Quadratic Hedging with Margin Requirements and Portfolio Constraints

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

We consider a mean-variance portfolio optimization problem, namely, a problem of minimizing the variance of the final wealth that results from trading over a fixed finite horizon in a continuous-time complete market in the presence of convex portfolio constraints, taking into account the cost imposed by margin requirements on trades and subject to the further constraint that the expected final wealth equal a specified target value. Market parameters are chosen to be random processes adapted to the information filtration available to the investor and asset prices are modeled by Itô processes.

To solve this problem we use an approach based on conjugate duality: we start by synthesizing a dual optimization problem, establish a set of optimality relations that describe an optimal solution in terms of solutions of the dual problem, thus giving necessary and sufficient conditions for the given optimization problem and its dual to each have a solution. Finally, we prove existence of a solution of the dual problem, and for a particular class of dual solutions, establish existence of an optimal portfolio and also describe it explicitly. The method elegantly and rather straightforwardly constructs a dual problem and its solution, as well as provides intuition for construction of the actual optimal portfolio.

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Contents

Chapter 1

Introduction

In this thesis we study a classical asset-allocation problem, namely the problem of mean-variance portfolio selection. The mean-variance approach to portfolio optimization was originally introduced by Markowitz [19], [20] and ever since then, together with the general approach of maximizing expected utility, has become one of the most popular ways to look at the problem of efficient asset-allocation. Mean-variance optimization has its roots in the highly intuitive idea of risk and reward tradeoff. As such, the mean-variance investor wishes to minimize the variance of his final wealth subject to achieving a desired minimum level of return. It is assumed that the investor starts with some initial level of wealth and trades continuously over a fixed finite horizon in a complete market. To make the problem more challenging various additional assumptions and restrictions can be added: for example, market parameters can be chosen to be random, there can be restrictions on the dollar amounts allocated to each stock, or short-selling might be prohibited; alternatively, we might wish to include transaction costs or payments into a margin account. In this thesis, with an attempt to consider a rather general problem, we include the following features, which make the problem quite non-trivial:

(a) we consider random market parameters;

(b) we account for margin payments by including a nonlinear term in the wealth dynamics;

(c) we consider numerous trading restrictions, such as prohibition of short selling, or dollar limitation on asset holdings, by constraining the portfolio to be in a fixed convex set.

Similar problems within a complete market framework have been previously discussed by, among others, Lim and Zhou [18] and Li, Zhou, and Lim [17]. The original work [18] considers the problem with random market parameters but for unconstrained portfolios using the approach of stochastic linear-quadratic (LQ) control. The follow-up work [17] included a no-short selling constraint, however, is restricted to the case of nonrandom market coefficients. Works by Korn and Trautmann [14], Korn [13], and Bielecki *et al* [2] instead consider a nonnegativity constraint on the wealth process but fail to incorporate convex portfolio constraints, or margin payment requirements.

In our attempt to solve the problem, we turn to the method of conjugate duality: the mean-variance optimization problem is intrinsically convex, and therefore is often amenable to the method of conjugate duality. According to this method, we treat the original problem as *primal* and formulate an associated *dual* optimization problem. If, as often happens, the dual problem is easier to solve, our next step is first to show the existence of a solution, and second, to construct an optimal portfolio, i.e. solution to primal problem, in terms of the solution of the dual problem. For the "weak duality principle" to hold, the sum of the primal and dual cost functionals must be nonnegative, while the infimal values of the primal and dual problems must sum to zero. In this thesis we are motivated by earlier works of Labbé and Heunis [15] and [16], which were in turn motivated by a recent approach to conjugate duality of Rogers [22] and, especially, by the definitive work of Bismut [3] on stochastic optimal control.

Conjugate duality methods have been previously applied to problems of optimal investment by, among others, Karatzas *et al* [9], Xu and Shreve [23], Cvitanic and Karatzas [5], Cuoco and Liu [4], Karatzas, Lehoczky and Shreve [8]. All of these authors focused on maximizing the expected utility of terminal wealth and showed the amazing effectiveness of the duality methods for a considerable variety of challenging optimal investment problems. The duality idea, however, was cleverly but non-transparently used by these authors, in such a way, that it is often difficult for the reader to get the intuition behind the formulation of a dual problem. Indeed, it is apparent that in these works the dual functional has essentially been arrived at by a process of patient experimentation, guesswork, and generalization from simpler cases. A synthetic approach of Rogers [22] and Klein and Rogers [12], which has its roots in a classic work of Bismut [3], offers an approach for constructively synthesizing a candidate dual cost function which to some extent eliminates this guesswork. However, the approach in these works ([22] and [12]) is limited to problems without any portfolio constraints and establishes only equality of the primal and dual values without any construction of the actual optimal portfolios. Accordingly, we turn for motivation to the classic work of Bismut [3], whose basic approach yields an elegant way of arriving at a dual problem and optimality relations by converting the given optimization problem into a Bolza problem in the stochastic calculus of variations. The optimality relations are then used to establish existence of a solution and to characterize the optimal portfolio and corresponding wealth process.

To summarize, we consider a mean-variance optimization problem with random market parameters, convex portfolio constraints and taking into account the cost imposed by margin requirements. It is this feature, namely the cost imposed by margin requirements, which distinguishes the problem of the thesis from the mean-variance problem studied by Labbé and Heunis [15], in which this cost is not taken into account. As we shall see, accounting for margin payments gives rise to serious technical challenges, which were not present in [15], and which we have been able to only partially resolve in the present thesis. Our basic approach is to establish a set of optimality relations that describe an optimal solution in terms of solutions of the dual problem, and give necessary and sufficient conditions for the problem and its dual to each have a solution. We then prove existence of a solution of the dual problem, and for a particular class of dual solutions, we tentatively establish existence of an optimal portfolio and describe it explicitly. We briefly discuss the possibility of constraining the problem studied in [15] even further: we not only consider margin requirements, but also require the wealth process to stay nonnegative throughout the trading period.

The thesis is organized as follows. In Chapter 2 we look at the mean-variance problem in the presence of convex portfolio constraints and considering margin requirements. In Chapter 3 we modify the original problem discussed in Chapter 2 to include an additional constraint that specifies that wealth should stay nonnegative over the trading interval. Chapter 4 ends the thesis with some concluding remarks.

Chapter 2

Mean-Variance Optimization with Convex Constraints

In this chapter we attack the classical problem of mean-variance portfolio optimization with convex portfolio constraints and taking into consideration margin requirements on trades. As such, the problem is similar to that addressed in [15], but includes the additional feature of accounting for margin payments in the course of trades, something that was not considered in [15]. As will become clear, this feature makes the problem quite challenging as compared with that of [15]. The chapter is organized as follows: we start by describing the market model in Section 2.1, then introduce the class of square integrable processes that will be used throughout this thesis in Section 2.2. We then formulate the optimization problem, including a weaker, partially constrained problem, in Section 2.3, synthesize the dual problem and optimality relations in Section 2.4, and show existence of a solution to the dual problem in Section 2.5. In Sections 2.6 - 2.8 we tentatively construct a solution of the optimality relations derived in Section 2.4, and use this solution to propose a candidate optimal portfolio for the constrained mean-variance problem with margin payments.

2.1 Market Model

Throughout the thesis we assume that we are given a complete probability space $(\Omega, \mathcal{F}, P)$ on which is defined some $\mathbb{R}^N$-valued standard Brownian motion $\{W(t),\ t \in [0,T]\}$ with a finite horizon $T \in (0, \infty)$. We put

$$\mathcal{F}_t := \sigma\{W(\tau),\ \tau \in [0,t]\} \ \vee\ \mathcal{N}(P), \tag{2.1.1}$$

where $\mathcal{N}(P)$ denotes the collection of all P-null events in $(\Omega, \mathcal{F}, P)$. We interpret $\{\mathcal{F}_t,\ t \in [0,T]\}$ as information available to investors at instant t, in a sense that the true state of nature is completely determined by the sample paths of W on $[0, T]$. Since, as we shall see next, all assets are progressively measurable with respect to $\{\mathcal{F}_t,\ t \in [0, T]\}$, this implies that at time $t \in [0, T]$ investors have complete knowledge about asset price movements up to and including instance t.

We choose market parameters to be random processes adapted to the information filtration available to the investor and we model asset prices by Itô processes. We assume the market consists of $N + 1$ assets that are traded continuously on the interval $[0, T]$, including a money market (or bond) with price $\{S_0(t)\}$ given by

$$dS_0(t) = r(t)S_0(t)\,dt, \qquad t \in [0, T], \qquad S_0(0) = 1, \tag{2.1.2}$$

and N stocks with prices $\{S_n(t)\}$, $n = 1, 2, ..., N$, given by

$$dS_n(t) = S_n(t)\left[b_n(t)\,dt + \sum_{m=1}^{N} \sigma_{nm}(t)\,dW_m(t)\right], \qquad 0 \le t \le T, \tag{2.1.3}$$

with initial values $S_n(0)$, $n = 1, 2, ..., N$, given by strictly positive constants. We will always assume the following conditions, which are standard in most work on portfolio optimization:

Condition 2.1.1. In (2.1.2) and (2.1.3) the interest rate $\{r(t)\}$, the entries $\{b_n(t)\}$ of the $\mathbb{R}^N$-valued process $\{b(t)\}$ of the mean rates of return on stocks, and the entries $\{\sigma_{nm}(t)\}$ of the $N \times N$ matrix-valued volatility process $\{\sigma(t)\}$ are uniformly bounded and $\{\mathcal{F}_t\}$-progressively measurable scalar processes on $\Omega \times [0, T]$. There exists a constant $\kappa \in (0, \infty)$ such that

$$z'\sigma(w, t)\sigma'(w, t)z \ge \kappa \|z\|^2, \qquad \forall (z, w, t) \in \mathbb{R}^N \times \Omega \times [0, T]. \tag{2.1.4}$$

◁

Remark 2.1.2. It follows from Condition 2.1.1 and Problem 5.8.1 of [10] (p. 372), there exists a constant $\kappa_1 \in (0, \infty)$ such that

$$\max\{\|(\sigma(w, t))^{-1}z\|, \|(\sigma'(w, t))^{-1}z\|\} \le \kappa_1 \|z\|, \qquad \forall (z, w, t) \in \mathbb{R}^N \times \Omega \times [0, T]. \tag{2.1.5}$$

The bound (2.1.5) will be frequently used throughout the thesis. ◁

Remark 2.1.3. We define the market price of risk

$$\theta(t) := (\sigma(t))^{-1}[b(t) - r(t)\mathbf{1}], \tag{2.1.6}$$

in which $\mathbf{1} \in \mathbb{R}^N$ has all unit entries. From Condition 2.1.1 and Remark 2.1.2 follows that the process $\{\theta(t)\}$ is uniformly bounded on $\Omega \times [0, T]$. ◁

Remark 2.1.4. To simplify notation, we will use $\mathcal{F}^*$ to denote the σ-algebra of $\mathcal{F}_t$-progressively measurable sets on $\Omega \times [0, T]$ and we will write $\eta \in \mathcal{F}^*$ to indicate that η is $\mathcal{F}^*$-measurable. ◁

To account for margin payments in the course of trading we introduce a function g as follows (see Remark 2.1.6 for further discussion on this):

Condition 2.1.5. The function $g : \Omega \times [0, T] \times \mathbb{R}^N \to \mathbb{R}$ is $\mathcal{F}^* \times \mathcal{B}(\mathbb{R}^N)$-measurable and such that $g(w, t, 0) = 0$ and $g(w, t, \cdot)$ is concave and continuous on $\mathbb{R}^N$ for each $(w, t) \in \Omega \times [0, T]$. Further, for each $(w, t) \in \Omega \times [0, T]$ function $g(w, t, \cdot)$ is globally Lipshitz-continuous on $\mathbb{R}^N$ with a Lipshitz constant $\zeta \in [0, \infty)$ that is uniform with respect to (w, t). ◁

Now consider a small investor with initial wealth x_0 and no further capital inflows throughout trading period. At any moment of time $t \in [0, T]$, based only on the information $\mathcal{F}_t$ available at the time, he can invest $\pi_n(t)$ dollars of his current wealth in asset with price $S_n(t)$. We assume that investor's actions do not affect market prices. More formally, given some non-random initial wealth $x_0 \in (0, \infty)$, and some $\{\mathcal{F}_t\}$-progressively measurable process $\pi : \Omega \times [0, T] \to \mathbb{R}^N$ of the set

$$\Pi := \left\{ \pi : \Omega \times [0, T] \to \mathbb{R}^N \middle| \; \pi \in \mathcal{F}^* \text{ and } \int_0^T \{ \, |g(t, \pi(t))| + \|\pi(t)\|^2 \} dt < \infty \text{ a.s.} \right\},$$

there exists a scalar-valued, continuous, and $\{\mathcal{F}_t\}$-progressively measurable process $\{X^\pi(t), t \in [0, T]\}$ such that

$$dX^\pi(t) = [r(t)X^\pi(t) + \pi'(t)\sigma(t)\theta(t) + g(t, \pi(t))] \, dt + \pi'(t)\sigma(t)dW(t), \qquad X^\pi(0) = x_0, \tag{2.1.7}$$

which is unique to within indistinguishability, P-strictly positive, and given by

$$X^\pi(t) = S_0(t) \Big\{ x_0 + \int_0^t S_0^{-1}(\tau)[\pi'(\tau)\sigma(\tau)\theta(\tau) + g(\tau, \pi(\tau))]d\tau + \int_0^t S_0^{-1}(\tau)\pi'(\tau)\sigma(\tau)dW(\tau) \Big\}. \tag{2.1.8}$$

Thus $X^\pi(t) = \pi_0(t) + \pi'(t)\mathbf{1}$ gives the investor's wealth at instant $t \in [0, T]$, where $\pi_0(t)$ denotes investor's wealth put into riskless asset at time $t \in [0, T]$.

Remark 2.1.6. The role of function g in (2.1.7) is to model the presence of margin requirements on trades, which oblige an investor to make payments to the broker when borrowing to buy securities or short selling assets. Cuoco and Liu [4] present a detailed discussion of the modelling of various types of margin payments, including examples of typical expression for margin function g, and show that Condition 2.1.5 formulates natural properties that one would expect of a term which modifies the wealth equation to account for margin payments. One example of such a function is given by

$$g(t, \pi(t)) := -r(t)(1+\lambda_-) \sum_{i=1}^{N} \pi_i^-(t), \quad \text{for all } t \in [0,T], \tag{2.1.9}$$

where $\lambda_- > 0$ is the proportional margin requirement on short positions and $\pi^-(t) := \max(0, -\pi(t))$ denote the negative part of $\pi(t)$. In other words, a cash amount equal to the $100(1+\lambda_-)\%$ of the market value of the total short position of the portfolio should be deposited with a broker at any time instant $t \in [0,T]$. ◁

2.2 A Class of Square Integrable Itô Processes

In the following section we define a class of square integrable Itô processes that will be used in future sections. Since we are interested in minimizing the variance of the terminal wealth $X^\pi(T)$, it is essential for this variance to be defined, in other words, we need to ensure that $\mathbb{E}[X^\pi(T)^2] < \infty$. The goal of this section is to identify conditions on the portfolio π that would guarantee square integrability of the terminal wealth $X^\pi(T)$.

Recall Remark 2.1.4 and put

$$\mathcal{H}_1 := \left\{ v : \Omega \times [0,T] \to \mathbb{R} \;\middle|\; v \in \mathcal{F}^* \text{ and } \mathbb{E}\left[\int_0^T |v(t)| dt\right]^2 < \infty \right\}, \tag{2.2.1}$$

$$\mathcal{H}_2 := \left\{ \xi : \Omega \times [0,T] \to \mathbb{R}^N \;\middle|\; \xi \in \mathcal{F}^* \text{ and } \mathbb{E}\int_0^T \|\xi(t)\|^2 dt < \infty \right\}, \tag{2.2.2}$$

$$\mathbb{I} := \mathbb{R} \times \mathcal{H}_1 \times \mathcal{H}_2. \tag{2.2.3}$$

Then set $\mathbb{I}$ is a collection of square integrable Itô processes with respect to the Brownian motion $\{W(t)\}$ in the following sense: let $(X_0, \dot{X}, \Lambda_X) \in \mathbb{I}$ and write $X \equiv (X_0, \dot{X}, \Lambda_X) \in \mathbb{I}$ to indicate that $\{(X(t), \mathcal{F}_t),\ t \in [0,T]\}$ is a continuous semimartingale of the form

$$X(t) = X_0 + \int_0^t \dot{X}(\tau) d\tau + \int_0^t \Lambda_X'(\tau) dW(\tau). \tag{2.2.4}$$

It is clear that the integrands $\dot{X}$ and Λ_X are uniquely determined a.e. for each $X \in \mathbb{I}$ on $\Omega \times [0,T]$. This useful representation of square integrable Itô processes was used in Labbé and Heunis [15] and is originally due to Bismut [3]. The next result shows the sense in which the process at (2.2.4) is square integrable:

Proposition 2.2.1. *If $X \in \mathbb{I}$, then*

$$\mathbb{E}[\sup_{0\le t\le T} |X(t)|^2] < \infty, \qquad \forall X \in \mathbb{I}. \tag{2.2.5}$$

Proof. The statement follows directly from Doob's L^2-inequality, see [3], p.390. □

The following proposition gives the necessary and sufficient conditions for wealth process X^π to be member of $\mathbb{I}$ (recall (2.1.7)):

Proposition 2.2.2. *Assume Condition 2.1.1 and Condition 2.1.5 and let $\pi \in \Pi$. Then $X^\pi \in \mathbb{I}$ if and only if $\pi \in \mathcal{H}_2$.*

Proof. Suppose $X^\pi \in \mathbb{I}$, then from the dynamics (2.1.7) of X^π we have $\Lambda'_{X^\pi}(t) = \pi'(t)\sigma(t)$ for all $t \in [0, T]$, therefore from the uniform boundedness of σ it follows that $\pi \in \mathcal{H}_2$.

Now suppose $\pi \in \mathcal{H}_2$. From Lipschitz-continuity of $g(w, t, \cdot)$ (recall Condition 2.1.5) follows that $|g(t, \pi)| \le \zeta\|\pi\|$ for all $\pi \in \mathbb{R}^N$ and $t \in [0, T]$, hence,

$$\mathbb{E}\left[\int_0^T |g(t, \pi(t))|dt\right]^2 < \infty. \tag{2.2.6}$$

Thus from (2.1.8), uniform boundedness of market parameters (Condition 2.1.1), (2.2.6), and Doob's L^2-inequality it follows that

$$\mathbb{E}[\sup_{0\le t\le T} |X^\pi(t)|^2] < \infty. \tag{2.2.7}$$

Therefore, $r(t)X^\pi(t) + \pi'(t)\sigma(t)\theta(t) + g(t, \pi(t)) \in \mathcal{H}_1$ and $\pi'(t)\sigma(t) \in \mathcal{H}_2$ due to uniform boundedness of the parameters. It follows that $X^\pi \in \mathbb{I}$. □

From Proposition 2.2.2 it follows that we should choose $\pi \in \mathcal{H}_2$ to ensure that $X^\pi(T)$ is square integrable.

