Gerber-Shiu analysis in some dependent Sparre Andersen risk models

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Gerber and Shiu (1998) first introduced the Gerber-Shiu expected discounted penalty function (or Gerber-Shiu function) which can be used to analyze the traditional quantities of interest in classical ruin theory, namely the time of ruin, the deficit at ruin, and the surplus immediately prior to ruin. Interestingly, the motivation of results therein was originally related to the problem of pricing American options, e.g, Gerber and Shiu (1997a, 1998). Subsequently this Gerber-Shiu function has been studied extensively in recent years in various risk models as a unified tool for the analysis of various risk models. In particular, Sparre Andersen risk models are often candidates for modeling the insurer's surplus process. For instance, Willmot (2007) and Landriault and Willmot (2008) assumed arbitrary interclaim times distribution. Li and Garrido (2005) considered the K_n family distribution for the interclaim times which includes the generalized Erlang renewal risk model studied by Gerber and Shiu (2005), the Erlang(n) renewal risk model considered by Li and Garrido (2004), and the well-known classical compound Poisson model (e.g. Gerber and Shiu (1998)) as a special case.

However, in the usual Sparre Andersen risk model, the assumption that the claim sizes and the interclaim times are independent is not reasonable to reflect some situations precisely (e.g. catastrophic insurance). Therefore, one approach is to consider time-dependent claim sizes rather than the traditional independent assumption between the interclaim times and the subsequent claim sizes. Indeed, there have been some papers analyzing ruin related quantities under certain dependent structures including Albrecher and Boxma (2004), Albrecher and Teugels (2006), Badescu et al. (2009), Boudreault et al. (2006), Cossette et al. (2008), and Marceau (2009). In this thesis, the insurer's process is analyzed by using the Gerber-Shiu function in the dependent (ordinary and delayed) Sparre Andersen risk models assuming an arbitrary dependence structure between the claim sizes and the interclaim times, and structural results are derived which provide some insights and qualitative aspects of the dependent nature of the surplus process.

As for the Gerber-Shiu function itself, we focus on the analysis of the generalized version of the Gerber-Shiu function by adding two more new variables in the traditional penalty function. These two variables, namely the surplus level immediately after the second last claim before ruin and the minimum surplus level before ruin, together with the other variables defined previously in the penalty function can provide more information regarding the surplus process before ruin occurs.

In Chapter 2, it is shown that that the generalized Gerber-Shiu function satisfies a defective renewal equation. In particular, an alternative expression for the Gerber-Shiu function obtained in Section 2.2.1 enables us to readily derive various discounted joint and marginal densities associated with the four variables in a penalty function. As a consequence, application of these general results from Chapter 2 is useful to obtain the explicit form of the densities in Sparre Andersen risk models in the subsequent chapters. In Chapter 3, we consider the large class of Coxian distribution for the interclaim times and identify the components of the defective renewal equation for the generalized Gerber-Shiu function. For reference, a more general class of distributions was considered by Dufresne (2001). The classical compound Poisson risk model is considered in detail in order to study the proper deficit distribution under the certain dependent structure introduced by Boudreault et al. (2006) in Section 3.2. Also, the analysis of the joint densities involving the time of ruin as in Dickson and Willmot (2005), and Landriault and Willmot (2009) is the subject matter of Section 3.3. In Chapter 4, the Gerber-Shiu function is analyzed in the delayed renewal risk model, where it is shown that many properties of the ordinary renewal risk process discussed in the previous chapters are carried over to these more general models. This modified ordinary process may enhance appropriateness of the modeling in the case where the first event has a significant impact on the subsequent events, its size is strongly dependent on the interclaim times, or this event is not observed at time 0. To illustrate such circumstances, a numerical example for earthquake insurance is provided in Section 4.4.1. The analysis of the classical Gerber-Shiu function with the traditional assumption (time-independent claim sizes) is done by Willmot (2004), Kim (2007), and Kim and Willmot (2010). As a special case, the stationary renewal risk model is considered by Willmot and Dickson (2003). This process is important in some cases because the limiting form of the recurrence time in the renewal process follows an equilibrium distribution, (e.g. Karlin and Taylor (1975)).

Furthermore, in Chapter 5 we consider the discrete risk model and provide a similar analysis for the generalized Gerber-Shiu function, analogous to the ordinary continuous time Sparre Andersen risk models. Also, in these models, numerous studies regarding the classical Gerber-Shiu function have been performed. For example, Li (2005a,b) considered a discrete K_n class distribution for the interclaim times. As a special case, the compound binomial model, a discrete analogue of the classical compound Poisson risk model, was first proposed by Gerber (1988) and further studied by Cheng et al. (2000), Cossette et al. (2003), Dickson (1994), Shiu (1989), Willmot (1993), Yeun and Guo (2001). Corresponding to the stationary renewal risk models in continuous time, the discrete stationary renewal risk models was studied by Pavlova and Willmot (2004). A recursive formula and a general expression for the generalized Gerber-Shiu function are provided in Section 5.2. As an application, a discrete Coxian interclaim time distribution is considered in Section 5.3. In addition, discrete delayed risk models are covered in Section 5.4.

Finally, in Chapter 6 some two-sided bounds for the renewal equation are obtained in terms of the tail of an arbitrary distribution, and to do so, it is convenient to apply reliability classifications as in Willmot and Lin (2001) and Willmot et al. (2001). Most of the bounds provided in this chapter improve some results of Willmot et al. (2001) and their application for ruin quantities and stochastic process are included here as well.

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I would express my gratitude to my supervisors, Professor Gordon E. Willmot and Professor Steve Drekic for their superb academic support during my whole PhD. Also, I am grateful to my committee, Professor Jun Cai, Professor David Landriault, and Professor Jean-François Renaud who offered a variety of constructive advice which makes this work more improved. Along with the committees, thanks go to again all mentioned above who taught me great ideas and various knowledge in ruin theory field through their lectures.

Dedication

This thesis is dedicated to my parents, SamTaeg Woo (Dad) and HaeSook Park (Mom) for supporting my study all the time. Also, I feel grateful and lucky to meet my husband, Eric Chi Kin Cheung who has always been on my side with great support and love. A special thanks as well to all great and nice friends I have met in Canada.

Contents

List of Tables

List of Figures

Chapter 1

Introduction and preliminaries

1.1 Risk models of interest

To begin, we introduce three risk models considered in the following chapters.

1.1.1 Dependent (ordinary) Sparre Andersen risk models

Let us consider the insurer's surplus process at time t defined as $\{U_t; t \geq 0\}$ with the initial surplus $u \geq 0$,

$$U_t = u + ct - \sum_{i=1}^{N_t} Y_i.$$

The number of claims process $\{N_t; t \geq 0\}$ is assumed to be a renewal process, with V_1 the time of the first claim and V_i the time between the $(i-1)$th and the ith claim for $i = 2, 3, 4, \ldots$. It is assumed that $\{V_i\}_{i=1}^{\infty}$ is an independent and identically distributed (iid) sequence of positive random variables with common probability density function (pdf) $k(t)$ and distribution function (df) $K(t) = 1 - \overline{K}(t)$. The claim sizes $\{Y_i\}_{i=1}^{\infty}$ are iid random variables with common pdf $p(y)$ and df $P(y) = 1 - \overline{P}(y)$. With these general modelling assumptions including independence between

$\{V_i\}_{i=1}^{\infty}$ and $\{Y_i\}_{i=1}^{\infty}$, the above surplus process $\{U_t; t \geq 0\}$ is referred as the Sparre Andersen risk model. As a special case of this model, we know the classical Poisson risk model when any arbitrary V_i is exponentially distributed. For the references, see Cramèr (1955), Gerber (1979), Grandell (1991), Seal (1969) and Sparre Andersen (1957). In the context of queueing theory, this model may be interpreted in terms of the equilibrium waiting time distribution in the G/G/1 queue (e.g. Cohen (1982), Prabhu (1998)).

In this thesis, we generalize the Sparre Andersen risk model as follows. We assume that the pairs $\{(V_i, Y_i); i = 1, 2, ...\}$ are iid, so that $\{cV_i - Y_i; i = 1, 2, ...\}$ is also an iid sequence which implies that the surplus process $\{U_t, t \geq 0\}$ retains the Sparre Andersen random walk structure (i.e. discrete time random walks, e.g. Asmussen (2000, p.33)). As for notation, it is convenient to specify the joint distribution of (V_i, Y_i) by the product of the marginal density $k(t)$ and the conditional density of Y_i given V_i. With (V, Y) being an arbitrary (V_i, Y_i), we let $P_t(y) = \Pr(Y \leq y \,|V = t) = 1 - \overline{P}_t(y)$ for $y > 0$. The usual Sparre Andersen model assumes independence between V and Y, and may be recovered with $P_t(y) = P(y)$ for all $t \geq 0$. Let $p_t(y) = P_t'(y)$ be the conditional density, so that the joint density of (V, Y) is given by $p_t(y)\,k(t)$. In what follows, it is also convenient to introduce the conditional Laplace transform $\widetilde{p}_t(s) = \int_0^{\infty} e^{-sy} p_t(y)\,dy$. It is instructive to note that the assumptions of absolute continuity are not necessary and are simply made for ease of exposition. To complete the definition of $\{U_t, t \geq 0\}$, we define c $(c > 0)$ to be the premium rate per unit time which is assumed to satisfy the positive security loading condition (PSLC), namely $E[cV - Y] > 0$.

1.1.2 Dependent delayed Sparre Andersen risk models

In the traditional delayed renewal risk model, the assumption of the traditional ordinary model regarding the first interclaim time V_1 is slightly modified. It is assumed that the distribution of the time (from 0) to the first event V_1 is different from that of V_i for $i = 2, 3, 4, \ldots$, and we assume that V_1 has pdf $k_1(t)$, df $K_1(t) = 1 - \overline{K}_1(t)$ and Laplace transform $\widetilde{k}_1(s) = \int_0^\infty e^{-st} k_1(t)dt$. The motivation of this modified model is that in some cases an event occurred some time in the past rather than at time 0 as implicitly assumed in the traditional ordinary renewal risk model. In other words, a business (or a system) might have been operating for some time before we start observing the process at time 0, and an event does not necessarily occur at time 0. Therefore, to enhance and improve the model to reflect these circumstances, we use different modelling assumption on the distribution of the time until the first claim V_1. In particular, if a process started in the past long time ago before it is first observed, then the time to the first claim has an equilibrium pdf given by $\overline{K}(t)/E[V]$. This special case of the traditional delayed renewal process is called the stationary (equilibrium) renewal process. It is emphasized that the limiting form of the forward recurrence time in the traditional ordinary renewal process follows an equilibrium distribution, and thus this model is important in some applications. Certainly, these modified processes revert to the traditional ordinary model upon the occurrence of the first claim. For further details of the traditional delayed and stationary renewal processes, see Cox (1962, Section 2.2), Grandell (1991), Rolski et al. (1999), Ross (1996, Section 3.5), and Willmot and Lin (2001, Section 11.4).

In a similar fashion to how a traditional delayed model extends a traditional ordinary model, the dependent delayed model can also be defined accordingly. Under the premise that a delayed model is characterized by modelling a different interclaim time distribution on the first event, we assume that a dependent delayed model is simply a dependent ordinary model except that the first pair (V_1, Y_1) has a different joint distribution from the other pairs (V_i, Y_i) for $i = 2, 3, 4, \ldots$. Thus, we let the conditional distribution of $Y_1|V_1$ be $P_{1,t}(y) = 1 - \overline{P}_{1,t}(y)$ and conditional density be $p_{1,t}(y) = P'_{1,t}(y)$. Except for the first pair, the same notations defined for the dependent ordinary model are used for the remaining pairs. An application of this model, for instance, is to earthquake insurance. Since larger earthquakes occur less frequently, and also the last observed earthquake may be occurred in the past rather than in the present, specific time-dependent structure for the claim sizes as well as the occurrence of the last main shock before time 0 are necessarily considered for modelling. For example, assume that the last claim before time 0 is known to have occurred at time $-t^*$. In such case, we simply let $k_1(t) = k(t+t^*)/\overline{K}(t^*)$ be the residual lifetime distribution corresponding to $k(t)$ and $p_{1,t}(y) = p_{t+t^*}(y)$.

1.1.3 Dependent discrete-time Sparre Andersen risk models

In risk theory, research regarding the insurer's surplus process in different Sparre Andersen renewal risk models has been done extensively by analyzing the Gerber-Shiu function first introduced by Gerber and Shiu (1998). Along with the continuous-time Sparre Andersen renewal risk model, some interesting results have also been derived in discrete-time Sparre Andersen renewal risk models which give us insights and approximation ideas for the continuous-time models.

First, let us consider the insurer's business $\{U(t); t \geq 0\}$ in the discrete renewal risk process as is now described. The surplus at time t defined as

$$U(t) = u + t - \sum_{i=1}^{N(t)} Y_i,$$

with the initial capital of the insurer being $u \geq 0$. Time is measured in discrete units $0, 1, 2, \ldots$, and premiums are payable at the rate of 1 per unit time. The claim number process $\{N(t); t \geq 0\}$ is assumed to be a renewal process, with independent and identically distributed (iid) positive interclaim times $\{W_i\}_{i=1}^{\infty}$ having common distribution function (df) $K(t) = 1 - \overline{K}(t)$ and probability function (pf) $k(t) = \overline{K}(t-1) - \overline{K}(t)$ for $t = 1, 2, \ldots$. The claim sizes $\{Y_i\}_{i=1}^{\infty}$ are iid positive random variables with common df $P(y) = 1 - \overline{P}(y)$ and pf $p(y) = \overline{P}(y-1) - \overline{P}(y)$ for $y = 1, 2, \ldots$. We denote an arbitrary pair of (W_i, Y_i) by (W, Y). If W and Y are assumed independent, the surplus process $\{U_t; t \geq 0\}$ is referred as the discrete time Sparre Andersen renewal risk model (e.g. Wu and Li (2008)).

As mentioned earlier, we shall generalize the above model by relaxing the independence assumption between the claim sizes and the interclaim times as follows. We only assume that the pairs $\{(W_i, Y_i); i = 1, 2, \ldots\}$ are iid, so that the increments $\{(W_i - Y_i); i = 1, 2, \ldots\}$ are also iid which implies the surplus process still possesses a discrete-time Sparre Andersen random walk property. Let us define the conditional pf of Y given W by $p_t(y) = \Pr(Y = y|W = t)$ and also its df by $P_t(y) = 1 - \overline{P}_t(y)$ for $y = 1, 2, \ldots$. Obviously, the joint distribution of (W, Y) is retrieved by the product of the marginal pf $k(t)$ and this conditional pf $p_t(y)$. It is convenient to introduce the conditional pgf $\hat{p}_t(s) = \sum_{y=1}^{\infty} s^y p_t(y)$. Lastly, PSLC is assumed, namely $E[W - Y] > 0$.

1.2 Generalized Gerber-Shiu penalty function

The classical Gerber-Shiu expected discounted penalty function (or Gerber-Shiu function) first studied by Gerber and Shiu (1998) is defined as

$$m_{\delta,12}(u) = E\left[e^{-\delta T} w_{12}\left(U_{T^-}, |U_T|\right) \mathrm{I}\left(T < \infty\right) \middle| U_0 = u\right], \qquad u \geq 0, \tag{1.1}$$

where $T = \inf\{t \geq 0 : U_t < 0\}$ with $T = \infty$ if $U_t \geq 0$ for all $t \geq 0$, i.e. T is the time of ruin. Also, U_{T^-} is the surplus immediately prior to ruin, $|U_T|$ is the deficit at ruin, $w_{12}(x, y)$ satisfies mild integrability conditions, $\mathrm{I}(A)$ is the usual indicator function of the event A, and δ (often interpreted as a force of interest) is assumed to be nonnegative.

The Gerber-Shiu function (1.1) has been studied extensively in recent years in models of dependent nature. Cossette et al. (2008) used $K(t) = 1 - e^{-\lambda t}$, but with $\Pr(Y \leq y|V \leq t) = C(P(y), 1 - e^{-\lambda t})/(1 - e^{-\lambda t})$, where $C(u, v)$ is a generalized Farlie-Gumbel-Morgenstern copula. Also, in the classical compound Poisson risk model, Zhao (2008) considered the Block and Basu's bivariate exponential distribution (e.g. Block and Basu (1974)) for joint density function of (V, Y). More generally, Badescu et al. (2009) assumed a bivariate phase-type distribution for

(V, Y). Albrecher and Teugels (2006) examined asymptotics for ruin probabilities for the present model, and a similar dependency structure is also examined by Albrecher and Boxma (2004). Boudreault et al. (2006) considered the dependent Poisson risk model with $K(t) = 1 - e^{-\lambda t}$ and $P_t(y) = e^{-\beta t} F_1(y) + (1 - e^{-\beta t}) F_2(y)$ where $F_1(y)$ and $F_2(y)$ are "usual" and "severe" claim size distribution functions, respectively. In particular, this dependent structure with Coxian interclaim times distribution is considered to illustrate how to obtain some joint and marginal distributions of ruin related quantities in Chapter 3. Recently, Marceau (2009) assumed a dependency structure via a bivariate geometric distribution in a discrete-time renewal risk process.

In the following, two new quantities regarding the above penalty function generalizes (1.1) are introduced by Cheung et al. (2010b). First define $X_t = \inf_{0 \leq s < t} U_s$ to be the minimum surplus before time t. Therefore, X_T is the minimum surplus before ruin. Second, let

$$R_n = u + \sum_{i=1}^{n} (cV_i - Y_i), \qquad n = 1, 2, ...,$$

and define $R_0 = u$. Clearly, R_n is the surplus immediately following the n-th claim if $n \geq 1$, and $R_{N_T - 1}$ is the surplus immediately after the second last claim before ruin occurs if $N_T > 1$, and $R_{N_T} = u$ if ruin occurs on the first claim (i.e. $N_T = 1$). Note that $R_{N_T - 1}$ may or may not equal X_T. Analysis involving X_T has been considered in a Lévy process setting by Doney and Kyprianou (2006). Also, we remark that the minimum surplus level variable added in the penalty function was also studied by Biffis and Kyprianou (2010), and by Biffis and Morales (2010) in the context of Lévy insurance risk processes. For reference, Cheung and Landriault (2010) consider an additional variable, namely the maximum surplus level before ruin, in the classical Gerber-Shiu penalty function, to analyze a taxation model (e.g. Albrecher and Hipp (2007)).

Three graphs below depicts the four variables in the generalized penalty function and some associated quantities under the ordinary (and delayed) Sparre Andersen renewal risk models as well as the discrete-time cases.

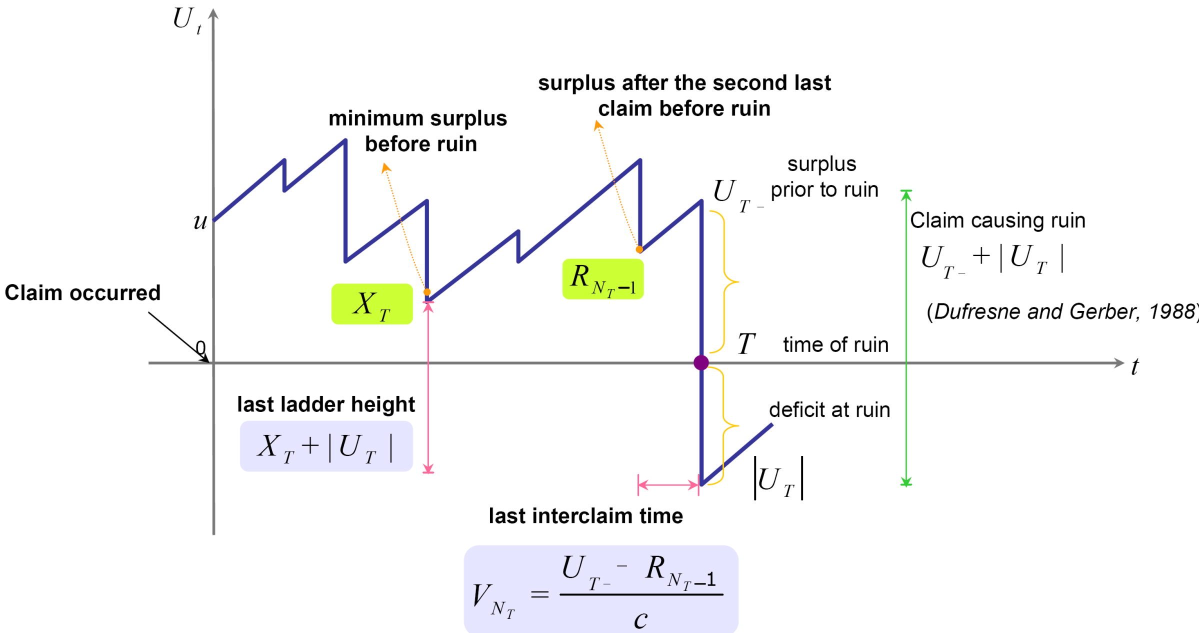

Figure 1.1: Ruin related quantities in the ordinary renewal risk models

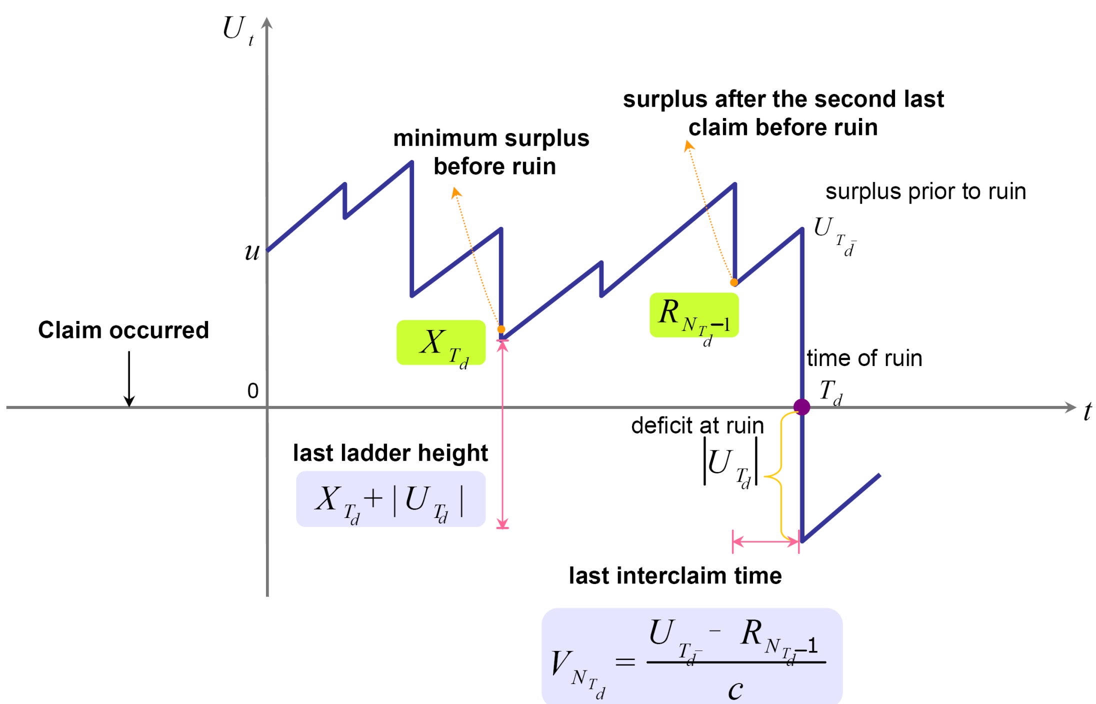

Figure 1.2: Ruin related quantities in the delayed renewal risk models

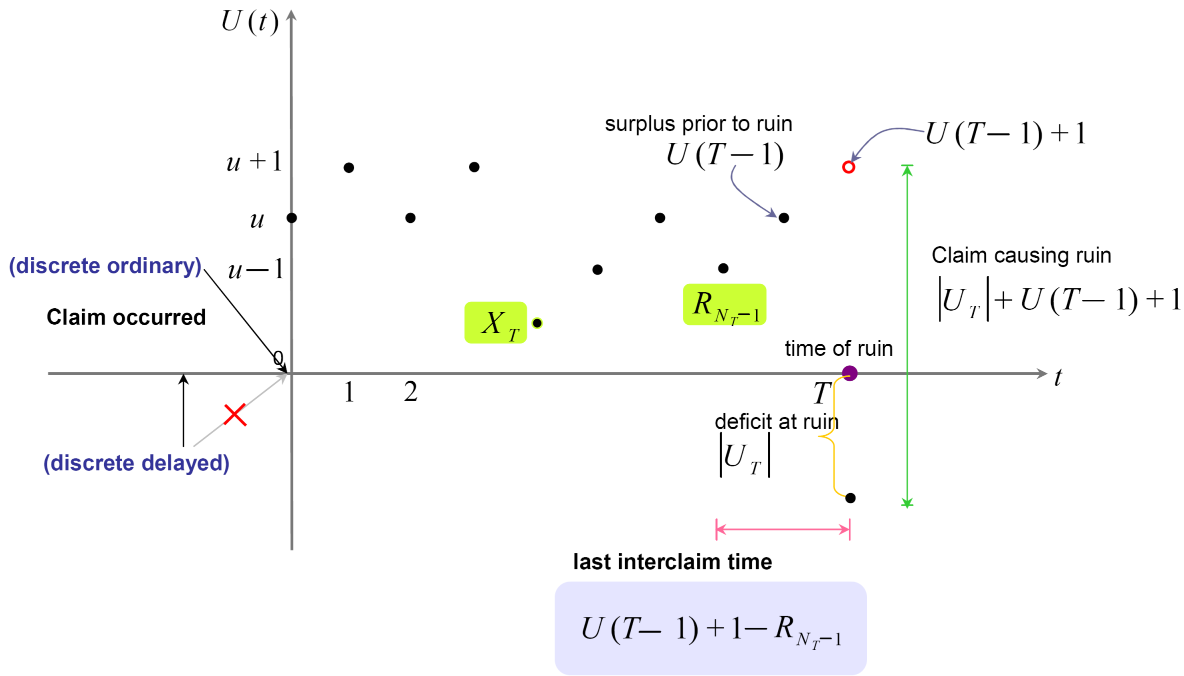

Figure 1.3: Ruin related quantities in discrete-time renewal risk models

Then we generalize (1.1) to

$$m_\delta^*(u) = E\left[e^{-\delta T} w^*\left(U_{T^-}, |U_T|, X_T, R_{N_T-1}\right) \mathrm{I}\left(T < \infty\right) \middle| U_0 = u\right], \qquad u \geq 0. \tag{1.2}$$

Remark that the introduction of these new quantities allows us to analyze the last ladder height before ruin $X_T + |U_T|$, and the last interclaim time before ruin $V_{N_T} = (U_{T^-} - R_{N_T-1})/c$. In addition, V_{N_T} has been studied by Cheung et al. (2010a) in the classical compound Poisson risk model (with $K(t) = 1 - e^{-\lambda t}$) via the Gerber-Shiu function

$$m_\delta(u) = E\left[e^{-\delta T} w\left(U_{T^-}, |U_T|, R_{N_T-1}\right) \mathrm{I}\left(T < \infty\right) \middle| U_0 = u\right], \qquad u \geq 0, \tag{1.3}$$

a special case of (1.2) with $w^*(x, y, z, v) = w(x, y, v)$. Thus (1.3) allows for the analysis of the last pair (V_{N_T}, Y_{N_T}) before ruin, and we remark that the claim causing ruin $Y_{N_T} = U_{T^-} + |U_T|$ has been studied on numerous occasions, beginning with Dufresne and Gerber (1988). Also, we will show that the analysis of this Gerber-Shiu function is essential to obtain the general case (1.2) in the following chapter.

In Chapter 2, we examine the mathematical structure of the above Gerber-Shiu functions as well as the particular special cases

$$m_{\delta,123}(u) = E\left[e^{-\delta T} w_{123}\left(U_{T^-}, \left|U_T\right|, X_T\right) \mathrm{I}\left(T < \infty\right) \middle| U_0 = u\right], \tag{1.4}$$

$$m_{\delta,23}(u) = E\left[e^{-\delta T} w_{23}\left(\left|U_T\right|, X_T\right) \mathrm{I}\left(T < \infty\right) \middle| U_0 = u\right], \tag{1.5}$$

$$m_{\delta,2}(u) = E\left[e^{-\delta T} w_2\left(\left|U_T\right|\right) \mathrm{I}\left(T < \infty\right) \middle| U_0 = u\right], \tag{1.6}$$

and

$$\overline{G}_\delta(u) = E\left[e^{-\delta T} \mathrm{I}\left(T < \infty\right) \middle| U_0 = u\right], \tag{1.7}$$

which correspond to the successively simplified penalty functions given by $w(x,y,z,v) = w_{123}(x,y,z)$, $w(x,y,z,v) = w_{23}(y,z)$, $w(x,y,z,v) = w_2(y)$, and $w(x,y,z,v) = 1$, respectively. Certainly, with $\delta = 0$, (1.7) reduces to the ruin probability $\psi(u)$ given by $\psi(u) = \Pr(T < \infty | U_0 = u)$. Under PSLC, it is known that $\psi(u) < 1$ for $u \geq 0$ (e.g. Asmussen (2000), Dufresne (2001), Feller (1971), Prabhu (1988)).

1.3 Mathematical preliminaries

We adopt the notational convention that the empty product is 1, and the empty sum is 0. Here are some mathematical preliminaries used later in this thesis.

(1) Dickson-Hipp operator and its property

An operator known as the *Dickson-Hipp operator* T_r (with $\mathrm{Re}(r) \geq 0$) defined as

$$T_r f(y) = \int_y^{\infty} e^{-r(x-y)} f(x)\, dx = \int_0^{\infty} e^{-rx} f(x+y) dx,$$

for an integrable function f (e.g. Dickson and Hipp (2001)) plays a role in the analysis of the expected discounted penalty function $m(u)$. Properties of the Dickson-Hipp operator T_r which notably include

$$T_{r_1} T_{r_2} f(y) = \frac{T_{r_1} f(y) - T_{r_2} f(y)}{r_2 - r_1}, \qquad r_1 \neq r_2,$$

for $\mathrm{Re}(r_i) \geq 0$ for $i = 1, 2$ are discussed in Li and Garrido (2004). If $\widetilde{h}(s) = \int_0^{\infty} e^{-sy} h(y) dy$ is the Laplace transform of $h(x)$, the Laplace transform of $T_r h(x)$ is given by

$$\int_0^{\infty} e^{-sx} \{T_r h(x)\} dx = T_s T_r h(0) = \frac{\widetilde{h}(r) - \widetilde{h}(s)}{s - r}.$$

Also, for the use in Chapter 4, a discrete version of Dickson-Hipp operation is defined as follows. For a function $h(y)$ defined on $y \in \mathbb{N}$, the discrete Dickson-Hipp operator denoted by $\mathcal{T}_r$ with $|r| \leq 1$ is defined as

$$\mathcal{T}_r h(y) = \sum_{x=y}^{\infty} r^{x-y} h(x) = \sum_{x=0}^{\infty} r^x h(x+y), \qquad y \in \mathbb{N}\ .$$

Clearly, $\mathcal{T}_r h(0) = \sum_{x=0}^{\infty} r^x h(x) = \hat{h}(r)$ is the generating function of $h(x)$, and if s and r are distinct then

$$\frac{\hat{h}(s)-\hat{h}(r)}{s-r} = \frac{\sum_{x=0}^{\infty} s^x h(x) - \sum_{x=0}^{\infty} r^x h(x)}{s-r} = \frac{\sum_{x=1}^{\infty} s^x h(x) + h(0) - \sum_{x=1}^{\infty} r^x h(x) - h(0)}{s-r}$$
$$= \frac{s\sum_{x=1}^{\infty} s^{x-1} h(x) - r\sum_{x=1}^{\infty} r^{x-1} h(x)}{s-r},$$

that is,

$$\frac{\hat{h}(s)-\hat{h}(r)}{s-r} = \frac{s\mathcal{T}_s h(1) - r\mathcal{T}_r h(1)}{s-r} = \mathcal{T}_s\mathcal{T}_r h(1) = \sum_{x=0}^{\infty} s^x \left\{\mathcal{T}_r h(x+1)\right\}. \tag{1.8}$$

For details regarding several nice properties of this operator, see Section 3 in Li (2005a) but the operator defined therein is for a function $h(x)$ on $x \in \mathbb{N}^+$.

(2) Laplace transform

The Laplace transform of a function $f(t)$, defined all real numbers $t \geq 0$, is the function $\widetilde{f}(s)$, defined by

$$\widetilde{f}(s) = \int_0^{\infty} e^{-st} f(t)dt.$$

In this thesis, we will use Laplace transforms, denoted by '$\sim$' above the letter. See Spiegel (1965a) for further details. Obviously, it is a special case of Dickson-Hipp operator introduced in (1), namely, $T_s f(0) = \widetilde{f}(s)$.

(3) Initial value theorem (e.g. Spiegel (1965a, p.5))

If the indicated limits exists, then

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} s\widetilde{f}(s).$$

Note that in order to apply this result, the function $f(t)$ is differentiable and the Laplace transform of $f'(t)$ is given by $\int_0^{\infty} e^{-st} f'(t)dt = s\widetilde{f}(s) - f(0)$.

(4) Dominated convergence theorem (e.g. Spiegel (1965b, p.74))

Let $\{f_n(x)\}$ be a sequence of functions measurable on set E such that $\lim_{n\to\infty} f_n(x) = f(x)$. Then if there exists a function $M(x)$ integrable on E such that $|f_n(x)| \le M(x)$ for all n, we have

$$\lim_{n\to\infty} \int_E f_n(x)dx = \int_E \lim_{n\to\infty} f_n(x)dx = \int_E f(x)dx.$$

The remainder of the present section contains a brief review of reliability properties and classifications for the analysis in Chapter 6. See Barlow and Proschan (1981), Fagiuoli and Pellerey (1993, 1994) and references therein for details. In particular, more applications of classifications of claim sizes and the number of claims distributions can be founded in Gerber (1979), Grandell (1997), Kalashnikov (1999), Lin (1996), Lin and Willmot (1999, 2000), and Willmot (1994).

(5) Reliability

Let $F_1(y) = 1 - \overline{F}(y)$ be the equilibrium df of $F(y)$ defined by $F_1(y) = \int_0^y \overline{F}(x)dx / \int_0^\infty \overline{F}(x)dx$. The df $F(y)$ is said to be decreasing (increasing) failure rate or DFR (IFR) if $\overline{F}(x+y)/\overline{F}(y)$ is nondecreasing (nonincreasing) in y for fixed $x \ge 0$.

Let the mean residual lifetime (MRL) be $r(y)$ given by

$$r(y) = \int_y^\infty \frac{(t-y)dF(t)}{\overline{F}(y)} = \int_0^\infty \frac{\overline{F}(y+t)}{\overline{F}(y)}dt, \qquad y \ge 0.$$

The df $F(y)$ is said to be increasing (decreasing) mean residual life time or IMRL (DMRL) if $r(y)$ is nondecreasing (nonincreasing) in y. We then know that DFR (IFR) implies IMRL (DMRL). Namely, IMRL (DMRL) class is larger than DFR (IFR) class. For reference, a related class of distribution is the used worse (better) than aged or UWA (UBA) class (e.g. Alzaid (1994), Willmot and Cai (2000)).

Another class larger than the DFR (IFR) class is the new worse (better) than used or NWU (NBU) for which

$$\overline{F}(x+y) \ge (\le)\overline{F}(x)\overline{F}(y), \qquad x \ge 0,\ y \ge 0.$$

Similarly, a larger class than the IMRL (DMRL) class is the 2-NWU (2-NBU) for which $F_1(y)$ is NWU (NBU).

Moreover, the df $F(y)$ is said to be the new worse (better) than used in convex ordering or NWUC (NBUC) if

$$\overline{F}_1(x+y) \geq (\leq) \overline{F}_1(x)\overline{F}(y), \qquad x \geq 0,\ y \geq 0.$$

The 2-NWU (2-NBU) and NWU (NBU) classes are contained in the NWUC (NBUC) class.

Finally, the df $F(y)$ is said to be the new worse (better) than used in expectation or NWUE (NBUE) if

$$\overline{F}_1(y) \geq (\leq) \overline{F}(y).$$

Thus, the NWUC (NBUC) is a subclass of NWUE (NBUE).

This thesis is organized as follows. In the next chapter we examine the structure of various Gerber-Shiu functions in dependent Sparre Andersen risk models. In Section 2.1, it is shown that the generalized Gerber-Shiu function (1.2) satisfies the defective renewal equation whose solution can be expressed in terms of a compound geometric tail (e.g. Lin and Willmot (1999), Resnick (1992)). As a result, defective joint and marginal distributions involving the quantities in the generalized penalty function are derived in Section 2.2. In particular, the case that the claim sizes are independent of the interclaim times is covered with an example of the exponential claim sizes and arbitrary interclaim times in Section 2.3. For the identification of the components in the defective renewal equation obtained in the previous chapter, we assume certain interclaim times distributions in Chapter 3. First, in Section 3.1 a Coxian interclaim times distribution is considered to analyze the generalized Gerber-Shiu function. In particular, the time-dependent claim size case is studied in Section 3.1.5. As a special case for a class of Coxian interclaim distributions, the classical compound Poisson risk models are assumed to obtain the proper deficit distribution given that ruin occurs for the time-dependent claims, which is the subject matter of

Section 3.2. Also, the joint defective density of the variables in the penalty function involving the time to ruin is derived in Section 3.3. While Chapter 2 and Chapter 3 are concerned with the ordinary Sparre Andersen risk models with some dependency examples, the modification of these models, namely the delayed Sparre Andersen risk models with dependency are considered in Chapter 4. Similar analysis related to the ordinary processes is considered in Chapter 4, whereas Chapter 5 is devoted to the discrete analog of the all previous models. That is, the general structural results under the discrete renewal risk models including the delayed case are derived followed by assuming some specific interclaim times distribution, for instance, a discrete K_n class distribution and a compound binomial distribution. Finally, the two-sides bounds for a renewal equation and their application are provided in Chapter 6.

Chapter 2

Structural properties

In the following section, we demonstrate that all Gerber-Shiu functions introduced in Chapter 1 satisfy defective renewal equations, each of which has associated compound geometric tail (in the sense of Willmot and Lin (2001, Section 9.1)) given by (1.7). In Section 2.2, the results of Section 2.1 are used to derive various joint and marginal distributions, and in particular an alternative expression for $m_\delta(u)$ in (1.3) is obtained as well. Finally, in Section 2.3, some further remarks concerning the independent case are made, and the case with exponential claims is considered in some detail. In particular, the joint Laplace transform of $(T, U_{T^-}, |U_T|, X_T, R_{N_T-1})$ is derived with exponential claim sizes, and the last interclaim time V_{N_T} before ruin is shown to have an Esscher transformed distribution which is stochastically dominated by a generic interclaim time distribution.

2.1 Defective renewal equations

To begin with, we first examine the nature of the joint distribution of the time of ruin T, the surplus prior to ruin U_{T^-}, the deficit at ruin $|U_T|$, and the surplus immediately after the second last claim before ruin R_{N_T-1}. If ruin occurs on the first claim, then the surplus (x) and the time (t) are related by $x = u + ct$, or equivalently $t = (x-u)/c$. Once the surplus x has been reached, a claim of size $x+y$ results in a deficit of y. The density is thus $k(t)p_t(x+y)$ where $t = (x-u)/c$. Therefore, a change of variables from t to x implies that the joint defective density of the surplus prior to ruin (x) and the deficit at ruin (y) for ruin occurring on the first claim is (e.g. Landriault and Willmot (2009) for the time-independent claims) given by

$$h_1(x, y\,|u) = \frac{1}{c}k\left(\frac{x-u}{c}\right)p_{\frac{x-u}{c}}(x+y), \tag{2.1}$$

and in this case the time of ruin T is $(x-u)/c$ and R_{N_T-1} equals u. If ruin occurs on claims subsequent to the first, T and R_{N_T-1} are no longer simple functions of U_{T^-} and $|U_T|$, and we denote the joint defective density of the time of ruin (t), the surplus before ruin (x), the deficit at ruin (y), and the surplus after the second last claim (v), by $h_2(t, x, y, v|u)$ for $v < x$. See Cheung et al. (2010a) for further discussion of this density in the classical compound Poisson risk model.

We now employ the argument of Gerber and Shiu (1998) to obtain an integral equation for $m^*_\delta(u)$. We will thus condition on the first drop in the surplus to a value below its initial level of u. The density of this first drop for a drop on the first claim is $h_1(x, y|0)$, where x represents the surplus level above u just before the drop (i.e. the surplus becomes $x+u$), and y is the drop below u, so that the surplus level after the drop is $u-y$. The time of this drop is x/c. If $y > u$, then ruin occurs on the first drop, and in this case $U_{T^-} = x+u$, $|U_T| = y-u$, $X_T = u$, and $R_{N_T-1} = u$. If $y < u$ then ruin does not occur, and the process begins anew (probabilistically) beginning at the surplus level $u-y$. If the drop in surplus below u does not occur on the first claim, then the density is $h_2(t, x, y, v|0)$. Again, ruin occurs if $y > u$, and in this case $U_{T^-} = x+u$, $|U_T| = y-u$, $X_T = u$, and $R_{N_T-1} = v+u$. Similarly, if $y < u$ then ruin does not occur, and the process continues from the new surplus level of $u-y$. Summing (integrating) over all values of t, x, y, and v results in the integral equation satisfied by (1.2), namely

$$m_\delta^*(u) = \int_0^u m_\delta^*(u-y)\left\{\int_0^\infty h_{1,\delta}\left(x,y\,|0\right)dx + \int_0^\infty \int_0^x h_{2,\delta}(x,y,v|0)dvdx\right\}dy + v_\delta^*(u), \quad (2.2)$$

where

$$h_{1,\delta}\left(x,y\,|u\right) = e^{-\frac{\delta(x-u)}{c}} h_1(x,y|u) \quad (2.3)$$

and

$$h_{2,\delta}(x,y,v|u) = \int_0^\infty e^{-\delta t} h_2(t,x,y,v|u)dt \quad (2.4)$$

are "discounted" densities. In this case, $v_\delta^*(u)$ is the contribution due to ruin on the first drop and is given by

$$v_\delta^*(u) = \left\{\int_u^\infty \int_0^\infty w^*(x+u,y-u,u,u)h_{1,\delta}(x,y|0) + \int_0^x w^*(x+u,y-u,u,v+u)h_{2,\delta}(x,y,v|0)dv\right\}dxdy. \quad (2.5)$$

Using (2.1), (2.5) may also be written as

$$\begin{aligned} v_\delta^*(u) &= \int_0^\infty e^{-\delta t}\left\{\int_0^\infty w^*(u+ct,y,u,u)p_t(y+ct+u)dy\right\}k(t)dt \\ &+ \int_u^\infty \int_0^\infty \int_0^x w^*(x+u,y-u,u,v+u)h_{2,\delta}^*(x,y,v|0)dvdxdy. \end{aligned} \quad (2.6)$$

In the following theorem, we now examine the structure of (2.2) in more detail.

Theorem 1 *The Gerber-Shiu function with the generalized penalty function in (1.2) satisfies the defective renewal equation given by*

$$m_\delta^*(u) = \phi_\delta \int_0^u m_\delta^*(u-y)f_\delta(y)dy + v_\delta^*(u), \qquad u \geq 0, \quad (2.7)$$

where ϕ_δ , $f_\delta(y)$ and $v_\delta^*(u)$ are given by (2.11), (2.12) and (2.6) respectively.

Proof. First, note that the discounted (marginal if $\delta = 0$) density of U_{T^-} and $|U_T|$ is obtained by summing and integrating over all values of t and v, yielding

$$h_\delta\left(x, y\,|u\right) = h_{1,\delta}\left(x, y\,|u\right) + \int_0^x h_{2,\delta}(x, y, v|u)dv. \tag{2.8}$$

Using (2.8) with $u = 0$, (2.2) may be re-expressed as

$$m_\delta^*(u) = \int_0^u m_\delta^*(u-y)\left\{\int_0^\infty h_\delta(x, y|0)dx\right\}dy + v_\delta^*(u). \tag{2.9}$$

It is not hard to see from (2.8) that (1.1) may be written as

$$m_{\delta,12}(u) = \int_0^\infty \int_0^\infty w_{12}(x, y)h_\delta(x, y|u)dxdy. \tag{2.10}$$

Thus, letting

$$\phi_\delta = \int_0^\infty \int_0^\infty h_\delta(x, y|0)dxdy, \tag{2.11}$$

it is clear from (2.10) with $w_{12}(x, y) = 1$ and (1.1) that $\phi_\delta = E\left[e^{-\delta T}\mathrm{I}\left(T < \infty\right)\big|U_0 = 0\right] < 1$. Also, define the ladder height density

$$f_\delta(y) = \frac{1}{\phi_\delta}\int_0^\infty h_\delta(x, y|0)dx, \tag{2.12}$$

which is clearly the same as the marginal discounted proper density of the deficit $|U_T|$ when $u = 0$. Thus, (2.9) may be written as (2.7). $\square$

It is clear from (2.7) that the generalized Gerber-Shiu function (1.2) satisfying a defective renewal equation only depends on the joint distribution of U_{T^-}, $|U_T|$, and R_{N_T-1} with zero initial surplus. Therefore, the analysis of $m_\delta(u)$ in (1.3) is essential to obtain various information regarding all variables in the generalized penalty function including the time to ruin. If

$w^*(x, y, z, v) = w(x, y, v)$, (2.5) and (2.7) reduces to

$$m_\delta(u) = \phi_\delta \int_0^u m_\delta(u-y) f_\delta(y) dy + v_\delta(u), \tag{2.13}$$

where

$$v_\delta(u) = \int_u^\infty \int_0^\infty \left\{ w(x+u, y-u, u) h_{1,\delta}(x, y|0) + \int_0^x w(x+u, y-u, v+u) h_{2,\delta}(x, y, v|0) dv \right\} dxdy.$$

Changing variables of integration yields

$$v_\delta(u) = \int_0^\infty \int_u^\infty \left\{ w(x, y, u) h_{1,\delta}(x, y|u) + \int_u^x w(x, y, v) h_{2,\delta}(x-u, y+u, v-u|0) dv \right\} dxdy, \tag{2.14}$$

since $h_{1,\delta}(x-u, y+u|0) = h_{1,\delta}(x, y|u)$. In Chapter 3, (2.13) is studied with identifications of ϕ_δ, $f_\delta(u)$, and $h_\delta(x, y|0)$ under the specific assumptions of the interclaim times.

Furthermore, the form of $v_\delta^*(u)$ and hence also (2.7) simplifies in some special cases. First, if $w^*(x, y, z, v) = w_{123}(x, y, z)$ so that (1.2) does not involve $R_{N_T - 1}$, then the right-hand side of (2.5) simplifies to

$$\int_u^\infty \int_0^\infty w_{123}(x+u, y-u, u) \left\{ h_{1,\delta}(x, y|0) + \int_0^x h_{2,\delta}(x, y, v|0) dv \right\} dxdy.$$

Thus, using (2.8), (1.4) satisfies the simpler defective renewal equation

$$m_{\delta,123}(u) = \phi_\delta \int_0^u m_{\delta,123}(u-y) f_\delta(y) dy + v_{\delta,123}(u), \tag{2.15}$$

where

$$v_{\delta,123}(u) = \int_u^\infty \int_0^\infty w_{123}(x+u, y-u, u) h_\delta(x, y|0) dxdy. \tag{2.16}$$

The special case (2.15) of (2.7) is analytically simpler due to the fact that it only involves $h_\delta(x, y|0)$.

Further simplification of (2.16) and hence (2.15) occurs if $w(x, y, z, v) = w_{23}(y, z)$, so that only

$|U_T|$ and X_T are involved. Clearly from (2.12) and (2.16), (1.5) satisfies the simpler defective renewal equation

$$m_{\delta,23}(u) = \phi_\delta \int_0^u m_{\delta,23}(u-y) f_\delta(y) dy + v_{\delta,23}(u), \tag{2.17}$$

where

$$v_{\delta,23}(u) = \phi_\delta \int_u^\infty w_{23}(y-u,u) f_\delta(y) dy, \tag{2.18}$$

and it is clear from (2.18) that $m_{\delta,23}(u)$ depends only on the ladder height density $f_\delta(y)$. Interestingly, the distribution of the last ladder height $X_T + |U_T|$ may be determined from that of the generic ladder height distribution.

Next, we note that if $w^* (x, y, z, v) = w_2 (y)$, then from (2.17) and (2.18), (1.6) satisfies the simpler defective renewal equation

$$m_{\delta,2} (u) = \phi_\delta \int_0^u m_{\delta,2} (u-y) f_\delta (y) dy + \phi_\delta \int_u^\infty w_2 (y-u) f_\delta (y) dy. \tag{2.19}$$

Equation (2.19) is the same defective renewal equation as in the independence case (see Willmot (2007, equation 2.11)), but with ϕ_δ and $f_\delta (y)$ defined by (2.11) and (2.12) respectively. Furthermore, with $w (x, y, z, v) = w_2 (y) = 1$, (1.7) satisfies

$$\overline{G}_\delta (u) = \phi_\delta \int_{0+}^u \overline{G}_\delta (u-y) f_\delta (y) dy + \phi_\delta \overline{F}_\delta (u), \tag{2.20}$$

and therefore $\overline{G}_\delta (u) = 1 - G_\delta (u)$ is (as the solution to (2.20) is well-known to be) a compound geometric tail, i.e.

$$\overline{G}_\delta (u) = \sum_{n=1}^\infty (1-\phi_\delta) (\phi_\delta)^n \overline{F}_\delta^{*n} (u), \qquad u \geq 0,$$

where $F_\delta (u) = 1 - \overline{F}_\delta (u) = \int_0^u f_\delta (y) dy$ and $1 - \overline{F}_\delta^{*n} (u)$ is the distribution function of the n-fold convolution. Of course, $\phi_\delta = \overline{G}_\delta(0)$, and the ruin probability is given by $\psi(u) = \Pr (T < \infty | U_0 = u)$ equivalent to $\overline{G}_0(u)$.

The general solution to (2.7) (or the special cases (2.13), (2.15), (2.17) or (2.19)) is expressible in terms of the compound geometric density $g_\delta(u) = -\overline{G}'_\delta(u)$ given by

$$g_\delta (u) = \sum_{n=1}^{\infty} (1 - \phi_\delta) (\phi_\delta)^n f_\delta^{*n} (u), \qquad u \geq 0, \tag{2.21}$$

where $f_\delta^{*n}(u) = -\frac{d}{du}\overline{F}_\delta^{*n}(u)$ is the density of the n-fold convolution of $f_\delta(u)$. It is well-known (e.g. Resnick (1992, Section 3.5)) that

$$m_\delta(u) = v_\delta(u) + \frac{1}{1-\phi_\delta} \int_0^u g_\delta(u-y) v_\delta(y) dy. \tag{2.22}$$

An alternative form of the solution which is convenient if $v_\delta(u)$ is differentiable is (e.g. Willmot and Lin (2001, p.154))

$$m_\delta(u) = \frac{1}{1-\phi_\delta} \left[v_\delta(u) - v_\delta(0)\overline{G}_\delta(u) - \int_0^u \overline{G}_\delta(u-y) v'_\delta(y) dy \right] . \tag{2.23}$$

As for the deficit itself, we remark that because (2.19) is functionally of the same form as in the more well-known independent case, it follows that any properties of the distribution of the deficit $|U_T|$ are formally the same as in the independent case, but with the present definitions of ϕ_δ and $f_\delta(y)$. In particular, it follows directly from Willmot (2002) that

$$\Pr\left(|U_T| > y \middle| T < \infty\right) = \frac{\int_{0^-}^u \left\{ \frac{\overline{F}_0(y+u-t)}{\overline{F}_0(u-t)} \right\} \overline{F}_0 (u-t) \, dG_0 (t)}{\int_{0^-}^u \overline{F}_0 (u-t) \, dG_0 (t)},$$

so that the conditional distribution of $|U_T|$ given $T < \infty$ remains a mixture of the residual lifetime distribution associated with F_0. In Section 3.2, this conditional distribution is derived under the certain dependent structure of $P_t(y)$ studied by Boudreault et al. (2006).

2.2 Associated defective distributions

As pointed out in Section 2.1, Theorem 1, trivariate "discounted" defective distribution of U_{T^-}, $|U_T|$ and R_{N_T-1} is sufficient to determine that of $U_{T^-}, |U_T|, R_{N_T-1}$, and X_T. Therefore, in the following we first derive this distribution by analyzing $m_\delta(u)$ in (1.3) with an alternative form of this. In turn, we study various joint and marginal distributions associated with these four variables. All results presented in this section are obtained with no specific assumptions on the claim sizes or the interclaim times.

