Stability and Performance for Two Classes of Time-Varying Uncertain Plants

by

Julie Vale

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In this thesis, we consider plants with uncertain parameters where those parameters may be time-varying; we show that, with reasonable assumptions, we can design a controller that stabilizes such systems while providing near-optimal performance in the face of persistent discontinuities in the time-varying parameters. We consider two classes of uncertainty. The first class is modeled via a (possibly infinite) set of linear time invariant plants - the uncertain time variation consists of unpredictable (but sufficiently slow) switches between those plants. We consider standard LQR performance, and, in the case of a finite set of plants, the more complicated problem of LQR step tracking. Our second class is a time-varying gain margin problem: we consider an reasonably general, uncertain, time-varying function at the input of an otherwise linear time invariant nominal plant. In this second context, we consider the tracking problem wherein the signal to be tracked is modeled by a (stable) filter at the exogenous input and we measure performance via a weighted sensitivity function. The controllers are periodic and mildly nonlinear, with the exception that the controller for the second class is linear.

Acknowledgements

I would like to thank my supervisor Professor Daniel Miller for his unending patience and without whom this research would not have been possible. I would also like to thank my committee for taking the time to provide such useful feedback. Additionally, I would like to thank Darrell Gaudette for his help with figuring out all of the little annoying things, like notation and style. Finally, thanks to my family for their continued support and understanding - I love you all.

Dedication

This is dedicated to my daughter.

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Chapter 1

Introduction

This thesis lies in the field of feedback control, in which the goal is to design a controller to force a given system (the plant) to achieve some desired objective; e.g., a car maintaining a specific speed down the highway, a ship following a desired path to its berth in a crowded port, or an airplane landing at a busy airport. It is clearly desirable to achieve these objectives while rejecting unexpected external disturbances, such as curves or hills, sea swells, or errant wind gusts, each of which could lead to disastrous consequences. In order to reject disturbances, such controllers typically measure system outputs and then use that feedback information to behave accordingly; e.g. if the car is going up a hill, then it will begin to slow down, the controller senses this and increases the power to the engine resulting in the car returning to the desired speed. Of course, there are limitations as to what can be accomplished: "You can't make a battleship act like a butterfly."

Typically, a Control Engineer begins the design of such a controller by quantifying the system's behaviour via a mathematical model. To complicate the issue of the ensuing design, imagine that there is some uncertainty as to how well the model behaviour matches the behaviour of the actual plant. Such uncertainty could arise, for example, from simplifications made during the modeling process or from unexpected system failures (e.g. catastrophic failure due to fatigue or an intermittent fault in a sensor or actuator). This issue of uncertainty naturally leads to the following questions: If we design a controller for a specific plant model, then how much uncertainty can there be in that model before the controlled system ceases to behave as expected? Can we design controllers that tolerate and perhaps even adapt to a plant that changes over time? If we can model the uncertainty a priori, then can we use that information to somehow improve the performance of the controlled system, e.g. can the performance be made optimal in some sense?

In this thesis we present a design technique that uses on the fly estimation to stabilize certain classes of uncertain systems, while obtaining some measure of (near-optimal) performance. We do so with the added wrinkle of allowing the uncertainty to be time-varying. We will present four problems and their associated solutions; these problems can be grouped into two classes of uncertainty.

Three of the problems can be grouped into one class: switching between a (possibly infinite) set of Linear Time Invariant (LTI) plants. This problem is inspired by fault analysis: there are many models, each of which represents the system operating in some fault mode, the goal is to provide stability and acceptable performance in the face of (unknown and possibly persistent) switching between fault modes. There are many different ways to approach this problem and to categorize what is meant by 'acceptable' performance; for an excellent (although slightly old) survey of the topic of fault tolerance, see [31]; a more recent, but less tractable, review can be found in [47]. Here, we are interested in the particular performance problem of Linear Quadratic Regulation (LQR) optimal performance in the sense that we wish to design a controller that provides the optimal performance for each LTI plant in the set. Our three sub-problems are

(i) stability and LQR performance for a finite set of plants,

(ii) stability and LQR step tracking performance for a finite set of plants, and

(iii) stability and LQR performance for a compact set of plants.

We will refer to these problems as the Finite Stability, Finite Tracking, and Compact Stability problems, respectively.

Observe that the time variation in all of the above problems is piecewise constant, so, in an effort to investigate a more general time variation, we turn to our second class. Here, we consider the tracking problem for an otherwise LTI plant with a fairly general time-varying gain at the plant input; we will refer to this as the Time Varying Gain Margin (TVGM) problem (the reason will become clear shortly). Performance in this context is often measured in terms of input output functions (such as weighted sensitivity), which is what we will do.

In the remainder of this chapter, we begin by introducing two classical uncertainty problems which are directly linked to our two classes of uncertainty and then discuss three approaches to solving uncertainty problems. We then move on to a relatively new approach to the problem and to the contributions of this thesis. Finally we conclude by providing an outline of the remainder of the thesis.

1.1 Two Classical Problems in Uncertainty

The Finite and Compact Stability problems lead directly to the classical Simultaneous Stabilization problem: given a (possibly infinite) set of plants, design a single controller which stabilizes every plant in the set. This problem can be traced back at least to [41] and [19], wherein, for the case of only two LTI plants, the authors provide easily checked necessary and sufficient conditions for the existence of a stabilizing LTI controller. The general problem is much harder: conditions for the existence of a stabilizing LTI controller in the case of three or more LTI plants is rationally undecidable [2]. A common example of the limitations of using LTI controllers, and one that we will use throughout this thesis to illustrate our approach's effectiveness, is the two plants

$$\frac{1}{s-1} \text{ and } \frac{-1}{s-1},$$

which **can not** be simultaneously stabilized by a single LTI controller.

Of course, even if a controller is able to provide simultaneous stability for a given set of plants, there is no guarantee that it will stabilize the system in the face of persistent switching between those plants; indeed, it is well known that (in general) switching too quickly between stable LTI systems leads to instability. For an excellent survey of some fundamental issues regarding stabilization of switching systems, see [17] and the references therein.

Our second problem of interest is the classical Gain Margin problem: with P_0 a fixed LTI plant and $[g^-, g^+] \in \mathbf{R}^+$, given the set

$$\mathcal{P} := \{gP_0 : g \in [g^-, g^+]\},$$

find a controller that stabilizes every plant in $\mathcal{P}$; furthermore, find a bound on the gain margin

$$GM := \frac{g^+}{g^-}$$

that ensures that such a controller exists. It is well known that if P_0 is unstable and non-minimum phase, then there *is* an upper bound on the gain margin achievable using an LTI controller [12]; in other words, for such a plant, if the ratio GM is large, then an LTI controller can *not* stabilize every plant in the set $\mathcal{P}$. If we allow the gain g to be time-varying and to be negative, then we obtain the TVGM problem.

1.2 Prior Work

We now turn to a discussion of three classical approaches to solving the problems outlined above: Robust Control, Adaptive Control, and Supervisory Control (or Logic Based Switching). Recall that we wish to obtain stability *and* performance; we examine the merits of these approaches accordingly.

1.2.1 Robust Control

The basic idea behind Robust Control is to design a single static (but not necessarily LTI) controller, which will provide stability and some measure of performance for every plant in the uncertainty set. Perhaps the most well known Robust Control method is the frequency domain approach in which the set of plant uncertainty is modeled via a nominal LTI plant in transfer function form with structured disturbances at its interfaces, also modeled via transfer functions; the goal is then to design an LTI controller to provide simultaneous stabilization and some measure of performance. In this context, performance is typically measured in the H_∞ sense; a common optimization goal is that of weighted sensitivity minimization.

Two significant benefits to this approach are that it yields an LTI controller (and all of the associated benefits) and that it is relatively straightforward for a non-expert to apply. There are three main drawbacks:

- It does not handle time variations well.
- Since the controller is LTI, the set of uncertainty that can be stabilized is limited by [41] and [19].
- Performance optimization is of the minimax type in the sense that, if we calculate the optimal performance for each plant individually, then the best performance that can be guaranteed for any plant is no better than the *worst* of these [46].

If we turn to Linear Time Varying (LTV) controllers and consider a finite set of plant uncertainty, then we can alleviate some of these drawbacks. Work in the 1980's and early 1990's, e.g. see [13], [10], [11], and [1] to name but a few of the approaches, has solved the general simultaneous stabilization problem; however, the approaches taken in these papers indicate that the performance may be quite poor. For example, in [13], the idea is to first design one deadbeat compensator for each plant, and then apply them (periodically) in sequence; unfortunately, there will (likely) be poor transient performance, at least until the controller reaches the compensator corresponding to the actual plant. In [11], an optimal generalized-hold is designed to achieve the stability objective; however, since generalized-holds typically have poor inter-sample behaviour [6], it is unclear whether or not good performance is obtained.

If we consider compact uncertainty sets, then the classical Gain Margin problem is perhaps the simplest problem that one could envision. Recall that the goal there was to stabilize the system in the face of a large range of uncertain gains (i.e. a large gain margin); performance in this context is typically measured via the size of the sensitivity function. If the nominal plant P_0 is unstable non-minimum phase and we restrict ourselves to LTI controllers, then it is shown in [43] that the size of the sensitivity function (when using the $2-$norm) tends to infinity as the gain margin required tends toward the maximum attainable[1]. If a linear periodic controller (LPC) is used, then the gain margin can be made arbitrarily large (e.g. see [7], [44], and [45]); however, there is some question as to the performance that can be attained via those approaches. For example, Yan, Anderson, and Bitmead [44] show that, under suitable assumptions, one can use an LPC to make

[1]In other words, for larger sets of uncertainty, the performance decreases.

the gain margin as large as desired while ensuring that the size of the sensitivity function remains bounded; however, we point out that (obviously) bounded is not the same as optimal. Finally, these approaches assume that the plant gain, although unknown, is nonetheless fixed; if we allow it to be time-varying, then there is no longer any guarantee that they will work.

1.2.2 Adaptive Control

The original goal of Adaptive Control was to provide stability and performance in the face of time-varying uncertainty. Unfortunately, obtaining even stability in this context turned out to be much more difficult than was originally expected; indeed, most results in this area do not prove anything explicit for the time-varying case, or, when they do, they require that the time variation be slow.

The main difference between this approach and Robust Control is that, here, the controller is not fixed; the idea is to design a controller that can 'adapt' to changing plant parameters. Typically, such a controller is realized via a linear compensator in tandem with a nonlinear estimator whose job is to 'tune' the compensator *during operation* to reflect any changes in the system parameters. Classically, the plant parameters are estimated on the fly and the controller is updated with the assumption that the current estimate is exactly correct. For a good review of this topic, see [21] and the references therein.

In general, the resulting controllers are typically shown to have the desired behaviour in only an asymptotic sense and are highly nonlinear; however, since the resulting controllers are time-varying, the LTI restrictions on simultaneous stabilization are not present, so we expect to be able to handle larger sets of uncertainty. Unfortunately, the estimation process is typically highly nonlinear and, at least in the initial stages, the control signals may become quite large. Furthermore, the estimation methods that are used typically require that the control signal be 'sufficiently exciting' in some sense (e.g. it must have a wide frequency spectrum); if it is not, then the estimate may not converge to the actual value, although it is still possible to achieve stability [20].

While there is (usually) no guarantee that the transient behaviour will be nice (indeed, it is typically quite poor), since the estimation is constantly being updated, there is a reasonable expectation that slow time variations can be tolerated; examples of classical Adaptive Controllers that directly address the issue of time-varying plants include [15], [20], and [36]. Indeed, in [36], the authors allow the time-variation to be fast, provided that it has some known structure.

A common formulation of the setup is as follows: design a controller so that, when it is attached to the actual plant, the output asymptotically approaches that of some known (stable) nominal plant which models the desired behaviour. This problem is called the Model Reference Adaptive Control (MRAC) problem. Typical assumptions on the uncertain plant are that

- the plant be minimum phase,
- an upper bound on the plant order is known, and
- an upper bound on the plant relative degree is known.

Some of these restrictions (e.g., the minimum phase assumption) can be lifted if, instead of requiring a complete model match, we simply insist that the plant poles match a desired set [9]. One of our results leverages some work in this area, so we have included this brief introduction for completeness; however, since we are interested in *all* LTI systems, including non-minimum phase ones, we will not pursue the MRAC formulation. For those who wish to investigate the topic further, we direct the reader to [22] for an excellent overview of the history of this approach.

1.2.3 Supervisory Control

Supervisory Control lies at the interface between Robust and Adaptive control: a significant amount of the design is performed offline, but some tuning is performed online. This approach arose in part to address the issue of time-varying uncertain systems (of which switching systems are a subset), which neither Robust nor Adaptive Control can handle very well. The idea here is to design (offline) a stabilizing compensator for each plant (as was done in [13]) along with a 'supervisor' that periodically chooses (online) which of the compensators to apply; the resulting closed loop system can be viewed as a complicated switching system.

It is not straightforward to analyze the stability of switched systems. Common areas of study include how quickly one can switch between otherwise stable systems while maintaining stability, and, if we could control the switching, what kinds of switching signals could stabilize an otherwise unstable set of systems. As stated earlier, an excellent review of the topic is [17] and the references therein. Unfortunately, in our case we do not know the switching information a priori, nor do we have control over it, so many of these tools do not apply to our problem.

Recent work in this area includes [28], [29], [30], [16], and [42]; in these papers, the uncertainty sets are quite large (e.g. compact sets of plants). These controllers often work by considering multiple adaptive identification models and then applying the one that minimizes some estimation performance index. While they are often shown to provide improved transient behaviour when compared to more classical Adaptive controllers, Supervisory Controllers are often highly nonlinear. Additionally, the focus of this approach is typically on stability and noise rejection rather than performance. A notable exception to this is [29], in which the proposed supervisory controller is shown to have some nice robustness characteristics; following the classical robust control result that stability robustness is linked to performance, the author argues that this avenue is a promising one with respect to providing optimal performance.

1.3 A New Approach to the Problem

We now turn to the approach adopted in this thesis. As with Supervisory Control, this approach lies at the boundary of Adaptive and Robust Control and is (at least partially) motivated by switching systems architecture. The main difference here is that we skirt many of the difficulties of the above approaches by **directly estimating the desired control signal** rather than estimating plant or controller parameters or determining a desired (supervisory/monitoring) switching signal. The resulting controllers provide performance that is robust to uncertainty and are able to adapt to variations in the plant parameters; as such, we will refer to this approach as Robust Adaptive Control Signal Estimation (RACE) Control. The resulting controller is periodic with period T.

The first instance of this approach appears in Miller and Rossi [26]. The idea behind the approach is simple, although only in hindsight: Why estimate plant or controller parameters when what you really want to know is which control signal to apply? Indeed, it turns out that by directly estimating the desired control signal, we can perform the estimation **linearly**, unlike in Adaptive or Supervisory Control. Furthermore, unlike in Robust Control, we can obtain performance that is as close to the optimal as desired for **every** plant in the uncertainty set. Of course, this is easier said than done. In this section we outline some of the prior work in this area and discuss the high level ideas behind the approach. We begin by discussing some papers which are key to the work presented in this thesis.

We begin with [26], where the authors proposed a RACE controller that was able to provide stability and near optimal LQR performance for every plant in a finite LTI set. A slight nonlinearity was introduced to deal with some of the adverse side effects of switching and the resulting controller was shown to tolerate a finite number of switches. Additionally, it was shown that the controller could achieve infinite gain margin and 60 degrees of phase margin for every plant in the set. Unfortunately, the tradeoff is that a small period T and fast sampling rates are required, leading to large controller gains and possibly poor noise tolerance; this drawback is common to all of Miller's earlier work[2].

The second paper of interest is [22]. The main contribution of that paper was the construction of a RACE controller which was shown to alleviate many of the classical MRAC drawbacks; i.e., it allows quite a general time variation, provides immediate performance (rather than asymptotic), removes the nonlinearities, and reduces the size of the control signal. This work required the common MRAC assumptions that the plant be minimum phase, the relative degree of the plants be known, and that the high frequency gains of the plants be bounded. In a followup conference paper [24], Miller redesigned this controller to solve the standard Gain Margin problem, including the case of negative gains and non-minimum phase nominal plants, while providing near optimal weighted sensitivity (in the $\mathcal{L}_1$ sense).

[2]Specifically, that work predating this author's work.

Finally, we look at [25], in which the RACE controller provides simultaneous stability and near optimal performance for a compact set of LTI plants. This work does not directly address the issue of switching, although it does indicate that, with the addition of a nonlinearity as in [26], it should be able to tolerate slow switching. Preliminary work on this problem, where the authors considered only first order systems, was presented in [18].

These papers pertain to this thesis in the following way. The ideas of [22] and [24] were leveraged to solve the TVGM problem, which was first presented in [40], while [26] was paramount to developing the solutions to the Finite Stability and Tracking problems ([38] and [37] respectively), and the ideas in [25] were critical to solving the Compact Stability problem [39]. This thesis compiles these four works (together with some logical additions and extensions), providing the following contributions:

- In all cases we provide stability in the face of persistent switches.
- In all but the Compact Stability problem, we provide near optimal performance for every plant in the uncertainty set in the face of persistent plant switches; in the Compact Stability problem we provide near optimal performance when there is no switching.
- Since it is not automatic, we explicitly consider the problem of stability in the face of noise for our nonlinear controllers[3].
- We explicitly prove that RACE can solve the Finite Tracking problem in the context of persistent plant switches, whereas previous work only hinted that it should be possible.
- In all but the TVGM problem, we allow a large period T, alleviating some of the noise and large controller gain concerns raised in [26] and [25] while providing a more aesthetically pleasing control signal (see Figure 1.1); in both of the Finite problems we also allow for slow sampling, which further alleviates these concerns.
- The RACE controller presented in [22] solves the TVGM problem, but only for minimum-phase nominal plants, while the controller in [24] allows non-minimum-phase plants, but does not allow time variation in the uncertain gain; our solution works for both a time-varying gain and any finite dimensional (FD) LTI plant.

1.4 Outline

We begin in Chapter 2 by discussing some mathematical preliminaries and notation that will hold throughout this thesis. In Chapter 3 we investigate the Finite Stabil-

[3]We do not do this in the Compact Stability problem (the analysis is significantly more complex), but we expect that our controller should be noise tolerant.

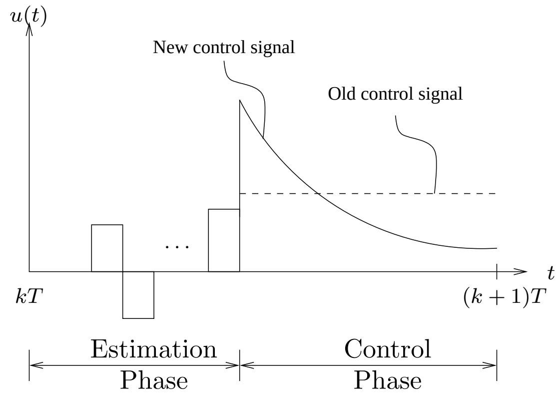

Figure 1.1: Comparison of our control signal versus those in prior work.

ity problem; in Chapter 4 we retain the majority of the structure from Chapter 3 and extend that result to the more difficult Finite Tracking problem. In Chapter 5 we turn back to simple LQR optimal performance, but allow for a compact set of LTI plants; since the set of uncertainty is infinite, we will require a new set of notation and a different estimation approach than in earlier chapters. In Chapter 6 we investigate the TVGM problem; it will turn out that the process will be similar to that of Chapter 5. Finally, in Chapter 7 we summarize our results and provide some ideas for possible directions of future work.

Chapter 2

Preliminaries

Each chapter in this thesis deals with a slightly different control problem, but there are some concepts, useful notation, and mathematical preliminaries that will be used throughout; we introduce those here.

2.1 Standard Definitions

Let $\mathbf{Z}$ denote the set of integers, $\mathbf{Z}^+$ the set of non-negative integers, $\mathbf{N}$ the set of natural numbers, $\mathbf{R}$ the set of real numbers, and $\mathbf{R}^+$ the set of non-negative real numbers. We denote the norm and the corresponding induced norm of a vector or matrix via $\|\cdot\|$ (we will use the 2-norm in Chapters 3 through 5 and the ∞-norm in Chapter 6). The set of real-valued bounded piecewise continuous signals on $\mathbf{R}^n$ is denoted $\mathcal{PC}_\infty(\mathbf{R}^n)$[1], when the dimension is clear from the context we will drop the $\mathbf{R}^n$. We measure the size of $f \in \mathcal{PC}_\infty$ by

$$\|f\|_\infty := \sup_{t\geq 0} \|f(t)\|,$$

and the induced norm of a linear operator $S : \mathcal{PC}_\infty \to \mathcal{PC}_\infty$ by

$$\|S\| = \sup_{f\in\mathcal{PC}_\infty,\, f\neq 0} \frac{\|Sf\|_\infty}{\|f\|_\infty}.$$

We say that $f \in \mathcal{PC}$ is **piecewise smooth on** $[a, b] \subset \mathbf{R}$ if there exists a finite set of points $\{x_i\}$,

$$a = x_1 < x_2 < \ldots < x_k = b,$$

such that, on each interval $(x_i, x_{i+1}),\ i = 1, .., k$, we have that f and $\dot{f}$ are continuous and bounded, and that they both have finite limits as $x \to x_i$ and $x \to x_{i+1}$. We say that $f \in \mathcal{PC}$ is **piecewise smooth** (and write $f \in PS$) if it is piecewise

[1] We do not use $\mathcal{L}_\infty(\mathbf{R}^n)$ since we are sampling signals without filtering them first.

smooth on every finite interval $[a,b] \subset \mathbf{R}^+$. We let PS_∞ denote the set of $f \in PS$ for which

$$\|f\|_\infty < \infty \textbf{ and } \operatorname{esssup}_{t\geq 0}\|\dot{f}(t)\| < \infty.$$

Finally, with $T > 0$, we let $PS_\infty(T)$ denote the set of $f \in PS_\infty$ for which every discontinuity in f and $\dot{f}$ are at least T time units apart.

It will be useful to analyze functions that depend on one particular interval of noise data; to that end, we adopt the sampled-data lifting notation from [3]: we partition the signal $w \in \mathcal{PC}_\infty$ into intervals of length T and use $\underline{w}_k$ to denote the piece of w which occurs in the k^{th} interval $[kT,(k+1)T)$ via

$$\underline{w}_k(t) := w(kT+t), \quad t \in [0,T).$$

A natural extension of this notation is the following: we let $\mathcal{PC}_\infty[0,T)$ denote the set of piecewise continuous bounded signals on the interval $[0,T)$, so $x \in \mathcal{PC}_\infty[0,T)$ if there exists a constant $c > 0$ such that

$$\|x(t)\| \leq c, \quad t \in [0,T);$$

additionally, we define

$$\|x\|_\infty := \sup_{t\in[0,T)} \|x(t)\|.$$

Furthermore, we say that the map

$$\begin{aligned} G \ : \ & \mathcal{PC}_\infty[0,T) \to \mathbf{R}^n \\ & x \mapsto G(x) \end{aligned}$$

has a **bounded gain** if there exists a constant $c > 0$ such that

$$\|G(x)\| \leq c\|x\|_\infty.$$

Clearly, if $w \in \mathcal{PC}_\infty$ then

$$\underline{w}_k \in \mathcal{PC}_\infty[0,T), \quad k \in \mathbf{Z}^+, \quad T > 0.$$

Finally, we will occasionally deal with maps of the form

$$G : \mathcal{PC}_\infty[0,T) \times \mathbf{R}^p \to \mathcal{PC}_\infty[0,T),$$

wherein the second argument plays the role of modulation; in this case we say that G has a **bounded gain** if there exists a constant $c > 0$ such that

$$\|G(x,y)\|_\infty \leq c\|x\|_\infty, \quad x \in \mathcal{PC}_\infty, \quad y \in \mathbf{R}^p,$$

with the smallest such c denoted by $\|G\|$. Similarly, we will occasionally deal with maps of the form

$$G : \mathcal{PC}_\infty[0,T) \times \mathbf{R}^p \to \mathbf{R}^p,$$

wherein the second argument still plays the role of modulation; again we say that G has a **bounded gain** if there exists a constant $c > 0$ such that

$$\|G(x, y)\| \leq c\|x\|_\infty, \quad x \in \mathcal{PC}_\infty, \quad y \in \mathbf{R}^p,$$

with the smallest such c denoted by $\|G\|$.

In this thesis we often wish to select the smaller of the two quantities $v_1 \in \mathbf{R}^n$ and $v_2 \in \mathbf{R}^n$. The standard notation for such an operation is

$$\operatorname*{argmin}_{v \in \{v_1, v_2\}} \|v\|;$$

unfortunately, the quantities v_1 and v_2 will sometimes be expressed only as complicated functions of system signals and parameters, in which case this notation will become highly cumbersome. To that end, we adopt the following slightly non-standard notation:

$$\operatorname{argmin}\{\|v_1\|, \|v_2\|\} := \operatorname*{argmin}_{v \in \{v_1, v_2\}} \|v\|.$$

Finally, to remove any ambiguity, for every $v_1 \in \mathbf{R}^n$ and $v_2 \in \mathbf{R}^n$ satisfying $\|v_1\| = \|v_2\|$, we set

$$\operatorname{argmin}\{\|v_1\|, \|v_2\|\} = v_1.$$

It will sometimes be useful to use order notation to express the size of some of our functions. To that end, we say that $f : \mathbf{R}^+ \to \mathbf{R}^{n \times m}$ is of order T^j and write $f = \mathcal{O}(T^j)$ if there exists a constant $\gamma > 0$ so that, for sufficiently small $T > 0$,

$$\|f(T)\| \leq \gamma T^j.$$

Occasionally, such a function f will also depend on a variable ϕ restricted to some set Φ; in this case, we say that $f = \mathcal{O}(T^j)$ if there exists a constant $\gamma > 0$ so that, for sufficiently small $T > 0$,

$$\|f(T, \phi)\| \leq \gamma T^j, \quad \phi \in \Phi.$$

We have two last pieces of notation. First, we define $\lfloor \rfloor$ to be the floor operator: for $x \in \mathbf{R}$,

$$\lfloor x \rfloor := \max\{y \in \mathbf{Z} \ : \ y \leq x\}.$$

Finally, we will use the Kronecker product $\otimes$ on vectors, so, with $x \in R^n$ and $y \in \mathbf{R}^m$ we have

$$x \otimes y = \begin{bmatrix} yx_1 \\ \vdots \\ yx_n \end{bmatrix} \in \mathbf{R}^{nm}.$$

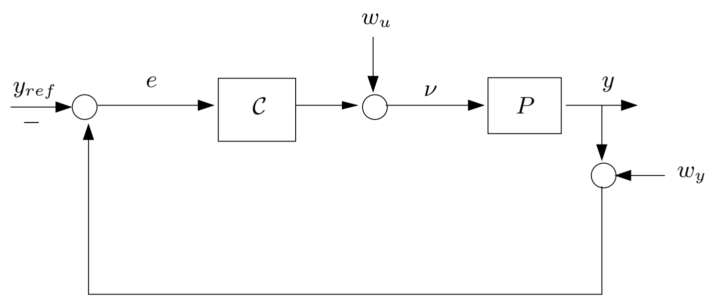

Figure 2.1: Simplified System Block Diagram

2.1.1 Feedback Structure and Stability

With P the plant and $\mathcal{C}$ the controller, noise introduced at the plant input and output, labeled w_u and w_y respectively, and an exogenous input y_{ref}, we have the feedback diagram shown in Figure 2.1.

We will investigate two standard types of closed loop stability: input/output (I/O) and asymptotic. In each chapter we will provide precise definitions of these concepts that are appropriate for the associated plant and controller structure under study. For now, we (loosely) say that

- The controller $\mathcal{C}$ **I/O stabilizes** the plant P if, when all initial conditions are zero, the map from the exogenous inputs to all outputs is well defined and bounded.

- The controller $\mathcal{C}$ **asymptotically stabilizes** the plant P if, when all of the exogenous inputs are zero, for every initial condition, the plant and controller's internal variables go to zero as time goes to infinity.

From this description we see that the former can be viewed as a kind of 'external' stability, while the latter can be viewed as a kind of 'internal' stability, Finally, we say that a controller **stabilizes** a set of plants $\mathcal{P}$ if it provides I/O stability *and* asymptotic stability for every plant $P \in \mathcal{P}$.

2.2 Notation Relating to Discontinuities

Recall that we are interested in time-varying sets of uncertainty and that we allow discontinuities in the plant parameters; we impose a minimum time between these discontinuities, which we label T_s. Furthermore, for simplicity, we insist that the plant parameters are always continuous from the right.

Starting with the first discontinuity at t_1, we gather the times at which these discontinuities occur to generate a strictly increasing, possibly finite, sequence of times $\{t_l\}$. This possibility of finiteness poses a notational problem and would require multiple special cases in our proofs. Since our results do not differentiate between systems with infinite and finite numbers of discontinuities, we circumvent this difficulty in the following way. If there are a finite number of discontinuities, then there exists a non-negative integer $\bar{l}$ which is such that $t_{\bar{l}}$ is the time at which the final discontinuity occurs. We then define

$$t_{\bar{l}+j} := t_{\bar{l}} + jT_s, \quad j \in \mathbf{N}$$

and consider a new strictly increasing sequence that is composed of the original finite sequence with these additional 'false' discontinuity times tacked onto the end. Finally, since we are only concerned with the system behavior for $t \geq 0$, for convenience we will insist that the system be continuous at $t = 0$ so

$$t_1 \neq 0.$$

Finally, we add the time

$$t_0 := 0$$

to the beginning of the sequence $\{t_l\}$, so we have

$$[0, \infty) = \bigcup_{l=0}^{\infty} [t_l, t_{l+1}).$$

To simplify our nomenclature refer to this new (infinite) sequence $\{t_l\}$ as the sequence of switching times.

It will be useful to define an associated sequence of (non-negative) integers $\{k_l\}$ that indicates which periods may contain switches: we would like k_l to satisfy

$$t_l \in [k_l T, (k_l + 1)T), \quad l \in \mathbf{Z}^+,$$

so it is natural to define

$$k_l := \left\lfloor \frac{t_l}{T} \right\rfloor, \quad l \in \mathbf{Z}^+. \tag{2.1}$$

Recall that we impose continuity at t_0; it follows immediately from this and (2.1) that there are *no* switches in the intervals

$$[kT, (k+1)T), \quad k \in \mathbf{Z}^+ \setminus \{k_l : l \in \mathbf{N}\};$$

we adopt the common set compliment notation and define

$$\{k_l : l \in \mathbf{N}\}^c := \mathbf{Z}^+ \setminus \{k_l : l \in \mathbf{N}\}.$$

Clearly, $\{k_l\}$ is a function of $\{t_l\}$ and T, to reduce clutter we do not make it explicit. Furthermore, if we insist that $T < T_s$, then

$$k_l \neq k_{l+1}, \quad l \in \mathbf{N};$$

observe that we may have $t_1 < T$ and therefore $k_1 = 0$, so we must allow $k_0 = k_1$ and be sure to account for this case in our proofs[2].

Finally, since some of our signals may not be continuous at the switching times, we adopt the standard notation to indicate the left hand limit of $x \in \mathcal{PC}_\infty$ at a point t: we say that

$$x(t^-) = a$$

if, for every $\varepsilon > 0$ there exists a constant $\delta > 0$ so that

$$\|x(\tau) - a\| < \varepsilon, \quad t - \tau \in (0, \delta).$$

We will not need the analogue $x(t^+)$.

2.3 The RACE Approach - A High Level Discussion

In the introduction, we outlined what RACE can accomplish. Before proceeding to the details of our problems and the proposed solutions, we provide a high level discussion on how RACE is implemented with the goal of simplifying our discussion in upcoming chapters; we will provide details and make all of the forthcoming notions more precise in following chapters.

Typically, the implementation of a RACE controller involves splitting each period into two parts, the Estimation Phase and the Control Phase[3]. In the Estimation Phase, we attempt to estimate the optimal control signal for the active plant (without directly estimating any plant or controller parameters) and then in the Control Phase we apply (a suitably scaled version of) that signal to the system. Of course we would prefer performing estimation and control simultaneously; however, our approach provides a very convenient structure with which to deal with discontinuities in the plant parameters. In the remainder of this section we will discuss two natural questions:

(i) What can we estimate?

(ii) How will discontinuities in the plant parameters affect the system behavior?

These notions will need to be made specific to each of our problems, as such they will be made more precise later on.

[2] For small T, this is no longer an issue; however, we are interested in large T, so the point is moot.

[3] Currently, there are two exceptions, we will briefly discuss one of these in Chapter 3 and the other in Chapter 6.

2.3.1 Estimation

Before we proceed, we need a piece of notation. We define the matrix $E_i \in \mathbf{R}^{nq\times n}$ to be analogous to the basis vector e_i in the following sense: with all q blocks of dimension $n \times n$ we define

$$E_i := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{th} \text{ block element.}$$

So, just as e_i can be used to obtain the i^{th} column of a matrix A (via Ae_i), we can use E_i to obtain the i^{th} block column element of a correspondingly partitioned matrix

$$A := \begin{bmatrix} A_1 & A_2 & \dots & A_q \end{bmatrix} \in \mathbf{R}^{r\times nq}$$

via

$$AE_i = A_i.$$

Recall that the controller is periodic with period T - we will discuss estimates over a single period $[kT, (k+1)T)$. We let $x \in \mathbf{R}^n$ be the plant state. We can (linearly) estimate the following:

(i) If the set of FDLTI plants is finite (i.e. $\{P_1, .., P_q\}$) and if the active plant is P_i, then we can estimate the quantity

$$E_i x[kT].$$

(ii) If the set of FDLTI plants is compact and p is the set of $2n$ Markov parameters associated with the active plant P and its observer canonical form state-space triple (A_p, B_p, C_p), then we can estimate the quantities

$$x[kT], \quad x[kT] \otimes p, \quad \underbrace{[x[kT] \otimes p] \otimes p}_{=:x[kT]\otimes^2 p}, \quad \text{etc..}$$

(iii) With $g(t)$ a time-varying parameter at a fixed FDLTI plant's input and a constant $\bar{u} \in \mathbf{R}$, we can estimate the quantities

$$\bar{u}, \quad \bar{u}g[kT], \quad \bar{u}g^2[kT], \quad \text{etc..}$$

Of course, there are certain restrictions on when we can obtain these estimates; we will discuss them at the appropriate points in the upcoming analysis. It may be possible to linearly estimate additional structures, but these are all that we will require in this thesis. While it is not obvious, these estimates can be leveraged to (linearly) construct estimates of a wide variety of control signals, including the desired optimal control signals.

2.3.2 Effect of Discontinuities on Estimation

Discontinuities will cause the control signal to be wrong for the remainder of the period in which they occur, so if there is a discontinuity in every period, then (in general) we will never be able to apply the desired control signal and the system will likely become unstable. To that end, we will insist that

$$T < \frac{T_s}{2},$$

so after every period *with* a discontinuity there is at least one *without* a discontinuity; furthermore, there is at most one discontinuity in any given period.

In the Control Phase we apply the estimate of the desired control signal; the behaviour there is essentially open loop, so the control signal will not be directly affected by the discontinuity. It will turn out that the behaviour of our controller over one period will be independent of the preceding periods, so we will be able to reassert the correct control in the period immediately following the switch; therefore, switching during the Control Phase is not a concern. On the other hand, if a discontinuity occurs in the Estimation Phase, then, due to the nature of our estimation method, the resulting estimate will likely be quite large - since this signal drives the Control Phase this would play havoc with the plant. To alleviate this problem we will obtain **two**[4] estimates *in sequence* and then use the smaller of the two with the view that, even though the choice may be wrong, at least it will be modest in size.

With this discussion in mind, we adopt the following notation: the duration of each estimate is T', so the Estimation Phase has duration $2T'$. Our controller is periodic of period T, so clearly we will need $T' < T/2$.

[4]Since we know that there is at most one switch in each period, two estimates will suffice.

Chapter 3

The Finite Stability Problem

In this chapter we propose a (mildly nonlinear) RACE controller which solves the Finite Stability problem; i.e. for a finite set of LTI plants $\{P_1, ..., P_q\}$, the controller provides the following:

- Stability in the face of persistent switching between those plants.
- Near optimal LQR performance when there is no switching.
- Near nominal LQR performance when there is switching (we will discuss what we mean by 'nominal' later on).

A preliminary version of this work was published in the conference paper [38]; in the first part (Section III) of that paper, we presented a highly intuitive and straightforward approach that performed estimation and control simultaneously and was able to achieve optimal performance after only a single period. Unfortunately, that controller has the major drawback that it can not handle even one plant change, so we do not present any of the details here.

This controller is motivated by the approach in [26]. Here we seek to alleviate the two main drawbacks in that work: we allow persistent plant changes and we no longer require that the controller period be small. Larger controller periods allow for smaller controller gains, so we expect improved noise performance. We investigate both asymptotic and input/output stability; since our controller is nonlinear, the latter is not automatic and has the effect of showing that our controller is noise tolerant. Furthermore, we will show that our nonlinearity can be (effectively) removed from the closed loop and put onto the noise signal, so our controller is almost linear.

A brief outline is as follows. In Section 3.1 we make the problem precise. In Section 3.2 we present the design of a RACE controller that is based on a generalized-hold and a generalized-sampler[1]; the design is intuitively appealing and the analysis

[1]A generalized sampler constructs a weighted integral of the measured signal rather than simply sampling this signal at a point.

straight-forward (as compared to [26]), we also analyze some noise related properties of this controller over a single period. In Section 3.3 we investigate the stability properties of this controller; in the context of plant switches, we construct a lower bound on the achievable rate of plant changes and a corresponding upper bound on the controller's period which, together, ensure that stability is maintained. In Section 3.4 we turn to the question of performance: when there are no switches we show that, if the Estimation Phase is sufficiently short, then we recover the optimal performance, while in the context of plant switches we show that, if the controller's period is sufficiently small, then we can recover the nominal performance (again, we will define what we mean by 'nominal' later). In Section 3.5 we present an illustrative example and we wrap up with a summary and concluding remarks in Section 3.6. In this chapter, we use the 2-norm to measure the size of a vector.

3.1 Problem Formulation

To define the problem, we will begin by obtaining a representation for our finite set of LTI plants and then impose some assumptions. The motivation for this problem is that of tolerating occasional faults, i.e. the plant switches from one LTI plant to another without warning. It will be convenient to express this behaviour in terms of a time varying plant that is defined via a switching signal, and then define a time varying uncertainty set composed of those time varying plants. Once we have done so, we will conclude by providing one of our two stability definitions and then defining what we mean by performance.