2.3 Formulation of Optimization Problem

2.3.1 Fully Constrained Problem

In this chapter we consider an investor who is interested in achieving an expected predetermined level of terminal wealth d while minimizing its variance $Var(X^\pi(T))$. This problem was previously addressed by Labbé and Heunis [15] but for the case where there are no margin requirements (that is $g(t, \pi(t)) = 0$ at (2.1.7)). As we follow the general approach of [15], it turns out that the presence of the nonlinear term g in (2.1.7) poses substantial additional challenges that were not present in [15]. These challenges will be pointed out specifically later in the thesis, at this point, we wish to emphasize that adding this additional term for margin requirements makes the problem non-trivial and not just a routine generalization of problem discussed in [15]. Before we attempt to solve this problem, we need to present it mathematically. We start by postulating

Condition 2.3.1. We are given a closed convex set $K \in \mathbb{R}^N$ with $0 \in K$, a random variable a on $(\Omega, \mathcal{F}, P)$ which is $\mathcal{F}_T$-measurable with

$$0 < \inf_{w \in \Omega} a(w) \leq \sup_{w \in \Omega} a(w) < \infty,$$

a constant $d \in (0, \infty)$, and $\mathcal{F}_T$-measurable square integrable random variable c_0 on $(\Omega, \mathcal{F}, P)$. ◁

Now define

$$\mathcal{A} := \{\pi \in \mathcal{H}_2 \mid \pi(t) \in K \text{ a.e.}\}, \tag{2.3.1}$$

$$\hat{J}(w, x) := \frac{1}{2}[a(w)x^2 + 2c_0(w)x], \qquad (w, x) \in \Omega \times \mathbb{R}, \tag{2.3.2}$$

$$G(\pi) := \mathbb{E}[X^\pi(T)] - d, \qquad \pi \in \mathcal{H}_2, \tag{2.3.3}$$

$$\hat{v} := \inf_{\substack{\pi \in \mathcal{A} \\ G(\pi)=0}} \mathbb{E}[\hat{J}(X^\pi(T))]. \tag{2.3.4}$$

We view set $\mathcal{A}$ as the set of all admissible portfolios, in other words, portfolios characterized by a closed convex set K and satisfying the square integrability condition of Proposition 2.2.2. Constraint $G(\pi) = 0$ represents a requirement on terminal wealth, guaranteeing the investor terminal wealth of d dollars on average. Thus the investor wishes to solve problem $(\hat{\mathcal{P}})$, i.e.

$$(\hat{\mathcal{P}}): \quad \text{determine some } \hat{\pi} \in \mathcal{A} \text{ such that } G(\hat{\pi}) = 0 \text{ and } \hat{v} = \mathbb{E}[\hat{J}(X^{\hat{\pi}}(T))]. \tag{2.3.5}$$

To make the problem meaningful we impose

Condition 2.3.2. The constant d and the set K are such that the set $\{G(\pi) \mid \pi \in \mathcal{A}\} \subset \mathbb{R}$ has nonempty interior which includes 0. ◁

Remark 2.3.3. If $0 \notin \{G(\pi) \mid \pi \in \mathcal{A}\}$, then the constraints in (2.3.4) are mutually contradictory and the problem (2.3.5) is meaningless. Condition 2.3.2 ensures that such a situation is avoided and moreover allows us to use the Lagrange Multiplier Theorem in Section 2.8 in order to deal with the constraint on the expected terminal wealth represented by $G(\pi) = 0$ at (2.3.4). ◁

Remark 2.3.4. Set $a = 2$ and $c_0 = 0$, then $\mathbb{E}[\hat{J}(X^\pi(T))] - d^2 = \text{Var}(X^\pi(T))$ when $G(\pi) = 0$. In other words, problem $(\hat{\mathcal{P}})$ becomes a problem of constrained mean-variance optimization, where we wish to minimize variance of terminal wealth subject to the terminal wealth constraint $\mathbb{E}[X^\pi(T)] = d$. ◁

2.3.2 Partially Constrained Problem

In order to solve the constrained problem $(\hat{\mathcal{P}})$ defined at (2.3.5) we introduce a simpler surrogate problem, namely a partially constrained optimization problem similar to (2.3.5) but simplified in the sense that the constraint on the expected value of the terminal wealth is removed from the problem. Later, in Section 2.8 we shall use the solution of the surrogate partially constrained problem introduced here, together with standard Lagrange multiplier methods, to deal with the fully constrained problem (2.3.5) in which there is a constraint on the expected terminal wealth. To define the partially constrained problem, we postulate

Condition 2.3.5. We are given a constant $q \in \mathbb{R}$ and an $\mathcal{F}_T$-measurable square integrable random variable c on $(\Omega, \mathcal{F}, P)$. ◁

Using the ingredients postulated at Condition 2.3.5 put

$$J(w, x) := \frac{1}{2}[a(w)x^2 + 2c(w)x] + q, \quad (w, x) \in \Omega \times \mathbb{R}, \tag{2.3.6}$$

$$\upsilon_{c,q} := \inf_{\pi \in \mathcal{A}} \mathbb{E}[J(X^\pi(T))]. \tag{2.3.7}$$

We now define the partially constrained optimization problem $(\mathcal{P}_{c,q})$ as:

$$(\mathcal{P}_{c,q}): \quad \text{determine some } \bar{\pi} \in \mathcal{A} \text{ such that } \upsilon_{c,q} = \mathbb{E}[J(X^{\bar{\pi}}(T))]. \tag{2.3.8}$$

Remark 2.3.6. It easily follows from the form of $x \mapsto J(w, x)$ in (2.3.6), Conditions 2.3.1, 2.3.5, and Proposition 2.2.2, that $-\infty < \upsilon_{c,q} < +\infty$. ◁

Remark 2.3.7. Condition 2.3.1 still holds with respect to parameter a and set K. Note that we distinguish between c_0 and c in the definitions of problems (2.3.5) and (2.3.8) to avoid confusion, as these coefficients play different roles. Notice also that the condition $G(\pi) = 0$ is missing from the definition of the value at (2.3.7) for the partially constrained problem, reflecting the fact that this constraint has been discarded (in contrast with the value at (2.3.4) for the fully constrained problem where the condition $G(\pi) = 0$ is present). We shall see in Section 2.8 how we can use the solution of the partially constrained problem (2.3.8) to address the fully constrained problem (2.3.5). ◁

Remark 2.3.8. The partially constrained problem (2.3.8) can also be interpreted in the following way: suppose an investor wishes to replicate some contingent claim with payoff B, but is unable to find a perfect hedge because either the market is incomplete, or, the investor has limitations on initial capital available for the purchase of the replicating portfolio. To determine a suitable hedging alternative the investor might choose to use quadratic hedging, that is to minimize the L^2-distance $\mathbb{E}[B - X^{\pi}(T)]^2$ under all attainable strategies. The L^2-hedging problem is achieved by setting $a \equiv 2$, $c \equiv -2B$, $q \equiv \mathbb{E}[B^2]$ at (2.3.8). ◁

2.4 Dual Problem and Optimality Relations

2.4.1 Redefining the Primal Problem

From now on we will put aside the constrained problem (2.3.5) and focus on solving the partially constrained problem (2.3.8). In this subsection we eliminate the portfolio π from the problem (2.3.8) to get an optimization problem over the set $\mathbb{I}$ defined in Section 2.2. The advantage of this reformulation is that the latter optimization problem is really a Bolza poblem in the calculus of variations to which we can apply a duality approach that is motivated by Bismut [3]. To do so, we introduce a number of penalty terms on $\mathbb{I}$ which account for the initial wealth and portfolio constraints: these terms give a zero penalty when constraints hold and infinite penalty otherwise.

Let $X \equiv (X_0, \dot{X}, \Lambda_X) \in \mathbb{I}$ and define $\mathcal{U}(X)$ as

$$\mathcal{U}(X) := \left\{ \pi \in \mathcal{A} \;\middle|\; \dot{X}(t) = r(t)X(t) + \pi'(t)\sigma(t)\theta(t) + g(t, \pi(t)) \text{ and } \Lambda_X(t) = \sigma'(t)\pi(t) \text{ a.e.} \right\}. \tag{2.4.1}$$

Then, recalling (2.1.7), it follows that for each $X \equiv (X_0, \dot{X}, \Lambda_X) \in \mathbb{I}$ we have

$$X(t) = X^{\pi}(t) \text{ a.e. } \text{ for some } \pi \in \mathcal{A} \iff X_0 = x_0 \text{ and } \mathcal{U}(X) \neq \emptyset, \tag{2.4.2}$$

from which, together with (2.3.7), we conclude that

$$v_{c,q} = \inf_{\substack{X \in \mathbb{I} \\ X_0 = x_0 \\ \mathcal{U}(X) \neq \emptyset}} \mathbb{E}[J(X(T))]. \tag{2.4.3}$$

Moreover, from (2.4.1) it is also clear that

$$\mathcal{U}(X) \neq \emptyset \iff \begin{array}{l} \dot{X}(t) = r(t)X(t) + \Lambda'_X(t)\theta(t) + g(t, [\sigma'(t)]^{-1}\Lambda_X(t)) \\ \text{and } [\sigma'(t)]^{-1}\Lambda_X(t) \in K \text{ a.e.} \end{array} \tag{2.4.4}$$

We now define penalty terms on $\mathbb{I}$:

$$l_0(x) := \begin{cases} 0 & \text{if } x = x_0, \\ \infty & \text{otherwise,} \end{cases} \tag{2.4.5}$$

for each $x \in \mathbb{R}$ to account for the initial wealth constraint and, motivated by (2.4.4), we also define a mapping $L : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \to \{0, \infty\}$ by

$$L(w,t,x,v,\xi) := \begin{cases} 0 & \text{if } v = r(w,t)x + \xi'\theta(w,t) + g(w,t,[\sigma'(w,t)]^{-1}\xi) \\ & \qquad \text{and } [\sigma'(w,t)]^{-1}\xi \in K, \\ \infty & \text{otherwise.} \end{cases} \tag{2.4.6}$$

It is clear from (2.4.6) that $L(t, X(t), \dot{X}(t), \Lambda_X(t))$ is $\mathcal{F}^*$-measurable, and

$$\mathbb{E} \int_0^T L(t, X(t), \dot{X}(t), \Lambda_X(t)) = \begin{cases} 0 & \text{if } \mathcal{U}(X) \neq \emptyset, \\ \infty & \text{otherwise.} \end{cases} \tag{2.4.7}$$

Finally, for each $X \equiv (X_0, \dot{X}, \Lambda_X) \in \mathbb{I}$ define (recall (2.3.6))

$$\Phi(X) := l_0(X_0) + \mathbb{E}[J(X(T))] + \mathbb{E} \int_0^T L(t, X(t), \dot{X}(t), \Lambda_X(t))dt. \tag{2.4.8}$$

Remark 2.4.1. With help of (2.4.5) and (2.4.7) we have successfully "encoded" constraints $X_0 = x_0$ and $\mathcal{U}(X) \neq \emptyset$, which represent constraints on initial wealth and admissibility of portfolio defined by $\pi := [\sigma']^{-1}\Lambda_X$. Thus when $l_0(X_0) = 0$ and $\mathbb{E} \int_0^T L(t, X(t), \dot{X}(t), \Lambda_X(t))dt = 0$, we have $\Phi(X) = \mathbb{E}[J(X(T))]$, with $\Phi(X) = +\infty$ otherwise. Hence, from (2.4.5), (2.4.6), and (2.4.7) it follows that

$$\inf_{\substack{X \in \mathbb{I} \\ X_0 = x_0 \\ \mathcal{U}(X) \neq \emptyset}} \mathbb{E}[J(X(T))] = \inf_{X \in \mathbb{I}} \Phi(X), \tag{2.4.9}$$

and combining (2.4.9) with (2.4.3), we have

$$\upsilon_{c,q} = \inf_{X \in \mathbb{I}} \Phi(X). \tag{2.4.10}$$

Moreover, if $\bar{X} \equiv (\bar{X}_0, \dot{\bar{X}}, \Lambda_{\bar{X}}) \in \mathbb{I}$ is such $\Phi(\bar{X}) = \inf_{X \in \mathbb{I}} \Phi(X)$, then $\bar{\pi} := [\sigma']^{-1}\Lambda_{\bar{X}}$ is a solution to problem (2.3.8). From (2.4.10) we conclude that solving the partially constrained problem (2.3.8) is equivalent to minimizing functional Φ defined at (2.4.8) over the set $\mathbb{I}$. ◁

Remark 2.4.1 thus shows that we have successfully eliminated portfolio π from the optimization problem, instead introducing a new optimization problem over set $\mathbb{I}$ that we shall name the primal problem, namely determining some $\bar{X} \in \mathbb{I}$ such that $\upsilon_{c,q} = \Phi(\bar{X})$. In the following sub-section 2.4.2 we shall see that the optimization problem over processes given by the right-hand side of (2.4.10) is ideally suited to the introduction of a dual problem.

2.4.2 Dual Problem

With the reformulated primal problem (2.4.10) in place, in this subsection we formulate an optimization problem that is dual to the primal problem (2.4.10). We start by defining the convex conjugate functions according to Definition A.1.1:

$$m_0(y) := l_0^*(y) := \sup_{x\in\mathbb{R}}\{xy - l_0(x)\}, \tag{2.4.11}$$

$$m_T(w,y) := J^*(w,-y) := \sup_{x\in\mathbb{R}}\{x(-y) - J(w,x)\}, \tag{2.4.12}$$

$$M(w,t,y,s,\gamma) := L^*(w,t,s,y,\gamma) := \sup_{\substack{x,v\in\mathbb{R},\\ \xi\in\mathbb{R}^N}}\{xs + vy + \xi'\gamma - L(w,t,x,v,\xi)\}, \tag{2.4.13}$$

for each $y \in \mathbb{R},\ s \in \mathbb{R},\ \gamma \in \mathbb{R}^N,\ w \in \Omega$ and $t \in [0,T]$. From definitions (2.4.5) and (2.3.6) we easily obtain

$$xy - l_0(x) := \begin{cases} x_0 y & \text{if } x = x_0, \\ -\infty & \text{otherwise,} \end{cases} \tag{2.4.14}$$

and

$$x(-y) - J(w,x) = \frac{-a(w)}{2}\left[x + \frac{(y+c(w))}{a(w)}\right]^2 + \frac{(y+c(w))^2}{2a(w)} - q. \tag{2.4.15}$$

From (2.4.14) and (2.4.15) together with Condition 2.3.1 it then follows that

$$m_0(y) = x_0 y, \qquad \forall y \in \mathbb{R}, \tag{2.4.16}$$

$$m_T(w,y) = \frac{(y+c(w))^2}{2a(w)} - q, \qquad \forall (w,y) \in \Omega\times\mathbb{R}. \tag{2.4.17}$$

For $M(w,t,y,s,\gamma)$ of (2.4.13) from (2.4.6) we find

$$\begin{aligned} M(w,t,y,s,\gamma) &:= \sup_{\substack{x,v\in\mathbb{R},\\ \xi\in\mathbb{R}^N}} \{xs+vy+\xi'\gamma - L(w,t,x,v,\xi)\} \\ &= \sup_{\substack{x\in\mathbb{R},\,\xi\in\mathbb{R}^N,\\ [\sigma'(w,t)]^{-1}\xi\in K}} \{xs+y[r(w,t)x+\xi'\theta(w,t)+g(w,t,[\sigma'(w,t)]^{-1}\xi)]+\xi'\gamma\} \\ &= \sup_{\substack{x\in\mathbb{R},\,\xi\in\mathbb{R}^N,\\ [\sigma'(w,t)]^{-1}\xi\in K}} \{x[s+yr(w,t)]+yg(w,t,[\sigma'(w,t)]^{-1}\xi)+\xi'[y\theta(w,t)+\gamma]\} \\ &= \sup_{x\in\mathbb{R}}\{x[s+yr(w,t)]\} + \sup_{\substack{\xi\in\mathbb{R}^N,\\ [\sigma'(w,t)]^{-1}\xi\in K}} \{yg(w,t,[\sigma'(w,t)]^{-1}\xi)+\xi'[y\theta(w,t)+\gamma]\} \\ &= \begin{cases} \sup_{\substack{\xi\in\mathbb{R}^N,\\ [\sigma'(w,t)]^{-1}\xi\in K}} \{yg(w,t,[\sigma'(w,t)]^{-1}\xi)+\xi'[y\theta(w,t)+\gamma]\} & \text{if } s+r(w,t)y=0, \\ \infty & \text{otherwise.} \end{cases} \end{aligned} \tag{2.4.18}$$

Recall Condition 2.1.5 and define

$$\delta(w,t,y,\Theta) := \sup_{u\in K}\{yg(w,t,u)-u'\Theta\}, \qquad (w,t,y,\Theta)\in\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^N \tag{2.4.19}$$

and

$$\Theta_y(w,t) := -\sigma(w,t)[y\theta(w,t)+\gamma], \qquad (w,t)\in\Omega\times[0,T]. \tag{2.4.20}$$

Take $u := \sigma'(w,t)]^{-1}\xi$ and recall that $K\in\mathbb{R}^N$, hence,

$$\sup_{\substack{\xi\in\mathbb{R}^N,\\ [\sigma'(w,t)]^{-1}\xi\in K}} \{yg(w,t,[\sigma'(w,t)]^{-1}\xi)+\xi'[y\theta(w,t)+\gamma]\} = \sup_{u\in K}\{yg(w,t,u)+u'\sigma(w,t)[y\theta(w,t)+\gamma]\}. \tag{2.4.21}$$

Then from (2.4.18) and (2.4.21) together with (2.4.19) and (2.4.20) we obtain

$$M(w,t,y,s,\gamma) = \begin{cases} \delta(w,t,y,\Theta_y(w,t)) & \text{if } s+r(w,t)y=0, \\ \infty & \text{otherwise.} \end{cases} \tag{2.4.22}$$

Now for each $Y=(Y_0,\dot{Y},\Lambda_Y)\in\mathbb{I}$ (recall (2.2.1) - (2.2.4)) put

$$\Psi(Y) := m_0(Y_0)+\mathbb{E}[m_T(Y(T))]+\mathbb{E}\int_0^T M(t,Y(t),\dot{Y}(t),\Lambda_Y(t))dt \tag{2.4.23}$$

and define the dual problem as

$$\text{determine some} \quad \bar{Y}\in\mathbb{I} \quad \text{such that} \quad \Psi(\bar{Y}) = \inf_{Y\in\mathbb{I}}\Psi(Y). \tag{2.4.24}$$

The motivation behind the definition of the function $\Psi(\cdot)$ at (2.4.23) and the dual problem at (2.4.24) will be given in sub-section 2.4.3 which follows.

Remark 2.4.2. At this point we would like to emphasize the importance of derivations in the present sub-section: it is the explicit formulae obtained in (2.4.16), (2.4.17) and (2.4.22) that allow us to define the dual problem (2.4.24) without preliminary guesswork and then derive optimality relations in the following sub-section. $\triangleleft$

Remark 2.4.3. For future developments it is useful to state some of the properties of function $\delta : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ defined in (2.4.19). From Condition 2.3.1 and Condition 2.1.5 follows that $\delta(w,t,y,\Theta) \geq 0$ for all $(w,t,y,\Theta) \in \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N$ since $0 \in K$. Further, δ is convex and lower semi-continuous, as well as positively homogeneous and subadditive. $\triangleleft$

Remark 2.4.4. From lower semi-continuouty of δ and (2.4.22) it follows that the mapping $M(t, Y(t), \dot{Y}(t), \Lambda_Y(t))$ is $\mathcal{F}^*$-measurable on $\Omega \times [0,T]$ (recall Remark 2.1.4) and $\Psi(Y) \in (-\infty, +\infty]$ for each $Y \equiv (Y_0, \dot{Y}, \Lambda_Y) \in \mathbb{I}$. $\triangleleft$

2.4.3 Necessary and Sufficient Optimality Conditions

We are now ready to attempt to solve the partially constrained problem (2.3.8). In this section we determine necessary and sufficient conditions for the portfolio to be a solution to problem (2.3.8). To this end we need the following basic result on square integrable Itô processes due to Bismut [3]:

Proposition 2.4.5. *Recall (2.2.1) - (2.2.4) and let* $X \equiv (X_0, \dot{X}, \Lambda_X) \in \mathbb{I}$, $Y \equiv (Y_0, \dot{Y}, \Lambda_Y) \in \mathbb{I}$ *and define*

$$\mathbf{M}(X,Y)(t) := X(t)Y(t) - X_0Y_0 - \int_0^t \{X(\tau)\dot{Y}(\tau) + \dot{X}(\tau)Y(\tau) + \Lambda'_X(\tau)\Lambda_Y(\tau)\}d\tau \quad (2.4.25)$$

for each $t \in [0,T]$. *Then* $\{(\mathbf{M}(X,Y)(t), \mathcal{F}_t),\ t \in [0,T]\}$ *is a continuous martingale with* $\mathbf{M}(X,Y)(0) = 0$.

