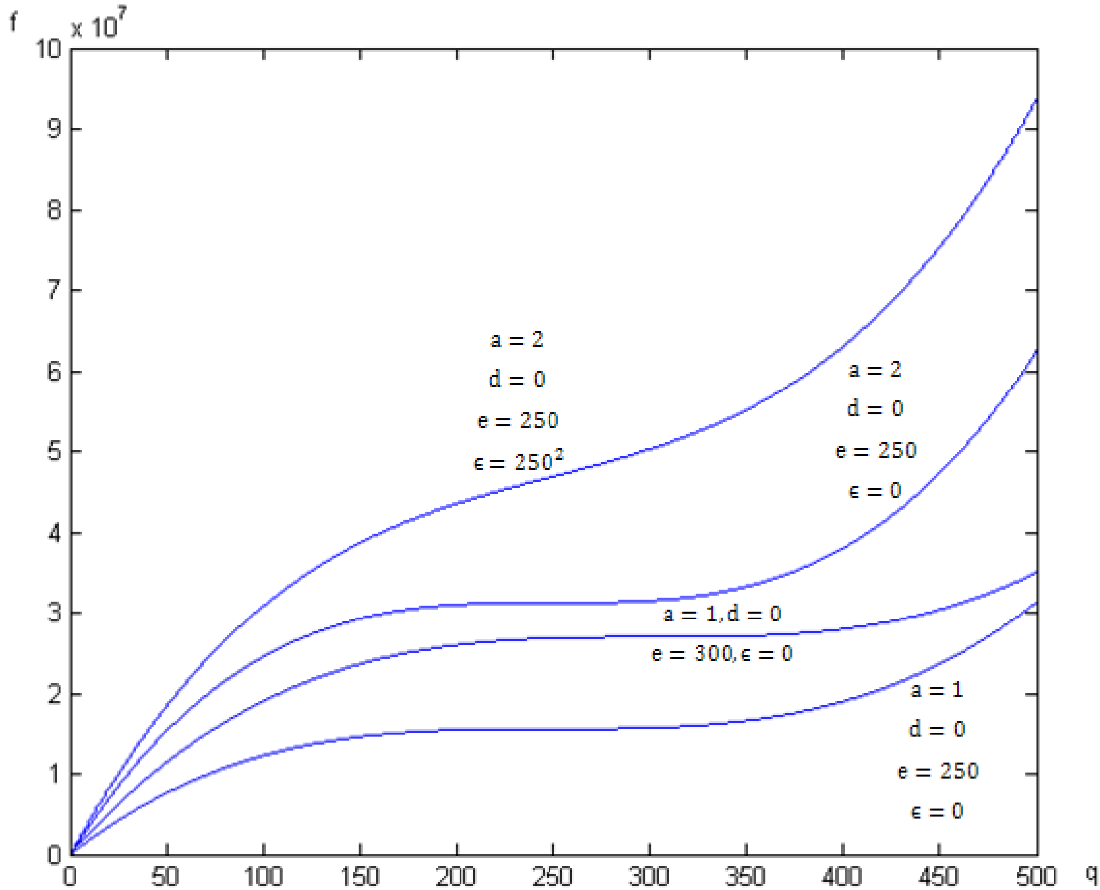


Figure 4.2: A plot of four cubic functions.

Instance	n m K/D	The number of facilities. The number of customers. Ratio of total capacity over total demand.
Initial Lag heuristic	$gap1\%$ $time1$ $itr1$	The gap between first feasible solution and Lagrangian bound in percentage. Computation time for the first feasible solution. Number of iterations for the first feasible solution.
Enhanced search	 $gap2\%$ $time2$ $itr2$	The gap between the current best feasible solution and Lagrangian bound in percentage. Computation time for the current best feasible solution. Number of iterations for the current best feasible solution.
Cost structure	$fixed\%$ $prd\%$ $var\%$	The percentage of fixed cost. The percentage of production cost. The percentage of variable cost.
Facility status	opn une	The number of open facilities in current best feasible solution. The number of facilities operating below the economic point in current best feasible solution.
Capacity utilization ($\frac{\text{production quantity}}{\text{capacity}}$ at each facility)	$max\%$ $min\%$ $avg\%$	The maximal utilization of all open facilities in current best feasible solution. The minimal utilization of all open facilities in current best feasible solution. The average utilization of all open facilities in current best feasible solution.

Table 4.1: Statistics tabulated for all tests.

	Production cost function	Cost structure	
1	$h_j(x) = \begin{cases} 40x_j^{0.5} & 0 < x_j \le e_j \\ 0.5(x_j - e_j)^2 + 40e_j^{0.5} & e_j < x_j \le K_j \end{cases}$ $\forall j = 1, .., n, e_j = 0.5K_j$	Original data from [18].	Table 4.3
2	Same as 1.	Fixed costs increased 10 times.	Table 4.4
3	$h_j(x) = \begin{cases} 400x_j^{0.5} & 0 < x_j \le e_j \\ 0.5(x_j - e_j)^2 + 400e_j^{0.5} & e_j < x_j \le K_j \end{cases}$ $\forall j = 1, .., n, e_j = 0.5K_j$	Original data from [18].	Table 4.5
4	Same as 1.	Variable costs increased 10 times.	Table 4.6

Table 4.2: Cost structures for function type 1.

4.2 Numerical Results

The numerical testing aims at evaluating the performance of the solution methodology under different function types and cost structures.

4.2.1 Performance of Function Type 1

For function type 1, Table 4.2 shows the production cost functions and cost structures.

Based on the Table 4.3, 4.4, 4.5 and 4.6, the best average gaps for the four cost structures are 0.94%, 0.88%, 0.56% and 0.30% respectively, which are always better than the average gaps without the enhanced search. The maximum gap across the four tables is 4.09%, arising from the case where fixed costs dominate. For each table, it is obvious that p25-p40 are the difficult instances, whose computation times are far longer than other sets of instances. It is also observed that for the last three cases in which one type of costs dominates, the maximum average gap

	Instance feature			Initial Lag heuristic			Enhanced search			Cost structure			Facility status		Capacity utilization		
	n	m	K/D	gap1%	time1	itr1	gap2%	time2	itr2	fixed%	prd%	var%	opn	une	max%	min%	avg%
p1	10	50	1.74	0.08	3.87	178	0.08	8.33	495	25.87	37.51	36.62	10	4	72.81	54.16	59.22
p2			1.74	0.09	2.62	145	0.09	7.52	474	19.52	40.66	39.81	10	4	72.81	53.62	59.22
p3			1.74	0.04	3.00	155	0.04	7.58	469	28.79	36.02	35.19	10	4	72.81	53.62	59.22
p4			1.74	0.06	3.54	172	0.06	8.63	498	36.15	32.26	31.59	10	4	72.81	54.16	59.21
p5			1.37	0.08	2.65	142	0.08	4.76	291	17.87	59.77	22.36	10	2	75.50	71.50	72.80
p6			1.37	0.10	2.67	145	0.10	4.37	284	13.14	63.22	23.64	10	2	75.00	71.00	72.80
p7			1.37	0.06	2.31	136	0.06	4.06	278	20.14	58.13	21.73	10	2	74.50	71.00	72.80
p8			1.37	0.10	1.95	129	0.10	3.53	260	26.09	53.79	20.12	10	2	74.50	71.00	72.80
p9			2.06	3.25	2.93	171	1.67	10.33	716	27.02	35.07	37.91	10	6	55.00	52.33	53.93
p10			2.06	2.25	2.57	159	2.04	9.28	673	20.42	38.26	41.32	10	6	55.00	52.33	53.93
p11			2.06	3.32	2.23	144	1.25	9.59	688	29.95	33.63	36.42	10	6	55.00	52.67	53.93
p12			2.06	4.18	3.73	204	0.75	11.00	736	37.44	30.06	32.50	10	6	55.00	52.67	53.93
p13	20	50	2.77	7.52	0.89	60	1.00	110.64	827	33.57	43.38	23.05	14	7	52.25	50.72	51.39
p14			2.77	3.70	0.86	67	1.41	101.68	868	25.56	48.68	25.76	17	11	51.77	34.88	49.64
p15			2.77	5.11	0.83	62	0.27	110.32	806	34.26	41.29	24.45	15	8	52.03	50.92	51.50
p16			2.77	7.47	0.89	65	0.06	22.82	602	42.18	36.30	21.52	13	6	51.74	51.13	51.49
p17			2.80	10.68	0.70	55	1.08	87.16	880	32.97	43.85	23.18	16	10	52.00	44.50	50.98
p18			2.80	5.50	0.92	71	0.43	18.17	849	25.65	49.15	25.21	16	10	51.50	49.50	50.98
p19			2.80	7.14	0.81	61	0.51	17.61	740	36.47	42.01	21.52	15	8	51.75	48.25	50.98
p20			2.80	8.34	0.90	64	0.73	43.62	744	44.43	36.81	18.75	15	8	52.00	47.75	50.98
p21			3.50	14.52	1.20	71	2.19	38.70	856	27.21	44.49	28.30	11	4	52.20	51.60	51.91
p22			3.50	10.07	1.20	73	2.70	55.01	1189	21.40	48.45	30.15	13	7	52.40	51.40	51.91
p23			3.50	12.06	1.40	83	2.11	97.97	913	31.22	42.37	26.41	11	4	52.20	51.60	51.91
p24			3.50	13.52	1.73	95	1.57	43.37	891	38.86	37.68	23.47	11	4	52.20	51.40	51.91
p25	30	150	4.12	7.27	151.76	409	1.56	3394.21	3099	25.21	35.85	38.94	10	1	54.23	36.75	50.45
p26			4.12	5.22	123.01	371	0.60	2019.50	2449	19.38	39.14	41.49	10	1	54.23	48.25	51.32
p27			4.12	4.52	138.25	390	0.81	1070.23	1810	28.56	34.79	36.65	9	0	54.23	46.20	51.33
p28			4.12	6.49	170.18	425	2.15	1029.89	1796	32.82	30.98	36.20	9	0	56.72	51.09	52.34
p29			3.03	4.17	145.56	447	0.22	488.95	1367	29.61	37.60	32.79	12	1	53.00	51.33	52.11
p30			3.03	5.63	121.40	416	0.17	478.19	1255	23.98	40.80	35.23	12	0	53.00	51.67	52.11
p31			3.03	5.10	134.35	430	0.38	254.45	869	34.38	35.34	30.28	12	0	53.00	47.33	52.11
p32			3.03	8.97	143.21	441	0.73	143.21	441	42.19	31.30	26.52	12	0	53.67	44.00	52.11
p33			4.04	7.46	162.71	454	1.65	2029.71	2378	24.89	37.34	37.77	10	1	52.00	39.50	49.50
p34			4.04	6.34	148.89	426	1.51	1472.63	2351	21.29	40.16	38.56	10	0	51.50	27.75	49.50
p35			4.04	7.83	184.69	486	1.85	2326.46	2348	28.97	37.10	33.93	9	0	53.50	52.50	53.04
p36			4.04	9.09	229.03	555	1.32	2732.97	2617	36.34	33.23	30.43	9	0	53.50	52.50	53.04
p37			6.06	7.64	500.53	755	1.07	856.65	1829	20.82	35.86	43.32	8	2	51.00	38.33	49.50
p38			6.06	6.86	608.75	803	1.55	989.48	1860	16.50	38.11	45.38	8	2	51.00	38.83	49.50
p39			6.06	8.54	606.58	776	1.64	945.76	1796	24.77	34.39	40.84	8	2	51.17	38.17	49.50
p40			6.06	9.91	597.28	757	1.83	998.61	1880	31.52	31.38	37.11	8	2	51.17	36.83	49.50
p41	10	90	2.12	4.30	28.72	293	1.03	67.67	984	28.71	35.87	35.42	10	4	54.26	43.80	51.14
p42	20	80	4.99	12.89	14.94	237	0.25	73.88	1488	27.21	34.45	38.34	14	9	52.17	50.81	51.12
p43	30	70	8.28	8.42	4.70	107	0.95	35.47	1156	26.80	34.91	38.29	14	10	51.93	51.29	51.56
p44	10	90	1.76	0.06	32.85	309	0.06	61.90	806	27.94	45.96	26.10	10	3	69.75	53.59	59.54
p45	20	80	4.14	7.23	14.38	256	1.21	302.28	1878	30.26	37.46	32.27	14	8	52.96	46.76	50.10
p46	30	70	7.10	15.26	3.09	96	2.67	255.67	1860	30.61	37.12	32.27	15	9	51.36	40.99	47.20
p47	10	90	1.76	0.10	91.96	549	0.10	113.16	978	29.01	46.73	24.26	10	3	69.42	53.16	59.41
p48	20	80	4.06	0.11	7.02	199	0.11	71.93	1579	31.78	39.68	28.55	14	9	51.78	50.46	50.83
p49	30	70	7.08	17.94	3.76	107	1.35	545.32	1647	31.40	39.46	29.15	13	8	52.57	47.43	50.86
p50	10	100	1.89	1.82	52.31	389	0.29	120.54	990	23.47	34.97	41.56	9	2	66.94	52.53	57.19
p51	20	100	3.98	11.97	25.37	220	2.24	124.32	1034	27.95	34.67	37.38	10	2	53.72	34.68	50.25
p52	10	100	1.60	0.11	103.62	506	0.11	103.62	506	17.58	48.11	34.31	8	0	85.95	56.75	67.20
p53	20	100	3.37	12.56	29.37	237	1.12	226.76	1377	27.07	34.72	38.21	11	4	61.17	48.21	53.61
p54	10	100	1.52	0.09	72.03	472	0.09	145.30	1063	16.11	57.83	26.05	10	2	89.47	58.65	71.00
p55	20	100	3.21	3.08	18.03	210	0.71	88.84	923	28.11	35.95	35.94	10	1	54.94	50.00	51.82
		Max	8.28	17.94	608.75	803	2.70	3394.21	3099	44.43	63.22	45.38	17	11	89.47	71.50	72.80
		Avg	3.32	5.93	85.72	277	0.94	443.70	1172	28.03	40.07	31.90	11	4	58.18	49.71	54.53
		Min	1.37	0.04	0.70	55	0.04	3.53	260	13.14	30.06	18.75	8	0	51.00	27.75	47.20

Table 4.3: Statistics for the basic case-function type 1.

