Simultaneously Embedding Planar Graphs at Fixed Vertex Locations

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

We discuss the problem of embedding planar graphs onto the plane with pre-specified vertex locations. In particular, we introduce a method for constructing such an embedding for both the case where the mapping from the vertices onto the vertex locations is fixed and the case where this mapping can be chosen. Moreover, the technique we present is sufficiently abstract to generalize to a method for constructing simultaneous planar embeddings with fixed vertex locations. In all cases, we are concerned with minimizing the number of bends per edge in the embeddings we produce.

In the case where the mapping is fixed, our technique guarantees embeddings with at most $8n - 7$ bends per edge in the worst case and, on average, at most $\frac{16}{3}n - 1$ bends per edge. This result improves previously known techniques by a significant constant factor.

When the mapping is not pre-specified, our technique guarantees embeddings with at most $O(n^{1-2^{1-k}})$ bends per edge in the worst case and, on average, at most $O(n^{1-\frac{1}{k}})$ bends per edge, where k is the number of graphs in the simultaneous embedding. This improves upon the previously known $O(n)$ bound on the number of bends per edge for k at least 2. Moreover, we give an average-case lower bound on the number of bends that has similar asymptotic behaviour to our upper bound.

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Chapter 1

Introduction

Planar graphs are defined by the class of graphs that can be drawn in the plane so that vertices map to unique points and edges map to simple continuous curves, no two of which cross. Such a drawing is referred to as a *planar embedding*.

Though defined geometrically, planar graphs have nice combinatorial characterizations. For instance, Kuratowski's theorem characterizes planar graphs in terms of *forbidden subgraphs*. That is, planar graphs can be defined purely in combinatorial terms without the notion of an embedding on some *topology* (such as the plane). In the area of graph drawing, however, we are primarily concerned with the actual drawings and understanding what properties can be enforced on the drawings.

Consider a drawing of a graph in the plane. Intuitively, there is a sense of complexity associated with such a drawing that is independent of the underlying graph. For example, consider the two drawings in Figure 1.1. The two drawings are planar embeddings of the same graph. Moreover, the two drawings can be thought of as being topologically equivalent. That is, there exists a *homeomorphism* of the plane mapping the first drawing onto the second. However, intuitively it seems natural to prefer the first drawing. To discuss theoretically the problem of giving *preferred* drawings of graphs, we need a kind of metric for measuring the *complexity* of a drawing. Thus, we proceed to discuss a few applications of graph drawing.

1.1 Problem and Motivation

1.1.1 Applications

In *information visualization* one studies different ways to represent information associated with large-scale systems. For example, Figure 1.2 shows a visualization of some websites *near* a search query for "Paul Erdős" on the World Wide Web. In this visualization, the

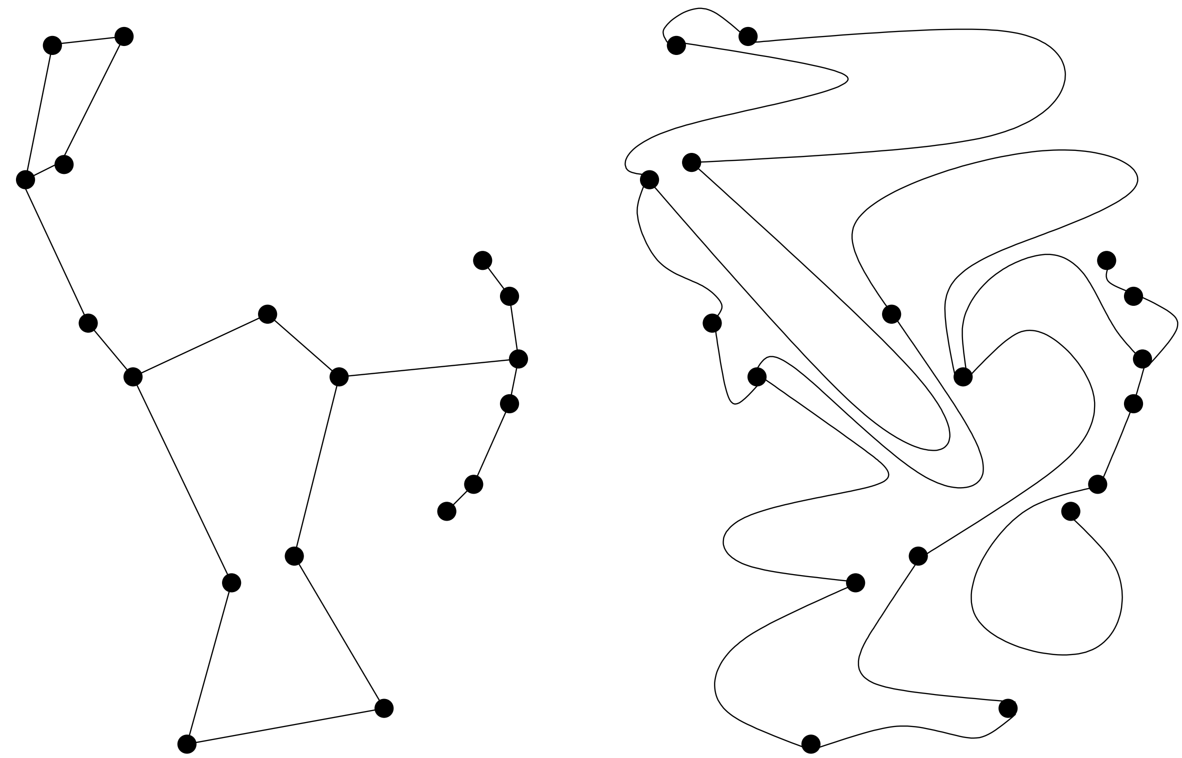

Fig. 1.1: Two homeomorphic embeddings of a planar graph with common vertex locations. The first drawing seems to exhibit less complexity.

nodes of the graph are *clustered* into neighbourhoods, and the edges are drawn as straight lines. The metric describing the complexity of the drawing is heuristically defined to combine several aesthetically preferable qualities. That is, the metric is an *energy function* defined over the drawing for which a lower energy state correlates with a preferred drawing. See [DBETT98] for additional applications of information visualization and examples of preferable properties of graph drawings.

A second application is concerned with the process of creating integrated circuits, referred to as *VLSI circuit design*. In the application of VLSI circuit design, ideal layouts correspond to planar embeddings in which vertices are preferred to be reasonably separated [MC79]. We can formalize this property by associating a grid (of an appropriate size) with the plane and requiring that vertices map to lattice points on this grid. Considering this restriction and additional restrictions on the edges led to the study of what is called a *visibility representation*. In such a representation, vertices are embedded as horizontal line segments and edges correspond to *visibilities* between the vertex segments; that is, two vertex segments are *visible* if they can be joined by a vertical line segment that intersects no other vertex segment. See Figure 1.3 for an example of a visibility representation of a planar graph. As was shown in [RT86] and [TT86], visibility representations of planar graphs can be constructed in linear time. We refer the reader to [DBETT94] for an extensive bibliography of papers concerned with the application of graph drawing to VLSI

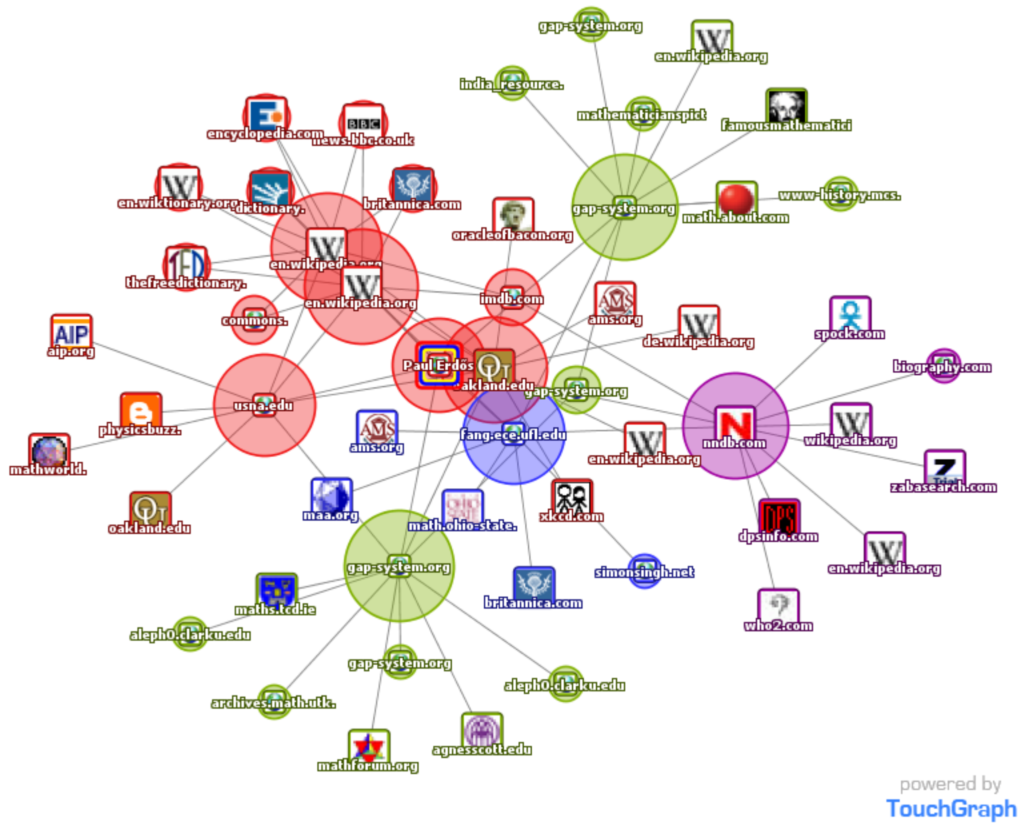

Fig. 1.2: A visualization of the World Wide Web around a Google query for "Paul Erdős" using the TouchGraph GoogleBrowser [Tou].

circuit design.

There are many additional applications of graph drawing. In bioinformatics, for example, representations of evolutionary trees, phylogenetic trees, molecular maps, genetic maps, biochemical pathways, and protein functions elicit graph drawing problems [HMM00]. See [Tam06] for a detailed description of bioinformatics-related applications of graph drawing, and see [DBETT98] for an extensive general discussion on a variety of applications of graph drawing and the particular metrics of interest in those applications.

1.1.2 Information-Based Complexity of a Drawing

In addition to the many applications, we can consider the complexity of a drawing of a graph from an information-theoretic perspective. We can define the complexity of a drawing by how much information is required to *encode* the drawing. Suppose that we are given a drawing of a graph in the plane in which edges map to *polygonal curves* (piecewise linear

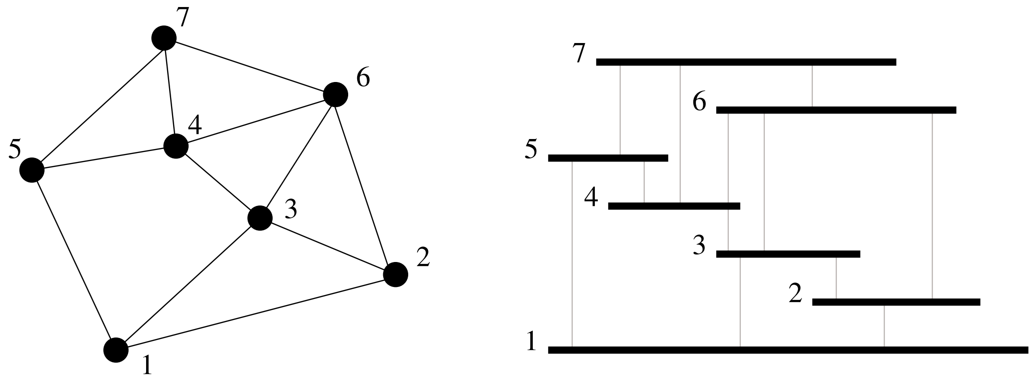

Fig. 1.3: A planar embedding of a graph (left) and visibility representation of the same graph (right). In the visibility representation, the vertices map to the horizontal segments and the vertical visibility lines are drawn to represent edges.

curves). Assume that the points at which the vertices are embedded can be represented in coordinates of finite precision. Furthermore, assume that the points at which each polygonal curve bends can also be represented in coordinates of finite precision. Then, it follows that we have a simple paradigm for encoding such a drawing. That is, in addition to the combinatorial information, we can encode the coordinates of the vertices and the bends. To avoid discussing this concept too deeply, we will simply assert this computationally based way of representing planar graphs and discuss a simple and natural metric for the complexity associated with such a representation. Specifically, we give preference to drawings that minimize the number of bends per edge, while maintaining the constraint that vertices and bends map to lattice points on a small grid. From here on, the *edge complexity* of a drawing of a graph is defined by the maximum number of times any edge bends. This metric appeals to a variety of the applications discussed previously and leads to several interesting theoretical questions, which we discuss next.

1.1.3 Embedding a Planar Graph with Fixed Vertex Locations

Consider the representation of a planar graph where vertices map to points and edges map to noncrossing polygonal curves. A fundamental question is whether a planar graph always admits such a representation in which the polygonal curves have no bends. This question was addressed by a classic result of Fáry [Fár48], which showed that all planar graphs can be embedded in the plane so that edges map to straight lines (a result independently proven by Wagner [Wag36] and Stein [Ste51]). Furthermore, concerning the restriction that vertices map to lattice points on a grid, a result of de Fraysseix et al. [dFPP90] showed that all planar graphs can be embedded in the plane in a manner such that edges map to straight lines and vertices map to lattice points on a $(2n-4) \times (n-2)$ grid. In the same year, Schnyder [Sch90] independently showed an equivalent result, for which the vertices map to lattice points on an $(n-2) \times (n-2)$ grid.

A second question, which is the primary focus of our work, is whether such a representation always exists for any choice of vertex locations. We consider two different variations of this problem. First, we assume that the mapping from the vertices onto the plane is fixed (has been pre-specified). This problem was addressed by Pach and Wenger [PW98], who showed that such an embedding always exists in which edges map to polygonal curves with at most $O(n)$ bends. In the second variation of this problem, we allow the mapping from the vertices onto the fixed points to be chosen freely, requiring only that each vertex map to a unique point. This problem was addressed by Kaufmann and Wiese [KW02], who showed that such an embedding always exists in which edges map to polygonal curves with at most a constant number of bends. Our results give a generalized method that encapsulates a solution to both of these problems and generalizes to the related problem of embedding many planar graphs simultaneously.

1.2 Simultaneous Planar Embeddings

Just as we considered the class of structures defined by graphs having planar embeddings, we can consider the class defined by the *superpositions* of such graphs. That is, we consider the combinatorial structures given by k planar graphs $G_1, G_2, \ldots, G_k$ over a common vertex set V. Generalizing the notion of a planar embedding, we can ask whether $G_1, G_2, \ldots, G_k$ admit a *simultaneous planar embedding*. That is, a simultaneous planar embedding corresponds to a planar embedding of each of $G_1, G_2, \ldots, G_k$ in which the vertices in V are commonly mapped. See Figure 1.4 for an example of a simultaneous planar embedding.

Simultaneous embeddings were considered by Tutte [Tut63], who described a method for embedding a planar graph simultaneously with its dual so that edges only cross their corresponding dual edge. As with planar embeddings, the problem of drawing a simultaneous planar embedding of a superposition of planar graphs with a variety of constraints has many important applications. The concept was introduced in [BCD$^+$03], where several of these applications are described. In particular, one can consider problems in VLSI cicuit design that concern embeddings of planar graphs, where edges are permitted to span multiple *layers*. In some models this corresponds precisely to the problem of constructing a simultaneous planar embedding of k graphs (each graph is embedded on a unique layer). Various metrics of interest for this particular problem and related models are discussed in [AKS91].

By a theorem of Pach and Wenger [PW98], any number of planar graphs can be simultaneously embedded in the plane in a way such that edges map to polygonal curves with at most $O(n)$ bends. For the special case of 2 graphs, a result of Erten and Kobourov [EK04] shows that 3 bends suffice to embed two graphs on a $O(n^2) \times O(n^2)$ grid, where vertices are located at lattice points. Moreover, their construction can be generalized to show that 2 bends suffice if the grid size is unconstrained.

If we consider the problem of simultaneously embedding k planar graphs where the vertex locations are fixed, then by the same result of Pach and Wenger [PW98], it follows

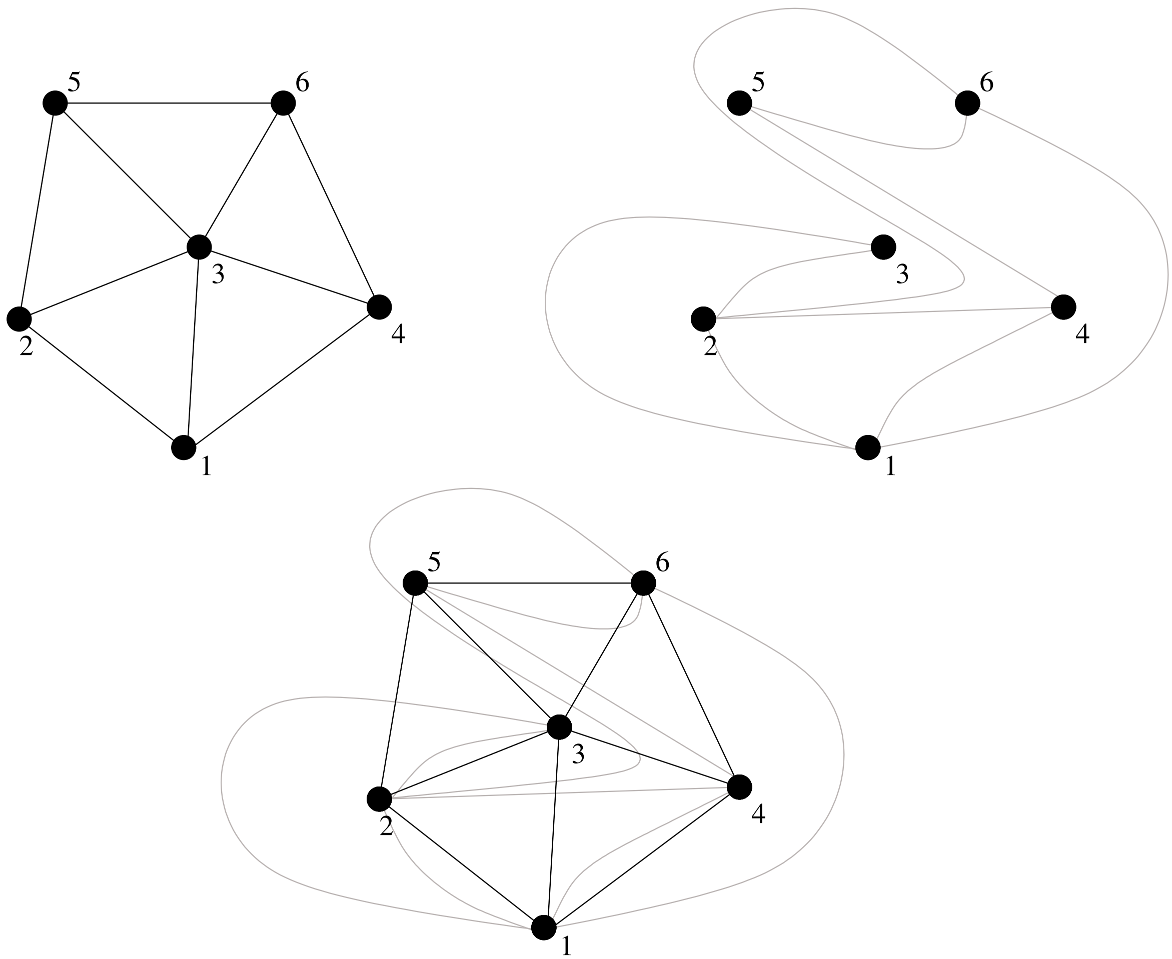

Fig. 1.4: Example of simultaneous planar embedding of two graphs. The first row shows two separate planar graphs over a common vertex set. The second row shows a simultaneous planar embedding of these two graphs.

that we can construct the simultaneous embedding even with a fixed vertex mapping. But, this result gives no guarantees on the complexity of the edges, other than that they map to polygonal curves with at most $O(n)$ bends. Our results show that this bound can be improved.

Considering lower bounds, one can show that not all pairs of planar graphs admit simultaneous planar embeddings in which edges map to straight lines. Furthermore, it was shown in [EBGJ$^+$07] that deciding whether two planar graphs permit such an embedding is **NP**-hard (and this decision problem is not known to be in **NP**). Our results give non-trivial lower bounds on the number of bends required given that the vertex locations are fixed.

1.3 Overview

The main result of this thesis is the general technique for embedding planar graphs described in Chapter 3. The number of bends in the embedding given by the technique is inherently adaptive in certain properties of the graph being embedded. Thus, in addition to the worst-case analysis, we consider how well the technique performs on average on uniformly random input. This analysis requires several preliminary results, which we discuss in Chapter 2.

In addition to the analysis, the method for constructing the embedding utilizes some well-studied graph drawing techniques. The required techniques are described in Chapter 2. We give proofs of almost all of the necessary results, and thus the material of this thesis is, for the most part, self-contained.