Consider the LTI plant P_i which is represented with the state-space model

$$\begin{array}{rcl} \dot{x}(t) & = & A_i\, x(t) + B_i\, u(t), \quad x(0) = x_0, \\ y(t) & = & C_i\, x(t), \end{array} \tag{3.1}$$

with $x(t) \in \mathbf{R}^n$ the state, $u(t) \in \mathbf{R}^m$ the control input, and $y(t) \in \mathbf{R}^r$ the plant output. Note that each of the LTI plants has the same number of inputs (m), outputs (r), and order (n)[2]. The finite set of LTI plants is given by

$$\mathcal{P} := \{P_i : i = 1, ..., q\}.$$

It is natural to impose the following assumptions:

Assumption 3.1 (C_i, A_i) *is observable for every* $i \in \{1, .., q\}$.

Assumption 3.2 (A_i, B_i) *is stabilizable for every* $i \in \{1, .., q\}$.

At this point we fix $\mathcal{P}$.

[2] If the orders are initially different, we let n be the largest order and augment the remaining admissible plant models with some additional observable but uncontrollable stable states.

We would like to fit the model (3.1) into the closed loop structure given by Figure 2.1. In this chapter we are not interested in tracking, so we set the exogenous reference signal y_{ref} to zero; incorporating the noise at the plant output yields

$$\begin{aligned} e &:= y + w_y \\ &= C_i x + w_y. \end{aligned}$$

If we also include the noise at the control input, then we can rewrite (3.1):

$$\begin{aligned} \dot{x}(t) &= A_i\, x(t) + B_i(u(t) + w_u(t)), \\ e(t) &= C_i\, x(t) + w_y(t). \end{aligned} \tag{3.2}$$

For our controller design method to work, we need the matrices $\{A_1, ..., A_q\}$ to have disjoint eigenvalues. Although this may seem restrictive, it turns out that, if we assume that the transfer functions

$$C_i(sI - A_i)^{-1}B_i, \quad i = 1, .., q$$

are distinct and that (A_i, B_i) and (A_j, B_j) share uncontrollable modes only if $j = i$, then the system can regularized by using output feedback of the form

$$u = \nu + Ke, \tag{3.3}$$

so (3.2) becomes

$$\left.\begin{aligned} \dot{x}(t) &= \underbrace{[A_i + B_i K C_i]}_{=:\hat{A}_i}\, x(t) + B_i \nu(t) + \underbrace{\begin{bmatrix} B_i & B_i K \end{bmatrix}}_{=:L_i} \underbrace{\begin{bmatrix} w_u \\ w_y \end{bmatrix}}_{=:w(t)}, \\ e(t) &= C_i x(t) + \begin{bmatrix} 0 & I \end{bmatrix} w(t), \end{aligned}\right\} \tag{3.4}$$

and it follows from [5] that, for almost all K, the matrices $\{\hat{A}_1, ..., \hat{A}_q\}$ will indeed enjoy the desired property. Henceforth we will impose

Assumption 3.3 $\{\hat{A}_1, ..., \hat{A}_q\}$ *have disjoint eigenvalues.*

The actual plant is not known exactly: at any given time it lies in $\mathcal{P}$. To define the set of admissible time varying plants we consider a piecewise constant switching signal

$$\sigma : \mathbf{R}^+ \to \{1, ..., q\}$$

which specifies the index of the plant at every time t; we assume that σ is continuous from the right[3]. Since $\mathcal{P}$ is fixed, we can use σ to uniquely define the time-varying plant P_σ which can be modeled via the following (time-varying) state-space model

$$\left.\begin{aligned} \dot{x}(t) &= \hat{A}_{\sigma(t)}\, x(t) + B_{\sigma(t)}\, \nu(t) + L_{\sigma(t)}\, w(t), \quad x(0) = x_0, \\ e(t) &= \underbrace{C_{\sigma(t)}\, x(t)}_{=y(t)} + \begin{bmatrix} 0 & I \end{bmatrix}\, w(t). \end{aligned}\right\} \tag{3.5}$$

[3]We require this for convenience; a more general σ can be handled with minor modifications.

With the minimum time between plant switches given by T_s, we can define the set of allowable switching signals: for any $T_s > 0$,

$$\Sigma_{T_s} := \{\sigma : \text{ there are at least } T_s \text{ time units between discontinuities in } \sigma\};$$

note that the first discontinuity is allowed to occur *before* $t = T_s$. We can also define the (time varying) uncertainty set: for any $T_s > 0$,

$$\mathcal{P}_{T_s} := \{P_\sigma : \sigma \in \Sigma_{T_s}\};$$

notice that $\mathcal{P}_\infty = \mathcal{P}$.

We now take a moment to investigate some useful properties of the switching signal σ together with some convenient notation. Each σ explicitly defines the times at which the plant P_σ switches, so using the (infinite) sequence of switching times that we defined in Section 2.2, we can also define the (infinite) sequence of LTI plant indices $\{i_l\}$ with $i_l \in \{1, ..., q\}$ which are such that

$$\sigma(t) = i_l, \quad t \in [t_l, t_{l+1}), \quad l \in \mathbf{N}.$$

Note that both $\{t_l\}$ and $\{i_l\}$ are implicit functions of σ.

3.1.1 Stability

We represent our controller $\mathcal{C}$ by (3.3) together with an as yet unspecified term given in input-output form:

$$\begin{aligned} \kappa \; &: \; \mathcal{PC}_\infty \to \mathcal{PC}_\infty \\ &: \; e \mapsto \nu. \end{aligned} \tag{3.6}$$

Combining this with the plant P_σ represented by (3.5) yields the closed loop system diagram shown in Figure 3.1 and leads naturally to the following I/O stability definition:

Definition 3.1 *With $T_s > 0$ and $x_0 = 0$, we say that the controller $\mathcal{C}$* **I/O stabilizes** $\mathcal{P}_{T_s}$ *if, for every $P_\sigma \in \mathcal{P}_{T_s}$, the map*

$$(w_u, w_y) \to (e, u, y)$$

is well defined and has bounded gain.

We will provide a definition of asymptotic stability once the structure of the compensator κ has been established in more detail.

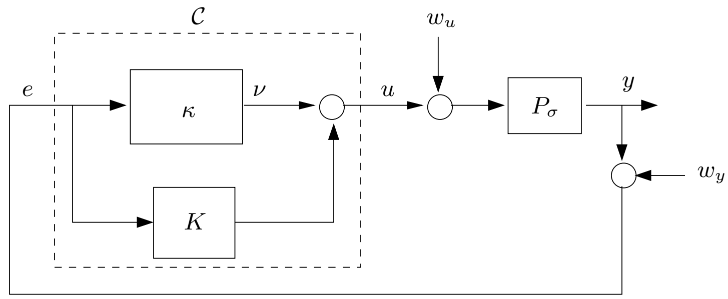

Figure 3.1: Block Diagram

3.1.2 LQR Performance

We would like to design a controller which not only provides stability but also near optimal LQR performance for each possible model $P_i \in \mathcal{P}$. To this end, for each $i \in \{1, ..., q\}$, we choose positive definite symmetric matrices $Q_i \in \mathbf{R}^{n\times n}$ and $R_i \in \mathbf{R}^{m\times m}$, set $w = 0$, and consider the classical quadratic performance index

$$J_i(x_0) = \int_0^\infty [x'(t)Q_i x(t) + u'(t)R_i u(t)]dt. \tag{3.7}$$

If we substitute using (3.3), then this cost function becomes

$$J_i(x_0) = \int_0^\infty x(t)'Q_i x(t) + (\nu(t) + Ke(t))'R_i(\nu(t) + Ke(t))dt, \tag{3.8}$$

which (since $e = C_i x$) is a standard form for the model (3.4). The LQR problem is to find, for each $x_0 \in \mathbf{R}^n$, the control signal ν which minimizes this cost. As is well-known, the optimal controller is state-feedback:

$$\nu = F_i x,$$

which gives rise to an optimal cost of the form

$$J_i^0(x_0) = x_0' V_i x_0$$

with V_i a positive definite solution of an associated Riccati equation. The closed loop matrix that arises from applying this state feedback is labeled

$$\bar{A}_i := \hat{A}_i + B_i F_i.$$

Remark 3.1 *Although (3.8) appears needlessly complicated compared to (3.7), this is exactly the structure that we will obtain when we analyze the step tracking case in the following chapter; if we retain this structure here, then the proofs in the following chapter will require minimal changes, so that is what we will do.*

Defining a cost function and associated optimal controller for plants that lie in $\mathcal{P}_{T_s}$ is harder than for those in $\mathcal{P}$ since we do not know the switching information a priori; we defer doing so until Section 3.4.

It will be useful to have uniform bounds on system parameters, so we define

$$a := \max_{i=1,..,q} \|\hat{A}_i\|,$$

$$b := \max_{i=1,..,q} \|B_i\|,$$

$$c := \max_{i=1,..,q} \|C_i\|,$$

$$\ell := \max_{i=1,..,q} \|L_i\|,$$

and

$$f := \max_{i=1,..,q} \|F_i\|.$$

Finally, since $\bar{A}_i$ is Hurwitz by design, there exist constants $\gamma_0 > 0$ and $\lambda_0 < 0$ such that

$$\|e^{\bar{A}_i t}\| \leq \gamma_0 e^{\lambda_0 t}, \quad i = 1, .., q, \quad t \geq 0. \tag{3.9}$$

3.2 The Controller

In this section we design the compensator κ which is periodic of period T. Recall that we wish to design a controller that provides optimal LQR performance for every LTI plant P_i and that, in Section 2.3.1, we indicated that it would be possible to find a good estimate of the optimal control signal

$$\nu(t) = F_i x(t) = \underbrace{F_i e^{\bar{A}_i(t-kT)}}_{=:H_i(t-kT)} x[kT], \quad t \in [kT, (k+1)T); \tag{3.10}$$

the main difficulty in doing so is that both i and $x[kT]$ are unknown. Although it would be sufficient to obtain these two pieces of information explicitly and independently, it is not necessary.

To achieve our objective, we begin by partitioning each period into two parts: the Estimation Phase followed by the Control Phase. In the Estimation Phase we will be able to use a generalized sampler to obtain an estimate of $E_i x[kT]$ and then, motivated by the properties of E_i, in the Control Phase we will use a generalized hold to apply (an estimate of) the desired control signal (3.10):

$$\nu(t) = \underbrace{\begin{bmatrix} H_1(t-kT) & \ldots & H_q(t-kT) \end{bmatrix}}_{=:H(t-kT)} E_i x[kT] = H_i(t-kT)x[kT].$$

The design discussion above gives rise to three minor issues. The first is that the 'optimal' control signal is applied only for part of the period (i.e. during the Control

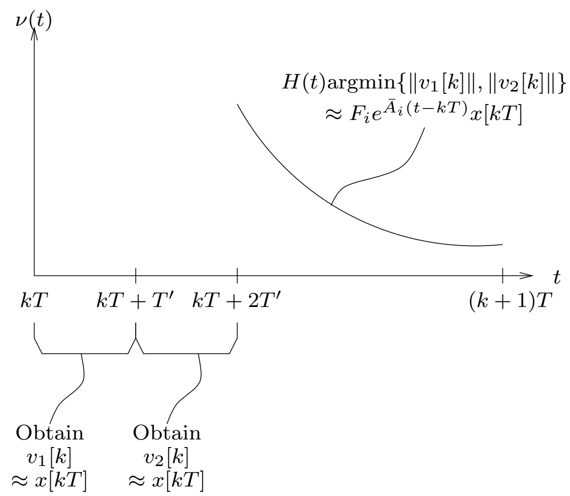

Figure 3.2: Input signal

Phase), so we do not use H directly; instead we use a suitably adjusted version of H which we label $\hat{H}$. Second, as discussed in Section 2.3.2, to mitigate the effect of a switch occurring during the Estimation Phase, we obtain two estimates in series; the duration of the Estimation Phase is $2T'$, so we require $T > 2T'$. Finally, we will be able to perform our estimation passively (i.e. the control signal is turned off during the Estimation Phase); to reflect this fact, we will set

$$\hat{H}(t) = 0, \quad t \in [kT, kT + 2T').$$

Figure 3.2 shows an example of the control signal ν over one period when there are no switches.

Finally, recall from Section 2.3.2 that we require $T < T_s/2$. We now state the final periodic compensator κ:

THE PROPOSED COMPENSATOR κ

With $T_s > 0$, $T \in (0, T_s/2)$, $T' \in (0, T/2)$, $S, \hat{H}$ periodic of period T, and $k \in \mathbf{Z}^+$, we define the controller by

$$v_1[k] := \int_{kT}^{kT+T'} S(t)e(t)\,dt, \tag{3.11}$$

$$v_2[k] := \int_{kT+T'}^{kT+2T'} S(t)e(t)\,dt, \tag{3.12}$$

$$\nu(t) = \begin{cases} 0 & t \in [kT, kT+2T') \\ \hat{H}(t)\mathrm{argmin}\{\|v_1[k]\|, \|v_2[k]\|\} & t \in [kT+2T', (k+1)T). \end{cases} \tag{3.13}$$

Remark 3.2 *Since the controller is nonlinear, there is no guarantee that the system will be well posed; however, it is routine to prove that, for every choice of* $\sigma \in \Sigma_{T_s}$ *and* $w \in \mathcal{L}_\infty$*, when (3.3) and* κ *are applied to the plant* P_σ*, every* $x_0 \in \mathbf{R}^n$ *yields a unique solution.*

Before proceeding we make three observations. First, this κ results in an overall controller that is nonlinear; however, the nonlinearity will turn out to be very mild. Second, since

$$\hat{H}(t) = 0, \ t \in [0, 2T'),$$

we can write (3.13) more compactly as

$$\nu(t) = \hat{H}(t)\text{argmin}\{\|v_1[k]\|, \|v_2[k]\|\} \quad t \in [kT, (k+1)T)$$

without worrying about causality issues. Finally, this compensator's behaviour depends only on the current period, so its 'initial condition' $v_i[-1]$ is irrelevant.

This choice of κ together with (3.3) leads naturally to the following definition of asymptotic stability:

Definition 3.2 *With* $T_s > 0$ *and* $w = 0$*, we say that the controller* $\mathcal{C}$ **asymptotically stabilizes** $\mathcal{P}_{T_s}$ *if, for every* $P_\sigma \in \mathcal{P}_{T_s}$ *we have that*

(i) for every $\varepsilon > 0$ *there exists a* $\delta > 0$ *so that, if* $\|x_0\| < \varepsilon$*, then*

$$\|x(t)\| < \delta, \quad t \geq 0,$$

$$\|v_1[k]\| < \delta, \ \textit{and} \ \|v_2[k]\| < \delta, \quad k \in \mathbf{Z}^+,$$

and

(ii) for every $x_0 \in \mathbf{R}^n$*, we have*

$$\lim_{t\to\infty} \|x(t)\| = 0, \ \lim_{k\to\infty} \|v_1[k]\| = 0, \ \textit{and} \ \lim_{k\to\infty} \|v_2[k]\| = 0.$$

Remark 3.3 *Observe that (ii) is a global convergence condition, rather than the typical local one. Furthermore, we will be able to prove a stronger condition than (i), namely: there exists a constant* $\gamma > 0$ *so that for every* $x_0 \in \mathbf{R}^n$ *we have*

$$\|x(t)\| \leq \gamma\|x_0\|, \quad t \geq 0,$$

$$\|v_1[k]\| \leq \gamma\|x_0\|, \ \textit{and} \ \|v_2[k]\| \leq \gamma\|x_0\|, \quad k \in \mathbf{Z}^+.$$

We now make the notions discussed above more precise. We begin by designing the sampler, then give the details of the hold, and last of all, we investigate some system properties in the presence of noise. The sampler and hold are both designed so that they have desirable properties when there are no plant switches and there is no noise. To that end, **in the next two sub-sections we set** $w = 0$ **and assume that the plant is** P_i **over the interval** $[kT, (k+1)T)$.

3.2.1 Designing the Generalized-Samplers and the Gain S

We would like to design our sampler's gain S such that, in the event that there is *no* plant switch and *no* noise, the generalized sampler provides exactly $E_i x[kT]$. We can state our estimation objective in the following way: if the plant is P_i and $w = 0$, then we wish S to be such that

$$v_1[k] = \int_{kT}^{kT+T'} S(t)e(t)\,dt = E_i x[kT] \tag{3.14}$$

and

$$v_2[k] = \int_{kT+T'}^{kT+2T'} S(t)e(t)\,dt = E_i x[kT]; \tag{3.15}$$

observe that $v_1[k], v_2[k] \in \mathbf{R}^{nq}$. To design such an S, observe that, since $w = 0$, we have that

$$e = y = C_i x,$$

so to achieve (3.14) for every admissible i we clearly need

$$\begin{aligned} \int_{kT}^{kT+T'} S(t)e(t)\,dt &= \int_{kT}^{kT+T'} S(t)C_i e^{\hat{A}_i(t-kT)} x[kT]\,dt \\ &= E_i x[kT], \qquad i = 1, .., q \end{aligned}$$

or equivalently

$$\int_{kT}^{kT+T'} S(t)C_i e^{\hat{A}_i(t-kT)}\,dt = E_i, \quad i = 1, .., q. \tag{3.16}$$

If we define the augmented matrices

$$\hat{A} := \text{diag}\{\hat{A}_1 ... \hat{A}_q\} \text{ and } C := \left[\begin{array}{ccc} C_1 & ... & C_q \end{array} \right],$$

then (3.16) collapses into

$$\int_0^{T'} S(t)Ce^{\hat{A}t}dt = I. \tag{3.17}$$

Finally, if we set

$$S(t) = \begin{cases} e^{-\hat{A}T'} S(t - T'), & t \in [T', 2T') \\ 0, & t \in [2T', T), \end{cases} \tag{3.18}$$

then (3.17) is also sufficient to ensure that (3.15) is satisfied for every admissible i. Assumptions 3.1 and 3.3 ensure that $(C, \hat{A})$ is observable, so there is a whole family of periodic functions which satisfy (3.17) and (3.18); we provide an example of such a function in Section 3.5.

We would like to ensure that v_1 and v_2 are bounded in the presence of noise. At first glance, it would seem that restricting S (on $[0, T)$) to $\mathcal{PC}_\infty[0, T)$ would be adequate; however, this would preclude the classical 'ideal sampler'. To that end, we restrict S (on $[0, T)$) to be the sum of an element of $\mathcal{PC}_\infty[0, T)$ and a sum of a finite sequence of impulses over $[0,T)$; all such maps are said to be **admissible**. Notice that an admissible S has the desirable property that the maps

$$\int_0^{T'} S(t)e(t)dt : \mathcal{PC}_\infty[0,T) \mapsto \mathbf{R}^{nq}$$

and

$$\int_{T'}^{2T'} S(t)e(t)dt : \mathcal{PC}_\infty[0,T) \mapsto \mathbf{R}^{nq}$$

have bounded gain.

Remark 3.4 *The particular choice of an admissible S only determines the effect of noise on the system. That being said, some choices of S will be easier to implement than others. Indeed, for implementation purposes, we will typically choose a finite string of impulses, equally spaced on $[0,T)$, since implementing this would be equivalent to constructing a weighted sequence of samples of the error, which is easy to carry out.*

At this point, with $T_s > 0$, for each $T \in (0, T_s/2)$ and $T' \in (0, T/2)$, we choose an admissible S that satisfies (3.17) and (3.18); to minimize clutter we do not write it as an explicit function of T and T'.

3.2.2 Designing the Generalized-Hold $\hat{H}$

Recall that the optimal control signal for the plant P_i is

$$\nu(t) = H_i(t)x[kT], \quad t \in [kT, (k+1)T);$$

as discussed at the beginning of this section, we can combine these H_i's to construct the optimal hold gain H. Also recall that, since the control signal is applied only for part of the period, we do not apply the optimal hold; instead we apply a suitably adjusted version, whose gain $\hat{H}$ is naturally partitioned as

$$\hat{H} := \begin{bmatrix} \hat{H}_1 & \ldots & \hat{H}_q \end{bmatrix}.$$

We would like $\hat{H}$ to have two properties:

(i) that it converges (in some sense) to H as T' goes to zero and

(ii) that, even if T' is non-zero, when there is *no* noise and *no* switches, the state exactly matches the optimal trajectory at the endpoints.

We first consider (ii). We would like

$$x[(k+1)T] = e^{\bar{A}_i T} x[kT], \tag{3.19}$$

but

$$e^{\bar{A}_i T} = e^{\hat{A}_i T} + \int_0^T e^{\hat{A}_i(T-\tau)} B_i H_i(\tau) d\tau$$

and

$$x[(k+1)T] \;=\; \left[e^{\hat{A}_i T} + \int_{2T'}^T e^{\hat{A}_i(T-\tau)} B_i \hat{H}_i(\tau)\; d\tau \right] x[kT],$$

so, since i is unknown, to satisfy (3.19) and thereby (ii) we need

$$\int_{2T'}^T e^{\hat{A}_i(T-\tau)} B_i \hat{H}_i(\tau) d\tau = \int_0^T e^{\hat{A}_i(T-\tau)} B_i H_i(\tau) d\tau, \quad i = 1, .., q.$$

This is clearly equivalent to

$$\int_{2T'}^T e^{\hat{A}_i(T-\tau)} B_i \underbrace{[\hat{H}_i(\tau) - H_i(\tau)]}_{=:\tilde{H}(\tau)} d\tau = \underbrace{\int_0^{2T'} e^{\hat{A}_i(T-\tau)} B_i H_i(\tau) d\tau,}_{=:\Psi_i(T,T')} \quad i = 1, .., q. \quad (3.20)$$

If $(\hat{A}_i, B_i)$ is controllable[4] then, for each i, (3.20) has a natural solution for $\tilde{H}_i$: with the controllability grammian defined by

$$W_i(t) := \int_0^t e^{-\hat{A}_i \tau} B_i B_i' e^{-\hat{A}_i' \tau} d\tau,$$

the least square solution to (3.20) is

$$\tilde{H}_i(t) = B_i' e^{-\hat{A}_i'(t-2T')} W_i^{-1}(T - 2T') e^{-\hat{A}_i(T-2T')} \Psi_i(T, T'), \quad t \in [2T', T). \quad (3.21)$$

Our choice of hold is then

$$\hat{H}(t) = \begin{cases} 0 & t \in [0, 2T') \\ \begin{bmatrix} H_1 + \tilde{H}_1 & H_2 + \tilde{H}_2 & \ldots & H_q + \tilde{H}_q \end{bmatrix}(t) & t \in [2T', T), \end{cases} \quad (3.22)$$

which clearly satisfies (ii). Note that each $\tilde{H}_i$ is an implicit function of T and T'; furthermore, we define

$$\tilde{H} := \begin{bmatrix} \tilde{H}_1 & \tilde{H}_2 & \ldots & \tilde{H}_q \end{bmatrix}.$$

We now prove that (i) holds for our choice of $\tilde{H}$; before doing so, observe that

$$\begin{aligned} \|H_i(t)\| = \|F_i e^{\bar{A}_i t}\| &\leq f\gamma_0 e^{\lambda_0 t} \\ &\leq f\gamma_0, \quad t \in [0, T), \quad i = 1, .., q. \end{aligned}$$

It will turn out that the following is a critical function:

$$\varepsilon_H(T, T') := 2b^2 f \gamma_0 e^{aT} T' \max_{i=1,..,q} \|W_i^{-1}(T - 2T')\|. \quad (3.23)$$

[4]If it is not, we do a similarity transformation to isolate the controllable part and then proceed in the same way.

Lemma 3.1 *With $T_s > 0$, for every $T \in (0, T_s/2)$ we have that*

(i) for every $T' \in (0, T/2)$

$$\|\tilde{H}_i(t)\| = \|\hat{H}_i(t) - H_i(t)\| \leq \varepsilon_H(T, T'), \quad t \in [2T', T), \quad i = 1, .., q,$$

(ii) and $\lim_{T' \to 0} \varepsilon_H(T, T') = 0.$

Proof: This result follows directly from (3.21) and is left to the reader. ∎

At this point we have defined $\hat{H}$ by (3.21) and (3.22), and chosen an admissible S that satisfies (3.17) and (3.18), so κ as written in (3.11)-(3.13) is well defined, as is the controller $\mathcal{C}$, which we relabel $\mathcal{C}(T, T')$ to emphasize its dependence on T and T'.

3.2.3 System Properties in the Presence of Noise

Now that we have designed the controller $\mathcal{C}(T, T')$, we would like to investigate how noise will affect the system; the presence of the nonlinearity (which arises by choosing the smaller of the two samples) complicates the analysis. It will turn out that, over periods where there is no switch, we can separate the effect of the nonlinearity from the effect of the state at the beginning of the period. In fact, it will turn out that we can write a state-space equation (over one period) in which the effect of the state is linear - the nonlinearities are contained entirely in the noise part of the equation[5]. Unfortunately, when there is a switch in the period we cannot isolate the nonlinearity in this way; however, we will be able to find a nice bound on the size of the sampler's outputs and hence the control signal.

The following proposition will make these two notions more precise, but first we remind the reader of the notation

$$\underline{w}_k(t) := w(kT + t), \quad t \in [kT, (k+1)T)$$

and that there are *no* plant switches on the intervals

$$[kT, (k+1)T), \quad k \in \{k_l : l \in \mathbf{N}\}^c.$$

Finally, we define

$$\rho(T') := \max_{i,j=1,..,q} \left\{ \max_{t \in [0,T')} \|e^{-\hat{A}_i t} e^{\hat{A}_j t}\| \right\};$$

notice that

$$\lim_{T' \to 0} \rho(T') = 1.$$

[5]The state does show up in the nonlinear part, but only in the sense that it selects between two functions of noise.

Proposition 3.1 *With $T_s > 0$, $T \in (0, T_s/2)$, and $T' \in (0, T/2)$, there exists a constant $\gamma_v(T, T') > 0$, $2q$ linear functions of noise*

$$\phi_{1,i} : \mathcal{L}_\infty[0, T) \to \mathbf{R}^{n+r}, \quad i = 1, .., q$$

$$\phi_{2,i} : \mathcal{L}_\infty[0, T) \to \mathbf{R}^{n+r}, \quad i = 1, .., q$$

with bounded gains, and q selector functions

$$\chi_i : \mathcal{L}_\infty[0, T) \times \mathbf{R}^{n+r} \to \{0, 1\}, \quad i = 1, .., q,$$

such that, for every $x_0 \in \mathbf{R}^n$, $\sigma \in \Sigma_{T_s}$, and $w \in \mathcal{PC}_\infty$, when $\mathcal{C}(T, T')$ is attached to P_σ, we have that:

(i) P_σ's state-space representation (3.5) satisfies

$$\begin{aligned} \dot{x}(t) \;\; = \;\; & \hat{A}_{\sigma(t)} x(t) + B_{\sigma(t)} \hat{H}_{\sigma(t)}(t) x[kT] + \begin{bmatrix} B_{\sigma(t)} \hat{H}(t) & L_{\sigma(t)} \end{bmatrix} \times \\ & \begin{bmatrix} \chi_{\sigma(t)}(\underline{w}_k, x[kT]) \phi_{1,\sigma(t)}(\underline{w}_k) + [1 - \chi_{\sigma(t)}(\underline{w}_k, x[kT])] \phi_{2,i}(\underline{w}_k) \\ w(t) \end{bmatrix}, \\ & \qquad t \in [kT, (k+1)T), \quad k \in \{k_l : l \in \mathbf{N}\}^c. \end{aligned}$$

(ii) The sampler outputs v_1 and v_2 satisfy

$$\min\{\|v_1[k_l]\|, \|v_2[k_l]\|\} \;\; \leq \;\; \rho(T') \|x[k_l T]\| + \gamma_v(T, T') \|w\|_\infty, \quad l \in \mathbf{Z}^+.$$

Proof: Please see Appendix A.

The above provides detailed structure on the behaviour of the closed loop system. Since the details of the way that the noise enters the system are irrelevant, the following corollary, which looks at periods with no plant switches, will be useful:

Corollary 3.1 *With $T_s > 0$, $T \in (0, T_s/2)$, and $T' \in (0, T/2)$, there exists $2q$ nonlinear, bounded gain functions*

$$\phi_i : \mathcal{L}_\infty[0, T) \times \mathbf{R}^{n+r} \to \mathcal{L}_\infty[0, T), \quad i = 1, .., q$$

$$\theta_i : \mathcal{L}_\infty[0, T) \times \mathbf{R}^{n+r} \to \mathbf{R}^{n+r}, \quad i = 1, .., q$$

so that, for every $x_0 \in \mathbf{R}^n$, $\sigma \in \Sigma_{T_s}$, $w \in \mathcal{PC}_\infty$, and $k \in \{k_l : l \in \mathbf{N}\}^c$, when $\mathcal{C}(T, T')$ is attached to P_σ, we have that (3.5) satisfies

$$\dot{x}(t) = \hat{A}_{\sigma(t)} x(t) + B_{\sigma(t)} \hat{H}_{\sigma(t)}(t) x[kT] + \phi_{\sigma(t)}(\underline{w}_k, x[kT]), \quad t \in [kT, (k+1)T),$$

$$x[(k+1)T] = e^{\bar{A}_{\sigma[kT]} T} x[kT] + \theta_{\sigma[kT]}(\underline{w}_k, x[kT]).$$

Proof: The result follows directly from Proposition 3.1(i) and the properties of $\hat{H}$ (most notably (3.19)) and is left to the reader. ∎

Remark 3.5 *We see that, on periods on which there are no plant switches, the closed loop system behaviour is quite regular: the effect of the noise can be separated out, in some sense, from the effect of the value of the state at the beginning of the period.*

3.3 Stability

In this section we will investigate stability. When there are no switches, we can prove stability for arbitrarily large T, which is not possible when we have plant switches; as such, we investigate these cases separately. We begin with the former.

Theorem 3.1 *For every $T > 0$ and $T' \in (0, T/2)$, the controller $\mathcal{C}(T, T')$ stabilizes $\mathcal{P}$.*

Proof:

Fix $T > 0$ and $T' \in (0, T/2)$. Let $x_0 \in \mathbf{R}^n$, $w \in \mathcal{PC}_\infty$, and $i = 1, .., q$ be arbitrary and assume that the plant is $P_i \in \mathcal{P}$. From Corollary 3.1 we have that there exists two nonlinear functions

$$\phi_i : \mathcal{L}_\infty[0, T) \times \mathbf{R}^{n+r} \to \mathcal{L}_\infty[0, T)$$

$$\theta_i : \mathcal{L}_\infty[0, T) \times \mathbf{R}^{n+r} \to \mathbf{R}^{n+r}$$

which have bounded gains and are such that, for every $k \in \mathbf{Z}^+$ we have

$$\dot{x}(t) = \hat{A}_i x(t) + B_i \hat{H}_i(t) x[kT] + \phi_i(\underline{w}_k, x[kT]), \quad t \in [kT, (k+1)T) \tag{3.24}$$

$$x[(k+1)T] = e^{\bar{A}_i T} x[kT] + \theta_i(\underline{w}_k, x[kT]); \tag{3.25}$$

since these functions have bounded gains, we can define

$$\gamma_\phi := \max_{i=1,..,q} \|\phi_i\|_\infty,$$

$$\gamma_\theta := \max_{i=1,..,q} \|\theta_i\|.$$

Finally, since T and T' are fixed, we have that $\|\hat{H}\|_\infty$ is well defined.

(Asymptotic Stability)

Let $x_0 \in \mathbf{R}^n$ remain arbitrary and set $w = 0$. In this context (3.24) and (3.25) reduce to

$$\dot{x}(t) = \hat{A}_i x(t) + B_i \hat{H}_i(t) x[kT], \quad t \in [kT, (k+1)T) \tag{3.26}$$

and

$$x[(k+1)T] = e^{\bar{A}_i T} x[kT]. \tag{3.27}$$

If we solve (3.27) and then substitute the result into the solution of (3.26), then we clearly have

$$x(t) = \left(e^{\hat{A}_i(t-kT)} + \int_{kT}^{t} e^{\hat{A}_i(t-\tau)} B_i \hat{H}_i(\tau) d\tau \right) e^{\bar{A}_i kT} x_0, \quad t \in [kT, (k+1)T), \quad k \in \mathbf{Z}^+$$

so

$$\begin{aligned} \|x(t)\| &\leq \left(e^{aT} + T e^{aT} b \|\hat{H}\|_\infty \right) \gamma_0 e^{\lambda_0 kT} \|x_0\|, \quad t \in [kT, (k+1)T), \quad k \in \mathbf{Z}^+ \\ &\leq \left(e^{aT} + T e^{aT} b \|\hat{H}\|_\infty \right) \gamma_0 e^{-\lambda_0 T} e^{\lambda_0 t} \|x_0\|, \quad t \geq 0; \end{aligned}$$

yielding both our desired bound on x and

$$\lim_{t \to \infty} \|x(t)\| = 0.$$

Since S is admissible and periodic, we have that there exists a constant $\gamma_s > 0$ so that

$$\begin{aligned} \|v_1[k]\| &= \left\| \int_{kT}^{kT+T'} S(t) e(t) dt \right\| \\ &\leq \gamma_s \max_{t \in [kT, kT+T')} \|e(t)\|, \quad k \in \mathbf{Z}^+ \end{aligned}$$

and

$$\begin{aligned} \|v_2[k]\| &= \left\| \int_{kT+T'}^{kT+2T'} S(t) e(t) dt \right\| \\ &\leq \gamma_s \max_{t \in [kT+T', kT+2T')} \|e(t)\|, \quad k \in \mathbf{Z}^+. \end{aligned}$$

Since $e = C_i x$, this clearly yields both the desired bounds on v_1 and v_2 and the limits

$$\begin{aligned} \lim_{k \to \infty} \|v_1[k]\| &\leq \lim_{t \to \infty} \gamma_s c \|x(t)\| \\ &= 0 \end{aligned}$$

and

$$\lim_{k \to \infty} \|v_2[k]\| = 0.$$

■

(I/O Stability)

Let $w \in \mathcal{PC}_\infty$ be arbitrary and set $x_0 = 0$. From the structure of $\mathcal{C}(T, T')$, the definitions of e and y, and since S is admissible and $\|\hat{H}\|_\infty$ is well defined, it is enough to find a bound on $\|x\|_\infty$ in terms of $\|w\|_\infty$.

We begin by solving (3.24) to find

$$x(t) = e^{\hat{A}_i(t-kT)}x(kT) + \int_{kT}^{t} e^{\hat{A}_i(t-\tau)}\big(B_i\hat{H}_i(\tau)x[kT] + \phi_i(\underline{w}_k, x[kT])\big)d\tau,$$
$$t \in [kT, (k+1)T), \quad k \in \mathbf{Z}^+,$$

so

$$\begin{aligned} \|x(t)\| &\le e^{aT}\|x[kT]\| + \int_{kT}^{t} e^{a(t-\tau)}\big(b\|\hat{H}\|_\infty\|x[kT]\| + \gamma_\phi\|w\|_\infty\big)d\tau \\ &\le \underbrace{\left[e^{aT} + Te^{aT}b\|\hat{H}\|_\infty\right]}_{=:\gamma_1}\|x[kT]\| + \underbrace{Te^{aT}\gamma_\phi}_{=:\gamma_2}\|w\|_\infty, \\ &\qquad t \in [kT, (k+1)T), \quad k \in \mathbf{Z}^+. \end{aligned} \tag{3.28}$$

Since $x_0 = 0$, solving (3.25) yields

$$x[kT] = \sum_{j=0}^{k-1} e^{\bar{A}_iT(k-1-j)}\theta_i(\underline{w}_j, x[jT]), \quad k \in \mathbf{Z}^+,$$

so

$$\begin{aligned} \|x[kT]\| &\le \sum_{j=0}^{k-1}\gamma_0 e^{\lambda_0 Tj}\gamma_\theta\|w\|_\infty \\ &\le \frac{\gamma_0\gamma_\theta}{1-e^{\lambda_0 T}}\|w\|_\infty, \quad k \in \mathbf{Z}^+, \end{aligned}$$

which combines with (3.28), yielding

$$\begin{aligned} \|x(t)\| &\le \gamma_1\|x[kT]\| + \gamma_2\|w\|_\infty, \quad t \in [kT, (k+1)T), \quad k \in \mathbf{Z}^+ \\ &\le \left[\gamma_1\frac{\gamma_0\gamma_\theta}{1-e^{\lambda_0 T}} + \gamma_2\right]\|w\|_\infty, \quad t \ge 0, \end{aligned}$$

so clearly

$$\|x\|_\infty \le \left[\gamma_1\frac{\gamma_0\gamma_\theta}{1-e^{\lambda_0 T}} + \gamma_2\right]\|w\|_\infty.$$

∎

We now turn to the case of (possibly persistent) plant switches; these plant switches introduce two main difficulties:

(i) It is well known that switching too quickly between stable LTI systems can lead to instability.

(ii) In periods with a plant switch, the incorrect control will (likely) be applied.

To address these two issues we will place bounds on T_s, T, and T':

(T_s) Even if (ii) was not an issue and we could immediately apply the correct control signal, (i) says that if T_s is too small then our controller may not be able to stabilize the system. To that end we will place a loose[6] **lower bound** on the choice of T_s. It will turn out that a critical value is

$$\underline{T_s} := \frac{\ln\left\{\gamma_0\left(1+\frac{2bf\gamma_0}{a}\right)^2\right\}}{|\lambda_0|}.$$

(T) As T increases, the duration and adverse effect of (ii) will also increase. The end result of this is twofold: first, the undesirable effect of the plant switch is further increased due to the application of incorrect control, and second, the total amount of time that the correct control signal is applied (before the next switch occurs) is decreased. To this end, we place an **upper bound** on T. It will turn out that a critical value is

$$\bar{T}(T_s) := \min\left\{\frac{T_s}{2}, \frac{|\lambda_0|}{3(a-\lambda_0)}\left(T_s - \underline{T_s}\right)\right\}.$$

(T') Finally, recall that the gain of our generalized hold is

$$\hat{H} = H + \tilde{H};$$

the size of $\tilde{H}$ and therefore $\hat{H}$ will (likely) increase with increasing T'. So, even if T is small, the adverse effect of (ii) will increase with increasing T'. To that end, we will also require that T' be small.