	Instance feature			Initial Lag heuristic			Enhanced search			Cost structure			Facility status		Capacity utilization		
	n	m	K/D	gap1%	time1	itr1	gap2%	time2	itr2	fixed%	prd%	var%	opn	une	max%	min%	avg%
p1	10	50	1.74	3.65	2.68	114	0.24	6.44	267	66.11	21.15	12.75	8	0	91.23	61.39	67.80
p2			1.74	2.43	2.93	127	0.58	6.75	288	58.20	23.82	17.98	8	0	71.51	59.57	64.37
p3			1.74	3.01	3.79	147	0.79	8.46	324	60.92	25.76	13.32	7	0	77.33	64.88	69.55
p4			1.74	0.07	6.65	199	0.07	6.65	199	68.59	20.71	10.70	7	0	76.89	64.61	69.55
p5			1.37	0.31	1.95	101	0.31	1.95	101	68.35	23.01	8.63	10	0	76.00	66.00	72.80
p6			1.37	0.09	2.64	121	0.09	2.64	121	60.21	28.95	10.84	10	0	74.50	71.00	72.80
p7			1.37	0.09	2.07	104	0.09	2.07	104	71.60	20.66	7.73	10	0	75.00	71.00	72.80
p8			1.37	3.84	1.56	84	0.23	3.82	195	70.57	23.30	6.14	9	0	83.50	79.00	80.89
p9			2.06	22.13	2.67	121	0.61	18.10	479	67.88	17.20	14.92	8	0	62.00	58.00	60.67
p10			2.06	12.51	1.92	103	1.23	9.64	332	63.91	21.38	14.71	8	0	61.33	59.67	60.67
p11			2.06	9.27	2.81	116	0.13	8.42	372	74.67	15.02	10.30	8	0	61.33	58.33	60.67
p12			2.06	13.15	2.09	104	1.70	10.86	343	69.73	21.68	8.59	7	0	71.33	68.33	69.33
p13	20	50	2.77	17.02	1.83	81	2.20	20.26	438	77.60	15.69	6.71	14	1	58.58	40.60	55.71
p14			2.77	5.34	2.51	99	0.14	7.16	329	71.48	18.21	10.31	12	0	55.09	53.39	54.29
p15			2.77	2.28	3.00	111	1.43	5.63	235	73.29	19.15	7.56	11	0	59.65	56.93	58.56
p16			2.77	5.30	3.20	114	0.27	7.91	345	79.34	14.80	5.87	11	0	59.50	57.08	58.55
p17			2.80	9.75	2.61	106	1.85	9.53	336	68.90	22.99	8.11	12	0	60.00	59.00	59.48
p18			2.80	16.24	2.00	87	0.30	52.63	588	73.40	18.56	8.04	13	0	55.25	54.50	54.90
p19			2.80	20.46	1.84	86	1.58	136.78	681	81.20	13.46	5.34	13	1	56.75	45.25	54.90
p20			2.80	11.10	1.84	84	0.41	30.51	624	80.75	14.83	4.42	12	0	60.00	59.00	59.48
p21			3.50	2.65	3.31	120	2.65	5.87	251	74.17	15.12	10.71	11	1	54.00	36.60	51.91
p22			3.50	22.41	2.17	95	1.85	84.79	754	72.32	17.56	10.12	11	1	53.40	41.20	51.91
p23			3.50	18.67	2.32	98	1.93	46.44	570	74.13	18.73	7.14	10	0	57.60	56.60	57.10
p24			3.50	21.62	2.28	93	0.65	43.95	581	80.05	14.45	5.50	10	0	57.60	56.80	57.10
p25	30	150	4.12	14.93	288.51	562	1.17	2000.95	2193	67.91	15.64	16.45	12	1	60.56	43.94	54.31
p26			4.12	12.78	172.52	405	1.16	1701.58	2315	63.46	16.52	20.03	11	1	55.50	45.59	52.87
p27			4.12	11.38	231.13	443	0.84	1240.80	1977	68.22	15.87	15.92	10	0	58.94	54.45	56.14
p28			4.12	7.00	276.84	471	0.24	718.77	1508	75.04	12.45	12.51	10	0	58.94	54.45	56.14
p29			3.03	11.58	76.25	256	0.79	1114.36	1336	67.39	21.25	11.36	16	0	64.00	60.00	61.88
p30			3.03	7.05	62.68	237	0.31	467.92	1044	69.99	18.58	11.43	17	0	59.33	57.00	58.24
p31			3.03	7.71	73.63	251	0.19	525.55	1365	75.29	16.94	7.77	16	0	62.67	61.00	61.88
p32			3.03	7.63	85.91	263	0.22	781.14	1463	76.17	17.87	5.96	15	0	66.33	65.00	66.00
p33			4.04	9.55	144.69	362	1.25	1050.70	1629	63.39	18.86	17.75	13	0	60.50	54.25	57.12
p34			4.04	17.95	118.48	342	1.58	2891.43	1980	65.25	21.13	13.62	13	0	57.50	56.50	57.12
p35			4.04	23.09	122.35	329	0.19	3506.51	2012	75.79	14.72	9.49	13	0	57.25	57.00	57.12
p36			4.04	14.71	136.44	336	0.98	995.04	1419	74.71	18.08	7.21	12	0	62.25	61.50	61.88
p37			6.06	9.56	251.77	423	3.45	7088.95	2953	58.63	19.78	21.58	9	0	55.83	54.17	55.00
p38			6.06	18.20	205.31	407	3.68	2981.96	2186	59.23	22.58	18.19	9	0	55.50	54.83	55.00
p39			6.06	22.09	281.27	425	1.25	2758.78	2154	70.77	16.19	13.05	9	0	55.33	54.67	55.00
p40			6.06	24.96	264.02	394	0.38	2752.06	2158	77.21	12.61	10.17	9	0	55.33	54.67	55.00
p41	10	90	2.12	8.71	56.00	405	0.08	156.50	977	70.39	17.15	12.45	6	0	63.24	56.54	59.20
p42	20	80	4.99	15.86	47.77	318	0.83	126.30	1185	72.71	12.81	14.48	5	0	56.27	50.82	54.59
p43	30	70	8.28	11.80	39.13	283	4.09	140.81	1043	64.92	22.78	12.30	4	0	61.93	59.70	60.47
p44	10	90	1.76	6.42	40.06	343	0.11	86.03	732	69.67	21.13	9.20	8	0	85.71	57.59	65.18
p45	20	80	4.14	11.71	36.61	289	0.19	62.45	633	73.72	12.46	13.82	6	0	55.65	54.13	54.68
p46	30	70	7.10	22.99	19.53	203	0.59	81.28	772	69.41	14.66	15.93	5	0	57.61	55.27	56.42
p47	10	90	1.76	6.22	54.32	386	0.10	95.94	750	70.41	20.64	8.95	8	0	85.12	57.17	64.80
p48	20	80	4.06	10.19	44.68	320	0.48	80.81	727	71.24	16.21	12.56	6	0	58.70	56.36	57.46
p49	30	70	7.08	13.47	22.31	214	0.47	73.60	956	72.91	12.98	14.11	5	0	55.79	53.65	54.83
p50	10	100	1.89	3.97	68.83	404	0.33	150.54	850	64.95	18.49	16.56	7	0	71.28	57.38	61.91
p51	20	100	3.98	7.34	112.79	399	1.02	279.80	1091	66.09	17.00	16.91	6	0	62.18	56.54	58.57
p52	10	100	1.60	3.70	48.38	347	0.21	103.01	727	62.86	23.73	13.41	9	0	94.21	59.28	70.42
p53	20	100	3.37	0.23	60.64	309	0.23	60.64	309	70.39	13.15	16.46	8	0	60.87	52.28	56.48
p54	10	100	1.52	0.11	44.69	334	0.11	44.69	334	65.75	23.58	10.67	10	0	89.47	58.65	71.07
p55	20	100	3.21	13.82	50.34	286	0.49	284.23	1248	70.95	16.79	12.25	8	0	64.43	55.49	58.55
		Max	8.28	24.96	288.51	562	4.09	7088.95	2953	81.20	28.95	21.58	17	1	94.21	79.00	80.89
		Avg	3.32	10.55	65.46	243	0.88	635.28	925	70.04	18.34	11.62	10	0	64.32	56.88	60.38
		Min	1.37	0.07	1.56	81	0.07	1.95	101	58.20	12.45	4.42	4	0	53.40	36.60	51.91

Table 4.4: Statistics for dominant fixed costs-function type 1.

	Instance feature			Initial Lag heuristic			Enhanced search			Cost structure			Facility status		Capacity utilization		
	n	m	K/D	gap1%	time1	itr1	gap2%	time2	itr2	fixed%	prd%	var%	opn	une	max%	min%	avg%
p1	10	50	1.74	2.84	2.29	108	0.21	5.40	245	6.30	81.56	12.14	8	0	92.11	60.59	67.89
p2			1.74	2.38	2.23	107	0.42	5.51	248	4.47	83.16	12.37	8	0	92.11	60.86	67.89
p3			1.74	2.72	2.95	126	0.25	6.47	270	7.23	80.74	12.03	8	0	91.23	60.86	67.82
p4			1.74	3.02	2.57	113	0.16	5.80	253	9.84	78.46	11.70	8	0	91.23	60.86	67.81
p5			1.37	0.09	2.82	125	0.09	2.82	125	7.17	83.86	8.97	10	0	75.00	71.00	72.80
p6			1.37	0.04	2.67	123	0.04	2.67	123	5.10	85.72	9.18	10	0	74.50	71.00	72.80
p7			1.37	0.09	2.82	127	0.09	2.84	127	8.22	82.90	8.88	10	0	74.50	71.00	72.80
p8			1.37	0.04	2.56	120	0.04	2.56	120	11.14	80.26	8.60	10	0	74.50	71.50	72.80
p9			2.06	4.79	2.29	110	0.07	8.13	371	6.42	82.99	10.59	8	0	61.33	60.00	60.67
p10			2.06	3.35	2.43	114	0.10	5.38	253	4.69	84.54	10.78	8	0	61.67	59.67	60.67
p11			2.06	3.98	2.23	108	0.16	5.23	244	7.57	81.93	10.50	8	0	62.00	59.67	60.67
p12			2.06	3.13	2.14	107	0.20	5.02	244	10.28	79.55	10.17	8	0	61.67	57.67	60.67
p13	20	50	2.77	4.76	2.18	88	1.10	13.14	526	6.63	86.97	6.41	11	0	60.31	57.82	58.99
p14			2.77	1.79	2.37	97	1.10	7.53	343	4.01	89.08	6.91	11	0	59.61	57.08	58.55
p15			2.77	4.55	2.50	98	0.98	15.90	432	6.51	86.77	6.73	11	0	59.65	57.08	58.56
p16			2.77	4.59	2.34	95	0.83	15.96	441	8.88	84.58	6.54	11	0	59.65	56.88	58.56
p17			2.80	14.04	1.93	86	1.02	305.26	859	5.82	88.67	5.51	12	0	60.00	58.75	59.48
p18			2.80	8.38	1.89	88	1.17	165.24	681	4.25	90.33	5.42	12	0	60.00	58.75	59.48
p19			2.80	7.83	1.84	88	1.04	72.59	720	6.89	87.85	5.27	12	0	60.00	58.75	59.48
p20			2.80	8.13	1.98	89	0.85	76.83	707	9.39	85.50	5.11	12	0	60.00	59.00	59.48
p21			3.50	10.31	2.51	101	1.10	60.12	745	5.21	88.47	6.32	10	0	57.60	56.40	57.10
p22			3.50	4.34	2.29	97	1.14	19.02	489	3.88	89.90	6.22	10	0	57.40	56.80	57.10
p23			3.50	13.00	2.11	90	1.01	111.81	741	6.30	87.64	6.07	10	0	57.60	56.60	57.10
p24			3.50	13.40	2.65	101	0.90	86.86	882	8.60	85.49	5.91	10	0	57.60	56.80	57.10
p25	30	150	4.12	5.88	194.66	402	1.48	1532.99	2438	5.21	82.83	11.96	10	0	60.55	56.30	57.95
p26			4.12	6.67	168.92	368	1.51	2326.41	2181	3.52	84.33	12.16	10	0	60.55	56.30	57.95
p27			4.12	5.11	189.57	397	1.35	1214.33	1949	5.73	82.40	11.87	10	0	60.27	56.30	57.93
p28			4.12	5.30	173.83	373	1.24	1128.17	1872	7.84	80.53	11.63	10	0	60.32	56.30	57.93
p29			3.03	6.52	69.06	238	0.23	1450.86	1596	5.99	85.67	8.34	16	0	62.67	61.00	61.88
p30			3.03	4.91	68.39	236	0.25	1224.95	1571	4.72	87.16	8.11	16	0	62.67	61.00	61.88
p31			3.03	4.42	67.49	235	0.17	574.29	1355	7.63	84.49	7.87	16	0	62.33	61.33	61.88
p32			3.03	4.61	71.14	241	0.14	512.98	1077	10.37	81.99	7.64	16	0	62.33	61.33	61.88
p33			4.04	10.44	127.76	327	0.17	6650.85	2805	5.33	85.38	9.29	13	0	58.00	55.25	57.12
p34			4.04	7.76	129.96	313	0.16	1982.52	1694	4.29	86.74	8.96	13	0	57.50	56.50	57.12
p35			4.04	8.05	144.61	320	0.21	2057.31	1728	6.95	84.35	8.70	13	0	57.50	54.50	57.12
p36			4.04	8.35	139.84	322	0.45	1963.91	1706	9.45	82.12	8.43	13	0	57.75	51.25	57.12
p37			6.06	8.66	238.53	383	0.39	2739.16	2134	4.82	84.62	10.57	9	0	55.33	54.67	55.00
p38			6.06	9.55	239.32	390	0.49	2849.42	2121	3.49	85.79	10.72	9	0	55.33	54.67	55.00
p39			6.06	9.80	292.35	401	0.45	2764.17	2115	5.68	83.84	10.48	9	0	55.33	54.67	55.00
p40			6.06	10.69	250.60	394	0.39	2742.53	2133	7.78	81.98	10.24	9	0	55.33	54.67	55.00
p41	10	90	2.12	0.09	115.89	530	0.09	115.89	530	6.10	83.12	10.78	6	0	62.85	56.75	59.17
p42	20	80	4.99	6.43	44.48	292	1.86	126.30	1141	6.17	81.44	12.39	5	1	57.87	45.65	54.44
p43	30	70	8.28	6.06	55.01	293	2.36	131.46	946	5.63	83.70	10.67	4	0	61.68	59.32	60.47
p44	10	90	1.76	2.03	47.38	339	0.13	96.41	724	6.16	85.67	8.17	8	0	84.87	57.59	65.06
p45	20	80	4.14	7.40	37.53	277	0.85	87.77	912	6.26	81.86	11.87	6	0	56.18	52.55	54.64
p46	30	70	7.10	5.78	25.46	228	0.22	72.52	934	5.96	80.32	13.71	5	0	57.61	55.49	56.43
p47	10	90	1.76	1.91	49.09	358	0.10	109.48	773	6.22	85.93	7.85	8	0	85.95	57.17	64.90
p48	20	80	4.06	3.65	49.80	329	0.12	91.56	752	6.11	83.13	10.76	6	0	58.67	56.14	57.45
p49	30	70	7.08	7.80	24.55	225	0.53	71.60	948	6.13	82.67	11.20	5	0	57.53	55.70	56.51
p50	10	100	1.89	2.36	78.19	422	0.59	149.76	867	5.65	80.60	13.75	7	0	89.26	58.23	65.56
p51	20	100	3.98	4.01	125.52	409	0.36	295.65	1106	5.73	79.58	14.69	6	0	62.18	56.75	58.58
p52	10	100	1.60	2.09	41.64	310	0.15	96.88	676	5.68	82.25	12.07	9	0	94.21	58.86	70.46
p53	20	100	3.37	3.41	71.87	313	0.14	179.70	812	5.99	80.75	13.27	8	0	83.47	55.91	61.27
p54	10	100	1.52	0.09	63.26	377	0.09	63.26	377	5.96	84.38	9.66	10	0	89.47	58.65	70.88
p55	20	100	3.21	4.04	62.01	298	0.23	184.41	1071	6.12	83.30	10.57	8	0	64.43	56.12	58.54
		Max	8.28	14.04	292.35	530	2.36	6650.85	2805	11.14	90.33	14.69	16	1	94.21	71.50	72.80
		Avg	3.32	5.37	63.90	234	0.56	665.37	968	6.43	84.00	9.58	10	0	66.02	58.22	60.88
		Min	1.37	0.04	1.84	86	0.04	2.56	120	3.49	78.46	5.11	4	0	55.33	45.65	54.44