As is described in Chapter 3, the general embedding technique can effectively be applied to solve both variations of the problem of embedding a planar graph with fixed vertex locations. Moreover, the technique is sufficiently abstract to generalize to a method for constructing simultaneous planar embeddings with fixed vertex locations. In the case where the mapping is pre-specified, our technique guarantees embeddings with at most $8n - 7$ bends per edge in the worst case and, on average, at most $\frac{16}{3}n - 1$ bends per edge. When the mapping is not pre-specified, our technique guarantees embeddings with at most $O(n^{1-2^{1-k}})$ bends per edge in the worst case and, on average, at most $O(n^{1-\frac{1}{k}})$ bends per edge, where k is the number of graphs in the simultaneous embedding. Furthermore, we give evidence that our upper bound for the problem of simultaneously embedding k graphs might be optimal by proving an average-case lower bound. Specifically, we show in Chapter 4 that at least $\Omega(n^{2-\frac{2}{k}})$ bends in total are necessary with high probability.

The lower bounds proven in Chapter 4 are the consequence of a general result on encoding planar graphs. Specifically, we give a method for encoding planar graphs that is adaptive in the number of bends in an embedding of the graph. The reasoning is that if a graph can be guaranteed (or even in some small probability) to be embeddable with some small number of bends, then this implies that the graph can always (or with a small probability) be encoded with fewer bits than the information theoretic lower bound. From this contradiction, the lower bound on the number of bends follows.

We conclude with a brief summary of the results and mention some possible future work and open questions.

Chapter 2

Preliminaries

The area of graph drawing is rich with techniques for embedding planar graphs with a variety of restrictions. A general heuristic for drawing graphs is to enforce as much structure on the underlying graph as is permitted. Typically, the problem of drawing a graph to meet certain restrictions reduces to the problem of drawing any supergraph of the original graph, while meeting the same restrictions. This rule applies to the problems we address, and our solutions take advantage of well-established results on augmenting planar graphs to enforce additional structure. This chapter outlines these fundamental techniques.

The analysis of our main graph drawing technique, which is introduced in Chapter 3, relates strongly to bounds on the size of subsequences of permutations in which elements occur in increasing order. In this chapter, we prove bounds on the size of such subsequences, both in probability and in the worst case. Furthermore, we are interested in the problem of partitioning the elements of one or more permutations into a minimum-sized set of increasing subsequences. Problems of this nature have a rich history, and we introduce the particular results that apply to our graph drawing technique in this chapter.

2.1 Methods for Drawing Planar Graphs

2.1.1 Enforcing Hamiltonicity

In 1956, Tutte [Tut56] proved that all 4-connected planar graphs contain a Hamiltonian cycle. A less general result was shown for 4-connected maximal planar graphs by Whitney [Whi31] in 1931. More general results of this nature have since been shown by Sanders [San97] and by Harant and Senitsch [HS09]. In addition to the existence of a Hamiltonian cycle, it has been shown that a Hamiltonian cycle can be found efficiently. We start by stating this result, which is due to Chiba and Nishizeki [CN89].

Lemma 2.1.1 ([CN89]). *Let G be a 4-connected planar graph with vertex set V. Then, G contains a Hamiltonian cycle, which can be found in $O(|V|)$ time.*

In the application of graph drawing, edges can typically be added to graphs to enforce both 3-connectivity and maximal planarity since drawings of graphs implicitly give drawings of their subgraphs. However, it is often helpful to assume additional structure on the input graphs. For instance, if we can assume the graphs are 4-connected, Lemma 2.1.1 guarantees the existence of an algorithm for finding a Hamiltonian cycle in each graph. In general, we cannot enforce Hamiltonicity by only adding edges, in which case we rely on the result of Lemma 2.1.2. The result depends on the linear-time enumeration of *triangles* (by which we mean the complete graph on 3 vertices) in planar graphs. Bar-Yehuda and Even [BYE82] showed that all triangles with one edge in a depth-first search tree can be enumerated in linear time. The result then follows by the bounded *arboricity* of planar graphs. That is, every planar graph can be decomposed into at most 3 spanning forests. The algorithm was later simplified by Chiba and Nishizeki [CN85].

Lemma 2.1.2. *Let G be a planar graph with vertex set V. A 4-connected planar supergraph G' of a subdivision of G, resulting from at most 1 subdivision per edge, can be constructed in $O(|V|)$ time.*

Proof. We assume that G is a maximal planar graph, or otherwise, we could add edges. Since maximal planar graphs are necessarily 3-connected, we assume that G is 3-connected but not 4-connected. Thus, G contains 3 vertices u, v, w whose removal separates G into more than 1 component. If we delete u and v, the resulting graph is connected but not 2-connected, and w is a cut vertex. By adding back v, the graph becomes 2-connected and so w and v lie in at least 2 common faces. Since G is assumed to be a maximal planar graph, each face is a triangle, and thus w and v must be adjacent. By symmetry, u, v, w are all mutually adjacent. Thus, G is not 4-connected if and only if it contains 3 mutually adjacent vertices whose deletion separates G into more than one component; that is, G contains a separating triangle.

Let u, v, w define a separating triangle. If we subdivide the edge uv we get two faces of degree 4, corresponding to the faces incident to uv in G. We can re-triangulate these faces without adding the edge uv. Thus, since we retain 3-connectivity and u is no longer adjacent to v, the vertices u, v, w no longer form a separating triangle. Moreover, we have not introduced any separating triangles since each triangle that is incident to the new subdivision vertex is a face and does not separate G. Thus, if we can iterate over all separating triangles in linear time, the claim holds.

To perform the iteration of triangles, we rely on the bounded *degeneracy* of G. That is, since G is from a minor-closed family of graphs, we can orient the edges of G so that the maximum outgoing degree is constant. In particular, we can assume that the average vertex degree is at most 5 by Euler's formula, and thus orient G so that the maximum outgoing degree is 5. Let D be such an orientation of G. We can therefore exhaustively enumerate all separating triangles by iterating over vertices in a topological order of D, and for each vertex v, check whether each pair of successors, together with v, form a triangle. Each such comparison can be done in $O(1)$ time by hashing all pairs of adjacent vertices.

Furthermore, we can verify whether a given triangle is a separating triangle by testing if v has a neighbour both inside and outside the triangle. Since, for each outgoing edge, we perform only a constant-time test with at most a constant number of other outgoing edges, the total runtime is linear in the number of edges. □

Corollary 2.1.3. *Let G be a planar graph with vertex set V. A Hamiltonian planar supergraph G' of a subdivision of G, resulting from at most 1 subdivision per edge, can be constructed in $O(|V|)$ time, and a Hamiltonian cycle C in G' can be found in $O(|V|)$ time.*

Proof. Follows immediately by combining Lemma 2.1.2 with Lemma 2.1.1. □

2.1.2 Book Embeddings on Two Pages

Given a set P of n points in the plane, it is natural to ask whether a planar graph can be embedded such that its vertices map to points in P. A result of Kaufmann and Wiese [KW02] shows that a planar graph G can be embedded on any point set of size $n = |V(G)|$, where edges are mapped to polygonal curves with at most 2 bends. Thus, we say that any point set of size n is *2-bend universal* for planar graphs on n vertices. The essential technique behind these results is used in the construction of Lemma 2.1.4 (a well-established drawing technique). One way to think of this construction is by way of considering the embeddings of graphs on a book, which were first studied by Ollmann [Oll73].

Informally, a *book embedding* of a graph G is an embedding of G onto the surface of a topology in the form of a book, defined by a set of half-planes, the *pages*, that intersect at their boundaries, the *spine*. Vertices are restricted to being drawn on the spine and edges are restricted to being drawn on exactly one page. The *book thickness* of a graph G refers to the minimum number of pages required by any book embedding of G. Any graph of book thickness at most 2 is clearly planar and, more specifically, is the subgraph of a Hamiltonian planar graph. This follows as we can always add the edges between consecutive vertices and the two outer vertices on the spine without introducing crossings. The converse is in fact also true. That is, if G is a subgraph of a Hamiltonian planar graph, then G has book thickness at most two, as is shown by the following lemma.

Lemma 2.1.4. *Suppose that G is a Hamiltonian maximal planar graph on n vertices V. Then, we can construct a planar embedding of G on a $(2n-2) \times (2n-2)$ grid such that*

1. *the vertices of G lie on the middle row of the grid at distinct lattice points,*
2. *all vertices consecutive along the middle row are joined by a straight edge,*
3. *all remaining edges lie strictly in the upper half or the lower half of the grid, except at their end points, and*
4. *edges are embedded as polygonal curves with at most 1 bend, each of which is at a lattice point.*

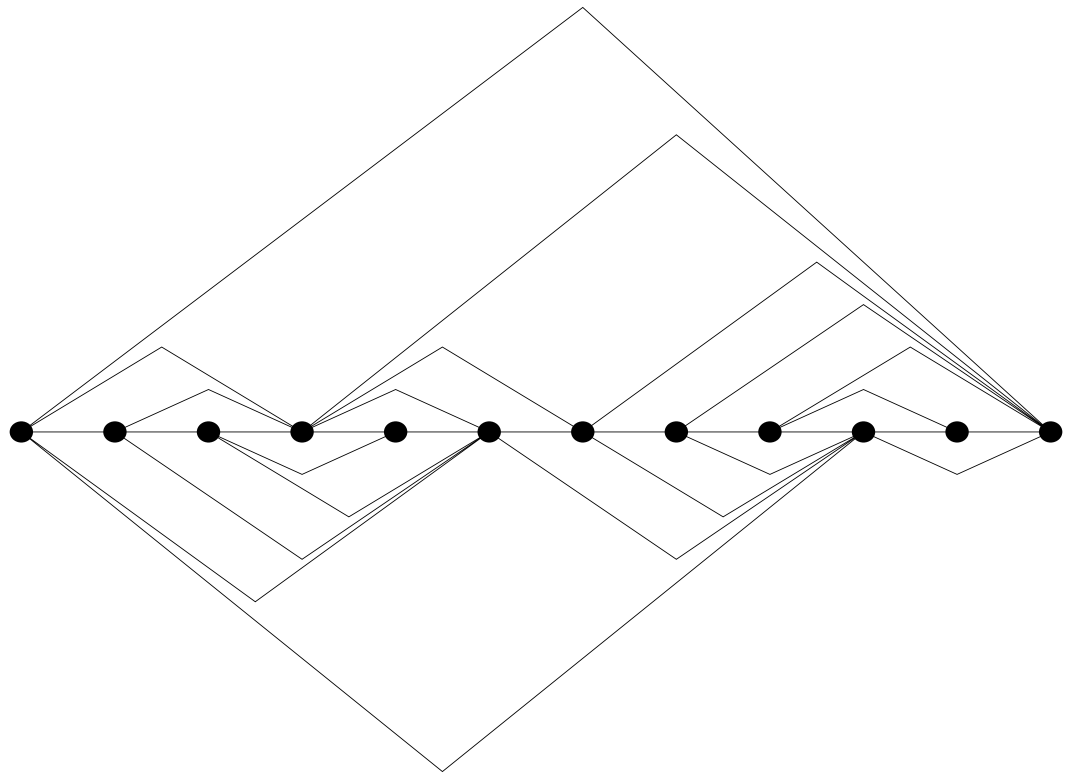

Fig. 2.1: An example of a book embedding of a Hamiltonian planar graph. The drawing was produced by an implementation of the method described in Lemma 2.1.4.

Proof. Let C be a Hamiltonian cycle in G. Let $v_1, v_2, \ldots, v_n$ be the vertices in V in the order they occur in C along one direction. For each $i \in \{1, \ldots, n\}$, embed v_i at the $(i-1)$-th integer coordinate on the middle row of the $(n-1) \times (2n-2)$ grid. Since these vertices are ordered by their occurrence in C, we can add the edges of C to satisfy conditions 1 and 2, where the edge from v_1 to v_n is drawn along the upper boundary of the grid. In any planar embedding of G, it follows that all remaining edges are either inside C or outside C. That is, G can be decomposed into 2 outerplanar graphs with C defining the outer face. Let (v_i, v_j), where $i < j$, be an edge inside C in the planar embedding of G with the edge (v_1, v_n) on the outer face. We can draw (v_i, v_j) in the upper half of the grid as a piecewise linear curve through the points $(i-1, 0), (\frac{1}{2}(i+j)-1, j-i-2), (j-1, 0)$, where the second coordinate is 0 along the middle row, positive above it, and negative below it. By the outerplanarity of the subgraph of G induced by the edges on or inside C, it follows that no edges can cross. Indeed, the endpoints of any two edges are non-overlapping, and the slopes of the line segments composing each edge are defined so that no edge crosses with one whose endpoints it contains. By symmetry, we can embed the edges outside C in the lower half in the same manner. All bend points are at lattice points if we refine the grid by replacing each column with 2 columns. That is, the vertices and bends are at lattice points on a $(2n-2) \times (2n-2)$ grid. See Figure 2.1 for an example of a drawing produced by an actual implementation of this construction on a random Hamiltonian planar graph. $\square$

Lemma 2.1.4 combined with Corollary 2.1.3 guarantees that any planar graph can be drawn such that its vertices map to points on a line and edges map to polygonal curves with at most 3 bends. Indeed, we can augment any planar graph so that it is Hamiltonian and draw the resulting graph using the book embedding technique, which gives the desired drawing of the original graph by treating the subdivision vertices as bends. By removing the grid restriction, the technique can be easily modified to guarantee at most 2 bends per edge. This result is the best possible for points on a line [KW02]. However, it was shown by Everett et al. [ELLW07] that there exists a 1-bend universal point set of size n for all planar graphs on n vertices. This result is optimal in the sense that there does not exist a universal point set of size n permitting a straight-line embedding of all planar graphs on n vertices when n is sufficiently large [CK89].

Using the results on universal point sets, we can simultaneously embed any number of planar graphs with a common set of vertex locations, such that edges map to polygonal curves with at most 1 bend. However, if the graphs share a common vertex set and we require that each vertex is mapped to a common location in the plane for each of its graphs, the problem remains open on how to construct a simultaneous planar embedding with few bends. For 2 graphs, Erten and Kobourov [EK04] showed that 2 bends suffice. Our results show that for k at least $\Omega(\log n)$, $O(n)$ bends are necessary if the points are restricted to be in convex position (we prove this in Chapter 4). But, for a uniformly random set of k planar graphs we can construct a simultaneous embedding where vertices are mapped onto any fixed point set and edges map to polygonal curves with at most $O(n^{1-\frac{1}{k}})$ bends. This upper bound is derived from properties of subsequences of random permutations, which are explained in the following sections.

2.2 Probability Concentration Bounds

2.2.1 The Second Moment

The upper bound described in Chapter 3 requires a few basic probabilistic techniques to analyze the average case performance. This section describes these general techniques, and in particular, the *second moment method*. We begin by stating and proving the following well-known inequality for the sake of completeness and to emphasize its simplicity.

Theorem 2.2.1 (Markov's Inequality). *Let X be a random variable. Then, for any $a > 0$,*

$$\mathbf{Pr}[|X| \geq a] \leq \frac{\mathbb{E}[|X|]}{a}.$$

Proof. Define an indicator variable $I_{(|X| \geq a)}$ that is 1 if $|X| \geq a$ and 0 otherwise. Then, since

$$aI_{(|X| \geq a)} \leq |X|$$

it follows by the linearity of expectation that

$$a\mathbb{E}\left[I_{(|X|\geq a)}\right] \leq \mathbb{E}\left[|X|\right]$$

or equivalently,

$$\frac{\mathbb{E}\left[|X|\right]}{a} \geq \mathbb{E}\left[I_{(|X|\geq a)}\right] = \mathbf{Pr}[|X| \geq a]$$

completing the proof. □

Markov's inequality can be used to establish concentration bounds on random variables. Specifically, we can use the second moment of the distribution of a random variable to bound the probability that the random variable deviates from its expectation by some positive value. This general bound is the well-known Chebyshev's inequality.

Theorem 2.2.2 (Chebyshev's Inequality)**.** *Let X be a random variable with mean μ and variance σ^2. Then, for any $a > 0$,*

$$\mathbf{Pr}[|X-\mu| \geq a] \leq \frac{\mathbf{Var}[X]}{a^2}.$$

Proof.

$$\mathbf{Pr}[|X-\mu| \geq a] = \mathbf{Pr}[(X-\mu)^2 \geq a^2],$$

which by Markov's inequality gives

$$\begin{aligned}\mathbf{Pr}[|X-\mu| \geq a] &\leq \frac{\mathbb{E}\left[(X-\mu)^2\right]}{a^2} \\ &= \frac{\mathbf{Var}[X]}{a^2}\end{aligned}$$

completing the proof. □

Substantially tighter bounds can be established for random variables with additional restrictions. For example, many distributions can be defined in terms of sums of independent random variables for which one can derive exponentially decreasing bounds such as the Chernoff bound. However, the polynomial concentration bounds given in terms of the second moment suffice for our results.

2.3 Longest Increasing Subsequences

2.3.1 Worst-Case Bounds

Results on the length of *monotonic subsequences* date as early as 1935 when Erdős and Szekeres [ES35] proved that any sequence of distinct numbers of length n^2+1 contains either an increasing or decreasing subsequence of length $n+1$. Various techniques for proving this result have since been discovered (see [Ste95] for six such proofs). The following particularly elegant proof is attributed to Hammersly [Ham72].

Theorem 2.3.1 (Erdős-Szekeres Theorem). *Any sequence of $(q-1)(r-1)+1$ distinct numbers contains either an increasing subsequence of length q or a decreasing subsequence of length r.*

Proof. Let $S = s_1, s_2, \ldots, s_n$ be any sequence of $n = (q-1)(r-1)$ distinct numbers. Partition the numbers of S into sets $P_1, P_2, \ldots$ as follows. Proceed inductively over the numbers of S in their sequence order, and assign each s_i to P_j, where j is the minimal index for which P_j does not contain a number greater than s_i, for $i = 1, 2, \ldots, n$. Let k be the maximal index for which P_k is non-empty. If $k \geq r$ the claim follows since, for each number in P_j, there must exist a greater number that was added to P_{j-1} in an earlier iteration (an earlier number in the sequence). Thus, assume $k < r$. It follows by the pigeonhole principle that, since $(q-1)(r-1)+1$ numbers are distributed to at most $r-1$ sets, some set must contain at least q numbers, which by construction forms an increasing subsequence. □

We consider two immediate consequences of Theorem 2.3.1. First, if one considers a sequence defined by a random permutation of the numbers $1, 2, \ldots, n^2+1$, then the probability that there exists an increasing subsequence of size at least $n+1$ is at least $\frac{1}{2}$. Second, by repeated application of the theorem, it follows that any k permutations of the numbers $1, 2, \ldots, n^{2^k}+1$ map a common set of at least $n+1$ numbers to increasing or decreasing order. It is a straightforward exercise to show that these results are the best possible. To circumvent the double exponential decay of the size of the longest increasing or decreasing subsequence, we turn to probabilistic analysis. That is, for random permutations we show that the length of the longest increasing subsequence can be bounded from below by a function that instead decays singly exponentially with respect to k.

2.3.2 Probabilistic Bounds

We begin by giving two simple proofs describing the probabilistic length of the longest increasing subsequence common to k permutations. It was established by Bollobás and Winkler [BW88] that the length of the longest increasing subsequence common to k permutations converges in probability to $c_k n^{\frac{1}{k+1}}$ for some constant $c_k > 0$. For completeness, we give a proof of an $\Omega(n^{\frac{1}{k+1}})$ bound that holds with probability 1 in the limit as $n \to \infty$. We then describe the expected value of the longest increasing subsequence common to k permutations.

Lemma 2.3.2. *Let $\pi_1, \pi_2, \ldots, \pi_k$, for $k \geq 2$, be uniformly random permutations over the set $S = \{1, 2, \ldots, n\}$. Then, there exists a set $\{a_1, a_2, \ldots, a_r\} \subseteq S$ such that $\pi_i(a_1) < \pi_i(a_2) < \cdots < \pi_i(a_r)$, for all $i = 1, 2, \ldots, k$, and $r \geq cn^{\frac{1}{k+1}}$ almost surely for any $c < e^{\frac{k-1}{k+1}}$. That is, $r \geq cn^{\frac{1}{k+1}}$ with probability that tends to 1 as $n \to \infty$ for any $c < e^{\frac{k-1}{k+1}}$.*

Proof. Let $\pi_1, \pi_2, \ldots, \pi_k$ be uniformly random permutations over $S = \{1, 2, \ldots, n\}$. Define X_r to be the number of r-element subsets of S that are mapped to increasing order in each permutation. Suppose each r-element subset has been uniquely numbered from 1 to $\binom{n}{r}$. Then, define $X_{r,i}$ to be an indicator variable that is 1 if subset i is mapped to increasing order in each permutation and 0 otherwise. By linearity of expectation,

$$\begin{aligned} X_r &= \sum_{i=1}^{\binom{n}{r}} X_{r,i} \\ \mathbb{E}[X_r] &= \sum_{i=1}^{\binom{n}{r}} \mathbb{E}[X_{r,i}] = \binom{n}{r}\left(\frac{1}{r!}\right)^k \end{aligned}$$

since each of the $r!$ orderings of a given subset are equally likely in each permutation. Furthermore, by defining T to be the set of all r-element subsets of S, we can bound the variance of X_r as

$$\begin{aligned} \mathbf{Var}[X_r] &= \mathbf{Var}[X_{r,1} + \cdots + X_{r,\binom{n}{r}}] \\ &= \sum_{i=1}^{\binom{n}{r}} \sum_{j=1}^{\binom{n}{r}} \mathbf{Cov}[X_{r,i}, X_{r,j}] \\ &= \sum_{i=1}^{\binom{n}{r}} \sum_{j=1}^{\binom{n}{r}} \mathbb{E}[X_{r,i} X_{r,j}] - \mathbb{E}[X_{r,i}]\,\mathbb{E}[X_{r,j}] \\ &= \sum_{i=0}^{r} \sum_{\substack{(A,B) \subseteq T^2, \\ |A \cap B| = i}} \left(\frac{i!}{r!r!}\right)^k - \left(\frac{1}{r!r!}\right)^k \\ &\leq \sum_{i=2}^{r} \sum_{\substack{(A,B) \subseteq T^2, \\ |A \cap B| = i}} \left(\frac{i!}{r!r!}\right)^k \end{aligned}$$

since the probability that two subsets with i common elements are both in increasing order is $(i!/r!^2)^k$.