2.2.1 Alternative expression for the generalized Gerber-Shiu function

Theorem 2 *If $h_{2,\delta}(x-u, y+u, v-u|0)$ may be expressed as*

$$h_{2,\delta}(x,y,v|0) = h_{1,\delta}(x,y|v)\nu_\delta(v), \tag{2.24}$$

where $\nu_\delta(v-u)$ for $v > u$ is a nonnegative function representing the discounted transition in the surplus from 0 to $v-u$, then we may find the Gerber-Shiu function (1.3) in the form as

$$m_\delta(u) = \beta_\delta(u) + \int_0^\infty \beta_\delta(v)\tau_\delta(u,v)dv, \tag{2.25}$$

where

$$\tau_\delta(u,v) = \begin{cases} \frac{1}{1-\phi_\delta}\left\{g_\delta(u-v) + \int_0^v \nu_\delta(v-t)g_\delta(u-t)dt\right\}, & v < u \\ \nu_\delta(v-u) + \frac{1}{1-\phi_\delta}\int_0^u \nu_\delta(v-t)g_\delta(u-t)\,dt, & v > u \end{cases}, \tag{2.26}$$

and $\nu_\delta(v) = \tau_\delta(0,v)$ is given by (2.24). In particular, for $\delta = 0$,

$$\tau_0(u,v) = \begin{cases} \frac{1}{1-\psi(0)}\left\{-\psi'(u-v) - \int_0^v \nu_0(v-t)\psi'(u-t)dt\right\}, & v < u \\ \nu_0(v-u) - \frac{1}{1-\psi(0)}\int_0^u \nu_0(v-t)\psi'(u-t)\,dt, & v > u \end{cases},$$

since $\phi_0 = \psi(0)$ *and* $g_0(u) = -\psi'(u)$.

Proof. For notational convenience, let

$$\beta_\delta(u) = \int_0^\infty \int_u^\infty w(x, y, u) h_{1,\delta}(x, y|u) dx dy, \tag{2.27}$$

and

$$\xi_\delta(u) = \int_u^\infty \int_0^\infty \int_v^\infty w(x, y, v) h_{2,\delta}(x - u, y + u, v - u|0) dx dy dv, \tag{2.28}$$

so that (2.14) can be expressed as $v_\delta(u) = \beta_\delta(u) + \xi_\delta(u)$. Then, using (2.22) we get the solution to $m_\delta(u)$ given by

$$\begin{aligned} m_\delta(u) &= v_\delta(u) + \frac{1}{1-\phi_\delta} \int_0^u g_\delta(u-t) v_\delta(t) dt \\ &= \beta_\delta(u) + \xi_\delta(u) + \frac{1}{1-\phi_\delta} \int_0^u g_\delta(u-t) \{\beta_\delta(t) + \xi_\delta(t)\} dt. \end{aligned} \tag{2.29}$$

Hence, (2.28) becomes, using (2.24) we obtain $\xi_\delta(u) = \int_u^\infty \beta_\delta(v)\nu_\delta(v-u)dv$, and with the above expression, the right-hand side of (2.29) except for the first term may be re-expressed as

$$\begin{aligned} &\xi_\delta(u) + \frac{1}{1-\phi_\delta} \int_0^u g_\delta(u-t) \{\beta_\delta(t) + \xi_\delta(t)\} dt \\ &= \int_u^\infty \beta_\delta(v)\nu_\delta(v-u)dv + \frac{1}{1-\phi_\delta} \int_0^u g_\delta(u-t) \left\{\beta_\delta(t) + \int_t^\infty \beta_\delta(v)\nu_\delta(v-t)dv\right\} dt. \end{aligned}$$

Interchanging the order of integration yields

$$\begin{aligned} &\xi_\delta(u) + \frac{1}{1-\phi_\delta} \int_0^u g_\delta(u-t) \{\beta_\delta(t) + \xi_\delta(t)\} dt \\ = &\int_u^\infty \beta_\delta(v)\nu_\delta(v-u)dv + \frac{1}{1-\phi_\delta} \int_0^u \beta_\delta(v) g_\delta(u-v) dv \\ + &\frac{1}{1-\phi_\delta} \left\{\int_0^u \beta_\delta(v) \int_0^v \nu_\delta(v-t) g_\delta(u-t) dt dv + \int_u^\infty \beta_\delta(v) \int_0^u \nu_\delta(v-t) g_\delta(u-t) dt dv\right\} \end{aligned} \tag{2.30}$$

Therefore, substituting (2.30) into (2.29) leads (2.25). Note that with $u = 0$ in (2.26), it is easily seen that $\tau_\delta(0, v) = \nu_\delta(v)$. Also the right-hand side of (2.24) is interpreted as: after the surplus reaches at level $v - u$, the next drop causes ruin explained by $h_{1,\delta}$ function with the surplus prior to ruin $x-u$ and the deficit at ruin $y+u$. $\square$

Remark 1 *It is clear from (2.24), (2.25) and (2.26) that the discounted joint density $h_{2,\delta}$ with zero initial surplus is sufficient to identify $\tau_\delta(u,v)$ in (2.26) which is essential to analyze the generalized Gerber-Shiu function $m_\delta(u)$ in (1.3), certainly further $m^*_\delta(u)$ in (1.2). $\tau_\delta(u,v)$ is obtained in the classical Poisson risk model by Cheung et al. (2010a), K_n-class interclaim time process by Willmot and Woo (2010). Also, in the semi-Markovian model, a matrix form of $\tau_\delta(u,v)$ is derived by Cheung and Landriault (2009). Further analysis regarding (2.26) with the surplus dependent premium rate (i.e. general premium rate) in various risk models is studied by Cheung (2010).*

Furthermore using (2.25), it is readily to obtain $h_{2,\delta}(x,y,v|u)$ as follows.

Corollary 1 *When ruin occurs not on the first claim, the joint density of the surplus prior to ruin U_{T^-}, the deficit at ruin $|U(T)|$, and the surplus after the second last claim before ruin $R_{N(T)-1}$ at (x,y,v) is given by*

$$h_{2,\delta}(x,y,v|u) = h_{1,\delta}(x,y|v)\tau_\delta(u,v), \qquad x>v, \tag{2.31}$$

where $\tau_\delta(u,v)$ is given by (2.26).

Proof. (1.3) may be viewed as an expectation so that it may be expressed as

$$m_\delta(u) = \int_0^\infty \int_u^\infty w(x,y,u)h_{1,\delta}(x,y|u)dxdy + \int_0^\infty \int_0^\infty \int_v^\infty w(x,y,v)h_{2,\delta}(x,y,v|u)dxdydv. \tag{2.32}$$

Thus, using (2.27) and comparing the above expression for $m_\delta(u)$ to (2.25) results in

$$\int_0^\infty \int_0^\infty \int_v^\infty w(x,y,v)h_{2,\delta}(x,y,v|u)dxdydv = \int_0^\infty \int_0^\infty \int_v^\infty w(x,y,v)h_{1,\delta}(x,y|v)\tau_\delta(u,v)dxdydv.$$

and the proof is completed. $\square$

Also, (2.25) may be interpreted probabilistically by regarding $\beta_\delta(u)$ as the contribution from ruin occurring on the first claim and the rest of term as the contribution for ruin on the other claims. See Cheung et al. (2010a, Section 3) for further details.

Integrating out y in (2.31) yields the marginal bivariate discounted defective density of (x, v),

$$h_\delta^{(2)}(x, v|u) = h_{1,\delta}^{(1)}(x|v)\tau_\delta(u, v), \qquad x > v, \tag{2.33}$$

where $\int_0^\infty h_{1,\delta}(x, y|v)dy = h_{1,\delta}^{(1)}(x|v)$.

Next, we remark that the *Lundberg's fundamental equation* (Lundberg (1932, p.144)) is given by

$$E\left[e^{-sY-(\delta-cs)V}\right] = 1, \tag{2.34}$$

which is important to analyze the Gerber-Shiu function in the present model, as we now demonstrate. First, Cheung et al. (2010b) considered the function

$$\eta(u) = \int_0^\infty e^{-\delta t}\omega_t(u + ct)\, dK(t) \tag{2.35}$$

for some function ω_t. The Laplace transform is

$$\begin{aligned}
\widetilde{\eta}(s) &= \int_0^\infty e^{-su} \int_0^\infty e^{-\delta t}\omega_t(u + ct)\, dK(t)\, du \\
&= \int_0^\infty e^{-(\delta-cs)t} \left\{\int_0^\infty e^{-s(u+ct)}\omega_t(u + ct)\, du\right\} dK(t) \\
&= \int_0^\infty e^{-(\delta-cs)t} \left\{\widetilde{\omega}_t(s) - \int_0^{ct} e^{-sx}\omega_t(x)\, dx\right\} dK(t) \\
&= \int_0^\infty e^{-(\delta-cs)t}\widetilde{\omega}_t(s)\, dK(t) - \int_0^\infty \int_0^{ct} e^{-\frac{1}{c}\{\delta x+(\delta-cs)(ct-x)\}}\omega_t(x)\, dx\, dK(t).
\end{aligned}$$

That is,

$$\widetilde{\eta}(s) = \int_0^\infty e^{-(\delta-cs)t}\widetilde{\omega}_t(s)\, dK(t) - \widetilde{\omega}_*(\delta - cs), \tag{2.36}$$

where

$$\widetilde{\omega}_*(s) = \int_0^\infty \int_0^{ct} e^{-\frac{1}{c}\{\delta x + s(ct-x)\}} \omega_t(x)\, dx\, dK(t).$$

In order to express ϕ_δ, $f_\delta(y)$, and $h_\delta(x, y|0)$ in (2.7) in terms of quantities related to the claim size distribution $P_t(y)$ and/or the interclaim time distribution $K(t)$, a common approach is to condition on the time and the amount of the first claim. By applying this approach to obtain the integral equation for $m_\delta(u)$ in (1.3), it follows that

$$m_\delta(u) = \beta_\delta(u) + \int_0^\infty e^{-\delta t} \sigma_{t,\delta}(u + ct) dK(t), \tag{2.37}$$

where

$$\beta_\delta(u) = \int_0^\infty e^{-\delta t} \alpha_t(u + ct, u) dK(t) \tag{2.38}$$

with

$$\alpha_t(x, u) = \int_x^\infty w(x, y - x, u) dP_t(y), \tag{2.39}$$

and

$$\sigma_{t,\delta}(x) = \int_0^x m_\delta(x - y) dP_t(y), \tag{2.40}$$

Note that $\beta_\delta(u)$ is the contribution to the penalty function due to ruin on the first claim, as is clear from the alternative representation given by (2.27).

The term on the right-hand side of (2.37) is of the form (2.35), and thus taking Laplace transforms of (2.37) yields, using (2.36)

$$\widetilde{m}_\delta(s) = \widetilde{\beta}_\delta(s) + \int_0^\infty e^{-(\delta - cs)t} \widetilde{\sigma}_{t,\delta}(s)\, dK(t) - \widetilde{\sigma}_w(\delta - cs), \tag{2.41}$$

where

$$\widetilde{\sigma}_w(s) = \int_0^\infty \int_0^{ct} e^{-\frac{1}{c}\{\delta x + s(ct-x)\}} \sigma_{t,\delta}(x)\, dx\, dK(t). \tag{2.42}$$

With an interchange of order of the integration, (2.42) becomes

$$\widetilde{\sigma}_w(s) = \int_0^\infty e^{-\frac{x}{c}(\delta - s)} \int_{\frac{x}{c}}^\infty e^{-st} \sigma_{t,\delta}(x)\, dK(t)\, dx. \tag{2.43}$$

But $\widetilde{\sigma}_{t,\delta}(s) = \widetilde{m}_\delta(s)\widetilde{p}_t(s)$ from (2.40), and thus (2.41) may be expressed as

$$\widetilde{m}_\delta(s) = \widetilde{\beta}_\delta(s) + \widetilde{m}_\delta(s) \int_0^\infty e^{-(\delta - cs)t} \widetilde{p}_t(s)\, dK(t) - \widetilde{\sigma}_w(\delta - cs),$$

and because $E[e^{-sY-(\delta-cs)V}] = \int_0^\infty e^{-(\delta-cs)t}\widetilde{p}_t(s)\, dK(t)$, it follows that

$$\left\{1 - E[e^{-sY-(\delta-cs)V}]\right\} \widetilde{m}_\delta(s) = \widetilde{\beta}_\delta(s) - \widetilde{\sigma}_w(\delta - cs). \tag{2.44}$$

Note that the left side of (2.44) is 0 if s is replaced by a root (with non-negative real part) of Lundberg's equation (2.34). This allows for identification of unknown quantities in the term $\widetilde{\sigma}_w(\delta - cs)$ on the right side of (2.44), a step generally needed to ultimately invert (either numerically or analytically under some additional conditions on the distributions of the interclaim time V and/or claim size Y) the Laplace transform $\widetilde{m}_\delta(s)$. We will revisit this expressions with specific assumptions on distributions of the claim sizes or the interclaim times in Section 2.3.1 and Chapter 3. In the next section we derive various joint and marginal densities involving $U_{T^-}, |U_T|, X_T$, and R_{N_T-1}.

2.2.2 Discounted joint and marginal densities

We will now express the joint discounted distribution of $(U_{T^-}, |U_T|, X_T, R_{N_T-1})$ in terms of the discounted densities $h_{1,\delta}(x, y|u)$ and $h_{2,\delta}(x, y, v|u)$ defined in (2.3) and (2.4) respectively. We first consider the penalty function $w^*(x, y, z, v) = w(x, y, v) = e^{-s_1 x - s_2 y - s_4 v}$ as in (1.3), and note that in this case (2.14) becomes

$$\begin{aligned} v_\delta(u) &= \int_0^\infty \int_u^\infty e^{-s_1 x - s_2 y - s_4 u} h_{1,\delta}(x-u, y+u|0) dx dy \\ &\quad + \int_0^\infty \int_u^\infty \int_u^x e^{-s_1 x - s_2 y - s_4 v} h_{2,\delta}(x-u, y+u, v-u|0) dv dx dy. \end{aligned} \tag{2.45}$$

Next consider the more general penalty function $w(x,y,z,v) = e^{-s_1 x - s_2 y - s_3 z - s_4 v}$. With this choice of penalty function, (2.5) becomes $v_\delta^*(u) = e^{-s_3 u} v_\delta(u)$ with $v_\delta(u)$ given by (2.45). Thus the Gerber-Shiu function

$$m_\delta^*(u) = E\left[e^{-\delta T - s_1 U_{T^-} - s_2 |U_T| - s_3 X_T - s_4 R_{N_T - 1}} \mathrm{I}(T < \infty) \middle| U_0 = u\right]$$

satisfies, from (2.22)

$$m_\delta^*(u) = e^{-s_3 u} v_\delta(u) + \int_0^u e^{-s_3 z} v_\delta(z) \frac{g_\delta(u-z)}{1-\phi_\delta} dz$$

which may be expressed using (2.45) as

$$\begin{aligned} m_\delta^*(u) &= \int_0^\infty \int_u^\infty e^{-s_1 x - s_2 y - s_3 u - s_4 u} h_{1,\delta}(x-u, y+u|0) dx dy \\ &\quad + \int_0^\infty \int_u^\infty \int_u^x e^{-s_1 x - s_2 y - s_3 u - s_4 v} h_{2,\delta}(x-u, y+u, v-u|0) dv dx dy \\ &\quad + \int_0^u \int_0^\infty \int_z^\infty e^{-s_1 x - s_2 y - s_3 z - s_4 z} \left\{ h_{1,\delta}(x-z, y+z|0) \frac{g_\delta(u-z)}{1-\phi_\delta} \right\} dx dy dz \\ &\quad + \int_0^u \int_0^\infty \int_z^\infty \int_z^x e^{-s_1 x - s_2 y - s_3 z - s_4 v} \left\{ h_{2,\delta}(x-z, y+z, v-z|0) \frac{g_\delta(u-z)}{1-\phi_\delta} \right\} dv dx dy dz. \end{aligned} \tag{2.46}$$

Therefore, by the uniqueness of the Laplace-Stieltjes transform, $(U_{T^-}, |U_T|, X_T, R_{N_T-1})$ has discounted defective densities on subspaces of $\mathbb{R}^4$ in the following corollary.

Corollary 2 *The discounted defective density of* $(U_{T^-}, |U_T|, X_T, R_{N_T-1})$ *at* (x,y,z,v) *is defined as*

1. $h^*_{12,\delta}(x,y|u) = h_{1,\delta}(x-u, y+u|0)$ *on* $\{(x,y,z,v)|\ x>u, y>0, z=u, v=u\}$ *corresponding to ruin on the first claim,*

2. $h^*_{124,\delta}(x,y,v|u) = h_{2,\delta}(x-u, y+u, v-u|0)$ *on* $\{(x,y,z,v)|\ x>u, y>0, z=u, u<v<x\}$ *corresponding to ruin on the first drop in surplus due to ruin on other than the first claim,*

3. $h^*_{123,\delta}(x,y,z|u) = h_{1,\delta}(x-z, y+z|0)g_\delta(u-z)/(1-\phi_\delta)$ *on* $\{(x,y,z,v)|\ x>z, y>0, 0<z<u, v=z\}$ *corresponding to a drop in surplus not causing ruin followed by ruin on the next claim, and*

4. $h^*_\delta(x,y,z,v|u) = h_{2,\delta}(x-z, y+z, v-z|0)g_\delta(u-z)/(1-\phi_\delta)$ *on* $\{(x,y,z,v)|\ z<v<x, y>0, 0<z<u\}$ *corresponding to a drop in surplus not causing ruin, followed by ruin occurring but not on the next claim after the drop.*

While it is possible to give probabilistic interpretations for the above four cases, we would like to comment on the quantity $h^*_{123,\delta}(x,y,z|u)$ in detail. Note that from (2.21), $g_\delta(u-z)/(1-\phi_\delta)$ can be expressed as $\sum_{n=1}^{\infty}(\phi_\delta)^n f^{*n}_\delta(u-z)$, and this can indeed be interpreted as the density for the surplus process, beginning with initial surplus u, being at level z after an arbitrary number of drops. Since the level z has to be the minimum level before ruin, the next drop (starting with level z) has to cause ruin and this is represented by the term $h_{1,\delta}(x,y|z)$. A similar interpretation can also be made for the quantity $h^*_\delta(x,y,z,v|u)$.

We now turn to the joint discounted defective density of $(U_{T^-}, |U_T|, X_T)$ in the following corollary.

Corollary 3 *The discounted defective density of* $(U_{T^-}, |U_T|, X_T)$ *is defined as*

1. $h^{**}_{12,\delta}(x,y|u) = h_\delta(x-u, y+u|0)$ *on* $\{(x,y,z)|\ x>u, y>0, z=u\}$ *corresponding to ruin on a first drop in surplus below* u*, and*

2. $h^{**}_{123,\delta}(x,y,z|u) = h_\delta(x-z, y+z|0)g_\delta(u-z)/(1-\phi_\delta)$ *on* $\{(x,y,z)|\ x>z, y>0, 0<z<u\}$ *corresponding to ruin occurring but not on the first drop in surplus.*

Proof. Using the same approach with the penalty function $w_{123}(x,y,z) = e^{-s_1 x - s_2 y - s_3 z}$, (2.16) becomes

$$v_{\delta,123}(u) = \int_u^\infty \int_0^\infty e^{-s_1(x+u)-s_2(y-u)-s_3 u} h_\delta(x,y|0)dxdy = \int_0^\infty \int_u^\infty e^{-s_1 x - s_2 y - s_3 u} h_\delta(x-u, y+u|0)dxdy,$$

and then from (2.15) and (2.22)

$$\begin{aligned} m_{\delta,123}(u) &= E\left[e^{-\delta T - s_1 U_{T^-} - s_2 |U_T| - s_3 X_T} \mathrm{I}\left(T<\infty\right) \middle| U_0 = u\right] \\ &= v_{\delta,123}(u) + \frac{1}{1-\phi_\delta}\int_0^u g_\delta(u-z) v_{\delta,123}(z)dz \\ &= \int_0^\infty \int_u^\infty e^{-s_1 x - s_2 y - s_3 u} h_\delta(x-u, y+u|0)dxdy \\ &\quad + \int_0^u \int_0^\infty \int_z^\infty e^{-s_1 x - s_2 y - s_3 z}\left\{h_\delta(x-z, y+z|0)\frac{g_\delta(u-z)}{1-\phi_\delta}\right\} dxdydz. \end{aligned}$$

Thus, by the uniqueness of the Laplace transform, we may obtain the discounted defective densities of $(U_{T^-}, |U_T|, X_T)$ on subspaces of $\mathbb{R}^3$. $\square$

Corollary 4 *For the time-independent claim sizes, the discounted joint density of $(U_{T^-}, |U_T|)$ at (x,y) is given by*

$$h_\delta(x,y|u) = \frac{1}{c} p(x+y)\gamma_\delta(u,x), \qquad u \geq 0, \tag{2.47}$$

where

$$\gamma_\delta(u,x) = \begin{cases} \int_0^x e^{-\delta\left(\frac{x-v}{c}\right)} k\left(\frac{x-v}{c}\right)\tau_\delta(u,v)dv, & x<u \\ e^{-\delta\left(\frac{x-u}{c}\right)} k\left(\frac{x-u}{c}\right) + \int_0^x e^{-\delta\left(\frac{x-v}{c}\right)} k\left(\frac{x-v}{c}\right)\tau_\delta(u,v)dv, & x>u \end{cases}. \tag{2.48}$$

Proof. Using (2.31), $h_\delta(x,y|u)$ in (2.8) may be rewritten as

$$h_\delta(x,y|u) = \mathrm{I}(x>u) h_{1,\delta}(x,y|u) + \int_0^x h_{1,\delta}(x,y|v)\tau_\delta(u,v)dv.$$

Thus, from (2.1) and (2.3), it follows that

$$h_\delta(x,y|u) = \frac{1}{c}e^{-\frac{\delta(x-u)}{c}}k\left(\frac{x-u}{c}\right)p_{\frac{x-u}{c}}(x+y)\,\mathrm{I}(x>u)+\int_0^x \frac{1}{c}e^{-\frac{\delta(x-v)}{c}}k\left(\frac{x-v}{c}\right)p_{\frac{x-v}{c}}(x+y)\,\tau_\delta(u,v)dv,$$

which is the result in the case of the time-dependent claim sizes. If the claim sizes are independent of the interclaim times, $p_t(y)$ simply reduces to $p(y)$ so that we may obtain (2.47) with (2.48). □

In particular, in the classical compound Poisson risk model we shall show that $h_\delta(x,y|u)$ in (2.47) is reduced to the result derived by Landriault and Willmot (2009).

Corollary 5 *In the classical compound Poisson risk model with $k(t)=\lambda e^{-\lambda t}$, the discounted joint density of $(U_{T^-},|U_T|)$ at (x,y) is given by*

$$h_\delta(x,y|u) = \frac{\lambda}{c}p(x+y)\gamma_\delta^*(u,x), \qquad u \geq 0,$$

where

$$\gamma_\delta^*(u,x) = \begin{cases} \int_0^x e^{-\rho(x-t)}\frac{g_\delta(u-t)}{1-\phi_\delta}dt, & x<u \\ e^{-\rho(x-u)} + \int_0^u e^{-\rho(x-t)}\frac{g_\delta(u-t)}{1-\phi_\delta}dt, & x>u \end{cases},$$

which is agreed with the result of Landriault and Willmot (2009).

Proof. First, for $x<u$, (2.48) becomes

$$\begin{aligned} &\lambda\int_0^x e^{-\frac{\lambda+\delta}{c}(x-v)}\tau_\delta(u,v)dv \\ &= \frac{\lambda}{1-\phi_\delta}\left\{\int_0^x e^{-\frac{\lambda+\delta}{c}(x-v)}g_\delta(u-v)dv+\left(\frac{\lambda+\delta}{c}-\rho\right)\int_0^x\int_0^v e^{-\frac{\lambda+\delta}{c}(x-v)}e^{-\rho(v-t)}g_\delta(u-t)dtdv\right\}. \end{aligned} \tag{2.49}$$

Interchanging the order of integration of the second term on the right hand side of (2.49) yields

$$\begin{aligned}
&\left(\frac{\lambda+\delta}{c}-\rho\right)\int_0^x\int_0^v e^{-\frac{\lambda+\delta}{c}(x-v)}e^{-\rho(v-t)}g_\delta(u-t)dtdv \\
&= \left(\frac{\lambda+\delta}{c}-\rho\right)\int_0^x e^{-\rho(x-t)}g_\delta(u-t)\left\{\int_t^x e^{-\left(\frac{\lambda+\delta}{c}-\rho\right)(x-v)}dv\right\}dt \\
&= \int_0^x e^{-\rho(x-t)}g_\delta(u-t)\left\{1-e^{-\left(\frac{\lambda+\delta}{c}-\rho\right)(x-t)}\right\}dt, \qquad (2.50)
\end{aligned}$$

and substituting (2.50) into (2.49) results in

$$\lambda\int_0^x e^{-\frac{\lambda+\delta}{c}(x-v)}\tau_\delta(u,v)dv = \lambda\int_0^x e^{-\rho(x-t)}\frac{g_\delta(u-t)}{1-\phi_\delta}dt.$$

Next, for $x > u$ it follows that (2.48) is

$$\lambda\left\{e^{-\frac{\lambda+\delta}{c}(x-u)}+\int_0^u e^{-\frac{\lambda+\delta}{c}(x-v)}\tau_\delta(u,v)dv+\int_u^x e^{-\frac{\lambda+\delta}{c}(x-v)}\tau_\delta(u,v)dv\right\}. \qquad (2.51)$$

Using (2.49) and (2.50), the second integral on the above may be rewritten as

$$\begin{aligned}
&\int_0^u e^{-\frac{\lambda+\delta}{c}(x-v)}\tau_\delta(u,v)dv \\
=\ & \frac{1}{1-\phi_\delta}\left[\int_0^u e^{-\frac{\lambda+\delta}{c}(x-v)}g_\delta(u-v)dv+\int_0^u e^{-\rho(x-t)}g_\delta(u-t)\left\{e^{-\left(\frac{\lambda+\delta}{c}-\rho\right)(x-u)}-e^{-\left(\frac{\lambda+\delta}{c}-\rho\right)(x-t)}\right\}dt\right] \\
&= e^{-\left(\frac{\lambda+\delta}{c}-\rho\right)(x-u)}\int_0^u e^{-\rho(x-t)}\frac{g_\delta(u-t)}{1-\phi_\delta}dt. \qquad (2.52)
\end{aligned}$$

Also,

$$\begin{aligned}
&\int_u^x e^{-\frac{\lambda+\delta}{c}(x-v)}\tau_\delta(u,v)dv \\
=\ & \left(\frac{\lambda+\delta}{c}-\rho\right)\left[\int_u^x e^{-\frac{\lambda+\delta}{c}(x-v)}e^{-\rho(v-u)}dv+\frac{1}{1-\phi_\delta}\int_u^x e^{-\frac{\lambda+\delta}{c}(x-v)}\int_0^u e^{-\rho(v-t)}g_\delta(u-t)dtdv\right] \\
=\ & \left(\frac{\lambda+\delta}{c}-\rho\right)\int_u^x e^{-\left(\frac{\lambda+\delta}{c}-\rho\right)(x-v)}\left\{e^{-\rho(x-u)}+\int_0^u e^{-\rho(x-t)}\frac{g_\delta(u-t)}{1-\phi_\delta}dt\right\}dv,
\end{aligned}$$

that is,

$$\int_u^x e^{-\frac{\lambda+\delta}{c}(x-v)}\tau_\delta(u,v)dv = \left\{1 - e^{-\left(\frac{\lambda+\delta}{c}-\rho\right)(x-u)}\right\}\left\{e^{-\rho(x-u)} + \int_0^u e^{-\rho(x-t)}\frac{g_\delta(u-t)}{1-\phi_\delta}dt\right\}. \tag{2.53}$$

Substituting (2.52) and (2.53) into (2.51), we have (2.48) for $x > u$ given by

$$\lambda\left\{e^{-\rho(x-u)} + \int_0^u e^{-\rho(x-t)}\frac{g_\delta(u-t)}{1-\phi_\delta}dt\right\}.$$

Thus, the proof is completed. □

We next turn our attention to the discounted density of the last ladder height before ruin, joint distribution of the last interclaim time before ruin and the claim causing ruin, and also their marginal distributions. In the classical compound Poisson risk model, these results were studied by Cheung et al. (2010a) and are thus generalized here.

Corollary 6 *The discounted defective density of the last ladder height before ruin is*

$$f_\delta(u,y) = \begin{cases} \frac{\phi_\delta}{1-\phi_\delta}\left[\overline{G}_\delta(u-y) - \overline{G}_\delta(u)\right] f_\delta(y), & y < u \\ \frac{\phi_\delta}{1-\phi_\delta}\left[1 - \overline{G}_\delta(u)\right] f_\delta(y), & y > u \end{cases}. \tag{2.54}$$

Proof. For the last ladder height $X_T + |U_T|$, the function

$$m_{\delta,5}(u) = E\left[e^{-\delta T} w_5\left(X_T + |U_T|\right) \mathrm{I}\left(T < \infty\right) \middle| U_0 = u\right]$$

satisfies, using (2.18)

$$m_{\delta,5}(u) = \phi_\delta \int_0^u m_{\delta,5}(u-y) f_\delta(y)dy + \phi_\delta \int_u^\infty w_5(y) f_\delta(y)dy$$

with solution, using (2.23)

$$m_{\delta,5}(u) = \frac{\phi_\delta}{1-\phi_\delta}\left[\left(1-\overline{G}_\delta(u)\right)\int_u^\infty w_5(y)f_\delta(y)dy + \int_0^u \left(\overline{G}_\delta(u-y)-\overline{G}_\delta(u)\right) w_5(y) f_\delta(y)dy\right]. \tag{2.55}$$

The Laplace transform of the discounted density is given by (2.55) with $w_5(y) = e^{-s_5 y}$ and therefore $X_T + |U_T|$ has defective discounted density (given $U_0 = u$) in (2.54) by inverting (2.55). $\square$

Note that with $\delta = 0$ in the classical compound Poisson model without dependency (i.e. $k(t) = \lambda e^{-\lambda t}$ and $p_t(y) = p(y)$), $h_0(x, y|0)$ in (2.8) equals $(\lambda/c)p(x + y)$ (e.g. Gerber and Shiu (1997b)). Thus, $v_{0,123}(u)$ in (2.16) becomes the same function with a different choice of the penalty function, namely, $w_{123}(x, y, z) = w_1(x) = e^{-sx}$ and $w_{123}(x, y, z) = w_{23}(y, z) = e^{-s(y+z)}$. Therefore, in this case the defective density of the last ladder height before ruin given by (2.54) is equivalent to the defective density of the surplus prior to ruin.

Corollary 7 *The bivariate Laplace transform of the last interclaim time before ruin V_{N_T} and the claim causing ruin*

$$E\left[e^{-\delta T - s_1 V_{N_T} - s_2 Y_{N_T}} \mathrm{I}\left(T < \infty\right) \middle| U_0 = u\right] = \int_0^\infty \int_0^\infty e^{-s_1 t - s_2 y} h_{4,\delta}(t, y|u) dy dt,$$

where the joint density of V_{N_T} and Y_{N_T} is given by

$$h_{4,\delta}\left(t, y \left| u\right.\right) = e^{-\delta t} k(t) p_t(y) \left\{\mathrm{I}(y > u + ct) + \mathrm{I}(y > ct) \int_0^{y-ct} \tau_\delta(u, v) dv\right\}. \tag{2.56}$$

Proof. Since $V_{N_T} = (U_{T^-} - R_{N_T - 1})/c$ and $Y_{N_T} = U_{T^-} + |U_T|$, we get the bivariate Laplace transform of V_{N_T} and Y_{N_T} with $w(x, y, v) = e^{-s_1((x-v)/c) - s_2(x+y)}$ in (1.2). In this case, from (2.27), we have

$$\beta_\delta(u) = \int_0^\infty \int_u^\infty e^{-s_1\left(\frac{x-u}{c}\right) - s_2(x+y)} \left\{e^{-\delta\left(\frac{x-u}{c}\right)} \frac{1}{c} k\left(\frac{x-u}{c}\right) p_{\frac{x-u}{c}}(x + y)\right\} dx dy.$$

Changing a variable from $(x - u)/c$ to (t) yields

$$\beta_\delta(u) = \int_0^\infty \int_{u+ct}^\infty e^{-s_1 t - s_2 y} \left\{e^{-\delta t} k(t) p_t(y)\right\} dy dt. \tag{2.57}$$

Thus from (2.25) and (2.57) we could get the bivariate Laplace transform of the last interclaim time before ruin V_{N_T} and the claim causing ruin Y_{N_T} and thus we have the joint density of these by inverting the transform with respect to s_1 and s_2. □

Corollary 8 *The Laplace transform of the last interclaim time V_{N_T} is given by*

$$E\left[e^{-\delta T - s_1 V_{N_T}} \mathrm{I}\left(T<\infty\right) \middle| U_0 = u\right] = \int_0^\infty e^{-s_1 t} h_{5,\delta}(t|u)dt,$$

where its density is given by

$$h_{5,\delta}(t|u) = e^{-\delta t}k(t)\left\{\overline{P}_t(u+ct) + \int_0^\infty \overline{P}_t(v+ct)\tau_\delta(u,v)dv\right\}, \qquad t>0. \tag{2.58}$$

Proof. The marginal Laplace transform of V_{N_T} is obtainable with $s_2 = 0$, in which case (2.57) becomes

$$\beta_\delta(u) = \int_0^\infty e^{-s_1 t}\left\{e^{-\delta t}k(t)\overline{P}_t(u+ct)\right\}dt,$$

and substituting the above result into (2.25) followed by inverting with respect to s_1 yields the marginal density of the last interclaim time V_{N_T} given by (2.58). □

Corollary 9 *The Laplace transform of the claim causing ruin Y_{N_T} is given by*

$$E\left[e^{-\delta T - s_2 Y_{N_T}} \mathrm{I}\left(T<\infty\right) \middle| U_0 = u\right] = \int_0^\infty e^{-s_2 y} h_{6,\delta}(y|u)dy,$$

where its density is given by

$$h_{6,\delta}(y|u) = \int_0^{\frac{y-u}{c}} p_t(y)k(t)dt + \int_0^y \tau_\delta(u,z)\int_0^{\frac{y-z}{c}} p_t(y)k(t)dtdz.$$

In particular, for the time-independent claim sizes it simplifies to

$$h_{6,\delta}(y|u) = p(y)\left[K_\delta\left(\frac{y-u}{c}\right)\mathrm{I}(y>u) + \int_0^y K_\delta\left(\frac{y-v}{c}\right)\tau_\delta(u,v)dv\right]. \tag{2.59}$$

Proof. Similarly, for Y_{N_T} with $s_1 = 0$, interchanging the order of integration in (2.57) yields

$$\beta_\delta(u) = \int_u^\infty e^{-s_2 y} \left\{ \int_0^{\frac{y-u}{c}} e^{-\delta t} k(t) p_t(y) dt \right\} dy,$$

in particular, if claim sizes are time-independent the above expression may be simplified as

$$\beta_\delta(u) = \int_0^\infty e^{-s_2 y} K_\delta \left(\frac{y-u}{c} \right) \mathrm{I}(y > u) p(y) dy$$

where $K_\delta(t) = \int_0^t e^{-\delta x} k(x) dx$ is a discounted df. Thus in this case from (2.25) we have the Laplace transform of the claim causing ruin and obtain its density given by (2.59) after an inversion of this transform. □

2.3 Arbitrary interclaim times without dependency

Here we assume that the claim sizes are independent, namely, $\overline{P}_t(y) = \overline{P}(y)$ and $p_t(y) = p(y)$. As in Gerber and Shiu (1998), the conditional density of $|U_T|$ given $U_{T^-} = x, R_{N_T - 1} = v, N_T \geq 2$, and $T = t$ is given by $p(x+y)/\overline{P}(x)$, so that one may write

$$h_2^*(t, x, y, v|u) = \frac{p(x+y)}{\overline{P}(x)} h^{(2)}(t, x, v|u) \tag{2.60}$$

where $h^{(2)}(t, x, v|u)$ represents the joint defective density of T, U_{T^-} and $R_{N_T - 1}$ for ruin occurring on claims subsequent to the first. Therefore, from (2.60)

$$h_{2,\delta}^*(x, y, v|u) = \frac{p(x+y)}{\overline{P}(x)} h_\delta^{(2)}(x, v|u) \tag{2.61}$$

where $h_\delta^{(2)}(x, v|u) = \int_0^\infty e^{-\delta t} h^{(2)}(t, x, v|u) dt$. Thus, using (2.1), (2.3), and (2.61), the discounted density (2.8) may be expressed as

$$h_\delta(x, y|u) = \frac{p(x+y)}{\overline{P}(x)} h_\delta(x|u), \tag{2.62}$$

where

$$h_\delta(x|u) = \frac{1}{c} e^{-\delta\left(\frac{x-u}{c}\right)} k\left(\frac{x-u}{c}\right) \overline{P}(x) + \int_0^x h_\delta^{(2)}(x, v|u) dv$$

is the discounted (marginal if $\delta = 0$) density of the surplus prior to ruin U_{T^-}.

Hence, (2.11) becomes, using (2.62),

$$\phi_\delta = \int_0^\infty h_\delta(x|0) \left\{ \int_0^\infty \frac{p(x+y)}{\overline{P}(x)} dy \right\} dx = \int_0^\infty h_\delta(x|0) dx,$$

and (2.12) may be expressed as the mixed density

$$f_\delta(y) = \int_0^\infty \left\{ \frac{h_\delta(x|0)}{\phi_\delta} \right\} \frac{p(x+y)}{\overline{P}(x)} dx. \tag{2.63}$$

The defective renewal equation may also be simplified. If $w(x, y, z, v) = w_{134}(x, z, v) w_2(y)$, then (2.6) may be expressed using (2.61) as

$$\begin{aligned} v_{\delta,134,2}(u) \quad = \quad & \int_0^\infty e^{-\delta t} w_{134}(u+ct, u, u) \left\{ \int_0^\infty w_2(y) p(y+ct+u) dy \right\} k(t) dt + \int_0^\infty \frac{1}{\overline{P}(x)} \\ & \times \left\{ \int_0^\infty w_2(y) p(x+y+u) dy \right\} \int_0^x w_{134}(x+u, u, v+u) h_\delta^{(2)}(x, v|0) dv dx. \end{aligned} \tag{2.64}$$

We now illustrate some of these ideas by deriving the joint Laplace transform of all these quantities in the case with exponential claim sizes.

2.3.1 Exponential claim sizes

We consider the joint Laplace transform of $(T, U_{T^-}, |U_T|, X_T, R_{N_T-1})$ when $p(y) = \beta e^{-\beta y}$. Letting $w(x,y,z,v) = e^{-s_1x-s_2y-s_3z-s_4v}$, (2.64) yields

$$\begin{aligned}
v_\delta^*(u) &= \int_0^\infty e^{-\delta t - s_1(u+ct)-s_3u-s_4u}\left\{\int_0^\infty e^{-s_2y}\beta e^{-\beta(y+ct+u)}dy\right\}k(t)dt \\
&\quad + \int_0^\infty \left\{\int_0^\infty e^{-s_2y}\beta e^{-\beta(y+u)}dy\right\}\int_0^x e^{-s_1(x+u)-s_3u-s_4(v+u)}h_\delta^{(2)}(x,v|0)dvdx \\
&= e^{-(\beta+s_1+s_3+s_4)u}\int_0^\infty e^{-(\delta+cs_1+c\beta)t}\left\{\int_0^\infty \beta e^{-(\beta+s_2)y}dy\right\}k(t)dt \\
&\quad + e^{-(\beta+s_1+s_3+s_4)u}\left\{\int_0^\infty \beta e^{-(\beta+s_2)y}dy\right\}\int_0^\infty\int_0^x e^{-s_1x-s_4v}h_\delta^{(2)}(x,v|0)dvdx \\
&= \frac{\beta e^{-(\beta+s_1+s_3+s_4)u}}{\beta+s_2}\left\{\widetilde{k}(\delta+cs_1+c\beta)+\widetilde{h}_\delta^{(2)}(s_1,s_4|0)\right\},
\end{aligned}$$

where $\widetilde{k}(s) = \int_0^\infty e^{-st}k(t)dt$ and $\widetilde{h}_\delta^{(2)}(s_1,s_4|0) = \int_0^\infty\int_0^x e^{-s_1x-s_4v}h_\delta^{(2)}(x,v|0)dvdx$ are the Laplace transforms of $k(t)$ and $h_\delta^{(2)}(x,v|0)$ respectively. For notational convenience, let $\gamma_\delta(s_1,s_4) = \widetilde{k}(\delta + cs_1 + c\beta) + \widetilde{h}_\delta^{(2)}(s_1,s_4|0)$, so that

$$v_\delta^*(u) = \frac{\beta\gamma_\delta(s_1,s_4)}{\beta+s_2}e^{-(\beta+s_1+s_3+s_4)u}. \tag{2.65}$$

It is clear from (2.63) that $f_\delta(y) = \beta e^{-\beta y} = p(y)$ in this case. Thus, from (2.7), the Gerber-Shiu function

$$m_\delta^*(u) = E\left[e^{-\delta T - s_1U_{T^-}-s_2|U_T|-s_3X_T-s_4R_{N_T-1}}\mathrm{I}(T<\infty)\,\middle|U_0=u\right]$$

satisfies

$$m_\delta^*(u) = \phi_\delta\int_0^u m_\delta^*(u-y)\beta e^{-\beta y}dy + v_\delta^*(u)$$

where $v_\delta^*(u)$ is given by (2.65). To solve this equation directly we will use Laplace transforms. Thus,

$$\widetilde{m}_\delta^*(z) = \phi_\delta\widetilde{m}_\delta^*(z)\frac{\beta}{\beta+z} + \frac{\beta\gamma_\delta(s_1,s_4)}{\beta+s_2}(\beta+s_1+s_3+s_4+z)^{-1},$$

and hence solving for $\widetilde{m}_\delta(z)$ yields

$$\begin{aligned}\widetilde{m}_\delta^*(z) &= \frac{\beta\gamma_\delta(s_1,s_4)}{\beta+s_2}\frac{(\beta+s_1+s_3+s_4+z)^{-1}}{1-\phi_\delta\beta(\beta+z)^{-1}} \\ &= \frac{\beta\gamma_\delta(s_1,s_4)}{(\beta+s_2)(\phi_\delta\beta+s_1+s_3+s_4)}\left\{\frac{s_1+s_3+s_4}{\beta+s_1+s_3+s_4+z}+\frac{\phi_\delta\beta}{\beta(1-\phi_\delta)+z}\right\}\end{aligned}$$

after a little algebra. Thus inversion with respect to z yields

$$m_\delta^*(u) = \frac{\beta\gamma_\delta(s_1,s_4)}{(\beta+s_2)(\phi_\delta\beta+s_1+s_3+s_4)}\left\{(s_1+s_3+s_4)e^{-(\beta+s_1+s_3+s_4)u}+\beta\overline{G}_\delta(u)\right\}, \tag{2.66}$$

where $\overline{G}_\delta(u)=\phi_\delta e^{-\beta(1-\phi_\delta)u}$ with ϕ_δ the solution to $\phi_\delta=\widetilde{k}(\delta+c\beta-\phi_\delta c\beta)$ (e.g. Willmot (2007)).

It is useful to be able to express $\widetilde{h}_\delta^{(2)}(s_1,s_4|0)$ or equivalently $\gamma_\delta(s_1,s_4)$ in terms of the interclaim time Laplace transform $\widetilde{k}(s)$. To do this, we will examine $m_\delta^*(u)$ by conditioning on the time and amount of the first claim, which simplifies if we ignore X_T by letting $s_3=0$ (and for simplicity we will also set $s_2=0$). Thus, let

$$m_{\delta,14}(u) = \frac{\gamma_\delta(s_1,s_4)}{\phi_\delta\beta+s_1+s_4}\left\{(s_1+s_4)e^{-(\beta+s_1+s_4)u}+\beta\overline{G}_\delta(u)\right\}, \tag{2.67}$$

which corresponds to the choice of the penalty function $w(x,y,z,v)=w_{14}(x,v)=e^{-s_1x-s_4v}$. Thus, in this case, $m_{\delta,14}(u)$ satisfies the integral equation (from (2.37))

$$m_{\delta,14}(u) = \int_0^\infty e^{-\delta t}\tau_\delta(u+ct,u)k(t)dt \tag{2.68}$$

where

$$\tau_\delta(t,u) = \int_0^t m_{\delta,14}(t-y)\beta e^{-\beta y}dy + \int_t^\infty e^{-s_1t-s_4u}\beta e^{-\beta y}dy.$$

Clearly,

$$\int_t^\infty e^{-s_1t-s_4u}\beta e^{-\beta y}dy = e^{-(\beta+s_1)t-s_4u},$$

and using (2.67)

$$
\begin{aligned}
&\int_0^t m_{\delta,14}(t-y)\beta e^{-\beta y}dy \\
&= \frac{\gamma_\delta(s_1,s_4)}{\phi_\delta\beta+s_1+s_4}\left\{(s_1+s_4)\int_0^t e^{-(\beta+s_1+s_4)(t-y)}\beta e^{-\beta y}dy+\beta\int_0^t \overline{G}_\delta(t-y)f_\delta(y)dy\right\} \\
&= \frac{\beta\gamma_\delta(s_1,s_4)}{\phi_\delta\beta+s_1+s_4}\left\{e^{-\beta t}[1-e^{-(s_1+s_4)t}]+\left[\frac{\overline{G}_\delta(t)}{\phi_\delta}-e^{-\beta t}\right]\right\} \\
&= \frac{\beta\gamma_\delta(s_1,s_4)}{\phi_\delta\beta+s_1+s_4}\left\{\frac{\overline{G}_\delta(t)}{\phi_\delta}-e^{-(\beta+s_1+s_4)t}\right\}.
\end{aligned}
$$

Thus,

$$
\tau_\delta(t,u)=\frac{\beta\gamma_\delta(s_1,s_4)}{\phi_\delta\beta+s_1+s_4}\left\{\frac{\overline{G}_\delta(t)}{\phi_\delta}-e^{-(\beta+s_1+s_4)t}\right\}+e^{-(\beta+s_1)t-s_4u},
$$

and therefore

$$
\begin{aligned}
&\int_0^\infty e^{-\delta t}\tau_\delta(u+ct,u)k(t)dt=\int_0^\infty e^{-\delta t-(\beta+s_1)(u+ct)-s_4u}k(t)dt \\
&+\frac{\beta\gamma_\delta(s_1,s_4)}{\phi_\delta\beta+s_1+s_4}\left\{\frac{1}{\phi_\delta}\int_0^\infty e^{-\delta t}\overline{G}_\delta(u+ct)k(t)dt-\int_0^\infty e^{-\delta t-(\beta+s_1+s_4)(u+ct)}k(t)dt\right\}.
\end{aligned}
$$

That is

$$
\begin{aligned}
&\int_0^\infty e^{-\delta t}\tau_\delta(u+ct,u)k(t)dt=e^{-(\beta+s_1+s_4)u}\widetilde{k}(\delta+c\beta+cs_1) \\
&+\frac{\beta\gamma_\delta(s_1,s_4)}{\phi_\delta\beta+s_1+s_4}\left\{\int_0^\infty e^{-\delta t-\beta(1-\phi_\delta)(u+ct)}k(t)dt-e^{-(\beta+s_1+s_4)u}\widetilde{k}(\delta+c\beta+cs_1+cs_4)\right\}.
\end{aligned}
$$

But $\int_0^\infty e^{-\delta t-c\beta(1-\phi_\delta)t}k(t)dt=\phi_\delta$, and thus

$$
\begin{aligned}
&\int_0^\infty e^{-\delta t}\tau_\delta(u+ct,u)k(t)dt=e^{-(\beta+s_1+s_4)u}\widetilde{k}(\delta+c\beta+cs_1) \\
&+\frac{\beta\gamma_\delta(s_1,s_4)}{\phi_\delta\beta+s_1+s_4}\left\{\overline{G}_\delta(u)-e^{-(\beta+s_1+s_4)u}\widetilde{k}(\delta+c\beta+cs_1+cs_4)\right\}, \qquad (2.69)
\end{aligned}
$$

which (by (2.68)) equals $m_{\delta,14}(u)$. Thus, equating (2.67) and (2.69), the terms involving $\overline{G}_\delta(u)$

cancel, and division by $e^{-(\beta+s_1+s_4)u}$ results in

$$\frac{\gamma_\delta(s_1,s_4)}{\phi_\delta\beta+s_1+s_4}(s_1+s_4)=\widetilde{k}(\delta+c\beta+cs_1)-\frac{\gamma_\delta(s_1,s_4)}{\phi_\delta\beta+s_1+s_4}\beta\widetilde{k}(\delta+c\beta+cs_1+cs_4),$$

which in turn implies that

$$\gamma_\delta(s_1,s_4)=\frac{(\phi_\delta\beta+s_1+s_4)\widetilde{k}(\delta+c\beta+cs_1)}{s_1+s_4+\beta\widetilde{k}(\delta+c\beta+cs_1+cs_4)}. \tag{2.70}$$

Finally, substitution of (2.70) into (2.66) yields

$$m_\delta^*(u)=C_\delta(s_1,s_2,s_3,s_4)\left\{(s_1+s_3+s_4)e^{-(\beta+s_1+s_3+s_4)u}+\phi_\delta\beta e^{-\beta(1-\phi_\delta)u}\right\}, \tag{2.71}$$

where

$$C_\delta(s_1,s_2,s_3,s_4)=\frac{\beta(\phi_\delta\beta+s_1+s_4)\widetilde{k}(\delta+c\beta+cs_1)}{(\beta+s_2)(\phi_\delta\beta+s_1+s_3+s_4)\{s_1+s_4+\beta\widetilde{k}(\delta+c\beta+cs_1+cs_4)\}}. \tag{2.72}$$

The last interclaim time before ruin $V_{N_T}=(U_{T^-}-R_{N_T-1})/c$ was analyzed in the classical compound Poisson risk model by Cheung et al. (2010a). For the present Sparre Andersen model with exponential claims, the Laplace transform of the defective distribution of V_{N_T} is given by (2.71) with $\delta=0$, $s_1=s/c$, $s_2=s_3=0$, and $s_4=-s/c$. Thus, using (2.72) also, it follows that

$$E\left[e^{-sV_{N_T}}\mathrm{I}\left(T<\infty\right)\middle|U_0=u\right]=\frac{\widetilde{k}(c\beta+s)}{\widetilde{k}(c\beta)}\psi(u),$$

and the proper distribution of $V_{N_T}|T<\infty$ is functionally independent of u with Laplace transform

$$E\left[e^{-sV_{N_T}}\middle|T<\infty\right]=\frac{\widetilde{k}(c\beta+s)}{\widetilde{k}(c\beta)}. \tag{2.73}$$

Clearly, (2.73) is the Laplace transform of an Esscher transformed distribution of $K(t)$, so that if $K_1(t)=1-\overline{K}_1(t)=\Pr\left(V_{N_T}\middle|T<\infty\right)$ is the distribution function, the density $k_1(t)=K_1'(t)$ is given by

$$k_1(t) = \frac{e^{-c\beta t}k(t)}{\widetilde{k}(c\beta)}. \tag{2.74}$$

Evaluation of $k_1(t)$ is straightforward for many choices of $k(t)$. In particular, if $k(t)$ is from the mixed Erlang, combination of exponentials, or phase-type classes, the same is easily seen to be true of $k_1(t)$.

Also, $V_{N_T}\big|T < \infty$ is stochastically dominated by the generic interclaim time random variable V, a result which agrees with intuition. For further details regarding the ordering result of the ruin related quantities including $V_{N_T}\big|T < \infty$, see Cheung et al. (2010b) and Cheung et al. (2010c). $\square$

For more general claim size distributions, a similar approach may be used to determine the joint Laplace transform as in Landriault and Willmot (2008).

Chapter 3

Sparre Andersen risk models

In this chapter, we study the generalized Gerber-Shiu function $m_\delta(u)$ in (1.3) in the Sparre Andersen risk model with certain distributions for the interclaim time. In Section 3.1, a K_n family distribution considered by Li and Garrido (2005) is assumed. As a special case of this model, the classical compound Poisson risk model is considered in Section 3.2 and Section 3.3.

3.1 Coxian interclaim time distributions

3.1.1 Introduction

In this section, we consider the model of Li and Garrido (2005), whereby $k(t)$ is a pdf from the K_n class of densities, whose Laplace-Stieltjes transform is the ratio of a polynomial of $k < n$ to a polynomial of degree n (see Cohen (1982), Tijms (1994), and Willmot (1999)) given by

$$\widetilde{k}(s) = \frac{\omega(s)}{\prod_{i=1}^{m}(\lambda_i + s)^{n_i}} \tag{3.1}$$

where $\lambda_i > 0$ for $i = 1, 2, \ldots, m$ with $\lambda_i \neq \lambda_j$ for $i \neq j$. Also, n_i is a nonnegative integer for $i = 1, 2, \ldots, m$, and $n = n_1 + \cdots + n_m > 0$, while $\omega(s)$ is a polynomial of degree $n-1$ or less (the denominator of (3.1) is a polynomial of degree n). The classical compound Poisson risk model (e.g. Gerber and Shiu (1998)) is recovered in the exponential case with $m = n = 1$, the Erlang(n) renewal risk model (e.g. Li and Garrido (2004)) with $m = 1$, and $n_m = n$, and the generalized Erlang renewal risk model (e.g. Gerber and Shiu (2005)) with $n_i = 1$ for $i = 1, 2, \ldots, n$, and $\omega(s) = \prod_{i=1}^{m} \lambda_i^{n_i}$ in all these cases. For reference, a wide class of distributions including (3.1), called the class $\mathcal{R}^f$ of distributions, was studied by Dufresne (2001). These have finite rational Laplace transforms includes the so-called phase-type distributions (e.g. Asmussen (1987, pp.74-76)).

