Pricing and Hedging the Guaranteed Minimum Withdrawal Benefits in Variable Annuities

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The Guaranteed Minimum Withdrawal Benefits (GMWBs) are optional riders provided by insurance companies in variable annuities. They guarantee the policyholders' ability to get the initial investment back by making periodic withdrawals regardless of the impact of poor market performance. With GMWBs attached, variable annuities become more attractive. This type of guarantee can be challenging to price and hedge. We employ two approaches to price GMWBs. Under the constant static withdrawal assumption, the first approach is to decompose the GMWB and the variable annuity into an arithmetic average strike Asian call option and an annuity certain. The second approach is to treat the GMWB alone as a put option whose maturity and payoff are random.

Hedging helps insurers specify and manage the risks of writing GMWBs, as well as find their fair prices. We propose semi-static hedging strategies that offer several advantages over dynamic hedging. The idea is to construct a portfolio of European options that replicate the conditional expected GMWB liability in a short time period, and update the portfolio after the options expire. This strategy requires fewer portfolio adjustments, and outperforms the dynamic strategy when there are random jumps in the underlying price. We also extend the semi-static hedging strategies to the Heston stochastic volatility model.

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Chapter 1

Introduction

This thesis investigates pricing and hedging issues of Guaranteed Minimum Withdrawal Benefits (GMWBs). A GMWB is a type of option or rider that can be added to a variable annuity. It provides protection against downside market risk by allowing the annuitant to withdraw a maximum percentage of their initial investment each year without penalty for a fixed term or for life. It is hoped that some of the techniques developed here will be applicable to other types of products.

A variable annuity is a retirement savings vehicle in which the benefits depend on the performance of the investment options selected by the owner of the contract. Typically variable annuities have a significant common stock or equity component. Since equity returns are volatile this market uncertainty creates investor demand for floors on returns. To meet this demand, insurers provide specific guaranteed benefits based on the occurrence of various events such as death, annuitization, and maturity. These guarantees protect the policyholder against both financial and mortality risk over a long period of time. They serve to make the basic variable annuities more attractive.

GMWBs are one of the most recent innovations in this suite of guarantees. Under this kind of rider, the policyholder has the option to begin withdrawing a certain amount from the account, and continue to make withdrawals no matter how poor the investment performance of the account. The guarantee could last 10-40 years. The withdrawal decision is made by the policyholder, and is undoubtedly influenced by the investment performance of the fund. That being said, however, there is not enough experience to establish accurately the determinants of this decision.

This thesis focuses on the methods of pricing and hedging, and assumes that withdrawals are taken statically from the first year at the maximum allowed amount without penalty. The layout of the rest of this chapter is as follows:

- Section 1 provides background information on variable annuities and the various types of guaranteed benefits;
- Section 2 describes GMWBs in more detail;
- Section 3 discusses the research motivation, provides a brief literature review, and outlines the structure of the thesis.

1.1 The Variable Annuity

Variable annuities can be purchased either by a single payment or a series of payments. In this thesis, we assume the initial deposit is the only payment. Under a deferred[1] variable annuity there is an accumulation period during which the account value changes in line with the investment performance of the assets in account. There is often some type of guarantee on the deposit (e.g., return of premiums based on some event), but the investment performance is not guaranteed. The investment options (sub-accounts)

[1]We will normally omit the word deferred in the sequel.

offered by the insurance companies are typically mutual funds of stocks, bonds, and money market instruments or some combination of the three.

Compared with fixed annuities, variable annuities are designed to protect against effect of inflation on fixed incomes over the long run. Sales of variable annuities tend to grow in a rising stock market. Sales of fixed annuities tend to move in the opposite direction. Table 1.1[2] shows individual annuity sales in the last few years in the US. Variable annuity sales rose by an average of 6.3% per annum over six years from 2001 to 2007, while fixed annuity sales grew by only 0.9% per year on average. In 2008, variable annuity sales decreased 15.4%, but fixed annuity sales increased 50.3%.

Year	**Variable Annuity**	**percent change**	**Fixed Annuity**	**percent change**	**Total Sales**	**percent change**
2001	$ 111.0		$ 74.3		$ 185.3	
2002	116.6	5%	103.3	39%	219.9	18.7%
2003	129.4	11%	89.4	-13.5%	218.8	-0.5%
2004	132.9	2.7%	87.9	-1.7%	220.8	0.9%
2005	137.6	3.5%	78.9	-10%	216.5	-1.9%
2006	160.4	16.6%	78.3	-0.8%	238.7	10.3%
2007	184.0	14.7%	72.8	-7.0%	256.8	7.6%
2008	155.6	-15.4%	109.4	50.3%	265.0	3.2%

Table 1.1: Sales of total individual annuities in the US, 2001-2008 ($ billions)

Policyholders can choose what date the payout phase begins. The retirement date is often recommended as being a good time to annuitize. Although policyholders can annuitize later than this, insurers usually specify a maximum annuitization date. For example, it could be the later of the policyholder's 90th birthday or the end of the 10th

[2] Source: http://www.iii.org/media/facts/statsbyissue/annuities/.

contract year. Upon annuitization, policyholders can choose to receive a lump-sum payment or a series of monthly annuity payments. There are a number of choices of how long the annuity payments will last. These are:

1. the lifetime of the policyholder;
2. the lifetime of the named beneficiary;
3. a specified period such as 20 years;
4. the longer of the policyholder's lifetime or a certain period.

During the payout phase, the periodic payments may be fixed or vary based on the performance of fund investment.

Variable annuities typically provide a guaranteed minimum death benefit (GMDB) if the policyholder dies before receiving any income. The death benefit often equals the greater of the account value and total premiums paid less any withdrawals. For example, a person had paid premiums totaling $100,000, and had made withdrawals equaling $15,000. The account value stands at $80,000 because of these withdrawals and investment losses. If he were to die, his beneficiary would receive $85,000.

Some variable annuities have optional death benefits such as roll-up or annual ratchet with extra charges. The roll-up feature provides a death benefit that equals the premium accumulated at a fixed interest rate. The annual ratchet feature allows for the guaranteed minimum to be reset to the account value as of a specified date if the underlying funds have performed well. Using the same example, if the current account value is $95,000, then the death benefit will be set to $95,000. The purpose of a stepped-up death benefit is to lock in the current high investment return and protect the death benefit against a subsequent decline in the value of the account.

Until a few years ago, the GMDB was the most popular rider for people buying variable annuities. In recent years Guaranteed Living Benefits (GLBs) have become more popular. Insurance companies added them to attract more customers. The major types of guaranteed benefits are: [3]

1. Guaranteed Minimum Income Benefit (GMIB)

   The GMIB guarantees a minimum compounding rate and a minimum level of annuity income payment. This benefit is only applicable if the policyholder annuitizes the contract.

2. Guaranteed Minimum Accumulation Benefit (GMAB)

   The GMAB guarantees a minimum account value at the end of a specified period. The policyholder has the right to renew the contract at a new guaranteed level at that time.

3. Guaranteed Minimum Withdrawal Benefit (GMWB)

   The GMWB guarantees the policyholder's ability to get the premium back by making periodic withdrawals regardless of the impact of poor market performance on the account value. There is a maximum annual withdrawal amount usually defined as 7% of premium, so the benefit period could last up to 15 years or more.

Among these benefits, the GMWB is the newest feature having been first introduced by Hartford Life Insurance Company in 2002. It has become the most popular benefit in the variable annuity market. More enhanced versions of GMWBs have subsequently come into the market in the last few years. For example, the *GMWB for life* (or *lifetime GMWB*) guarantees that the policyholder can withdraw a percentage (e.g. 5%) of the premium from age 65 for the rest of his or her life. The *joint life GMWB* guarantees benefit payments to the surviving spouse. The next section will describe the GMWB

[3] See Hardy (2003) for more information on some of these guarantees.

in greater detail.

Variable annuities have many types of charges which reduce the account value, most of which are deducted periodically from the account itself. Types of charges include: mortality and expense risk; administrative; sub-account fees; optional guaranteed benefits (e.g., GMIB, GMAB, GMWB, and ratchet GMDB). Other charges, such as surrender charges and fund-transferring charges, are based on specific transactions that the policyholders make.

The surrender charge is assessed when the policyholder withdraws money from the contract within a certain period after a purchase. This typically occurs within six to eight years, but may even do so within ten years. Generally, the surrender charge is a percentage of the amount withdrawn, and declines gradually over the period. For example, a 7% charge might apply in the first year after a purchase, 6% in the second year, 5% in the third year, and so on until the eighth year, when the surrender charge no longer applies. A withdrawal amount below a certain level each year is often free of charge (e.g., 10% of the account value).

1.2 The Guaranteed Minimum Withdrawal Benefit

By providing the policyholder with downside income protection from investment risk and flexible withdrawals, the GMWB has become the most attractive optional benefit in the variable annuity market. Two companies (AmerUs Group and American National) even added similar lifetime income benefits on their fixed indexed annuities in 2006. Although variable annuities are suitable saving instruments for retirement, they involve investment risk. In addition, the policyholders face uncertainties about future income needs. The GMWB helps to reduce these uncertainties. The GMWB buyers have the flexibility to start or stop withdrawing, and keep the contract value growing in the sub-accounts at the same time. Variable annuities may increase income to keep up with inflation if their returns exceed the inflation rate. However, the GMWB does not protect against decreasing purchasing power due to inflation.

To describe the benefit features of the GMWB clearly, we define two terms: Guaranteed Withdrawal Balance (GWB) and Maximum Annual Withdrawal Amount (MAWA). GWB is the total benefit amount that the policyholder can withdraw from the contract. The initial GWB equals the single premium if the GMWB is elected at issuance of the variable annuity. Some companies may provide a benefit amount more than the initial investment. For example, Phoenix Life and Annuity Company offers a benefit amount percentage of 105%. GWB will be adjusted upon additional premium payments and withdrawals. If the GMWB is elected later, GWB equals the greater of the contract value and the premiums paid. MAWA equals a percentage (normally 5-12%, and 7% is common) of the initial GWB, and remains constant if no adjustments are made. The minimum guaranteed benefit period is the initial GWB divided by the MAWA. Both the GWB and the contract value are reduced by the amount of withdrawal when the amount is no more than the MAWA. The policyholder can withdraw a larger amount than the MAWA, which may reduce the GWB by more than the amount of the withdrawal. The excess withdrawal amount will be subject to applicable surrender charges. The MAWA may also be reduced based on the new account value.

It is helpful to use a numerical example to illustrate the calculation of the GMWB benefit. Assume a GMWB guarantees a maximum annual withdrawal amount at 7% of premiums paid, and the variable annuity is a single premium deferred annuity. A policyholder elects the GMWB at the beginning of the contract with an initial investment of $ 100,000. The GWB is $100,000, and the MAWA is $7,000. We assume the policyholder withdraws $7,000 at the end of each year. Hence, the total guaranteed payments last for a period of 15 years. In Table 1.2, we have assumed one set of annual net investment returns to highlight the structure of the GMWB. Figure 1.1 shows how the account value and the GWB change every year. In reality, there could be thousands of different scenarios.

Contract year	Investment return	Fund before withdrawal	Annual withdrawal	Fund after withdrawal	Remaining benefit(GWB)
1	5%	105,000	7,000	98,000	93,000
2	5%	102,900	7,000	95,900	86,000
3	10%	105,490	7,000	98,490	79,000
4	5%	103,415	7,000	96,415	72,000
5	10%	106,056	7,000	99,056	65,000
6	-20%	79,245	7,000	72,245	58,000
7	-10%	65,020	7,000	58,020	51,000
8	-10%	52,218	7,000	45,218	44,000
9	5%	47,479	7,000	40,479	37,000
10	-20%	32,383	7,000	25,383	30,000
11	-10%	22,845	7,000	15,845	23,000
12	-20%	12,676	7,000	5,676	16,000
13	5%	5,960	7,000	0	9,000
14	5%	0	7,000	0	2,000
15	5%	0	2,000	0	0

Table 1.2: Example of the GMWB, full utilization, hypothetical returns

The market's performance affects the policyholder differently under the GMWB. If the market performs strongly, the policyholder can end up with positive account value and GWB. The GMWB put option has no payoff in this case. If the market performs poorly, the policyholder can get the guaranteed withdrawal benefit even though the total investment return is negative. The insurer is responsible for the shortfall, that

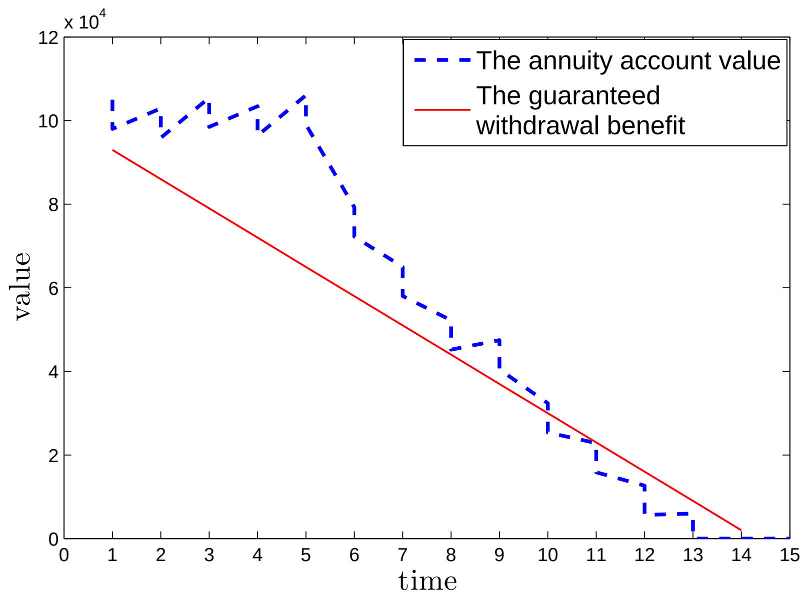

Figure 1.1: A sample path of the annuity account value process

is, the payoff of the GMWB option. The GMWB protects policyholders' investments in a long drawn-out bear market. The above example shows the latter of the two scenarios. After the first year net return of 5%, the fund grows to \$105,000, then the policyholder withdraws \$7,000. The account value is reduced to \$98,000 by the amount of withdrawal, and the GWB is reduced to \$93,000. The next year the fund grows from \$98,000 to \$102,900. We assume very poor returns in some later years, and the account value becomes zero at the end of year 13, then the variable annuity contract terminates and the policyholder continues to receive the remaining guaranteed withdrawal benefits for 2 years. The GMWB starts paying off from contract year 13. In the real world, the payoff time of the GMWB is random. It could be any contract year before annuitization.

To encourage policyholders to defer taking advantage of the GMWB, insurers sometimes offer bonuses to policyholders who make no withdrawals during the first five years. For example, if the policyholder waits for three years before making a withdrawal, then a 5% simple interest per annum will be credited to the GWB, which increases the GWB from \$100,000 to \$115,000. In this case, the maximum annual withdrawal amount increases to \$8,050 per year. If he waits for five years, then the GWB increases to \$125,000, and he can withdraw \$8,750 per year after year five.

Another contract feature, known as the step-up option, enables the policyholder to reset the guaranteed withdrawal balance to the current higher account value when investment performance is strong. By choosing to step up, the policyholder is able to increase the total benefit amount and the maximum annual withdrawal amount. Accordingly, the period of time over which withdrawals can be taken is extended. Insurance companies may have different rules about the time of exercising the step-up option. For example, the policyholder can only choose to step up the benefit amount every five years with a possible 30-day window. The option may reduce the inflation effect on incomes when the account value goes up and the step-up option is available. The charge will increase upon step-up election by 20 to 40 basis points. Assume the step-up option is available every five years. At the end of year five, the account value after withdrawal totaling \$99,056 exceeds the GWB \$65,000, so the policyholder chooses to step up. The GWB is reset to the account value \$99,056, and the MAWA remains at \$7,000. It is not optimal to step up after that. Hence, the policyholder keeps withdrawing \$7,000 every year till the benefit is depleted in year 20. Table 1.3 shows the cash flows for this case.

The fee structure has an impact on the GMWB price. The GMWB charge is deducted from the contract value periodically. It could be a percentage of the current account value. It could also be a percentage of the initial premium or a percentage of the remaining guaranteed benefit amount, or the greater of these two. The annual charge ranges from 20 to 75 basis points depending on the nature of the benefit. Typ-

Contract year	Investment return	Fund before withdrawal	Annual withdrawal	Fund after withdrawal	Remaining benefit(GWB)
1	5%	105,000	7,000	98,000	93,000
2	5%	102,900	7,000	95,900	86,000
3	10%	105,490	7,000	98,490	79,000
4	5%	103,415	7,000	96,415	72,000
5	10%	106,056	7,000	99,056	99,056
6	-20%	79,245	7,000	72,245	92,056
7	-10%	65,020	7,000	58,020	85,056
8	-10%	52,218	7,000	45,218	78,056
9	5%	47,479	7,000	40,479	71,056
10	-20%	32,383	7,000	25,383	64,056
11	-10%	22,845	7,000	15,845	57,056
12	-20%	12,676	7,000	5,676	50,056
13	5%	5,960	7,000	0	43,056
14	r%	0	7,000	0	36,056
15	r%	0	7,000	0	29,056
16	r%	0	7,000	0	22,056
17	r%	0	7,000	0	15,056
18	r%	0	7,000	0	8,056
19	r%	0	7,000	0	1,056
20	r%	0	1,056	0	0

Table 1.3: Example of the GMWB with step-up feature

ically, the price in basis points based on the account value is lower than the price in basis points based on guaranteed benefit if there is no step-up option. With the step-up feature, the relationship would be reversed. The charge is subtracted from each sub-account in the same proportion as the sub-account investment to the account value. If the charge is based on the account value, insurers typically deduct it on a daily basis. Some do so on an annual basis of the average daily net asset value of sub-accounts. If the charge is based on the GWB, insurers deduct it quarterly or yearly since the GWB is less volatile than the account value.

The newly-introduced lifetime GMWBs provide guaranteed annual income until death. The policyholders are also able to access potentially increased account values and control the asset allocation in ways that the traditional variable annuitization normally does not allow. Consequently, variable annuities with lifetime GMWBs are becoming very popular. The lifetime GMWBs usually have two options: single life or joint spousal life. For the single life option, the benefit payments end at the death of the person covered. For the joint spousal life option, the benefit payments end when the remaining spouse dies. The charges for the lifetime GMWBs are usually based on the benefit balance or the greater of account value and benefit balance. The fee rate for the single life option ranges from 40 to 75 basis points, while the spousal life option tends to be 10-20 basis points higher.

For the single life option, a spouse continuation option is available upon the first death with the same charge, but the account value and the benefit amount may be adjusted. For the joint spousal life, there will be no recalculation of the benefit amount when the first death occurs. The annual benefit payment amount is a percentage of the initial guaranteed benefit amount. Although normally ranging from 4% to 7%, it often lies at 5%. Many companies vary the percentage with the policyholders' age when they take the first withdrawal. The older the policyholder is, the larger the withdrawals will be. For example, the MAWA is 5% if the attained age is 60 upon first withdrawal, and 6% if the attained age is 70. The lifetime GMWB could also include a GMDB. For example, the Hartford Insurance Company offers these riders. The lifetime GMWB may include an automatic annual step-up option to provide inflation protection. But for some contracts, step-up option has to be elected.

To limit the risks undertaken, the insurers impose several limits on the rider: maximum/minimum issue age limits (e.g., 80/50), required annuitization age limits (90), and policy size limits ($5,000 \sim 1$ million).

1.3 Motivation and Outline of the Thesis

The motivation for this thesis is that GMWBs give rise to many interesting problems. Basically, a GMWB is a specialized long-term put option sold by insurance companies. The payoff is triggered when the account value is not enough to pay the withdrawal amount. The payoff is the remaining guaranteed benefit which is distributed in equal periodic payments at the maximum withdrawal amount without penalty. The payoff time depends on the path of the fund value and the withdrawal pattern. On the other hand, the total value of the GMWB charges also depends on the path of the account value. The GMWB charge rate affects the total charge value, the account value, and the GMWB benefit payoff. The fair price is the charge rate that makes the expected benefit value equal to the expected charge value. Since both the expected value of the GMWB benefit and charges are path-dependent, hedging GMWBs is a complex problem.

There has been little academic research analyzing this product. Milevsky and Salisbury (2006) price the GMWB under both the deterministic withdrawal assumption and the dynamic lapse assumption. They claim that the true value of the GMWB lies somewhere between the two prices based on the two assumptions. The prices they calculate are much higher than the prevalent rates in the market. We use the same static assumptions and a similar decomposition approach, and obtain prices that are close to the market prices. This discrepancy motivates us to study the product more thoroughly. Bauer et al. (2008) present a general framework to value several types of guarantees in variable annuities. They take mortality and surrender into consideration and give numerical examples with several withdrawal patterns. Their assumptions are different than ours, and the calculated guarantee fees are lower than the market prices. They claim that the guarantee fee would be much higher with an optimal withdrawal strategy, but they did not describe the optimal strategy. Dai et al. (2008) construct singular stochastic control models for pricing GMWBs under both continuous and discrete frameworks. They analyze the impact of various model parameters on the GMWB fee and the optimal withdrawal behaviors of the policyholders who either withdraw a discrete amount or withdraw at the guaranteed rate continuously. Chen et al. (2008) have studied the effect of mutual fund fees and sub-optimal withdrawal behavior on the value of the GMWB, and the effects of various modeling assumptions on the optimal withdrawal strategy. Their conclusions are that only if several unrealistic modeling assumptions are made is it possible to obtain GMWB fees in the same range as is normally charged. In all other cases, typical fees are not enough to cover the cost of hedging these guarantees. Peng et al. (2009) have developed model formulations of the price function and ruin probability, and derived analytic approximation solutions to the pricing formulations, with interest rate risk.

In this thesis, we focus on hedging strategies for GMWBs as well as pricing methods. More specifically, we propose semi-static strategies that are not affected by price jumps and are robust to model misspecification risk. The idea of static hedging has been discussed by Carr and Wu (2004) for European options, but very little work has been done for the case of path-dependent options. GMWBs have some path dependence so it is more challenging.

The stock market was very volatile in October 2008. The realized daily volatility of the S&P500 index over that month was 78%, and the interest rates fell to historic lows. Turnbull (2008) shows that the delta-hedging strategy for GMWBs could incur a big hedging loss in this short time period, which is equivalent to several years of the anticipated profit from the variable annuities. The semi-static hedging strategies can overcome the limitations of the delta-hedging strategy. In addition, we propose two semi-static hedging strategies under the Heston model. There is evidence showing that stock price volatilities change over time,[4] but there are few publications on static hedging under a stochastic volatility model.

The remaining balance of the thesis is organized as follows:

In Chapter 2, we explore two approaches to price the GMWB under the Black-Scholes model. The first approach decomposes the variable annuity and the GMWB into an annuity certain and a floating-strike Asian call option. The call option value is a function of the GMWB charge rate. The price of the GMWB is obtained by solving the decomposition equation numerically. The second approach treats the GMWB as a put option with a random payoff time. Both the benefit put option value and the GMWB charge value depend on the GMWB charge rate. The price is the charge rate that makes the two values equal. Prices that we get from the two approaches are consistent with those in the market.

In Chapter 3, we propose a semi-static hedging strategy which provides protection against random jumps in the fund value. The idea is to replicate the expected GMWB loss with standard options in a short time period. We assume the underlying fund can be

[4]See Hull (2006, Ch. 15), Hardy (2003, Ch. 2). This can also be seen from a stock market volatility index (VIX). VIX is the expected return volatility of the S&P 500 index over the next 30 days. It is implied from the prices of S&P 500 index options. The following table quotes the VIX values in 2007 and 2008. The current VIX values are high, but historically they have not been persistently high.

2008	Dec	Nov	Oct	Sep	Aug	Jul	Jun	May	Apr	Mar	Feb	Jan
Value of VIX	40.0	55.3	59.9	39.3	20.7	22.9	24.0	17.8	20.8	25.6	26.5	26.2
2007	Dec	Nov	Oct	Sep	Aug	Jul	Jun	May	Apr	Mar	Feb	Jan
Value of VIX	22.5	22.9	18.5	18.0	23.4	23.5	16.2	13.1	14.2	14.6	15.4	10.4

modeled by an index. We explain how to simulate the expected GMWB loss conditional on the index value at a future time, and how to obtain the optimal portfolio weights. At the beginning of the hedging period, we set up this replicating portfolio. Within the hedging horizon, there is no need to rebalance the portfolio. At the end of the hedging period, we construct a new replicating portfolio based on the new account value. Our studies show that the semi-static hedging strategy is comparable with delta hedging under the Black-Scholes model, and outperforms delta hedging under a jump-diffusion model.

In Chapter 4, we implement two semi-static hedging strategies under the Heston stochastic volatility model. The first strategy utilizes a replicating portfolio with a maturity that is longer than the hedging period but shorter than the target maturity. The second strategy uses a replicating portfolio that will expire at the end of the hedging period. Under strategy one, the volatility risk is mitigated to some extent because the portfolio value changes with the volatility at the end of the hedging period. The values of the hedging target and the replicating portfolio are computed conditional on the index value and the volatility at the end of the hedging period. Under strategy two, the portfolio payoff depends on the index value only. The expected GMWB loss is computed based on the index value and the mean variance conditional on the index value at the end of the hedging period. For European options, semi-analytical formulas exist under the Heston model. For GMWBs, we apply an efficient simulation method to estimate the hedging target at the end of the hedging period. Our conclusion is that semi-static hedging strategies perform well under the Heston model.

We end with suggested topics for future research in Chapter 5.

Chapter 2

Pricing a Plain Vanilla GMWB

This chapter sets up a basic framework to price a plain vanilla GMWB, the simplest version of GMWBs. As described in Chapter 1, most insurance companies charge for the GMWB rider by deducting an ongoing fee as a percentage of invested assets instead of an up-front fee. Both the GMWB benefit value and the total amount of fee change with the asset value and many other factors. We start with simple assumptions to identify factors that influence the price of the GMWB and analyze their relationship.

Pricing a plain vanilla GMWB helps us understand the structure in a simple setting, and provide a benchmark for future analysis. The plain vanilla GMWB has no bonus or step-up feature. This contract is designed for an individual annuitant, and the total dollar amount that can be withdrawn is fixed at inception. In addition, we make the following assumptions in this chapter:

- No lapses or mortality decrements considered here.
- The underlying reference portfolio has a lognormal distribution.
- Interest rates are constant.

- The amount withdrawn each period is equal to the maximum amount permitted under the contract that does not attract any surrender penalty.
- The maturity of the contract is determined as the time when the guaranteed total benefit amount is withdrawn.

Under these assumptions, we can price the GMWB in terms of a call option. A variable annuity with a GMWB can be viewed as a term certain annuity, plus a call option on the positiveness in the account value at maturity. Since we assume that all policyholders fully utilize the GMWB from the first contract year until the end, and that there are no surrenders or deaths, the call option has a fixed exercise time at maturity, which makes it easier to value. In addition, we are able to use the control variate technique to improve simulation efficiency based on the assumption of constant withdrawals. The cash flows from the term certain annuity are fixed, making its present value easily calculable.

More generally, the GMWB is a put option on the variable annuity account with a random exercise time. When the current account value is higher than the forthcoming withdrawal amount, the amount withdrawn is from the policyholder's account itself. In other words, the insurer has no payoff liability under the GMWB at this time. But once the account value is not sufficient, it will be set to zero after the withdrawal. The contract is then closed, and the remaining guaranteed payments will be paid by the insurance company. The payoff of the GMWB is the present value of those remaining guaranteed payments when the account value becomes zero. This put option approach allows for dynamic withdrawals, but we can not use the control variate technique any more.

This chapter is organized as follows: We begin with notation and assumptions in

Section 2.1. In Section 2.2, we explain the call option approach to price the plain vanilla GMWB. The call option price can be simulated efficiently using a control variate method. In Section 2.3, we describe the put option approach to price the GMWB. This approach is straightforward, and can be used to check the call option approach. In Section 2.4, we examine the consistency between these two pricing approaches with numerical examples. We end this chapter with a brief summary in Section 2.5.

2.1 Notation and Assumptions

Let us first define notation and make assumptions. Denote the issuance time of the annuity contract as time 0. We assume the initial investment is the only premium payment. The account value is denoted by A_t at time t. The initial value of the account balance is A_0. The initial deposit is assumed to be invested in a fund whose value at time t is S_t. We ignore any up-front charges. The initial account value can be expressed as α units of the fund, $A_0 = \alpha S_0$. For simplicity, we let $\alpha = 1$ from now on.

We use the assumptions of the Black-Scholes model. Under the real-world probability measure P, the fund value process $\{S_t\}$ is assumed to satisfy the following stochastic differential equation

$$dS_t = (\mu - q)S_t dt + \sigma S_t dW_t, \tag{2.1}$$

where μ is the instantaneous expected return, q is the charge rate of the GMWB, σ is the volatility, and $\{W_t\}$ is a standard Brownian motion with mean 0 and variance t. It is appropriate to price the GMWB under the risk-neutral measure because it gives a unique arbitrage-free price in a complete market. The drift term of the stochastic differential equation is then changed to risk-free interest rate minus the GMWB fee rate, $(r-q)$, under the risk-neutral measure Q.

We assume the GMWB option is elected at time 0. The GMWB guaranteed withdrawal rate is denoted by g, and the guaranteed MAWA is denoted by $w = g * A_0$. The actual annual withdrawal amount is assumed to be equal to w and the withdrawals last for T years from time 0,

$$T = \frac{A_0}{w} = \frac{1}{g}. \tag{2.2}$$

We assume that GMWB charges are continuously deducted from the account value. Since all the withdrawal amounts are assumed not to exceed w, there will be no withdrawal penalty or surrender charge. We assume that policyholders make a withdrawal at the end of each time period, and that there are no surrenders or deaths during the term of the contract. If we use quarterly time steps, then the length of a single period is $h = \frac{1}{4}$. Each withdrawal amount is equal to $w \cdot h$, and the total number of time steps is

$$N = \left[\frac{T}{h}\right].$$

Note that the account value is reduced by each withdrawal, but it can never fall below zero. Once withdrawals deplete the annuity account, the account value will be set to zero and remain zero for the rest of the term. To price the GMWB, it is helpful to introduce a so called shadow account B_t which keeps track of the fund performance until the maturity regardless of whether the actual account has terminated or not. When the actual account value A_t is larger than zero, the shadow account has the same value as the actual account. The shadow account value becomes negative when the actual account value A_t is set to zero. This will make it easy to recognize the liability after the annuity account value reaches zero. Their relation can be expressed as:

$$A_t = \max(B_t, 0). \tag{2.3}$$

For both pricing approaches, the fair charge rate of the GMWB will satisfy an equation of the form

$$f(q^*) = 0. \tag{2.4}$$

For the call option approach, the function f is equal to the initial contract value minus the present value of the decomposed *annuity certain* and the price of the Asian call option. For the put option approach, the function is the difference between the expected present values of the benefit payoff and charges. The explicit forms of f are given by equations (2.14) and (2.32) respectively.

There is no analytic solution for the pricing equation, but we can use numerical methods to solve for the price. For example, the bisection method is simple and will not fail. If the function f is monotone and approximately linear around the true price, we can use the interpolation method. As shown in Sections 2.2 and 2.3, the pricing functions do have these features. We first pick two charge rates q_1 and q_2 that satisfy

$$f(q_1)f(q_2) < 0.$$

We then search for the fair value of the charge q^* by linear interpolation between q_1 and q_2.

2.2 Pricing the GMWB in Terms of a Call Option

In this section, we use a call option decomposition approach to price the GMWB. A similar decomposition was also used by Milevsky and Salisbury (2006). In Section 2.2.1, we show how to decompose the contract into a call option, plus an annuity certain, and how to price the call option. In Section 2.2.2, we introduce a control variate technique to compute the price of a floating strike Asian call option. Section 2.2.3 provides numerical examples, and compares our results with those obtained by Milevsky and Salisbury (2006).

The variable annuity and the GMWB together are equivalent to an annuity certain, plus a call option. We assume the policyholder withdraws wh at the end of each time step if the account value is positive. When the account value becomes zero, the policyholder will continue to receive wh at the end of each time step for N times in total. This series of N payments is equivalent to the cash flows from a T-year annuity certain with a periodical payment of wh. At maturity, the policyholder may have a positive account balance

$$A_T = \max\left(B_T, 0\right). \tag{2.5}$$

The balance can be seen as a call option payoff.

The account value A behaves differently from a normal equity investment. To understand this, suppose that the initial account value is invested in a non-dividend-paying stock S ($S_0 = A_0$). A European call option on this stock with a strike price of zero has the same payoff as $\max(S_T, 0)$ at maturity T, which is equal to S_T. Therefore, this call option should have a market value that is equal to the initial stock price S_0 and the initial account value A_0. But for the call option that we are interested in, this is not the case. The market value of the option is certainly less than the initial account value A_0. This is because the annuity account has fees and withdrawals deducted periodically. The fee rate q is similar to a continuous dividend rate on an equity portfolio. But withdrawals are taken by the policyholder at a fixed amount that does not change with the account value.

The present value of the annuity certain is equal to

$$wa_{\overline{N}|}^{(1/h)} = \sum_{i=1}^{N} wh\ e^{-rih} = wh\frac{1-e^{-rT}}{e^{rh}-1}. \tag{2.6}$$

Note that this annuity certain does not depend on the account value. Under the risk-neutral measure Q, the market value of the call option at time zero is given by

$$V_C = E_Q\big[e^{-rT}\max(B_T, 0)\big]. \tag{2.7}$$

All the cash flows from the annuity certain and the call option are financed by the initial premium. From the no-arbitrage pricing theory, the present value of the annuity certain, plus the expected present value of the option payoff, should be equal to the initial account value. We have the following equation:

$$wa_{\overline{N}|}^{(1/h)} + E_Q\big[e^{-rT}\max(B_T, 0)\big] = A_0. \tag{2.8}$$

The charge rate q affects the account value A_t, so the expected present value of the call option V_C is a function of q. The fair value of charge q^* must satisfy the following equation:

$$wa_{\overline{N}|}^{(1/h)} + V_C(q^*) - A_0 = 0. \tag{2.9}$$

2.2.1 The Evolution of the Annuity Account

In this subsection, we model the evolution of the account value to explain the call option decomposition. We will show how to simplify some aspects of the call option formula by using the fact that the returns in each period are assumed to be independently and identically distributed.