Define Y to be the size of the largest subset of S whose elements are in increasing order in each permutation. $Y \geq r$ if and only if $X_r > 0$. Thus,

$$\begin{aligned}
\mathbf{Pr}[Y \geq r] &= \mathbf{Pr}[X_r > 0] \\
&= 1 - \mathbf{Pr}[X_r \leq 0] \\
&= 1 - \mathbf{Pr}[X_r - \mathbb{E}[X_r] \leq -\mathbb{E}[X_r]] \\
&\geq 1 - \mathbf{Pr}[|X_r - \mathbb{E}[X_r]| \geq \mathbb{E}[X_r]] \\
&\geq 1 - \frac{\mathbf{Var}[X_r]}{\mathbb{E}[X_r]^2}
\end{aligned}$$

where the last bound follows by Chebyshev's inequality. Thus, we proceed to bound $\mathbf{Var}[X_r]/\mathbb{E}[X_r]^2$ in terms of $r = cn^{\frac{1}{k+1}}$ as $n \to \infty$.

From the approximation for the variance of X_r,

$$\begin{aligned}
\frac{\mathbf{Var}[X_r]}{\mathbb{E}[X_r]^2} &\leq \frac{1}{\mathbb{E}[X_r]^2} \sum_{i=2}^{r} \sum_{\substack{(A,B)\subseteq T^2, \\ |A\cap B|=i}} \left(\frac{i!}{r!^2}\right)^k \\
&= \frac{r!^{2k}}{\binom{n}{r}^2} \sum_{i=2}^{r} \binom{n}{r}\binom{r}{i}\binom{n-r}{r-i}\left(\frac{i!}{r!^2}\right)^k \\
&= \sum_{i=2}^{r} \left(\frac{r!(n-r)!}{(r-i)!}\right)^2 \frac{i!^{k-1}}{n!(n-2r+i)!} \\
&= \sum_{i=2}^{r} \frac{i!^{k+1}}{n^i}\binom{r}{i}^2 \frac{(1-\frac{r}{n})\cdots(1-\frac{2r-i-1}{n})}{(1-\frac{1}{n})\cdots(1-\frac{r-1}{n})}
\end{aligned}$$

which, for $r = cn^{\frac{1}{k+1}}$ and $n \to \infty$, tends to

$$\begin{aligned}
\sum_{i=2}^{r} \frac{i!^{k+1}}{n^i}\binom{r}{i}^2 &\leq \sum_{i=2}^{r} (\sqrt{2\pi i})^{k+1} \frac{i^{i(k-1)}}{e^{i(k+1)}} \frac{(er)^{2i}}{n^i} \\
&= \sum_{i=2}^{r} (\sqrt{2\pi i})^{k+1} \left(\frac{c^{\frac{k+1}{k-1}} i}{er}\right)^{i(k-1)}
\end{aligned}$$

by replacing n with $\left(\frac{r}{c}\right)^{k+1}$. Substitute $b = c^{\frac{k+1}{k-1}}$. The terms in this summation can be shown to tend to zero by observing their first derivatives with respect to i. That is,

$$\frac{d}{di}\left[(\sqrt{2\pi i})^{k+1}\left(\frac{bi}{er}\right)^{i(k-1)}\right] = (\sqrt{2\pi})^{k+1}\left[(k-1)i\ln\frac{bi}{r} + \frac{k+1}{2i}\right] e^{(k-1)i\ln\frac{bi}{er} + \frac{k+1}{2}\ln i}$$

which is 0 if and only if

$$(1-k)i\ln\frac{bi}{r} = \frac{k+1}{2}. \tag{2.1}$$

Solutions to this equation can be defined using the *Lambert W function*. That is, if i satisfies the above equation then

$$i = \frac{k+1}{2(1-k)W(\frac{b(k+1)}{2r(1-k)})}$$

where $W(x)$ is the Lambert W function, defined implicitly as $W(x)e^{W(x)} = x$. As discussed in [CGH+96], $W(x)$ is a double-valued function for real x. Therefore, (2.1) has at most 2 solutions for $2 \le i \le r$; that is, the function $f(i) = (\sqrt{2\pi i})^{k+1}\left(\frac{bi}{er}\right)^{i(k-1)}$ has at most 2 critical points over $[2, r]$.

From (2.1), it follows that $f(i)$ is decreasing at $i = 2$ as long as $r \ge 2be^{\frac{1-k}{4(k-1)}}$. Since this function is initially increasing from 0 and is increasing as $i \to \infty$, it achieves its maximum over $[2, r]$ at either $i = 2$ or $i = r$ for sufficiently large r. Furthermore, by comparing $f(2)$ with $f(r)$ and assuming that $b = c^{\frac{k+1}{k-1}} < e$, we conclude that the terms in the summation achieve their maximum at $i = 2$, when r is sufficiently large. Thus, it follows that

$$\begin{aligned}\mathbf{Var}[X_r]/\mathbb{E}[X_r]^2 &\le r(\sqrt{4\pi})^{k+1}\left(\frac{2b}{er}\right)^{2(k-1)} \\ &\le (2\sqrt{\pi})^{k+1}c^{2k+3}n^{\frac{3-2k}{k+1}}\end{aligned}$$

which tends to 0 as $n \to \infty$ for any fixed $k \ge 2$. □

Corollary 2.3.3. *Let $\pi_1, \ldots, \pi_k$ be uniformly random permutations over $S = \{1, 2, \ldots, n\}$. Let S^* be the largest subset of S whose elements are mapped to increasing order in each permutation. Then, the expected size of S^* is $\Theta(n^{\frac{1}{k+1}})$.*

Proof. Let $\pi_1, \ldots, \pi_k$ be uniformly random permutations over $S = \{1, 2, \ldots, n\}$, and let X be the length of the largest subset of S whose elements are mapped to increasing order in each permutation. By combining Theorem 2.3.1 with Lemma 2.3.2, it follows that for sufficiently large n the $\mathbf{Pr}[X \ge n^{\frac{1}{k+1}}] \ge \frac{1}{2}$. Thus,

$$\mathbb{E}[X] \ge \frac{1}{2}n^{\frac{1}{k+1}}$$

by Markov's inequality. To complete the proof, we upper bound the expectation of X. Let Y_r be the number of r-element subsets of S mapped to increasing order in each permutation. Then,

$$Y_r \ge \binom{X}{r}$$

and therefore

$$\mathbb{E}[Y_r] \ge \mathbb{E}\left[\binom{X}{r}\right]$$

which by definition equals

$$\sum_{i=r}^{n} \binom{i}{r} \mathbf{Pr}[X = i] \geq \mathbf{Pr}[X \geq r].$$

Recall from the proof of Lemma 2.3.2 that $\mathbb{E}[Y_r] = \binom{n}{r}\left(\frac{1}{r!}\right)^k$. Thus,

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{i=1}^{n} i\mathbf{Pr}[X = i] \\
&\leq r\mathbf{Pr}[X < r] + n\mathbf{Pr}[X \geq r] \\
&\leq r + n\binom{n}{r}\left(\frac{1}{r!}\right)^k
\end{aligned}$$

which, for $r = 2en^{\frac{1}{k+1}}$ and sufficiently large n is

$$\begin{aligned}
&\leq r + n\left(\frac{ne}{r}\right)^r\left(\frac{e}{r}\right)^{kr} \\
&= r + n\frac{n^r e^{r(k+1)}}{r^{r(k+1)}} \\
&\leq 2en^{\frac{1}{k+1}} + \frac{n}{2^{2e(k+1)n^{\frac{1}{k+1}}}} \\
&\sim 2en^{\frac{1}{k+1}}
\end{aligned}$$

completing the proof. □

As was previously mentioned, the result of Corollary 2.3.3 was originally due to Bollobás and Winkler [BW88]. The exact asymptotic behaviour of the expected length of the longest increasing subsequence has been the subject of much literature. Concerning the case when $k = 1$, computations led Baer and Brock [BB68] to conjecture that the expected length was $\sim 2\sqrt{n}$. Work by Logan and Shepp [LS77] proved that the asymptotic size was at least $\sim 2\sqrt{n}$, and in the same year Vershik and Kerov [VK77] proved that the asymptotic behaviour was indeed $\sim 2\sqrt{n}$. This result was followed by a sequence of papers concerning the concentration of the length of the longest increasing subsequence about its expectation.

Frieze [Fri91] gave a concentration bound showing that the length of the longest increasing subsequence is within $n^{1/3+\epsilon}$, for $\epsilon > 0$, with probability at least $1 - e^{-n^{\beta}}$ for some constant $\beta > 0$. Bollobás and Brightwell [BB92] improved the concentration to $n^{1/4+\epsilon}$. Tighter bounds were later given by Talagrand [Tal95], and the asymptotic distribution of the longest increasing subsequence was determined by Baik et al. [BDJ99]. See [AD99] for an extensive survey on results of this nature. Concentration bounds have also been given for the case where $k \geq 2$ by Bollobás and Brightwell [BB92].

2.3.3 Algorithmic Problem and Its Variations

Given that non-trivial bounds on the length of the longest increasing subsequence exist, an obvious question is whether such sequences can be found efficiently. We refer to the algorithmic problem of finding the longest increasing subsequence in a sequence of n numbers as the *longest increasing subsequence* problem. Using a straightforward dynamic programming approach, the problem can be solved in $O(n^2)$ time. It was shown by Fredman [Fre75] that this technique can be improved to a $O(n \log n)$ algorithm and that this is optimal in the comparison-based model. However, on the RAM model, the runtime can be reduced to $O(n \log \log n)$, by the use of efficient sorting techniques and van Emde Boas trees (see [vEBKZ76] and [HS77]).

A related problem asks for the longest subsequence common to a set of k sequences. This classic computer science problem is often referred to as the *longest common subsequence* problem and has many applications in bioinformatics; for example, the longest common subsequence in the DNA of two or more organisms can be used as a metric for similarity [CLRS01]. Though the dynamic programming solution to this problem is well studied, it is exponential in the number of sequences. Specifically, if we assume all k sequences are of size n, then the standard dynamic programming solution has runtime at least $\Omega(n^k)$. Furthermore, it was shown by Maier [Mai78] that this problem is **NP**-hard for general values of k. We refer the reader to [BHR00] for a survey on algorithmic results pertaining to the longest common subsequence problem.

For our purposes, we are interested in solving a variant of these problems called the *longest common increasing subsequence* problem. That is, given k sequences $S_1, S_2, \ldots, S_k$ of numbers, we want to find an increasing sequence of numbers that is a subsequence of each of $S_1, S_2, \ldots, S_k$. By requiring that the longest common sequence is increasing seemingly reduces the complexity of the problem. In particular, this restriction admits a straightforward polynomial-time algorithm for finding such a sequence. Yang et al. [YHC05] showed that for two sequences, of size n and m, the longest common increasing subsequence can be found in $O(mn)$ time and space. Adaptive improvements by Brodal et al. [BKKK05] and Chan et al. [CZF$^+$07] reduced the runtime for cases when the size of the output is small and when the number of k-tuples of positions at which the k sequences have the same element is small. We state one such result (from [BKKK05]) for when the k input sequences are permutations over $1, 2, \ldots, n$, which we will use in Chapter 3.

Theorem 2.3.4 ([BKKK05]). *Let $\pi_1, \pi_2, \ldots, \pi_k$ be permutations over $S = \{1, 2, \ldots, n\}$. Then, the largest subset S^* of S whose elements are mapped to increasing order in each of $\pi_1, \pi_2, \ldots, \pi_k$ can be found in $O(\min\{kn^2, n \log^{k-1} n \log \log n + kn\})$ time.*

2.4 Partially Ordered Sets, Chains, and Antichains

Order theory provides a convenient and formal method for discussing properties of subsets with some notion of order. To help the discussion of monotonic subsequences, we give a brief definition of a few order-theoretic concepts.

A binary relation $\leq$ on a set S is referred to as a *partial order* if it is *reflexive*, *anti-symmetric*, and *transitive*. That is, for all $a, b, c \in S$,

- $a \leq a$,
- if $a \leq b$ and $b \leq a$ then $a = b$, and
- if $a \leq b$ and $b \leq c$ then $a \leq c$.

A set coupled with a partial order is referred to as a *partially ordered set*. Two elements a and b are *comparable* if either $a \leq b$ or $b \leq a$, otherwise they are *incomparable*. For a partially ordered set S, if either $a \leq b$ or $b \leq a$ for all $a, b \in S$, then the order is *total*, and S is referred to as a *totally ordered set*.

A set S' is called a *chain* in a partially ordered set S if $S' \subseteq S$ and S' is a totally ordered set. That is, for all $a, b \in S' \subseteq S$, it is the case that a and b are comparable. If it is the case that neither $a \leq b$ nor $b \leq a$ for all $a, b \in S' \subseteq S$, then S' is called an *antichain*. The cardinality of the largest chain $S' \subseteq S$ is called the *height* of S. The *width* of S is defined by the minimum number of chains that partition S; that is, the width of S is the minimum w for which there exists chains $S_1, S_2, \ldots S_w \subseteq S$ such that $S = S_1 \cup S_2 \cup \cdots \cup S_w$. See [DP02] for a general reference on order theoretic concepts.

2.4.1 Dilworth's Theorem

One of the fundamental properties of chains and antichains in partially ordered sets is summarized by Dilworth's theorem [Dil50]. For completeness, we will state and prove this well-known result using essentially the technique from [LP09].

Theorem 2.4.1 (Dilworth's Theorem)**.** *Let S be a partially ordered set, and let $S' \subseteq S$ be an antichain in S of maximum cardinality. Then, the width of S is equal to $|S'|$.*

Proof. König's theorem states that in any bipartite graph, the number of edges in the maximum matching equals the number of vertices in the minimum vertex cover. We can prove Dilworth's theorem from this well-known property of bipartite graphs as follows. Given a partially ordered set S, construct a bipartite graph G_S with vertex sets A and B, each with a vertex for each element of S. Add edges between a vertex v_a in A and a vertex v_b in B if their corresponding elements in a and b satisfy $a \leq b$ but not $b \leq a$. That is, $v_a \in A$ and $v_b \in B$ are adjacent in G_S if $a < b$. By König's theorem, there exists a maximum matching M with k edges and a vertex cover C with k vertices. Consider the set of elements $X \subseteq S$ whose corresponding vertices are not in C. Since C is a vertex cover, these corresponding vertices must all be pairwise nonadjacent, and thus X is an antichain of size at least $n - k$. Partition the elements of S into sets as follows. Initially, partition each element into its own set, then merge two partitions P_1 and P_2 if there exists $a \in P_1$ and $b \in P_2$ such that the corresponding vertices v_a and v_b define an edge in M. Since the number of partitions decreases by one for each edge in M, we partition S into $n - k$ sets. Moreover, each set in the partition defines a chain in S. This can be seen inductively as we start with a set of chains and merging occurs only when an extremal end of one chain is joined to the opposite extremal end of another chain. Thus, we have shown that the size of the largest antichain in S is at least the width of S. Furthermore, the size of the largest antichain cannot exceed the width since it can contain at most one element in each chain, and we have therefore proven the claim. □

A useful consequence of this proof is a constructive method for finding the minimum set of chains that partition a partially ordered set. That is, assume that, for a partially ordered set S, we can evaluate $a \leq b$, for $a, b \in S$ in $O(1)$ time. Then, we can construct the bipartite graph G_S as defined above, compute a maximum matching, and partition the vertices into chains based on the maximum matching. The partitioning can be done trivially in $O(n)$ time, and thus the complexity of finding the minimum set of chains that partition a partially ordered set reduces to the complexity of finding a maximum matching in a bipartite graph. We summarize this idea in the following theorem.

Theorem 2.4.2. *Given a partially ordered set S, define G_S by a bipartition of vertices A and B, each with a vertex v_a for each element in $a \in S$, and edges $(v_a, v_b) \in A \times B$ for each $a, b \in S$ such that $a < b$. Then, we can find a minimum set of chains that partition S in $O(\sum_{a,b \in S} f_s(a,b) + \text{max\_matching}(G_S))$ time, where $\text{max\_matching}(G_S)$ is the time to compute a maximum matching in the bipartite graph G_S and $f_s(a,b)$ is the time to evaluate $a < b$ for $a, b \in S$.*

2.4.2 Relation to Monotonic Subsequences

Let π be a permutation over $S = \{1, 2, \ldots, n\}$. Define a partial order $\leq_\pi$ over the elements in S such that $a \leq_\pi b$, for $a, b \in S$, if and only if $a \leq b$ and a precedes b in π when read from left to right (where $\leq$ is the natural order on S). Recall that a subset $C \in S$ in which every $a, b \in C$ satisfy either that $a \leq_\pi b$ or $b \leq_\pi a$ is called a *chain*. Furthermore, an *antichain* refers to a subset $A \in S$ in which every $a, b \in A$ satisfy neither that $a \leq_\pi b$ nor $b \leq_\pi a$. It is easy to see that chains in S correspond to increasing subsequences in π and antichains correspond to decreasing subsequences. We proceed to generalize this idea to higher dimensions.

Let $\Pi = \pi_1, \pi_2, \ldots, \pi_k$ be permutations over $S = \{1, 2, \ldots, n\}$. Define a partial order $\leq_\Pi$ over the elements in S such that $a \leq_\Pi b$, for $a, b \in S$, if and only if $a \leq b$ and a precedes b in each of $\pi_1, \pi_2, \ldots, \pi_k$ when read from left to right. Under this definition, chains in S correspond precisely to increasing subsequences common to all of $\pi_1, \pi_2, \ldots, \pi_k$. The *height* of S corresponds to the longest increasing subsequence common to all of $\pi_1, \pi_2, \ldots, \pi_k$.

Furthermore, the *width* of S is the minimum cardinality over all partitions of S in which each part is a chain.

This definition permits a useful geometric interpretation. That is, for each element $i \in S$, define a $(k+1)$-dimensional point $\mathbf{p}_i$ for which the first entry is i and the j-th entry is the index from the left of the occurrence of i in π_{j-1} for $j = 2, \ldots, k+1$. Let P be the set of points defined in this way for each element in S. We can define a partial order $\leq_P$ on the points in P such that $\mathbf{p} = (p_1, \ldots, p_{k+1})$, $\mathbf{q} = (q_1, \ldots, q_{k+1}) \in P$ satisfy $\mathbf{p} \leq_P \mathbf{q}$ if and only if $p_i \leq q_i$ for all $i = 1, \ldots, k+1$. Under this definition, a chain corresponds to a sequence of points such that, for each point $\mathbf{p}$ in the sequence, all points preceding $\mathbf{p}$ fall strictly in the inferior orthant with origin at $\mathbf{p}$. The algorithm of Theorem 2.3.4 essentially works by computing the longest such geometric interpretation of a chain in 2-dimensions in $O(n \log \log n)$ time using a type of Cartesian tree, and then extends to higher dimensions using a range tree (see [GBT84] for the solution to the geometric problem).

2.4.3 Minimum Partitions into Chains

In Chapter 3, we give a method for embedding planar graphs for which the bound on the number of bends depends on how well we can partition elements from k permutations into sets that form increasing subsequences common to each of the k permutations. We give probabilistic bounds on the minimum size of such a partition in this section, as well as bounds on the longest common increasing subsequence. Similar probabilistic results have been discussed in [Bri92] and [BW88], which include bounds on the convergence in probability and the concentration of measure. The proofs we give in this section establish identical asymptotic bounds (up to constant factors) and adhere to common encoding-based arguments.

Let $\Pi = \pi_1, \pi_2, \ldots, \pi_k$ be permutations over $S = \{1, 2, \ldots, n\}$. Define a partial order $\leq_\Pi$ over the elements in S such that $a \leq_\Pi b$, for $a, b \in S$, if and only if $a \leq b$ and a precedes b in each of $\pi_1, \pi_2, \ldots, \pi_k$ when read from left to right. The following results give probabilistic bounds on the width of S.

Lemma 2.4.3. *If $\pi_1, \pi_2, \ldots, \pi_k$ are chosen uniformly at random, then the width of S is at least $\Omega(n^{1-\frac{1}{k+1}})$ with high probability. That is, the width of S is at least $\Omega(n^{1-\frac{1}{k+1}})$ with probability at least $1 - n^{-c}$, for any constant c.*

Proof. To prove the claim, we give an encoding argument to derive a contradiction unless the width of S is at least $cn^{1-\frac{1}{k+1}}$ with high probability for some constant c. In particular, we utilize the information theoretic lower bound on the number of bits required to encode k permutations chosen uniformly at random. That is, if $\pi_1, \pi_2, \ldots, \pi_k$ are permutations chosen uniformly at random, then the size A of the unique encoding of $\pi_1, \pi_2, \ldots, \pi_k$ given by any fixed encoding scheme must satisfy the inequality

$$A \geq k \lg n! - \Delta$$

with probability at least $1 - 2^{1-\Delta}$. This follows since 2^H is the number of k-tuples of permutations, where $H = k \lg n!$, and thus the probability that a random k-tuple were one of the at most $2^{\lceil H - \Delta \rceil}$ with an encoding of size less than or equal to A is at most

$$2^{\lceil H-\Delta \rceil - H} \leq 2^{1-\Delta}$$

giving the desired probability.