As pointed out by Li and Garrido (2005), a partial fraction expression of (3.1) results in

$$\widetilde{k}(s) = \sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{a_{i,j}}{(\lambda_i + s)^j} \tag{3.2}$$

where

$$a_{i,j} = \frac{1}{(n_i - j)!} \frac{d^{n_i - j}}{ds^{n_i - j}} \left\{ \prod_{k=1, k \neq i}^{m} \frac{\omega(s)}{(\lambda_k + s)^{n_k}} \right\} \Bigg|_{s=-\lambda_i}.$$

Inversion of (3.2) results in

$$k(t) = \sum_{i=1}^{m} \sum_{j=1}^{n_i} a_{i,j} \frac{t^{j-1} e^{-\lambda_i t}}{(j-1)!}, \tag{3.3}$$

and the K_n class may be viewed in terms of finite combinations of Erlangs. Also, it is assumed that the claim sizes $\{Y_i\}_{i=1}^{\infty}$ with Y_i the size of the ith claim are iid positive random variables which implies that $p_t(y) = p(y)$, $\overline{P}_t(y) = \overline{P}(y)$, and Laplace transform $\widetilde{p}(s) = \int_0^\infty e^{-sy} p(y) dy$.

Also, Lundberg's (generalized) fundamental equation (2.34), that is,

$$\widetilde{p}(s) \widetilde{k}(\delta - cs) = 1 \tag{3.4}$$

is of central importance in the ensuing analysis, and Li and Garrido (2005) showed that (3.4) has exactly n roots $\rho_1, \rho_2, \cdots, \rho_n$ with nonnegative real part $\mathrm{Re}(\rho_j) \geq 0$ in the complex plane. We shall henceforth assume (as did Li and Garrido (2005)) that these roots are distinct, i.e. $\rho_i \neq \rho_j$ for $i \neq j$.

It follows from (3.1) and (3.2) that

$$\omega(\delta - cs) = \left\{ \prod_{k=1}^{m} (\lambda_k + \delta - cs)^{n_k} \right\} \sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{a_{i,j}}{(\lambda_i + \delta - cs)^j},$$

is still a polynomial in s of degree $n-1$ or less. More generally, if $\theta_{i,j}$ are constants, then as pointed out by Li and Garrido (2005),

$$q(s) = \left\{ \prod_{k=1}^{m} (\lambda_k + \delta - cs)^{n_k} \right\} \sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{\theta_{i,j}}{(\lambda_i + \delta - cs)^j}, \tag{3.5}$$

is a polynomial in s of degree $n-1$ or less. Therefore, from the theory of Lagrange polynomials, (3.5) may be re-expressed as

$$q(s) = \sum_{i=1}^{n} q(\rho_i) \left\{ \prod_{j=1, j\neq i}^{n} \frac{s - \rho_j}{\rho_i - \rho_j} \right\}. \tag{3.6}$$

The Laplace transform relationship in (2.44) is used in Section 3.1.2 to derive a defective renewal equation for (1.3), and to show that this is a generalization of that obtained by Li and Garrido (2005) for its special case (1.3). In Section 3.1.3, the results of Sections 2.2.1, 2.2.2 and 3.1.2 are used to obtained the trivariate "discounted" defective distribution of $U_{T^-}, |U_T|$, and $R_{N_T - 1}$. Joint and marginal distributions of the claim causing ruin (e.g. Dufresne and Gerber (1988)) given by $Y_{N_T} = U_{T^-} + |U_T|$, and the last interclaim time before ruin are also obtained. The asymptotic result for these densities are considered in Section 3.1.4. Up to Section 3.1.4, we assume that the claim sizes are independent on the interclaim times whereas this assumption is relaxed to allowance of the dependency between these two random variables in Section 3.1.5. In particular, the model studied by Boudreault et al. (2006) is considered.

3.1.2 Defective renewal equations

In this section we identify the components in the defective renewal equation (2.13) under the present model. More precisely, the function $v_\delta(u)$ will be identified. Much information can be obtained from (2.13), including solutions for $m_\delta(u)$ which are discussed further in subsequent sections of the paper.

In this case, in order to express (2.37) in the form (2.13), consider $\sigma_{t,\delta}(x) = \sigma_\delta(x)$ in (2.40) (i.e. $P_t(y) = P(y)$), then in this case (2.43) is

$$\widetilde{\sigma}_w(\delta - cs) = \int_0^\infty e^{-sx} \left\{ \int_{\frac{x}{c}}^\infty e^{-(\delta-cs)t} k(t)dt \right\} \sigma_\delta(x)dx. \tag{3.7}$$

Using (3.3),

$$\begin{aligned}
&\int_{\frac{x}{c}}^\infty e^{-(\delta-cs)t} k(t)dt = \sum_{i=1}^m \sum_{j=1}^{n_i} a_{i,j} \int_{\frac{x}{c}}^\infty \frac{t^{j-1} e^{-(\lambda_i+\delta-cs)t}}{(j-1)!} dt \\
&= \sum_{i=1}^m \sum_{j=1}^{n_i} \frac{a_{i,j}}{c^j (j-1)!} \int_x^\infty t^{j-1} e^{-\left(\frac{\lambda_i+\delta}{c} - s\right)t} dt \\
&= \sum_{i=1}^m \sum_{j=1}^{n_i} \sum_{k=0}^{j-1} \frac{a_{i,j} x^k e^{-\left(\frac{\lambda_i+\delta}{c} - s\right)x}}{c^j k!} \int_0^\infty \frac{t^{j-k-1} e^{-\left(\frac{\lambda_i+\delta}{c} - s\right)t}}{(j-k-1)!} dt = \sum_{i=1}^m \sum_{j=1}^{n_i} \sum_{k=0}^{j-1} \frac{a_{i,j} x^k e^{-\left(\frac{\lambda_i+\delta}{c} - s\right)x}}{c^k k! (\lambda_i + \delta - cs)^{j-k}}.
\end{aligned} \tag{3.8}$$

Substitution of the above expression into (3.7) followed by an interchange of order of the summation yields

$$\begin{aligned}
\widetilde{\sigma}_w(\delta - cs) &= \sum_{i=1}^m \sum_{j=1}^{n_i} \sum_{k=0}^{j-1} \frac{a_{i,j}}{c^k k! (\lambda_i + \delta - cs)^{j-k}} \int_0^\infty x^k e^{-\left(\frac{\lambda_i+\delta}{c}\right)x} \sigma_\delta(x)dx \\
&= \sum_{i=1}^m \sum_{j=1}^{n_i} \sum_{k=0}^{j-1} \frac{(-1)^k a_{i,j} \widetilde{\sigma}_\delta^{(k)}\left(\frac{\lambda_i+\delta}{c}\right)}{c^k k! (\lambda_i + \delta - cs)^{j-k}} = \sum_{i=1}^m \sum_{k=0}^{n_i-1} \sum_{j=k+1}^{n_i} \frac{(-1)^k a_{i,j} \widetilde{\sigma}_\delta^{(k)}\left(\frac{\lambda_i+\delta}{c}\right)}{c^k k! (\lambda_i + \delta - cs)^{j-k}} \\
&= \sum_{i=1}^m \sum_{k=0}^{n_i-1} \sum_{j=1}^{n_i-k} \frac{(-1)^k a_{i,j+k} \widetilde{\sigma}_\delta^{(k)}\left(\frac{\lambda_i+\delta}{c}\right)}{c^k k! (\lambda_i + \delta - cs)^j} = \sum_{i=1}^m \sum_{j=1}^{n_i} \sum_{k=0}^{n_i-j} \frac{(-1)^k a_{i,j+k} \widetilde{\sigma}_\delta^{(k)}\left(\frac{\lambda_i+\delta}{c}\right)}{c^k k! (\lambda_i + \delta - cs)^j}.
\end{aligned}$$

Thus, we get

$$\widetilde{\sigma}_w(\delta - cs) = \sum_{i=1}^{m}\sum_{j=1}^{n_i} \frac{\theta^*_{i,j}}{(\lambda_i + \delta - cs)^j}, \tag{3.9}$$

where

$$\theta^*_{i,j} = \sum_{k=j}^{n_i} \frac{(-1)^{k-j} a_{i,k} \widetilde{\sigma}_\delta^{(k-j)}\left(\frac{\lambda_i+\delta}{c}\right)}{(k-j)! c^{k-j}}. \tag{3.10}$$

With the above $\widetilde{\sigma}_w(\delta - cs)$, (2.44) becomes

$$\widetilde{m}_\delta(s)\left\{1 - \widetilde{p}(s)\widetilde{k}(\delta - cs)\right\} = \widetilde{\beta}_\delta(s) - \sum_{i=1}^{m}\sum_{j=1}^{n_i} \frac{\theta^*_{i,j}}{(\lambda_i + \delta - cs)^j},$$

where $\theta^*_{i,j}$ are constants given by (3.9). And we may rewrite the above equation as

$$\widetilde{m}_\delta(s)\left\{1 - \widetilde{p}(s)\widetilde{k}(\delta - cs)\right\} = \widetilde{\beta}_\delta(s) - \frac{q_*(s)}{\prod_{k=1}^{m}(\lambda_k + \delta - cs)^{n_k}}, \tag{3.11}$$

where

$$q_*(s) = \left\{\prod_{k=1}^{m}(\lambda_k + \delta - cs)^{n_k}\right\} \sum_{i=1}^{m}\sum_{j=1}^{n_i} \frac{\theta^*_{i,j}}{(\lambda_i + \delta - cs)^j}$$

is of the form (3.5) and is a polynomial of degree $n-1$ or less. Therefore, from (3.4) and (3.6), $q_*(s)$ may be expressed as

$$q_*(s) = \sum_{i=1}^{n}\left\{\widetilde{\beta}_\delta(\rho_i)\prod_{k=1}^{m}(\lambda_k + \delta - c\rho_i)^{n_k}\right\}\prod_{j=1, j\neq i}^{n}\left(\frac{s-\rho_j}{\rho_i - \rho_j}\right), \tag{3.12}$$

where it is tacitly assumed that $\widetilde{m}_\delta(\rho_i) < \infty$ for $i = 1, 2, \ldots, n$, to ensure that the left side of (3.11) vanishes when $s = \rho_i$.

Multiplication of (3.11) by $\prod_{k=1}^{m}(\lambda_k+\delta-cs)^{n_k}$ and using (3.12) results in

$$\widetilde{m}_\delta(s)\left\{\prod_{k=1}^{m}(\lambda_k+\delta-cs)^{n_k}\right\}\left\{1-\widetilde{p}(s)\widetilde{k}(\delta-cs)\right\}$$
$$=\widetilde{\beta}_\delta(s)\left\{\prod_{k=1}^{m}(\lambda_k+\delta-cs)^{n_k}\right\}-\sum_{i=1}^{n}\left\{\widetilde{\beta}_\delta(\rho_i)\prod_{k=1}^{m}(\lambda_k+\delta-c\rho_i)^{n_k}\right\}\prod_{j=1,j\neq i}^{n}\left(\frac{s-\rho_j}{\rho_i-\rho_j}\right). \tag{3.13}$$

But from Li and Garrido (2005, equations 19 and 22), the defective density $\phi_\delta f_\delta(y)$ has Laplace transform

$$\phi_\delta\widetilde{f}_\delta(s)=1-\frac{\prod_{k=1}^{m}(\lambda_k+\delta-cs)^{n_k}}{c^n\prod_{j=1}^{n}(\rho_j-s)}\left\{1-\widetilde{p}(s)\widetilde{k}(\delta-cs)\right\},$$

which may be rearranged as

$$\left\{\prod_{k=1}^{m}(\lambda_k+\delta-cs)^{n_k}\right\}\left\{1-\widetilde{p}(s)\widetilde{k}(\delta-cs)\right\}=\left\{1-\phi_\delta\widetilde{f}_\delta(s)\right\}(-c)^n\prod_{j=1}^{n}(s-\rho_j).$$

Substitution of this expression into the left side of (3.13) followed by division of both sides of (3.13) by $(-c)^n\prod_{j=1}^{n}(s-\rho_j)$ results in

$$\widetilde{m}_\delta(s)\left\{1-\phi_\delta\widetilde{f}_\delta(s)\right\}=\widetilde{v}_\delta(s) \tag{3.14}$$

where

$$\widetilde{v}_\delta(s)=\widetilde{\beta}_\delta(s)\frac{\prod_{k=1}^{m}(s-\frac{\lambda_k+\delta}{c})^{n_k}}{\prod_{j=1}^{n}(s-\rho_j)}-\sum_{i=1}^{n}\frac{\widetilde{\beta}_\delta(\rho_i)\prod_{k=1}^{m}(\rho_i-\frac{\lambda_k+\delta}{c})^{n_k}}{(s-\rho_i)\prod_{j=1,j\neq i}^{n}(\rho_i-\rho_j)}. \tag{3.15}$$

Inversion of (3.14) yields (2.13), and it remains to identify $v_\delta(u)$ by inverting (3.15).

As $\prod_{k=1}^{m}(s-(\lambda_k+\delta)/c)^{n_k}$ and $\prod_{j=1}^{n}(s-\rho_j)$ are polynomials of degree n with leading coefficient equal to unity, the polynomial

$$q_0(s)=\left\{\prod_{k=1}^{m}\left(s-\frac{\lambda_k+\delta}{c}\right)^{n_k}\right\}-\prod_{j=1}^{n}(s-\rho_j)$$

is of degree $n-1$. Because $q_0(\rho_i) = \prod_{k=1}^m (\rho_i - (\lambda_k + \delta)/c)^{n_k}$, (3.6) yields

$$q_0(s) = \sum_{i=1}^n \left\{ \prod_{k=1}^m \left(\rho_i - \frac{\lambda_k + \delta}{c} \right)^{n_k} \right\} \prod_{j=1, j \neq i}^n \left(\frac{s - \rho_j}{\rho_i - \rho_j} \right).$$

Therefore, equating these two expressions for $q_0(s)$ and rearranging yields the identity

$$\frac{\prod_{k=1}^m \left(s - \frac{\lambda_k+\delta}{c}\right)^{n_k}}{\prod_{j=1}^n (s - \rho_j)} = 1 + \sum_{i=1}^n \frac{\prod_{k=1}^m \left(\rho_i - \frac{\lambda_k+\delta}{c}\right)^{n_k}}{(s - \rho_i) \prod_{j=1, j \neq i}^n (\rho_i - \rho_j)}.$$

Replacement of the coefficient of $\widetilde{\beta}_\delta(s)$ in (3.15) by the right hand side of this expression yields

$$\widetilde{v}_\delta(s) = \widetilde{\beta}_\delta(s) + \sum_{i=1}^n \frac{\prod_{k=1}^m \left(\rho_i - \frac{\lambda_k+\delta}{c}\right)^{n_k}}{\prod_{j=1, j \neq i}^n (\rho_i - \rho_j)} \left\{ \frac{\widetilde{\beta}_\delta(s) - \widetilde{\beta}_\delta(\rho_i)}{s - \rho_i} \right\},$$

i.e.,

$$\widetilde{v}_\delta(s) = \widetilde{\beta}_\delta(s) + \sum_{i=1}^n a_i^* \left\{ \frac{\widetilde{\beta}_\delta(\rho_i) - \widetilde{\beta}_\delta(s)}{s - \rho_i} \right\} \tag{3.16}$$

where

$$a_i^* = \frac{\prod_{k=1}^m \left(\frac{\lambda_k+\delta}{c} - \rho_i\right)^{n_k}}{\prod_{j=1, j \neq i}^n (\rho_j - \rho_i)}. \tag{3.17}$$

Inversion of (3.16) yields

$$v_\delta(u) = \beta_\delta(u) + \sum_{i=1}^n a_i^* T_{\rho_i} \beta_\delta(u). \tag{3.18}$$

Moreover, an alternative approach to obtain (3.18) by the initial value theorem as did Li and Garrido (2005) is available. In order to identify $m_\delta(0)$, we need to consider differentiability of $m_\delta(u)$. In this case, from (2.37) it is sufficient that $\beta_\delta(u)$ in (2.38), namely $w(x, y, v)$ in (2.39) is differentiable. From (3.3) $k(t)$ is shown to be differentiable. Hence, let us assume that the form of penalty function is differentiable. Now, from (3.11), we have

$$\widetilde{m}_\delta(s) = \frac{\widetilde{\beta}_\delta(s) - q_*(s) \prod_{k=1}^m (\lambda_k + \delta - cs)^{-n_k}}{1 - \widetilde{p}(s) \widetilde{k}(\delta - cs)}. \tag{3.19}$$

Because $q_*(s)$ is a polynomial of degree $n-1$ or less, let $q_*(s) = \sum_{j=0}^{n-1} q_j s^j$ then

$$\lim_{s\to\infty} \frac{sq_*(s)}{\prod_{k=1}^m (\lambda_k + \delta - cs)^{n_k}} = \lim_{s\to\infty} \frac{\sum_{j=1}^n q_{j-1} s^j}{\prod_{k=1}^m (\lambda_k + \delta - cs)^{n_k}}$$
$$= \lim_{s\to\infty} \frac{\sum_{j=1}^n q_{j-1} s^{j-n}}{\prod_{k=1}^m \left(\frac{\lambda_k+\delta}{s} - c\right)^{n_k}} = \frac{q_{n-1}}{(-c)^n}.$$

Therefore, using the above result and (3.19) we may obtain $m_\delta(0)$ by the initial value theorem,

$$\lim_{s\to\infty} s\widetilde{m}_\delta(s) = \lim_{s\to\infty} \frac{s\widetilde{\beta}_\delta(s) - sq_*(s)\prod_{k=1}^m (\lambda_k + \delta - cs)^{-n_k}}{1 - \widetilde{p}(s)\widetilde{k}(\delta - cs)}$$
$$= \frac{\lim_{s\to\infty} s\widetilde{\beta}_\delta(s) - \frac{q_{n-1}}{(-c)^n}}{1 - \left\{\lim_{s\to\infty} \frac{\widetilde{p}(s)}{s}\right\}\left\{\lim_{s\to\infty} s\widetilde{k}(\delta - cs)\right\}} = \beta_\delta(0) - \frac{q_{n-1}}{(-c)^n},$$

namely,

$$m_\delta(0) = \beta_\delta(0) - \frac{q_{n-1}}{(-c)^n}. \tag{3.20}$$

Then, we need to identify q_{n-1} which is the coefficient of s^{n-1} in $q_*(s)$. From (3.12),

$$q_*(s) = -(-c)^n \sum_{i=1}^n \left\{ \widetilde{\beta}_\delta(\rho_i) \frac{\prod_{k=1}^m \left(\frac{\lambda_k+\delta}{c} - \rho_i\right)^{n_k}}{\prod_{j=1, j\neq i}^n (\rho_j - \rho_i)} \right\} \prod_{j=1, j\neq i}^n (s - \rho_j),$$

thus we get q_{n-1} given by

$$q_{n-1} = -(-c)^n \sum_{i=1}^n a_i^* \widetilde{\beta}_\delta(\rho_i), \tag{3.21}$$

where

$$a_i^* = \frac{\prod_{k=1}^m \left(\frac{\lambda_k+\delta}{c} - \rho_i\right)^{n_k}}{\prod_{j=1, j\neq i}^n (\rho_j - \rho_i)}$$

which is equivalent to (3.17). Substitution of (3.21) into (3.20) yields

$$m_\delta(0) = \beta_\delta(0) + \sum_{i=1}^n a_i^* \widetilde{\beta}_\delta(\rho_i). \tag{3.22}$$

Using (2.27), (3.22) becomes

$$m_\delta(0) = \beta_\delta(0) + \sum_{i=1}^{n} a_i^* \int_0^\infty \int_0^\infty \int_v^\infty e^{-\rho_i v} w(x,y,v) h_{1,\delta}(x,y|v) dxdydv. \tag{3.23}$$

Clearly, from (2.13), (3.22) is equal to $v_\delta(0)$. Using (2.27), (2.14) may be rewritten as

$$v_\delta(0) = \beta_\delta(0) + \int_0^\infty \int_0^\infty \int_v^\infty w(x,y,v) h_{2,\delta}(x,y,v|0) dxdydv,$$

and comparing the above equation to (3.23) we may obtain

$$h_{2,\delta}(x,y,v|0) = \sum_{i=1}^{n} a_i^* e^{-\rho_i v} h_{1,\delta}(x,y|v), \qquad x > v, \; y > 0. \tag{3.24}$$

Therefore, combining (2.27) and (3.24) it is easy to find $v_\delta(u)$ in (2.14) given by

$$v_\delta(u) = \beta_\delta(u) + \sum_{i=1}^{n} a_i^* T_{\rho_i} \beta_\delta(u),$$

which is agreed with (3.18).

For the special case with the penalty function given by $w_{12}(x,y)$, Willmot and Woo (2010) recovered the defective renewal equation of Li and Garrido (2005) for (1.1). In this case, $m_{\delta,12}(u)$ satisfies the defective renewal equation (2.7) with $v_\delta^*(u) = v_{\delta,12}(u)$, where

$$v_{\delta,12}(u) = \sum_{i=1}^{n} b_i T_{\rho_i} \alpha_{12}(u), \tag{3.25}$$

where

$$\alpha_{12}(x) = \int_x^\infty w_{12}(x, y-x) p(y) dy, \tag{3.26}$$

and

$$b_i = \frac{\omega(\delta - c\rho_i)}{c^n \prod_{j=1, j\neq i}^{n} (\rho_j - \rho_i)}. \tag{3.27}$$

See Willmot and Woo (2010) for further details.

We remark that in the special case of (3.1) with $\omega(s) = \prod_{k=1}^{m} \lambda_k^{n_k}$, (3.27) simplifies to

$$b_i = \frac{\prod_{k=1}^{m} (\lambda_k/c)^{n_k}}{\prod_{j=1, j\neq i}^{n} (\rho_j - \rho_i)},$$

and using Equation 4 of Li and Garrido (2004), (3.25) may be expressed as

$$v_{\delta,12}(u) = \left\{ \prod_{k=1}^{m} \left(\frac{\lambda_k}{c} \right)^{n_k} \right\} T_{\rho_1} T_{\rho_2} \cdots T_{\rho_n} \alpha_{12}(u).$$

Also, when $w_{12}(x, y) = 1$, (3.26) becomes $\overline{P}(x)$, and $m_{\delta,12}(u)$ reduces to (2.20). But this implies that $v_{\delta,12}(y) = \phi_\delta \int_y^\infty f_\delta(x)dx$, which from (3.25) yields immediately that

$$\phi_\delta \int_y^\infty f_\delta(x)dx = \sum_{i=1}^{n} b_i T_{\rho_i} \overline{P}(y) = \sum_{i=1}^{n} b_i \int_0^\infty e^{-\rho_i x} \overline{P}(x+y)dx.$$

This implies in turn for $y = 0$ that

$$\phi_\delta = \sum_{i=1}^{n} b_i \int_0^\infty e^{-\rho_i x} \overline{P}(x)dx,$$

and also (by differentiating with respect to y) that

$$f_\delta(y) = \frac{1}{\phi_\delta} \sum_{i=1}^{n} b_i \int_0^\infty e^{-\rho_i x} p(x+y)dx = \frac{1}{\phi_\delta} \sum_{i=1}^{n} b_i T_{\rho_i} p(y).$$

Furthermore using (3.1), (3.27) may be expressed as

$$b_i = \frac{\widetilde{k}(\delta - c\rho_i) \prod_{k=1}^{m} \left(\frac{\lambda_k + \delta}{c} - \rho_i \right)^{n_k}}{\prod_{j=1, j\neq i}^{n} (\rho_j - \rho_i)} = \widetilde{k}(\delta - c\rho_i) a_i^*$$

where a_i^* is given by (3.17).

3.1.3 Associated densities

From (2.24) in Section 2.2.1, note that once we get $\nu_\delta(v-u)$ from $h_{2,\delta}(x,y,v|0)$ which is obtained by $v_\delta(0)$ (basically, from $m_\delta(0)$), we readily find the various joint and marginal distribution involving $U_{T^-}, |U_T|, X_T$, and R_{N_T-1}.

In this case, it is clear that from (3.24), $\nu_\delta(v-u)$ is given by

$$\nu_\delta(v-u) = \sum_{i=1}^{n} a_i^* e^{-\rho_i(v-u)},$$

and thus we may express $m_\delta(u)$ as the form of (2.25) where $\tau_\delta(u,v)$ in (2.26) is given by

$$\tau_\delta(u,v) = \begin{cases} \frac{1}{1-\phi_\delta}\left\{g_\delta(u-v) + \sum_{i=1}^n a_i^* \int_0^v e^{-\rho_i(v-t)} g_\delta(u-t)\,dt\right\}, & v < u \\ \sum_{i=1}^n a_i^* \left\{e^{-\rho_i(v-u)} + \frac{1}{1-\phi_\delta}\int_0^u e^{-\rho_i(v-t)} g_\delta(u-t)\,dt\right\}, & v > u \end{cases}. \tag{3.28}$$

Also, for $\delta = 0$,

$$\tau_0(u,v) = \begin{cases} \frac{1}{1-\psi(0)}\left\{-\psi'(u-v) - \sum_{i=1}^n a_i^* \int_0^v e^{-\rho_i(v-t)} \psi'(u-t)dt\right\}, & v < u \\ \sum_{i=1}^n a_i^* \left\{e^{-\rho_i(v-u)} - \frac{1}{1-\psi(0)}\int_0^u e^{-\rho_i(v-t)} \psi'(u-t)\,dt\right\}, & v > u, \end{cases} \tag{3.29}$$

since $\phi_0 = \psi(0)$ and $g_0(u) = -\psi'(u)$. In (3.29), the ρ_i are the roots of (3.4) when $\delta = 0$. Clearly, the classical compound Poisson risk model with $K(t) = 1 - e^{-\lambda t}$ is the special case of the present model with $m = n = 1$ in (3.1), and (3.28) and (3.29) easily simplify to the result given by Cheung et al. (2010a).

From (2.31), the joint density of $(U_{T^-}, |U_T|, R_{N_T-1})$ at (x,y,v) for $N_T \geq 2$ is obtained by

$$h_{2,\delta}(x,y,v|u) = h_{1,\delta}(x,y|v)\tau_\delta(u,v), \qquad x > v,$$

where from (2.3) with (3.3),

$$h_{1,\delta}\left(x,y\left|u\right.\right)=\frac{1}{c}p(x+y)\sum_{i=1}^{m}\sum_{j=1}^{n_i}a_{i,j}\frac{\left(\frac{x-u}{c}\right)^{j-1}e^{-(\lambda_i+\delta)\left(\frac{x-u}{c}\right)}}{(j-1)!},\qquad x>u, \tag{3.30}$$

and $\tau_\delta(u,v)$ is given by (3.28).

Furthermore, by applying the general results provided in Section 2.2.2, we may easily obtain the joint discounted densities of $(U_{T^-},\left|U_T\right|,X_T,R_{N_T-1})$, joint discounted density of $(U_{T^-},\left|U_T\right|)$ (also studied by Li and Garrido (2005)), the joint discounted density of the last interclaim time before ruin V_{N_T} and the claim causing ruin Y_{N_T}, and also their marginal distributions. In particular, from (2.59) the marginal density of the claim causing ruin Y_{N_T} is given by

$$h_{6,\delta}(y|u)=p(y)\left[K_\delta\left(\frac{y-u}{c}\right)\mathrm{I}(y>u)+\int_0^y K_\delta\left(\frac{y-v}{c}\right)\tau_\delta(u,v)dv\right], \tag{3.31}$$

where $K_\delta(t)=\int_0^t e^{-\delta x}k(x)dx$.

To evaluate $K_\delta(t)$, if follows from (3.1) and (3.3) that

$$\begin{aligned}K_\delta(t) &= \widetilde{k}(\delta)-\int_t^\infty e^{-\delta x}k(x)dx\\ &= \frac{\omega(\delta)}{\prod_{i=1}^m(\lambda_i+\delta)^{n_i}}-\sum_{i=1}^m\sum_{j=1}^{n_i}a_{i,j}\int_t^\infty\frac{x^{j-1}e^{-(\lambda_i+\delta)x}}{(j-1)!}dx\\ &= \frac{\omega(\delta)}{\prod_{i=1}^m(\lambda_i+\delta)^{n_i}}-\sum_{i=1}^m e^{-(\lambda_i+\delta)t}\sum_{j=1}^{n_i}a_{i,j}\sum_{k=0}^{j-1}\frac{t^k}{k!(\lambda_i+\delta)^{j-k}},\end{aligned}$$

i.e.,

$$K_\delta(t)=\frac{\omega(\delta)}{\prod_{i=1}^m(\lambda_i+\delta)^{n_i}}-\sum_{i=1}^m e^{-(\lambda_i+\delta)t}\sum_{k=0}^{n_i-1}\frac{t^k}{k!}\sum_{j=k+1}^{n_i}\frac{a_{i,j}}{(\lambda_i+\delta)^{j-k}}. \tag{3.32}$$

Thus, (3.32) may be substituted into (3.31). We note that the proper pdfs corresponding to (2.56), (2.58), and (3.31) may be obtained by appropriate normalization.

In the next section we consider the asymptotic forms of $\tau_\delta(u,v)$ and associated densities.

3.1.4 Asymptotic results

Evaluation of the defective joint and marginal densities for a large u essentially requires specification of the asymptotic behavior of $\tau_\delta(u,v)$ in (3.28). First, we may reexpress a form of (3.28) for $v<u$ as follows (ignore the other case, i.e. for $v>u$ since we analyze (3.28) when $u\to\infty$). Using a integration by parts, (3.28) for $v<u$ becomes

$$\tau_\delta(u,v) = \frac{1}{1-\phi_\delta}\left\{g_\delta\left(u-v\right)+\sum_{i=1}^n a_i^*\int_0^v e^{-\rho_i(v-t)}g_\delta\left(u-t\right)dt\right\}$$
$$= \frac{1}{1-\phi_\delta}\left[g_\delta\left(u-v\right)+\sum_{i=1}^n a_i^*\left\{\overline{G}_\delta(u-v)-e^{-\rho_i v}\overline{G}_\delta(u)-\rho_i\int_0^v e^{-\rho_i(v-t)}\overline{G}_\delta\left(u-t\right)dt\right\}\right]. \tag{3.33}$$

Because the form of $\tau_\delta(u,v)$ contains the compound geometric density $g_\delta(u)$ and the compound geometric tail $\overline{G}_\delta(u)$, recall the asymptotic results for those functions. From Willmot et al. (2001), we know that the compound geometric density satisfies the defective renewal equation

$$g_\delta(u) = \phi_\delta\int_0^u g_\delta(u-y)f_\delta(y)dy + \phi_\delta(1-\phi_\delta)f_\delta(u).$$

If $e^{\kappa_\delta y}f_\delta(y)$ is directly Riemann integrable on $(0,\infty)$ (one of the sufficient condition provided by Willmot and Lin (2001, p.157) is that we can show whether $\widetilde{f}_\delta(-\kappa_\delta-\epsilon)=\int_0^\infty e^{(\kappa_\delta+\epsilon)y}dF_\delta(y)<\infty$ for some $\epsilon>0$, see also Feller (1971, pp.362-263) and Resnick (1992, Section 3.10)), then using the famous Cramér-Lundberg result yields (using the notation $a(x)\sim b(x)$ for $x\to\infty$, to mean $\lim_{x\to\infty}a(x)/b(x)=1$)

$$g_\delta(u)\sim C_\delta^* e^{-\kappa_\delta u}, \qquad u\to\infty, \tag{3.34}$$

where $\kappa_\delta>0$ satisfies $\int_0^\infty e^{\kappa_\delta y}dF_\delta(y)=\phi_\delta^{-1}$ and

$$C_\delta^* = \frac{1-\phi_\delta}{\phi_\delta\int_0^\infty y e^{\kappa_\delta y}f_\delta(y)dy}. \tag{3.35}$$

Also, if F is non-arithmetic (and thus not a discrete counting) then it is well-known that (e.g. Willmot and Lin (2001, p.158))

$$\overline{G}_\delta(u) \sim \frac{C_\delta^*}{\kappa_\delta} e^{-\kappa_\delta u}, \qquad u \to \infty, \tag{3.36}$$

where C_δ^* is given by (3.35). Then, using (3.33) one finds

$$\begin{aligned}
&\lim_{u\to\infty} e^{\kappa_\delta u}\tau_\delta(u,v) \\
&= \lim_{u\to\infty} \frac{e^{\kappa_\delta u}}{1-\phi_\delta}\left[g_\delta\left(u-v\right) + \sum_{i=1}^n a_i^* \left\{\overline{G}_\delta(u-v) - e^{-\rho_i v}\overline{G}_\delta(u) - \rho_i \int_0^v e^{-\rho_i(v-t)}\overline{G}_\delta\left(u-t\right)dt\right\}\right].
\end{aligned}$$

We know that $\overline{G}_\delta(u) \le e^{-\kappa_\delta u}$ for $u \ge 0$, by dominated convergence it follows that

$$\begin{aligned}
&\lim_{u\to\infty} e^{\kappa_\delta u}\tau_\delta(u,v) \\
&= C_\delta e^{\kappa_\delta v} + \sum_{i=1}^n a_i^* \left[\frac{C_\delta}{\kappa_\delta}\left(e^{\kappa_\delta v} - e^{-\rho_i v}\right) - \rho_i \int_0^v \left\{\lim_{u\to\infty} \frac{e^{\kappa_\delta(u-t)}\overline{G}_\delta(u-t)}{1-\phi_\delta}\right\} e^{\kappa_\delta t} e^{-\rho_i(v-t)}dt\right] \\
&= C_\delta \left\{e^{\kappa_\delta v} + \sum_{i=1}^n \frac{a_i^*}{\kappa_\delta}\left(e^{\kappa_\delta v} - e^{-\rho_i v} - \rho_i \int_0^v e^{\kappa_\delta t} e^{-\rho_i(v-t)}dt\right)\right\}
\end{aligned}$$

where $C_\delta = C_\delta^*(1-\phi_\delta)^{-1}$ with (3.35).

Therefore,

$$\tau_\delta(u,v) \sim C_\delta l_\delta(v) e^{-\kappa_\delta u}, \qquad u \to \infty, \tag{3.37}$$

where

$$l_\delta(v) = e^{\kappa_\delta v} + \sum_{i=1}^n a_i^* \left(\frac{e^{\kappa_\delta v} - e^{-\rho_i v}}{\rho_i + \kappa_\delta}\right). \tag{3.38}$$

In particular, for the compound Poisson risk model with $K(t) = 1 - e^{-\lambda t}$, only one root ρ exists and thus $l_\delta(v)$ in (3.38) reduces to

$$l_\delta^*(v) = e^{\kappa_\delta v} + \left(\frac{\lambda+\delta}{c} - \rho\right)\frac{e^{\kappa_\delta v} - e^{-\rho v}}{\rho + \kappa_\delta}.$$

For the joint density of $(U_{T^-}, |U_T|, R_{N_T-1})$, from (2.31) and (3.37) with (3.30) the discounted joint density of $(U_{T^-}, |U_T|, R_{N_T-1})$ for $x > v$ is asymptotically distributed as

$$h_{2,\delta}(x, y, v|u) \sim C_\delta l_\delta(v) h_{1,\delta}(x, y|v) e^{-\kappa_\delta u}, \qquad u \to \infty,$$

where C_δ and $l_\delta(v)$ are given by (3.35) and (3.38) respectively.

3.1.5 Time-dependent claim sizes case

Similar to the Section 3.1.2 but with time-dependent claim sizes, we shall demonstrate that (1.3) satisfies the defective renewal equation (2.7) with the identification of $v_\delta(u)$ To illustrate that, in the following we shall assume the dependency model structure introduced by Boudreault et al. (2006). The (conditional) density of $Y|V$ with an exponential mixing weight function with rate β is assumed by

$$p_t(y) = e^{-\beta t} f_1(y) + (1 - e^{-\beta t}) f_2(y), \qquad y \geq 0, \tag{3.39}$$

where $f_1(y) = -\overline{F}_1'(y)$ and $f_2(y) = -\overline{F}_2'(y)$ are claim sizes distributions with mean μ_1 and μ_2 respectively. In this case, we have the PSLC as

$$E[cV - Y] = cE[V] - \left\{ \widetilde{k}(\beta)\mu_1 + (1 - \widetilde{k}(\beta))\mu_2 \right\} > 0. \tag{3.40}$$

This model is more appropriate to reflect natural catastrophes (e.g. earthquakes). Numerical example assuming (3.39) in the delayed risk models is provided in Section 4.4.1.

Letting $\sigma_{\delta,i}(u) = \int_0^u m_\delta(u - y) f_i(y) dy$ for $i = 1, 2$, it follows that (2.40) may be expressed as

$$\sigma_{t,\delta}(u) = e^{-\beta t}\{\sigma_{\delta,1}(u) - \sigma_{\delta,2}(u)\} + \sigma_{\delta,2}(u).$$

With the above $\sigma_{t,\delta}(u)$, (2.43) is

$$\widetilde{\sigma}_w(\delta-cs)=\int_0^\infty e^{-sx}\int_{\frac{x}{c}}^\infty e^{-(\delta-cs)t}\sigma_{t,\delta}(x)\,dK(t)\,dx.$$

$$=\int_0^\infty e^{-sx}\left[\{\sigma_{\delta,1}(x)-\sigma_{\delta,2}(x)\}\left\{\int_{\frac{x}{c}}^\infty e^{-(\delta+\beta-cs)t}dK(t)\right\}-\sigma_{\delta,2}(x)\int_{\frac{x}{c}}^\infty e^{-(\delta+\beta-cs)t}dK(t)\right]dx.$$

Thus, again the distribution of intereclaim times is given by (3.3), using (3.9) and (3.10) yields

$$\widetilde{\sigma}_w(\delta-cs)=\sum_{i=1}^m\sum_{j=1}^{n_i}\left\{\frac{q_{1,i,j}}{(\lambda_i+\beta+\delta-cs)^j}+\frac{q_{2,i,j}}{(\lambda_i+\delta-cs)^j}\right\}, \tag{3.41}$$

where

$$q_{1,i,j}=\sum_{k=j}^{n_i}\frac{(-1)^{k-j}a_{i,k}}{(k-j)!c^{k-j}}\left\{\widetilde{\sigma}_{\delta,1}^{(k-j)}\left(\frac{\lambda_i+\beta+\delta}{c}\right)-\widetilde{\sigma}_{\delta,2}^{(k-j)}\left(\frac{\lambda_i+\beta+\delta}{c}\right)\right\}, \tag{3.42}$$

$$q_{2,i,j}=\sum_{k=j}^{n_i}\frac{(-1)^{k-j}a_{i,k}}{(k-j)!c^{k-j}}\left\{\widetilde{\sigma}_{\delta,2}^{(k-j)}\left(\frac{\lambda_i+\delta}{c}\right)\right\}, \tag{3.43}$$

and $\widetilde{a}^{(j)}(s)=\int_0^\infty(-x)^je^{-sx}a(x)dx$.

Hence, we obtain (2.44) with (3.41) as

$$\widetilde{m}_\delta(s)\left(1-E\left[e^{-sY-(\delta-cs)V}\right]\right)=\widetilde{\beta}_\delta(s)-\sum_{i=1}^m\sum_{j=1}^{n_i}\left\{\frac{q_{1,i,j}}{(\lambda_i+\beta+\delta-cs)^j}+\frac{q_{2,i,j}}{(\lambda_i+\delta-cs)^j}\right\},$$

where $q_{1,i,j}$ and $q_{2,i,j}$ are constants. Also, it may be expressed as

$$\widetilde{m}_\delta(s)\left(1-E\left[e^{-sY-(\delta-cs)V}\right]\right)=\widetilde{\beta}_\delta(s)-\frac{q^*(s)}{l(s)}, \tag{3.44}$$

where

$$l(s)=\prod_{k=1}^m\{(\lambda_k+\beta+\delta-cs)(\lambda_k+\delta-cs)\}^{n_k} \tag{3.45}$$

is a polynomial of degree $2n$ for $\beta\neq 0$, and

$$q^*(s)=l(s)\sum_{i=1}^m\sum_{j=1}^{n_i}\left\{\frac{q_{1,i,j}}{(\lambda_i+\beta+\delta-cs)^j}+\frac{q_{2,i,j}}{(\lambda_i+\delta-cs)^j}\right\}$$

is again of the form (3.5) and is a polynomial of degree $2n-1$ or less. Division of both sides of (3.44) by $\left(1-E\left[e^{-sY-(\delta-cs)V}\right]\right)$ and then multiplication of the numerator and the denominator on the right hand side of (3.44) by $l(s)$ results in

$$\widetilde{m}_\delta(s)=\frac{\widetilde{\beta}_\delta(s)-q^*(s)l(s)^{-1}}{1-r(s)l(s)^{-1}} \tag{3.46}$$

where $q^*(s)$ and $l(s)$ are respectively given by (3.48), (3.45), and

$$r(s)=l(s)\left[\left\{\widetilde{f}_1(s)-\widetilde{f}_2(s)\right\}\widetilde{k}(\delta+\beta-cs)+\widetilde{f}_2(s)\widetilde{k}(\delta-cs)\right]. \tag{3.47}$$

Now let us discuss the roots of the generalized Lundberg equation (2.34) in this case. With (3.39) we have

$$E\left[e^{-sY-(\delta-cs)V}\right]=\left\{\widetilde{f}_1(s)-\widetilde{f}_2(s)\right\}\widetilde{k}(\delta+\beta-cs)+\widetilde{f}_2(s)\widetilde{k}(\delta-cs)=1.$$

Note that the roots of the denominator on the right-hand side in (3.46) solve the above equation.

Proposition 1 *For $\delta>0$, $l(s)-r(s)$ has exactly $2n$ roots denoted by $\rho_1,\rho_2,\dots,\rho_{2n}$, which have a positive real part $\mathrm{Re}(\rho_j)>0$ for $j=1,2,\dots,2n$. In particular, for $\delta=0$, $l(s)-r(s)$ has exactly $2n-1$ roots with a positive real part and one zero root.*

Proof. By using the Rouche's theorem and applying the result in Klimenok (2001) (e.g. Boudreault et al. (2006, Proposition 1.2)) we can determine the number of roots of the equation $1-r(s)/l(s)=0$. The details are ommitted here. □

Then, $q^*(s)$ in (3.46) may be expressed as

$$q^*(s)=\sum_{i=1}^{2n}\widetilde{\beta}_\delta(\rho_i)l(\rho_i)\prod_{j=1,j\neq i}^{2n}\left(\frac{s-\rho_j}{\rho_i-\rho_j}\right). \tag{3.48}$$

Now, again by the initial value theorem, we may identify $m_\delta(0)$. To do so, we assume that the penalty function $w(x, y, v)$ in (1.3) is differentiable. Since $q^*(s)$ is a polynomial of degree $2n-1$ or less, assuming $q^*(s) = \sum_{j=0}^{2n-1} q_j^* s^j$ then from (3.45), we get

$$\lim_{s\to\infty} \frac{sq^*(s)}{l(s)} = \lim_{s\to\infty} \frac{\sum_{j=1}^{2n} q_{j-1}^* s^j}{\prod_{k=1}^{m} \{(\lambda_k + \beta + \delta - cs)(\lambda_k + \delta - cs)\}^{n_k}}$$
$$= \lim_{s\to\infty} \frac{\sum_{j=1}^{2n} q_{j-1}^* s^{j-2n}}{\prod_{k=1}^{m} \left\{\left(\frac{\lambda_k+\beta+\delta}{s} - c\right)\left(\frac{\lambda_k+\delta}{s} - c\right)\right\}^{n_k}} = \frac{q_{2n-1}^*}{(-c)^{2n}},$$

and, from (3.47)

$$\lim_{s\to\infty} \frac{r(s)}{l(s)} = \lim_{s\to\infty} \left[\left\{\widetilde{f}_1(s) - \widetilde{f}_2(s)\right\} \widetilde{k}(\delta + \beta - cs) + \widetilde{f}_2(s)\widetilde{k}(\delta - cs)\right]$$
$$= \left\{\lim_{s\to\infty} \frac{\widetilde{f}_1(s) - \widetilde{f}_2(s)}{s}\right\} \left\{\lim_{s\to\infty} s\widetilde{k}(\delta + \beta - cs)\right\} + \left\{\lim_{s\to\infty} \frac{\widetilde{f}_2(s)}{s}\right\} \left\{\lim_{s\to\infty} s\widetilde{k}(\delta - cs)\right\} = 0.$$

Therefore, from the above results by taking the limit on (3.46), it follows that

$$\lim_{s\to\infty} s\widetilde{m}_\delta(s) = \lim_{s\to\infty} \frac{s\widetilde{\beta}_\delta(s) - sq^*(s)l(s)^{-1}}{1 - r(s)l(s)^{-1}}$$
$$= \lim_{s\to\infty} s\widetilde{\beta}_\delta(s) - \frac{q_{2n-1}^*}{(-c)^{2n}} = \beta_\delta(0) - \frac{q_{2n-1}^*}{(-c)^{2n}},$$

namely,

$$m_\delta(0) = \beta_\delta(0) - \frac{q_{2n-1}^*}{(-c)^{2n}}. \tag{3.49}$$

Then, we need to identify q_{2n-1}^* which is the coefficient of s^{2n-1} in $q^*(s)$. From (3.48),

$$q_*(s) = (-1)^{2n-1} \sum_{i=1}^{2n} \left\{\frac{\widetilde{\beta}_\delta(\rho_i)l(\rho_i)}{\prod_{j=1,j\neq i}^{2n}(\rho_j - \rho_i)}\right\} \prod_{j=1,j\neq i}^{2n} (s - \rho_j),$$

thus we get q_{n-1} given by

$$q_{2n-1}^* = -(-c)^{2n} \sum_{i=1}^{2n} a_i^{**} \widetilde{\beta}_\delta(\rho_i), \tag{3.50}$$

where

$$a_i^{**} = \frac{\prod_{k=1}^{m} \left(\frac{\lambda_k+\delta}{c} - \rho_i\right)^{n_k} \left(\frac{\lambda_k+\beta+\delta}{c} - \rho_i\right)^{n_k}}{\prod_{j=1,j\neq i}^{2n} (\rho_j - \rho_i)}.$$

Substitution of (3.50) into (3.49) yields

$$m_\delta(0) = \beta_\delta(0) + \sum_{i=1}^{2n} a_i^{**} \widetilde{\beta}_\delta(\rho_i). \tag{3.51}$$

Similar to Section 3.1.2, we may obtain

$$h_{2,\delta}(x,y,v|0) = \sum_{i=1}^{2n} a_i^{**} e^{-\rho_i v} h_{1,\delta}(x,y|v).$$

Thus, we may easily find $v_\delta(u)$ in (2.14) from $\beta_\delta(u)$ and $h_{2,\delta}(x,y,v|0)$

$$v_\delta(u) = \beta_\delta(u) + \sum_{i=1}^{2n} a_i^{**} T_{\rho_i} \beta_\delta(u).$$

In particular, if $w(x,y,v) = w_{12}(x,y)$ as a classical Gerber-Shiu penalty function, $\beta_\delta(u)$ in (2.38) becomes a form of (2.35), Thus, using (2.36) and (3.41) followed by substituting (2.39) and (3.39), we get $\widetilde{\beta}_\delta(s)$ as

$$\begin{aligned} \widetilde{\beta}_{\delta,12}(s) &= \int_0^\infty e^{-(\delta-cs)t} \widetilde{\alpha}_t(s) dK(t) - \sum_{i=1}^{m} \sum_{j=1}^{n_i} \left\{ \frac{q_{1,i,j}^*}{(\lambda_i+\beta+\delta-cs)^j} + \frac{q_{2,i,j}^*}{(\lambda_i+\delta-cs)^j} \right\} \\ &= \{\widetilde{\alpha}_1(s) - \widetilde{\alpha}_2(s)\} \widetilde{k}(\delta+\beta-cs) + \widetilde{\alpha}_2(s) \widetilde{k}(\delta-cs) \\ &\quad - \sum_{i=1}^{m} \sum_{j=1}^{n_i} \left\{ \frac{q_{1,i,j}^*}{(\lambda_i+\beta+\delta-cs)^j} + \frac{q_{2,i,j}^*}{(\lambda_i+\delta-cs)^j} \right\} \end{aligned}$$

where

$$\widetilde{\alpha}_i(s) = \int_0^\infty \int_0^\infty e^{-sx} w_{12}(x,y) f_i(x+y) dy dx, \tag{3.52}$$

for $i = 1, 2$, and constants $q_{1,i,j}^*$ and $q_{2,i,j}^*$ are respectively given by (3.42) and (3.43) but $\widetilde{\sigma}_{\delta,i}$ replaced by $\widetilde{\alpha}_i$.

Therefore, with the above expression for $\widetilde{\beta}_{\delta,12}(s)$, (3.44) becomes

$$\widetilde{m}_{\delta,12}(s)\left(1 - E\left[e^{-sY-(\delta-cs)V}\right]\right) = \gamma_\delta(s) - \frac{q^{**}(s)}{l(s)}, \tag{3.53}$$

where

$$\gamma_\delta(s) = \{\widetilde{\alpha}_1(s) - \widetilde{\alpha}_2(s)\}\,\widetilde{k}(\delta+\beta-cs) + \widetilde{\alpha}_2(s)\widetilde{k}(\delta-cs), \tag{3.54}$$

$l(s)$ is given by (3.45), a polynomial of degree $2n$ for $\beta \neq 0$, and

$$q^{**}(s) = l(s)\sum_{i=1}^{m}\sum_{j=1}^{n_i}\left\{\frac{q_{1,i,j}+q^{*}_{1,i,j}}{(\lambda_i+\beta+\delta-cs)^j} + \frac{q_{2,i,j}+q^{*}_{2,i,j}}{(\lambda_i+\delta-cs)^j}\right\}$$

is again of the form (3.5) and is a polynomial of degree $2n-1$ or less. Again, $q^{**}(s)$ may be expressed as

$$q^{**}(s) = \sum_{i=1}^{2n}\gamma_\delta(\rho_i)l(\rho_i)\prod_{j=1,j\neq i}^{2n}\left(\frac{s-\rho_j}{\rho_i-\rho_j}\right), \tag{3.55}$$

and thus, from (3.53) it follows

$$\widetilde{m}_{\delta,12}(s) = \frac{\gamma_\delta(s) - q^{**}(s)l(s)^{-1}}{1-r(s)l(s)^{-1}}$$

where $r(s)$ is given by (3.47). Similar to the previous case, we know that

$$\lim_{s\to\infty}\frac{sq^{**}(s)}{l(s)} = \frac{q^{**}_{2n-1}}{(-c)^{2n}},$$

where q^{**}_{2n-1} is the coefficient of $q^{**}(s)$ in (3.55). Also, from (3.52) and (3.54), $\lim_{s\to\infty} s\gamma_\delta(s) = 0$ since $\lim_{s\to\infty}\widetilde{\alpha}_i(s) = 0$. Hence, again by the initial value theorem we may obtain $m_{\delta,12}(0)$ as follows:

$$\lim_{s\to\infty} s\widetilde{m}_{\delta,12}(s) = \frac{s\gamma_\delta(s) - sq^{**}(s)l(s)^{-1}}{1-r(s)l(s)^{-1}} = -\frac{q^{**}_{2n-1}}{(-c)^{2n}}.$$

From (3.55) we may find

$$q_{2n-1}^{**} = -(-c)^{2n} \sum_{i=1}^{2n} \left\{ b_{1,i}\widetilde{\alpha}_1(\rho_i) + b_{2,i}\widetilde{\alpha}_2(\rho_i) \right\}, \tag{3.56}$$

where

$$b_{1,i} = \frac{\prod_{k=1}^{m} \left(\frac{\lambda_k+\delta}{c} - \rho_i\right)^{n_k} \omega(\delta+\beta-c\rho_i)}{c^n \prod_{j=1,j\neq i}^{2n}(\rho_j - \rho_i)}$$

and

$$b_{2,i} = \frac{\prod_{k=1}^{m} \left(\frac{\lambda_k+\beta+\delta}{c} - \rho_i\right)^{n_k} \omega(\delta-c\rho_i)}{c^n \prod_{j=1,j\neq i}^{2n}(\rho_j - \rho_i)} - b_{1,i}.$$

Therefore,

$$m_{\delta,12}(0) = \sum_{i=1}^{2n} \left\{ b_{1,i}\widetilde{\alpha}_1(\rho_i) + b_{2,i}\widetilde{\alpha}_2(\rho_i) \right\}. \tag{3.57}$$

In addition, by comparing (2.10) and (3.57) with (3.52) we immediately obtain

$$h_\delta(x,y|0) = \sum_{i=1}^{2n} \left\{ b_{1,i}e^{-\rho_i x} f_1(x+y) + b_{2,i}e^{-\rho_i x} f_2(x+y) \right\},$$

consequently we find (2.11) and (2.12) respectively in this case,

$$\phi_\delta = \int_0^\infty \int_0^\infty h_\delta(x,y|0)dxdy = \sum_{i=1}^{2n} \left\{ b_{1,i}\frac{1-\widetilde{f}_1(\rho_i)}{\rho_i} + b_{2,i}\frac{1-\widetilde{f}_2(\rho_i)}{\rho_i} \right\}, \tag{3.58}$$

and

$$f_\delta(y) = \frac{1}{\phi_\delta}\int_0^\infty h_\delta(x,y|0)dx = \frac{1}{\phi_\delta}\sum_{i=1}^{2n} \left\{ b_{1,i}T_{\rho_i}f_1(y) + +b_{2,i}T_{\rho_i}f_2(y) \right\}. \tag{3.59}$$

In particular, for the classical compound Poisson risk model with $m = n = 1$ and $\omega(s) = \lambda$, (3.57) reduces to

$$m_{\delta,12}(0) = \frac{\lambda}{c}\sum_{i=1}^{2} \frac{\left(\frac{\lambda+\delta}{c} - \rho_i\right)\widetilde{\alpha}_1(\rho_i) + \frac{\beta}{c}\widetilde{\alpha}_2(\rho_i)}{\prod_{j=1,j\neq i}^{2}(\rho_j - \rho_i)},$$

namely

$$m_{\delta,12}(0) = \frac{\lambda}{c} \frac{\left(\frac{\lambda+\delta}{c} - \rho_2\right)\{\widetilde{\alpha}_1(\rho_1) - \widetilde{\alpha}_1(\rho_2)\} + \frac{\beta}{c}\{\widetilde{\alpha}_2(\rho_1) - \widetilde{\alpha}_2(\rho_2)\} + (\rho_2 - \rho_1)\widetilde{\alpha}_1(\rho_1)}{\rho_2 - \rho_1},$$

which is equivalent to the result by Boudreault et al. (2006, Equation 37),

$$m_{\delta,12}(0) = \frac{\lambda}{c}\left[\left(\frac{\lambda+\delta}{c} - \rho_2\right) T_{\rho_2} T_{\rho_1}\alpha_1(0) + \frac{\beta}{c} T_{\rho_2} T_{\rho_1}\alpha_2(0) + T_{\rho_1}\alpha(0)\right].$$

3.2 Deficit at ruin with time-dependent claims

As mentioned earlier in Chapter 2, it follows that any properties of the distribution of the deficit $|U_T|$ are formally the same as in the independent case, but with the present definitions of ϕ_δ in (2.11) and $f_\delta(y)$ in (2.12). Therefore, now we illustrate how to obtain proper distribution of deficit under the dependency model studied by Boudreault et al. (2006).