Theorem 3.2 *For every $T_s > \underline{T_s}$ and $T \in (0, \bar{T}(T_s))$, if T' is sufficiently small then $\mathcal{C}(T, T')$ stabilizes $\mathcal{P}_{T_s}$.*

Proof:

Fix $T_s > \underline{T_s}$, $T \in (0, \bar{T}(T_s))$, $\sigma \in \Sigma_{T_s}$, and let $x_0 \in \mathbf{R}^n$, $w \in \mathcal{PC}_\infty$, and $T' \in (0, T/2)$ be arbitrary. From Corollary 3.1, there exist $2q$ nonlinear functions

$$\phi_i : \mathcal{L}_\infty[0,T) \times \mathbf{R}^{n+r} \to \mathcal{L}_\infty[0,T), \quad i = 1, .., q$$

and

$$\theta_i : \mathcal{L}_\infty[0,T) \times \mathbf{R}^{n+r} \to \mathbf{R}^{n+r}$$

that have bounded gains and are such that, for every $k \in \{k_l : l \in \mathbf{N}\}^c$ we have

$$\dot{x}(t) = \hat{A}_{\sigma(t)}x(t) + B_{\sigma(t)}\hat{H}_{\sigma(t)}(t)x[kT] + \phi_{\sigma(t)}(\underline{w}_k, x[kT]), \quad t \in [kT, (k+1)T) \quad (3.29)$$

[6] This bound is loose in the sense that our controller can stabilize uncertainty sets with somewhat smaller T_s, but to do so we must force T to be small.

and

$$x[(k+1)T] = e^{\bar{A}_{\sigma(t)}T} x[kT] + \theta_{\sigma(t)}(\underline{w}_k, x[kT]). \tag{3.30}$$

Since these functions have bounded gains and are implicit functions of T', we can define

$$\gamma_\phi(T') := \max_{i=1,..,q} \|\phi_i\|_\infty$$

and

$$\gamma_\theta(T') := \max_{i=1,..,q} \|\theta_i\|.$$

Finally, recall that switches are confined to $\{k_l : l \in \mathbf{N}\}$ and

$$t_l \in [k_l, (k_l+1)T), \quad l \in \mathbf{Z}^+;$$

however, observe that, even though the plant parameters are discontinuous at t_l, the state x is not.

Before turning to the particular details of each of our two stability types, it will be useful to perform some preliminary analysis. To proceed, it is important to note that $\|\hat{H}(t)\|$ and $\|\hat{H}\|_\infty$ are implicit functions of T'; indeed, there is no uniform upper bound on $\|\hat{H}\|_\infty$; however, by definition, for every $i = 1,..,q$ we clearly have that

$$\|\hat{H}_i(t)\| = \begin{cases} 0 & t \in [0, 2T') \\ \|H_i(t) - \tilde{H}_i(t)\| & t \in [2T', T), \end{cases}$$

so

$$\begin{aligned} \|\hat{H}_i\|_\infty &\le \max_{t\in[2T',T)} \|H_i(t) - \tilde{H}_i(t)\| \\ &\le f\gamma_0 + \max_{t\in[2T',T)} \|\tilde{H}_i(t)\|, \quad i = 1,..,q. \end{aligned}$$

If we use the definition of ε_H given in (3.23) and Lemma 3.1 to bound the rightmost term, then we find

$$\|\hat{H}_i\|_\infty \le f\gamma_0 + \varepsilon_H(T, T'), \quad i = 1,..,q. \tag{3.31}$$

Since T is fixed, we write $\varepsilon_H(T')$ instead of $\varepsilon_H(T, T')$.

We first address the $l = 0$ case in the sense that we investigate the interval $[0, k_1 T]$[7]. We begin by solving (3.30):

$$x[kT] = e^{\bar{A}_{i_0}kT} x_0 + \sum_{j=0}^{k-1} e^{\bar{A}_{i_0}(k-1-j)T} \theta_{i_0}(\underline{w}_j, x[jT]), \quad k = 0,..,k_1,$$

so

$$\begin{aligned} \|x[kT]\| &\le \gamma_0 e^{\lambda_0 kT} \|x_0\| + \sum_{j=0}^{k-1} \gamma_0 e^{\lambda_0 jT} \gamma_\theta(T') \|w\|_\infty \\ &\le \gamma_0 \|x_0\| + \gamma_0 \gamma_\theta(T') \frac{1}{1 - e^{\lambda_0 T}} \|w\|_\infty, \quad k = 0,..,k_1. \end{aligned} \tag{3.32}$$

[7]Observe that this includes the degenerate case of $t_1 < T$, in which case the interval consists of only the point 0.

Additionally, we can solve (3.29):

$$x(t) = e^{\hat{A}_{i_0}(t-kT)}x[kT] + \int_{kT}^{t} e^{\hat{A}_{i_0}(t-\tau)}\big[B_{i_0}\hat{H}_{i_0}(\tau)x[kT] + \phi_{i_0}(\underline{w}_k, x[kT])\big]d\tau,$$
$$t \in [kT,(k+1)T], \quad k = 0,..,k_1-1,$$

so

$$\|x(t)\| \leq \underbrace{\big[e^{aT} + Te^{aT}b\big(f\gamma_0 + \varepsilon_H(T')\big)\big]}_{=:\alpha_0(T')}\|x[kT]\| + Te^{aT}\gamma_\phi(T')\|w\|_\infty,$$
$$t \in [kT,(k+1)T], \quad k = 0,..,k_1-1,$$

which combines with (3.32) to provide a nice bound over the interval of interest:

$$\begin{aligned}\|x(t)\| \quad \leq \quad & \alpha_0(T')\gamma_0\|x_0\| + \\ & \underbrace{\left[\alpha_0(T')\gamma_0\gamma_\theta(T')\frac{1}{1-e^{\lambda_0 T}} + Te^{aT}\gamma_\phi(T')\right]}_{=:\bar{\gamma}(T')}\|w\|_\infty, \quad t \in [0, k_1 T]. \end{aligned} \tag{3.33}$$

We now turn to $t \geq k_1 T$. We will present two claims: the first examines the behavior of x on intervals where there *is* a switch and the second uses Corollary 3.1 to examine the behavior of x on intervals where there is *no* switch. Together with (3.32), these claims will provide the necessary tools for proving both types of stability.

Claim 1: There exist constants $\gamma_1(T') > 0$ and $\bar{\gamma}_1(T') > 0$ such that

$$\|x(t)\| \leq \gamma_1(T')\|x[k_l T]\| + \bar{\gamma}_1(T')\|w\|_\infty, \quad t \in [k_l T,(k_l+1)T], \quad l \in \mathbf{N}.$$

Proof:

Let $l \in \mathbf{N}$ be arbitrary. Since the interval of interest contains a plant switch, we use Proposition 3.1(ii) to bound the size of the sampler output: there exists a constant $\gamma_v(T') > 0$ such that

$$\min\{\|v_1[k_l]\|, \|v_2[k_l]\|\} \leq \rho(T')\|x[k_l T]\| + \gamma_v(T')\|w\|_\infty$$

so, by definition of ν and using (3.31) to bound $\|\hat{H}\|_\infty$, we have

$$\begin{aligned}\|\nu(t)\| \quad \leq \quad & \big(f\gamma_0 + \varepsilon_H(T')\big) \times \big(\rho(T')\|x[k_l T]\| + \gamma_v(T')\|w\|_\infty\big), \\ & t \in [k_l T,(k_l+1)T). \end{aligned} \tag{3.34}$$

The plant switch causes a discontinuity in the plant parameters, so we must be careful when solving for x in this period. To that end, we split the period into two parts: before the switch $[k_l T, t_l]$, and after the switch $[t_l,(k_l+1)T]$.

We start by investigating the first interval. Solving (3.5) yields

$$x(t) = e^{\hat{A}_{i_l}(t-k_lT)}x[k_lT] + \int_{k_lT}^{t} e^{\hat{A}_{i_l}(t-\tau)}\left[B_{i_l}\nu(\tau) + L_{i_l}w(\tau)\right]d\tau, \quad t \in [k_lT, t_{l+1}],$$

so

$$\|x(t)\| \;\leq\; e^{a(t-k_lT)}\|x[k_lT\| + \frac{\overbrace{(e^{a(t-k_lT)}-1)}^{\leq e^{a(t-k_lT)}}}{a}\left(b \max_{t\in[t_l,t_{l+1})}\|\nu(\tau)\| + \ell\|w\|_\infty\right), \quad t \in [k_lT, t_{l+1}].$$

We can then use (3.34) to find

$$\begin{aligned}
\|x(t)\| &\leq e^{aT}\|x[k_lT]\| + \frac{e^{aT}}{a}b\big(f\gamma_0 + \varepsilon_H(T')\big)\big(\rho(T')\|x[k_lT]\| + \gamma_v(T')\|w\|_\infty\big) + \\
&\quad \frac{e^{aT}\ell}{a}\|w\|_\infty \\
&= \underbrace{e^{aT}\left[1 + \frac{b\big(f\gamma_0 + \varepsilon_H(T')\big)\rho(T')}{a}\right]}_{=:\alpha_1(T')}\|x[k_lT]\| + \\
&\quad \underbrace{\frac{e^{aT}}{a}\left[b\big(f\gamma_0 + \varepsilon_H(T')\big)\gamma_v(T') + \ell\right]}_{=:\alpha_2(T')}\|w\|_\infty, \quad t \in [k_lT, t_l].
\end{aligned} \tag{3.35}$$

Similarly, in the second interval $[t_l, (k_l+1)T)$ we find

$$\|x(t)\| \;\leq\; e^{aT}\|x(t_l)\| + \frac{e^{aT}}{a}b\big(f\gamma_0 + \varepsilon_H(T')\big)\rho(T')\|x[k_lT]\| + \alpha_2(T')\|w\|_\infty, \quad t \in [t_l, (k_l+1)T],$$

In the case where $k_lT = t_l$ this provides

$$\|x(t)\| \leq \alpha_1(T')\|x[k_lT]\| + \alpha_2(T')\|w\|_\infty, \quad t \in [k_lT, (k_l+1)T]. \tag{3.36}$$

otherwise, we use (3.35) to find

$$\begin{aligned}
\|x(t)\| &\leq e^{aT}\left(\alpha_1(T')\|x[k_lT]\| + \alpha_2(T')\|w\|_\infty\right) + \\
&\quad \frac{e^{aT}}{a}b\big(f\gamma_0 + \varepsilon_H(T')\big)\rho(T')\|x[k_lT]\| + \alpha_2(T')\|w\|_\infty \\
&= \underbrace{e^{aT}\left(\alpha_1(T') + \frac{b\big(f\gamma_0 + \varepsilon_H(T')\big)\rho(T')}{a}\right)}_{=:\gamma_1(T')}\|x[k_lT]\| + \\
&\quad \underbrace{\alpha_2(T')\left(e^{aT}+1\right)}_{=:\bar{\gamma}_1(T')}\|w\|_\infty, \quad t \in [k_lT, t_l].
\end{aligned} \tag{3.37}$$

Since it is clear that

$$\gamma_1(T') \geq \alpha_1(T')$$

and

$$\bar{\gamma}_1(T') \geq \alpha_2(T'),$$

we can combine (3.37) with (3.36) and (3.35) to find a bound on the entire interval:

$$\|x(t)\| \leq \gamma_1(T')\|x[k_l T]\| + \bar{\gamma}_1(T')\|w\|_\infty, \quad t \in [k_l T, (k_l+1)T].$$

$\blacksquare$

We now turn to intervals where there is no switch: $[(k_l+1)T, k_{l+1}T)$. Note that we have only one degenerate case: it may be that $t_1 < 2T$, so $p_1 T \leq 1$, in which case the interval $[(p_0+1)T, p_1 T] = [T, p_1 T]$ contains only the point T.

Claim 2: There exists $\gamma_2(T') > 0$ and $\bar{\gamma}_2(T') > 0$ such that, for all $l \in \mathbf{N}$ we have

$$\|x(t)\| \leq e^{\lambda_0(t-(k_l+1)T)}\gamma_2(T')\|x[k_l T]\| + \bar{\gamma}_2(T')\|w\|_\infty, \quad t \in [(k_l+1)T, k_{l+1}T].$$

Proof:

Let $l \in \mathbf{N}$ be arbitrary. To reduce notational clutter we write

$$[\underline{k}T, \bar{k}T) := [(k_l+1)T, k_{l+1}T)$$

and write i instead of i_l (so the plant is P_i over the interval). We begin by solving (3.30):

$$x[kT] = e^{\bar{A}_i(k-\underline{k})T}x[\underline{k}T] + \sum_{j=\underline{k}}^{k-1} e^{\bar{A}_i(k-1-j)T}\theta_i(\underline{w}_j, x[jT]), \quad \underline{k} \leq k \leq \bar{k}-1;$$

so

$$\begin{aligned} \|x[kT]\| &\leq \gamma_0 e^{\lambda_0(k-\underline{k})T}\|x[\underline{k}T]\| + \sum_{j=0}^{\infty} \gamma_0 e^{\lambda_0 jT}\gamma_\theta(T')\|w\|_\infty \\ &\leq \gamma_0 e^{\lambda_0(k-\underline{k})T}\|x[\underline{k}T]\| + \frac{\gamma_0\gamma_\theta(T')}{1-e^{\lambda_0 T}}\|w\|_\infty. \end{aligned} \tag{3.38}$$

Next, we solve (3.29):

$$\begin{aligned} x(t) &= e^{\hat{A}_i(t-kT)}x[kT] + \int_{kT}^{t} e^{\hat{A}_i(t-\tau)}\left[B_i\hat{H}_i(t)x[kT] + \phi_i(\underline{w}_k, x[kT])\right]d\tau, \\ &\qquad t \in [kT, (k+1)T], \quad k = \underline{k}, .., \bar{k}-1, \end{aligned}$$

so

$$\begin{aligned} \|x(t)\| &\leq e^{aT}\|x[kT]\| + \frac{e^{aT}}{a}\left[b\big(f\gamma_0 + \varepsilon_H(T')\big)\|x[kT]\| + \gamma_\phi(T')\|w\|_\infty\right] \\ &= e^{aT}\left(1 + \frac{b\big(f\gamma_0 + \varepsilon_H(T')\big)}{a}\right)\|x[kT]\| + \frac{e^{aT}}{a}\gamma_\phi(T')\|w\|_\infty, \\ &\qquad t \in [kT, (k+1)T], \quad k = \underline{k}, .., \bar{k}-1. \end{aligned}$$

Combined with (3.38), this yields

$$\begin{aligned}
\|x(t)\| &\leq e^{aT}\left(1+\frac{b\big(f\gamma_0+\varepsilon_H(T')\big)}{a}\right)\left(\gamma_0 e^{\lambda_0(k-\underline{k})T}\|x[\underline{k}T]\|+\frac{\gamma_0\gamma_\theta(T')}{1-e^{\lambda_0 T}}\|w\|_\infty\right)\\
&\quad +\frac{e^{aT}}{a}\gamma_\phi(T')\|w\|_\infty\\
&= e^{\lambda_0(k-\underline{k})T}\underbrace{\gamma_0 e^{aT}\left(1+\frac{b\big(f\gamma_0+\varepsilon_H(T')\big)}{a}\right)}_{=:\alpha_3(T')}\|x[\underline{k}T]\|+\\
&\quad \underbrace{e^{aT}\left[\left(1+\frac{b\big(f\gamma_0+\varepsilon_H(T')\big)}{a}\right)\frac{\gamma_0\gamma_\theta(T')}{1-e^{\lambda_0 T}}+\frac{\gamma_\phi(T')}{a}\right]}_{=:\alpha_4(T')}\|w\|_\infty,\\
&\qquad\qquad t\in[kT,(k+1)T],\quad k=\underline{k},..,\bar{k}-1.
\end{aligned}$$

We now turn to Claim 1, which says that, in particular,

$$\|x[\underline{k}T]\|\leq\gamma_1(T')\|x[k_l T]\|+\bar{\gamma}_1(T')\|w\|_\infty,$$

so we have

$$\begin{aligned}
\|x(t)\| &\leq e^{\lambda_0(k-\underline{k})T}\alpha_3(T')\|x[\underline{k}T]\|+\alpha_4(T')\|w\|_\infty\\
&\leq e^{\lambda_0(k-\underline{k})T}\alpha_3(T')\left(\gamma_1(T')\|x[k_l T]\|+\bar{\gamma}_1(T')\|w\|_\infty\right)+\alpha_4(T')\|w\|_\infty,\\
&\qquad\qquad t\in[kT,(k+1)T],\quad k=\underline{k},..,\bar{k}-1;
\end{aligned}$$

it follows that

$$\begin{aligned}
\|x(t)\| &\leq e^{\lambda_0(t-\underline{k})T}\underbrace{e^{-\lambda_0 T}\alpha_3(T')\gamma_1(T')}_{=:\gamma_2(T')}\|x[k_l T]\|+\\
&\quad \underbrace{[\alpha_3(T')\bar{\gamma}_1(T')+\alpha_4(T')]}_{=:\bar{\gamma}_2(T')}\|w\|_\infty,\qquad t\in[\underline{k}T,\bar{k}T].
\end{aligned}$$

■

We now assemble our results to find a bound over the entire interval $[0,\infty)$. To do so, notice that, in particular, Claim 2 says that

$$\|x[k_{l+1}T]\|\leq e^{\lambda_0(k_{l+1}-(k_l+1))T}\gamma_2(T')\|x[k_l T]\|+\bar{\gamma}_2(T')\|w\|_\infty,\quad l\in\mathbf{N}$$

so, since

$$t_{l+1}-t_l\geq T_s,\qquad l\in\mathbf{N},$$

we have that

$$\|x[k_{l+1}T]\leq e^{\lambda_0(T_s-2T)}\gamma_2(T')\|x[k_l T]\|+\bar{\gamma}_2(T')\|w\|_\infty,\quad l\in\mathbf{N}.$$

If

$$e^{\lambda_0(T_s-2T)}\gamma_2(T') < 1 \tag{3.39}$$

then it follows immediately that

$$\|x[k_lT]\| \le \underbrace{\left(e^{\lambda_0(T_s-2T)}\gamma_2(T')\right)^{(l-1)}}_{\le 1}\|x[k_1T]\| + \frac{\gamma_2(T')}{1-e^{\lambda_0(T_s-2T)}}\|w\|_\infty, \quad l \in \mathbf{N},$$

which combines with (3.33) to yield

$$\begin{aligned}\|x[k_lT]\| \;\le\; & \left(e^{\lambda_0(T_s-2T)}\gamma_2(T')\right)^{(l-1)}\alpha_0(T')\gamma_0\|x_0\| + \\ & \underbrace{\left[\bar{\gamma}(T') + \frac{\gamma_2(T')}{1-e^{\lambda_0(T_s-2T)}}\bar{\gamma}_2(T')\right]}_{=:\bar{\gamma}_3(T')}\|w\|_\infty, \quad l \in \mathbf{N}.\end{aligned} \tag{3.40}$$

It will turn out that our hypothesis ensures that (3.39) holds (for sufficiently small T'); to maintain the flow of the proof, we defer showing this until the end. In the meantime, we assume that (3.39) holds (and restrict T' accordingly) and proceed. From Claims 1 and 2 we have that

$$\begin{aligned}\|x(t)\| \;\le\; & \max\left\{\gamma_1(T'),\gamma_2(T')\right\}\|x[k_lT]\| + \max\left\{\bar{\gamma}_1(T'),\bar{\gamma}_2(T')\right\}\|w\|_\infty, \\ & \qquad t \in [k_lT, k_{l+1}T], \quad l \in \mathbf{N};\end{aligned}$$

if we combine this with (3.40), then it follows immediately that

$$\begin{aligned}\|x(t)\| \;\le\; & \max\left\{\gamma_1(T'),\gamma_2(T')\right\}\left(e^{\lambda_0(T_s-2T)}\gamma_2(T')\right)^{(l-1)}\alpha_0(T')\gamma_0\|x_0\| + \\ & \underbrace{\left(\max\left\{\gamma_1(T'),\gamma_2(T')\right\}\times\bar{\gamma}_3(T') + \max\left\{\bar{\gamma}_1(T'),\bar{\gamma}_2(T')\right\}\right)}_{=:\bar{\gamma}_4(T')}\|w\|_\infty, \\ & \qquad t \in [k_lT, k_{l+1}T], \quad l \in \mathbf{N},\end{aligned} \tag{3.41}$$

which we will use in conjunction with (3.33) to prove our stability results.

(Asymptotic Stability)

If we set $w = 0$ and let $x_0 \in \mathbf{R}^n$ remain arbitrary, then by (3.33) we have

$$\|x(t)\| \le \alpha_0(T')\gamma_0\|x_0\|, \quad t \in [0, k_1T],$$

and by (3.41) we have

$$\begin{aligned}\|x(t)\| \;\le\; & \max\left\{\gamma_1(T'),\gamma_2(T')\right\}\left(e^{\lambda_0(T_s-2T)}\gamma_2(T')\right)^{(l-1)}\alpha_0(T')\gamma_0\|x_0\|, \\ & \qquad t \in [k_lT, k_{l+1}T], \quad l \in \mathbf{N}.\end{aligned}$$

Clearly, for each admissible T', using (3.39) yields the desired bound

$$\|x(t)\| \le \max\left\{\gamma_1(T'),\gamma_2(T')\right\}\alpha_0(T')\gamma_0\|x_0\|, \quad t \ge 0, \quad l \in \mathbf{N};$$

furthermore, we also have

$$\lim_{t\to\infty}\|x(t)\| \le \lim_{l\to\infty}\left(\max\left\{\gamma_1(T'),\gamma_2(T')\right\}\left(e^{\lambda_0(T_s-2T)}\gamma_2(T')\right)^{(l-1)}\alpha_0(T')\gamma_0\right),$$

so, again using (3.39), it follows that

$$\lim_{t\to\infty}\|x(t)\| = 0.$$

As in the proof of Theorem 3.1, since S is periodic and admissible and (with the noise turned off) $e = C_i x$, these clearly yield both the desired bounds on v_1 and v_2 and the limit

$$\lim_{k\to\infty}\|v_1[k]\| = 0, \text{ and } \lim_{k\to\infty}\|v_2[k]\| = 0.$$

■

(I/O Stability)

Set $x_0 = 0$ and let $w \in \mathcal{PC}_\infty$ be arbitrary. Observe that, for each T, T' pair, $\|\hat{H}\|_\infty$ is well defined, so, as in the proof of Theorem 3.1, from the structure of $\mathcal{C}(T,T')$, the definitions of e and y, and since S is admissible, it is enough to find a bound on $\|x\|_\infty$ in terms of $\|w\|_\infty$. From (3.41), we have

$$\|x(t)\| \le \bar{\gamma}_4(T')\|w\|_\infty, \quad t \ge k_1 T,$$

so using (3.33) to provide a bound on $\|x(t)\|$ over the interval $[0, k_1T]$ yields

$$\|x\|_\infty \le \max\left\{\bar{\gamma}_0(T'),\bar{\gamma}_4(T')\right\}\|w\|_\infty.$$

■

It remains to show that our hypothesis ensures that (3.39) holds for small T'. We begin by using the explicit formula for $\gamma_2(T')$ derived in this proof:

$$\gamma_2(T') = e^{-\lambda_0 T}\alpha_3(T')\gamma_1(T'),$$

with

$$\alpha_3(T') = \gamma_0 e^{aT}\left(1 + \frac{b(f\gamma_0 + \varepsilon_H(T'))}{a}\right),$$

$$\gamma_1(T') = e^{aT}\left(\alpha_1(T') + \frac{b(f\gamma_0 + \varepsilon_H(T'))\rho(T')}{a}\right),$$

and

$$\alpha_1(T') = e^{aT}\left(1 + \frac{b(f\gamma_0 + \varepsilon_H(T'))\rho(T')}{a}\right);$$

back-substituting we find

$$\begin{aligned}
\gamma_2(T') &= e^{-\lambda_0 T}\gamma_0 e^{2aT}\left(1 + \frac{b(f\gamma_0 + \varepsilon_H(T'))}{a}\right)\times \\
&\quad \left[e^{aT}\left(1 + \frac{b(f\gamma_0 + \varepsilon_H(T'))\rho(T')}{a}\right) + \frac{b(f\gamma_0 + \varepsilon_H(T'))\rho(T')}{a}\right] \\
&\le e^{-\lambda_0 T}\gamma_0 e^{3aT}\left(1 + \frac{b(f\gamma_0 + \varepsilon_H(T'))}{a}\right)\left(1 + \frac{2b(f\gamma_0 + \varepsilon_H(T'))\rho(T')}{a}\right).
\end{aligned}$$

But

$$\lim_{T'\to 0} \rho(T') = 1$$

and, from Lemma 3.1,

$$\lim_{T'\to 0} \varepsilon_H(T') = 0$$

so

$$\begin{aligned}
\lim_{T'\to 0} e^{\lambda_0(T_s-2T)}\gamma_2(T') &\leq e^{\lambda_0(T_s-2T)}e^{-\lambda_0 T}\gamma_0 e^{3aT}\gamma_0\left(1+\frac{bf\gamma_0}{a}\right)\left(1+\frac{2bf\gamma_0}{a}\right) \\
&\leq e^{\lambda_0 T_s}e^{3(a-\lambda_0)T}\gamma_0\left(1+\frac{2bf\gamma_0}{a}\right)^2.
\end{aligned}$$

To simplify this expression, we will use the hypothesis that $T_s > \underline{T_s}$ and $T < \bar{T}(T_s)$; it follows directly from this and the definition of $\bar{T}(T_s)$ that

$$\begin{aligned}
T &< \tfrac{|\lambda_0|}{3(a-\lambda_0)}(T_s - \underline{T_s}) \\
\Rightarrow\quad 3(a-\lambda_0)T &< |\lambda_0|(T_s - \underline{T_s}) \\
\Rightarrow\quad e^{3(a-\lambda_0)T} &< e^{|\lambda_0|(T_s-\underline{T_s})} \\
&= e^{\lambda_0(\underline{T_s}-T_s)},
\end{aligned}$$

so

$$\lim_{T'\to 0} e^{\lambda_0(T_s-2T)}\gamma_2(T') < e^{\lambda_0\underline{T_s}}\gamma_0\left(1+\frac{2bf\gamma_0}{a}\right)^2.$$

If we now apply the definition of $\underline{T_s}$, then we find that

$$\lim_{T'\to 0} e^{\lambda_0(T_s-2T)}\gamma_2(T') < 1,$$

and therefore, for sufficiently small T' we have that

$$e^{\lambda_0(T_s-2T)}\gamma_2(T') < 1.$$

■

Remark 3.6 *We indicated earlier that $\underline{T_s}$ is a loose bound; indeed, it is easy to argue that a more reasonable (and much tighter) requirement is*

$$T_s > \frac{\ln(\gamma_0)}{|\lambda_0|}.$$

For such a T_s, we claim (without proof) that the controller $\mathcal{C}(T,T')$ will stabilize the set $\mathcal{P}_{T_s}$[8]; unfortunately, the only way that we have found to prove stability for such a bound involves forcing T to be small, which is undesirable. On the other hand, using the bound $\underline{T_s}$ allows for the derivation of the bound $\bar{T}(T_s)$ which has the nice property that $\bar{T}(T_s) \to \infty$ as $T_s \to \infty$, which is consistent with Theorem 3.1.

[8]Indeed, this is the bound that we will use in the (more general) context of a compact set of plants (Chapter 5).

3.4 Performance

Here we investigate the performance of the proposed controller; as such, we set the noise to zero throughout the remainder of this section. We will consider two cases: performance with no plant switches, and performance in the face of plant switches.

We begin with the first case: the plant is unknown, but lies in the finite set of LTI plants $\mathcal{P}$; to that end, we fix the plant index, say at i. For the plant P_i, the corresponding LQR-optimal signals are

$$x^0(t) := e^{\bar{A}_i t}x_0, \quad t \geq 0, \tag{3.42}$$

$$\nu^0(t) := F_i e^{\bar{A}_i t}x_0, \quad t \geq 0, \tag{3.43}$$

and

$$e^0(t) := C_i x^0(t), \quad t \geq 0,$$

so the optimal LQR cost is

$$J_i^0(x_0) := \int_0^\infty \underbrace{(x^0(t))'\bar{Q}_i x^0(t) + (\nu^0(t) + Ke^0(t))'R_i(\nu^0(t) + Ke^0(t))}_{=:M_i(x^0(t),\nu^0(t),e^0(t))} dt$$

and the actual cost is

$$J_i(x_0) := \int_0^\infty M_i(x(t), \nu(t), e(t))dt.$$

Observe that, since there is no noise and there are no plant switches, both samplers will always produce the same result, so the closed loop system is linear time varying (LTV). We will show that, by making T' small, the difference between J_i and J_i^0 can be made as small as desired for every plant P_i.

Theorem 3.3 *For every $\varepsilon > 0$ and $T > 0$, if T' is sufficiently small, then the controller $\mathcal{C}(T,T')$ stabilizes $\mathcal{P}$ and, when $\mathcal{C}(T,T')$ is attached to $P_i \in \mathcal{P}$, we have that*

$$\begin{aligned} |J_i(x_0) - J_i^0(x_0)| &\leq \int_0^\infty \left\| M_i(x(t),\nu(t),e(t)) - M_i(x^0(t),\nu^0(t),e^0(t)) \right\| dt \\ &\leq \varepsilon\|x_0\|^2, \quad x_0 \in \mathbf{R}^{n+r}. \end{aligned}$$

Proof:

Fix $\varepsilon > 0$, $T > 0$, and $w = 0$. Let $x_0 \in \mathbf{R}^n$ and $i = 1, .., q$ be arbitrary. Stability follows directly from Theorem 3.1. With ε_H given by (3.23), from Lemma 3.1 we have that

$$\|\tilde{H}_i(t)\| \leq \varepsilon_H(T,T'), \quad t \in [2T', T), \quad i = 1, .., q.$$

Since T is fixed, we drop it and write $\varepsilon_H(T')$; furthermore, Lemma 3.1 also says that

$$\lim_{T'\to 0} \varepsilon_H(T', T) = 0,$$

so there exists a constant $\bar{T}_1' \in (0, T/2)$ so that

$$\varepsilon_H(T') < 1, \quad T' \in (0, \bar{T}_1'),$$

so henceforth we let $T' \in (0, \bar{T}_1')$ be arbitrary. From Corollary 3.1 we have that

$$\dot{x}(t) = \hat{A}_i x(t) + B_i \hat{H}_i(t) x[kT], \quad t \in [kT, (k+1)T) \tag{3.44}$$

and

$$x[(k+1)T] = e^{\bar{A}_i T} x[kT]. \tag{3.45}$$

We define

$$\tilde{x} := x - x^0$$

and

$$\tilde{\nu} := \nu - \nu^0,$$

so from (3.45) we have that

$$\tilde{x}[kT] = 0, \quad k \in \mathbf{Z}^+. \tag{3.46}$$

We will investigate the cost function over one period T, and then extend the result to the entire time range. It follows immediately from the definitions of J_i and J_i^0 that

$$\begin{aligned}
&\left| J_i(x_0) - J_i^0(x_0) \right| \\
&\qquad \le \int_0^\infty \left\| M_i(x(t), \nu(t), e(t)) - M_i(x^0(t), \nu^0(t), e^0(t)) \right\| dt \\
&\qquad = \sum_{k=0}^\infty \left(\int_{kT}^{(k+1)T} \left\| M_i(x(t), \nu(t), e(t)) - M_i(x^0(t), \nu^0(t), e^0(t)) \right\| dt \right). \quad (3.47)
\end{aligned}$$

We now find a relationship between the actual and optimal cost functions over a single period T. With

$$\gamma_{lqr} := 2 \max_{i=1,..,q} \left\{ \max\{\|\bar{Q}_i + C_i' K' R_i K C_i\|, \|R_i\|, \|R_i K C_i\|\} \right\} \tag{3.48}$$

it is straight-forward to check that

$$\begin{aligned}
&\int_{kT}^{(k+1)T} \left\| M_i(x(t), \nu(t), e(t)) - M_i(x^0(t), \nu^0(t), e^0(t)) \right\| dt \le \\
&\quad \gamma_{lqr} \int_{kT}^{(k+1)T} \Big[\|x^0(t)\| \|\tilde{x}(t)\| + \|\tilde{x}(t)\|^2 + \|\nu^0(t)\| \|\tilde{\nu}(t)\| + \|\tilde{\nu}(t)\|^2 + \\
&\qquad \|\tilde{\nu}(t)\| \|\tilde{x}(t)\| + \|\nu^0(t)\| \|\tilde{x}(t)\| + \|\tilde{\nu}(t)\| \|x^0(t)\| \Big] dt, \\
&\qquad\qquad k \in \mathbf{Z}^+. \quad (3.49)
\end{aligned}$$

We now use the definitions of x^0 and ν^0 given in (3.42) and (3.43) respectively, then apply (3.9) and simplify, to find

$$\begin{aligned}
&\int_{kT}^{(k+1)T} \left\|M_i(x(t),\nu(t),e(t)) - M_i(x^0(t),\nu^0(t),e^0(t))\right\| dt \\
&\qquad\qquad \leq \ \gamma_{lqr} \int_{kT}^{(k+1)T} \Big[\|e^{\bar{A}_i(t-kT)}\| \times \|x[kT]\| \big(\|\tilde{x}(t)\| + \|\tilde{\nu}(t)\|\big) + \\
&\qquad\qquad\qquad \|F_i e^{\bar{A}_i(t-kT)}\| \times \|x[kT]\| \big(\|\tilde{x}(t)\| + \|\tilde{\nu}(t)\|\big) + \\
&\qquad\qquad\qquad \|\tilde{\nu}(t)\|^2 + \|\tilde{x}(t)\|^2 + \|\tilde{\nu}(t)\|\|\tilde{x}(t)\|\Big] dt \\
&\qquad\qquad \leq \ \gamma_{lqr} \int_{kT}^{(k+1)T} \Big[\gamma_0(1+f)\|x[kT]\| \big(\|\tilde{x}(t)\| + \|\tilde{\nu}(t)\|\big) + \\
&\qquad\qquad\qquad \|\tilde{\nu}(t)\|^2 + \|\tilde{x}(t)\|^2 + \|\tilde{\nu}(t)\|\|\tilde{x}(t)\|\Big] dt, \\
&\qquad\qquad\qquad\qquad k \in \mathbf{Z}^+. \quad (3.50)
\end{aligned}$$

The upshot of this is that, if we can bound $\int_{kT}^{(k+1)T} \|\tilde{\nu}(t)\| dt$, $\int_{kT}^{(k+1)T} \|\tilde{\nu}(t)\|^2 dt$, and $\|\tilde{x}(t)\|$ by a suitably scaled version of $\|x[kT]\|$, then we can leverage (3.45) to obtain the desired result. We begin with $\int_{kT}^{(k+1)T} \|\tilde{\nu}(t)\| dt$ and $\int_{kT}^{(k+1)T} \|\tilde{\nu}(t)\|^2 dt$:

Claim 1: There exists a constant $\gamma_1 > 0$ satisfying

$$\int_{kT}^{(k+1)T} \|\tilde{\nu}(t)\| dt \leq \gamma_1(T' + \varepsilon_H(T'))\|x[kT]\|, \quad k \in \mathbf{Z}^+,$$

and

$$\int_{kT}^{(k+1)T} \|\tilde{\nu}(t)\|^2 dt \leq \gamma_1(T' + \varepsilon_H(T'))\|x[kT]\|^2, \quad k \in \mathbf{Z}^+.$$

Proof:

By definition and (3.46) we have that

$$\begin{aligned}
\int_{kT}^{(k+1)T} \|\tilde{\nu}(t)\| dt &= \int_{kT}^{(k+1)T} \|\nu(t) - \nu^0(t)\| dt \\
&= \int_0^T \|(\hat{H}_i(t) - H_i(t))x[kT]\| dt \\
&= \int_0^{2T'} \|-H_i(t)x[kT]\| dt + \int_{2T'}^T \|\tilde{H}_i(t)x[kT]\| dt
\end{aligned}$$

so

$$\begin{aligned}
\int_{kT}^{(k+1)T} \|\tilde{\nu}(t)\| dt &\leq \left[\int_0^{2T'} f\gamma_0 \, dt + \int_{2T'}^T \varepsilon_H(T') dt\right] \|x[kT]\| \\
&\leq [2f\gamma_0 T' + \varepsilon_H(T')T] \, \|x[kT]\|, \quad k \in \mathbf{Z}^+.
\end{aligned}$$

Similarly,

$$\begin{aligned}\int_{kT}^{(k+1)T} \|\tilde{\nu}(t)\|^2 dt &\leq \left[\int_0^{2T'} (f\gamma_0)^2 dt + \int_{2T'}^{T} \underbrace{\varepsilon_H(T')^2}_{<\varepsilon_H(T')} dt\right] \|x[kT]\|^2 \\ &\leq \left[2(f\gamma_0)^2 T' + \varepsilon_H(T')T\right] \|x[kT]\|^2.\end{aligned}$$

Set $\gamma_1 := \max\{2f\gamma_0, 2(f\gamma_0)^2, T\}$ to obtain the desired result. ∎

Now we turn to $\|\tilde{x}(t)\|$:

Claim 2: There exists a constant $\gamma_2 > 0$ satisfying

$$\|\tilde{x}(t)\| \leq \gamma_2(T' + \varepsilon_H(T'))\|x[kT]\|, \quad t \in [kT, (k+1)T), \quad k \in \mathbf{Z}^+.$$

Proof:

Using the definitions of $\bar{A}_i$, x^0, and ν^0 combined with (3.44) we obtain

$$\dot{\tilde{x}}(t) = \hat{A}_i \tilde{x}(t) + B_i \tilde{\nu}(t), \quad t \geq 0.$$

Solving this and using (3.46), we find that

$$\tilde{x}(t) = \int_{kT}^{t} e^{\hat{A}_i(t-\tau)} B_i \tilde{\nu}(t) d\tau, \quad t \in [kT, (k+1)T), \quad k \in \mathbf{Z}^+$$

so

$$\begin{aligned}\|\tilde{x}(t)\| &\leq \int_{kT}^{t} \|e^{\hat{A}_i(t-\tau)} B_i\| \|\tilde{\nu}(t)\| d\tau \\ &\leq be^{aT} \int_{kT}^{(k+1)T} \|\tilde{\nu}(t)\| d\tau,\end{aligned}$$

to which we apply Claim 1, to obtain

$$\|\tilde{x}(t)\| \leq \underbrace{be^{aT}\gamma_1}_{=:\gamma_2}(T' + \varepsilon_H(T'))\|x[kT]\|, \quad t \in [kT, (k+1)T), \quad k \in \mathbf{Z}^+.$$

∎

Now we return to (3.50), to which we apply Claims 1 and 2 to yield

$$\begin{aligned}&\int_{kT}^{(k+1)T} \left\|M_i(x(t), \nu(t), e(t)) - M_i(x^0(t), \nu^0(t), e^0(t))\right\| dt \\ \leq\ & \gamma_{lqr}\Big\{\gamma_0(1+f)\|x[kT]\| \left[T\gamma_2(T' + \varepsilon_H(T'))\|x[kT]\| + \gamma_1(T' + \varepsilon_H(T'))\|x[kT]\|\right] \\ &\quad +\gamma_1(T' + \varepsilon_H(T'))\|x[kT]\|^2 + T\gamma_2^2(T' + \varepsilon_H(T'))^2\|x[kT]\|^2 + \\ &\quad \gamma_1(T' + \varepsilon_H(T'))\|x[kT]\|\gamma_2(T' + \varepsilon_H(T'))\|x[kT]\|\Big\} \\ =\ & \gamma_{lqr}\Big\{\gamma_0(1+f)\left[T\gamma_2 + \gamma_1\right] + \gamma_1 + T\gamma_2^2(T' + \varepsilon_H(T')) + \gamma_1\gamma_2(T' + \varepsilon_H(T'))\Big\} \times \\ & (T' + \varepsilon_H(T'))\|x[kT]\|^2, \quad k \in \mathbf{Z}^+.\end{aligned}$$

Observe that

$$(T' + \varepsilon_H(T')) \leq \frac{T}{2} + 1,$$

so this reduces to

$$\begin{aligned}
&\int_{kT}^{(k+1)T} \left\| M_i(x(t), \nu(t), e(t)) - M_i(x^0(t), \nu^0(t), e^0(t)) \right\| dt \\
&\quad \leq \underbrace{\gamma_{lqr}\Big\{ \big[\gamma_0(1+f) + \gamma_2(T/2+1)\big]\big[T\gamma_2 + \gamma_1\big] + \gamma_1 \Big\}}_{=:\gamma_3} (T' + \varepsilon_H(T'))\|x[kT]\|^2, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k \in \mathbf{Z}^+.
\end{aligned}$$

We can now combine this with the solution to (3.45) to yield

$$\begin{aligned}
&\sum_{k=0}^{\infty} \int_{kT}^{(k+1)T} \left\| M_i(x(t), \nu(t), e(t)) - M_i(x^0(t), \nu^0(t), e^0(t)) \right\| dt \leq \\
&\qquad\qquad\qquad \gamma_3(T' + \varepsilon_H(T')) \sum_{k=0}^{\infty} \|e^{\bar{A}_i kT} x_0\|^2
\end{aligned}$$

and then use (3.9) to obtain

$$\begin{aligned}
&\sum_{k=0}^{\infty} \int_{kT}^{(k+1)T} \left\| M_i(x(t), \nu(t), e(t)) - M_i(x^0(t), \nu^0(t), e^0(t)) \right\| dt \\
&\qquad \leq \gamma_3(T' + \varepsilon_H(T')) \sum_{k=0}^{\infty} \gamma_0^2 e^{2\lambda_0 kT} \|x_0\|^2 \\
&\qquad = (T' + \varepsilon_H(T')) \underbrace{\gamma_3 \gamma_0^2 (1 - e^{2\lambda_0 T})^{-1}}_{=:\gamma_4} \|x_0\|^2.
\end{aligned}$$

Finally, using Lemma 3.1, we have

$$\lim_{T' \to 0} (T' + \varepsilon_H(T'))\gamma_4 = 0,$$

so, for every sufficiently small T', with (3.47), we obtain

$$\begin{aligned}
\left| J_i(x_0) - J_i^0(x_0) \right| &\leq \int_0^{\infty} \left\| M_i(x(t), \nu(t), e(t)) - M_i(x^0(t), \nu^0(t), e^0(t)) \right\| dt \\
&\leq \varepsilon \|x_0\|^2.
\end{aligned}$$

■

We now consider the performance when the plant is allowed to switch. Unfortunately, even though we set the noise to zero as in the previous section, if a switch occurs during the Estimation Phase, then the output of the samplers will typically be different, so the nonlinearity will directly affect the analysis.