Suppose that $\pi_1, \pi_2, \ldots, \pi_k$ are permutations over $S = \{1, \ldots, n\}$ chosen uniformly at random and that the partially ordered set induced by these permutations has width w. Let $C_1, C_2, \ldots, C_w$ be a set of disjoint chains that partition $\{1, \ldots, n\}$. We can encode each permutation as follows. Iterate over the elements in left-to-right order and, for each element, encode the index of the chain in which it is contained. Since each index can be encoded using $\lg w + 1$ bits, we encode at most $kn \lg w + kn$ bits. Finally, encode the chains $C_1, C_2, \ldots, C_w$. The number of partitions of $\{1, \ldots, n\}$ into w parts is at most

$$\frac{w^n}{w!}$$

and we can therefore encode the chains using at most $n \lg w - \lg w! + 1$ bits. Thus, w satisfies

$$kn \lg w + kn \lg e + n \lg w - \lg w! + 1 \geq k \lg n! - (\Delta + 1)$$

with probability at least $1 - \left(\frac{1}{2}\right)^{\Delta}$. Collecting the terms on the left-hand side of this inequality shows that, with probability at least $1 - \left(\frac{1}{2}\right)^{\Delta}$, w satisfies

$$(k+1)n \lg 2^{2 + \frac{\Delta+2}{(k+1)n}} w \geq kn \lg n$$

since we can upper bound $kn \lg e - \lg w!$ with $2(k+1)n$. Introducing an $n \lg n$ term to both sides gives

$$(k+1)n \lg 2^{2 + \frac{\Delta+2}{(k+1)n}} w + n \lg n \geq (k+1)n \lg n$$

or equivalently,

$$(k+1)n \lg 2^{2 + \frac{\Delta+2}{(k+1)n}} w n^{\frac{1}{k+1}} \geq (k+1)n \lg n$$

which, by dividing a factor of $(k+1)n$ and exponentiating both sides, gives

$$2^{2 + \frac{\Delta+2}{(k+1)n}} w n^{\frac{1}{k+1}} \geq n$$

or equivalently,

$$w \geq \frac{n^{1 - \frac{1}{k+1}}}{2^{2 + \frac{\Delta+2}{(k+1)n}}}$$

with probability at least $1 - \left(\frac{1}{2}\right)^{\Delta}$. □

As an aside, we can extend the result of Lemma 2.3.2 to the case when $k = 1$ from the previous lemma and Dilworth's theorem.

Lemma 2.4.4. *If $\pi_1, \pi_2, \ldots, \pi_k$ are chosen uniformly at random, then the height of S is almost surely $\Theta(n^{1-\frac{1}{k+1}})$.*

Proof. First, we claim that the height of S is almost surely at least $\Omega(n^{1-\frac{1}{k+1}})$. This claim holds when $k \geq 2$ by Lemma 2.4.3, thus we assume that $k = 1$. By Lemma 2.4.3 and Dilworth's theorem (Theorem 2.4.1), the largest antichain in S almost surely has size $\Omega(\sqrt{n})$. Since antichains occur in equal probability with chains in the case when $k = 1$ (that is, increasing subsequences occur in equal probability to decreasing subsequences), it follows that the height is $\Omega(\sqrt{n})$ almost surely.

Suppose that the largest chain C in S had size r. Then, we could encode the permutations $\pi_1, \pi_2, \ldots, \pi_k$ by encoding the elements in C, their positions in each of $\pi_1, \pi_2, \ldots, \pi_k$, and the order of the remaining elements. That is, we can encode the k permutations using

$$(k+1)\lg\binom{n}{r} + k\lg(n-r)!$$

bits, which by Stirling's approximation, is at most

$$(k+1)r\lg\frac{n}{r} + (k+1)r\lg e + k(n-r)\lg(n-r) - kn\lg e + kr\lg e + \frac{k}{2}\lg(2\pi n) + 1$$

bits. Thus, by the same argument that was used in the proof of Lemma 2.4.3, it follows that the inequality

$$(k+1)r\lg\frac{n}{r} + (2k+1)r\lg e + (n-r)\lg(n-r) + 1 \geq kn\lg n - (\Delta+1)$$

holds with probability at least $1 - \left(\frac{1}{2}\right)^{\Delta}$, from which we can conclude that

$$r\lg n - (k+1)r\lg r + (2k+1)r\lg e + \Delta + 2 \geq 0$$

or equivalently,

$$r \leq e^{\frac{2k+1}{k+1}} 2^{\frac{\Delta+2}{r(k+1)}} n^{\frac{1}{k+1}}$$

with probability at least $1 - \left(\frac{1}{2}\right)^{\Delta}$. Thus, it follows that

$$r > e^{\frac{2k+1}{k+1}} 2^{\frac{\Delta+2}{(k+1)n^{\frac{1}{k+1}}}} n^{\frac{1}{k+1}}$$

with probability at most $2^{-\Delta}$, and thus, r is almost surely $O(n^{1-\frac{1}{k+1}})$. By combining this derivation with the previous bound, we can conclude that r is almost surely $\Theta(n^{1-\frac{1}{k+1}})$. $\square$

The result of Lemma 2.4.3 shows that, with high probability, we cannot expect to be able to partition the elements into fewer than $\Omega(n^{1-\frac{1}{k+1}})$ chains. The following result shows that a partition of this size is in fact achievable with high probability. The technique used to prove the following result uses the geometric interpretation described in the previous section (2.4.2). That is, we consider a set of n points distributed in a $(k+1)$-dimensional hypercube according to the k uniformly random permutations. From this representation, we can utilize an observation of Brightwell [Bri92] to upper bound the number of antichains of a given size.

Lemma 2.4.5. *If $\pi_1, \pi_2, \ldots, \pi_k$ are chosen uniformly at random, then the width of S is at most $O(n^{1-\frac{1}{k+1}})$ with high probability.*

Proof. Let C be the d-dimensional hypercube $[0, n-1]^d$. Let P be a set of n points in C at integer coordinates. Define a partial order $\leq_P$ over the points in P such that points $\mathbf{p}, \mathbf{q}$ satisfy $\mathbf{p} \leq_P \mathbf{q}$ if and only if each pair of coordinates p_i of $\mathbf{p}$ and q_i of $\mathbf{q}$ satisfy $p_i \leq q_i$, where $\leq$ is the natural order of the integers. We will proceed to upper bound the number of bits required to encode P under the assumption that P is an antichain under the partial order $\leq_P$.

Suppose we partition the hypercube C into a maximal set of subcubes W of side length w. Without loss of generality, we can assume that n is a multiple of w, and thus, the subcubes are all aligned with C and have unique origins spanning $\{0, 1, \ldots, \frac{n}{w} - 1\}^d$. For each coordinate $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \{0, 1, \ldots, \frac{n}{w} - 1\}^d$ with at least one zero entry (x is the origin of a subcube that intersects one of the faces of C through the origin), define a *tower* consisting of the subcubes with origins at coordinates $(x_1 + iw, x_2 + iw, \ldots, x_d + iw)$, for each $i = 0, \ldots, \frac{n}{w} - 1$ (see Figure 2.2).

Consider a partial order $\leq_W$ over the subcubes such that two subcubes with origins $\mathbf{x} = (x_1, x_2, \ldots, x_d)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_d)$ are comparable if and only if $x_i \leq y_i$, for each $i = 1, \ldots, d$, or $y_i \leq x_i$, for each $i = 1, \ldots, d$. Let A be a maximal antichain of subcubes under the partial order $\leq_W$. As observed in [Bri92], each tower in C contains exactly one subcube in A. Thus, each maximal antichain consists of at most $d\left(\frac{n}{w}\right)^{d-1}$ subcubes since the number of towers is at most $d\left(\frac{n}{w}\right)^{d-1}$. Moreover, for a given subcube, there are at most two incomparable subcubes in each neighbouring tower. Thus, we can enumerate all antichains in W by considering the two possible subcube choices when iterating through the towers in a neighbour-to-neighbour order. Hence, the total number of antichains in W is at most $2^{d\left(\frac{n}{w}\right)^{d-1}}$.

Now, suppose P is a point set of size n in which the entire set of points form an antichain with respect to the partial order $\leq_P$. Then, it follows that all of the points in P lie inside the subcubes of some maximal antichain of W. Therefore, the number of such point sets is at most

$$d\left(\frac{n}{w}\right)^{d-1} \binom{dwn^{d-1}}{n}$$

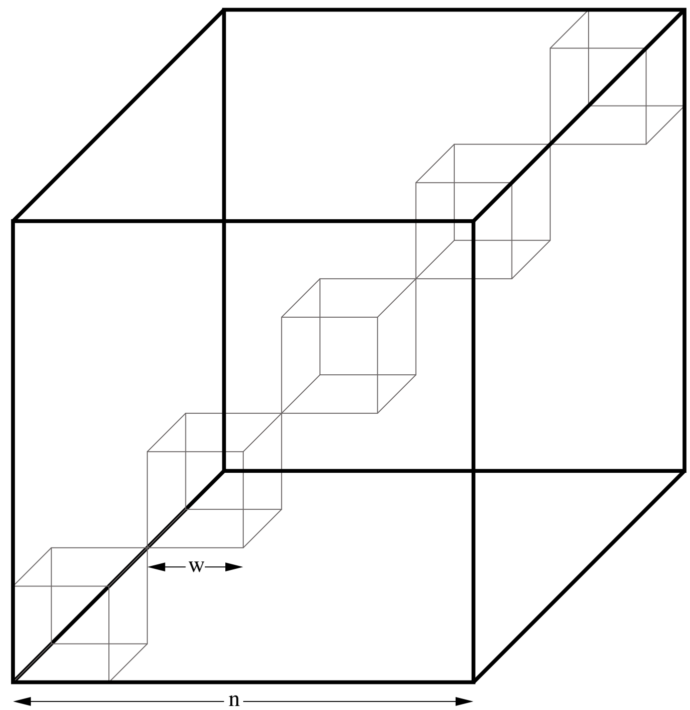

Fig. 2.2: A d-dimensional hypercube with side length n. The points associated with the permutations are located at integer coordinates inside this cube. The subcubes of side length w along the diagonal show a *tower* of cubes starting at the bottom-left corner.

and thus, it follows that we can encode P using at most

$$\lg\left(d\left(\frac{n}{w}\right)^{d-1}\right) + \lg\binom{dwn^{d-1}}{n}$$

bits. If we set $w = d^{\frac{1}{d-1}} n^{\frac{d-2}{d-1}}$ (which is a valid choice), then it follows that we can encode P using at most

$$\lg\left(d\left(\frac{n}{d^{\frac{1}{d-1}} n^{\frac{d-2}{d-1}}}\right)^{d-1}\right) + \lg\binom{dd^{\frac{1}{d-1}} n^{\frac{d-2}{d-1}} n^{d-1}}{n}$$
$$= n + \lg\binom{d^{\frac{d}{d-1}} n^{\frac{d^2-d-1}{d-1}}}{n}$$

bits, which by Stirling's approximation, is

$$\le n + n\lg\left(\frac{ed^{\frac{d}{d-1}} n^{\frac{d^2-d-1}{d-1}}}{n}\right)$$
$$= \lg(ed^{\frac{d}{d-1}})n + \left(\frac{d(d-2)}{d-1}\right) n \lg n$$

bits.

Now suppose that the largest antichain in S (in terms of the partial order defined by $\pi_1, \ldots, \pi_k$) has size r. We can encode which elements are in this antichain and the positions in each of $\pi_1, \ldots, \pi_k$ corresponding to these elements using $(k+1)\lg\binom{n}{r}$ bits. We can then encode the order of the remaining elements using $k \lg (n-r)!$ bits. Treating the subsequences of $\pi_1, \ldots, \pi_k$ over the elements in the antichain as a set of $(k+1)$-dimensional points in C, it follows that they can be encoded using at most $\lg(e(k+1)^{\frac{k+1}{k}})n + \left(\frac{(k+1)(k-1)}{k}\right) r \lg r$ bits. Thus, by the same argument that was used in the proof of Lemma 2.4.3, it follows that the inequality

$$(k+1)\lg\binom{n}{r} + k\lg(n-r)! + c_k n + \left(\frac{(k+1)(k-1)}{k}\right) r \lg r \geq k \lg n! - (\Delta + 1)$$

holds with probability at least $1 - \left(\frac{1}{2}\right)^{\Delta}$, where $c_k = \lg(e(k+1)^{\frac{k+1}{k}})$. By Stirling approximation, it then follows that

$$(k+1) r \lg \frac{n}{r} - kr \lg n + \left(\frac{(k+1)(k-1)}{k}\right) r \lg r + ((k+1)\lg e + c_k + 1) r + \Delta + 1 \geq 0$$

or equivalently,

$$r \lg n + ((k+1)\lg e + c_k + 1) r + \Delta + 1 \geq \left(\frac{k+1}{k}\right) r \lg r$$

with probability at least $1 - \left(\frac{1}{2}\right)^{\Delta}$. By dividing out a factor of r and exponentiating both sides, we conclude that

$$e^{k+1} 2^{c_k+1} 2^{\frac{\Delta+1}{r}} n \geq r^{\frac{k+1}{k}}$$

or equivalently,

$$r \leq e^k 2^{\frac{k(c_k+1)}{k+1}} 2^{\frac{k(\Delta+1)}{r(k+1)}} n^{1-\frac{1}{k+1}} \tag{2.2}$$

with probability at least $1 - \left(\frac{1}{2}\right)^{\Delta}$. We can therefore conclude that

$$r > e^k 2^{\frac{k(c_k+1)}{k+1}} 2^{\frac{\frac{k(\Delta+1)}{k}}{(k+1)n^{\frac{k}{k+1}}}} n^{1-\frac{1}{k+1}}$$

with probability at most $2^{-\Delta}$ since in the event that this inequality is satisfied it follows that the inequality 2.2 is not satisfied.

That is, we have shown that r is at most $O(n^{1-\frac{1}{k+1}})$ with high probability. By Dilworth's theorem (Theorem 2.4.1), it follows that the width of S is $O(n^{1-\frac{1}{k+1}})$ with high probability. □

Corollary 2.4.6. *If* $\pi_1, \pi_2, \ldots, \pi_k$ *are chosen uniformly at random, then the width of* S *is* $\Theta(n^{1-\frac{1}{k+1}})$ *with high probability.*

Proof. Follows immediately by Lemma 2.4.5 and Lemma 2.4.3. □

The result of Lemma 2.4.5 shows that we can expect, with high probability, to be able to partition the elements of S into at most $O(n^{1-\frac{1}{k+1}})$ chains. It is possible to contrive a set of permutations for which no pair of elements form a chain. Thus, we cannot give any non-trivial guarantees in the worst case on the width. But, we can always compute the optimal solution exactly in polynomial time by use of Theorem 2.4.2.

Chapter 3

Edge Complexity Upper Bounds

In this chapter, we describe a general technique for constructing an embedding of a planar graph in which vertices are mapped to pre-specified points and edges are mapped to polygonal curves that are disjoint except at their endpoints. The result utilizes a partitioning of the point set in which the vertices mapped to a given part in the partition satisfy certain constraints. The number of times an edge bends in the embedding is proportional to the number of parts in the partition. Thus, the complexity of the drawings produced by our method is determined by how effectively we can partition an arbitrary point set to meet the required constraints.

The best partitioning of the point set is largely determined by how the vertices are mapped onto the point set. We first consider the situation where this mapping is fixed. In this case, we give both average-case and worst-case upper bounds on the number of times each edge bends, which we show are optimal up to constant factors in Chapter 4. In the situation where the mapping is not fixed, we give bounds on how well the point set can be partitioned by our choice of mapping. Again, we consider both average-case and worst-case bounds on the number of times each edge bends, for which we prove near optimality in Chapter 4. Moreover, the results apply to the problem of constructing a simultaneous embedding of many planar graphs. The average-case results we consider are defined under the assumption that the input graph is chosen uniformly at random from the set of planar graphs on a given number of vertices.

The general outline of this chapter is as follows. Section 3.1 contains a formal statement of the problem where the mapping from the vertices onto the plane is pre-specified and summarizes our results in regards to this problem. Section 3.2 contains a formal statement of the problem where only the vertex locations are fixed; that is, the range of the bijective vertex mapping is pre-specified, but the actual permutation can be chosen freely. We state our results for this problem and mention how these results relate to the general problem where the vertex locations can also be chosen arbitrarily. Section 3.3 includes a description and proof of the main drawing algorithm, which we then use to derive the results for the two problems we consider.

3.1 Embedding Problem with a Pre-Specified Vertex Mapping

3.1.1 Problem Statement

We first consider the problem of constructing an embedding of a planar graph where the vertices map to pre-specified locations. That is, we are given a planar graph G with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and a set of points in the plane $P = \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ and are required to construct a planar embedding of G in which vertex v_i is mapped to the point $\mathbf{p}_i$ for all $i = 1, 2, \ldots, n$. It is fairly straightforward to observe that such an embedding always exists if the edges can be embedded as any curve. However, embeddings where edges are mapped to curves with less complexity are preferred. For this reason, we require that edges are represented as *polygonal curves*. Assuming such a representation, we measure the complexity of an edge in an embedding by the number of vertices in the polygonal curve representing the edge. That is, we are concerned with how many times an edge *bends*. Thus, we consider the problem of embedding G, with the pre-specified vertex mapping, such that edges map to polygonal curves with a bounded number of bends.

3.1.2 Summary of Results

By a result of Pach and Wenger [PW98], any planar graph G can be embedded, with a pre-specified mapping of the vertices, such that edges map to polygonal curves with at most $O(n)$ bends per edge. Moreover, they proved that this result is optimal in the strong sense that $\Omega(n)$ bends are almost surely necessary for $\Omega(n)$ edges if G was chosen uniformly at random from the class of perfect matchings. Using the technique described in Section 3.3, we give an alternative method for constructing such an embedding that also guarantees at most $O(n)$ bends per edge. However, our construction is fairly simple in that it admits a straightforward implemention and, up to constant factors, gives fewer bends in the worst case than the method of Pach and Wenger. The exact bound given in [PW98] on the number of bends per edge was $120n$, where n is the number of vertices. Though their bound can easily be improved by small constant factors, it is not obvious how to substantially reduce the bound. Our results use an entirely different technique that shows each edge can be drawn with at most $8n - 7$ bends in the worst case. Furthermore, if the planar graph is chosen uniformly at random (from the set of all planar graphs), then the number of bends is at most $\frac{16}{3}n - 1$ on expectation. The algorithm used to construct the embedding can be implemented to run in $O(n^2)$ time. As a total of $\Omega(n^2)$ bends are almost surely necessary, one cannot guarantee better performance. Furthermore, the problem of minimizing the total number of bends has been shown to be **NP**-hard in the case where the graph G is a perfect matching [BF96].

3.2 Simultaneous Embeddings Problem with Fixed Vertex Locations

3.2.1 Problem Statement

The second problem we consider concerns the construction of a simultaneous planar embedding of a set of graphs with fixed vertex locations. Given planar graphs $G_1, \ldots, G_k$, over a common vertex set $V = \{v_1, v_2, \ldots, v_n\}$, and a set of points in the plane $P = \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$, we are required to construct a simultaneous planar embedding of $G_1, G_2, \ldots, G_k$ in which each vertex v_i is mapped to a unique point in P for all $i = 1, 2, \ldots, n$. That is, we are required to embed each of $G_1, G_2, \ldots, G_k$ onto the plane with a common vertex mapping onto P. The mapping from vertices onto points is not pre-specified, and thus, the fundamental problem is choosing the best mapping that simultaneously admits an embedding with small edge complexity for each of the graphs. We limit our consideration to embeddings where edges map to polygonal curves and are therefore concerned with minimizing the number of bends.

3.2.2 Summary of Results

The version of this problem in which we have only 1 graph was first considered by Kaufmann and Wiese [KW02]. Their results showed that a planar graph can be embedded with vertices mapping to any fixed point set so that edges have at most 2 bends. However, their construction can lead to drawings where the size of the smallest bounding rectangle is exponentially larger than the size of the smallest bounding rectangle for the point set. By permitting 3 or 4 bends per edge they reduce this bound to $O(W^3)$ and $O(W)$, respectively, where W is the size of the smallest bounding rectangle for the original fixed point set. Our results generalize this result to the version of simultaneously embedding k graphs, for any positive integer k, for which we give both worst-case and average-case bounds on the number of bends. The average case corresponds to the problem where the graphs are chosen uniformly at random from the set of planar graphs on n vertices (with repetition), in which we are concerned with the expected number of bends. Furthermore, we prove in Chapter 4 that our average-case bound is close to optimal.

If we modify the problem to allow the point set to be chosen arbitrarily, the problem reduces to the general problem of constructing a simultaneous planar embedding of k graphs, minimizing the number of bends. For two graphs, a result of Erten and Kobourov [EK04] shows that 2 bends suffices. However, their method relies on the existence of an orthogonal direction in the plane for each graph, and thus does not generalize to 3 or more graphs. Clearly, our technique for constructing simultaneous embeddings on any point set extends to this problem for any number of graphs, but the number of bends is far from optimal in the case of having 2 graphs.

Alternatively, one can consider the problem where the graphs $G_1, G_2, \ldots, G_k$ do not share a common vertex set but are still required to be embedded on a common point set of size n, where $n = \max_{1 \le i \le k} |V(G_i)|$. Everett et al. [ELLW07] showed that, for all n, there exists a universal point set P of size n for which any planar graph with n vertices can be embedded so that its vertices map to points in P and its edges map to polygonal curves with at most 1 bend. Thus, if the point set can be chosen arbitrarily, $G_1, G_2, \ldots, G_k$ can be simultaneously embedded with at most 1 bend per edge. If the point set is pre-specified, then we can simply use the result of Kaufmann and Wiese to guarantee at most 2 bends per edge.