3.2.1 Introduction

Boudreault et al. (2006) consider a dependence structure given by (3.39). For this model with $\overline{K}(t) = e^{-\lambda t}$, the PSLC is

$$\frac{c}{\lambda} - \frac{\lambda\mu_1 + \beta\mu_2}{\beta + \lambda} > 0. \tag{3.60}$$

In addition, we need the solutions to the generalized Lundberg equation to analyze the Gerber-Shiu function. In this case, (2.34) is given by

$$\frac{\lambda(\lambda+\delta-cs)\widetilde{f}_1(s) + \lambda\beta\widetilde{f}_2(s)}{(\lambda+\delta+\beta-cs)(\lambda+\delta-cs)} = 1,$$

and two roots exist denoted by ρ_1 and ρ_2.

For $\delta = 0$ (assumed $\rho_1 = \rho$ and $\rho_2 = 0$), letting $\phi_0 = \phi$ and $f_0(y) = f(y)$ be respectively, from Theorem 5 in Boudreault et al. (2006) we have

$$\phi = \frac{\lambda}{c}\left(\frac{\lambda}{c}T_0T_\rho\overline{F}_1(0) + \frac{\beta}{c}T_0T_\rho\overline{F}_2(0) + T_\rho\overline{F}_1(0)\right), \tag{3.61}$$

and

$$f(y) = q_1\frac{T_\rho\overline{F}_1(y)}{T_0T_\rho\overline{F}_1(0)} + q_2\frac{T_\rho\overline{F}_2(y)}{T_0T_\rho\overline{F}_2(0)} + (1-q_1-q_2)\frac{T_\rho f_1(y)}{T_\rho\overline{F}_1(0)}, \tag{3.62}$$

with

$$q_1 = \frac{\frac{\lambda}{c}\frac{\lambda}{c}T_0T_\rho\overline{F}_1(0)}{\kappa_0}, \qquad q_2 = \frac{\frac{\lambda}{c}\frac{\beta}{c}T_0T_\rho\overline{F}_2(0)}{\kappa_0}, \tag{3.63}$$

and $0 \le q_1, q_2 \le 1$ with $0 \le q_1 + q_2 \le 1$. Also, the defective renewal equation for $m_{\delta,12}(u)$ in (1.1) is given by

$$m_{0,12}(u) = \phi\int_0^u m_{0,12}(u-y)f(y)dy + \xi(u), \tag{3.64}$$

where

$$\xi(u) = \frac{\lambda}{c}\left(\frac{\lambda}{c}T_0T_\rho\gamma_1(u) + \frac{\beta}{c}T_0T_\rho\gamma_2(u) + T_\rho\gamma_1(u)\right), \tag{3.65}$$

and $\gamma_i(t) = \int_t^\infty w_{12}(t, y-t)f_i(y)dy$.

In order to rewrite a form of $f(y)$ which is a mixture of three distributions, introduce the tail of ladder height distribution of $f_i(y)$ denoted by $\overline{H}_i(y)$ and defined by

$$\overline{H}_i(y) = \frac{\int_0^\infty e^{-\rho t}\overline{F}_i(y+t)dt}{\int_0^\infty e^{-\rho t}\overline{F}_i(t)dt} = \frac{T_\rho\overline{F}_i(y)}{T_\rho\overline{F}_i(0)}, \qquad i = 1, 2, \tag{3.66}$$

which may viewed as a mixture over t of the df $1-\overline{F}_i(y+t)/\overline{F}_i(t)$ with mixing density proportional to $e^{-\rho t}\overline{F}_i(t)$. Also, with $\rho = 0$ in (3.66) we get the equilibrium distribution of $f_i(y)$ denoted by $f_{i,1}(y)$ for $i = 1, 2$, then

$$\frac{T_\rho\overline{F}_i(y)}{T_0T_\rho\overline{F}_i(0)} = \frac{\mu_1\ T_\rho f_{i,1}(y)}{\mu_1 T_0T_\rho f_{i,1}(0)} = \frac{\int_0^\infty e^{-\rho t}f_{i,1}(y+t)dt}{\int_0^\infty e^{-\rho t}\overline{F}_{i,1}(t)dt} = h_{i,1}(y), \tag{3.67}$$

where $h_{i,1}(y)$ is the equilibrium pdf of $1 - \overline{H}_i(y)$ and $\overline{F}_{i,1}(t) = \int_t^\infty f_{i,1}(y)dy$ for $i = 1, 2$. Note that the equilibrium distribution of the residual life time distribution is the residual lifetime of the equilibrium distribution. See Willmot and Lin (2001, p.22) for further details.

Therefore, by using (3.66) and (3.67) $f(y)$ in (3.62) may be re-expressed as a mixture of three ladder height given by

$$f(y) = q_1 h_{1,1}(y) + q_2 h_{2,1}(y) + (1 - q_1 - q_2) h_1(y), \tag{3.68}$$

where $h_{i,1}(y)$ is given by (3.67), $h_i(y) = -\overline{H}_i'(y)$ and q_1, q_2 are given by (3.63).

We remark that $f(y)$ is a DFR if claim sizes distributions f_1 and f_2 are DFR since the ladder height distribution h_i and $h_{i,1}$ of f_i and $f_{i,1}$ respectively for $i = 1, 2$ hold the reliability class implications and mixing preserves the DFR property (i.e. generating heavy tailed distributions). See Willmot and Lin (2001) and Barlow and Proschan (1975). Thus, we may apply the existing result to obtain bounds for the defective renewal equation based on the reliability property of the ladder height distribution $f(y)$ as in Willmot (2002).

3.2.2 Proper distribution of the deficit at ruin

First, we know that if $w_{12}(x, y) = 1$, then $m_{0,12}(u)$ in (1.1) is the ruin probability $\psi(u)$. In this case, (3.64) becomes

$$\psi(u) = \phi \int_0^u \psi(u - y) f(y) dy + \phi \overline{F}(u), \tag{3.69}$$

where $\overline{F}(y) = \int_y^\infty f(t)dt$ and $\phi\overline{F}(u)$ is equivalent to $\xi(u) = \frac{\lambda}{c}\left(\frac{\lambda}{c} T_0 T_\rho \overline{F}_1(u) + \frac{\beta}{c} T_0 T_\rho \overline{F}_2(u) + T_\rho \overline{F}_1(u)\right)$.
Also, From Willmot (2000, equation 2.3) $\psi(u)$ is given by

$$\psi(u) = \frac{\phi}{1 - \phi} \int_{0-}^u \overline{F}(u - t) dG_0(t). \tag{3.70}$$

Now, let us consider $w_{12}(x_1, x_2) = \mathrm{I}(x_2 \geq y)$ in (1.6), then we may obtain the tail of the deficit at ruin distribution denoted by $\overline{G}(u, y)$ as

$$\overline{G}(u, y) = E\left[\mathrm{I}(|U_T| \geq y)\mathrm{I}(T < \infty)|U_0 = u\right] = \Pr\left(|U_T| \geq y, T < \infty | U_0 = u\right). \tag{3.71}$$

In this case, we also find (3.65) is

$$\xi(u) = \frac{\lambda}{c}\left(\frac{\lambda}{c}T_0T_\rho\overline{F}_1(u+y) + \frac{\beta}{c}T_0T_\rho\overline{F}_2(u+y) + T_\rho\overline{F}_1(u+y)\right), \tag{3.72}$$

which is equivalent to the product of ϕ and the tail of $f(y)$ given by (3.61) and (3.62) respectively. Therefore, we may find the defective renewal equation for $\overline{G}(u,y)$ from (3.64) given by (see Willmot (2002, equation 1.3))

$$\overline{G}(u,y) = \phi\int_0^u \overline{G}(u-t,y)f(t)dt + \phi\overline{F}(u+y). \tag{3.73}$$

Furthermore, let us consider the (proper) conditional distribution of the deficit given that ruin occurs denoted by $G_u(y) = 1 - \overline{G}_u(y)$. From Theorem 2.1 in Willmot (2002),

$$\overline{G}_u(y) = \frac{\overline{G}(u,y)}{\psi(u)} = \frac{\int_{0^-}^u \overline{F}(u+y-t)dG_0(t)}{\int_{0^-}^u \overline{F}(u-t)dG_0(t)} = \frac{\int_{0^-}^u \overline{F}_{u-t}(y)\overline{F}(u-t)dG_0(t)}{\int_{0^-}^u \overline{F}(u-t)dG_0(t)}, \qquad y \geq 0, \tag{3.74}$$

where $\overline{F}_{0,x}(y)$ is a residual lifetime (excess loss) tail df associated with a mixture of ladder height df f_0 given by $\overline{F}_x(y) = \frac{\overline{F}(x+y)}{\overline{F}(x)} = 1 - F_x(y)$. Then we can interpret $\overline{G}_u(y)$ as a mixture of residual tails $\overline{F}_u(y)$, mixed over u. Differentiation of $-\overline{G}_u(y)$ in (3.74) with respect to y yields

$$g_u(y) = \frac{\int_{0^-}^u f_{u-t}(y)\overline{F}(u-t)dG_0(t)}{\int_{0^-}^u \overline{F}(u-t)dG_0(t)}, \qquad y \geq 0. \tag{3.75}$$

See Gerber et al. (1987) and Willmot (2000) for further details.

But $\overline{G}_0(0) = \phi$, using (3.70), (3.75) may be rewritten as

$$g_u(y) = \frac{\phi f_u(y)\overline{F}(u)}{\psi(u)} + \frac{\phi}{(1-\phi)\psi(u)}\int_{0^+}^u f_{u-t}(y)\overline{F}(u-t)G_0'(t)dt, \qquad y \geq 0.$$

For $u = 0$, it follows that

$$g_0(y) = \frac{\phi f(y)}{\phi} = \frac{-\xi'(y)}{\phi},$$

where ϕ and $\xi(y)$ are given by (3.61) and (3.72) respectively. This is equivalent to the result for the ladder height distribution $g_{2,0}(y|0)/\psi(0)$ derived by Boudreault et al. (2006, Section 5).

Now we consider the simpler representation the deficit distribution given that ruin occurs $g_u(y)$ which is the associated density as a mixture of the densities $f_x(y)$ if the claim size distribution has a certain form. From Theorem 1 in Willmot (2000), it follows that suppose

$$f_x(y) = \sum_{k=1}^{r} \alpha_k(x)\phi_k(y), \qquad y \geq 0, \tag{3.76}$$

where weight functions $\{\alpha_k(x); k = 1, 2, \ldots, r\}$ and density functions $\{\phi_k(y); k = 1, 2, \ldots, r\}$ for some positive integer r for $x \geq 0$, then

$$g_u(y) = \sum_{k=1}^{r} \overline{\alpha}_k(u)\phi_k(y), \qquad y \geq 0, \tag{3.77}$$

where

$$\overline{\alpha}_k(u) = \frac{\phi \int_0^u \alpha_k(u-t)\overline{F}(u-t)dG(t)}{(1-\phi)\psi(u)}. \tag{3.78}$$

So we know that $g_u(y)$ is a mixture of the same functions $\phi_k(y)$ as $f_x(y)$, but with different mixing weight functions given by (3.78). Then we consider some examples to illustrate how to obtain explicit form of $g_u(y)$ by using this result.

3.2.3 Examples

Example 1 (*Exponential claim sizes*)

Let us assume that both claim size are exponentially distributed with mean $1/\alpha_1$ and $1/\alpha_2$, namely $\overline{F}_1(y) = e^{-\alpha_1 y}$ and $\overline{F}_2(y) = e^{-\alpha_2 y}$, then the tail of the marginal distribution of the claim sizes is $\overline{F}(y) = p\overline{F}_1(y) + (1-p)\overline{F}_2(y)$, $y \geq 0$ where $p = \frac{\lambda}{\lambda+\beta}$. In this case, note that $\overline{F_i}(y) = \overline{F}_{i,1}(y) = \overline{H_i}(y) = \overline{H}_{i,1}(y)$ for $i = 1, 2$ so $\overline{F}(y)$ becomes a mixture of two exponentials which are the same as the claim size distribution with different mixing weights, namely

$$\overline{F}(y) = q_1\overline{F}_1(y) + q_2\overline{F}_2(y) + (1-q_1-q_2)\overline{F}_1(y) = (1-q_2)\overline{F}_1(y) + q_2\overline{F}_2(y), \qquad y \geq 0,$$

where q_2 is given by (3.63), and

$$\overline{F}_x(y) = \frac{\overline{F}(x+y)}{\overline{F}(x)} = \{1-q_2(x)\}\overline{F}_1(y) + q_2(x)\overline{F}_2(y), \qquad y \geq 0,$$

where $q_2(x) = \frac{q_2\overline{F}_2(x)}{(1-q_2)\overline{F}_1(x)+q_2\overline{F}_2(x)}$. Then $g_u(y)$ in (3.75) becomes

$$g_u(y) = \frac{\int_0^u [\{1-q_2(u-t)\}f_1(y) + q_2(u-t)f_2(y)]\overline{F}(u-t)dG_0(t)}{\int_0^u \overline{F}(u-t)dG_0(t)},$$

that is

$$g_u(y) = q(u)f_1(y) + (1-q(u))f_2(y), \qquad y \geq 0, \tag{3.79}$$

where

$$q(u) = \frac{\int_0^u \{1-q_2(u-t)\}\overline{F}(u-t)dG_0(t)}{\int_0^u \overline{F}(u-t)dG_0(t)}, \tag{3.80}$$

which implies that $g_u(y)$ is also a mixture of the same two exponentials as for the claim size distribution $\overline{F}(y)$, but with p replaced by $q(u)$ given by (3.80). To evaluate $q(u)$, we may use (3.70) and re-express $q(u)$ as

$$q(u) = \frac{\phi(1-q_2)}{(1-\phi)\psi(u)}\left\{e^{-\alpha_1 u}(1-\phi) + e^{-\alpha_1 u}\int_{0+}^u e^{\alpha_1 t}dG_0(t)\right\}, \tag{3.81}$$

since $(1-q_2(x))\overline{F}(x) = (1-q_2)\overline{F}_1(x)$ and $\overline{G}_0(0) = 1-\phi$. In this case, we know that

$$\psi(u) = C_1e^{-R_1u} + C_2e^{-R_2u}, \qquad u \geq 0, \tag{3.82}$$

where R_1 and R_2 are the distinct roots of the equation,

$$\int_0^\infty e^{R_jy}f(y)dy = (1-q_2)\frac{\alpha_1}{\alpha_1-R_j} + q_2\frac{\alpha_2}{\alpha_2-R_j} = \frac{1}{\phi},$$

and C_1 , C_2 are constants given by Gerber et al. (1987). Otherwise it might be easier way to use the Tijms approximation method to find out constants C_1 , C_2 since Tijms approximation returns the exact value of the ultimate ruin probability in this case, namely

$$C_1 = \frac{\phi}{1-\phi}\left\{R_1 \int_0^{\infty} y e^{R_1 y} dF(y)\right\}^{-1}, \qquad C_2 = \phi - C_1.$$

So we may easily calculate $q(u)$ in (3.81) using (3.82),

$$q(u) = \frac{\phi(1-q_2)}{(C_1 e^{-R_1 u} + C_2 e^{-R_2 u})}\left\{e^{-\alpha_1 u} + \frac{C_1 R_1 (e^{-R_1 u} - e^{-\alpha_1 u})}{(1-\phi)(\alpha_1 - R_1)} + \frac{C_2 R_2 (e^{-\alpha_1 u} - e^{-R_2 u})}{(1-\phi)(R_2 - \alpha_1)}\right\}. \quad (3.83)$$

Next, we calculate the conditional probabilities of the deficit given that ruin occurs for simple forms of the claim size distributions including exponentials and combinations of exponentials. In the following numerical examples, we shall assume $\lambda = 1$ and $\beta = 1/3$ and choose appropriate values of c with satisfying the positive security loading condition given by (3.60). Assume that $\overline{F}_1(y) = e^{-2.5y}$ and $\overline{F}_2(y) = e^{-0.5y}$ for $y \geq 0$. Then we may obtain the ruin probabilities from (3.82),

$$\psi(u) = 0.690472 e^{-0.166667u} + 0.0847093 e^{-1.68614u}, \qquad u \geq 0,$$

and we may readily get $g_u(y)$ for the initial surplus $u = 0.25, 0.5, 1, 2, 4$ and the deficit amount $y = 0.25, 0.5, 1, 2, 4, 8$.

$g_u(y)$	y=0.25	0.5	1	2	4	8
u=0.25	0.78411	0.51435	0.26578	0.12006	0.04184	0.00566
0.50	0.71398	0.48879	0.27345	0.13313	0.04713	0.00637
1	0.62872	0.45772	0.28277	0.14901	0.05355	0.00724
2	0.56718	0.43529	0.28950	0.16048	0.05818	0.00787
4	0.55033	0.42915	0.29134	0.16362	0.05945	0.00804

Table 3.1: Exponential claim sizes

Example 2 (*Combination of exponentials claim sizes*)

Here, we consider more general distribution class, which is a combination of exponentials (e.g. Willmot (2000), Dufresne (2007)) given by

$$\overline{F_1}(y) = \sum_{k=1}^{r_1} p_{1,k} e^{-\beta_{1,k} y}, \qquad \overline{F_2}(y) = \sum_{k=1}^{r_2} p_{2,k} e^{-\beta_{2,k} y}, \qquad y \geq 0,$$

where $\sum_{k=1}^{r_1} p_{1,k} = 1$ and $\sum_{k=1}^{r_2} p_{2,k} = 1$. Then, the tail of the marginal distributions of F_1 and F_2 are $\overline{P}(y) = \alpha \overline{F}_1(y) + (1-\alpha)\overline{F}_2(y) = \sum_{k=1}^{r_1} \alpha p_{1,k} e^{-\beta_{1,k} y} + \sum_{k=1}^{r_2} (1-\alpha) p_{2,k} e^{-\beta_{2,k} y}$ for $y \geq 0$, where $\alpha = \frac{\lambda}{\lambda+\beta}$. The means of each distribution are given by $\mu_1 = \sum_{k=1}^{r_1} p_{1,k}/\beta_{1,k}$,$\mu_2 = \sum_{k=1}^{r_2} p_{2,k}/\beta_{2,k}$ respectively. The equilibrium distributions (also tails) of F_1 and F_2 are given by

$$f_{1,1}(y) = \frac{1}{\mu_1} \sum_{k=1}^{r_1} p_{1,k} e^{-\beta_{1,k} y}, \qquad \overline{F}_{1,1}(y) = \frac{1}{\mu_1} \sum_{k=1}^{r_1} \frac{p_{1,k}}{\beta_{1,k}} e^{-\beta_{1,k} y},$$

and

$$f_{2,1}(y) = \frac{1}{\mu_2} \sum_{k=1}^{r_2} p_{2,k} e^{-\beta_{2,k} y}, \qquad \overline{F}_{2,1}(y) = \frac{1}{\mu_2} \sum_{k=1}^{r_2} \frac{p_{2,k}}{\beta_{2,k}} e^{-\beta_{2,k} y}.$$

Then, $f(y)$ given by (3.68) is composed of the three ladder height distributions $h_{1,1}(y), h_{2,1}(y)$ and $h_1(y)$ which are

$$h_1(y) = \sum_{k=1}^{r_1} p_{1,k}^* \beta_{1,k} e^{-\beta_{1,k} y}, \qquad h_{i,1}(y) = \sum_{k=1}^{r_i} p_{i,k}^{**} \beta_{i,k} e^{-\beta_{i,k} y}, \qquad i = 1, 2, \tag{3.84}$$

where

$$p_{1,k}^* = \frac{\frac{p_{1,k}}{s_1+\beta_{1,k}}}{\sum_{j=1}^{r_1} \frac{p_{1,j}}{s_1+\beta_{1,j}}}, \qquad p_{i,k}^{**} = \frac{\frac{p_{i,k}}{\beta_{i,k}(s_1+\beta_{i,k})}}{\sum_{j=1}^{r_i} \frac{p_{i,j}}{\beta_{i,j}(s_1+\beta_{i,j})}}, \qquad i = 1, 2, \tag{3.85}$$

and $\sum_{k=1}^{r_1} p_{1,k}^* = 1$, $\sum_{k=1}^{r_i} p_{i,k}^{**} = 1$. Thus combining (3.84) and (3.85) yields

$$f(y) = \sum_{i=1}^{2} \sum_{k=1}^{r_i} q_{i,k} \beta_{i,k} e^{-\beta_{i,k} y}, \qquad y \geq 0, \tag{3.86}$$

where

$$q_{1,k} = q_1 p_{1,k}^{**} + (1 - q_1 - q_2) p_{1,k}^*, \qquad q_{2,k} = q_2 p_{2,k}^{**}, \tag{3.87}$$

with q_1 and q_2 are given by (3.63) and $p^{**}_{1,k}, p^{**}_{2,k}, p^{*}_{1,k}$ are given by (3.85). Then the residual lifetime distribution of $f(y)$ becomes also a combination of exponentials as

$$f_x(y) = \frac{f(x+y)}{\overline{F}(x)} = \sum_{i=1}^{2}\sum_{k=1}^{r_i} q_{i,k}(x)\beta_{i,k}e^{-\beta_{i,k}y}, \tag{3.88}$$

where

$$q_{1,k}(x) = \frac{q_{1,k}e^{-\beta_{1,k}x}}{\sum_{i=1}^{2}\sum_{j=1}^{r_i} q_{i,j}e^{-\beta_{i,j}x}}, \qquad q_{2,k}(x) = 1 - q_{1,k}(x), \tag{3.89}$$

and $q_{1,k}$ and $q_{2,k}$ are given by (3.87). Obviously, f_x(y) in (3.88) is of the same form as (3.76), namely

$$f_x(y) = \sum_{i=1}^{2}\sum_{k=1}^{r_i} q_{i,k}(x)\beta_{i,k}e^{-\beta_{i,k}y} = \sum_{k=1}^{r} q_k(x)f_k(y), \qquad y \geq 0,\ x \geq 0, \tag{3.90}$$

where $r = r_1 + r_2$, $q_k(x) = \mathrm{I}(k \leq r_1)q_{1,k}(x) + \mathrm{I}(k > r_1)q_{2,k}(x)$, and $f_k(y) = \mathrm{I}(k \leq r_1)\beta_{1,k}e^{-\beta_{1,k}y} + \mathrm{I}(k > r_1)\beta_{2,k}e^{-\beta_{2,k}y}$.

Hence, we may obtain $g_u(y)$ from (3.90) with applying the results in (3.76),(3.77) and (3.78), it is a mixture of the same distributions $f_k(y)$ with different weight functions,

$$g_u(y) = \sum_{k=1}^{r} \overline{q}_k(u)f_k(y), \qquad y \geq 0,\ u \geq 0, \tag{3.91}$$

where

$$\overline{q}_k(u) = \frac{\phi\int_0^u q_k(u-t)\overline{F}(u-t)dG_0(t)}{(1-\phi)\psi(u)}. \tag{3.92}$$

To evaluate the weights $\overline{q}_k(u)$, first we may obtain $\psi(u)$ in this case which is similar to (3.82) for exponential example but with more roots (e.g. Willmot (2000, example 1)),

$$\psi(u) = \sum_{k=1}^{r} C_k e^{-R_k u}, \qquad u \geq 0, \tag{3.93}$$

where $\{R_i;\ i = 1, 2, \ldots, r\}$ are the $r(= r_1 + r_2)$ distinct roots of the equation,

$$\int_0^\infty e^{R_j y}f(y)dy = \sum_{k=1}^{r_1} q_{1,k}\frac{\beta_{1,k}}{\beta_{1,k} - R_j} + \sum_{k=1}^{r_2} q_{2,k}\frac{\beta_{2,k}}{\beta_{2,k} - R_j} = \frac{1}{\phi}, \tag{3.94}$$

and $\{C_k;\ k = 1, 2, \ldots, r\}$ are constants (see Gerber et al. (1987) for further details). We know the tail of $f(y)$ from (3.86),

$$\overline{F}(y) = \sum_{i=1}^{2}\sum_{k=1}^{r_i} q_{i,k}e^{-\beta_{i,k}y} = \sum_{k=1}^{r} q_k e^{-\beta_k y}, \qquad y \geq 0, \tag{3.95}$$

where $q_k = \mathrm{I}(k \leq r_1)q_{1,k} + \mathrm{I}(k > r_1)q_{2,k}$ and $\beta_k = \mathrm{I}(k \leq r_1)\beta_{1,k} + \mathrm{I}(k > r_1)\beta_{2,k}$.

Then, by using (3.70), (3.93), (3.95) and $g_0(t) = -\psi'(t)$, the denominator in (3.92) becomes

$$\begin{aligned}
\frac{1-\phi}{\phi}\sum_{j=1}^{r} C_j e^{-R_j u} &= (1-\phi)\sum_{k=1}^{r} q_k e^{-\beta_k u} + \sum_{j=1}^{r} C_j R_j \sum_{k=1}^{r} q_k e^{-\beta_k u}\int_{0+}^{u} e^{-(R_j-\beta_k)t}dt \\
&= (1-\phi)\sum_{k=1}^{r} q_k e^{-\beta_k u} + \sum_{j=1}^{r} C_j R_j \sum_{k=1}^{r} \frac{q_k}{R_j-\beta_k}(e^{-\beta_k u} - e^{-R_j u}).
\end{aligned}$$

Rearranging this equation, it follows that

$$\sum_{j=1}^{r} C_j e^{-R_j u}\left\{\frac{1-\phi}{\phi} - R_j\sum_{k=1}^{r}\frac{q_k}{\beta_k - R_j}\right\} = \sum_{k=1}^{r} q_k e^{-\beta_k u}\left\{(1-\phi) - \sum_{j=1}^{r}\frac{C_j R_j}{\beta_k - R_j}\right\}. \tag{3.96}$$

From (3.94), (3.95) note that $\sum_{k=1}^{r} q_k \frac{\beta_k}{\beta_k - R_j} = \phi^{-1}$ and it may expressed as

$$\sum_{k=1}^{r} q_k\left(\frac{\beta_k}{\beta_k - R_j} - 1\right) = \frac{1-\phi}{\phi},$$

since $\sum_{k=1}^{r} q_k = 1$. The right-hand side of (3.96) also become

$$\sum_{k=1}^{r} q_k\left\{(1-\phi) - \sum_{j=1}^{r}\frac{C_j R_j}{\beta_k - R_j}\right\}e^{-\beta_k u} = 0,$$

so the coefficients must be 0 since this is an identity for all distinct u and the β_k, namely

$$(1-\phi) - \sum_{j=1}^{r}\frac{C_j R_j}{\beta_k - R_j} = 0. \tag{3.97}$$

Next consider the numerator in (3.92), it follows from (3.89),(3.90),(3.95) and (3.97),

$$\int_0^u q_k(u-t)\overline{F}(u-t)dG_0(t) = q_k e^{-\beta_k u}\left\{(1-\phi)+\sum_{j=1}^r C_j R_j \int_0^u e^{(\beta_k - R_j)t}dt\right\}$$

$$= \quad q_k\left\{\left((1-\phi)-\sum_{j=1}^r \frac{C_j R_j}{\beta_k - R_j}\right)e^{-\beta_k u}+\sum_{j=1}^r \frac{C_j R_j}{\beta_k - R_j}e^{-R_j u}\right\} = q_k \sum_{j=1}^r \frac{C_j R_j}{\beta_k - R_j}e^{-R_j u}.$$

Therefore, we may obtain the closed form of the weights $\overline{q}_k(u)$ given by

$$\overline{q}_k(u) = \frac{q_k \sum_{j=1}^r \frac{C_j R_j}{\beta_k - R_j}e^{-R_j u}}{\frac{1-\phi}{\phi}\sum_{j=1}^r C_j e^{-R_j u}}, \qquad k = 1, 2, \ldots, r,$$

where q_k and β_k are given by (3.95), which is the exactly same result as (2.21) in Willmot (2000).

Let us consider that the claim sizes distributions are combination of two exponentials given by $\overline{F}_1(y) = 2e^{-2y} - e^{-4y}$ and $\overline{F}_2(y) = 2e^{-0.25y} - e^{-0.5y}$ for $y \geq 0$, with mean 0.75 and 6 respectively, and assume $c = 3$ with satisfying the condition given by (3.60). Then the ruin probabilities from (3.93) become

$$\psi(u) = 0.583962e^{-0.105429u} - 0.0325012e^{-0.545127u} + 0.154227e^{-1.25257u} - 0.0181871e^{-4.34103u},$$

for $u \geq 0$, and $g_u(y)$ is also easily obtained in Table 3.2.

$g_u(y)$	y=0.25	0.5	1	2	4	8
u=0.25	0.59849	0.43239	0.22983	0.10241	0.05669	0.02265
0.50	0.53560	0.38692	0.21646	0.10924	0.06344	0.02526
1	0.42943	0.32294	0.20258	0.12127	0.07392	0.02925
2	0.32336	0.26420	0.19293	0.13419	0.08402	0.03285
4	0.28418	0.24428	0.19193	0.14047	0.08761	0.03371

Table 3.2: Combination of exponentials claim sizes

Example 3 (*Mixtures of Erlangs with the same scale parameter*)

We shall derive $g_u(y)$ which has the same form as the claim sizes distributions which densities are mixtures of Erlangs with the same scale parameter, that is

$$f_1(y) = \sum_{k=1}^{r_1} p_{1,k} \frac{\beta_1^k y^{k-1} e^{-\beta_1 y}}{(k-1)!}, \qquad f_2(y) = \sum_{k=1}^{r_2} p_{2,k} \frac{\beta_2^k y^{k-1} e^{-\beta_2 y}}{(k-1)!}, \qquad y \geq 0, \tag{3.98}$$

where $\{p_{1,1}, p_{1,2}, \ldots, p_{1,r_1}\}$ and $\{p_{2,1}, p_{2,2}, \ldots, p_{2,r_2}\}$ are probability measures, and the tail of f_1 and f_2 are

$$\overline{F_1}(y) = e^{-\beta_1 y} \sum_{k=0}^{r_1-1} \overline{P}_{1,k} \frac{(\beta_1 y)^k}{k!}, \qquad \overline{F_2}(y) = e^{-\beta_2 y} \sum_{k=0}^{r_2-1} \overline{P}_{2,k} \frac{(\beta_2 y)^k}{k!}, \qquad y \geq 0, \tag{3.99}$$

where $\overline{P}_{i,k} = \sum_{j=k+1}^{r_i} p_{i,j}$ for $i = 1, 2$ and $k = 0, 1, \ldots, r_i - 1$. Then the tail of the marginal distributions of the claim sizes are

$$\overline{P}(y) = e^{-\beta_1 y} \sum_{k=0}^{r_1-1} \alpha \overline{P}_{1,k} \frac{(\beta_1 y)^k}{k!} + e^{-\beta_2 y} \sum_{k=0}^{r_2-1} (1-\alpha) \overline{P}_{2,k} \frac{(\beta_2 y)^k}{k!}, \qquad y \geq 0,$$

where $\alpha = \lambda/(\lambda + \beta)$. In this case, the means of each claim sizes are $\mu_i = \frac{d}{ds} \widetilde{f}_i(-s)|_{s=0} = \sum_{j=1}^{r_i} j p_{i,j}/\beta_i$ for $i = 1, 2$. Letting $\overline{P}^*_{i,k} = \sum_{j=k+1}^{r_i} p_{i,j}$ for $i = 1, 2$ and $k = 0, 1, \ldots, r_i - 1$, the equilibrium distributions (also tails) corresponding to F_1 and F_2 are respectively given by

$$f_{i,1}(y) = \sum_{k=1}^{r_i} p^*_{i,k} \frac{\beta_i^k y^{k-1} e^{-\beta_i y}}{(k-1)!}, \qquad \overline{F}_{i,1}(y) = e^{-\beta_i y} \sum_{k=0}^{r_i-1} \overline{P}^*_{i,k} \frac{(\beta_i y)^k}{k!}, \qquad i = 1, 2, \tag{3.100}$$

where $\{p^*_{i,1}, p^*_{i,2}, \ldots, p^*_{i,r_i}\}$ is a probability measure with

$$p^*_{i,k} = \frac{\sum_{j=k}^{r_i} p_{i,j}}{\sum_{j=1}^{r_i} j p_{i,j}}, \qquad k = 1, 2, \ldots, r_i.$$

In order to obtain the ladder height distributions $f(y)$ given by (3.68), we need to know $h_{1,1}(y), h_{2,1}(y)$

and $h_1(y)$. Using (3.98),(3.99) and (3.100), it follows

$$\begin{aligned}
h_1(y) &= \frac{\int_0^\infty e^{-s_1 t} f_1(y+t)dt}{\int_0^\infty e^{-s_1 t}\overline{F}_1(t)dt} = \frac{e^{-\beta_1 y}\sum_{k=1}^{r_1} p_{1,k}\frac{\beta_1^k}{(k-1)!}\int_0^\infty (y+t)^{k-1}e^{-(s_1+\beta_1)t}dt}{\sum_{j=0}^{r_1-1}\overline{P}_{1,j}\frac{(\beta_1)^j}{j!}\int_0^\infty t^j e^{-(s_1+\beta_1)t}dt} \\
&= \frac{e^{-\beta_1 y}\sum_{k=1}^{r_1} p_{1,k}\frac{\beta_1^k}{(k-1)!}\sum_{j=0}^{k-1}\binom{k-1}{j}y^{k-1-j}\int_0^\infty t^j e^{-(s_1+\beta_1)t}dt}{\frac{1}{s_1+\beta_1}\sum_{j=0}^{r_1-1}\overline{P}_{1,j}\left(\frac{\beta_1}{s_1+\beta_1}\right)^j} \\
&= \frac{e^{-\beta_1 y}\sum_{k=1}^{r_1} p_{1,k}\frac{\beta_1^k}{(k-1)!}\sum_{j=1}^{k}\binom{k-1}{j-1}y^{k-j}\frac{(j-1)!}{(s_1+\beta_1)^j}}{\frac{1}{s_1+\beta_1}\sum_{j=0}^{r_1-1}\overline{P}_{1,j}\left(\frac{\beta_1}{s_1+\beta_1}\right)^j}.
\end{aligned}$$

After changing the order of summations in the numerator, one yields

$$\begin{aligned}
&e^{-\beta_1 y}\sum_{j=1}^{r_1}\frac{\beta_1^{j-1}}{(s_1+\beta_1)^j}\sum_{k=j}^{r_1} p_{1,k}\frac{\beta_1^{k-j+1}y^{k-j}}{(k-j)!} = e^{-\beta_1 y}\sum_{j=1}^{r_1}\frac{\beta_1^{j-1}}{(s_1+\beta_1)^j}\sum_{k=1}^{r_1-j+1} p_{1,k+j-1}\frac{\beta_1^k y^{k-1}}{(k-1)!} \\
&= \frac{e^{-\beta_1 y}}{s_1+\beta_1}\sum_{k=1}^{r_1}\frac{\beta_1^k y^{k-1}}{(k-1)!}\left\{\sum_{j=1}^{r_1-k+1} p_{1,k+j-1}\left(\frac{\beta_1}{s_1+\beta_1}\right)^{j-1}\right\} = \frac{e^{-\beta_1 y}}{s_1+\beta_1}\sum_{k=1}^{r_1}\frac{\beta_1^k y^{k-1}}{(k-1)!}\left\{\sum_{j=0}^{r_1-k} p_{1,k+j}\left(\frac{\beta_1}{s_1+\beta_1}\right)^{j}\right\}.
\end{aligned}$$

Then,

$$h_1(y) = \sum_{k=1}^{r_1} q_{1,k}^* \frac{\beta_1^k y^{k-1} e^{-\beta_1 y}}{(k-1)!}, \tag{3.101}$$

where $\{q_{1,1}^*, q_{1,2}^*, \ldots, q_{1,r_1}^*\}$ is a probability measure with

$$q_{1,k}^* = \frac{\sum_{j=k}^{r_1} p_{1,j}\left(\frac{\beta_1}{s_1+\beta_1}\right)^{j-k}}{\sum_{j=0}^{r_1-1}\overline{P}_{1,j}\left(\frac{\beta_1}{s_1+\beta_1}\right)^{j}} = \frac{\sum_{j=k}^{r_1} p_{1,j}\left(\frac{\beta_1}{s_1+\beta_1}\right)^{j-k}}{\sum_{j=1}^{r_1} p_{1,j}\sum_{i=0}^{j-1}\left(\frac{\beta_1}{s_1+\beta_1}\right)^{i}}. \tag{3.102}$$

In the above equation, the second expression for $q_{1,k}^*$ is agreed with the result in Willmot and Lin (2001, p.163). It is easy to obtain $h_{i,1}$ for $i = 1, 2$ since $f_{i,1}$ and $\overline{F}_{i,1}$ are same as the f_i and $\overline{F}_i$ with different weights,

$$h_{i,1}(y) = \sum_{k=1}^{r_i} q_{i,k}^{**} \frac{\beta_i^k y^{k-1} e^{-\beta_i y}}{(k-1)!}, \qquad i = 1, 2, \tag{3.103}$$

where $\{q^{**}_{i,1}, q^{**}_{i,2}, \ldots, q^{**}_{i,r_1}\}$ is a probability measure with

$$q^{**}_{i,k} = \frac{\sum_{j=k}^{r_i} p^*_{i,j}\left(\frac{\beta_i}{s_1+\beta_i}\right)^{j-k}}{\sum_{j=0}^{r_i-1} \overline{P}^*_{i,j}\left(\frac{\beta_i}{s_1+\beta_i}\right)^{j}}, \qquad i = 1, 2. \tag{3.104}$$

From (3.101) and (3.103), the ladder height distribution $f(y)$ in (3.68) becomes

$$f(y) = \sum_{i=1}^{2}\sum_{k=1}^{r_i} q_{i,k}\frac{\beta_i^k y^{k-1} e^{-\beta_i y}}{(k-1)!}, \qquad y \geq 0, \tag{3.105}$$

where $q_{1,k} = q_1 q^{**}_{1,k} + (1 - q_1 - q_2)q^*_{1,k}$ and $q_{2,k} = q_2 q^{**}_{2,k}$ with q_1 and q_2 are given by (3.63), $q^*_{1,k}$ is given by (3.102) and $q^{**}_{i,k}$ for $i = 1, 2$ are given by (3.104).

In order to make it available to apply the result in Theorem 2 in Willmot (2000), we shall re-express the form of $f(y)$ to be a mixture distribution but with the same scale parameter. First, assuming $\beta_1 < \beta_2$ and using the results provided by Willmot and Woo (2007, Section 2.2), the Laplace transform of $f(y)$ in (3.105) may be expressed as

$$Q(z) = \sum_{i=1}^{2}\sum_{k=1}^{r_i} q_{i,k} z^k \left\{\frac{\beta_i/\beta_2}{1-(1-\beta_i/\beta_2)z}\right\}^k, \tag{3.106}$$

where $Q(z) = \sum_{j=1}^{\infty} q_j z^j$ and $z = \frac{\beta_2}{\beta_2+s}$. Then we may find out q_j the coefficient of z^j in the pgf (3.106) given by

$$q_j = \sum_{i=1}^{2}\sum_{k=1}^{\min(j,\ r_i)} q_{i,k}\binom{j-1}{k-1}\left(\frac{\beta_i}{\beta_2}\right)^k\left(1-\frac{\beta_i}{\beta_2}\right)^{j-k}, \qquad j = 1, 2, \ldots. \tag{3.107}$$

Thus, with (3.107) we may rewrite (3.105) as

$$f(y) = \sum_{j=1}^{\infty} q_j \frac{\beta_2^j y^{j-1} e^{-\beta_2 y}}{(j-1)!}, \qquad y \geq 0,$$

and the residual lifetime distribution of $f(y)$ becomes

$$\begin{aligned} f_x(y) &= \frac{f(x+y)}{\overline{F}(x)} = \frac{\sum_{j=1}^{\infty} q_j \frac{\beta_2^j (x+y)^{j-1} e^{-\beta_2 (x+y)}}{(j-1)!}}{e^{-\beta_2 x} \sum_{j=0}^{\infty} \overline{Q}_j \frac{(\beta_2 x)^j}{j!}} \\ &= \frac{e^{-\beta_2 y} \sum_{j=1}^{\infty} q_j \frac{\beta_2^j}{(j-1)!} \sum_{k=1}^{j} \binom{j-1}{k-1} y^{k-1} x^{j-k}}{\sum_{j=0}^{\infty} \overline{Q}_j \frac{(\beta_2 x)^j}{j!}} = \frac{e^{-\beta_2 y} \sum_{k=1}^{\infty} \frac{\beta_2^k y^{k-1}}{(k-1)!} \sum_{j=k}^{\infty} q_j \frac{(\beta_2 x)^{j-k}}{(j-k)!}}{\sum_{j=0}^{\infty} \overline{Q}_j \frac{(\beta_2 x)^j}{j!}}, \end{aligned}$$

where q_j is given by (3.107) and $\overline{Q}_k = \sum_{j=k+1}^{\infty} q_{i,j}$ for $k = 0, 1, \ldots$. Therefore,

$$f_x(y) = \sum_{k=1}^{\infty} q_k(x) \frac{\beta_2^k y^{k-1} e^{-\beta_2 y}}{(k-1)!},$$

where $\{q_1(x), q_2(x), \ldots\}$ is a probability measure satisfying

$$q_k(x) = \sum_{j=k}^{\infty} q_j \frac{(\beta_2 x)^{j-k}}{(j-k)!} \left\{ \sum_{j=0}^{\infty} \overline{Q}_j \frac{(\beta_2 x)^j}{j!} \right\}^{-1}.$$

It is the same result as Lemma 1 in Willmot (2000) when the claim size distribution is a mixture of Erlangs with same scale parameter. Clearly, f_x(y) is of the same form as (3.76) with $\alpha_k(x) = q_k(x)$ and $\phi_k(y) = f_k(y)$ which is the Erlang-k densities, namely

$$p_k(y) = \frac{\beta_2^k y^{k-1} e^{-\beta_2 y}}{(k-1)!},$$

and from (3.76),(3.77) and (3.78), we may readily have $g_u(y)$ which is also a mixture of the same $p_k(y)$ with different weight functions,

$$g_u(y) = \sum_{k=1}^{\infty} \overline{q}_k(u) p_k(y), \qquad y \geq 0, \ u \geq 0,$$

where

$$\overline{q}_k(u) = \frac{\phi \int_0^u q_k(u-t) \overline{F}(u-t) dG_0(t)}{(1-\phi)\psi(u)}. \tag{3.108}$$

Similar to the previous examples, in order to evaluate the coefficient $\overline{q}_k(u)$, we first need to find out the function $\psi(u)$. The ultimate ruin probabilities in this case are provided by (e.g. Klugman, Panjer and Willmot (2008, pp.294-295))

$$\psi(u) = e^{-\beta_2 u} \sum_{k=0}^{\infty} \overline{C}_k \frac{(\beta_2 u)^k}{k!}, \qquad u \geq 0,$$

where $\overline{C}_k = \sum_{j=k+1}^{\infty} c_j$ and the distribution $\{c_j;\ j = 0, 1, \ldots\}$ may be obtained with $c_0 = 1 - \phi$,

$$c_k = \phi \sum_{j=1}^{k} q_j c_{k-j}, \qquad k = 1, 2, \ldots, \tag{3.109}$$

where q_j is given by (3.107). Also, we may calculate $\overline{C}_k$ using the following equation with $\overline{C}_0 = \phi$,

$$\overline{C}_k = \phi \sum_{j=1}^{k} q_j \overline{C}_{k-j} + \phi \sum_{j=k+1}^{\infty} q_j, \qquad k = 1, 2, \ldots.$$

Then, using Theorem 2 in Willmot (2000), we may obtain $\overline{q}_k(u)$ in (3.108) as follows.

$$\overline{q}_k(u) = \frac{\sum_{j=k}^{\infty} q_j \tau_{j-k}(\beta_2 u)}{\sum_{j=1}^{\infty} q_j \sum_{m=0}^{j-1} \tau_m(\beta_2 u)}, \qquad k = 1, 2, \ldots,$$

with $\tau_m(x) = \sum_{i=0}^{\infty} c_i x^{i+m}/(i+m)!$ for $m = 0, 1, \ldots,$ where q_j and c_i are given by (3.107) and (3.109) respectively.

Example 4 (*Mixtures of Erlangs with the different scale parameter*)

Furthermore, we may follow the similar approach as shown previously to obtain the distribution of the deficit ar ruin when the claim sizes distributions are mixtures of Erlang with different scale parameter, that is

$$f_1(y) = \sum_{i=1}^{n_1} \sum_{k=1}^{r_1} p_{1,i,k} \frac{\beta_{1,i}^k y^{k-1} e^{-\beta_{1,i} y}}{(k-1)!}, \qquad f_2(y) = \sum_{i=1}^{n_2} \sum_{k=1}^{r_2} p_{2,i,k} \frac{\beta_{2,i}^k y^{k-1} e^{-\beta_{2,i} y}}{(k-1)!}, \qquad y \geq 0, \tag{3.110}$$

where $\{p_{m,i,k}; i = 1, 2, ..., n_1,\ k = 1, 2, ..., r_1\}$ for $m = 1, 2$ are probability measures. We shall assume $\beta_{1,i} < \beta_{n_1}$ for $i = 1, 2, ..., n_1 - 1$ and $\beta_{2,i} < \beta_{n_2}$ for $i = 1, 2, ..., n_2 - 1$. Then, again using the results provided by Willmot and Woo (2007), we may rewrite f_1 and f_2 as mixtures of Erlangs with the same scale parameter β_{n_1} and β_{n_2} respectively. The Laplace transform of f_1 is

$$\widetilde{f}_1(s) = \sum_{i=1}^{n_1} \sum_{k=1}^{r_1} p_{1,i,k} \left(\frac{\beta_{1,i}}{\beta_{1,i} + s} \right)^k ,$$

and it may be expressed in the form (3.106) with $z = \left(\frac{\beta_{1,n_1}}{\beta_{1,n_1}+s} \right)$ given by

$$\begin{aligned} Q(z) &= \sum_{i=1}^{n_1} \sum_{k=1}^{r_1} p_{1,i,k} z^k \left\{ \frac{\beta_{1,i}/\beta_{1,n_1}}{1-(1-\beta_{1,i}/\beta_{1,n_1})z} \right\}^k \\ &= \sum_{i=1}^{n_1} \sum_{k=1}^{r_1} p_{1,i,k} \sum_{m=0}^{\infty} h_m \left(k, \frac{\beta_{1,i}}{\beta_{1,n_1}} \right) z^{m+k} = \sum_{i=1}^{n_1} \sum_{k=1}^{r_1} p_{1,i,k} \sum_{j=k}^{\infty} h_{j-k} \left(k, \frac{\beta_{1,i}}{\beta_{1,n_1}} \right) z^j \\ &= \sum_{i=1}^{n_1} \sum_{j=1}^{\infty} z^j \sum_{k=1}^{\min(j,\ r_1)} p_{1,i,k} h_{j-k} \left(k, \frac{\beta_{1,i}}{\beta_{1,n_1}} \right) = \sum_{j=1}^{\infty} q_{1,j} z^j , \end{aligned}$$

where $h_m(k,\varphi) = \frac{\Gamma(k+m)}{\Gamma(k)m!} \varphi^k (1-\varphi)^m$ for $m = 0, 1, ...$ and $\sum_{m=0}^{\infty} h_m(k,\varphi) z^m = \left\{ \frac{\varphi}{1-(1-\varphi)z} \right\}^k$. Thus, we may obtain the coefficient of z^j in $Q(z)$,

$$q_{1,j} = \sum_{i=1}^{n_1} \sum_{k=1}^{\min(j,\ r_1)} p_{1,i,k} h_{j-k} \left(k, \frac{\beta_{1,i}}{\beta_{1,n_1}} \right) = \sum_{i=1}^{n_1} \sum_{k=1}^{\min(j,\ r_1)} p_{1,i,k} \binom{j-1}{k-1} \left(\frac{\beta_{1,i}}{\beta_{1,n_1}} \right)^k \left(1 - \frac{\beta_{1,i}}{\beta_{1,n_1}} \right)^{j-k} , \tag{3.111}$$

for $j = 1, 2, ...$, then f_1 in (3.110) becomes

$$f_1(y) = \sum_{j=1}^{\infty} q_{1,j} \frac{\beta_{1,n_1}^j y^{j-1} e^{-\beta_{1,n_1} y}}{(j-1)!}, \qquad y \geq 0,$$

where $q_{1,j}$ is given by (3.111). Similarly, f_2 in (3.110) is also expressed as

$$f_2(y) = \sum_{j=1}^{\infty} q_{2,j} \frac{\beta_{2,n_2}^j y^{j-1} e^{-\beta_{2,n_2} y}}{(j-1)!}, \qquad y \geq 0,$$

where $q_{2,j}$ is given by

$$q_{2,j} = \sum_{i=1}^{n_2} \sum_{k=1}^{\min(j,\ r_2)} p_{2,i,k} \binom{j-1}{k-1} \left(\frac{\beta_{2,i}}{\beta_{2,n_2}} \right)^k \left(1 - \frac{\beta_{2,i}}{\beta_{2,n_2}} \right)^{j-k} , \qquad j = 1, 2, \ldots .$$

And the tail distributions of f_1 and f_2 becomes

$$\overline{F}_i(y) = e^{-\beta_{i,n_i} y} \sum_{k=0}^{\infty} \overline{Q}_{i,k} \frac{(\beta_{i,n_i} y)^k}{k!}, \qquad y \geq 0,$$

where $\overline{Q}_{i,k} = \sum_{j=k+1}^{\infty} q_{i,j}$ for $i = 1, 2$. Therefore, we may apply the same approach to obtain the deficit at ruin $g_u(y)$ when the f_1 and f_2 are in the form of (3.98) but with infinite mixtures case.

3.3 Joint defective densities involving the time to ruin

In this section, we derive the joint defective distribution of four variables in the generalized penalty function involving the time of ruin in the classical compound Poisson risk model.

3.3.1 Joint defective densities of $(T, U_{T^-}, |U(T)|, X_T, R_{N(T)-1})$

In Section 2.2, we have already obtained the discounted joint densities of $(U_{T^-}, |U(T)|, X_T, R_{N(T)-1})$. Here, by inverting these results with respect to δ we derive the joint defective densities of the previous four quantities including the time to ruin as well. To do so, Lagrange's implicit function theorem is applied (see Dickson and Willmot (2005), Landriault and Willmot (2009)).