Table 4.5: Statistics for dominant production costs-function type 1.

	Instance feature			Initial Lag heuristic			Enhanced search			Cost structure			Facility status		Capacity utilization		
	n	m	K/D	gap1%	time1	itr1	gap2%	time2	itr2	fixed%	prd%	var%	opn	une	max%	min%	avg%
p1	10	50	1.74	0.07	2.09	115	0.07	5.09	321	6.38	16.42	77.20	10	3	100.00	33.13	62.95
p2			1.74	0.07	2.71	141	0.07	5.43	336	4.53	16.75	78.73	10	3	100.00	33.13	62.89
p3			1.74	0.03	3.06	156	0.03	5.85	357	7.33	16.32	76.36	10	3	100.00	33.13	62.87
p4			1.74	0.07	2.37	124	0.07	5.12	319	9.96	15.94	74.10	10	3	100.00	33.13	62.90
p5			1.37	0.10	1.90	109	0.10	1.90	109	6.07	22.59	71.34	10	0	88.50	61.50	72.80
p6			1.37	0.09	2.26	123	0.09	2.26	123	4.30	23.10	72.60	10	0	89.00	61.50	72.80
p7			1.37	0.04	1.67	101	0.04	1.68	101	6.97	22.45	70.58	10	0	88.50	61.00	72.80
p8			1.37	0.06	2.25	122	0.06	2.25	122	9.49	21.81	68.70	10	0	88.00	61.00	72.80
p9			2.06	0.10	2.29	130	0.10	6.79	440	7.25	10.26	82.49	10	4	62.33	31.33	48.53
p10			2.06	0.08	2.32	129	0.08	7.05	448	5.16	10.49	84.36	10	4	61.67	31.00	48.53
p11			2.06	0.07	2.25	125	0.07	6.44	420	8.31	10.12	81.57	10	4	62.00	30.67	48.53
p12			2.06	0.09	1.87	106	0.09	6.85	432	11.26	9.86	78.88	10	4	62.33	31.00	48.53
p13	20	50	2.77	0.07	1.01	66	0.07	10.23	691	15.00	17.58	67.42	18	12	53.69	18.24	40.02
p14			2.77	0.07	0.94	58	0.07	10.62	707	11.50	19.00	69.51	19	13	53.69	12.39	37.71
p15			2.77	0.07	0.97	59	0.07	8.83	604	15.99	16.93	67.08	17	10	53.69	27.06	41.98
p16			2.77	0.34	1.01	63	0.34	8.81	590	20.99	15.86	63.15	17	9	53.69	27.06	41.89
p17			2.80	0.05	0.97	59	0.05	10.94	722	14.94	17.51	67.55	18	12	53.75	14.00	39.65
p18			2.80	0.11	1.14	64	0.11	11.45	741	11.44	18.93	69.63	19	13	53.75	13.50	37.57
p19			2.80	0.09	0.97	59	0.09	8.55	554	15.92	16.88	67.20	17	9	54.00	27.00	41.99
p20			2.80	0.13	1.08	62	0.13	8.94	575	20.95	15.84	63.21	17	9	53.75	27.00	41.99
p21			3.50	0.04	1.36	78	0.04	17.14	1013	15.61	17.58	66.81	18	14	51.60	8.60	31.72
p22			3.50	0.09	1.26	74	0.09	17.05	1072	11.99	18.96	69.05	19	16	51.60	7.20	30.05
p23			3.50	0.10	1.22	73	0.10	14.88	948	16.63	16.90	66.47	17	13	51.60	8.60	33.59
p24			3.50	0.70	1.36	79	0.66	46.07	943	20.44	15.33	64.23	16	11	51.60	8.60	35.69
p25	30	150	4.12	2.06	95.22	334	0.86	4851.29	2687	6.06	9.31	84.63	14	1	64.68	33.26	52.17
p26			4.12	1.08	108.22	359	0.66	1942.84	2157	4.44	9.11	86.45	14	3	63.68	27.98	49.31
p27			4.12	1.42	99.84	348	0.67	930.86	1750	6.68	8.87	84.45	13	1	64.68	34.91	52.06
p28			4.12	1.79	90.39	315	0.64	939.63	1650	9.11	8.69	82.20	13	1	64.68	33.88	52.07
p29			3.03	1.55	54.76	253	0.39	2278.74	1771	7.72	10.53	81.75	19	3	58.33	39.33	52.11
p30			3.03	1.44	57.49	278	0.35	2145.34	1860	5.79	10.69	83.52	19	3	58.33	38.00	52.11
p31			3.03	2.00	59.44	284	0.35	2346.24	1871	9.28	10.32	80.40	19	3	58.33	39.67	52.11
p32			3.03	2.02	56.71	275	0.35	1196.12	1600	11.88	10.54	77.59	18	0	59.00	51.67	55.00
p33			4.04	2.54	82.51	328	0.40	7183.58	3083	6.79	9.10	84.11	15	2	53.25	23.50	49.50
p34			4.04	1.15	79.65	315	0.30	1811.33	2062	4.80	9.27	85.93	15	2	53.00	23.50	49.50
p35			4.04	1.68	87.33	311	0.49	1720.97	2448	7.74	9.02	83.24	15	3	53.00	23.50	49.50
p36			4.04	2.18	84.65	317	0.73	2158.79	2309	10.48	8.77	80.74	15	3	53.50	26.25	49.50
p37			6.06	0.72	201.94	492	0.72	1211.74	3480	5.48	8.09	86.44	12	7	51.33	14.83	41.25
p38			6.06	0.40	212.72	499	0.40	988.31	3083	3.97	8.23	87.80	12	7	51.00	15.67	41.25
p39			6.06	1.80	221.41	484	0.82	4559.68	3353	6.42	7.98	85.60	12	6	50.83	15.67	41.25
p40			6.06	1.86	224.14	473	1.22	2289.03	2860	8.03	7.51	84.46	11	3	51.17	14.50	45.00
p41	10	90	2.12	0.09	28.47	297	0.09	49.84	724	9.12	11.60	79.28	10	3	66.94	30.38	51.74
p42	20	80	4.99	1.06	9.13	169	0.33	159.67	1945	17.18	16.11	66.71	15	11	64.57	15.25	34.86
p43	30	70	8.28	7.75	3.62	82	0.45	927.43	1693	18.34	16.05	65.60	14	11	61.17	12.10	32.21
p44	10	90	1.76	0.06	32.01	322	0.06	36.04	426	8.54	16.52	74.94	10	1	89.92	49.16	62.52
p45	20	80	4.14	0.08	6.21	132	0.08	76.14	1625	19.12	18.59	62.29	17	11	65.05	10.75	34.02
p46	30	70	7.10	1.75	3.15	88	0.08	85.75	1438	21.44	18.90	59.66	16	11	59.22	6.08	34.55
p47	10	90	1.76	0.11	37.80	354	0.11	54.55	664	9.72	24.97	65.31	10	2	100.00	36.50	65.61
p48	20	80	4.06	3.10	4.56	127	1.10	331.85	1695	23.83	23.55	52.62	16	10	55.37	5.43	39.89
p49	30	70	7.08	2.61	3.00	89	0.51	295.67	2517	22.17	21.18	56.65	12	10	55.46	5.95	33.50
p50	10	100	1.89	0.09	48.78	367	0.09	58.06	521	5.10	11.37	83.53	9	1	100.00	36.08	62.14
p51	20	100	3.98	1.69	21.28	205	0.58	329.10	1463	8.59	10.18	81.23	12	5	64.71	21.43	47.67
p52	10	100	1.60	0.09	40.64	326	0.09	57.89	538	4.47	16.23	79.30	10	1	100.00	39.25	69.63
p53	20	100	3.37	0.07	21.18	202	0.07	87.16	997	7.63	12.37	80.00	13	4	99.17	16.80	51.20
p54	10	100	1.52	0.08	96.83	489	0.08	96.83	489	4.91	19.24	75.85	10	0	100.00	54.17	72.62
p55	20	100	3.21	2.32	13.95	183	0.63	284.14	1036	8.83	11.72	79.45	13	2	65.79	26.16	53.29
		Max	8.28	7.75	224.14	499	1.22	7183.58	3480	23.83	24.97	87.80	19	16	100.00	61.50	72.80
		Avg	3.32	0.90	40.50	207	0.30	758.45	1245	10.58	14.51	74.91	14	5	67.18	27.68	49.45
		Min	1.37	0.03	0.94	58	0.03	1.68	101	3.97	7.51	52.62	9	0	50.83	5.43	30.05

Table 4.6: Statistics for dominant variable costs-function type 1.

	Production cost function	Cost structure	
1	$h_j(x) = \begin{cases} 40x_j^{0.5} & 0 < x_j \le e_j \\ 20e_j^{0.5}(x - e_j)/(K_j - x) & e_j < x_j \le K_j \end{cases}$ $\forall j = 1, ..., n, e_j = 0.5K_j$	Original data from [18].	Table 4.8
2	Same as 1.	Fixed costs increased 10 times.	Table 4.9
3	$h_j(x) = \begin{cases} 400x_j^{0.5} & 0 < x_j \le e_j \\ 200e_j^{0.5}(x - e_j)/(K_j - x) & e_j < x_j \le K_j \end{cases}$ $\forall j = 1, ..., n, e_j = 0.5K_j$	Original data from [18].	Table 4.10
4	Same as 1.	Variable costs increased 10 times.	Table 4.11

Table 4.7: Cost structures for function type 2.

happens when fixed costs dominate while the minimum average gap happens when transportation costs dominate. Moreover, according to column *une* (the number of facilities operating under the economic point) when production costs dominate, no open facilities operate under the economic point. When fixed costs dominate, only 6 instances have one facility operating under the economic point. When transportation costs dominate, 49 of the 55 instances have at least one facility operating under the economic point. This is due to transportation costs being more important than other costs, thus diminishing the importance of economies of scale.

4.2.2 Performance of Function Type 2

For function type 2, Table 4.7 shows the production cost functions and cost structures.