To proceed, we must first precisely define the performance goal. If $\sigma \in \Sigma_{T_s}$ was known in advance, then we could simply solve for the optimal control signal, corresponding state trajectories, and corresponding optimal cost, and use these quantities as our goal. However, this is unrealistic since σ is not known in advance and we are evaluating a causal control law. Instead we select the nominal (possibly not optimal) controller in the following way: for a given plant $P_\sigma \in \mathcal{P}_{T_s}$, we define the nominal control signal on the interval $[t_l, t_{l+1})$ to be the one that corresponds to the LTI optimal controller for the plant P_{i_l}. More precisely, if the plant is $P_\sigma \in \mathcal{P}_{T_s}$, then the nominal control signal ν^0, nominal state x^0, and nominal error e^0 are given by

$$\nu^0(t) := F_{i_l} e^{\bar{A}_{i_l}(t-t_l)} x(t_l), \quad t \in [t_l, t_{l+1}), \quad l \in \mathbf{N}, \tag{3.51}$$

$$x^0(t) := e^{\bar{A}_{i_l}(t-t_l)} x(t_l), \quad t \in [t_l, t_{l+1}) \quad l \in \mathbf{N}, \tag{3.52}$$

and

$$e^0(t) := C_{\sigma(t)} x^0(t), \quad t \geq 0.$$

with a corresponding nominal cost of

$$J^0_{[t_l,t_{l+1})}(x(t_l)) := \int_{t_l}^{t_{l+1}} \left[(x^0(t))' \bar{Q}_{i_l} x^0(t) + (\nu^0(t) + Ke^0(t))' R_{i_l} (\nu^0(t) + Ke^0(t)) \right] dt, \quad l \in \mathbf{N}.$$

Note that this is exactly the optimal cost for the LTI plant P_{i_l} over the interval $[t_l, t_{l+1})$. Similarly, our actual cost is

$$J_{[t_l,t_{l+1})}(x(t_l)) := \int_{t_l}^{t_{l+1}} \left[x'(t) \bar{Q}_{i_l} x(t) + (\nu(t) + Ke(t))' R_{i_l} (\nu(t) + Ke(t)) \right] dt, \quad l \in \mathbf{N}.$$

If $[\underline{t}, \bar{t}) \subset [t_l, t_{l+1})$, then we define $J_{[\underline{t},\bar{t})}(\xi(t_l), y_{ref})$ and $J^0_{[\underline{t},\bar{t})}(\xi(t_l), y_{ref})$ in the natural way.

The following result shows that, by making T and T' small, for every $P_\sigma \in \mathcal{P}_{T_s}$ the performance can be made as close to the nominal one as desired.

Theorem 3.4 *For every $\varepsilon > 0$ and $T_s > \underline{T_s}$ there exists a constant $\bar{T}_1 \in (0, \bar{T}(T_s))$ such that for every $T \in (0, \bar{T}_1)$, if $\overline{T'}$ is sufficiently small, then the controller $\mathcal{C}(T, T')$ stabilizes $\mathcal{P}_{T_s}$ and, for every $\sigma \in \Sigma_{T_s}$, attaching the controller $\mathcal{C}(T, T')$ to the plant P_σ yields*

$$\left| J^0_{[t_l,t_{l+1})}(x(t_l)) - J_{[t_l,t_{l+1})}(x(t_l)) \right| \leq \varepsilon \|x(t_l)\|^2, \quad l \in \mathbf{N}, \quad x_0 \in \mathbf{R}^n.$$

Proof: Stability follows directly from Theorem 3.2. The proof of performance is complicated and is provided in Appendix A.

3.5 An Example

To illustrate the proposed design and some of the closed loop system properties we will discuss the following example problem: we allow the plant to switch between the two LTI plants

$$P_1 \ : \ \begin{cases} \dot{x}(t) &= x(t) + u(t) \\ y(t) &= x(t) \end{cases}$$

and

$$P_2 \ : \ \begin{cases} \dot{x}(t) &= x(t) + u(t) \\ y(t) &= -x(t), \end{cases}$$

which are not simultaneously stabilizable using an LTI controller. Observe that $n = 1$. We discuss this problem in two different settings: in the first, we concentrate on performance and ignore noise, while in the second we investigate the response of the system when noise is present.

Notice that the eigenvalues are not disjoint, so our controller design begins with the selection of the regularization gain

$$K = 2,$$

which results in the state-space representations

$$\hat{A}_1 = 3 \text{ and } \hat{A}_2 = -1;$$

there is no change to B_i and C_i:

$$B_1 = B_2 = 1, \quad C_1 = 1, \text{ and } C_2 = -1.$$

We select the LQR variables

$$Q_i = R_i = 1, \quad i = 1, 2;$$

using the cost function (3.8) yields the optimal gains

$$F_1 = -4.414 \text{ and } F_2 = 0.414^9.$$

For this problem, we can calculate $\underline{T_s} = 1.94s$; to illustrate that this bound is loose, we will switch plants every 0.5 seconds[10].

The hold is explicitly defined via (3.21) and (3.22), but we must still design our sampler function S. To do so we need some additional definitions. We choose an additional parameter $j \in \mathbf{Z}^+$ satisfying

$$j \geq 2n = 2$$

[9] These are the gains for the optimal control signal $\nu = F_i x$; to find the gains for $u = \bar{F}_i x$ simply set $\bar{F}_i = F_i + KC_i$.

[10] Observe that, as discussed in Remark 3.6 since these plants are first order, we have that $\gamma_0 = 1$, so we expect that the tight lower bound on T_s is actually zero.

and then set

$$h := \frac{T'}{j};$$

then we define

$$\mathcal{O}_j(C, e^{\hat{A}h}) := \begin{bmatrix} C \\ \tilde{C}e^{\hat{A}h} \\ \vdots \\ Ce^{\hat{A}(j-1)h} \end{bmatrix}.$$

Since $j \geq 2n = 2$ and $(\hat{A}, C)$ is assumed to be observable, by [7], for sufficiently small h, the matrix $\mathcal{O}_j(C, e^{\hat{A}h})$ has full column rank. If we define

$$\begin{bmatrix} \bar{S}_0 & \bar{S}_1 & \dots & \bar{S}_{j-1} \end{bmatrix} := \left(\mathcal{O}_j(C, e^{\hat{A}h})'\mathcal{O}_j(C, e^{\hat{A}h})\right)^{-1} \mathcal{O}_j(C, e^{\hat{A}h})'$$

and let $S(t)$ (on $[0, T)$) be a weighted sum of j impulses given by

$$S(t) = \sum_{k=0}^{j-1} \bar{S}_k \delta(t - kh),$$

then it follows that

$$\int_0^{T'} S(t)e(t)dt = \sum_{k=0}^{j-1} \bar{S}_k e(kh); \tag{3.53}$$

it is routine to check that the sampler (3.53) satisfies the desired properties. Observe that the generalized sampler can be implemented using a weighted sum of j samples of e, spaced h time units apart. For this reason, we refer to h as the sample time.

We will present two controllers which use different choices of T' and j, so to conserve space, we provide only one set of example calculations. Here, we select the period, Estimation Phase duration, and sampler parameter to be

$$T = 0.1s, \ 2T' = 0.02s, \ \text{and } j = 10,$$

respectively; with these choices the sample time is

$$h = 0.001s.$$

The corresponding sampler gains are

$$\bar{S}[0] = \begin{bmatrix} -13.39 \\ -13.74 \end{bmatrix}, \dots, \ \bar{S}[9] = \begin{bmatrix} 13.51 \\ 13.66 \end{bmatrix}.$$

Completing the controller design, we find that the generalized hold gains are

$$\hat{H}_1(t) = -4.414e^{-1.414t} - 1.795e^{-3(t-0.04)}$$

and

$$\hat{H}_2(t) = -0.414e^{-1.414t} - 0.086e^{(t-0.04)}.$$

Together, S and H are two orders of magnitude smaller than the gains of [26].

3.5.1 Performance

Here we turn the noise off and investigate the performance of the system. The controller period is $T = 0.1s$ and the sample time is $h = 0.001s$; we could use a longer period, but then we would need to switch more slowly (since we need $2T < T_s$) and our figures would be highly uninteresting. As stated above, the plant switches every 0.5 seconds - the switch occurs in the middle of the first sample during the Estimation Phase. We investigate two sampling durations, $T' = 0.01s$ and $T' = 0.03s$[11], shown in Figure 3.3 and Figure 3.4 respectively; the time axis indicates the locations where the plant switches. These results show that both the state and control signals are close to the nominal ones; the jumps in u and corresponding increases in the state correspond to the Estimation Phase, where $\nu = 0$[12]. In the case where the Estimation Phase is shorter (i.e. $T' = 0.01s$) the oscillations in u become smaller and both the control signal and the state get closer to the nominal ones, as expected; given the scale of our figures, it is somewhat difficult to see the improvement, so we include a zoomed figure of the state x with the two cases overlaid (Figure 3.5).

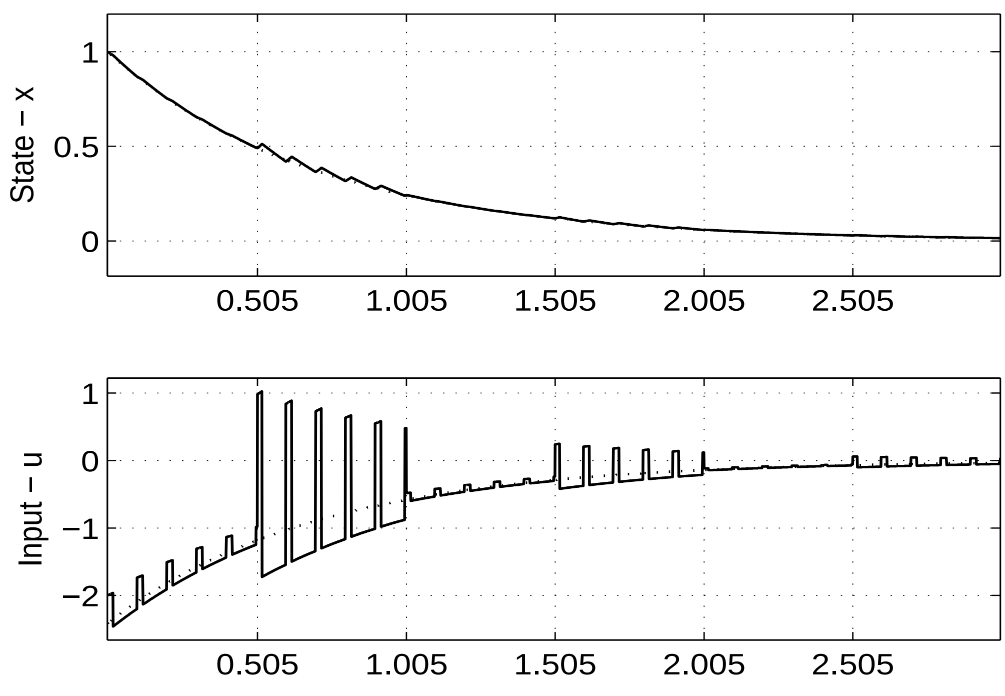

Figure 3.3: Example - solid is actual, dashed is nominal - $T' = 0.01s$.

[11] Observe that, in this second case, the Estimation Phase takes up more than half of the period.

[12] Recall that we have regularized the system, hence the input u is not zero there, instead it is Ke.

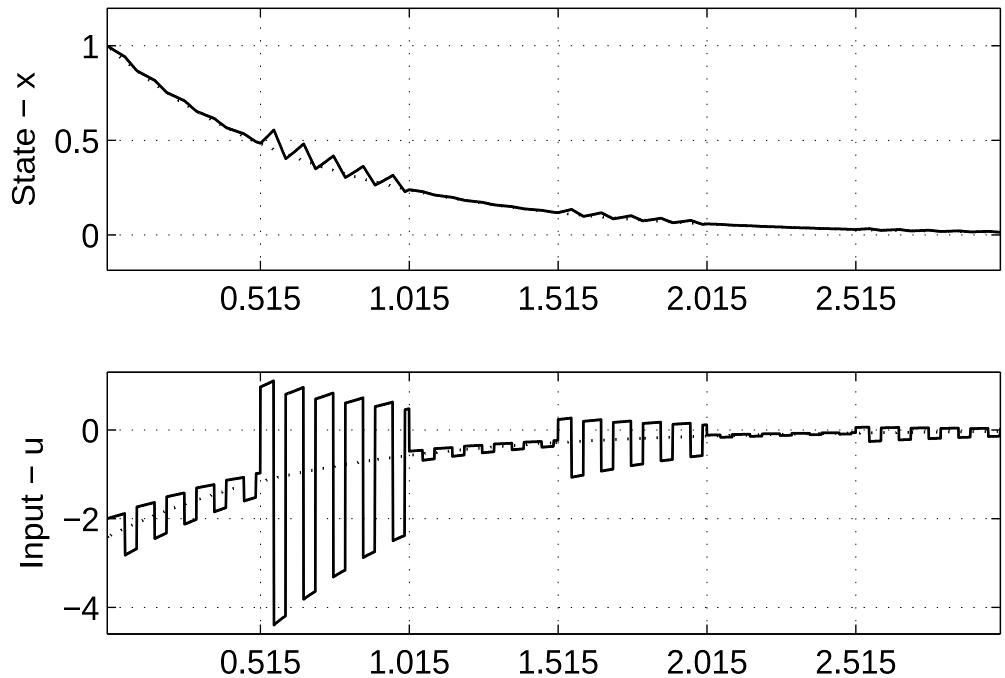

Figure 3.4: Example - solid is actual, dashed is nominal - $T' = 0.03s$.

3.5.2 Noise Rejection

Here we consider the same setup as above, but we add noise at the plant output and set the initial conditions to zero. Since smaller T's lead to larger controller gains (even thought we have not proven it) we expect our noise tolerance to improve as T' gets *larger* (as opposed to performance, which improves as T' gets *smaller*). Results are shown in Figures 3.6 and 3.7; in the state and input components of these figures, the time axis indicates locations where plant switches occur. We see that the state experiences significantly more of an adverse affect in the $T' = 0.01s$ case as compared to $T' = 0.03s$, supporting our hypothesis. Finally, in both cases we see that the system does not go unstable in the presence of noise and is able to recover the desired behaviour once noise is turned off (i.e. the state goes to zero).

Finally, for completeness, we combine these two cases: we consider $T' = 0.01s$ and set the initial condition on x back to 1 (as it was in the performance examples above). The result is Figure 3.8; again, observe that the system does not go unstable in the presence of noise and is able to recover the desired behaviour once noise is turned off (i.e. the state goes to zero).

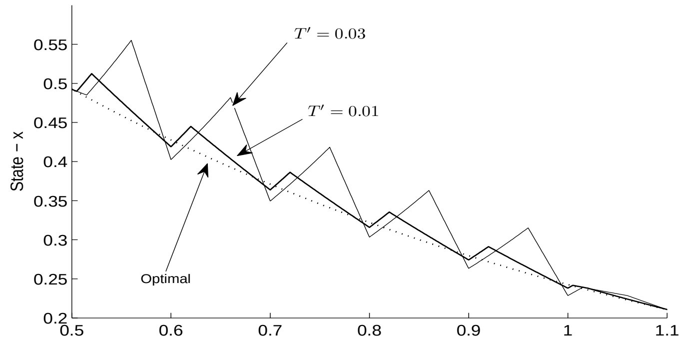

Figure 3.5: Comparing the state x - zoomed in.

3.6 Summary and Concluding Remarks

In this chapter we consider the problem of simultaneous stabilization and LQR optimal performance. The set of time-varying plants is modeled by allowing (sufficiently slow but possibly persistent) switches between a finite number of LTI models; we characterize the class of time varying plants by the quantity T_s, which is the smallest admissible time between switches, and provide an easily computable bound on how often a switch is allowed. For any sufficiently large T_s, we design a periodic, mildly nonlinear controller to achieve the objective; it consists of two parts: a generalized sampler which estimates the state at the beginning of each period and a generalized hold which applies the desired control signal.

In comparison to earlier work on a related problem [26][13], here the controller's period can be large, so we expect to have relatively smaller controller gains and relatively improved noise rejection. One drawback shared with [26] is that, the closer to optimal that we require the performance to become, the larger the controller gains will be and the poorer the noise behaviour becomes. Further work is required to analyze and continue improving upon this tradeoff, although that is beyond the scope of this thesis. It is clearly desirable to extend this result to the more demanding objective of step tracking; the next chapter will focus on this.

[13]The major difference is that [26] only allows a finite number of plant switches.

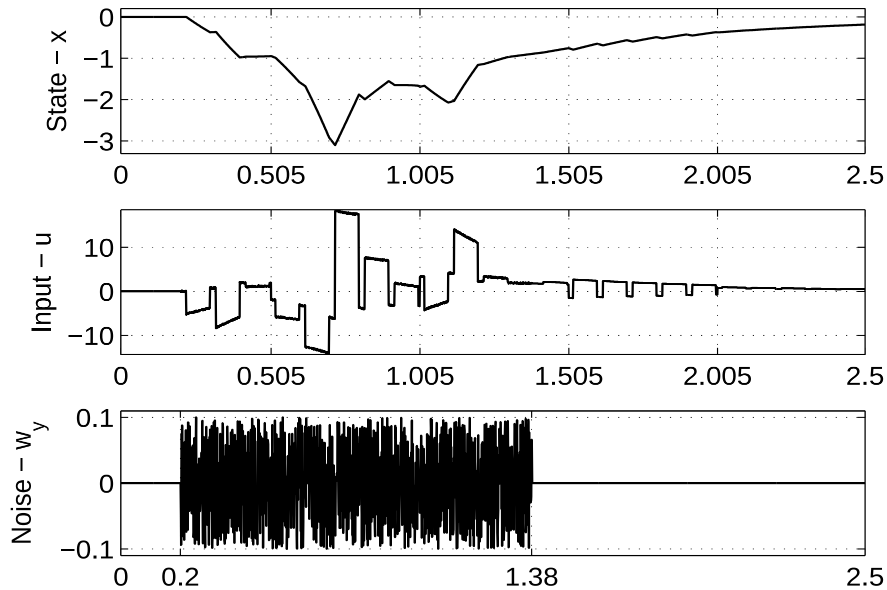

Figure 3.6: Example with noise - $T' = 0.01s$.

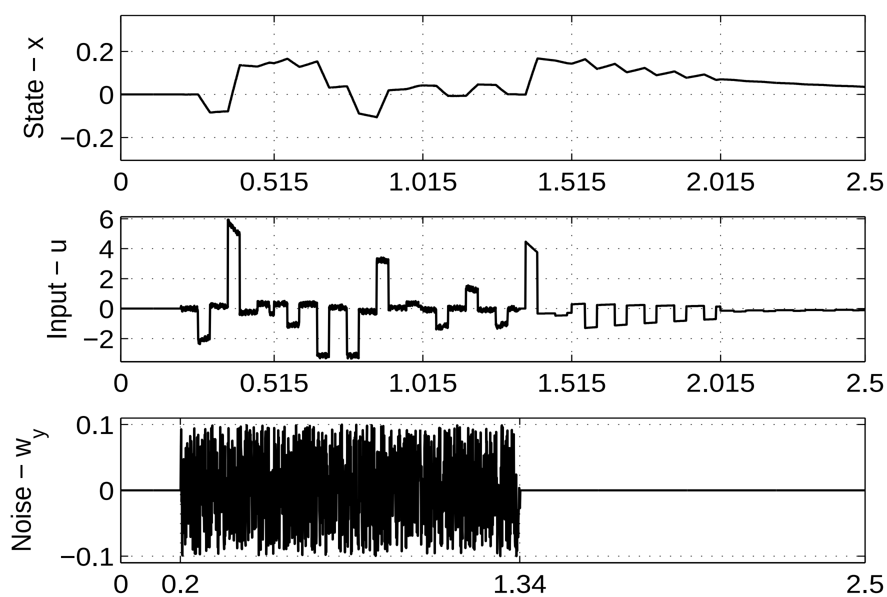

Figure 3.7: Example with noise - $T' = 0.03s$.

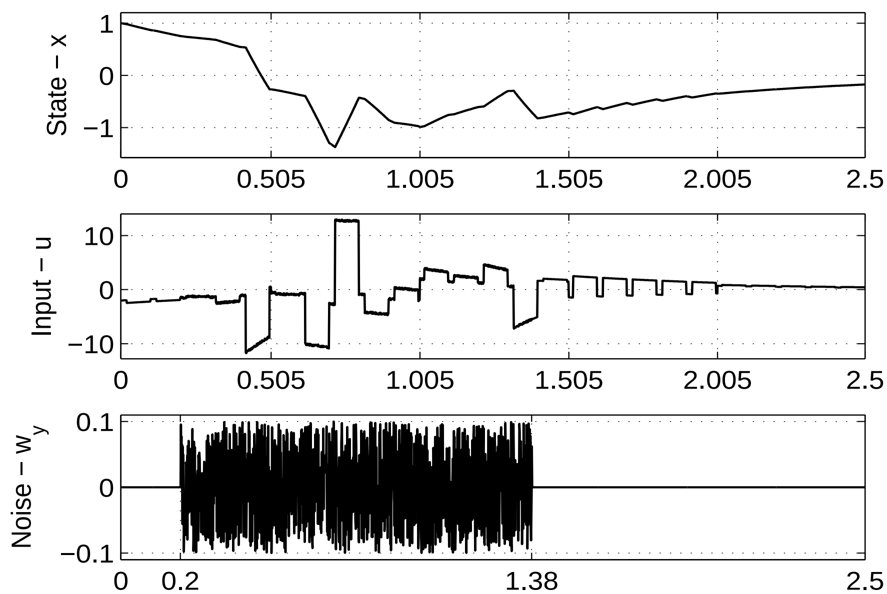

Figure 3.8: Example with noise and initial conditions - $T' = 0.01s$.

Chapter 4

The Finite Tracking Problem

In this chapter the objective is to revisit the problem of our previous chapter with the goal of providing the more demanding control objective of optimal step tracking. At first glance one would think that step tracking is no harder than stabilization; the main difficulty arises from having to achieve this in the face of plant parameter changes: the augmented state is generally *not* continuous when the plant switches. This leads to issues when incorporating noise in the plant model and involves the addition of a mildly restrictive assumption that allows us to bound the size of the discontinuity in the augmented state. This chapter's controller design and analysis will be very similar to that of the previous chapter; we point out any differences and will try to minimize repetition. This work was first presented in [37]. In that paper, we only proved I/O stability; for completeness, here we will also prove asymptotic stability.

This chapter follows the same structure as Chapter 3; a brief outline is as follows. In Section 4.1 we provide the problem definition and some mathematical preliminaries, including definitions of LQR tracking, stability, and some tools for handling noise. In Section 4.2 we present the RACE controller and analyze some noise related properties of the controller over a single period. In Section 4.3 we investigate closed loop stability, while in Section 4.4 we turn to the question of performance. In Section 4.5 we revisit the example of Section 3.5 in the context of step tracking, and we wrap up with a summary and concluding remarks in Section 4.6. As before, we use the 2-norm to measure the size of a vector. Since the proofs in this chapter are quite similar to those in the previous one (with appropriate changes to deal with the discontinuity in the augmented plant state discussed above), we relegate them all to Appendix B.

4.1 Problem Formulation

We would like to extend the controller of Chapter 3 to provide LQR step tracking in the face of (possibly persistent) switching between LTI plants. To that end, we retain the notation for the LTI plant P_i given by (3.1), the sets $\mathcal{P}$ and Σ_{T_s}, and the switching signal σ outlined in Section 3.1 as well as Assumptions 3.1 and 3.2; recall that we can use $\sigma \in \Sigma_{T_s}$ to write down the state-space representation of the time varying plant P_σ:

$$\begin{aligned} \dot{x}(t) &= A_{\sigma(t)}\, x(t) + B_{\sigma(t)}\, u(t), \quad x(0) = x_0, \\ y(t) &= C_{\sigma(t)}\, x(t). \end{aligned}$$

Before we add noise to our model, it will be useful to present a preliminary discussion regarding our performance goal. Indeed, this discussion will lead to an augmented model for which the inclusion of noise is non-trivial. To begin, we let $y_{ref} \in \mathbf{R}^r$ be the set-point to be tracked and define the tracking error by

$$e(t) := y(t) - y_{ref}.$$

There is more than one way to define optimal step tracking, we choose the following approach. Motivated by the internal model principle [8], we augment the plant with an integrator at the input, yielding a state-space representation of the augmented plant P_i with $\dot{u}$ as the input and e as the output:

$$\begin{aligned} \begin{bmatrix} \ddot{x}(t) \\ \dot{e}(t) \end{bmatrix} &= \underbrace{\begin{bmatrix} A_i & 0 \\ C_i & 0 \end{bmatrix}}_{=:\hat{A}_i} \underbrace{\begin{bmatrix} \dot{x}(t) \\ e(t) \end{bmatrix}}_{=:\eta(t)} + \underbrace{\begin{bmatrix} B_i \\ 0 \end{bmatrix}}_{=:\hat{B}_i} \dot{u}(t), \qquad (4.1) \\ & \qquad \eta_0 := \eta(0) = \begin{bmatrix} A_i x_0 + B_i u(0) \\ C_i x_0 - y_{ref} \end{bmatrix} \in \mathbf{R}^{n+r} \\ e(t) &= \underbrace{\begin{bmatrix} 0 & I \end{bmatrix}}_{=:\hat{C}_i} \eta(t). \end{aligned}$$

With this in hand, for each $P_i \in \mathcal{P}$ and with $Q_i > 0$ and $R_i > 0$, we choose the following natural cost criterion:

$$\int_0^\infty [e(t)' Q_i e(t) + \dot{u}(t)' R_i \dot{u}(t)] dt, \qquad (4.2)$$

which is in the standard form for the model (4.1). Recall that sufficient conditions for the existence of a solution to this optimal control problem are that

(i) $(Q_i^{1/2} \hat{C}_i, \hat{A}_i)$ be detectable and

(ii) $(\hat{A}_i, \hat{B}_i)$ be stabilizable.

To satisfy (i) observe that Assumption 3.1 implies that $(\hat{C}_i, \hat{A}_i)$ is observable, which, coupled with the fact that $Q_i > 0$, means that $(Q_i^{1/2} \hat{C}_i, \hat{A}_i)$ is observable as well. Item (ii) is not automatic; since the objective is to track steps asymptotically, it is natural to impose

Assumption 4.1 *For every $i = 1, .., q$, (A_i, B_i, C_i) has no transmission zeros at zero and has at least as many inputs as outputs:*

$$rank \begin{bmatrix} A_i & B_i \\ C_i & 0 \end{bmatrix} = n + r.$$

Together, Assumptions 3.2 and 4.1 guarantee that (ii) holds, so we have satisfied the two requirements.

Finally, recall that, in Chapter 3, for the controller design method to work, we required that the A matrices have disjoint eigenvalues; if they did not, then we introduced a regularizing output feedback to solve the problem. The final result of that analysis was the model (3.4) and Assumption 3.3. Observe that here we will *never* have disjoint eigenvalues since all of the augmented LTI plants share at least the integrator poles at zero, so we again assume that the transfer functions

$$C_i(sI - A_i)^{-1}B_i, \quad i = 1, .., q$$

are distinct and that (A_i, B_i) and (A_j, B_j) share uncontrollable modes only if $j = i$, and then apply some regularizing output feedback, this time of the form

$$\dot{u} = Ke + \nu,$$

so (4.1) becomes

$$\begin{aligned} \dot{\eta}(t) &= \underbrace{\begin{bmatrix} A_i & B_iK \\ C_i & 0 \end{bmatrix}}_{=:\tilde{A}_i} \eta(t) + \underbrace{\begin{bmatrix} B_i \\ 0 \end{bmatrix}}_{=:\tilde{B}_i} \nu(t), \\ e(t) &= \underbrace{\begin{bmatrix} 0 & I \end{bmatrix}}_{=:\tilde{C}} \eta(t). \end{aligned} \tag{4.3}$$

As before, it follows from [5] that, for almost all K, the matrices $\{\tilde{A}_1, ..., \tilde{A}_q\}$ will enjoy the desired property; we replace Assumption 3.3 with

Assumption 4.2 $\{\tilde{A}_1, ..., \tilde{A}_q\}$ *have disjoint eigenvalues.*

4.1.1 Noise

Observe that, if we incorporate noise into (4.3) then we will have $\dot{w}_u$ and $\dot{w}_y$ terms, which are highly undesirable since the noise signals need not be differentiable; we shall carry out a change of variables (in the general context of the time varying plant P_σ) to avoid this problem. To proceed, first recognize that, for every $\sigma \in \Sigma_{T_s}$

and $y_{ref} \in \mathbf{R}^r$, the state η satisfies

$$\eta(t) = \begin{bmatrix} \dot{x}(t) \\ e(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A_{\sigma(t)} & B_{\sigma(t)} \\ C_{\sigma(t)} & 0 \end{bmatrix}}_{=:G_{\sigma(t)}} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} + \underbrace{\begin{bmatrix} B_{\sigma(t)} & 0 \\ 0 & I \end{bmatrix}}_{=:L_{\sigma(t)}} \underbrace{\begin{bmatrix} w_u(t) \\ w_y(t) \end{bmatrix}}_{=:w(t)} - \begin{bmatrix} 0 \\ I \end{bmatrix} y_{ref}, \quad t \geq 0. \tag{4.4}$$

Now define

$$\xi(t) := \eta(t) - L_{\sigma(t)} w(t), \quad \sigma \in \Sigma_{T_s}, \quad t \geq 0. \tag{4.5}$$

We would like to write the differential equation for ξ, but we must be careful: while x and u are clearly continuous, both $\dot{x}$ and e typically jump when the plant changes, which means that both η and ξ do as well. Since the set of switching times are given by $\{t_l : l \in \mathbf{N}\}$, it is easy to verify that

$$\left.\begin{array}{rcl} \dot{\xi}(t) & = & \tilde{A}_{\sigma(t)}\xi(t) + \tilde{B}_{\sigma(t)}\nu(t) + \tilde{A}_{\sigma(t)} L_{\sigma(t)} w(t), \\ e(t) & = & \tilde{C}\xi(t) + \tilde{C} L_{\sigma(t)} w(t), \qquad t \notin \{t_l : l \in \mathbf{N}\}. \end{array}\right\} \tag{4.6}$$

At the jump points we use (4.4) and (4.5) to describe the change in the state from $\xi(t_l^-)$ to $\xi(t_l)$:

$$\left.\begin{array}{rcl} \xi(t_l) & = & \eta(t_l) - L_{\sigma(t_l)} w(t_l) \\ & = & G_{\sigma(t_l)} \begin{bmatrix} x(t_l) \\ u(t_l) \end{bmatrix} - \begin{bmatrix} 0 \\ I \end{bmatrix} y_{ref}, \quad l \in \mathbf{N}; \end{array}\right\} \tag{4.7}$$

we can also describe the change in the opposite direction:

$$\left.\begin{array}{rcl} \xi(t_l^-) & = & \eta(t_l^-) - L_{\sigma(t_l^-)} w(t_l^-) \\ & = & G_{\sigma(t_l^-)} \begin{bmatrix} x(t_l) \\ u(t_l) \end{bmatrix} - \begin{bmatrix} 0 \\ I \end{bmatrix} y_{ref}, \quad l \in \mathbf{N}. \end{array}\right\} \tag{4.8}$$

Finally, we provide the special case that defines the initial condition

$$\xi_0 := \xi(0) = G_{\sigma(0)} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} - \begin{bmatrix} 0 \\ I \end{bmatrix} y_{ref};$$

recall that we insist that the plant parameters be continuous at $t_0 = 0$, so we define

$$\xi(0^-) := \xi_0 \text{ and } \sigma(0^-) := \sigma(0).$$

Observe that, since both x and u are continuous, the jump in ξ comes directly from the change in the plant parameters, encapsulated in G_σ. We would like to bound the size of this jump, but it is not clear how to do this unless G_σ is square and invertible, so we impose

Assumption 4.3 *The system has the same number of inputs as outputs: $m = r$.*

Remark 4.1 *If $m > r$ then we can always regularize the system to make it square without losing any observability/stabilizability properties, but at the cost of losing optimality.*

From Assumption 4.3 we have that G_σ is square, coupled with Assumption 4.1, we find that G_σ is invertible. We can now state a technical result that provides a bound on the size of the jump in the state ξ; to do so, it will be useful to define

$$\bar{g} := \max_{i,j=1,..,q} \|G_i G_j^{-1}\| \geq 1.$$

Lemma 4.1 *With $T_s > 0$, $P_\sigma \in \mathcal{P}_{T_s}$, and $l \in \mathbf{N}$ we have*

$$\|\xi(t_l)\| \leq \bar{g}\|\xi(t_l^-)\| + (\bar{g}+1)\|y_{ref}\|,$$

$$\left\|\xi(t_l^-)\right\| \leq \bar{g}\|\xi(t_l)\| + (\bar{g}+1)\|y_{ref}\|.$$

4.1.2 Stability

We would like to define what we mean by closed loop stability, but first we need to specify the closed-loop system of interest. The actual plant is P_σ and the controller $\mathcal{C}$ consists of an integrator of the form

$$\dot{u} = Ke + \nu, \quad u(0) = u_0 \tag{4.9}$$

together with an as yet unspecified term given in input-output form:

$$\begin{aligned} \kappa \ &: \ \mathcal{PC}_\infty \to \mathcal{PC}_\infty \\ &: \ e \mapsto \nu. \end{aligned} \tag{4.10}$$

Together, these yield the closed loop system given in Figure 4.1 and lead naturally to

Definition 4.1 *With $T_s > 0$, $x_0 = 0$, and $u_0 = 0$, we say that the controller $\mathcal{C}$* **I/O stabilizes** *$\mathcal{P}_{T_s}$ if, for every $P_\sigma \in \mathcal{P}_{T_s}$ the map*

$$(w_u, w_y, y_{ref}) \to (u, y, \nu)$$

is well defined and has bounded gain.

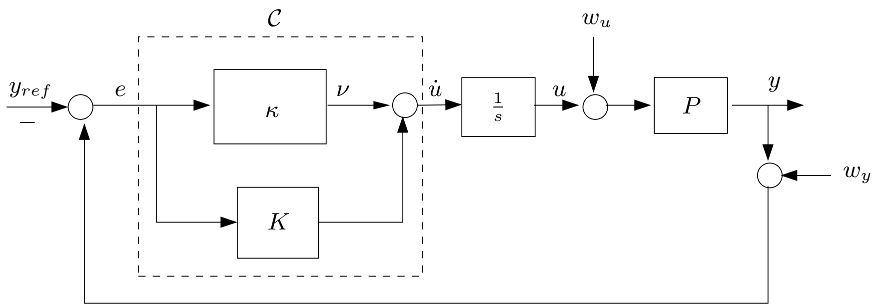

Figure 4.1: System Block Diagram

We will provide a definition of asymptotic stability once the structure of the compensator κ has been established in more detail.

Remark 4.2 *Let V denote the closed loop map*

$$V : (w_u, w_y, y_{ref}) \to (u, y, \nu).$$

To prove closed loop stability, it will be convenient to use the preliminary analysis culminating in (4.6)-(4.8); the problem is that this analysis proceeds on the assumption that y_{ref} is a constant. Fortunately, it is clear from Figure 4.1 that

$$V(w_u, w_y, y_{ref}) = V(w_u, w_y - y_{ref}, 0)$$

which means that, to prove stability, it is sufficient to prove that

$$(w_u, w_y, 0) \to (u, y, \nu)$$

has bounded gain; i.e., we can set y_{ref} to zero in the analysis, which is what we will do in the upcoming proofs of closed loop stability.