Proof. See Proposition I-1 in [3] (p. 387). $\square$

Propositition 2.4.5 is in fact our primary motivation for the introduction of the sets $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathbb{I}$ at (2.2.1) - (2.2.3).

Proposition 2.4.6. *Suppose Conditions 2.1.1, 2.1.5, 2.3.1, 2.3.5. Functions* Φ *and* Ψ *given by (2.4.8) and (2.4.23) respectively are well-defined, with values in* $(-\infty, +\infty]$ *for each* $X \equiv (X_0, \dot{X}, \Lambda_X) \in \mathbb{I}$, $Y \equiv (Y_0, \dot{Y}, \Lambda_Y) \in \mathbb{I}$, *and satisfy the "weak duality" relation*

$$\Phi(X) + \Psi(Y) \geq 0, \qquad \forall (X,Y) \in \mathbb{I} \times \mathbb{I}. \qquad (2.4.26)$$

Further, for arbitrary $\bar{X} \equiv (\bar{X}_0, \dot{\bar{X}}, \Lambda_{\bar{X}}) \in \mathbb{I}$ *and* $\bar{Y} \equiv (\bar{Y}_0, \dot{\bar{Y}}, \Lambda_{\bar{Y}}) \in \mathbb{I}$*, we have the equality* $\Phi(\bar{X}) + \Psi(\bar{Y}) = 0$ *if and only if all of the following conditions hold:*

(A) $l_0(\bar{X}_0) + m_0(\bar{Y}_0) = \bar{X}_0 \bar{Y}_0$,

(B) $J(\bar{X}(T)) + m_T(\bar{Y}(T)) = -\bar{X}(T)\bar{Y}(T)$ *a.s.,*

(C) $L(t, \bar{X}(t), \dot{\bar{X}}(t), \Lambda_{\bar{X}}(t)) + M(t, \bar{Y}(t), \dot{\bar{Y}}(t), \Lambda_{\bar{Y}}(t))$
$$= \bar{X}(t)\dot{\bar{Y}}(t) + \dot{\bar{X}}(t)\bar{Y}(t) + \Lambda'_{\bar{X}}(t)\Lambda_{\bar{Y}}(t) \text{ a.e. on } \Omega \times [0, T].$$

Proof. The proof is identical to that of Proposition 5.2 of [15] (p. 84) and is therefore omitted. □

Similarly to definition (2.4.20) of Θ_y, put

$$\Theta_Y(t) := -\sigma(t)[\theta(t)Y(t) + \Lambda_Y(t)], \qquad \forall\, Y \equiv (Y_0, \dot{Y}, \Lambda_Y) \in \mathbb{I}. \tag{2.4.27}$$

Proposition 2.4.6 is a general and abstract result which relies only on the convex conjugates at (2.4.11) - (2.4.13), and does not make any use of the explicit formulae that we have available for l_0, m_0, J, m_T, L, and M (see (2.4.5), (2.3.6), (2.4.6), (2.4.16), (2.4.17), and (2.4.22)). We now shall refine Proposition 2.4.6 by taking into account these explicit expressions in the next proposition:

Proposition 2.4.7. *Assume Condition 2.1.1, 2.1.5, 2.3.1, 2.3.5. For arbitrary* $(\bar{X}, \bar{Y}) \in \mathbb{I} \times \mathbb{I}$ *with* $\bar{X} \equiv (\bar{X}_0, \dot{\bar{X}}, \Lambda_{\bar{X}})$ *and* $\bar{Y} \equiv (\bar{Y}_0, \dot{\bar{Y}}, \Lambda_{\bar{Y}})$*, we have*

$$\Phi(\bar{X}) = \inf_{X \in \mathbb{I}} \Phi(X) = \sup_{Y \in \mathbb{I}} [-\Psi(Y)] = -\Psi(\bar{Y}) \tag{2.4.28}$$

if and only if

(1) $\bar{X}_0 = x_0$,

(2) $\bar{X}(T) = -\frac{\bar{Y}(T)+c}{a}$ *a.s.,*

(3) $\dot{\bar{Y}}(t) + r(t)\bar{Y}(t) = 0$ *a.e.,*

(4) $\bar{\pi} \in \mathcal{U}(\bar{X})$ *and* $\delta(t, \bar{Y}(t), \Theta_{\bar{Y}}(t)) + \bar{\pi}'(t)\Theta_{\bar{Y}}(t) = \bar{Y}(t) g(t, \bar{\pi}(t))$ *a.e.*
for $\bar{\pi}(t) := [\sigma'(t)]^{-1}\Lambda_{\bar{X}}(t)$.

Proof. From (2.4.5) and (2.4.16) follows that for arbitrary $x, y \in \mathbb{R}$

$$l_0(x) + m_0(y) = xy \quad \text{if and only if} \quad x = x_0.$$

Similarly, from (2.3.6) and (2.4.17) we find that for arbitrary $x, y \in \mathbb{R}$ and $w \in \Omega$

$$J(w,x) + m_T(w,y) = -xy \quad \text{if and only if} \quad x = -\frac{y + c(w)}{a(w)}.$$

Hence, conditions (A) and (B) of Proposition 2.4.6 hold if and only if $\bar{X}_0 = x_0$ and $\bar{X}(T) = -\frac{\bar{Y}(T)+c}{a}$ a.s. Further, for arbitrary $(w,t) \in \Omega \times [0,T]$, and $(x,v,\xi), (y,s,\gamma) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$, we have:

$$\begin{aligned}
&L(w,t,x,v,\xi) + M(w,t,y,s,\gamma) = xs + vy + \xi'\gamma \\
&\iff v = r(w,t)x + \xi'\theta(w,t) + g(w,t,[\sigma'(w,t)]^{-1}\xi),\ [\sigma'(w,t)]^{-1}\xi \in K,\ s + r(w,t)y = 0, \\
&\qquad \text{and } \delta(w,t,y,\Theta_y) = xs + vy + \xi'\gamma \\
&\iff v = r(w,t)x + \xi'\theta(w,t) + g(w,t,[\sigma'(w,t)]^{-1}\xi),\ [\sigma'(w,t)]^{-1}\xi \in K,\ s + r(w,t)y = 0, \\
&\qquad \text{and } \delta(w,t,y,\Theta_y) = x(-r(w,t)y) + (r(w,t)x + \xi'\theta(w,t) + g(w,t,[\sigma'(w,t)]^{-1}\xi))y + \xi'\gamma \\
&\iff v = r(w,t)x + \xi'\theta(w,t) + g(w,t,[\sigma'(w,t)]^{-1}\xi),\ [\sigma'(w,t)]^{-1}\xi \in K,\ s + r(w,t)y = 0, \\
&\qquad \text{and } \delta(w,t,y,\Theta_y) = \xi'[\theta(w,t)y + \gamma] + yg(w,t,[\sigma'(w,t)]^{-1}\xi).
\end{aligned}$$

Hence, condition (C) of Proposition 2.4.6 holds if and only if $\dot{\bar{Y}}(t) + r(t)\bar{Y}(t) = 0$ a.e., $\bar{\pi} \in \mathcal{U}(\bar{X})$ and $\delta(t, \bar{Y}(t), \Theta_{\bar{Y}}(t)) + \bar{\pi}'(t)\Theta_{\bar{Y}}(t) = \bar{Y}(t)g(t,\bar{\pi}(t))$ a.e. for $\bar{\pi}(t) := [\sigma'(t)]^{-1}\Lambda_{\bar{X}}(t)$ (recall definition (2.4.1) of $\mathcal{U}(\bar{X})$).

Combining, we conclude that for arbitrary $(\bar{X}, \bar{Y}) \in \mathbb{I} \times \mathbb{I}$

$$\text{(A) - (C) of Proposition 2.4.6 hold} \iff \text{(1) - (4) hold.}$$

From inequality (2.4.26) then follows that equality $\Phi(\bar{X}) + \Psi(\bar{Y}) = 0$ is equivalent to (2.4.28), and hence, (2.4.28) holds if and only if conditions (1) - (4) are satisfied. □

Remark 2.4.8. From Proposition 2.4.7 we find that if the pair $(\bar{X}, \bar{Y}) \in \mathbb{I} \times \mathbb{I}$ satisfies conditions (1) - (4), then $\bar{X}$ minimizes functional Φ and hence, as follows from Remark 2.4.1, $\bar{\pi} := [\sigma']^{-1}\Lambda_{\bar{X}}$ defines the solution to the partially constrained problem $(\mathcal{P}_{c,q})$ at (2.3.8). Thus Proposition 2.4.7 gives us necessary and sufficient conditions for portfolio $\bar{\pi}$ (with associated wealth process $\bar{X}$) to be the solution to problem (2.3.8), providing us with a dual approach to solving the problem (2.3.8):

$$\begin{aligned}
\text{Step I}:&\ \text{find some solution } \bar{Y} \text{ to the dual problem (2.4.24);} \\
\text{Step II}:&\ \text{construct some } \bar{X} \text{ related to } \bar{Y} \text{ through items (1) - (4) of Proposition 2.4.7;} \\
\text{Step III}:&\ \text{ensure constructed } \bar{X} \text{ belongs to the set } \mathbb{I} \text{ and corresponding } \bar{\pi} \text{ to the set } \mathcal{H}_2; \\
\text{Step IV}:&\ \text{ensure constructed } \bar{X} \text{ satisfy items (1) - (4) of Proposition 2.4.7.}
\end{aligned} \tag{2.4.29}$$

◁

Remark 2.4.9. Items (1) - (4) of Proposition 2.4.7 are quite similar to those in Labbé and Heunis [15], but there is an essential difference in item (4), namely the complementary slackness relation

$$\delta(t, \bar{Y}(t), \Theta_{\bar{Y}}(t)) + \bar{\pi}'(t)\Theta_{\bar{Y}}(t) = \bar{Y}(t)g(t, \bar{\pi}(t)).$$

Although this is a perfectly natural complementary slackness relation expressing the usual "Pontryagin maximum condition" that one expects (recall (2.4.19)), the presence of the margin-payment term g (recall (2.1.7)) nevertheless makes it a good deal more complicated than the corresponding complementary slackness condition in [15] (for which $g = 0$), and in fact this relation will pose the main challenge when we construct a tentative solution to the primal problem in Section 2.6. ◁

2.5 Solution to the Dual Problem

From Proposition 2.4.7 it follows that to solve problem (2.3.8) we need to construct a pair $(\bar{X}, \bar{Y}) \in \mathbb{I} \times \mathbb{I}$ that satisfies the optimality relations (1) - (4) of Proposition 2.4.7. Motivated by item (3) of Proposition 2.4.7, define

$$\mathbb{I}_1 := \{Y \equiv (Y_0, \dot{Y}, \Lambda_Y) \in \mathbb{I} \mid \dot{Y}(t) = -r(t)Y(t) \text{ a.e.}\}, \tag{2.5.1}$$

then from (2.4.22) and (2.4.23) follows that Ψ takes the value $+\infty$ on $\mathbb{I} - \mathbb{I}_1$, hence,

$$\inf_{Y \in \mathbb{I}} \Psi(Y) = \inf_{Y \in \mathbb{I}_1} \Psi(Y), \tag{2.5.2}$$

and thus we can restrict our choice of Y to $Y \in \mathbb{I}_1$. This is useful because, as we shall see (in Remark 2.5.1 forthcoming) members of the set at (2.5.1) have a simple and explicit parameterization. To establish this parameterization recall (2.1.2) and define

$$\mathcal{J}(\gamma)(t) := \int_0^t S_0(\tau)\gamma'(\tau)dW(\tau), \qquad \gamma \in \mathcal{H}_2, \quad t \in [0, T], \tag{2.5.3}$$

and a mapping $\Xi(\cdot) : \mathbb{R} \times \mathcal{H}_2 \to \mathbb{I}_1$ defined by

$$\Xi(y, \gamma)(t) := S_0^{-1}(t)\,[y + \mathcal{J}(\gamma)(t)], \qquad (y, \gamma) \in \mathbb{R} \times \mathcal{H}_2, \quad t \in [0, T], \tag{2.5.4}$$

which establishes a linear bijection between $\mathbb{R} \times \mathcal{H}_2$ and the set $\mathbb{I}_1$. Indeed, from $Y \equiv (Y_0, \dot{Y}, \Lambda_Y) \in \mathbb{I}_1$ together with (2.2.4) it follows that Y satisfies

$$Y(t) = Y_0 - \int_0^t r(\tau)Y(\tau)d\tau + \int_0^t \Lambda_Y'(\tau)dW(\tau), \qquad t \in [0, T], \tag{2.5.5}$$

and then, from an easy calculation using (2.5.5), (2.1.2) and Itô's formula, we obtain

$$Y(t)S_0(t) = Y_0 + \int_0^t S_0(\tau)\Lambda'_Y(\tau)dW(\tau), \qquad t \in [0,T]. \tag{2.5.6}$$

Combining (2.5.6) with (2.5.3) and (2.5.4) we see that $Y \equiv (Y_0, \dot{Y}, \Lambda_Y) \in \mathbb{I}_1$ has the form

$$Y(t) = \Xi(y,\gamma) \text{ with } y = Y_0 \text{ and } \gamma = \Lambda_Y. \tag{2.5.7}$$

On the other hand, let $Y := \Xi(y,\gamma)$ for some fixed pair $(y,\gamma) \in \mathbb{R} \times \mathcal{H}_2$, then

$$Y(t)S_0(t) = y + \int_0^t S_0(\tau)\gamma'(\tau)dW(\tau),$$

and from Itô's formula it follows that Y satisfies (2.5.5), while Doob's L^2-inequality yields $\mathbb{E}[\sup_{0\le t\le T}|Y(t)|^2] < \infty$ (since $\gamma \in \mathcal{H}_2$), thus combining with (2.5.5), we have $Y \in \mathbb{I}_1$.

Remark 2.5.1. With $(y,\gamma) \in \mathbb{R} \times \mathcal{H}_2$ and $Y := \Xi(y,\gamma)$ we have an equivalent way to write $\mathbb{I}_1$ as

$$\mathbb{I}_1 = \{\Xi(y,\gamma) \mid (y,\gamma) \in \mathbb{R} \times \mathcal{H}_2\}. \tag{2.5.8}$$

Further note that for $Y = \Xi(y,\gamma)$, we have

$$\begin{aligned} &Y_0 = y, \;\; \dot{Y}(t) = -r(t)\Xi(y,\gamma)(t), \;\; \Lambda_Y(t) = \gamma(t), \\ &\text{and } \;\Theta_Y(t) = -\sigma(t)[\theta(t)\Xi(y,\gamma)(t) + \gamma(t)] \;\text{ a.e.} \end{aligned} \tag{2.5.9}$$

◁

With the new representation for $\mathbb{I}_1$ at (2.5.8) in place, define

$$\tilde{\Psi}(y,\gamma) := \Psi(\Xi(y,\gamma)), \qquad (y,\gamma) \in \mathbb{R} \times \mathcal{H}_2, \tag{2.5.10}$$

then from (2.4.16), (2.4.17), (2.4.22), and (2.4.23) we have

$$\tilde{\Psi}(y,\gamma) = x_0 y + \mathbb{E}\left[\frac{(Y(T)+c)^2}{2a}\right] + \mathbb{E}\int_0^T \delta(t, Y(t), \Theta_Y(t))dt - q \tag{2.5.11}$$

for each $(y,\gamma) \in \mathbb{R} \times \mathcal{H}_2$ and $Y := \Xi(y,\gamma)$. From (2.5.10) and (2.5.8) it follows that

$$\inf_{(y,\gamma)\in\mathbb{R}\times\mathcal{H}_2} \tilde{\Psi}(y,\gamma) = \inf_{Y\in\mathbb{I}_1} \Psi(Y). \tag{2.5.12}$$

Remark 2.5.2. It is immediate from (2.5.12) and (2.5.2) that

$$\inf_{Y \in \mathbb{I}} \Psi(Y) \;=\; \inf_{(y,\gamma) \in \mathbb{R}\times\mathcal{H}_2} \tilde{\Psi}(y,\gamma),$$

in other words, the dual problem (2.4.24) can be alternatively viewed as a minimization problem of $\tilde{\Psi}$ over $(y,\gamma) \in \mathbb{R} \times \mathcal{H}_2$. From Remark 2.5.1 it follows that if $(\bar{y},\bar{\gamma}) \in \mathbb{R} \times \mathcal{H}_2$ are such that $\inf_{(y,\gamma) \in \mathbb{R}\times\mathcal{H}_2} \tilde{\Psi}(y,\gamma) = \tilde{\Psi}(\bar{y},\bar{\gamma})$, then $\bar{Y} := \Xi(\bar{y},\bar{\gamma})$ defined as in (2.5.9) solves the dual problem (2.4.24). ◁

Motivated by Remark 2.5.2 we next give the following essential result which guarantees existence of a minimizer in the dual problem:

Proposition 2.5.3. *Assume Condition 2.1.1, 2.1.5, 2.3.1, 2.3.5. Then functional $\tilde{\Psi}(y,\gamma)$ is proper, convex, lower semi-continuous and coercive. Hence, there exists some $(\bar{y},\bar{\gamma}) \in \mathbb{R} \times \mathcal{H}_2$ such that*

$$\inf_{(y,\gamma) \in \mathbb{R}\times\mathcal{H}_2} \tilde{\Psi}(y,\gamma) = \tilde{\Psi}(\bar{y},\bar{\gamma}) \in \mathbb{R}. \tag{2.5.13}$$

Proof. From Remark 2.4.3 follows that function $\delta(w,t,y,\Theta)$ is non-negative, convex, and semi-continuous for all $(\omega,t,y,\Theta) \in \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N$. Thus the proof is identical to the proof of Proposition 5.4 of [15] (p. 86), with substitution of (2.4.19) for $\delta(\cdot)$ in [15], which in its turn first shows that $\tilde{\Psi}$ is proper, convex, lower semi-continuous and coercive and then uses Theorem A.1.3 to assert existence of the optimal solution $(\bar{y},\bar{\gamma}) \in \mathbb{R} \times \mathcal{H}_2$. □

Corollary 2.5.4. *Let $(\bar{y},\bar{\gamma}) \in \mathbb{R} \times \mathcal{H}_2$ be given by Proposition 2.5.3 and put*

$$\bar{Y}(t) := \Xi(\bar{y},\bar{\gamma})(t), \qquad \forall t \in [0,T]. \tag{2.5.14}$$

Then $\bar{Y} \in \mathbb{I}_1 \subset \mathbb{I}$ and from Remark 2.5.2 follows that

$$\Psi(\bar{Y}) = \inf_{Y \in \mathbb{I}} \Psi(Y),$$

in particular, $\bar{Y}$ solves the dual problem (2.4.24).