The results from Table 4.8, 4.9, 4.10 and 4.11 show similar characteristics to function type 1. First of all, the enhanced search always gives a better solution

	Instance feature			Initial Lag heuristic			Enhanced search			Cost structure			Facility status		Capacity utilization		
	n	m	K/D	gap1%	time1	itr1	gap2%	time2	itr2	fixed%	prd%	var%	opn	une	max%	min%	avg%
p1	10	50	1.74	3.64	3.37	115	0.03	9.84	315	22.87	34.71	42.42	8	0	77.91	65.73	72.45
p2			1.74	4.39	2.34	102	0.66	19.61	542	17.69	37.11	45.20	8	0	77.52	64.49	72.53
p3			1.74	6.78	2.96	121	0.59	14.07	391	23.03	33.76	43.21	7	0	81.01	72.59	76.97
p4			1.74	4.31	3.09	121	0.20	7.33	284	29.51	30.88	39.60	7	0	80.23	72.59	77.02
p5			1.37	0.07	2.42	106	0.07	2.42	106	28.12	37.57	34.31	10	0	78.00	67.00	72.80
p6			1.37	0.05	2.54	106	0.05	2.54	106	21.39	41.14	37.47	10	0	79.00	66.50	72.80
p7			1.37	0.24	2.51	108	0.24	2.51	108	31.15	36.02	32.83	10	0	78.50	64.50	72.80
p8			1.37	1.48	2.31	101	1.48	5.96	248	38.39	32.54	29.07	10	1	80.00	48.50	72.80
p9			2.06	2.86	2.62	114	1.32	11.09	445	21.29	34.47	44.24	7	0	76.33	55.67	69.33
p10			2.06	3.98	2.62	115	1.22	13.32	392	19.34	36.78	43.89	8	1	71.67	38.00	60.67
p11			2.06	4.15	2.62	112	0.96	10.62	422	25.19	33.46	41.36	7	0	78.67	57.33	69.33
p12			2.06	6.91	2.98	114	2.04	15.10	429	31.67	30.51	37.82	7	1	79.00	47.00	69.33
p13	20	50	2.77	7.56	2.04	67	1.70	27.27	366	25.81	46.78	27.41	10	0	76.11	66.58	72.50
p14			2.77	9.79	2.25	72	1.79	61.21	540	19.47	48.67	31.85	10	0	75.12	50.62	66.80
p15			2.77	13.21	2.23	76	0.82	63.59	536	26.12	44.23	29.65	9	0	75.84	69.50	73.24
p16			2.77	8.48	2.67	87	0.43	16.43	467	33.10	40.03	26.87	9	0	76.08	68.44	73.22
p17			2.80	12.65	2.46	85	1.40	96.91	618	24.51	47.27	28.22	10	0	77.50	58.75	71.38
p18			2.80	7.26	2.01	72	1.84	24.93	381	19.34	50.47	30.19	10	0	77.25	66.75	71.38
p19			2.80	10.93	2.29	79	1.41	58.36	497	28.50	44.76	26.74	10	0	77.25	63.00	71.38
p20			2.80	10.07	2.28	80	2.36	34.35	543	35.49	40.64	23.87	10	1	77.75	48.50	71.38
p21			3.50	1.93	3.79	114	0.92	7.97	273	21.72	45.85	32.42	8	0	75.60	65.00	71.38
p22			3.50	11.21	3.40	103	0.70	121.18	808	19.08	48.33	32.59	9	1	70.60	49.20	63.44
p23			3.50	11.41	3.85	111	0.78	51.07	657	25.15	43.96	30.89	8	0	75.80	66.40	71.38
p24			3.50	14.63	4.10	115	0.29	61.04	806	31.99	39.94	28.07	8	0	75.60	66.00	71.38
p25	30	150	4.12	2.80	415.46	600	0.24	1178.24	1420	18.44	37.30	44.26	9	0	74.26	66.29	70.63
p26			4.12	3.17	323.12	549	0.55	921.54	1716	16.00	39.77	44.23	10	0	70.96	56.37	66.94
p27			4.12	3.19	342.87	571	0.26	626.30	1212	21.90	36.11	41.99	9	0	73.62	63.32	70.81
p28			4.12	1.27	607.66	663	1.27	607.66	663	27.95	33.03	39.02	9	0	75.46	51.74	69.50
p29			3.03	8.15	106.47	339	1.06	1965.58	1746	22.22	41.05	36.73	14	0	79.67	50.00	70.71
p30			3.03	6.56	114.02	362	0.59	2397.97	2177	19.35	43.03	37.61	14	0	73.67	60.33	70.71
p31			3.03	9.35	99.65	325	0.69	2597.20	2097	26.53	40.30	33.16	13	0	78.00	74.33	76.15
p32			3.03	11.88	93.85	302	0.34	3004.88	1997	33.59	36.43	29.98	13	0	77.33	74.67	76.15
p33			4.04	9.21	198.53	443	2.68	6081.45	3073	20.40	39.52	40.07	11	1	73.50	32.25	67.50
p34			4.04	6.35	171.27	419	1.58	2338.17	2471	16.89	41.52	41.60	11	1	72.75	38.25	67.50
p35			4.04	5.16	182.52	432	1.08	1407.10	2112	23.16	38.70	38.13	10	0	78.25	72.00	74.25
p36			4.04	12.72	205.64	456	1.15	1886.79	2644	29.65	35.46	34.89	10	0	78.50	68.75	74.25
p37			6.06	5.59	460.94	595	1.89	6954.52	3801	15.70	36.37	47.93	7	0	77.83	67.33	70.71
p38			6.06	8.80	368.18	583	1.18	1941.21	3047	13.97	37.62	48.41	8	1	65.67	49.83	61.88
p39			6.06	2.56	540.78	654	1.85	2659.93	2795	18.51	35.34	46.14	7	0	74.67	58.83	70.71
p40			6.06	4.26	486.96	644	1.91	2531.71	2786	24.03	32.86	43.11	7	0	78.33	61.33	70.71
p41	10	90	2.12	6.10	45.93	346	0.69	562.71	2056	20.84	32.64	46.51	5	0	70.78	61.83	66.36
p42	20	80	4.99	3.03	32.42	240	3.03	58.06	593	23.39	34.69	41.92	5	1	68.72	40.47	62.18
p43	30	70	8.28	19.94	97.13	476	0.92	395.31	2501	22.90	34.20	42.90	4	0	68.78	59.27	63.12
p44	10	90	1.76	0.82	71.68	406	0.82	71.68	406	26.27	38.87	34.86	8	0	72.27	52.89	65.58
p45	20	80	4.14	8.91	18.88	189	2.34	202.55	1593	27.20	38.41	34.40	7	2	72.73	47.03	60.56
p46	30	70	7.10	22.49	40.90	321	1.58	350.63	2689	27.16	38.39	34.45	6	1	70.43	34.26	58.66
p47	10	90	1.76	4.23	42.28	367	0.48	118.14	883	29.20	41.28	29.53	9	0	73.55	52.69	64.64
p48	20	80	4.06	15.53	14.15	169	0.69	646.66	1666	27.27	39.09	33.64	7	0	67.51	52.43	61.88
p49	30	70	7.08	1.25	44.27	343	1.25	138.61	1237	28.60	38.54	32.86	6	2	64.52	33.83	54.88
p50	10	100	1.89	4.30	83.63	456	0.13	358.93	1196	20.05	33.28	46.67	7	0	74.47	66.90	71.83
p51	20	100	3.98	3.14	75.79	358	1.49	278.06	1103	20.45	36.14	43.41	6	0	77.47	70.78	73.27
p52	10	100	1.60	3.45	74.37	424	0.14	425.57	1362	19.37	37.91	42.72	8	0	81.03	73.95	78.39
p53	20	100	3.37	9.65	46.08	266	2.27	739.26	1355	21.95	36.14	41.91	8	1	76.28	37.63	68.74
p54	10	100	1.52	0.07	92.59	496	0.07	92.59	496	21.03	36.16	42.81	8	0	78.25	64.58	72.50
p55	20	100	3.21	9.37	40.97	258	0.45	501.42	1727	22.21	38.17	39.62	7	0	75.65	63.47	71.05
		Max	8.28	22.49	607.66	663	3.03	6954.52	3801	38.39	50.47	48.41	14	2	81.03	74.67	78.39
		Avg	3.32	6.75	101.90	281	1.05	814.17	1216	24.11	38.51	37.38	9	0	75.33	58.48	69.64
		Min	1.37	0.05	2.01	67	0.03	2.42	106	13.97	30.51	23.87	4	0	64.52	32.25	54.88

Table 4.8: Statistics for the basic case-function type 2.

	Instance feature			Initial Lag heuristic			Enhanced search			Cost structure			Facility status		Capacity utilization		
	n	m	K/D	gap1%	time1	itr1	gap2%	time2	itr2	fixed%	prd%	var%	opn	une	max%	min%	avg%
p1	10	50	1.74	9.72	4.66	152	4.14	14.99	443	51.29	23.50	25.22	5	0	92.57	92.10	92.32
p2			1.74	1.40	5.34	178	1.40	5.34	178	57.58	17.97	24.45	6	0	85.27	68.00	81.09
p3			1.74	7.94	4.52	152	4.59	23.54	553	56.97	23.12	19.91	5	0	93.30	92.57	92.96
p4			1.74	10.09	4.35	148	2.58	17.49	485	64.96	18.83	16.21	5	0	93.23	92.71	92.98
p5			1.37	6.61	2.34	106	1.95	5.35	235	64.49	21.27	14.24	8	0	91.50	90.50	91.00
p6			1.37	11.05	1.97	97	3.29	10.12	320	67.71	18.50	13.79	9	1	86.00	47.00	80.89
p7			1.37	6.59	2.22	103	1.16	6.71	270	71.39	18.73	9.88	8	0	91.50	90.50	91.00
p8			1.37	17.54	2.11	100	0.32	12.28	347	77.74	14.57	7.69	8	0	91.50	90.50	91.00
p9			2.06	4.85	6.13	188	4.85	6.13	188	63.82	16.53	19.65	6	0	85.67	58.33	80.89
p10			2.06	0.83	3.99	141	0.83	3.99	141	60.81	18.41	20.78	6	0	84.67	72.33	80.89
p11			2.06	4.12	3.40	123	4.12	3.40	123	70.57	14.54	14.88	6	0	87.00	54.33	80.89
p12			2.06	6.38	3.09	118	6.38	3.93	181	76.00	12.40	11.60	6	1	88.67	44.67	80.89
p13	20	50	2.77	29.70	2.34	78	0.68	34.24	429	69.09	21.26	9.65	9	0	86.95	84.47	85.95
p14			2.77	2.27	4.56	123	2.27	4.56	123	61.92	22.59	15.49	8	0	83.26	55.91	79.18
p15			2.77	6.08	6.27	155	1.61	10.62	319	65.09	22.88	12.03	7	0	89.98	89.12	89.65
p16			2.77	8.71	6.33	153	0.50	12.79	360	72.31	18.16	9.53	7	0	89.93	89.12	89.64
p17			2.80	16.07	4.38	124	2.10	14.65	390	57.91	26.70	15.40	8	0	89.50	89.00	89.22
p18			2.80	16.37	2.93	93	2.70	22.64	327	65.50	22.75	11.75	9	0	83.75	52.75	79.31
p19			2.80	21.81	2.76	90	1.26	32.12	422	69.24	22.09	8.67	8	0	89.50	88.75	89.22
p20			2.80	35.00	2.76	90	0.41	70.29	517	75.91	17.30	6.79	8	0	89.50	88.75	89.22
p21			3.50	0.55	6.96	173	0.55	6.96	173	58.78	21.06	20.17	7	0	83.60	72.80	81.57
p22			3.50	23.75	2.64	86	1.15	13.45	287	59.96	23.70	16.35	7	0	84.20	77.60	81.57
p23			3.50	52.02	2.62	82	2.35	136.31	654	70.64	17.58	11.79	7	0	85.80	71.80	81.57
p24			3.50	46.86	2.59	84	4.25	67.36	572	76.18	14.59	9.23	7	0	87.60	60.60	81.57
p25	30	150	4.12	8.46	932.17	871	1.63	3013.78	2177	56.91	21.81	21.28	7	0	88.83	85.21	86.84
p26			4.12	7.39	672.99	705	0.30	2156.59	2006	52.81	21.39	25.80	7	0	83.82	77.46	81.84
p27			4.12	10.89	955.69	846	1.80	2150.58	1978	64.41	16.78	18.81	7	0	86.03	64.75	82.12
p28			4.12	25.13	1055.14	769	3.13	8695.09	3579	71.21	13.81	14.99	7	0	87.50	67.46	82.07
p29			3.03	21.29	167.58	344	1.95	1960.67	1879	64.37	19.20	16.42	12	0	87.67	55.00	82.50
p30			3.03	11.55	145.21	353	0.81	534.44	1118	64.59	20.92	14.49	12	0	84.67	68.67	82.50
p31			3.03	15.79	139.01	331	3.12	384.90	1069	74.21	15.76	10.04	12	0	87.33	62.67	82.50
p32			3.03	11.86	133.58	315	5.00	267.15	706	79.23	13.12	7.64	12	0	89.67	58.67	82.50
p33			4.04	6.65	308.07	433	0.91	991.82	1271	59.89	19.71	20.40	9	0	86.25	69.75	82.50
p34			4.04	21.02	293.61	453	0.60	3323.79	2612	59.67	22.00	18.32	9	0	84.25	74.75	82.50
p35			4.04	19.34	328.40	473	2.93	2491.82	2350	69.95	17.19	12.86	9	0	86.50	52.75	82.50
p36			4.04	22.46	437.75	495	3.50	2897.87	2125	68.00	21.90	10.10	8	0	93.25	92.25	92.81
p37			6.06	10.57	1917.69	723	0.12	8838.16	2541	54.69	19.77	25.54	6	0	83.83	79.83	82.50
p38			6.06	7.72	1262.22	777	0.35	6275.27	2472	51.39	22.94	25.66	6	0	84.00	81.33	82.50
p39			6.06	11.62	1297.40	608	1.03	3435.50	1906	63.33	17.53	19.14	6	0	85.50	72.50	82.50
p40			6.06	14.49	1361.87	562	2.39	2643.31	1706	70.14	14.58	15.28	6	0	86.83	67.50	82.50
p41	10	90	2.12	17.73	104.96	540	1.82	395.84	1613	64.15	15.76	20.08	4	0	85.44	70.13	81.02
p42	20	80	4.99	49.02	51.31	304	1.61	1093.75	1804	57.44	18.79	23.77	3	0	88.98	87.20	87.98
p43	30	70	8.28	2.04	64.87	347	2.04	64.87	347	64.39	15.00	20.61	3	0	84.27	71.57	79.84
p44	10	90	1.76	3.70	182.61	695	3.70	182.61	695	65.19	15.77	19.04	5	0	85.42	56.17	78.80
p45	20	80	4.14	34.03	50.15	299	5.03	200.02	1048	62.57	14.59	22.84	4	0	84.95	56.36	77.44
p46	30	70	7.10	41.37	20.61	179	8.61	216.22	1287	63.75	15.09	21.16	4	1	85.84	44.21	75.00
p47	10	90	1.76	4.90	135.07	627	4.90	142.29	772	64.33	16.11	19.56	5	1	86.02	48.70	78.09
p48	20	80	4.06	15.44	106.33	445	0.22	345.60	1452	62.60	18.41	18.99	4	0	87.13	86.07	86.37
p49	30	70	7.08	33.80	80.29	417	4.12	557.24	1675	55.94	24.03	20.02	3	0	91.59	91.10	91.38
p50	10	100	1.89	8.27	166.47	611	4.38	1493.58	2204	50.67	22.80	26.53	4	0	92.19	91.23	91.87
p51	20	100	3.98	27.73	216.97	499	3.67	1863.20	2289	55.68	23.90	20.42	4	0	91.95	91.03	91.41
p52	10	100	1.60	11.60	142.77	577	0.31	371.49	1364	61.53	19.14	19.33	6	0	88.44	86.78	87.78
p53	20	100	3.37	10.47	123.30	415	2.53	515.24	1885	59.06	15.13	25.81	5	0	86.23	65.61	81.69
p54	10	100	1.52	0.41	164.24	630	0.41	164.24	630	64.19	16.97	18.84	6	0	86.36	79.17	84.50
p55	20	100	3.21	17.28	105.66	387	4.34	315.56	1528	68.11	17.28	14.61	6	0	87.04	53.42	80.27
		Max	8.28	52.02	1917.69	871	8.61	8838.16	3579	79.23	26.70	26.53	12	1	93.30	92.71	92.98
		Avg	3.32	15.46	240.28	345	2.41	1064.03	1093	64.19	18.84	16.97	7	0	87.50	73.15	84.38
		Min	1.37	0.41	1.97	78	0.12	3.40	123	50.67	12.40	6.79	3	0	83.26	44.21	75.00

Table 4.9: Statistics for dominant fixed costs-function type 2.