To begin with, we derive an expression for the compound geometric density in order to invert with respect to δ. From equations (2.19) and (2.20) in Landriault and Willmot (2009) which are

$$f_\delta^{*n}(u) = \left(\frac{\lambda}{c\phi_\delta}\right)^n \int_0^\infty e^{-\rho t} \xi_n(u,t) dt, \tag{3.112}$$

where

$$\xi_n(u,t) = \frac{u^{n-1}}{\Gamma(n)} p^{*n}(t+u) + \sum_{j=1}^{n-1} \binom{n}{j} \frac{(-1)^j}{\Gamma(n)} \int_0^u x^{n-1} p^{*j}(u-x) p^{*(n-j)}(t+x) dx. \tag{3.113}$$

Using these expression for the n-fold convolution $f_\delta^{*n}(u)$, from (2.21) we have now

$$\sum_{n=1}^{\infty} (\phi_\delta)^n f_\delta^{*n}(u) = \frac{g_\delta(u)}{1-\phi_\delta} = \int_0^\infty e^{-\rho t}\chi(u,t)dt,$$

where

$$\chi(u,t) = \sum_{n=1}^{\infty} \left(\frac{\lambda}{c}\right)^n \xi_n(u,t). \tag{3.114}$$

Now, we need to find $g(u,t)$ satisfying

$$\frac{g_\delta(u)}{1-\phi_\delta} = \int_0^\infty e^{-\rho t}\chi(u,t)dt = \int_0^\infty e^{-\delta t} g(u,t)dt. \tag{3.115}$$

From equation 4 in Dickson and Willmot (2005), we may obtain

$$g(u,t) = ce^{-\lambda t}\left\{\chi(u,ct) + \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!}\int_0^{ct} \frac{y}{ct} p^{*n}(ct-y)\chi(u,y)dy\right\}. \tag{3.116}$$

Then, inversion of the results in Corollary 2 with respect to δ, we may obtain the joint defective density of $(T, U_{T^-}, |U(T)|, X_T, R_{N(T)-1})$ as follows.

Corollary 10 *In the classical compound Poisson model, the joint defective density of $(T, U_{T^-}, |U(T)|, X_T, R_{N(T)-1})$ is defined as*

1. $h^*_{12}(t,x,y|u) = h_1(x,y|u)$ *on* $\{(t,x,y,z,v)|\ t=(x-u)/c, x>u, y>0, z=u, v=u\}$ *corresponding to ruin on the first claim,*

2. $h^*_{124}(t,x,y,v|u) = h_2(t,x-u,y+u,v-u|0)$ *on* $\{(t,x,y,z,v)|\ t>0, x>u, y>0, z=u, u<v<x\}$ *corresponding to ruin on the first drop in surplus due to ruin on other than the first claim,*

3. $h^*_{123}(t,x,y,z|u) = h_1(x,y|z)g(u-z,t-(x-z)/c)$ *on* $\{(t,x,y,z,v)|\ t>(x-z)/c, x>z, y>0, 0<z<u, v=z\}$ *corresponding to a drop in surplus not causing ruin followed by ruin on the next claim, and*

4. $h^*(t,x,y,z,v|u) = \int_0^t g(u-z,t-r)h_2(r,x-z,y+z,v-z|0)dr$ *on* $\{(t,x,y,z,v)|\ t>0, z<v<$

$x, y > 0, 0 < z < u\}$ corresponding to a drop in surplus not causing ruin, followed by ruin occurring but not on the next claim after the drop,

where $g(u,t)$ is given by (3.116).

Proof: From Corollary 2, the first and the second densities above are easily inverted with respect to δ by using (2.3) and (2.4). For the third case, the discounted density is given by

$$\int_0^\infty e^{-\delta t} h_{123}^*(t,x,y,z|u)dt = \left\{ e^{-\frac{\delta(x-z)}{c}} \frac{g_\delta(u-z)}{1-\phi_\delta} \right\} h_1(x,y|z).$$

Using (3.115) and changing a variable $(t + (x-z)/c)$ to (t), the right-hand side of the above equation becomes

$$\int_0^\infty e^{-\delta\left(t+\frac{x-z}{c}\right)} g(u-z,t)h_1(x,y|z)dt = \int_{\frac{x-z}{c}}^\infty e^{-\delta t} g\left(u-z, t-\frac{x-z}{c}\right) h_1(x,y|z)dt.$$

Thus, inversion of the above expression with respect to δ is equivalent to $h_{123}^*(t,x,y,z|u)$. For the last case, we have the discounted density given by

$$\int_0^\infty e^{-\delta t} h^*(t,x,y,z,v|u)dt = \int_0^\infty e^{-\delta t} \frac{g_\delta(u-z)}{1-\phi_\delta} h_2(t,x-z,y+z,v-z|0)dt.$$

Again, with the aid of (3.115) followed by interchanging the order of integration the we rearrange the right-hand side of the above equation as

$$\begin{aligned}
&\int_0^\infty e^{-\delta r} \frac{g_\delta(u-z)}{1-\phi_\delta} h_2(r,x-z,y+z,v-z|0)dr \\
&= \int_0^\infty e^{-\delta r} \left\{ \int_0^\infty e^{-\delta t} g(u-z,t)dt \right\} h_2(r,x-z,y+z,v-z|0)dr \\
&= \int_0^\infty \left\{ \int_r^\infty e^{-\delta t} g(u-z,t-r)dt \right\} h_2(r,x-z,y+z,v-z|0)dr \\
&= \int_0^\infty e^{-\delta t} \left\{ \int_0^t g(u-z,t-r)h_2(r,x-z,y+z,v-z|0)dr \right\} dt.
\end{aligned}$$

Therefore the joint density in the last case is recovered. □

Next, we derive $h_2^*(t, x, y, v|u)$, the joint defective densities of $(T, U_{T^-}, |U(T)|, R_{N(T)-1})$.

3.3.2 Joint defective densities of $(T, U_{T^-}, |U(T)|, R_{N(T)-1})$

In this section, we derive the explicit form of the joint distribution of $T, U_{T^-}, |U(T)|$, and $R_{N(T)-1}$ (denoted by $h_2(t, x, y, v|u)$) for ruin on more than one claim. We shall use the approach as in Landriault and Willmot (2009) which studied the joint distribution of T, U_{T^-} and $|U(T)|$ in the classical compound Poisson risk model.

To begin, recall $\tau_\delta(u, v)$ in the classical compound Poisson risk model given by (Cheung et al. (2010a))

$$\tau_\delta(u,v) = \begin{cases} \frac{1}{1-\phi_\delta}\left\{g_\delta(u-v) + \left(\frac{\lambda+\delta}{c} - \rho\right)\int_0^v e^{-\rho(v-y)}g_\delta(u-y)\,dy\right\}, & v < u \\ \left(\frac{\lambda+\delta}{c} - \rho\right)\left\{e^{-\rho(v-u)} + \frac{1}{1-\phi_\delta}\int_0^u e^{-\rho(v-y)}g_\delta(u-y)\,dy\right\}, & v > u \end{cases}.$$

Since $\left(\frac{\lambda+\delta}{c} - \rho\right) = \frac{\lambda}{c}\widetilde{p}(\rho)$,

$$\tau_\delta(u,v) = \begin{cases} \frac{1}{1-\phi_\delta}\left\{g_\delta(u-v) + \frac{\lambda}{c}\int_0^\infty\int_0^v e^{-\rho(x+v-y)}g_\delta(u-y)\,p(x)dydx\right\}, & v < u \\ \frac{\lambda}{c}\left\{\int_0^\infty e^{-\rho(x+v-u)}p(x)dx + \frac{1}{1-\phi_\delta}\int_0^\infty\int_0^u e^{-\rho(x+v-y)}g_\delta(u-y)\,p(x)dydx\right\}, & v > u \end{cases}.$$

Also using the form of $g_\delta(u)$ given by (2.21) it follows that

$$\tau_\delta(u,v) = \begin{cases} \sum_{n=1}^\infty(\phi_\delta)^n\left\{f_\delta^{*n}(u-v) + \frac{\lambda}{c}\int_0^\infty\int_0^v e^{-\rho(x+v-y)}f_\delta^{*n}(u-y)\,p(x)dydx\right\}, & v < u \\ \frac{\lambda}{c}\left\{\int_0^\infty e^{-\rho(x+v-u)}p(x)dx + \sum_{n=1}^\infty(\phi_\delta)^n\int_0^\infty\int_0^u e^{-\rho(x+v-y)}f_\delta^{*n}(u-y)\,p(x)dydx\right\}, & v > u \end{cases}.$$

Then using (3.112) and (3.113), we may rewrite the above expression of $\tau_\delta(u, v)$ as

$$\tau_\delta(u,v) = \begin{cases} \sum_{n=1}^{\infty}\left(\frac{\lambda}{c}\right)^n \left\{\int_0^\infty e^{-\rho t}\xi_n(u-v,t)dt + \frac{\lambda}{c}\int_0^\infty\int_0^v\int_0^\infty e^{-\rho(t+x+v-y)}\xi_n(u-y,t)p(x)dtdydx\right\}, \\ \frac{\lambda}{c}\left\{\int_0^\infty e^{-\rho(t+v-u)}p(t)dt + \sum_{n=1}^{\infty}\left(\frac{\lambda}{c}\right)^n\int_0^\infty\int_0^u\int_0^\infty e^{-\rho(t+x+v-y)}\xi_n(u-y,t)p(x)dtdydx\right\} \end{cases}. \tag{3.117}$$

From (3.114), it is clear that (3.117) reduces to

$$\tau_\delta(u,v) = \begin{cases} \int_0^\infty e^{-\rho t}\chi(u-v,t)dt + \frac{\lambda}{c}\int_0^v\int_0^\infty\int_0^\infty e^{-\rho(t+x+v-y)}\chi(u-y,t)p(x)dtdxdy, & v<u \\ \frac{\lambda}{c}\left\{\int_0^\infty e^{-\rho(t+v-u)}p(t)dt + \int_0^u\int_0^\infty\int_0^\infty e^{-\rho(t+x+v-y)}\chi(u-y,t)p(x)dtdxdy\right\}, & v>u \end{cases}. \tag{3.118}$$

In order to apply Lagrange's implicit function theorem on the analytic function $e^{-\rho t}$ as Landriault and Willmot (2009) did, we first need to rearrange (3.118) in the form of $\int e^{-\rho t}\cdot dt$ as follows.

For $v<u$ in (3.118), changing a variable from $(t+x+v-y)$ to (t) on the second integral on the right hand side yields

$$\int_0^v\int_0^\infty\int_0^\infty e^{-\rho(t+x+v-y)}\chi(u-y,t)p(x)dtdxdy = \int_0^v\int_0^\infty\int_{x+v-y}^\infty e^{-\rho t}\chi(u-y,t-x-v+y)p(x)dtdxdy,$$

and interchanging the order of integration two times results in

$$\begin{aligned} &\int_0^v\int_0^\infty\int_{x+v-y}^\infty e^{-\rho t}\chi(u-y,t-x-v+y)p(x)dtdxdy \\ &\int_0^v\int_{v-y}^\infty\int_0^{t-v+y} e^{-\rho t}\chi(u-y,t-x-v+y)p(x)dxdtdy \\ &= \left(\int_0^v\int_{v-t}^v + \int_v^\infty\int_0^v\right)\left\{\int_0^{t-v+y} e^{-\rho t}\chi(u-y,t-v+y-x)p(x)dxdydt\right\}. \end{aligned}$$

That is,

$$\int_0^\infty e^{-\rho t}\left(\int_{\max(v-t,0)}^v\int_0^{t-v+y}\chi(u-y,x)p(t-v+y-x)dxdy\right)dt. \tag{3.119}$$

Similarly, for $v>u$ in (3.118) by a change of a variable from $(t+x+v-y)$ to (t) followed by interchanging the order of integration,

$$\int_0^u \int_0^\infty \int_{x+v-y}^\infty e^{-\rho t}\chi(u-y,t-x-v+y)p(x)dtdxdy$$
$$\int_0^u \int_{v-y}^\infty \int_0^{t-v+y} e^{-\rho t}\chi(u-y,t-x-v+y)p(x)dxdtdy$$
$$= \left(\int_{v-u}^v \int_{v-t}^u + \int_v^\infty \int_0^u\right)\left\{\int_0^{t-v+y} e^{-\rho t}\chi(u-y,t-v+y-x)p(x)dxdydt\right\}.$$

Also, it may be rewritten as

$$\int_{v-u}^\infty e^{-\rho t}\left(\int_{\max(v-t,0)}^u \int_0^{t-v+y} \chi(u-y,x)p(t-v+y-x)dxdy\right)dt. \tag{3.120}$$

Therefore, combining the expressions (3.119) and (3.120) leads (3.118) to

$$\tau_\delta(u,v) = \int_{\max(v-u,0)}^\infty e^{-\rho t}\beta(u,t,v)dt, \tag{3.121}$$

where

$$\beta(u,t,v) = \begin{cases} \chi(u-v,t) + \frac{\lambda}{c}\int_{\max(v-t,0)}^v r(u,t,v,y)dy, & v<u,\ t>0, \\ \frac{\lambda}{c}\left\{p(t-v+u) + \int_{\max(v-t,0)}^u r(u,t,v,y)dy\right\}, & v>u,\ t>v-u \end{cases}, \tag{3.122}$$

and $r(u,t,v,y) = \int_0^{t-v+y}\chi(u-y,x)p(t-v+y-x)dx$.

Now, we would like to apply the result of Lagrange's implicit function theorem on the analytic function $e^{-\rho t}$ (Landriault and Willmot (2009, equation 33)), that is

$$e^{-\rho t} = e^{-\frac{\lambda+\delta}{c}t} + \sum_{n=1}^\infty \frac{\left(\frac{\lambda}{c}\right)^n}{n!} t\int_t^\infty a^{n-1}e^{-\frac{\lambda+\delta}{c}a}p^{*n}(a-t)da. \tag{3.123}$$

Replacement of the expression for $e^{-\rho t}$ in (3.123) by the right hand side of (3.121) yields

$$\tau_\delta(u,v) = \int_{\max(v-u,0)}^\infty e^{-\frac{\lambda+\delta}{c}t}\beta(u,t,v)dt + \sum_{n=1}^\infty \frac{\left(\frac{\lambda}{c}\right)^n}{n!}\int_{\max(v-u,0)}^\infty \int_t^\infty ta^{n-1}e^{-\frac{\lambda+\delta}{c}a}p^{*n}(a-t)\beta(u,t,v)dadt. \tag{3.124}$$

For $v<u$, interchanging the order of integration and variables between t and a in the second term on the right hand side of (3.124), one finds

$$
\begin{aligned}
& e^{-\frac{\lambda+\delta}{c}(x-v)}\tau_\delta(u,v) \\
&= \int_0^\infty e^{-\frac{\lambda+\delta}{c}(t+x-v)}\beta(u,t,v)dt + \sum_{n=1}^\infty \frac{\left(\frac{\lambda}{c}\right)^n}{n!}\int_0^\infty \int_0^a ta^{n-1}e^{-\frac{\lambda+\delta}{c}(a+x-v)}p^{*n}(a-t)\beta(u,t,v)dtda \\
&= \int_0^\infty e^{-\frac{\lambda+\delta}{c}(t+x-v)}\beta(u,t,v)dt + \sum_{n=1}^\infty \frac{\left(\frac{\lambda}{c}\right)^n}{n!}\int_0^\infty \int_0^t at^{n-1}e^{-\frac{\lambda+\delta}{c}(t+x-v)}p^{*n}(t-a)\beta(u,a,v)dadt.
\end{aligned}
$$

Changing a variable from (t/c) to (t) yields

$$
\begin{aligned}
& \int_0^\infty e^{-\delta\left(t+\frac{x-v}{c}\right)}\left\{ce^{-\lambda\left(t+\frac{x-v}{c}\right)}\beta(u,ct,v)\right\}dt \\
& + \sum_{n=1}^\infty \frac{\left(\frac{\lambda}{c}\right)^n}{n!}\int_0^\infty e^{-\delta\left(t+\frac{x-v}{c}\right)}\left\{ce^{-\lambda\left(t+\frac{x-v}{c}\right)}\int_0^{ct} a(ct)^{n-1}p^{*n}(ct-a)\beta(u,a,v)da\right\}dt \\
&= \int_{\frac{x-v}{c}}^\infty e^{-\delta t}\left\{ce^{-\lambda t}\beta(u,ct-x+v,v)\right\}dt \\
& + \int_{\frac{x-v}{c}}^\infty e^{-\delta t}\left\{ce^{-\lambda t}\sum_{n=1}^\infty \frac{\left\{\lambda(t-\frac{x-v}{c})\right\}^n}{n!}\int_0^{ct-x+v}\frac{a}{ct-x+v}p^{*n}(ct-x+v-a)\beta(u,a,v)da\right\}dt.
\end{aligned} \tag{3.125}
$$

Thus substituting (3.125) into the right hand side of (2.4) and comparing the coefficient of $e^{-\delta t}$ results in

$$
\begin{aligned}
h_2(t,x,y,v|u) &= \lambda e^{-\lambda t}\left\{\beta(u,ct-x+v,v) + \sum_{n=1}^\infty \frac{\left\{\lambda(t-\frac{x-v}{c})\right\}^n}{n!}\right. \\
& \left.\times \int_0^{ct-x+v}\frac{a}{ct-x+v}p^{*n}(ct-x+v-a)\beta(u,a,v)da\right\}p(x+y),
\end{aligned} \tag{3.126}
$$

for $v < u$ and $t > (x-v)/c$.

Similarly, for $v > u$, by interchanging the order of integration and changing variables (3.124) becomes

$$
\begin{aligned}
& e^{-\frac{\lambda+\delta}{c}(x-v)}\tau_\delta(u,v) \\
&= \int_{v-u}^{\infty} e^{-\frac{\lambda+\delta}{c}(t+x-v)}\beta(u,t,v)dt + \sum_{n=1}^{\infty}\frac{\left(\frac{\lambda}{c}\right)^n}{n!}\int_{v-u}^{\infty}\int_{v-u}^{a} ta^{n-1}e^{-\frac{\lambda+\delta}{c}(a+x-v)}p^{*n}(a-t)\beta(u,t,v)dtda \\
&= \int_{v-u}^{\infty} e^{-\frac{\lambda+\delta}{c}(t+x-v)}\beta(u,t,v)dt + \sum_{n=1}^{\infty}\frac{\left(\frac{\lambda}{c}\right)^n}{n!}\int_{v-u}^{\infty}\int_{v-u}^{t} at^{n-1}e^{-\frac{\lambda+\delta}{c}(t+x-v)}p^{*n}(t-a)\beta(u,a,v)dadt.
\end{aligned}
$$

By changing variables from (t/c) to (t) and then $\{t+(x-v)/c\}$ to (t), it follows that

$$
\begin{aligned}
& \int_{\frac{v-u}{c}}^{\infty} e^{-\delta\left(t+\frac{x-v}{c}\right)}\left\{ce^{-\lambda\left(t+\frac{x-v}{c}\right)}\beta(u,ct,v)\right\}dt \\
& + \sum_{n=1}^{\infty}\frac{\left(\frac{\lambda}{c}\right)^n}{n!}\int_{\frac{v-u}{c}}^{\infty} e^{-\delta\left(t+\frac{x-v}{c}\right)}\left\{ce^{-\lambda\left(t+\frac{x-v}{c}\right)}\int_{v-u}^{ct} a(ct)^{n-1}p^{*n}(ct-a)\beta(u,a,v)da\right\}dt \\
&= \int_{\frac{x-u}{c}}^{\infty} e^{-\delta t}\left\{ce^{-\lambda t}\beta(u,ct-x+v,v)\right\}dt \\
& + \int_{\frac{x-u}{c}}^{\infty} e^{-\delta t}\left\{ce^{-\lambda t}\sum_{n=1}^{\infty}\frac{\left\{\lambda(t-\frac{x-v}{c})\right\}^n}{n!}\int_{v-u}^{ct-x+v}\frac{a}{ct-x+v}p^{*n}(ct-x+v-a)\beta(u,a,v)da\right\}dt.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
h_2(t,x,y,v|u) &= \lambda e^{-\lambda t}\left\{\beta(u,ct-x+v,v) + \sum_{n=1}^{\infty}\frac{\left\{\lambda(t-\frac{x-v}{c})\right\}^n}{n!}\right. \\
&\quad \left. \times \int_{v-u}^{ct-x+v}\frac{a}{ct-x+v}p^{*n}(ct-x+v-a)\beta(u,a,v)da\right\}p(x+y), \quad (3.127)
\end{aligned}
$$

for $v > u$ and $t > (x-u)/c$. Combining (3.126) and (3.127) summarizes the explicit form of $h_2(u,t,x,y,v)$ in the following corollary.

Corollary 11 *In the classical compound Poisson model, the joint defective density of $(T, U_{T^-}, |U(T)|, R_{N(T)-1})$ is defined as*

$$
h_2(t,x,y,v|u) = \lambda e^{-\lambda t}p(x+y)\eta(u,t,x,v), \qquad x > v,
$$

where

$$
\eta(u,t,x,v) = \beta(u,ct-x+v,v) + \sum_{n=1}^{\infty}\frac{\left\{\lambda(t-\frac{x-v}{c})\right\}^n}{n!}\int_{\max(v-u,0)}^{ct-x+v}\frac{a}{ct-x+v}p^{*n}(ct-x+v-a)\beta(u,a,v)da,
$$

for $t > \{x-\min(v,u)\}/c$.

Chapter 4

Delayed renewal risk models

In this chapter, we analyze the delayed risk model which is similar to the Sparre Andersen model except for the assumption on the first interclaim time distribution.

4.1 Introduction

For the dependent delayed renewal risk process the two Gerber-Shiu functions in (1.2) and (1.3) are respectively replaced by

$$m^*_{d,\delta}(u) = E[e^{-\delta T_d} w^*(U_{T_d^-}, |U_{T_d}|, X_{T_d}, R_{N_{T_d}-1}) \mathrm{I}(T_d < \infty) \big| U_0 = u], \tag{4.1}$$

and

$$m_{d,\delta}(u) = E[e^{-\delta T_d} w(U_{T_d^-}, |U_{T_d}|, R_{N_{T_d}-1}) \mathrm{I}(T_d < \infty) \big| U_0 = u], \tag{4.2}$$

where T_d is the time to ruin in the delayed model. If $w^* \equiv 1$ or $w \equiv 1$ in (4.1) or (4.2) respectively, the Gerber-Shiu funtion is reduced to

$$\overline{G}_{d,\delta}(u) = E[e^{-\delta T_d}\ \mathrm{I}(T_d < \infty) \big| U_0 = u], \tag{4.3}$$

and again (4.3) with $\delta = 0$ is equivalent to the ruin probability in the delayed model denoted by $\psi^d(u) = \Pr(T_d < \infty | U_0 = u)$.

In the following section, it is demonstrated that the Gerber-Shiu functions in (4.1), (4.2) and (4.3) may be expressed in terms of the (1.2), (1.3) and (1.7). Given these results, in Section 4.3, the discounted joint densities of $(U_{T_d^-}, |U_{T_d}|, X_{T_d}, R_{N_{T_d}-1})$ are derived using the results in the ordinary risk model. Interestingly, it is sufficient to examine the discounted joint densities of $(U_{T_d^-}, |U_{T_d}|, R_{N_{T_d}-1})$ with $U_0 = 0$ to obtain any other quantities of interest involving those four variables in the penalty function. Therefore, the general form of these joint densities are studied subsequently. In Section 4.4, we consider some examples assuming specific claim sizes. For the case of time-dependent claims we assume earthquake insurance and compare the last ladder height under the present model to the ordinary renewal risk model. In addition, we also consider the usual delayed model with time-independent claim sizes including exponentially distributed claim sizes with arbitrary interclaim times. Finally, some asymptotic results with regard to (4.3) are the subject matter of Section 4.5.

4.2 General structure

To begin the analysis, we first define the joint distribution of the time of ruin (t), the surplus prior to ruin (x), the deficit at ruin (y), and the surplus immediately after the second last claim before ruin occurs (v) in the delayed model, given $U_0 = u$. If ruin occurs on the first claim, then the surplus (x) and the time (t) are related by $x = u + ct$, or equivalently $t = (x - u)/c$. Therefore, the joint defective pdf of the surplus (x) and the deficit (y) is given by

$$h_1^d(x, y|u) = \frac{1}{c} k_1 \left(\frac{x-u}{c} \right) p_{1, \frac{x-u}{c}}(x+y), \qquad x > u, y > 0, \tag{4.4}$$

and in this case $R_{N_{T_d}-1}$ equals u. If ruin occurs on the second or subsequent claims, there is no such linear relationship between the time of ruin and the surplus prior to ruin, and we simply let $h_2^d(t,x,y,v|u)$ be the joint defective pdf of $(T_d, U_{T_d^-}, |U_{T_d}|, R_{N_{T_d}-1})$ for ruin on subsequent claims. From Cheung et al. (2010b), these joint defective densities in the ordinary renewal risk model with dependent structure are respectively $h_1(x,y|u)$ and $h_2(t,x,y,v|u)$ given by (2.1).

We now employ the arguments of Gerber and Shiu (1998) to obtain an expression for $m^*_{d,\delta}(u)$ in (4.1). By conditioning on the first drop in surplus below u, get the following equation for $m^*_{d,\delta}(u)$ is obtained (e.g. Gerber and Shiu (1998, 2005), Li and Garrido (2005), Kim (2007), Kim and Willmot (2010), Willmot (2007)):

$$m^*_{d,\delta}(u) = \int_0^u m^*_\delta(u-y)\left\{\int_0^\infty h^d_{1,\delta}\left(x,y\,|0\right)dx + \int_0^\infty \int_0^x h^d_{2,\delta}(x,y,v|0)dvdx\right\}dy + v^*_{d,\delta}(u), \quad (4.5)$$

where

$$h^d_{1,\delta}\left(x,y\,|u\right) = e^{-\frac{\delta(x-u)}{c}} h^d_1(x,y|u), \quad (4.6)$$

and

$$h^d_{2,\delta}(x,y,v|u) = \int_0^\infty e^{-\delta t} h^d_2(t,x,y,v|u)dt \quad (4.7)$$

are "discounted" joint densities. In this case, $v^*_{d,\delta}(u)$ is the contribution due to ruin on the first drop and is given by

$$\begin{aligned} v^*_{d,\delta}(u) &= \int_u^\infty \int_0^\infty w^*(x+u, y-u, u, u) h^d_{1,\delta}(x,y|0)dxdy \\ &+ \int_u^\infty \int_0^\infty \int_0^x w^*(x+u, y-u, u, v+u) h^d_{2,\delta}(x,y,v|0)dvdxdy. \end{aligned} \quad (4.8)$$

Let us introduce the discounted joint density of the surplus and the deficit

$$h^d_\delta(x,y|u) = h^d_{1,\delta}\left(x,y\,|u\right) + \int_0^x h^d_{2,\delta}(x,y,v|u)dv, \quad (4.9)$$

$$\phi_{d,\delta} = \int_0^\infty \int_0^\infty h^d_\delta(x,y|0)dxdy, \quad (4.10)$$

and

$$f_{d,\delta}(y) = \frac{1}{\phi_{d,\delta}} \int_0^\infty h_\delta^d(x, y|0)dx, \tag{4.11}$$

which allows $m^*_{d,\delta}(u)$ in (4.5) to be expressed as

$$m^*_{d,\delta}(u) = \phi_{d,\delta} \int_0^u m^*_\delta(u-y) f_{d,\delta}(y)dy + v^*_{d,\delta}(u). \tag{4.12}$$

In particular, for $w^*(x, y, z, v) = w(x, y, v)$, (4.12) becomes

$$m_{d,\delta}(u) = \phi_{d,\delta} \int_0^u m_\delta(u-y) f_{d,\delta}(y)dy + v_{d,\delta}(u), \tag{4.13}$$

where

$$v_{d,\delta}(u) = \int_u^\infty \int_0^\infty \left\{ w(x+u, y-u, u) h^d_{1,\delta}(x, y|0) + \int_0^x w(x+u, y-u, v+u) h^d_{2,\delta}(x, y, v|0)dv \right\} dxdy. \tag{4.14}$$

Furthermore, if $w(x, y, v) = 1$, (4.3) satisfies

$$\overline{G}_{d,\delta}(u) = \phi_{d,\delta} \int_0^u \overline{G}_\delta(u-y) f_{d,\delta}(y)dy + \phi_{d,\delta} \overline{F}_{d,\delta}(u), \tag{4.15}$$

where $\overline{F}_{d,\delta}(u) = \int_u^\infty f_{d,\delta}(y)dy$.

4.3 Associated defective densities

In this section, we study, using the integral relationship result of $m^*_{d,\delta}(u)$ given by (4.12), the discounted joint densities of various variables in the penalty function. We begin with a discussion of the discounted joint density of $(U_{T_d^-}, |U_{T_d}|, X_{T_d}, R_{N_{T_d}-1})$.

Corollary 12 *In the delayed renewal risk model, the discounted joint density of the surplus prior to ruin $U_{T_d^-}$, the deficit at ruin $|U_{T_d}|$, the minimum surplus before ruin X_{T_d}, and the surplus immediately after second last claim before ruin $R_{N_{T_d}-1}$ at (x, y, z, v) is defined as follows:*

1. *If ruin occurs on the first drop caused by*

(a) *the first claim :* $h^d_{1,\delta}(x-u, y+u|0)$ *for* $x > u, y > 0, z = u, v = u,$ *and*

(b) *claims other than the first :* $h^d_{2,\delta}(x-u, y+u, v-u|0)$ *for* $x > u, y > 0, z = u, u < v < x.$

2. *If ruin occurs on the second drop caused by*

(a) *the next claim after the first drop :* $\phi_{d,\delta} f_{d,\delta}(u-z) h_{1,\delta}(x-z, y+z|0)$ *for* $x > z, y > 0, 0 < z < u, v = z,$ *and*

(b) *subsequent claims after the first drop :* $\phi_{d,\delta} f_{d,\delta}(u-z) h_{2,\delta}(x-z, y+z, v-z|0)$ *for* $x > z, y > 0, 0 < z < u, z < v < x.$

3. *If ruin occurs on drops (other than the first two drops) caused by*

(a) *the next claim after the drop :* $\left\{\int_z^u \phi_{d,\delta} f_{d,\delta}(u-l) g_\delta(l-z)/(1-\phi_\delta) dl\right\} h_{1,\delta}(x-z, y+z|0)$ *for* $x>z, y>0,\ 0<z<u, v=z,$ *and*

(b) *subsequent claims after the drop :* $\left\{\int_z^u \phi_{d,\delta} f_{d,\delta}(u-l) g_\delta(l-z)/(1-\phi_\delta) dl\right\}\ h_{2,\delta}(x-z, y+z, v-z|0)$ *for* $x>z, y>0, 0<z<u, z<v<x.$

Proof: First, with a choice of $w^*(x, y, z, v) = e^{-s_1 x - s_2 y - s_3 z - s_4 v}$ as in (4.1), from (4.12) and (4.8) the Gerber-Shiu function satisfies

$$m^*_{d,\delta}(u) = \phi_{d,\delta} \int_0^u m^*_\delta(u-y) f_{d,\delta}(y) dy + e^{-s_3 u} v_{d,\delta}(u), \tag{4.16}$$

where $v_{d,\delta}(u)$ from (4.14) is given by

$$v_{d,\delta}(u) = \int_0^\infty \int_u^\infty e^{-s_1 x - s_2 y} \left\{ e^{-s_4 u} h^d_{1,\delta}(x-u, y+u|0) + \int_u^x e^{-s_4 v} h^d_{2,\delta}(x-u, y+u, v-u|0) dv \right\} dx dy. \tag{4.17}$$

Using the expression for $m_\delta^*(u)$ given by Cheung et al. (2010b, Section 3) leads the integral on the right-hand side in (4.16) to

$$\begin{aligned}
&\phi_{d,\delta}\int_0^u m_\delta^*(u-y)f_{d,\delta}(y)dy = \int_0^u \Bigg[\int_0^\infty\int_l^\infty e^{-s_1x-s_2y-s_3l-s_4l}h_{1,\delta}(x-l,y+l|0)dxdy \\
&+ \int_0^\infty\int_l^\infty\int_l^x e^{-s_1x-s_2y-s_3l-s_4v}h_{2,\delta}(x-l,y+l,v-l|0)dvdxdy \\
&+ \int_0^l\int_0^\infty\int_z^\infty e^{-s_1x-s_2y-s_3z-s_4z}\left\{h_{1,\delta}(x-z,y+z|0)\frac{g_\delta(l-z)}{1-\phi_\delta}\right\}dxdydz \\
&+ \int_0^l\int_0^\infty\int_z^\infty\int_z^x e^{-s_1x-s_2y-s_3z-s_4v}\left\{h_{2,\delta}(x-z,y+z,v-z|0)\frac{g_\delta(l-z)}{1-\phi_\delta}\right\}dvdxdydz\Bigg]\phi_{d,\delta}f_{d,\delta}(u-l)dl.
\end{aligned} \tag{4.18}$$

Combining the above and (4.17) with a multiplication of e^{-s_3u} yields the Laplace-Stieltjes transform of $(T_d, U_{T_d^-}, |U_{T_d}|, X_{T_d}, R_{N_{T_d}-1})$. With an interchange of the order of integration followed by Laplace-Stieltjes transform inversion with respect to (s_1, s_2, s_3, s_4), Corollary 12 is proved. We distinguish between the three cases according to the number of drops causing ruin. If ruin occurs on the first drop in surplus below an initial level u, then there are two possibilities; ruin occurs on the first claim or the subsequent claims. The second term on the right-hand side in (4.16), namely $e^{-s_3u}v_{d,\delta}(u)$ represents these two cases. Hence from (4.14) with $w(x,y,v) = e^{-s_1x-s_2y-s_4v}$, it follows that 1(a) and 1(b) are obtained respectively. If ruin occurs not on the first drop, then these cases are explained by the integral terms on the right-hand side in (4.16). Thus from (4.18), we can obtain four different situations corresponding to ruin on the drop (second or subsequent to this) caused by the (next or not next) claim after the drop. And the joint densities in these four cases are given by 2(a),2(b) and 3(a),3(b) respectively. See Figure below for graphs depicting the six different cases contributing to this discounted joint density. □

Note that probabilistic interpretations for the above cases are also available. For example, in cases 3(a) and 3(b), $\phi_{d,\delta}f_{d,\delta}(u-l)$ appears in common which can be interpreted as the size of the first drop being $(u-l)$ not causing ruin. After this first drop, the surplus process is same as the

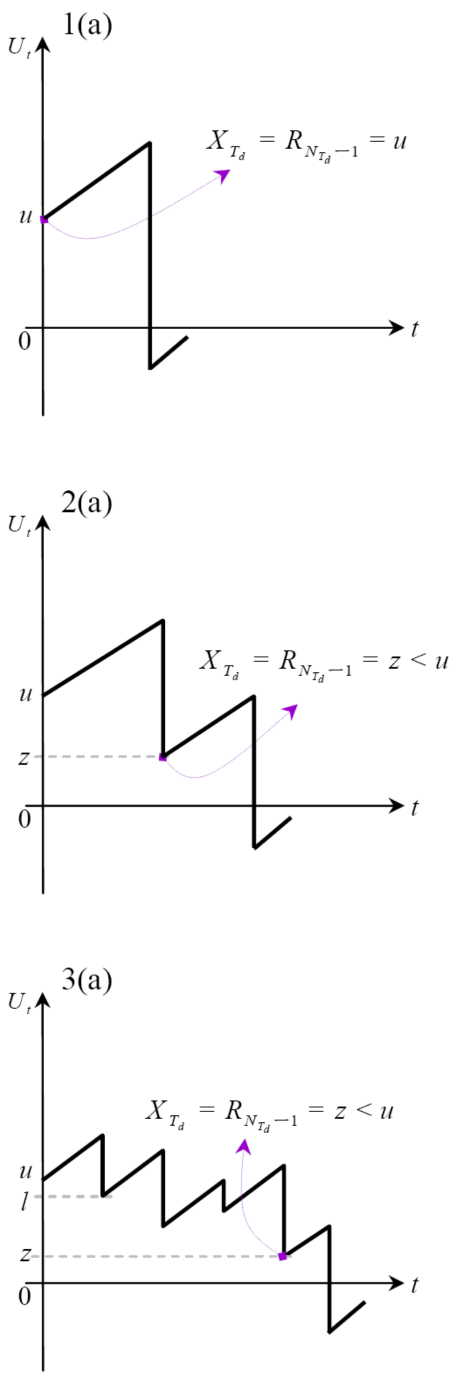

Figure 4.1: The discounted joint density of $U_{T_d^-}$, $|U_{T_d}|$, X_{T_d}, and $R_{N_{T_d}-1}$ at (x, y, z, v)

ordinary process with an initial surplus l. This is followed by an arbitrary number of drops (≥ 1) which bring the surplus process from l to z, as explained by the term $g_\delta(l-z)/(1-\phi_\delta)$. Here, l is arbitrary for $z < l < u$ and with a level of surplus z, ruin immediately occurs on the next claim represented by $h_{1,\delta}$ for 3(a) or on the subsequent claim represented by $h_{2,\delta}$ for 3(b).

Furthermore, we know that $\phi_{d,\delta}$ in (4.10) and $f_{d,\delta}(y)$ in (4.11) can be obtained by $h^d_{2,\delta}(x,y,v|0)$ since $h^d_{1,\delta}(x,y|u)$ is readily known by using (4.4) and (4.6). Therefore, from Corollary 12, note that $h^d_{2,\delta}(x,y,v|0)$ is sufficient to obtain the joint densities of four variables in the penalty function under the delayed risk model as in the ordinary risk model (see Cheung et al. (2010b)). Thus, this discounted joint distribution is derived in the following corollary.

Corollary 13 *In the delayed renewal risk model, the discounted joint density of* $(U_{T_d^-}, |U_{T_d}|, R_{N_{T_d}-1})$ *at* (x,y,v) *is defined as:*

$$h^d_{2,\delta}(x,y,v|u) = h_{1,\delta}(x,y|v)\xi_\delta(u,v), \qquad 0<v<x, y>0, \tag{4.19}$$

where

$$\xi_\delta(u,v) = A_\delta(u,v) + \int_0^\infty A_\delta(u,z)\tau_\delta(z,v)dz, \tag{4.20}$$

and

$$A_\delta(u,z) = \begin{cases} \int_0^\infty e^{-\delta t} p_{1,t}(u+ct-z)dK_1(t), & 0<z<u \\ \int_{(z-u)/c}^\infty e^{-\delta t} p_{1,t}(u+ct-z)dK_1(t), & z>u \end{cases}. \tag{4.21}$$

Proof: By conditioning on the time and the amount of the first claim in order to identify the components in (4.13), we have

$$m_{d,\delta}(u) = \beta_{d,\delta}(u) + \int_0^\infty e^{-\delta t}\sigma_{\delta,t}(u+ct)dK_1(t), \tag{4.22}$$

where $\sigma_{\delta,t}(x) = \int_0^x m_\delta(x-y)dP_{1,t}(y)$, and

$$\beta_{d,\delta}(u) = \int_0^\infty e^{-\delta t}\int_{u+ct}^\infty w(u+ct, y-u-ct, u)dP_{1,t}(y)dK_1(t). \tag{4.23}$$

In other words, by using (4.6), (4.23) may be rewritten as

$$\beta_{d,\delta}(u) = \int_u^\infty \int_0^\infty e^{-\delta(\frac{x-u}{c})} w(x,y,u) h_1^d(x,y|u) dy dx, \tag{4.24}$$

or equivalently $\beta_{d,\delta}(u) = \int_u^\infty \int_0^\infty w(x,y,u) h_{1,\delta}^d(x,y|u) dy dx$. Note that $\beta_{d,\delta}(u)$ may be interpreted as the contribution to the penalty function due to ruin on the first claim. Since $m_{d,\delta}(u)$ in (4.2) is an expectation, it follows directly that it may be represented as

$$m_{d,\delta}(u) = \int_0^\infty \int_u^\infty w(x,y,u) h_{1,\delta}^d(x,y|u) dx dy + \int_0^\infty \int_0^\infty \int_0^x w(x,y,v) h_{2,\delta}^d(x,y,v|u) dv dy dx. \tag{4.25}$$

Then, using (4.24), it may be reexpressed as

$$m_{d,\delta}(u) = \beta_{d,\delta}(u) + \int_0^\infty \int_0^\infty \int_0^x w(x,y,v) h_{2,\delta}^d(x,y,v|u) dv dy dx. \tag{4.26}$$

Then, comparing (4.22) and (4.26) followed by a change of integration leads us to

$$\begin{aligned} \int_0^\infty \int_0^\infty \int_0^x w(x,y,v) h_{2,\delta}^d(x,y,v|u) dv dx dy &= \int_0^\infty e^{-\delta t} \sigma_{\delta,t}(u+ct) dK_1(t) \\ = \int_0^\infty e^{-\delta t} \left\{ \int_0^{u+ct} m_\delta(z) p_{1,t}(u+ct-z) dz \right\} dK_1(t) &= \int_0^\infty m_\delta(z) A_\delta(u,z) dz, \end{aligned} \tag{4.27}$$

where $A_\delta(u,z)$ given by (4.21). Similar to (4.25), $m_\delta(u)$ is also be expressed in terms of the joint defective densities and thus we get

$$\begin{aligned} &\int_0^\infty m_\delta(z) A_\delta(u,z) dz \\ = &\int_0^\infty \left\{ \int_0^\infty \int_z^\infty w(x,y,z) h_{1,\delta}(x,y|z) dx dy + \int_0^\infty \int_0^\infty \int_0^x w(x,y,v) h_{2,\delta}(x,y,v|z) dv dx dy \right\} A_\delta(u,z) dz. \end{aligned}$$

When $w(x,y,v) = e^{-s_1 x - s_2 y - s_3 v}$ on the left-hand side of (4.27) and on the above equation, equating coefficients of $e^{-s_1 x - s_2 y - s_3 v}$ results in

$$h_{2,\delta}^d(x,y,v|u) = h_{1,\delta}(x,y|v) A_\delta(u,v) + \int_0^\infty h_{2,\delta}(x,y,v|z) A_\delta(u,z) dz, \qquad 0<v<x, y>0. \tag{4.28}$$

But $h_{2,\delta}(x,y,v|z) = h_{1,\delta}(x,y|v)\tau_\delta(z,v)$ for $0 < v < x$ (explicit forms for $\tau_\delta(z,v)$ under certain assumptions on the interclaim time and its probabilistic interpretations are provided by Cheung et al. (2010a), and Willmot and Woo (2010)), we may express (4.28) as (4.19). □

As with $\tau_\delta(u,z)$, the function $\xi_\delta(u,v)$ in (4.20) can also be probabilistically interpreted in the following manner. If $R_{N_{T_d}-1} = v$, the delayed process starting with an initial level u should reach the surplus level v, just after the second last claim before ruin. This transition from u to v in the current process is represented by the function $\xi_\delta(u,v)$ as seen from (4.19). However, since the first pair (V_1, Y_1) is assumed different from the other pairs, $\xi_\delta(u,v)$ may also be obtained by conditioning on the time and amount of the first claim which is expressible in terms of $A_\delta(u,z)$ in (4.21). By the definition of $h^d_{2,\delta}$, ruin occurs at $N_{T_d} \geq 2$ and thus if $N_{T_d} = 2$ then the process would be at level v after the first claim explaining the term $A_\delta(u,v)$. Otherwise, for $N_{T_d} > 2$, the process would be at some arbitrary level z after the first claim and then moves from z to v like in the ordinary process with $A_\delta(u,z)\tau_\delta(z,v)$.

Furthermore, using Corollary 13 results in an alternative representation for $m_{d,\delta}(u)$ as follows.

Corollary 14 *In the delayed renewal risk model, the Gerber-Shiu function $m_{d,\delta}(u)$ defined by (4.2) satisfies*

$$m_{d,\delta}(u) = \beta_{d,\delta}(u) + \int_0^\infty \beta_\delta(v)\xi_\delta(u,v)dv, \tag{4.29}$$

where $\beta_{d,\delta}(u)$, $\beta_\delta(u)$ and $\xi_\delta(u,v)$ are given by (4.24), (2.27) and (4.20) respectively.

Proof: Substitution of (4.19) into (4.26) directly yields the result above. □

We point out that Corollary 14 may also make sense intuitively based on the numbers of the claims which cause ruin. If ruin occurs on the first claim with an initial level u, this case may be represented by $\beta_{d,\delta}(u)$. Or if the process first moves from u to v after an arbitrary number of claims (≥ 1) followed by ruin on the subsequent claim from an initial level v, this case may be represented by $\xi_\delta(u,v)\beta_\delta(v)$. In particular, for the ordinary model we know that there is no difference between $\xi_\delta(u,v)$ and $\tau_\delta(u,v)$ while $\beta_{d,\delta}(u)$ is equivalent to $\beta_\delta(u)$ given by (2.27), so that (4.29) is reduced to (2.25).

In particular, we may readily find some ruin related quantities with appropriate choices of the penalty functions in (4.29) since only $\beta_{d,\delta}(u)$ and $\beta_\delta(u)$ contain the penalty function. For example, if $w(x,y,v) = e^{-s(x-v)/c}$, we have the Laplace transform of $V_{N_{T_d}}$ given by $m_{d,0}(u)$ in (4.2). We remark that this quantity represents the last inerclaim time when $N_{T_d} > 1$ or the time until the first claim causing ruin occurs when $N_{T_d} = 1$. In this case, (4.23) becomes $\beta_{d,0}(u) = \int_0^\infty e^{-st}k_1(t)\overline{P}_{1,t}(u+ct)dt$ and $\beta_0(u) = \int_0^\infty e^{-st}k(t)\overline{P}_t(u+ct)dt$. Thus, substituting these expressions into (4.29) and inverting with respect to s yields the marginal defective density of $V_{N_{T_d}}$ (denoted by h_3^d) given by

$$h_3^d(t|u) = a_{1,u}(t)k_1(t) + a_{2,u}(t)k(t), \qquad t > 0,$$

where $a_{1,u}(t) = \overline{P}_{1,t}(u+ct)/\psi^d(u)$ and $a_{2,u}(t) = \{\int_0^\infty \xi_\delta(u,v)\overline{P}_t(v+ct)dv\}/\psi^d(u)$.

In addition, we may obtain bounds for the last interclaim time when $P_{1,t}(y) = P_t(y) = P(y)$ as follows. First, define $\overline{H}_V^d(t|u) = \int_t^\infty h_V^d(y|u)dy$ and introduce two reliability classes, a new worse (better) than used or NWU (NBU) (i.e. $\overline{K}_1(x+y) \geq (\leq)\overline{K}_1(x)\overline{K}_1(y)$ for $x, y \geq 0$). See Barlow and Proschan (1981). From Cheung et al. (2010c, Theorem 7), if $K_1(t)$ is NWU (NBU), $\overline{K}_1(t) \geq (\leq)\overline{K}(t)$ for $t > 0$, and there exists a function $\overline{F}(y)$ on $[0,\infty)$ such that $\overline{P}(x+y) \leq (\geq)\overline{P}(x)\overline{F}(y)$ for $x, y \geq 0$, then the survival function of $V_{N_{T_d}}|T_d < \infty$ satisfies $\overline{H}_V^d(t|u) \leq (\geq)\overline{F}(ct)\overline{K}_1(t)$. Depending on the properties of $P(y)$, Cheung et al. (2010c) provided three possible choices of $\overline{F}(y)$.

We next turn our attention to the last ladder height $Y_{N_{T_d}} = X_{T_d} + |U_{T_d}|$. As mentioned previously, if $w^*(x,y,z,v) = w_{23}(y,z) = e^{-s(y+z)}$ in (4.12) and (4.8), with the aid of the Laplace transform of the last ladder height in the ordinary model given by (2.55), inverting with respect to s yields the defective discounted density of $Y_{N_{T_d}}$ (denoted by $f_{d,\delta}(u,y)$) given by

$$f_{d,\delta}(u,y) = \begin{cases} \frac{\phi_\delta}{1-\phi_\delta}[\overline{G}_{d,\delta}(u-y) - \overline{G}_{d,\delta}(u)]f_\delta(y), & y<u \\ \frac{\phi_\delta}{1-\phi_\delta}[\phi_{d,\delta} - \overline{G}_{d,\delta}(u)]f_\delta(y) + \phi_{d,\delta}f_{d,\delta}(y), & y>u \end{cases}. \tag{4.30}$$

Then the proper survival function $Y_{N_{T_d}}$ given that ruin occurs denoted by $\overline{F}^*_{d,u}(y)$ can be obtained as $\int_y^\infty f_{d,0}(u,x)dx/\psi^d(u)$. Clearly, in the ordinary model (4.30) reduces to (2.54).

In the following section, we illustrate a numerical example in case of the time-dependent claims in the delayed model which contains a comparison of the last ladder height with the ordinary model. And the usual delayed model with the time-independent claims is also presented.

4.4 Examples

4.4.1 Time-dependent claims : Earthquake insurance

Let us consider the dependency model in (3.39) (Boudreault et al. (2006)). Suppose that $f_1(y) = 2.5e^{-2.5y}$, $f_2(y) = 0.5e^{-0.5y}$, $\beta = 1/3$, and $k(t) = te^{-t}$ (i.e. Erlang (2) interclaim times) with $c=2$ and $\delta=0$. In this example, if the interclaim time t is large then the time-dependent claim size distribution $p_t(y)$ is more likely to be determined by f_2 than f_1.

Here, if the last earthquake before time 0 has occurred 5 years ago, we simply let $k_1(t) = k(t+5)/\overline{K}(5)$ be the residual lifetime distribution corresponding to $k(t)$ and $p_{1,t}(y) = p_{t+5}(y)$. Then from (3.58) and (3.59), ϕ_0, $\overline{F}_0(y)$, and thus $\psi(u)$ can be computed. In turn, an application of Equation 32 and Box I in Cheung et al. (2010b) gives $\overline{F}^*_u(y)$ (the proper survival function of the last ladder height in the ordinary model). For the present model, if $w(x,y,v) = w_2(y)$ and

$u=0$ in (4.22) and (4.23), we may obtain the defective density of the deficit as

$$h_0^d(y|0) = \frac{1}{c} k_1 \left(\frac{x-u}{c} \right) \overline{P}_{1,\frac{x-u}{c}}(y) + \int_0^\infty \int_0^{ct} h_0(y|z) p_{1,t}(ct-z) k_1(t) dz dt,$$

where $h_0(y|z)$ is the same as $h_0^d(y|u)$ but defined in the ordinary model. With this $h_0^d(y|0)$, from (4.10) and (4.11) we get $\phi_{d,0}$ and $\overline{F}_{d,0}(y)$, and hence $\psi^d(u)$ from (4.15). Then with the aid of (4.30) one ultimately finds $\overline{F}^*_{d,u}(y)$. When $u=0.5$, the comparison of $\overline{F}^*_{d,u}(y)$ with $\overline{F}^*_u(y)$, and also with the generic ladder heights $\overline{F}_{d,0}(y)$ and $\overline{F}_0(y)$ is summarized in Figure 4.2. In the graph, 'D' and 'O' indicates the delayed model and the ordinary model respectively.