Let the accumulation factor on the fund during time interval $\big[(i-1)h,\ ih\big)$ be X_i,

$$X_i = \frac{S_i}{S_{i-1}},\ i = 1, 2, \ldots, N.$$

We first ignore the GMWB fee and consider the guaranteed withdrawal payments only. The fee can be deducted continuously by modifying the drift term of the fund process. If the insurer deducts the fee with different frequencies, then the account value needs to be modified accordingly. When the policyholder withdraws, the insurer sells some units of the fund to make the payment. The total value of funds sold is equal to the amount withdrawn. The annuity account balance right after each withdrawal is expressed as:

$$A_i = \max\left(A_{i-1}X_i - wh, 0\right), \quad i = 1, 2, \ldots, N. \tag{2.10}$$

where $wh = \frac{A_0}{N} = \frac{S_0}{N}$.

The shadow account value has a close relationship with the actual account value as stated in equation (2.5). For the purpose of pricing, it is convenient for us to deal with the shadow account process B. The shadow account value after each withdrawal is given by

$$B_i = B_{i-1}X_i - wh = B_{i-1}X_i - \frac{A_0}{N}, \quad i = 1, 2, \ldots, N. \tag{2.11}$$

Note that the shadow account can be negative. For example, if the withdrawal amount is \$1,000 and the current account balance is \$ 600, then the account value after withdrawal will be set to zero while the insurance company has a liability of \$400. This example can be expressed by the following notation:

$$wh = 1000,$$
$$A_{i-1}X_i = B_{i-1}X_i = 600,$$
$$A_i = 0,$$
$$B_i = -400.$$

The shadow account balance at maturity can be written as:

$$\begin{aligned} B_N &= B_{N-1}X_N - \frac{A_0}{N} \\ &= \big(B_{N-2}X_{N-1} - \frac{A_0}{N}\big)X_N - \frac{A_0}{N} \\ &\vdots \\ &= A_0\prod_{i=1}^{N} X_i - \frac{A_0}{N}(1 + X_N + X_N X_{N-1} + \cdots + \prod_{i=2}^{N} X_i) \\ &= S_0\prod_{i=1}^{N} X_i - \frac{S_0}{N}(1 + X_N + X_N X_{N-1} + \cdots + \prod_{i=2}^{N} X_i). \end{aligned} \tag{2.12}$$

The first term on the right-hand side corresponds to the maturity value of the underlying fund. The second term has the form of an arithmetic average of the fund values, but the random variables are not in the usual order. This order causes no difficulty if we just want to simulate the payoff using Monte Carlo. However, we now show how to replace this expression with a similar one that will be more convenient for use with a control variate. We now define a new vector of independent random variables Y whose elements have the same values as X except with reverse order:

$$
\begin{aligned}
Y_1 &= X_N, \\
Y_2 &= X_{N-1}, \\
&\vdots \\
Y_N &= X_1.
\end{aligned}
$$

That is, random variables Y and X have the same distribution. Then we can treat the random vector Y as another set of accumulation factors on the fund over all the time steps

$$
\begin{aligned}
Y_1 &= \frac{S_1}{S_0}, \\
Y_2 &= \frac{S_2}{S_1}, \\
&\vdots \\
Y_N &= \frac{S_N}{S_{N-1}}.
\end{aligned}
$$

Furthermore, the product

$$\prod_{k=1}^{M-1} X_{M-k}$$

is equal in distribution to the product

$$\prod_{k=1}^{M-1} Y_k, \quad \text{for } 2 \leq M \leq N+1.$$

The second term of equation (2.12) can be rewritten as an average of the fund unit values from time zero to time $T-h$. Thus, the option we are interested in becomes a floating strike price Asian call option. The strike price is the arithmetic average of N fund values from time zero to time $(N-1)h$. The call option is easier to value in terms of this form:

$$\begin{aligned} & E_Q\big[\max(B_N, 0)\big] \\ &= E_Q\Big[\max\big(S_0 \prod_{i=1}^{N} Y_i - \frac{S_0}{N}(1 + Y_1 + Y_1Y_2 + \cdots + \prod_{i=1}^{N-1} Y_i),\ 0\big)\Big] \\ &= E_Q\big[\max(S_N - \frac{1}{N}\sum_{i=0}^{N-1} S_i,\ 0).\big] \end{aligned} \tag{2.13}$$

Note that the account value is reduced by the GMWB charge. If we assume the charge is deducted continuously, then we can modify the return of the fund, and the above derivation still applies. To obtain the fair price of the GMWB, we only need to solve the following pricing equation:

$$A_0 - wa_{\overline{N}|}^{(1/h)} - e^{-rT} E_Q\big[\max(S_N - \bar{S_A}, 0)\big] = 0, \tag{2.14}$$

$$\text{where} \quad \bar{S_A} = \frac{1}{N}\sum_{i=0}^{N-1} S_i.$$

The first two terms in equation (2.14) are constant; the call option price changes with the charge rate q. As the charge rate increases, the account value will be lower, so the call option value will decrease. Suppose that the return on the underlying fund follows a log normal distribution

$$ln\frac{S_T}{S_0} \sim N\left(\mu T, \sigma\sqrt{T}\right),$$

where μ and σ are constant parameters. Under the risk-neutral measure Q, we have $\mu = r - q - \frac{1}{2}\sigma^2$. As far as we know, there is no closed-form formula to compute the price of this call option. We will use a simulation approach to approximate it. Moreover, we can reduce the simulation error by using the control variate method.

2.2.2 The Control Variate Technique

In this subsection, we explain how to compute the call option price using the control variate technique. As one of the variance reduction techniques in Monte Carlo simulation method, this technique has been discussed in many books, for example, Hull (2006); Glasserman (2003).

To price the GMWB, we estimate the price of the floating strike arithmetic Asian option using Monte Carlo simulation. Kemna and Vorst (1990) first proposed to price Asian options based on the geometric average price. Boyle et al. (1997) compared the performances of several variance reduction methods (the antithetic variate, the control variate and the moment matching methods) to that of the traditional Monte Carlo method. They showed by simulation that the control variate method is the most efficient for valuing an Asian call option.

Boyle (1993) explicitly gave the pricing formula for continuous average Asian option. For our problem, the geometric mean strike Asian call option has a payoff at maturity T given by

$$\max\Big(S_N - \bar{S_G},\ 0\Big),$$

where $\bar{S_G} = \prod_{i=0}^{N-1} S_i^{\frac{1}{N}}$. The GMWB fee can be incorporated in the Black-Scholes formula as a continuous dividend. That is, we can deduct the charge rate q from the expected return of the underlying fund. Then the pricing formula for the floating strike geometric Asian call is given by

$$
\begin{aligned}
C_G &= S_0 e^{-qT} N(d_1) - K N(d_2) && (2.15)\\
\text{where} & && (2.16)\\
d_1 &= \frac{\ln \frac{S_0}{K} + (-q + \frac{\sigma_3^2}{2})T}{\sigma_3 \sqrt{T}}, \\
d_2 &= d_1 - \sigma_3 \sqrt{T},
\end{aligned}
$$

The formula is based on the moments of the underlying price and the moments of the geometric average of the asset price. The inputs to the formula are calculated as

$$
\begin{aligned}
K &= S_0 e^{-rT} \mu_2, \\
\sigma_3 &= \sqrt{\Sigma_1^2 + \Sigma_2^2 - 2\rho \Sigma_1 \Sigma_2} = \sigma \Big(\frac{(N+1)(2N+1)}{6N^2} \Big)^{\frac{1}{2}}, \\
\text{where} \quad \Sigma_1 &= \sigma, \\
\Sigma_2^2 &= \frac{1}{T} \ln \left(1 + \frac{\sigma_2^2}{\mu_2^2} \right) = \sigma^2 \frac{(N-1)(2N-1)}{6N^2}, \\
\rho &= \frac{1}{T \Sigma_1 \Sigma_2} \ln \left(1 + \frac{\text{Cov}_{12}}{\mu_1 \mu_2} \right), \\
\text{Cov}_{12} &= \mu_1 \mu_2 \big(\exp \big\{ \frac{\sigma^2}{2} (N-1) h \big\} - 1 \big), \\
\mu_1 &= E\Big[\frac{S_T}{S_0} \Big] = e^{T(r-q)}, \\
\sigma_1^2 &= Var\Big[\frac{S_T}{S_0} \Big] = \mu_1^2 \big(e^{T\sigma^2} - 1 \big), \\
\mu_2 &= E\Big[\Big(\prod_{i=1}^{N-1} Y_i \Big)^{1/N} \Big] = \exp \Big\{ \frac{(N-1)}{2} \mu h + \frac{\sigma^2}{2} \frac{(N-1)(2N-1)}{6N} h \Big\}, \\
\sigma_2^2 &= Var\Big[\Big(\prod_{i=1}^{N-1} Y_i \Big)^{1/N} \Big] = \mu_2^2 \Big(\exp \Big\{ \sigma^2 \frac{(N-1)(2N-1)}{6N} h \Big\} - 1 \Big), \\
\mu &= r - q - \frac{\sigma^2}{2}.
\end{aligned}
$$

For a given value of the GMWB charge rate, we calculate the formula value of the geometric mean strike call option. The second step is to simulate the present values of the geometric mean strike call option $\hat{C}_G$ and the arithmetic mean strike call option $\hat{C}_A$. Suppose we generate M scenarios of fund returns over T years. The value of the arithmetic mean strike call option simulated using Control Variate technique is denoted by C_A^*,

$$C_A^* = \hat{C}_A + \beta(C_G - \hat{C}_G), \tag{2.17}$$

where β can be estimated from the first $M_1(M_1 \ll M)$ simulated option values. We then use the estimated $\hat{\beta}$ and the remaining $(M - M_1)$ simulated option values to get C_A^*. The standard error of C_A^* is much lower than that of $\hat{C}_A$.

We are now ready to solve the pricing equation for q:

$$A_0 - w\, a_{\overline{N}|}^{(1/h)} - C_A^*(q) = 0. \tag{2.18}$$

Note that A_0 and $w\, a_{\overline{N}|}^{(1/h)}$ are constant, the shape of the left hand side of the pricing equation (2.18) is determined by $C_A^*(q)$. From no arbitrage considerations, the price of the floating strike arithmetic Asian option is a decreasing function of the fee rate q. Hence, we should be able to find a unique solution for equation (2.18). There is no explicit expression for the solution q^* since the option value is calculated by simulation and it depends on q. The bisection method and interpolation method can be used.

The Monte Carlo simulation method is accurate but time-consuming. Levy (1992) uses a moment-matching approach to approximate the sum of lognormal random variables with a log-normal random variable. By assuming the average strike price $\bar{S_A}$ is a lognormal random variable, we can use the pricing formula (2.16) to obtain a quick approximation. The inputs are given as follows

$$
\begin{aligned}
K &= S_0 e^{-rT}\mu_2, && (2.19)\\
\sigma_3^2 &= \Sigma_1^2 + \Sigma_2^2 - 2\rho\Sigma_1\Sigma_2, && (2.20)\\
\mu_2 &= E\Big[\frac{\bar{S_A}}{S_0}\Big] = \frac{1}{N}\sum_{i=0}^{N-1} e^{(r-q)ih} = \frac{1-e^{(r-q)T}}{N(1-e^{(r-q)h})},\\
\Sigma_2^2 &= \frac{1}{T}\ln E\Big[(\frac{\bar{S_A}}{S_0})^2\Big] - \frac{2}{T}\ln\mu_2,\\
E\Big[(\frac{\bar{S_A}}{S_0})^2\Big] &= \frac{1}{N^2}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\exp\big\{(r-q)(i+j)h + \sigma^2(i\wedge j)h\big\},\\
\rho\Sigma_1\Sigma_2 &= \frac{1}{T}\ln E\Big[\frac{S_N\bar{S_A}}{S_0^2}\Big] - \frac{1}{T}\ln(\mu_1\mu_2),\\
E\Big[\frac{S_N\bar{S_A}}{S_0^2}\Big] &= \frac{\mu_1}{N}\sum_{i=0}^{N-1}\exp\big\{(\mu+\frac{\sigma^2}{2})ih + \sigma^2 ih\big\}.
\end{aligned}
$$

2.2.3 Numerical Examples

This section applies the method that has been discussed to some simple GMWB examples. The following assumptions are made:

Initial premium	$A_0 = S_0 = 100$,
Risk-free interest rate	$r = 5\%$,
Volatility of fund return	$\sigma = 20\%$,
Guaranteed annual withdrawal rate	$g = 5\%, 6.66\dot{6}\%, 10\%$,
Maturity	$T = 20, 15, 10$,
Length of time interval	$h = 1, 1/4, 1/12$.

For more frequent withdrawals, we divide the maximum annual withdrawal amount into equal amounts according to the length of time interval, then deduct that amount from the account value at the end of each time interval.

The arithmetic mean strike Asian call option is a monotone decreasing function of the GMWB fee rate q. For example, Figure 2.1 shows their relationship for the ten-year contract with monthly withdrawals. The fair price can be solved numerically by a standard function such as *fsolve* in Matlab.

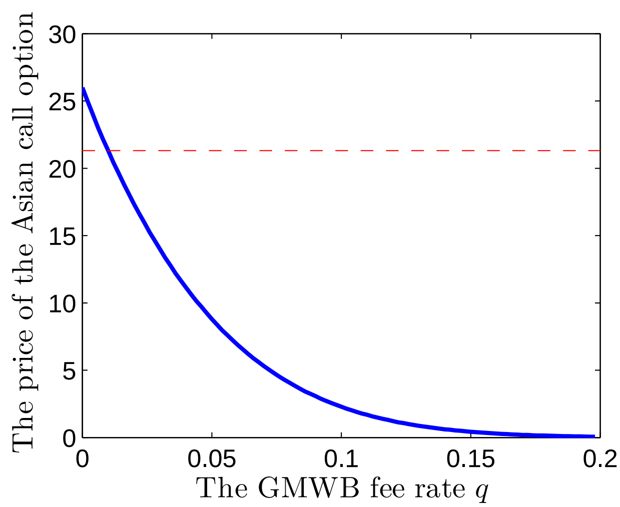

Figure 2.1: The arithmetic mean strike Asian call option price C_A^* vs. the GMWB fee rate q. ($S_0 = 100,\ T = 10,\ w = 10,\ h = 1/12,\ r = 5\%,\ \sigma = 20\%$)

Table 2.1 gives the charge rates for each GMWB computed from Monte Carlo simulation (MC Simu) method and the lognormal approximation (LN Appr) method. The lognormal approximation method is used to get estimates because the simulation method is very time-consuming. As the guaranteed withdrawal rate g rises, the GMWB benefit value increases, and therefore the charge q goes up too. As we can see, the charge rate becomes a little higher as the frequency increases. This is because the present value of the guaranteed withdrawal benefit becomes larger as more values are withdrawn earlier. The standard deviation of the charge rate q, reported in parentheses, is obtained from a sample of q by repeating the pricing process for several times (e.g., 1000). The lognormal approximation method overestimates the left tail of the distribution of the sum of lognormal variables. Therefore, the arithmetic average floating strike Asian call option is overpriced by the lognormal approximation method, and the GMWB fee is underestimated. The error for a 15-year quarterly withdrawal GMWB is about 10 basis points. The error goes up as the maturity T and withdrawal frequency h increase.

GMWB rate	**Maturity**	**Computing Approach**	**the Length of the Time Step**					
			$h = 1$		$h = 1/4$		$h = 1/12$	
g	T (yrs)		$a_{\overline{N}\rceil}$	q (bps)	$a^{(4)}_{\overline{N}\rceil}$	q (bps)	$a^{(12)}_{\overline{N}\rceil}$	q (bps)
5%	20	MC Simu	61.64	27.65	62.82	28.33	63.08	28.49
				(0.05)		(0.05)		(0.05)
		LN Appr		19.95		19.85		19.83
$6.66\dot{6}$%	15	MC Simu	68.61	47.52	69.91	48.89	70.20	49.21
				(0.05)		(0.05)		(0.05)
		LN Appr		38.72		38.95		39.01
10%	10	MC Simu	76.74	92.41	78.20	95.80	78.53	96.63
				(0.06)		(0.06)		(0.06)
		LN Appr		83.15		84.83		85.24

Table 2.1: The fair charge $\boldsymbol{q}$ of the GMWB solved by the call option approach. "MC Simu" represents the Monte Carlo simulation method with control variate technique. We use $\mathbf{10^6}$ scenarios. The lognormal approximation method is denoted by "LN Appr". $\boldsymbol{(r = 5\%,\ A_0 = 100,\ \sigma = 0.2)}$

2.2.4 Comparison with work by Milevsky and Salisbury

Milevsky and Salisbury (2006) provide a similar static pricing approach. One of their assumptions is different from ours. They assume that withdrawals are continuous at rate $w = \frac{A_0}{T}$. Their pricing equation is:

$$e^{-rT} E_Q\Big[\max\big(0, S_T - \frac{S_0}{T}\int_0^T \frac{S_T}{S_t} dt\big)\Big] + \frac{w}{r}(1 - e^{-rT}) = S_0. \tag{2.21}$$

We assume discrete withdrawals, so our pricing equation is:

$$e^{-rT} E_Q\Big[\max\big(0, S_N - \frac{1}{N}\sum_{i=0}^{N-1} S_i\big)\Big] + \frac{w(1 - e^{-rT})}{\frac{e^{rh}-1}{h}} = S_0. \tag{2.22}$$

In continuous time, our pricing equation becomes:

$$e^{-rT} E_Q\Big[\max\big(0, S_T - \frac{1}{T}\int_0^T S_t dt\big)\Big] + \frac{w}{r}(1 - e^{-rT}) = S_0. \tag{2.23}$$

The only difference between expressions (2.21) and (2.23) is the second term in the big square bracket. We can show that they are equal. Under the risk-neutral measure Q, we can write:

$$\frac{S_T}{S_t} = \exp\{-(r - \frac{\sigma^2}{2})(T-t) + \sigma(W_T - W_t)\}. \tag{2.24}$$

Using the independence property of Brownian motion we have:

$$\begin{aligned} \frac{S_0}{T}\int_0^T \frac{S_T}{S_t} dt &= \frac{S_0}{T}\int_0^T \exp\{-(r - \frac{\sigma^2}{2})(T-t) + \sigma W_{T-t}\} dt, \\ &= \frac{1}{T}\int_0^T S_0 \exp\{-(r - \frac{\sigma^2}{2})u + \sigma W_u\} du, \quad \text{let} \quad u = T - t, \\ &= \frac{1}{T}\int_0^T S_u du. \end{aligned} \tag{2.25}$$

Hence, our continuous pricing equation is the same as that used by Milevsky and Salisbury (2006). The solutions of the equations should also be the same.

In Table 2.2, we compare our pricing results, q_Y, with those obtained by Milevsky and Salisbury (2006), q_{MS}. Under the same interest rate and volatility assumptions, we simulate in discrete time with monthly time steps. This should be very close to the continuous case. However, our numerical results are significantly lower than those obtained by Milevsky and Salisbury (2006). For a 5% GMWB with a 20% volatility, our price is 28.5 basis points, and their price is 37 basis points. They argue that the current market price, between 30 and 50 basis points, for a typical 7% GMWB is not sufficient to cover the hedging cost.

GMWB MAWA	**Maturity**	**Annuity Certain**	**Annuity Certain**	**Fund Volatility**			
				$\sigma = 20\%$		$\sigma = 30\%$	
g	T(yrs)	**discrete**	**continuous**	q_Y	q_{MS}	q_Y	q_{MS}
5%	20	63.08	63.21	28.51	37	76.54	90
6%	$16.6\dot{6}$	67.71	67.85	40.61	54	103.68	123
7%	14.28	71.34	71.46	53.78	73	132.25	158
10%	10	78.53	78.69	96.65	140	221.2	271

Table 2.2: **Comparison of the GMWB charge rates q with Milevsky-Salisbury's results (static full withdrawal utilization) ($r = 5\%,\ A_0 = 100,\ h = 1/12$)**

To further investigate the problem, we examine how the price changes with the length of time step. Increasing the number of time steps per annum will approximate the continuous case. Table 2.1 shows that the price will move up a little bit when the length of time step is shortened. For example, Table 2.3 gives the convergence pattern of the price for the 10% GMWB with a 20% volatility. As we increase the number of

Steps per annum	Charge rate	Annuity
$\frac{1}{h}$	q	a
12	96.6501	78.5300
100	97.0496	78.6742
2000	97.1693	78.6929
4000	97.2781	78.6934

Table 2.3: Approximating the price for continuous withdrawal $(r = 5\%,\ A_0 = 100,\ \sigma = 0.2,\ w = 10\%,\ T = 10)$

steps, the price rises very slowly. Based on these results, we can say that the price under the continuous withdrawal assumption will not reach the 140 basis points that is given by Milevsky and Salisbury (2006). This difference indicates inconsistency. On this basis, we conclude that the numbers obtained by Milevsky and Salisbury (2006) are on average 28% too high. We will use a different approach to value the GMWB rider in the next section, and we find that under this approach we obtain the same prices as we did in this section. Recently, Blamont and Sagoo (2009) use the same decomposition approach and give the fair prices that are close to our results under the same assumptions.

2.3 Pricing the GMWB as a Put Option

In this section, we present an alternative pricing approach for the GMWB. We will use it to check the previous call option approach. The GMWB resembles a put option in providing downside protection. Different from the standard option, the GMWB has a random payoff time and a price that is deducted from the account continuously. We will explain how to compute the benefit payoff value and the charge value by Monte Carlo simulation. The advantage of this put approach over the call option approach is its flexibility to deal with any withdrawal strategy, including dynamic withdrawals and lapses. On the other hand, we are not able to employ the control variate technique to simulate so efficiently as we did with the call option approach.

Denote the maturity of the annuity contract by T. The maturity is usually the annuitization date. It is often different from the length of the payment period of the GMWB. Suppose the policyholder makes equal withdrawals wh at the end of each time step. Note that the annuity account value will not reach zero unless withdrawals are made, so this only occurs at the end of a time step. For simplicity, we use the number of time step as the subscript to denote the account value at the end of that time step. Define a random variable k^* as the time step at the end of which the GMWB payoff is triggered:

$$k^* = \begin{cases} k, & \text{if} \quad A_{k-1} > 0, \quad A_k = 0, \ 0 < k \leq N \\ N, & \text{if} \quad A_N > 0. \end{cases} \tag{2.26}$$

The shadow account value B, defined in Section 2.1, is equal to the actual account value A when A is positive. But B can be negative when A changes to zero. This is because we continue to deduct withdrawals from B, but A will not change once it is set to zero. Let G denote the present value of the remaining guaranteed benefit amount. Initially, there are $N = \frac{T}{h}$ guaranteed benefit payments of wh. At time zero, we have:

$$G_0 = \sum_{i=1}^{N} whe^{-rhi}. \tag{2.27}$$

At the end of each time step, the guaranteed benefit amount is reduced by the withdrawal amount wh,

$$G_i = wh\sum_{j=1}^{N-i} e^{-rjh}, \quad i = 1, \cdots, N-1, \tag{2.28}$$
$$G_N = 0.$$

Then, at the end of time step k^*,

$$G_{k^*} = wh \sum_{i=1}^{N-k^*} e^{-rih}. \tag{2.29}$$

The payoff of the GMWB option at time k^*h can be expressed as:

$$\max\big(0, G_{k^*} - B_{k^*}\big).$$

If the policyholder delays annuitization, the account value at maturity may be larger than the withdrawal amount and there is guaranteed benefit remaining, and then the payoff is:

$$\max(0, G_N - A_N).$$

Under the risk-neutral measure Q, we can calculate the present value of the GMWB benefit as:

$$\begin{aligned} V_B(q) &= E_Q\left[e^{-rk^*h}\max(0, G_{k^*} - B_{k^*})\mathbf{1}_{\{0<k^*\leq N\}}\right] \\ &\quad + E_Q\left[e^{-rT}\max(0, G_N - B_N)\mathbf{1}_{\{G_N>0, B_N>0\}}\right]. \end{aligned} \tag{2.30}$$

Since the charge rate q affects the account value, V_B is a function of q.

We now discuss how to obtain the charge value. Recall that the charges are levied at a rate q on the account value as long as it is positive. For a stock S paying a continuous dividend at rate q, we know that the expected present value of future dividends is equal to $(S_0 - S_0 e^{-qT})$. But for the annuity account, the present value of the continuous charges cannot be calculated using this formula since there are discrete withdrawals deducted over time. Assume withdrawals are deducted at the end of each time period. Then the account value behaves the same as the stock price with dividends within each unit period. The present value at time $(i-1)h$ of the expected charges in the period $\big[(i-1)h,\ ih\big)$ is equal to $A_{i-1}(1-e^{-qh})$. The expected present value of the total charges is expressed as:

$$V_C(q) = E_Q\Big[\sum_{i=1}^{k^*} e^{-r(i-1)h} A_{i-1}(1-e^{-qh})\Big]. \tag{2.31}$$

The fair charge rate q^* should be set at a level such that the total charges can exactly cover the payoff of the GMWB. The charge rate has a direct impact on the total value of charges. The GMWB benefit value is affected by the charge rate because the account value is reduced by the charges. Hence, we have the following pricing equation:

$$V_C(q^*) - V_B(q^*) = 0. \tag{2.32}$$

The total value of charges, V_C, is an increasing function of the charge rate q. A higher fee rate results in a lower account value, which makes the GMWB more valuable. Thus, the value of the GMWB benefit V_B increases as q increases. The total value of charges V_C is more sensitive to the fee rate q than the value of benefit. Based on our simulation results, V_B goes up slower than V_C does as shown in Figure 2.2.

To find the fair price, we use numerical methods, such as the bisection method or interpolation method. Pick a fee rate q_1 that satisfies $V_B(q_1) > V_C(q_1)$, and a higher

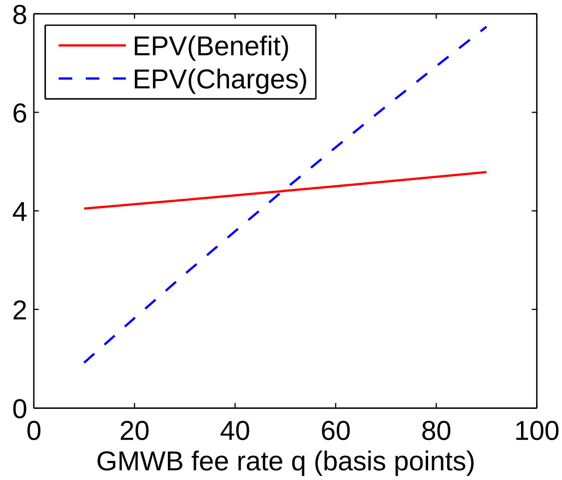

Figure 2.2: The expected present values of GMWB benefit and charges vs. the GMWB fee rate q. $(S_0 = 100,\ T = 15,\ w = 6.6667,\ h = 1/4,\ r = 5\%,\ \sigma = 20\%)$

value q_2 that satisfies $V_B(q_2) < V_C(q_2)$. It is important to use the same random numbers in these simulations. This will increase the correlation between V_C and V_B, and thus reduce the variance of the $(V_C - V_B)$. Assuming an approximately linear relation between the present values and the fee rates within a small interval, we can estimate the fair price q^* by interpolating the two rates. Let $V_{B1} = V_B(q_1),\ V_{B2} = V_B(q_2),\ V_{C1} = V_C(q_1)$, and $V_{C2} = V_C(q_2)$, then we have

$$q^* = q_1 + \frac{(V_{B1} - V_{C1})(q_2 - q_1)}{V_{B1} - V_{C1} + V_{C2} - V_{B2}} \tag{2.33}$$

2.4 Consistency between the two Pricing Approaches

In the variable annuity contract with the GMWB, we are interested in four values:

1. V_C, the expected present value under the risk-neutral measure Q of the total GMWB charges;
2. V_B, the expected present value under the risk-neutral measure Q of the GMWB put option benefit;
3. V_W, the expected present value under the risk-neutral measure Q of the withdrawal amount taken by the policyholder before the account value becomes zero;
4. $E_Q[e^{-rT}A_T]$, the expected present value under the risk-neutral measure Q of the account balance at maturity.

The insurance company charges an ongoing fee for the GMWB, and pays the GMWB put option benefit out. The net present value to the insurance company is equal to the expected present value under the risk-neutral measure Q of the cash inflows minus that of cash outflows.

$$NPV_{ins} = V_C - V_B$$

The only cash outflow for the policyholder is the initial investment S_0. The policyholder takes withdrawals before the account value is set to zero. If the account balance becomes zero, the policyholder is guaranteed to receive the remaining GMWB benefit payments. The policyholder also gets the account balance at maturity if it is positive. These are cash inflows for the policyholder. The net present value to the policyholder is

$$NPV_{pol} = V_B + V_W + E_Q\left[e^{-rT}A_T\right] - S_0$$

Note that the charges have been reflected in the account values. If the policyholder starts to withdraw the maximum annual withdrawal amount from the first year (i.e. full utilization), then the withdrawal value, plus the GMWB put benefit, $V_B + V_W$, is equal to the annuity certain defined in Section 2.2.

Under the equilibrium condition, both the net present values to the insurer and the policyholder should be equal to zero. The call option pricing approach ensures the net value to the policyholder is zero. The put option pricing approach lets the net value to the insurer be zero. Hence, they should give the same price.

In theory, the put option approach and the call option approach should give the same results under the same assumptions. Through comparing the results under static withdrawal assumption, we are able to cross check the two approaches. We recalculate the GMWB examples given in Section 2.2.3 using the put option approach. Table 2.4 lists the fair charge rates for all the GMWB examples. The results are the same as those in Table 2.1 which are solved by the call option approach except for minor random errors.

GMWB	**Term**	**Time step**					
rate		$h = 1$		$h = 1/4$		$h = 1/12$	
g	T (yrs)	q (bps)	**Put**	q (bps)	**Put**	q (bps)	**Put**
5%	20	27.65 (0.02)	3.55	28.32 (0.02)	3.53	28.49 (0.02)	3.53
$6.66\dot{6}$%	15	47.51 (0.04)	4.41	48.90 (0.04)	4.36	49.20 (0.04)	4.34
10%	10	92.44 (0.07)	5.50	95.85 (0.08)	5.37	96.65 (0.08)	5.34

Table 2.4: **The GMWB Fee Rates q solved by the put option approach ($r = 5\%$, $\sigma = 20\%$, $A_0 = 100$).**

2.5 Summary

Milevsky and Salisbury (2006) argued that insurers underprice GMWBs in variable annuities. We employ a similar decomposition approach to price the GMWB. In the discrete time, the variable annuity with the GMWB is equivalent to an annuity certain, plus a floating strike arithmetic Asian call option. The price of the Asian call option can be computed efficiently by using the control variate technique. The control variate is the price of the corresponding geometric Asian call option which has a closed-form solution. The GMWB prices we obtained are comparable to the prices in the market.

The GMWB can also be priced as a put option with a random expiration date. The payoff time is the time when the account value is not sufficient to pay the withdrawal amount or the guaranteed benefit is depleted. The payoff is equal to the present value of the remaining future guaranteed benefit payments. We simulated the expected present values of the GMWB benefit and charges, and searched for the price that makes the benefit equal to the charge. Our numerical results show consistency between the two pricing approaches.

Chapter 3

Semi-static Hedging for GMWBs

In this chapter, we propose a semi-static approach to hedge GMWBs. The net liability of a GMWB at a future time can be replicated by a portfolio of short-term European put options. In this chapter, we assume the portfolio weights are static over the life of the replicating options. That is, we only rebalance the portfolio when the short-term options expire. But we will relax this assumption later in Chapter 4.

This semi-static approach offers several advantages over a dynamic hedging strategy. Dynamic hedging here is referred to as a technique to hedge delta, gamma or vega exposures of a financial derivative (see Hull 2006). A dynamic strategy is based on the assumed ability to trade continuously. However, it is impossible to rebalance the portfolio continuously. In practice, a dynamic hedging strategy has to balance transaction cost and quality of the hedge.[1] In times of high volatility, dynamic hedging is expensive. If there are random jumps in the underlying price, then the market is incomplete and dynamic hedges often result in large errors. In contrast, the semi-static strategy we are

[1]Several hedging techniques have been developed, such as Risk-Minimizing hedging (see Møller 1998), Mean-Variance hedging (see Föllmer and Sondermann 1986; Föllmer and Schweizer 1991; Duffie and Richardson 1991; Schweizer 1992, 2001), and Quantile hedging (see Föllmer and Leukert 1999).

proposing requires fewer portfolio adjustments, and random jumps in the underlying price will not affect its performance. Semi-static hedging also reduces the risk of severe liquidity events. Execution errors and operational risk can also be mitigated.

In the limit, as the length of the hedging interval for semi-static hedging decreases to zero, semi-static hedging will converge to dynamic hedging. For longer time intervals, the two hedging strategies behave differently. Based on our implementation, the hedging errors from semi-static hedging can be small even for time intervals as long as one year. In contrast, dynamic hedging requires more frequent position adjustments to avoid large discrepancies. For other models or instruments, the rebalancing intervals must be determined using numerical experiments, but this is also true for dynamic hedging.