	Instance feature			Initial Lag heuristic			Enhanced search			Cost structure			Facility status		Capacity utilization		
	n	m	K/D	gap1%	time1	itr1	gap2%	time2	itr2	fixed%	prd%	var%	opn	une	max%	min%	avg%
p1	10	50	1.74	1.96	3.84	135	0.07	8.13	296	5.63	80.27	14.10	7	0	70.54	66.87	68.93
p2			1.74	4.06	3.68	132	0.10	17.14	461	3.83	81.79	14.38	7	0	70.16	66.87	68.95
p3			1.74	2.20	3.59	130	0.11	7.10	276	6.23	79.72	14.05	7	0	70.16	67.48	68.95
p4			1.74	0.09	3.92	137	0.09	3.92	137	8.51	77.79	13.69	7	0	70.16	67.48	68.92
p5			1.37	0.08	3.78	152	0.08	3.78	152	6.48	85.35	8.16	10	0	73.50	72.00	72.80
p6			1.37	0.09	3.24	133	0.09	3.24	133	4.60	87.07	8.33	10	0	74.00	71.50	72.80
p7			1.37	0.08	2.95	126	0.08	2.95	126	7.44	84.46	8.11	10	0	73.50	72.00	72.80
p8			1.37	0.09	2.71	119	0.09	2.71	119	10.11	82.02	7.87	10	0	74.00	72.00	72.80
p9			2.06	1.54	2.76	118	1.54	5.91	258	6.20	83.36	10.44	8	1	65.00	46.33	60.67
p10			2.06	4.25	2.51	111	1.37	11.56	381	4.53	84.88	10.59	8	1	65.00	47.33	60.67
p11			2.06	4.84	2.68	116	1.60	13.96	388	6.40	82.59	11.01	7	0	71.33	68.00	69.33
p12			2.06	2.21	2.79	121	1.38	7.22	296	8.74	80.52	10.74	7	0	71.00	68.00	69.33
p13	20	50	2.77	6.49	3.32	102	0.35	18.92	438	6.30	86.78	6.91	10	0	65.88	56.91	63.95
p14			2.77	5.80	3.37	102	0.14	16.85	436	3.75	89.16	7.10	10	0	65.22	60.14	63.97
p15			2.77	5.95	3.99	115	0.33	18.14	447	6.09	86.99	6.92	10	0	65.67	56.91	63.96
p16			2.77	4.31	3.68	108	0.61	9.70	360	8.30	84.95	6.75	10	0	66.74	53.23	63.94
p17			2.80	10.38	2.98	99	0.30	108.65	802	5.39	89.02	5.59	11	0	65.50	63.75	64.89
p18			2.80	10.01	2.76	96	0.31	101.40	691	3.96	90.51	5.54	11	0	66.25	61.75	64.89
p19			2.80	9.06	2.65	92	0.44	48.10	604	6.42	88.21	5.38	11	0	66.25	57.25	64.89
p20			2.80	16.66	2.59	89	0.30	260.66	917	8.77	86.00	5.23	11	0	66.00	60.00	64.89
p21			3.50	10.90	3.24	100	0.21	89.05	673	4.80	88.73	6.47	9	0	64.80	59.80	63.44
p22			3.50	7.39	3.76	111	0.39	30.15	742	3.60	89.99	6.41	9	0	65.20	57.80	63.44
p23			3.50	9.62	2.82	93	0.50	73.82	823	5.85	87.85	6.29	9	0	65.20	57.40	63.44
p24			3.50	8.13	3.10	100	0.89	22.40	496	7.98	85.90	6.12	9	0	65.80	50.80	63.44
p25	30	150	4.12	5.42	384.75	540	0.37	4383.50	2615	4.75	83.34	11.90	9	0	65.83	59.18	64.48
p26			4.12	1.20	500.50	645	1.20	678.07	1125	3.28	83.67	13.06	9	1	66.06	43.33	62.32
p27			4.12	1.44	657.11	690	1.44	1228.54	1519	5.34	81.93	12.73	9	1	66.36	40.59	62.33
p28			4.12	1.70	614.66	681	1.43	1407.97	1695	6.51	80.89	12.61	8	0	71.51	69.24	70.23
p29			3.03	5.19	103.29	313	0.17	1307.88	1586	5.41	86.37	8.22	15	0	67.67	65.00	66.00
p30			3.03	3.71	97.19	303	0.26	744.20	1568	4.35	87.71	7.94	15	0	66.67	65.33	66.00
p31			3.03	4.10	106.00	321	0.19	837.62	1582	7.04	85.25	7.71	15	0	66.67	65.33	66.00
p32			3.03	4.65	95.43	300	0.16	821.63	1570	9.59	82.92	7.49	15	0	66.67	65.33	66.00
p33			4.04	6.76	172.76	366	0.76	2399.79	2020	4.60	85.94	9.46	11	0	68.50	64.75	67.50
p34			4.04	4.24	133.33	320	1.10	1185.31	1499	3.96	87.11	8.93	12	1	65.25	47.25	61.88
p35			4.04	3.46	163.36	364	0.97	767.01	1136	5.91	84.94	9.15	11	0	70.00	66.25	67.50
p36			4.04	3.70	148.56	337	0.85	756.45	1143	8.08	82.96	8.96	11	0	69.75	66.25	67.50
p37			6.06	13.64	411.83	473	3.36	4156.97	2990	4.46	84.56	10.97	8	0	64.17	52.83	61.88
p38			6.06	35.50	349.24	408	8.18	1871.06	2264	2.74	85.63	11.63	7	0	71.33	69.67	70.71
p39			6.06	37.29	427.30	425	7.92	2941.93	2523	4.48	84.12	11.40	7	0	71.33	69.50	70.71
p40			6.06	48.49	390.11	417	9.17	4278.72	2877	6.17	82.63	11.20	7	0	71.50	69.67	70.71
p41	10	90	2.12	0.09	178.42	634	0.09	178.42	634	5.32	82.67	12.01	5	0	67.72	65.05	66.20
p42	20	80	4.99	4.72	96.78	430	0.14	266.75	1282	5.19	81.05	13.76	4	0	66.74	65.07	66.06
p43	30	70	8.28	0.85	66.50	361	0.85	66.50	361	5.79	82.87	11.34	4	0	62.72	54.79	60.35
p44	10	90	1.76	2.05	86.97	465	0.48	199.88	1061	4.97	83.97	11.05	6	0	73.15	67.59	70.61
p45	20	80	4.14	38.33	36.66	257	4.29	335.37	2052	5.37	80.89	13.74	5	0	64.84	62.63	63.98
p46	30	70	7.10	3.83	39.76	285	1.44	193.04	1114	4.81	78.84	16.35	4	0	69.49	67.09	68.63
p47	10	90	1.76	0.79	80.96	456	0.79	80.96	456	5.63	84.27	10.09	7	0	68.35	52.96	63.72
p48	20	80	4.06	4.01	70.20	367	0.50	133.37	909	5.18	82.64	12.18	5	0	68.78	66.67	67.37
p49	30	70	7.08	5.30	59.16	337	0.64	128.25	985	5.10	82.30	12.60	4	0	68.43	67.09	67.86
p50	10	100	1.89	0.15	126.03	522	0.15	126.03	522	5.12	78.54	16.34	6	0	67.47	59.68	65.51
p51	20	100	3.98	58.90	78.37	331	13.14	600.04	1779	4.73	80.06	15.21	5	0	71.23	69.23	70.44
p52	10	100	1.60	0.96	90.95	420	0.96	227.23	1006	5.14	80.69	14.17	8	1	71.73	44.68	66.62
p53	20	100	3.37	3.95	91.85	358	0.25	463.60	1453	4.84	77.13	18.03	6	0	68.64	66.67	67.32
p54	10	100	1.52	0.10	131.51	489	0.10	131.51	489	5.05	84.16	10.80	8	0	73.63	64.46	70.82
p55	20	100	3.21	2.89	113.49	414	0.10	313.97	1038	5.46	83.85	10.69	7	0	67.80	64.03	65.67
		Max	8.28	58.90	657.11	690	13.14	4383.50	2990	10.11	90.51	18.03	15	1	74.00	72.00	72.80
		Avg	3.32	7.88	112.36	287	1.32	619.95	1006	5.71	83.96	10.33	9	0	68.31	61.80	66.53
		Min	1.37	0.08	2.51	89	0.07	2.71	119	2.74	77.13	5.23	4	0	62.72	40.59	60.35

Table 4.10: Statistics for dominant production costs-function type 2.