4.1.3 LQR Step Tracking Performance

We now turn the noise off (i.e. set $w = 0$) and, for each $P_i \in \mathcal{P}$, rewrite the cost function (4.2) using the new state ξ and the input ν:

$$J_i(\xi_0) = \int_0^\infty \xi(t)' \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & Q_i \end{bmatrix}}_{=:\bar{Q}_i} \xi(t) + (\nu(t) + Ke(t))' R_i (\nu(t) + Ke(t)) dt.$$

Assuming that $\mathcal{C}$ stabilizes P_i, the LQR problem is to find, for each $\xi_0 \in \mathbf{R}^{n+r}$, the control signal ν which minimizes this cost. As is well-known, the optimal controller is state-feedback of the form

$$\nu = F_i \xi,$$

which gives rise to an optimal cost of the form

$$J_i(\xi_0) = \xi_0^T V_i \xi_0$$

with V_i a positive definite solution of an associated Riccati equation. The closed loop matrix which arises from applying this state feedback is labeled

$$\bar{A}_i := \tilde{A}_i + \tilde{B}_i F_i.$$

As before, we defer defining a cost function and associated optimal controller for plants that lie in $\mathcal{P}_{T_s}$ until Section 4.4.

Our proofs often require uniform bounds on system parameters, so we define

$$a := \max_{i=1,..,q} \|\tilde{A}_i\|,$$

$$b := \max_{i=1,..,q} \|\tilde{B}_i\|,$$

$$f := \max_{i=1,..,q} \|F_i\|,$$

and

$$\ell := \max_{i=1,..,q} \|L_i\|;$$

note that $\|\tilde{C}\| = 1$. Finally, since $\bar{A}_i$ is Hurwitz by design, there exist constants $\gamma_0 > 1$ and $\lambda_0 < 0$ such that

$$\|e^{\bar{A}_i t}\| \leq \gamma_0 e^{\lambda_0 t}, \quad i = 1, .., q, \quad t \geq 0. \tag{4.11}$$

4.2 The Controller

In this section we design the compensator κ; it is periodic of period T. The design of this compensator closely mirrors that of the previous chapter, so we provide only the highlights. Recall that we would like to design the overall controller so that both u and e are LQR optimal on each period, i.e., so that if the plant is P_i on $[kT, (k+1)T)$, then

$$\nu(t) = F_i \xi(t) = \underbrace{F_i e^{\bar{A}_i (t-kT)}}_{=:H_i(t-kT)} \xi[kT], \quad t \in [kT, (k+1)T). \tag{4.12}$$

As before, we do not use H directly; instead we use a suitably adjusted version of H, which we label $\hat{H}$. Furthermore, to reflect the fact that the control signal is turned off during the Estimation Phase, we set

$$\hat{H}(t) = 0, \quad t \in [kT, kT + 2T').$$

Although the details of the gains S and H have slight modifications, there is no change to the structure of κ:

THE PROPOSED COMPENSATOR κ

With $T_s > 0$, $T \in (0, T_s/2)$, $T' \in (0, T/2)$, $S, \hat{H}$ periodic of period T, and $k \in \mathbf{Z}^+$, we define the controller by

$$v_1[k] := \int_{kT}^{kT+T'} S(t)e(t)\,dt, \tag{4.13}$$

$$v_2[k] := \int_{kT+T'}^{kT+2T'} S(t)e(t)\,dt, \tag{4.14}$$

$$\nu(t) = \begin{cases} 0 & t \in [kT, kT+2T') \\ \hat{H}(t)\text{argmin}\{\|v_1[k]\|, \|v_2[k]\|\} & t \in [kT+2T', (k+1)T). \end{cases} \tag{4.15}$$

Remark 4.3 *Since the controller is nonlinear, there is no guarantee that the system will be well posed; however, it is routine to prove that, for every choice of $y_{ref} \in \mathcal{L}_\infty$, $\sigma \in \Sigma_{T_s}$, and $w \in \mathcal{L}_\infty$, when (4.9) and κ are applied to the plant P_σ, every $x_0 \in \mathbf{R}^n$ and $u_0 \in \mathbf{R}^m$ yields a unique solution.*

As before, the nonlinearity will turn out to be very mild, we can rewrite (4.15) more compactly as

$$\nu(t) = \hat{H}(t)\text{argmin}\{\|v_1[k]\|, \|v_2[k]\|\} \quad t \in [kT, (k+1)T)$$

without worrying about causality issues, and the 'initial condition' $v_i[-1]$ is irrelevant.

This choice of κ together with (4.9) leads naturally to the following definition of asymptotic stability:

Definition 4.2 *With $T_s > 0$, $w = 0$, and $y_{ref} = 0$, we say that the controller $\mathcal{C}$* **asymptotically stabilizes** *$\mathcal{P}_{T_s}$ if, for every $P_\sigma \in \mathcal{P}_{T_s}$ we have that*

(i) for every $\varepsilon > 0$ there exists a $\delta > 0$ so that, if $\|x_0\| < \varepsilon$ and $\|u_0\| < \varepsilon$, then

$$\|x(t)\| < \delta, \ \|u(t)\| < \delta, \quad t \geq 0,$$

$$\|v_1[k]\| < \delta, \ \text{and} \ \|v_2[k]\| < \delta, \quad k \in \mathbf{Z}^+,$$

and

(ii) for every $x_0 \in \mathbf{R}^n$ and $u_0 \in \mathbf{R}^m$, we have

$$\lim_{t\to\infty} \|x(t)\| = 0, \quad \lim_{t\to\infty} \|u(t)\| = 0,$$

$$\lim_{k\to\infty} \|v_1[k]\| = 0, \ \textit{and} \ \lim_{k\to\infty} \|v_2[k]\| = 0.$$

.

Remark 4.4 *As before, observe that (ii) is a global convergence condition, rather than the typical local one. Furthermore, we will be able to prove a stronger condition than (i), namely: there exists a constant $\gamma > 0$ so that for every $x_0 \in \mathbf{R}^n$ we have*

$$\|x(t)\| \leq \gamma(\|x_0\| + \|u_0\|), \ \|u(t)\| \leq \gamma(\|x_0\| + \|u_0\|), \quad t \geq 0,$$

$$\|v_1[k]\| \leq \gamma(\|x_0\| + \|u_0\|), \ \textit{and} \ \|v_2[k]\| \leq \gamma(\|x_0\| + \|u_0\|), \quad k \in \mathbf{Z}^+.$$

We now explain how to choose the sampler gain S and the hold gain H and then investigate some system properties in the presence of noise.

4.2.1 Designing the Gains S and H

We begin with S. If the plant is P_i and **there is no noise** then we wish S to be such that

$$v_1[k] = \int_{kT}^{kT+T'} S(t)e(t)\,dt = E_i\xi[kT] \tag{4.16}$$

and

$$v_2[k] = \int_{kT+T'}^{kT+2T'} S(t)e(t)\,dt = E_i\xi[kT]; \tag{4.17}$$

observe that $v_1[k], v_2[k] \in \mathbf{R}^{(n+r)q}$. If we define the augmented matrices

$$\tilde{A} := \mathrm{diag}\{\tilde{A}_1 ... \tilde{A}_q\} \text{ and } \bar{C} := \begin{bmatrix} \tilde{C} & ... & \tilde{C} \end{bmatrix},$$

and require

$$\int_0^{T'} S(t)\bar{C}e^{\tilde{A}t}dt = I \tag{4.18}$$

and

$$S(t) = \begin{cases} e^{-\tilde{A}T'}S(t-T'), & t \in [T', 2T') \\ 0, & t \in [2T', T), \end{cases} \tag{4.19}$$

then (4.16) and (4.17) are satisfied for every admissible i. Assumptions 3.1 and 4.2 ensure that $(\bar{C}, \tilde{A})$ is observable, so there is a whole family of periodic functions which satisfy (4.18) and (4.19). As before, we restrict ourselves to only those gains S that are admissible.

We now turn to H. Observe that, with the controllability grammian defined by

$$W_i(t) := \int_0^t e^{-\tilde{A}_i\tau}\tilde{B}_i\tilde{B}_i' e^{-\tilde{A}_i'\tau} d\tau,$$

if we define

$$\tilde{H}_i(t) := \tilde{B}_i' e^{-\tilde{A}_i'(t-2T')} W_i^{-1}(T-2T') e^{-\tilde{A}_i(T-2T')} \Psi_i(T,T'), \quad t \in [2T', T) \tag{4.20}$$

then

$$\hat{H}(t) = \begin{cases} 0 & t \in [0, 2T') \\ \begin{bmatrix} H_1 + \tilde{H}_1 & H_2 + \tilde{H}_2 & \ldots & H_q + \tilde{H}_q \end{bmatrix}(t) & t \in [2T', T), \end{cases} \tag{4.21}$$

yields a hold with the desired characteristic that

$$\xi[(k+1)T] = e^{\bar{A}_i T}\xi[kT]. \tag{4.22}$$

Furthermore, if we define

$$\varepsilon_H(T,T') := 2b^2 f \gamma_0 e^{aT} T' \max_{i=1,..,q} \|W_i^{-1}(T-2T')\|, \tag{4.23}$$

then we can easily prove the following:

Lemma 4.2 *With $T_s > 0$, for every $T \in (0, T_s/2)$:*

(i) for every $T' \in (0, T/2)$

$$\|\tilde{H}_i(t)\| = \|\hat{H}_i(t) - H_i(t)\| \leq \varepsilon_H(T,T'), \quad t \in [2T', T), \quad i = 1,..,q,$$

(ii) and $\lim_{T' \to 0} \varepsilon_H(T,T') = 0$.

Proof: This result follows directly from (4.20) and is left to the reader. ∎

At this point, with $T_s > 0$, for each $T \in (0, T_s/2)$ and $T' \in (0, T/2)$, we choose an admissible S that satisfies (4.18) and (4.19); to minimize clutter we do not write it as an explicit function of T and T'. We have also defined $\hat{H}$ by (4.20) and (4.21), so κ as written in (4.13)-(4.15) is well defined, as is the controller $\mathcal{C}$, which we relabel $\mathcal{C}(T,T')$ to emphasize its dependence on T and T'.

4.2.2 System Properties in the Presence of Noise

As before, over periods where there is no switch, the nonlinearity in the generalized sampler can be moved to a nonlinearity on the noise w and, when there is a switch in the period, we have a nice bound on the size of the sampler outputs and hence the control signal. Due to the inclusion of the reference signal y_{ref} and the discontinuity in ξ, this result is more complicated than before.

We remind the reader of the notation

$$\underline{w}_k(t) := w(kT + t), \quad t \in [kT, (k+1)T)$$

and we define

$$\rho(T') := \max_{i,j=1,..,q} \left\{ \max_{t \in [0,T')} \| e^{-\tilde{A}_i t} G_i G_j^{-1} e^{\tilde{A}_j t} \| \right\}$$

and

$$\rho_y(T') := \max \left\{ \max_{i,j=1,..,q} \left\{ \max_{t \in [0,T')} \| e^{-\tilde{A}_i t} (G_i G_j^{-1} - I) \| \right\}, \bar{g} + 1 \right\};$$

notice that

$$\lim_{T' \to 0} \rho(T') = \bar{g} \text{ and } \lim_{T' \to 0} \rho_y(T') = \bar{g} + 1.$$

With these in hand, we find the analogues to Proposition 3.1 and Corollary 3.1:

Proposition 4.1 *With $T_s > 0$, $T \in (0, T_s/2)$, and $T' \in (0, T/2)$, there exists a constant $\gamma_v(T, T') > 0$, $2q$ linear functions of noise*

$$\phi_{1,i} : \mathcal{L}_\infty[0,T) \to \mathbf{R}^{n+r}, \quad i = 1, .., q$$

$$\phi_{2,i} : \mathcal{L}_\infty[0,T) \to \mathbf{R}^{n+r}, \quad i = 1, .., q$$

that have bounded gain, and q selector functions

$$\chi_i : \mathcal{L}_\infty[0,T) \times \mathbf{R}^{n+r} \to \{0,1\}, \quad i = 1, .., q$$

such that, for every $\sigma \in \Sigma_{T_s}$ and $w \in \mathcal{L}_\infty$, when $\mathcal{C}(T, T')$ is attached to the plant P_σ, we have that

(i) For every $k \in \{k_l : l \in \mathbf{N}\}^c$, P_σ's state-space representation (4.6) satisfies

$$\begin{aligned} \dot{\xi}(t) &= \tilde{A}_{\sigma(t)}\xi(t) + \tilde{B}_{\sigma(t)}\hat{H}_{\sigma(t)}(t)\xi[kT] + \begin{bmatrix} \tilde{B}_{\sigma(t)}\hat{H}(t) & \tilde{A}_{\sigma(t)}L_{\sigma(t)} \end{bmatrix} \times \\ & \quad \begin{bmatrix} \chi_{\sigma(t)}(\underline{w}_k, \xi[kT])\phi_{1,\sigma(t)}(\underline{w}_k) + [1 - \chi_{\sigma(t)}(\underline{w}_k, \xi[kT])]\phi_{2,\sigma(t)}(\underline{w}_k) \\ w(t) \end{bmatrix}, \\ & \qquad t \in [kT, (k+1)T), \quad x_0 \in \mathbf{R}^n, \ u_0 \in \mathbf{R}^m, \ y_{ref} \in \mathbf{R}^r. \end{aligned}$$

(ii) The sampler outputs v_1 and v_2 satisfy

$$\begin{aligned} \min\{\|v_1[k_l]\|, \|v_2[k_l]\|\} &\le \rho(T')\|\xi[k_l T^-]\| + \rho_y(T')\|y_{ref}\| + \gamma_v(T,T')\|w\|_\infty, \\ & \qquad l \in \mathbf{N}, \ x_0 \in \mathbf{R}^n, \ u_0 \in \mathbf{R}^m, \ y_{ref} \in \mathbf{R}^r. \end{aligned}$$

Corollary 4.1 *With $T_s > 0$, $T \in (0, T_s/2)$, and $T' \in (0, T/2)$, there exists a set of $2q$ nonlinear functions*

$$\phi_i : \mathcal{L}_\infty[0,T) \times \mathbf{R}^{n+r} \to \mathcal{L}_\infty[0,T), \quad i = 1,..,q$$

$$\theta_i : \mathcal{L}_\infty[0,T) \times \mathbf{R}^{n+r} \to \mathbf{R}^{n+r}, \quad i = 1,..,q$$

which have bounded gain and are such that, for every $\sigma \in \Sigma_{T_s}$, $w \in \mathcal{L}_\infty$, and $k \in \{k_l : l \in \mathbf{N}\}^c$, if $\mathcal{C}(T,T')$ is attached to P_σ, then (4.6) satisfies

$$\dot{\xi}(t) = \tilde{A}_{\sigma(t)}\xi(t) + \tilde{B}_{\sigma(t)}\hat{H}_{\sigma(t)}(t)\xi[kT] + \phi_{\sigma(t)}(\underline{w}_k, \xi[kT]), \quad t \in [kT, (k+1)T),$$

and

$$\begin{aligned}\xi[(k+1)T^-] \;\; &= \;\; e^{\bar{A}_{\sigma[kT]}T}\xi[kT] + \theta_{\sigma[kT]}(\underline{w}_k, \xi[kT]),\\ &\qquad\qquad\qquad x_0 \in \mathbf{R}^n,\; u_0 \in \mathbf{R}^m,\; y_{ref} \in \mathbf{R}^r.\end{aligned}$$

Proof: The result follows directly from Proposition 4.1(i) and the properties of $\hat{H}$ (most notably (4.22)) and is left to the reader. ■

4.3 Stability

As before, when there are no switches, we can prove stability for arbitrarily large T, which is not possible when we have plant switches. We begin with the case of no switching; the proof of this theorem is a simple extension of the proof of Theorem 3.1.

Theorem 4.1 *For every $T > 0$ and $T' \in (0, T/2)$ the controller $\mathcal{C}(T,T')$ stabilizes $\mathcal{P}$.*

We now turn to the case of (possibly persistent) plant switches; as before, these plant switches introduce two main difficulties:

(i) It is well known that switching too quickly between stable LTI systems can lead to instability.

(ii) In periods with a plant switch, the incorrect control will (likely) be applied.

Observe that (i) is exacerbated since plant switches cause an undesirable jump in the state ξ, so our lower bound on T_s will be more complicated:

$$\underline{T_s} := \frac{\ln\left\{\bar{g}\gamma_0\left(1 + \frac{bf\gamma_0}{a}\right)\left(1 + \frac{bf\gamma_0}{a}(1+\bar{g})\right)\right\}}{|\lambda_0|};$$

we have the same upper bound on T:

$$\bar{T}(T_s) := \min\left\{\frac{T_s}{2}, \frac{|\lambda_0|}{3(a-\lambda_0)}\left(T_s - \underline{T_s}\right)\right\}.$$

Theorem 4.2 *For every $T_s > \underline{T_s}$ and $T \in (0, \bar{T}(T_s))$, if T' is sufficiently small then $\mathcal{C}(T, T')$ stabilizes $\mathcal{P}_{T_s}$.*

Remark 4.5 *In this case, it is easy to argue that a tighter lower bound on T_s is in fact*

$$\frac{\ln\{\bar{g}\gamma_0\}}{|\lambda_0|};$$

again, we do not investigate this bound since to do so, we would need T to be small.

4.4 Step Tracking Performance

Here we investigate the step tracking performance of the proposed controller, so we set the noise to zero. We consider the case of no plant switches and then the case where we allow plant switches.

In the first case, the plant is unknown, but lies in the finite set of LTI plants $\mathcal{P}$. To that end, we fix the plant index, say at i; hence, for a given plant and integrator initial conditions $x_0 \in \mathbf{R}^n$ and $u_0 \in \mathbf{R}^m$, and a constant reference $y_{ref} \in \mathbf{R}^n$, we have a corresponding initial condition

$$\xi_0 = \begin{bmatrix} A_i x_0 + B_i u_0 \\ C_i x_0 - y_{ref} \end{bmatrix}, \tag{4.24}$$

which plays an important role in the upcoming discussion. Indeed, for the plant P_i, the corresponding LQR-optimal signals are

$$\xi^0(t) := e^{\bar{A}_i t}\xi_0, \quad t \geq 0, \tag{4.25}$$

$$\nu^0(t) := F_i e^{\bar{A}_i t}\xi_0, \quad t \geq 0, \tag{4.26}$$

and

$$e^0(t) := C_i \xi^0(t), \quad t \geq 0,$$

so the optimal LQR cost is

$$J_i^0(\xi_0) := \int_0^\infty \underbrace{(\xi^0(t))'\bar{Q}_i\xi^0(t) + (\nu^0(t) + Ke^0(t))'R_i(\nu^0(t) + Ke^0(t))}_{=:M_i(\xi^0(t),\nu^0(t),e^0(t))}\, dt$$

and the actual cost is

$$J_i(\xi_0) := \int_0^\infty M_i(\xi(t), \nu(t), e(t))dt.$$

As in Chapter 3, here the closed loop system is LTV and, by making T' small, for every plant P_i the difference between J_i and J_i^0 can be made as small as desired.

Theorem 4.3 *For every $\varepsilon > 0$ and $T > 0$, if T' is sufficiently small then the controller $\mathcal{C}(T, T')$ stabilizes $\mathcal{P}$ and, when $\mathcal{C}(T, T')$ is attached to $P_i \in \mathcal{P}$, we have that*

$$\begin{aligned} |J_i(\xi_0) - J_i^0(\xi_0)| &\leq \int_0^\infty \left\| M_i(\xi(t), \nu(t), e(t)) - M_i(\xi^0(t), \nu^0(t), e^0(t)) \right\| dt \\ &\leq \varepsilon \|\xi_0\|^2, \quad \xi_0 \in \mathbf{R}^{n+r}. \end{aligned}$$

We now consider the step tracking performance when the plant is allowed to switch. Recall from Chapter 3 that switches can cause the output of the samplers to be different, so the nonlinearity will affect the analysis.

As in the Section 3.4, we define the nominal control signal in the following way: if the plant is $P_\sigma \in \mathcal{P}_{T_s}$, then the nominal control signal ν^0 is given by

$$\nu^0(t) := F_{i_l} e^{\bar{A}_{i_l}(t - t_l)} \xi(t_l), \quad t \in [t_l, t_{l+1}), \quad l \in \mathbf{N}; \tag{4.27}$$

similarly, we define

$$\xi^0(t) := e^{\bar{A}_{i_l}(t - t_l)} \xi(t_l), \quad t \in [t_l, t_{l+1}) \;\; l \in \mathbf{N} \tag{4.28}$$

and

$$\begin{aligned} e^0(t) &:= \tilde{C} \xi^0(t) \\ &= \begin{bmatrix} 0 & I \end{bmatrix} \xi^0(t), \quad t \geq 0. \end{aligned}$$

Since, in general, the state $\xi(t)$ jumps at $t = t_l$, the actual control signal on the interval $[t_l, (k_l + 1)T)$ will (likely) be wrong - this error is related to $\xi(t_l)$ and to y_{ref}, so for each $\sigma \in \Sigma_{T_s}$ and $l \in \mathbf{Z}^+$, our nominal cost (which is defined only over the interval $[t_l, t_{l+1})$) will be a function of both of these terms and can be expressed via

$$\begin{aligned} & J^0_{[t_l, t_{l+1})}(\xi(t_l), y_{ref}) := \\ & \qquad \int_{t_l}^{t_{l+1}} \left[(\xi^0(t))' \bar{Q}_{i_l} \xi^0(t) + (\nu^0(t) + K e^0(t))' R_{i_l} (\nu^0(t) + K e^0(t)) \right] dt; \end{aligned}$$

this is exactly the optimal cost for the LTI plant P_{i_l} over the interval $[t_l, t_{l+1})$. Similarly, our actual cost is

$$J_{[t_l, t_{l+1})}(\xi(t_l), y_{ref}) := \int_{t_l}^{t_{l+1}} \left[\xi(t)' \bar{Q}_{i_l} \xi(t) + (\nu(t) + K e(t))' R_{i_l} (\nu(t) + K e(t)) \right] dt.$$

If $[\underline{t}, \bar{t}) \subset [t_l, t_{l+1})$, then we define $J_{[\underline{t},\bar{t})}(\xi(t_l), y_{ref})$ and $J^0_{[\underline{t},\bar{t})}(\xi(t_l), y_{ref})$ in the natural way.

We now show that, by making T and T' small, for every $P_\sigma \in \mathcal{P}_{T_s}$ the performance can be made as close to the nominal one as desired.

Theorem 4.4 *For every $\varepsilon > 0$ and $T_s > \underline{T_s}$ there exists a constant $\bar{T}_1 \in (0, \bar{T}(T_s))$ such that for every $T \in (0, \bar{T}_1)$, if T' is sufficiently small, then the controller $\mathcal{C}(T, T')$ stabilizes $\mathcal{P}_{T_s}$ and, for every $\sigma \in \Sigma_{T_s}$, attaching the controller $\mathcal{C}(T, T')$ to the plant P_σ yields*

$$\left| J^0_{[t_l,t_{l+1})}(\xi(t_l), y_{ref}) - J_{[t_l,t_{l+1})}(\xi(t_l), y_{ref}) \right| \le \varepsilon \left[\|\xi(t_l)\| + \|y_{ref}\| \right]^2, \quad l \in \mathbf{N},$$

$$x_0 \in \mathbf{R}^n, \quad u_0 \in \mathbf{R}^m, \quad y_{ref} \in \mathbf{R}^r.$$

4.5 An Example

We now revisit the example of Section 3.5:

$$P_1 \; : (A_1, B_1, C_1) = (1, 1, 1) \text{ and } P_2 \; : (A_2, B_2, C_2) = (1, 1, -1).$$

Here, our controller design begins with the selection of the regularization gain

$$K = 1,$$

which results in the augmented state-space matrices

$$\tilde{A}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \quad \tilde{B}_1 = \tilde{B}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ and } \tilde{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

We select the LQR variables

$$Q_i = R_i = 1, \quad i = 1, 2,$$

which yield the optimal gains

$$F_1 = \begin{bmatrix} -2.7321 & -2 \end{bmatrix} \text{ and } F_2 = \begin{bmatrix} -2.7321 & 0 \end{bmatrix}.$$

We choose a reference signal of

$$y_{ref} = 1.$$

With this in hand, we can calculate $\underline{T_s} \approx 7.2s$; as in Chapter 3, to illustrate that this is a loose bound, we will switch at a faster rate. Observe that the bound given in Remark 4.5 yields $\approx 1.5s$.

The hold is explicitly defined via (4.20) and (4.21), but we must still design our sampler function S; we will do so in the same way as in Section 3.5. To that end, we choose an additional parameter $j \in \mathbf{Z}^+$ satisfying

$$j \geq 2n = 2$$

and then set

$$h := \frac{T'}{j}$$

and define

$$\mathcal{O}_j(\tilde{C}, e^{\tilde{A}h}) := \begin{bmatrix} \tilde{C} \\ \tilde{C}e^{\tilde{A}h} \\ \vdots \\ \tilde{C}e^{\tilde{A}(j-1)h} \end{bmatrix}.$$

Since $j \geq 2n = 2$ and $(\tilde{A}, \tilde{C})$ is assumed to be observable, by [7], for sufficiently small h, the matrix $\mathcal{O}_j(\tilde{C}, e^{\tilde{A}h})$ has full column rank. If we define the weights

$$\begin{bmatrix} \bar{S}_0 & \bar{S}_1 & \ldots & \bar{S}_{j-1} \end{bmatrix} := \left(\mathcal{O}_j(\tilde{C}, e^{\tilde{A}h})'\mathcal{O}_j(\tilde{C}, e^{\tilde{A}h})\right)^{-1} \mathcal{O}_j(\tilde{C}, e^{\tilde{A}h})'$$

and we let $S(t)$ (on $[0, T)$) be a weighted sum of j impulses

$$\int_0^{T'} S(t)e(t)dt = \sum_{k=0}^{j-1} \bar{S}_k e(kh); \tag{4.29}$$

then it is routine to check that the sampler (4.29) satisfies the desired properties.

Again, we present only one set of example calculations. Here, we select the period, Estimation Phase duration, and sampler parameter to be

$$T = 1s,\ 2T' = 0.4s,\ \text{and}\ j = 20,$$

respectively; with these choices the sample time is

$$h = 0.01s.$$

The corresponding sampler gains are

$$\bar{S}[0] = 1000 \begin{bmatrix} -1.7950 \\ 0.2121 \\ -1.7743 \\ -0.2116 \end{bmatrix},\ \ldots,\ \bar{S}[19] = 1000 \begin{bmatrix} 1.8122 \\ -0.1493 \\ 1.8033 \\ 0.1492 \end{bmatrix}.$$

Completing the controller design, we find that the generalized hold gains are

$$\tilde{H}_1(t) = \tilde{B}_1' e^{-\tilde{A}_1(t-2T')} \begin{bmatrix} -13.1072 & -9.0308 \\ -34.2661 & -23.6544 \end{bmatrix}$$

and

$$\tilde{H}_2(t) = \tilde{B}_2' e^{-\tilde{A}_2(t-2T')} \begin{bmatrix} -7.6305 & -1.3209 \\ 30.5702 & 5.5256 \end{bmatrix},$$

which are many orders of magnitude smaller than those of [26] (the gains are not explicitly stated in [26], they can be found in [27]). Observe that these gains are much larger than those in Chapter 3, so we expect the noise rejection to degrade accordingly; this is due to the inclusion of the integrator. A similar relationship is found in the examples of [26].

4.5.1 Performance

Here we turn the noise off and investigate the performance of the system; since they have minimal effect, we set the initial conditions $x_0 = u_0 = 0$. The controller period is $T = 1$ and the sample time is $h = 0.01$. The plant switches every seven seconds - the switch occurs in the middle of the first sample during the Estimation Phase. We investigate two sampling durations, $T' = 0.2$ and $T' = 0.1$, shown in Figure 4.2 and Figure 4.3 respectively. These results show that both the error and control signal are close to the nominal ones; the dips in u correspond to the Estimation Phase, where $\nu = 0$. In the case where the Estimation Phase is shorter (i.e. $T' = 0.1$) the oscillations in u become smaller and both the control and error signals get closer to the nominal ones, as expected. The time axis indicates the locations where the plant switches.

Observe that, as compared to Figures 3.3 and 3.4, here the control signal u is very smooth - this is the effect of the integrator. If we were to look at the signal $\dot{u}$ instead, then we would see a behaviour very similar to that of the control signal of Chapter 3. Furthermore, observe that the system takes much longer to settle here; this is a direct consequence of the phase lag introduced by the integrator.

To better illustrate the behavior during the Estimation and Control Phases, Figure 4.4 shows a closeup view of the control signal from Figure 4.3 over a single period wherein a plant switch occurs.

4.5.2 Noise Rejection

Here we consider the same setup as above, but set $h = 0.001$[1], add noise at the plant output, and set y_{ref} to zero. As in Chapter 3, we expect our noise tolerance to improve as T' gets larger. Results are shown in Figures 4.6 and 4.5; in the error and input components of these figures, the time axis indicates locations where plant switches occur. We see that the system experiences significantly more of an adverse

[1]We use a smaller h since our simulation runs in MATLAB and is actually a discretized approximation of the system which uses h as its sampling rate, including during the Control Phase - the addition of noise means that we require a finer resolution for this approximation to be accurate. This is also why we use a smaller noise signal as compared to Chapter 3.

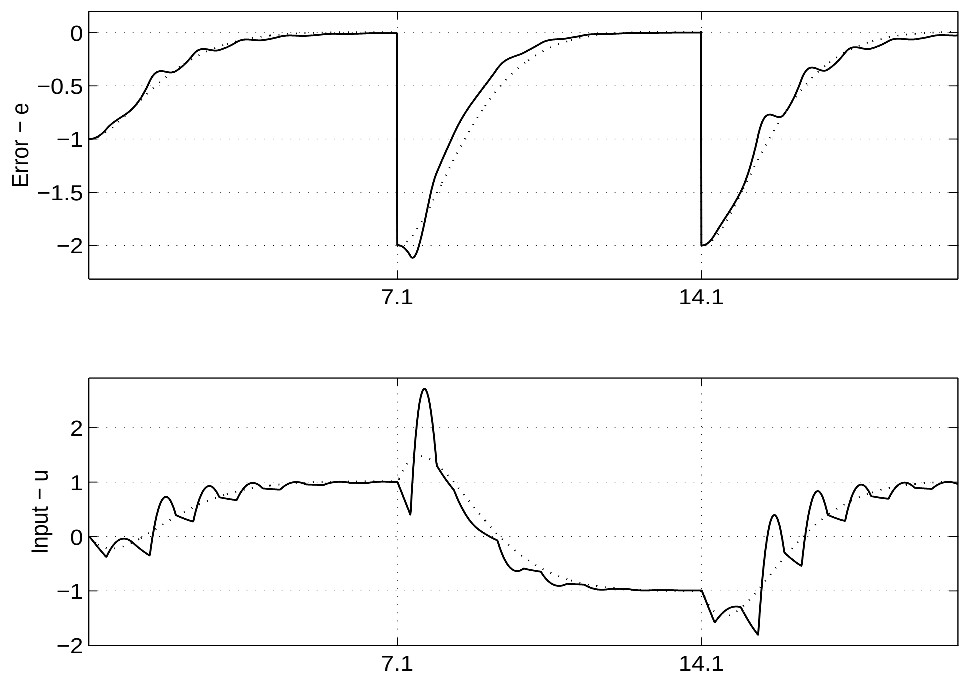

Figure 4.2: Example - solid is actual, dashed is nominal - $T' = 0.2s$.

affect in the $T' = 0.01s$ case as compared to $T' = 0.02s$. Finally, in both cases we see that the system does not go unstable in the presence of noise and is able to recover the desired tracking behaviour once noise is turned off (i.e. the error goes to zero).

Remark 4.6 *Since the controller is essentially open loop during a single period, although we have not explicitly shown it here, as the period T increases, we expect the noise tolerance to decrease. Since T' is bounded above by $T/2$, forcing T to be small also forces T' to be small. Hence, we expect a tradeoff in the size of the noise gain between a small T and a large T'.*

Finally, for completeness, we combine these two cases with a twist: we set the initial condition on x and u to one. We chose $T' = 0.02s$, $h = 0.001s$, and $y_{ref} = 1$. The result is Figure 4.7; again, observe that the system does not go unstable in the presence of noise and is able to recover the desired tracking behaviour once noise is turned off (i.e. the error goes to zero).

4.6 Summary and Conclusions

In this chapter we extend the result of Chapter 3 to the more challenging problem of step tracking. We follow a similar analysis to that of Chapter 3, with one important

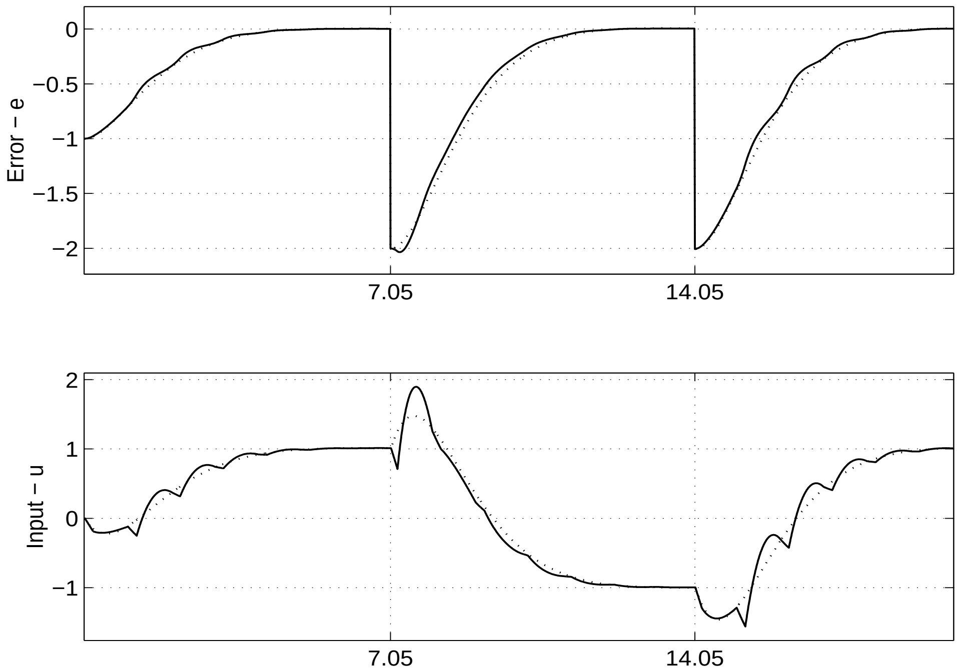

Figure 4.3: Example - solid is actual, dashed is nominal - $T' = 0.1s$.

difference: here the state ξ can jump at the switching times. This difference has many ramifications; for example, we must bound the size of the jump and be sure to include its effect in the lower bound $\underline{T_s}$; we must also take care when writing down and solving our system equations since the derivative of the state is not well defined at times when the plant parameters switch. A second (and less far-reaching) difference is that incorporating the noise signals in the augmented plant's state-space realization is non-trivial and requires a change of variables.

As with Chapter 3, in comparison to earlier work by Miller on a related (but easier) problem [26], here the controller's period can be large, so, as in Chapter 3 we expect to have relatively smaller controller gains and relatively improved noise rejection; our example above indicates that this is indeed the case. Unfortunately, the inclusion of the integrator to solve the tracking problem leads to much larger gains than those of Chapter 3. One drawback shared with [26], and shown in our simulations, is that there is a tradeoff between performance and noise tolerance.

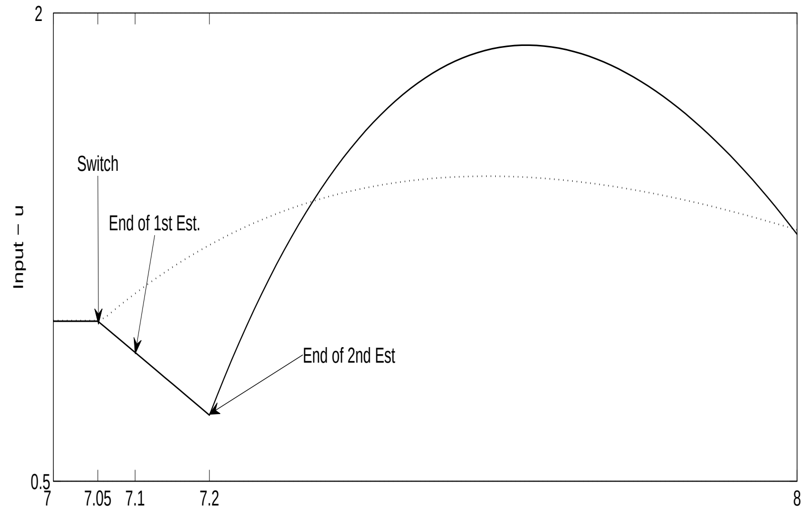

Figure 4.4: Closeup of u from Figure 4.3.

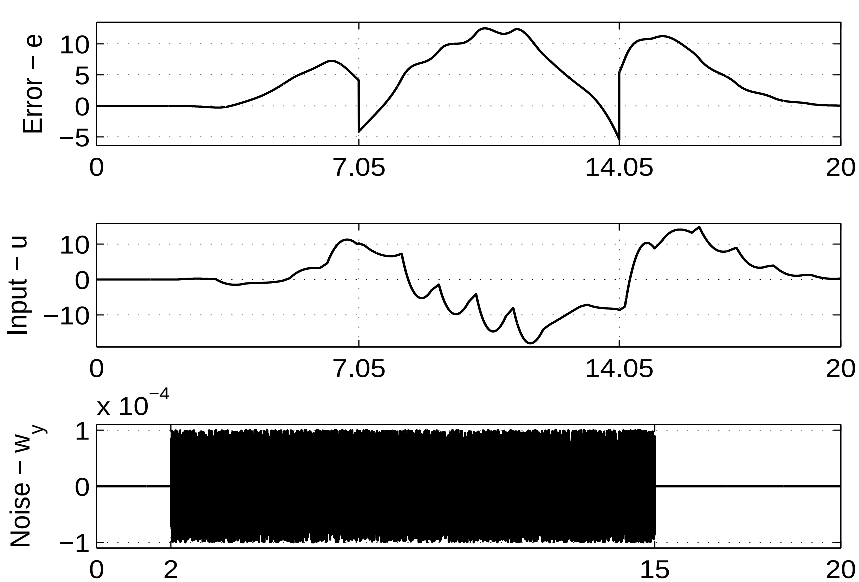

Figure 4.5: Example with noise - $T' = 0.01s$.