2.6 Construction of the Optimal Portfolio

In the previous section we have successfully completed the first step of our plan outlined in Remark 2.4.8, namely, we have shown that a solution $\bar{Y}$ to the dual problem exists and identified it in Corollary 2.5.4. In this section we focus on the second step of our plan, that is constructing some $\bar{X} \in \mathbb{I}$ that together with above-mentioned $\bar{Y}$ satisfies optimality relations (1) - (4) of Proposition 2.4.7. While relations (1) - (4) give useful information about optimal $\bar{X}$, they do not provide us with an explicit "recipe" for constructing one, therefore, some "educated guesswork" is needed. Solving the rather complicated optimality relations (1) - (4), in a way, is analogous to solving a complicated differential equation: we start by making an informed guess or assumption about the solution, and then try to prove that our assumption was indeed correct.

We will now focus our attention on particular class of optimal dual solutions $\bar{Y} \in \mathbb{I}^+$, where we define

$$\mathbb{I}^+ := \left\{ Y \in \mathbb{I}_1 \,\middle|\, \inf_{t\in[0,T]}\{Y(t)\} > 0 \;\text{ or }\; \inf_{t\in[0,T]}\{-Y(t)\} > 0 \right\}, \tag{2.6.1}$$

in other words, optimal solutions that are either strictly positive or strictly negative processes that satisfy item (3) of Proposition 2.4.7. From now on we are going to make the simplifying assumption that

Condition 2.6.1. The optimal solution $(\bar{y}, \bar{\gamma}) \in \mathbb{R} \times \mathcal{H}_2$ of Proposition 2.5.3 is such that $\bar{Y} = \Xi(\bar{y}, \bar{\gamma}) \in \mathbb{I}^+$. ◁

Remark 2.6.2. Effectively Condition 2.6.1 amounts to the assumption that the solution $\bar{Y}$ of the dual problem (recall Corollary 2.5.4) is either a strictly positive or strictly negative process. ◁

Now define

$$H(y,\nu)(t) := y \exp\left\{ -\int_0^t r(\tau)d\tau \right\} \cdot \mathcal{E}(-[\theta + \sigma^{-1}\nu]' \bullet W)(t), \quad (y,\nu) \in \{\mathbb{R} - \{0\}\} \times \mathcal{G}, \tag{2.6.2}$$

where we use

$$\mathcal{E}(M)(t) := \exp\left\{ M(t) - \frac{1}{2}\langle M \rangle(t) \right\}$$

for a continuous $\{\mathcal{F}_t\}$–local martingale M, and

$$\mathcal{G} := \left\{ \nu : \Omega \times [0,T] \to \mathbb{R}^N \,\middle|\, \nu \in \mathcal{F}^* \text{ and } H(y,\nu)[\theta + \sigma^{-1}\nu] \in \mathcal{H}_2 \right\}. \tag{2.6.3}$$

Lemma 2.6.3. *Assume Conditions 2.1.1, 2.1.5. Then*

$$Y \equiv (Y_0, \dot{Y}, \Lambda_Y) \in \mathbb{I}^+ \iff Y = H(y, \nu_Y) \text{ for some } (y, \nu_Y) \in \{(-\infty, 0) \cup (0, \infty)\} \times \mathcal{G}, \tag{2.6.4}$$

in which case $Y_0 = y$ *and* $\Theta_Y(t) = Y(t)\nu_Y(t)$ *a.e.*

Proof. Fix some $Y \equiv (Y_0, \dot{Y}, \Lambda_Y) \in \mathbb{I}^+ \subset \mathbb{I}_1$ and define (recall (2.4.27))

$$\nu_Y(t) := \frac{\Theta_Y(t)}{Y(t)} = -\sigma(t)\left[\theta(t) + \frac{\Lambda_Y(t)}{Y(t)}\right], \tag{2.6.5}$$

then

$$\Lambda_Y(t) = -Y(t)[\theta(t) + \sigma^{-1}(t)\nu_Y(t)], \tag{2.6.6}$$

from which combining with (2.2.4) and (2.5.1) it follows that $Y(t)$ has dynamics

$$dY(t) = -Y(t)\{r(t)dt + [\theta(t) + \sigma^{-1}(t)\nu_Y(t)]'dW(t)\} \tag{2.6.7}$$

and thus can be written as $Y = H(y, \nu_Y)$ with $y := Y_0$. From (2.6.6), (2.6.3) and fact that $\Lambda_Y \in \mathcal{H}_2$ we easily conclude that $\nu_Y \in \mathcal{G}$.

Now let $Y(t) = H(y, \nu_Y)(t)$ for some $(y, \nu_Y) \in \{(-\infty, 0) \cup (0, \infty)\} \times \mathcal{G}$. Expanding (2.6.2) using Itô's formula, we obtain

$$dH(y, \nu_Y)(t) = -H(y, \nu_Y)(t)\{r(t)dt + [\theta(t) + \sigma^{-1}(t)\nu_Y(t)]'dW(t)\}, \tag{2.6.8}$$

from which follows that $Y_0 = y$, $\dot{Y}(t) = -r(t)Y(t)$ and $\Lambda_Y(t) = -Y(t)[\theta(t) + \sigma^{-1}(t)\nu_Y(t)]$. From definition (2.6.3) of $\mathcal{G}$ and the fact that $\nu_Y \in \mathcal{G}$, we have $\Lambda_Y = -H(y, \nu_Y)[\theta + \sigma^{-1}\nu_Y] \in \mathcal{H}_2$, from which together with uniform boundedness of θ and r (recall Condition 2.1.1) we see that $\mathbb{E}\left[\sup_{0\le t\le T} |Y(t)|^2\right] < \infty$, hence, $\dot{Y} = -rY \in \mathcal{H}_1$ and from non-negativity of exponentials in (2.6.2) it follows that $Y \in \mathbb{I}^+$. □

Remark 2.6.4. Recall that Remark 2.5.1 established that members of the set $\mathbb{I}_1$ defined at (2.5.1) have a simple and explicit parameterization defined through $(y, \gamma) \in \mathbb{R} \times \mathcal{H}_2$. Lemma 2.6.3 shows that the set $\mathbb{I}^+ \subset \mathbb{I}_1$ also has an alternative parameterization

$$\mathbb{I}^+ = \{H(y, \nu_Y) \mid (y, \nu_Y) \in \{(-\infty, 0) \cup (0, \infty)\} \times \mathcal{G}\}. \tag{2.6.9}$$

Moreover, for $\bar{Y}$ defined in Corollary 2.5.4 we have an equivalence

$$\begin{aligned} &\Xi(\bar{y}, \bar{\gamma}) \equiv H(\bar{y}, \nu_{\bar{Y}}) \quad \text{with} \\ &\bar{\gamma}(t) \equiv -H(\bar{y}, \nu_{\bar{Y}}(t))[\theta(t) + \sigma^{-1}(t)\nu_{\bar{Y}}(t)] \in \mathcal{H}_2 \quad \text{and} \quad \nu_{\bar{Y}}(t) \equiv \frac{\Theta_{\bar{Y}}(t)}{\bar{Y}(t)} \in \mathcal{G} \end{aligned} \tag{2.6.10}$$

whenever Condition 2.6.1 is satisfied. ◁

Our goal is now to construct some $\bar{X} \equiv (\bar{X}_0, \dot{\bar{X}}, \Lambda_{\bar{X}}) \in \mathbb{I}$ that satisfies conditions (1), (2), and (4) of Proposition 2.4.7 together with $\bar{Y} \in \mathbb{I}_1$ defined in Corollary 2.5.4. Since we already know that $\bar{Y}$ satisfies condition (3), it then follows from Proposition 2.4.7 that such $\bar{X}$ minimizes the functional Φ over $\mathbb{I}$, and therefore, $\bar{\pi} := [\sigma(t)]^{-1}\Lambda_{\bar{X}} \in \mathcal{U}(\bar{X})$ solves the partially constrained problem $(\mathcal{P}_{c,q})$ defined at (2.3.8).

We motivate the construction of $\bar{X}$ in the following way: take $\bar{\pi} \in \mathcal{U}(\bar{X})$ (recall 2.4.1)) for some $\bar{X} \in \mathbb{I}$ (as required by item (4) of Proposition 2.4.7) and let $\bar{Y} \in \mathbb{I}_1$ be as defined in Corollary 2.5.4, then recalling (2.4.27) and item (4) of Proposition 2.4.7, particularly, $\delta(t, \bar{Y}(t), \Theta_{\bar{Y}}(t)) + \bar{\pi}'(t)\Theta_{\bar{Y}}(t) = \bar{Y}(t)g(t, \bar{\pi}(t))$, we find

$$\begin{aligned}\bar{X}(t)\dot{\bar{Y}}(t) + \dot{\bar{X}}(t)\bar{Y}(t) + \Lambda'_{\bar{X}}(t)\Lambda_{\bar{Y}}(t) &= -r(t)\bar{X}(t)\bar{Y}(t) + \bar{Y}(t)[r(t)\bar{X}(t) + \bar{u}'(t)\sigma(t)\theta(t) \\ &\quad + g(t, \bar{u}(t))] + \bar{u}'(t)\sigma(t)\Lambda_{\bar{Y}}(t) \\ &= \bar{u}'(t)\sigma(t)[\theta(t)\bar{Y}(t) + \Lambda_{\bar{Y}}(t)] + \bar{Y}(t)g(t, \bar{u}(t)) \\ &= -\bar{u}'\Theta_{\bar{Y}}(t) + \bar{Y}(t)g(t, \bar{u}(t)) \\ &= \delta(t, \bar{Y}(t), \Theta_{\bar{Y}}(t)).\end{aligned} \tag{2.6.11}$$

Since we assumed that $\bar{X}$ is a member of the set $\mathbb{I}$, we can then use Proposition 2.4.5 together with (2.6.11) to conclude that $\{(\mathbf{M}(\bar{X}, \bar{Y}), \mathcal{F}_t),\ t \in [0, T]\}$ defined at (2.4.25), or,

$$\left\{\left(\bar{X}(t)\bar{Y}(t) - \bar{X}_0\bar{Y}_0 - \int_0^t \delta(\tau, \bar{Y}(\tau), \Theta_{\bar{Y}}(\tau))d\tau,\ \mathcal{F}_t\right), \quad t \in [0, T]\right\} \tag{2.6.12}$$

is a continuous martingale. Then we must have

$$\bar{X}(t)\bar{Y}(t) - \int_0^t \delta(\tau, \bar{Y}(\tau), \Theta_{\bar{Y}}(\tau))d\tau = \mathbb{E}\left[\bar{X}(T)\bar{Y}(T) - \int_0^T \delta(\tau, \bar{Y}(\tau), \Theta_{\bar{Y}}(\tau))d\tau \,\middle|\, \mathcal{F}_t\right],$$

which is equivalent to

$$\bar{X}(t)\bar{Y}(t) = \mathbb{E}\left[\bar{X}(T)\bar{Y}(T) - \int_t^T \delta(\tau, \bar{Y}(\tau), \Theta_{\bar{Y}}(\tau))d\tau \,\middle|\, \mathcal{F}_t\right]. \tag{2.6.13}$$

Condition 2.6.1 together with (2.6.13) and item (2) of Proposition 2.4.7 motivate the following definition of $\bar{X}$ in terms of $\bar{Y} := \Xi(\bar{y}, \bar{\gamma})$ defined in Corollary 2.5.4:

$$\bar{X}(t) := -\frac{1}{\bar{Y}(t)}\mathbb{E}\left[\frac{\bar{Y}(T)(\bar{Y}(T) + c)}{a} + \int_t^T \delta(\tau, \bar{Y}(\tau), \Theta_{\bar{Y}}(\tau))d\tau \,\middle|\, \mathcal{F}_t\right]. \tag{2.6.14}$$

Observe that the $\mathcal{F}_T$-measurable random variable $\frac{\bar{Y}(T)(\bar{Y}(T)+c)}{a}$ is square integrable since $\bar{Y} \in \mathbb{I}$, a has a strictly lower bound and c is square integrable as follows from Conditions 2.3.1 and 2.3.5, therefore conditional expectation in (2.6.14) is well defined. Further, from the martingale representation theorem applied to (2.6.12) it follows that there exists some $\mathbb{R}^N$-valued and a.e.-unique process $\psi \in \mathcal{F}^*$, with $\int_0^T \|\psi(t)\|^2 dt < \infty$ a.s., such that

$$\bar{X}(t)\bar{Y}(t) - \int_0^t \delta(\tau, \bar{Y}(\tau), \Theta_{\bar{Y}}(\tau))d\tau = \bar{X}(0)\bar{Y}(0) + \int_0^t \psi'(\tau)dW(\tau). \tag{2.6.15}$$

Remark 2.6.5. Observe that the process $\bar{X}$ defined at (2.6.14) is determined entirely by the solution $\bar{Y}$ of the dual problem (2.4.24) (recall Corollary 2.5.4). Likewise the integrand $\psi \in \mathcal{F}^*$ given by the martingale representation theorem (MRT) is also determined entirely by $\bar{Y}$. Our goal is to establish that $\bar{X}$ defined at (2.6.14) is actually the optimal wealth process for the problem $(\mathcal{P}_{c,q})$ defined at (2.3.8). To this end the expansion at (2.6.15) given by the MRT turns out to be very useful. ◁

Remark 2.6.6. Condition 2.6.1 enables us to define $\bar{X}$ as in (2.6.14). Without Condition 2.6.1 we would not be able proceed beyond (2.6.13) and at this point we see no way to avoid imposing Condition 2.6.1. ◁

2.7 Verifying Optimality

In the previous section we have restricted our attention to a particular class of optimal dual solutions $\bar{Y} \in \mathbb{I}^+$ and constructed a tentative optimal wealth process $\bar{X}$ through optimality relations (1) - (4) of Proposition 2.4.7. Following the plan outlined in Remark (2.4.8), the goal of this section is to complete steps III and IV, in particular, verify that proposed $\bar{X}$ satisfies set membership $\mathbb{I}$ and optimality relations (1) - (4) of Proposition 2.4.7, hence, $\bar{\pi} \in \mathcal{U}(\bar{X})$ solves the partially constrained problem $(\mathcal{P}_{c,q})$ defined at (2.3.8).

For future developments it is important, however, to identify the explicit formula for $\bar{\pi} \in \mathcal{U}(\bar{X})$. To this end, we start by showing that $\bar{X}$ satisfies the wealth dynamics (2.1.7), in other words for some $\bar{\pi} \in \mathcal{U}(\bar{X})$ we have

$$d\bar{X}(t) = \left[r(t)\bar{X}(t) + \bar{\pi}'(t)\sigma(t)\theta(t) + g(t, \bar{\pi}(t))\right] dt + \bar{\pi}'(t)\sigma(t)dW(t). \tag{2.7.1}$$

Observe that (2.6.14) expresses $\bar{X}$ as a quotient of two semimartingales, so that $\bar{X}$ is also a semimartingale (by Itô's formula which guarantees that a quotient of semimartingales is again a semimartingale), hence, there exist some $\dot{\bar{X}}$ and $\Lambda_{\bar{X}}$ such that $\bar{X}$ has representation $\bar{X} \equiv (\bar{X}_0, \dot{\bar{X}}, \Lambda_{\bar{X}})$, or (2.2.4) with

$$\bar{X}_0 := \bar{X}(0). \tag{2.7.2}$$

From Itô's formula and the fact that $\bar{Y} \equiv (\bar{y}, -r\bar{Y}, \bar{\gamma})$ (as follows from Remark 2.5.1) we find

$$\begin{aligned} d(\bar{X}\bar{Y})(t) &= [\bar{X}(t)\dot{\bar{Y}}(t) + \dot{\bar{X}}(t)\bar{Y}(t) + \Lambda'_{\bar{X}}(t)\Lambda_{\bar{Y}}(t)]dt + [\bar{X}(t)\Lambda_{\bar{Y}}(t) + \bar{Y}(t)\Lambda_{\bar{X}}(t)]'dW_t \\ &= [-r(t)\bar{X}(t)\bar{Y}(t) + \dot{\bar{X}}(t)\bar{Y}(t) + \Lambda'_{\bar{X}}(t)\bar{\gamma}(t)]dt + [\bar{X}(t)\bar{\gamma}(t) + \bar{Y}(t)\Lambda_{\bar{X}}(t)]'dW_t. \end{aligned}$$

Rearranging the terms, as well as subtracting and adding $\bar{Y}(t)\Lambda'_{\bar{X}}(t)\theta(t)$ we obtain

$$d(\bar{X}\bar{Y})(t) = \bar{Y}(t)\left(-r(t)\bar{X}(t) + \dot{\bar{X}}(t) - \Lambda'_{\bar{X}}(t)\theta(t) + \frac{1}{\bar{Y}(t)}\Lambda'_{\bar{X}}(t)[\bar{Y}(t)\theta(t) + \bar{\gamma}(t)]\right)dt$$
$$+ [\bar{X}(t)\bar{\gamma}(t) + \bar{Y}(t)\Lambda_{\bar{X}}(t)]'dW_t,$$

from which together with definition (2.4.27) of $\Theta_{\bar{Y}}$ we find

$$d(\bar{X}\bar{Y})(t) = \bar{Y}(t)\left(-r(t)\bar{X}(t) + \dot{\bar{X}}(t) - \Lambda_{\bar{X}}(t)\theta(t) - \frac{1}{\bar{Y}(t)}\Lambda'_{\bar{X}}(t)[\sigma(t)]^{-1}\Theta_{\bar{Y}}(t)\right)dt$$
$$+ [\bar{X}(t)\bar{\gamma}(t) + \bar{Y}(t)\Lambda_{\bar{X}}(t)]'dW_t. \quad (2.7.3)$$

From (2.7.3) and (2.7.2) it then follows that

$$\bar{X}(t)\bar{Y}(t) - \int_0^t \delta(\tau, \bar{Y}(\tau), \Theta_{\bar{Y}}(\tau))d\tau \ = \ \bar{X}_0\bar{y}$$
$$+ \int_0^t \bar{Y}(\tau)\Bigg(- r(\tau)\bar{X}(\tau) + \dot{\bar{X}}(\tau) - \Lambda'_{\bar{X}}(\tau)\theta(\tau) - \frac{1}{\bar{Y}(\tau)}\Bigg[\Lambda'_{\bar{X}}(\tau)[\sigma(\tau)]^{-1}\Theta_{\bar{Y}}(\tau)$$
$$+ \delta(\tau, \bar{Y}(\tau), \Theta_{\bar{Y}}(\tau))\Bigg]\Bigg)d\tau + \int_0^t [\bar{X}(\tau)\bar{\gamma}(\tau) + \bar{Y}(\tau)\Lambda_{\bar{X}}(\tau)]'dW_\tau. \quad (2.7.4)$$

Comparing terms of (2.7.4) with (2.6.15) and recalling that $\bar{Y}(t) \neq 0$ a.e. for all $t \in [0, T]$ (Condition 2.6.1) we find

$$\dot{\bar{X}}(t) = r(t)\bar{X}(t) + \Lambda'_{\bar{X}}(t)\theta(t) + \frac{1}{\bar{Y}(t)}\left[\Lambda'_{\bar{X}}(t)[\sigma(t)]^{-1}\Theta_{\bar{Y}}(t) + \delta(t, \bar{Y}(t), \Theta_{\bar{Y}}(t))\right] \quad (2.7.5)$$

and

$$\Lambda_{\bar{X}}(t) := \frac{\psi(t)}{\bar{Y}(t)} - \frac{\bar{X}(t)\bar{\gamma}(t)}{\bar{Y}(t)}. \quad (2.7.6)$$