	Instance feature			Initial Lag heuristic			Enhanced search			Cost structure			Facility status		Capacity utilization		
	n	m	K/D	gap1%	time1	itr1	gap2%	time2	itr2	fixed%	prd%	var%	opn	une	max%	min%	avg%
p1	10	50	1.74	0.08	4.06	162	0.08	9.38	459	7.10	10.21	82.69	10	4	88.60	29.61	61.47
p2			1.74	0.09	4.06	172	0.09	10.87	520	5.04	10.46	84.50	10	4	88.60	27.10	61.71
p3			1.74	0.05	4.34	174	0.05	10.28	484	8.13	10.10	81.77	10	4	88.60	27.10	61.83
p4			1.74	0.08	3.45	142	0.08	10.31	468	11.02	9.64	79.34	10	4	88.60	29.61	61.42
p5			1.37	0.03	3.59	140	0.03	5.10	217	7.11	11.50	81.39	10	1	89.00	49.00	72.80
p6			1.37	0.08	2.87	124	0.08	2.87	124	5.06	11.71	83.24	10	0	89.00	50.00	72.80
p7			1.37	0.03	2.90	126	0.03	4.20	194	8.15	11.36	80.48	10	1	89.00	49.50	72.80
p8			1.37	0.07	3.34	139	0.07	4.73	211	11.05	10.99	77.96	10	1	89.00	49.00	72.80
p9			2.06	0.07	3.43	154	0.07	12.14	615	7.61	8.81	83.58	10	5	75.00	17.67	48.53
p10			2.06	0.06	2.89	138	0.06	12.17	621	5.42	9.01	85.57	10	5	74.67	17.67	48.53
p11			2.06	0.03	3.29	144	0.03	14.20	684	8.72	8.64	82.64	10	5	71.67	17.67	48.53
p12			2.06	0.09	3.04	138	0.09	13.01	668	11.79	8.38	79.83	10	5	73.33	17.67	48.53
p13	20	50	2.77	0.11	1.59	70	0.11	19.95	946	15.69	18.48	65.82	18	13	76.66	9.23	40.70
p14			2.77	0.29	1.73	78	0.19	49.28	984	11.40	19.55	69.05	18	13	76.66	9.23	40.40
p15			2.77	0.11	1.79	77	0.11	18.33	871	16.71	17.78	65.50	17	11	76.66	9.23	42.73
p16			2.77	1.05	1.95	80	0.25	56.33	866	20.64	16.47	62.89	16	9	76.66	9.23	44.88
p17			2.80	0.10	1.73	73	0.10	61.11	1052	15.70	18.51	65.79	18	13	76.25	9.00	39.65
p18			2.80	0.57	1.95	80	0.09	71.60	1076	11.40	19.58	69.02	18	13	76.25	9.00	39.65
p19			2.80	0.06	2.17	87	0.06	31.57	1013	16.72	17.81	65.47	17	12	76.25	9.00	41.99
p20			2.80	0.73	2.20	80	0.51	73.62	1001	20.57	16.57	62.85	16	10	78.50	9.00	44.61
p21			3.50	0.50	2.04	85	0.20	49.20	1343	15.23	17.26	67.51	17	14	62.80	7.20	33.59
p22			3.50	0.01	1.98	87	0.01	29.27	1338	11.53	18.77	69.70	18	15	62.80	7.20	31.72
p23			3.50	0.95	2.04	86	0.32	75.77	1382	15.87	16.54	67.59	16	12	66.80	7.20	35.69
p24			3.50	1.40	2.43	97	0.03	50.48	1230	20.90	15.55	63.55	16	12	66.80	7.20	35.69
p25	30	150	4.12	1.96	196.92	506	0.64	4692.78	3393	4.79	8.78	86.43	11	3	80.60	21.11	63.73
p26			4.12	1.21	190.20	521	0.67	3780.50	3250	3.25	8.67	88.08	10	0	80.60	50.97	69.20
p27			4.12	2.60	157.64	416	1.10	2667.26	2739	5.31	8.56	86.14	10	1	81.09	48.07	69.26
p28			4.12	1.38	340.33	654	1.08	1827.94	2332	7.25	8.41	84.35	10	1	80.60	40.35	69.33
p29			3.03	2.98	141.54	319	0.57	7570.96	2823	6.29	10.37	83.34	14	1	82.67	31.33	70.71
p30			3.03	2.42	94.91	330	0.96	5208.41	2952	4.46	10.45	85.09	14	1	82.33	31.33	70.71
p31			3.03	2.10	92.37	325	0.76	1810.38	1918	7.20	10.34	82.45	14	2	83.00	31.33	70.71
p32			3.03	3.62	97.55	333	0.83	3540.83	2332	9.10	10.40	80.50	13	1	83.33	45.67	76.15
p33			4.04	2.98	271.80	547	0.87	7634.44	3543	5.43	8.98	85.59	12	2	76.00	23.50	61.88
p34			4.04	1.84	187.50	512	0.51	3971.16	3115	3.93	9.03	87.04	12	2	75.50	23.50	61.88
p35			4.04	1.54	168.68	436	0.56	1700.38	1986	6.38	8.87	84.75	12	2	76.25	24.00	61.88
p36			4.04	1.74	219.65	523	0.92	969.83	1669	7.97	8.76	83.27	11	1	83.25	31.25	67.50
p37			6.06	0.99	344.84	670	0.45	4097.68	3737	4.41	7.64	87.95	10	5	71.50	7.83	49.50
p38			6.06	1.19	201.46	529	0.45	3233.09	3310	3.33	7.84	88.82	10	5	71.17	28.50	49.50
p39			6.06	1.62	269.38	591	0.80	2107.78	3353	4.88	7.32	87.80	9	3	69.83	34.33	55.00
p40			6.06	1.33	361.45	691	1.01	2522.99	3171	6.68	7.24	86.09	9	3	71.17	30.83	55.00
p41	10	90	2.12	0.20	30.45	312	0.20	74.05	1001	9.36	12.10	78.54	10	4	81.82	24.68	55.82
p42	20	80	4.99	2.41	10.50	158	0.68	450.34	2079	16.05	15.25	68.70	14	9	67.72	14.58	37.64
p43	30	70	8.28	4.91	5.38	95	0.30	476.32	1755	18.22	16.36	65.42	14	9	66.42	12.10	35.22
p44	10	90	1.76	0.08	41.67	351	0.08	63.73	767	9.05	14.06	76.89	10	3	87.39	43.52	64.43
p45	20	80	4.14	5.33	9.08	144	0.49	1161.02	2304	18.65	18.33	63.02	16	11	79.79	8.06	37.11
p46	30	70	7.10	0.10	5.63	116	0.10	59.34	1482	21.48	19.27	59.25	16	11	70.87	6.08	35.93
p47	10	90	1.76	0.08	227.75	918	0.08	250.29	1314	10.89	19.12	69.98	10	3	90.08	27.43	66.27
p48	20	80	4.06	1.36	6.21	124	1.36	561.01	1954	23.84	23.37	52.79	16	10	66.14	5.43	39.93
p49	30	70	7.08	1.04	5.26	102	1.04	210.98	2302	22.12	20.94	56.94	12	10	68.91	5.95	33.97
p50	10	100	1.89	0.31	62.81	387	0.31	114.49	853	5.33	8.08	86.59	9	2	88.43	28.48	62.53
p51	20	100	3.98	1.63	36.47	264	1.26	426.74	1859	8.05	9.98	81.97	11	4	76.28	16.27	52.07
p52	10	100	1.60	0.61	98.53	455	0.08	242.82	1046	4.15	9.98	85.87	8	0	91.74	66.46	81.57
p53	20	100	3.37	0.38	37.81	258	0.38	178.14	1483	7.24	10.43	82.33	12	5	89.26	16.67	56.83
p54	10	100	1.52	0.09	176.80	700	0.09	216.36	1003	5.63	10.20	84.18	10	1	89.26	34.72	70.94
p55	20	100	3.21	1.50	37.30	310	0.59	132.21	928	7.61	10.63	81.77	11	1	78.95	26.09	61.04
		Max	8.28	5.33	361.45	918	1.36	7634.44	3737	23.84	23.37	88.82	18	15	91.74	66.46	81.57
		Avg	3.32	1.06	76.23	279	0.40	1139.09	1566	10.31	12.61	77.08	12	5	78.55	24.07	54.84
		Min	1.37	0.01	1.59	70	0.01	2.87	124	3.25	7.24	52.79	8	0	62.80	5.43	31.72

Table 4.11: Statistics for dominant variable costs-function type 2.

and the average gaps are all within 2.41%. For cost structures 2, 3, 4, where one cost component dominates, the average gaps are 2.41%, 1.32% and 0.40%. This relationship is same as that showed in the Table 4.3, 4.4, 4.5 and 4.6, i.e. when transportation cost dominates, the average gap gives the minimum percentage. The hard instances are p25-p40. The values in column *une* indicate the same relation discussed in the cases of function type 1 that when transportation costs dominate, more facilities operate under the economic points whereas when production cost dominates, the number of facilities operating under the economic points are the minimum. Moreover, from the average capacity utilization across 55 instances, we can see that open facilities are more congested when fixed costs dominate, with an average capacity utilization of 84.38%, compared to 66.53% when production costs dominate.

When comparing the solution methodology for function types 1 and 2, it is apparent that type 1 leads to lower computation time and average gap. This difference may be attributed to the difference in the function shapes under congestion. From the shape of the functions in Figure 4.1, we can see that function type 1 starts to increase after the economic point, while function type 2 only begins to increase dramatically when closer to the capacity.

4.2.3 Performance of Cubic Function

For the cubic function, Table 4.12 shows the production cost functions and cost structures.

The results from Table 4.13, 4.14, 4.15 and 4.16 show similar characteristics as

	Production cost function	Cost structure	
1	$h_j(x) = 2 \times 10^{-4}x^3 - 3 \times 10^{-4}e_j x^2 + 3 \times 10^{-4}e^2 x$ $\forall j = 1, ..., n, e_j = 0.5K_j$	Original data from [18].	Table 4.13
2	Same as 1.	Fixed costs increased 10 times.	Table 4.14
3	$h_j(x) = 3 \times 10^{-4}x^3 - 3 \times 10^{-3}e_j x^2 + 3 \times 10^{-3}e^2 x$ $\forall j = 1, ..., n, e_j = 0.5K_j$	Original data from [18].	Table 4.15
4	Same as 1.	Variable costs increased 10 times.	Table 4.16

Table 4.12: Cost structures for cubic function.

that of function type 1 and 2. The effectiveness of the enhanced search is supported by the improvement again with average gaps of 1.22%, 3.79%, 1.28% and 0.76%. The maximum gap is from the case where fixed costs dominate. Instances p25-p55 requires more time to solve, but the gap is not necessarily inferior to other instances, e.g. in Table 4.14, p5 gives a gap of 10.25% after 88 seconds whereas p43 takes more than 2.5 hours but results in a gap of 2.60%. An even deeper search that allows closing facilities operating above the economic points is tested on both instances, but the gaps do not change. For cost structures 2, 3, 4, the best average gap 0.76% still shows in the case where transportation costs dominate while the case where fixed costs dominate gives the worst average gap with 3.79%. By looking at the number of open facilities operating under the economic points, we see similar results to the other two function types, i.e. when transportation costs dominate, the importance of economies of scale is diminished. Although most facilities operates above the economic points when fixed costs and production costs dominate, the average capacity utilization across 55 instances indicates that the open facilities are more congested when fixed costs dominate with an average capacity utilization of 92.57%, compared to 84.68% when production costs dominate.

	Instance feature			Initial Lag heuristic			Enhanced search			Cost structure			Facility status		Capacity utilization		
	n	m	K/D	gap1%	time1	itr1	gap2	time2	itr2	fixed%	prd%	var%	opn	une	max%	min%	avg%
p1	10	50	1.74	1.20	2.93	136	1.20	2.93	136	27.44	29.32	43.24	8	0	100.00	61.89	81.71
p2			1.74	0.06	4.76	184	0.06	4.76	184	20.44	32.04	47.53	8	0	100.00	71.89	82.43
p3			1.74	0.81	3.42	151	0.81	3.42	151	29.77	28.33	41.90	8	0	100.00	65.72	81.95
p4			1.74	2.88	2.89	136	0.85	7.05	313	32.71	28.36	38.93	7	0	100.00	75.68	90.17
p5			1.37	8.66	1.65	99	0.21	9.03	363	29.62	22.26	48.11	8	0	98.57	81.15	91.00
p6			1.37	4.22	1.56	96	0.87	3.76	216	26.21	23.08	50.71	9	0	94.29	53.58	80.89
p7			1.37	3.80	1.65	98	0.21	4.26	231	33.46	21.06	45.48	8	0	98.40	81.10	91.00
p8			1.37	10.81	1.62	98	0.81	7.16	322	41.02	18.83	40.15	8	0	100.00	71.88	91.00
p9			2.06	7.98	2.40	116	0.37	18.69	539	18.61	37.05	44.34	6	0	84.78	74.34	80.89
p10			2.06	9.52	2.17	110	0.90	26.71	456	13.99	39.13	46.88	6	0	84.48	75.19	80.89
p11			2.06	12.18	2.57	121	0.40	24.79	579	21.33	35.84	42.83	6	0	85.53	74.72	80.89
p12			2.06	5.47	2.93	129	0.13	8.25	342	27.52	33.02	39.46	6	0	85.88	74.74	80.89
p13	20	50	2.77	4.16	1.81	85	0.37	4.10	211	15.93	59.57	24.50	10	0	88.87	77.54	82.52
p14			2.77	5.25	1.67	80	0.57	6.24	231	13.31	61.43	25.26	10	0	88.83	77.56	82.52
p15			2.77	4.34	1.53	74	0.28	4.21	209	20.37	56.38	23.25	10	0	88.72	77.42	82.55
p16			2.77	5.06	1.31	66	0.08	3.59	187	26.38	52.14	21.49	10	0	88.59	77.31	82.53
p17			2.80	17.57	1.50	74	0.48	24.69	432	14.11	67.05	18.84	9	0	82.42	77.27	79.31
p18			2.80	10.87	1.23	59	1.00	42.70	423	10.90	69.97	19.14	9	0	81.86	75.80	79.31
p19			2.80	11.38	1.26	61	0.68	32.84	433	16.93	65.23	17.84	9	0	82.08	75.87	79.31
p20			2.80	12.42	1.36	67	0.53	18.88	350	22.20	61.10	16.70	9	0	81.92	76.08	79.31
p21			3.50	11.68	2.36	91	1.88	24.01	550	9.26	76.16	14.57	8	0	76.33	61.47	71.38
p22			3.50	8.40	1.53	69	2.12	7.27	287	6.89	78.52	14.59	8	0	76.82	60.92	71.38
p23			3.50	13.07	1.40	65	2.51	23.81	449	10.95	75.08	13.97	8	0	77.52	60.34	71.38
p24			3.50	9.95	1.61	73	2.75	7.27	293	14.67	71.88	13.46	8	0	78.05	59.82	71.38
p25	30	150	4.12	1.71	232.94	442	0.59	569.14	1017	13.74	52.99	33.27	11	0	100.00	71.27	87.84
p26			4.12	1.22	265.50	458	0.78	615.89	1052	11.26	54.53	34.21	11	0	100.00	71.41	87.83
p27			4.12	1.76	242.63	458	0.43	531.34	1007	17.47	50.73	31.80	11	0	100.00	71.28	87.81
p28			4.12	2.29	237.32	437	0.25	473.54	963	22.85	47.79	29.37	11	0	100.00	70.90	85.62
p29			3.03	7.73	125.58	343	2.22	1186.29	2041	20.67	44.17	35.16	13	0	85.77	53.08	76.15
p30			3.03	7.12	124.08	374	0.44	910.47	1490	16.36	46.96	36.68	12	0	84.45	80.46	82.50
p31			3.03	6.11	114.99	350	0.24	423.36	1038	24.57	42.39	33.04	12	0	84.60	75.74	82.50
p32			3.03	7.10	108.69	332	0.63	406.20	1063	31.21	38.83	29.96	12	0	85.97	69.61	82.50
p33			4.04	8.52	221.40	457	1.11	2127.96	2059	12.44	60.42	27.15	9	0	84.66	80.96	82.50
p34			4.04	10.46	193.92	436	1.58	1896.58	1905	9.21	62.64	28.16	9	0	84.64	81.10	82.50
p35			4.04	11.77	200.04	416	1.06	1972.51	1939	14.46	59.02	26.53	9	0	84.65	81.01	82.50
p36			4.04	12.97	203.82	435	0.74	1754.17	1888	19.13	55.78	25.09	9	0	84.65	80.86	82.50
p37			6.06	12.74	719.32	570	2.42	5999.10	2439	4.64	78.30	17.06	6	0	83.89	81.52	82.50
p38			6.06	11.42	822.20	628	2.71	11583.65	2934	3.45	79.27	17.28	6	0	83.81	81.58	82.50
p39			6.06	10.86	651.69	564	2.42	5938.38	2589	5.63	77.48	16.90	6	0	83.90	81.39	82.50
p40			6.06	11.12	886.02	607	2.16	6603.41	2546	7.70	75.78	16.52	6	0	83.82	81.48	82.50
p41	10	90	2.12	11.96	45.63	365	0.61	121.32	863	23.72	33.92	42.36	7	0	100.00	75.42	92.37
p42	20	80	4.99	8.02	14.40	172	0.11	34.79	447	32.47	23.34	44.20	8	0	100.00	71.15	91.79
p43	30	70	8.28	13.04	6.26	112	3.93	21.90	566	39.60	14.22	46.18	9	1	100.00	34.17	89.82
p44	10	90	1.76	0.07	29.98	281	0.07	29.98	281	23.36	39.92	36.71	8	0	100.00	68.63	90.02
p45	20	80	4.14	12.79	13.03	168	1.16	42.26	653	30.55	23.81	45.64	9	0	100.00	76.32	93.64
p46	30	70	7.10	9.02	11.93	178	0.78	38.08	537	38.99	19.89	41.12	9	0	100.00	69.17	94.33
p47	10	90	1.76	15.85	38.19	339	1.07	229.21	1455	23.87	40.34	35.79	8	0	100.00	69.03	89.49
p48	20	80	4.06	8.54	14.54	209	0.50	32.54	474	33.49	31.12	35.39	9	0	100.00	73.53	92.44
p49	30	70	7.08	22.09	7.53	124	1.04	82.27	738	37.55	21.11	41.34	9	0	100.00	67.57	91.97
p50	10	100	1.89	5.73	55.19	357	4.80	174.42	925	17.40	35.68	46.92	7	0	100.00	95.43	99.35
p51	20	100	3.98	11.83	32.59	222	4.14	107.16	831	27.25	28.22	44.53	9	1	100.00	43.32	89.22
p52	10	100	1.60	7.67	68.30	416	0.38	196.53	953	17.47	39.01	43.52	8	0	100.00	78.62	93.95
p53	20	100	3.37	9.47	42.59	284	2.62	123.51	935	22.15	26.84	51.00	10	0	100.00	56.97	90.86
p54	10	100	1.52	17.61	61.95	375	5.40	277.76	1193	16.97	42.48	40.55	8	0	100.00	92.79	98.76
p55	20	100	3.21	12.09	36.33	257	0.82	175.02	947	25.46	33.04	41.50	10	0	100.00	75.06	88.76
		Max	8.28	22.09	886.02	628	5.40	11583.65	2934	41.02	79.27	51.00	13	1	100.00	95.43	99.35
		Avg	3.32	8.59	106.87	246	1.22	818.17	866	20.89	45.87	33.24	9	0	91.61	72.37	84.84
		Min	1.37	0.06	1.23	59	0.06	2.93	136	3.45	14.22	13.46	6	0	76.33	34.17	71.38

Table 4.13: Statistics for the basic case-cubic function.