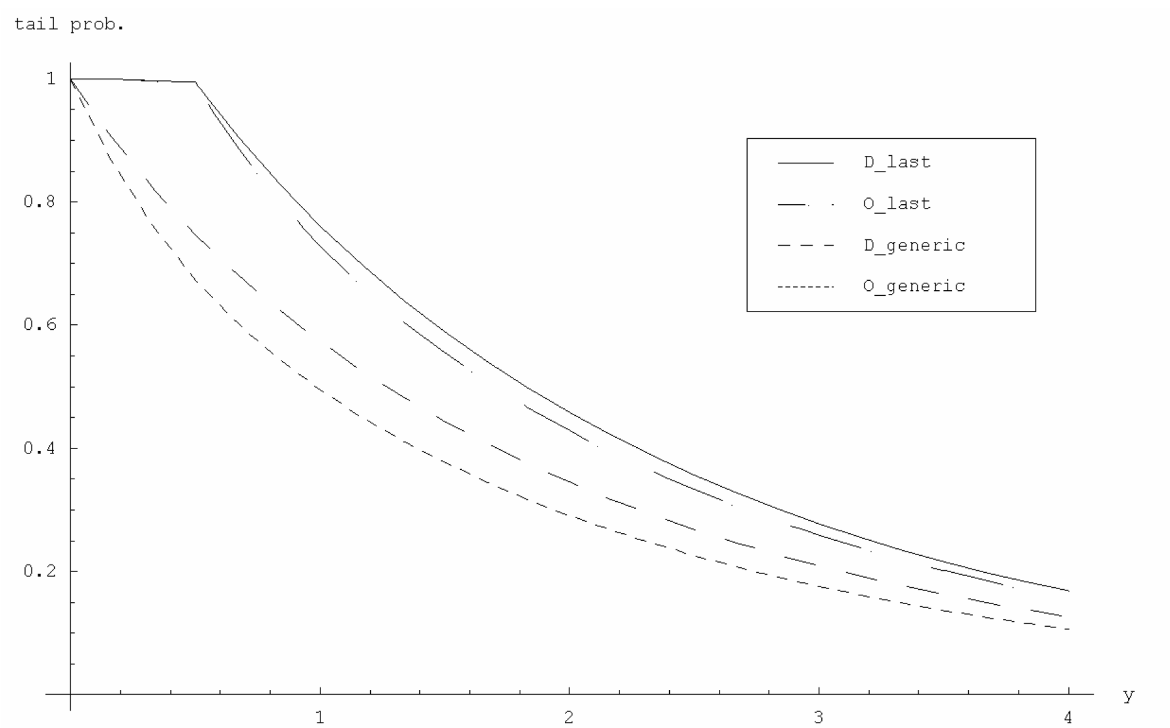

Figure 4.2: The last ladder heights and the generic ladder heights in the delayed and the ordinary models

From Figure 4.2, there is a distinctive difference among the four ladder heights of our interest. In particular, they can be ordered as $\overline{F}^*_{d,u}(y) \geq \overline{F}^*_u(y) \geq \overline{F}_{d,0}(y) \geq \overline{F}_0(y)$. We remark that the stochastic ordering $\overline{F}^*_u(y) \geq \overline{F}_0(y)$ has been proved by Cheung et al. (2010b) in the ordinary model. More interestingly, under this dependent structure, we may conclude that the insurer is more likely to face the larger severity (drop under the minimum surplus level) in the delayed model compared to the ordinary model. In other words, with the model having no adjustment for the pair of the first event (i.e. ordinary model), the insurer may suffer bigger loss than expected. In addition, from Figure 4.3 we can also check that $\psi^d(u) \geq \psi(u)$ for $u \geq 0$, and the difference between these ruin probabilities may not be significant for a large u.

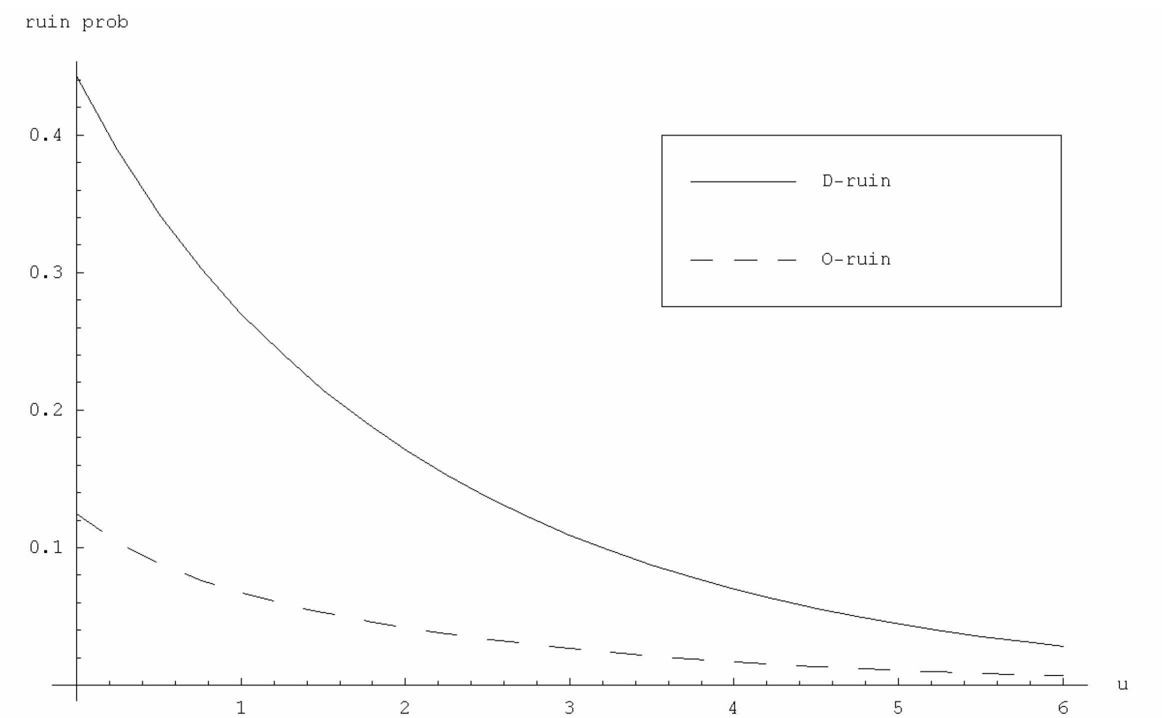

Figure 4.3: The ruin probabilities in the delayed and the ordinary models

4.4.2 Time-independent claims

In this section, we demonstrate how to obtain $m^*_{d,\delta}(u)$ in (4.1) with a specific assumption on the claim sizes. First, let us consider some simplified situation concerning interclaim-independent claim sizes. Suppose that $p_{1,t}(y) = p_1(y), \overline{P}_{1,t}(y) = \overline{P}_1(y)$, and $p_t(y) = p(y), \overline{P}_t(y) = \overline{P}(y)$. Then as in Gerber and Shiu (1998), the conditional density of $|U_{T_d}|$ given $T_d = t, U_{T_d^-} = x, R_{N_{T_d}-1} = v$ for $N_{T_d} \geq 2$ is given by $p(x+y)/\overline{P}(x)$. By using this, one finds that the joint defective density of $(T_d, U_{T_d^-}, |U_{T_d}|, R_{N_{T_d}-1})$ may be expressed as (see Section 2.3 for the ordinary model)

$$h_2^d(t,x,y,v|u) = \frac{p(x+y)}{\overline{P}(x)} h_{(2)}^d(t,x,v|u) \tag{4.31}$$

where $h_{(2)}^d(t,x,v|u)$ is the joint defective density of $(T_d, U_{T_d^-}, R_{N_{T_d}-1})$ for $N_{T_d} \geq 2$. Thus the discounted form of (4.31) is

$$h_{2,\delta}^d(x,y,v|u) = \frac{p(x+y)}{\overline{P}(x)} h_{(2),\delta}^d(x,v|u) \tag{4.32}$$

where $h_{(2),\delta}^d(x,v|u) = \int_0^\infty e^{-\delta t} h_{(2)}^d(t,x,v|u)dt$. Then, by substituting (4.32) into the integral on the right-hand side of (4.9) one may write

$$h_\delta^d(x,y|u) = \frac{p_1(x+y)}{\overline{P}_1(x)} h_{1,\delta}^d(x|u) + \frac{p(x+y)}{\overline{P}(x)} h_{2,\delta}^d(x|u), \tag{4.33}$$

where

$$h_{1,\delta}^d(x|u) = \frac{1}{c} e^{-\delta\left(\frac{x-u}{c}\right)} k_1\left(\frac{x-u}{c}\right) \overline{P}_1(x) \tag{4.34}$$

and $h_{2,\delta}^d(x|u) = \int_0^x h_{(2),\delta}^d(x,v|u)dv$.

Therefore, using (4.33), $\phi_{d,\delta}$ in (4.10) may be expressed as

$$\phi_{d,\delta} = \int_0^\infty \int_0^\infty h_\delta^d(x,y|0)dydx = \int_0^\infty h_\delta^d(x|0)dx, \tag{4.35}$$

where $h_\delta^d(x|0) = h_{1,\delta}^d(x|0) + h_{2,\delta}^d(x|0)$, and also $f_{d,\delta}(y)$ in (4.11) may be expressed as the mixed density (see Kim (2007) and Willmot (2007))

$$f_{d,\delta}(y) = \int_0^\infty \left\{ \frac{h_{1,\delta}^d(x|0)}{\phi_{d,\delta}} \right\} \frac{p_1(x+y)}{\overline{P}_1(x)} dx + \int_0^\infty \left\{ \frac{h_{2,\delta}^d(x|0)}{\phi_{d,\delta}} \right\} \frac{p(x+y)}{\overline{P}(x)} dx. \tag{4.36}$$

The following example illustrate how to derive the joint Laplace transform of five variable, namely $m_{d,\delta}^*(u)$ in (4.1) with a proper choice of the penalty function, if claim sizes are exponentially distributed.

Example (*Exponential claim sizes with arbitrary interclaim times*)
The joint Laplace transform of $(T, U_{T^-}, |U_T|, X_T, R_{N_T-1})$ with exponential claim sizes under the ordinary renewal risk model was considered in Section 2.3.1. Here, by using the results therein, the joint Laplace transform of those five variables under the delayed renewal risk model is revisited. Suppose that the first and the subsequent claims are exponentially distributed with the rate β, i.e. $p_1(y) = p(y) = \beta e^{-\beta y}$.

In this case, $v^*_{d,\delta}(u)$ in (4.8) can be obtained, by some simple algebra, as

$$v^*_{d,\delta}(u) = \frac{\beta \gamma_{d,\delta}(s_1, s_4)}{\beta + s_2} e^{-(\beta+s_1+s_3+s_4)u}, \tag{4.37}$$

where

$$\gamma_{d,\delta}(s_1, s_4) = \widetilde{k}_1(\delta + c\beta + cs_1) + \int_0^\infty \int_0^x e^{-s_1 x - s_4 v} h^d_{(2),\delta}(x, v|0) dv dx, \tag{4.38}$$

and $\widetilde{k}_1(s) = \int_0^\infty e^{-st} k_1(t) dt$. Then, combining $m^*_\delta(u)$ in (2.71) with (2.72), and $f_{d,\delta}(y) = p(y) = \beta e^{-\beta y}$ from (4.36), $m^*_{d,\delta}(u)$ in (4.12) becomes

$$\begin{aligned} m^*_{d,\delta}(u) &= C_\delta(s_1, s_2, s_3, s_4)\phi_{d,\delta}\beta e^{-\beta u} \int_0^u \left\{ (s_1 + s_3 + s_4) e^{-(s_1+s_3+s_4)y} + \phi_\delta \beta e^{\phi_\delta \beta y} \right\} dy + v^*_{d,\delta}(u) \\ &= C_\delta(s_1, s_2, s_3, s_4)\phi_{d,\delta}\beta \left\{ e^{-\beta(1-\phi_\delta)u} - e^{-(\beta+s_1+s_3+s_4)u} \right\} + v^*_{d,\delta}(u), \end{aligned} \tag{4.39}$$

where $v^*_{d,\delta}(u)$ is given by (4.37).

Similar to Section 2.3.1, $\gamma_{d,\delta}(s_1, s_4)$ (or the Laplace transform of $h^d_{(2),\delta}(x, v|0)$) may be expressed in terms of the Laplace transform of the interclaim times $\widetilde{k}(s)$. We simply consider $m^*_{d,\delta}(u)$ with $s_2 = s_3 = 0$. Namely, with $w^*(x, y, z, v) = e^{-s_1 x - s_4 v}$ we denote (4.39) by

$$m_{d,\delta,14}(u) = C_\delta(s_1, 0, 0, s_4)\phi_{d,\delta}\beta \left\{ e^{-\beta(1-\phi_\delta)u} - e^{-(\beta+s_1+s_4)u} \right\} + v_{d,\delta,14}(u) \tag{4.40}$$

where

$$v_{d,\delta,14}(u) = \gamma_{d,\delta}(s_1, s_4)e^{-(\beta+s_1+s_4)u} \tag{4.41}$$

from (4.37). Then, from (4.22) by conditioning on the time and amount of the first claim one finds the alternative integral expression for $m_{d,\delta,14}(u)$. First, in this case, $\beta_{d,\delta}(u)$ in (4.23) with a choice of $w(x,y,v) = e^{-s_1 x - s_4 v}$ reduces to

$$\int_0^\infty e^{-\delta t}\left\{\int_{u+ct}^\infty e^{-cs_1 t}e^{-(s_1+s_4)u}\beta e^{-\beta y}dy\right\}k_1(t)dt = \widetilde{k}_1(\delta + c\beta + cs_1)e^{-(\beta+s_1+s_4)u}. \tag{4.42}$$

Thus, $m_{d,\delta,14}(u)$ satisfies the integral equation from (4.22)

$$m_{d,\delta,14}(u) = \beta_{d,\delta,14}(u) + \int_0^\infty e^{-\delta t}\sigma_{\delta,t}(u+ct)dK_1(t), \tag{4.43}$$

where $\beta_{d,\delta,14}(u)$ is given by (4.42), and $\sigma_{\delta,t}(x) = \int_0^x m_{\delta,14}(x-y)\beta e^{-\beta y}dy$.

With substitution of (2.67) into the above equation, we can express the integral on the right-hand side of (4.43) as

$$\begin{aligned}
&\int_0^\infty e^{-\delta t}\sigma_{\delta,t}(u+ct)dK_1(t) \qquad (4.44)\\
= \;& C_\delta(s_1,0,0,s_4)\int_0^\infty e^{-\delta t}k_1(t)\int_0^{u+ct}\beta e^{-\beta(u+ct-y)}\left\{(s_1+s_4)e^{-(\beta+s_1+s_4)y} + \phi_\delta\beta e^{-\beta(1-\phi_\delta)y}\right\}dydt\\
= \;& C_\delta(s_1,0,0,s_4)\beta e^{-\beta u}\left[\int_0^\infty e^{-(\delta+c\beta)t}\int_0^{u+ct}\left\{(s_1+s_4)e^{-(s_1+s_4)y} + \phi_\delta\beta e^{\phi_\delta\beta y}\right\}dydt\right]\\
= \;& C_\delta(s_1,0,0,s_4)\beta\left\{\widetilde{k}_1(\delta+c\beta-\phi_\delta c\beta)e^{-\beta(1-\phi_\delta)u} - \widetilde{k}_1(\delta+c\beta+cs_1+cs_4)e^{-(\beta+s_1+s_4)u}\right\}. \qquad (4.45)
\end{aligned}$$

Thus, combining (4.42) and (4.44) leads (4.43) to

$$\begin{aligned}
&m_{d,\delta,14}(u) = \widetilde{k}_1(\delta+c\beta+cs_1)e^{-(\beta+s_1+s_4)u}\\
+ \;& C_\delta(s_1,0,0,s_4)\beta\left\{\widetilde{k}_1(\delta+c\beta-\phi_\delta c\beta)e^{-\beta(1-\phi_\delta)u} - \widetilde{k}_1(\delta+c\beta+cs_1+cs_4)e^{-(\beta+s_1+s_4)u}\right\}. \qquad (4.46)
\end{aligned}$$

But, with $s_1 = s_4 = 0$, $C_\delta(0,0,0,0) = \beta^{-1}$ from (2.72), and (4.38) reduces to (from (4.35))

$$\gamma_{d,\delta}(0,0) = \widetilde{k}_1(\delta + c\beta) + \int_0^\infty h^d_{2,\delta}(x|0)dx = \phi_{d,\delta}$$

since $\int_0^\infty h^d_{1,\delta}(x|0)dx = \widetilde{k}_1(\delta + c\beta)$ from (4.34) in this case. Consequently, using these $C_\delta(0,0,0,0)$ and $\gamma_{d,\delta}(0,0)$, from (4.40) and (4.41) with $s_1 = s_4 = 0$ we obtain (4.3) as

$$\overline{G}_{d,\delta}(u) = \phi_{d,\delta} e^{-\beta(1-\phi_\delta)u},$$

where

$$\phi_{d,\delta} = \widetilde{k}_1(\delta + c\beta - \phi_\delta c\beta), \tag{4.47}$$

from (4.46) with $s_1 = s_4 = 0$ and $u = 0$ (e.g. Kim (2007)). Evidently, in the ordinary model, $\overline{G}_\delta(u) = \phi_\delta e^{-\beta(1-\phi_\delta)u}$ with $\phi_\delta = \widetilde{k}(\delta + c\beta - \phi_\delta c\beta)$ (e.g. Willmot (2007)).

Now, equating (4.40) and (4.46) followed by rearranging the equation yields

$$\begin{aligned}\gamma_{d,\delta}(s_1,s_4)e^{-(\beta+s_1+s_4)u} &= \widetilde{k}_1(\delta + c\beta + cs_1)e^{-(\beta+s_1+s_4)u} + C_\delta(s_1,0,0,s_4)\beta \\ \times\Big[\Big\{\widetilde{k}_1(\delta+c\beta-\phi_\delta c\beta)-\phi_{d,\delta}\Big\}\, e^{-\beta(1-\phi_\delta)u} &+ \Big\{\phi_{d,\delta}-\widetilde{k}_1(\delta+c\beta+cs_1+cs_4)\Big\}\, e^{-(\beta+s_1+s_4)u}\Big].\end{aligned} \tag{4.48}$$

Then, application of (4.47) to (4.48) followed by division by $e^{-(\beta+s_1+s_4)u}$ leads to

$$\gamma_{d,\delta}(s_1,s_4) = \widetilde{k}_1(\delta + c\beta + cs_1) + C_\delta(s_1,0,0,s_4)\beta\left\{\phi_{d,\delta} - \widetilde{k}_1(\delta + c\beta + cs_1 + cs_4)\right\}.$$

In other words, using (2.72),

$$\gamma_{d,\delta}(s_1,s_4) = \frac{\phi_{d,\delta}\beta\widetilde{k}(\delta+c\beta+cs_1)+(s_1+s_4)\widetilde{k}_1(\delta+c\beta+cs_1)+\beta k_\delta(s_1,s_4)}{s_1+s_4+\beta\widetilde{k}(\delta+c\beta+cs_1+cs_4)}, \tag{4.49}$$

where

$$k_\delta(s_1,s_4) = \widetilde{k}_1(\delta + c\beta + cs_1)\widetilde{k}(\delta + c\beta + cs_1 + cs_4) - \widetilde{k}(\delta + c\beta + cs_1)\widetilde{k}_1(\delta + c\beta + cs_1 + cs_4). \tag{4.50}$$

Finally, substitution of (4.37) into (4.39) together with the use of (4.49) yields

$$m^*_{d,\delta}(u) = C_{d,\delta}(s_1,s_2,s_3,s_4)(s_1+s_3+s_4)e^{-(\beta+s_1+s_3+s_4)u} + C_\delta(s_1,s_2,s_3,s_4)\phi_{d,\delta}\beta e^{-\beta(1-\phi_\delta)u},$$

where

$$
\begin{aligned}
&C_{d,\delta}(s_1,s_2,s_3,s_4)\\
&=\frac{\beta\left[\phi_{d,\delta}\beta s_3\widetilde{k}(\delta+c\beta+cs_1)+(\phi_\delta\beta+s_1+s_3+s_4)\left\{(s_1+s_4)\widetilde{k}_1(\delta+c\beta+cs_1)+\beta k_\delta(s_1,s_4)\right\}\right]}{(\beta+s_2)(s_1+s_3+s_4)(\phi_\delta\beta+s_1+s_3+s_4)\left\{s_1+s_4+\beta\widetilde{k}(\delta+c\beta+cs_1+cs_4)\right\}}. \quad (4.51)
\end{aligned}
$$

But $k_\delta(s_1,s_4)$ in (4.50) equals to 0 in the ordinary model, $\gamma_{d,\delta}(s_1,s_4)$ in (4.49) and $C_{d,\delta}(s_1,s_2,s_3,s_4)$ in (4.51) are equivalent respectively to $\gamma_\delta(s_1,s_4)$ in (2.70) and $C_\delta(s_1,s_2,s_3,s_4)$ in (2.72).

Similar to Section 2.3.1, for example, we may readily obtain the Laplace transform of $V_{N_{T_d}}=(U_{T_d^-}-R_{N_{T_d}-1})/c$ with a choice of $s_1=s/c, s_4=-s/c$, and $\delta=0$ from (4.40) with (2.72), (4.41) and (4.49) as

$$
E[e^{-sV_{N_{T_d}}}\mathrm{I}(T_d<\infty)|U_0=u]=\frac{\widetilde{k}(c\beta+s)}{\widetilde{k}(c\beta)}\phi_{d,0}e^{-\beta(1-\phi_0)u}+\frac{\widetilde{k}_1(c\beta+s)\widetilde{k}(c\beta)-\widetilde{k}(c\beta+s)\widetilde{k}_1(c\beta)}{\widetilde{k}(c\beta)}e^{-\beta u}, \quad (4.52)
$$

and inversion (4.52) with respect to s followed by dividing by $\psi^d(u)=\phi_{d,0}e^{-\beta(1-\phi_0)u}$ yields the proper density of $V_{N_{T_d}}$ in the delayed renewal risk model as a mixture of Esscher transformed distributions of $K_1(t)$ and $K(t)$, namely

$$
h_V^d(t|u)=(1-a_u)k_e(t)+a_u k_{1,e}(t)
$$

where $a_u=\widetilde{k}_1(c\beta)e^{-\beta u}/\psi^d(u)$, $k_e(t)=e^{-c\beta t}k(t)/\widetilde{k}(c\beta)$, and $k_{1,e}(t)=e^{-c\beta t}k_1(t)/\widetilde{k}_1(c\beta)$. Since $k_1(t)=k(t)$ and $\phi_{d,0}=\phi_0$ for the ordinary renewal risk model, the second term on the right-hand side of (4.52) is cancelled out and thus agrees with (2.74).

4.5 Asymptotic results for the compound geometric tail

In this section, we consider some asymptotic results regarding the compound geometric tail in the delayed renewal process, consequently ruin probabilities are also obtained. First, suppose that $\kappa_\delta > 0$ which is the adjustment coefficient satisfying $\int_0^\infty e^{\kappa_\delta y} f_\delta(y)dy = 1/\phi_\delta$ then we know that the asymptotic result for the compound geometric tail for the ordinary model is given by (3.36), i.e. $\lim_{u\to\infty} e^{\kappa_\delta u}\overline{G}_\delta(u) = C_\delta$, with $C_\delta = (1-\phi_\delta)^{-1}\phi_\delta\kappa_\delta \int_0^\infty y e^{\kappa_\delta y} dF_\delta(y)$, and that

$$\overline{G}_\delta(u) \le e^{-\kappa_\delta u}, \qquad u \ge 0 \tag{4.53}$$

by a Lundberg inequality. Here, suppose that $\tilde{p}_{1,t}(-\kappa_\delta) = \int_0^\infty e^{\kappa_\delta y} dP_{1,t}(y) < \infty$, implying that $\lim_{x\to\infty} e^{\kappa_\delta x}\overline{P}_{1,t}(x) = 0$. Also, as (4.53) holds, by dominated convergence it follows that

$$\begin{aligned} &\lim_{u\to\infty} e^{\kappa_\delta(u+ct)} \left\{ \overline{P}_{1,t}(u+ct) + \int_0^{u+ct} \overline{G}_\delta(u+ct-y)dP_{1,t}(y) \right\} \\ = &\int_0^\infty \left\{ \lim_{u\to\infty} e^{\kappa_\delta(u+ct-y)}\overline{G}_\delta(u+ct-y) \right\} e^{\kappa_\delta y} dP_{1,t}(y) = C_\delta \widetilde{p}_{1,t}(-\kappa_\delta). \end{aligned}$$

Namely,

$$\lim_{u\to\infty} e^{\kappa_\delta u}\overline{W}_{\delta,t}(u) = C_\delta \widetilde{p}_{1,t}(-\kappa_\delta), \tag{4.54}$$

where $\overline{W}_{\delta,t}(u) = \overline{G_\delta * P_{1,t}}(u) = \overline{P}_{1,t}(u) + \int_0^u \overline{G}_\delta(u-y)dP_{1,t}(y)$.

Now, from (4.22) with $w(x,y,v) = 1$, (4.3) has an integral expression as

$$\overline{G}_{d,\delta}(u) = \int_0^\infty e^{-\delta t}\overline{W}_{\delta,t}(u+ct)dK_1(t). \tag{4.55}$$

Since (4.54) holds which implies that $e^{\kappa_\delta u}\overline{W}_{\delta,t}(u)$ is a bounded function of u on $(0,\infty)$. Thus, again by dominated convergence one finds from (4.55)

$$
\begin{aligned}
\lim_{u\to\infty} e^{\kappa_\delta u}\overline{G}_{d,\delta}(u) &= \lim_{u\to\infty}\int_0^\infty \left\{e^{\kappa_\delta(u+ct)}\overline{W}_{\delta,t}(u+ct)\right\} e^{-(\delta+c\kappa_\delta)t}dK_1(t) \\
&= \int_0^\infty \left\{\lim_{u\to\infty} e^{\kappa_\delta(u+ct)}\overline{W}_{\delta,t}(u+ct)\right\} e^{-(\delta+c\kappa_\delta)t}dK_1(t) \\
&= \int_0^\infty C_\delta \widetilde{p}_{1,t}(-\kappa_\delta)e^{-(\delta+c\kappa_\delta)t}dK_1(t) = C_\delta E[e^{\kappa_\delta Y_1-(\delta+c\kappa_\delta)V_1}],
\end{aligned}
$$

where

$$
E[e^{\kappa_\delta Y_1-(\delta+c\kappa_\delta)V_1}] = \int_0^\infty e^{-(\delta+c\kappa_\delta)t}\widetilde{p}_{1,t}(-\kappa_\delta)dK_1(t).
$$

Hence,

$$
\overline{G}_{d,\delta}(u) \sim C_\delta E[e^{\kappa_\delta Y_1-(\delta+c\kappa_\delta)V_1}]e^{-\kappa_\delta u}, \qquad u\to\infty. \tag{4.56}
$$

In particular, for interclaim-independent claim sizes, i.e. $p_{1,t}(y) = p(y)$, we know that

$$
E[e^{\kappa_\delta Y_1-(\delta+c\kappa_\delta)V_1}] = \widetilde{p}(-\kappa_\delta)\widetilde{k}_1(\delta+c\kappa_\delta) = \frac{\widetilde{k}(\delta+c\kappa_\delta)}{\widetilde{k}_1(\delta+c\kappa_\delta)}
$$

since κ_δ satisfies $\widetilde{p}(-\kappa_\delta)\widetilde{k}(\delta+c\kappa_\delta) = 1$ (see Cheung et al. (2010b, Section 4)). Therefore, in this case (4.56) reduces to

$$
\overline{G}_{d,\delta}(u) \sim \frac{C_\delta\widetilde{k}(\delta+c\kappa_\delta)}{\widetilde{k}_1(\delta+c\kappa_\delta)}e^{-\kappa_\delta u}, \qquad u\to\infty.
$$

And further for $\delta = 0$ we know that $\overline{G}_{d,0}(u) = \psi^d(u)$, and the asymptotic result for $\psi^d(u)$ from the above is agreed with Theorem 11.4.3 in Willmot and Lin (2001).

Alternatively, we may directly obtain the asymptotic form in (4.54) by using the result for the tail of a compound geometric convolution $\overline{W}_{\delta,t}(u)$ which satisfies the defective renewal equation, (see Willmot and Cai (2004) and references therein)

$$
\overline{W}_{\delta,t}(x) = \phi_\delta\int_0^x \overline{W}_{\delta,t}(x-y)dF_\delta(y) + \phi_\delta\overline{F}_\delta(x) + (1-\phi_\delta)\overline{P}_{1,t}(x).
$$

It is shown to be

$$
\lim_{u\to\infty} e^{\kappa_\delta u}\overline{W}_{\delta,t}(u) = \frac{(1-\phi_\delta)\int_0^\infty e^{\kappa_\delta y}dP_{1,t}(y)}{\phi_\delta\kappa_\delta\int_0^\infty ye^{\kappa_\delta y}dF_\delta(y)} = C_\delta\widetilde{p}_{1,t}(-\kappa_\delta). \tag{4.57}
$$

Furthermore, if $w(x, y, v) = 1$ in (4.13), then it is clear that from (4.15) $\overline{G}_{d,\delta}(u)/\phi_{d,\delta}$ is also the tail of a compound geometric convolution, and thus if $\tilde{f}_{d,\delta}(-\kappa_\delta) = \int_0^\infty e^{\kappa_\delta y} dF_{d,\delta}(y) < \infty$, the same argument used to drive (4.57) results in

$$\overline{G}_{d,\delta}(u) \sim C_\delta \phi_{d,\delta} \widetilde{f}_{d,\delta}(-\kappa_\delta) e^{-\kappa_\delta u}, \qquad u \to \infty. \tag{4.58}$$

Curiously, comparison of (4.56) with (4.58) results in the identity

$$\phi_{d,\delta} \widetilde{f}_{d,\delta}(-\kappa_\delta) = E[e^{\kappa_\delta Y_1 - (\delta + c\kappa_\delta) V_1}], \tag{4.59}$$

and obviously both sides of (4.59) equal 1 in the nondelayed case.

Chapter 5

Discrete renewal risk models

In this chapter, analysis of a generalized Gerber-Shiu function is considered in a discrete-time (ordinary) Sparre Andersen renewal risk process with time-dependent claim sizes. The results are then applied to obtain ruin related quantities under some renewal risk processes assuming specific interclaim distributions such as a discrete K_n distribution and a truncated geometric distribution (i.e. compound binomial process). Furthermore, the discrete delayed renewal risk process is considered and results related to the ordinary process are derived as well.

5.1 Introduction

First, let us introduce the classical Gerber-Shiu discounted penalty function defined in a discrete-time Sparre Andersen renewal risk model (e.g. Li (2005a,b), Wu and Li (2008))

$$m_{v,12}(u) = E\left[v^T w_{12}\left(U(T-1), |U(T)|\right) \mathrm{I}\left(T<\infty\right) \middle| U(0)=u\right], \qquad u \in \mathbb{N} . \tag{5.1}$$

where T is the time of ruin defined as $T = \min\{t \in \mathbb{N}^+ : U(t) < 0\}$ with $T = \infty$ if $U(t) \geq 0$ for all $t \geq 1$. Also, $U(T-1)$ is the surplus before ruin, $|U(T)|$ is the deficit at ruin, $w_{12}(x, y)$ is the penalty function, and $v \in (0,1]$ is interpreted as a discount factor.

As in continuous-time models considered in the previous chapters, in order to generalize the above Gerber-Shiu function, we first define $X_t = \min_{0 \le s < t} U(s)$ to be the minimum surplus before time t. Thus, X_T is the minimum level of surplus before ruin occurs. Second, let us define $R_n = u + \sum_{i=1}^{n}(W_i - Y_i)$ for $n = 1, 2, \ldots$, and $R_0 = u$, i.e. R_n is the surplus just after the n-th claim if $n \ge 1$. Therefore, $R_{N(T)-1}$ is the surplus immediately after the second last claim before ruin occurs if $N(T) > 1$, and $R_0 = u$ if ruin occurs on the first claim (i.e. $N(T) = 1$). Note that these two new quantities X_T and $R_{N(T)-1}$ may or may not be the same depending on a given sample path. Then (5.1) may be generalized to

$$m_v^*(u) = E\left[v^T w^*\left(U(T-1), |U(T)|, X_T, R_{N(T)-1}\right) \mathrm{I}\left(T < \infty\right) \middle| U(0) = u\right], \qquad u \in \mathbb{N} . \tag{5.2}$$

We may analyze the last ladder height before ruin $X_T + |U(T)|$, and the last interclaim time before ruin $W_{N(T)} = U(T-1) - R_{N(T)-1} + 1$ from (5.2). As a special case of (5.2) with $w^*(x,y,z,r) = w(x,y,r)$, it follows that

$$m_v(u) = E\left[v^T w\left(U(T-1), |U(T)|, R_{N(T)-1}\right) \mathrm{I}\left(T < \infty\right) \middle| U(0) = u\right], \qquad u \in \mathbb{N} . \tag{5.3}$$

Using (5.3) we may study the last pair $(W_{N(T)}, Y_{N(T)})$ where $Y_{N(T)}$ is the claim causing ruin given by $Y_{N(T)} = U(T-1) + |U(T)| + 1$. Note that the actual surplus level prior to ruin is $R_{N(T)-1} + W_{N(T)}$ which is equivalent to $U(T-1)+1$. Cheung et al. (2010a) studied this quantity in the classical compound Poisson risk process.

Also, we consider the particular special cases of the above Gerber-Shiu functions with the successively simplified penalty functions respectively defined by $w^*(x,y,z,r) = w_{123}(x,y,z)$, $w^*(x,y,z,r) = w_{23}(y,z)$, $w^*(x,y,z,r) = w_2(y)$, and $w^*(x,y,z,r) = 1$, i.e.

$$m_{v,123}(u) = E\left[v^T w_{123}\left(U(T-1), |U(T)|, X_T\right) \mathrm{I}\left(T < \infty\right) \middle| U(0) = u\right], \qquad u \in \mathbb{N} , \tag{5.4}$$

$$m_{v,23}(u) = E\left[v^T w_{23}\left(|U(T)|, X_T\right) \mathrm{I}\left(T<\infty\right) \Big| U(0)=u\right], \qquad u \in \mathbb{N}, \tag{5.5}$$

$$m_{v,2}(u) = E\left[v^T w_2\left(|U(T)|\right) \mathrm{I}\left(T<\infty\right) \Big| U(0)=u\right], \qquad u \in \mathbb{N}, \tag{5.6}$$

and

$$\overline{G}_v(u) = E\left[v^T \mathrm{I}\left(T<\infty\right) \Big| U(0)=u\right], \qquad u \in \mathbb{N}. \tag{5.7}$$

Certainly, with $v=1$, (5.7) reduces to the ruin probability $\psi(u) = \Pr\left(T<\infty | U(0)=u\right)$.

Furthermore, Lundberg's (generalized) fundamental equation is given by (e.g. Li (2005a,b))

$$E[v^W s^{Y-W}] = 1, \tag{5.8}$$

and in latter sections the roots of this equation play an important role for analyzing the Gerber-Shiu functions just introduced.

For the analysis of (5.3) in Section 5.3, we shall define an auxiliary function and the discrete Dickson-Hipp operator (see Dickson and Hipp (2001), Li and Garrido (2004) for a continuous version of this operator) as follows. First, suppose

$$\eta_v(u) = \sum_{t=1}^{\infty} v^t \omega_t(u+t) k(t), \qquad u \in \mathbb{N}, \tag{5.9}$$

for some function $\omega_t(u)$, with generating function $\hat{\eta}_v(s) = \sum_{x=0}^{\infty} s^x \eta_v(x)$. Then, taking the summation from 0 to ∞ yields

$$\begin{aligned}
\hat{\eta}_v(s) &= \sum_{u=0}^{\infty} s^u \sum_{t=1}^{\infty} v^t \omega_t(u+t) k(t) = \sum_{t=1}^{\infty} s^{-t} v^t k(t) \left\{ \sum_{u=0}^{\infty} s^{u+t} \omega_t(u+t) \right\} \\
&= \sum_{t=1}^{\infty} \left(\frac{v}{s}\right)^t k(t) \left\{ \hat{\omega}_t(s) - \sum_{u=0}^{t-1} s^u \omega_t(u) \right\},
\end{aligned}$$

where $\hat{\omega}_t(s) = \sum_{x=0}^{\infty} s^x \omega_t(x)$. That is,

$$\hat{\eta}_v(s) = \sum_{t=1}^{\infty} \left(\frac{v}{s}\right)^t k(t)\hat{\omega}_t(s) - \hat{\omega}_{*,v}(s), \tag{5.10}$$

where $\hat{\omega}_{*,v}(s) = \sum_{t=1}^{\infty} \sum_{u=0}^{t-1} s^{u-t} v^t k(t) \omega_t(u)$.

In Section 5.2.1, analogous to a Sparre Andersen renewal risk process in continuous-time studied in Section 2.1, structural properties of the generalized Gerber-Shiu function under the present model are derived. Consequently, an alternative form of solution for the generalized Gerber-Shiu function and various joint and marginal distributions of ruin related quantities can be obtained. In Section 5.3, to identify the quantities involved in the recursive formulas for the generalized Gerber-Shiu function, we assume a wide class of distributions, called the discrete K_n class for the interclaim times. The discrete K_n class of distributions studied by Li (2005a,b) has a probability generating function (pgf) which is a ratio of two polynomials of order n. The compound binomial model can be easily retrieved as a special case, and has been widely considered by many researchers (e.g. Cheng et al. (2000), Dickson (1994), Gerber (1988), Shiu (1989), Willmot (1993), Yuen and Guo (2001)). Finally, in Section 5.4, a modified discrete ordinary renewal process (i.e. discrete delayed renewal process) is considered. In particular, we assume time-dependent claim sizes so that this model may be more reasonable as a model in which a pair of the first event follows different distributional assumption from the subsequent pairs. For the generalized Gerber-Shiu function in this model, a recursive formula for the discounted joint probability function (pf) is derived in terms of the corresponding generalized Gerber-Shiu function in the ordinary model.

5.2 General structure

In the present section we explore the structure of the generalized Gerber-Shiu functions.

5.2.1 Recursive formulas

To begin the analysis, if ruin occurs on the first claim, the joint defective pf of the surplus prior to ruin (x) and the deficit at ruin (y) is given by

$$h_1(x, y|u) = k(x - u + 1)p_{x-u+1}(x + y + 1), \qquad x \in \mathbb{N}, y \in \mathbb{N}^+ \tag{5.11}$$

where $T = x - u + 1$ and $R_{N(T)-1} = u$. For the subsequent claims causing ruin (i.e. $N(T) = 2, 3, \ldots$), since there is no longer a linear relationship between the time of ruin and the surplus prior to ruin, the joint defective pf of $(T, U(T-1), |U(T)|, R_{N(T)-1})$ at (t, x, y, r) is given by $h_2(t, x, y, r|u)$ for $t = 2, 3, \ldots$, $x, r \in \mathbb{N}$ and $y \in \mathbb{N}^+$. Also, the discounted joint pf corresponding to h_1 and h_2 are respectively given by

$$h_{1,v}(x, y|u) = v^{x-u+1}h_1(x, y|u), \tag{5.12}$$

and

$$h_{2,v}(x, y, r|u) = \sum_{t=2}^{\infty} v^t h_2(t, x, y, r|u). \tag{5.13}$$

As in Li (2005a), and Wu and Li (2008), let us consider the drop below the surplus u to obtain a recursive equation for $m_v^*(u)$ defined in (5.2). The pf of this first drop caused by a first claim is governed by $h_1(x, y|0)$, in this case, the surplus level above an initial capital u before the drop is $x + 1$, and the drop amount below u is y, so that the surplus level after this drop becomes $u - y$ and the time of this drop is $x + 1$. If the drop below u is caused by any subsequent claims to the first one, then the pf is governed by $h_2(x, y, r|0)$. There are two possibilities depending on whether the first drop causes ruin or not. If $y \leq u$ (i.e. the surplus level after the drop $u - y$ is still nonnegative), then the process begins anew (probabilistically) with the new initial surplus $u - y$. If $y \geq u + 1$ (i.e ruin occurs on the first drop), then in case of the drop occurring on the first claim, $U(T-1) = x + u$, $|U(T)| = y - u$, $X_T = u$, and $R_{N(T)-1} = u$. While in case of the drop occurring on other than the first claim, $U(T-1) = x + u$, $|U(T)| = y - u$, $X_T = u$, and $R_{N(T)-1} = r + u$.

Summing over all values of t, x, y, and r results in the recursive formula as follow. For $u \in \mathbb{N}$,

$$m_v^*(u) = \sum_{y=1}^{u} m_v^*(u-y) \left\{ \sum_{x=0}^{\infty} h_{1,v}(x, y|0) + \sum_{x=0}^{\infty} \sum_{r=0}^{x} h_{2,v}(x, y, r|0) \right\} + l_v^*(u), \tag{5.14}$$

where $h_{1,v}(x, y|0)$ and $h_{2,v}(x, y, r|0)$ are given by (5.12) and (5.13) respectively, and

$$l_v^*(u) = \sum_{y=u+1}^{\infty} \sum_{x=0}^{\infty} \left\{ w^*(x+u, y-u, u, u) h_{1,v}(x, y|0) + \sum_{r=0}^{x} w^*(x+u, y-u, u, r+u) h_{2,v}(x, y, r|0) \right\}, \tag{5.15}$$

represents the contribution due to ruin on the first drop.

By summing over all values of t and r, we may obtain the discounted (marginal if $v = 1$) joint pf of the surplus prior to ruin (x) and the deficit at ruin (y) given by

$$h_v(x, y|u) = h_{1,v}(x, y|u) + \sum_{r=0}^{x} h_{2,v}(x, y, r|u). \tag{5.16}$$

With $u = 0$ in the above pf, (5.14) may be rewritten as

$$m_v^*(u) = \sum_{y=1}^{u} m_v^*(u-y) \left\{ \sum_{x=0}^{\infty} h_v(x, y|0) \right\} + l_v^*(u). \tag{5.17}$$

Then, letting

$$\phi_v = \sum_{x=0}^{\infty} \sum_{y=1}^{\infty} h_v(x, y|0),$$

and the ladder height pf be

$$f_v(y) = \frac{1}{\phi_v} \sum_{x=0}^{\infty} h_v(x, y|0), \qquad y \in \mathbb{N}^+, \tag{5.18}$$

one may express (5.17) as

$$m_v^*(u) = \phi_v \sum_{y=1}^{u} m_v^*(u-y) f_v(y) + l_v^*(u). \tag{5.19}$$

Clearly, the discounted joint pf of $U(T-1)$, $|U(T)|$, and $R_{N(T)-1}$ for ruin occurring on claims subsequent to the first with zero initial surplus (i.e. $h_{2,v}(x, y, r|0)$) is essential for analysis of the generalized Gerber-Shiu function $m_v^*(u)$ in (5.2), since the discounted pf $h_{1,v}(x,y|0)$ in (5.12) is known explicitly. Now, we consider the generalized Gerber-Shiu function (5.3) which also satisfies the recursive expressions from (5.19) with $w^*(x,y,z,r) = w(x,y,r)$ in (5.15), namely

$$m_v(u) = \phi_v \sum_{y=1}^{u} m_v(u-y) f_v(y) + l_v(u), \tag{5.20}$$

where

$$l_v(u) = \sum_{y=u+1}^{\infty} \sum_{x=0}^{\infty} \left\{ w(x+u, y-u, u) h_{1,v}(x,y|0) + \sum_{r=0}^{x} w(x+u, y-u, r+u) h_{2,v}(x,y,r|0) \right\}.$$

A change of the variable of summation yields

$$l_v(u) = \sum_{y=1}^{\infty} \sum_{x=u}^{\infty} \left\{ w(x,y,u) h_{1,v}(x,y|u) + \sum_{r=u}^{x} w(x,y,r) h_{2,v}(x-u, y+u, r-u|0) \right\}, \tag{5.21}$$

since $h_{1,v}(x-u, y+u|0) = h_{1,v}(x,y|u)$.

Furthermore, the Gerber-Shiu functions (5.4), (5.5), (5.6) and (5.7) also satisfy recursive equations respectively as follows. If $w^*(x,y,z,r) = w_{123}(x,y,z)$, then (5.15) becomes

$$l_{v,123}(u) = \sum_{y=u+1}^{\infty} \sum_{x=0}^{\infty} w_{123}(x+u, y-u, u) \left\{ h_{1,v}(x,y|0) + \sum_{r=0}^{x} h_{2,v}(x,y,r|0) \right\},$$

that is,

$$l_{v,123}(u) = \sum_{y=u+1}^{\infty} \sum_{x=0}^{\infty} w_{123}(x+u, y-u, u) h_v(x,y|0), \tag{5.22}$$

using (5.16). With this $l_{v,123}(u)$ in this case, (5.19) reduces to

$$m_{v,123}(u) = \phi_v \sum_{y=1}^{u} m_{v,123}(u-y) f_v(y) + l_{v,123}(u).$$

For the simpler case of $w^*(x,y,z,r) = w_{23}(y,z)$, (5.5) may be obtainable from (5.19) with (5.15) and (5.18)

$$m_{v,23}(u) = \phi_v \sum_{y=1}^{u} m_{v,23}(u-y) f_v(y) + l_{v,23}(u),$$

where

$$l_{v,23}(u) = \phi_v \sum_{y=u+1}^{\infty} w_{23}(y-u, u) f_v(y). \tag{5.23}$$

In this case, $m_{v,23}(u)$ depends only on the ladder height pf $f_v(y)$ and thus the pf of the last ladder height $X_T + |U_T|$ can be obtained from the generic ladder height pf (see Section 5.2.3). Further, if the penalty function is only dependent on the deficit at ruin, i.e. $w^*(x, y, z, r) = w_2(y)$, then from (5.19) with (5.15), (5.6) satisfies

$$m_{v,2}(u) = \phi_v \sum_{y=1}^{u} m_{v,2}(u-y) f_v(y) + \phi_v \sum_{y=u+1}^{\infty} w_2(y-u) f_v(y).$$

If $w^*(x, y, z, r) = 1$, then (5.7) also has a recursive equation

$$\overline{G}_v(u) = \phi_v \sum_{y=1}^{u} \overline{G}_v(u-y) f_v(y) + \phi_v \overline{F}_v(u). \tag{5.24}$$

The solution to (5.24) is known as the discrete compound geometric tail with $\overline{G}_v(0) = \phi_v$ (e.g. Willmot and Lin (2001), Wu and Li (2008)) given by

$$\overline{G}_v(u) = \sum_{n=1}^{\infty} (1-\phi_v)(\phi_v)^n \overline{F}_v^{*n}(u), \qquad u \in \mathbb{N}\ ,$$

where $F_v(u) = 1 - \overline{F}_v(u) = \sum_{y=1}^{u} f_v(y)$ and $1 - \overline{F}_v^{*n}(u)$ is the df of the n-fold convolution of $f_v(u)$. The general solution to (5.19) (and its special cases with (5.20) of particular interest) may be expressed as (e.g. Wu and Li (2008, Theorem 1))

$$m_v(u) = \frac{1}{1-\phi_v} \sum_{y=0}^{u} g_v(u-y) l_v(y), \qquad u \in \mathbb{N}\ , \tag{5.25}$$

where $g_v(u)$ is the compound geometric pf $g_v(u) = \overline{G}_v(u-1) - \overline{G}_v(u)$ for $u \in \mathbb{N}^+$ with $g_v(0) = 1 - \phi_v$. Note that $g_v(u)$ is given by

$$g_v(u) = \sum_{n=0}^{\infty} (1-\phi_v)(\phi_v)^n f_v^{*n}(u), \qquad u \in \mathbb{N}\ ,$$

where $f_v^{*n}(u)$ is the pf of the n-fold convolution of $f_v(u)$ with the usual convention $f_v^{*0}(u)=\mathrm{I}(u=0)$.

5.2.2 Analysis of $m_v(u)$

As we demonstrated in Section 5.2.1, determination of the discounted joint pf $h_{2,v}(x,y,r|0)$ is sufficient to study $m_v^*(u)$ in (5.2), thus we may find an alternative form of solution to $m_v(u)$ which leads to $h_{2,v}(x,y,r|0)$ in the following proposition (similar to Theorem 2 in the continuous-time model).

Proposition 2 *Assume that the discounted pf $h_{2,v}(x,y,r|0)$ admits the representation*

$$h_{2,v}(x,y,r|0) = h_{1,v}(x,y|r)\nu_v(r), \qquad x,r \in \mathbb{N}, y \in \mathbb{N}^+, \tag{5.26}$$

for some function $\nu_v(r)$. Then the Gerber-Shiu function in (5.3) may be expressed as

$$m_v(u) = \beta_v(u) + \sum_{r=0}^{\infty} \beta_v(r)\tau_v(u,r), \qquad u \in \mathbb{N}\ , \tag{5.27}$$

where

$$\beta_v(u) = \sum_{x=u}^{\infty}\sum_{y=1}^{\infty} w(x,y,u)h_{1,v}(x,y|u), \tag{5.28}$$

and

$$\tau_v(u,r) = \begin{cases} \frac{1}{1-\phi_v}\left\{g_v\left(u-r\right) + \sum_{y=0}^{r}\nu_v(r-y)g_v(u-y)\right\}, & r=0,1,\ldots,u-1 \\ \frac{1}{1-\phi_v}\sum_{y=0}^{u}\nu_v(r-y)g_v\left(u-y\right), & r=u,u+1,\ldots \end{cases}, \tag{5.29}$$

with $\tau_v(0,r) = \nu_v(r)$.

Proof. For notational convenience, let

$$\xi_v(u) = \sum_{y=1}^{\infty}\sum_{x=u}^{\infty}\sum_{r=u}^{x} w(x,y,r)h_{2,v}(x-u,y+u,r-u|0) = \sum_{r=u}^{\infty}\sum_{y=1}^{\infty}\sum_{x=r}^{\infty} w(x,y,r)h_{2,v}(x-u,y+u,r-u|0), \tag{5.30}$$

then (5.21) may be rewritten as $l_v(u) = \beta_v(u) + \xi_v(u)$ where $\beta_v(u)$ is given by (5.28). With this expression for $l_v(u)$, it follows from (5.25) one has

$$m_v(u) = \frac{1}{1-\phi_v} \sum_{y=0}^{u} g_v(u-y) \left\{ \beta_v(y) + \xi_v(y) \right\}. \tag{5.31}$$

Under the assumption (5.26), one has

$$h_{2,v}(x-u, y+u, r-u|0) = h_{1,v}(x, y|r)\nu_v(r-u).$$

Utilizing the above equation with (5.28), (5.30) becomes $\sum_{r=u}^{\infty} \beta_v(r)\nu_v(r-u)$, and thus the right-hand side of (5.31) may be expressed as

$$\frac{1}{1-\phi_v} \sum_{y=0}^{u} g_v(u-y) \left\{ \beta_v(y) + \xi_v(y) \right\} = \frac{1}{1-\phi_v} \sum_{y=0}^{u} g_v(u-y) \left\{ \beta_v(y) + \sum_{r=y}^{\infty} \beta_v(r)\nu_v(r-y) \right\}.$$

After an interchange of the order of summation on the above equation, (5.31) may be rewritten as

$$\begin{aligned} m_v(u) &= \frac{1}{1-\phi_v} \sum_{r=0}^{u} \beta_v(r) g_v(u-r) + \frac{1}{1-\phi_v} \left(\sum_{r=0}^{u-1} \sum_{y=0}^{r} + \sum_{r=u}^{\infty} \sum_{y=0}^{u} \right) \beta_v(r)\nu_v(r-y) g_v(u-y) \\ &= \beta_v(u) + \sum_{r=0}^{\infty} \beta_v(r)\tau_v(u,r), \end{aligned}$$

since $g_v(0) = 1 - \phi_v$. Thus, we have shown that (5.27) holds true with $\tau_v(u,r)$ given by (5.29). $\square$

It is instructive to note that the form of solution in (5.27) is convenient to study the ruin related quantities with a proper choice of the penalty function since in the solution, the penalty function only appears in the function $\beta_v(u)$ in (5.28). The assumption (5.26) is explicitly considered in detail in connection with discrete K_n interclaim times in Section 5.3. For the remainder of this paper, we shall assume that (5.26) holds as it is true probabilistically in general. For further details regarding this issue in a continuous-time model, see Cheung (2009). Also, the solution (5.27) is a more appealing form in the sense that it distinguishes the contribution on the penalty function based on whether ruin occurs on the first claim or the subsequent ones. As we know, based on our previous discussion, the time of ruin and the surplus prior to ruin are directly linked when ruin event happens on the first claim.

In the next subsection, with a proper choice of the penalty function, we demonstrate how to derive the general forms of discounted joint and marginal pf of ruin related quantities. In other words, to obtain the following results, no specific assumptions on the interclaim times or the claim sizes are assumed.

5.2.3 Discounted probability functions of ruin related quantities

First, the discounted joint pf of $U(T-1)$, $|U(T)|$ and $R_{N(T)-1}$ is readily obtained from Proposition 2 as follows. Note that (5.3) may be viewed as an expectation of the penalty function so it may be expressed using the discounted joint pfs (5.12) and (5.13) as

$$m_v(u) = \sum_{y=1}^{\infty}\sum_{x=u}^{\infty} w(x,y,u)h_{1,v}(x,y|u) + \sum_{y=1}^{\infty}\sum_{r=0}^{\infty}\sum_{x=r}^{\infty} w(x,y,r)h_{2,v}(x,y,r|u). \tag{5.32}$$

Using (5.28) and comparing the above expression to (5.27), one obtains

$$h_{2,v}(x,y,r|u) = h_{1,v}(x,y|r)\tau_v(u,r), \qquad x = r, r+1, \ldots. \tag{5.33}$$

Therefore (5.29) may also be interpreted as the discounted transition function in a surplus from u to r.