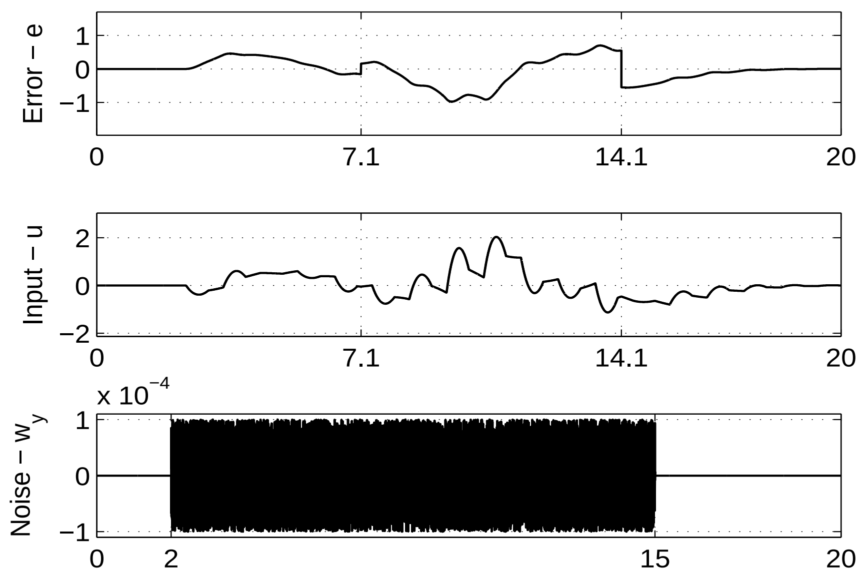

Figure 4.6: Example with noise - $T' = 0.02s$.

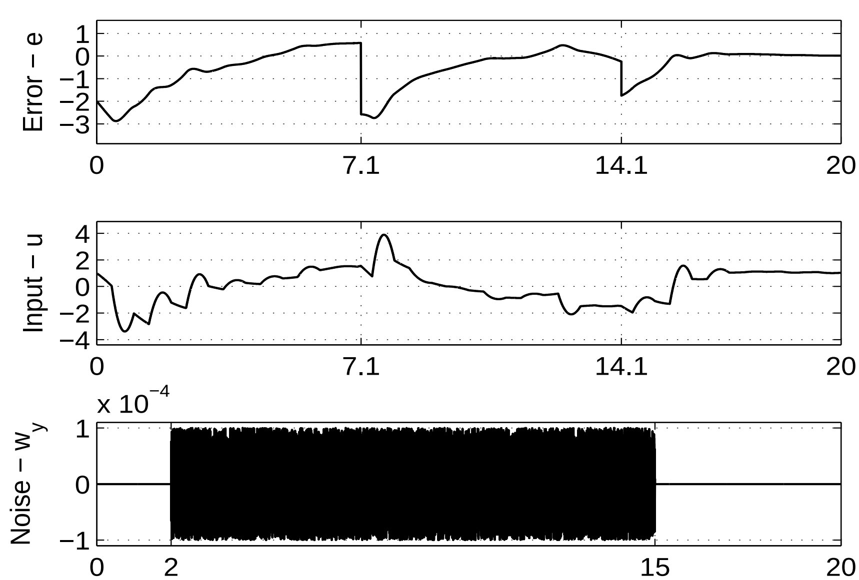

Figure 4.7: Example with noise, initial conditions, and $y_{ref} = 1$ - $T' = 0.02s$.

Chapter 5

The Compact Stability Problem

We now turn away from the question of tracking and back to the original Chapter 3 problem of stability with LQR performance; here we seek to extend the result of Chapter 3 to the case of a *compact* set of LTI plants. Since there may be an infinite number of plants, we no longer turn the control signal off when performing our estimation; a side effect of this is that our actual state will no longer match the optimal one at the period endpoints. Since the analysis here is significantly more complex, in this chapter we consider only the Single Input Single Output (SISO) case, ignore noise, and consider stability only in the asymptotic sense; furthermore, we do not explicitly analyze performance in the face of plant switches, although we do provide an illustrative simulation. These results draw heavily on [25]; an early version of this work was presented in the conference paper [39].

Before we proceed, we remind the reader of the contribution of this design. Recall that the RACE controller in [25] provides simultaneous stability and near optimal LQR performance for a compact set of LTI plants; however, it does not directly address the issue of switching and the controller period is small. As in the previous two chapters, we provide a redesign that provides stability in the face of persistent plant switching and allows the controller period to be large. Although we do not analyze performance in the face of plant switches, we expect that our controller yields a similar result to that of the previous two chapters.

A brief outline of this chapter is as follows. In Section 5.1 we make the problem precise. In Section 5.2 we address the question of estimation. In Section 5.3 we leverage the results of Section 5.2 to design a RACE controller and present some results that will be useful in proving the main theorems of this chapter. In Section 5.4 we investigate the stability properties of this controller both with and without plant switches, while in Section 5.5 we turn to the question of performance when there are no switches. In Section 5.6 we present two illustrative examples and we wrap up with a summary and concluding remarks in Section 5.7. In this chapter, we use the 2-norm to measure the size of a vector.

5.1 Problem Formulation

Here we will use transfer functions to define the set of LTI plants of interest. We wish to consider a compact set of transfer functions that all have the same order n; these transfer functions are of the form

$$P : \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}, \quad a_i, b_i \in \mathbf{R}, \quad i = 0, .., n-1. \tag{5.1}$$

We say that such a transfer function is **minimal** if the numerator and denominator polynomials are coprime; furthermore, we say that a set of such transfer functions is compact if the set composed of all of the corresponding coefficients

$$\left(\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}, \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix} \right) \in \mathbf{R}^n \times \mathbf{R}^n$$

is compact.

To define the set of admissible transfer functions, we fix $n \in \mathbf{Z}^+$ and define Γ to be the set of all transfer functions of the form (5.1) that are minimal and of order n. We then assume that the set of admissible LTI plants lies in a compact subset of Γ, which we label $\mathcal{P}$[1].

We would like to evaluate performance in the LQR sense, so we will need a state-space representation of $P \in \mathcal{P}$. It will turn out that our analysis will be much simpler if we adopt a state-space representation that is highly structured. To that end, for each plant $P \in \mathcal{P}$, we adopt the state-space representation corresponding to the (minimal) observable canonical form:

$$\left.\begin{aligned} \dot{\xi}(t) &= \underbrace{\begin{bmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ -a_0 & \cdots & \cdots & -a_{n-1} \end{bmatrix}}_{=:A_P} \xi(t) + \underbrace{\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix}}_{=:B_P} u(t) \\ y(t) &= \underbrace{\begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}}_{=:C} \xi(t); \end{aligned}\right\} \tag{5.2}$$

we associate this model with the triple (A_P, B_P, C). Observe that, since the representation is assumed to be minimal, the pairs (C, A_P) and (A_P, B_P) are automatically controllable and observable, respectively. Furthermore, since $\mathcal{P}$ is compact,

[1] Clearly this includes the case of a finite number of plants similar to that in Chapter 3. While the approach in this chapter will work for a finite number of plants, here, as opposed to previous chapters, we will always require a short Estimation Phase, which leads to larger controller gains and correspondingly poor noise tolerance.

so is the associated set of observer canonical triples

$$\tilde{\mathcal{P}} := \{(A_P, B_P, C) \in \mathbf{R}^{n\times n} \times \mathbf{R}^n \times \mathbf{R}^{1\times n} : P \in \mathcal{P}\}.$$

Recall that we wish to provide near optimal LQR performance for each possible model in $\mathcal{P}$. With $R > 0$[2], we consider the classical performance index

$$J(\xi_0) := \int_0^\infty [y^2(t) + Ru^2(t)]\, dt.$$

As is well-known, for each $P \in \mathcal{P}$, the optimal controller for (A_P, B_P, C) is state-feedback and of the form

$$u = F_P \xi,$$

which gives rise to a cost of the form

$$J_P(\xi_0) = \xi_0^T V_P \xi_0$$

with V_p the positive definite solution associated with a Riccati equation. The associated closed loop system is

$$\dot{\xi} = \underbrace{(A_P + B_P F_P)}_{\bar{A}_P} \xi.$$

It is not straightforward to estimate such a feedback signal; however, it will turn out that we can estimate nice functions of the first $2n$ Markov parameters

$$\{CB_P, CA_PB_P, ..., CA_P^{2n-1}B_P\}.$$

To that end, observe that B_P is composed of the first n Markov parameters while C is constant; we can prove that A_P and F_P are nice functions of the first $2n$ Markov parameters:

Lemma 5.1 *(Parametrization Lemma) [25] A_P and F_P are analytic functions of the first $2n$ Markov parameters*

$$\{CB_P, CA_PB_P, ..., CA_P^{2n-1}B_P\}$$

for all $P \in \Gamma$.

Remark 5.1 *Since*

$$\bar{A}_P = A_P + B_P F_P,$$

it follows immediately from this lemma that the function $e^{\bar{A}_P t}$ is an analytic function of t and the first $2n$ Markov parameters.

[2] Unlike previous chapters, here we use the same r for every plant.

Observe that, since every transfer function in $\mathcal{P}$ is assumed to be minimal, for every $P, \bar{P} \in \mathcal{P}$, we have that

$$(A_P, B_P, C) = (A_{\bar{P}}, B_{\bar{P}}, C) \Leftrightarrow P = \bar{P}$$

and we can choose $m \in \{n, .., 2n\}$ to ensure that

$$\begin{bmatrix} CB_P \\ CA_P B_P \\ \vdots \\ CA_P^{m-1} B_P \end{bmatrix} = \begin{bmatrix} CB_{\bar{P}} \\ CA_{\bar{P}} B_{\bar{P}} \\ \vdots \\ CA_{\bar{P}}^{m-1} B_{\bar{P}} \end{bmatrix} \Leftrightarrow (A_P, B_P, C) = (A_{\bar{P}}, B_{\bar{P}}, C).^3$$

Therefore, the first m Markov parameters can be used to uniquely define each plant; the converse is also true. **We now fix $m \in \{n, .., 2n\}$ to ensure that this uniqueness property holds.** Furthermore, we define

$$\mathcal{M} := \left\{ \begin{bmatrix} CB_P \\ CA_P B_P \\ \vdots \\ CA_P^{m-1} B_P \end{bmatrix} : P \in \mathcal{P} \right\};$$

since $\tilde{\mathcal{P}}$ is compact, $\mathcal{M}$ is compact as well.

Since our estimation method hinges on the Markov parameters, it will be more convenient to analyze our plant in terms of $p \in \mathcal{M}$ (instead of $P \in \mathcal{P}$), so we relabel the matrices A_P, B_P, F_P, and $\bar{A}_P$ by A_p, B_p, F_p, and $\bar{A}_p$. Finally, if the transfer function $P \in \mathcal{P}$ is associated with the Markov parameters $p \in \mathcal{M}$, then we label it P_p.

Since the set of plants in not required to be finite, in this chapter we consider a slightly different switching signal σ: instead of having σ specify an index, we consider the (piecewise constant) switching signal

$$\sigma : \mathbf{R}^+ \to \mathcal{M}$$

which specifies the Markov Parameters of the time-varying input/output map P_σ at every time t; as usual, we assume that σ is continuous from the right. With $T_s > 0$, we define

$$\begin{aligned} \Sigma_{T_s} := \{\sigma(t) \ : \quad & 1) \quad \sigma(t) \in \mathcal{M}, \quad t \geq 0, \\ & 2) \quad \sigma(t) \text{ is a piecewise constant function of } t \text{, and} \\ & 3) \quad \text{there is at least } T_s \text{ time units between discontinuities}\}. \end{aligned}$$

We can then define

$$\mathcal{P}_{T_s} := \{P_\sigma \ : \sigma \in \Sigma_{T_s}\}.$$

[3] In general, we need $m = 2n$; however, if there is a lot of structure in the problem (e.g. a gain margin problem), then it may be that $m < 2n$ is sufficient.

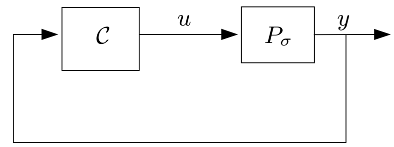

Figure 5.1: Block diagram

As usual, notice that $\mathcal{P}_\infty = \mathcal{P}$. We can also express the time-varying plant P_σ in observer canonical state-space form via

$$\begin{aligned} \dot{\xi}(t) &= A_{\sigma(t)}\, \xi(t) + B_{\sigma(t)}\, u(t), \quad \xi(0) = \xi_0, \\ y(t) &= C\, \xi(t); \end{aligned} \tag{5.3}$$

which we associate with the triple (A_σ, B_σ, C). We define F_σ and $\bar{A}_\sigma$ in an analogous way. Since there is no noise and no reference signal and since, in this chapter, we will not require a regularization feedback, plants of this form together with the controller $\mathcal{C}$ yield the (very simple) block diagram of Figure 5.1.

In our previous chapters, the compensator κ had rich structure that led to a natural definition of asymptotic stability. That will not be the case here; however, it will turn out that our controller $\mathcal{C}$ can be modeled via a (sampled data) state-space representation. As in our previous chapters, to handle the switching we will use a mild nonlinearity, so, with G and J periodic functions of k, we will adopt a state space representation of $\mathcal{C}$ of the form

$$\begin{aligned} z[k+1] &= G(k, z[k], y(kh)), \quad z[0] = z_0 \in \mathbf{R}^l, \\ u(kh+\tau) &= J(k, z[k], \tau), \qquad\quad \tau \in [0, h). \end{aligned} \tag{5.4}$$

If we label the period of G and J by ℓ, then the period of the controller is $T := \ell h$. We associate this system with the 4-tuple (G, J, h, ℓ). As usual, it will turn out that our compensator's behaviour will depend only on the current period, so the initial condition of the controller $z[0]$ will have no bearing on the stability of the system. The following definition of asymptotic stability follows naturally from the above discussion:

Definition 5.1 *With $T_s > 0$, the controller $\mathcal{C}$* **asymptotically stabilizes** $\mathcal{P}_{T_s}$ *if, for every $P_\sigma \in \mathcal{P}_{T_s}$ we have that*

(i) for every $\varepsilon > 0$ there exists a $\delta > 0$ so that, if $\|x_0\| < \varepsilon$, then

$$\|x(t)\| < \delta, \quad t \geq 0, \;\; \textit{and} \;\; \|z[k]\| < \delta, \quad k \in \mathbf{Z}^+,$$

and

(ii) for every $x_0 \in \mathbf{R}^n$, we have

$$\lim_{t\to\infty} \xi(t) = 0 \;\; \textit{and} \;\; \lim_{k\to\infty} z[k] = 0.$$

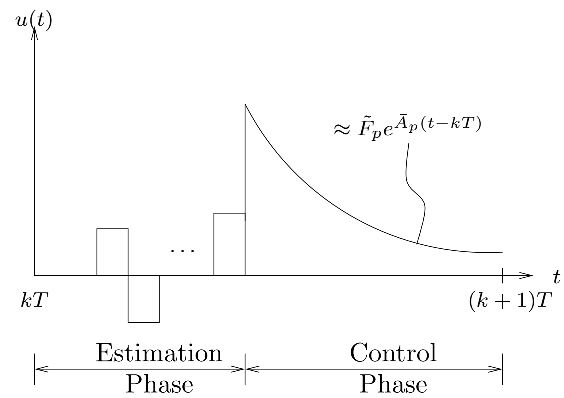

Figure 5.2: Input signal.

Remark 5.2 *As usual, observe that (ii) is a global convergence condition, rather than the typical local one. Furthermore, we will be able to prove a stronger condition than (i), namely: there exists a constant* $\gamma > 0$ *so that for every* $x_0 \in \mathbf{R}^n$ *we have*

$$\|x(t)\| \leq \gamma \|x_0\|, \quad t \geq 0, \ \textit{and} \ \|z[k]\| \leq \gamma \|x_0\|, \quad k \in \mathbf{Z}^+.$$

We now present some uniform bounds on our system parameters. Since $\tilde{\mathcal{P}}$ is compact, the following are well defined:

$$a := \sup_{p \in \mathcal{M}} \|A_p\|,$$

$$b := \sup_{p \in \mathcal{M}} \|B_p\|.$$

Furthermore, since $\mathcal{M}$ is compact and F_p is an analytic function of p, the following is well defined:

$$f := \sup_{p \in \mathcal{M}} \|F_p\|.$$

Observe that $\|C\| = 1$. Finally, since we do not have a finite number of plants, it is not straightforward to find a nice, exponentially decaying bound on $\|e^{\bar{A}_p t}\|$; we defer doing so until the next section.

As usual, we split the period of the controller into two parts, the Estimation Phase, where we estimate $F_p e^{\bar{A}_p(t-kT)} \xi[kT]$, and the Control Phase, where we apply the estimate of $F_p e^{\bar{A}_p(t-kT)} \xi[kT]$. Unlike in previous chapters, here, the estimation method requires the application of a sequence of test signals (or probes), which are constructed on the fly; a typical control signal is illustrated in Figure 5.2. As in our previous two chapters, we will carry out each phase in (almost) a linear fashion in order to end up with (almost) a linear controller.

5.2 Estimation

We indicated in Section 2.3.1 that we can estimate the desired control signal by estimating polynomials in p; in this section we will show how to do this. This process is not straightforward: we will require an additional approximation as well as a special way of writing down polynomials whose arguments are vector valued.

5.2.1 Estimating Polynomials in p

We begin by presenting a technical lemma that provides the basis for our polynomial estimation approach. To do so, it will be useful to define several matrices:

$$S_m := \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2^2 & \cdots & 2^m \\ & & \vdots & & \\ 1 & m & m^2 & \cdots & m^m \end{bmatrix},$$

$$O_m(C, A_p) := \begin{bmatrix} C \\ CA_p \\ \vdots \\ CA_p^m \end{bmatrix},$$

$$X_m(h) = \mathrm{diag}\{1,\, h,\, h^2/(2!), ..., h^m/(m!)\}.$$

Note that S is a Vandermonde matrix and h is non-zero, so both S_m and $X_m(h)$ are invertible. We will be using a sequence of samples of y; to that end, we define

$$\mathcal{Y}(t) := \begin{bmatrix} y(t) \\ y(t+h) \\ \vdots \\ y(t+mh) \end{bmatrix}.$$

Lemma 5.2 *(Key Estimation Lemma) [25] Let $\bar{h} \in (0,1)$. There exists a constant $\gamma > 0$ so that for every $t_0 \geq 0$, $\xi_0 \in \mathbf{R}^n$, $h \in (0, \bar{h})$, $\bar{u} \in \mathbf{R}$, and $p \in \mathcal{M}$, the solution of (5.2) with*

$$u(t) = \bar{u}, \quad t \in [t_0, t_0 + mh)$$

satisfies

$$\left\| X_m(h)^{-1} S_m^{-1} \mathcal{Y}(t_0) - O_m(C, A_p)\xi(t_0) - \begin{bmatrix} 0 \\ p \end{bmatrix} \bar{u} \right\| \leq \gamma h(\|\xi(t_0)\| + \|\bar{u}\|),$$

$$\|\xi(t) - \xi(t_0)\| \leq \gamma h(\|\xi(t_0)\| + \|\bar{u}\|), \quad t \in [t_0, t_0 + mh].$$

Observe that the second result of this lemma says that the effect of probing can be made small by making h small.

To see how the Key Estimation Lemma (KEL) can be applied, suppose that we first set

$$u(t) = 0, \quad t \in [t_0, t_0 + (m+1)h)^4;$$

observe that

$$O_m(C, A_p) = \begin{bmatrix} I_n \\ \begin{bmatrix} I_{m+1-n} & 0 \end{bmatrix} A_p^n \end{bmatrix},$$

so a good estimate of $\xi(t_0)$ is

$$\begin{bmatrix} I_n & 0 \end{bmatrix} X_m(h)^{-1} S_m^{-1} \mathcal{Y}(t_0).$$

Since we may have a plant switch in the interval $[t_0, t_0 + (m+1)h)$, we proceed as in our previous chapters: we set

$$u(t) = 0, \quad t \in [t_0 + (m+1)h, t_0 + 2(m+1)h)$$

as well, so another good estimate of $\xi(t_0)$ is

$$\begin{bmatrix} I_n & 0 \end{bmatrix} X_m(h)^{-1} S_m^{-1} \mathcal{Y}(t_0 + (m+1)h).$$

As usual, we choose the smaller of the two estimates:

$$\begin{aligned} \mathrm{Est}\{\xi(t_0)\} \quad := \quad & \mathrm{argmin}\{\| \begin{bmatrix} I_n & 0 \end{bmatrix} X_m(h)^{-1} S_m^{-1} \mathcal{Y}(t_0) \|, \\ & \qquad \| \begin{bmatrix} I_n & 0 \end{bmatrix} X_m(h)^{-1} S_m^{-1} \mathcal{Y}(t_0 + (m+1)h) \| \}. \end{aligned}$$

If we insist that

$$2(m+1)h < T_s$$

(recall that T_s is the minimum time between switches), then at least one interval will **not** contain a plant switch; if h is also small, then we will be guaranteed that the above estimate will be accurate when there is no plant switch and will be modest in size when there is one. We define

$$h_m := (m+1)h,$$

so each application of the Estimation Lemma requires an input of duration h_m, which we can view as a kind of internal probing period.

We estimate additional terms in the following way. With $g \in \mathbf{R}^{1\times(m+1)}$ and $\rho > 0$ a scaling factor, suppose we define a test signal to be a linear functional of our above estimate:

$$\bar{u} = \rho g \mathrm{Est}\{\xi(t_0)\}.$$

If we now set

$$u(t) = \bar{u}, \quad t \in [t_0 + 2\bar{h}_m, t_0 + 4\bar{h}_m),$$

[4] We use $m+1$ instead of m to avoid initialization issues when constructing a state space representation of our controller; this has no effect on our proofs.

then by the KEL we should define

$$\begin{aligned}\mathrm{Est}\{p\bar{u}\} \ := \ & \frac{1}{\rho}\left[\begin{array}{cc} 0 & I \end{array}\right] \times \\ & \operatorname{argmin}\{\|X_m(h)^{-1}S_m^{-1}[\mathcal{Y}(t_0+2h_m)-\mathcal{Y}(t_0)]\|, \\ & \qquad \|X_m(h)^{-1}S_m^{-1}[\mathcal{Y}(t_0+3h_m)-\mathcal{Y}(t_0+h_m)]\|\}.\end{aligned}$$

Of course, this can be repeated a number of times (for different choices of g) so it should be possible to estimate terms of the form $\phi(p)\xi(t_0)$ with $\phi : \mathbf{R} \longmapsto \mathbf{R}^{m+1}$ a polynomial in its arguments.

Unfortunately, this is not enough to estimate to desired control signal. To see why, recall that the optimal control law is

$$u(t) = F_p e^{\bar{A}_p(t-kT)}\xi[kT], \quad t \in [kT,(k+1)T), \quad k \in \mathbf{Z}^+,$$

with F_p and $\bar{A}_p$ *analytic* functions of p (i.e. F_p and $\bar{A}_p$ are not necessarily polynomials in p). The process outlined above can only estimate polynomials, so we must perform an additional approximation.

5.2.2 Approximation by a Sampled-Data Controller

The optimal control signal is of the form

$$u(t) = \underbrace{F_p e^{\bar{A}_p(t-kT)}}_{=:H(p,t-kT)}\xi[kT], \quad t \in [kT,(k+1)T), \quad k \in \mathbf{Z}^+;$$

the problem is that $H(p,t-kT)$ is not a polynomial in p and t and so it can *not* be estimated via the Estimation Lemma. However, H *is* an analytical function of its two arguments, so, if we fix an upper bound on the period, say T_{max}, then by the Stone-Weierstrass Approximation Theorem [32], for every $\varepsilon > 0$ there exists a polynomial H^ε satisfying

$$\|H(p,t) - H^\varepsilon(p,t)\| \le \varepsilon, \ \ p \in \mathcal{M}, \ \ t \in [0,T_{max}]; \tag{5.5}$$

we can use the Estimation Lemma to estimate $H^\varepsilon(p,t)\xi[kT]$.

There are three technical issues that we must address before we move on. First, for each $\varepsilon > 0$ there are many polynomials H^ε that satisfy (5.5); to that end we adopt the following convention: if we fix $\varepsilon > 0$, then we implicitly mean that we fix H^ε so that it satisfies (5.5). Second, observe that, since $\mathcal{M}$ is compact, for every $\varepsilon > 0$ there exists a constant $\gamma(\varepsilon) > 0$ such that

$$\|H^\varepsilon(p,t)\| \le \gamma(\varepsilon), \quad p \in \mathcal{M}, \quad t \in [0,T_{max}];$$

for each $\varepsilon > 0$, we let $\bar{\gamma}(\varepsilon)$ be the smallest such $\gamma(\varepsilon)$ and adopt the notation

$$\|H^\varepsilon\|_\infty := \bar{\gamma}(\varepsilon).$$

Finally, it is natural to define

$$H^0(p,t) := H(p,t).$$

We now wish to analyze the closed loop system behaviour when the control law

$$u(t) = H^\varepsilon(p, t-kT)\xi[kT], \quad t \in [kT,(k+1)T), \quad k \in \mathbf{Z}^+ \tag{5.6}$$

is applied to the system, but first, we need some notation. When the optimal control law is applied to the plant (5.2), we label the corresponding state response, output, and control signal by ξ^0, y^0, and u^0, respectively; similarly, when (5.6) is applied to the plant, we label the corresponding responses by ξ^ε, y^ε, and u^ε. In both cases, we omit the dependence of ξ, y, and u on the parameter $p \in \mathcal{M}$. Furthermore, when we apply (5.6) to the plant (A_p, B_p, C) we obtain

$$\xi^\varepsilon(t) = \underbrace{\left[e^{A_p(t-kT)} + \int_0^{t-kT} e^{A_p(t-kT-\tau)} H^\varepsilon(p,\tau)d\tau\right]}_{=:\Phi_p^\varepsilon(t-kT,0)} \xi^\varepsilon[kT], \quad t \in [kT,(k+1)T);$$

observe that, by our definition of H^0, we have

$$\Phi_p^0(t-kT,0) = e^{\bar{A}_p(t-kT)}.$$

In our previous chapters, we showed that the actual cost can be made as close as desired to the optimal one. In this chapter, we will investigate a related problem: we show that the actual trajectories can be made as close as desired to the optimal ones in the sense that

$$\int_0^\infty \Big([y(t) - y^0(t)]^2 + R[u(t) - u^0(t)]^2\Big)dt$$

can be made small. We adopt the standard notation

$$\|x\|_2^2 := \int_0^\infty \big(x(t)\big)^2 dt.$$

Proposition 5.1 *There exist constants $\bar{\varepsilon} > 0$, $\gamma_0 > 1$, and $\lambda_0 < 0$ so that, for every $\varepsilon \in [0,\bar{\varepsilon})$, $\xi_0 \in \mathbf{R}^n$, $T \in (0, T_{max})$ and $p \in \mathcal{M}$, we have that*

(i) $$\|\xi^0(t) - \xi^\varepsilon(t)\| \le \varepsilon\gamma_0 e^{\lambda_0 t}\|\xi_0\|, \quad t \ge 0,$$

(ii) $$\|u^0(t) - u^\varepsilon(t)\| \le \varepsilon\gamma_0 e^{\lambda_0 t}\|\xi_0\|, \quad t \ge 0,$$

(iii) $$\|\Phi_p^\varepsilon(T,0)^k\| \le \gamma_0 e^{\lambda_0 kT}, \quad k \in \mathbf{Z}^+,$$

(iv) $$\left\| \begin{bmatrix} y^\varepsilon \\ R^{1/2}u^\varepsilon \end{bmatrix} - \begin{bmatrix} y^0 \\ R^{1/2}u^0 \end{bmatrix} \right\|_2^2 \le \varepsilon^2\gamma_0^2\|\xi_0\|^2.$$

Proof: Please see Appendix C.

At this point we apply Proposition 5.1 and fix the constants $\bar{\varepsilon} > 0$, $\gamma_0 > 0$, and $\lambda_0 < 0$ so that they have the required properties.

Remark 5.3 *This proposition says that a controller that uses the polynomial approximation H^ε instead of the optimal function H provides behaviour that can be made arbitrarily close to the optimal one by making ε small.*

Remark 5.4 *Observe that (iii) says that the closed loop system's behaviour at the period endpoints is nice in the sense that it experiences exponential decay. Furthermore, since the proposition allows $\varepsilon = 0$, (iii) also gives us our nice bound:*

$$\|e^{\bar{A}_p t}\| \leq \gamma_0 e^{\lambda_0 t}, \quad p \in \mathcal{M}, \quad t \geq 0.$$

Now that we have a control signal that can be written in the form of a polynomial in p, it remains to show how to apply the KEL to obtain a good estimate of this control signal. To do so, we will need to re-parameterize our polynomial.

5.2.3 Polynomial Notation

Here we adopt the notation of [25] and modify it to our needs, which we now quickly summarize. The goal is to parametrize the polynomial $H^\varepsilon(p,t)$ in such a way that we can estimate the various terms in a straight-forward and systematic fashion. We adopt the standard notation: for $x \in \mathbf{R}^n$, we have

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Following Rudin [32], we introduce the notion of a multi-index, which is an ordered $m+1$-tuple

$$\alpha = (\alpha_1, ..., \alpha_{m+1}), \quad \alpha_i \in \mathbf{Z}^+.$$

For such a multi-index, we can define

$$|\alpha| := \alpha_1 + \cdots + \alpha_{m+1},$$

$$(p,t)^\alpha := p_1^{\alpha_1} \cdots p_m^{\alpha_m} t^{\alpha_{m+1}};$$

since we are dealing with integer exponents, we define $0^0 := \lim_{x \to 0} x^0 = 1$. Hence, given that H^ε is a polynomial which maps $\mathbf{R}^{m+1} \longmapsto \mathbf{R}^{1\times n}$, it follows that there exists a finite index set $I \subset (\mathbf{Z}^+)^{m+1}$ and constant matrices $c_\alpha \in \mathbf{R}^{1\times n}$, $\alpha \in I$, so that we can write H^ε in the form

$$H^\varepsilon(p,t) = \sum_{\alpha \in I} (p,t)^\alpha c_\alpha, \quad (p,t) \in \mathcal{M} \times [0, T_{max}].$$

Now we turn to the estimation problem. From Section 5.2.1, $\xi(t_0)$ can be easily estimated using the KEL as motivation. Now consider the problem of estimating the polynomial

$$\sum_{\alpha \in I} (p,t)^\alpha c_\alpha \xi(t). \tag{5.7}$$

We define q to be the largest multi-index of the first p elements:

$$q := \max_{\alpha \in I} \sum_{i=1}^{m} |\alpha_i|$$

and $\bar{q}$ to be the largest index of the $m+1^{th}$ element:

$$\bar{q} := \max_{\alpha \in I} |\alpha_{m+1}|.$$

From the KEL we know that for each $j \in \{1, ..., n\}$, it is possible to estimate $p\xi_j(t)$ by carrying out a simple experiment, so, by doing a succession of n experiments, we can estimate

$$\xi(t) \otimes p = \begin{bmatrix} p\xi_1(t) \\ p\xi_2(t) \\ \vdots \\ p\xi_n(t) \end{bmatrix} \in \mathbf{R}^{np}.$$

Using the same logic, we can estimate

$$(\xi(t) \otimes p) \otimes p \in \mathbf{R}^{np^2}$$

using a succession of nm experiments. Of course, this can be repeated as many times as desired. To this end, we now define

$$\xi(t) \otimes^0 p := \xi(t)$$

and

$$\xi(t) \otimes^{i+1} p := (\xi(t) \otimes^i p) \otimes p, \quad i \in \mathbf{N};$$

notice that $\xi(t) \otimes^i p$ is a column vector of height nm^i. It is easy to see that the vector $\xi(t) \otimes^i p$ contains all possible terms of the form

$$\{p^\alpha \xi_j(t) : |\alpha| = i, \quad j = 1, ..., n\},$$

so (5.7) can be rewritten: we can choose row vectors $d_{i,j}$ of length nm^i so that

$$\sum_{\alpha \in I} (p,t)^\alpha c_\alpha \xi(t) = \sum_{j=0}^{\bar{q}} t^j \sum_{i=0}^{q} d_{i,j} (\xi(t) \otimes^i p),$$

which we can estimate using the KEL. Observe that fixing ε and thereby H^ε has the effect of fixing the limits $\bar{q}$ and q and the coefficients $d_{i,j}$.

Remark 5.5 *Perhaps the most problematic feature of this approach is that of obtaining a closed form description of $H(p,t)$ and constructing the approximation $H^{\varepsilon}(p,t)$. As discussed in [25], unless special structure is available, the best approach is a numerical one: grid the $\mathcal{M}$ parameter space, compute the optimal gain at each point on the grid, and then fit a good polynomial approximation to it; unfortunately this will be difficult to do if m or the set of parameter uncertainty is large.*

5.2.4 Applying The KEL

Following Section 5.2.1, envision setting

$$u(t) = 0, \quad t \in [kT, kT + 2h_m),$$

so it follows from the KEL that a good estimate of $\xi[kT]$ is given by

$$\begin{aligned} \mathrm{Est}\big\{\xi[kT]\big\} &= \mathrm{Est}\big\{\xi[kT] \otimes^0 p\big\} \\ &:= \mathrm{argmin}\Big\{ \big\| \left[\begin{array}{cc} I_n & 0 \end{array}\right] X_m(h)^{-1} S_m^{-1} \mathcal{Y}(kT) \big\|, \\ &\qquad\qquad \big\| \left[\begin{array}{cc} I_n & 0 \end{array}\right] X_m(h)^{-1} S_m^{-1} \mathcal{Y}(kT + h_m) \big\| \Big\} \\ &= \xi[kT] + \mathcal{O}(h)\xi[kT], \end{aligned}$$

with the last equality holding if there is no plant switch on $[kT, kT + 2h_m)$. To estimate terms of the form

$$\xi[kT] \otimes^i p, \quad i = 1, ..., q,$$

recall that $\xi[kT] \otimes^i p$ are column vectors of height np^i. With $\rho > 0$ a scaling factor, set

$$u(t) = \begin{cases} \rho \mathrm{Est}\big\{\xi[kT]\big\}_1 & t \in [kT + 2h_m, kT + 4h_m), \\ \quad\vdots & \quad\vdots \\ \rho \mathrm{Est}\big\{\xi[kT]\big\}_n & t \in [kT + 2nh_m, kT + 2(n+1)h_m). \end{cases}$$

It follows from the KEL and the discussion of Section 5.2.1 that we should define

$$\begin{aligned} \mathrm{Est}\big\{p\xi_i[kT]\big\} &:= \frac{1}{\rho} \left[\begin{array}{cc} 0 & I_p \end{array}\right] \times \\ &\quad \mathrm{argmin}\{\|X_m(h)^{-1} S_m^{-1}[\mathcal{Y}(kT + 2h_m) - \mathcal{Y}(kT)]\|, \\ &\qquad\qquad \|X_m(h)^{-1} S_m^{-1}[\mathcal{Y}(kT + 3h_m) - \mathcal{Y}(kT + h_m)]\|\} \\ &= p\xi_i(kT) + \mathcal{O}(h)\xi[kT], \qquad i = 1, .., n, \end{aligned}$$

with the last equality holding if there is no plant switch on $[kT, kT + 2h_m)$. By stacking these estimates we can obtain an estimate of $\xi[kT] \otimes p$, which we label $\mathrm{Est}\big\{\xi[kT] \otimes p\big\}$. We can then estimate $\xi[kT] \otimes^2 p$ in an analogous way (by probing with successive elements of $\mathrm{Est}\big\{\xi[kT] \otimes p\big\}$); since $\xi[kT] \otimes p$ is of dimension

$$n_1 := nm,$$

this will take n_1 experiments, each of length $2h_m$, yielding a total of $2n_1h_m$ units of time. This can be repeated in a similar fashion to yield estimates of $\xi[kT] \otimes^i p$, $i = 3, ..., q$, with the i^{th} term taking

$$2n_{i-1}h_m := 2nm^{i-1}h_m$$

units of time. In this fashion, we can construct a good estimate of

$$H^\varepsilon(p, t - kT)\xi[kT] = \sum_{j=0}^{\bar{q}} (t - kT)^j \sum_{i=0}^{q} d_{i,j}(\xi[kT] \otimes^i p)$$

to be applied during the Control Phase.

5.2.5 A More General From

As mentioned above, it is often difficult to obtain the polynomial H^ε and the coefficients $d_{i,j}$; as the order of the required polynomial increases, the problem becomes even more difficult. To that end, we now discuss a more general form for polynomial which can, if there is sufficient structure in the problem, have a lower order than H^ε (we illustrate this in the second example of Section 5.6).

If we choose $W \in \mathbf{R}^{n\times(m+1)}$ to be such that $WO_m(C, A_p)$ is invertible (which we can do since $O_m(C, A_p)$ has full column rank) and define

$$w(t) := WO_m(C, A_p)\xi(t), \quad t \geq 0$$

and

$$\bar{H}^\varepsilon(p, \tau) := H^\varepsilon(p, \tau) \left(WO_m(C, A_p)\right)^{-1}, \quad p \in \mathcal{M}, \quad \tau \in [0, T),$$

then

$$\bar{H}^\varepsilon(p, \tau)w[kT] \equiv H^\varepsilon(p, \tau)\xi[kT], \quad p \in \mathcal{M}, \quad \tau \in [0, T), \quad k \in \mathbf{Z}^+.$$

It is sometimes possible to choose W so that $\bar{H}^\varepsilon$ has a lower order than H^ε. To estimate such a polynomial, observe that, by the KEL,

$$\begin{aligned} \text{Est}\big\{w(t_0)\big\} \quad := \quad & \text{argmin}\{\|WX_m(h)^{-1}S_m^{-1}\mathcal{Y}(t_0)\|, \\ & \|WX_m(h)^{-1}S_m^{-1}\mathcal{Y}(t_0 + h_m)\|\} \end{aligned}$$

is a good estimate of $w(t_0)$, which we can feed back into the system (in the same way as we did with $\text{Est}\big\{\xi(t_0)\big\}$ in Section 5.2.4) to obtain the required components to construct a good estimate of the polynomial

$$\bar{H}^\varepsilon(p, t - kT)w[kT] = \sum_{j=0}^{\bar{q}} (t - kT)^j \sum_{i=0}^{\hat{q}} \bar{d}_{i,j}(w[kT] \otimes^i p),$$

which we can use as our control signal in the Control Phase.

In an effort to keep our proofs simpler and easier to follow, we do not present our controller using this general form. That being said, given the definition of w and the fact that $\tilde{\mathcal{P}}$ is compact (and therefore $WO_m(C, A_p)$ and its inverse are uniformly bounded), it is reasonable to expect that the general form would yield the same results.

5.3 The Controller

This proposed control law is periodic of period T; we begin by describing its open loop behaviour on a single period. To proceed, we require several definitions. First, we define certain important points in time:

$$T_1 := 2h_m = \text{ the time to estimate } \xi[kT],$$

$$T_{i+1} = T_i + 2n_{i-1}h_m = \text{ the time to estimate } p \otimes^i \xi[kT], \quad i = 1, ..., q.$$

The idea is that on the interval $[kT, kT + T_1)$ we estimate $\xi[kT]$, while on the interval $[kT + T_i, kT + T_{i+1})$ we estimate $\xi[kT] \otimes^i p$. Note that, from our earlier definition of T', we have

$$T_{q+1} = 2T'.$$

Our last important time is the period T, which we require to be an integer multiple of h; on the interval $[kT + 2T', (k + 1)T)$ we implement the Control Phase. To reduce clutter, we define

$$\bar{T}(T_s) := \min\{T_s/2, T_{max}\}.$$

Finally, with $i \in \mathbf{N}$, we let the matrix $V_i(h) \in \mathbf{R}^{n_i \times (n_i + n_{i-1})}$ consist of n_{i-1} copies of $\begin{bmatrix} O & I_m \end{bmatrix} X_m(h)^{-1} S_m^{-1}$ arranged in a block diagonal form.