	Instance feature			Initial Lag heuristic			Enhanced search			Cost structure			Facility status		Capacity utilization		
	n	m	K/D	gap1%	time1	itr1	gap2%	time2	itr2	fixed%	prd%	var%	opn	une	max%	min%	avg%
p1	10	50	1.74	28.50	2.39	111	5.76	17.66	486	63.69	10.44	25.87	5	0	100.00	67.74	93.42
p2			1.74	11.98	3.63	141	0.24	18.25	523	57.94	12.95	29.11	5	0	100.00	98.93	99.63
p3			1.74	18.44	3.04	134	4.30	12.85	471	67.25	9.52	23.23	5	0	100.00	68.85	93.66
p4			1.74	20.89	3.46	141	4.94	7.35	329	74.08	7.50	18.41	5	0	100.00	68.85	93.64
p5			1.37	25.43	1.42	88	10.25	4.54	288	78.44	3.62	17.94	8	1	100.00	32.50	91.00
p6			1.37	18.54	1.06	74	7.60	2.37	165	76.86	4.48	18.66	8	0	100.00	59.00	91.00
p7			1.37	20.40	1.28	82	8.48	2.71	176	84.63	2.94	12.42	8	0	100.00	60.00	91.00
p8			1.37	21.07	1.09	74	8.91	3.96	268	88.47	2.21	9.32	8	0	100.00	59.50	91.00
p9			2.06	37.56	4.40	166	2.22	33.84	686	59.39	10.89	29.72	5	0	100.00	85.33	97.07
p10			2.06	13.99	2.14	104	1.44	6.49	306	59.90	12.61	27.49	5	0	100.00	85.33	97.07
p11			2.06	16.87	1.92	93	2.04	6.26	290	71.25	9.00	19.76	5	0	100.00	85.33	97.07
p12			2.06	34.65	1.89	94	2.17	10.76	371	77.71	6.96	15.33	5	0	100.00	87.67	97.07
p13	20	50	2.77	16.47	1.36	69	5.24	4.27	257	67.95	17.41	14.65	9	0	100.00	51.48	94.61
p14			2.77	20.53	0.86	49	2.92	6.08	212	60.84	25.41	13.75	8	0	100.00	79.46	91.74
p15			2.77	19.21	1.31	61	1.98	4.60	152	65.51	23.31	11.18	7	0	100.00	83.97	94.86
p16			2.77	5.35	1.75	76	5.35	1.75	76	71.14	19.57	9.28	7	0	100.00	59.33	91.30
p17			2.80	9.27	2.78	104	9.27	2.79	104	58.43	24.67	16.90	8	1	100.00	14.25	89.22
p18			2.80	11.61	1.09	55	3.58	2.51	149	60.94	25.70	13.36	8	0	99.75	65.00	89.22
p19			2.80	35.78	1.20	64	6.22	12.28	303	71.84	18.14	10.02	8	0	100.00	66.25	89.22
p20			2.80	39.23	1.08	59	7.42	8.39	295	78.16	14.12	7.71	8	0	100.00	64.00	89.22
p21			3.50	6.88	5.88	176	0.58	13.40	442	45.61	36.56	17.84	6	0	97.00	94.00	95.17
p22			3.50	47.87	1.62	66	1.97	102.84	537	44.31	40.13	15.56	6	0	97.20	93.60	95.17
p23			3.50	69.79	1.20	59	0.16	284.33	633	57.14	31.03	11.83	6	0	96.60	94.00	95.17
p24			3.50	79.38	1.14	56	1.65	118.56	540	64.35	25.40	10.25	6	0	100.00	86.80	95.17
p25	30	150	4.12	21.61	680.06	586	1.18	10106.32	2929	48.79	30.05	21.16	7	0	100.00	90.40	96.69
p26			4.12	16.69	581.65	559	2.23	5067.08	3169	45.83	32.16	22.01	7	0	100.00	77.53	92.32
p27			4.12	15.28	1214.33	753	5.67	2608.21	2543	56.98	26.94	16.08	7	1	100.00	46.89	88.91
p28			4.12	23.49	779.65	594	1.44	7173.24	3609	58.52	26.38	15.10	6	0	100.00	89.00	97.61
p29			3.03	7.40	283.22	469	2.41	581.29	1074	66.68	12.59	20.73	10	0	100.00	90.00	99.00
p30			3.03	16.43	150.82	356	1.48	997.19	1353	66.46	14.57	18.98	10	0	100.00	91.67	99.00
p31			3.03	18.55	163.10	340	1.19	904.93	1510	76.88	10.12	13.00	10	0	100.00	90.67	99.00
p32			3.03	18.99	149.99	322	1.29	877.41	1454	82.22	7.72	10.06	10	0	100.00	92.33	99.00
p33			4.04	20.07	254.08	417	4.08	1285.81	1662	56.32	21.92	21.76	8	0	100.00	71.75	92.81
p34			4.04	10.58	420.25	542	3.73	716.87	1201	54.43	24.81	20.76	8	0	100.00	75.75	92.81
p35			4.04	5.37	411.14	501	5.37	411.14	501	66.12	18.72	15.16	8	0	100.00	69.75	92.81
p36			4.04	5.63	364.36	470	5.63	364.36	470	73.37	14.81	11.82	8	0	100.00	69.00	92.81
p37			6.06	2.16	2303.65	804	2.16	2303.65	804	40.94	38.26	20.79	6	0	86.17	75.17	82.50
p38			6.06	18.03	1315.03	629	1.55	5514.59	2375	37.26	43.05	19.69	6	0	86.67	76.17	82.50
p39			6.06	12.76	1472.28	620	4.24	2412.68	1644	48.68	34.99	16.33	6	0	89.67	66.83	82.50
p40			6.06	17.05	1898.64	648	6.76	3206.32	1728	56.11	30.41	13.48	6	0	92.83	62.00	82.50
p41	10	90	2.12	54.96	42.90	309	0.26	984.55	1530	63.43	14.00	22.57	4	0	100.00	98.39	99.60
p42	20	80	4.99	40.87	76.94	333	2.12	447.22	1634	57.92	19.38	22.71	3	0	99.77	97.33	98.40
p43	30	70	8.28	21.74	114.88	364	6.08	167.86	1145	60.08	16.12	23.79	3	0	100.00	83.95	94.06
p44	10	90	1.76	22.53	63.88	412	5.74	109.48	843	63.99	14.06	21.95	5	0	100.00	66.90	91.32
p45	20	80	4.14	65.82	54.09	286	3.48	304.81	1529	70.34	11.45	18.21	5	0	100.00	81.72	94.66
p46	30	70	7.10	64.09	19.80	169	5.64	382.08	910	65.61	12.62	21.78	4	0	100.00	80.38	90.39
p47	10	90	1.76	23.12	63.24	394	4.22	252.89	1557	64.69	14.93	20.39	5	0	100.00	71.77	94.00
p48	20	80	4.06	40.48	29.80	223	2.98	313.14	1303	60.88	16.25	22.88	5	1	100.00	94.93	98.58
p49	30	70	7.08	56.94	17.63	163	4.09	261.50	1124	67.06	14.33	18.61	4	0	100.00	74.28	93.57
p50	10	100	1.89	20.93	105.97	476	2.27	285.84	1232	63.34	12.09	24.57	5	0	100.00	81.38	96.28
p51	20	100	3.98	33.33	80.00	303	0.74	445.32	1294	68.66	11.59	19.75	5	0	100.00	99.47	99.89
p52	10	100	1.60	6.51	119.04	539	6.51	119.04	539	64.75	14.89	20.36	6	0	100.00	51.90	91.67
p53	20	100	3.37	20.32	98.25	367	0.74	361.39	1300	60.37	16.56	23.07	5	0	100.00	91.20	96.73
p54	10	100	1.52	11.15	120.42	529	7.46	120.42	529	56.27	18.35	25.38	6	1	100.00	93.52	98.70
p55	20	100	3.21	33.65	159.51	479	0.86	786.31	1950	63.86	19.74	16.41	6	1	100.00	96.40	98.74
		Max	8.28	79.38	2303.65	804	10.25	10106.32	3609	88.47	43.05	29.72	10	1	100.00	99.47	99.89
		Avg	3.32	25.02	248.22	294	3.79	918.24	977	63.98	18.20	17.82	6	0	99.01	70.80	92.57
		Min	1.37	2.16	0.86	49	0.16	1.75	76	37.26	2.21	7.71	3	0	86.17	4.35	76.14

Table 4.14: Statistics for dominant fixed costs-cubic function.