Also, if $w_{123}(x,y,z) = s_1^x s_2^y s_3^z$, then using (5.25) with $l_{v,123}(u)$ in (5.22) given by

$$l_{v,123}(u) = \sum_{y=1}^{\infty}\sum_{x=u}^{\infty} s_1^x s_2^y s_3^u h_v(x-u, y+u|0)$$

in this case, one finds the joint generating function

$$E\left[v^T s_1^{U(T-1)} s_2^{|U(T)|} s_3^{X_T} \mathrm{I}\left(T<\infty\right) \big| U(0)=u\right]$$
$$=\frac{1}{1-\phi_v}\sum_{z=0}^{u} g_v(u-z) l_{v,123}(z)=\sum_{z=0}^{u}\sum_{y=1}^{\infty}\sum_{x=z}^{\infty} s_1^x s_2^y s_3^z\left\{\frac{g_v(u-z)}{1-\phi_v}h_v(x-z,y+z|0)\right\}.$$

Thus, by the uniqueness of the generating function, it follows that the discounted joint pf of $(U(T-1),|U(T)|,X_{N(T)})$ at (x,y,z) is given by

$$h_{3,v}(x,y,z|u)=\frac{g_v(u-z)}{1-\phi_v}h_v(x-z,y+z|0), \qquad x=z,z+1,\ldots,y\in\mathbb{N}^+, z=0,1,\ldots,u.$$

Also, we obtain the joint generating function of the time of ruin T and the last ladder height $|U(T)|+X_T$ with a choice of $w_{23}(y,z)=s^{y+z}$ by using (5.25) with $l_{v,23}(u)$ in (5.23) given by $\phi_v\sum_{y=u+1}^{\infty} s^y f_v(y)$, that is

$$E\left[v^T s^{|U(T)|+X_T} \mathrm{I}\left(T<\infty\right) \big| U(0)=u\right]$$
$$=\frac{1}{1-\phi_v}\sum_{z=0}^{u} g_v(u-z) l_{v,23}(z)=\sum_{z=0}^{u}\sum_{y=z+1}^{\infty} s^y\left\{\frac{\phi_v g_v(u-z)}{1-\phi_v}f_v(y)\right\}.$$

An interchange of the order of summation in the above equation followed by equating coefficients of s^y yields the discounted pf of the last ladder height $|U(T)|+X_T$, namely,

$$f_{1,v}(u,y)=\begin{cases}\frac{\phi_v}{1-\phi_v}\left\{\overline{G}_v(u-y)-\overline{G}_v(u)\right\}f_v(y), & y=1,2,\ldots,u\\ \frac{\phi_v}{1-\phi_v}\left\{1-\overline{G}_v(u)\right\}f_v(y), & y=u+1,u+2,\ldots\end{cases}$$

since $\sum_{z=0}^{y-1} g_v(u-z)=\overline{G}_v(u-y)-\overline{G}_v(u)$ and $\sum_{z=0}^{u} g_v(u-z)=1-\overline{G}_v(u)$.

Furthermore, the joint pf of the last pair of the interclaim time and the claim size, i.e. $(W_{N(T)},Y_{N(T)})$ is obtainable as well. Recall that the last interclaim time is $W_{N(T)}=U(T-1)-R_{N(T)-1}+1$ and the claim causing ruin is $Y_{N(T)}=U(T-1)+|U(T)|+1$. In this case, if $w(x,y,r)=s_1^{x-r+1}s_2^{x+y+1}$, then $\beta_v(u)$ in (5.28) with (5.11) and (5.12) followed by changing a variable from $(x-u+1)$ to t leads to

$$\sum_{x=u}^{\infty}\sum_{y=1}^{\infty} s_1^{x-u+1}s_2^{x+y+1}\left\{v^{x-u+1}k(x-u+1)p_{x-u+1}(x+y+1)\right\} = \sum_{t=1}^{\infty}\sum_{y=u+t+1}^{\infty} s_1^t s_2^y\left\{v^t k(t)p_t(y)\right\}.$$

With the above $\beta_v(u)$, the bivariate generating function of $(W_{N(T)}, Y_{N(T)})$ is obtainable from (5.27) as

$$\begin{aligned}
E&\left[v^T s_1^{W_{N(T)}} s_2^{Y_{N(T)}} \mathrm{I}\left(T<\infty\right)\middle| U(0)=u\right] = \beta_v(u) + \sum_{r=0}^{\infty}\beta_v(r)\tau_v(u,r)\\
&= \sum_{t=1}^{\infty}\sum_{y=u+t+1}^{\infty} s_1^t s_2^y\left\{v^t k(t)p_t(y)\right\} + \sum_{r=0}^{\infty}\sum_{t=1}^{\infty}\sum_{y=r+t+1}^{\infty} s_1^t s_2^y\left\{v^t k(t)p_t(y)\tau_v(u,r)\right\} \qquad (5.34)\\
&= \sum_{t=1}^{\infty}\sum_{y=u+t+1}^{\infty} s_1^t s_2^y\left\{v^t k(t)p_t(y)\right\} + \sum_{t=1}^{\infty}\sum_{y=t+1}^{\infty}\sum_{r=0}^{y-t-1} s_1^t s_2^y\left\{v^t k(t)p_t(y)\tau_v(u,r)\right\},
\end{aligned}$$

and thus, the discounted joint pf of the last interclaim time $W_{N(T)}$ and the claim causing ruin $Y_{N(T)}$ is given by

$$h_{4,v}(t,y|u) = v^t k(t)p_t(y)\left\{\mathrm{I}(y\geq u+t+1) + \mathrm{I}(y\geq t+1)\sum_{r=0}^{y-t-1}\tau_v(u,r)\right\}, \qquad t\in\mathbb{N}^+.$$

For the generating function of $W_{N(T)}$, (5.34) with $s_2=1$ becomes

$$E\left[v^T s_1^{W_{N(T)}}\mathrm{I}\left(T<\infty\right)\middle| U(0)=u\right] = \sum_{t=1}^{\infty} s_1^t\left\{v^t k(t)\overline{P}_t(u+t) + v^t k(t)\sum_{r=0}^{\infty}\overline{P}_t(r+t)\tau_v(u,r)\right\},$$

and in turn we obtain the discounted pf of the last interclaim time $W_{N(T)}$ given by

$$h_{5,v}(t|u) = v^t k(t)\left\{\overline{P}_t(u+t) + \sum_{r=0}^{\infty}\overline{P}_t(r+t)\tau_v(u,r)\right\}, \qquad t\in\mathbb{N}^+.$$

Similarly, the discounted pf of the claim causing ruin $Y_{N(T)}$ can be derived with $s_1=1$ in (5.34). The details are omitted here. For reference, Li (2005b) studied joint and marginal distributions of the claim causing ruin together with the surplus before ruin and the deficit at ruin assuming a discrete K_n distribution for the interclaim times. We also assume this specific class in the following section.

5.3 A class of discrete K_n distributions

To begin, we introduce a class of discrete K_n (or Coxian) distributions for the interclaim times as follows. In Willmot (1993), the mixed Poisson connection between the classical continuous-time compound Poisson model and the discrete-time compound binomial model was discussed in detail. Similarly, to determine the representation of a discrete K_n distribution, we utilize the structure of a truncated mixed Poisson distribution when the mixing distribution is in the class of continuous K_n distributions.

First, we define a class of discrete K_n family distribution for interclaim times having pgf which is a ratio of two polynomials of order n given by (e.g. Li (2005a,b))

$$\hat{k}(s) = \frac{s\varepsilon(s)}{\prod_{i=1}^{m}(1-sq_i)^{n_i}}, \tag{5.35}$$

where $0 < q_i < 1$ for $i = 1, 2, \ldots, m$ with $q_i \neq q_j$ for $i \neq j$. Also, n_i is a nonnegative integer for $i = 1, 2, \ldots, m$, and $n = \sum_{i=1}^{m} n_i > 0$, while $\varepsilon(s)$ is a polynomial of degree $n-1$ or less (the denominator of (5.35) is a polynomial of degree n). It contains many distributions as special cases, for instance, if $m = n = 1$, $q_i = q$ and $\varepsilon(s) = 1-q$, then it is a shifted or truncated geometric distribution with pgf $\hat{k}(s) = s(1-q)/(1-sq)$. In other words, the claim number process $\{N(t); t \geq 0\}$ reduces to a binomial process which is further studied at the end of this section.

Now, to recover a class of discrete K_n distributions with pgf in the form of (5.35), let us define the pgf of the truncated mixed Poisson probabilities as

$$P(s) = \frac{\widetilde{a}(1-s) - \widetilde{a}(1)}{1-\widetilde{a}(1)}, \tag{5.36}$$

where $\widetilde{a}(s) = \int_0^\infty e^{-st} dA(t)$ and the associated mixing df $A(t)$ belongs to the continuous K_n class of distributions with Laplace transform (e.g. Willmot and Woo (2010, Equation 1 with $q_i = (1+\lambda_i)^{-1}$))

$$\widetilde{a}(s) = \frac{\zeta(s)}{\prod_{i=1}^m (1 - q_i + sq_i)^{n_i}}, \tag{5.37}$$

with $\zeta(s)$ is a polynomial of degree $n-1$ or less. With (5.37), (5.36) can be represented as

$$P(s) = \frac{\zeta(1-s) - \zeta(1)\prod_{i=1}^m (1-sq_i)^{n_i}}{C\prod_{i=1}^m (1-sq_i)^{n_i}}, \tag{5.38}$$

where C is a constant given by $C = 1 - \zeta(1)$. Since $P(s)$ is the truncated pgf at zero (i.e. $P(0) = 0$), (5.38) may be rewritten as

$$P(s) = \frac{s\Upsilon(s)}{\prod_{i=1}^m (1-sq_i)^{n_i}},$$

where $\Upsilon(s)$ a polynomial of degree $n-1$ or less, which is the same form of (5.35).

As just shown, a discrete distribution $k(t)$ with the pgf (5.35) is the truncated mixed Poisson when the mixing distribution is in a class of continuous Coxian distributions. Thus, using the results in Willmot and Woo (2010), one finds the truncated mixed Poisson probabilities, namely pf of the interclaim times $k(t)$, can be obtained as

$$k(t) = \sum_{i=1}^m \sum_{j=1}^{n_i} a^*_{i,j} \frac{(1-q_i)^j q_i^t}{1-(1-q_i)^j} \binom{j+t-1}{t}, \qquad t = 1, 2, \ldots, \tag{5.39}$$

where $a^*_{i,j}$ are constants. Therefore, a discrete K_n class may be viewed in terms of finite combinations of truncated negative binomial (Pascal) distributions. The pgf (5.35) is thus given by

$$\hat{k}(s) = \sum_{i=1}^{m} \sum_{j=1}^{n_i} a_{i,j}^{**} \frac{1-(1-sq_i)^j}{(1-sq_i)^j},$$

where $a_{i,j}^{**}$ are constants.

To begin the analysis, we consider the time-independent claim sizes (i.e. $p_t(y) = p(y)$) and condition on the time and the amount of the first claim. The recursive expression for $m_v(u)$ in (5.3) may be obtained as

$$m_v(u) = \beta_v(u) + \sum_{t=1}^{\infty} v^t \sigma_v(u+t)k(t), \qquad u \in \mathbb{N}\ , \tag{5.40}$$

where

$$\beta_v(u) = \sum_{t=1}^{\infty} \sum_{y=u+t+1}^{\infty} v^t w(u+t-1, y-u-t, u)p(y)k(t), \tag{5.41}$$

and

$$\sigma_v(x) = \sum_{y=1}^{x} m_v(x-y)p(y). \tag{5.42}$$

The above $\beta_v(u)$ may be regarded as the contribution to the penalty function due to ruin on the first claim, which is the same as (5.28) from (5.11) and (5.12). Because the second term on the right-hand side of (5.40) is in the form of (5.9) with $\omega_t(u) = \sigma_v(u)$ being independent of t, using (5.10) one obtains the generating function for $m_v(u)$ in (5.40), namely $\hat{m}_v(s) = \sum_{u=0}^{\infty} s^u m_v(u)$ as

$$\hat{m}_v(s) = \hat{\beta}_v(s) + \hat{\sigma}_v(s)\hat{k}\left(\frac{v}{s}\right) - \hat{\sigma}_{*,v}(s), \tag{5.43}$$

where $\hat{\beta}_v(s) = \sum_{u=0}^{\infty} s^u \beta_v(u)$, $\hat{\sigma}_v(s) = \sum_{u=1}^{\infty} s^u \sigma_v(u)$ and

$$\hat{\sigma}_{*,v}(s) = \sum_{t=1}^{\infty} \sum_{u=0}^{t-1} s^{u-t} v^t k(t)\sigma_v(u).$$

In this case, using (5.39) followed by a interchange of summation yields

$$\begin{aligned}\hat{\sigma}_{*,v}(s) &= \sum_{t=1}^{\infty}\sum_{u=0}^{t-1} s^{u-t}v^t\sigma_v(u)\left\{\sum_{i=1}^{m}\sum_{j=1}^{n_i} a_{i,j}^{*}\frac{(1-q_i)^j q_i^t}{1-(1-q_i)^j}\binom{j+t-1}{t}\right\} \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n_i} a_{i,j}^{**}\sum_{u=0}^{\infty} s^u\sigma_v(u)\left\{\sum_{t=u+1}^{\infty}\left(\frac{vq_i}{s}\right)^t\binom{j+t-1}{t}\right\}, \qquad (5.44)\end{aligned}$$

where $a_{i,j}^{**} = a_{i,j}^{*}(1-q_i)^j\left\{1-(1-q_i)^j\right\}^{-1}$. To simplify the right-hand side on the above equation, first note that (see Klugman et al. (2008, p.154))

$$\begin{aligned}\sum_{t=u+1}^{\infty}\left(\frac{vq_i}{s}\right)^t\binom{j+t-1}{t} &= \left(1-\frac{vq_i}{s}\right)^{-j}\frac{\frac{vq_i}{s}}{1-\frac{vq_i}{s}}\sum_{k=1}^{j}\left(1-\frac{vq_i}{s}\right)^k\left(\frac{vq_i}{s}\right)^u\binom{u+k-1}{u} \\ &= \left(1-\frac{vq_i}{s}\right)^{-j}\sum_{k=0}^{j-1}\left(1-\frac{vq_i}{s}\right)^k\left(\frac{vq_i}{s}\right)^{u+1}\binom{u+k}{u}.\end{aligned}$$

Then using the above result, (5.44) can be rewritten as

$$\hat{\sigma}_{*,v}(s) = \sum_{i=1}^{m}\sum_{j=1}^{n_i}\sum_{k=0}^{j-1} a_{i,j}^{**}\sigma_v(i,k)\frac{\frac{1}{s}}{\left(1-\frac{vq_i}{s}\right)^{j-k}} = \sum_{i=1}^{m}\sum_{j=1}^{n_i}\theta_{i,j}\frac{\frac{1}{s}}{\left(1-\frac{vq_i}{s}\right)^{j}},$$

where $\sigma_v(i,k) = \sum_{u=0}^{\infty}\sigma_v(u)(vq_i)^{u+1}\binom{u+k}{u}$ and $\theta_{i,j} = \sum_{k=j}^{n_i} a_{i,j}^{**}\sigma_v(i,k-j)$. In other words, one obtains the representation of (5.44) as

$$\hat{\sigma}_{*,v}(s) = \sum_{i=1}^{m}\sum_{j=1}^{n_i}\theta_{i,j}\frac{s^{j-1}}{(s-vq_i)^j}, \qquad (5.45)$$

where $\theta_{i,j}$ are constants.

Using $\hat{\sigma}_v(s) = \hat{m}_v(s)\hat{p}(s)$ from (5.42) and (5.45) followed by rearranging (5.43) yields

$$\hat{m}_v(s)\left\{1-\hat{p}(s)\hat{k}(v/s)\right\} = \hat{\beta}_v(s) - \frac{Q_v(s)}{\prod_{k=1}^{m}(s-vq_k)^{n_k}}, \qquad (5.46)$$

where

$$Q_v(s) = \left\{ \prod_{k=1}^{m} (s - vq_k)^{n_k} \right\} \sum_{i=1}^{m} \sum_{j=1}^{n_i} \theta_{i,j} \frac{s^{j-1}}{(s - vq_i)^j}, \tag{5.47}$$

is a polynomial in s of degree $n-1$ or less. Li (2005a, Theorem 1) showed that the Lundberg equation (5.8), namely $\hat{p}(s)\hat{k}(v/s) = 1$, has exactly n solutions $\rho_1, \rho_2, \ldots, \rho_n$ with $0 < |\rho_j| < 1$ for $0 < v < 1$. In what follows, we also assume that these roots are distinct. Then, by the theory of Lagrange polynomials, (5.47) may be expressed as

$$Q_v(s) = \sum_{i=1}^{n} \left\{ \hat{\beta}_v(\rho_i) \prod_{k=1}^{m} (\rho_i - vq_k)^{n_k} \right\} \prod_{j=1, j\neq i}^{n} \left(\frac{s - \rho_j}{\rho_i - \rho_j} \right),$$

where it is assumed that $\hat{m}_v(\rho_i) < \infty$ for $i = 1, 2, \ldots, n$. Using the above expression for $Q_v(s)$ and multiplying (5.46) by $\prod_{k=1}^{m}(s - vq_k)^{n_k}$ results in

$$\begin{aligned} &\hat{m}_v(s) \left\{ \prod_{k=1}^{m} (s - vq_k)^{n_k} \right\} \left\{ 1 - \hat{p}(s)\hat{k}(v/s) \right\} \\ &= \hat{\beta}_v(s) \left\{ \prod_{k=1}^{m} (s - vq_k)^{n_k} \right\} - \sum_{i=1}^{n} \left\{ \hat{\beta}_v(\rho_i) \prod_{k=1}^{m} (\rho_i - vq_k)^{n_k} \right\} \prod_{j=1, j\neq i}^{n} \left(\frac{s - \rho_j}{\rho_i - \rho_j} \right). \end{aligned} \tag{5.48}$$

From Lemma 1 in Li (2005a), we know that the defective pf $\phi_v f_v(y)$ has generating function given by

$$\phi_v \hat{f}_v(s) = 1 - \frac{\prod_{k=1}^{m}(s - vq_k)^{n_k}}{\left(\prod_{l=1}^{n} \frac{vq_l}{\rho_l}\right) \prod_{i=1}^{n}(s - \rho_i)} \left\{ 1 - \hat{p}(s)\hat{k}(v/s) \right\},$$

and thus (5.48) may be expressed as

$$\hat{m}_v(s) \left\{ 1 - \phi_v \hat{f}_v(s) \right\} = \hat{l}_v(s),$$

where

$$\hat{l}_v(s) = \frac{\hat{\beta}_v(s) \prod_{k=1}^{m}(s - vq_k)^{n_k}}{\left(\prod_{l=1}^{n} \frac{vq_l}{\rho_l}\right) \prod_{i=1}^{n}(s - \rho_i)} - \sum_{i=1}^{n} \frac{\hat{\beta}_v(\rho_i) \prod_{k=1}^{m}(\rho_i - vq_k)^{n_k}}{\left(\prod_{l=1}^{n} \frac{vq_l}{\rho_l}\right) (s - \rho_i) \prod_{j=1, j\neq i}^{n}(\rho_i - \rho_j)}. \tag{5.49}$$

Then, similar to Willmot and Woo (2010) we can find the expression for $\prod_{k=1}^{m}(s-vq_k)^{n_k}/\prod_{i=1}^{n}(s-\rho_i)$ as

$$\frac{\prod_{k=1}^{m}(s-vq_k)^{n_k}}{\prod_{i=1}^{n}(s-\rho_i)} = 1+\sum_{i=1}^{n}\frac{\prod_{k=1}^{m}(\rho_i-vq_k)^{n_k}}{(s-\rho_i)\prod_{j=1,j\neq i}^{n}(\rho_i-\rho_j)}.$$

Substitution of this expression into the first term on the right-hand side of (5.49) results in

$$\hat{l}_v(s) = \left(\prod_{l=1}^{n}\frac{\rho_l}{vq_l}\right)\left[\hat{\beta}_v(s)+\sum_{i=1}^{n}\frac{\prod_{k=1}^{m}(\rho_i-vq_k)^{n_k}}{\prod_{j=1,j\neq i}^{n}(\rho_i-\rho_j)}\left\{\frac{\hat{\beta}_v(s)-\hat{\beta}_v(\rho_i)}{s-\rho_i}\right\}\right].$$

By the uniqueness of the generating function and the property of the discrete Dickson-Hipp operator in (1.8), we identify $l_v(u)$ in (5.20) as

$$l_v(u) = \left(\prod_{l=1}^{n}\frac{\rho_l}{vq_l}\right)\left\{\beta_v(u)+\sum_{i=1}^{n}a_i^*\mathcal{T}_{\rho_i}\beta_v(u+1)\right\}, \tag{5.50}$$

where

$$a_i^* = \frac{\prod_{k=1}^{m}(\rho_i-vq_k)^{n_k}}{\prod_{j=1,j\neq i}^{n}(\rho_i-\rho_j)}. \tag{5.51}$$

As in Willmot and Woo (2010) who studied a continuous-time risk model with a class of Coxian interclaim time distributions, now we recover the case of the classical Gerber-Shiu function in (5.1) studied by Li (2005a) in the present discrete model. If $w(w,y,r) = w_{12}(x,y)$, then $\beta_v(u)$ in (5.41) reduces to

$$\beta_{v,12}(u) = \sum_{t=1}^{\infty}v^t\alpha_{12}(u+t)k(t),$$

where

$$\alpha_{12}(x) = \sum_{y=x+1}^{\infty}w_{12}(x-1,y-x)p(y) = \sum_{y=1}^{\infty}w_{12}(x-1,y)p(x+y), \qquad x\in\mathbb{N}^+. \tag{5.52}$$

Since $\beta_{v,12}(u)$ is again of the form (5.9), using (5.10) and (5.45) with $\alpha_{12}(u)$ in place of $\sigma_v(u)$ yields its generating function as

$$\hat{\beta}_{v,12}(s) = \hat{\alpha}_{12}(s)\hat{k}\left(\frac{v}{s}\right)-\sum_{i=1}^{m}\sum_{j=1}^{n_i}\theta_{i,j}^*\frac{s^{j-1}}{(s-vq_i)^j},$$

where $\theta^*_{i,j}$ are constants. Then, using (5.35) and (5.47), we obtain

$$\hat{\beta}_{v,12}(s)\left\{\prod_{k=1}^{m}(s-vq_k)^{n_k}\right\} = \hat{\alpha}_{12}(s)Q_1^*(s) - Q_2^*(s), \tag{5.53}$$

where

$$Q_1^*(s) = \left\{\prod_{k=1}^{m}(s-vq_k)^{n_k}\right\}\hat{k}\left(\frac{v}{s}\right), \tag{5.54}$$

and $Q_2^*(s)$ are both polynomials of degree $n-1$ or less. Thus, again by the theory of Lagrange polynomials, $Q_k^*(s)$ for $k=1,2$ may be expressed as

$$Q_k^*(s) = \sum_{i=1}^{n} Q_k^*(\rho_i)\prod_{j=1,j\neq i}^{n}\left(\frac{s-\rho_j}{\rho_i-\rho_j}\right). \tag{5.55}$$

Then, substituting (5.53) into (5.49) followed by application of (5.55), it follows that

$$\begin{aligned}\hat{l}_{v,12}(s) &= \frac{\hat{\alpha}_{12}(s)Q_1^*(s)-Q_2^*(s)}{\left(\prod_{l=1}^{n}\frac{vq_l}{\rho_l}\right)\prod_{i=1}^{n}(s-\rho_i)} - \sum_{i=1}^{n}\frac{\hat{\alpha}_{12}(\rho_i)Q_1^*(\rho_i)-Q_2^*(\rho_i)}{\left(\prod_{l=1}^{n}\frac{vq_l}{\rho_l}\right)(s-\rho_i)\prod_{j=1,j\neq i}^{n}(\rho_i-\rho_j)} \\ &= \left(\prod_{l=1}^{n}\frac{\rho_l}{vq_l}\right)\sum_{i=1}^{n}\frac{Q_1^*(\rho_i)}{\prod_{j=1,j\neq i}^{n}(\rho_i-\rho_j)}\left\{\frac{\hat{\alpha}_{12}(s)-\hat{\alpha}_{12}(\rho_i)}{s-\rho_i}\right\}\end{aligned}$$

Hence, by the uniqueness of the generating function and (1.8) one finds

$$l_{v,12}(u) = \left(\prod_{l=1}^{n}\frac{\rho_l}{vq_l}\right)\sum_{i=1}^{n} b_i\mathcal{T}_{\rho_i}\alpha_{12}(u+1),$$

where $b_i = Q_1^*(\rho_i)/\prod_{j=1,j\neq i}^{n}(\rho_i-\rho_j)$ is same as $a_i^*\hat{k}(v/\rho_i)$ with a_i^* given by (5.51) due to (5.54). This result agrees with Equation 37 in Li (2005a). Also, if $w_{12}(x,y)=1$, then from (5.24) we know that $l_{v,12}(u) = \phi_v\overline{F}_v(u)$ which is

$$\phi_v\overline{F}_v(u) = \left(\prod_{l=1}^{n}\frac{\rho_l}{vq_l}\right)\sum_{i=1}^{n} b_i\mathcal{T}_{\rho_i}\overline{P}(u+1) = \left(\prod_{l=1}^{n}\frac{\rho_l}{vq_l}\right)\sum_{i=1}^{n} b_i\sum_{x=0}^{\infty}\rho_i^x\overline{P}(x+u+1), \qquad u\in\mathbb{N},$$

since $\alpha_{12}(x) = \sum_{y=1}^{\infty}p(x+y) = \overline{P}(x)$ from (5.52). Because $f_v(y) = \overline{F}_v(y-1)-\overline{F}_v(y)$, we immediately get

$$\phi_v f_v(y) = \phi_v \left(\prod_{l=1}^{n} \frac{\rho_l}{vq_l}\right) \sum_{i=1}^{n} b_i \sum_{x=0}^{\infty} \rho_i^x p(x+y+1) = \phi_v \left(\prod_{l=1}^{n} \frac{\rho_l}{vq_l}\right) \sum_{i=1}^{n} b_i \mathcal{T}_{\rho_i} p(y+1), \qquad y \in \mathbb{N}^+ ,$$

which agrees with Equation 26 in Li (2005a).

Furthermore, as we know, to analyze (5.2) we need to derive a (discounted) joint pf of $U(T-1), |U(T)|$ and $R_{N(T)-1}$ with a zero initial surplus as follows. From (5.21) and (5.28) at $u=0$, it is obvious that

$$l_v(0) = \beta_v(0) + \sum_{y=1}^{\infty} \sum_{x=0}^{\infty} \sum_{r=0}^{x} w(x,y,r) h_{2,v}(x,y,r|0).$$

Comparing the above result with (5.50) at $u=0$ yields

$$\sum_{r=0}^{\infty} \sum_{x=r}^{\infty} \sum_{y=1}^{\infty} w(x,y,r) h_{2,v}(x,y,r|0) = \left(\prod_{l=1}^{n} \frac{\rho_l}{vq_l} - 1\right) \beta_v(0) + \left(\prod_{l=1}^{n} \frac{\rho_l}{vq_l}\right) \sum_{i=1}^{n} a_i^* \mathcal{T}_{\rho_i} \beta_v(1), \qquad (5.56)$$

and using (5.28), the right-hand side of (5.56) can be rewritten as

$$\begin{aligned} &\sum_{x=0}^{\infty} \sum_{y=1}^{\infty} w(x,y,0) \left(\prod_{l=1}^{n} \frac{\rho_l}{vq_l} - 1\right) h_{1,v}(x,y|0) \\ &\quad + \sum_{r=0}^{\infty} \sum_{x=r+1}^{\infty} \sum_{y=1}^{\infty} w(x,y,r+1) \left(\prod_{l=1}^{n} \frac{\rho_l}{vq_l}\right) \sum_{i=1}^{n} a_i^* \rho_i^r h_{1,v}(x,y|r+1). \end{aligned} \qquad (5.57)$$

If $w(x,y,r) = s_1^x s_2^y s_3^r$, then equating coefficients of $s_1^x s_2^y s_3^r$ in the left-hand side of (5.56) and (5.57) results in

$$h_{2,v}(x,y,r|0) = \left\{ \mathrm{I}(r=0) \left(\prod_{l=1}^{n} \frac{\rho_l}{vq_l} - 1\right) + \mathrm{I}(r \neq 0) \left(\prod_{l=1}^{n} \frac{\rho_l}{vq_l}\right) \sum_{i=1}^{n} a_i^* \rho_i^{r-1} \right\} h_{1,v}(x,y|r), \qquad (5.58)$$

where $h_{1,v}(x,y|r)$ is given by (5.12) with (5.11) and (5.39) in this case.

Thus, under the discrete K_n interclaim time distribution, (5.26) is satisfied according to (5.58) with the obvious $\nu_v(r)$ in the bracket, and we can apply (5.29) in Proposition 1 to obtain the discounted transition function $\tau_v(u, r)$. Therefore, from (5.33) along with such an expression for $\tau_v(u, r)$, one readily obtains the discounted joint pf of $U(T-1), |U(T)|$ and $R_{N(T)-1}$ with an arbitrary initial surplus (i.e. $h_{2,v}(x, y, r|u)$) as well. Certainly, any joint or marginal pfs studied in Section 5.2.3 can be founded in this model by applying the general results therein.

Special case : Compound binomial process

Here, we consider the compound binomial process as a special case of the discrete K_n class. This is a discrete analogue of the classical compound Poisson process. In this case, the interclaim times follow a zero-truncated geometric distribution $k(t) = (1-q)q^{t-1}$ for $t = 1, 2, \ldots$, with pgf $\hat{k}(s) = s(1-q)/(1-sq)$. Denote the unique positive root of the Lundberg equation in (5.8) to be ρ (e.g. Li (2005a)). Then, in this case, a_1^* in (5.51) reduces to $\rho - vq$ and from (5.50) the Gerber-Shiu function in (5.3) satisfies the recursive expression (5.20) with

$$l_v(u) = \frac{\rho}{vq} \left\{ \beta_v(u) + (\rho - vq) \mathcal{T}_\rho \beta_v(u+1) \right\}.$$

Also, the discounted joint pf of $U(T-1), |U(T)|$ and $R_{N(T)-1}$ in (5.58) simplifies to

$$h_{2,v}(x, y, r|0) = \left(\frac{\rho}{vq} - 1 \right) \left[\mathrm{I}(r=0) + \mathrm{I}(r \neq 0)\rho^r \right] h_{1,v}(x, y|r).$$

As mentioned earlier, this function determines the analysis of the generalized Gerber-Shiu function in (5.2) and its special cases.

5.4 Discrete delayed renewal risk process

As in Chapter 4, with the same assumptions considered in Section 5.1, but assuming the process has been running for some time before it is first observed, then the interclaim time for the first event is assumed to be different from the subsequent ones. Without the assumption of time-dependent claim size, this delayed model was considered by Alfa and Drekic (2008) and Drekic and Mera (2010) using matrix algorithms to compute various joint probabilities of ruin related quantities.

In this model, let us assume that the pf of the first interclaim time is $k_1(t)$ and the joint pf of the first pair (W_1, Y_1) is defined as $k_1(t)p_{1,t}(y)$ where $p_{1,t}(y)$ is a conditional pf of $Y_1 = y$ given $W_1 = t$ which implies the dependency structure of the first claim size is different from the subsequent ones. Corresponding to (5.2) and (5.3) in the discrete ordinary model, we consider a generalization of the classical Gerber-Shiu function in the present model

$$m^*_{d,v}(u) = E\left[v^{T_d} w^*\left(U(T_d-1), |U(T_d)|, X_{T_d}, R_{N(T_d)-1}\right) \mathrm{I}\left(T_d < \infty\right) \big| U(0) = u\right], \qquad u \in \mathbb{N}\,, \tag{5.59}$$

where T_d is the time of ruin in the delayed model. If $w^*(x,y,z,r) = w(x,y,r)$, (5.59) reduces

$$m_{d,v}(u) = E\left[v^{T_d} w\left(U(T_d-1), |U(T_d)|, R_{N(T_d)-1}\right) \mathrm{I}\left(T_d < \infty\right) \big| U(0) = u\right], \qquad u \in \mathbb{N}\,. \tag{5.60}$$

As in Chapter 4, the structural results of the generalized Gerber-Shiu functions (5.59) and (5.60) in a discrete delayed renewal risk model can be derived in terms of the discrete ordinary renewal risk model studied in Section 5.2.1.

First, the joint defective pf of the surplus prior to ruin (x) and the deficit at ruin (y) when ruin occurs on the first claim (i.e. $N(T_d) = 1$) is defined as

$$h^d_1(x, y|u) = k_1(x-u+1)p_{1,x-u+1}(x+y+1), \qquad x \in \mathbb{N}, y \in \mathbb{N}^+ \tag{5.61}$$

where $T_d = x - u + 1$ and $R_{N(T_d)-1} = u$. For other cases (i.e. $N(T_d) \geq 2$), the joint defective pf of $(T_d, U(T_d-1), |U(T_d)|, R_{N(T_d)-1})$ at (t, x, y, r) is denoted by $h^d_2(t, x, y, r|u)$ for $t = 2, 3, \ldots$, $x, r \in \mathbb{N}$ and $y \in \mathbb{N}^+$. The discounted joint pf of these h^d_1 and h^d_2 are assumed respectively to be

$$h^d_{1,v}(x, y|u) = v^{x-u+1} h^d_1(x, y|u), \tag{5.62}$$

and

$$h^d_{2,v}(x,y,r|u) = \sum_{t=2}^{\infty} v^t h^d_2(t,x,y,r|u). \tag{5.63}$$

Again, conditioning on the first drop below the initial surplus level u, we may obtain the following recursive formula for (5.59) involving (5.2) with (5.62) and (5.63)

$$m^*_{d,v}(u) = \sum_{y=1}^{u} m^*_v(u-y)\left\{\sum_{x=0}^{\infty} h^d_{1,v}(x,y|0) + \sum_{x=0}^{\infty}\sum_{r=0}^{x} h^d_{2,v}(x,y,r|0)\right\} + l^*_{d,v}(u), \tag{5.64}$$

where

$$l^*_{d,v}(u) = \sum_{y=u+1}^{\infty}\sum_{x=0}^{\infty}\left\{w^*(x+u,y-u,u,u)h^d_{1,v}(x,y|0) + \sum_{r=0}^{x} w^*(x+u,y-u,u,r+u)h^d_{2,v}(x,y,r|0)\right\}, \tag{5.65}$$

which is the case when ruin occurs on the first drop. From (5.62) and (5.63), the joint pf of the surplus prior to ruin (x) and the deficit at ruin (y) is obtainable as

$$h^d_v(x,y|u) = h^d_{1,v}(x,y|u) + \sum_{r=0}^{x} h^d_{2,v}(x,y,r|u). \tag{5.66}$$

Using (5.66), (5.64) may be expressed as

$$m^*_{d,v}(u) = \phi_{d,v}\sum_{y=1}^{u} m^*_v(u-y)f_{d,v}(y) + l^*_{d,v}(u),$$

where $\phi_{d,v} = \sum_{x=0}^{\infty}\sum_{y=1}^{\infty} h^d_v(x,y|0)$ and $f_{d,v}(y) = \phi^{-1}_{d,v}\sum_{x=0}^{\infty} h^d_v(x,y|0)$. We note that (5.59) only depends on (5.63) with $u=0$ as in the discrete ordinary model studied in Section 5.2.1. Hence, in order to derive this joint pf, we consider (5.60) in the following. If $w^*(x,y,z,r) = w(x,y,r)$, then (5.65) becomes

$$l_{d,v}(u) = \sum_{y=u+1}^{\infty}\sum_{x=0}^{\infty}\left\{w(x+u,y-u,u)h^d_{1,v}(x,y|0) + \sum_{r=0}^{x} w(x+u,y-u,r+u)h^d_{2,v}(x,y,r|0)\right\},$$

and changing variables of summations yields

$$l_{d,v}(u) = \sum_{y=1}^{\infty}\sum_{x=u}^{\infty}\left\{w(x,y,u)h^d_{1,v}(x,y|u) + \sum_{r=u}^{x} w(x,y,r)h^d_{2,v}(x-u,y+u,r-u|0)\right\}.$$

Next, by conditioning on the time and the amount of the first claim, one finds

$$m_{d,v}(u) = \beta_{d,v}(u) + \sum_{t=1}^{\infty} v^t \sigma_{v,t}(u+t) k_1(t), \tag{5.67}$$

where

$$\beta_{d,v}(u) = \sum_{t=1}^{\infty} \sum_{y=u+t+1}^{\infty} v^t w(u+t-1, y-u-t, u) p_{1,t}(y) k_1(t), \tag{5.68}$$

and

$$\sigma_{v,t}(x) = \sum_{y=0}^{x} m_v(x-y) p_{1,t}(y). \tag{5.69}$$

As in (5.41), (5.68) may be interpreted as the contribution on the penalty function due to ruin on the first claim in the present model. From (5.61) and (5.62), it may be reexpressed as

$$\beta_{d,v}(u) = \sum_{x=u}^{\infty} \sum_{y=1}^{\infty} w(x, y, u) h_{1,v}^{d}(x, y|u). \tag{5.70}$$

Similar to (5.32), using (5.70) and (5.63) we can express (5.60) as

$$m_{d,v}(u) = \beta_{d,v}(u) + \sum_{x=0}^{\infty} \sum_{y=1}^{\infty} \sum_{r=0}^{x} w(x, y, r) h_{2,v}^{d}(x, y, r|u). \tag{5.71}$$

Comparing the above expression to (5.67) with (5.69), one deduces that

$$\sum_{x=0}^{\infty} \sum_{y=1}^{\infty} \sum_{r=0}^{x} w(x, y, r) h_{2,v}^{d}(x, y, r|u) = \sum_{t=1}^{\infty} v^t \sum_{z=0}^{u+t} m_v(z) p_{1,t}(u+t-z) k_1(t). \tag{5.72}$$

Interchanging the order of summation on the right-hand side of (5.72) yields

$$\left(\sum_{z=0}^{u} \sum_{t=1}^{\infty} + \sum_{z=u+1}^{\infty} \sum_{t=z-u}^{\infty} \right) m_v(z) \left\{ v^t p_{1,t}(u+t-z) k_1(t) \right\} = \sum_{z=0}^{\infty} m_v(z) A_v(u, z), \tag{5.73}$$

where

$$A_v(u, z) = \begin{cases} \sum_{t=1}^{\infty} v^t p_{1,t}(u+t-z) k_1(t), & z = 0, 1, \ldots, u \\ \sum_{t=z-u}^{\infty} v^t p_{1,t}(u+t-z) k_1(t), & z = u+1, u+2, \ldots \end{cases}.$$

Using (5.32) and (5.33), it follows that the right-hand side of (5.73) is

$$\begin{aligned}
&\sum_{z=0}^{\infty} m_v(z) A_v(u,z) \\
&= \sum_{z=0}^{\infty} A_v(u,z) \left\{ \sum_{y=1}^{\infty} \sum_{x=z}^{\infty} w(x,y,z) h_{1,v}(x,y|z) + \sum_{y=1}^{\infty} \sum_{r=0}^{\infty} \sum_{x=r}^{\infty} w(x,y,r) h_{1,v}(x,y|r) \tau_v(z,r) \right\}.
\end{aligned} \tag{5.74}$$

Thus, if $w(x,y,r) = s_1^x s_2^y s_3^r$ in (5.72) and (5.74), by the uniqueness of the generating function, we obtain the discounted joint pf of $U(T_d - 1), |U(T_d)|$ and $R_{N(T_d)-1}$ given by

$$h_{2,v}^d(x,y,r|u) = h_{1,v}(x,y|r)\xi_v(u,r), \qquad x = r, r+1, \ldots, y \in \mathbb{N}^+, r \in \mathbb{N} \ ,$$

where

$$\xi_v(u,r) = A_v(u,r) + \sum_{z=0}^{\infty} A_v(u,z)\tau_v(z,r)$$

represents the discounted transition in a surplus from u to r in the discrete delayed process. In turn, a substitution of the above expression of $h_{2,v}^d$ into (5.71) with (5.28), one finds

$$m_{d,v}(u) = \beta_{d,v}(u) + \sum_{r=0}^{\infty} \beta_v(r)\xi_v(u,r).$$

Based on the above expression for (5.60), the parts containing the penalty function are only in the functions $\beta_{d,v}$ and β_v, so that it is more useful for analyzing various ruin related quantities as in Proposition 2 in the case of a discrete ordinary renewal risk process.

Chapter 6

Two-sided bounds for a renewal equation

Many quantities of interest in the study of renewal processes may be expressed as a special type of integral equation known as a renewal equation. The main purpose of this chapter is to provide bounds on the solution of renewal equations based on various reliability classifications. It contains exponential and nonexponential inequalities by depending on the type of renewal equation.

6.1 Introduction

In this section, we derive two-sided bounds for renewal equations. Most of the bounds obtained in the present paper are based on the results developed by Willmot et al. (2001) but all are improved. Let us start with the renewal equation (e.g. Ross (1996), Karlin and Taylor (1975), Tijms (1994), Willmot et al. (2001))

$$m(x) = \phi \int_0^x m(x-y)\, dF(y) + \phi\, r(x), \qquad x \geq 0, \tag{6.1}$$

where $\phi > 0$, $F(y) = 1 - \overline{F}(y)$ is a proper df with $F(0) = 0$ and $r(x) \geq 0$ is locally bounded. As in Willmot et al. (2001), we define $x_0 = \inf\{x : F(x) = 1\}$ and obviously $x_0 = \infty$ if $\overline{F}(x) > 0$ for $x > 0$. The renewal equation (6.1) is said to be proper if $\phi = 1$, defective if $\phi < 1$, and excessive if $\phi > 1$. In particular, for $\phi < 1$ and $r(x) = \overline{F}(x)$, we obtain the special case in which the solution $m(x)$ is the tail of a compound geometric distribution. A wide variety of quantities in insurance risk theory and in applied probability are known to satisfy (defective) renewal equations of the form (6.1). For example, Willmot and Lin (2001) showed that the convolutions of a compound geometric distribution with another random variable may be solution to a defective renewal equation. Within this formulation, they provide examples including ruin model perturbed by a diffusion and an approximation to the equilibrium waiting time distribution in the M/G/c queues. Also, see Feller (1971) and Resnick (1992) for further detailed discussion of the application. The general solution to (6.1) is (e.g. Resnick (1992, Section 3.5))

$$m(x) = \phi r(x) + \sum_{n=1}^{\infty} \phi^{n+1} \int_0^x r(x-y) dF^{*(n)}(y) = \sum_{n=0}^{\infty} \phi^{n+1} (r * F^{*(n)})(x), \qquad x \geq 0, \tag{6.2}$$

where $F^{*(n)}(y)$ is the df of the n-fold convolution of F with itself. We also introduce the Lundberg condition that κ exists such that

$$\int_0^{\infty} e^{\kappa y} dF(y) = \frac{1}{\phi}. \tag{6.3}$$

The main method applied to find tighter bounds in Section 6.2 and Section 6.3 is in the following. Basically, we substitute the existing results given by Willmot et al. (2001) into the integrand on the right side of (6.1) is utilized. Then we may obtain improved bounds by using the Lundberg condition (6.3) and assuming some specific reliability classifications. To find new tighter bounds, we repeat the above procedure, that is, replacement of $m(x-y)$ in the integral term in (6.1) by the bounds obtained in the previous step. By mathematical induction, we show that iteration of the above steps yields better bounds than Willmot et al. (2001). Note that similar ideas to those used here, but only for the exponential bounds, have also appeared in studies regarding specific ruin-related quantities. Chadficonstantinidis and Politis (2007) have studied the two-sided bounds for the distribution of the deficit at ruin in the Sparre Andersen model which improve and generalize similar results studied by Cai and Garrido (1998, 1999), Willmot (2002), Willmot et al. (2001), and Willmot and Lin (2001). Psarrakos and Politis (2008) have also derived improved tail bounds for the joint distribution of the surplus prior to and at ruin in the classical risk model. Furthermore, concerning the approach based on the reliability classification for finding bounds, see Willmot (1994), Willmot and Lin (2001) and references therein.

The following sections are organized as follows. If $\phi \geq 1$ then there is always $\kappa \leq 0$ satisfying (6.3). Then, in this case, it is convenient to find exponential bounds associated with κ which is the subject matter of Section 6.2. For a defective renewal equation (i.e. $\phi < 1$), however, the previous types of bounds are not generally available since we may not find κ satisfying (6.3). Thus, in Section 6.3, we discuss nonexponential bounds by introducing some useful bounding functions provided by Willmot et al. (2001). Finally, in Section 6.4 some examples including various ruin related quantities are provided to illustrate the applications of the results given by the previous sections. These results show the gradual refinement of the two-sided bounds by increasing the number of iterations.

6.2 Exponential bounds

In this section, we establish improved bounds corresponding to the results in Section 3 of Willmot et al. (2001).

First, from Theorem 3.1 in Willmot et al. (2001), it follows that

$$\alpha_L(x)e^{-\kappa x} \leq m(x) \leq \alpha_U(x)e^{-\kappa x}, \tag{6.4}$$

where

$$\alpha(z) = \frac{e^{\kappa z} r(z)}{\int_z^\infty e^{\kappa y} dF(y)}, \qquad z \geq 0,$$

and

$$\alpha_U(x) = \sup_{0\le z\le x,\ \overline{F}(z)>0} \alpha(z), \qquad \alpha_L(x) = \inf_{0\le z\le x,\ \overline{F}(z)>0} \alpha(z).$$

Theorem 3 *Suppose that κ satisfies (6.3). If $r(x) = 0$ for $x \ge x_0$, then for $n = 1, 2, 3\ldots,$*

$$m(x) \le \alpha_U(x)e^{-\kappa x} - \sum_{m=1}^{n} \phi^m (c_U * F^{*(m-1)})(x), \qquad x \ge 0, \tag{6.5}$$

where $c_U(x) = \alpha_U(x)e^{-\kappa x}\int_x^\infty e^{\kappa y}dF(y) - r(x)$. Similarly, for $n = 1, 2, 3\ldots,$

$$m(x) \ge \alpha_L(x)e^{-\kappa x} + \sum_{m=1}^{n} \phi^m (c_L * F^{*(m-1)})(x), \qquad x \ge 0, \tag{6.6}$$

where $c_L(x) = r(x) - \alpha_L(x)e^{-\kappa x}\int_x^\infty e^{\kappa y}dF(y)$.

Proof. First, for the upper bound, we shall show by mathematical induction on n that (6.5) holds true for all $n = 1, 2, 3, \ldots$. For $n = 1$, inserting the upper bound for $m(x)$ in (6.4) into the integrand on the right side of (6.1), we obtain

$$\begin{aligned} m(x) &\le \phi \int_0^x \alpha_U(x-y)e^{-\kappa(x-y)}dF(y) + \phi r(x) \\ &\le \phi\alpha_U(x)e^{-\kappa x}\left\{\frac{1}{\phi} - \int_x^\infty e^{\kappa y}dF(y)\right\} + \phi r(x) \\ &= \alpha_U(x)e^{-\kappa x} - \phi(c_U * F^{*(0)})(x). \end{aligned}$$

Thus, (6.5) holds true for $n = 1$. Assuming that (6.5) holds true for some $n \ge 1$ followed by substitution into the integrad in (6.1) yields

$$
\begin{aligned}
m(x) &\leq \phi \int_0^x \left[\alpha_U(x-y)e^{-\kappa(x-y)} - \sum_{m=1}^{n} \phi^m (c_U * F^{*(m-1)})(x-y) \right] dF(y) + \phi r(x) \\
&\leq \alpha_U(x)e^{-\kappa x} - \phi(c_U * F^{*0})(x) - \sum_{m=1}^{n} \phi^{m+1} \int_0^x (c_U * F^{*(m-1)})(x-y) dF(y) \\
&= \alpha_U(x)e^{-\kappa x} - \sum_{m=1}^{n+1} \phi^m (c_U * F^{*(m-1)})(x).
\end{aligned}
$$

Therefore, (6.5) holds for all $n = 1, 2, 3, \ldots$.

For the lower bound (6.6), we also apply a similar argument to show that it holds true for $n = 1, 2, 3, \ldots$. For $n = 1$,

$$
\begin{aligned}
m(x) &\geq \phi \int_0^x \alpha_L(x-y)e^{-\kappa(x-y)} dF(y) + \phi r(x) \geq \phi \alpha_L(x) e^{-\kappa x} \left\{ \frac{1}{\phi} - \int_x^\infty e^{\kappa y} dF(y) \right\} + \phi r(x) \\
&= \alpha_L(x)e^{-\kappa x} + \phi(c_L * F^{*(0)})(x),
\end{aligned}
$$

and thus (6.6) is true. Assume that (6.6) holds for some $n \geq 1$ and inserting such lower bound into the integrad in (6.1) yields

$$
\begin{aligned}
m(x) &\geq \phi \int_0^x \left\{ \alpha_L(x-y)e^{-\kappa(x-y)} + \sum_{m=1}^{n} \phi^m (c_L * F^{*(m-1)})(x-y) \right\} dF(y) + \phi r(x) \\
&\geq \alpha_L(x)e^{-\kappa x} + \phi(c_L * F^{*(0)})(x) + \sum_{m=1}^{n} \phi^{m+1} \int_0^x (c_L * F^{*(m-1)})(x-y) dF(y) \\
&= \alpha_L(x)e^{-\kappa x} + \sum_{m=1}^{n+1} \phi^m (c_L * F^{*(m-1)})(x).
\end{aligned}
$$

Thus, (6.6) holds for all $n = 1, 2, 3, \ldots$. By mathematical induction, we have the desired result. We remark that the two-sided bounds given by (6.5) and (6.6) are getting tighter as n increases since $c_U(x)$ and $c_L(x)$ are nonnegative by definitions. $\square$

Next, we find the improve bounds corresponding to the results from Corollary 3.1 in Willmot et al. (2001) in the following corollary. First, from Corollary 3.1 in Willmot et al. (2001), we know that

$$\sigma_L(x)\psi_L(x)e^{-\kappa x} \leq m(x) \leq \sigma_U(x)\psi_U(x)e^{-\kappa x}, \qquad x \geq 0, \tag{6.7}$$

where

$$\psi_U(x) = \sup_{0\leq z\leq x, \overline{F}(z)>0} \frac{r(z)}{\overline{F}(z)}, \qquad \psi_L(x) = \inf_{0\leq z\leq x, \overline{F}(z)>0} \frac{r(z)}{\overline{F}(z)}, \tag{6.8}$$

and

$$\sigma_U(x) = \sup_{0\leq z\leq x, \overline{F}(z)>0} \frac{e^{\kappa z}\overline{F}(z)}{\int_z^\infty e^{\kappa y}dF(y)}, \qquad \sigma_L(x) = \inf_{0\leq z\leq x, \overline{F}(z)>0} \frac{e^{\kappa z}\overline{F}(z)}{\int_z^\infty e^{\kappa y}dF(y)}. \tag{6.9}$$

Corollary 15 *Suppose that κ satisfies (6.3). If $r(x) = 0$ for $x \geq x_0$, then for $n = 1, 2, 3 \ldots,$*

$$m(x) \leq \sigma_U(x)\psi_U(x)e^{-\kappa x} - \sum_{m=1}^{n} \phi^m (h_U * F^{*(m-1)})(x), \qquad x \geq 0, \tag{6.10}$$

where $h_U(x) = \sigma_U(x)\psi_U(x)e^{-\kappa x} \int_x^\infty e^{\kappa y}dF(y) - r(x)$. Similarly, for $n = 1, 2, 3 \ldots,$

$$m(x) \geq \sigma_L(x)\psi_L(x)e^{-\kappa x} + \sum_{m=1}^{n} \phi^m (h_L * F^{*(m-1)})(x), \qquad x \geq 0, \tag{6.11}$$

where $h_L(x) = r(x) - \sigma_L(x)\psi_L(x)e^{-\kappa x} \int_x^\infty e^{\kappa y}dF(y)$.

Proof. By mathematical induction applied in Theorem 1, we may prove the bounds (6.10) and (6.11) as well. First, for the upper bound, we are going to show that (6.10) holds true for all $n = 1, 2, 3, \ldots$. For $n = 1$, putting the upper bound for $m(x)$ in (6.7) into the integrand on the right side of (6.1), we obtain

$$\begin{aligned} m(x) &\leq \phi \int_0^x \sigma_U(x-y)\psi_U(x-y)e^{-\kappa(x-y)}dF(y) + \phi r(x) \\ &= \sigma_U(x)\psi_U(x)e^{-\kappa x} - \phi(h_U * F^{*(0)})(x), \end{aligned}$$

and thus (6.10) is true for $n = 1$. Suppose that (6.10) holds for some $n \geq 1$. Replacing the integrand part in (6.1) by such induction assumption yields

$$\begin{aligned} m(x) &\leq \phi \int_0^x \left\{ \sigma_U(x-y)\psi_U(x-y)e^{-\kappa(x-y)} - \sum_{m=1}^{n} \phi^m (h_U * F^{*(m-1)})(x-y) \right\} dF(y) + \phi r(x) \\ &\leq \sigma_U(x)\psi_U(x)e^{-\kappa x} - \sum_{m=1}^{n+1} \phi^m (h_U * F^{*(m-1)})(x). \end{aligned}$$

Hence, (6.10) holds true for all $n = 1, 2, 3, \ldots$. Similarly, the lower bound may be obtained as (6.11). Again, noting that $h_U(x)$ and $h_L(x)$ are nonnegative by definitions, it follows that the bounds given by (6.10) and (6.11) may be improved as n increases. □

Remark 2 *Note that from Corollary 3.1 in Willmot et al. (2001), for $\phi \geq 1$, if F is NWUC, then $\sigma_L(x) = \phi$ and if F is NBUC, then $\sigma_U(x) = \phi$. Similarly, for $\phi \leq 1$, if F is NWUC, then $\sigma_U(x) = \phi$ and if F is NBUC, then $\sigma_L(x) = \phi$.*

In particular, if $\kappa = 0$ (i.e. $\phi = 1$) from (6.3), then $\sigma_U(x) = \sigma_L(x) = 1$ in (6.9). Thus the two-sided bounds in (6.10) and (6.11) become, for $n = 1, 2, 3, \ldots$,

$$\psi_L(x) + \sum_{m=1}^{n} (h_{1,L} * F^{*(m-1)})(x) \leq m(x) \leq \psi_U(x) - \sum_{m=1}^{n} (h_{1,U} * F^{*(m-1)})(x), \qquad x \geq 0,$$

where $h_{1,L}(x) = r(x) - \psi_L(x)\overline{F}(x)$ and $h_{1,U}(x) = \psi_U(x)\overline{F}(x) - r(x)$.