3.3 General Technique and Its Application

In the following sections, we describe a general technique for embedding planar graphs with fixed vertex locations. We begin by establishing a method for drawing the edges of a planar graph assuming that the graph's vertices have already been mapped to their corresponding points. More specifically, we assume that the vertices, and their associated points, have been partitioned into sets, each satisfying certain conditions on the underlying graph and the locations of the points. Edges are embedded as polygonal curves and the number of times an edge bends is proportional to the size of this provided partition. The description and proof of this technique is in Section 3.3.1. The subsequent sections describe how to effectively partition the vertices in the required way and give bounds on the size of this partition that in turn gives bounds on the edge complexity. In Section 3.3.2, we describe a method for partitioning the vertices with the assumption that the mapping from the vertices onto the points is pre-specified. In Section 3.3.3, we describe a method for partitioning the vertices common to k graphs with the assumption that the vertex locations are fixed. In both cases, we analyze the edge complexity guaranteed by using the drawing technique with these partitions.

3.3.1 The Main Lemma

Let P be an arbitrary point set of size n. In the case that the points in P can be specified in finite precision, we will bound the precision required to specify the drawings we produce. That is, we will first assume that there exists a $W \times H$ grid having the points in P at integer coordinates. When W and H are smaller, fewer bits are needed to specify the locations of the points. Thus, our goal is to produce a drawing in which both the points in P and the locations of bends lie at integer coordinates on a small grid (one with dimensions bounded in terms of W and H). In the case that the points in P require infinite precision, we will instead be concerned with minimizing the size of the smallest rectangle bounding the drawing we produce.

Consider an arbitrary direction $\hat{\mathbf{u}}$ for which the points in P occur at distinct distances, and let $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ be the points in P in the order they occur along this direction. Such

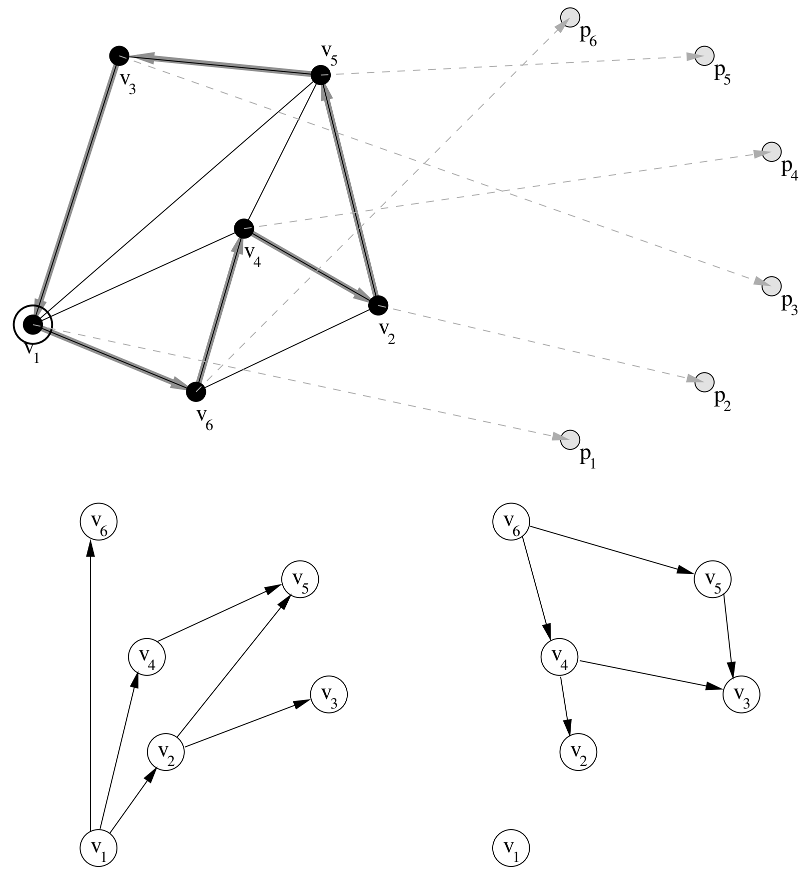

Fig. 3.1: An example pair of partial orders defined in terms of the orientation of a planar graph's Hamiltonian cycle and its pre-specified vertex mapping. The vertex labels are defined by the vertical order of their corresponding points. The top row shows the graph, the oriented Hamiltonian cycle, with starting point at v_1, from which the partial orders are defined, and the pre-specified mapping of the vertices onto their corresponding points. The second row shows a transitive reduction of the two partial orders, which are defined by the vertex labels and the order these vertices occur along the Hamiltonian cycle.

a direction always exists, but for most directions, the points will not occur at distances that can specified in finite precision. However, assuming the points were initially specified in finite precision, we can always find a direction for which the points occur at distinct distances and lie on a finite grid that is aligned with this direction. It is also clear that if specifying the points initially required infinite precision, all directions may preserve this requirement. To avoid discussing how to choose the best direction $\hat{\mathbf{u}}$, we will bound the precision/area requirements of the drawings we produce assuming that the direction $\hat{\mathbf{u}}$ has been fixed.

Consider a planar graph G over the vertex set V with a Hamiltonian cycle C. Let $f : V \to P$ be a fixed bijective mapping from the vertices in V to the points in P. Assume that $v_1, v_2, \ldots, v_n$ are the vertices in V, ordered so that v_i is required to map to $\mathbf{p}_i$ for

$i = 1, \dots, n$. Fix a cyclic orientation of C, and identify a unique vertex in C as the starting point. The order in which the vertices occur along the orientation of C from its starting point corresponds to a permutation $\pi : V \to [n]$. Define the rank of a vertex v_i, denoted $\text{rank}_C(v_i)$, to be equal to $\pi(v_i)$; that is, $\text{rank}_C(v_i)$ is the index of the occurence of v_i along the orientation of C from its starting point. Define two partial orders $\leq_1$,$\leq_2$ over the vertex set V such that, for $v_i, v_j \in V$,

1. $v_i \leq_1 v_j$ if and only if $i \leq j$ and $\text{rank}_C(v_i) \leq \text{rank}_C(v_j)$, and
2. $v_i \leq_2 v_j$ if and only if $j \leq i$ and $\text{rank}_C(v_i) \leq \text{rank}_C(v_j)$.

Thus, in the context of $\leq_1$, a chain refers to a set of vertices that occur along C in the same order their corresponding points occur along the direction $\hat{\mathbf{u}}$. Similarly, in the context of $\leq_2$, a chain refers to a set of vertices that occur along C in the opposite order their corresponding points occur along the direction $\hat{\mathbf{u}}$. See Figure 3.1 for an example pair of partial orders defined in terms of a Hamiltonian planar graph with a pre-specified vertex mapping.

Let $\lambda : V \to \{1, \dots, r\}$ be an r-labeling of the vertices in V. Define the i-th equivalence class of λ to be the set of vertices $\Lambda_i = \{v \in V : \lambda(v) = i\}$. We say that λ is an *embeddable labeling* if

1. each equivalence class Λ_i is non-empty,
2. the vertices in Λ_i are mapped by f to the points $P_i = \mathbf{p}_{j+1}, \mathbf{p}_{j+2}, \dots, \mathbf{p}_{j+|\Lambda_i|}$, where $j = \sum_{k<i} |\Lambda_k|$,
3. the vertices in Λ_i form a chain under the partial order $\leq_1$ or the partial order $\leq_2$.

It is clear that such a labeling always exists for any fixed vertex mapping f since, if we give each vertex a unique label, the corresponding singleton sets all form chains (for both partial orders). If two equivalence classes Λ_i, Λ_j have the property that they are chains under different partial orders, then we say that the equivalence classes Λ_i, Λ_j are *alternating*. We refer to the equivalence classes for which the elements form a chain under the partial order $\leq_1$ as *forward* classes, and we refer to the remaining equivalence classes as *backward* classes.

Lemma 3.3.1. *Let G be a Hamiltonian planar graph over the vertex set V. Suppose that, for a fixed vertex mapping f, λ is an embeddable labeling of V with equivalence classes $\Lambda_1, \Lambda_2, \dots, \Lambda_r$. Then, we can construct an embedding of G with the vertex mapping f, such that edges map to polygonal curves with at most $8r - 4\alpha - 4$ bends, where α is the number of consecutive pairs of alternating equivalence classes.*

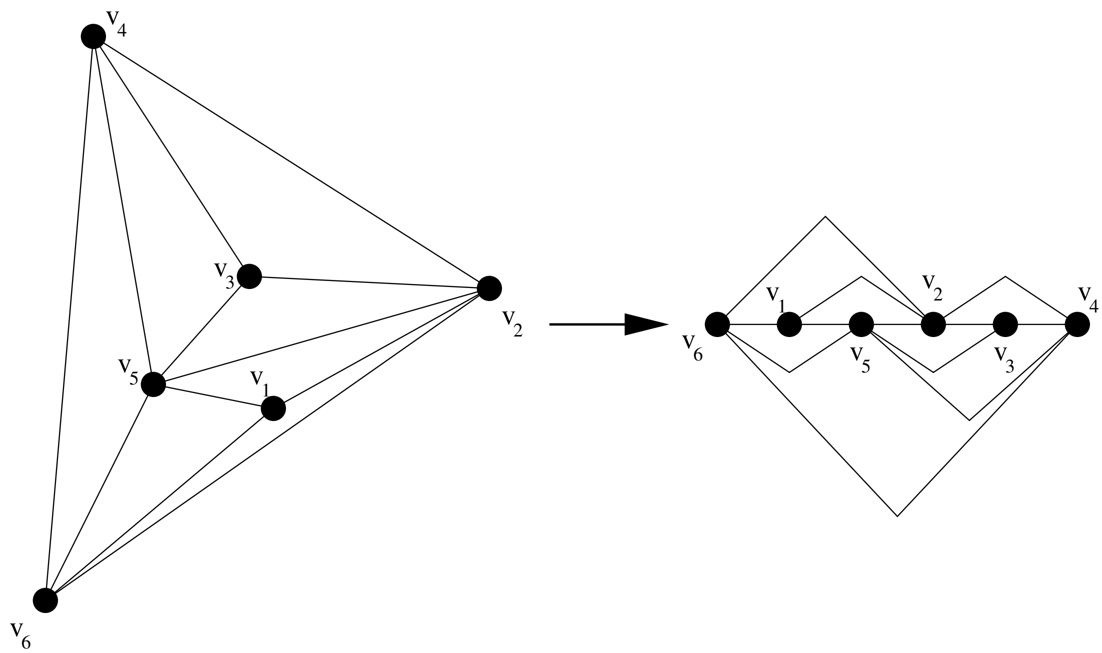

Fig. 3.2: The first step of the construction in Lemma 3.3.1. A book embedding is constructed using the technique described in Lemma 2.1.4 for the provided Hamiltonian planar graph.

Proof. Let C be the Hamiltonian cycle in C for which the embeddable labeling λ is defined. For convenience, assume that the plane is rotated such that $\hat{\mathbf{u}}$ is directed horizontally from left to right. Thus, the points $P = \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ are indexed by the order they occur from left to right. From the labeling λ, we can also assume that V can be partitioned into $V_1, \ldots, V_r$ such that V_i is mapped to the points $P_i = \mathbf{p}_{j+1}, \mathbf{p}_{j+2}, \ldots, \mathbf{p}_{j+|V_i|}$, where $j = \sum_{k<i} |V_k|$. Furthermore, the vertices in V_i are mapped to the points P_i so that they either occur from left to right or from right to left in the order they occur along the fixed orientation of C.

We begin by introducing the notion of a band. Formally, we define a band by the *Minkowski sum* of a curve Γ and a ball of radius δ in the l_1 metric. That is, the band corresponds to the set of points at a distance less than or equal to δ from a point in Γ under the l_1 metric. Suppose that two points $\mathbf{p_1}$ and $\mathbf{p_2}$ are *joined* by a band of radius δ, by which we mean that we are given a band defined by the Minkowski sum of a curve joining $\mathbf{p_1}$ to $\mathbf{p_2}$ and a ball of radius δ. It follows that as long as $\delta > 0$ we can draw any number of internally noncrossing curves that join $\mathbf{p_1}$ to $\mathbf{p_2}$ and are subsets of the point set defining the band. That is, the curves lie on the boundary or the interior of the band.

Since G is planar and Hamiltonian, we can construct a book embedding of G of thickness 2 by Lemma 2.1.4. Specifically, we can embed G on two pages separated by a Hamiltonian path H through the vertices in V. The Hamiltonian path H corresponds to the Hamiltonian cycle C less one of its edges. From this book embedding, we can separately embed each page (assigning the edges in H to the top page) in the plane. Each vertex is mapped

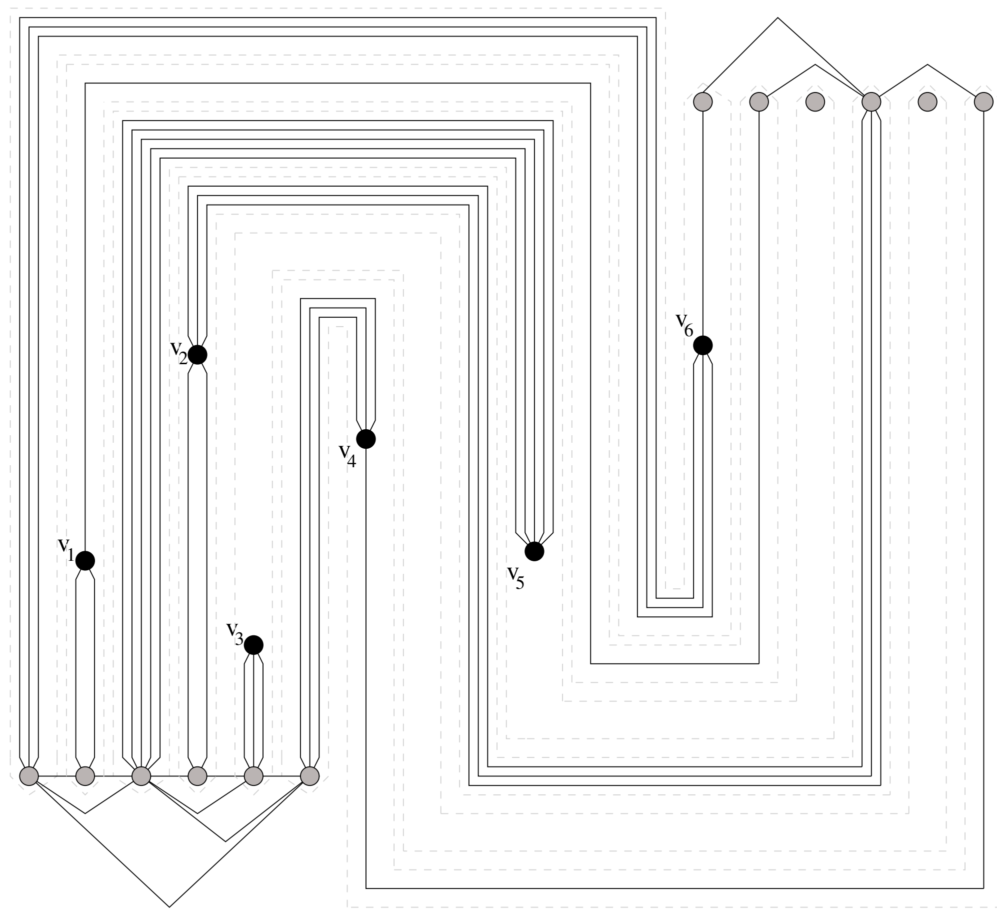

Fig. 3.3: The second step of the construction in Lemma 3.3.1. For each vertex v, a band $b(v)$ (shown with dashed grey lines) is routed through the fixed point on which v is required to be mapped (shown as a black dot). The band $b(v)$ starts and ends at the points on each page of the book embedding corresponding to v (shown as a gray dot). The edges incident to each vertex are routed through their corresponding bands, no two of which cross. The vertex partition used in the construction is $V_1 = \{v_1, v_2, v_3\}, V_2 = \{v_4, v_5\}, V_3 = \{v_6\}$.

twice, once to a point on the first page and once to a point on the second page. Suppose that, for each vertex v, the two images of v are joined by a band $b(v)$ of some non-zero radius $\delta(v)$. Furthermore, assume that no two of these bands intersect and that the only edges intersecting the band $b(v)$ are those incident to v. We claim that each vertex v can be mapped to any point in the interior of $b(v)$ so that we can modify the separate embeddings of each page into a complete embedding of G with these new vertex locations. This follows since we can stop each edge on each page at the point it crosses the boundary of the band corresponding to its incident vertices and then continue the embedding of the edge through these bands to the arbitrary vertex locations inside each band. Since no two bands intersect, only edges incident to a given vertex can cross. But, these crossings can easily be avoided as the edges come from separate ends of the band (from the separate

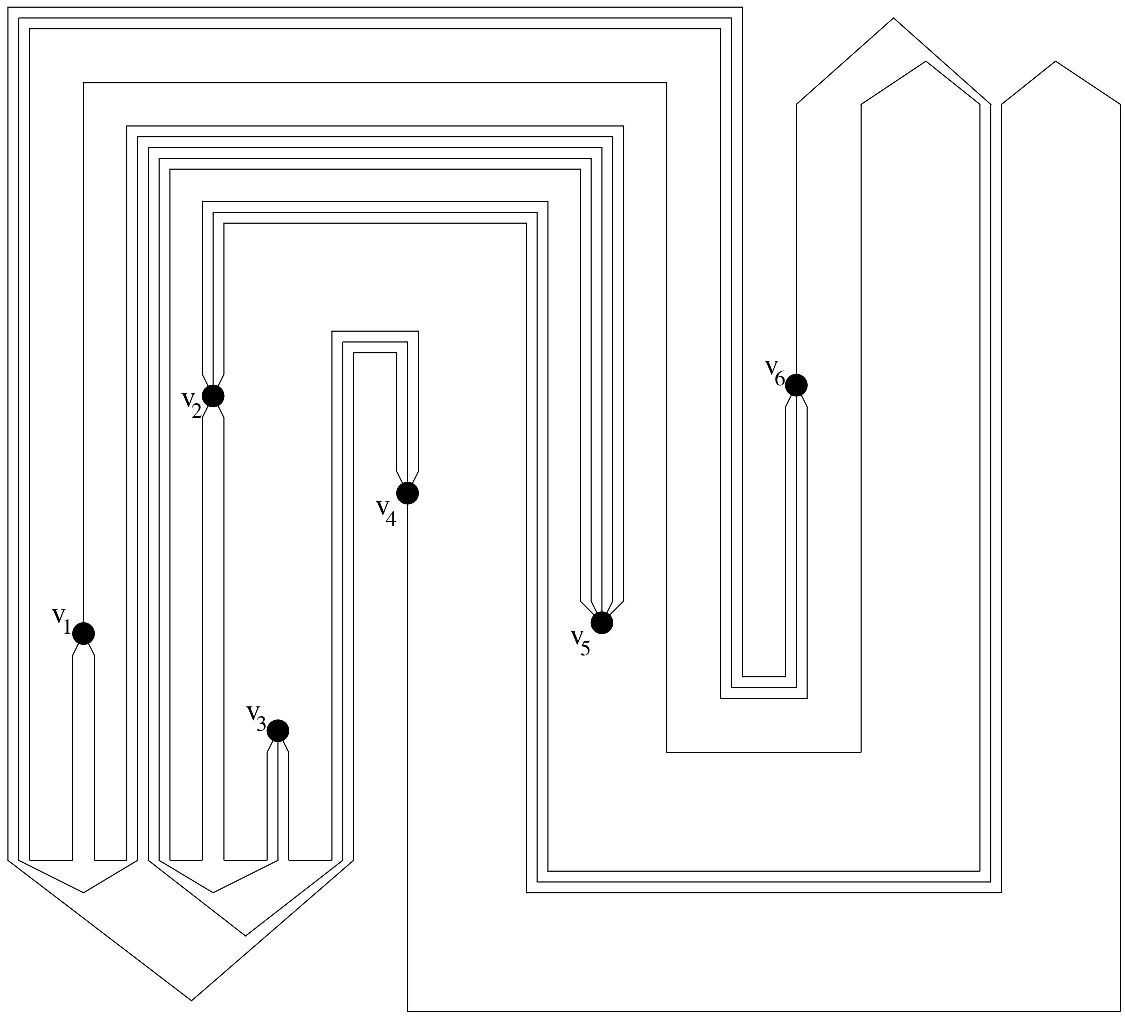

Fig. 3.4: The final drawing produced by the construction in Lemma 3.3.1 for the provided Hamiltonian planar graph shown in Figure 3.2.

pages).

Using the previously described idea, we can construct the desired embedding of G with the fixed vertex mapping as follows. In the first step, we construct a book embedding of the provided graph G. See Figure 3.2 for an example. We then separately embed each page in the plane (assigning the edges in H to the top page). Let $s(v)$ be the point v is mapped to in the top page's embedding, and let $t(v)$ be the point v is mapped to in the bottom page's embedding. Furthermore, let $p(v)$ be the point at which the pre-specified mapping f maps the vertex v. The second step of our procedure routes a band $b(v)$, for each vertex v, from $s(v)$ to $t(v)$ through $p(v)$ in such a way that no two bands intersect and the only edges intersecting the band $b(v)$ are those incident to v. See Figure 3.3 for an example. As was previously argued, it follows from these bands that we can produce the desired embedding of G. See Figure 3.4 for an example of the final embedding resulting from this technique. It remains to show how to route these bands in the desired manner and bound the number of bends of the resulting edges.