	Instance feature			Initial Lag heuristic			Enhanced search			Cost structure			Facility status		Capacity utilization		
	n	m	K/D	gap1%	time1	itr1	gap2	time2	itr2	fixed%	prd%	var%	opn	une	max%	min%	avg%
p1	10	50	1.74	1.26	3.57	148	1.26	3.57	148	23.95	37.88	38.17	8	0	100.00	62.78	82.98
p2			1.74	0.16	4.38	172	0.16	4.38	172	17.60	41.37	41.03	8	0	100.00	70.62	82.66
p3			1.74	0.83	4.34	169	0.83	4.34	169	26.08	36.88	37.04	8	0	100.00	64.66	82.93
p4			1.74	1.61	3.00	131	1.61	3.00	131	32.87	33.31	33.82	8	0	100.00	65.01	83.16
p5			1.37	8.06	1.36	83	0.64	8.39	302	26.67	29.92	43.41	8	0	96.26	83.52	91.00
p6			1.37	3.62	1.83	102	0.64	4.21	226	23.61	30.29	46.10	9	0	91.52	61.71	80.89
p7			1.37	2.67	1.42	85	0.63	4.52	221	30.28	28.42	41.30	8	0	96.00	84.25	91.00
p8			1.37	4.87	1.78	100	0.16	4.51	229	37.81	25.35	36.85	8	0	96.00	84.10	91.00
p9			2.06	11.46	2.12	103	0.57	30.33	537	15.70	46.79	37.50	6	0	84.86	76.49	80.89
p10			2.06	8.14	1.83	101	1.00	14.02	421	11.70	48.98	39.32	6	0	83.67	76.20	80.89
p11			2.06	11.57	1.83	100	0.50	29.36	551	18.09	45.43	36.48	6	0	83.67	76.33	80.89
p12			2.06	9.53	2.22	102	0.27	17.67	437	23.62	42.36	34.02	6	0	83.96	76.56	80.89
p13	20	50	2.77	4.31	1.78	80	0.57	4.46	211	12.28	68.64	19.07	10	0	87.84	77.38	82.61
p14			2.77	3.94	2.32	96	0.76	4.63	222	10.18	70.29	19.53	10	0	87.70	77.25	82.61
p15			2.77	4.43	1.93	86	0.44	4.79	222	15.89	65.81	18.30	10	0	87.90	77.43	82.61
p16			2.77	4.83	1.61	73	0.24	3.35	167	20.92	61.89	17.19	10	0	87.56	77.27	82.61
p17			2.80	19.52	1.59	71	0.61	80.82	515	10.57	75.27	14.16	9	0	82.04	77.79	79.31
p18			2.80	11.68	1.28	62	1.02	24.80	395	8.07	77.67	14.26	9	0	81.26	77.18	79.31
p19			2.80	13.28	1.39	64	0.79	40.97	430	12.77	73.71	13.52	9	0	81.16	77.00	79.31
p20			2.80	14.76	1.39	66	0.59	72.84	512	17.00	70.11	12.89	9	0	81.06	77.04	79.31
p21			3.50	11.40	2.01	81	1.78	13.06	467	6.71	82.68	10.60	8	0	75.50	63.14	71.38
p22			3.50	9.47	2.00	80	2.21	16.66	387	4.94	84.57	10.49	8	0	76.26	60.41	71.38
p23			3.50	10.87	1.67	73	2.40	24.69	455	7.97	81.82	10.21	8	0	76.67	59.85	71.38
p24			3.50	9.05	1.81	75	2.68	8.38	308	10.79	79.30	9.91	8	0	77.12	61.26	71.38
p25	30	150	4.12	12.88	176.09	364	0.77	585.78	1290	12.06	50.14	37.80	12	0	100.00	64.70	84.62
p26			4.12	12.34	206.05	415	0.34	626.69	1353	9.87	51.27	38.86	12	0	100.00	71.85	84.42
p27			4.12	12.84	206.79	402	1.09	891.14	1339	15.33	48.40	36.27	12	0	100.00	60.61	84.71
p28			4.12	12.12	202.65	394	1.86	931.33	1296	18.42	47.32	34.26	11	0	100.00	80.84	93.42
p29			3.03	7.95	115.89	321	1.57	859.08	1537	17.08	53.58	29.34	13	0	82.49	55.29	76.15
p30			3.03	8.83	105.85	315	0.76	1825.82	1728	13.25	57.02	29.72	12	0	83.78	81.23	82.50
p31			3.03	8.17	116.67	341	0.37	953.01	1445	20.29	52.39	27.31	12	0	83.81	81.19	82.50
p32			3.03	6.91	114.89	330	0.18	423.78	1006	26.27	48.45	25.28	12	0	83.58	81.55	82.50
p33			4.04	9.14	239.41	448	1.54	2109.94	1989	9.55	69.55	20.90	9	0	83.88	81.61	82.50
p34			4.04	11.54	213.10	428	1.92	1847.80	1909	7.01	71.52	21.47	9	0	83.80	81.34	82.50
p35			4.04	12.67	199.68	414	1.47	1720.47	1815	11.16	68.32	20.52	9	0	83.82	81.50	82.50
p36			4.04	12.79	199.20	404	1.11	1824.77	1845	14.96	65.40	19.64	9	0	83.89	81.44	82.50
p37			6.06	12.75	1012.32	572	2.62	5787.72	2326	3.33	84.39	12.27	6	0	83.49	81.68	82.50
p38			6.06	11.77	977.17	566	2.82	5762.79	2341	2.47	85.13	12.40	6	0	83.30	81.93	82.50
p39			6.06	11.60	1258.58	605	2.60	7795.40	2434	4.06	83.75	12.19	6	0	83.35	81.94	82.50
p40			6.06	11.18	1068.06	544	2.40	6775.25	2231	5.59	82.43	11.99	6	0	83.53	81.82	82.50
p41	10	90	2.12	13.79	54.85	373	0.54	114.16	801	20.28	43.42	36.30	7	0	100.00	75.41	92.32
p42	20	80	4.99	7.12	26.63	234	0.18	50.44	539	29.07	28.36	42.57	8	0	100.00	75.04	93.46
p43	30	70	8.28	3.71	11.82	166	3.71	11.82	166	37.16	19.48	43.36	9	1	100.00	38.65	89.91
p44	10	90	1.76	0.09	30.61	281	0.09	30.61	281	19.48	49.67	30.85	8	0	100.00	70.43	89.92
p45	20	80	4.14	7.25	13.20	164	0.13	26.19	380	27.32	31.83	40.85	9	0	100.00	78.03	93.64
p46	30	70	7.10	3.29	9.13	139	1.14	20.84	365	35.47	26.75	37.78	9	0	100.00	71.51	94.23
p47	10	90	1.76	11.39	54.55	405	0.30	155.61	984	19.88	50.21	29.91	8	0	100.00	70.08	89.47
p48	20	80	4.06	10.79	12.99	179	0.39	32.31	436	29.02	39.76	31.22	9	0	100.00	73.38	92.82
p49	30	70	7.08	10.60	8.75	146	0.99	38.14	612	33.97	28.63	37.41	9	0	100.00	67.21	91.94
p50	10	100	1.89	6.18	72.10	413	6.18	242.05	1040	17.38	43.65	38.96	8	1	100.00	39.93	86.19
p51	20	100	3.98	6.68	34.12	229	0.61	83.68	583	24.27	31.76	43.98	9	0	100.00	84.85	96.25
p52	10	100	1.60	10.33	65.43	378	0.35	355.31	1218	14.62	48.90	36.48	8	0	100.00	79.57	93.95
p53	20	100	3.37	5.52	39.61	258	1.13	90.39	599	19.48	31.24	49.28	10	0	100.00	84.78	95.15
p54	10	100	1.52	9.94	62.10	381	6.07	284.17	1211	14.00	52.54	33.46	8	0	100.00	93.08	98.76
p55	20	100	3.21	6.22	28.31	224	2.98	99.51	752	23.37	40.18	36.46	11	1	100.00	47.49	86.21
		Max	8.28	19.52	1258.58	605	6.18	7795.40	2434	37.81	85.13	49.28	13	1	100.00	93.08	98.76
		Avg	3.32	8.54	127.04	239	1.28	777.89	831	17.85	53.17	28.99	9	0	90.92	73.18	84.68
		Min	1.37	0.09	1.28	62	0.09	3.00	131	2.47	19.48	9.91	6	0	75.50	38.65	71.38

Table 4.15: Statistics for dominant production costs-cubic function.

	Instance feature			Initial Lag heuristic			Enhanced search			Cost structure			Facility status		Capacity utilization		
	n	m	K/D	gap1%	time1	itr1	gap2%	time2	itr2	fixed%	prd%	var%	opn	une	max%	min%	avg%
p1	10	50	1.74	0.07	3.67	178	0.07	8.00	495	7.68	5.23	87.09	10	4	100.00	27.10	63.27
p2			1.74	0.07	2.59	145	0.07	7.25	474	5.47	5.38	89.15	10	4	100.00	27.10	63.27
p3			1.74	0.07	2.98	155	0.07	7.39	469	8.79	5.18	86.03	10	4	100.00	27.10	63.27
p4			1.74	0.04	3.42	172	0.04	8.16	498	11.89	4.99	83.12	10	4	100.00	27.10	63.27
p5			1.37	0.05	2.51	142	0.05	4.62	291	8.07	2.62	89.31	10	2	100.00	26.50	72.80
p6			1.37	0.04	2.62	145	0.04	4.46	284	5.75	2.69	91.56	10	2	100.00	26.50	72.80
p7			1.37	0.05	2.45	136	0.05	4.35	278	9.23	2.59	88.18	10	2	100.00	26.50	72.80
p8			1.37	0.01	2.29	129	0.01	3.99	260	12.47	2.48	85.05	10	2	100.00	26.50	72.80
p9			2.06	0.06	3.04	171	0.06	10.94	716	7.85	6.16	85.99	10	6	84.00	17.67	48.53
p10			2.06	0.09	2.75	159	0.09	9.80	673	5.59	6.31	88.10	10	6	84.00	17.67	48.53
p11			2.06	0.07	2.32	144	0.07	9.89	688	8.98	6.11	84.91	10	6	87.00	17.67	48.53
p12			2.06	0.08	3.82	204	0.08	11.26	736	12.14	5.87	81.99	10	6	84.00	17.67	48.53
p13	20	50	2.77	2.33	0.86	60	0.47	113.16	827	11.06	19.41	69.53	14	7	97.09	7.41	53.82
p14			2.77	1.69	0.95	67	0.55	136.97	868	9.94	23.90	66.15	17	11	97.09	7.41	43.87
p15			2.77	2.82	0.94	62	0.45	128.92	806	13.77	20.03	66.19	15	8	97.09	7.41	50.07
p16			2.77	2.03	1.01	65	0.29	25.94	602	15.88	17.73	66.39	13	6	97.09	7.41	57.51
p17			2.80	1.99	0.78	55	0.17	106.75	880	13.42	22.52	64.06	16	10	83.50	9.00	44.61
p18			2.80	0.78	1.08	71	0.23	20.25	858	9.41	23.84	66.75	16	10	83.50	9.00	44.61
p19			2.80	1.80	0.92	61	0.15	19.87	740	13.83	21.31	64.85	15	8	83.50	9.00	47.58
p20			2.80	2.57	1.14	64	0.09	49.92	744	18.35	20.20	61.45	15	8	83.50	9.00	47.58
p21			3.50	1.69	1.31	71	0.16	44.10	856	7.87	26.83	65.30	11	4	81.40	7.20	51.91
p22			3.50	0.61	1.40	73	0.16	63.55	1189	6.50	30.31	63.19	13	7	77.60	7.20	43.92
p23			3.50	4.18	1.65	83	0.17	109.62	913	8.79	26.56	64.65	11	4	81.40	7.20	51.91
p24			3.50	2.37	1.81	95	0.07	47.86	891	11.89	25.66	62.45	11	4	81.40	7.20	51.91
p25	30	150	4.12	4.28	163.85	409	0.16	3710.69	3099	4.11	13.54	82.35	10	1	100.00	32.53	73.46
p26			4.12	2.32	131.59	371	0.89	5304.61	3188	3.38	13.53	83.08	11	2	100.00	24.22	68.23
p27			4.12	2.52	153.30	390	1.34	1145.14	1810	4.99	13.30	81.70	10	1	100.00	24.57	73.12
p28			4.12	8.12	174.55	425	0.79	3362.49	2826	6.21	13.30	80.49	9	0	100.00	52.37	80.17
p29			3.03	4.66	161.41	447	2.64	1328.24	1887	5.45	6.30	88.25	12	1	100.00	37.67	82.50
p30			3.03	5.16	129.34	416	2.77	1055.32	1699	3.94	5.94	90.12	12	0	100.00	64.00	82.50
p31			3.03	6.74	145.64	430	3.37	565.91	1315	6.36	5.85	87.79	12	0	98.33	58.00	82.50
p32			3.03	3.66	162.04	441	3.66	162.04	441	8.66	5.89	85.45	12	0	100.00	50.00	82.50
p33			4.04	5.28	174.38	454	0.87	2242.37	2378	4.00	10.25	85.76	9	0	97.50	67.75	82.50
p34			4.04	4.88	166.84	426	2.11	1544.35	2351	3.17	10.19	86.63	10	1	93.50	33.25	74.25
p35			4.04	9.01	185.88	486	0.52	4843.05	3015	4.73	10.13	85.14	9	0	96.00	70.00	82.50
p36			4.04	8.63	223.38	555	0.96	2670.83	2617	6.46	9.92	83.62	9	0	97.00	69.25	82.50
p37			6.06	2.07	492.93	755	2.07	1545.74	2430	3.10	20.37	76.53	8	2	71.00	34.50	61.88
p38			6.06	1.02	585.46	803	1.02	1040.18	2007	2.23	20.77	77.00	8	2	70.50	36.33	61.88
p39			6.06	3.15	514.97	776	1.62	4190.72	4206	3.64	20.31	76.05	8	2	71.33	40.17	61.88
p40			6.06	3.23	516.61	757	3.09	3254.20	3087	4.33	19.39	76.28	7	1	91.00	44.50	70.71
p41	10	90	2.12	0.11	28.80	293	0.11	67.86	984	9.69	10.18	80.13	10	4	100.00	24.89	58.42
p42	20	80	4.99	0.05	14.70	237	0.05	72.82	1488	16.29	10.67	73.04	14	9	95.28	8.65	43.70
p43	30	70	8.28	0.04	4.63	107	0.04	35.29	1156	18.00	7.15	74.85	14	10	89.08	15.43	43.59
p44	10	90	1.76	0.07	31.79	309	0.07	60.28	806	9.72	11.18	79.09	10	3	100.00	35.42	68.38
p45	20	80	4.14	1.95	14.21	256	0.86	340.66	1878	15.90	12.69	71.41	14	8	100.00	8.06	49.55
p46	30	70	7.10	6.94	3.01	96	2.84	824.89	2003	20.48	11.64	67.88	15	9	100.00	8.33	45.36
p47	10	90	1.76	0.22	88.95	549	0.22	109.87	978	12.60	14.18	73.21	10	3	100.00	27.64	69.37
p48	20	80	4.06	0.07	6.74	199	0.07	67.83	1579	22.72	15.92	61.35	14	9	97.09	5.43	49.16
p49	30	70	7.08	7.76	3.21	107	2.06	349.55	1936	20.53	13.89	65.59	12	7	100.00	5.95	52.74
p50	10	100	1.89	0.07	50.64	389	0.07	115.64	990	5.55	6.20	88.25	9	2	100.00	13.50	66.37
p51	20	100	3.98	2.80	24.15	220	1.35	135.92	1105	7.21	5.61	87.18	10	2	99.21	37.90	66.90
p52	10	100	1.60	0.34	94.21	506	0.34	94.21	506	4.42	6.11	89.47	8	0	100.00	59.07	82.89
p53	20	100	3.37	1.11	26.44	237	0.59	216.76	1356	6.26	7.18	86.57	10	2	100.00	19.64	72.11
p54	10	100	1.52	0.28	69.19	472	0.28	140.42	1066	5.94	7.16	86.91	10	2	100.00	30.09	75.31
p55	20	100	3.21	4.50	17.32	210	1.37	224.31	1130	7.02	7.72	85.26	10	1	100.00	40.59	74.50
		Max	8.28	9.01	585.46	803	3.66	5304.61	4206	22.72	30.31	91.56	17	11	100.00	70.00	82.89
		Avg	3.32	2.30	83.79	277	0.76	759.62	1334	9.12	12.37	78.51	11	4	93.64	26.53	62.71
		Min	1.37	0.01	0.78	55	0.01	3.99	260	2.23	2.48	61.35	7	0	70.50	5.43	43.59

Table 4.16: Statistics for dominant variable costs-cubic function.

Chapter 5

Conclusion

This thesis considers a general class of capacitated facility location problems where the production costs follow an S-shape to capture economies of scale and congestion. A nonlinear integer programming formulation is provided. Given the special structure of the model, Lagrangian relaxation is applied to the demand constraints, decomposing the problem by facility. The resulting subproblem is further decomposed into three cases, depending on the production status of the facility. This amounts to solving a nonlinear knapsack problem using a specialized procedure. The Lagrangian multipliers are updated based on a Lagrangian master problem. The algorithm iterates between the master problem that updates μ, and the subproblem until the Lagrangian lower bound is found.

At a second step, a Lagrangian heuristic is presented that uses the solution of the subproblems. The heuristic is enhanced by some facilities operating under the economic point and iterating further.

Numerical testing on three function types with different cost structures reveals the efficiency of the proposed solution approach with average gaps of 0.94%, 2.41% and 3.79%, respectively. In terms of solution structure, we found that when fixed costs dominate, most open facilities operate under congestion, but when production costs dominate, most open facilities operate under economies of scale.

For future research, using other function forms and embedding the approach within branch-and-bound are worth exploring.

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