If the plant was fixed, say (A_p, B_p, C), then it would be natural to define our estimates via

$$\begin{aligned} \mathrm{Est}\big\{\xi[kT] \otimes^0 p\big\} \quad := \quad & \mathrm{argmin}\{\| \begin{bmatrix} I_n & 0 \end{bmatrix} X_m(h)^{-1} S_m^{-1} \mathcal{Y}(kT) \|, \\ & \| \begin{bmatrix} I_n & 0 \end{bmatrix} X_m(h)^{-1} S_m^{-1} \mathcal{Y}(kT + h_m) \| \}. \end{aligned}$$

and, with $i = 1, ..., q$ and $j = 1, ..., n_{i-1}$,

$$\begin{aligned} \mathrm{Est}\big\{\xi[kT] \otimes^i p\big\} := \frac{1}{\rho} \mathrm{argmin} & \left\{ \left\| V_i(h) \begin{bmatrix} \mathcal{Y}(kT + T_i) - \mathcal{Y}(kT) \\ \vdots \\ \mathcal{Y}(kT + T_{i+1} - 2h_m) - \mathcal{Y}(kT) \end{bmatrix} \right\|, \right. \\ & \left. \left\| V_i(h) \begin{bmatrix} \mathcal{Y}(kT + T_i + h_m) - \mathcal{Y}(kT + h_m) \\ \vdots \\ \mathcal{Y}(kT + T_{i+1} - h_m) - \mathcal{Y}(kT + h_m) \end{bmatrix} \right\| \right\}; \end{aligned}$$

however, in general, our plant will not be fixed. To that end, we recall that the switching signal $\sigma(t)$ takes on the value in $\mathcal{M}$ corresponding to the active plant at time t and instead we define

$$\begin{aligned} \mathrm{Est}\big\{\xi[kT] \otimes^0 \sigma[kT]\big\} \quad := \quad & \mathrm{argmin}\{\| \begin{bmatrix} I_n & 0 \end{bmatrix} X_m(h)^{-1} S_m^{-1} \mathcal{Y}(kT) \|, \\ & \| \begin{bmatrix} I_n & 0 \end{bmatrix} X_m(h)^{-1} S_m^{-1} \mathcal{Y}(kT + h_m) \| \}. \end{aligned}$$

and, with $i = 1, ..., q$ and $j = 1, ..., n_{i-1}$,

$$\text{Est}\big\{\xi[kT] \otimes^i \sigma[kT]\big\} :=$$
$$\frac{1}{\rho}\text{argmin}\left\{\left\| V_i(h)\begin{bmatrix} \mathcal{Y}(kT+T_i) - \mathcal{Y}(kT) \\ \vdots \\ \mathcal{Y}(kT+T_{i+1}-2h_m) - \mathcal{Y}(kT) \end{bmatrix}\right\|, \left\| V_i(h)\begin{bmatrix} \mathcal{Y}(kT+T_i+h_m) - \mathcal{Y}(kT+h_m) \\ \vdots \\ \mathcal{Y}(kT+T_{i+1}-h_m) - \mathcal{Y}(kT+h_m) \end{bmatrix}\right\|\right\}.$$

We can now write down our proposed controller $\mathcal{C}$, which we relabel $\mathcal{C}(\varepsilon, T, T')$ to emphasize its dependence on T, T', and ε, presented in open loop form and given in three phases:

THE PROPOSED CONTROLLER $\mathcal{C}(\varepsilon, T, T')$

With $T_s > 0$, $T \in (0, \bar{T}(T_s))$, $T' \in (0, T/2)$, $\varepsilon \in (0, \bar{\varepsilon})$, and $k \in \mathbf{Z}^+$ we define the controller by

Stage 1 - State Estimation Phase: $[kT, kT + T_1)$

$$u(t) = 0, \ \ t \in [kT, kT + T_1) = [kT, kT + 2h_m),$$

Stage 2 - Control Estimation Phase: $[kT + T_1, kT + 2T')$

$$u(t) = \rho\text{Est}\big\{\xi[kT] \otimes^{i-1} \sigma[kT]\big\}_j,$$
$$t \in [kT + T_i + 2(j-1)h_m, kT + T_i + 4(j-1)h_m),$$
$$j = 1, ..., n_{i-1}, \quad i = 1, ..., q,$$

Stage 3 - Control Phase: $[kT + 2T', (k+1)T)$

$$u(t) = \sum_{j=0}^{\bar{q}} (t - kT)^j \sum_{i=0}^{q} d_{i,j}\text{Est}\big\{\xi[kT] \otimes^i \sigma[kT]\big\}, \quad t \in [kT + 2T', (k+1)T).$$

Remark 5.6 *Recall that $\mathcal{C}(\varepsilon, T, T')$ can be described by the state-space representation (5.4). Since the controller is nonlinear, there is no guarantee that the system will be well posed; however, it is routine to prove that, for every choice of $p \in \mathcal{M}$, when $\mathcal{C}(\varepsilon, T, T')$ is applied to (A_p, B_p, C), every $\xi_0 \in \mathbf{R}^n$ and $z_0 \in \mathbf{R}^l$ yields a unique solution.*

At this point we examine the behaviour of the closed loop system over a single period. We will consider two cases: one in which there is no plant switch on the interval and one in which there is at most one such switch. We remind the reader of the switching notation outlined in Section 2.2: recall that switches are confined to the times $\{t_l\}$ *excluding* $t_0 = 0$ and that we define

$$k_l := \left\lfloor \frac{t_l}{T} \right\rfloor, \quad l \in \mathbf{Z}^+,$$

so there are **no** plant switches on intervals of the form

$$[kT, (k+1)T), \quad k \in \{k_l : l \in \mathbf{N}\}^c;$$

furthermore, recall that (for simplicity) we insist that $\sigma \in \Sigma_{T_s}$ be continuous from the right. With this in hand, we can state our result:

Proposition 5.2 *With $T_s > 0$ and $\varepsilon \in (0, \bar{\varepsilon})$, there exists a constant $\gamma > 0$ so that:*

(i) For every $T \in (0, \bar{T}(T_s))$, if T' is sufficiently small, then, for every $\xi_0 \in \mathbf{R}^n$, and $\sigma \in \Sigma_{T_s}$, if the controller $\mathcal{C}(\varepsilon, T, T')$ is attached to the plant P_σ then, for every $k \in \{k_l : l \in \mathbf{N}\}^c$, we have that

(a) $\|\xi(t) - \Phi^{\varepsilon}_{\sigma(t)}(t - kT, 0)\xi[kT]\| \le \gamma T' \|\xi[kT]\|, \quad t \in [kT, (k+1)T),$

(b) $|u(t)| \le \gamma \|\xi[kT]\|, \qquad t \in [kT, kT + 2T'),$ *and*

$$|u(t) - H^{\varepsilon}(\sigma(t), t)\xi[kT]| \le \gamma T' \|\xi[kT]\|, \quad t \in [kT + 2T', (k+1)T).$$

(ii) There exists a constant $\bar{T}_1 \in (0, \bar{T}(T_s))$ so that, for every $T \in (0, \bar{T}_1)$, if T' is sufficiently small, then, for every $\xi_0 \in \mathbf{R}^n$, $\sigma \in \Sigma_{T_s}$, and $k \in \mathbf{Z}^+$, when the controller $\mathcal{C}(\varepsilon, T, T')$ is attached to the plant P_σ, we have that

(a) $\|\xi(t) - \xi(kT)\| \le \gamma T \|\xi[kT]\|, \quad t \in [kT, (k+1)T),$

(b) $|u(t)| \le \gamma \|\xi(kT)\|, \quad t \in [kT, (k+1)T).$

Proof: Please see Appendix C.

The first part of this proposition states that the actual state ξ behaves much like ξ^{ε}, at least over periods in which there is no switch; furthermore, this match improves as T' gets smaller. The second part of the proposition says that, in all periods, even ones with a switch, we can find a nice bound on the size of the state and the input over that period, but the bound requires that T be small and T' be even smaller.

It will turn out that, over intervals in which there are no switches, ξ exhibits exponential decay over the entire time interval, which will be useful when proving all of our major theorems. To that end, we state the following technical lemma:

Lemma 5.3 *(Boundedness Lemma) With $T_s > 0$ and $\varepsilon \in (0, \bar{\varepsilon})$, for every $T \in (0, \bar{T}(T_s))$, there exists a constant $\varepsilon_\lambda \in (\lambda_0, 0)$ so that, for every $\lambda \in (\lambda_0, \varepsilon_\lambda)$, there exists a constant $\gamma(T, \lambda) > 0$ so that, for every sufficiently small T' and every $\sigma \in \Sigma_{T_s}$, if the controller $\mathcal{C}(\varepsilon, T, T')$ is attached to the plant P_σ then, for every interval of the form $[\underline{k}T, \bar{k}T)$ over which there are* **no** *switches, we have that*

$$\|\xi(t)\| \leq \gamma(T, \lambda) e^{\lambda(t - \underline{k}T)} \|\xi[\underline{k}T]\|, \quad t \in [\underline{k}T, \bar{k}T], \quad \xi_0 \in \mathbf{R}^n;$$

furthermore,

$$\lim_{T \to 0} \gamma(T, \lambda) = \gamma_0.$$

Proof: Please see Appendix C.

The above lemma says that we can make $\gamma(T, \lambda) e^{\lambda(t - \underline{k}T)}$ as close as we wish to $\gamma_0 e^{\lambda_0 (t - \underline{k}T)}$, which will be very useful when we are proving stability in the face of plant switches. When there are no switches, we do not need this level of generality; to that end, we present the following Corollary:

Corollary 5.1 *With $T \in (0, T_{max})$ and $\varepsilon \in (0, \bar{\varepsilon})$, there exist constants $\lambda \in (\lambda_0, 0)$ and $\gamma > 0$ so that, for every $p \in \mathcal{M}$ and sufficiently small T', if the controller $\mathcal{C}(\varepsilon, T, T')$ is attached to the plant P_p then*

$$\|\xi(t)\| \leq \gamma e^{\lambda t} \|\xi_0\|, \quad t \geq 0, \quad \xi_0 \in \mathbf{R}^n.$$

Proof: This result follows directly from Lemma 5.3 and is left to the reader. ■

Finally, we look at a useful property of the controller's state-space representation:

Lemma 5.4 *The control law $\mathcal{C}(\varepsilon, T, T')$ has a representation of the form (5.4) given by (G, J, h, ℓ) which is deadbeat in the following sense:*

$$G(0, z, y) = G(0, 0, y).$$

Proof: Please see Appendix C.

Henceforth, when we discuss properties provided by the controller $\mathcal{C}(\varepsilon, T, T')$, we implicitly mean properties provided by (G, J, h, ℓ).

5.4 Asymptotic Stability

In this section we will investigate asymptotic stability; since this problem is significantly more complex than our previous ones, for time reasons we will not investigate I/O stability. Recall that our estimation process now requires active probing. The effect of this is twofold: first, for the estimates to be good, we require fast sampling, and second, to keep the adverse effects of these probes to a minimum, we require the duration of the Estimation Phase to be small. To this end, unlike in previous chapters, even when there are no switches, we cannot prove stability for arbitrarily large T' (we can still allow T to be large).

Theorem 5.1 *For every $T \in (0, T_{max})$ and $\varepsilon \in (0, \bar{\varepsilon})$, if $T' > 0$ is sufficiently small, then the controller $\mathcal{C}(\varepsilon, T, T')$ asymptotically stabilizes $\mathcal{P}$.*

Proof:

Fix $T \in (0, T_{max})$ and $\varepsilon \in (0, \bar{\varepsilon})$ and let $p \in \mathcal{M}$ and $\xi_0 \in \mathbf{R}^n$ be arbitrary. Observe that, by Lemma 5.4 we have that $G(0, z, y) = G(0, 0, y)$; therefore, since the controller is periodic, if $y(t) \to 0$ then $z[k] \to 0$ as well. By the structure of $\mathcal{C}(\varepsilon, T, T')$ we have that u is bounded by ξ; furthermore, $y = C\xi$. The upshot of all of this is that, to prove stability, it is enough to show that $\xi(t) \to 0$ as $t \to \infty$. To that end, we now invoke Corollary 5.1: there exist constants $\bar{\lambda} \in (\lambda_0, 0)$ and $\bar{\gamma} > 0$ so that, for every sufficiently small T' we have

$$\|\xi(t)\| \leq \bar{\gamma} e^{\bar{\lambda} t} \|\xi_0\|, \quad t \geq 0,$$

providing the desired bound; furthermore, clearly

$$\begin{aligned} \lim_{t \to \infty} \|\xi(t)\| &= \lim_{t \to \infty} \left(\bar{\gamma} e^{\bar{\lambda} t} \|\xi_0\| \right) \\ &= 0. \end{aligned}$$

■

We now allow for persistent plant changes and show that our controller is stabilizing under the condition that switches occur slowly enough. We suspect that, as in the previous two chapters, we could (loosely) bound T_s below and then T above in such a way that, as T_s gets big, so does the bound on T. Since the analysis here is significantly more complex and we wish to leverage Proposition 5.2(ii) (which requires that T be small) instead, we will force T to be small and use the tighter lower bound on T_s.

To motivate such a bound, consider the following simplified example. We have two LTI plants, P_p and $P_{\bar{p}}$, and the time-varying plant P_σ switches back and forth between them, spending τ_1 time units at P_p and τ_2 time units at $P_{\bar{p}}$ before repeating:

for every $k \in \mathbf{Z}^+$,

$$P_{\sigma(t)} = \begin{cases} P_p, & t = [k(\tau_1+\tau_2), k(\tau_1+\tau_2)+\tau_1) \\ P_{\bar{p}}, & t = [k(\tau_1+\tau_2)+\tau_1, (k+1)(\tau_1+\tau_2)) \end{cases}$$

Stability is clearly dictated by the matrix

$$e^{\bar{A}_p\tau_1}e^{\bar{A}_{\bar{p}}\tau_2}; \tag{5.8}$$

the closed loop system is stable iff the eigenvalues of this matrix all lie within the open unit disk, with a sufficient condition being that

$$\|e^{\bar{A}_p\tau_1}e^{\bar{A}_{\bar{p}}\tau_2}\| < 1. \tag{5.9}$$

Of course, if one knows in advance that the time-varying plant is as indicated, then one can always stabilize the system with a more cleverly designed controller even if (5.8) has eigenvalues outside the open unit disk; unfortunately, in our case no such a priori information is available.

This brings us to the general case. Using (5.9) as motivation, a sufficient condition for stability should be

$$\sup_{p,\bar{p}\in\mathcal{M}}\ \sup_{\tau_i > T_s} \|e^{\bar{A}_p\tau_1}e^{\bar{A}_{\bar{p}}\tau_2}\| < 1. \tag{5.10}$$

To simplify this condition, observe that Proposition 5.1(iii) says that

$$\|e^{\bar{A}_p t}\| \le \gamma_0 e^{\lambda_0 t}, \ \ t \ge 0, \ \ p \in \mathcal{M},$$

so (5.10) holds if

$$\gamma_0 e^{\lambda_0 T_s} < 1$$

or equivalently if

$$T_s > \frac{\ln(\gamma_0)}{|\lambda_0|} =: \underline{T_s}.$$

Theorem 5.2 *With $\varepsilon \in (0,\bar{\varepsilon})$, if $T_s > \underline{T_s}$, then there exists a constant $\bar{T} > 0$ so that, for every $T \in (0,\bar{T})$, if T' is sufficiently small, then the controller $\mathcal{C}(\varepsilon,T,T')$ stabilizes $\mathcal{P}_{T_s}$.*

Proof:

Fix $\varepsilon \in (0,\bar{\varepsilon})$ and $T_s > \underline{T_s}$ and let $\xi_0 \in \mathbf{R}^n$, $\sigma \in \Sigma_{T_s}$, and $T \in (0, T_s/2)$ be arbitrary. As in the proof of Theorem 5.1, observe that, by Lemma 5.4 we have that $G(0,z,y) = G(0,0,y)$; therefore, since the controller is periodic, if $y(t) \to 0$ then $z[k] \to 0$ as well. By the structure of $\mathcal{C}(\varepsilon,T,T')$ we have that u is bounded by ξ; furthermore, $y = C\xi$. The upshot of all of this is that, to prove stability, it is enough to show that $\xi(t) \to 0$ as $t \to \infty$.

Before we proceed, we invoke Lemma 5.3: there exists a constant $\varepsilon_\lambda \in (\lambda_0, 0)$ such that, for every $\lambda \in (\lambda_0, \varepsilon_\lambda)$ there exist constants $\bar{T}'(T, \lambda) > 0$ and $\gamma(T, \lambda) > 0$ so that, for every $T' \in (0, \bar{T}'(T, \lambda))$ we have

$$\|\xi(t)\| \leq \gamma(T,\lambda)e^{\lambda(t-(k_l+1)T)}\|\xi[(k_l+1)T]\|, \quad t \in [(k_l+1)T, k_{l+1}T], \quad l \in \mathbf{N} \tag{5.11}$$

and the special case[5]

$$\|\xi(t)\| \leq \gamma(T,\lambda)e^{\lambda t}\|\xi_0\|, \quad t \in [0, k_1 T), \tag{5.12}$$

At this point we fix $\lambda \in (\lambda_0, \varepsilon_\lambda)$ so that it satisfies

$$T_s > \frac{\ln(\gamma_0)}{|\lambda|} > \frac{\ln(\gamma_0)}{|\lambda_0|} = \underline{T_s} \tag{5.13}$$

Since λ is now fixed, we drop it from $\bar{T}'(T, \lambda)$ and $\gamma(T, \lambda)$ and simply write $\bar{T}'(T)$ and $\gamma(T)$.

We now pause for a moment to discuss the structure of the remainder of this proof. Observe that it is sufficient to prove that, for every sufficiently small $T > 0$ there exists a constant $\bar{T}'(T) > 0$ so that, for every $T' \in (0, \bar{T}'(T))$ we have asymptotic stability. We have just established the existence of a constant $\bar{T}'$ which will turn out to be sufficient to achieve the desired objective; we let $T' \in (0, \bar{T}'(T))$ be arbitrary and devote the remainder of this proof to showing that we obtain the desired result for every sufficiently small T.

As usual, we begin by investigating the first interval $[0, k_1 T)$. Observe that, if $t_1 < T$ then $k_1 = 0$ and there is nothing to prove, so we assume that this is not the case. From (5.12) we know that nothing untoward happens in this initial period, furthermore, since ξ is continuous, for all cases of k_1, we have that

$$\|\xi[k_1 T]\| \leq \gamma(T)e^{\lambda k_1 T}\|\xi_0\|. \tag{5.14}$$

We now investigate $t > k_1 T$. The bound on T_s is designed to ensure that there is enough time for the system to recover from the adverse affect of a plant switch before the next switch occurs. To that end, we proceed by finding an update equation for $\|\xi[k_{1+1}T]\|$ in terms of $\|\xi[k_l T]\|$ and then show that, because of the bound on T_s, the state at the endpoints $\|\xi[k_l T]\|$ tends to zero as l tends to infinity; we then conclude by showing that, between these endpoints, the state is well behaved.

Obtaining the desired update equation requires two steps. First, we observe that (5.11) gives us a nice bound over intervals with no switch; in particular, it says that

$$\|\xi[k_{l+1}T]\| \leq \gamma(T)e^{\lambda(k_{l+1}-(k_l+1))T}\|\xi[(k_l+1)T]\|, \quad l \in \mathbf{N}.$$

[5]Observe that, if $t_1 < T$, then the interval of interest is empty, so implicitly, this inequality only applies if $t_1 > T$.

The second step is to obtain a bound on $\|\xi[(k_l+1)T]\|$, which we do via Proposition 5.2(ii)(a), which says that, if $T>0$ is sufficiently small, then there exists a constant $\alpha_1(T)>0$ such that

$$\|\xi(t)\| \leq \alpha_1(T)\|\xi[k_lT]\|, \quad t\in[k_lT,(k_l+1)T], \quad l\in\mathbf{N}, \tag{5.15}$$

in particular we have

$$\|\xi[(k_l+1)T]\| \leq \alpha_1(T)\|\xi[k_lT]\|, \quad l\in\mathbf{N};$$

furthermore, observe that, since ξ is continuous, we clearly have that

$$\lim_{T\to 0}\alpha_1(T)=1.$$

We now combine this with our first step to find that, for every sufficiently small $T>0$, we have

$$\|\xi[k_{l+1}T]\| \leq \gamma(T)\alpha_1(T)e^{\lambda(k_{l+1}-(k_l+1))T}\|\xi[k_lT]\|, \quad l\in\mathbf{N}.$$

Since

$$t_{l+1}-t_l<T_s, \quad l\in\mathbf{N},$$

we have

$$\|\xi[k_{l+1}T]\| \leq \gamma(T)\alpha_1(T)e^{-2\lambda T}e^{\lambda T_s}\|\xi[k_lT]\|, \quad l\in\mathbf{N}, \tag{5.16}$$

so, solving iteratively yields

$$\|\xi[k_lT]\| \leq \left(\gamma(T)\alpha_1(T)e^{-2\lambda T}e^{\lambda T_s}\right)^{l-1}\|\xi[k_1T]\|, \quad l\in\mathbf{N}$$

and, using (5.14), (5.11), and (5.15), for every sufficiently small $T>0$, we have

$$\|\xi(t)\| \leq \gamma^2(T)\alpha_1(T)\left(\gamma(T)\alpha_1(T)e^{-2\lambda T}e^{\lambda T_s}\right)^{l-1}\|\xi_0\|,\ t\in[k_lT,k_{l+1}T),\ l\in\mathbf{N}. \tag{5.17}$$

Clearly, if

$$\gamma(T)\alpha_1(T)e^{-2\lambda T}e^{\lambda T_s}<1$$

then

$$\lim_{l\to 0}\left(\gamma(T)\alpha_1(T)e^{-2\lambda T}e^{\lambda T_s}\right)^{l-1}=0,$$

and therefore we have the desired limit

$$\lim_{t\to 0}\|\xi(t)\|=0;$$

furthermore, if we combine (5.17) with the bound on the interval $[0,k_1T]$ given by (5.14), then we have the desired bound on the entire interval

$$\|\xi(t)\| \leq \max\{\gamma^2(T)\alpha_1(T),\gamma(T)\}\|\xi_0\|, \quad t\geq 0.$$

It remains to show that our assumption on T_s ensures that, for every sufficiently small T and every $T' \in (0, \bar{T}'(T))$, we have

$$\gamma(T)\alpha_1(T)e^{-2\lambda T}e^{\lambda T_s} < 1.$$

To do so, observe that (5.13) yields

$$\gamma(T)\alpha_1(T)e^{-2\lambda T}e^{\lambda T_s} < \gamma(T)\alpha_1(T)e^{-2\lambda T}\frac{1}{\gamma_0},$$

and furthermore, from Lemma 5.3 we have

$$\lim_{T\to 0}\gamma(T) = \gamma_0,$$

so

$$\begin{aligned}\lim_{T\to 0}\left(\gamma(T)\alpha_1(T)e^{-2\lambda T}e^{\lambda T_s}\right) &< \lim_{T\to 0}\left(\gamma(T)\alpha_1(T)e^{-2\lambda T}\frac{1}{\gamma_0}\right)\\ &= 1;\end{aligned}$$

therefore, for every sufficiently small T and every $T' \in (0, \bar{T}'(T))$, we have

$$\gamma(T)\alpha_1(T)e^{-2\lambda T}e^{\lambda T_s} < 1.$$

■

5.5 Performance

We now turn to the question of performance. In this chapter, we will only investigate the case where there is no plant switching; here we find that, for arbitrary T, we can get as close as we wish to the optimal trajectories, provided that we make both ε and T' small.

Theorem 5.3 *For every $T \in (0, T_{max})$ and $\delta > 0$, if $\varepsilon \in (0, \bar{\varepsilon})$ and $T' \in (0, T/2)$ are sufficiently small, then $\mathcal{C}(\varepsilon, T, T')$ asymptotically stabilizes $\mathcal{P}$ and, when $\mathcal{C}(\varepsilon, T, T')$ is attached to any plant $P \in \mathcal{P}$, the closed loop system response satisfies*

$$\left\| \begin{bmatrix} y \\ R^{1/2}u \end{bmatrix} - \begin{bmatrix} y^0 \\ R^{1/2}u^0 \end{bmatrix} \right\|_2^2 \leq \delta\|\xi_0\|^2, \quad \xi_0 \in \mathbf{R}^n.$$

Proof:

We begin by fixing $\delta > 0$ and $T \in (0, T_{max})$. Let $\xi_0 \in \mathbf{R}^n$, $p \in \mathcal{M}$, and $T' \in (0, T/2)$ be arbitrary. From Proposition 5.1(iv), for every $\varepsilon \in (0, \bar{\varepsilon})$, we have that

$$\left\| \begin{bmatrix} y^\varepsilon \\ R^{1/2}u^\varepsilon \end{bmatrix} - \begin{bmatrix} y^0 \\ R^{1/2}u^0 \end{bmatrix} \right\|_2^2 \leq \varepsilon^2\gamma_0^2\|\xi_0\|^2,$$

so fixing

$$\varepsilon \in \left(0, \min\left\{\sqrt{\frac{\delta}{2}}\frac{1}{\gamma_0}, \bar{\varepsilon}\right\}\right)$$

yields

$$\left\|\begin{bmatrix} y^\varepsilon \\ R^{1/2}u^\varepsilon \end{bmatrix} - \begin{bmatrix} y^0 \\ R^{1/2}u^0 \end{bmatrix}\right\|_2^2 \leq \frac{\delta}{2}\|\xi_0\|^2; \tag{5.18}$$

therefore,

$$\begin{aligned}
\left\|\begin{bmatrix} y \\ R^{1/2}u \end{bmatrix} - \begin{bmatrix} y^0 \\ R^{1/2}u^0 \end{bmatrix}\right\|_2^2 \;&\leq\; \left\|\begin{bmatrix} y \\ R^{1/2}u \end{bmatrix} - \begin{bmatrix} y^\varepsilon \\ R^{1/2}u^\varepsilon \end{bmatrix}\right\|_2^2 + \\
&\quad\; \left\|\begin{bmatrix} y^\varepsilon \\ R^{1/2}u^\varepsilon \end{bmatrix} - \begin{bmatrix} y^0 \\ R^{1/2}u^0 \end{bmatrix}\right\|_2^2 \\
&\leq\; \left\|\begin{bmatrix} y \\ R^{1/2}u \end{bmatrix} - \begin{bmatrix} y^\varepsilon \\ R^{1/2}u^\varepsilon \end{bmatrix}\right\|_2^2 + \frac{\delta}{2}\|\xi_0\|^2.
\end{aligned}$$

Stability now follows immediately from Theorem 5.1. It remains to show that, for sufficiently small T', we have

$$\left\|\begin{bmatrix} y \\ R^{1/2}u \end{bmatrix} - \begin{bmatrix} y^\varepsilon \\ R^{1/2}u^\varepsilon \end{bmatrix}\right\|_2^2 \leq \frac{\delta}{2}\|\xi_0\|^2;$$

the remainder of this proof is devoted to this.

Since $y = C\xi$, it suffices to show that $\|u - u^\varepsilon\|_2^2$ and $\|\xi - \xi^\varepsilon\|_2^2$ are both bounded by (a scaled version of) $T'\|\xi_0\|^2$, we can then obtain the desired result by making T' small. To that end, we define

$$\tilde{\xi}^\varepsilon := \xi^\varepsilon - \xi$$

and

$$\tilde{u}^\varepsilon := u^\varepsilon - u.$$

We begin with $\|\tilde{\xi}\|_2^2$:

Claim 1: There exists a constant $\gamma_1 > 0$ such that, if $T' > 0$ is sufficiently small, then we have

$$\|\tilde{\xi}^\varepsilon\|_2^2 \leq \gamma_1 T'\|\xi_0\|^2.$$

Proof:

We start by investigating the period endpoints. Using Proposition 5.2(i)(a), it is easy to show that, for every sufficiently small $T' > 0$, there exists a function $\Delta_p[k] \in \mathcal{PC}_\infty$ and a constant $\alpha_1 > 0$[6] such that

$$\|\Delta_p[k]\| \leq \alpha_1 T'\|\xi[kT]\|, \quad k \in \mathbf{Z}^+, \quad p \in \mathcal{M}$$

[6]We will invoke Proposition 5.2 many times in this proof. To reduce notation, we will use α_1 as our constant each time.

and

$$\tilde{\xi}^\varepsilon[(k+1)T] = \Phi_p^\varepsilon(T,0)\tilde{\xi}^\varepsilon[kT] + \Delta_p[k], \quad k \in \mathbf{Z}^+$$

whose solution satisfies

$$\tilde{\xi}^\varepsilon[kT] = \sum_{i=0}^{k-1} (\Phi_p^\varepsilon(T,0))^{k-1-i}\Delta_p[i], \quad k \in \mathbf{Z}^+.$$

Using Proposition 5.1(iii) to bound $\|(\Phi_p^\varepsilon(T,0))^k\|$ together with our bound on $\|\Delta\|$ yields

$$\|\tilde{\xi}^\varepsilon[kT]\| \le \sum_{i=0}^{k-1} \gamma_0 e^{\lambda_0(k-1-i)T}\alpha_1 T'\|\xi[iT]\|, \quad k \in \mathbf{Z}^+. \tag{5.19}$$

If we invoke Corollary 5.1, then we find that there exist constants $\bar{\lambda} \in (\lambda_0, 0)$ and $\bar{\gamma} > 0$ so that, for every sufficiently small T' we have

$$\|\xi(t)\| \le \bar{\gamma}e^{\bar{\lambda}t}\|\xi_0\|, \quad t \ge 0; \tag{5.20}$$

applying this to (5.19) yields

$$\begin{aligned}
\|\tilde{\xi}^\varepsilon[kT]\| &\le \sum_{i=0}^{k-1} \gamma_0 e^{\lambda_0(k-1-i)T}\alpha_1 T'\bar{\gamma}e^{\bar{\lambda}iT}\|\xi_0\| \\
&= \sum_{i=0}^{k-1} \gamma_0 e^{\lambda_0 iT}\alpha_1\bar{\gamma}T' e^{\bar{\lambda}(k-1-i)T}\|\xi_0\| \\
&= \gamma_0\alpha_1\bar{\gamma}T' e^{\bar{\lambda}(k-1)T}\sum_{i=0}^{k-1} e^{(\lambda_0-\bar{\lambda})iT}\|\xi_0\| \\
&\le \underbrace{\frac{\gamma_0\alpha_1\bar{\gamma}e^{-\bar{\lambda}T}}{1-e^{(\lambda_0-\bar{\lambda})T}}}_{=:\alpha_2} T' e^{\bar{\lambda}kT}\|\xi_0\|, \quad k \in \mathbf{Z}^+.
\end{aligned} \tag{5.21}$$

We now investigate the behaviour inside the period. Returning to Proposition 5.2(i)(a), it is also easy to show that, for every sufficiently small $T' > 0$, we have

$$\|\tilde{\xi}^\varepsilon(t) - (\Phi_p^\varepsilon(t-kT,0))\tilde{\xi}^\varepsilon[kT]\| \le \alpha_1 T'\|\xi[kT]\|, \quad t \in [kT,(k+1)T), \quad k \in \mathbf{Z}^+.$$

Since T and ε are fixed, $\|H^\varepsilon\|_\infty$ is well defined; furthermore, it is straightforward to show that

$$\|\Phi_p^\varepsilon(t,0)\| \le \underbrace{e^{aT} + Te^{aT}b\|H^\varepsilon\|_\infty}_{=:\gamma_\phi}, \quad t \in [0,T]$$

so

$$\|\tilde{\xi}^\varepsilon(t)\| \le \gamma_\phi\|\tilde{\xi}^\varepsilon[kT]\| + \alpha_1 T'\|\xi[kT]\|, \quad t \in [kT,(k+1)T), \quad k \in \mathbf{Z}^+.$$

Using (5.21) to bound $\|\tilde{\xi}^{\varepsilon}[kT]\|$ and (5.20) to bound $\|\xi[kT]\|$ yields (for small T')

$$\begin{aligned}
\|\tilde{\xi}^{\varepsilon}(t)\| &\leq \gamma_{\phi}\alpha_2 T' e^{\bar{\lambda}kT}\|\xi_0\| + \alpha_1 T'\bar{\gamma}e^{\bar{\lambda}t}\|\xi_0\|, \quad t \in [kT,(k+1)T), \quad k \in \mathbf{Z}^+ \\
&= \underbrace{\left(\gamma_{\phi}\alpha_2 e^{-\bar{\lambda}T} + \alpha_1\bar{\gamma}\right)}_{=:\alpha_3} e^{\bar{\lambda}t}T'\|\xi_0\|, \quad t \geq 0,
\end{aligned}$$

which we use to find that, for every sufficiently small T' we have

$$\begin{aligned}
\int_0^{\infty}\|\tilde{\xi}^{\varepsilon}(t)\|^2 dt &\leq \int_0^{\infty}\alpha_3^2 e^{2\bar{\lambda}t}(T')^2\|\xi_0\|^2 dt \\
&= \frac{\alpha_3^2}{2|\bar{\lambda}|}(T')^2\|\xi_0\|^2 \\
&\leq \frac{\alpha_3^2}{2|\bar{\lambda}|}T'\|\xi_0\|^2.
\end{aligned}$$

■

We now turn to $\tilde{u}^{\varepsilon}$:

Claim 2: There exists a constant $\gamma_2 > 0$ such that, for sufficiently small T', we have

$$\|\tilde{u}^{\varepsilon}\|_2^2 \leq \gamma_2 T'\|\xi_0\|^2.$$

Proof:

By definition,

$$\begin{aligned}
\tilde{u}^{\varepsilon}(t+kT) &= H^{\varepsilon}(p,t)\xi^{\varepsilon}[kT] - u(t-kT) \\
&= H^{\varepsilon}(p,t)\xi[kT] + H^{\varepsilon}(p,t)\tilde{\xi}^{\varepsilon}[kT] - u(t-kT), \quad t \in [0,T), \quad k \in \mathbf{Z}^+,
\end{aligned}$$

so, for every $k \in \mathbf{Z}^+$ we have,

$$\begin{aligned}
\int_0^T \|\tilde{u}^{\varepsilon}(t+kT)\|^2 dt &= \int_0^{2T'}\|H^{\varepsilon}(p,t)\xi^{\varepsilon}[kT] - u(t-kT)\|^2 dt + \\
&\quad \int_{2T'}^{T}\|H^{\varepsilon}(p,t)\xi[kT] + H^{\varepsilon}(p,t)\tilde{\xi}^{\varepsilon}[kT] - u(t-kT)\|^2 dt.
\end{aligned}$$

It will be convenient to look at each term on the RHS independently. We start with the first. Since $\|H^{\varepsilon}\|_{\infty}$ is well defined, by the first result of Proposition 5.2(i)(b), for sufficiently small T', we have

$$\begin{aligned}
&\int_0^{2T'}\|H^{\varepsilon}(p,t)\xi^{\varepsilon}[kT] - u(t-kT)\|^2 dt \\
&\qquad \leq 2T'\left(\|H^{\varepsilon}\|_{\infty}\|\xi^{\varepsilon}[kT]\| + \alpha_1\|\xi[kT]\|\right)^2 \\
&\qquad \leq 2T'\left(\|H^{\varepsilon}\|_{\infty}(\Phi_p^{\varepsilon}(T,0))^k\|\xi_0\| + \alpha_1\|\xi[kT]\|\right)^2.
\end{aligned}$$

To bound the second term, we use the second result of Proposition 5.2(i)(b) together with (5.21) to find that, for sufficiently small T', we have

$$\int_{2T'}^{T} \left\| \left[H^{\varepsilon}(p,t)\xi[kT] - u(t-kT) \right] + H^{\varepsilon}(p,t)\tilde{\xi}^{\varepsilon}[kT] \right\|^2 dt \leq$$
$$\underbrace{(T - 2T')}_{\leq T} \left(\alpha_1 T' \|\xi[kT]\| + \|H^{\varepsilon}\|_{\infty} \alpha_2 T' e^{\bar{\lambda} kT} \|\xi_0\| \right)^2.$$

Putting these together and using (5.20) to bound $\|\xi[kT]\|$ and our bound on $\|\Phi_p^{\varepsilon}\|$ yields (for small T')

$$\begin{aligned}
\int_0^T \|\tilde{u}^{\varepsilon}(t+kT)\|^2 dt &\leq \Big[2 \left(\|H^{\varepsilon}\|_{\infty} \gamma_0 e^{\lambda_0 kT} + \alpha_1 \bar{\gamma} e^{\bar{\lambda} kT} \right)^2 + \\
&\qquad T e^{2\bar{\lambda} kT} \left(\alpha_1 \bar{\gamma} + \|H^{\varepsilon}\|_{\infty} \alpha_2 \right)^2 \Big] T' \|\xi_0\|^2 \\
&\leq \underbrace{\left[2 \left(\|H^{\varepsilon}\|_{\infty} \gamma_0 + \alpha_1 \bar{\gamma} \right)^2 + T \left(\alpha_1 \bar{\gamma} + \|H^{\varepsilon}\|_{\infty} \alpha_2 \right)^2 \right]}_{=:\alpha_4} \times \\
&\qquad T' e^{2\bar{\lambda} kT} \|\xi_0\|^2, \quad k \in \mathbf{Z}^+
\end{aligned}$$

so, for every sufficiently small T' we have

$$\begin{aligned}
\int_0^{\infty} \|\tilde{u}^{\varepsilon}(t)\|^2 dt &\leq \sum_{i=0}^{\infty} \alpha_4 T' e^{2\bar{\lambda} kT} \|\xi_0\|^2 \\
&= \frac{\alpha_4}{1 - e^{2\bar{\lambda} T}} T' \|\xi_0\|^2.
\end{aligned}$$

■

Combining Claims 1 and 2, for sufficiently small T', we clearly have that

$$\begin{aligned}
\left\| \begin{bmatrix} y \\ R^{1/2} u \end{bmatrix} - \begin{bmatrix} y^{\varepsilon} \\ R^{1/2} u^{\varepsilon} \end{bmatrix} \right\|_2^2 &\leq \|y - y^{\varepsilon}\|_2^2 + R\|u - u^{\varepsilon}\|_2^2 \\
&\leq \|C\| \times \|\xi - \xi^{\varepsilon}\|_2^2 + R\|u - u^{\varepsilon}\|_2^2 \\
&\leq (\gamma_1 + R\gamma_2) T' \|\xi_0\|^2
\end{aligned}$$

so clearly, for sufficiently small T', we have

$$\left\| \begin{bmatrix} y \\ R^{1/2} u \end{bmatrix} - \begin{bmatrix} y^{\varepsilon} \\ R^{1/2} u^{\varepsilon} \end{bmatrix} \right\|_2^2 \leq \frac{\delta}{2} \|\xi_0\|^2.$$

■

5.6 Examples

Here we will consider two example uncertainty sets, each of which admits the pair of plants $\frac{1}{s-1}$ and $\frac{-1}{s-1}$. In the first example, the uncertainty will be entirely encapsulated at the plant input, yielding the gain margin problem; we present this example so that we have a basis for comparison in Chapter 6. The second example is a reinvestigation of the example of [25]; we will use this example to illustrate how to use the more general form given in Section 5.2.5.