To route the desired bands, we rely on the partition $V_1, V_2, \ldots, V_r$ of the vertices in V. Consider the first part V_1. If Λ_1 is a forward class, it follows that the left-to-right order of the points in P_1 corresponding to the vertices in V_1 agrees with their order along C (in the fixed orientation). If Λ_1 is a backward class, the left-to-right order of the points in P_1 is the reverse order of the vertices in V_1 as they occur along C. Thus, we start by embedding the bottom page above P_1 if Λ_1 is a forward or below P_1 if Λ_1 is a backward class. We can position the bottom page so that, for each v in V_1, the points $p(v)$ and $s(v)$ have the same horizontal coordinate. Extend vertical bands joining the vertex locations on the bottom page to positions that are ordered consistently from left to right above or below the point set P_1 (the opposite side of P_1 from where the bands had started). We can continue these bands to a set of locations aligned either above or below the point set P_2 by introducing one orthogonal bend in the case that Λ_1, Λ_2 are alternating classes, and two bends otherwise. We can therefore repeat the above procedure maintaining that locations of the bands lie above or below the succeeding point set in the same (or reversed) left-to-right order as they had initially. In the final step, we position the embedding of the top page of our book embedding so that each band $b(v)$ joins $s(v)$ to $t(v)$.

The edges in G can then be embedded as polygonal curves extending from their ends in the two halves through these orthogonal bands to the vertices they join. An edge consists of at most 3 bends on its respective book page and passes through at most 2 bands. Each band bends at most 4 times for each consecutive pair of sets in the partition $V_1, V_2, \ldots, V_r$. When each band is routed from a set of points P_i to a set of points P_{i+1} for which Λ_i, Λ_{i+1} are alternating, 2 bends are saved. Thus, the total number of times an edge bends is at most $8r - 4\alpha - 4$, where α is the number of consecutive pairs of alternating equivalence classes. □

We now consider how the precision/area requirements of the drawing produced by Lemma 3.3.1 can be bounded. We assume first that the points $P = \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ occured at integer coordinates on a $W \times H$ grid aligned with the direction of $\hat{\mathbf{u}}$. Thus, each x-coordinate is in $\{0, 1, \ldots, W\}$ and each y-coordinate is in $\{0, 1, \ldots, H\}$. The construction used in Lemma 3.3.1 embeds edges as polygonal curves such that each piece is either orthogonal to or in the direction of $\hat{\mathbf{u}}$, with the exception of the end pieces. The y-coordinates of the bends corresponding to these curves can easily be located above the points P using $O(n)$ additional coordinates above the points. However, it might be the case that $\Omega(n)$ curves pass between two points in P at consecutive x-coordinates. But, to position the x-coordinates of the bends on a grid, it suffices to scale the x-coordinates of each point by a factor of $O(n)$. Thus, we can construct our drawing on a $O(nW) \times O(H+n)$ grid aligned with the direction $\hat{\mathbf{u}}$.

In the case that the original locations of the points required infinite precision to specify, we can bound the size of the smallest axis-aligned rectangle containing our drawing as $(W + \epsilon) \times (H + \epsilon)$. Indeed, since the bands are orthogonal they can be compressed into a small region of size ϵ outside of the original drawing as they are routed, and the edges in the book embedding can be similarly compressed by simply scaling the vertical coordinates of each bend until they are within a distance sufficiently smaller than ϵ from the spine.

3.3.2 Embedding with a Pre-Specified Mapping of the Vertices

Let G be a planar graph with vertex set V. Suppose we are given a set of points in the plane P and are required to construct an embedding of G in the plane where the mapping from vertices in V to the points in P is fixed. We proceed to describe how to solve this problem using Lemma 3.3.1.

Theorem 3.3.2. *A planar graph G can be embedded with a pre-specified vertex mapping onto the points P such that edges map to polygonal curves with at most $8n - 7$ bends.*

Proof. Choose an arbitrary direction $\hat{\mathbf{u}}$ for which the points in P occur at distinct distances and assume that $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ are the points in P in the order they occur along the direction $\hat{\mathbf{u}}$. Furthermore, assume that $v_1, v_2, \ldots v_n$ are the vertices in V ordered so that v_i is the vertex required to be mapped to the point $\mathbf{p}_i$ in the embedding.

We assume first that G contains a Hamiltonian cycle C and will later consider the case where we must augment G to enforce a Hamiltonian cycle. Fix an orientation of C and a starting point. We can thus define the partial order $\leq_C$ (equivalent to the partial order $\leq_1$ used in Lemma 3.3.1). Greedily construct a partition $V_1, V_2, \ldots, V_r$ of the vertices in V as follows.

Let $V_1 = \{v_1, v_2, \ldots, v_{j_1}\}$, where j_1 is chosen as large as possible while satisfying that $\{v_1, v_2, \ldots, v_{j_1}\}$ forms a chain with respect to the partial order $\leq_1$. Similarly, define $V_2 = \{v_{j_1+1}, v_{j_1+2}, \ldots, v_{j_2}\}$, where j_2 is chosen as large as possible while satisfying that $\{v_{j_1+1}, v_{j_1+2}, \ldots, v_{j_2}\}$ form a chain. Repeat this process until we have partitioned V into sets $V_1, V_2, \ldots, V_r$, where $V_i = \{v_{j_{i-1}+1}, \ldots, v_{j_i}\}$ for all $i = 1, \ldots, r$. Since the singleton set $\{v_i\}$ is a chain, for all $i = 1, \ldots, n$, this process will terminate and $r \leq n$. Define the r-labeling λ by assigning each vertex $v \in V_i$ the label i, for $i = 1, \ldots, r$. It follows by our construction that λ is an embeddable labeling.

By Lemma 3.3.1, we can therefore construct the desired embedding of G having at most $8n - 4$ bends per edge. However, we can reduce the edge complexity by guaranteeing a smaller partition of the vertices into chains. Suppose that instead of choosing an arbitrary orientation of C, we choose the one which resulted in the smaller partition (the smaller value of r). Let $V_1, V_2, \ldots, V_r$ and $W_1, W_2, \ldots, W_q$ be the two partitions generated by the greedy algorithm for the two possible orientations of C, and without loss of generality, assume that $r \leq q$. We can think of the vertices in each V_i and W_j as being ordered by their order along the respective orientation of C. In terms of this order, counting the first vertex in each V_i, for $i = 1, \ldots, r$, and the last vertex in each W_j, for $j = 1, \ldots, q$ counts $r + q$ vertices. We consider how many times a given vertex can be counted. Suppose u is the first vertex in V_i. Then, if $V_i \neq V_1$, the last element v in V_{i-1} must be contained in some W_j with u since u must follow v in the alternative orientation (otherwise the greedy algorithm would have included u in V_{i-1}). Thus, each vertex counted from $V_1, V_2, \ldots, V_r$ will be counted at most once, except for possibly the first element. Thus, it follows that

$$r + q \leq n + 1$$

and therefore

$$r \leq \left\lfloor \frac{n}{2} \right\rfloor$$

since we assumed that $r \leq q$. Thus, by choosing the best of the two orientations, we can construct the desired embedding of G having at most $4n - 4$ bends per edge.

If G did not contain a Hamiltonian cycle (which we have assumed), then we can construct a supergraph G' of a subdivision of G that contains a Hamiltonian cycle by Corollary 2.1.3. Let V' be the vertices in G'. After choosing the locations to map the introduced subdivision vertices, we can then repeat the above contruction on G'. Let C' be an oriented Hamiltonian cycle in G' with a fixed starting point. Let C be a cycle defined by the induced cyclic ordering of C' over the vertices in G. That is, C contains only the vertices in G, and they occur along C in the same order as in the orientation of C' (this cycle need not exist in G). Define a partial order $\leq_{C'}$ in terms of C' and a partial order $\leq_C$ in terms of C, both in a similar manner to the partial order $\leq_1$ used in Lemma 3.3.1.

By using the above greedy technique to partition V based on the partial order $\leq_C$ and the point set P, we can construct sets $V_1, \ldots, V_r$, each which forms a chain with respect to the partial order $\leq_C$ and corresponds to consecutive points along the direction $\hat{\mathbf{u}}$. Furthermore, we can guarantee that r is at most $\left\lfloor \frac{n}{2} \right\rfloor$.

Consider each subdivision vertex v in G' in order along the Hamiltonian cycle C'. Let u be the vertex preceding v in the orientation of C (we can assume that u has already been mapped). We can map v to any point as it is a subdivision vertex. Thus, we can map v to a point that occurs consecutive to u along the direction $\hat{\mathbf{u}}$. If V_i is the chain containing u, it follows that v can be mapped so that it can be added to V_i to give a chain with respect to the partial order $\leq_{C'}$. It follows that we can map each subdivision vertex so that we can use the above greedy technique to partition V' into set $V'_1, \ldots, V'_r$, each which forms a chains with respect to the partial order $\leq_{C'}$ and corresponds to consecutive points along the direction $\hat{\mathbf{u}}$. That is, we can find an r-labeling of V' that defines an embeddable labeling for G' in terms of C' and the points at which the vertices in V' are mapped.

By Lemma 3.3.1, it follows that we can construct an embedding of G' having at most $4n - 4$ bends per edge. Treating this embedding as an embedding of G proves that we can construct an embedding of an arbitrary planar graph with a fixed vertex mapping having at most $2(4n - 4) + 1 = 8n - 7$ bends per edge. This follows as each edge in G is represented as two polygonal curves in G', each with at most $4n - 4$ bends, and an internal vertex which we treat as a bend. □

We now suppose that the graph G was chosen uniformly at random from the set of all planar graphs over the n vertices V. The goal here is to show that the adaptive nature of Lemma 3.3.1 can be used to give fewer bends on average. If $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ are the points in P in the order they occur along the direction $\hat{\mathbf{u}}$, then we define $v_1, v_2, \ldots, v_n$ to be the vertices in V such that v_i is mapped to $\mathbf{p_i}$ under the fixed mapping f. We claim that we can modify our technique for enforcing and finding a Hamiltonian cycle C in a supergraph G' of a subdivision of a G so that the vertices $v_1, v_2, \ldots, v_n$ occur in a uniformly random cyclic permutation along either orientation of C.

Let $\Phi_1, \Phi_2, \ldots, \Phi_q$ be the *isomorphism classes* of the planar graphs on n vertices. That is, Φ_i is a unique maximal set of mutually isomorphic planar graphs on n vertices, for all $i = 1, \ldots, q$, and each planar graph is in some class. A uniformly random planar graph G on n vertices can be sampled in the following manner. First, we select an isomorphism class at random so that the i-th isomorphism class Φ_i is sampled with probability $\frac{1}{N}|\Phi_i|$, where N is the number of planar graphs on n vertices. Then, a unique representive of this class is assigned a uniformly random permutation of its vertex labeling. Call the resulting labeling λ_1. Suppose we construct a supergraph G' of a subdivision of G using Corollary 2.1.3 by first assigning G a second uniformly random labeling λ_2. Thus, the cyclic order of the resulting Hamiltonian cycle C has some probability distribution. But, observe that this distribution depends only on the procedure in Corollary 2.1.3 and λ_2. That is, the resulting cyclic order is independent of λ_1. Thus, in terms of the labeling λ_1, each cyclic permutation of the resulting Hamiltonian cycle C occurs with equal probability. The next result follows by an analysis of how well on expectation we can find an embeddable labeling of V with the previously described assumption on the order of $v_1, v_2, \ldots, v_n$.

Theorem 3.3.3. *A uniformly random planar graph G can be embedded in the plane with a pre-specified vertex mapping onto the points P, such that edges map to polygonal curves with at most $\frac{16}{3}n - 2$ bends on expectation.*

Proof. We use the same approach as in the proof of Theorem 3.3.2, but with a modified greedy algorithm for defining the embeddable labeling of the vertices. Suppose that the vertices $v_1, \ldots, v_n$ are the preimages of the points $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$, ordered from left to right. Again, we initially assume that G contains the Hamiltonian cycle C, which visits the vertices in a uniformly random cyclic permutation. Fix a starting point and an orientation of C. We can therefore consider the partial orders $\leq_1$,$\leq_2$ used in Lemma 3.3.1, defined in terms of the orientation of C.

Consider a consecutive pair of vertices v_i, v_{i+1}. Either $\text{rank}_C(v_i) < \text{rank}_C(v_{i+1})$ or $\text{rank}_C(v_i) > \text{rank}_C(v_{i+1})$ in terms of the fixed orientation of C. Thus, $\{v_i, v_{i+1}\}$ is a chain under either the partial order $\leq_1$ or the partial order $\leq_2$. Thus, we can greedily partition the vertices $v_1, \ldots, v_n$ into sets $V_1, \ldots, V_r$ as follows. Choose the partial order $\leq_*$ for which $\{v_1, v_2\}$ is a chain. Then, define $V_1 = \{v_1, v_2, \ldots, v_{j_1}\}$ where j_1 is chosen as large as possible maintaining that $\{v_1, v_2, \ldots, v_{j_1}\}$ is a chain under the partial order $\leq_*$. Repeat this process on the remaining vertices, possibly alternating the choice of partial order, until we have partitioned $v_1, \ldots, v_n$ into $V_1, \ldots, V_r$. Define the r-labeling λ by assigning each vertex $v \in V_i$ the label i, for $i = 1, \ldots, r$. It follows by our construction that λ is an embeddable labeling. Thus, we proceed to compute the expected size of r and the expected number of alternating equivalence classes the number of bends then follows by Lemma 3.3.1.

We can think of the vertices as a random permutation $\pi : [n] \to [n]$, where the i-th number in the permutation corresponds to the order of the vertex v_i along C. Let X be the size of the greedy partition of V that is constructed in terms of C. Each part corresponds to either an increasing or decreasing contiguous subsequence in π. Call these increasing or decreasing contiguous subsequences produced by the greedy algorithm *blocks*. We wish to bound the number of blocks produced by the greedy algorithm. Let X be a random variable describing the number of blocks. Let a, b, c be a consecutive triple of numbers in the permutation π. We say that b is an *extremal point* if either $a, c < b$ or $a, c > b$. That is, in terms of either the partial order $\leq_1$ or the partial order $\leq_2$, b corresponds to a vertex that cannot be in a chain with both of the vertices corresponding to a and c. Let Y_h be the number of contiguous subsequences of length h containing $h-2$ extremal points. Clearly, Y_3 counts the number of extremal points in π. Similarly, Y_4 counts the number of pairs of consecutive extremal points in π.

We can compute the number of blocks X in terms of extremal points since X counts the numbers in π that end blocks, and each such number is necessarily an extremal point, except for the last number in π. Thus, $X \leq 1 + Y_3$. However, not all extremal points end blocks since they can be preceded by an extremal point that ends a block, in which case they would start a block and necessarily be followed by a number in this block. Thus, we need to discount from the sum $1 + Y_3$ the extremal points that are preceded by extremal points ending blocks. If we subtract all pairs of consecutive extremal points, that is, compute $1 + Y_3 - Y_4$, then we eliminate all those such extremal points. However, we have additionally eliminated extremal points preceded by two extremal points, which could possibly end chains. We can account for these extremal points by counting the number of triples of consecutive extremal points. That is, we have shown that $X \leq 1 + Y_3 - Y_4 + Y_5$. Repeating this line of reasoning (the inclusion-exclusion principle), it follows that we have the exact solution $X = 1 + Y_3 - Y_4 + Y_5 - Y_6 + \ldots$, which we can terminate at the n-th term Y_n. Thus, it follows that

$$\mathbb{E}[X] = \mathbb{E}[1 + Y_3 - Y_4 + Y_5 - Y_6 + \ldots]$$

or equivalently,

$$\mathbb{E}[X] = 1 + \mathbb{E}[Y_3] - \mathbb{E}[Y_4] + \mathbb{E}[Y_5] - \mathbb{E}[Y_6] + \ldots$$

by linearity of expectation. We can compute the expectation of the random variable Y_h as follows. Define an indicator variable $Y_{h,i}$ that is 1 if the i-th contiguous subsequence of length h from the left contains $h-2$ extremal points and 0 otherwise. Then, it follows that

$$Y_h = \sum_{i=1}^{n-h+1} Y_{h,i}$$

and therefore

$$\mathbb{E}[Y_h] = \sum_{i=1}^{n-h+1} \mathbb{E}[Y_{h,i}]$$

by linearity of expectation, which gives

$$\mathbb{E}[Y_h] = \frac{(n-h+1)}{h!}\phi(h)$$

where $\phi(h)$ counts the number of alternating permutations of $\{1, 2, \ldots, h\}$. The number of such permutations is well studied and can be shown (see [Slo73]) to satisfy

$$2(\tan x + \sec x) = \phi(0) + \phi(1)x + \phi(2)\frac{x^2}{2!} + \phi(3)\frac{x^3}{3!} + \ldots$$

and therefore

$$2(\tan x - \sec x) = -\phi(0) + \phi(1)x - \phi(2)\frac{x^2}{2!} + \phi(3)\frac{x^3}{3!} - \ldots \tag{3.1}$$

where we define $\phi(0) = \phi(1) = 1$. Thus, it follows that in the limit as $n \to \infty$

$$\begin{aligned}\mathbb{E}[X] &= 1 + \frac{(n-2)}{3!}\phi(3) - \frac{(n-3)}{4!}\phi(4) + \frac{(n-4)}{5!}\phi(5) - \ldots \\ &= 1 + (n+1)\left(\frac{\phi(3)}{3!} - \frac{\phi(4)}{4!} + \frac{\phi(5)}{5!}\ldots\right) - \left(\frac{\phi(3)}{2!} - \frac{\phi(4)}{3!} + \frac{\phi(5)}{4!}\ldots\right)\end{aligned}$$

which by (3.1) gives

$$\begin{aligned}\mathbb{E}[X] &= 1 + (n+1)\left(2\tan(1) - 2\sec(1) + 1\right) - \left(1 + \frac{d}{dx}\bigg|2(\tan x - \sec x)\bigg|_{x=1}\right) \\ &= (n+1)\left(2\tan(1) - 2\sec(1) + 1\right) - 2\sec(1)(\sec(1) - \tan(1))\end{aligned}$$

Observe that if n is even this infinite series upper bounds the expectation of X. Furthermore, since the expectation of X is necessarily increasing, it follows that

$$\begin{aligned}\mathbb{E}[X] &\leq (n+2)\left(2\tan(1) - 2\sec(1) + 1\right) - 2\sec(1)(\sec(1) - \tan(1)) \\ &= (2\tan(1) - 2\sec(1) + 1)n + 4\tan(1) - 4\sec(1) - 2\sec(1)(\sec(1) - \tan(1)) \\ &\leq (2\tan(1) - 2\sec(1) + 1)n\end{aligned}$$

for all possible n.

Next, we consider the expected number of pairs of consecutive equivalence classes Λ_i, Λ_{i+1}, defined in terms of the embeddable labeling λ, that are alternating. Using the definition of blocks given above, we say that two blocks in π are alternating if their corresponding equivalence classes are alternating. Let the random variable Q count the number of consecutive pairs of alternating blocks. We are thus interested in computing $\mathbb{E}[Q]$. For convenience, we will ignore counting the consecutive pair that occurs when the last block is of size 1.

Observe that Q counts precisely the numbers in π ending blocks that are the first block in a consecutive pair of alternating blocks. Define these numbers as *alternating points*. Let b be an alternating point and suppose that a, b, c, d is the contiguous subsequence in π around b. The subsequence a, b, c, d can occur in any permutation for which a, b, c is an alternating permutation but a, b, c, d is not. Let R_4 be the number of contiguous subsequences in π in the previously described form. It follows that $\mathbb{E}[Q] \leq \mathbb{E}[R_4]$.

Consider a number u that is counted by R_4 but is not an alternating point. It follows that the contiguous subsequence in π around u is of the form s, t, u, v, w, where s, t, u, v is an alternating permutation but s, t, u, v, w is not. Indeed, if u is counted by R_4 but is not an alternating point, then u must start a block and therefore s, t, u must be alternating in addition to s, t, u, v. If we therefore define R_5 to be the number of contiguous subsequences in π of form $s_1, s_2, \ldots, s_5$, for which $s_1, s_2, \ldots, s_4$ is an alternating permutation but $s_1, s_2, \ldots, s_5$ is not, then it follows that $\mathbb{E}[Q] \geq \mathbb{E}[R_4 - R_5]$. By the inclusion-exclusion principle, it follows that repeating this argument gives the exact solution

$$\mathbb{E}[Q] = \mathbb{E}[R_4 - R_5 + R_6 - R_7 + \ldots]$$

which by linearity of expectation, gives

$$\mathbb{E}[Q] = \mathbb{E}[R_4] - \mathbb{E}[R_5] + \mathbb{E}[R_6] - \mathbb{E}[R_7] + \ldots$$

where R_h counts the number of contiguous subsequences in π of the form $s_1, s_2, \ldots, s_h$, for which $s_1, s_2, \ldots, s_{h-1}$ is an alternating permutation but $s_1, s_2, \ldots, s_h$ is not. Using $\phi(h)$ to count the number of alternating permutations of length h, it follows that

$$\mathbb{E}[R_h] = \frac{(n-h+1)}{h!}(h\phi(h-1) - \phi(h))$$

and therefore,

$$\mathbb{E}[Q] = \frac{(n-3)}{4!}(4\phi(3) - \phi(4)) - \frac{(n-4)}{5!}(5\phi(4) - \phi(5)) + \frac{(n-5)}{6!}(6\phi(5) - \phi(6)) - \ldots$$

which, in the limit as $n \to \infty$, is

$$\sum_{h=3}^{\infty} \frac{(-1)^{h+1}(n-h)}{(h+1)!}((h+1)\phi(h) - \phi(h+1))$$

which simplifies to give

$$\mathbb{E}[Q] = \frac{2}{3}n - 2 + 2n\sum_{h=4}^{\infty}(-1)^{h+1}\frac{\phi(h)}{h!} - 2\sum_{h=4}^{\infty}(-1)^{h+1}\frac{\phi(h)}{(h-1)!} + \sum_{h=4}^{\infty}(-1)^{h+1}\frac{\phi(h)}{h!}. \quad (3.2)$$

Recall that

$$2(\tan(1) - \sec(1)) = \sum_{h=0}^{\infty} (-1)^{h+1} \frac{\phi(h)}{h!}$$

and that

$$2\sec(1)(\sec(1) - \tan(1)) = \sum_{h=0}^{\infty} (-1)^{h} \frac{\phi(h+1)}{h!}$$

which we can substitute into (3.2) to conclude that

$$\mathbb{E}[Q] = (\frac{4}{3} + 4(\tan(1) - \sec(1)))n + (4\sec(1) + 2)(\tan(1) - \sec(1)) + 1/3$$

in the limit as $n \to \infty$. Observe that this value is a lower bound on the the expectation of Q in the case that n is odd. Furthermore, since the expectation of Q is nondecreasing, we therefore have the bound

$$\mathbb{E}[Q] \geq \left(\frac{4}{3} + 4(\tan(1) - \sec(1))\right) n - 3$$

for all n.