Most literature on static hedging focuses on European options or barrier options. Derman et al. (1994) introduces a static replicating approach to hedge barrier options in a binomial tree model using standard options with varying maturities. Bowie and Carr (1995), Carr and Chou (1997), and Carr et al. (1998) develop static hedges for barrier options in the Black-Scholes model using options with the same maturity, but with multiple strikes. The method relies on a relationship between European puts and calls with different strike prices. Carr and Wu (2004) propose a static hedging strategy for European options. It is based on a spanning relation between the value of a given option and the values of a continuum of short-term options. They show that static hedging is robust to model misspecification. Allen and Padovani (2002) extend the Derman-Ergener-Kani approach to achieve greater model independency. The algorithm searches for a portfolio of vanilla options which minimize hedging errors under scenarios of the future stock prices and volatilities. Andersen et al. (2002), Fink (2003), Nalholm and Poulsen (2006), and Giese and Maruhn (2007) extend static hedging for barrier options to general asset dynamics with jumps and stochastic volatilities. Cassano (2001) considers how close one can get to a perfect hedge using just a small number of options. He shows that under standard assumptions, it only takes a handful of different options to achieve quite good replication.

Given the long-term feature of GMWBs, a perfect static hedging strategy is not available or can be too expensive. The current standard approach in the industry is dynamic hedging with daily rebalancing intervals. As discussed above, the semi-static hedging strategy would be a very good alternative. GMWBs give rise to mild path dependency. Any type of path dependency complicates the application of semi-static hedging. In this chapter, we suggest a way to handle this path dependency in semi-static hedging. Its implementation is based on Monte Carlo simulations, so it can be applied to other path-dependent options too. In Section 3.1, we show step by step how to compute the hedging target and how to construct the hedging portfolio. In Section 3.2, we compare the performance of semi-static hedging and dynamic hedging under both the Black-Scholes model and the Kou's jump-diffusion model.

3.1 Implementation of the Semi-static Hedging Strategy

We develop the semi-static hedging strategy step by step in this section. First of all, let us clarify what needs to be hedged in GMWBs. If the GMWB fee rate is fairly set, the GMWB put option and the total charges should have the same market value at issuance. As time goes by, the two values may be different. For example, when the fund performs well during time interval $(0, t)$, the insurer receives higher charges without incurring the GMWB option payoff. The expected value of future charges becomes larger than the GMWB put option value at time t. That is, the insurer will have a profit. On the other hand, if the fund performs poorly, the expected value of all future charges would be less than the GMWB option value. To prevent a loss, insurers need to hedge the GMWB option value that exceeds the charge value. We call it the expected GMWB net liability.

The idea of semi-static hedging is to set up a portfolio of standard options at time zero to replicate the expected GMWB in a short time period, and then update the hedging portfolio for another period. It involves the following steps:

1. **Model the underlying funds using market indexes.** Each underlying fund can be mapped to several equity and bond indices using regression. For simplicity, we assume an equity index can successfully mimic the fund unit value, and the fund return is equal to the index return minus the GMWB fee rate. That is, we assume that the GMWB fee is deducted from the fund, and the fund return is actually the net return (ignoring all other charges). For a particular trajectory of index returns within the hedging horizon, we can calculate the annuity account values assuming deterministic withdrawal amounts and times.

2. **Obtain the expectation of future GMWB net liability conditional on the index value at a future time**, such as in one year. For a given future index value, there will be different possible annuity account values, because the account value is path-dependent due to withdrawals. For example, assume the initial index value is $I_0 = 100$. Denote the fund value by S and the account value by A. After one year, the index value is assumed to be $I_1 = 100$. Let us consider two possible index values in the middle of the year, 120 and 80. Suppose the

GMWB fee, q, is 50 basis points deducted continuously. Assume the policyholder withdraws 10 from the account in half a year. The fund values and account values are given by

$$S_{\frac{1}{2}} = I_{\frac{1}{2}} e^{-\frac{q}{2}}, \qquad S_1 = I_1 e^{-q} = 99.50,$$
$$A_{\frac{1}{2}} = S_{\frac{1}{2}} - 10, \qquad A_1 = A_{\frac{1}{2}} \frac{S_1}{S_{\frac{1}{2}}}.$$

The two paths will lead to two different account values as illustrated in Figure 3.1. The expected future GMWB net liability at time 1 will have two different

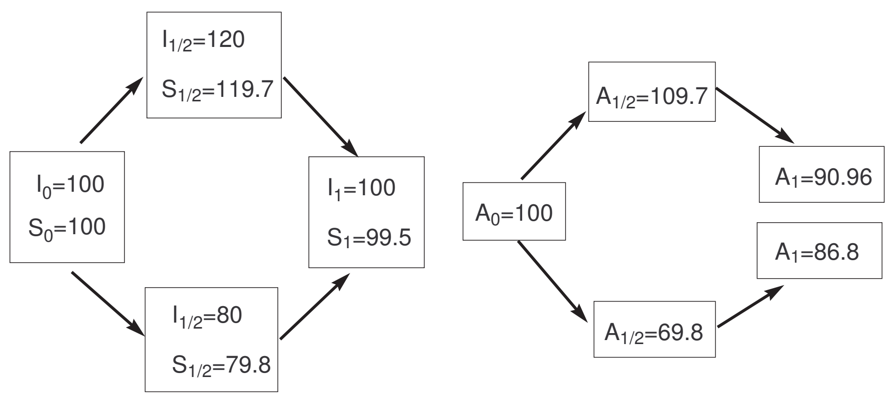

Figure 3.1: Illustration of different account values conditional on a particular index value. Assume the policyholder withdraws 10 from the account in the middle of the year. We have assumed that the GMWB fee is 50 basis points deducted continuously.

values given the two account values. This path-dependent feature cannot be perfectly replicated with standard options. However, if we approximate the expected future GMWB net liability with a variable that is not path-dependent, then replicating the approximation is feasible. As we demonstrate below, an approximation is provided by the conditional mean of the expected GMWB net liability given the index value at the end of the hedging period. That is, we are hedging the conditional expectation of the expected GMWB net liability as a function of the index value at the end of the hedging period. In fact, the approximation error is quite small for a hedging period of one year. To simplify the exposition, later in the text we shall refer to the conditional expectation of the expected GMWB net liability as the conditional expected GMWB net liability.

3. **Search for a portfolio of put options on the index to replicate the expected GMWB net liability.** Because GMWBs are put options, the conditional expected GMWB net liability is a convex function of the index value and can be replicated by a linear combination of several European put option payoff functions. These put options have different strike prices. The optimal weights are obtained by minimizing an appropriate risk measure with some constraints. For example, we can minimize the sum of squared differences between the portfolio payoff and the conditional expected GMWB net liability, subject to the condition that the cost of the replicating portfolio is not larger than the present value of the expected GMWB net liability.

4. **When the replicating portfolio expires, repeat Steps 2-3 based on the new index value and account value** to set up another replicating portfolio. GMWBs typically last more than 10 years. At the end of the first hedging period, the replicating portfolio may generate payoffs, and there may be GMWB claims. The balance is kept in a bank account. Based on the updated index value and account value, we will purchase a new portfolio of put options whose weights are determined by the procedures described in Steps 2 and 3.

We will first illustrate the concept of semi-static hedging using the same set of assumptions as in Chapter 2. That is, the interest rate is assumed to be constant;

the fund value follows a univariate lognormal distribution with a constant volatility; the withdrawal amount per annum is equal to the maximum allowed amount without penalty; and there are no lapse or mortality decrements.

In Step 2, the expected GMWB net liability is calculated under the risk-neutral measure. For a given value of the index at the end of the hedging period, the conditional expected GMWB net liability is not affected by a change of measure. This is because the drift of the conditional path of the fund value is determined by the fixed terminal value rather than the risk-neutral rate of return. Therefore, it is not necessary to use the real-world measure for our semi-static hedging strategy as long as we choose a suitably large range of possible index values at the end of the hedging period. The size of this range can be determined such that the real-world probability that the future index value falls in the range is close to one.

In Step 3, the replicating portfolio can be obtained using the Least Squares method, where the sum of squared replicating errors at selected future index values is minimized. If the distribution of future index values under the real-world measure is known, we could also use Weighted Least Squares method. However, we do not know the true distribution in the real world. The risk-neutral measure usually assigns more weights to lower index values than the real world measure does. A lower index value typically leads to a higher expected GMWB net liability. Hence, we will use the risk-neutral measure to obtain a conservative hedging portfolio for a plain vanilla GMWB. Note that using the risk-neutral measure is not conservative for GMWBs with a step-up option. In order to be conservative in this case, we may over-estimate the index rate of return and put more weight on higher index values.

To compute efficiently the conditional expected GMWB net liability in Step 2, in

Section 3.1.1, we use a mathematical model called the Brownian bridge. By definition, a Brownian bridge is a Brownian motion conditioned on its end value. Because of our assumption that a geometric Brownian motion provides an adequate description of the dynamics of the index value, a Brownian bridge is the right model for the conditioned process and will allow us to find the conditional expected GMWB net liability. In particular, by constructing Brownian bridges, we can easily generate a desired number of paths of the fund value conditional on the same index value at the end of the hedging period.

In the following sections we provide more detailed description of the proposed method. In particular, in Section 3.1.3, we derive the replicating target, which is equal to the conditional expected GMWB net liability. Step 3 of the proposed method is discussed in Section 3.1.4, where we list four approaches to obtain a replicating portfolio. In Section 3.1.5, we discuss how to implement the semi-static hedging strategy in multiple periods and how to modify the strategy by using put options whose maturities are shorter or longer than the hedging period.

3.1.1 Conditional paths of the fund value

In Step 2 of the proposed hedging method, we are facing the problem of generating paths of fund values given prescribed values at the beginning and end of a time period. The fund value process is assumed to solve the following stochastic differential equation under the risk-neutral measure:

$$dS_t = (r - q)S_t dt + \sigma S_t dW_t, \tag{3.1}$$

where $\{W_t\}$ is a standard Brownian motion. The problem is to simulate S_s, conditional on S_0 and S_t, where $s \in (0, t)$. The problem is not difficult if the joint distribution of S_s and S_t is explicitly known. Under the Black-Scholes model, the problem can be solved by constructing a Brownian bridge, which is also called a tied down Brownian motion (see Karlin and Taylor 1981; Glasserman 2003; Shreve 2004). In the next chapter, we assume the volatility of the fund value to be stochastic. In this case, the joint distribution is not known, so the simple Brownian bridge method will not work. We will use a simulation method that is based on a recently developed acceptance-rejection sampling method for diffusion processes.

Brownian Bridge

The Brownian bridge is a useful tool in parameter estimation of diffusions and implementation of variance reduction techniques (see Glasserman 2003). By definition, a Brownian bridge is a standard Brownian motion conditioned on the initial state and the end state of the process. The conditional distribution of the Brownian motion is known to be a normal distribution. Note that the increments of this process are no longer independent. A brownian bridge can also be characterized as the solution of the following stochastic differential equation (Karlin and Taylor 1981):

$$dY_t = -\frac{Y_t - y}{1-t}dt + dW_t, \quad 0 \le t < 1. \tag{3.2}$$

Shreve (2004, p. 175-8) provides another representation of the Brownian bridge

$$Y_t = (1-t)\int_0^t \frac{dW_s}{1-s} + ty, \quad 0 \le t < 1. \tag{3.3}$$

Simulating a Brownian bridge in discrete time amounts to filling in the intermediate values when the endpoints are known. Given the values of a Brownian motion at time t and $t+\Delta$, the value at times $t+\lambda\Delta$, $0<\lambda<1$, is normally distributed with mean (Glasserman 2003, p. 83)

$$E\left[Y_{t+\lambda\Delta} \mid Y_t, Y_{t+\Delta}\right] = \lambda Y_{t+\Delta} + (1-\lambda)Y_t, \tag{3.4}$$

and variance

$$\Delta\lambda(1-\lambda). \tag{3.5}$$

Therefore, the following sample can be simulated from this conditional distribution:

$$Y_{t+\lambda\Delta} = \lambda Y_{t+\Delta} + (1-\lambda)Y_t + \sqrt{\Delta\lambda(1-\lambda)}\, z, \tag{3.6}$$

where $z \sim N(0,1)$, and z is independent of Y_t and $Y_{t+\Delta}$.

Monte Carlo simulation of conditional paths of the fund value

We now explain how the Brownian bridge construction can be used to simulate conditional paths of the fund value for the purpose of finding a static hedging portfolio. In this chapter, we assume withdrawals are taken quarterly, so it is convenient to use a quarterly time step to simulate Brownian bridge paths. Denote the length of the time step by $h=\frac{1}{4}$. Let I and S denote the index value and the fund value, respectively. We assume that under the risk neutral measure the index value follows a geometric Brownian motion with drift r, volatility σ, and the initial value I_0. Assume the hedging horizon is one year, $T_h=1$. Therefore, the index values on a path conditional on the terminal value I_1 are given by

$$I_{ih} = I_0 \exp\{\sigma Y_{ih} + (r - \frac{\sigma^2}{2})ih\},\ i = 1, 2, 3. \tag{3.7}$$

A conditional path of the index value can be constructed using a path of Brownian bridge Y_t. For a given index value in one year, I_1, the end value of the Brownian motion, Y_1, is fixed at

$$Y_1 = \frac{1}{\sigma}\big(\ln\frac{I_1}{I_0} - (r - \frac{\sigma^2}{2})\big). \tag{3.8}$$

Once we have determined Y_1, we may sample the components of (Y_h, Y_{2h}, Y_{3h}) in any order. The first intermediate value is sampled by conditioning on Y_1 and $Y_0 = 0$. The rest of the points are sampled by conditioning on the two closest points already sampled. In our implementation, we sample (Y_h, Y_{2h}, Y_{3h}) in the order of time. First, we generate three independent standard normal random variables $z_1,\ z_2,\ z_3$. Then, we set $t = 0,\ \Delta = 4h, \lambda = \frac{h}{\Delta} = \frac{1}{4}$ and plug them into equation (3.6) to obtain

$$\begin{aligned} Y_h &= \lambda Y_1 + \sqrt{(1-\lambda)h} z_1 \\ &= \frac{1}{4} Y_1 + \frac{\sqrt{3h}}{2} z_1. \end{aligned} \tag{3.9}$$

Conditioning on Y_h and Y_1, we sample Y_{2h} by updating $t = h, \Delta = 3h, \lambda = \frac{1}{3}$.

$$\begin{aligned} Y_{2h} &= \frac{1}{3} Y_1 + \frac{2}{3} Y_h + \frac{\sqrt{6h}}{3} z_2 \\ &= \frac{1}{2} Y_1 + \frac{\sqrt{3h}}{3} z_1 + \frac{\sqrt{6h}}{3} z_2. \end{aligned} \tag{3.10}$$

Similarly, Y_{3h} is sampled by conditioning on Y_{2h} and Y_1. We plug in $t = 2h, \Delta = 2h, \lambda =$

$\frac{1}{2}$ to get

$$
\begin{aligned}
Y_{3h} &= \frac{1}{2}Y_1 + \frac{1}{2}Y_{2h} + \frac{\sqrt{2h}}{2}z_3 \\
&= \frac{3}{4}Y_1 + \frac{\sqrt{3h}}{6}z_1 + \frac{\sqrt{6h}}{6}z_2 + \frac{\sqrt{2h}}{2}z_3. \qquad (3.11)
\end{aligned}
$$

The fund return is assumed to be equal to the index return less the GMWB fee rate. From equation (3.7), we can express the fund values in terms of the simulated Brownian bridge Y:

$$
\begin{aligned}
S_0 &= I_0, \\
S_{ih} &= I_{ih}e^{-qih} \qquad (3.12) \\
&= S_0 e^{(\mu - q - \frac{\sigma^2}{2})ih + \sigma Y_{ih}}, \quad i = 1, 2, 3. \qquad (3.13)
\end{aligned}
$$

3.1.2 The Account Values

In Section 3.1.1, we have explained how to generate paths of fund values with the same end index value by constructing Brownian bridges. The account value is dependent on the path of the fund value. In this subsection, we calculate the account values based on the simulated paths given that the end value of the index is fixed. Based on the account value at a given index value, in Section 3.1.3, we will compute the conditional expected GMWB net liability.

Assume the maximum annual withdrawal amount is w, $w = \frac{A_0}{T}$. An amount of wh is withdrawn from the account at the end of each time step. We assume that the account is initially invested in one unit of the fund, that is, $A_0 = S_0$. After the first withdrawal at time h, $\frac{wh}{S_h}$ units of fund are sold, and there are $(1 - \frac{wh}{S_h})$ units of fund remaining in the account. Similarly, at time ih, $i = 2, 3, 4$, $\frac{wh}{S_{ih}}$ units of fund will be deducted from the account. The shadow account value immediately after the withdrawal can be expressed in terms of the fund values:

$$\begin{aligned}
B_h &= S_h\big(1 - \frac{wh}{S_h}\big), \\
B_{2h} &= S_{2h}\big(1 - \frac{wh}{S_h} - \frac{wh}{S_{2h}}\big), \\
B_{3h} &= S_{3h}\big(1 - \frac{wh}{S_h} - \frac{wh}{S_{2h}} - \frac{wh}{S_{3h}}\big), \\
B_1 &= S_1\big(1 - \frac{wh}{S_h} - \frac{wh}{S_{2h}} - \frac{wh}{S_{3h}} - \frac{wh}{S_1}\big).
\end{aligned} \tag{3.14}$$

The account values immediately after withdrawals are given by

$$A_{ih} = \max\big(0, B_{ih}\big), \; i = 1, 2, 3, 4. \tag{3.15}$$

During the first policy year, the account value is unlikely to reach zero because the withdrawal amount is very small compared with the account value.

The account value is positively correlated with the intermediate fund values. Given the same index value at the end of the hedging horizon, a higher end account value corresponds to a concave-shaped path where the intermediate fund values sit above the straight line that connects the initial and the end fund values; and a lower end account value comes from a convex-shaped path where the intermediate fund values lie below the line. (see Figure 3.2) This is because fewer units of the fund need to be sold when the fund value is high at the time of withdrawal.

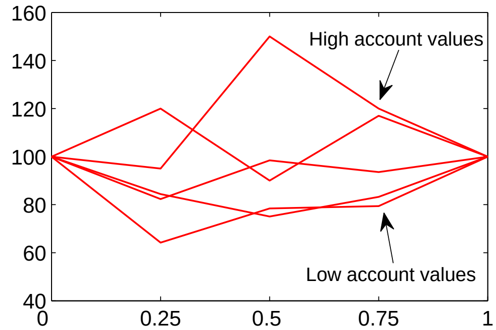

Figure 3.2: Sample paths of the fund value given that both the initial and end fund values are 100.

3.1.3 The Expected GMWB net liability Conditional on the Index Value

Our semi-static hedging strategy is to search for a portfolio of standard put options whose payoff closely replicates the expected GMWB net liability. We denote the expected GMWB net liability by l. At a future time T_h, l depends both on the future fund value and on the path arriving at that value. But the payoff of a standard put option is not path-dependent. In order to obtain the optimal portfolio weights, we choose to replicate the conditional expected GMWB net liability. In this section, we discuss methods of calculating this value.

We have explained how to compute the GMWB benefit and charges in Section 2.3. From equations (2.30) and (2.31), the values of the GMWB benefit and charges at the end of the hedging horizon (i.e. time T_h) under a risk neutral measure are given by

$$b = E^Q_{T_h}\Big[\Big(\frac{wh(1-e^{-r(T-k^*h)})}{e^{rh}-1} - B_{k^*h}\Big)e^{r(T_h-k^*h)}\Big], \quad (3.16)$$

$$c = E^Q_{T_h}\Big[\sum_{u=0}^{k^*} A_{uh}(1-e^{-qh})e^{r(T_h-uh)}\Big], \quad (3.17)$$

where k^* is the time step corresponding to the GMWB payoff, and is defined by the equation (2.26). For a specific path that is associated with a given index value, the GMWB put payoff may be incurred within the hedging horizon. This is very unlikely during the first policy year, but it may happen during a future hedging period. If this occurs, we will accumulate the values of benefit and charges to the end of the hedging period. The expected GMWB net liability at the end of hedging period is defined as the difference of the two values:

$$l = b - c. \tag{3.18}$$

Initially, the fee rate of the GMWB, q, is set to satisfy

$$l = 0, \quad \text{if} \quad T_h = 0.$$

For a given index value I_{T_h} at time T_h, the expected net liability l depends on the path of account values. The account values are determined by the current and previous fund values. Thus, we can express l as a function of the path of fund values and the index value at time T_h, $(S_h, S_{2h}, \cdots, S_{T_h-h}, I_{T_h})$. Denote it by $l(I_{T_h}, D)$, where D is the discrete path, $D = (S_h, S_{2h}, \cdots, S_{T_h-h})$.

The hedging instruments we are going to use are standard put options, whose payoffs depend on the index value at time T_h only. For the purpose of the semi-static hedging, we would like to approximate $l(I_{T_h}, D)$ by a function of I_{T_h} only. It is well known, that the conditional expectation of $l(I_{T_h}, D)$ given I_{T_h},

$$L(I_{T_h}) = E\big[l(I_{T_h}, D) \mid I_{T_h}\big], \tag{3.19}$$

is the best approximation of $l(I_{T_h}, D)$ by a function of I_{T_h} in the L^2-sense. It has the following property:

$$E\Big[\big(l(I_{T_h}, D) - L(I_{T_h})\big)^2\Big] \leq E\Big[\big(l(I_{T_h}, D) - f(I_{T_h})\big)^2\Big], \tag{3.20}$$

for any measurable function f of I_{T_h}.

The conditional expected GMWB net liability $L(I_{T_h})$ can be replicated by payoffs of standard put options. Formally this can be justified by the fact that any continuous function could be uniformly closely approximated by linear splines (see Hammerlin and Hoffmann 1991, Ch. 6). In this subsection, we discuss methods of computing $L(I_{T_h})$ and analyze some properties of this conditional expectation. In our proposed semi-static hedging strategy, we construct a portfolio whose payoff at time T_h matches $L(I_{T_h})$. Some extension of this approach will be discussed in Section 3.1.5.

There is no analytic expression for $L(I_{T_h})$. However, using Monte Carlo simulation techniques we are able to evaluate it at an arbitrary number of values of I_{T_h}. The more end points we investigate, the easier it is to reconstruct L by interpolation.

The region of possible index values is determined based on the assumed distribution of the index value at time T_h. We consider two methods of selecting the end points of the index. These methods will be used to search for the optimal hedging portfolio in the next subsection. The first method is to choose m equally-spaced points in a chosen interval. If we believe that each point is equally important, then the optimal portfolio weights can be solved using the Least Squares method. That is, we minimize the sum of squared differences between the target function $L(I_{T_h})$ and the replicating portfolio at these index values. The second method is to determine the index values from their inverse cumulative distribution function. Using these points for the Least Squares method is equivalent to using the probability function as the weight function for the Weighted Least Squares method. In this chapter, we assume the index value follows a lognormal distribution. As a result of this approach, more points will be selected in an interval associated with a higher probability. To implement the method, we choose m equally spaced points in the interval $(0, 1)$ and then use the inverse function to get the index values. We recommend using this sampling method in cases where points with higher probability densities are considered to be more important, and we want to ensure that the replicating errors at these points are small.

To obtain a value of $L(I_{T_h})$ by simulation, we first generate paths of index values within the hedging horizon, and then evaluate $l(I_{T_h}, D)$ for each path D. We use the average of these l-values as an estimate of $L(I_{T_h})$. Note that if the account value reaches zero within the hedging period, we will accumulate the GMWB net liability to the end of the hedging period. In summary, numerical computation of $L(I_{T_h})$ may go as follows:

1. Choose a set of m index values at time T_h, $(I_{T_h,1}, \cdots, I_{T_h,m})$;
2. Simulate a path of the fund value conditional on the index value $I_{T_h,i},\ i = 1$, and calculate the end value of the account $A_{T_h,j},\ j = 1$;
3. Based on the account value $A_{T_h,j}$, compute l_j by equation (3.18);
4. Repeat step (2) to (3) for $j = 2, \cdots, M$; take $\widehat{L(I_{T_h,i})} = \frac{1}{M}\sum_{j=1}^{M} l_j$;
5. Repeat step (2) to (4) for $i = 2, \cdots, m$.

The above procedure involves simulating Mm scenarios over 15 years, which is time-consuming. We use two approaches to speed up the process. The first approach is to compute l using a Control Variate technique. The second approach is to approximate $L(I_{T_h})$ using the Taylor expansion of l about the average account value at time T_h. They are explained in the following two subsections. In step (3) above, we replace equation (3.18) with equation (3.21). In step (4), L is estimated by equation (3.25), which only requires three values of l instead of M. The total number of scenarios needed reduces from Mm to $3m$.

A Control Variate technique

To improve the simulation efficiency, we propose to use the control variate technique to compute l. As discussed in Section 2.4, a policyholder's gain is an insurer's net liability. We can write l as

$$l = C_{T_h} + wh \sum_{i=1}^{N-T_h/h} e^{-rih} - A_{T_h} - c, \tag{3.21}$$

where

$$\begin{aligned} C_{T_h} = &\ E^Q_{T_h}\left[e^{-r(T-T_h)} A_T \mid A_{T_h}\right], \\ c = &\ \sum_{u=0}^{3} A_{uh}(1-e^{-qh})e^{r(T_h-uh)}. \end{aligned} \tag{3.22}$$

Here C_{T_h} represents the time-T_h value of the account value at maturity, and c is the accumulated value of GMWB charges in the first year. On the left hand side of equation (3.21), A_{T_h} is known at time T_h, and the annuity certain is constant. For the rest of this subsection, we assume $T_h = 1,\ h = 1/4$. To compute C_{T_h}, we rewrite the account value at maturity in the same way as in equation (2.12):

$$\begin{aligned} A_T &= \max(0, B_T), \\ &= \max\Big(0, A_1 \prod_{i=5}^{N} X_{ih} - \frac{S_0}{N}(1 + X_T + \cdots + \prod_{i=6}^{N} X_{ih})\Big), \\ &= \frac{N-4}{N} \max\Big(0, \frac{N}{N-4} A_1 \prod_{i=5}^{N} X_{ih} - \frac{S_0}{N-4}(1 + X_T + \cdots + \prod_{i=6}^{N} X_{ih})\Big), \end{aligned}$$

where X_{ih} represents the fund return during the i-th time step. Reverse the order of the returns, and let

$$(x_h, \cdots, x_T) = (X_T, \cdots, X_h).$$

We get another expression that is equal to the above expression in probability:

$$\begin{aligned}
&\frac{N-4}{N}\max\Big(0, \frac{N}{N-4}A_1\prod_{i=1}^{N-4}x_{ih} - \frac{S_0}{N-4}(1+x_h+\cdots+\prod_{i=1}^{N-5}x_{ih})\Big) \\
&= \frac{N-4}{N}\max\Big(0, \frac{NA_1}{(N-4)S_0}s_{(N-4)h} - \frac{1}{N-4}\sum_{i=0}^{N-5}s_{ih}\Big),
\end{aligned}$$

where $s_{ih},\ i \leq N-4$, represents the fund value corresponding to $(x_h, \cdots, x_{ih})$ independent of A_1. Equation (3.22) is equal to the price of an arithmetic Asian call option at time 1

$$\begin{aligned}
& E_1^Q\big[e^{-r(T-T_h)}A_T \mid A_1\big] \\
= & \frac{N-4}{N}E_1^Q\Big[e^{-r(T-T_h)}\max\Big(0, \frac{NA_1}{(N-4)S_0}s_{T-1} - \frac{1}{N-4}\sum_{i=0}^{N-5}s_{ih}\Big) \mid A_1\Big]
\end{aligned}$$

The control variate that we use is the geometric Asian option whose price C_G has a Black-Scholes type formula:

$$C_G = \frac{N-4}{N}\left(\frac{NA_1}{N-4}e^{-q(T-T_h)}N(d_1) - KN(d_2)\right), \tag{3.23}$$

where,

$$\begin{aligned}
d_1 &= \frac{\ln\frac{NA_1}{(N-4)K} + (-q+\frac{\sigma_3^2}{2})(T-T_h)}{\sigma_3\sqrt{T-T_h}}, \\
d_2 &= d_1 - \sigma_3\sqrt{T-T_h}, \\
K &= S_0 e^{-r(T-T_h)}\mu_2, \\
\sigma_3 &= \sigma\Big[\frac{(N-3)\big(2(N-4)+1\big)}{6(N-4)^2}\Big]^{\frac{1}{2}},
\end{aligned}$$

$$\mu_2 = \exp\Big\{\frac{(N-5)}{2}(r-q-\frac{\sigma^2}{2})h + \frac{\sigma^2}{2}\frac{(N-5)\big(2(N-4)-1\big)}{6(N-4)}h\Big\}.$$

It is similar to equation (2.16) except that we replace N with $N-4$, and replace S_0 with $\frac{NA_1}{N-4}$. The other steps of simulation are the same as in Section 2.2.2.

Additional way to improve efficiency

Another way to improve simulation efficiency is to approximate the conditional expected GMWB net liability $L(I_1)$ with only three values of l for each given index value. If we treat the path-dependent expected GMWB net liability $l(I_1, D)$ as a function of the end value of the account A_1, then we can approximate the expectation of l conditional on I_1 by the Taylor expansion of l at the mean value of A_1. Two additional l values are needed to estimate the first and second order differentials of l. Thus, the number of l values needed to compute L is reduced from Mm to $3m$.

Equation (3.23) tells us that the option value is dependent on A_1, so the expected net liability l is a function of A_1 given I_1. We write it as $l(A_1^I)$ for the purpose of explaining the approximation idea. Denote the conditional expectation of the account values given the index value at time 1 by $\bar{A}_1^I = E(A_1 \mid I_1)$. The variance of the account values is very small for a given index value. Thus, $l(A_1^I)$ is well approximated by its Taylor series evaluated at $\bar{A}_1^I$ for all A_1^I values that are sufficiently close to $\bar{A}_1^I$. The Taylor expansion of $l(A_1^I)$ about $\bar{A}_1^I$ is given by

$$l(A_1^I) = l(\bar{A}_1^I) + \frac{\partial l(\bar{A}_1^I)}{\partial A}(A_1^I - \bar{A}_1^I) + \frac{\partial^2 l(\bar{A}_1^I)}{2\partial A^2}(A_1^I - \bar{A}_1^I)^2 + \cdots . \qquad (3.24)$$

$L(I_1)$ can be approximated by taking expectations of both sides of equation (3.24)

$$\begin{aligned} L(I_1) &= E\big[l(A_1^I) \mid I_1\big] \\ &\approx l(\bar{A}_1^I) + \frac{\partial l(\bar{A}_1^I)}{\partial A} E(A_1^I - \bar{A}_1^I) + \frac{\partial^2 l(\bar{A}_1^I)}{2\partial A^2} E(A_1^I - \bar{A}_1^I)^2 \\ &= l(\bar{A}_1^I) + \frac{\partial^2 l(\bar{A}_1^I)}{2\partial A^2} Var(A_1^I). \end{aligned} \quad (3.25)$$

The variance of $l(I_1, A_1^I)$ can be approximated by

$$\begin{aligned} & Var\big[l(A_1^I) \mid I_1\big] \\ = \;& Var(A_1^I)\Big(\frac{\partial l(\bar{A}_1^I)}{\partial A}\Big)^2 + \frac{1}{4}\Big(E[(A_1^I - \bar{A}_1^I)^4] - [Var(A_1^I)]^2\Big)\Big(\frac{\partial^2 l(\bar{A}_1^I)}{\partial A^2}\Big)^2 \\ \approx \;& Var(A_1^I)\Big(\frac{\partial l(\bar{A}_1^I)}{\partial A}\Big)^2. \end{aligned} \quad (3.26)$$

The derivatives of l at $\bar{A}_1^I$ can be computed using the finite difference method. If we use equation (3.21), then $L(I_1)$ can be approximated as follows:

$$\begin{aligned} L(I_1) &= E\big[l(A_1^I) \mid I_1\big], \\ &= E[C_A \mid I_1] + wh\sum_{i=1}^{N-4} e^{-rih} - \bar{A}_1^I - E[c \mid I_1], \\ &= C_A(\bar{A}_1^I) + wh\sum_{i=1}^{N-4} e^{-rih} - \bar{A}_1^I - E[c \mid I_1] + \frac{\partial^2 C_A(\bar{A}_1^I)}{2\partial A^2} Var(A_1^I) \end{aligned} \quad (3.27)$$

Discussion of the results

Let us consider an example. For a 15-year contract, assume the policyholder withdraws quarterly. The following assumptions about the market and the contract are used:

$$r = 5\%, \qquad \sigma = 20\%,$$

$$A_0 = S_0 = 100, \qquad w = 6.66\dot{6},$$

$$T = 15,\ N = 60, \qquad q = 0.0048886.$$

The fair charge q of this GMWB has been calculated in Chapter 2. Given that the index value at time 1 is equal to 100, $I_1 = 100$, we simulate a set of paths of the fund value, and calculate the account values at the end of the year. Figure 3.3 depicts the histogram of 10,000 simulated account values at time 1. The variation of the account values is small because the index value is fixed at the end of the year. The left tail is slightly heavier than the right one. Lower account values at a given time imply higher values of the expected GMWB net liability in the future. We expect a heavier right tail for the expected GMWB net liability.