Motivated by (2.4.1), we let

$$\bar{\pi}(t) := [\sigma(t)]^{-1}\Lambda_{\bar{X}}(t), \quad (2.7.7)$$

in other words define

$$\bar{\pi}(t) := [\sigma(t)]^{-1}\left[\frac{\psi(t)}{\bar{Y}(t)} - \frac{\bar{X}(t)\Lambda_{\bar{Y}}(t)}{\bar{Y}(t)}\right]. \quad (2.7.8)$$

Observe that (2.7.5) then becomes

$$\dot{\bar{X}}(t) = r(t)\bar{X}(t) + \bar{\pi}'(t)\sigma(t)\theta(t) + \frac{1}{\bar{Y}(t)}\left[\bar{\pi}'(t)\Theta_{\bar{Y}}(t) + \delta(t, \bar{Y}(t), \Theta_{\bar{Y}}(t))\right]. \quad (2.7.9)$$

Remark 2.7.1. We have shown that the process $\bar{X}$ defined by (2.6.14) necessarily satisfies (2.7.9) for $\bar{\pi}$ defined at (2.7.8). This means that once we have verified that item (4) of Proposition 2.4.7 holds, namely the complementary slackness relation

$$\delta(t, \bar{Y}(t), \Theta_{\bar{Y}}(t)) + \bar{\pi}'(t)\Theta_{\bar{Y}}(t) = \bar{Y}(t)g(t, \bar{\pi}(t)), \tag{2.7.10}$$

then from (2.7.10) and (2.7.9) we obtain

$$\dot{\bar{X}}(t) = r(t)\bar{X}(t) + \bar{\pi}'(t)\sigma(t)\theta(t) + g(t, \bar{\pi}(t)). \tag{2.7.11}$$

This shows that (2.7.9) is consistent with optimality relations (1) - (4) of Proposition 2.4.7. ◁

Remark 2.7.2. In view of Remark 2.7.1 our goal is to show that the pair $(\bar{X}, \bar{Y})$ given by Corollary 2.5.4 and (2.6.14) satisfies conditions (1) - (4) of Proposition 2.4.7. From the definition of $\bar{Y}$ in Corollary 2.5.4 it follows that $\bar{Y} \in \mathbb{I}_1$, hence, item (3) of Proposition 2.4.7 is verified (recall (2.5.1)). Moreover, it is immediate from (2.6.14) with $t := T$ that

$$\bar{X}(T) = -\frac{\bar{Y}(T) + c}{a} \text{ a.s.} \tag{2.7.12}$$

(since the r.h.s. of (2.7.12) is $\mathcal{F}_T$-measurable), thus (2) of Proposition 2.4.7 is also verified. It therefore remains to verify items (1) and (4) of Proposition 2.4.7 as well as the function-space memberships $\bar{X} \in \mathbb{I}$ and $\bar{\pi} \in \mathcal{H}_2$.

We will by-pass checking the function-space memberships; assuming these to be true, we procede with verifying items (1) and (4) of Proposition 2.4.7, that is we will suppose that

$$\bar{X} \in \mathbb{I} \text{ and } \bar{\pi} \in \mathcal{H}_2. \tag{2.7.13}$$

◁

The next result, which is attained by use of a variational analysis of the dual cost function $\tilde{\Psi}$ around the optimal pair $(\bar{y}, \bar{\gamma})$ (recall Proposition 2.5.3), will be used to show that items (1) and (4) of Proposition 2.4.7 are satisfied.

Proposition 2.7.3. *Assume Conditions 2.1.1, 2.1.5, 2.3.1, 2.3.5, 2.6.1. Let $(\bar{y}, \bar{\gamma})$ be given by Proposition 2.5.3 and $\bar{Y} := \Xi(\bar{y}, \bar{\gamma})$ be the solution of the dual problem as established in Corollary 2.5.4. Then for arbitrary $(\alpha, \eta) \in \mathbb{R} \times \mathcal{H}_2$ and $R := \Xi(\alpha, \eta)$ we have*

$$\begin{aligned} 0 \leq \lim_{\epsilon \searrow 0} \frac{\tilde{\Psi}(\bar{y} + \epsilon\alpha, \bar{\gamma} + \epsilon\eta)(t) - \tilde{\Psi}(\bar{y}, \bar{\gamma})(t)}{\epsilon} &= (x_0 - \bar{X}_0)\alpha \\ + \lim_{\epsilon \searrow 0} \mathbb{E} \int_0^T \Bigg\{ \frac{\delta(t, \bar{Y}(t) + \epsilon R(t), \Theta_{\bar{Y}}(t) + \epsilon\Theta_R(t)) - \delta(t, \bar{Y}(t), \Theta_{\bar{Y}}(t))}{\epsilon} \\ + \bar{\pi}'(t)\Theta_R(t) - \frac{R(t)}{\bar{Y}(t)} \left[\delta(t, \bar{Y}, \Theta_{\bar{Y}}(t)) + \bar{\pi}'(t)\Theta_{\bar{Y}}(t)\right] \Bigg\} dt, \end{aligned} \tag{2.7.14}$$

with Ξ defined at (2.5.4), $\Theta_{\bar{Y}}$ and Θ_R as in Remark 2.5.1, $\bar{\pi} := [\sigma']^{-1}\Lambda_{\bar{X}}$ and $\tilde{\Psi}$ given at (2.5.11).

Proof. Define $\bar{Y}^\epsilon = \Xi(\bar{y}+\epsilon\alpha, \bar{\gamma}+\epsilon\eta)$ then from (2.5.4) and (2.4.27) one can easily see that $\bar{Y}^\epsilon(t) = \bar{Y}(t) + \epsilon R(t)$ and $\Theta_{\bar{Y}^\epsilon}(t) = \Theta_{\bar{Y}}(t) + \epsilon\Theta_R(t)$. Hence, from (2.5.11) we find

$$
\begin{aligned}
\frac{\tilde{\Psi}(\bar{y}+\epsilon\alpha, \bar{\gamma}+\epsilon\eta)(t) - \tilde{\Psi}(\bar{y},\bar{\gamma})(t)}{\epsilon} &= x_0\alpha + \mathbb{E}\Bigg[\frac{(\bar{Y}^\epsilon(T)+c)^2}{2a\epsilon} - \frac{(\bar{Y}(T)+c)^2}{2a\epsilon} \\
&\quad + \int_0^T \frac{\delta(t,\bar{Y}^\epsilon(t),\Theta_{\bar{Y}^\epsilon}(t)) - \delta(t,\bar{Y}(t),\Theta_{\bar{Y}}(t))}{\epsilon}dt\Bigg] \\
&= x_0\alpha + \mathbb{E}\left[\left(\frac{\bar{Y}(T)+c}{a}\right)R(T)\right] + \epsilon\mathbb{E}\left[\frac{R^2(T)}{2a}\right] \\
&\quad + \mathbb{E}\int_0^T \frac{\delta(t,\bar{Y}(t)+\epsilon R(t),\Theta_{\bar{Y}}(t)+\epsilon\Theta_R(t)) - \delta(t,\bar{Y}(t),\Theta_{\bar{Y}}(t))}{\epsilon}dt.
\end{aligned}
$$

Recall $\frac{\bar{Y}(T)+c}{a} = -\bar{X}(T)$ (as follows from definition (2.6.14) when $t=T$), then taking limit as $\epsilon \searrow 0$ we obtain

$$
\begin{aligned}
\lim_{\epsilon\searrow 0}\frac{\tilde{\Psi}(\bar{y}+\epsilon\alpha, \bar{\gamma}+\epsilon\eta)(t) - \tilde{\Psi}(\bar{y},\bar{\gamma})(t)}{\epsilon} &= x_0\alpha - \mathbb{E}[\bar{X}(T)R(T)] \\
+ \lim_{\epsilon\searrow 0}\mathbb{E}\int_0^T \frac{\delta(t,\bar{Y}(t)+\epsilon R(t),\Theta_{\bar{Y}}(t)+\epsilon\Theta_R(t)) - \delta(t,\bar{Y}(t),\Theta_{\bar{Y}}(t))}{\epsilon}dt.
\end{aligned}
\tag{2.7.15}
$$

From definition of $R := \Xi(\alpha,\eta) \in \mathbb{I}$ along the lines of Remark 2.5.1 follows that $R_0 = \alpha$, $\dot{R}(t) = -r(t)R(t)$, and $\Lambda_R(t) = \eta(t)$. Since $\bar{X} \in \mathbb{I}$ according to (2.7.13), for $\mathbf{M}(\bar{X}, R)$ of

Proposition 2.4.5 we have (recall (2.7.2), (2.7.9) and (2.7.7))

$$\begin{aligned}\mathbf{M}(\bar{X},R)(t) &= \bar{X}(t)R(t) - \bar{X}_0 R_0 - \int_0^t \Big\{ \bar{X}(\tau)\dot{R}(\tau) + \dot{\bar{X}}(\tau)R(\tau) + \Lambda'_{\bar{X}}(\tau)\Lambda_R(\tau)\Big\} d\tau \\ &= \bar{X}(t)R(t) - \bar{X}_0\alpha - \int_0^t \Big\{ - r(\tau)\bar{X}(\tau)R(\tau) + \Big(r(\tau)\bar{X}(\tau) + \bar{\pi}'(\tau)\sigma(\tau)\theta(\tau) \\ &\quad + \frac{1}{\bar{Y}(\tau)}[\delta(\tau,\bar{Y}(\tau),\Theta_{\bar{Y}}(\tau)) + \bar{\pi}'(\tau)\Theta_{\bar{Y}}(\tau)]\Big) R(\tau) + \bar{\pi}'(\tau)\sigma(\tau)\eta(\tau)\Big\} d\tau \\ &= \bar{X}(t)R(t) - \bar{X}_0\alpha + \int_0^t \Big\{ - \bar{\pi}'(\tau)\sigma(\tau)[\theta(\tau)R(\tau) + \eta(\tau)] \\ &\quad - \frac{R(\tau)}{\bar{Y}(\tau)}\Big[\delta(\tau,\bar{Y}(\tau),\Theta_{\bar{Y}}(\tau)) + \bar{\pi}'(\tau)\Theta_{\bar{Y}}(\tau)\Big]\Big\} d\tau. \end{aligned} \tag{2.7.16}$$

With Θ_R as in Remark 2.5.1, (2.7.16) becomes

$$\mathbf{M}(\bar{X},R)(t) = \bar{X}(t)R(t) - \bar{X}_0\alpha + \int_0^t \Big\{ \bar{\pi}'(\tau)\Theta_R(\tau) - \frac{R(\tau)}{\bar{Y}(\tau)}\Big[\delta(\tau,\bar{Y}(\tau),\Theta_{\bar{Y}}(\tau)) + \bar{\pi}'(\tau)\Theta_{\bar{Y}}(\tau)\Big]\Big\} d\tau. \tag{2.7.17}$$

From Proposition 2.4.5 it follows that $\mathbb{E}[\mathbf{M}(\bar{X},R)(t)] = 0$ for each $t \in [0,T]$, therefore,

$$\mathbb{E}[\bar{X}(T)R(T)] = \bar{X}_0\alpha - \mathbb{E}\left[\int_0^T \Big\{ \bar{\pi}'(\tau)\Theta_R(\tau) - \frac{R(\tau)}{\bar{Y}(\tau)}\Big(\delta(\tau,\bar{Y}(\tau),\Theta_{\bar{Y}}(\tau)) + \bar{\pi}'(\tau)\Theta_{\bar{Y}}(\tau)\Big)\Big\} d\tau\right]. \tag{2.7.18}$$

Substituting (2.7.18) in the limit expression (2.7.15) we finally obtain (2.7.14). □

We now are ready to show that item (1) of Proposition 2.4.7 holds.

Proposition 2.7.4. *Suppose Conditions 2.1.1, 2.1.5, 2.3.1, 2.3.5, 2.6.1. Then item (1) of Proposition 2.4.7 holds, i.e.* $\bar{X}_0 = x_0$.

Proof. Take some $\epsilon \in (0,1)$. Now set $\alpha = \bar{y}$ and $\eta = \bar{\gamma}$, then $R(t) = \bar{Y}(t)$ and from Proposition 2.7.3 together with positive homogeneity of δ we obtain

$$\begin{aligned} 0 &\le (x_0 - \bar{X}_0)\bar{y} + \mathbb{E}\int_0^T \delta(t,\bar{Y}(t),\Theta_{\bar{Y}}(t)) + \bar{\pi}'(t)\Theta_{\bar{Y}}(t) - \big[\delta(t,\bar{Y}(t),\Theta_{\bar{Y}}(t)) + \bar{\pi}'(t)\Theta_{\bar{Y}}(t)\big]\, dt \\ &= (x_0 - \bar{X}_0)\bar{y}. \end{aligned}$$

Alternatively, set $\alpha = -\bar{y}$ and $\eta = -\bar{\gamma}$, then $R(t) = -\bar{Y}(t)$ and once again from positive homogeneity of δ (note $1 - \epsilon > 0$) we find

$$0 \leq (x_0 - \bar{X}_0)(-\bar{y}) + \mathbb{E} \int_0^T -\delta(t, \bar{Y}(t), \Theta_{\bar{Y}}(t)) - \bar{\pi}'(t)\Theta_{\bar{Y}}(t) + \big[\delta(t, \bar{Y}(t), \Theta_{\bar{Y}}(t)) + \bar{\pi}'(t)\Theta_{\bar{Y}}(t)\big]\, dt$$

$$= (x_0 - \bar{X}_0)(-\bar{y}).$$

From this we conclude that either $\bar{X}_0 = x_0$ or $\bar{y} = 0$. From Condition 2.6.1 it follows that $\bar{y} \neq 0$, hence, $\bar{X}_0 = x_0$. □

The following simple result from Karatzas and Shreve [11] will be used to show that $\bar{\pi} \in K$ a.e., that is, $\bar{\pi}$ satisfies the portfolio constraint in problem (2.3.8) (recall (2.3.1)).

Lemma 2.7.5. *Define*

$$\tilde{\delta}(v) := \sup_{\pi \in K}\{-\pi' v\}, \qquad v \in \mathbb{R}^N, \tag{2.7.19}$$

and

$$B := \{(w,t) \in \Omega \times [0,T] \mid \bar{u}(w,t) \in K\},$$

for some $\mathcal{F}^*$-measurable process $\bar{u} : \Omega \times [0,T] \to \mathbb{R}^N$. Then there exists some $\mathcal{F}^*$-measurable mapping $\xi : \Omega \times [0,T] \to \mathbb{R}^N$, such that

$$\|\xi(t)\| \leq 1, \ \ |\tilde{\delta}(\xi(t))| \leq 1, \ \ a.e. \ on \ \Omega \times [0,T]$$

$$\tilde{\delta}(\xi(t)) + \bar{u}'(t)\xi(t) = 0, \ \ a.e. \ on \ B$$

$$\tilde{\delta}(\xi(t)) + \bar{u}'(t)\xi(t) < 0, \ \ a.e. \ on \ \{\Omega \times [0,T]\} - B.$$

Proof. See Lemma 5.4.2 of [11] (p. 207). □

Remark 2.7.6. To save ourselves from writing out two sets of proofs for cases when the dual solution is strictly positive or strictly negative, recalling Condition 2.6.1, we introduce a simple "book-keeping" variable Υ defined by

$$\Upsilon := \begin{cases} 1 & \text{if } \bar{Y}(t) > 0 \text{ for all } t \in [0,T], \\ -1 & \text{if } \bar{Y}(t) < 0 \text{ for all } t \in [0,T], \end{cases} \tag{2.7.20}$$

in other words, Υ specifies whether $\bar{Y}$ of Corollary 2.5.4 is a strictly positive or a strictly negative process. From (2.7.20) it is immediate that $\Upsilon\bar{Y} > 0$ a.e. Further, recall Remark 2.6.4, and define

$$\nu_{\bar{Y}} := \frac{\Theta_{\bar{Y}}}{\bar{Y}}, \tag{2.7.21}$$

then $\nu_{\bar{Y}} \in \mathcal{G}$ and $\bar{Y} = H(\bar{y}, \nu_{\bar{Y}})$ (recall (2.6.2)), as follows from Remark 2.6.4. Now let $R := H(\bar{y}, \nu_R)$ for arbitrary $\nu_R \in \mathcal{G}$ and set

$$\eta(t) := -H(\bar{y}, \nu_R(t))[\theta(t) + \sigma^{-1}(t)\nu_R(t)], \tag{2.7.22}$$

then $\eta \in \mathcal{H}_2$ and $R = \Xi(\bar{y}, \eta)$ by Remark 2.6.4, moreover,

$$\nu_R \equiv \frac{\Theta_R}{R} \tag{2.7.23}$$

for $\Theta_R(t) := -\sigma(t)[\theta(t)R(t) + \eta(t)]$ as in (2.4.27). Observe that $R(0) = \bar{y}$ (recall (2.6.2)), thus R is strictly positive or negative whenever Υ is positive or negative respectively, hence, $\Upsilon R > 0$ and $\bar{Y} R > 0$ a.e. ◁

With the preceding preparations in place we are finally ready to prove that $\bar{\pi}$ satisfies the portfolio constraint:

Proposition 2.7.7. *Suppose Conditions 2.1.1, 2.1.5, 2.3.1, 2.3.5, 2.6.1. Then $\bar{\pi}(t) \in K$ for all $t \in [0, T]$ a.e.*

Proof. Let $\epsilon > 0$. From subadditivity and positive homegeneity of function δ mentioned in Remark 2.4.3 it follows that

$$\delta(t, \bar{Y}(t) + \epsilon R(t), \Theta_{\bar{Y}}(t) + \epsilon \Theta_R(t)) \leq \delta(t, \bar{Y}(t), \Theta_{\bar{Y}}(t)) + \epsilon \delta(t, R(t), \Theta_R(t)). \tag{2.7.24}$$

Therefore, from (2.7.24) together with Propositions 2.7.3 and 2.7.4 we conclude that

$$0 \leq \mathbb{E} \int_0^T \left\{ \delta(t, R(t), \Theta_R(t)) - \frac{R(t)}{\bar{Y}(t)} \delta(t, \bar{Y}(t), \Theta_{\bar{Y}}(t)) + \bar{\pi}'(t) \left[\Theta_R(t) - \frac{R(t)}{\bar{Y}(t)} \Theta_{\bar{Y}}(t) \right] \right\} dt. \tag{2.7.25}$$

Since the inequality (2.7.25) holds for arbitrary $(\alpha, \eta) \in \mathbb{R} \times \mathcal{H}_2$, we can always restrict our choice to R, such that $R(t)\bar{Y}(t) > 0$ a.e. Therefore, take R as in Remark 2.7.6, then $R(t)/\bar{Y}(t) > 0$ a.e. and using positive homogeneity of δ again we transform (2.7.25) into

$$0 \leq \mathbb{E} \int_0^T \left\{ \delta(t, R(t), \Theta_R(t)) - \delta(t, R(t), \frac{R(t)}{\bar{Y}(t)} \Theta_{\bar{Y}}(t)) + \bar{\pi}'(t) \left[\Theta_R(t) - \frac{R(t)}{\bar{Y}(t)} \Theta_{\bar{Y}}(t) \right] \right\} dt. \tag{2.7.26}$$