Thus, by Lemma 3.3.1 it follows that we can embed G onto a fixed set of points so that the edges have at most $8\mathbb{E}[X] - 4\mathbb{E}[Q] - 4 \leq \frac{8}{3}n - 1$ bends on expectation, assuming that G contained a Hamiltonian cycle. Had G not contained a Hamiltonian cycle, the number of bends is instead at most $\frac{16}{3}n - 1$ per edge on expectation. □

We have given a constructive method for embedding a planar graph with a fixed vertex mapping. The general outline of the procedure is as follows.

1. Using the linear-time algorithm described in Lemma 2.1.2, construct a 4-connected supergraph G' of a subdivision G.

2. Using the linear-time algorithm described in Lemma 2.1.1, find a Hamiltonian cycle C in G'.

3. Generate an embeddable labeling of the vertices using the linear-time greedy algorithm described in Theorem 3.3.3.

4. Construct the embedding of G using the technique described in Lemma 3.3.1. This construction can be straightforwardly implemented in quadratic time.

The number of bends guaranteed by this technique (which can easily be shown to be within an additive constant of the bounds in Theorem 3.3.2) is optimal up to constant factors (as is shown in Chapter 4). Moreover, as was shown in Theorem 3.3.3, the average-case number of bends is at most $\frac{16}{3}n - 1$ bends on expectation. Moreover, the precision required of the embedding is only a factor of $O(n)$ larger than the precision of the original point set. If one is concerned only with the number of bends, the constant factor can be reduced by modifying Lemma 3.3.1 to use non-orthogonal bands. However, for the sake of simplicity, we refrain from discussing these possible improvements.

3.3.3 Average-Case Algorithm for Simultaneously Embedding with Fixed Vertex Locations and Analysis

Let $G_1, G_2, \ldots, G_k$ be planar graphs over a common vertex set V. Suppose we are given an arbitrary set of n points in the plane P, where $n = |V|$, and are required to construct a simultaneous planar embedding of $G_1, G_2, \ldots, G_k$ in which the vertices in V map to the points in P. We proceed to describe how to solve this problem using Lemma 3.3.1, with the assumption that $G_1, G_2, \ldots, G_k$ are sampled uniformly at random from the set of all planar graphs on n vertices.

Theorem 3.3.4. *Uniformly random planar graphs $G_1, G_2, \ldots, G_k$ over the vertex set V can be simultaneously embedded on any pre-specified point set P such that edges map to polygonal curves that have at most $O(n^{1-\frac{1}{k}})$ bends per edge with high probability.*

Proof. Choose an direction $\hat{\mathbf{u}}$ and assume that $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ are the points in P in the order they occur along the direction $\hat{\mathbf{u}}$, each of which occurs at a distinct distance along the direction $\hat{\mathbf{u}}$. We are primarily concerned with how to map the vertices in V to the points $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$.

Assume first that $G_1, G_2, \ldots, G_k$ are all Hamiltonian. That is, $C_1, C_2, \ldots, C_k$ are Hamiltonian cycles such that $C_1 \subseteq G_1, C_2 \subseteq G_2, \ldots, C_k \subseteq G_k$. Fix an orientation and a starting point in each of $C_1, \ldots, C_k$. Define a partial order $\leq$ on the vertex set V such that, for $u, v \in V$, $u \leq v$ if and only if u precedes v in each of $C_1, \ldots, C_k$. Thus, a set of vertices $\{v_1, v_2, \ldots, v_j\}$ is a chain if these vertices all occur in the same relative order along each of $C_1, \ldots, C_k$.

Assign an index to each vertex by its order in C_1. That is, the vertex with index i corresponds to the i-th vertex visited along the orientation of C_1 from its fixed starting point. We can think of the orientations of $C_2, C_3, \ldots, C_k$ as permutations $\pi_1, \ldots, \pi_{k-1}$ of these indices. In terms of these indices, a chain of vertices under the partial order $\leq$ corresponds to an increasing subsequence common to each of $\pi_1, \ldots, \pi_{k-1}$. Thus, it follows by Lemma 2.4.5 that the vertices in V can be partitioned into at most $O(n^{1-\frac{1}{k}})$ chains with high probability. That is, the width of V is $O(n^{1-\frac{1}{k}})$ with high probability.

Let $V_1, V_2, \ldots, V_r$ be a minimum cardinality partition of V into chains. Define the r-labeling λ by assigning each vertex $v \in V_i$ the label i, for $i = 1, \ldots, r$. We can choose a mapping of the vertices in V to the points in P such that the vertices in V_i map to the points $P_i = \mathbf{p}_{j+1}, \mathbf{p}_{j+2}, \ldots, \mathbf{p}_{j+|V_i|}$, where $j = \sum_{l<i} |V_l|$, and occur along the direction $\hat{\mathbf{u}}$ in ascending order with respect to $\leq$. It follows by our construction that λ is an embeddable labeling for this vertex mapping. Thus, by Lemma 3.3.1 each of the graphs $G_1, G_2, \ldots, G_k$ can be simultaneously embedded on the point set P such that edges map to polygonal curves with at most $8r - 6$ bends. As it was previously argued, this gives at most $O(n^{1-\frac{1}{k}})$ bends per edge with high probability.

If any of $G_1, G_2, \ldots, G_k$ did not contain a Hamiltonian cycle, say G_i for $1 \leq i \leq k$, then we could have constructed a supergraph G_i' of a subdivision of G_i containing a Hamiltonian cycle with at most 1 subdivision per edge by Corollary 2.1.3. As was argued in Section 3.3.2, we can modify this construction so that the each possible cyclic order of the vertices in V in each of the Hamiltonian cycles $C_1, C_2, \ldots, C_k$ found by this construction occur with equal probability. By adding arbitrary points to P for the subdivision vertices and applying the above drawing technique, it follows that $G_1, G_2, \ldots, G_k$ can still be simultaneously embedded on P with at most $O(n^{1-\frac{1}{k}})$ bends per edge. □

The proof of Theorem 3.3.4 describes an algorithm for constructing a simultaneous embedding of the planar graphs $G_1, G_2, \ldots, G_k$ over the vertex set V onto the pre-specified point set P. The general outline of the procedure is as follows.

1. Using the linear-time algorithm described in Lemma 2.1.2, construct a 4-connected supergraph G_i' of a subdivision of G_i for each $i = 1, \ldots, k$.

2. Using the linear-time algorithm described in Lemma 2.1.1, find a Hamiltonian cycle C_i in G_i' for each $i = 1, \ldots, k$.

3. Using Theorem 2.4.2 and a partial order on V defined in terms of $C_1, \ldots, C_k$, partition the vertices in V into chains $V_1, V_2, \ldots, V_r$ in polynomial time.

4. Use the partition $V_1, V_2, \ldots, V_r$ to generate an embeddable labeling of the vertices.

5. Construct the embedding of G_i onto P using the technique described in Lemma 3.3.1 for each $i = 1, \ldots, k$, producing the desired simultaneous embedding.

As is argued by Theorem 3.3.4, this procedure constructs a simultaneous planar embedding of $G_1, G_2, \ldots, G_k$ such that edges map to polygonal curves having at most $O(n^{1-\frac{1}{k}})$ bends with high probability. However, it is possible in the worst case that the Hamiltonian cycles discovered for $G_1, G_2, \ldots, G_k$ give a partial order resulting in V having width n. We can improve this worst-case behaviour by using the fact that the equivalence classes in an embeddable labeling can be chains under two possible partial orders. We discuss these details in the next section.

3.3.4 Worst-Case Algorithm for Simultaneous Embeddings with Fixed Vertex Locations and Analysis

Let $G_1, G_2, \ldots, G_k$ be planar graphs over a common vertex set V. As in the previous section, we assume we are given a fixed set P of n points in the plane, where $n = |V|$, and are required to construct a simultaneous planar embedding of $G_1, G_2, \ldots, G_k$ in which the vertices in V map to the points in P. We proceed to describe a technique that gives better worst-case performance than the method described in the previous section. The analysis will use the following worst-case bound.

Lemma 3.3.5. *Let $\pi_1, \pi_2, \ldots, \pi_k$ be permutations over $S = \{1, 2, \ldots, n\}$. Then, we can partition the elements of S into sets $T_1, \ldots, T_r$ such that the elements in each part form increasing or decreasing subsequences in each of $\pi_1, \pi_2, \ldots, \pi_k$, where r is $O(n^{1-2^{-k}})$.*

Proof. Applying the Erdős-Szekeres theorem (Theorem 2.3.1), we can find a set of elements of size at least $n^{\left(\frac{1}{2}\right)^k}$ that form an increasing or decreasing subsequence in each of $\pi_1, \pi_2, \ldots, \pi_k$. Thus, S can be partitioned into a set of monotonic subsequences by repeatedly removing the longest monotonic subsequence. If there are at least $\frac{n}{2^i}$ elements at some iteration, we remove at least $\left(\frac{n}{2^i}\right)^{\left(\frac{1}{2}\right)^k}$ elements. Thus, we can bound the number of iterations it takes to reduce $\frac{n}{2^{i-1}}$ elements to $\frac{n}{2^i}$ as follows. Choose j smallest such that

$$\frac{n}{2^{i-1}} - j\left(\frac{n}{2^i}\right)^{\left(\frac{1}{2}\right)^k} \quad \leq \quad \frac{n}{2^i}$$

or equivalently,

$$\frac{n}{2^i} \leq j\left(\frac{n}{2^i}\right)^{\left(\frac{1}{2}\right)^k}$$

which gives

$$j \geq \left(\frac{n}{2^i}\right)^{1-\left(\frac{1}{2}\right)^k}$$

where j is the number of iterations needed to reduce the $\frac{n}{2^{i-1}}$ elements down to $\frac{n}{2^i}$. It follows then that the total number of iterations is at most

$$\sum_{i=0}^{\lg n} \left(\frac{n}{2^i}\right)^{1-\left(\frac{1}{2}\right)^k} \leq 4n^{1-\left(\frac{1}{2}\right)^k}$$

completing the proof. □

The proof of Lemma 2.4.5 is constructive in that it describes a greedy algorithm for partitioning the elements into sets that occur in increasing or decreasing order in each permutation when read from left to right. The algorithm can be thought of as an approximation to computing the minimum such partition of the elements that guarantees sublinear performance in the worst case. Moreover, this algorithm can be implemented to have a runtime that is fixed-parameter tractable in k. To achieve this, we need a method for finding the largest set of elements that are either an increasing or decreasing subsequence in each of the k permutations. We can solve this problem by finding the longest increasing subsequence in each of the 2^k possible combinations of original/reversed orderings of the k permutations. By Theorem 2.3.4, the longest increasing subsequence common to one of these 2^k sets of permutations can be found in time $O(\min\{kn^2, n\log^{k-1} n \log\log n + kn\})$. By choosing the largest set over the 2^k possible choices solves the desired problem. Furthermore, since the number of iterations required to completely partition the elements is at most $O(n)$, the entire partition can be computed in time $O(\min\{2^k kn^3, 2^k n^2 \log^{k-1} n \log\log n + 2^k kn\})$, which is fixed-parameter tractable in k.

We will reduce the problem of embedding k planar graphs with vertices mapping to a fixed point set to this permutation problem. In this case, we can generally assume that k is at most $O(\log n)$ and thus this algorithm can be considered polynomial time. This assumption is founded on the fact that if k is $\omega(\log n)$ then, as we will see in Chapter 4, $\Omega(n^2)$ bends are almost surely necessary (assuming the k graphs were sampled uniformly at random from a general class of planar graphs). When only $\Omega(n^2)$ bends can be guaranteed, we can just use the technique described in the previous section, which is always polynomial in n and k.

Theorem 3.3.6. *Planar graphs $G_1, G_2, \ldots, G_k$ over the vertex set V can be simultaneously embedded on any pre-specified point set P such that edges map to polygonal curves with at most $O(n^{1-(\frac{1}{2})^{k-1}})$ bends.*

Proof. Choose an arbitrary direction $\hat{\mathbf{u}}$ and assume that $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ are the points in P in the order they occur along the direction $\hat{\mathbf{u}}$. We are primarily concerned with how to map the vertices in V to the points $\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n}$. Assume first that $G_1, G_2, \ldots, G_k$ are all Hamiltonian. That is, $C_1, C_2, \ldots, C_k$ are Hamiltonian cycles such that $C_1 \subseteq G_1, C_2 \subseteq G_2, \ldots, C_k \subseteq G_k$. Fix an orientation and a starting point in each of $C_1, \ldots, C_k$. Assign an index to each vertex in V by its order in C_1. That is, the vertex with index i corresponds to the i-th vertex visited along the orientation of C from its fixed starting point. We can think of the orientations of $C_2, C_3, \ldots, C_k$ as permutations $\pi_1, \ldots, \pi_{k-1}$ of these indices. In the context of the graph G_i, for each $i = 2, \ldots, k$, chains under the partial orders $\leq_1$ and $\leq_2$ used in Lemma 3.3.1 correspond to increasing or decreasing subsequences of the permutation π_{i-1}. Thus, it follows by Lemma 3.3.5, that the vertices in V can be partitioned into at most $O(n^{1-(\frac{1}{2})^{k-1}})$ sets, each of which in the individual context of any G_i, for $i = 1, \ldots, k$, is a chain under the either the partial order $\leq_1$ or the partial order $\leq_2$.

Let $V_1, V_2, \ldots, V_r$ be a minimum cardinality partition of V into sets that are chains under either the partial order $\leq_1$ or the partial order $\leq_2$. Define an r-labeling λ by assigning each vertex $v \in V_i$ the label i, for $i = 1, \ldots, r$. We can choose a mapping of the vertices in V to the points in P such that the vertices in V_i map to the points $P_i = \mathbf{p}_{j+1}, \mathbf{p}_{j+2}, \ldots, \mathbf{p}_{j+|V_i|}$, where $j = \sum_{l<i} |V_l|$, and occur along the direction $\hat{\mathbf{u}}$ in ascending order. It follows by our construction that λ is an embeddable labeling for this vertex mapping. Thus, by Lemma 3.3.1 each of the graphs $G_1, G_2, \ldots, G_k$ can be simultaneously embedded on the point set P such that edges map to polygonal curves with at most $8r - 6$ bends. From the previous argument on the size of r, this gives at most $O(n^{1-(\frac{1}{2})^{k-1}})$ bends per edge in the worst case.

If any of $G_1, G_2, \ldots, G_k$ did not contain a Hamiltonian cycle, say G_i for $1 \leq i \leq k$, then we could have constructed a supergraph G_i' of a subdivision of G_i containing a Hamiltonian cycle with at most 1 subdivision per edge by Corollary 2.1.3. By adding arbitrary points to P for the subdivision vertices and applying the above construction, it follows that $G_1, G_2, \ldots, G_k$ can still be simultaneously embedded on P with at most $O(n^{1-(\frac{1}{2})^{k-1}})$ bends per edge. □

Chapter 4

Edge Complexity Lower Bounds

In this section, we prove a lower bound on the minimum edge complexity of a simultaneous planar embedding of k graphs with fixed vertex locations. In particular, if the fixed vertex locations correspond to any convex point set, we show that k randomly chosen graphs, sampled uniformly from a subclass of planar graphs, almost surely cannot be drawn with fewer than $O(n^{2-\frac{2}{k}})$ total bends. This result gives some evidence for the optimality (or near optimality) of the probabilistic upper bound described in Chapter 3.

To prove the lower bound, we will use an encoding argument to show that if a simultaneous planar embedding of k graphs from a class $\mathcal{C}$ can always be constructed such that each graph's embedding contains fewer than β bends, then β must be sufficiently large to not contradict an information theoretic lower bound. For convenience, we derive an encoding scheme for the class of *perfect matchings*. Perfect matchings are always planar, and thus lower bounds that apply to perfect matchings also apply to planar graphs. Moreover, embeddings of perfect matchings somewhat capture the complexity of drawing planar graphs to minimize bends.

We begin in Section 4.1 to identify encoding-related properties of perfect matchings. We show straightforward bounds on the minimum number of bits required to encode perfect matchings. In Section 4.1.4, we describe the fundamental encoding technique that encodes perfect matchings using fewer bits when they can be drawn with fewer bends. Using this technique, we prove lower bounds on the edge complexity of drawings of perfect matchings in Section 4.1.5. In Section 4.2, we generalize the encoding technique and lower bounds.

4.1 Encoding Perfect Matchings

4.1.1 An Encoding Lower Bound by Enumeration

We define a *perfect matching* on n (labeled) vertices as a graph where each vertex has degree exactly one. There is an obvious one-to-one correspondence between perfect matchings on

n vertices and partitions of $\{1, 2, \ldots, n\}$ into parts of size two. Thus, there are no perfect matchings on an odd number of vertices, and when n is even there are

$$\frac{n!}{(n/2)!2^{n/2}}$$

perfect matchings. Thus, we have the following lower bound on the encoding size of perfect matchings.

Proposition 4.1.1. *Let P be a uniformly chosen random perfect matching on n vertices. Furthermore, let A be the encoding size of P given by any fixed encoding scheme. Then, $A \geq \frac{n}{2} \lg \frac{n}{e} - \Delta$ with probability at least $1 - 2^{1-\Delta}$.*

Proof. First, we use Stirling's approximation to derive that

$$\lg \frac{n!}{(n/2)!2^{n/2}} \geq \frac{n}{2} \lg \frac{n}{e}$$

for all n. Then, for any fixed encoding scheme, it follows that the number of unique perfect matchings that can be encoded using A bits, for $A \leq \frac{n}{2} \lg \frac{n}{e} - \Delta$ is at most $2^{\lceil H - \Delta \rceil}$, for $H = \frac{n}{2} \lg \frac{n}{e}$. Thus, the probability that a perfect matching chosen uniformly at random can be encoded using at most A bits is at most $2^{1-\Delta}$. □

4.1.2 Noncrossing Perfect Matchings

We define *noncrossing perfect matchings* as the subset of perfect matchings on n vertices that can be embedded in the plane with vertices mapping to points in convex position and edges mapping to straight lines. Noncrossing perfect matchings correspond precisely to the noncrossing partitions of $\{1, 2, \ldots, n\}$ into parts of size two. Observe then that there are no noncrossing perfect matchings when the number of vertices n is odd. Furthermore, we can conclude from this one-to-one correspondence that, when n is even, there are

$$\frac{1}{m+1}\binom{2m}{m}$$

noncrossing perfect matchings, where $m = n/2$ is the number of edges in the matching. The following encoding bound uses this result to efficiently encode planar graphs that can be drawn sufficiently close to noncrossing perfect matchings on a convex point set.

Lemma 4.1.2. *Suppose that a connected graph G can be embedded with a fixed mapping of the vertices onto a convex point set such that the total number of crossings between the edges in G and the boundary of the convex hull of the point set is at most σ. Then, G can be encoded with at most $6|E(G)| + 4\sigma$ bits.*

Proof. Let G be a planar graph with n vertices and m edges. Consider an embedding of G on a convex point set where the total number of crossings with the boundary of the convex hull of the point set is at most σ. Let C be the boundary of the convex hull. Introduce a vertex at each point where an edge in G crosses C. Furthermore, replace each original vertex with a set of vertices, one for each incident edge. That is, a vertex v of degree d is replaced with d vertices (located consecutive on C) each incident to a unique one of the original edges incident to v (maintaining the same edge ordering). Thus, we have replaced G with two noncrossing perfect matchings, one on the inside of C and one on the outside. Vertices corresponding to crossings are incident to edges on both the inside and outside, and all remaining $2m$ vertices are exclusively on one side. Using Stirling's approximation to bound the Catalan number, it follows that we can encode these two noncrossing perfect matchings using at most $2m + 2\sigma$ bits. To recover the original graph from the separate encodings of the inside and outside, we encode a sequence $S \in \{0, 1, 2, 3\}^*$ corresponding to the order of vertices from the inside, outside, and the crossing vertices along C. Specifically, we output a 0 for internal vertices, a 1 for external vertices, a 2 for crossing vertices in the order the vertices occur along C. Furthermore, to distinguish which consecutive vertices map back to an original vertex, we encode a 3 after each such block. This adds at most an additional $4m + 2\sigma$ bits. Thus, G can be encoded using $6m + 4\sigma$ bits. □

4.1.3 Planar Separators

The result of Lemma 4.1.2 does not give an efficient encoding when a graph can be embedded only with a superlinear number of bends (on a convex point set). Furthermore, the bound can easily be improved by, for example, using *arithmetic encoding* to encode the sequence used to reconstruct the graph. However, Lemma 4.1.2 will be useful for providing a base case in the inductive proof of Lemma 4.1.5. We must additionally establish some results on planar separators and their relation to bends. We define an edge separator on a graph G, with nonnegative weights assigned to its vertices, to be a set of edges whose removal partitions G into two disjoint subgraphs each with weight at most 2/3 of the total weight. The following edge separator theorem for planar graphs is a result of Gazit and Miller [GM90].