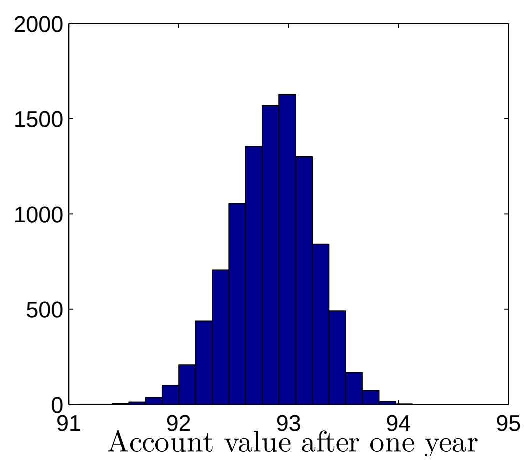

Figure 3.3: The histogram of 10,000 account values at time 1 conditional on $I_0 = 100$. The withdrawal amount is 1.6667 per quarter. $(I_0 = 100,\ I_1 = 100,\ r = 5\%,\ \sigma = 20\%)$

The conditional expectation of the account values has an approximately linear relationship with the index value at time 1, as shown in Figure 3.4. Figure 3.5 shows that

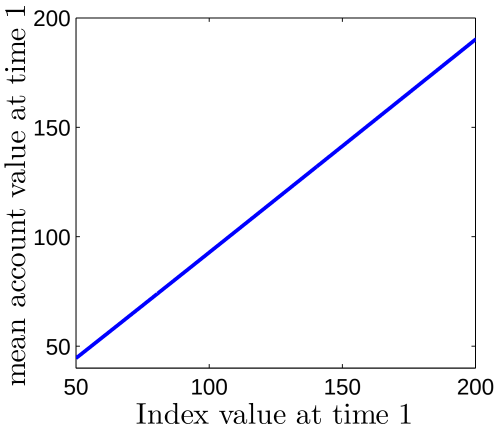

Figure 3.4: The conditional expectation of the account values has an approximately linear relation with the index value at time 1.

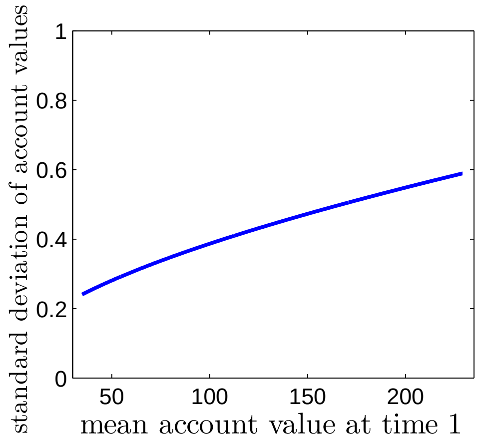

Figure 3.5: At time 1, the standard deviation of the account value, conditional on the index value, increases slowly with the average account value.

the standard deviation of the account value increases slowly with the mean account value at time 1.

Conditional on the index value of $I_1 = 100$, we have

$$\bar{A}_1^I = 92.3870, \quad Var(A_1^I) = 0.1398, \quad E(c \mid I_1) = 0.4923.$$

$$C_A(\bar{A}_1^I) + wh \sum_{i=1}^{N-4} e^{-rih} - \bar{A}_1^I - E[c \mid I_1] = 0.0213.$$

The "delta" and "gamma" are estimated as follows:

$$\begin{aligned} \frac{\partial l(\bar{A}_1^I)}{\partial A} &= \frac{l(\bar{A}_1^I + \varepsilon) - l(\bar{A}_1^I - \varepsilon)}{2\varepsilon} = -0.2940, \quad \varepsilon = 0.1, \\ \frac{\partial^2 C_A(\bar{A}_1^I)}{\partial A^2} &= \frac{C_A(\bar{A}_1^I + \varepsilon) + C_A(\bar{A}_1^I - \varepsilon) - 2C_A(\bar{A}_1^I)}{\varepsilon^2} = 0.0061. \end{aligned}$$

Based on equation (3.27), the expected net liability for $I_1 = 100$ is estimated as

$$L(100) = 0.0213 + 0.1398 * 0.0061/2 = 0.0217.$$

The variance of l is estimated as

$$Var(l \mid I_1 = 100) = 0.1398 * 0.2940^2 = 0.0121.$$

Using direct simulation approach, we have

$$L(100) = 0.0216, \quad Var(l \mid I_1 = 100) = 0.0126.$$

The approximations are very close to the simulation results. Figure 3.6 shows the histogram of the simulated l-values conditional on the index value of 100 at time 1.

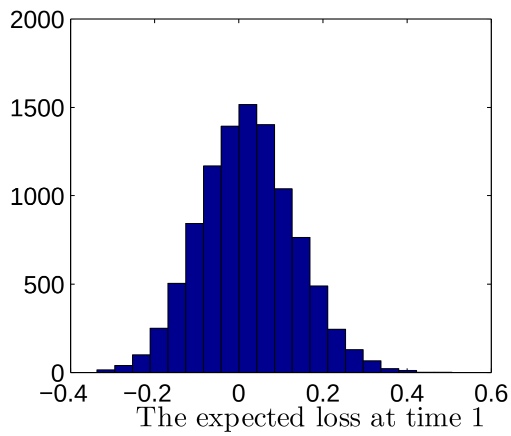

Figure 3.6: The histogram of 10,000 simulated expected GMWB net liabilities l given the index value of 100 at time 1. ($r = 5\%,\ \sigma = 20\%,\ I_0 = 100,\ w = 6.6667,\ h = 1/4.$)

For any other value of I_1, the conditional expected GMWB net liability $L(I_1)$ can be obtained by the same methods. Figure 3.8 shows that, under the risk neutral measure Q, the index values after one year fall in the interval of $[40, 200]$ with probability close to one. We uniformly choose $m = 1800$ values from this interval, and simulate $M = 10,000$ paths for each index value.

Figure 3.7 plots the curves of the expected GMWB benefit, charges and net liability with respect to I_1. The benefit decreases to zero as the index value increases. The value of charge is nearly proportional to the index value. The expected net liability falls below zero when the index value becomes larger than 100.18. The positive part of the solid curve represents the expected GMWB net liability to insurers, and that is what we would like to hedge. This part of the curve corresponds to low index values after one year. If the expected GMWB net liability is negative, it means that the insurer will gain profit at a high future index value. Therefore, there is no need to hedge the negative part of the net liability curve. The negatively-sloped convex curve indicates that standard put options on the index could be used as hedging instruments for the GMWB.

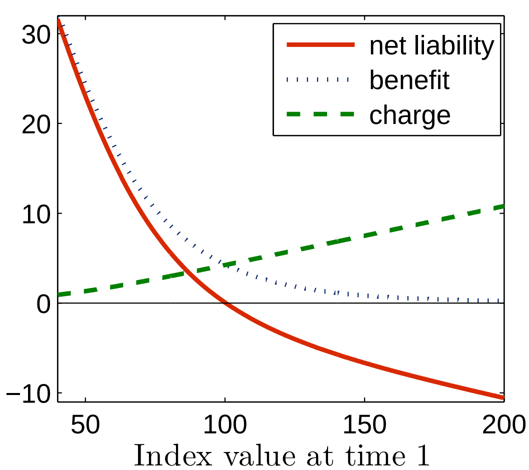

Figure 3.7: The conditional expected GMWB net liability $L(I_1)$ is a convex decreasing function of the index value at time 1.

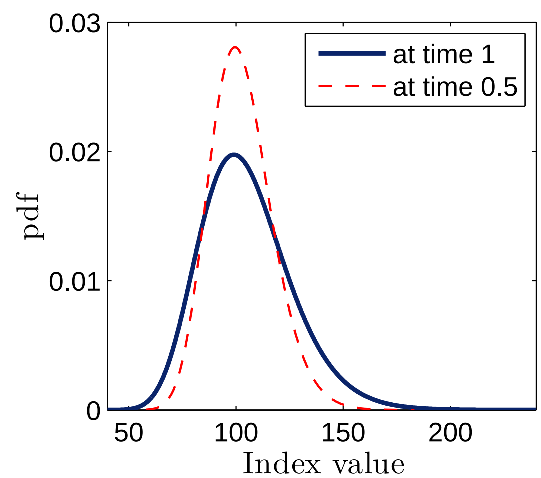

Figure 3.8: The probability density functions of the index value at time 1 and time 0.5 ($S_0 = 100, r = 5\%, \ \sigma = 20\%$)

3.1.4 The Optimal Hedging Portfolio

The main step of the semi-static hedging strategy is to construct a portfolio whose payoff replicates the conditional expected net liability L at the end of the hedging period T_h. Figure 3.7 shows that L is a convex function of the index value at time $T_h = 1$. The payoff of a European index put option is a piece-wise linear convex function of the index value at time 1. It is possible to fit the conditional expected GMWB net liability curve using piece-wise linear segments.

In a model that is complete and arbitrage free, there exists a perfect replicating portfolio that has the same value and sensitivities as the expected GMWB net liability. The perfect replicating portfolio includes an infinite number of standard options. However, for our problem, we can achieve adequate replication using only a small number of options. If there are a few fixed strikes available, we can find an optimal weight for each strike. If options with many different strikes are available in the market, then there are methods to choose both the optimal strike prices and the optimal weight under each strike. In this section, we first describe methods to search for the optimal weights given the strikes, and then explain methods to obtain optimal strikes.

In the context of European contracts, a similar static replication idea has been given by Carr and Madan (2001). They show that for any fixed value S^*, any twice differentiable function of the terminal stock price S can be replicated by a unique initial position of $\left[f(S^*) - f'(S^*)S^*\right]$ unit discount bonds, $f'(S^*)$ shares, and $f''(K)dK$ out-of-the-money options of all strikes K:

$$f(S) = \left[f(S^*) - f'(S^*)S^*\right] + f'(S^*)S + \int_0^{S^*} f''(K)(K-S)^+ dK + \int_{S^*}^{\infty} f''(K)(S-K)^+ dK \tag{3.28}$$

The positions in the bond and the stock create a tangent to the payoff at a stock price S^*. The positions in the options are used to bend the tangent line so as to match the payoff at all price levels. For $S^* \to \infty$, if the terms on the right hand side are all finite, then we can replicate using only bonds, stocks, and puts. Equation (3.28) may be used to replicate the conditional expected GMWB net liability.

Our hedging approach is to numerically search for the optimal replicating portfolio given a set of strike prices. In the previous subsection, we have calculated the expected GMWB net liability $L(I_1)$ at m index values at time 1. Denote by m^* the number of the sampled index values at which the expected GMWB net liability is above zero at time 1. We will use the values of the expected net liability at these m^* index points to search for the replicating portfolio. Suppose we are able to buy one-year index put options with n different strike prices. In this section, the value of n is assumed to be fixed. As the number of puts with different strikes increases, the accuracy of the replication will increase. Based on our numeric experiments on four, five, and six option strikes, we believe that the replicating errors, using only six options, are small enough for many practical applications. We will use a $n \times 1$ vector K to denote the strike prices. The payoff functions of these put options are

$$F(I_{1,i}, K_j) = \max(0, K_j - I_{1,i}), \quad i = 1, \cdots, m^*, \; j = 1, \cdots, n. \tag{3.29}$$

The quantities of these puts needed are denoted by a $n \times 1$ vector θ. If $n = m^*$, then there is a θ that make the portfolio payoff and the expected net liability equal. We usually have $n < m^*$, and θ is obtained by minimizing the difference between $F(I_1, K)\theta$ and $L(I_1)$. Denote the differences between the portfolio payoff and the expected net liability by a vector g whose elements are given by

$$g_i = L(I_{1,i}) - \sum_{j=1}^{n} \theta_j F(I_{1,i}, K_j), \quad i = 1, \cdots, m^*. \tag{3.30}$$

The purpose is to make these differences as small as possible. There are many different criteria that may be used to judge whether g is small. We consider four approaches to determine the portfolio weights:

- Least Squares (LS),
- Weighted Least Squares (WLS),
- Optimal over-replicating,
- Optimal under-replicating.

The objective of the **Least Squares** method is to minimize the sum of squared errors. This approach treats each point equally, trying to limit large fitting discrepancy at any point. When we do not know exactly the distribution of the future index values, it is better to use the LS approach rather than the WLS approach. However, we still need to identify the range of possible values.

Using the Least Squares approach, the problem may be written as follows:

$$\min_{\theta>0} \sum_{i=1}^{m^*} g_i^2 = \min_{\theta>0} \sum_{i=1}^{m^*} \Big(L(I_{1,i}) - \sum_{j=1}^{n} \theta_j F(I_{1,i}, K_j) \Big)^2. \tag{3.31}$$

We can add another constraint that the initial cost of the hedging portfolio is less than or equal to the time 0 value of the pure net liability V. Denote by a $1 \times n$ vector P the price of the n put options. Define V as

$$V = E_0^Q \left[e^{-rT_h} L(I_1) \mathbf{1}_{\{L(I_1)>0\}} \right] \tag{3.32}$$

Then the constraint is written as $P\theta \le V, \theta > 0$.

The Weighted Least Squares approach also minimizes the mean squared error, but it allows a weighting function to determine the contribution of each point to the final variable determination. The index value at time 1 is not uniformly distributed, so it makes sense to assign more weights to points with high probabilities of occurring. As a result, the fit is tighter in the relevant area. The disadvantage of this approach is that it is based on the assumption that the weights are known exactly. This is almost never the case in real applications, so estimated weights must be used instead. For our problem, the weights should be given by the probability density function of the underlying fund values under the physical probability measure P. If this cannot be estimated accurately, we can use the density under the risk neutral measure Q as an approximation. Since the risk-free rates are typically lower than the real average fund returns, using the density under Q measure will give conservative results.

Assume $(I_{1,1}, \cdots, I_{1,m^*})$ is a sample of index values at time 1 in increasing order. We define the probability p_i at point $I_{1,i}$ under the risk-neutral measure Q as

$$\begin{aligned} p_1 &= \frac{1}{2}\text{Prob}(I_1 \le I_{1,1}), \\ p_i &= \frac{1}{2}\text{Prob}(I_{1,i-1} \le I_1 \le I_{1,i+1}), \quad i = 2, \cdots, m^* - 1, \\ p_{m^*} &= \frac{1}{2}\text{Prob}(I_1 \ge I_{1,m^*-1}). \end{aligned} \tag{3.33}$$

The optimization problem can be expressed as follows:

$$\begin{aligned} \min_{\theta} \sum_{i=1}^{m^*} g_i^2 p_i &= \min_{\theta} \sum_{i=1}^{m^*} \Big(L(I_{1,i}) - \sum_{j=1}^{n} \theta_j F(I_{1,i}, K_j) \Big)^2 p_i, \\ \text{subject to} \quad & P\theta \le V, \ \theta > 0. \end{aligned}$$

In the previous subsection, we introduced two approaches to get a sample of index values. The equally spaced index values can be used for both optimization approaches. The probability inverted index values can be used for the WLS approach directly, yielding accurate approximation. In this case, the mean squared error is the arithmetic average of the squared errors valued at chosen points.

Consider the same GMWB example as in Section 3.1.2. Assume the hedging instruments are chosen to be one-year put options with six different strike prices.

$$K = (50;\ 60;\ 70;\ 80;\ 90;\ 100)$$

The maximum strike price is slightly less than the index value of 100.1835 at which the expected GMWB net liability is zero. Ideally, the maximum strike price in the portfolio should be equal to 100.1835. We choose 100 with the understanding that the error will not be significant. The more option strikes we use, the better the replicating effect. For a given a number of strikes, the optimal portfolio weights will tell us which strikes are important.

Table 3.1 summarizes the optimization results for Least Squares and Weighted Least Squares approach with and without constraints. The at-the-money put option dominates the hedging portfolio. This is a desirable result since these options are very liquid in the market. This is another sign showing that the semi-static hedging strategy is practical. The WLS gives more weights to the at-the-money put option. The expected present value of the net liability V under the risk-neutral measure is estimated as $V = 1.6391$. The costs of the replicating portfolios are slightly higher when we do not add any constraints. Portfolio weights with constraints are very close to those without any. The cost constraint does not have too much impact on the replicating results.

		Least Squares		**Weighted Least Squares**	
Strike	**Put price**	$\boldsymbol{\theta_c^{LS}}$	$\boldsymbol{\theta^{LS}}$	$\boldsymbol{\theta_c^{WLS}}$	$\boldsymbol{\theta^{WLS}}$
50	0.0003	0.1931	0.1931	0.1440	0.1440
60	0.0113	0.1343	0.1343	0.1559	0.1559
70	0.1262	0.1437	0.1437	0.1373	0.1372
80	0.6872	0.1054	0.1054	0.1068	0.1069
90	2.3101	0.0802	0.0800	0.0798	0.0796
100	5.5735	0.2443	0.2445	0.2444	0.2446
Cost		1.6391	1.6394	1.6391	1.6395

Table 3.1: Semi-static hedging portfolios (of one-year puts) at time 0 solved by Least Squares and Weighted Least Squares approach. We denote the optimal weights under constraints by θ_c^{LS} and θ_c^{WLS}. The other two columns are obtained without constraints. ($r = 5\%,\ \sigma = 0.2,\ A_0 = 100,\ h = 1/4,\ \ w = 6.66\dot{6}\%,\ T_h = 1.$)

We can see from Figure 3.9 that obtained portfolios from the LS and WLS approaches replicate the expected GMWB net liability very well. Figure 3.10 and 3.11 show the replicating errors g at different index values. The cusps in the figure correspond to the turning points of the portfolio payoff function. The turning points are lower than the hedging target where the index values are equal to the strikes. A positive hedging error indicates that the target is under-hedged. The LS approach tries to minimize the replicating errors at all index values. The errors from the WLS approach are higher at index values with small probabilities, and lower at index values with high probabilities. The method recognizes that the high loss events are less likely to happen, and it allows bigger under-fitting at lower index values. For index values that are close to 100, the replicating errors are smaller than those obtained from the LS method. The difference between these results are not significant because the approximation is based on a large number ($m^* = 1800$) of index values in the interval. If we use fewer points, the difference will be larger. If we include enough put options in the replicating portfolio, the initial cost of this portfolio will approach the value of the replicating target. Hence, the constraint $P\theta \leq V$ will have quite minimal effect on the construction of the replicating portfolio as seen in Table 3.1.

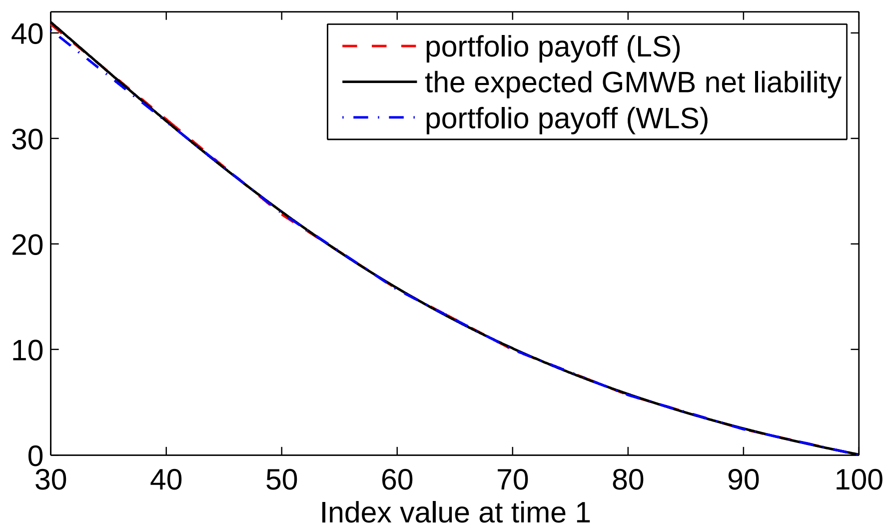

Figure 3.9: The replicating portfolio payoff and the hedging target

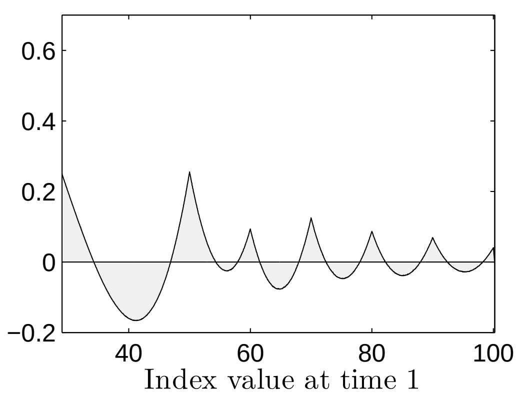

Figure 3.10: The replicating error g using Least Squares with constraint vs. Index value I_1 at time 1

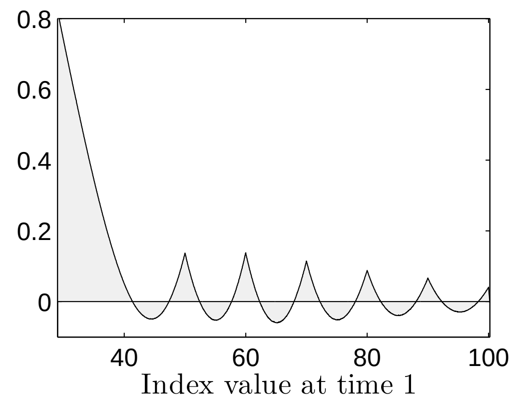

Figure 3.11: The replicating error g using Weighted Least Squares with constraint vs. Index value I_1 at time 1

Least Squares and Weighted Least Squares generate both positive and negative offsets. In contrast, over-replicating and under-replicating aim to approximate the GMWB net liability from the above and from the below respectively. Hence, the costs of the two hedging strategies are higher and lower than the LS and WLS approaches, respectively. We are interested in finding over-hedging and under-hedging portfolios whose payoffs are as close to the net liability curve as possible. The payoff functions of the portfolios are piece-wise linear. The net liability curve is convex as shown in Figure 3.7. For the over-replicating portfolio, the turning points of its payoff should lie on the GMWB net liability curve. For the under-replicating portfolio, each linear piece of the payoff should be tangent to the net liability curve, and the turning points lie below the curve. These turning points are determined by the strike prices of the put options. Figure 3.12 illustrates how the payoffs of the two hedging portfolios replicate the GMWB net liability curve.

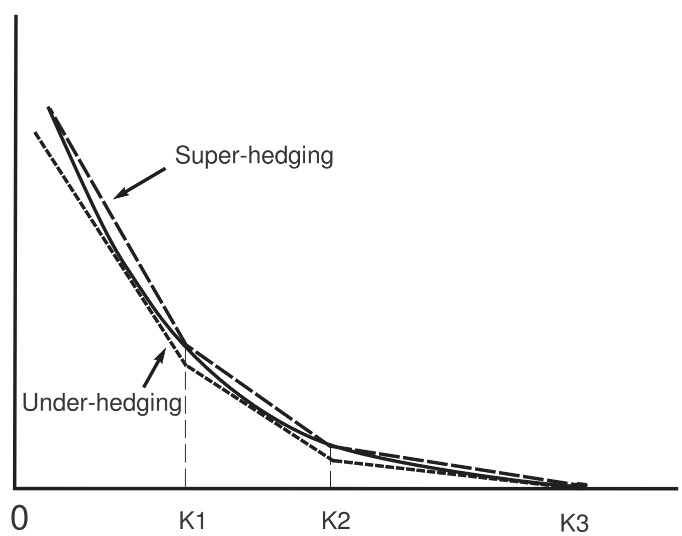

Figure 3.12: Illustration of the over and under hedging strategies

The quantity of each put option can be easily solved from the slope of each payoff segment. Remember that the portfolio includes put options with n different strike prices. Denote by a $n \times 1$ vector θ^o the over-replicating portfolio weights. The strike prices, in increasing order, are denoted by $K_i, i = 1, \cdots, n$. The maximum strike K_n is set such that $L(K_n) = 0$. If such a strike price is not available, we choose the minimum strike available that is larger than the ideal strike for the over-hedging portfolio. For the under-hedging portfolio, the maximum strike price K_n is set to be the maximum strike available that is less than the ideal strike price. We work from the put option with the highest strike price back to the lowest. It is easy to verify that the weights can be calculated as follows:

$$\theta_n^o = \frac{L(K_{n-1})}{K_n - K_{n-1}}, \tag{3.34}$$

$$\theta_i^o = \frac{L(K_{i-1}) - L(K_i)}{K_i - K_{i-1}} - \sum_{j=i+1}^{n} \theta_j^o, \quad i = n-1, \cdots, 2. \tag{3.35}$$

$$\theta_1^o = \frac{L(I_{1,1}) - L(K_1)}{K_1 - I_{1,1}} - \sum_{j=2}^{n} \theta_j^o. \tag{3.36}$$

Denote by θ^u the under-replicating portfolio weights. To solve for θ^u, we need to find the tangent point for each linear payoff segment. The method is to seek for a point on the net liability curve that gives the minimum slope from the turning point at a higher strike. Again, it is easy to verify the following formulas:

$$\theta_n^u = \min_{\substack{I_{1,i} > K_{n-1} \\ I_{1,i} < K_n}} \frac{L(I_{1,i})}{K_n - I_{1,i}}, \tag{3.37}$$

$$\theta_j^u = \min_{\substack{I_{1,i} > K_{j-1} \\ I_{1,i} < K_j}} \frac{L(I_{1,i}) - \sum_{k=j+1}^{n} \theta_k^u (K_k - K_j)}{K_j - I_{1,i}} - \sum_{k=j+1}^{n} \theta_k^u, \quad j = n-1, \cdots, 2. \tag{3.38}$$

$$\theta_1^u = \min_{\substack{I_{1,i} > K_1 \\ I_{1,i} < I_{1,1}}} \frac{L(I_{1,i}) - \sum_{k=2}^{n} \theta_k^u (K_k - K_1)}{K_1 - I_{1,i}} - \sum_{k=2}^{n} \theta_k^u. \tag{3.39}$$

We apply the above formulas to the previous example, and compare the replicating effect. When evaluating the results, we keep the following point in mind. The maximum strike price in the portfolio, 100, is less than the price where $L = 0$. This will cause the over-hedging portfolio slightly under-hedges the GMWB for index values close to the strike 100. Table 3.2 gives weights of the under- and over-hedging portfolios for the GMWB example. The portfolio costs from LS/WLS, as shown in Table 3.1, are in between the costs of the under- and over-hedging portfolios. The at-the-money put option has again the highest weight.

		Under-hedging		**Over-hedging**	
Strike	**Put price**	**Quantity**	**Cost**	**Quantity**	**Cost**
50	0.0003	0.1573	0.0001	0.1767	0.0001
60	0.0113	0.1412	0.0016	0.1535	0.0017
70	0.1262	0.1541	0.0194	0.1368	0.0173
80	0.6872	0.0920	0.0632	0.1074	0.0738
90	2.3101	0.0926	0.2138	0.0751	0.1735
100	5.5735	0.2375	1.3239	0.2512	1.4002
Cost			\$1.6220		\$1.6666

Table 3.2: Optimal over and under semi-static hedging portfolios at time 0 ($r = 5\%,\ \sigma = 0.2,\ A_0 = 100,\ h = 1/4,\ \ g = 6.66\dot{6}\%.$)

Figure 3.13 and 3.14 show the replicating errors as a function of the index values at time 1. For the under-hedging portfolio, the hedging error reduces to zero when the portfolio payoff is tangent to the target curve, otherwise, it is positive. For the over-hedging portfolio, the hedging error becomes zero where the index values are equal to the strikes.

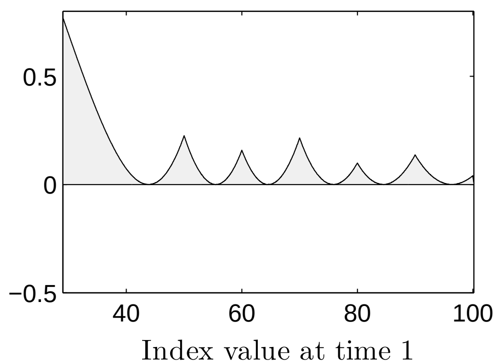

Figure 3.13: The under-hedging portfolio replicating error vs. the index value at time 1

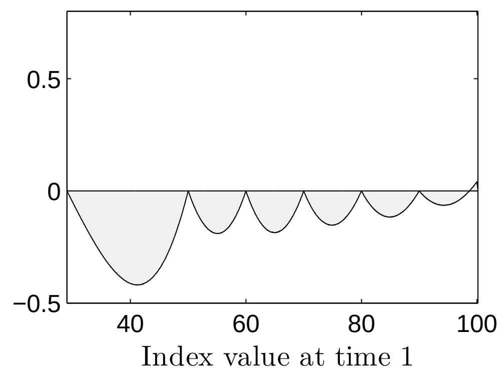

Figure 3.14: The over-hedging portfolio replicating error vs. the index value at time 1

In Table 3.3, we summarize the maximum absolute error and mean absolute error for the four approaches. The LS and WLS approaches provide better fit. The maximum absolute error is larger under the WLS approach, but the probability weighted mean absolute error is lower than that under the LS approach. The over-hedging portfolio offers the security that the expected net liability will be covered on average. The under-hedging portfolio reduces the initial cost while trying to replicate the expected net liability.

Optimization	$\lvert g_{i^*}\rvert = \max_i \lvert g_i\rvert$	$\lvert g_{i^*}\rvert / L(I_{1,i^*})$	$(1) = \frac{1}{m^*}\sum_{i=1}^{m^*} \lvert g_i\rvert$	$(1)/V$	$(2) = \sum_{i=1}^{m^*} \lvert g_i\rvert p_i$	$(2)/V$
LS w. const.	0.2554	1.1%	0.0619	3.8%	0.0127	0.8%
WLS w. const.	0.8229	1.9%	0.0936	5.7%	0.0126	0.8%
Over-hedging	0.4193	1.4%	0.1460	8.9%	0.0300	1.8%
Under-hedging	0.7714	1.8%	0.1038	6.3%	0.0180	1.1%

Table 3.3: Comparison of the semi-static hedging errors from four portfolio construction approaches. g_i is the replicating error at index value $I_{1,i}$. $L(I_{1,i})$ is the conditional expected GMWB net liability at $I_{1,i}$. V is defined by equation (3.32). ($r = 5\%,\ \sigma = 0.2,\ A_0 = 100,\ h = 1/4,\ \ w = 6.666,\ T_h = 1.$)

Given the strikes, the optimal portfolio weights can be uniquely determined. If there are many strikes available in the market, the optimal strikes can be found by a random search method. Once we set the number of different strikes to use, we can search for the optimal strikes. They are determined as the set of strikes that minimize the conditional mean squared error with optimal portfolio weights. There are some efficient minimization methods that can be used to solve this problem, such as the accelerated random search method. More specifically, the process involves the following steps:

1. Generate many sets of candidate strikes from a uniform distribution.
2. For each set of strikes, solve for the optimal weights by the approaches described above.
3. Search for the optimal strikes by minimizing the conditional mean squared replicating errors over all sets of strikes.

3.1.5 Possible Extensions

Semi-static hedging in multiple periods

As the maturities of GMWBs are usually more than 10 years, the semi-static hedging portfolio needs to be rolled over for multiple periods. At the end of the first hedging period, if the fund value goes down, the initial hedging portfolio will generate a payoff to cover the expected GMWB net liability. But the expected net liability may not be realized at this time. If the fund value goes high, there is no option payoff. Based on the new account value and index value, a new portfolio is constructed to hedge the future GMWB net liability in the second period. The possible positive option payoff may be used to pay the cost of the new hedging portfolio. This hedging process continues until maturity. All cash flows will be accumulated to the end, where we can then analyze the profit or loss.

We assume the maturity of the GMWB is T, and the length of the unit hedging period is $d = \frac{T}{N}$. The replicating portfolio is rebalanced at the end of each hedging period. The optimal portfolio can be constructed with d-year put options using the methods described in the Section 3.1.4. We denote the price of the put portfolio at the beginning of each period by $V_k,\ k = 1, \cdots, \frac{T}{d}$. The portfolio payoff at the end of each period is denoted by $F_k,\ k = 1, \cdots, \frac{T}{d}$. The portfolio should be self-financed, so we need to invest in or borrow from a bank account. Denote the bank account balance at kd by B_{kd}.

To hedge the T-year GMWB over N time interval of $d = \frac{T}{N}$, we purchase a portfolio of d-year European put options at inception, and then purchase a new portfolio at the beginning of each future period. The semi-static hedging strategy takes the following steps to implement:

1. At time kd, sample a set of future index values $\{I^s_{(k+1)d,j}\}_{j=1}^m$ based on the current value $I_{kd},\ k = 0$. The superscript s represents that they are not the realized index values but the sampled index values to be used for constructing the hedging portfolio.
2. Simulate the expected GMWB net liability $L(I^s_{(k+1)d})$ conditional on each of the sampled index value $I^s_{(k+1)d}$.
3. Solve for the optimal portfolio weights θ_k by minimizing the difference between the portfolio payoff and the expected GMWB net liability obtained in step (2).
4. Borrow from the bank an amount of $\theta_k V_k$ which is the portfolio cost at time kd.
5. At time $(k+1)d$, the portfolio payoff $\theta_k F_{k+1}$ is added to the bank account balance.
6. Repeat step (1) to (5) for $k = 1, \cdots, N-1$.

The bank account balance evolves as follows:

$$B_0 = -\theta_0 V_0, \tag{3.40}$$

$$B_{kd} = B_{(k-1)d}e^{rd} + \theta_{k-1}F_k - \theta_k V_k, \quad k = 1, 2, \cdots, N. \tag{3.41}$$

From the insurer's point of view, the entire hedged position at time kd has a value of H_k which is given by

$$H_k = B_{(k-1)d}e^{rd} + \theta_{k-1}F_k - l_{kd}, \tag{3.42}$$

$$= B_{kd} + \theta_k V_k - l_{kd}, \tag{3.43}$$

where l_{kd} is the real expected GMWB net liability at time kd. Equation (3.42) is the value before rebalancing, and equation (3.43) is the value after rebalancing.

For a specific index value at the end of one hedging period, the variation of the expected GMWB net liability l is smaller if the length of the period d is shorter. In the previous section, we let $d = 1$. Figure 3.15 shows 1% and 99% quantiles of $l(I_1, D)$ conditional on the index value at time 1. They are very close to the conditional mean value $L(I_1)$. This means that the hedging error would be small with a replicating portfolio that exactly replicates L. If we hedge more frequently, the discrepancy becomes even smaller. For example, Figure 3.16 shows the 1% and 99% quantiles of $l(I_{0.5}, D)$ conditional on the index value in six months. The standard deviations of l for one year hedging horizon are within the range of $(0.07, 0.25)$. If we reduce the hedging horizon to 6 months, the standard deviations fall in the range of $(0.02, 0.07)$. These are shown in Figure 3.17. When L is positive, the standard deviation of l comes mainly from the uncertainty of the GMWB benefit. As the index value increases, the standard deviation of l decreases because the value of L drops. When L becomes negative, the standard deviation is from the randomness of the GMWB fee. Comparing Figure 3.15 and 3.16, we see that the relation between $L(I_{0.5})$ and $I_{0.5}$ in 6 months is almost the same as that in one year. But the distributions of $I_{0.5}$ and I_1 are different as shown in Figure 3.8. The index value in 6 months is less widely spread. Thus, by shortening the hedging step we can avoid including some deep out-of-the-money options in the portfolio.