Corollary 16 *If $\kappa \leq (>)\ 0$, then the upper (lower) bounds in Theorem 3 and Corollary 15 equal the exact solution (6.2) when $n \to \infty$. In particular, for NWUC df with $r(x) = \overline{F}(x)$, the lower (upper) bound in Corollary 15 also becomes (6.2) for $n \to \infty$, namely, the bounds are sharp.*

Proof. Let us define $l_U(x) = \alpha_U(x)e^{-\kappa x}$. First, using $l_U(x)$ we reexpress the upper bound in (6.5) as

$$
\begin{aligned}
m(x) &\leq \alpha_U(x)e^{-\kappa x} - \sum_{m=1}^{n} \phi^m (c_U * F^{*(m-1)})(x) \\
&= l_U(x) - \sum_{m=1}^{n} \phi^m \left\{ l_U(\cdot) \int_{\cdot}^{\infty} e^{\kappa y} dF(y) - r(\cdot) \right\} * F^{*(m-1)})(x) \\
&= \sum_{m=1}^{n} \phi^m \left(r * F^{*(m-1)} \right)(x) + l_U(x) - \sum_{m=1}^{n} \phi^m l_U(\cdot) \left\{ \frac{1}{\phi} - \int_0^{\cdot} e^{\kappa y} dF(y) \right\} * F^{*(m-1)}(x).
\end{aligned}
\tag{6.12}
$$

If $\kappa \leq 0$ (i.e. $\phi \geq 1$), then $\int_0^{\cdot} e^{\kappa y} dF(y) \leq \int_0^{\cdot} dF(y) = F(\cdot)$. Using this, (6.12) becomes

$$
m(x) \leq \sum_{m=1}^{n} \phi^m \left(r * F^{*(m-1)} \right)(x) + l_U(x) - \sum_{m=1}^{n} \phi^{m-1} \left(l_U * F^{*(m-1)} \right)(x) + \sum_{m=1}^{n} \phi^m \left(l_U * F^{*(m)} \right)(x),
$$

and thus, for $n \to \infty$

$$
m(x) \leq \sum_{m=0}^{\infty} \phi^{m+1} \left(r * F^{*(m)} \right)(x).
$$

Simiarly, if $\kappa > 0$ then $\int_0^{\cdot} e^{\kappa y} dF(y) > \int_0^{\cdot} dF(y) = F(\cdot)$ and thus, one finds the the lower bound in (6.6) for $n \to \infty$,

$$
m(x) \geq \sum_{m=0}^{\infty} \phi^{m+1} \left(r * F^{*(m)} \right)(x).
$$

By the similar argument used abvoe, we readily prove that the bounds in Corollary 15 become equivalent to the exact solution in (6.2). In addtion, if $\kappa \leq 0$ (i.e. implying $\phi \geq 1$), F is NWUC and $r(x) = \overline{F}(x)$, then we have $\sigma_L(x) = \phi$ from Remark 2 and $\psi_L(x) = 1$. Therefore, the lower bound in (6.11) reduces to

$$
\begin{aligned}
m(x) &\geq \phi(x)e^{-\kappa x} + \sum_{m=1}^{n} \phi^m (h_L * F^{*(m-1)})(x) \\
&= \sum_{m=1}^{n} \phi^m \left(r * F^{*(m-1)}\right)(x) + \phi e^{-\kappa x} - \sum_{m=1}^{n} \phi^m \left[\phi e^{-\kappa \cdot} \left\{\frac{1}{\phi} - \int_0^{\cdot} e^{\kappa y} dF(y)\right\}\right] * F^{*(m-1)}(x) \\
&= \sum_{m=1}^{n} \phi^m \left(r * F^{*(m-1)}\right)(x) - \sum_{m=1}^{n} \phi^m \left(e^{-\kappa \cdot} * F^{*(m-1)}\right)(x) + \sum_{m=0}^{n} \phi^{m+1} \left(e^{-\kappa \cdot} * F^{*(m)}\right)(x),
\end{aligned}
$$

and in turn, as $n \to \infty$,

$$m(x) \geq \sum_{m=0}^{\infty} \phi^{m+1} \left(r * F^{*(m)}\right)(x).$$

Therefore, two-sided bounds in Corollary 15 are sharp and also equivalent to the solution for renewal equations given by (6.2). For the other case ($\kappa > 0$), the same argument can be applied to prove that the bounds are sharp. □

6.3 Nonexponential bounds

Our main object in this section is to obtain another type of bound instead of the exponential type considered in the previous section in the case of a defective renewal equation. Up to now we have only discussed how to construct the bounds as long as κ exists satisfying (6.3). In this entire section, it is assumed $\phi \in (0, 1)$, which implies that κ satisfying (6.3) may not be found. Consider the case that $e^{\kappa y}$ is replaced by $\{\overline{B}(y)\}^{-1}$ in (6.3) where $B(y) = 1 - \overline{B}(y)$ is a df, thereby yielding

$$\int_0^{\infty} \{\overline{B}(y)\}^{-1} dF(y) = \frac{1}{\phi}. \tag{6.13}$$

Some choices of $\overline{B}(x)$, for example, $\overline{B}(x) = (1 + \kappa^* x)^{-n}$ where the distribution F has the moments up to the n-th order, are appropriate and useful if F has no moment generating function, but has finite moments. See Willmot and Lin (2001) and references therein for further discussion. For the analysis of nonexponential bounds in Willmot et al. (2001), it is convenient to introduce the function

$$\tau(x,z) = \frac{r(z)\overline{V}(x-z)}{\int_z^\infty \{\overline{B}(y)\}^{-1}\, dF(y)}, \qquad 0 \le z \le x, \tag{6.14}$$

as long as $\overline{F}(z) > 0$, and assume that $V(y) = 1 - \overline{V}(y)$ is either a B-NWU df, that is

$$\overline{B}(y)\overline{V}(x) \le \overline{V}(x+y), \qquad x \ge 0,\ y \ge 0, \tag{6.15}$$

or $V(y) = 1 - \overline{V}(y)$ is a B-NBU df, that is

$$\overline{B}(y)\overline{V}(x) \ge \overline{V}(x+y), \qquad x \ge 0,\ y \ge 0. \tag{6.16}$$

The following theorem provides the tighter bounds than Theorem 4.1 in Willmot et al. (2001)

Theorem 4 *Suppose that the df $B(y)$ satisfies (6.13), and the df $V(y)$ satisfies (6.15). If $r(x) = 0$ for $x \ge x_0$, then for $n = 1, 2, 3, \ldots,$*

$$m(x) \le \frac{\tau_U(x)}{\overline{V}(0)} - \sum_{m=1}^{n} \phi^m (q_U * F^{*(m-1)})(x), \qquad x \ge 0, \tag{6.17}$$

where

$$\tau_U(x) = \sup_{0 \le z \le x,\ \overline{F}(z) > 0} \tau(x,z), \qquad x \ge 0, \tag{6.18}$$

and $q_U(x) = \frac{\tau_U(x)}{\overline{V}(0)} \int_x^\infty \{\overline{B}(y)\}^{-1} dF(y) - r(x)$. Conversely, if the df $B(y)$ satisfies (6.13), but the df $V(y)$ satisfies (6.16), then for $n = 1, 2, 3, \ldots,$

$$m(x) \ge \frac{\tau_L(x)}{\overline{V}(0)} + \sum_{m=1}^{n} \phi^m (q_L * F^{*(m-1)})(x), \qquad x \ge 0, \tag{6.19}$$

where

$$\tau_L(x) = \inf_{0 \le z \le x,\ \overline{F}(z) > 0} \tau(x,z), \qquad x \ge 0, \tag{6.20}$$

and $q_L(x) = r(x) - \frac{\tau_L(x)}{\overline{V}(0)} \int_x^\infty \{\overline{B}(y)\}^{-1} dF(y)$.

Proof. First, consider the upper bound (6.17). We shall prove it holds true for all $n = 1, 2, 3, \ldots$. From Theorem 4.1 in Willmot et al. (2001), if the df $B(y)$ satisfies (6.13), and the df $V(y)$ satisfies (6.15), then

$$m(x) \leq \frac{\tau_U(x)}{\overline{V}(0)}, \qquad x \geq 0, \tag{6.21}$$

where $\tau_U(x)$ is given by (6.18). From (6.1) and (6.21), it follows that

$$\begin{aligned} m(x) &\leq \phi \int_0^x \frac{\tau_U(x-y)}{\overline{V}(0)} dF(y) + \phi r(x) \leq \phi \frac{\tau_U(x)}{\overline{V}(0)} \int_0^x dF(y) + \phi r(x) \\ &\leq \phi \frac{\tau_U(x)}{\overline{V}(0)} \int_0^x \frac{1}{\overline{B}(y)} dF(y) + \phi r(x) \leq \phi \frac{\tau_U(x)}{\overline{V}(0)} \left\{ \frac{1}{\phi} - \int_x^\infty \frac{1}{\overline{B}(y)} dF(y) \right\} + \phi r(x) \\ &= \frac{\tau_U(x)}{\overline{V}(0)} - \phi(q_U * F^{*(0)})(x), \end{aligned}$$

thus, (6.17) is true for $n = 1$. Assuming that (6.17) holds for some $n \geq 1$, its substitution into the integrand in (6.1) yields

$$\begin{aligned} m(x) &\leq \phi \int_0^x \left\{ \frac{\tau_U(x-y)}{\overline{V}(0)} - \sum_{m=1}^{n} \phi^m (q_U * F^{*(m-1)})(x-y) \right\} dF(y) + \phi r(x) \\ &\leq \frac{\tau_U(x)}{\overline{V}(0)} - \sum_{m=1}^{n+1} \phi^m (q_U * F^{*(m-1)})(x). \end{aligned}$$

Therefore, (6.17) holds true for all $n = 1, 2, 3, \ldots$ by induction. Also, we obtain the tighter bound as n increases since $q_U(x)$ is nonnegative.

In contrast, if the df $B(y)$ satisfies (6.13), but the df $V(y)$ satisfies (6.16), then Willmot et al. (2001) showed that

$$m(x) \geq \frac{\tau_L(x)}{\overline{V}(0)}, \qquad x \geq 0, \tag{6.22}$$

where $\tau_L(x)$ is given by (6.20). Then, to find the improved bound to (6.22), the following line of logic is essentially due to Willmot et al. (2001) and Willmot and Lin (2001). Let

$$m_0(z) = \phi r(z), \qquad z \geq 0, \tag{6.23}$$

and for $k = 1, 2, 3, ...$

$$m_k(z) = \phi r(z) + \sum_{n=1}^{k} \phi^{n+1} \int_0^z r(z-y) dF^{*(n)}(y), \qquad z \geq 0. \tag{6.24}$$

Then $\{m_k(z); k = 0, 1, 2, ...\}$ satisfies the recursive relationship

$$m_{k+1}(z) = \phi r(z) + \phi \int_0^z m_k(z-y) dF(y), \qquad z \geq 0. \tag{6.25}$$

We shall use an inductive approach as in Cai and Wu (1997). Let $A_1(z) = 1 - \overline{A}_1(z) = \phi \int_0^z \{\overline{B}(y)\}^{-1} dF(y)$, which is a df since (6.13) holds. Let $A_k(z) = 1 - \overline{A}_k(z)$ be the df of the sum of k independent random variables, each with the df $A_1(z)$ where, by the law of total probability, for $k = 1, 2, 3, \ldots$

$$\overline{A}_{k+1}(z) = \overline{A}_1(z) + \int_0^z \overline{A}_k(z-y) dA_1(y), \qquad z \geq 0. \tag{6.26}$$

We shall show by induction that for $k = 0, 1, 2, \ldots,$

$$m_k(z) \geq \frac{\tau_L(x)}{\overline{V}(x-z)} \overline{A}_{k+1}(z) + \sum_{m=1}^{k+1} \phi^m (q_{L,x} * F^{*(m-1)})(z), \qquad 0 \leq z \leq x, \tag{6.27}$$

where $q_{L,x}(z) = r(z) - \frac{\tau_L(x)}{\overline{V}(x-z)} \int_z^\infty \{\overline{B}(y)\}^{-1} dF(y)$. By (3.28) and (6.20),

$$r(z) \geq \frac{\tau_L(x)}{\overline{V}(x-z)} \int_z^\infty \{\overline{B}(y)\}^{-1} dF(y), \qquad 0 \leq z \leq x.$$

Then, for $k = 0$, (6.23) implies that

$$\begin{aligned} m_0(z) &= \phi r(z) + \phi \frac{\tau_L(x)}{\overline{V}(x-z)} \int_z^\infty \frac{1}{\overline{B}(y)} dF(y) - \phi \frac{\tau_L(x)}{\overline{V}(x-z)} \int_z^\infty \frac{1}{\overline{B}(y)} dF(y) \\ &= \frac{\tau_L(x)}{\overline{V}(x-z)} \overline{A}_1(z) + \phi \left\{ r(z) - \frac{\tau_L(x)}{\overline{V}(x-z)} \int_z^\infty \frac{1}{\overline{B}(y)} dF(y) \right\} \\ &= \frac{\tau_L(x)}{\overline{V}(x-z)} \overline{A}_1(z) + \phi (q_{L,x} * F^{*(0)})(z), \end{aligned} \tag{6.28}$$

since $\overline{A}_1(z) = \phi \int_z^\infty \{\overline{B}(y)\}^{-1} dF(y)$. Thus, (6.27) holds true for $k = 0$.

Now assume that (6.27) holds true for some $k \geq 0$. Then, from (6.25), replacement of $\phi r(z)$ and $m_k(z-y)$ by (6.28) and (6.27) respectively results in

$$\begin{aligned} m_{k+1}(z) \quad &\geq \quad \frac{\tau_L(x)}{\overline{V}(x-z)}\overline{A}_1(z) + \phi(q_{L,x} * F^{*(0)})(z) + \phi \int_0^z \frac{\tau_L(x)}{\overline{V}(x+y-z)}\overline{A}_{k+1}(z-y)dF(y) \\ &+ \quad \sum_{m=1}^{k+1} \phi^{m+1} \int_0^z (q_{L,x} * F^{*(m-1)})(z-y)dF(y). \end{aligned}$$

But $\overline{V}(x+y-z) \leq \overline{B}(y)\overline{V}(x-z)$ for $0 \leq z \leq x$ from (6.16), the above inequality may be rewritten as

$$\begin{aligned} m_{k+1}(z) \quad &\geq \quad \frac{\tau_L(x)}{\overline{V}(x-z)}\left\{\overline{A}_1(z) + \int_0^z \overline{A}_{k+1}(z-y)dA_1(y)\right\} + \sum_{m=1}^{k+2} \phi^m (q_{L,x} * F^{*(m-1)})(z) \\ &= \quad \frac{\tau_L(x)}{\overline{V}(x-z)}\overline{A}_{k+2}(z) + \sum_{m=1}^{k+2} \phi^m (q_{L,x} * F^{*(m-1)})(z). \end{aligned}$$

Hence, (6.27) holds for all $k = 0, 1, 2, \ldots$ by induction.

It follows from Ross (1996, pp.99-101) that $\sum_{k=1}^{\infty} A_k(x) < \infty$, implying that $\lim_{k\to\infty} A_k(x) = 0$, or equivalently, $\lim_{k\to\infty} \overline{A}_k(x) = 1$. Therefore, now combining (6.2), (6.24) and (6.27) with $z = x$ we may write,

$$\begin{aligned} m(x) \quad &= \quad \lim_{k\to\infty} m_k(x) \geq \frac{\tau_L(x)}{\overline{V}(0)} \lim_{k\to\infty} \overline{A}_{k+1}(x) + \sum_{m=1}^{\infty} \phi^m (q_L * F^{*(m-1)})(x) \\ &= \quad \frac{\tau_L(x)}{\overline{V}(0)} + \sum_{m=1}^{\infty} \phi^m (q_L * F^{*(m-1)})(x) = \sum_{m=0}^{\infty} \phi^{m+1} \left(r * F^{*(m)}\right)(x), \end{aligned}$$

where $q_L(x) = r(x) - \frac{\tau_L(x)}{\overline{V}(0)} \int_x^{\infty} \{\overline{B}(y)\}^{-1} dF(y)$. Since $q_L(x)$ is nonnegative, by truncating the above expression, the lower bound (6.19) is proved. Certainly, increasing n in (6.19) yields tighter and tighter lower bound, and as $n \to \infty$ it becomes the exact solution in (6.2) as proved in Corollary 16. $\square$

In particular, if the df $B(y)$ is NWU satisfying (6.13), then the upper bound in Theorem 4 implies to give, for $n = 1, 2, 3, \ldots$,

$$m(x) \leq \frac{\tau_U(x)}{\overline{B}(0)} - \sum_{m=1}^{n} \phi^m (q_{1,U} * F^{*(m-1)})(x), \qquad x \geq 0,$$

where $q_{1,U}(x) = \frac{\tau_U(x)}{\overline{B}(0)} \int_x^\infty \{\overline{B}(y)\}^{-1} dF(y) - r(x)$, since (6.15) holds true for NWU df $B(y)$ with $V(y) = B(y)$.

If the df $B(y)$ is NBU satisfying (6.13), then Theorem 4 yields, for $n = 1, 2, 3, \ldots$,

$$m(x) \geq \tau_L(x) + \sum_{m=1}^{n} \phi^m (q_{1,L} * F^{*(m-1)})(x), \qquad x \geq 0,$$

where $q_{1,L}(x) = r(x) - \tau_L(x) \int_x^\infty \{\overline{B}(y)\}^{-1} dF(y)$, since the df $B(y)$ is NBU and hence $\overline{B}(0)\overline{B}(0) \geq \overline{B}(0)$ which implies that $\overline{B}(0) = 1$, and (6.16) holds with $V(y) = B(y)$.

Next, we may improve the upper bound for $m(x)$ given by Willmot et al. (2001, Corollary 4.1), which is simpler and generalizes (6.7).

Corollary 17 *Suppose that the df $B(y)$ satisfies (6.13), and the df $V(y)$ satisfies (6.15), then for $n = 1, 2, 3 \ldots$,*

$$m(x) \leq \frac{\psi_U(x)}{\overline{V}(0)} \overline{V}(x) - \sum_{m=1}^{n} \phi^m (w_U * F^{*(m-1)})(x), \qquad x \geq 0, \tag{6.29}$$

where $w_U(x) = \frac{\psi_U(x)}{\overline{V}(0)} \overline{V}(x) \int_x^\infty \{\overline{B}(y)\}^{-1} dF(y) - r(x)$, and $\psi_U(x)$ is given by (6.8).

Proof. To prove (6.29) by induction. We first consider $n = 1$. From Corollary 4.1 in Willmot et al. (2001), we have

$$m(x) \leq \frac{\psi_U(x)}{\overline{V}(0)} \overline{V}(x), \qquad x \geq 0. \tag{6.30}$$

Substituting the upper bound in (6.30) into the integrand on the right side of (6.1), we obtain

$$\begin{aligned} m(x) &\leq \phi \int_0^x \frac{\psi_U(x-y)}{\overline{V}(0)} \overline{V}(x-y) dF(y) + \phi r(x) \leq \phi \frac{\psi_U(x)}{\overline{V}(0)} \overline{V}(x) \left\{ \frac{1}{\phi} - \int_x^\infty \frac{1}{\overline{B}(y)} dF(y) \right\} + \phi r(x) \\ &= \frac{\psi_U(x)}{\overline{V}(0)} \overline{V}(x) - \phi(w_U * F^{*(0)})(x), \end{aligned}$$

since $\overline{V}(x-y)\overline{B}(y) \leq \overline{V}(x)$ for $0 \leq y \leq x$ by assumption (6.15). Thus, (6.29) holds true for $n = 1$. The induction step can be shown similarly. Hence (6.29) is true for all $n = 1, 2, 3, \ldots$. □

Note that in applying Theorem 4, since (6.15) is assumed to hold, one has $\overline{V}(x) \geq \overline{V}(x+y-z) \geq \overline{B}(y)\overline{V}(x-z)$ for $0 \leq z \leq y$ and $0 \leq z \leq x$. Combining (6.8) and (6.14), one finds

$$\psi_U(x)\overline{V}(x) = \sup_{0 \leq z \leq x,\ \overline{F}(z) > 0} \frac{r(z)\overline{V}(x)}{\int_z^\infty dF(y)} \geq \sup_{0 \leq z \leq x,\ \overline{F}(z) > 0} \frac{r(z)\overline{V}(x-z)}{\int_z^\infty \{\overline{B}(y)\}^{-1} dF(y)} \geq \frac{r(x)\overline{V}(0)}{\int_z^\infty \{\overline{B}(y)\}^{-1} dF(y)},$$

which implies $w_U(x)$ is nonnegative. Therefore, increasing n in (6.29) yields increasingly tighter bound.

Moreover, based on Corollary 17, different types of bounds depending on other reliability properties of the df $B(y)$ are proposed in the following two corollaries motivated by Willmot and Lin (2001, pp.71-72).

Corollary 18 *Suppose that the df $B(y)$ is NWUC satisfying (6.13). Then for $n = 1, 2, 3, \ldots$,*

$$m(x) \leq \psi_U(x) \frac{\overline{B}_1(x)}{\overline{B}_1(0)} - \sum_{m=1}^{n} \phi^m (w_{1,U} * F^{*(m-1)})(x), \qquad x \geq 0, \tag{6.31}$$

where $w_{1,U}(x) = \psi_U(x) \frac{\overline{B}_1(x)}{\overline{B}_1(0)} \int_x^\infty \{\overline{B}(y)\}^{-1} dF(y) - r(x)$.

Proof. If the df $B(y)$ is NWUC, then (6.15) is satisfied with $V(y) = B_1(y)$ where $B_1(y) = 1 - \overline{B}_1(y)$ is the equilibrium df of $B(y)$ defined by $B_1(y) = \int_0^y B(x)dx\{\int_0^\infty \overline{B}(x)dx\}^{-1}$. Thus, from Corollary

17, (6.31) can be obtained. □

Note that for the heavy tail claim sizes, Corollary 18 is convenient to calculate the ruin probability since $\psi_U(x)$ equals 1 in this case. In the next section, we will illustrate how to compute the bounds for the ultimate ruin probability by using (6.31).

Next, under a larger reliability class assumption, for example, if the df $B(y)$ is NWUE, we have the following result.

Corollary 19 *Suppose that the df $B(y)$ is NWUE satisfying (6.13). Then for $n = 1, 2, 3, \ldots$,*

$$m(x) \leq \psi_U(x)\overline{V}(x) - \sum_{m=1}^{n} \phi^m (w_{2,U} * F^{*(m-1)})(x), \qquad x \geq 0, \tag{6.32}$$

where $w_{2,U}(x) = \psi_U(x)\overline{V}(x)\int_x^\infty \{\overline{B}(y)\}^{-1} dF(y) - r(x)$ *and* $\overline{V}(x) = \int_0^\infty \overline{B}(y)dy \left\{x + \int_0^\infty \overline{B}(y)dy\right\}^{-1}$.

Proof. From equation (2.4.3) given by Willmot and Lin (2001), it follows that $\overline{B}(y) \leq \overline{V}(y)$. Clearly, $V(x)$ is a Pareto df which is DFR, and therefore NWU as well. Hence, from Corollary 17, the result follows since $\overline{B}(y) \leq \overline{V}(y) \leq \overline{V}(x+y)/\overline{V}(x)$, i.e. (6.15) is satisfied with $V(0) = 0$. □

For the lower bound (6.19), we may also find the simple and improved bound corresponding to Corollary 4.2 in Willmot et al. (2001) given by

$$m(x) \geq \frac{\phi}{\overline{V}(0)} \gamma(x) \overline{V}(x), \qquad x \geq 0,$$

where

$$\gamma(x) = \max\left\{ \frac{\overline{B}(x_0)}{\phi} \psi_L(x), \inf_{0 \leq z \leq x\ , \overline{F}(z) > 0} r(z) \right\}, \tag{6.33}$$

with $\psi_L(x)$ is given by (6.8) and $x_0 = \inf\{x : F(x) = 1\}$.

Corollary 20 *Suppose that the df $B(y)$ satisfies (6.13), and the df $V(y)$ satisfies (6.16). Then,*

$$m(x) \geq \frac{\phi}{\overline{V}(0)} \gamma(x) \overline{V}(x) + \sum_{m=1}^{n} \phi^m (q_L * F^{*(m-1)})(x), \qquad x \geq 0,\ n = 1, 2, \ldots, \tag{6.34}$$

where $q_L(x) = r(x) - \frac{\tau_L(x)}{\overline{V}(0)} \int_x^\infty \{\overline{B}(y)\}^{-1} dF(y)$, $\gamma(x)$ *is given by (6.33) and* $\psi_L(x)$ *is given by (6.8).*

Proof. The result follows immediately from Theorem 4 since $\tau(x,z) \geq \{r(z)/\overline{F}(z)\}\overline{B}(x_0)\overline{V}(x)$ or $\tau(x,z) \geq \phi r(z)\overline{V}(x)$. See Corollary 4.2 in Willmot et al. (2001) for the detail of proof. $\square$

Now, we obtain the bounds by considering the reliability assumptions for F itself in the following.

Corollary 21 *Suppose that* $F(y)$ *is an absolutely continuous NWU df. Then for* $n = 1, 2, 3, \ldots,$

$$m(x) \leq \phi\psi_U(x)\{\overline{F}(x)\}^{1-\phi} - \sum_{m=1}^{n} \phi^m (l_U * F^{*(m-1)})(x), \qquad x \geq 0, \tag{6.35}$$

where $l_U(x) = \psi_U(x)\overline{F}(x) - r(x)$. *Conversely, if* $F(y)$ *is an absolutely continuous NBU df, then for* $n = 1, 2, 3, \ldots,$

$$m(x) \geq \phi\psi_L(x)\{\overline{F}(x)\}^{1-\phi} + \sum_{m=1}^{n} \phi^m (l_L * F^{*(m-1)})(x), \qquad x \geq 0, \tag{6.36}$$

where $l_L(x) = r(x) - \psi_L(x)\overline{F}(x)$.

Proof. From Corollary 4.4 in Willmot et al. (2001), if $F(y)$ is an absolutely continuous NWU df, then

$$m(x) \leq \phi\psi_U(x)\{\overline{F}(x)\}^{1-\phi}, \qquad x \geq 0, \tag{6.37}$$

where $\psi_U(x)$ is given by (6.8). Conversely, if $F(y)$ is an absolutely continuous NBU df, then

$$m(x) \geq \phi\psi_L(x)\{\overline{F}(x)\}^{1-\phi}, \qquad x \geq 0,$$

where $\psi_L(x)$ is given by (6.8).

For the upper bound, to prove (6.35) for all $n = 1, 2, 3, \ldots$, we first consider the case $n = 1$. Inserting (6.37) into the integrand on the right side of (6.1), we readily obtain

$$m(x) \leq \phi^2 \int_0^x \psi_U(x-y)\{\overline{F}(x-y)\}^{1-\phi} dF(y) + \phi r(x).$$

Because $\overline{F}(x-y)\overline{F}(y) \leq \overline{F}(x)$ for $0 \leq y \leq x$ by NWU assumption, it follows that

$$\begin{aligned} m(x) &\leq \phi^2 \psi_U(x) \int_0^x \left\{ \frac{\overline{F}(x)}{\overline{F}(y)} \right\}^{1-\phi} dF(y) + \phi r(x) \\ &= \phi \psi_U(x) \left\{ \overline{F}(x) \right\}^{1-\phi} \left\{ 1 - \left\{ \overline{F}(x) \right\}^{\phi} \right\} + \phi r(x) \\ &= \phi \psi_U(x) \{\overline{F}(x)\}^{1-\phi} - \phi (l_U * F^{*(0)})(x). \end{aligned}$$

Thus, it holds for $n = 1$. Suppose that (6.35) holds true for some $n \geq 1$, then we get

$$\begin{aligned} m(x) &\leq \phi \int_0^x \left\{ \phi \psi_U(x-y)\{\overline{F}(x-y)\}^{1-\phi} - \sum_{m=1}^{n} \phi^m (l_U * F^{*(m-1)})(x-y) \right\} dF(y) + \phi r(x) \\ &\leq \phi \psi_U(x)\{\overline{F}(x)\}^{1-\phi} - \sum_{m=1}^{n+1} \phi^m (l_U * F^{*(m-1)})(x). \end{aligned}$$

By induction, we can obtain the improved bound (6.35) compared to (6.37) since $l_U(x)$ is a nonnegative function from (6.8).

In addition, with similar argument, using the fact that $\overline{F}(x-y)\overline{F}(y) \geq \overline{F}(y)$ for $0 \leq y \leq x$ under NBU assumption on F, the lower bound may be easily obtained as (6.36). □

6.4 Applications

In this section, we illustrate the two-sided bounds in various examples. The first four examples are related to insurance risk theory and the last one involves alternating renewal processes.

Example 1 (*The severity of ruin*)

Let us consider the classical compound Poisson model with $\overline{K}(t) = e^{-\lambda t}$ as in Section 3.2 and Section 3.3, and the claim size has a mean μ. Also, we assume that $c = (1+\theta)\lambda\mu$ where $\theta > 0$ is the premium loading factor. Here, the interest is the probability that the deficit at the time of ruin $|U_T|$ is at most y and ruin has occurred with initial surplus x defined as $G(x, y) = \psi(u) - \overline{G}(x, y)$ where $\overline{G}(x, y)$ is given by (3.71), i.e.

$$G(x, y) = \Pr\left(|U_T| \leq y, T < \infty | U_0 = x\right).$$

From (3.73), we know that $G(x, y)$ satisfies defective renewal equation (e.g. Gerber et al. (1987))

$$G(x, y) = \frac{1}{1+\theta} \int_0^x G(x-t, y)dF(t) + \frac{1}{1+\theta}\left\{\overline{F}(x) - \overline{F}(x+y)\right\}. \tag{6.38}$$

It is obvious that (6.38) is equivalent in form to (6.1) with $\phi = 1/(1+\theta) < 1$ and $r(x) = \overline{F}(x) - \overline{F}(x+y)$. Thus, if $\kappa > 0$ exists such that $1 + \theta = \int_0^\infty e^{\kappa t} dF(t)$, then from Theorem 3 with $r(x) = \overline{F}(x) - \overline{F}(x+y)$, the upper bound is given by, for $n = 1, 2, 3, \ldots,$

$$G(x, y) \leq \alpha_U(x)e^{-\kappa x} - \sum_{m=1}^{n} \left(\frac{1}{1+\theta}\right)^m (c_U * F^{*(m-1)})(x), \qquad x \geq 0, \tag{6.39}$$

where $c_U(x) = \alpha_U(x)e^{-\kappa x} \int_x^\infty e^{\kappa y} dF(y) - [\overline{F}(x) - \overline{F}(x+y)]$ and $\alpha_U(x)$ is given by (6.2), and the lower bound is given by, for $n = 1, 2, 3, \ldots,$

$$G(x, y) \geq \alpha_L(x)e^{-\kappa x} + \sum_{m=1}^{n} \left(\frac{1}{1+\theta}\right)^m (c_L * F^{*(m-1)})(x), \qquad x \geq 0, \tag{6.40}$$

where $c_L(x) = [\overline{F}(x) - \overline{F}(x+y)] - \alpha_L(x)e^{-\kappa x} \int_x^\infty e^{\kappa y} dF(y)$ and $\alpha_L(x)$ is given by (6.2). For $n = 1$ in (6.39) and (6.40), these results are in agreement with Theorem 3.2 in Chadficonstantinidis and Politis (2007). Clearly, for $n \geq 2$, we may obtain more improved results.

Also, from Corollary 15, we may easily obtain the simple bounds, for $n = 1, 2, 3, \ldots,$

$$G(x, y) \leq \sigma_U(x)\psi_U(x)e^{-\kappa x} - \sum_{m=1}^{n} \left(\frac{1}{1+\theta}\right)^m (h_U * F^{*(m-1)})(x), \qquad x \geq 0, \tag{6.41}$$

where $h_U(x) = \sigma_U(x)\psi_U(x)e^{-\kappa x}\int_x^\infty e^{\kappa y}dF(y) - [\overline{F}(x) - \overline{F}(x+y)]$, and for $n = 1, 2, 3, \ldots$,

$$G(x, y) \geq \sigma_L(x)\psi_L(x)e^{-\kappa x} + \sum_{m=1}^{n}\left(\frac{1}{1+\theta}\right)^m (h_L * F^{*(m-1)})(x), \qquad x \geq 0, \tag{6.42}$$

where $h_L(x) = [\overline{F}(x) - \overline{F}(x+y)] - \sigma_L(x)\psi_L(x)e^{-\kappa x}\int_x^\infty e^{\kappa y}dF(y)$. In this case, (6.8) becomes

$$\psi_U(x) = 1 - \left\{\inf_{0\leq z\leq x,\ \overline{F}(z)>0} \frac{\overline{F}(z+y)}{\overline{F}(z)}\right\}, \qquad \psi_L(x) = 1 - \left\{\sup_{0\leq z\leq x,\ \overline{F}(z)>0} \frac{\overline{F}(z+y)}{\overline{F}(z)}\right\}. \tag{6.43}$$

and $\sigma_U(x)$ and $\sigma_L(x)$ are given by (6.9). Again, for $n = 1$, the above inequalities (6.41) and (6.42) are consistent with those in Theorem 3.3 of Chadficonstantinidis and Politis (2007) and it is also easy to improve the above bounds (6.41) and (6.42) by increasing n where $n \geq 2$.

Example 2 (*The ultimate ruin probability*)

Let us denote the ultimate (i.e. infinite time) ruin probability with initial surplus x by $\psi(x)$. Consequently, (6.38) becomes

$$\psi(x) = \frac{1}{1+\theta}\int_0^x \psi(x-t)dF(t) + \frac{1}{1+\theta}\overline{F}(x), \qquad x \geq 0,$$

since $\psi(x) = \lim_{y\to\infty} G(x, y)$. See e.g. Willmot and Lin (2001, equation 10.1.7).

Hence, if $\kappa > 0$ exists satisfying (6.3), from Corollary 15 with $\phi = 1/(1+\theta) < 1$ and $r(x) = \overline{F}(x)$ in equation (6.1) we can obtain the bounds. In particular, if $P(y)$ is 2-NWU (2-NBU) df which implies NWUC (NBUC) df $F(y)$, then we get, for $n = 1, 2, 3, \ldots$,

$$\psi(x) \leq \frac{1}{1+\theta}e^{-\kappa x} - \sum_{m=1}^{n}\left(\frac{1}{1+\theta}\right)^m (h_{U(L)} * F^{*(m-1)})(x), \qquad x \geq 0, \tag{6.44}$$

where $h_U(x) = \frac{1}{1+\theta}e^{-\kappa x}\int_x^\infty e^{\kappa y}dF(y) - \overline{F}(x)$ and $h_L(x) = -h_U(x)$.

However, if no κ satisfying (6.3) exists and $P(y)$ is 2-NWU df (i.e. implying $F(y)$ is NWUC df), then from Corollary 18 gives the upper bound, for $n = 1, 2, 3, \ldots$,

$$\psi(x) \le \frac{\overline{B}_1(x)}{\overline{B}_1(0)} - \sum_{m=1}^{n} \left(\frac{1}{1+\theta} \right)^m (w_{1,U} * F^{*(m-1)})(x), \qquad x \ge 0, \tag{6.45}$$

where $w_{1,U}(x) = \frac{\overline{B}_1(x)}{\overline{B}_1(0)} \int_x^\infty \{\overline{B}(y)\}^{-1} dF(y) - \overline{F}(x)$.

In what follows, the security loading θ is assumed to be 0.1 and $\lambda = 1$. The bound for $n = 0$ is calculated by using the corresponding result given by Willmot et al. (2001). For the first two claim size distribution, $\kappa > 0$ exists in (6.3) and thus, using (6.44), we calculate the bounds for ruin probability. When the claim sizes are in the class of the combinations of exponentials, so the exact ruin probabilities can be found based on the results in Gerber et al. (1987, Section 3).

First the claim size distribution is assumed to be a mixture of two exponentials given by

$$P(x) = 1 - \frac{1}{3} e^{-1/2x} - \frac{2}{3} e^{-2x}, \qquad x \ge 0.$$

From Corollary 15 with $\sigma_U(x) = \phi$ since a mixture of exponentials is a long-tailed, namely P is NWUC, one yields the upper bounds for the ruin probabilities as in Table 6.1. The values on the third column marked with $*$ in Table 6.1 are obtained by adding more terms of convolutions in the exact solution given by (6.2). Altough the resulting bounds here contain convolution and thus look similar to the exact solution in (6.2). By comparing the numbers for $u = 10$ as below, it is clear that the upper bound works much better than the method by using the exact solution. It is obvious that a large number of n results in a tighter bound. Second, we may calculate the lower bounds for the ruin probabilities when the claim size distribution is light-tailed such as a sum of exponentials. In this illustration the claim size distribution is assumed to be

$$P(x) = 1 - 2e^{-1.5x} + e^{-3x}, \qquad x \ge 0.$$

Again using Corollary 15 with $\sigma_L(x) = \phi$ since P is NBUC, we may obtain the lower bounds shown in Table 6.2. The last demonstration illustrates the evaluation of the upper bounds when

	$u = 1$	$u = 10$	$u = 10^*$
Exact	0.8425516	0.4913739	
n = 0	0.8566998	0.5021355	0.004084
1	0.8476895	0.5020066	0.017208
2	0.8439231	0.5015902	0.044707
3	0.8428390	0.5007778	0.087981
4	0.8426010	0.4995784	0.143615
5	0.8425588	0.4981179	0.205298

Table 6.1: Bohman distribution (mixture of two exponentials)

	$u = 1$	$u = 10$
Exact	0.8143244	0.2821805
n = 0	0.8080771	0.2799380
1	0.8109008	0.2799381
2	0.8131127	0.2799381
3	0.8140137	0.2799388
4	0.8142623	0.2799418
5	0.8143143	0.2799525

Table 6.2: Sum of two exponentials

there is no κ satisfying (6.3). We consider the Pareto claim size distribution given by

$$P(x) = 1 - (1+x)^{-4}, \qquad x \geq 0.$$

Then we may use the result in Corollary 18 with $\overline{B}(x) = (1 + k^* x)^{-2}$ since the equilibrium distribution of the claim size distribution has up to the second moment in this case. Since both $F(y)$ and $B(y)$ are NWUC, from (6.45), the nonexponential-type upper bounds for the ruin probabilities can be obtained as in Table 6.3. For comparison, the result given by Ramsay (2003) is used to calculate the exact value as well. Note that the bounds for the ruin probability with arbitrary claim sizes distribution are obtainable as long as ϕ in (6.1) is known.

Furthermore, we consider a dependency model in which the claim sizes are dependent on their respective interclaim times and also compute the bounds for ruin probability.

	$u = 1$	$u = 10$
Exact	0.8383994	0.4751918
n = 0	0.9703688	0.7660722
1	0.9062603	0.7616415
2	0.8672447	0.7559901
3	0.8522605	0.7489817
4	0.8481224	0.7406976
5	0.8472357	0.7055629

Table 6.3: Pareto distribution

Example 3 (*Dependency model*)

From (3.64) in Section 3.2, we know that the Gerber-Shiu function under the dependent model studied by Boudreault et al. (2006) satisfies the defective renewal equation. Thus we may apply Corollary 15 to obtain the upper bound for the ruin probabilities when two claim size distributions are exponentials given by $\overline{F}_1(y) = e^{-2.5y}$ and $\overline{F}_2(y) = e^{-0.5y}$ for $y \geq 0$. After finding $F = g_\delta$, $\phi = \kappa_\delta$ and $r = \xi_\delta$ in equation (6.1) from (28),(29),(30) and (31) Boudreault et al. (2006), we get the upper bound for the ruin probabilities as in Table 6.4. Furthermore, we consider the claim size

	$u = 1$	$u = 10$
Exact	0.6001594	0.1304086
n = 0	0.6561768	0.1464128
1	0.6224251	0.1459792
2	0.6070369	0.1449589
3	0.6018794	0.1433618
4	0.6005205	0.1413565
5	0.6002266	0.1391894

Table 6.4: Dependency model (exponentials)

distributions are mixtures of two exponentials given by

$$\overline{F}_1(y) = 0.7e^{-2x} + 0.3e^{-0.5y}, \qquad \overline{F}_2(y) = 0.5e^{-0.3y} + 0.5e^{-0.1y}, \qquad y \geq 0,$$

where $\overline{F}_1(y) = \int_y^\infty f_1(x)dx$ and $\overline{F}_2(y) = \int_y^\infty f_2(x)dx$, then upper bounds for the ruin probabilities are computed as in Table 6.5. From (3.68), the ladder height distribution f is a mixture of three ladder height of the claim size distributions. Therefore we may get the upper bound for this example since the mixing regenerates a heavy-tailed distribution.

	$u=1$	$u=10$
Exact	0.7253252	0.4902439
n = 0	0.7663269	0.5628704
1	0.7328309	0.5376456
2	0.7263444	0.5175386
3	0.7254343	0.5043506
4	0.7253348	0.4968704
5	0.7253259	0.4931003

Table 6.5: Dependency model (mixture of two exponentials)

The following example contains some bounds for the joint distribution of the surplus prior to and at ruin in the classical compound Poisson model.

Example 4 (*The joint distribution of the surplus prior to and at ruin*)
Let us consider the joint distribution of the surplus prior to U_{T-} and the deficit at ruin $|U_T|$ (denoted by $H(u,x,y)$) studied by Dickson (1992) and Gerber and Shiu (1997b). Our interest is the tail df of $H(u,x,y)$, that is, $\overline{H}(u,x,y) = 1 - H(u,x,y)$ defined by

$$\overline{H}(u,x,y) = \Pr\left(U_{T-} > x,\ |U_T| > y,\ T < \infty | U_0 = u\right). \tag{6.46}$$

We know that the function (6.46) satisfies the defective renewal equation which was proved by Gerber and Shiu (1998). See also Schmidli (1999) and Dickson (1992) for a discussion on this. Here, in the classical compound Poisson risk model, we follow the result which is presented in Proposition 2.1 in Psarrakos and Politis (2008) given by

$$\overline{H}(u,x,y) = \frac{1}{1+\theta}\int_0^u \overline{H}(u-t,x,y)dF(t) + \frac{1}{1+\theta}\overline{F}(\max\{u+y, x+y\}).$$

Let us define $\alpha_{U,x,y}(u)$ and $\alpha_{L,x,y}(u)$ as follows:

$$\alpha_{x,y}(z) = \frac{e^{\kappa z}\overline{F}(\max\{z+y, x+y\})}{\int_z^\infty e^{\kappa t}dF(t)}, \qquad z \geq 0,$$

and

$$\alpha_{U,x,y}(u) = \sup_{0 \leq z \leq u,\ \overline{F}(z)>0} \alpha_{x,y}(z), \qquad \alpha_{L,x,y}(u) = \inf_{0 \leq z \leq u,\ \overline{F}(z)>0} \alpha_{x,y}(z).$$

Then, Theorem 3 with $r(u) = r_{x,y}(u) = \overline{F}(\max\{u+y, x+y\})$ yields, for $n = 1, 2, 3, \ldots,$

$$\overline{H}(u,x,y) \leq \alpha_{U,x,y}(u)e^{-\kappa u} - \sum_{m=1}^{n} \phi^m (c_{U,x,y} * F^{*(m-1)})(u), \qquad u \geq 0, \tag{6.47}$$

where $c_{U,x,y}(u) = \alpha_{U,x,y}(u)e^{-\kappa u} \int_u^\infty e^{\kappa t} dF(t) - \overline{F}(\max\{u+y, x+y\})$, and

$$\overline{H}(u,x,y) \geq \alpha_{L,x,y}(u)e^{-\kappa u} + \sum_{m=1}^{n} \phi^m (c_{L,x,y} * F^{*(m-1)})(u), \qquad u \geq 0, \tag{6.48}$$

where $\alpha_{L,x,y}(u) = \overline{F}(\max\{u+y, x+y\}) - \alpha_{L,x,y}(u)e^{-\kappa u} \int_u^\infty e^{\kappa t} dF(t)$.

The above bounds (6.47) and (6.48) for $n = 1$ agrees with the results in Theorem 6.1 in Psarrakos and Politis (2008). Also, we may easily obtain tighter bounds by applying a larger value of n where $n \geq 2$.

Furthermore, using other generalized forms of bounds obtained in this chapter, we may readily find various types of two-sided bounds. For example, Corollary 21 yields, if F is an absolutely continuous NWU df then, for $n = 1, 2, 3, \ldots,$

$$\overline{H}(u,x,y) \leq \phi\psi_{U,x,y}(u)\{\overline{F}(u)\}^{1-\phi} - \sum_{m=1}^{n} \phi^m (l_{U,x,y} * F^{*(m-1)})(u), \qquad x \geq 0,$$

where $l_{U,x,y}(u) = \psi_{U,x,y}(u)\overline{F}(u) - \overline{F}(\max\{u+y, x+y\})$. Conversely, if F is an absolutely continuous NBU df, then for $n = 1, 2, 3, \ldots,$

$$\overline{H}(u,x,y) \geq \phi\psi_{L,x,y}(u)\{\overline{F}(u)\}^{1-\phi} + \sum_{m=1}^{n} \phi^m (l_{L,x,y} * F^{*(m-1)})(u), \qquad x \geq 0,$$

where $l_{L,x,y}(u) = \overline{F}(\max\{u+y, x+y\} - \psi_{L,x,y}(u)\overline{F}(u))$, $\psi_{U,x,y}(u)$ and $\psi_{L,x,y}(u)$ are equivalent to $\psi_U(u)$ and $\psi_L(u)$ respectively in (6.8) with $r(u) = r_{x,y}(u) = \overline{F}(\max\{u+y, x+y\})$

Finally, we consider the example related to the alternating renewal processes.

Example 5 (*The excess lifetime*)

The interarrival time between $(n-1)$th and nth event is denoted by X_n. It is assumed that $\{X_n\}_{n=1}^{\infty}$ is a sequence of iid non-negative random variables with df $F(x) = 1 - \overline{F}(x)$ and $F(0) = 0$. Then we consider the time of nth event denoted by $S_n = \sum_{i=1}^{n} X_i$ for $n = 1, 2, \ldots$. and $S_0 = 0$. Let $N(t) = \sup\{n : S_n \leq t\}$ be the number of events that occur before time t. The counting process $\{N(t); t \geq 0\}$ is called a renewal process (e.g. Ross (1996, p.98)). We now consider that the excess or residual lifetime at t denoted by $Y(t) = S_{N(t)+1} - t$ is the time until the next renewal. By conditioning on X_1, it turns out that $g(t, x) = \Pr\{Y(t) \geq x\}$ satisfies the renewal equation (e.g. Resnick (1992, pp.199-201) and Ross (1996, pp.114-118))

$$g(t, x) = \overline{F}(t + x) + \int_0^t g(t - y, x) dF(y), \qquad t \geq 0,\ x \geq 0. \tag{6.49}$$

Evidently, (6.49) is in the form of (6.1) with $\phi = 1$ (i.e. $\kappa = 0$ from (6.3)) and $r(x) = r_t(x) = \overline{F}(t + x)$. Therefore, from Corollary 15 with $\sigma_U(x) = \sigma_L(x) = 1$, we may obtain the two-sided bounds as follows. For $n = 1, 2, 3, \ldots$,

$$\psi_{L,t}(x) + \sum_{m=1}^{n} (h_{L,t} * F^{*(m-1)})(x) \leq g(t, x) \leq \psi_{U,t}(x) - \sum_{m=1}^{n} (h_{U,t} * F^{*(m-1)})(x), \qquad t \geq 0,\ x \geq 0, \tag{6.50}$$

where $h_{L,t}(x) = \overline{F}(t + x) - \psi_{L,t}(x)\overline{F}(x)$, $h_{U,t}(x) = \psi_{U,t}(x)\overline{F}(x) - \overline{F}(t + x)$, and

$$\psi_{U,t}(x) = \sup_{0 \leq z \leq x,\ \overline{F}(z) > 0} \frac{\overline{F}(t + z)}{\overline{F}(z)}, \qquad \psi_{L,t}(x) = \inf_{0 \leq z \leq x,\ \overline{F}(z) > 0} \frac{\overline{F}(t + z)}{\overline{F}(z)}. \tag{6.51}$$

If the df $F(x)$ is NWU, then $\overline{F}(t+z)/\overline{F}(z) \geq \overline{F}(t)$, and thus from (6.51), $\psi_{L,t}(x) = \overline{F}(t)$. Similarly, if the df $F(x)$ is NBU, then $\psi_{U,t}(x) = \overline{F}(t)$. Hence, we may simplify one of the bounds (6.50) for each case.

Chapter 7

Concluding remarks and future research

In this thesis, we consider a generalization of the classical Gerber-Shiu function in various risk models. The generalization involves introduction of two new variables in the original penalty function which includes the surplus prior to ruin U_{T^-} and the deficit at ruin $|U_T|$. These new variables are the minimum surplus level before ruin occurs X_T and the surplus immediately after the second last claim before ruin occurs R_{N_T-1}. Although these quantities can not be observed until ruin occurs, we can still identify their distributions in advance because they do not functionally depend on the time of ruin, but only depend on known quantities including the initial surplus allocated to the business. In addition, even if they are not directly connected to real world applications, our understanding of the analysis of the random walk and the resultant risk management can only be improved by a deeper knowledge of any and all associated quantities.

In Chapter 2, we demonstrate that the generalized Gerber-Shiu functions satisfy defective renewal equations with the some associated compound geometric distribution in the ordinary Sparre Andersen renewal risk models (continuous time). As a result, the forms of joint and marginal distributions associated with the variables in the generalized penalty function are derived for an arbitrary distribution of interclaim/interarrival times. Because the identification of the compound geometric components is difficult without any specific conditions on the interclaim times, in Chapter 3 we consider the special case when the interclaim time distribution is from the Coxian class of distributions including the special case involving the classical compound Poisson model. Note that the analysis of the generalized Gerber-Shiu function involving the triplet $(U_{T^-}, |U_T|, R_{N_T-1})$ is sufficient to study of the four $(U_{T^-}, |U_T|, X_T, R_{N_T-1})$. This is shown to be true even in cases where the interclaim of the first event is assumed to be different from the subsequent interclaims (i.e. delayed renewal risk models) in Chapter 4 and the counting (the number of claims) process is defined in discrete time (i.e. discrete renewal risk models) in Chapter 5. It is clear to me that proper modelling of various real world insurance phenomena dealing with natural disasters such as earthquakes needs to address both dependencies between claim sizes and claim times as well as the delayed nature arising from beginning the insurance coverage at a specific point in time (identified as $t = 0$ in our model). In addition, we further analyze various ruin related quantities obtained from a discrete analogue of the generalized Gerber-Shiu function. To do so, we introduce a nonnegative function representing a transition in the surplus (denoted by $\tau_\delta(u, z)$ in a continuous time process) which is an integral component of the analysis. References for the discrete renewal risk model includes Shiu (1989) and Willmot (1993). Application of these results are provided in cases when claim sizes depends on a discrete interclaim time, for instance, the bivariate compound geometric distribution studied by Marceau (2009).

In Chapter 6 two-sided bounds for a renewal equation are studied. These results may be used in cases involving various ruin quantities from the generalized Gerber-Shiu function analyzed in the previous chapters. Note that the larger the number of iterations in computing the bound produces the closer result to the exact value. However, for the nonexponential bounds the form of the bound contains a convolution involving heavy-tailed distributions (e.g. heavy-tailed claims, extreme events), we need to find an alternative method to implement the convolution computation in this case. This would be one of the future research topics on the bounds for a renewal equation. For this problem, some recursive results for convolutions were studied by De Pril (1985), Sundt

(2002) and Hipp (2006) and references therein. Alternatively, an asymptotic approach using the results in Albrecher et al. (2010) may be also considered. Furthermore, comparison with the existing works (e.g. Kalashnikov (1999)) would be interesting. In addition, an extension to Markov renewal equations in Markovian random environments (e.g. Miyazawa (2002)), which includes a regime switching model as its special case, may be studied as well. Finally, further application of the results obtained here may be possible for dependent extreme events in insurance business by considering various copulas (e.g. Embrechts et al. (1997)).

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