Theorem 4.1.3 ([GM90])**.** *Let G be a planar graph with nonnegative vertex weights that sum to at most 1 and do not individually exceed 2/3. Then, G has an edge separator of size*

$$1.58\sqrt{\sum_{v \in V(G)} deg^2(v)}.$$

For a graph G, its crossing number $\mathrm{cr}(G)$ is defined as the minimum number of crossings in a drawing of the graph. We will use the approach of Pach and Wenger [PW98] to relate bounds on the crossing number to the minimum size of an edge separator by using the following result, established by Leighton [Lei83].

Lemma 4.1.4. *Let G be a graph with nonnegative vertex weights that sum to at most 1 and do not individually exceed 2/3. Then, G has an edge separator of size*

$$1.58\sqrt{16\text{cr}(G) + \sum_{v \in V(G)} deg^2(v)}$$

where $\text{cr}(G)$ *is the crossing number of* G*.*

Proof. Fix a drawing of G with $\text{cr}(G)$ crossings such that no two edges cross more than once (such a drawing always exists [PSS96]). Construct G' by replacing each crossing with a vertex weighted 0. By Theorem 4.1.3, G' has a separator of size

$$1.58\sqrt{16\text{cr}(G) + \sum_{v \in V(G)} \deg^2(v)}$$

since each crossing vertex has degree 4. Removing all edges in G that were partially removed in the separator for G' gives an edge separator for G of at most the same size. □

This result correlates the crossing number of a graph to the size of an edge separator. By relating the number of bends to the crossing number, we can effectively give an encoding of a graph using the small separator size. We formalize this result in the next section.

4.1.4 Encoding Perfect Matchings with Small Edge Complexity

Lemma 4.1.5. *Let P be a perfect matching with n vertices. If P can be embedded on a convex point set of size n with at most β total bends, then P can be encoded using at most*

$$\frac{n}{4}\lg(\beta + n) + cn,$$

bits, where c is a constant.

Proof. Let P be a perfect matching on n vertices that can be embedded on a convex point set of size n with at most β bends. Let C be a cycle corresponding to the boundary of the convex hull of the point set. Observe that each line segment composing an edge can cross C at most twice. If we define $G = P \cup C$ (allowing multi-edges), it follows that G can be drawn with $\sigma \leq 2\beta$ crossings. Given an encoding of G, we can recover P because we know which edges belong to C. Thus, to prove the claim, it suffices to show that we can encode G using $\frac{n}{4}\lg(\sigma + n) + cn$ bits, for some constant c.

As the claim holds trivially when n is a constant, we can therefore proceed by induction over the number of vertices. We can assume that $\sigma \geq n$ since we could have otherwise encoded G using cn bits, for some $c > 11$, by Lemma 4.1.2. Fix a drawing of G with σ crossings such that no two edges cross more than once. Weight each vertex by 1 plus half the number of crossings associated with its incident edge from P (normalized so the sum of all weights is 1). By Lemma 4.1.4, the vertices of G can be partitioned into two sets V_1, V_2, for which the graphs G_1, G_2 induced by V_1 and V_2 can be drawn with σ_1 and σ_2 crossings, respectively, such that $\sigma_1 + |V_1| \leq \sigma_2 + |V_2| \leq 2/3(\sigma + n)$ without loss of generality. Furthermore, at most $1.58\sqrt{16\sigma + 9n}$ edges in G join a vertex in V_1 to a vertex in V_2 since each vertex has degree 3. Let α be defined such that $\alpha n = |V_1|$ (and thus, $(1-\alpha)n = |V_2|$). Since each of these graphs have necessarily fewer vertices, it follows by induction that they can be encoded using at most

$$\begin{aligned} & 1/4(\alpha n - r)\lg(2/3(\sigma+n)) + c(\alpha n - r) \\ & + 1/4((1-\alpha)n - r)\lg(2/3(\sigma+n)) + c((1-\alpha)n - r) \\ = \; & 1/4(n-2r)\lg(2/3(\sigma+n)) + c(n-2r) \\ = \; & \frac{n}{4}\lg(\sigma+n) + cn + \frac{r}{2}\lg(3/2) - \frac{r}{2}\lg(\sigma+n) - \frac{n}{4}\lg(3/2) - 2cr \end{aligned}$$

bits, where $r \leq 1.58\sqrt{16\sigma + 9n}$ is the number of edges from P that were in the separator. To encode the edges from P that were in the separator, we specify which vertices in each partition were incident to edges in the separator, the permutation defining their adjacencies, and the size of the separator, all of which uses at most

$$A = \lg\binom{n}{2r} + \lg r! + \lg r$$

bits. Furthermore, to reconstruct G from the encoding of the separator and the two smaller perfect matchings, we just need to encode how C was partitioned, giving a total encoding size of

$$\frac{n}{4}\lg(\sigma+n) + cn + \frac{r}{2}\lg 3/2 - \frac{r}{2}\lg(\sigma+n) - \frac{n}{4}\lg 3/2 - 2cr + \lg\binom{n}{q} + A + O(1)$$

where q is the number of edges from C in the separator. By Stirling's approximation, we can bound this size as

$$\begin{aligned} & \frac{n}{4}\lg(\sigma+n) + cn + \frac{r}{2}\lg 3/2 - \frac{r}{2}\lg(\sigma+n) - \frac{n}{4}\lg 3/2 - 2cr \\ & + q\lg\frac{n}{q} + \lg(e)q + 2r\lg\frac{n}{r} + r\lg r + r + O(1) \end{aligned}$$

which is at most

$$\begin{aligned} & \frac{n}{4}\lg(\sigma+n) + cn + q\lg\frac{n}{q} + \lg(e)q + 2r\lg\frac{n}{r} + r\lg r \\ & - r\lg\sqrt{\sigma+n} - \frac{n}{8} - 2(c-1)r + O(1) \end{aligned}$$

and, since r is at most $1.58\sqrt{16\sigma + 9n}$, the number of bits required is at most

$$\begin{aligned} & \frac{n}{4}\lg(\sigma+n) + cn + q\lg\frac{n}{q} + \lg(e)q + 2r\lg\frac{n}{r} - \frac{n}{8} - 2(c-3)r + O(1) \\ \leq \; & \frac{n}{4}\lg(\sigma+n) + cn + q\lg\frac{n}{q} + \lg(e)q + 2r\lg\frac{n}{r} - \frac{n}{8} + O(1) \end{aligned}$$

for $c \geq 6$. Now, suppose that $\sigma \leq \frac{n^2}{e^{16}}$, which implies that $1.58\sqrt{16\sigma + 9n} \leq \frac{n}{e^6}$ for any $\sigma \geq n$, assuming that n is sufficiently large (which we can assume as in the situation where n is a constant we can achieve the desired encoding trivially by choosing c large enough). Thus, it follows that in this case both r and q are at most $\frac{n}{e^6}$. Since the function $f(x) = x\ln\frac{a}{x}$ achieves its maximum when $x = a/e$, the number of bits required to encode G is at most

$$\begin{aligned} & \frac{n}{4}\lg(\sigma+n) + cn + \frac{n}{e^6}\lg e^6 + \lg(e)\frac{n}{e^6} + 2\frac{n}{e^6}\lg e^6 - \frac{n}{8} + O(1) \\ \leq \; & \frac{n}{4}\lg(\sigma+n) + cn \end{aligned}$$

as we were required to show. Thus, G can be encoded using $\frac{n}{4}\lg(\sigma+n) + cn$ bits, for some constant c, in all cases when $\sigma \leq \frac{n^2}{e^{16}}$. In the case that $\sigma > \frac{n^2}{e^{16}}$, we can encode G by simply specifying which of the $\frac{n!}{(n/2)!2^{n/2}}$ perfect matchings corresponds to P. That is, we can encode G using

$$\frac{n}{2}\lg\frac{n}{e} + 1$$

bits. Since $\sigma > \frac{n^2}{e^{16}}$, it follows that

$$\frac{n}{2}\lg\frac{n}{e} + 1 \leq \frac{n}{4}\lg(\sigma+n) + 11n,$$

and we can therefore encode G in all cases using the desired number of bits. □

4.1.5 Lower Bounds on the Edge Complexity

We have established an adaptive encoding scheme that uses fewer bits to encode perfect matchings with fewer bends. By use of Proposition 4.1.1, we can use this encoding scheme to give a lower bound on the probability of randomly choosing a perfect matching that requires many bends in any drawing. We first give a general lower bound on the number of bends needed to embed a random perfect matching on a convex point set with a fixed vertex mapping.

Theorem 4.1.6. *Let P be a perfect matching on n vertices, where n is even, chosen uniformly at random. Furthermore, assume that we are given n points in convex position and a pre-specified one-to-one mapping of the vertices in P to the points. Then, all embeddings of P with vertex locations on this convex point set, in which edges map to polygonal curves, have at least $\Omega(n^2)$ total bends with high probability.*

Proof. Let P be a perfect matching on n vertices chosen uniformly at random. Suppose that P can be embedded with β bends with a pre-specified mapping of its vertices onto a fixed convex point set. Then by Lemma 4.1.5, we can encode P using $\frac{n}{4}\lg(\beta+n)+cn$ bits, for some constant c. By Proposition 4.1.1,

$$\frac{n}{4}\lg(\beta+n)+cn \quad \geq \quad \frac{n}{2}\lg\frac{n}{e}-(\Delta+1)$$

with probability at least $1-\left(\frac{1}{2}\right)^{\Delta}$. Thus, for n sufficiently large, it clearly must hold that $\beta \geq n$ with probability at least $1-\left(\frac{1}{2}\right)^{\Delta}$. Thus, we can conclude that

$$\frac{n}{4}\lg(2\beta)+cn \geq \frac{n}{2}\lg\frac{n}{e}-(\Delta+1)$$

and therefore we have

$$\frac{n}{4}\lg(2\beta)+\left(c+\frac{\lg e}{2}\right)n+\Delta+1 \geq \frac{n}{2}\lg n$$

or equivalently,

$$\frac{n}{4}\lg(2^{4c+2\lg e+1+\frac{4\Delta+1}{n}}\beta) \geq \frac{n}{2}\lg n$$

with probability at least $1-\left(\frac{1}{2}\right)^{\Delta}$. By removing the factor of n and exponentiating both sides, it follows that

$$\sqrt{2^{4c+2\lg e+1+\frac{4\Delta+1}{n}}\beta} \geq n$$

or equivalently, we have

$$\beta \geq \frac{n^2}{e^2 2^{4c+1+\frac{4\Delta+1}{n}}}$$

with probability at least $1-\left(\frac{1}{2}\right)^{\Delta}$. □

A similar result was shown by Pach and Wenger [PW98], who gave a counting argument that utilized the relation between the the crossing number and the bisection width of a graph. Specifically, their technique was used to show that almost surely at least $\Omega(n)$ edges have at least $\Omega(n)$ bends. The encoding scheme of Lemma 4.1.5 however is adaptive in the number of bends and can be used to give lower bounds for simultaneous planar embeddings. Specifically, we have the following result on perfect matchings.

Theorem 4.1.7. *Let $P_1,\ldots,P_k$ be perfect matchings on the same n vertices V, where n is even, chosen uniformly at random. Furthermore, assume that we are given n fixed points in convex position. Then, in all simultaneous embeddings of $P_1,\ldots,P_k$ with vertex locations on this convex point set, in which edges map to polygonal curves, each of $P_1,\ldots,P_k$ has at least $\Omega(n^{2-\frac{4}{k}})$ total bends with high probability.*

Proof. Let $P_1, \ldots, P_k$ be perfect matchings on n vertices V chosen uniformly at random. Suppose that $P_1, \ldots, P_k$ can all be embedded with at most β bends on some mapping of V to a pre-specified convex set of n points. Then by Lemma 4.1.5, we can encode any k-tuple of perfect matchings using $\frac{kn}{4} \lg(\beta + n) + ckn + \lg n!$ bits, for some constant c. By Proposition 4.1.1,

$$\frac{kn}{4} \lg(\beta + n) + ckn + \lg n! \quad \geq \quad \frac{kn}{2} \lg \frac{n}{e} - (\Delta + 1)$$

and therefore, as in the proof of 4.1.6, we can conclude that

$$\frac{kn}{4} \lg(2\beta) + \left(c + \frac{\lg e}{2}\right) kn + \Delta + 1 \quad \geq \quad \frac{kn}{2} \lg n - \lg n!$$

with probability at least $1 - \left(\frac{1}{2}\right)^{\Delta}$. Using Stirling's approximation, it follows that

$$\frac{kn}{4} \lg(2\beta) + (c + \frac{\lg e}{2})kn + \Delta + 1 \geq \frac{kn}{2} \lg n - n \lg n + O(n)$$

which, for n sufficiently large, implies that

$$\frac{kn}{4} \lg(e^2 2^{\frac{4c}{k} + \frac{4(\Delta+1)}{kn} + 1} \beta) \geq (1 - \frac{2}{k}) \frac{kn}{2} \lg n$$

with probability at least $1 - \left(\frac{1}{2}\right)^{\Delta}$. By dividing by a factor of $\frac{kn}{2}$ and exponentiating both sides, it follows that

$$\sqrt{e^2 2^{\frac{4c}{k} + \frac{4(\Delta+1)}{kn} + 1} \beta} \geq n^{1 - \frac{2}{k}}$$

or equivalently, we have

$$\beta \geq \frac{n^{2 - \frac{4}{k}}}{e^2 2^{\frac{4c}{k} + \frac{4(\Delta+1)}{kn} + 1}}$$

with probability at least $1 - \left(\frac{1}{2}\right)^{\Delta}$. □

4.2 Generalized Encoding Method and Lower Bounds

4.2.1 Encoding Planar Graphs with Small Edge Complexity

In the previous section we derived lower bounds on the edge complexity of embeddings of perfect matchings. These results can be immediately generalized to other classes of graphs. In particular, we can generalize the results to a class of graphs that contain more information, and as a result, give a tighter lower bound on the edge complexity of planar graphs. The essential technique simply converts a graph into a perfect matching by separating each vertex into a set of vertices, one for each incident edge. We can then apply the encoding techniques of the previous section to derive a more general bound.

Theorem 4.2.1. *Let G be a connected planar graph with n vertices. If G can be embedded so that its vertices map onto a fixed convex point set of size n with at most β bends, then G can be encoded using at most*

$$\frac{m}{2}\lg(\beta + 2m) + cm$$

bits, where c is a constant.

Proof. Let G be a connected planar graph with n vertices that can be embedded with at most β bends, such that the vertices map onto a convex point set of size n. Replace each vertex v in G with a contiguous sequence of vertices, each incident to a unique edge that was incident to v. This gives a perfect matching with $2m$ vertices that can be drawn with at most β bends on a convex point set. We can therefore encode this perfect matching using

$$\frac{m}{2}\lg(\beta + 2m) + c'm$$

bits, for some constant c', by Lemma 4.1.5. To recover G from this encoding, we can simply encode a binary string with m zeros and n ones, where each run of zeros describes which blocks of vertices in the perfect matching map to vertices in G. Since G is connected, this binary string can be encoded using at most $2m$ bits. Thus, for $c = 2 + c'$, G can be encoded using

$$\frac{m}{2}\lg(\beta + 2m) + cm$$

bits. □

4.2.2 Lower Bounds on the Edge Complexity for Paths

To improve the bounds derived for perfect matchings, we consider a class of graphs that are still planar but require more information to describe. In particular, we consider the class of *paths* on n vertices, which are essentially defined by a permutation that can be read in either left or right order. The following claim gives a lower bound on the amount of information necessary to describe a path chosen uniformly at random.

Proposition 4.2.2. *Let P be a path on n vertices chosen uniformly at random. Furthermore, let A be the encoding size of P given by any fixed encoding scheme. Then, $A \geq \lg \frac{n!}{2} - \Delta$ with probability at least $1 - 2^{1-\Delta}$.*

Proof. Follows essentially by the same argument as Proposition 4.1.1 and the fact that the number of paths on n vertices is $\frac{1}{2}n!$. □

By using this lower bound and Theorem 4.2.1, we apply a similar reasoning to that used in the proof of Theorem 4.1.7 to achieve the following result.

Theorem 4.2.3. *Let $P_1, \ldots, P_k$ be paths on the same n vertices V, where n is even, chosen uniformly at random. Furthermore, assume we are given n points in convex position. Then, in all simultaneous embeddings of $P_1, \ldots, P_k$ in which vertices map onto this convex point set and edges map to polygonal curves, each of $P_1, \ldots, P_k$ has at least $\Omega(n^{2-\frac{2}{k}})$ bends with high probability.*

Proof. Let $P_1, \ldots, P_k$ be paths on n vertices V chosen uniformly at random. Suppose $P_1, \ldots, P_k$ can all be embedded with at most β bends on some mapping of V to a pre-specified convex set of n points. Then by Lemma 4.2.1, we can encode any k-tuple of paths using $\frac{kn}{2} \lg(\beta + 2n) + cn + \lg n!$ bits, for some constant c. By Proposition 4.2.2,

$$\frac{kn}{2} \lg(\beta + 2n) + ckn + \lg n! \quad \geq \quad k \lg \frac{1}{2} n! - (\Delta + 1)$$

with probability at least $1 - \left(\frac{1}{2}\right)^{\Delta}$. Thus, for n sufficiently large, it clearly must hold that $\beta \geq 2n$ with probability at least $1 - \left(\frac{1}{2}\right)^{\Delta}$. Thus, we can conclude that

$$\frac{kn}{2} \lg(2\beta) + ckn + k + \Delta + 1 \quad \geq \quad (k-1) \lg n!$$

with probability at least $1 - \left(\frac{1}{2}\right)^{\Delta}$. Using Stirling's approximation, it follows that

$$\frac{kn}{2} \lg(2\beta) + ckn + k + \Delta + 1 \geq (k-1) n \lg n + O(kn)$$

which, for n sufficiently large, implies that

$$\frac{kn}{2} \lg(2^{2c + \frac{2\Delta+1}{kn} + \frac{2}{n}} \beta) \geq (2 - \frac{2}{k}) \frac{kn}{2} \lg n$$

with probability at least $1 - \left(\frac{1}{2}\right)^{\Delta}$. By dividing by a factor of $\frac{kn}{2}$ and exponentiating both sides, it follows that

$$2^{2c + \frac{2\Delta+1}{kn} + \frac{2}{n}} \beta \geq n^{2 - \frac{2}{k}}$$

or equivalently, we have

$$\beta \geq \frac{n^{2-\frac{2}{k}}}{2^{2c + \frac{2\Delta+1}{kn} + \frac{2}{n}}}$$

with probability at least $1 - \left(\frac{1}{2}\right)^{\Delta}$. $\square$

Chapter 5

Conclusion

We have described a general technique for embedding planar graphs with fixed vertex locations. Our results have shown that this technique encapsulates a solution to both the problem when the mapping between vertices is pre-specified and when the mapping can be chosen freely.

In the first problem, our technique was shown to give a simple constructive method that guarantees the same edge complexity as the original solution due to Pach and Wenger [PW98] up to constant factors. However, the constant factor is substantially improved by our technique both in the worst case and in the average case.

In the second problem, our technique agrees with the number of bends guaranteed by the original solution due to Kaufmann and Wiese [KW02]. However, the additional structure was shown to admit a generalized technique for solving the problem of constructing a simultaneous planar embedding with fixed vertex locations. Our results included tight worst-case and average-case bounds on the number of bends that our technique gives for this generalized problem.

In conjunction with the tight worst-case and average-case upper bounds given by our technique, we discussed related lower bounds. Our results gave average-case lower bounds; that is, we proved lower bounds that hold almost surely on uniformly random input, a stronger notion than simply showing the existence of some input for which the lower bound holds. Utilizing a technique derived from [PW98], we developed a general method for encoding a planar graph that is adaptive in the the number of bends required by any embedding of the graph with fixed vertex locations. This method allowed us to give a lower bound (essentially, the same as the lower bound in [PW98]) for the first problem. However, the technique we have described was shown to generalize to the problem of constructing a simultaneous planar embedding. For this problem, our lower bound gives a function that grows similarly to our upper bound.

There are several avenues of future work that could be explored from these results. We conjecture that, for the problem of constructing a simultaneous planar embedding with fixed vertex locations, our upper bound is optimal up to constant factors. Furthermore, we conjecture that a tighter analysis of the encoding method we describe suffices to prove this optimality. If this optimality is in fact true, then this result shows a deep connection between the number of bends in a planar embedding and monotonic subsequences in permutations.

Second, granted that our technique already gives near optimal results for the problem of constructing a simultaneous planar embedding with fixed vertex locations, it is reasonable to consider generalizing this technique to the case where the vertex locations are not fixed. Given k planar graphs over a common vertex set, for $k > 2$, is there an ideal point set onto which the graphs can be simultaneously embedded? Or, can the results of Erten and Kobourov [EK04] be generalized to construct simultaneous planar embeddings of more than k graphs with few bends?

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