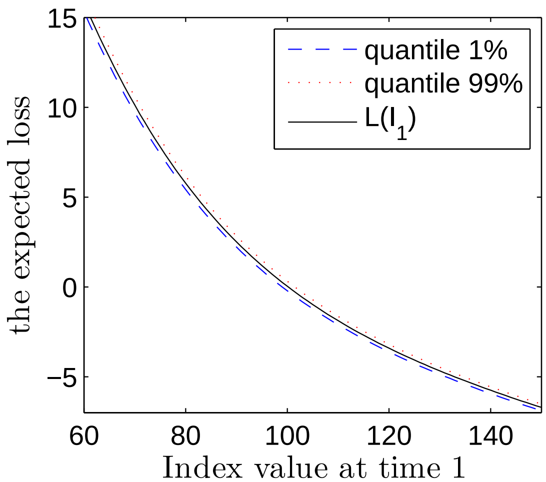

Figure 3.15: Quantiles of the expected net liability l at time 1

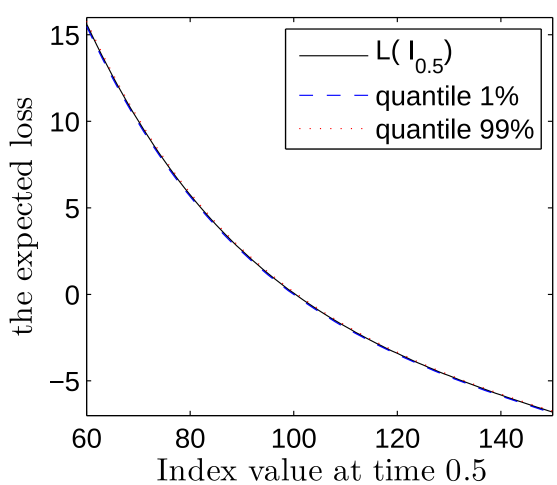

Figure 3.16: Quantiles of the expected net liability l at time 0.5

The strikes of the portfolio are determined based on the curvature of the expected GMWB net liability after d years. If d is shorter than one year, then fewer strikes will be needed. This is because the variation of fund values after d years is smaller as shown in Figure 3.8. For example, based on the estimated payoff function as shown in Figure 3.16 where $d = \frac{1}{2}$, we assume that the hedging portfolio is composed of put options with strikes of $(60, 70, 80, 90, 100)$.

As an example, we semi-statically hedge the GMWB quarterly for one year, and compare the hedging results with those from hedging only once a year. In Table 3.4, our simulation results indicate that when we hedge the GMWB quarterly, the aver-

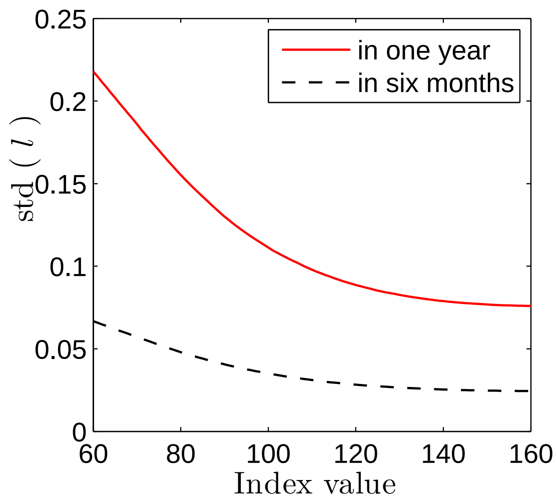

Figure 3.17: The standard deviation of the expected net liability l

Profit/Loss	Hedge Yearly ($d = 1$)	Hedge Quarterly ($d = \frac{1}{4}$)
mean	0.6576	1.1881
standard deviation	2.3252	2.6075

Table 3.4: Comparison of hedging errors in one year with yearly and quarterly frequencies. (Assume risk-free rate $r = 0.05$; real world expected rate $r_p = 0.1$; contract maturity $T = 15$; quarterly withdrawal frequency $h = 1/4$; fund volatility $\sigma = 0.2$; initial account value $S_0 = 100$; the Guaranteed annual withdrawal rate $g = 1/T$; GMWB charge $q = 48.886$bps; portfolio strikes $K = (60, 70, 80, 90, 100)$.)

age profit/loss after one year improves significantly, but the standard deviation only increases a little. This is because the standard deviation of the expected GMWB net liability has two sources: the hedging period and the period beyond the hedging horizon. When we increase the hedging frequency, the standard deviation from the first source is reduced. However, the standard deviation from the second source is increased more because the second period is much longer than the first period. The overall standard deviation rises when we increase hedging frequency in the first year. We only hedge the conditional expected net liability, so the hedging errors have a larger mean value.

In practice, the insurer has thousands of contracts to hedge at the same time. These contracts were issued at different points in time, and may have different account values and time to maturity. We can search for a replicating portfolio for each individual contract as explained previously. Then these portfolios can be combined and managed together.

Hedging with additional shorter maturity options

In Section 3.1.3, we use the conditional expectation, $L(I_1)$, to approximate the true net liability, and the replicating portfolio has a maturity of one year. A better approximation is a function of index values at two or more time points within the hedging horizon. The replicating portfolio would include options with different maturities.

For example, the new function could be the conditional expectation of $l(I_1, D)$ given index values at time $\frac{1}{2}$ and 1, namely I_1 and $I_{0.5}$:

$$L(I_1, I_{0.5}) = E_1\big[l(I_1, D) \mid I_1, I_{0.5}\big], \tag{3.44}$$

Instead of a net liability curve, we have a surface to be replicated. The replicating portfolio is composed by a set of 6-month puts and a set of one-year puts. Denote the strike prices for each set by a $n_1 \times 1$ vector K^1 and a $n_2 \times 1$ vector K^2. The option weights are denoted by θ^1 and θ^2. The payoffs of two sets of options are

$$F(I_{0.5,i}, K_j^1) = \max(0, K_j^1 - I_{0.5,i}), \quad i = 1, \cdots, m, \; j = 1, \cdots, n_1. \tag{3.45}$$

$$F(I_{1,i}, K_j^2) = \max(0, K_j^2 - I_{1,i}), \quad i = 1, \cdots, m, \; j = 1, \cdots, n_2. \tag{3.46}$$

Denote the differences between the portfolio payoff and the simulated expected net liability by g. At a given index value $I_{1,i}$, the difference is

$$g_i = L(I_{1,i}, I_{0.5,i}) - \sum_{j=1}^{n_1} \theta_j^1 F(I_{0.5,i}, K_j^1) - \sum_{j=1}^{n_2} \theta_j^2 F(I_{1,i}, K_j^2), \quad i = 1, \cdots, m. \tag{3.47}$$

The optimization methods described in Section 3.1.4 may be used to solve for θ^1 and θ^2.

Hedging with longer maturity options

The previous strategies use options with maturities that are equal or less than the hedging horizon. We now consider strategies that use options with longer maturities than the hedging horizon. For example, we may use two-year put options to construct a one-year semi-static hedging portfolio. The strategy is to match the future value of the portfolio with the expected GMWB net liability at time 1.

The advantage of the previous strategies is that the replicating instruments have known payoff functions at the end of the hedging period. The disadvantage is that we need options with many different strikes to well approximate the expected GMWB net liability curve because the portfolio payoff is a piece-wise linear function.

Options with longer maturities have values at the end of the hedging period that are smooth functions of the index value. Therefore, we may be able to achieve good approximation with a smaller number of different strikes. The drawback of this kind of strategy is that the replicating effect relies on future volatility. If the volatility diverges from the assumption, the future value of the option may become a different function of the index value.

Full replication of the GMWB net liability

The cost of hedging can sometimes make risk managers reluctant to hedge. But the cost has to be compared to the potential loss the company will suffer if the market moves in an unfavorable direction. For our proposed semi-static hedging strategy, there are initial costs of buying put options in exchange for protection against the potential loss. In this subsection, we show an alternative replicating strategy that has no immediate costs but the implicit cost of giving up potential profits if equity prices move upwards. This strategy indicates a pessimistic and conservative view of the market.

To reduce the initial cost, we replicate both the positive and negative parts of the GMWB net liability. That is, both the expected GMWB charge and the expected GMWB liability are completely hedged. The expected present value of the GMWB net liability is zero at the beginning, and the cost of the replicating portfolio is also zero. The replicating portfolio for the positive part of the net liability includes put options only. To replicate the negative part, we take a short position on a call option. This call option will cancel out the expected GMWB charge when the index value increases. The expected GMWB liability can be replicated by a set of put options as explained in Section 3.1.4.

The strike price and quantity of the call option are obtained by linear regression for the expected GMWB charges conditional on the end index values where the expected GMWB liabilities are close to zero. For the example in Section 3.1.3, the estimated strike is 36.3, and the quantity of the call option is calculated as -0.0658. If this strike is not available, we can choose the closest price available and recalculate the quantity. The expected GMWB net liability, deducted by the call option payoff, is replicated by put options with strikes of (50; 60; 70; 80; 90; 100; 110; 150). Based on the expected GMWB liability curve, we first chose 11 strikes (50; 60; $\cdots$; 150). The portfolio weights for strikes (120; 130; 140) are very small, so we eliminate these three strikes.

Table 3.5 shows the portfolio weights θ solved by the Least Squares method and Weighted Least Squares method. The theoretical present value of expected net liability at time 0 is zero, but the estimated net liability based on simulation is 0.0129. The initial cost of the full-replicating portfolio is comparable with that number.

Figure 3.18 plots the replicating errors with respect to index values at time 1. For index values with high probabilities, the replicating errors are small. This feature is more prominent for the results following the Weighted Least Squares method.

Strike	Price	θ^{LS}	θ^{WLS}
Call 36.3	65.4556	-0.0658	-0.0658
Put 50	0.0003	0.2861	0.1444
Put 60	0.0113	0.0829	0.1560
Put 70	0.1262	0.1565	0.1363
Put 80	0.6872	0.1027	0.1073
Put 90	2.3101	0.0789	0.0794
Put 100	5.5735	0.0583	0.0513
Put 110	10.6753	0.0482	0.0551
Put 150	43.0440	0.0740	0.0728
Portfolio Cost		-0.0105	-0.0224

Table 3.5: The full-replicating portfolios at time 0 solved by the Least Squares method and Weighted Least Squares method. ($r = 5\%$, $\sigma = 0.2$, $A_0 = 100$, $h = 1/4$, $g = 6.66\dot{6}\%$, $T_h = 1$.)

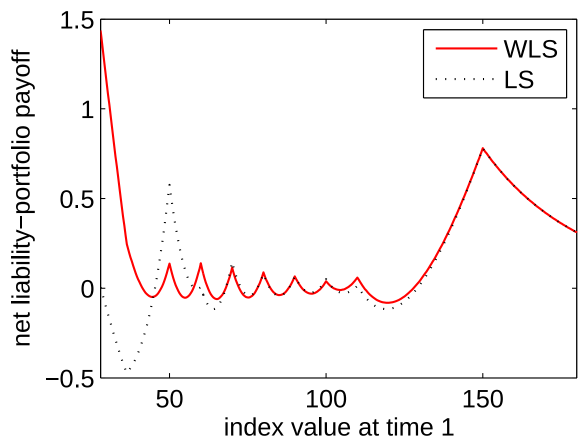

Figure 3.18: The replicating errors at time 1 from the full-replicating portfolios.

3.2 The Effectiveness of the Semi-static Hedging Strategies

In this section, we examine the effectiveness of the semi-static hedging strategies through simulation. Under the constant interest rate and volatility assumptions, the delta and delta-gamma hedging strategies would be good representations of dynamic hedging strategies. We compare the semi-static hedging strategy with these dynamic hedging strategies in two cases: the hedging target is the loss part of the expected GMWB net liability and the hedging target is the expected GMWB net value (positive or negative). The hedging error is defined as the difference between the value of the hedging portfolio and the value of the target being hedged at the end of the hedging period. By simulating a large number of paths of the index prices, we estimate the distributional characteristics of the hedging errors.

In our simulation studies we have found that the performance of the proposed semi-static strategy is comparable to delta hedging under the Black-Scholes model. However, when there are random jumps in the index value process, the semi-static strategy outperforms delta hedging strategy. In the real world, random price jumps do happen, especially in a volatile market where hedging is more important. Delta hedging may lead to a possible large loss when random price jumps can occur. This cannot be eliminated or mitigated by increasing the rebalancing frequency (see Naik and Lee 1990). The delta-gamma hedging strategy can not hedge the jump risk either. In contrast, the semi-static strategy can still perform well in the appearance of price jumps.

3.2.1 Comparison of the Semi-static Hedging with Dynamic Hedging under the Black-Scholes Model

Consider the problem faced by the insurer who wrote a GMWB on a certain variable annuity. The insurer would like to ensure that the expected GMWB liability be covered in the following one year. During the year, there are liquid exchange traded assets such as the underlying stocks, futures and options on the index, which can be used to hedge the GMWB. Below we compare three strategies:

- a semi-static hedging using one-year European put options,
- a dynamic delta-hedging strategy with the underlying index (stocks),
- a delta-gamma hedging strategy with the underlying index (stocks) and one-year put options.[2]

The semi-static strategy is based on the Least Squares approach that is described in Section 3.1.4. The dynamic strategies have been discretized by rebalancing the underlying position weekly. In practice, direct trading in the hundreds of stocks comprising the index is not employed. Practically all delta-hedging is done using the liquid index futures. However, we choose to use the index directly for simplicity. Given our assumption of constant interest rates and GMWB fee rate, the simulated performances of the delta hedges based on the index or its futures are very close.[3] Hence, this choice does not affect our results.

[2] When the time-to-maturity of the one-year put decreases to two months, we switch to four-month put options so that the gamma of the put will not be close to zero.

[3] The required position in the futures, H_F, has a deterministic relation with the required position in the underlying, H_I. $H_F = e^{-(r-q)(T_h-t)}H_I$.

We assume that under the real-world measure P the price of the underlying fund follows the stochastic differential equation

$$dS_t = (\mu - q)S_t dt + \sigma S_t dW_t. \tag{3.48}$$

To price the GMWB and to compute the delta, we use the dynamics under the respective risk-neutral measure Q:

$$dS_t = (r - q)S_t dt + \sigma S_t dW_t^*, \tag{3.49}$$

where $\{W_t^*\}$ is a standard Brownian motion under the measure Q.

Our simulation results are based on the example that we presented in Section 3.1.4 with the following assumptions:

real return	$\mu = 0.10,$	risk-free rate	$r = 0.05,$
volatility	$\sigma = 20\%,$	initial fund value	$S_0 = 100,$
annual withdrawal	$w = 6.667,$	withdrawal time step	$h = \frac{1}{4},$
GMWB fee rate	$q = 0.0048886.$		

Suppose the hedging horizon is one year, $T_h = 1$, and the hedging frequency is once a week, $h' = \frac{1}{52}$, $N_h = \frac{T_h}{h'} = 52$. We generate each trajectory of weekly fund prices based on the process (3.48). The expected GMWB net liability (net contract value) is computed using weekly fund prices and the risk-neutral dynamics. The delta of the GMWB is the sensitivity of the expected GMWB net liability to the change of the index value, that is, its partial derivative.[4] The delta is approximated by a

[4]The underlying fund return is a bit lower than the index return because of the GMWB fee. This causes a small basis risk for hedging.

finite difference method. At a time step u, $u = 0, \cdots, N_h - 1$, the delta, denoted by Δ_u, is numerically calculated through changing the index value up and down by a small percentage ϵ, and dividing the change of the expected GMWB net liability, $l(I_{uh'}, D), D = (I_0, \cdots, I_{(u-1)h'})$, by the change of the index value:

$$\Delta_u = \frac{l\big((1+\epsilon)I_{uh'}, D\big) - l\big((1-\epsilon)I_{uh'}, D\big)}{2\epsilon I_{uh'}}, \ \epsilon > 0. \tag{3.50}$$

The delta represents the amount of the index that we should hold at each time step. The GMWB has put option feature, so the delta is negative. Negative delta means to short the index, and deposit the income in a bank account. The discrete hedged position, H^D, consists of three components:

- a short GMWB position, that is, the expected GMWB net liability at time step u, given by $-l(I_{uh'}, D)$;
- a short position in the index, $\Delta_u I_{uh'}$;
- a risk-free bank account balance, $B_{uh'}$. The balance includes the income from writing the GMWB, less the cost of initiating and rebalancing the hedge portfolio.

Initially, the hedged position is

$$H_0^D = -l(I_0) + \Delta_0 I_0 + B_0 = 0, \tag{3.51}$$

$$B_0 = l(I_0) - \Delta_0 I_0. \tag{3.52}$$

At the time step u, $u = 1, \cdots, N_h - 1$, the hedge is updated by adding $(\Delta_u - \Delta_{u-1})$ units of the index. This requires us to borrow an amount of $(\Delta_u - \Delta_{u-1})I_{uh'}$ from the bank, so the bank balance becomes

$$B_{uh'} = B_{(u-1)h'} e^{rh'} - (\Delta_u - \Delta_{u-1}) I_{uh'}. \quad (3.53)$$

After the rebalancing, the portfolio value is

$$\begin{aligned} H_u^D &= -l(I_{uh'}, D) + \Delta_u I_{uh'} + B_{uh'}, \quad u = 1, \cdots, N_h - 1. && (3.54) \\ &= -l(I_{uh'}, D) + \Delta_{u-1} I_{uh'} + B_{(u-1)h'} e^{rh'}. && (3.55) \end{aligned}$$

At the end of the hedging period, we will not update the hedge

$$H_{N_h}^D = -l(I_1, D) + \Delta_{N_h-1} I_1 + B_{1-h'} e^{rh'}. \quad (3.56)$$

The portfolio value H^D determines the hedging error (profit or loss after hedging), which is defined as the difference between the value of the hedge portfolio and the value of the target option being hedged. In addition to this final step hedging error, we are also interested in the discounted hedging error:

$$HE_u^D = e^{-ruh'} H_u^D, \quad u = 1, \cdots, N_h.$$

For the semi-static hedging strategy, there is a cash outflow related to purchasing the replicating portfolio $V_0\theta$ at time zero. Hence, the initial bank balance B_0 is negative. No rebalancing is needed after that. At the end of the hedging period, the portfolio may generate positive payoff. Since we use a finite number of options to approximate the expected GMWB net liability, the portfolio payoff is not guaranteed to be equal to the expected GMWB net liability. In addition, due to its static nature, the hedge typically will not be delta neutral all the time. The hedging error can be computed as follows:

$$
\begin{aligned}
B_0 &= l(I_0) - V_0\theta, &(3.57)\\
H_u^S &= -l(I_{uh'}, D) + V_u\theta + B_0 e^{ruh'}, \quad u = 1, \cdots, N_h - 1, &(3.58)\\
H_{N_h}^S &= -l(I_1, D) + F\theta + B_0 e^{rT_h}. &(3.59)
\end{aligned}
$$

Note that the hedging error for the static strategy, $H_u^S,\ u < N_h$, is equal to unrealized profit or loss, because there is no trading activity until maturity.

The delta in equation (3.50) is computed based on the expected GMWB net liability l which could be negative. The semi-static hedging strategy that we first proposed only replicates the loss part of the net liability. To be consistent, we compare the dynamic hedging and semi-static hedging in two cases: full hedge and partial hedge. In the first case, we use the semi-static hedge portfolio constructed in Table 3.5. In the second case, we compute the delta based on the loss part of the GMWB net liability l^+, which is defined as

$$
l^+(I_{uh'}, D) = E^Q[e^{-r(t^*-uh')}(G_{t^*} - C_{t^*})^+ | I_{uh'}, D], \tag{3.60}
$$

where t^* is the first time that the account values reaches zero after withdrawal,

$$
t^* = \begin{cases} t, & \text{if} \quad A_{t-h} > 0, \quad A_t = 0,\ 0 < t \le T, \\ T, & \text{if} \quad A_T > 0, \end{cases} \tag{3.61}
$$

G_{t^*} is the value of the remaining guaranteed benefits at time t^*, and C_{t^*} is the value of the GMWB charges at time t^*. Then, the delta Δ_u^+ at time uh' is computed as

$$
\Delta_u^+ = \frac{l^+\big((1+\epsilon)I_{uh'}, D\big) - l^+\big((1-\epsilon)I_{uh'}, D\big)}{2\epsilon I_{uh'}},\ \epsilon > 0. \tag{3.62}
$$

The semi-static hedging strategy uses put options as replicating instrument, but delta hedging strategy does not. To examine the difference, we also implement a delta-gamma strategy by adding a second put option to the dynamic hedging portfolio. The strike price is equal to the initial account value. The maturity of the put option is equal to the current hedging horizon (e.g. one year). Denote the put price by P, the long position of the put by D_P, and the number of shares of the underlying by D_S. By matching the gamma and delta of the portfolio with that of the GMWB net liability, we obtain the following results:

$$D_P = \frac{\partial^2 l}{\partial S^2} \Big/ \frac{\partial^2 P}{\partial S^2}, \tag{3.63}$$

$$D_S = \frac{\partial l}{\partial S} - \frac{\partial P}{\partial S} D_P. \tag{3.64}$$

There is a problem with this hedging strategy using the one-year put option. Under the downward scenarios, the hedging errors are extremely large. This is caused by the near-zero gamma value of the put and the high value of the put as the time approaches the end of the hedging period. It makes the strategy incomparable with the other hedging strategies. We modify the strategy by replacing the one-year put with another put whose maturity is four months when the time-to-maturity decreases to two months. Then, the delta-gamma hedging strategy will give a better hedging result.

Using the previous GMWB example, we compare the three hedging strategies based on 2,000 simulations. Table 3.6 lists the simulated delta, gamma and vega values of the GMWB and the replicating portfolio at time zero. The absolute delta value in the full hedge case is larger than that in the partial hedge case. But the gamma and vega values are lower than those in the partial hedge case. The semi-static replicating portfolio does not have the exact same Greek values as the GMWB does. Their delta and gamma values are close, but the replicating portfolio has a much lower vega than the GMWB.

	Delta		Gamma		Vega	
	GMWB	Static Port	GMWB	Static Port	GMWB	Static Port
Loss only	-0.1332	-0.1142	0.0056	0.0067	29.7907	13.4572
Full hedge	-0.2141	-0.2091	0.0047	0.0045	23.2607	9.0590

Table 3.6: Greeks comparison of the GMWB net liability and the Semi-static portfolio at time zero. The hedging target is l for the full hedge. To hedge the loss part only, it is l^+ defined in equation (3.60).

Figure 3.19 and 3.20 show the estimated probability densities of the discounted profit/loss at the end of hedging period in the partial and full hedge cases respectively. These densities are estimated by the kernel smoothing method. The distribution of the profit/loss before hedging has a large variance and a longer left tail. The mean value is 0.9086, which represents a profit. This is because we assume a 10% drift term for the index value process.

In the partial hedge case, the left tails of the profit/loss distributions are substantially shortened after hedging. For the semi-static hedging, the variation is largely reduced on the loss (left) side. The initial replicating portfolio cost is 1.6394 (see Table 3.1). The left side would be truncated at 1.6394 if the payoff from the hedging portfolio had completely offset the expected GMWB net liability. The tail on the left side of 1.6394 is caused by the randomness from the fund performance and the replicating errors between the hedging portfolio payoff and the conditional expected GMWB net liability. Since we do not hedge the expected GMWB profit, the distribution of the hedging errors has a fat right tail. Under the delta-neutral strategy, the variation of the hedging errors is slightly smaller than that under the semi-static strategy. However, the left tail is fatter than that after the semi-static hedging. The estimated density function under the delta-gamma hedging is very close to that under delta hedging. This indicates that the gamma-neutral strategy does not improve significantly the delta hedging result in this case. We conclude that the semi-static strategy performs as well we the dynamic hedging strategies in the partial hedge case.

In the full hedge case, the variation of the hedging errors is substantially reduced. The semi-static strategy and delta hedging produce similar profit/loss distributions. The difference is that the density function under the semi-static strategy has a higher peak and a slightly longer left tail. The delta-gamma hedging strategy outperforms the other two strategies. Note that this is based on the modification of switching to a longer-maturity put option near the end of the hedging period.

Table 3.7 and 3.8 report the summary statistics of the last step hedging errors in the partial and full hedge cases respectively. The statistics include mean, median, standard deviation, maximum, minimum, quantiles 1%, 10%, skewness, and kurtosis. When we only hedge the loss part of the GMWB net value, the average hedging errors are positive, and small losses are still possible. In the full hedge case, the mean hedging errors are close to zero, and the standard deviations are much smaller than those in the first case. The statistics indicate that the semi-static hedging strategy performs as well as the dynamic strategy under the Black-Scholes model.

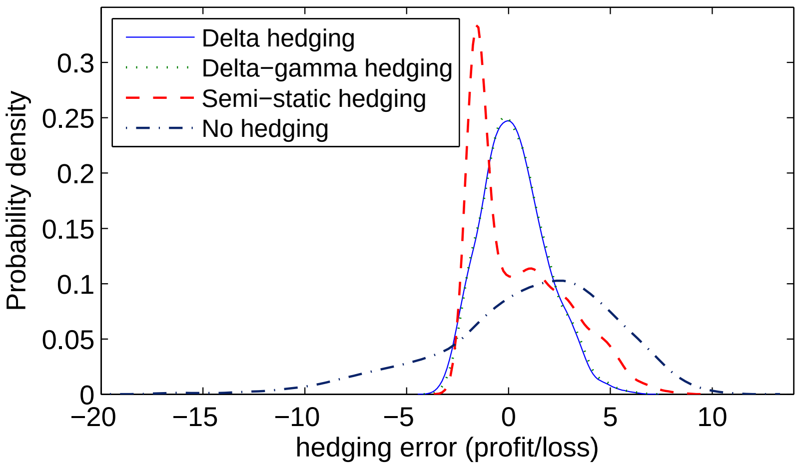

Figure 3.19: Estimated probability densities of profit/loss in one year. The hedging target is the loss part of the GMWB net liability. Assume withdrawals occur quarterly. The dynamic hedging position is updated weekly. The semi-static replicating portfolio consists of six puts and is obtained by Least Squares. ($A_0 = 100$, $\mu = 10\%$, $r = 5\%$, $\sigma = 20\%$, 2,000 scenarios)

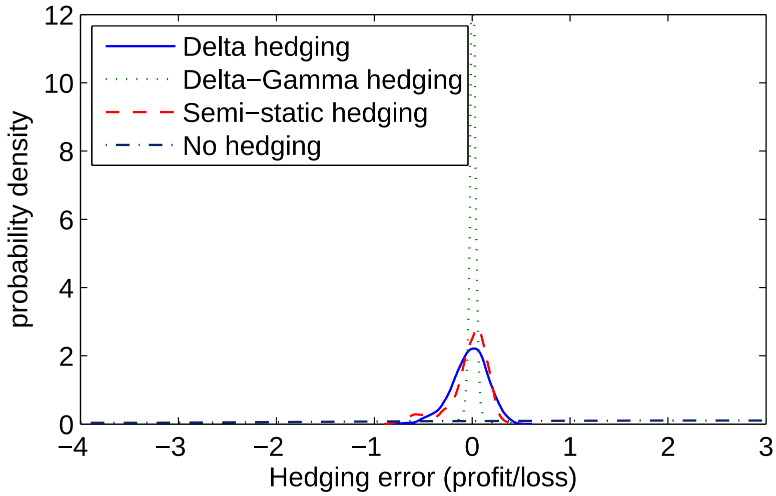

Figure 3.20: Estimated probability densities of profit/loss in one year. The hedging target is the GMWB net value, that is, both the profit and the loss. Assume withdrawals occur quarterly. The dynamic hedging position is updated weekly. The semi-static replicating portfolio consists of eight puts and one call. The portfolio weights are obtained by the Least Squares method. ($A_0 = 100$, $\mu = 10\%$, $r = 5\%$, $\sigma = 20\%$, 2,000 scenarios)

Profit/loss	No hedging	Delta-hedging	Delta-gamma hedging	Semi-static hedging
mean	0.9086	0.2853	0.3464	0.4586
median	1.4463	0.1699	0.1702	-0.2639
standard deviation	4.2609	1.6064	1.6071	2.2772
maximum	10.6167	6.2780	6.4719	8.4592
minimum	-16.8649	-3.3558	-3.0969	-2.2297
quantile 1%	-11.3618	-2.7814	-2.6150	-1.9299
quantile 10%	-5.1458	-1.7400	-1.6784	-1.7224
skewness	-0.8051	0.4328	0.4981	0.839
kurtosis	3.8322	2.9927	3.0382	2.6994

Table 3.7: Simulated hedge performance comparison of the semi-static and dynamic strategies in the first year. The hedging target is the loss part of the GMWB net liability. Assume withdrawals occur quarterly. The semi-static replicating portfolio consists of six puts. The dynamic hedging position is updated weekly. ($\mu = 10\%,\ r = 5\%,\ \sigma = 0.2,\ A_0 = 100,\ \ w = 6.66\dot{6},\ h = 1/4.$)

Profit/loss	No hedging	Delta-hedging	Delta-gamma hedging	Semi-static hedging
mean	0.9086	-0.0252	0.0077	-0.0336
median	1.4463	-0.0123	0.0063	0.0151
standard deviation	4.2609	0.1850	0.0358	0.2070
maximum	10.6167	0.4885	0.4003	0.4163
minimum	-16.8649	-0.7368	-0.1829	-0.7967
quantile 1%	-11.3618	-0.5199	-0.0848	-0.6965
quantile 10%	-5.1458	-0.2664	-0.0299	-0.3038
skewness	-0.8051	-0.4662	0.7064	-1.3556
kurtosis	3.8322	3.3905	14.031	4.8622

Table 3.8: Simulated hedge performance comparison of the semi-static and dynamic strategies in the first year. The hedging target is the GMWB net value, that is, both the profit and the loss. Assume withdrawals occur quarterly. The semi-static replicating portfolio consists of eight puts and one call. The dynamic hedging position is updated weekly. ($\mu = 10\%,\ r = 5\%,\ \sigma = 0.2,\ A_0 = 100,\ \ w = 6.66\dot{6},\ h = 1/4.$)

3.2.2 Comparison of the Semi-static Hedging with Dynamic Hedging under the Jump-Diffusion Model

It is known that in the real world equity prices exhibit jumps. In this subsection, we compare the semi-static and delta hedging strategies under a jump-diffusion model. For long-dated options, jumps tend to get averaged out so that the implied equity price distribution when there are jumps is almost indistinguishable from the one obtained when there are no jumps (see Hull 2006, p. 379). We assume that jumps are not priced because we are interested in the effect they have on the costs of hedging of different hedging strategies that have the same initial costs. At the same time, we know that there is evidence that jumps constitute nondiversifiable risk. For example, Chen et al. (2008) show that jumps do affect the fair price of the GMWB.

In comparison with the Merton's normal jump-diffusion model (Merton 1976), Kou's double exponential jump diffusion model offers better fit to the real market data (Kou 2002). We choose to use Kou's model to compare the hedging performance of the two hedging strategies. The simulation results show that the semi-static hedging strategy outperform delta hedging strategy when there are jumps in the fund values.

Kou's jump-diffusion model

Under the double exponential jump diffusion model, the dynamics of the fund value S_t under the physical measure P is given by

$$d\frac{S_t}{S_{t-}} = (\mu - q)dt + \sigma dW_t + d\Big(\sum_{i=1}^{N_t}(V_i - 1)\Big), \tag{3.65}$$

where $\{W_t\}$ is a standard Brownian motion, and $\{N_t\}$ is the number of jumps until time t. $\{N_t\}$ is assumed to be a Poisson process with rate λ, and V_i is the size of the i-th jump in the form of percentage of S_{t-}. We assume that V_i's are independent and identically distributed (i.i.d.) non-negative random variables such that $Y = \ln(V)$ has a mixed exponential distribution with the density:

$$f_Y(y) = p\eta_1 e^{-\eta_1 y} I_{(y\geq 0)} + (1-p)\eta_2 e^{\eta_2 y} I_{(y<0)},\ 0 < p < 1,\ \eta_1 > 1, \eta_2 > 0, \tag{3.66}$$

where p and $1-p$ are probabilities of jump up and down respectively, and $\frac{1}{\eta_1}$, $\frac{1}{\eta_2}$ represent mean jump sizes in percentage form. W_t, N_t and Y are assumed to be independent.

There is no unique risk-neutral probability measure in the presence of random jumps. Kou (2002) shows that a particular risk-neutral measure Q can be obtained using the rational expectations argument with a HARA-type utility function.[5] The equilibrium price of an option is given by the expectation under this risk-neutral measure of the discounted option payoff. Under this risk-neutral probability measure, S_t still follows a double exponential jump diffusion process, but of the form

$$d\frac{S_t}{S_{t-}} = (r - q - \lambda^*\zeta^*)dt + \sigma dW_t^* + d\Big(\sum_{i=1}^{N_t^*}(V_i^* - 1)\Big), \tag{3.67}$$

$$\ln(\frac{S_t}{S_0}) = (r - q - \frac{\sigma^2}{2} - \lambda^*\zeta^*)t + \sigma W_t^* + \sum_{i=1}^{N_t^*} Y_i^*, \tag{3.68}$$

$$\zeta^* = E^Q[V^*] - 1 = \frac{p^*\eta_1^*}{\eta_1^* - 1} + \frac{(1-p)^*\eta_2^*}{\eta_2^* + 1} - 1, \tag{3.69}$$

[5]The utility functions have Hyperbolic Absolute Risk Aversion.

$$U(c,t) = \begin{cases} e^{-\theta t}\frac{c^\alpha}{\alpha}, & 0 < \alpha < 1,\ \theta > 0; \\ e^{-\theta t}\ln(c), & \alpha = 0,\ \theta > 0. \end{cases}$$

where $\{W_t^*\}$ is a standard Brownian motion under Q, and $\{N_t^*\}$ is a Poisson process with intensity λ^*. The log jump size $Y^* = \ln(V^*)$ follows a new double exponential distribution with parameters $(p^*,\ \eta_1^*,\ \eta_2^*)$ that satisfy the same conditions as before. They all depend on the utility function.