Further, from subadditivity of supremums we see that

$$\begin{aligned} \sup_{\pi \in K} \{g(t, \pi)R(t) - \pi' \Theta_R(t)\} \leq & \sup_{\pi \in K} \left\{ g(t, \pi)R(t) - \pi' \frac{R(t)}{\bar{Y}(t)} \Theta_{\bar{Y}}(t) \right\} \\ & + \sup_{\pi \in K} \left\{ -\pi' \left[\Theta_R(t) - \frac{R(t)}{\bar{Y}(t)} \Theta_{\bar{Y}}(t) \right] \right\} \end{aligned}$$

which is equivalent to (recall (2.4.19) and (2.7.19))

$$\delta(t, R(t), \Theta_R(t)) \leq \delta(t, R(t), \frac{R(t)}{\bar{Y}(t)}\Theta_{\bar{Y}}(t)) + \tilde{\delta}(\Theta_R(t) - \frac{R(t)}{\bar{Y}(t)}\Theta_{\bar{Y}}(t)). \tag{2.7.27}$$

From (2.7.27) then follows that we can rewrite the inequality (2.7.26) as

$$0 \leq \mathbb{E}\int_0^T \left\{\tilde{\delta}(\Theta_R(t) - \frac{R(t)}{\bar{Y}(t)}\Theta_{\bar{Y}}(t)) + \bar{\pi}'(t)\left[\Theta_R(t) - \frac{R(t)}{\bar{Y}(t)}\Theta_{\bar{Y}}(t)\right]\right\} dt. \tag{2.7.28}$$

With help of (2.7.20) and positive homogeneity of $\tilde{\delta}$ with respect to $\Upsilon R > 0$ (as mentioned in Remark 2.7.6), (2.7.28) becomes

$$0 \leq \mathbb{E}\int_0^T \Upsilon R(t)\left\{\tilde{\delta}\left(\Upsilon\frac{\Theta_R(t)}{R(t)} - \Upsilon\frac{\Theta_{\bar{Y}}(t)}{\bar{Y}(t)}\right) + \bar{\pi}'(t)\left[\Upsilon\frac{\Theta_R(t)}{R(t)} - \Upsilon\frac{\Theta_{\bar{Y}}(t)}{\bar{Y}(t)}\right]\right\} dt. \tag{2.7.29}$$

Finally, recalling definitions (2.7.21) and (2.7.23) of $\nu_{\bar{Y}}$ and ν_R of Remark 2.7.6, with the new reparameterization (2.7.29) is equivalent to

$$0 \leq \mathbb{E}\int_0^T \Upsilon R(t)\left\{\tilde{\delta}\left(\Upsilon\left[\nu_R(t) - \nu_{\bar{Y}}(t)\right]\right) + \bar{\pi}'(t)\left(\Upsilon\left[\nu_R(t) - \nu_{\bar{Y}}(t)\right]\right)\right\} dt. \tag{2.7.30}$$

Now put

$$B := \{(w, t) \in \Omega \times [0, T] \mid \bar{\pi}(w, t) \in K\}. \tag{2.7.31}$$

By Lemma 2.7.5, there exists some $\mathcal{F}^*$-measurable mapping ξ: $\Omega \times [0, T] \mapsto \mathbb{R}^N$ such that $\|\xi(t)\|$ is bounded on $\Omega \times [0, T]$, and

$$\tilde{\delta}(\xi(t)) + \bar{\pi}'(t)\xi(t) = 0, \text{ a.e. on } B$$

$$\tilde{\delta}(\xi(t)) + \bar{\pi}'(t)\xi(t) < 0, \text{ a.e. on } \{\Omega \times [0, T] - B\}.$$

Now suppose that $(P \otimes \lambda)\{(\Omega \times [0, T]) - B\} > 0$ and put (recall (2.7.20))

$$\nu_{R_\xi}(t) := \nu_{\bar{Y}}(t) + \Upsilon\xi(t). \tag{2.7.32}$$

Then $\nu_{R_\xi} \in \mathcal{G}$ since $\nu_{\bar{Y}} \in \mathcal{G}$ and ξ is uniformly bounded (recall (2.6.3)). Moreover, rearranging (2.7.32), we obtain

$$\Upsilon\left[\nu_{R_\xi}(t) - \nu_{\bar{Y}}(t)\right] = \xi(t) \tag{2.7.33}$$

and for $R_\xi := H(\bar{y}, \nu_{R_\xi})$ we have $\Upsilon R_\xi(t) > 0$ a.e., as evident from Remark 2.7.6. Hence, from (2.7.33) and choice of ξ it follows

$$\mathbb{E}\int_0^T \Upsilon R_\xi(t)\left\{\tilde{\delta}\left(\Upsilon\left[\nu_{R_\xi}(t) - \nu_{\bar{Y}}(t)\right]\right) + \bar{\pi}'(t)\left(\Upsilon\left[\nu_{R_\xi}(t) - \nu_{\bar{Y}}(t)\right]\right)\right\} dt$$

$$= \mathbb{E}\int_0^T \Upsilon R_\xi(t)\{\tilde{\delta}(\xi(t)) + \bar{\pi}'(t)\xi(t)\}dt \;<\; 0. \tag{2.7.34}$$

The last inequality (2.7.34) contradicts (2.7.30) and therefore $(P\otimes\lambda)\{(\Omega\times[0,T])-B\}=0$ as evident from definition of the set B at (2.7.31). It follows then that $\bar{\pi}(t)\in K$ a.e. $\square$

Recall definitions (2.4.19) and (2.7.20) and set

$$\check{\delta}_K(w,t,\Upsilon,\nu):=\delta(w,t,\Upsilon,\Upsilon\nu)=\sup_{\pi\in K}\{\Upsilon g(w,t,\pi)-\Upsilon\pi'\nu\}. \tag{2.7.35}$$

Remark 2.7.8. Recall δ defined at (2.4.19) and Remark 2.7.6, particularly, (2.7.21), and observe that from positive homogeneity of $\check{\delta}_K$ at (2.7.35) it follows that

$$\Upsilon\bar{Y}(t)\check{\delta}_K(t,\Upsilon,\nu_{\bar{Y}}(t))=\delta(t,\bar{Y}(t),\Theta_{\bar{Y}}(t)), \tag{2.7.36}$$

since

$$\Upsilon\bar{Y}(t)\sup_{\pi\in K}\{\Upsilon g(t,\pi)-\Upsilon\pi'\nu_{\bar{Y}}(t)\}=\sup_{\pi\in K}\{\bar{Y}(t)g(t,\pi)-\pi'\Theta_{\bar{Y}}(t)\}.$$

$\triangleleft$

The following lemma will be used to show that $\bar{X}$ and $\bar{\pi}$ satisfy condition (4) of Proposition 2.4.7.

Lemma 2.7.9. *There exists some $R^*:=\Xi(\bar{y},\eta^*)$ with $\eta^*\in\mathcal{H}_2$ such that $R^*(t)\bar{Y}(t)>0$ for all $t\in[0,T]$ a.e. and $\check{\delta}_K(t,\Upsilon,\nu_{R^*}(t))=\Upsilon g(t,\bar{\pi}(t))-\Upsilon\bar{\pi}'(t)\nu_{R^*}(t)$ a.e. for $\bar{Y}$ of Corollary 2.5.4, Υ defined at (2.7.20) and $\nu_R^*:=\frac{\Theta_{R^*}}{R^*}$.*

Proof. Fix an arbitrary pair $(w,t)\in\Omega\times[0,T]$ and let

$$f(u):=\psi_K(-\Upsilon u)-\Upsilon g(w,t,-\Upsilon u),\qquad u\in\mathbb{R}^N, \tag{2.7.37}$$

where

$$\psi_K(u)=0 \text{ if } u\in K \quad\text{and}\quad \psi_K(u)=+\infty \text{ if } u\notin K. \tag{2.7.38}$$

The convex conjugate of function f is then given by (recall definition (2.7.20) of Υ)

$$\begin{aligned} f^*(\eta) :&= \sup_{u\in\mathbb{R}^N}\{u'\eta-[\psi_K(-\Upsilon u)-\Upsilon g(w,t,-\Upsilon u)]\} \\ &= \begin{cases} \sup_{u\in\mathbb{R}^N}\{u'\eta-[\psi_K(-u)-g(w,t,-u)]\} & \text{if } \Upsilon=1,\\ \sup_{u\in\mathbb{R}^N}\{u'\eta-[\psi_K(u)+g(w,t,u)]\} & \text{if } \Upsilon=-1\end{cases} \\ &= \begin{cases} \sup_{u\in\mathbb{R}^N}\{-u'\eta-[\psi_K(u)-g(w,t,u)]\} & \text{if } \Upsilon=1,\\ \sup_{u\in\mathbb{R}^N}\{u'\eta-[\psi_K(u)+g(w,t,u)]\} & \text{if } \Upsilon=-1\end{cases} \\ &= \sup_{u\in\mathbb{R}^N}\{\Upsilon g(w,t,u)-\Upsilon u'\eta-\psi_K(u)\} \\ &= \sup_{u\in K}\{\Upsilon g(w,t,u)-\Upsilon u'\eta\},\quad \text{for all } \eta\in\mathbb{R}^N. \end{aligned} \tag{2.7.39}$$

Recalling definition (2.7.35), we see that (2.7.39) is in fact

$$f^*(\eta) := \check{\delta}_K(w,t,\Upsilon,\eta), \quad \text{for all } \eta \in \mathbb{R}^N. \tag{2.7.40}$$

From Proposition I.5.1 and I.5.6 of [7] (p. 21 and p. 26), (2.7.37) and (2.7.40) it follows

$$\begin{aligned} \partial f(u) &= \{\eta \in \mathbb{R}^N : \; f^*(\eta) + f(u) = u'\eta\} \\ &= \{\eta \in \mathbb{R}^N : \; \check{\delta}_K(w,t,\Upsilon,\eta) = -\psi_K(-\Upsilon u) + \Upsilon g(w,t,-\Upsilon u) + u'\eta\}, \end{aligned} \tag{2.7.41}$$

and at the same time

$$\begin{aligned} \partial f(u) &= \begin{cases} (-1)\partial\psi_K(-u) + (-1)\partial(-g)(w,t,-u) & \text{if } \Upsilon = 1, \\ \partial\psi_K(u) + \partial g(w,t,u) & \text{if } \Upsilon = -1 \end{cases} \\ &= (-\Upsilon)\partial\psi_K(-\Upsilon u) + (-\Upsilon)\partial(-\Upsilon g)(w,t,-\Upsilon u) \end{aligned} \tag{2.7.42}$$

for each $u \in \mathbb{R}^N$ and where $\partial\psi_K(u)$ and $\partial(-\Upsilon g)(w,t,u)$ denote the subgradients at $u \in \mathbb{R}^N$ of the convex functions $\psi_K(\cdot)$ and $-\Upsilon g(w,t,\cdot)$. Hence, together (2.7.41) and (2.7.42) give

$$\begin{aligned} &(-\Upsilon)\partial\psi_K(u) + (-\Upsilon)\partial(-\Upsilon g)(w,t,u) \\ &\qquad = \{\eta \in \mathbb{R}^N : \check{\delta}_K(w,t,\Upsilon,\eta) = -\psi_K(u) + \Upsilon g(w,t,u) - \Upsilon u'\eta\}. \end{aligned} \tag{2.7.43}$$

From Condition 2.1.5 and Corollary I.5.1 and Proposition I.5.2 of [7] (p.21 and p.22) it follows that $\partial(-\Upsilon g)(w,t,u)$ is non-empty, convex and closed for each $(w,t,u) \in \Omega \times [0,T] \times \mathbb{R}^N$. Further, since $0 \in \partial\psi_K(u)$ whenever $u \in K$, choose $u := \bar{\pi}$, then $\bar{\pi} \in K$ as follows from Proposition 2.7.7 and from (2.7.43) we have

$$(-\Upsilon)\partial(-\Upsilon g)(w,t,\bar{\pi}(w,t)) \subset \{\eta \in \mathbb{R}^N : \; \check{\delta}_K(w,t,\Upsilon,\eta) = \Upsilon g(w,t,\bar{\pi}(w,t)) - \Upsilon\bar{\pi}'(w,t)\eta\} \text{ a.e.} \tag{2.7.44}$$

From Condition 2.1.5 we then conclude that $-\Upsilon g(\cdot)$ is a normal convex integrand, and from Corollaries 4.6 and 1.1 of [21] (p. 23 and p. 8) follows that the mapping $(w,t) \to \partial(-\Upsilon g)(w,t,\bar{\pi}(w,t))$ is measurable and there exists some $\mathcal{F}^*$-measurable mapping $(w,t) \to \nu_{R^*}(w,t) : \Omega \times [0,T] \to \mathbb{R}^N$ such that

$$\nu_{R^*}(t) \in (-\Upsilon)\partial(-\Upsilon g)(t,\bar{\pi}(t)) \text{ a.e.,} \tag{2.7.45}$$

and

$$\check{\delta}_K(t,\Upsilon,\nu_{R^*}(t)) = \Upsilon g(t,\bar{\pi}(t)) - \Upsilon\bar{\pi}'(t)\nu_{R^*}(t) \text{ a.e.} \tag{2.7.46}$$

as a consequence of (2.7.44). From Condition 2.1.5 it follows that $\|\eta\| \leq \zeta$ for each $\eta \in (-\Upsilon)\partial(-\Upsilon g)(w,t,u)$ and $(w,t,u) \in \Omega \times [0,T] \times \mathbb{R}^N$, hence, $\|\nu_{R^*}(t)\| \leq \zeta$ a.e and therefore $\nu_{R^*} \in \mathcal{G}$ (recall (2.6.3)).

Finally, recall definition (2.6.2) and set

$$\eta^* := -H(\bar{y}, \nu_{R^*}(t))[\theta(t) + \sigma^{-1}(t)\nu_{R^*}(t)], \tag{2.7.47}$$

then from Remark 2.6.4 it follows that $\eta^* \in \mathcal{H}_2$, and for $R^* := \Xi(\bar{y}, \eta^*)$ we have $\nu_{R^*} \equiv \frac{\Theta_{R^*}}{R^*}$, as well as $R^*(t)\bar{Y}(t) > 0$ and $\Upsilon R^*(t) > 0$ for each $t \in [0, T]$ a.e., and

$$\check{\delta}_K(t, \Upsilon, \nu_{R^*}(t)) = \Upsilon g(t, \bar{\pi}(t)) - \Upsilon \bar{\pi}'(t)\nu_{R^*}(t) \text{ a.e.}$$

as follows from (2.7.46). □

We now are ready to complete the program outlined in Remark 2.7.2:

Proposition 2.7.10. *Suppose Conditions 2.1.1, 2.1.5, 2.3.1, 2.3.5, 2.6.1. Then for $\bar{\pi}$ defined at (2.7.8) item (4) of Proposition 2.4.7 holds, particularly, $\bar{\pi} \in \mathcal{U}(\bar{X})$ and*

$$\delta(t, \bar{Y}(t), \Theta_{\bar{Y}}(t)) + \bar{\pi}'(t)\Theta_{\bar{Y}}(t) = \bar{Y}(t)g(t, \bar{\pi}(t)) \text{ a.e.}$$

for $\bar{Y}$ of Corollary 2.5.4 and $\Theta_{\bar{Y}}$ as in Remark 2.5.1.

Proof. Take $R := \Xi(\bar{y}, \eta)$ with $\eta \in \mathcal{H}_2$ as in Remark 2.7.6, then with help of (2.7.20) and using positive homogeneity of δ with respect to $\Upsilon R > 0$ a.e. we rewrite (2.7.25) as

$$0 \le \mathbb{E}\int_0^T \Upsilon R(t) \left\{ \delta\left(t, \Upsilon, \Upsilon\frac{\Theta_R(t)}{R(t)}\right) - \delta\left(t, \Upsilon, \Upsilon\frac{\Theta_{\bar{Y}}(t)}{\bar{Y}(t)}\right) + \Upsilon\bar{\pi}'(t)\left[\frac{\Theta_R(t)}{R(t)} - \frac{\Theta_{\bar{Y}}(t)}{\bar{Y}(t)}\right]\right\} dt. \tag{2.7.48}$$

Recalling definitions (2.7.21) and (2.7.23) of $\nu_{\bar{Y}}$ and ν_R of Remark 2.7.6, as well as (2.7.35) of $\check{\delta}_K$, we find (2.7.48) to be equivalent to

$$0 \le \mathbb{E}\int_0^T \Upsilon R(t)\{\check{\delta}_K(t, \Upsilon, \nu_R(t)) - \check{\delta}_K(t, \Upsilon, \nu_{\bar{Y}}(t)) + \Upsilon\bar{\pi}'(t)\nu_R(t) - \Upsilon\bar{\pi}'(t)\nu_{\bar{Y}}(t)\}dt. \tag{2.7.49}$$

Since $\bar{\pi} \in K$ (as follows from Proposition 2.7.7), we know that

$$\check{\delta}_K(t, \Upsilon, \nu_{\bar{Y}}(t)) + \Upsilon\bar{\pi}'(t)\nu_{\bar{Y}}(t) \ge \Upsilon g(t, \bar{\pi}(t)) \text{ a.e.} \tag{2.7.50}$$

and from Lemma 2.7.9 follows that there exists some R^* such that $R^*(t)\bar{Y}(t) > 0$ a.e. for all $t \in [0, T]$ and $\check{\delta}_K(t, \Upsilon, \nu_{R^*}(t)) + \Upsilon\bar{\pi}'(t)\nu_{R^*}(t) = \Upsilon g(t, \bar{\pi}(t))$. From (2.7.49) and (2.7.50) then follows

$$\begin{aligned} 0 &\le \mathbb{E}\int_0^T \Upsilon R^*(t)\{\, \check{\delta}_K(t, \Upsilon, \nu_{R^*}(t)) - \check{\delta}_K(t, \Upsilon, \nu_{\bar{Y}}(t)) + \Upsilon\bar{\pi}'(t)\nu_{R^*}(t) - \Upsilon\bar{\pi}'(t)\nu_{\bar{Y}}(t)\}dt \\ &= \mathbb{E}\int_0^T \Upsilon R^*(t)\{\, \Upsilon g(t, \bar{\pi}(t)) - \check{\delta}_K(t, \Upsilon, \nu_{\bar{Y}}(t)) - \Upsilon\bar{\pi}'(t)\nu_{\bar{Y}}(t)\}dt \\ &\le \mathbb{E}\int_0^T \Upsilon R^*(t)\{\Upsilon g(t, \bar{\pi}(t)) - \Upsilon g(t, \bar{\pi}(t))\}dt = 0. \end{aligned} \tag{2.7.51}$$

From (2.7.51) it is immediate that

$$\mathbb{E}\int_0^T \Upsilon R^*(t)\{\ \Upsilon g(t,\bar{\pi}(t)) - \check{\delta}_K(t,\Upsilon,\nu_{\bar{Y}}(t)) - \Upsilon\bar{\pi}'(t)\nu_{\bar{Y}}(t)\}dt = 0. \tag{2.7.52}$$

Moreover, from (2.7.50) and strict positivity of $\Upsilon R^*(t)$ for all $t \in [0,T]$ a.e. (recall Remark 2.7.6) it follows that (2.7.52) implies

$$\Upsilon g(t,\bar{\pi}(t)) - \check{\delta}_K(t,\Upsilon,\nu_{\bar{Y}}(t)) - \Upsilon\bar{\pi}'(t)\nu_{\bar{Y}}(t) = 0, \quad \text{for all } t \in [0,T]. \tag{2.7.53}$$