We choose to use the parameter values given by Kou (2002), who argues that they are reasonable for the U.S. stock market:

$$\lambda = \lambda^* = 10, \quad p = p^* = 0.3, \quad \eta_1 = \eta_1^* = 50, \quad \eta_2 = \eta_2^* = 25.$$

The average number of jumps per annum is 10. The probability of an upward jump is 0.3, and the probability of jumping down is 0.7. The average size of an upward jump is 2%, and the average size of a downward jump is 4%. With these parameters, the average jump size is −2.2%, and the standard deviation of jump sizes is 4.47%.

These jumps will change the fund value and account value. We still use the previous method to calculate the delta, and assume that there are no jumps when determining the GMWB future liability and charges. The fund value after a small time period h' is simulated as follows:

$$S_{t+h'} = S_t \exp\Big\{(\mu - q - \frac{\sigma^2}{2})h' + \sigma\sqrt{h'}z + \sum_{N_t+1}^{N_{t+h'}} Y_i\Big\}. \tag{3.70}$$

Hedging comparison under the jump-diffusion model

Merton (1976) explains that an option writer will lose money if a jump occurs regardless of the size and direction of the change. Naik and Lee (1990) show that the Black-Scholes hedging techniques fail to be self-financing in the presence of random price jumps. Jump risk in the jump-diffusion model with an infinite number of possible jump sizes can only be eliminated by an infinite number of hedging instruments. With a finite number of instruments, there is no perfect hedge even in the theoretical case of continuous rebalancing.[6] Hence, we do not specifically hedge jump risk.

Our simulation results agree with the theory. The dynamic hedging allows for large potential losses because of random price jumps. Figures 3.21 and 3.22 depict paths of the discounted delta-hedging errors over the hedging period. We can see that there are sudden large losses because of random jumps in the underlying prices. As the fund value rises or falls, the value of delta hedging portfolio is always below the contract value because of the option convexity feature of the contract. For the delta-gamma hedging (Figure 3.23 and 3.24), there are both positive and negative hedging error changes. In contrast, the semi-static hedging is much less sensitive to price jumps. The semi-static hedging portfolio can generate a payoff that covers the expected GMWB net liability.

Figure 3.25 and 3.26 show the estimated probability densities of the discounted profit/loss at the end of the hedging period using kernel smoothing method. Under the jump-diffusion model, it is clear that the semi-static hedging outperforms the delta and delta-gamma hedging. In both the partial hedge and the full hedge cases, the distributions of the profit/loss under the delta and delta-gamma hedging exhibit a longer left tail than the semi-static hedging. The semi-static hedging is able to prevent disastrous losses that the delta and delta-gamma hedging cannot do. In the partial hedge case, the distribution from the semi-static hedging has a relatively longer right tail. But the distribution from delta hedging does not show this feature. The delta-gamma hedging result is very sensitive to gamma of the put option. The large hedging

[6] He et al. (2006) develop a dynamic hedging strategy to minimize a measure of the jump risk, and show through simulation that both this strategy and a semi-static strategy can sharply reduce the standard deviation of the profit or loss.

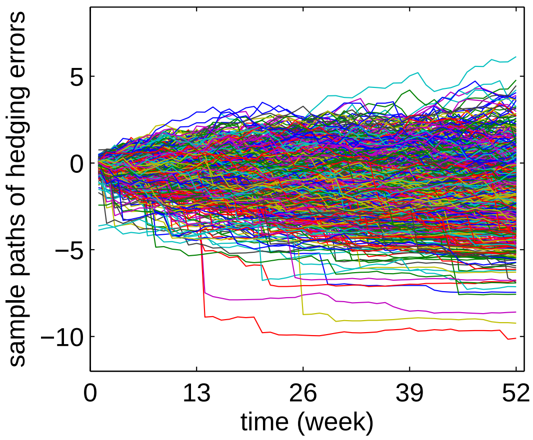

Figure 3.21: Sample paths of delta-hedging errors under the Kou's jump diffusion model in the partial hedge case. ($A_0 = 100$, $\mu = 10\%$, $\sigma = 0.2$, 1,000 scenarios)

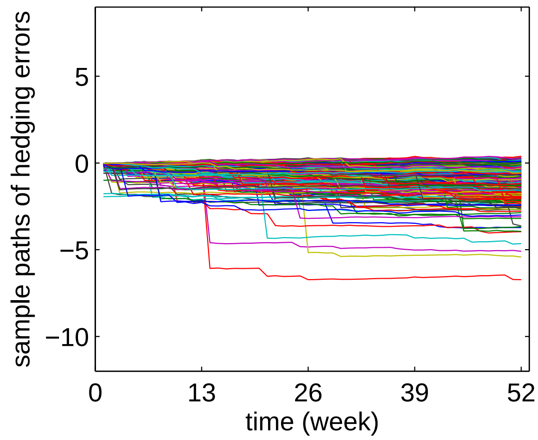

Figure 3.22: Sample paths of delta-hedging errors under the Kou's jump diffusion model in the full hedge case. ($A_0 = 100$, $\mu = 10\%$, $\sigma = 0.2$, 1,000 scenarios)

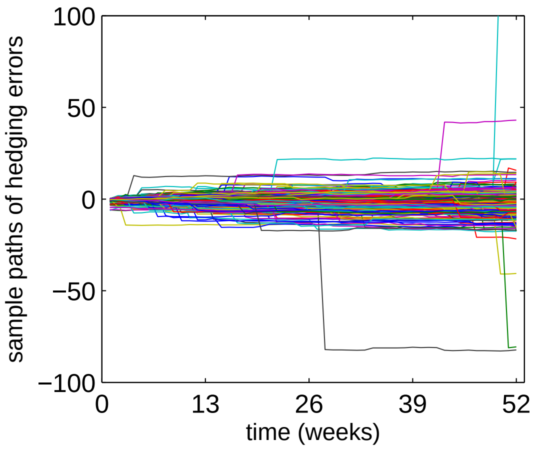

Figure 3.23: Sample paths of delta-gamma hedging errors under the Kou's jump diffusion model in the partial hedge case. ($A_0 = 100$, $\mu = 10\%$, $\sigma = 0.2$, 1,000 scenarios)

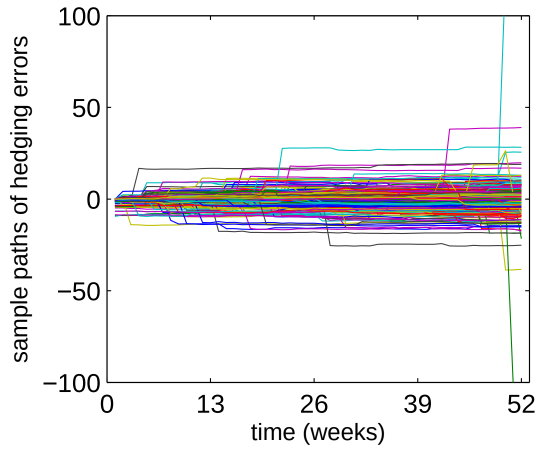

Figure 3.24: Sample paths of delta-gamma hedging errors under the Kou's jump diffusion model in the full hedge case. ($A_0 = 100$, $\mu = 10\%$, $\sigma = 0.2$, 1,000 scenarios)

errors are associated with scenarios where the gamma of the put option becomes near zero. If a put option with a longer maturity is used, the delta-gamma hedging result may be improved.

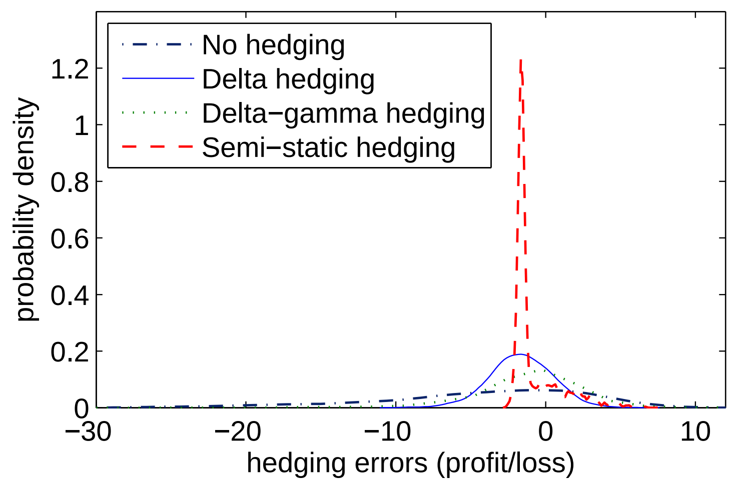

Figure 3.25: Estimated probability densities of the profit/loss at time 1 in the presence of random jumps. The hedging target is the loss part of the GMWB net liability. withdrawals occur quarterly. The dynamic hedging position is updated weekly. The semi-static replicating portfolio consists of six puts and is obtained by the Least Squares.

Table 3.9 and 3.10 show the statistics of the hedging errors under the Kou's jump-diffusion model in partial hedge and full hedge cases respectively. The average loss before hedging is 4.24 under the jump-diffusion model. After the semi-static hedging, the standard deviation of the hedging errors, in the partial hedge case, is smaller than that under the Black-Scholes model. In the full hedge case, the standard deviation increases from 0.21 to 0.23. In contrast, the performance of delta hedging deteriorates under the jump-diffusion model, and all the statistics become worse. Both the average loss and the standard deviation are larger than those after the semi-static hedging. The maximum loss and quantile measures show that the chance of a large loss after delta hedging is higher than that after the semi-static hedging. The kurtosis value for the

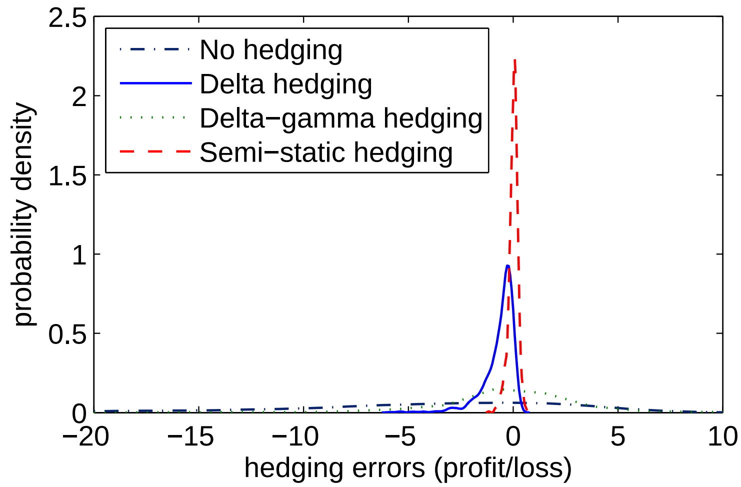

Figure 3.26: Estimated probability densities of the profit/loss at time 1 in the presence of random jumps. The hedging target is the GMWB net value, that is, both the profit and the loss. Assume withdrawals occur quarterly. The dynamic hedging position is updated weekly. The semi-static replicating portfolio consists of eight puts and one call.

Profit/loss	**No hedging**	**Delta-hedging**	**Delta-gamma hedging**	**Semi-static hedging**
mean	-4.2412	-1.6404	-1.0665	-0.7664
median	-2.8820	-1.5961	-0.7126	-1.5201
standard deviation	7.2487	2.0936	7.1680	1.6438
maximum	9.7954	5.9302	138.7238	7.6780
minimum	-32.2215	-9.4061	-82.2782	-2.5595
quantile 1%	-25.8274	-6.8227	-15.3967	-2.2193
quantile 10%	-14.1513	-4.2241	-5.4706	-1.8544
skewness	-2.4675	-0.1301	4.6295	1.9814
kurtosis	12.0982	3.4625	179.7347	6.5983

Table 3.9: Simulated hedge performance comparisons of static and dynamic strategies when there are random jumps under Kou's model. The hedging target is the loss part of the GMWB net liability. ($r = 5\%,\ \sigma = 0.2,\ A_0 = 100,\ \ g = 6.66\dot{6}\%$.)

Profit/loss	No hedging	Delta-hedging	Delta-gamma hedging	Semi-static hedging
mean	-4.2412	-0.6586	-0.1831	0.0107
median	-2.8820	-0.4207	-0.0112	0.0390
standard deviation	7.2487	0.7859	6.7454	0.2252
maximum	9.7954	0.4038	123.6294	0.9194
minimum	-32.2215	-5.8923	-104.2825	-1.1619
quantile 1%	-25.8274	-3.7803	-15.0800	-0.6831
quantile 10%	-14.1513	-1.5548	-4.1943	-0.2602
skewness	-0.9681	-2.4675	2.5062	-0.7249
kurtosis	3.8120	12.0982	174.3477	5.1412

Table 3.10: Simulated hedge performance comparisons of static and dynamic strategies when there are random jumps under Kou's model. The hedging target is the GMWB net value, that is, both the profit and the loss. ($r = 5\%,\ \sigma = 0.2,\ A_0 = 100,\ \ g = 6.66\dot{6}\%$.)

delta-gamma hedging errors becomes very large. A high kurtosis means that infrequent extreme deviations are unavoidable.

3.3 Summary

GMWBs are long-term path-dependent put options. The dynamic hedging strategy requires frequent and intensive simulation of future cash flows over a large number of scenarios. It can be very expensive when the market is very volatile. In addition, the dynamic strategy fails to hedge the risk that arises from random price jumps. The semi-static hedging strategy we propose can guard the insurer from the jump risk. The replicating portfolio only needs to be rebalanced a few times a year, so computing time is largely reduced.

We assume that the underlying fund can be mimicked by an index for simplicity, and that the withdrawal amount is constant at the maximum level without incurring penalty. Under the Black-Scholes model, we set the hedging target as the positive expected GMWB net liability conditional on the index value at the end of the hedging period. Then we search for a portfolio of put options on the index whose payoff replicates the hedging target. One optimization criteria is to minimize the sum of the squared differences between the portfolio payoff and the hedging target. When the portfolio expires, we construct a new portfolio based on the current account value. If we set the hedging target as the conditional expectation of the future GMWB net value, then the initial cost of the replicating portfolio will be reduced. The full-replicating portfolio includes several put options and one call option. Our simulation results show that the semi-static hedging strategy is comparable with the delta and delta-gamma hedging strategies under the Black-Scholes model. When the index value follows a jump-diffusion process, the replicating portfolio obtained under the Black-Scholes model still works well. But the delta and delta-gamma hedging strategies leave large losses behind.

Chapter 4

Hedging GMWBs Under the Heston Model

In the previous chapters, we have modeled the fund value process using a geometric Brownian motion with a constant volatility. However, it is known that this model typically does not provide a satisfactory fit for long-term fund returns. The empirical distribution of equity returns is highly peaked and fat-tailed relative to the normal distribution. This feature indicates that the empirical distribution is a mixture of distributions with different variances. The phenomenon that large moves follow large moves and small moves follow small moves is called volatility clustering. It implies that volatilities over different time periods are not independent because of market inefficiency. By assuming that the volatility of the equity price is a stochastic process, derivatives can be valued more accurately. Traders use the Black-Sholes formula to price options by allowing the implied volatilities to vary with respect to strikes and time to maturities. Stochastic volatility (SV) models can explain the volatility smile and the term structure in a self-consistent way (see Gatheral 2005). In this chapter, we assume the volatility of the fund returns follows the Heston SV model, and we explain how to hedge GMWBs semi-statically under this assumption.

The following SV models have been proposed in the literature: Hull and White (1987) derive a series-form option valuation formula if the asset prices and volatilities are uncorrelated. They assume that the volatility risk can be diversified. Scott (1987) points out that the option price depends on the risk premium related to the random volatility, and use simulation to compute option prices. Wiggins (1987) derives statistical estimators for volatility process parameters, and numerically prices the call option. Stein and Stein (1991) model the volatility as an OrnsteinUhlenbeck process. They assume that the volatility is uncorrelated with the asset price, and derive an option pricing formula conditional on the path of volatility. Heston (1993) develops a closed form solution for options based on characteristic functions. The model allows arbitrary correlation between volatility and asset returns. Stochastic interest rates can also be introduced into the model. These attractive features make the Heston model the most popular SV model.

This chapter is organized as follows:

In Section 4.1, we briefly introduce the Heston model, European option pricing formulas under the model, and some numerical implementation issues.

In Section 4.2, we propose two semi-static hedging strategies under the Heston model for European put options. As far as we know, there is only one paper (Takahashi and Yamazaki 2008) on static hedging of European options under the Heston model. Our simulation results indicate that our proposed semi-static strategies outperform the minimum-variance hedging strategy.

In Section 4.3, we apply the semi-static hedging strategies to GMWBs. The future value of the GMWB is dependent on the path of the fund value. To simulate the fund value process conditioned on the end value under the Heston model, we use a recently proposed acceptance-rejection sampling method and a time-changed Brownian motion.

4.1 The Heston Stochastic Volatility Model

For a stochastic volatility model, the constant volatility σ in the Black-Scholes model is replaced with a square root of variance ν_t that is stochastic,

$$dS_t = \mu S_t \, dt + \sqrt{\nu_t} S_t \, dW_t. \tag{4.1}$$

It is convenient to write it in terms of the logarithm of the price $x = ln(S)$

$$dx_t = (\mu - \frac{\nu_t}{2})dt + \sqrt{\nu_t} dW_t. \tag{4.2}$$

The form of ν_t is specified by a particular diffusion process. In the Heston model, the stochastic differential equation for variance takes the form of the square root mean-reverting process:

$$d\nu_t = \kappa(\theta - \nu_t)dt + \sigma_\nu \sqrt{\nu_t} \, dW_t^v, \tag{4.3}$$

where

- θ is the mean long-term variance,
- κ is the rate at which the variance reverts toward its long-term mean, (The larger κ, the more rapidly the variance converges to θ.)
- σ_ν is the volatility of the variance process,

- $\{W_t^v\}$ is a standard Brownian motion with $dW_t dW_t^v = \rho dt$. Typically, ρ is negative, which indicates that a down-move in the equity price is correlated with an up-move in the variance.

It is known that for any $t > 0$, the random variable ν_t is non-centrally chi-square distributed and the volatility process $\sqrt{\nu_t}$ is Rayleigh distributed (see Miller et al. 1958).

When volatility becomes random and there are no traded assets on volatility, the market is not complete. Scott (1987) shows that the option value must satisfy a partial differential equation whose solution depends on an unspecified parameter called the market price of volatility risk. Heston (1993) specifies a volatility risk premium that is proportional to the variance ν. Once the market price of volatility risk is determined, a certain risk-neutral measure Q can be derived uniquely, and the price of the option is equal to the expected value of the discounted future payoffs under the measure Q. The fund value and the volatility processes that we are going to use in this chapter become

$$dS_t = rS_t\,dt + \sqrt{\nu_t}S_t\left[\sqrt{1-\rho^2}dW_{1t} + \rho dW_{2t}\right] \tag{4.4}$$

$$d\nu_t = \kappa^*(\theta^* - \nu_t)dt + \sigma_\nu\sqrt{\nu_t}\,dW_{2t} \tag{4.5}$$

$$\kappa^* = \kappa + \lambda$$

$$\theta^* = \frac{\kappa\theta}{\kappa+\lambda} \tag{4.6}$$

where $\{W_{1t}\}$ and $\{W_{2t}\}$ are independent standard Brownian motions under the measure Q, and ρ is the instantaneous correlation coefficient between S and ν. To simplify the exposition, we will still use κ and θ to represent κ^* and θ^* under the measure Q. The parameters can be estimated using market prices of options on the same underlying S. (see Bakshi et al. 1997)

4.1.1 Pricing European Options Under the Heston Model

Heston (1993) shows that the price at time t of a European call option with a maturity of $T-t$ and a strike of K is given by

$$C(S_t,\nu_t,t) = S_tP_1 - Ke^{-r(T-t)}P_2, \tag{4.7}$$

$$P_j\big(x,\nu,T,t;\ln(K)\big) = \frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi\ln(K)}f_j(x,\nu,T,t;\phi)}{i\phi}\right]d\phi, \quad j=1,2, \tag{4.8}$$

$$f_j(x,\nu,T,t;\phi) = e^{C_j(T,\phi)+D_j(T,\phi)\nu+i\phi x}, \quad j=1,2, \tag{4.9}$$

$$C_j(T,t,\phi) = r\phi i(T-t)+\frac{a}{\sigma_\nu^2}\left\{(b_j-\rho\sigma_\nu\phi i+d)(T-t)-2\ln\Big[\frac{1-ge^{d(T-t)}}{1-g}\Big]\right\}, \tag{4.10}$$

$$D_j(T,t,\phi) = \frac{b_j-\rho\sigma_\nu\phi i+d}{\sigma_\nu^2}\Big[\frac{1-e^{d(T-t)}}{1-ge^{d(T-t)}}\Big], \tag{4.11}$$

$$g = \frac{b_j-\rho\sigma_\nu\phi i+d}{b_j-\rho\sigma_\nu\phi i-d},$$

$$d = \sqrt{(b_j-\rho\sigma_\nu\phi i)^2-\sigma_\nu^2(2u_j\phi i-\phi^2)},$$

$$a=\kappa\theta,\ u_1=\frac{1}{2},\ u_2=-\frac{1}{2},\ b_1=\kappa-\rho\sigma_\nu,\ b_2=\kappa.$$

where Re[ ] denotes the real part of a complex variable. The corresponding put option price is given by

$$P(S_t,\nu_t,t) = Ke^{-r(T-t)}(1-P_2)-S_t(1-P_1). \tag{4.12}$$

An alternative equivalent formula for the characteristic function is used in Bakshi et al. (1997); Duffie et al. (2000); Gatheral (2005), where the authors assume the form (4.9) with

$$C_j(T,t,\phi) \;=\; r\phi i(T-t)+\frac{a}{\sigma_\nu^2}\Big\{(b_j-\rho\sigma_\nu\phi i-d)(T-t)-2\ln\Big[\frac{1-\frac{1}{g}e^{-d(T-t)}}{1-\frac{1}{g}}\Big]\Big\}, \tag{4.13}$$

$$D_j(T,t,\phi) = \frac{b_j - \rho\sigma_\nu\phi i - d}{\sigma_\nu^2}\Big[\frac{1-e^{-d(T-t)}}{1-\frac{1}{g}e^{-d(T-t)}}\Big]. \qquad (4.14)$$

This different representation of the characteristic function leads to procedures that avoid some numerical problems that may occur when using the previous formula. In the next subsection, we explain the problem in detail.

Fourier inversion methods can be applied in different ways to get closed-form formulas for European call and put prices. Sepp (2003) proposes an option pricing formula that only involves one integral, so the price can be computed very efficiently. Under this approach, we first get the analytic expression of the forward Fourier transform of the payoff function, and then invert the transform using the characteristic function of x_T to obtain the option prices. The prices of the European call and put options are given by

$$C(S_t,\nu_t,t) = S_t - E_t^Q[e^{-r(T-t)}\min(S_T,K)], \qquad (4.15)$$

$$P(S_t,\nu_t,t) = Ke^{-r(T-t)} - E_t^Q[e^{-r(T-t)}\min(S_T,K)], \qquad (4.16)$$

where $E_t^Q[e^{-r(T-t)}\min(S_T,K)]$ has the following representation:

$$\begin{aligned}
E_t^Q[e^{-r(T-t)}\min(S_T,K)] &= Ke^{-r(T-t)}\frac{1}{\pi}\int_0^\infty \mathrm{Re}\Big[\frac{O(X,\nu_t,T;\phi)}{\phi^2+1/4}\Big]d\phi, \qquad (4.17)\\
O(X,\nu_t,T;\phi) &= e^{(-i\phi+0.5)X+A(\phi,T)+B(\phi,T)\nu_t},\\
X &= \ln(S_t/K)+r(T-t),\\
A(\phi,T) &= \frac{\kappa\theta}{\sigma_{\nu_t}^2}\left[c_1(T-t)+2\ln\Big(\frac{c_2+c_1e^{-d(T-t)}}{2d}\Big)\right],\\
B(\phi,T) &= -(\phi^2+\frac{1}{4})\frac{1-e^{-d(T-t)}}{c_2+c_1e^{-d(T-t)}}.\\
c_1 &= -u-\rho\sigma_\nu\phi i+d, \quad c_2 = u+\rho\sigma_\nu\phi i+d,
\end{aligned}$$

$$
\begin{aligned}
d &= \sqrt{\phi^2\sigma_\nu^2(1-\rho^2) + 2i\phi\rho\sigma_\nu u + u^2 + \frac{\sigma_\nu^2}{4}}, \\
u &= \kappa - \frac{\rho\sigma_\nu}{2}.
\end{aligned}
$$

It can be shown that the integrals are uniformly convergent, so the partial derivatives of the option price with respect to S and ν can be obtained by differentiating the integrand. In Section 4.2, we will use this approach to value European put options.

4.1.2 Numerical Implementation

The integrals in the pricing formulas (4.8) and (4.17) have to be evaluated numerically with great precision for a wide range of parameters. Adaptive Simpson and adaptive Gauss-Lobatto quadrature have been suggested in the literature. Kahl and Jackel (2005) point out that there are numerical problems when using characteristic function with equation (4.10), whereas using equation (4.13) always seemed to lead to a stable procedure. Equation (4.16) also gives stable results. The problem with equation (4.10) is due to the fact that the integrand involves multi-valued functions such as the complex logarithm and square root. The complex square root d has two values with opposite signs. The value with positive sign is the principal branch. The complex logarithm of $z = re^{i\varphi}$ is given by

$$
\ln z = \ln r + i(\varphi + 2n\pi), \quad \text{where } n \text{ is an integer.}
$$

The principal branch is obtained by limiting $\varphi \in [-\pi, \pi]$ and setting $n = 0$. Most software packages use the principal branch of the function. Then the characteristic function can become discontinuous, which results in wrong option prices. For options with short or middle term maturities, the pricing error is not noticeable. But long-term options can be mispriced significantly. Albrecher et al. (2007) show that for nearly any choice of parameters in the Heston model, these instabilities occur for large enough maturity. They also prove that these problems do not occur at all when using (4.13) under the full dimensional and unrestricted parameter space. Lord and Kahl (2008) also investigate the problem and prove the stability of the equation (4.13) with certain parameter constraints. By formulating the characteristic function properly like in (4.13), we can get the correct price using the principal branch.

Carr and Madan (1999) have developed an efficient method based on Fast Fourier Transform to compute option prices for a range of strikes. The basic idea is to derive an analytic expression for the Fourier transform of the option price, and then to get the price by Fourier inversion. The Fourier transform and its inversion work for square-integrable functions (see Rudin 1991, Plancherel's theorem). However, the call option price function

$$\int_k^\infty e^{-rT}(e^s - e^k)p_T(s)ds$$

is not square integrable, where $k = \ln(K)$ and $p_T(s)$ is the risk-neutral density of $s_T = \ln(S_T)$.[1] Carr and Madan (1999) suggest to multiply the price function by a factor $e^{\alpha k}$ with a suitable $\alpha > 0$. Cont and Tankov (2004) propose an alternative approach by subtracting an intrinsic value from the option price. This approach does not require choosing an appropriate value for α. But for short maturities and strikes near the spot price, this approach overprices the options. It is because the intrinsic value of an option is not differentiable at the spot price. Another approach is to subtract the Black-Scholes price with appropriate volatility from the option price instead of subtracting

[1]The call price goes to S_0 when k approaches $-\infty$.

the intrinsic value. This approach is used in Takahashi and Yamazaki (2008).

4.1.3 The Simulation Method

The simulation-based approach under the Heston model is useful for path-dependent products. We use the following representation and a discretization scheme proposed by Andersen (2007) to value options and the GMWB:

$$\ln(\frac{S_t}{S_0}) = rt - \frac{\int_0^t \nu_s ds}{2} + \frac{\rho}{\sigma_\nu}\big[\nu_t - \nu_0 - \kappa(t\theta - \int_0^t \nu_s ds)\big] + \sqrt{1-\rho^2}\int_0^t \sqrt{\nu_s} dW_{1s} \quad (4.18)$$

Kolkiewicz and Tan (2006) have used equation (4.18) in valuing unit-linked life insurance products. The distribution of $\ln \frac{S_t}{S_0}$ conditional on ν_t and $\int_0^t \nu_s ds$ is normal with mean $\tilde{\mu}_t$ and variance $\tilde{\sigma}_t^2$, where

$$\tilde{\mu}_t = \big(rt - \frac{\int_0^t \nu_s ds}{2}\big) + \frac{\rho}{\sigma_\nu}\big[\nu_t - \nu_0 - \kappa(t\theta - \int_0^t \nu_s ds)\big] \quad (4.19)$$

$$\tilde{\sigma}_t^2 = (1-\rho^2)\int_0^t \nu_s ds \quad (4.20)$$

This allows us to price the put option by

$$\begin{aligned} P(S_0, K, \nu, T) &= E\big[e^{-rT}\max(0, K - S_T) \mid (\nu_t)_{0\le t\le T}\big] \\ &= e^{-rT}K\Phi(-d_2) - S_0 e^{\tilde{\mu}_T - rT + \frac{\tilde{\sigma}_T^2}{2}}\Phi(-d_1), \quad (4.21) \\ d_1 &= \frac{\ln\frac{S_0}{K} + \tilde{\mu}_T + \tilde{\sigma}_T^2}{\tilde{\sigma}_T}, \\ d_2 &= d_1 - \tilde{\sigma}_T. \end{aligned}$$

There are many discretization schemes for the Heston model (4.4)∼(4.5) in the literature. We do not need to simulate both processes but rather the end value ν_T and the integral $\int_0^T \nu_s ds$. Broadie and Kaya (2006) develop a scheme to simulate the variance process from the exact distribution based on an acceptance-rejection method. The integrated variance process is generated from numerical Fourier inversion of its conditional characteristic function, which is time-consuming and complex. Lord et al. (2008) consider an Euler scheme with a full truncation rule to deal with negative variance values in a direct discretization. The authors show that the computational efficiency of the scheme exceeds that of the more complicated schemes in Broadie and Kaya (2006). However, the scheme is largely heuristic and uses none of the known analytical results for the Heston model. To obtain a reasonably low discretization bias, the grid has to be very small. Smith (2008) approximates the Fourier inversions required to simulate the integrated variance process. Andersen (2007) develops two schemes for the variance process based on moment-matching technique, and uses drift interpolation, instead of Fourier inversion, to approximate the integrated variance process. Both schemes outperform all other schemes in terms of computational efficiency, and the Quadratic-Exponential scheme is the best. The algorithm approximates the new variance which follows a non-central chi-square distribution by a squared normal random variable if the current variance is large. If the current variance is close to zero, use a random variable from a modified exponential distribution to approximate the new variance. We will use this scheme to price and hedge GMWBs later in this chapter.

The algorithm proposed by Andersen (2007) can be formulated as follows: Assume we use N equally-spaced time steps, and the step length is h. We first simulate ν_{i+1} given ν_i, $i = 0, \cdots, N-1$. Then S_{i+1} can be simulated from equation (4.18), (4.19), and (4.20). Knowing ν_i, we can simulate the next value, ν_{i+1}, in the following way:

1. Select an arbitrary level $\Psi_c \in [1, 2]$, e.g. $\Psi_c = 1.5$.

2. Given ν_i, compute x and s^2,

$$x = \theta + (\nu_i - \theta)e^{-\kappa h}, \tag{4.22}$$

$$s^2 = \frac{\nu_i \sigma_\nu^2 e^{-\kappa h}}{\kappa}(1 - e^{-\kappa h}) + \frac{\theta \sigma_\nu^2}{2\kappa}(1 - e^{-\kappa h})^2. \tag{4.23}$$

3. Compute $\Psi = \frac{s^2}{x^2}$.

4. If $\Psi \leq \Psi_c$:

   (a) Compute a and b,

$$b^2 = 2\Psi^{-1} - 1 + \sqrt{2\Psi^{-1}(2\Psi^{-1} - 1)}, \tag{4.24}$$

$$a = \frac{x}{1 + b^2}. \tag{4.25}$$

   (b) Set $\nu_{i+1} = a(b + z_i)^2$, where z_i is a standard normal random number.

5. If $\Psi > \Psi_c$:

   (a) Compute β and p,

$$p = \frac{\Psi - 1}{\Psi + 1}, \tag{4.26}$$

$$\beta = \frac{2}{x(\Psi + 1)}. \tag{4.27}$$

   (b) Draw a uniform random number U.

   (c) Set $\nu_{i+1} = \begin{cases} 0, & 0 \leq U \leq p \\ \beta^{-1} \ln \frac{1-p}{1-U}, & p < U \leq 1. \end{cases}$

Conditional on ν_i and $\int_{ih}^{(i+1)h} \nu_s ds$, $\ln(\frac{S_{i+1}}{S_i})$ follows a normal distribution with mean

$\tilde{\mu}_{i+1}$ and variance $\tilde{\sigma}^2_{i+1}$, where

$$\tilde{\mu}_{i+1} = \left(rh - \frac{\int_{ih}^{(i+1)h} \nu_s ds}{2}\right) + \frac{\rho}{\sigma_\nu}\left[\nu_{i+1} - \nu_i - \kappa(h\theta - \int_{ih}^{(i+1)h} \nu_s ds)\right], \quad (4.28)$$

$$\tilde{\sigma}^2_{i+1} = (1-\rho^2)\int_{ih}^{(i+1)h} \nu_s ds. \quad (4.29)$$

For a small time step h, we can approximate $\int_{ih}^{(i+1)h} \nu_s ds$ with $(\nu_i + \nu_{i+1})h/2$. Then, S_{i+1} can be simulated using the above results and a standard normal random number.

4.2 Semi-static Hedging of European Put Options under the Heston Model

The guarantees provided by insurers in variable annuity products embed complicated put options. It is beneficial and interesting to study the static hedging problem of the standard European options before we deal with GMWBs. In Section 4.2.1, we will explore semi-static hedging strategies for a long-term European put option on an index whose volatility follows the Heston model. We will also compare the semi-static hedging strategies with the minimum-variance hedging strategy in Section 4.2.2.

Suppose our hedging target is a T-year European put option with a strike price of K. At time 0, we want to set up a hedging portfolio such that by time t, $0 < t < T$, the portfolio can replicate the time-t value of the T-year put option. That is, our initial hedging horizon is t years. Assuming the strike prices of the portfolio are fixed, we aim to search for the optimal portfolio weights that make time-t-values of the portfolio and the T-year put as close as possible. After t years, we will rebalance the portfolio based on the new underlying price. In this section, we propose two approaches to hedge the

T-year put semi-statically.

- The first strategy is to replicate the time-t-value of the target T-year put option using a portfolio of put options with a shorter maturity $T_p \in (t, T)$ in the initial hedging period. The purpose of choosing $T_p > t$ is to take advantage of the fact that the time-t-value of the portfolio is also a function of the volatility at time t. Thus, the volatility risk of the T-year put is mitigated to some extent. In fact, the replicating effect becomes better as T_p gets closer to T. We compute the time-t values of the T-year put and the T_p-year puts conditional on each pair of the underlying price and volatility $(S_t, \sqrt{\nu_t})$. The portfolio weights are solved by minimizing the mean squared difference between the target option value and the portfolio value at time t.

- The second strategy is to replicate the target option using a portfolio of puts that will expire at the end of the initial hedging period. We observe that the time-t value of a put option with maturity T varies largely with the underlying price at time t, but not that significantly with the volatility at time t. Thus, we can value the put options at the points $(S_t, \sqrt{\bar{\nu}_t})$ where $\bar{\nu}_t = E(\,\nu_t \mid S_t)$. That is, the conditional mean value of the volatility at time t is used as a representative of all volatilities associated with a given underlying price at time t. The problem is reduced to one dimension. The portfolio weights are obtained by minimizing the sum of squared differences between the target option value and the portfolio payoff weighted by the estimated marginal probability density of S_t. This method reduces the amount of computation at the cost of accuracy.

In their recent paper, Takahashi and Yamazaki (2008) propose a static hedging strategy for European options under a SV model by transforming it into a local volatility model.

They obtain the local volatility from the option price computed under the SV model, which is based on the result developed by Dupire (1994). Then they approximate the original asset price with a shadow asset price that follows the local volatility model. The static hedging approach proposed by Carr and Madan (2001) is then applied to the shadow asset price. Using a numerical example under the Heston model, the authors demonstrate that this static strategy outperforms the minimum-variance hedging strategy. The efficiency of this method relies on the existence of closed-form solution of the option price. Thus, it is difficult to apply their method to path-dependent contracts. As far as we know, this thesis is the first work that applies semi-static hedging to path-dependent contracts like GMWBs. In fact, the approaches that we propose in this section are suitable for both path-independent and path-dependent options.

4.2.1 The Two Semi-static Hedging Strategies

The idea of semi-static hedging is to replicate the intermediate value of the long-term T-year put option with a portfolio of short-term put options. In this subsection, we introduce two approaches to construct the semi-static hedging portfolio.

Strategy One: Two-dimension

The value of the put option at a future time before the maturity depends on the future underlying price and volatility. Denote the time-t value of the target put option by $V^T(\nu_t, S_t)$. The replicating portfolio consists of T_p-year put options, $T_p \in (t, T)$, with k different strikes. We denote the time-t values of these T_p-year put options by a $1 \times k$ vector-valued function of two arguments $V_p^T(\nu_t, S_t)$, and denote the portfolio weights by a $k \times 1$ vector θ. We also assume the interest rate is constant at r. The vector of optimal portfolio weights is obtained by minimizing the average squared difference between the time-t values of the hedging portfolio and the T-year put option. The joint probability density of the underlying price and variance at time t, $p(\nu_t, S_t)$, is used as the weight in the Least Squares method. Then our optimization problem can be represented as

$$\min_{\theta} E_t \|V^T - V_p^T \theta\|^2 = \min_{\theta} \int_0^{\infty} \int_0^{\infty} \Big[V^T(\nu_t, S_t) - V_p^T(\nu_t, S_t)\theta \Big]^2 p(\nu_t, S_t) dS_t d\nu_t. \quad (4.30)$$

The joint distribution of the underlying price and variance can be derived from the joint characteristic function in the same way as in Heston (1993). The characteristic function can be obtained from a partial differential equation, and then the density function can be represented in terms of the characteristic function by the inversion theorem. However, numerical integration is computationally demanding. Zhylyevskyy (2005) uses Fast Fourier Transform with kernel smoothing to approximate the integral. Since the densities are used as weights only, they do not have to be very accurate. We use the simulation and histogram method to estimate the probability densities.

The implementation of this semi-static hedging strategy takes the following steps:

1. Choosing a set of underlying prices at time t, $S_{t,j},\ j = 1, \cdots, n_s$, and a set of variances $\nu_{t,i},\ i = 1, \cdots, n_v$. The range of the values can be determined from simulation.
2. Estimating the joint probability densities at the chosen points by an empirical distribution obtained through simulations, $p(\nu_{t,i},\ S_{t,j})$.
3. Calculating the time-t value of the T-year put option using equation (4.12) or (4.16) at each point $(\nu_{t,i},\ S_{t,j})$.
4. Determining the time-t values of the T_p-year put options at points $(\nu_{t,i},\ S_{t,j})$.
5. Obtaining the portfolio weights by Weighted Least Squares, i.e. minimizing the weighted squared difference between the value of the portfolio and the T-year put.

Strategy Two: One-dimension

The replicating portfolio expires at time t. Denote the portfolio payoff at time t by $F(S_t)$. The time-t value of the target put option is also dependent on the variance at time t. To replicate the option that depends on volatility with the ones that do not, we value all the options using the conditional mean of the variance $\bar{\nu}_t = E[\nu_t \mid S_t]$. The squared differences between the payoff of the hedging portfolio and the value of the target put option are weighted by the marginal probability densities of the underlying price, $p^*(S_t)$. The optimization problem becomes

$$\min_{\theta} \int_0^{\infty} \Big[V^T(\bar{\nu}_t, S_t) - F(S_t)\theta \Big]^2 p^*(S_t) dS_t. \tag{4.31}$$

For the second strategy, the implementation procedure is similar. The option value at $(\bar{\nu}_{t,j}, S_{t,j})$ is simulated in the same way as before. The optimal portfolio is obtained as in Section 3.1.4 by following the steps:

1. Choosing a set of underlying prices at time t, $S_{t,j},\ j = 1, \cdots, n_s$, and estimating the mean of the variance conditional on the price, $\bar{\nu}_{t,j} = E[\nu_t \mid S_{t,j}]$.
2. Estimating the marginal probability densities at the chosen points by the empirical distribution obtained through simulations, $p^*(S_{t,j})$.
3. Calculating the time-t value of the T-year put option using equation (4.12) or (4.16) at each point $(\bar{\nu}_{t,i},\ S_{t,j})$.
4. Determining the payoff of the t-year put portfolio.
5. Solving the portfolio weights by Weighted Least Squares.

A numerical example

We use the parameters as given by Bakshi et al. (1997) who estimated them from the S&P500 index:

$$\kappa = 1.15, \quad \theta = 0.04, \quad \sigma_\nu = 0.39, \quad \rho = -0.64.$$

The parameters represent the mean reversion speed, the long-run variance, the volatility of the variance process, and the correlation between the underlying return changes and volatility changes.

In addition, we have the following assumptions:

- The hedging horizon is one year, $t = 1$;
- The initial underlying value is $S_0 = 100$;
- The risk-free interest rate is $r = 0.05$;
- The strike of the target put option is $K = 100$;
- The maturity of the target put option is $T = 10$;
- The hedging portfolio for the first strategy has a maturity of $T_p = 2$.

Figure 4.1 shows the joint probability density of the underlying values and variances at time t. The conditional distribution of the underlying value is lognormal. The underlying value and the variance are negatively correlated.

The value of the 10-year put option at time $t = 1$ is a convex decreasing function of the underlying price conditional on a specific variance at time 1. This is the same as that under a constant volatility assumption. Under the Heston model, when we fix the underlying price at time 1, then the value of the 10-year put is a slowly increasing function of the variance at time 1. Overall, the put value does not change with the variance as significantly as with the underlying price. Figure 4.2 shows the surface of

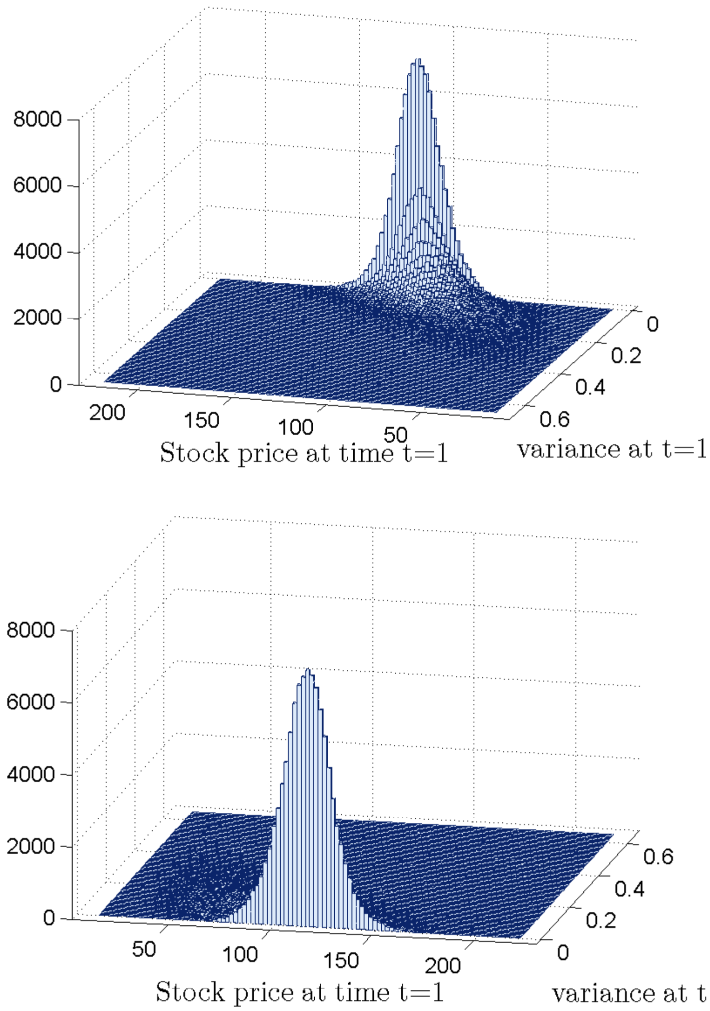

Figure 4.1: Histogram of the underlying values and variances at time t=1 under the Heston model ($S_0 = 100,\ r = 0.05,\ \kappa = 1.15,\ \theta = 0.04,\ \sigma_\nu = 0.39,\ \rho = -0.64$)

the 10-year put values. In Figure 4.3, the value of the 2-year at-the-money put option has a similar shape. If the strike price is higher, the surface bends up starting from a higher underlying price.

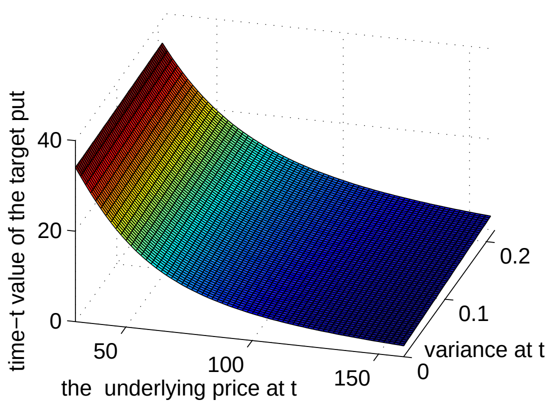

Figure 4.2: The time-t value of the target put option, V_t^T, conditional on the underlying price and variance under the Heston model ($S_0 = 100,\ r = 0.05,\ t = 1,\ T = 10,\ K = 100,\ \kappa = 1.15,\ \theta = 0.04,\ \sigma_\nu = 0.39,\ \rho = -0.64$)

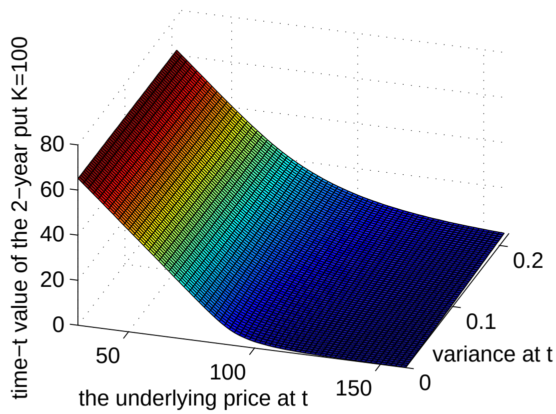

Figure 4.3: The time-t value of the 2-year at-the-money put conditional on the underlying price and variance under the Heston model ($S_0 = 100,\ r = 0.05,\ t = 1,\ T = 10,\ K = 100,\ \kappa = 1.15,\ \theta = 0.04,\ \sigma_\nu = 0.39,\ \rho = -0.64$)

The strikes of the puts in the replicating portfolio are assumed to be 50, 60, 70, 80, 90, 100, 110, 120, 130, 140, and 150. To get a better fit, we also need to borrow from a bank account. We limit our position on the puts to be long only. That is, the minimum weights for the put options are set to zero. This constraint could be relaxed in practice. It would reduce hedging effectiveness, but also reduce up-front costs. Table 4.1 lists the optimal portfolio weights for the two hedging strategies obtained by solving equations (4.30) and (4.31) respectively. For the first strategy, we also compute the weights of the replicating portfolio using the Least Squares approach to compare with that using Weighted Least Squares approach. For this, options with strikes of 100, 130 and 140 are not used. The in-the-money option struck at 150 is expensive, and it costs more than other options in the portfolio. For the second strategy, we only calculate the portfolio weights under the WLS approach. The weights are more uniformly spread.

	Strategy One			**Strategy Two**	
Strikes	**Price**	**WLS**	**LS**	**Price**	**WLS**
Cash	0.9512	2.7015	2.8401	0.9512	2.8665
50	0.2655	0.3192	0.4680	0.0640	0.1802
60	0.6052	0.0942	0	0.2081	0.1061
70	1.2196	0.1109	0.0968	0.5578	0.0840
80	2.2486	0.0667	0.1311	1.3010	0.0685
90	3.8754	0.0734	0	2.7334	0.0497
100	6.3283	0	0	5.2974	0.0432
110	9.8630	0.0627	0.0159	9.5801	0.0320
120	14.6999	0.0093	0.0932	16.0211	0.0241
130	20.9058	0	0	24.2532	0.0146
140	28.2927	0	0	33.3529	0.0019
150	36.4873	0.0618	0.0419	42.7415	0.0450
Cost		6.2927	6.2927		6.2927

Table 4.1: Semi-static replicating portfolio for a European put option under the Heston model ($r = 5\%,\ S_0 = 100,\ t = 1,\ T = 10,\ K = 100,\ \kappa = 1.15,\ \theta = 0.04,\ \sigma_\nu = 0.39,\ \rho = -0.64$)

Figure 4.4 and 4.5 predict the replicating errors for the first strategy at time t as a function of the underlying price and the variance. The surfaces are relatively flat and close to zero. The maximum of absolute errors under the LS approach are smaller, but the mean absolute error is larger than that under the WLS approach. Figure 4.6 shows that the replicating errors under the second strategy are very large at some underlying prices and variances at time t. Those points tend to have low probability densities. Overall, the first strategy gives a closer replication than the second strategy.

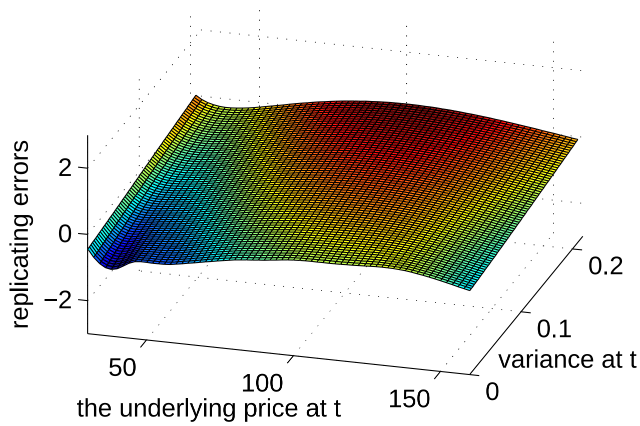

Figure 4.4: The replicating errors at time $t = 1$ for the first strategy using WLS approach($S_0 = 100,\ r = 0.05,\ \kappa = 1.15,\ \theta = 0.04,\ \sigma_\nu = 0.39,\ \rho = -0.64,\ T_p = 2$)

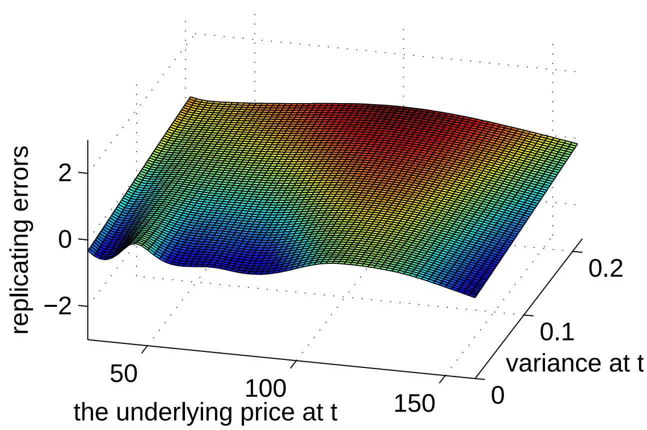

Figure 4.5: The replicating errors at time $t = 1$ for the first strategy using LS approach ($S_0 = 100,\ r = 0.05,\ \kappa = 1.15,\ \theta = 0.04,\ \sigma_\nu = 0.39,\ \rho = -0.64,\ T_p = 2$)

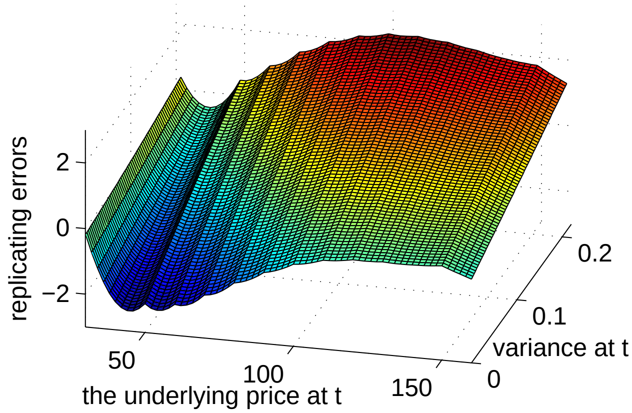

Figure 4.6: The replicating errors at time $t = 1$ with one-year puts for the second strategy using WLS approach($S_0 = 100,\ r = 0.05,\ \kappa = 1.15,\ \theta = 0.04,\ \sigma_\nu = 0.39,\ \rho = -0.64,\ T_p = 1$)

Table 4.2 summarizes three statistics of replicating errors: the maximum absolute error, the sum of absolute errors and the mean absolute error. Note that the joint probability density is estimated based on a histogram method. It is only an approximation, but it is proportional to the true value and can be used for comparison of the two strategies. Thus, the mean absolute errors in the table are not comparable to the maximum absolute errors.

	Strategy One		**Strategy Two**
	WLS	LS	WLS
max abs err	0.9841	0.6221	2.4958
sum abs err	1199.4	1008.8	4449.7
mean abs err	22.9854	28.7494	83.9413

Table 4.2: Comparison of replicating errors under the Heston model

4.2.2 Comparison with Dynamic Hedging

In this subsection, we compare our semi-static hedging strategies with two types of dynamic hedging strategy: the minimum-variance (MV) and delta hedging strategy. The MV hedging strategy uses the underlying as the only hedging instrument. Under the stochastic volatility model, the market is incomplete, so there is no perfect hedge by trading only the underlying.[2] But this strategy is more practical in the presence of untraded risks or model misspecifications and transaction costs. If the hedging instruments are not limited to the underlying, we can take a position in another put option to achieve a complete delta-neutral hedge under the stochastic volatility model.

The minimum-variance delta Δ^{mv} is computed as the amount of the underlying asset that minimizes the instantaneous variance of a delta-hedged portfolio. For a put option, it solves the optimization problem

$$\min_{\Delta} \mathrm{Var}(\Delta dS - dP) = \min_{\Delta} \big[\Delta^2 \mathrm{Var}(dS) - 2\mathrm{Cov}(dS, dP)\Delta + \mathrm{Var}(dP)\big]. \tag{4.32}$$

The solution of this problem is given by the ratio of the instantaneous covariance between increments in the option price and the underlying price and the instantaneous variance of the increments in the underlying price. Under the Heston model (4.3), the MV delta is obtained as follows:

$$\begin{aligned} \Delta^{mv} &= \frac{\mathrm{Cov}(dS, dP)}{\mathrm{Var}(dS)} = \frac{\mathrm{Cov}(dS, \frac{\partial P}{\partial S} dS + \frac{\partial P}{\partial \nu} d\nu)}{\mathrm{Var}(dS)} \quad (4.33) \\ &= \frac{\partial P}{\partial S} + \frac{\partial P}{\partial \nu} \frac{\mathrm{Cov}(dS, d\nu)}{\mathrm{Var}(dS)} \quad (4.34) \end{aligned}$$

[2] Schweizer (1991) proposes to use locally risk-minimizing hedges, which aim at minimizing the variance of the cost process of non-self-financing hedges. Bakshi et al. (1997), Frey (1997) and others have applied this MV hedging method to the stochastic volatility model.

$$= \frac{\partial P}{\partial S} + \frac{\rho \sigma_\nu}{S} \frac{\partial P}{\partial \nu}. \tag{4.35}$$

In the Black-Scholes model, the MV delta is the same as the standard delta. When the volatility is correlated with the asset price, the MV delta is equal to the standard delta plus an additional term. The volatility risk is partially hedged through the correlation between the underlying price and the volatility.

Delta hedging strategy involves another put option P_2 with a short maturity. In the hedging portfolio, the amount of the underlying is denoted by D_S and the amount of the put option is denoted by D_P. They are determined by the following equations

$$D_P = \frac{\frac{\partial P}{\partial \nu}}{\frac{\partial P_2}{\partial \nu}}, \tag{4.36}$$

$$D_S = \frac{\partial P}{\partial S} - \frac{\partial P_2}{\partial S} D_P. \tag{4.37}$$

To continue with the previous example, we would like to hedge the 10-year put option with a one-year horizon. Assume the return of the underlying asset under the physical measure is $r_p = 10\%$. The dynamic hedging position is adjusted at a time step of $h = \frac{1}{100}$. We compare the two semi-static hedging strategies with the MV strategy based on 10,000 simulations.

Figure 4.7 depicts the estimated probability densities of the hedging errors at the end of the hedging period. It shows that both of the semi-static strategies outperform the MV strategy, and that the delta-neutral strategy gives the best result. The distribution of hedging errors from the MV strategy has a much longer left tail. The variations of hedging errors from both semi-static strategies are smaller than that from the MV hedging. The second semi-static strategy does not explicitly hedge over volatilities, so the distribution of hedging errors still has a longer left tail. The first semi-static hedging portfolio is obtained from two-dimension optimization, and the distribution of hedging errors is almost symmetric.

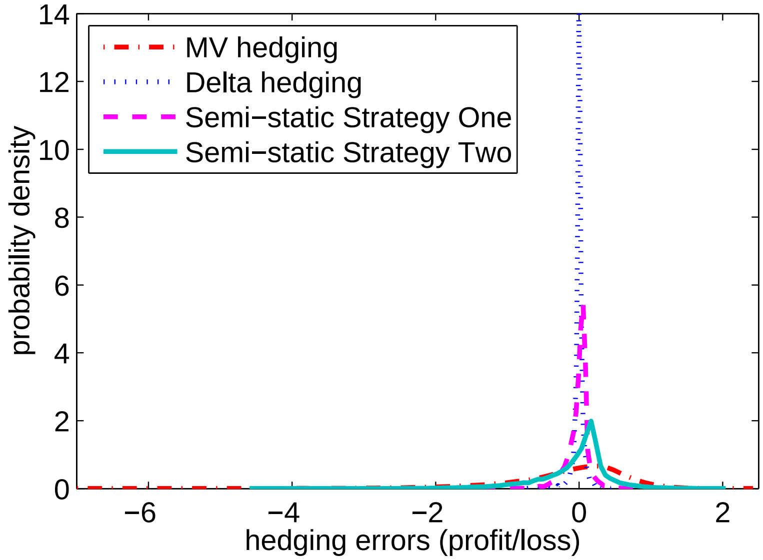

Figure 4.7: The estimated probability densities of hedging errors in one year under the Heston model. The drift term of the underlying price under the physical measure is assumed to be $r_p = 10\%$. The MV and delta hedging portfolios are rebalanced at each time interval $h = 1/100$. The static portfolios for both strategies are obtained by WLS approach. ($S_0 = 100,\ r = 0.05,\ r_p = 0.1,\ \kappa = 1.15,\ \theta = 0.04,\ \sigma_\nu = 0.39,\ \rho = -0.64,\ t = 1$)

Table 4.3 summarizes the statistics of the hedging errors based on the simulation. Based on the simulated mean and standard deviation of the hedging errors, we test whether the true mean of the hedging errors is zero. Only the delta-neutral hedging strategy gives zero mean hedging error. The other three strategies result in small losses, and both the semi-static strategies give smaller losses than the MV strategy. The negative skewness measures for the MV and Semi-static hedging strategies indicate heavier left tails of the distributions. The minimum loss, 1% 10% quantiles and mean shortfalls show that the distribution from the MV strategy has the heaviest left tail. The kurtosis of the hedging errors from the delta-neutral strategy is larger than that from all the other strategies. This means that the distribution of hedging errors from the delta strategy is more likely to have outliers.

	MV Hedging	**Delta Hedging**	**Static Strategy 1**	**Static Strategy 2**
mean	-0.1655	-0.0004	-0.0089	-0.0644
standard deviation	0.8489	0.0703	0.1498	0.5203
maximum	2.1029	0.8427	0.9787	1.9085
minimum	-7.9127	-0.7101	-2.1271	-4.4599
quantile 1%	-3.0851	-0.2271	-0.4855	-1.8701
quantile 10%	-1.2323	-0.0555	-0.1904	-0.7094
mean shortfall	-0.7748	-0.0390	-0.1309	-0.4809
skewness	-1.5557	0.1352	-0.9727	-1.5354
kurtosis	7.5715	17.5193	11.6744	8.8088

Table 4.3: Comparison of dynamic and static hedging strategies under the Heston model ($S_0 = 100,\ r = 0.05,\ r_p = 0.1,\ \kappa = 1.15,\ \theta = 0.04,\ \sigma_\nu = 0.39,\ \rho = -0.64,\ t = 1,\ h = \frac{1}{100}$)

Despite the fact that the delta-neutral hedging strategy produces the most desirable result, there are practical issues with this strategy in the downward equity scenario. For example, we can see from Figure 4.8 that the delta strategy requires to buy a huge amount of the 2-year put option and the underlying equity when the underlying price drops intensely. The cost of the hedging portfolio becomes very high. The problem is that there may not be enough put options available or the funding resources may be very limited. The third graph in Figure 4.8 shows that the MV hedging strategy does not have this issue. The semi-static strategies require no transaction during the hedging period, and the cost of the replicating portfolio is limited to the expected present value of the GMWB net liability. Hence, the semi-static strategies are more practical to adopt in unfavorable scenarios.

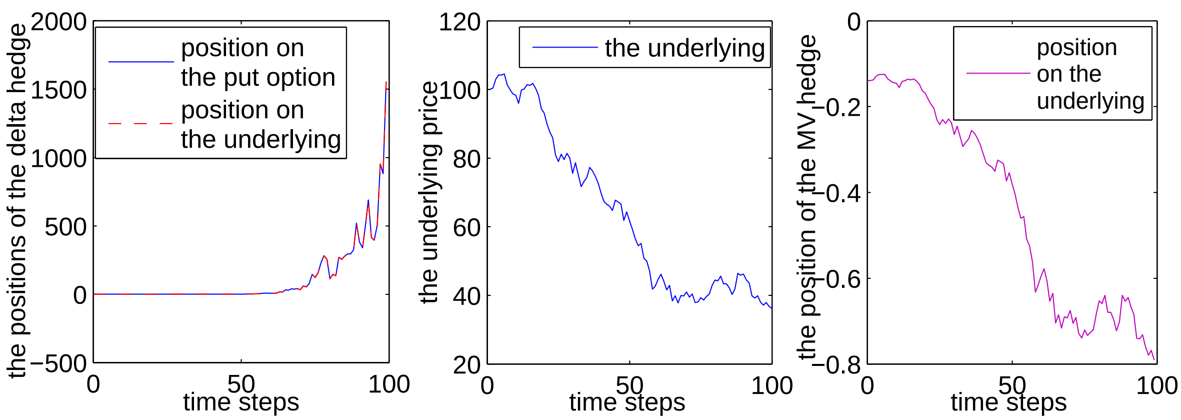

Figure 4.8: A downward scenario of the underlying and the corresponding hedge positions for the delta-neutral strategy and the MV strategy. In the first graph, the two lines overlap. The length of rebalancing time step is $h = \frac{1}{100}$. ($S_0 = 100,\ r = 0.05,\ r_p = 0.1,\ \kappa = 1.15,\ \theta = 0.04,\ \sigma_\nu = 0.39,\ \rho = -0.64,\ t = 1$)

4.3 Semi-static Hedging for GMWBs Under the Heston Model

In this section, we demonstrate that the proposed in Section 4.2 static hedging strategies can also be applied to path dependent options when the volatility of the underlying index is stochastic. In particular, we apply the two static strategies to GMWBs. The idea is similar, that is, to use a portfolio of standard put options to replicate the expected GMWB loss in a short time period. But hedging GMWBs is more complicated than hedging European put options because of their path-dependent feature.

Before discussing the hedging strategies, in Section 4.3.1, we compute the fair prices of GMWBs under the Heston model. The parameters of the Heston model that we use are based on those estimated by Bakshi et al. (1997). There is no semi-analytical formula to value GMWBs because of their path-dependent feature. We rely on the simulation method described in Subsection 4.1.3 to estimate the future GMWB benefit and charges. Then we will discuss the semi-static hedging by assuming the price is given.

When we allow the volatility to be stochastic, the fund value is impacted by the path of volatilities, and the expected GMWB loss depends on the path of the fund values over time. Under the Heston model, the expected GMWB loss can be approximated by the expectation of the GMWB loss conditional on the index value and the volatility at the end of the hedging period. We can also approximate it with the expectation of the GMWB loss conditional on the index value only. Both approaches require generating the conditional paths of the fund value given the end value of the fund. This can only be done by simulation. Fortunately, there is a method to do this.

In Section 4.3.2, we use the acceptance-rejection sampling method to simulate paths of the variance process conditional on the end value. This method is an efficient simulation method recently proposed by Beskos et al. (2006) and applied to the pricing problem by DiCesare and Mcleish (2008). In Section 4.3.3, we use time-changed Brownian motion method to construct conditional paths of the fund value process. Finally, we explain the two semi-static hedging strategies in Section 4.3.4. The semi-static replicating portfolio can be obtained by solving an optimization problem on two dimensions (index values and variances) or one dimension (index values with corresponding conditional mean variances).

4.3.1 Pricing GMWBs under the Heston model

To price the GMWB under the Heston model, we first simulate the fund value process using the Quadratic-Exponential method that is explained in Section 4.1.3. Then we estimate the expected present value of the GMWB benefit based on equation (2.30), and the expected present value of the GMWB charge by equation (2.31). The fair price that makes the two values equal can be obtained numerically in the same way as in Section 2.3.

Using the parameters specified in Subsection 4.2.1, we have computed the prices of the GMWB with quarterly withdrawals under the Heston model (Table 4.4). We have also computed the prices with a reduced volatility of the variance process. This value of σ_ν will be used in Sections from 4.3.2 to 4.3.4, and we will explain how this value is determined in Section 4.3.2. We can see that changes in this parameter do not affect the price too much. When the volatility of the variance becomes smaller, the GMWB price drops a little bit.

GMWB rate	**Maturity**	**Price *q*** (bps)	
g	T	$\sigma_\nu = 0.39$	$\sigma_\nu = 0.2476557$
10%	10	97.5336	96.4967
6.667%	15	54.0684	53.3282
5%	20	33.3235	32.3959

Table 4.4: The GMWB prices under the Heston model. Assume static quarterly withdrawals. The standard deviation is obtained from 1000 repeated simulations. ($S_0 = 100,\ r = 0.05,\ \kappa = 1.15,\ \theta = 0.04,\ \rho = -0.64,\ h = \frac{1}{4}$)

4.3.2 Simulating conditional paths of the variance process

In order to estimate the expected GMWB loss conditional on the index value at the end of the hedging period, we need to simulate or sample the terminal index value in some region and subsequently bridge the initial value to this terminal value at the time points of interest which affect the account value. Formally this means that we have to find the dynamic of the fund value conditional on the terminal values of the fund and the volatility. Under the Black-Scholes model, the Brownian bridge can be easily simulated as explained in Section 3.1.1. Under the Heston model, the previous technique is not applicable to the mean-reverting square root process. A direct method where we simulate paths in a forward manner and then bundle them according to their terminal values, would lead to an algorithm that is prone to a significant approximation error. However, an efficient imputation method for general processes, based on acceptance-rejection sampling, has recently been proposed by Beskos et al. (2006). The advantage of this method over Euler approximation is that it produces exact simulations of a diffusion process. The algorithm generates a bridge path at a finite number of points. If the path is accepted, extra points can be filled in at arbitrary time points using only

Brownian bridge interpolation. Below we briefly explain this method and apply it to the Heston model.