Multiplying both side of (2.7.53) by $\Upsilon\bar{Y}(t) > 0$ (recall (2.7.20)) and rearranging we obtain (recall Remark 2.7.8 and (2.7.21))

$$\delta(t,\bar{Y}(t),\Theta_{\bar{Y}}(t)) + \bar{\pi}'(t)\Theta_{\bar{Y}}(t) = \bar{Y}(t)g(t,\bar{\pi}(t)).$$

This together with Proposition 2.7.7 shows that $\bar{\pi} \in \mathcal{U}(\bar{X})$ (recall (2.4.1)) and therefore we conclude that item (4) of Proposition 2.4.7 holds. □

Remark 2.7.11. Observe the crucial role of Condition 2.6.1 in Propositions 2.7.4, 2.7.7 and 2.7.10 and note that Proposition 2.7.10 completes step IV outlined in Remark 2.4.8. ◁

We conclude this section by summarizing the preceding results in Theorem 2.7.12:

Theorem 2.7.12. *Suppose Conditions 2.1.1, 2.1.5, 2.3.1, 2.3.5. Then there exists a pair* $(\bar{y},\bar{\gamma}) \in \mathbb{R}\times\mathcal{H}_2$ *that minimizes the proper convex functional* $\tilde{\Psi}(\cdot,\cdot)$ *over* $\mathbb{R}\times\mathcal{H}_2$*. Moreover, if* $(\bar{y},\bar{\gamma})$ *are such that Condition 2.6.1 holds, then* $\bar{Y}$*,* $\bar{X}$*,* $\bar{\pi}$ *defined by (2.5.14), (2.6.14), (2.6.15), (2.7.8) satisfy*

$$\inf_{\pi\in\mathcal{A}} \mathbb{E}[J(X^{\pi}(T))] = \mathbb{E}[J(X^{\bar{\pi}}(T))] = -\inf_{(y,\gamma)\in\mathbb{R}\times\mathcal{H}_2} \tilde{\Psi}(y,\gamma) = -\tilde{\Psi}(\bar{y},\bar{\gamma})$$

with $\bar{\pi} \in \mathcal{A}$*,* $\bar{X} = X^{\bar{\pi}}$ *a.e., and in particular,* $\bar{\pi}$ *solves the partially constrained problem* $(\mathcal{P}_{c,q})$ *defined at (2.3.8).*

Proof. Proposition 2.5.3 establishes existences of the optimal dual pair $(\bar{y},\bar{\gamma}) \in \mathbb{R}\times\mathcal{H}_2$, Propositions 2.7.4, 2.7.7, and 2.7.10 show that the proposed $\bar{\pi}$ and $\bar{X}$ satisfy conditions (1) - (4) of Proposition 2.4.7 whenever $(\bar{y},\bar{\gamma})$ satisfy Condition 2.6.1,which in its turn shows that $\bar{\pi}$ solves problem $(\mathcal{P}_{c,q})$ defined at (2.3.8). □

2.8 Fully Constrained Optimization Problem

In this section we return to the main goal of the paper, that is solving the fully constrained problem $(\hat{\mathcal{P}})$ defined at (2.3.5). In this section we postulate Conditions 2.1.1, 2.1.5, 2.3.1, 2.3.2. Then

(i) $\mathcal{A}$ is a convex subset of $\mathcal{H}_2$ (Condition 2.3.2);

(ii) G (defined in (2.3.3)) is an affine functional on $\mathcal{H}_2$ (as follows from (2.1.8));

(iii) mapping $\pi \longrightarrow \mathbb{E}[\hat{J}(X^\pi(T))]$ (with $\hat{J}$ defined in (2.3.2)) defines an $\mathbb{R}$-valued convex mapping on $\mathcal{A} \subset \mathcal{H}_2$ (as follows from Proposition 2.2.2, Conditions 2.3.1, 2.3.2, convexity of $\hat{J}$ and (2.1.8)).

Now define the Lagrangian function which enforces the constraint $G(\pi) = 0$ on the expected terminal wealth. We recall that this constraint is the critical additional feature of the fully constrained problem $(\hat{\mathcal{P}})$ defined at (2.3.5):

$$\mathcal{L}(\mu;\pi) := \mathbb{E}[\hat{J}(X^\pi(T))] + \mu G(\pi), \qquad \pi \in \mathcal{H}_2, \ \ \mu \in \mathbb{R}. \tag{2.8.1}$$

Theorem A.1.4 and Condition 2.3.2 establish existence of a Lagrange multiplier $\bar{\mu} \in \mathbb{R}$ such that

$$\hat{v} = \sup_{\mu \in \mathbb{R}} \inf_{\pi \in \mathcal{A}} \mathcal{L}(\mu;\pi) = \inf_{\pi \in \mathcal{A}} \mathcal{L}(\bar{\mu};\pi). \tag{2.8.2}$$

Put

$$J_1(\mu; w, x) := \frac{1}{2}[a(w)x^2 + 2c_\mu(w)x] - \mu d, \qquad c_\mu(w) := c_0(w) + \mu, \tag{2.8.3}$$

and observe that

$$\mathbb{E}[\hat{J}(X^\pi(T))] + \mu G(\pi) = \mathbb{E}[J_1(\mu; X^\pi(T))],$$

hence,

$$\mathcal{L}(\mu;\pi) = \mathbb{E}[J_1(\mu; X^\pi(T))], \quad \pi \in \mathcal{H}_2, \quad \mu \in \mathbb{R}. \tag{2.8.4}$$

Since for every fixed $\mu \in \mathbb{R}$ function $J_1(\mu;\cdot,\cdot)$ is identical to function J of (2.3.6), problem (2.8.2) corresponds to problem (2.3.8) with $c := c_\mu$ and $q := -\mu d$. Define

$$\tilde{\Psi}_\mu(\mu; y, \gamma) := x_0 y + \mathbb{E}\left[\frac{(Y(T) + c_\mu)^2}{2a}\right] + \mathbb{E}\int_0^T \delta(t, Y(t), \Theta_Y(t))dt + \mu d, \tag{2.8.5}$$

for each $(\mu; y, \gamma) \in \mathbb{R} \times \mathbb{R} \times \mathcal{H}_2$ with $Y := \Xi(y,\gamma)$ by analogy with (2.5.11) in the partially constrained case. By Proposition 2.5.3 for each $\mu \in \mathbb{R}$ there exists a pair $(\bar{y}(\mu), \bar{\gamma}(\mu)) \in \mathbb{R} \times \mathcal{H}_2$ that minimizes the function (2.8.5). Following Theorem 2.7.12, for each $\mu \in \mathbb{R}$ and

$t \in [0,T]$ we introduce

$$\bar{Y}(\mu;t) := \Xi(\bar{y}(\mu), \bar{\gamma}(\mu))(t), \tag{2.8.6}$$

$$\Lambda_{\bar{Y}}(\mu;t) := \bar{\gamma}(\mu)(t), \quad \Theta_{\bar{Y}}(\mu;t) := -\sigma(t)[\theta(t)\bar{Y}(\mu;t) + \Lambda_{\bar{Y}}(\mu;t)], \tag{2.8.7}$$

$$\bar{X}(\mu;t) := -\frac{1}{\bar{Y}(\mu;t)}\mathbb{E}\left[\frac{\bar{Y}(\mu;T)(\bar{Y}(\mu;T)+c_\mu)}{a} + \int_t^T \delta(\tau, \bar{Y}(\mu;\tau), \Theta_{\bar{Y}}(\mu;\tau))d\tau \,\middle|\, \mathcal{F}_t\right], \tag{2.8.8}$$

$$\bar{\pi}(\mu;t) := [\sigma(t)]^{-1}\left[\frac{\psi(\mu;t)}{\bar{Y}(\mu;t)} - \frac{\bar{X}(\mu;t)\Lambda_{\bar{Y}}(\mu;t)}{\bar{Y}(\mu;t)}\right], \tag{2.8.9}$$

where process $\psi(\mu;t)$ in (2.8.9) is the a.e. unique $\mathbb{R}^N$-valued $\mathcal{F}^*$-measurable process such that

$$\int_0^T \|\psi(\mu;t)\|^2 dt < \infty \text{ a.s.}$$

and

$$\bar{X}(\mu;t)\bar{Y}(\mu;t) = \bar{X}(\mu;0)\bar{Y}(\mu;0) + \int_0^t \delta(\tau, \bar{Y}(\mu;\tau), \Theta_{\bar{Y}}(\mu;\tau))d\tau + \int_0^t \psi'(\mu;\tau)dW(\tau). \tag{2.8.10}$$

Lemma 2.8.1. *We have $G(\bar{\pi}(\bar{\mu})) = 0$.*

Proof. Proof is identical to that on p. 93 of [15] and therefore is omitted. □

With Lemma 2.8.1 in place, we now are ready to present the main result of this Chapter, Theorem 2.8.2:

Theorem 2.8.2. *Suppose the market as described in Section 2.1 and assume Conditions 2.1.1, 2.1.5, 2.3.1, 2.3.2. Then for each $\mu \in \mathbb{R}$, there exists a pair $(\bar{y}(\mu), \bar{\gamma}(\mu)) \in \mathbb{R} \times \mathcal{H}_2$ such that*

$$h(\mu) := \tilde{\Psi}_\mu(\mu; \bar{y}(\mu), \bar{\gamma}(\mu)) = \inf_{(y,\gamma)\in\mathbb{R}\times\mathcal{H}_2} \tilde{\Psi}_\mu(\mu; y, \gamma). \tag{2.8.11}$$

Further, there exists some $\bar{\mu} \in \mathbb{R}$ such that

$$h(\bar{\mu}) = \inf_{\mu\in\mathbb{R}} h(\mu) = \inf_{\mu\in\mathbb{R}} \tilde{\Psi}_\mu(\mu, \bar{y}(\mu), \bar{\gamma}(\mu)) \tag{2.8.12}$$

and if the optimal pair $(\bar{y}(\bar{\mu}), \bar{\gamma}(\bar{\mu}))$ is such that it satisfies Condition 2.6.1, then $\hat{\pi} := \bar{\pi}(\bar{\mu})$, given by (2.8.9) and (2.8.10), is solution of problem $(\hat{\mathcal{P}})$ defined at (2.3.5).

Proof. Proposition 2.5.3 establishes existence of optimal pairs $(\bar{y}(\mu), \bar{\gamma}(\mu)) \in \mathbb{R} \times \mathcal{H}_2$, Theorem A.1.4 establishes existence of an optimal Lagrange multiplier $\bar{\mu} \in \mathbb{R}$, Lemma 2.8.1 ensures constraint is fulfilled, and finally, Theorem 2.7.12 specifies that $\hat{\pi}$ is a solution of problem $(\hat{\mathcal{P}})$ whenever $(\bar{y}(\bar{\mu}), \bar{\gamma}(\bar{\mu}))$ satisfy Condition 2.6.1. $\square$

Chapter 3

Nonnegative Wealth Case

3.1 Introduction to the Problem

In the previous chapter we have not imposed any additional constraints on the optimal wealth process $\bar{X}$. In reality, however, an investor would most likely prefer the wealth process to be nonnegative, thus avoiding bankruptcy over the trading interval. In this chapter we modify the original optimization problem $(\hat{\mathcal{P}})$ defined at (2.3.5) to include this additional requirement and use the same approach as in Chapter 2 to address the problem. We must stress at the outset that adding a non-negativity constraint on the wealth to the problem of Chapter 2 results in a stochastic control problem with both a portfolio constraint (or "control" constraint) and an almost-sure constraint on the wealth process (that is a "state" constraint). Such problems, with combined state and control constraints, are known to be particularly challenging, even in the setting of purely deterministic control problems. In the stochastic case these challenges are compounded, to the extent that there are effectively no results of any kind in the established literature on stochastic control problems with a combination of state and control constraints. As we shall see, in the present chapter we do not succeed in solving this problem either, but our efforts at least suggest the next natural step in achieving a complete solution to this problem. The matter is discussed at greater length in Remark 3.2.4.

We postulate Conditions 2.1.1, 2.1.5, 2.3.1, 2.3.2, 2.3.5 and start by modifying definition (2.3.1) of the set $\mathcal{A}$ and put

$$\mathcal{A}^+ := \{\pi \in \mathcal{H}_2 \ | \ \pi(t) \in K \ \text{ a.e. and } X^\pi(t) \geq 0 \text{ a.e.}\}, \tag{3.1.1}$$

$$\hat{v} := \inf_{\substack{\pi \in \mathcal{A}^+ \\ G(\pi)=0}} \mathbb{E}[\hat{J}(X^\pi(T))]. \tag{3.1.2}$$

and now we wish to solve the following constrained problem $(\mathcal{P})$ (recall definitions (2.3.3) and (2.3.2)):

$$(\mathcal{P}): \quad \text{determine some } \hat{\pi} \in \mathcal{A}^+ \text{ such that } G(\hat{\pi}) = 0 \text{ and } \hat{\upsilon} = \mathbb{E}[\hat{J}(X^{\hat{\pi}}(T))]. \tag{3.1.3}$$

Once again we first introduce a partially constrained problem $(\mathcal{P}^+_{c,q})$ (recall definition (2.3.6)):

$$(\mathcal{P}^+_{c,q}): \quad \text{determine some } \bar{\pi} \in \mathcal{A}^+ \text{ such that } \upsilon_{c,q} = \mathbb{E}[J(X^{\bar{\pi}}(T))], \tag{3.1.4}$$

where

$$\upsilon_{c,q} := \inf_{\pi \in \mathcal{A}^+} \mathbb{E}[J(X^{\pi}(T))]. \tag{3.1.5}$$

Remark 3.1.1. The only difference between problems (2.3.5) and (3.1.3) and (2.3.8) and (3.1.4) respectively is in the definition of the sets $\mathcal{A}$ and $\mathcal{A}^+$, particularly, condition $X^{\pi} \geq 0$ a.e. We will soon see how this new constraint affects the primal and dual problems defined in the following section. $\triangleleft$

3.2 Solving Partially Constrained Problem

We now closely follow the steps of Sections 2.3 - 2.5 and try to solve the partially constrained problem 3.1.4. We start as in Chapter 2 by introducing set $\mathcal{U}(X)$. So let $X \equiv (X_0, \dot{X}, \Lambda_X) \in \mathbb{I}$, and recalling (2.1.7) define

$$\mathcal{U}(X) := \left\{ \pi \in \mathcal{A}^+ \;\middle|\; \begin{array}{r} \dot{X}(t) = r(t)X(t) + \pi'(t)\sigma(t)\theta(t) + g(t, \pi(t)) \\ \text{and } \Lambda_X(t) = \sigma'(t)\pi(t) \text{ a.e.} \end{array} \right\} \tag{3.2.1}$$

for $\mathcal{A}^+$ defined at (3.1.1). It then follows that for each $X \equiv (X_0, \dot{X}, \Lambda_X) \in \mathbb{I}$

$$X(t) = X^{\pi}(t) \text{ a.e. for some } \pi \in \mathcal{A}^+ \iff X_0 = x_0, \; X(t) \geq 0 \text{ a.e.}, \; \text{ and } \; \mathcal{U}(X) \neq \emptyset, \tag{3.2.2}$$

from which, together with (3.1.5), it follows that

$$\upsilon_{c,q} = \inf_{\substack{X \in \mathbb{I} \\ X_0 = x_0 \\ X \geq 0 \text{ a.e.} \\ \mathcal{U}(X) \neq \emptyset}} \mathbb{E}[J(X(T))]. \tag{3.2.3}$$

We now define penalty functions for the constraints in (3.2.3):

$$l_0(x) := \begin{cases} 0 & \text{if } x = x_0, \\ \infty & \text{otherwise,} \end{cases} \tag{3.2.4}$$

for each $x \in \mathbb{R}$ to account for the initial wealth constraint and (recall (2.3.6))

$$l_T(w,x) := \begin{cases} J(w,x) & \text{if } x \geq 0, \\ \infty & \text{otherwise,} \end{cases} \tag{3.2.5}$$

to account for the nonnegativity of wealth constraint. Finally, motivated by (3.2.1) we define a mapping $L : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \to \{0,\infty\}$ by

$$L(w,t,x,v,\xi) := \begin{cases} 0 & \text{if } v = r(w,t)x + \xi'\theta(w,t) + g(w,t,[\sigma'(w,t)]^{-1}\xi), \\ & \qquad x \geq 0, \text{ and } [\sigma'(w,t)]^{-1}\xi \in K, \\ \infty & \text{otherwise.} \end{cases} \tag{3.2.6}$$

From (3.2.6) it follows that $L(t, X(t), \dot{X}(t), \Lambda_X(t))$ is $\mathcal{F}^*$-measurable, and

$$\mathbb{E}\int_0^T L(t,X(t),\dot{X}(t),\Lambda_X(t)) = \begin{cases} 0 & \text{if } X(t) \geq 0 \text{ a.e. and } \mathcal{U}(X) \neq \emptyset, \\ \infty & \text{otherwise.} \end{cases} \tag{3.2.7}$$

Now put

$$\Phi(X) := l_0(X_0) + \mathbb{E}[l_T(X(T))] + \mathbb{E}\int_0^T L(t,X(t),\dot{X}(t),\Lambda_X(t))dt, \tag{3.2.8}$$

for each $X \equiv (X_0, \dot{X}, \Lambda_X) \in \mathbb{I}$, then combining (3.2.3), (3.2.4), (3.2.5), (3.2.6), and (3.2.7), we have

$$\upsilon_{c,q} = \inf_{X \in \mathbb{I}} \Phi(X). \tag{3.2.9}$$

Remark 3.2.1. Recall (2.4.8) and (2.4.10) of Chapter 2 and observe that (3.2.8) together with (3.2.9) define the primal problem associated with (3.1.4). ◁

With the primal problem in place, we now define the convex conjugate functions. From (2.4.11) we find

$$m_0(y) = x_0 y, \tag{3.2.10}$$

for each $y \in \mathbb{R}$. Analogous to (2.4.12) and recalling (3.2.5) define

$$\begin{aligned} m_T(w,y) &:= \sup_{x\in\mathbb{R}}\{x(-y) - l_T(w,x)\} \\ &= \begin{cases} \frac{(y+c(w))^2}{2a(w)} - q & \text{if } y + c(w) \leq 0, \\ -q & \text{otherwise} \end{cases} \end{aligned} \tag{3.2.11}$$

for each $y \in \mathbb{R}$ and $w \in \Omega$. For $M(w,t,y,s,\gamma)$ of (2.4.13) from (3.2.6) for each $y \in \mathbb{R},\ s \in \mathbb{R},\ \gamma \in \mathbb{R}^N,\ w \in \Omega$ and $t \in [0,T]$ we find

$$
\begin{aligned}
M(w,t,y,s,\gamma) &= \sup_{\substack{x,v\in\mathbb{R},\\ \xi\in\mathbb{R}^N}} \{xs + vy + \xi'\gamma - L(w,t,x,v,\xi)\} \\
&= \sup_{\substack{x\in\mathbb{R},\, x\geq 0,\, \xi\in\mathbb{R}^N,\\ [\sigma'(w,t)]^{-1}\xi\in K}} \{xs + y[r(w,t)x + \xi'\theta(w,t) + g(w,t,[\sigma'(w,t)]^{-1}\xi)] + \xi'\gamma\} \\
&= \sup_{\substack{x\geq 0,\, \xi\in\mathbb{R}^N,\\ [\sigma'(w,t)]^{-1}\xi\in K}} \{x[s + yr(w,t)] + yg(w,t,[\sigma'(w,t)]^{-1}\xi) + \xi'[y\theta(w,t)+\gamma]\} \\
&= \sup_{x\geq 0}\{x[s+yr(w,t)]\} + \sup_{\substack{\xi\in\mathbb{R}^N,\\ [\sigma'(w,t)]^{-1}\xi\in K}} \{yg(w,t,[\sigma'(w,t)]^{-1}\xi) + \xi'[y\theta(w,t)+\gamma]\} \\
&= \begin{cases} \sup_{\substack{\xi\in\mathbb{R}^N,\\ [\sigma'(w,t)]^{-1}\xi\in K}} \{yg(w,t,[\sigma'(w,t)]^{-1}\xi) + \xi'[y\theta(w,t)+\gamma]\} & \text{if } s + r(w,t)y \leq 0, \\ \infty & \text{otherwise.} \end{cases}
\end{aligned}
\tag{3.2.12}
$$

Combining (3.2.12) with definitions (2.4.19) and (2.4.20) of δ and Θ_y we find

$$
M(w,t,y,s,\gamma) = \begin{cases} \delta(w,t,y,\Theta_y(w,t)) & \text{if } s + r(w,t)y \leq 0, \\ \infty & \text{otherwise.} \end{cases} \tag{3.2.13}
$$

We can easily see from lower semi-continuouty of δ and (3.2.13) that $M(t,Y(t),\dot{Y}(t),\Lambda_Y(t))$ is $\mathcal{F}^*$-measurable. Now for each $Y = (Y_0,\dot{Y},\Lambda_Y) \in \mathbb{I}$ define

$$
\Psi(Y) := m_0(Y_0) + \mathbb{E}[m_T(Y(T))] + \mathbb{E}\int_0^T M(t,Y(t),\dot{Y}(t),\Lambda_Y(t))dt, \tag{3.2.14}
$$

and note that $\Psi(Y) \in (-\infty,+\infty]$ for each $Y \equiv (Y_0,\dot{Y},\Lambda_Y) \in \mathbb{I}$.