Suppose that we wish to generate a path of the variance process given the values $\nu_0 = x_0$ and $\nu_T = x_T$. A well-known transformation reduces the variance process (4.5) to a process Y_t with unit diffusion term. Let

$$Y_t = \int_0^{\nu_t} \frac{1}{\sigma_\nu \sqrt{\nu_s}} d\nu_s = \frac{2}{\sigma_\nu}\sqrt{\nu_t}. \tag{4.38}$$

By Itô's formula, the process Y_t satisfies the diffusion equation

$$dY_t = a(Y_t)dt + dW_{2t}, \tag{4.39}$$

where,

$$\begin{aligned} a(Y_t) &= \frac{\kappa(\theta - \frac{\sigma_\nu^2}{4}Y_t^2)}{\frac{\sigma_\nu^2}{2}Y_t} - \frac{1}{2}Y_t^{-1} \\ &= (\frac{2\kappa\theta}{\sigma_\nu^2} - \frac{1}{2})Y_t^{-1} - \frac{\kappa}{2}Y_t. \end{aligned} \tag{4.40}$$

Therefore, to generate a path of the variance process, it is sufficient to generate a path of Y_t conditional on the values $Y_0 = y_0 = \frac{2}{\sigma_\nu}\sqrt{x_0}$ and $Y_T = y_T = \frac{2}{\sigma_\nu}\sqrt{x_T}$.

Before describing the method, we introduce the following notation. Denote by $\{Y_t^{(y_0,y_T)}\}$ the conditional process of $\{Y_t\}$ given the endpoints y_0 and y_T, and by $\{Z_t^{(y_0,y_T)}\}$ the conditional process of a standard Brownian motion $\{Z_t\}$ given the values $Z_0 = y_0$ and $Z_T = y_T$. The probability space $C[0,\ T]$ is defined as the space of real-valued continuous functions over the interval $[0,\ T]$, together with $\mathcal{B}$ the corresponding Borel sigma algebra. Let P_Y denote the probability measure induced by the process

$\{Y_t\}$ on $C[0,\ T]$. Denote by $p_Y(y_0, y_T)$ the transition probability density function of the process $\{Y_t\}$. Assuming the basic conditions of Girsanov Theorem are satisfied, DiCesare and Mcleish (2008) proved that the Radon-Nikodym derivative of the induced measures for the tied-down processes on $C[0,\ T]$ is given by

$$\frac{dP_{Y^{(y_0,y_T)}}}{dP_{Z^{(y_0,y_T)}}}(Z) = \frac{p_Z(y_0, y_T)}{p_Y(y_0, y_T)} \exp\left[\int_{y_0}^{y_T} a(y)dy - \frac{1}{2}\int_0^T [a^2(Z_t) + a'(Z_t)]dt\right]. \tag{4.41}$$

The transition density of the Brownian motion $\{Z_t\}$ is the standard normal probability density, but we do not know the transition density of $\{Y_t\}$. However, we only need to know the Radon-Nikodym derivative up to a constant of proportionality to perform the acceptance-rejection sampling, as long as it is bounded. This is because we are usually interested in a weighted average, in which case the weights are normalized, and knowing the proportional weights is sufficient. We can write

$$\frac{dP_{Y^{(y_0,y_T)}}}{dP_{Z^{(y_0,y_T)}}}(Z) \propto \exp\left[-\frac{1}{2}\int_0^T [a^2(Z_t) + a'(Z_t)]dt\right] \tag{4.42}$$

Suppose there exists $c > 0$ such that $a^2 + a' + c > 0$. Then we have

$$\frac{dP_{Y^{(y_0,y_T)}}}{dP_{Z^{(y_0,y_T)}}}(Z) \propto \exp\left[-\frac{1}{2}\int_0^T [a^2(Z_t) + a'(Z_t) + c]dt\right] \leq 1. \tag{4.43}$$

The acceptance-rejection method is to simulate a continuous path according to a Brownian bridge process, $Z_t \sim P_{Z^{(y_0,y_T)}}$, and accept it with probability (4.43). But we can only sample a path at discrete times. Beskos et al. (2006) have proposed an algorithm that only requires finite information about the path. If we accept a path on the basis of finite information about it, then any path from a Brownian bridge agreeing with the accepted points can also be accepted. The algorithm also requires that the integrand $a^2 + a'$ is bounded above, that is,

$$\frac{d}{2} > a^2 + a' + c > 0,\ d > 0,\ c > 0.$$

Denote the integrand by

$$h(Z_t) = a^2(Z_t) + a'(Z_t) + c,$$

where Z_t represents a path drawn from $P_{Z^{(y_0, y_T)}}$. For a Poisson process $\{N_t\}$ with intensity $\frac{1}{2}h(Z_t)$, the probability that no event occurs is equal to the sampling ratio (4.43). That is,

$$P[N_T = 0] = \exp\left[-\int_0^T \frac{1}{2}h(Z_t)dt\right] = \exp\left[-\frac{1}{2}\int_0^T \left[a^2(Z_t) + a'(Z_t) + c\right]dt\right].$$

If, using only finite information about a path, we determine that the event $\{N_T = 0\}$ has occurred, then we can accept the path. A realization of $\{N_t\}$ can be produced by thinning a homogeneous Poisson process $\{N_t^*\}$ with constant intensity $\frac{1}{2}d$. We first generate a path of $\{N_t^*\}$, and accept the points on the path that occur at times $\tau_1 < \cdots < \tau_n$ with respective probabilities $h(\tau_1)/d, ..., h(\tau_n)/d$.

The algorithm goes as follows:

1. Simulate a path of N^* over the time interval [0,T]. Denote the arrival times by $\tau_1 < \tau_2 < \cdots < \tau_n$.
2. Simulate a standard Brownian bridge at arrival times, $Z_{\tau_j},\ j = 1, \cdots, n$, conditional on $y_0,\ y_T$.
3. Generate n independent uniform random numbers $u_j,\ j = 1, \cdots, n$. If $u_j > \frac{h(Z_{\tau_j})}{d}$

for all $j = 1, \cdots, n$, we reject that the arrival times $\tau_j,\ j = 1, \cdots, n$ are the arrival times for process N, and determine that $N_T = 0$. In this case, we accept the simulated skeleton path of Z as a path of Y conditional on y_0, y_T. Otherwise, we reject the simulated path and return to step 1 if any $u_j \leq \frac{h(Z_{\tau_j})}{d}$.

4. Simulate Brownian bridges at desired time points conditional on the accepted skeleton path $y_0, y_T, Z_{\tau_j},\ j = 1, \cdots, n$.
5. Convert the Brownian bridges of y_t into conditional paths of the variance by $\nu_t = \frac{\sigma_\nu^2 y_t^2}{4}$.

In some cases, $a^2 + a'$ is unbounded, where a is defined in equation (4.40). Then we may truncate it over an appropriate low probability region for the underlying Brownian bridge. The constants c and d are estimated by finding the minimum and maximum values of the function $a^2 + a'$ over a region in which the process resides with probability close to 1. The transition distribution of the variance process $\{\nu_t\}$ is known to be non-central chi-square. Let $\epsilon = 10^{-10}$, we can solve for the quantiles $(y_\epsilon, y_{1-\epsilon})$ at probability ϵ and $1 - \epsilon$. Then, set

$$d = \sup_{Y_s \in [y_\epsilon, y_{1-\epsilon}]} \left[a^2(Y_s) + a'(Y_s)\right], \quad 0 < s < T. \tag{4.44}$$

$$c = -\inf_{Y_s \in [y_\epsilon, y_{1-\epsilon}]} \left[a^2(Y_s) + a'(Y_s)\right], \quad 0 < s < T. \tag{4.45}$$

The shape of the function $a^2 + a'$ varies dramatically with parameters $\kappa, \theta, \sigma_\nu$. For example, Figures 4.9 and 4.10 depict the function with different values of σ_ν. Figure 4.9 corresponds to the parameters that are estimated by Bakshi et al. (1997),

$$(\kappa, \theta, \sigma_\nu) = (1.15, 0.04, 0.39).$$

With these parameters, the function $a^2 + a'$ is unbounded at zero. The simulation method is not guaranteed to work in this case, and to use it we need to truncate the function $a^2 + a'$ with a larger ϵ. This increases the number of pathes being rejected, and reduces the simulation efficiency. Thus, we have decided to reduce σ_ν so that $2\kappa\theta > \sigma_\nu^2$ and the variance is above zero for sure under the Heston model. In Figure 4.9, the parameter $\sigma_\nu = 0.2476557$ is chosen such that

$$\frac{2\kappa\theta}{\sigma_\nu^2} = \frac{3}{2} \tag{4.46}$$

$$a^2(y) + a'(y) = 0y^{-2} + 0.3306y^2 - 1.7250. \tag{4.47}$$

Using this parameter value, we can obtain reasonable bounds for the function $a^2 + a'$. Assume the time period is one year, $T = 1$. The quantiles of the variance at time T are $\nu_{T,\epsilon} = 0.0000$ and $\nu_{T,1-\epsilon} = 0.5583$, $\epsilon = 10^{-10}$. The corresponding Y_T values are $y_\epsilon = 0.0007$ and $y_{1-\epsilon} = 6.0340$. We truncate the function $a^2 + a'$ at these points, and set the bounds to be $c = 1.7250$ and $d = 10.3130$ accordingly.

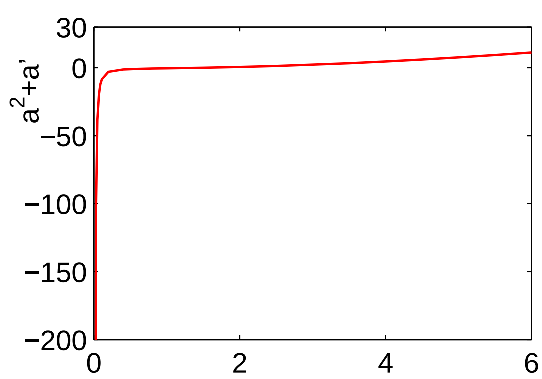

Figure 4.9: The density ratio function $a^2(y) + a'(y)$ ($\kappa = 1.15,\ \theta = 0.04,\ \sigma_\nu = 0.39,\ \rho = -0.64$)

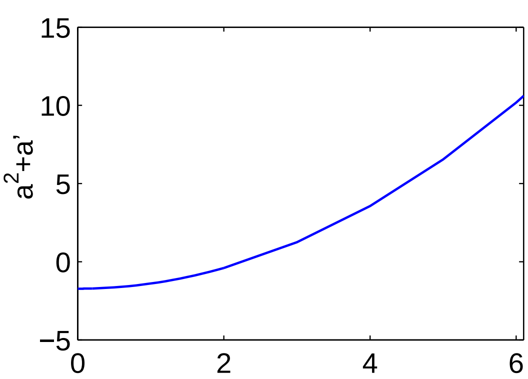

Figure 4.10: The density ratio function $a^2(y) + a'(y)$ ($\kappa = 1.15,\ \theta = 0.04,\ \sigma_\nu = 0.25,\ \rho = -0.64$)

4.3.3 Simulating conditional paths of the fund value process

Based on the simulated conditional paths of the variance process, we can use equation (4.18) and time-changed Brownian motion to construct paths of the fund value conditional on the end fund value and the path of the variance. In equation (4.18), the fund value, conditional on the variance path, involves one stochastic integral, which we denote by X_t:

$$X_t = \int_0^t \sqrt{\nu_s} dW_{1s}. \tag{4.48}$$

Define a time change process

$$\beta_t = \int_0^t \nu_s ds. \tag{4.49}$$

If $\nu_s > 0$, then β_t is strictly increasing. It is known that there is a Brownian motion $\{B\}$ such that

$$X_t = B_{\beta_t} \tag{4.50}$$

in distribution (see Oksendal 1998, p. 146).

Given the end values of $X_0 = x_0$ and $X_T = x_T$, we can generate the bridge at time t, $X_t^{(x_0,x_T)}$, using the standard Brownian bridge interpolation at time change β_t:

$$X_t^{(x_0,x_T)} = B_{\beta_t}^{(x_0,x_T)}, \quad B_{\beta_0} = x_0,\ B_{\beta_T} = x_T,\ 0 < t < T. \tag{4.51}$$

The fund value conditional on the end points can be obtained by the following equation:

$$S_t^{(S_0,S_T)} = S_0 \exp\Big\{(r-q)t - \frac{1}{2}\int_0^t \nu_s ds + \frac{\rho}{\sigma_\nu}\big[v_t - v_0 - \kappa(\theta t - \int_0^t \nu_s ds)\big] + \sqrt{1-\rho^2} X_t^{(x_0,x_T)}\Big\}. \tag{4.52}$$

We may compute the integral $\int_0^t \nu_s ds$ numerically from the simulated variances at a large number of equally spaced time points.

4.3.4 The two semi-static hedging strategies for the GMWB

Suppose that we want to hedge the GMWB in the time period of $[0,t]$. We will modify the two semi-static strategies that are introduced in Section 4.2.1 for the European put option. In Section 4.2.1, we have estimated the joint probability density of the index value and the variance at time t under the Heston model. Then we can choose a grid of index values and variances that cover most of the region with positive densities. The expected GMWB loss conditional on a specific index value and a variance at time t is simulated by constructing conditional paths of the variance process and the fund value process. The first semi-static strategy uses all the points on the grid, while the second strategy uses the conditional mean of the variance paired with each index value. The replicating portfolio is obtained by minimizing the weighted squared difference between the expected GMWB loss and the value of the portfolio.

Strategy One: Two-dimensional Optimization

Within a short time period, such as one year, the GMWB can be hedged by a portfolio of put options with an intermediate maturity T_p, such as two years. This strategy takes the following steps:

1. Choosing a set of underlying prices at time t, $S_{t,j},\ j=1,\cdots,n_s$, and a set of variances $\nu_{t,i},\ i=1,\cdots,n_v$. The range of the values can be determined from simulation.
2. Estimating the joint probability densities at the chosen points, $p(\nu_{t,i},\ S_{t,j})$.
3. Simulating conditional paths of the variance process that end at the same value $\nu_{t,i}$ using the acceptance-rejection method.

4. Simulating conditional paths of the index value process that end at $(\nu_{t,i},\ S_{t,j})$ using equation (4.52).
5. Calculating the annuity account value at t for each path of the index value, and compute the average account value corresponding to $(\nu_{t,i},\ S_{t,j})$.
6. Approximating the conditional expected GMWB net liability using a Taylor expansion as explained in Section 3.1.3.
7. Calculating the time-t values of the T_p-year put options using equation (4.12) or (4.16) at each point $(\nu_{t,i},\ S_{t,j})$.
8. Solving the portfolio weights by Weighted Least Squares, i.e. minimizing the weighted squared difference between the value of the portfolio and the expected GMWB loss.

Strategy Two: One-dimensional Optimization

The expected GMWB loss does not change with the index variance as significantly as with the index value. For each index value at time t, we can estimate the conditional mean of variances. The problem is then simplified to one dimension if we replicate the expected GMWB loss at the index values paired with the conditional mean variances. The maturity of the replicating portfolio is equal to the length of the hedging period. The strategy can be implemented by the following steps:

1. Choosing a set of underlying prices at time t, $S_{t,j},\ j = 1, \cdots, n_s$, and estimating the conditional mean of the variances $\bar{\nu}_{t,j}$ from simulation.
2. Estimating the marginal probability densities at the chosen points, $p^*(S_{t,j})$.
3. Simulating conditional paths of the variance process that end at $\bar{\nu}_{t,j}$ using the acceptance-rejection method.
4. Simulating conditional paths of the index value process that end at $(\bar{\nu}_{t,j},\ S_{t,j})$

using equation (4.52).

5. Calculating the annuity account value at t for each conditional path of the index value, and compute the average account value corresponding to $(\bar{\nu}_{t,j},\ S_{t,j})$.
6. Approximating the conditional expected GMWB net liability using a Taylor expansion as explained in Section 3.1.3.
7. Calculating the payoff of the replicating portfolio that includes several t-year put options.
8. Solving the portfolio weights by Weighted Least Squares, i.e. minimizing the weighted squared difference between the payoff of the portfolio and the expected GMWB loss.

Below we present a numerical example. Suppose that we want to hedge a 15-year 6.6667% GMWB in the first contract year. Assume withdrawals are taken quarterly. We use the following parameters:

$$r = 5\%,\ \kappa = 1.15,\ \theta = 0.04,\ \sigma_\nu = 0.2477,\ \rho = -0.64,$$

$$S_0 = 100,\ w = 6.6667,\ h = 1/4,\ T = 15,\ t = 1,\ T_p = 2.$$

The first step is to select a set of index values and variances in one year. We choose 81 index values of $(35, 36, 37, \cdots, 115)$, and 22 variance values of $(0.001, 0.011, 0.021, \cdots, 0.211)$. Then we proceed with Step 2 to 4, and obtain the risk-neutral probabilities, paths of variances and index values for all the pairs of chosen index values and variances. Based on the path of the index values, we calculate the account value as described in Section 3.1.2. Conditional on the account value and variance at time one, we simulate the future variances and account values using

Quadratic-Exponential scheme that is explained in Section 4.1.3. The expected GMWB loss can be obtained by approaches that are introduced in Section 3.1.3.

Figure 4.11 shows the surface of the expected GMWB loss. The GMWB loss is high when the index value is low. This is similar to the European put option. But the GMWB loss has a different relationship with the index variance. If we increase the variance for a low index value, the GMWB loss first decreases and then increases slightly. However, for a high index value, the GMWB loss goes up as the variance increases. The reason is that the GMWB value is path-dependent.

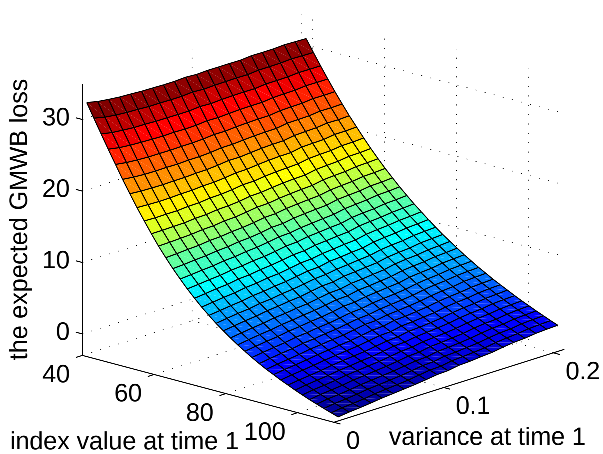

Figure 4.11: The expected GMWB loss after one year under the Heston model ($r = 5\%,\ \kappa = 1.15,\ \theta = 0.04,\ \sigma_\nu = 0.2476557,\ \rho = -0.64,\ S_0 = 100,\ w = 6.6667,\ h = 1/4,\ T = 15,\ t = 1$)

To explain this phenomenon, we examine paths of the fund value in the hedging period. We pick three end fund values of (40, 90, 140), and three end fund variances of (0.01, 0.05, 0.15). Conditional on each pair of the end values, we construct several conditional paths of the fund value, and then average the paths to obtain one path of fund values for each pair of the end fund value and variance. Figure 4.12 shows the

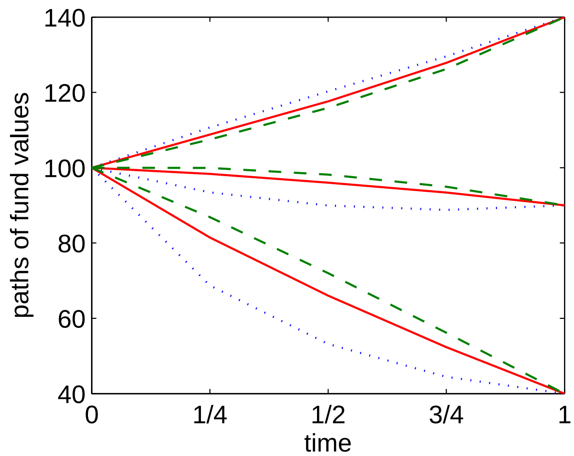

Figure 4.12: The paths of average fund values given the index value and the variance at the end of the year. The dotted lines correspond to a variance of 0.01 at time 1. The solid lines correspond to a variance of 0.05 at time 1. The dashed lines correspond to a variance of 0.15 at time 1.

paths of the average fund values in the first year. The dotted lines correspond to a variance of 0.01 at time 1. The solid lines correspond to a variance of 0.05 at time 1. The dashed lines correspond to a variance of 0.15 at time 1. For a low index value at time 1, the path of the average fund values that corresponds to the low variance, is convex shaped; the path corresponding to the high variance is concave shaped. But the difference becomes smaller as the variance continuously increases. Therefore, for a low index value at time 1, the average account value increases with the variance at a decreasing speed. The GMWB loss is negatively related to the account value, so this effect mitigates the increasing trend of the expected future GMWB loss (like a put option) as we increase the variance at time 1. For a higher index value at time 1, the paths with different end variances are close, so the average account values do not vary largely, and the expected GMWB loss constantly increases with the variance.

The expected GMWB loss in Figure 4.11 decreases with the index value. When the index value is larger than 110, the expected loss becomes negative, that is, an expected profit. Only the real GMWB loss needs to be hedged, so we replicate the part of the expected GMWB loss that corresponds to the index values smaller than 110. The strikes of the put options in the hedging portfolio are chosen to be 50, 60, 70, 80, 90, 100, and 110. Table 4.5 shows that optimal portfolio weights obtained by (Weighted) Least Squares. The options used by the first strategy are struck at 60, 70 and 110 only. The second strategy uses options with all the strikes. The first strategy costs less than the second strategy.

Put	**Strategy One**			**Strategy Two**	
Strikes	**Price**	**LS**	**WLS**	**Price**	**WLS**
Cash	0.9512	-3.7190	-3.7791	0.9512	-1.9779
50	0.1529	0	0	0.0252	0.1293
60	0.4309	0.4815	0.1370	0.1160	0.1593
70	1.1039	0.1583	0.4163	0.4019	0.1207
80	2.0871	0	0	1.1260	0.1142
90	3.8689	0	0	2.6696	0.0639
100	6.5832	0	0	5.5102	0.0701
110	10.4144	0.3483	0.3333	10.0795	0.1888
Initial Cost		0.4573	0.3575		0.7778

Table 4.5: Semi-static replicating portfolios for the GMWB under the Heston model. The initial cost of the replicating portfolio is the sum of the prices of put options and the cash value. ($r = 5\%,\ S_0 = 100,\ t = 1,\ T = 15,\ h = 1/4,\ w = 6.6667,\ \kappa = 1.15,\ \theta = 0.04,\ \sigma_\nu = 0.2476557,\ \rho = -0.64$)

Figures 4.13 and 4.14 depict the replicating errors of the two replicating strategies. They suggest that for path-dependent contracts, the two methods are also applicable. The second semi-static strategy uses less information, and thus produces relatively larger errors than the first strategy.

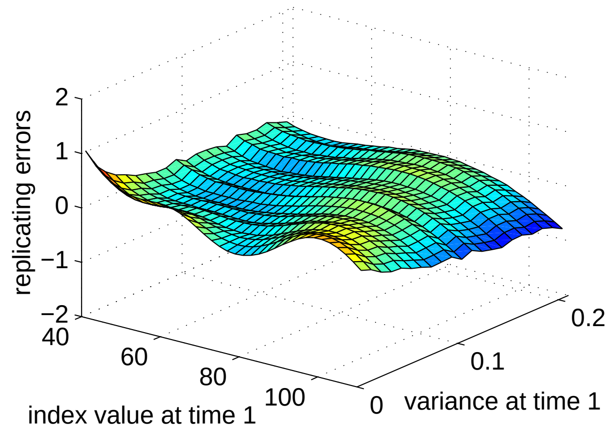

Figure 4.13: The replicating errors using the first semi-static strategy for the GMWB($\kappa = 1.15,\ \theta = 0.04,\ \sigma_\nu = 0.2476557,\ \rho = -0.64$)

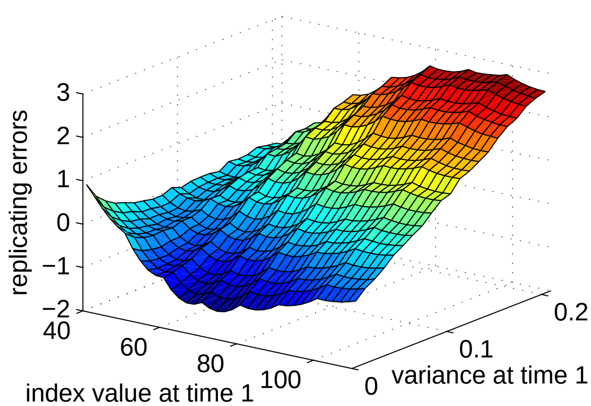

Figure 4.14: The replicating errors using the second semi-static strategy($\kappa = 1.15,\ \theta = 0.04,\ \sigma_\nu = 0.2476557,\ \rho = -0.64$)

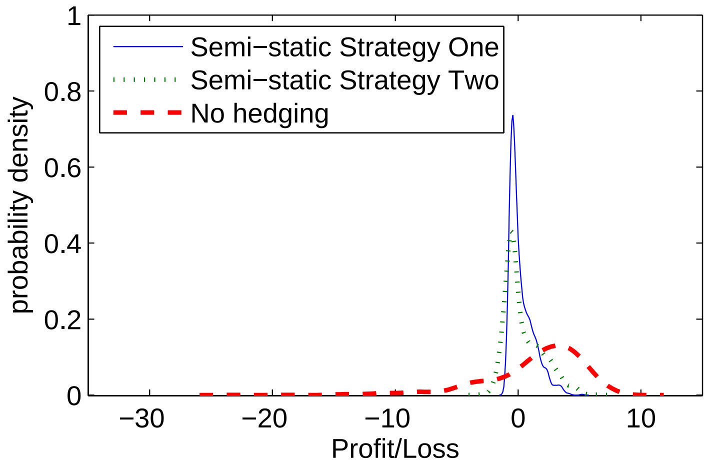

Figure 4.15: Estimated probability density of profit/loss after hedging the GMWB in one year under the Heston model. The drift term of the index value under the real world measure is assumed to be $r_p = 0.10$. The replicating portfolio weights are given in Table 4.5. ($S_0 = 100,\ \kappa = 1.15,\ \theta = 0.04,\ \sigma_\nu = 0.2476557,\ \rho = -0.64,\ \nu_0 = 0.04,\ t = 1,\ r = 0.05$)

We now simulate the fund value scenarios under the real world measure by assuming a drift term of $r_p = 10\%$. Then we compute the expected GMWB net liabilities in one year over these scenarios. Employing the two semi-static hedging strategies respectively, we compare the simulated profits or losses. Figure 4.15 depicts the estimated probability density functions of the profit/loss after the two hedging strategies. The distribution before hedging has a very long left tail, which indicates that large losses are likely to happen. The 5% quantile of the profit/loss is -6.68. After applying the two strategies, the left tails of the distributions become very small. With Strategy One, the 5% quantile increases to -0.72. With Strategy Two, the 5% quantile increases to -1.39. The standard deviation of the profit/loss is 4.21 before hedging. It decreases to 0.99 after hedging under Strategy One, and drops to 1.52 after hedging under Strategy Two. By and large, the two semi-static hedging strategies work well under the Heston model, though

Strategy One provides better risk-protection than Strategy Two. Note that there are still some residual errors over time after hedging. They can be managed by setting a reserve.

4.4 Summary

The idea of semi-static hedging can be extended to the stochastic volatility case. We choose the Heston model to illustrate this because many researchers have developed results on option pricing and discretization for simulation under this model. Takahashi and Yamazaki (2008) propose an approach to implement static hedging for European call options under the Heston model. But the efficiency of their approach relies on the existence of a closed-form solution of the option price. We propose two semi-static hedging strategies that work well for both European options and path-dependent options such as GMWBs.

The semi-static hedging idea is to set up a portfolio of standard put options to replicate the expected GMWB loss in a short time period, and then update the portfolio for another period. The expected GMWB loss, conditional on a specific index value and a variance at the end of the current hedging period, is our hedging target. To compute the hedging target, we construct conditional paths of the variance process using the acceptance-rejection sampling method, and obtain conditional paths of the fund value process by time changed Brownian motion method. The first strategy uses a portfolio whose maturity is a little longer than the hedging horizon. This will help to partially hedge the volatility risk. The value of the portfolio and the hedging target at the end of the hedging period depend on the end values of the index and the variance. The portfolio weights are solved by minimizing the mean squared difference between the two values.

The second strategy uses a portfolio that expires at the end of the hedging period, so the replicating portfolio payoff only varies with the end value of the index. We compute the hedging target based on the end value of the index and the mean of the variance conditional on the index value. The portfolio weights are obtained similarly. Although our simulation results show that both strategies can reduce the expected GMWB loss significantly, the first strategy does perform better than the second one.

Chapter 5

Summary and Future Research

GMWBs are becoming popular in the variable annuity market worldwide. This product is basically an embedded put option with random payoff and random maturity. Pricing and hedging of the GMWB is not an easy task for either practitioners or academics.

Under the static continuous withdrawal assumption, Milevsky and Salisbury (2006) decompose the product into a quanto Asian put option and an annuity certain. In this thesis, we assume static discrete withdrawal, and decompose the GMWB with a variable annuity into an arithmetic average strike Asian call option and an annuity certain. The prices for a popular GMWB given by Milevsky and Salisbury (2006) range from 73 to 160 basis points. In contrast, the products in the market are charging 30 to 50 basis points. The price we obtain is about 54 basis points. To confirm our pricing results, we treat the GMWB as a put option and get almost the same price using direct simulation.

Hedging is often more important than pricing. The popular hedging strategies are dynamic hedging. The performance of dynamic strategies is restricted by transaction costs and market liquidity. What is more, dynamic strategies perform poorly in volatile periods when large price jumps occur frequently. We propose a semi-static hedging strategy that can overcome these drawbacks. A portfolio of short-term put options is used to hedge the long-term GMWB statically within a short time period. The portfolio is then updated when the options expire or before they expire. We illustrate this idea using a simple setting. The withdrawal amounts over the life of the contract are assumed to be fixed. Our simulation results show that the semi-static hedging performs well both under the Black-Scholes model and under a jump diffusion model.

Finally, we extend the semi-static hedging to the Heston model. We propose two hedging strategies that can be applied to both European options and path-dependent options like GMWBs. The first strategy is to construct a portfolio with a maturity longer than the hedging horizon, but shorter than the GMWB maturity. The second strategy is to use a portfolio that expires at the end of the hedging period. The first strategy performs better than the second, but computations for the second strategy is less time-consuming. Both strategies work reasonably well based on the simulation results.

The following future work can be considered:

- Include barrier options in the semi-static hedging portfolio.

  Barrier options can be replicated by standard European options, and they are traded in the market. It is possible to add them into the replicating portfolio for GMWBs. For example, the down-and-in put options can be used to prevent large losses at a low cost.

- Study the *lifetime GMWBs* where interest rate and mortality become important factors.

  The *lifetime GMWBs* serve as an alternative to annuitization, and become an important option in the retirement income system. The policyholders are able to enjoy the growth of the fund and a guaranteed constant income which mitigates the longevity risk. The value of this benefit is largely affected by the interest rates and mortality rates because the maturity is life-dependent and can be very long (e.g., over 30 years). The valuation of this product will be based on simulation. Mortality levels at older ages cannot be projected accurately because of fewer observations. Even if the life expectancy can be calculated correctly, it's only good for overall population and very large groups. Unlike immediate annuities, the lifetime GMWBs cannot diversify mortality risk after the policyholders' account values are depleted. In single premium immediate annuities, the unused account value of deceased policyholders can subsidize living policyholders' income payments. With the lifetime GMWBs, after the account values go to zero, mortality risk cannot be diversified among those policyholders. What is more, mortality improvement is another random factor and very hard to predict. Death rates for adults and seniors show a decreasing pattern, so the survival time of policyholders may be longer than the insurers' assumptions. Uncertainty about the extent of future mortality improvement forces insurers to accept the risk of higher than expected guaranteed benefit payments.

- Pricing and hedging with dynamic utilization.

  Policyholder behavior brings significant uncertainty to product profitability and hedging cost. During the life of the contract, policyholders may surrender to exit the position in the contract. They may take any amount of partial withdrawals anytime under the GMWB. Transferring fund between different types of assets will change the volatility of the fund return, which directly affects the GMWB put option value. The prospective policyholders are heterogeneous, but the insurer cannot completely determine which class they belong to at issuance, so every policyholder is charged the same rate. To fairly price and hedge the GMWB, certain assumptions are needed. Assumptions on policyholder behavior are subjective and need a long time period to be validated. Factors affecting decisions may include economic variables (such as GDP, inflation, unemployment), in-the-moneyness of the benefit (the value of the remaining guaranteed benefit relative to the contract value), personal situations (age, other income etc.), and product features (policy duration, benefit features). For example, if the account value falls below the remaining guaranteed benefit amount, then the policyholders who contemplate to surrendering may change their mind and opt for the withdrawal benefit. On the other hand, the policyholders tend to surrender when the market performs strongly. With enough data, multi-state model and logistic regression model can be used to model these behaviors and quantify their effects on the GMWB net value.

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