Remark 3.2.2. Observe that (3.2.10) stayed unchanged, while (3.2.11) and (3.2.13) are slightly different from (2.4.17) and (2.4.22). It is the new form of (3.2.11) that will become a crucial challenge when we attempt to show that the dual problem corresponding to (3.2.9) has a solution. ◁

We now turn to determining necessary and sufficient conditions for the portfolio to be a solution to problem $(\mathcal{P}^+_{c,q})$ defined at (3.1.4) by presenting a result similar to that of Proposition 2.4.7:

Proposition 3.2.3. *Assume Conditions 2.1.1, 2.1.5, 2.3.1, 2.3.5. For arbitrary* $(\bar{X}, \bar{Y}) \in \mathbb{I} \times \mathbb{I}$ *with* $\bar{X} \equiv (\bar{X}_0, \dot{\bar{X}}, \Lambda_{\bar{X}})$ *and* $\bar{Y} \equiv (\bar{Y}_0, \dot{\bar{Y}}, \Lambda_{\bar{Y}})$, *we have*

$$\Phi(\bar{X}) = \inf_{X \in \mathbb{I}} \Phi(X) = \sup_{Y \in \mathbb{I}} [-\Psi(Y)] = -\Psi(\bar{Y}) \tag{3.2.15}$$

if and only if

(1) $\bar{X}_0 = x_0$,

(2) $\bar{X}(T) = \begin{cases} -\frac{\bar{Y}(T)+c}{a} & \text{if } \bar{Y}(T) + c \leq 0, \\ 0 & \text{otherwise} \end{cases}$ *a.s.,*

(3) $\bar{X}(t) \, [\dot{\bar{Y}}(t) + r(t)\bar{Y}(t)] = 0$ *a.e.,*

(4) $\bar{X}(t) \geq 0$ *a.e.,*

(5) $\bar{\pi} \in \mathcal{U}(\bar{X})$ *and* $\delta(t, \bar{Y}(t), \Theta_{\bar{Y}}(t)) + \bar{\pi}'(t)\Theta_{\bar{Y}}(t) = \bar{Y}(t) g(t, \bar{\pi}(t))$ *a.e.*

for $\bar{\pi}(t) := [\sigma'(t)]^{-1}\Lambda_{\bar{X}}(t)$.

Proof. From (3.2.4) and (3.2.10) it follows that for arbitrary $x \geq 0,\ y \in \mathbb{R}$ one has

$$l_0(x) + m_0(y) = xy \quad \text{if and only if} \quad x = x_0.$$

Similarly, from (2.3.6), (3.2.5), and (3.2.11) we find that for arbitrary $x \geq 0,\ y \in \mathbb{R}$ and $w \in \Omega$

$$l_T(w, x) + m_T(w, y) = -xy \quad \text{if and only if} \quad x = \begin{cases} -\frac{y + c(w)}{a(w)} & \text{if } y + c(w) \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, conditions (A) and (B) of Proposition 2.4.6 hold if and only if items (1) and (2) hold. Further, for arbitrary $(w, t) \in \Omega \times [0, T]$, and $(x, v, \xi), (y, s, \gamma) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$, we have:

$$\begin{aligned}
& L(w, t, x, v, \xi) + M(w, t, y, s, \gamma) = xs + vy + \xi'\gamma \\
& \iff v = r(w,t)x + \xi'\theta(w,t) + g(w, t, [\sigma'(w,t)]^{-1}\xi),\ [\sigma'(w,t)]^{-1}\xi \in K,\ s + r(w,t)y \leq 0, \\
& \qquad x \geq 0,\ \text{and } \delta(w, t, y, \Theta_y) = xs + vy + \xi'\gamma \\
& \iff v = r(w,t)x + \xi'\theta(w,t) + g(w, t, [\sigma'(w,t)]^{-1}\xi),\ [\sigma'(w,t)]^{-1}\xi \in K,\ s + r(w,t)y \leq 0, \\
& \qquad x \geq 0,\ \text{and } \delta(w, t, y, \Theta_y) = xs + (r(w,t)x + \xi'\theta(w,t) + g(w, t, [\sigma'(w,t)]^{-1}\xi))y + \xi'\gamma \\
& \iff v = r(w,t)x + \xi'\theta(w,t) + g(w, t, [\sigma'(w,t)]^{-1}\xi),\ [\sigma'(w,t)]^{-1}\xi \in K,\ s + r(w,t)y \leq 0, \\
& \qquad x \geq 0,\ \text{and } \delta(w, t, y, \Theta_y) = x(s + r(w,t)y) + \xi'[\theta(w,t)y + \gamma] + yg(w, t, [\sigma'(w,t)]^{-1}\xi). \\
& \iff v = r(w,t)x + \xi'\theta(w,t) + g(w, t, [\sigma'(w,t)]^{-1}\xi),\ [\sigma'(w,t)]^{-1}\xi \in K,\ x(s + r(w,t)y) = 0,
\end{aligned}$$

$$x \geq 0, \text{ and } \delta(w,t,y,\Theta_y) = \xi'[\theta(w,t)y+\gamma] + yg(w,t,[\sigma'(w,t)]^{-1}\xi),$$

where the last equivalence comes from the fact that $x(s+r(w,t)y) \leq 0$ and

$$\delta(w,t,y,\Theta_y) \geq \xi'[\theta(w,t)y+\gamma] + yg(w,t,[\sigma'(w,t)]^{-1}\xi)$$

for all $[\sigma'(w,t)]^{-1}\xi \in K$ as follows from definition of δ (recall (2.4.19)).

Hence, condition (C) of Proposition 2.4.6 holds if and only if $\bar{X}(t) \geq 0$ a.e., $\bar{X}(t)[\dot{\bar{Y}}(t) + r(t)\bar{Y}(t)] = 0$ a.e., $\bar{\pi} \in \mathcal{U}(\bar{X})$, and $\delta(t,\bar{Y}(t),\Theta_{\bar{Y}}(t)) + \bar{\pi}'(t)\Theta_{\bar{Y}}(t) = \bar{Y}(t)g(t,\bar{\pi}(t))$ a.e. for $\bar{\pi}(t) := [\sigma'(t)]^{-1}\Lambda_{\bar{X}}(t)$ (recall definition (3.2.1) of $\mathcal{U}(\bar{X})$).

Combining, we conclude that for arbitrary $(\bar{X},\bar{Y}) \in \mathbb{I} \times \mathbb{I}$

$$\text{(A) - (C) of Proposition 2.4.6 hold} \iff \text{(1) - (5) hold.}$$

It is clear that Proposition 2.4.6 also holds for Φ and Ψ defined by (3.2.8) and (3.2.14), therefore from inequality (2.4.26) then follows that equality $\Phi(\bar{X}) + \Psi(\bar{Y}) = 0$ is equivalent to (3.2.15), and hence, (3.2.15) holds if and only if conditions (1) - (5) are satisfied. □

Again, as in Chapter 2 we see from Proposition 3.2.3 that to solve problem (3.1.4) we need to construct a pair $(\bar{X},\bar{Y}) \in \mathbb{I} \times \mathbb{I}$ that satisfies the optimality relations (1) - (5) of Proposition 3.2.3. Motivated by Proposition 3.2.3 we now focus our attention on solving the *dual problem*:

$$\text{determine some} \quad \bar{Y} \in \mathbb{I} \quad \text{such that} \quad \Psi(\bar{Y}) = \inf_{Y \in \mathbb{I}} \Psi(Y). \tag{3.2.16}$$

We notice that similarly to derivations in Chapter 2 we can restrict our choice of Y to those from the set

$$\mathbb{I}_1 := \{Y \equiv (Y_0, \dot{Y}, \Lambda_Y) \in \mathbb{I} \mid \dot{Y}(t) = -r(t)Y(t) \text{ a.e.}\}, \tag{3.2.17}$$

since from (3.2.13), (3.2.14) and item (3) of Proposition 3.2.3 it follows that

$$\inf_{Y \in \mathbb{I}} \Psi(Y) = \inf_{Y \in \mathbb{I}_1} \Psi(Y). \tag{3.2.18}$$

Recall from Chapter 2 (Remark 2.5.1) that members of set $\mathbb{I}_1$ have the parameterization

$$Y(t) = \Xi(y,\gamma) \text{ with } y = Y_0 \text{ and } \gamma = \Lambda_Y. \tag{3.2.19}$$

Thus define

$$\tilde{\Psi}(y,\gamma) := \Psi(\Xi(y,\gamma)), \qquad (y,\gamma) \in \mathbb{R} \times \mathcal{H}_2, \tag{3.2.20}$$

then from (3.2.10), (3.2.11), (3.2.13), and (3.2.14) follows that for each $(y,\gamma) \in \mathbb{R} \times \mathcal{H}_2$ and $Y := \Xi(y,\gamma)$ we find

$$\tilde{\Psi}(y,\gamma) = x_0 y + \mathbb{E}\left[\frac{(Y(T)+c)^2}{2a}\mathbb{1}_{Y(T)+c\le 0}\right] + \mathbb{E}\int_0^T \delta(t, Y(t), \Theta_Y(t))dt - q. \tag{3.2.21}$$

Moreover, by arguments exactly as in Remark 2.5.2 we obtain

$$\inf_{(y,\gamma)\in\mathbb{R}\times\mathcal{H}_2} \tilde{\Psi}(y,\gamma) = \inf_{Y\in\mathbb{I}} \Psi(Y). \tag{3.2.22}$$

Remark 3.2.4. Comparing (3.2.21) to its analogue (2.5.11) of problem $(\mathcal{P}_{c,q})$ in Chapter 2, we see that while $\tilde{\Psi}$ of (3.2.21) is still proper, convex, and lower semi-continuous, it is no longer coercive (recall Definition A.1.2). Indeed, comparing the dual function (3.2.21) with the dual function (2.5.11) (for the problem without a nonnegative wealth constraint) we see that there is one small but crucial difference, namely the indicator of the event $Y(T) + c \le 0$ occurring at the second term on the right side of (3.2.21), but which is not present at the second term on the right of (2.5.11). This indicator function completely ruins the coercivity of the dual function (3.2.21); since coercivity of the dual function is the entire basis of the proof of existence of a dual solution at Proposition 2.5.3 using Theorem A.1.3 (for the case where there is no constraint on the wealth process), we cannot expect to establish an analog of Proposition 2.5.3 which guarantees existence of a minimizer of the dual function (3.2.21). Without this minimizer we have nothing in terms of which to construct a solution of the primal problem in terms of a minimizer of the dual function using the optimality relations (in this case given by Proposition 3.2.3). What is at the root of this rather fundamental difficulty? In Chapter 2 we used the space of dual variables $\mathbb{I}$ (recall (2.2.1) - (2.2.3)) to provide the Lagrange multipliers which enforce the portfolio constraint in problem (2.3.8). In the present chapter we are trying to use the same space of dual variables to provide Lagrange multipliers which enforce not only the portfolio constraint but also the nonnegativity constraint on the wealth process as well. This is simply asking too much, as is demonstrated by non-existence of a dual solution within this space of dual variables. The way out of this impasse seems to be to pair the space $\mathbb{I}$ with another vector space of variables for enforcing the wealth constraint, and then take the space of dual variables to be the product of these spaces. A classic work of Dubovitskii and Milyutin [6] on deterministic optimal control with combined control and state constraints - which is one of the very few works to make definite progress on this problem, albeit in a deterministic setting - suggests that the vector space of variables appropriate to the wealth constraint is the adjoint of the space of essentially bounded functions. A goal of future research is to try to adapt this idea to the considerably more challenging setting of a stochastic (instead of a deterministic) control problem such as the one studied in the present chapter.

$\triangleleft$

Chapter 4

Conclusion

In this thesis we consider the problem of quadratic hedging and mean-variance portfolio selection in the presence of margin requirements and pointwise portfolio constraints. Our approach and goals can be summarized as follows:

(i) identify a set of optimality relations that describe an optimal solution in terms of solutions of the dual problem, in other words, determine the necessary and sufficient conditions for the problem and its dual to each have a solution;

(ii) show existence of a solution of the dual problem;

(iii) use the solution of the dual problem established in (ii) to characterize the solution of the original (or primal) problem.

In this thesis we have followed the same general approach used in Labbé and Heunis [15], originally motivated by works of Rogers [22] and Bismut [3], and have successfully completed steps (i) and (ii): Proposition 2.4.7 introduces optimality relations and Proposition 2.5.3 establishes existence of a solution to the dual problem. The method allowed us to essentially eliminate the necessity for guesswork and systematically arrive at the dual problem (2.4.24) and then derive necessary and sufficient optimality conditions for Proposition 2.4.7.

The optimality relations (1) - (4) of Proposition 2.4.7 are quite different from the optimality relations of [15], this difference arising from the presence of a nonlinear term in the wealth equation which accounts for margin payments (recall Remark 2.1.6), a term which was not present in the corresponding wealth equation of [15], where margin payments are not included in the model. A consequence of this difference is that the optimality relations are in fact considerably more difficult to solve than the corresponding optimality relations of [15]. In particular, in order to solve these relations (that is complete step (iii)) we have to resort to introducing simplifying assumptions (none of which are necessary in [15]). Specifically, these assumptions are Condition 2.6.1, as well as the function-space membership assumptions at (2.7.13). At this point we see no way of proceeding without Condition 2.6.1, which essentially asserts that the optimal dual process is either strictly positive or strictly negative, since this assumption is essential to our solution method. The situation may be compared, for example, with assumptions made when trying to solve a partial differential equation analytically, where one often assumes at the outset that the solution is actually in special product form, and without this assumption one cannot go any further. On the other hand, we feel that it may be possible to actually demonstrate the function-space memberships at (2.7.13), as was done in [15]. However, the methods used in [15] simply do not extend to our case, and the establishment of methods which do cover our case seems to present challenges which we cannot resolve at present. Thus showing that (2.7.13) indeed holds and removing Condition 2.6.1 can be viewed as possible extensions of the present work.

In Chapter 3 we add another feature to the problem studied in Chapter 2 by restricting the wealth process to stay nonnegative, effectively inducing a no-bancruptcy restriction. We are able to complete step (i) and derive optimality relations (1) - (5) in Proposition 3.2.3 that are very close to those of Proposition 2.4.7. This again demonstrates the power of the basic approach used in the thesis for synthesizing optimality relations and a dual problem. We are not able, however, to complete step (ii) and derive a result analogous to Proposition 2.5.3 of Chapter 2 since it is impossible to establish existence of a dual solution for a class of square integrable dual processes $Y \in \mathbb{I}$ (recall (2.2.1)-(2.2.3)). To overcome this situation, one would have to expand the class of dual solutions beyond $\mathbb{I}$.

To conclude, in this thesis we have studied a very difficult problem of quadratic hedging with pointwise portfolio constraints and margin requirements. We were able to derive the necessary and sufficient conditions for the problem of interest and its dual to each have a solution and constructed a tentative optimal portfolio for a particular class of dual solutions.

APPENDICES

Appendix A

Background

A.1 Elements of Convex Analysis

In this appendix we recall a few important definitions and results from classical convex analysis.

Definition A.1.1. *Consider the functional* $f\colon \mathbb{R}^m \mapsto \mathbb{R} \cup \{\pm\infty\}$*. Its conjugate function* $f^*\colon \mathbb{R}^m \mapsto \mathbb{R} \cup \{\pm\infty\}$ *is defined as*

$$f^*(y) := \sup_{x \in \mathbb{R}^m} \{x'y - f(x)\}.$$

Definition A.1.2. *Consider the functional* $f\colon \mathcal{T} \mapsto \mathbb{R} \cup \{\pm\infty\}$*. We say that* f *is coercive if* $f(x) \to \infty$ *whenever* $\|x\| \to \infty$*.*

The following theorem is stated and proved in Ekeland and Temam [7], p. 35:

Theorem A.1.3. *Let* $\mathcal{T}$ *be a reflexive Banach space (with norm* $\|\cdot\|$*) and* f *some functional defined on* $\mathcal{T}$*. Suppose that* f *is proper (that is* $f(x) \in \mathbb{R} \cup \{\infty\}$ *for all* $x \in \mathcal{T}$ *and* $\mathcal{F}(\tilde{x} \in \mathcal{T})$*, convex, lower semi-continuous and coercive. The there exists* $\bar{x} \in \mathcal{T}$ *such that*

$$f(\bar{x}) = \inf_{x \in \mathcal{T}} f(x).$$

The following result is stated and proved in Aubin [1], p.36:

Theorem A.1.4. ***(Lagrange Multiplier Theorem).*** *Let* $\mathcal{T}$ *be a vector space and suppose that (i)* $\mathcal{A}$ *is a convex subset of* $\mathcal{T}$*;*

(ii) $f\colon \mathcal{A} \mapsto \mathbb{R}$ *is a convex function;*

(iii) $G : \mathcal{T} \mapsto \mathbb{R}^m$ *is an affine mapping;*

(iv) 0 is in the interior of $\{G(u) \mid u \in \mathcal{A}\}$*.*

Define $\mathcal{L}(\mu; u) := f(u) + \mu G(u)$*. then there is a Lagrange multiplier* $\bar{\mu} \in \mathbb{R}$ *such that*

$$\inf_{\substack{u \in \mathcal{A} \\ G(u)=0}} f(u) = \sup_{\mu \in \mathbb{R}} \inf_{u \in \mathcal{A}} \mathcal{L}(\mu; u) = \inf_{u \in \mathcal{A}} \mathcal{L}(\bar{\mu}; u).$$

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