In the following examples, we allow the plant to switch and show that we maintain stability. For comparison purposes, we also include a nominal signal which is defined to be the LQR optimal trajectory for whichever plant is active (as in our previous chapters):

$$\xi^{nominal}(t) := e^{\bar{A}_{\sigma(t)}t}\xi_0, \quad t \geq 0,$$
$$y^{nominal}(t) := C\xi^{nominal}(t), \quad t \geq 0,$$

and

$$u^{nominal}(t) := F_{\sigma(t)}\xi^{nominal}(t), \quad t \geq 0.$$

5.6.1 Example 1 - Gain Margin

Here we consider transfer functions of the form

$$P : \frac{b}{s+a}$$

and the compact set

$$\mathcal{P} = \{P : (a, b) \in (1, [-1, -1/2] \cup [1/2, 1])\};$$

observe that this is the gain margin problem (with gain margin 1/2) and that it is enough to set $m = n = 1$, so

$$p = b.$$

We choose the LQR parameter

$$R = 1,$$

which yields the feedback gain

$$f_b = -\frac{1}{b}(1 + \sqrt{1 + b^2})$$

and the closed loop matrix

$$\bar{a}_b = \sqrt{1 + b^2}.$$

It turns out that the polynomial[7]

$$\big[-0.0048\, b^7 + 0.0119\, b^5 + 0.1056\, b^3 + 0.1250\, b\big]\tau^2 +$$

[7]To find this polynomial, we use MATLAB's 'polyfit' function to approximate f_b and $e^{-\phi}$ and then use $\phi = \sqrt{1 + b^2}\tau \approx (\frac{1}{2}b^2 + 1)\tau$.

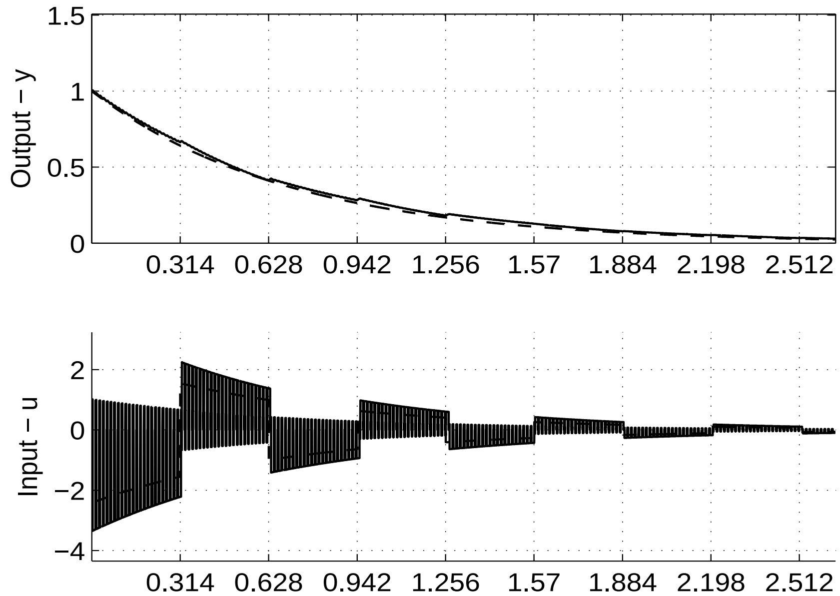

Figure 5.3: Example 1 - solid is actual, dashed is nominal - $h = 0.0001s$.

$$\left[0.0333\, b^5 - 0.1483\, b^3 - 0.4298\, b\right]\tau - 0.0747\, b^3 + 0.4818\, b$$

provides a reasonably good approximation to

$$f_b e^{\bar{a}_b \tau}.$$

We perform two experiments: $h = 0.0001s$ and $h = 0.0005s$, shown in Figure 5.3 and 5.4, respectively. In both cases, the Estimation Phase has duration $32h$, the Control Phase has duration $100h$, and we switch between $b = 1$ and $b = -1$ every $\pi/10s$. As usual, we see improved performance for smaller sampling rates.

Since it is somewhat difficult to see what is going on in the control signal, we provide a zoomed in version of the $h = 0.0001s$ case in Figure 5.5; observe that at the beginning of each Estimation phase, the control signal jumps to zero - this is the state estimation phase of the controller. Furthermore, observe that, because the switch occurs during the Estimation Phase, the resulting control signal is the wrong sign; however, the correct control is reasserted in the following period.

5.6.2 Example 2

Here we provide an example which will clearly benefit from the more general form provided in Section 5.2.5. We have the same structure of first order transfer functions:

$$P : \frac{b}{s+a},$$

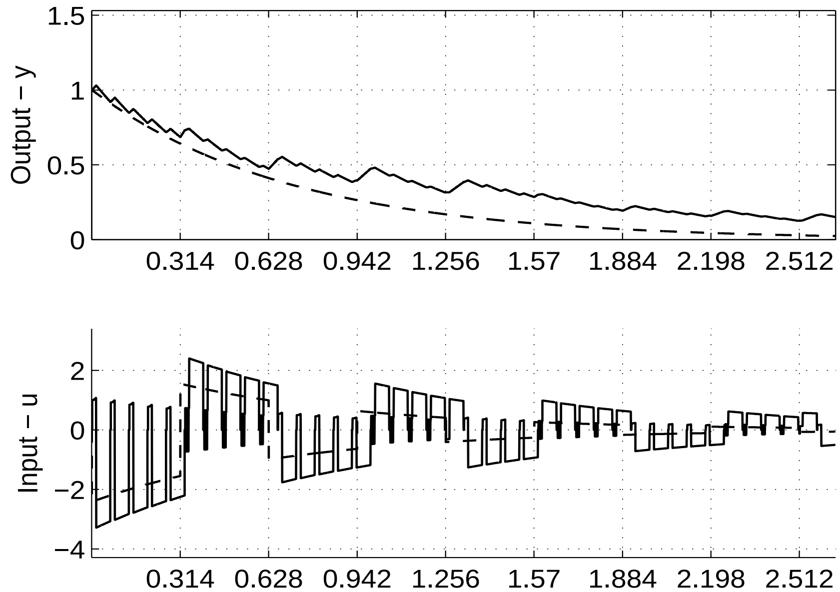

Figure 5.4: Example 1 - solid is actual, dashed is nominal - $h = 0.0005s$.

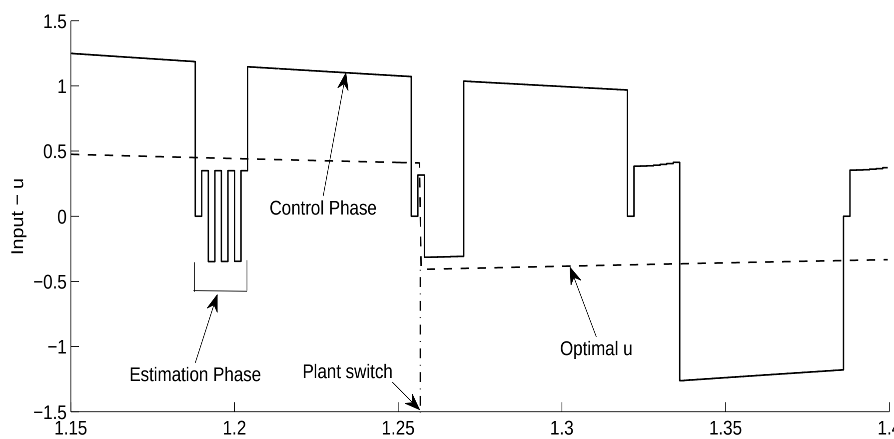

Figure 5.5: Example 1 - zoom of u from $h = 0.0001s$.

but this time we consider the compact set

$$\mathcal{P} = \{P : (a, b) \in ([-1, 1], \{-1, 1\})\};$$

here we require $m = 2n = 2$, so

$$p = \begin{bmatrix} b \\ ab \end{bmatrix}.$$

We choose the LQR parameter

$$r = 1,$$

which yields the feedback gain

$$f_p = -b(a + \sqrt{1 + a^2}),$$

the closed loop matrix

$$\bar{a}_p = \sqrt{1 + a^2},$$

and the extended observability matrix

$$\mathcal{O}_m(1, a) = \begin{bmatrix} 1 \\ a \\ a^2 \end{bmatrix}.$$

Observe that

$$\sqrt{1 + a^2} \approx 1 + \frac{1}{2}a^2,$$

so if we set

$$W = \begin{bmatrix} 1 & 1 & 1/2 \end{bmatrix},$$

then

$$\begin{aligned} f_p e^{\bar{a}_p \tau} &\approx -b\left(1 + a + \frac{1}{2}a^2\right) e^{\sqrt{1+a^2}\tau} \\ &= -be^{\sqrt{1+a^2}\tau} W \mathcal{O}_m(1, a), \end{aligned}$$

so if we approximate

$$w[kT] = W\mathcal{O}_m(1, a)\xi[kT]$$

instead of $\xi[kT]$ in the state estimation phase, then we can reduce the order of the required polynomial estimate by two. Indeed, the polynomial[8]

$$\Big[0.0640\, b^5 a^4 + 0.2561\, b^3 a^2 + 0.2561\, b\Big]\tau^2 + \Big[-0.4403\, b^3 a^2 - 0.8805\, b\Big]\tau + 0.9872\, b$$

provides a reasonably good approximation to $-be^{\bar{a}_b \tau}$.

It turns out that the structure of the arguments $b^j a^{j-1}$ allows us to significantly reduce the number of probes. As usual, we begin with the state estimation phase, wherein we obtain our estimate of w. To see how to reduce the number of probes, observe that we can proceed in the following way:

[8]To find this polynomial, we again use MATLAB's 'polyfit' function to approximate $e^{-\phi}$ and then substitute with $\phi = \sqrt{1 + a^2}\tau \approx (\frac{1}{2}a^2 + 1)\tau$.

(i) Probe with $\text{Est}\{w[kT]\}$ to obtain a good estimate of

$$pw[kT] = \begin{bmatrix} b \\ ba \end{bmatrix} w[kT].$$

(ii) Probe with the estimate of $\left(\begin{bmatrix} b \\ ba \end{bmatrix} w[kT] \right)_2$ to obtain a good estimate of

$$p(ba)w[kT] = \begin{bmatrix} b^2 a \\ b^2 a^2 \end{bmatrix} w[kT].$$

$$\vdots$$

(iv) Probe with the estimate of $\left(\begin{bmatrix} b^4 a^3 \\ b^4 a^4 \end{bmatrix} w[kT] \right)_2$ to obtain a good estimate of

$$p(b^4 a^4)w[kT] = \begin{bmatrix} b^5 a^4 \\ b^5 a^5 \end{bmatrix} w[kT].$$

We then use

$$\text{estimate of} \left(\begin{bmatrix} b \\ ba \end{bmatrix} w[kT] \right)_1 \approx bw[kT]$$
$$\vdots$$
$$\text{estimate of} \left(\begin{bmatrix} b^5 a^4 \\ b^5 a^5 \end{bmatrix} w[kT] \right)_1 \approx b^5 a^4 w[kT]$$

to construct the desired polynomial. Since we never probe with the first element of the estimated vectors, the total number of probes is significantly reduced; indeed, if we had not taken advantage of the structure, then we would have needed 15 probes in the control estimation phase instead of only four.

We perform two experiments: $h = 0.0005s$ and $h = 0.0008s$, shown in Figure 5.6 and 5.7, respectively. In both cases, the Estimation Phase has duration $36h$, the Control Phase has duration $200h$, we fix $a = 1$, and we switch between $b = 1$ and $b = -1$ every $\pi/3s$. As expected, we see improved performance for smaller sampling rates.

5.7 Summary and Concluding Remarks

In this chapter, we revisit the problem of Chapter 3 in the context of a compact (rather than finite) set of LTI plants that is given in transfer function form. We provide a canonical state-space representation for each plant and design a mildly nonlinear RACE controller that provides asymptotic stability in the face of (possibly persistent) switching between plants in the set. Furthermore, the proposed

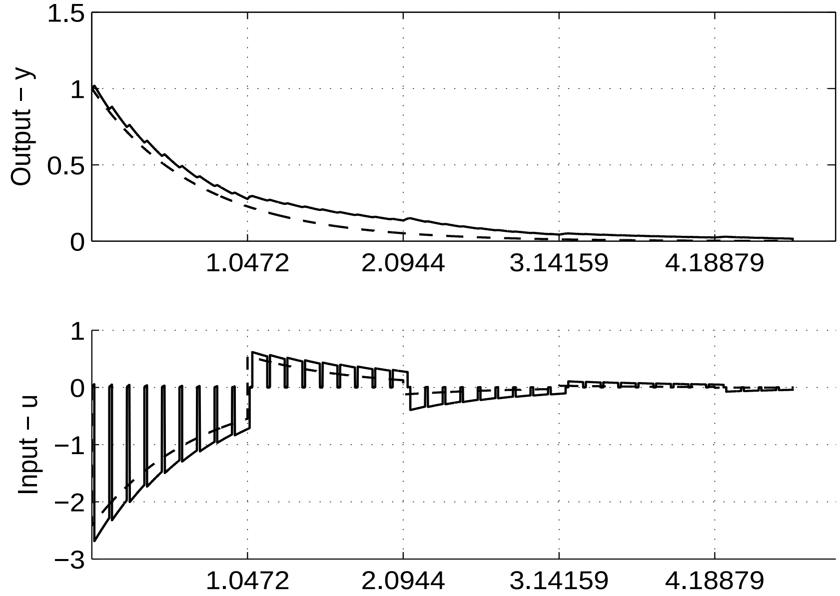

Figure 5.6: Example 2 - solid is actual, dashed is nominal - $h = 0.0005s$.

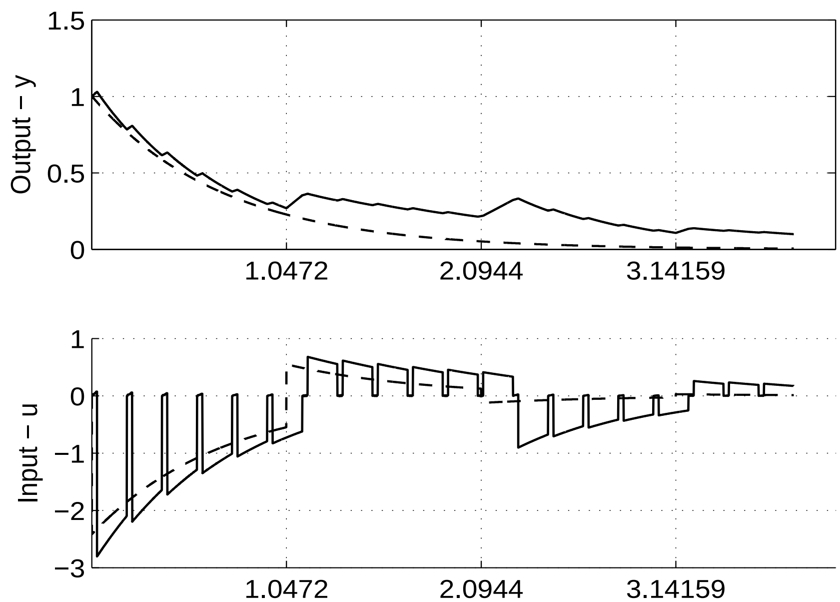

Figure 5.7: Example 2 - solid is actual, dashed is nominal - $h = 0.0008s$.

controller is shown to provide near optimal LQR performance for each LTI plant. Although we have not discussed it here, we expect that near nominal performance (in the same sense as in the previous two chapters) can be obtained for sufficiently small T, T', and ε.

The main difference between this controller and those of Chapters 3 and 4 is that, here, our estimation method requires active probing, which significantly complicates the analyses. A major drawback to this approach is that, even though we know that they exist, the polynomial H^ε and the coefficients $d_{i,j}$ can be quite difficult to find, especially for higher order plants.

Chapter 6

The Time Varying Gain Margin Problem

At this point, we change gears and consider a problem that is both simpler and more complex, depending on perspective. We consider a nominal LTI plant P_1 with an unknown time-varying (TV) multiplicative gain g at the input; together, these comprise the actual plant P_g[1]. This problem is simpler in the sense that we have a single LTI plant with a single unknown parameter; however, it is more complicated in the sense that the uncertainty is allowed to be a reasonably general time-varying function, rather than piecewise constant.

As has been the case throughout this thesis, we are interested in more than just stability. Here we will consider the tracking problem that can be modeled via a (stable) filter W at the exogenous input. In keeping with some of the standard results in this area, we will use a weighted sensitivity function to measure performance; furthermore, in this chapter we use the infinity norm to measure the size of a vector (instead of the 2-norm as was used in all previous chapters). This controller is motivated by those of [24] and [22]; we seek to alleviate the restrictions of those papers by allowing for both rapidly time varying gains (which [24] does not do) *and* non-minimum phase plants (which [22] does not do). A preliminary version of this work was presented in the conference paper [40].

Recall that, in [24] it was shown that one can achieve any desired gain margin while ensuring that the size of the weighted sensitivity function[2] is near LTI-optimal. There, the gain was constant, but here it is time-varying, so a natural question is: can one provide better performance for a given LTI plant by using a nonlinear time-varying (NLTV) controller instead of an LTI one? When using the 2-norm to measure the signal size, the answer is NO, at least in the simplest setup [14]. When using the ∞-norm to measure signal size, as is done in [24] as well as here, the answer is YES, at least for certain multi-input multi-output cases [34]. However,

[1]Since we will denote the plant associated with the gain g as P_g, our nominal plant is P_1 (instead of the familiar P_0).

[2]It is critical that the weight be strictly proper.

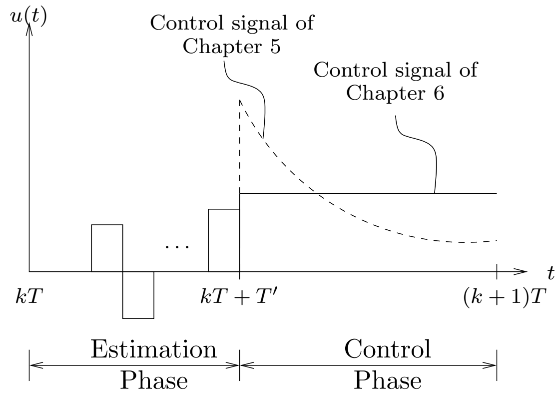

Figure 6.1: Comparison of Input Signal from Chapter 5 and Chapter 6

here our goal is to design a LTV controller and, at least in the discrete time case, it has been proven [33] that there is no advantage to LTV over LTI controllers. Based on this argument, our performance goal will be LTI optimal weighted sensitivity.

In all of the previous chapters, we used a generalized hold to apply a control signal that looked much like the optimal one. The main reason behind using a generalized hold (rather than a zero order hold) was to allow for longer controller periods. Unfortunately, we will not be able to allow large periods here: we would like to allow the gain g to be rapidly time varying, so, since the controller is essentially open loop over each period, intuitively we will need short periods to ensure stability. To that end, we will use a (easier to analyze) piecewise constant control signal (see Figure 6.1).

If we assume that the TV gain g lies in a compact set, then the set composed of all possible plants P_g is also compact, so it is not surprising that the discussion here will be reminiscent of that of Chapter 5. There is one exception: since the gain is always time varying, in this chapter, we will *not* differentiate between gains that do or do not contain discontinuities.

A brief outline of this chapter is as follows. In Section 6.1 we make the problem precise. In Section 6.2 we address the question of estimation. In Section 6.3 we leverage the results of Section 6.2 to design a RACE controller and present some results that will be useful in proving the main theorem of this chapter. In Section 6.4 we present the main result: we show that our controller design can provide stability and near optimal (LTI) performance in the face of time varying uncertainty in the gain g. In Section 6.5 we present two illustrative examples. In Section 6.6 we discuss a promising avenue for future research in this area and we wrap up with a summary and concluding remarks in Section 6.7.

Before proceeding, we briefly introduce some notation that will be exclusive to this chapter. We use the notation $\Phi_A(t, t_0)$ to denote the state transition matrix corresponding to the square matrix A; furthermore, since we are dealing with time variations and LTI systems, it will be convenient (and natural) to write gK to indicate an LTI function K with a time varying function g on the output.

6.1 Problem Formulation

We are interested in a SISO nominal plant P_1 with an unknown gain g at the input; together, these form the time-varying plant P_g. Furthermore, recall that we model the signal to be tracked via a (stable) filter W at the exogenous input. In this section, we provide state-space models for P_1, P_g, and W, some assumptions on the set of admissible gains, definitions of stability and optimal performance, and some useful notation.

The nominal SISO plant P_1 can be modeled via the state-space representation

$$\begin{array}{rcl} \dot{x}_p(t) & = & A_p\, x(t) + B_p\, u(t), \quad x_p(0) = x_{p0}, \\ y(t) & = & C_p\, x_p(t), \end{array} \tag{6.1}$$

which we assume to be stabilizable and detectable with relative degree m. It follows naturally that, for $g \in \mathcal{PC}_\infty$, the TV plant P_g is given by

$$\begin{array}{rcl} \dot{x}_p(t) & = & A_p\, x_p(t) + B_p\, g(t)u(t), \quad x_p(0) = x_{p0}, \\ y(t) & = & C_p\, x_p(t). \end{array} \tag{6.2}$$

We now turn to the class of admissible gains. To achieve the tracking objective

- it is reasonable to require a bound on the derivative of g, since otherwise it would be hard to track;
- we need $g(t)$ to be bounded away from zero to ensure that $\frac{1}{g(t)}$ is bounded;
- we need $g(t)$ to be bounded (so that we can prove a uniform type of result); and last of all,
- we require that discontinuities in g be sufficiently far apart.

These requirements can be encapsulated via three parameters:

- a constant $c_g > 0$,
- a compact set $G \subset \mathbf{R}$ not including zero, and
- a fixed time T_s.

We can then define the set of admissible gains by

$$\mathcal{G}(G, T_s, c_g) := \{g \in PS_\infty(T_s) : \begin{array}{l} 1)\ g(t) \in G,\ t \in \mathbf{R} \text{ and} \\ 2)\ \text{esssup}_{t\geq 0} \|\dot{g}(t)\| \leq c_g\}; \end{array}$$

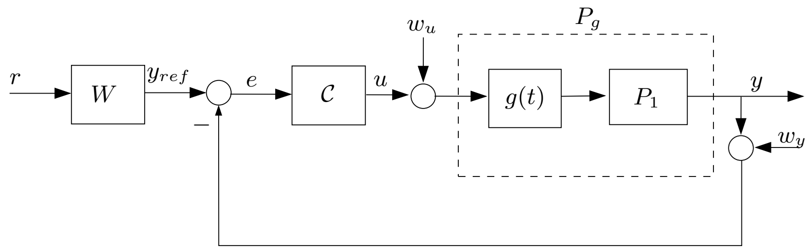

Figure 6.2: Block Diagram

hence, the set of admissible TV plants is given by

$$\mathcal{P} := \{P_g : g \in \mathcal{G}(G, T_s, c_g)\}.$$

The class of reference signals to be tracked is modeled by the stable, finite dimensional, LTI, low-pass filter W, which is driven by an exogenous input r. We model this filter via the following minimal representation:

$$\begin{aligned} \dot{x}_w(t) &= A_w\, x_w(t) + B_w\, r(t), \quad x_w(0) = x_{w0}, \\ y_{ref}(t) &= C_w\, x_w(t) \end{aligned} \tag{6.3}$$

and we make the following key assumption:

Assumption 6.1 *The relative degree of W is* **greater than** *the relative degree of P_1.*

Remark 6.1 *Tracking problems tend to be concerned with the low frequency component of the signal to be tracked, so if W does not satisfy Assumption 6.1, it can be made to do so by rolling off the high frequency with minimal effect on the problem.*

To incorporate this model into the structure of Figure 2.1, we introduce noise at the plant input and output (w_u and w_y respectively), define the tracking error by

$$e := y_{ref} - (y + w_y),$$

and label the (as yet undefined) controller by $\mathcal{C}$. Together, these yield the closed loop block diagram shown in Figure 6.2.

To achieve our objective, observe that if the FDLTI controller K_{lti} stabilizes[3] P_1, then the controller $\frac{1}{g}K_{lti}$ stabilizes P_g and provides the same weighted sensitivity. Hence, if we first choose K_{lti} such that it stabilizes and provides near (LTI) optimal performance for the nominal plant P_1, then, since g is unknown, the problem becomes that of designing $\mathcal{C}$ such that it behaves close to $\frac{1}{g}K_{lti}$ when attached to the plant P_g. Of course, this is easier said than done. In the course of the design

[3]We will define what we mean by stability in a moment.

process, we will consider three different controllers, each of which has K_{lti} as a component. To that end, it will be useful to adopt the following minimal representation of K_{lti}:

$$\begin{array}{rcl} \dot{x}_k(t) & = & A_k\ x_k(t) + B_k\ e(t), \quad x_k(0) = x_{k0}, \\ u^0(t) & = & C_k\ x_k(t) + D_k\ e(t). \end{array} \tag{6.4}$$

It will be convenient to combine the state-space representations of the plant, filter, and LTI controller, yielding a representation of the augmented open loop system:

$$\begin{array}{rcl} \begin{bmatrix} \dot{x}_p(t) \\ \dot{x}_k(t) \\ \dot{x}_w(t) \end{bmatrix} & = & \underbrace{\begin{bmatrix} A_p & 0 & 0 \\ -B_kC_p & A_k & B_kC_w \\ 0 & 0 & A_w \end{bmatrix}}_{=:A} \underbrace{\begin{bmatrix} x_p(t) \\ x_k(t) \\ x_w(t) \end{bmatrix}}_{=:x(t)} + \underbrace{\begin{bmatrix} B_p \\ 0 \\ 0 \end{bmatrix}}_{=:B_u} g(t)u(t) + \\ & & \underbrace{\begin{bmatrix} 0 \\ 0 \\ B_w \end{bmatrix}}_{=:B_r} r(t), \qquad x(0) = x_0 := \begin{bmatrix} x_{p0} \\ x_{k0} \\ x_{w0} \end{bmatrix} \in \mathbf{R}^n, \\ u^0(t) & = & \underbrace{\begin{bmatrix} -D_kC_p & C_k & D_kC_w \end{bmatrix}}_{=:F} x(t), \\ e(t) & = & \underbrace{\begin{bmatrix} -C_p & 0 & C_w \end{bmatrix}}_{=:C} x(t). \end{array} \tag{6.5}$$

With this in hand, we can express our controller $\mathcal{C}$ as a combination of K_{lti} together with an (as yet unspecified) compensator κ given in input-output form:

$$\begin{array}{rcl} \kappa & : & \mathcal{PC}_\infty \to \mathcal{PC}_\infty \\ & : & \begin{bmatrix} e \\ u^0 \end{bmatrix} \mapsto u. \end{array}$$

This yields the closed loop system shown in Figure 6.3 and the following natural definition of stability:

Definition 6.1 *With $x_0 = 0$, we say that the controller $\mathcal{C}$* **I/O stabilizes** *$\mathcal{P}$ if, for every $P_g \in \mathcal{P}$ the map*

$$(w_u, w_y, r) \to (u, y, e)$$

is well defined and has bounded norm.

Observe that, since our closed loop system is linear, finite dimensional, and stabilizable, asymptotic stability is an automatic result of I/O stability. Furthermore, in the remainder of this chapter, we restrict ourselves to only those FDLTI controllers K_{lti} that stabilize P_1 and label the set of all such controllers by $\mathcal{S}(P_1)$.

Since we will be investigating the performance of different controller and gain combinations we would like the notation for the weighted sensitivity to emphasize this:

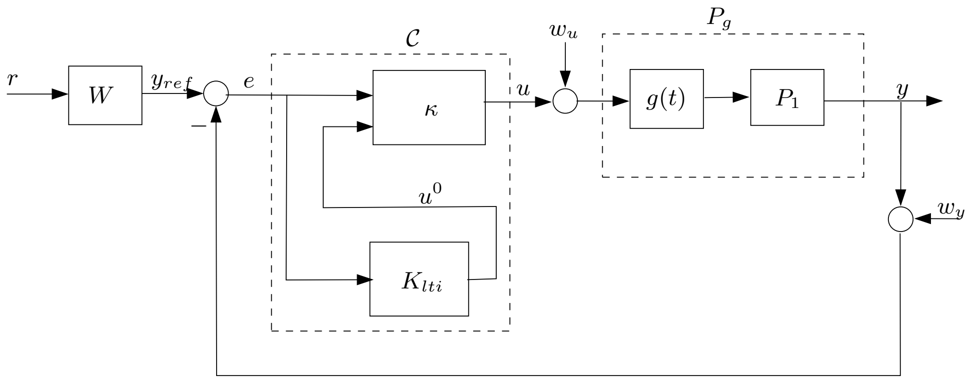

Figure 6.3: Expanded Block Diagram

Definition 6.2 *With $x_0 = 0$, $w_y = 0$, and $w_u = 0$, for a given gain g and controller $\mathcal{C}$, we define the weighted sensitivity by*

$$S(g, \mathcal{C})W : r \mapsto e,$$

so

$$S(g, \mathcal{C})W := (I + P_g\mathcal{C})^{-1}W$$

and the cost associated with g and $\mathcal{C}$ is therefore

$$\|S(g, \mathcal{C})W\| := \sup_{\substack{r \in \mathcal{PC}_\infty, \\ r \neq 0}} \frac{\|\left((I + P_g\mathcal{C})^{-1}W\right)(r)\|_\infty}{\|r\|_\infty}.$$

This leads naturally to the following definition of the optimal cost achievable by an LTI controller:

$$\alpha_{lti} := \inf_{K \text{ is LTI and stabilizes } P_1} \|S(1, K)W\|.$$

We can now restate our performance objective: we would like to design $\mathcal{C}$ so that $\|S(g, \mathcal{C})W\|$ can be made as close as desired to α_{lti}, independent of $g \in \mathcal{G}(G, T_s, c_g)$. As we indicated above, to do so it would clearly be enough to design $K_{lti} \in \mathcal{S}(P_1)$ so that it provides a cost close to α_{lti} and then apply the controller $\frac{1}{g}K_{lti}$; however, we can not do so since g is unknown.

6.2 Estimation

As usual, we must perform some estimation. We implied in Section 2.3.1 that we can estimate the quantities $g^i[kT]u^0[kT]$ reasonably well and then use them to apply an estimate of the desired control signal. This section is devoted to explaining how to do this. We begin by outlining the estimation method, then perform an additional approximation, and finally, show how these can be combined to provide (an estimate of) the desired control signal. In this section, we assume that the noise is turned off.

6.2.1 Estimating Polynomials in g

Before providing the details, we use a simple (first order) example to outline the idea behind a single iteration of the estimation process. For simplicity, we will assume that g is constant. Consider the following nominal plant P_1 given by

$$\dot{y}(t) = ay(t) + gu(t)$$

together with the reference model W given by

$$\begin{aligned} \dot{x}_w(t) &= \underbrace{\begin{bmatrix} 0 & 1 \\ -\alpha_0 & -\alpha_1 \end{bmatrix}}_{=:A_w} x_w(t) + \underbrace{\begin{bmatrix} 0 \\ \beta \end{bmatrix}}_{=:B_w} r(t) \\ y_{ref}(t) &= \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{=:C_w} x_w(t). \end{aligned}$$

Recall that our controller's input is the signal $e = y_{ref} - y$. We will begin by investigating y. Consider the constant input

$$u(t) = \bar{u}, \quad t \in [t_0, t_0 + h),$$

which yields

$$y(t_0 + h) = (1 + ah)y(t_0) + hg\bar{u} + \mathcal{O}(h^2)y(t_0) + \mathcal{O}(h^2)\bar{u}.$$

Unfortunately, the effect of the initial conditions are overriding that of $g\bar{u}$. We can begin to address this problem by rearranging the above equation[4]:

$$\frac{1}{h}\left[y(t_0 + h) - y(t_0)\right] = ay(t_0) + g\bar{u} + \mathcal{O}(h)y(t_0) + \mathcal{O}(h)\bar{u}. \tag{6.6}$$

We now have a situation where, although the initial conditions are no longer dominating, neither is the effect of $g\bar{u}$. To combat this, we use a more complicated u; we first turn off u to get our hands on the initial conditions, and then turn it back on:

$$u(t) = \begin{cases} 0 & t \in [t_0, t_0 + 2h) \\ \bar{u} & t \in [t_0 + 2h, t_0 + 4h). \end{cases}$$

With this control signal, we can subtract off (an approximation of) the initial conditions to obtain the desired signal. Indeed, with this more complicated input we find the effect of the initial conditions:

$$y(t_0 + h) = (1 + ah)y(t_0) + \mathcal{O}(h^2)y(t_0)$$

[4]This procedure corresponds to a discrete-time approximation of the derivative of the plant output, with the estimate improving but noise tolerance worsening as $h \to 0$. This is similar to the trade-off that arises in PID design with regards to the high frequency roll off of the derivative term: the higher the frequency, the better the approximation to a pure differentiator, but the worse the noise tolerance. To make our noise problems even more challenging, as the relative degree of P_1 increases, we will need additional derivative approximations; indeed, we will need m of them.

and the effect of $\bar{u}$:

$$y(t_0+3h) = (1+ah)y(t_0+2h) + hg\bar{u} + \mathcal{O}(h^2)y(t_0+2h) + \mathcal{O}(h^2)\bar{u}$$

which we combine and rearrange as in (6.6) to provide

$$\begin{aligned}
&\frac{1}{h}\left\{[y(t_0+3h) - y(t_0+2h)] - [y(t_0+h) - y(t_0)]\right\} = \\
&\qquad = a[y(t_0+2h) - y(t_0)] + g\bar{u} + \mathcal{O}(h)y(t_0) + \mathcal{O}(h)y(t_0+2h) + \mathcal{O}(h)\bar{u} \\
&\qquad = a[e^{2ah}y(t_0) - y(t_0)] + g\bar{u} + \mathcal{O}(h)y(t_0) + \mathcal{O}(h)e^{2ah}y(t_0) + \mathcal{O}(h)\bar{u} \\
&\qquad = g\bar{u} + \mathcal{O}(h)y(t_0) + \mathcal{O}(h)\bar{u}
\end{aligned}$$

which is a good approximation of $g\bar{u}$. However, our controller's input is the signal e, so we must also consider the y_{ref} part. Performing a similar analysis, and using the fact that the relative degree of the reference model is two (so $C_wB_w = 0$) we obtain

$$\begin{aligned}
&\frac{1}{h}\left\{[y_{ref}(t_0+3h) - y_{ref}(t_0+2h)] - [y_{ref}(t_0+h) - y_{ref}(t_0)]\right\} \\
&\qquad = \frac{1}{h}C_w\left[e^{3A_wh} - e^{2A_wh} - e^{A_wh} + I\right]x_w(t_0) + \mathcal{O}(h)x_w(t_0) + \mathcal{O}(h)\|r\|_\infty \\
&\qquad = \mathcal{O}(h)x_w(t_0) + \mathcal{O}(h)\|r\|_\infty;
\end{aligned}$$

therefore, looking at the error signal $e = y_{ref} - y$ we see that

$$\begin{aligned}
&\frac{1}{h}\left\{[e(t_0+h) - e(t_0)] - [e(t_0+3h) - e(t_0+2h)]\right\} = \\
&\qquad\qquad g\bar{u} + \mathcal{O}(h)y(t_0) + \mathcal{O}(h)\bar{u} + \mathcal{O}(h)x_w(t_0) + \mathcal{O}(h)\|r\|_\infty, \qquad (6.7)
\end{aligned}$$

which is a good approximation of $g\bar{u}$.

We now consider the general case. To proceed, we will require four useful matrices:

$$S_m := \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2^2 & \cdots & 2^m \\ & & \vdots & & \\ 1 & m & m^2 & \cdots & m^m \end{bmatrix},$$

$$O_m(C, A_p) := \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A}_p \\ \vdots \\ \tilde{C}\tilde{A}_p^m \end{bmatrix},$$

$$X_m(h) = \mathrm{diag}\{1,\, h,\, h^2/(2!), ..., h^m/(m!)\},$$

and

$$\mathcal{E}(t) := \begin{bmatrix} e(t) \\ e(t+h) \\ \vdots \\ e(t+mh) \end{bmatrix}.$$

The matrices S_m and H_m arise naturally from the structure of the solution to (6.1) and (6.3); we show this in Appendix D in the proof of the up-coming Estimation Lemma. Note that S_m is a Vandermonde matrix and h is non-zero, so both S_m and $X_m(h)$ are invertible; observe that $X_m(h)^{-1} = \mathcal{O}(h^{-m})$. Finally, since m is the relative degree of P_1, we are guaranteed that $[C_p(A_p)^{m-1}B_p]^{-1}$ exists. Motivated by the structure of u and the size of these matrices, we define the probing period to be

$$h_m := (m+1)h.$$

With $e_{m+1} \in \mathbf{R}^{m+1}$ used to represent the $(m+1)^{\text{th}}$ basis vector, we can now rewrite (6.7) using this notation:

$$\begin{aligned}
&\frac{1}{h}\{[e(t_0+h)-e(t_0)]-[e(t_0+3h)-e(t_0+2h)]\} \\
&\quad = \frac{1}{h}\{[e(t_0+2h)-e(t_0)]-[e(t_0+3h)-e(t_0+h)]\} \\
&\quad = \begin{bmatrix} \frac{-1}{h} & \frac{1}{h} \end{bmatrix} \left\{ \begin{bmatrix} e(t_0) \\ e(t_0+h) \end{bmatrix} - \begin{bmatrix} e(t_0+2h) \\ e(t_0+3h) \end{bmatrix} \right\} \\
&\quad = e_{m+1}^T X_m(h)^{-1} S_m^{-1} [\mathcal{E}(t_0) - \mathcal{E}(t_0+h_m)] \\
&\quad = [C_p(A_p)^{m-1}B_p]^{-1} e_{m+1}^T X_m(h)^{-1} S_m^{-1} [\mathcal{E}(t_0) - \mathcal{E}(t_0+h_m)].
\end{aligned}$$

which is a good approximation to $g\bar{u}$.

We now formally state the estimation result:

Lemma 6.1 *[Estimation Lemma] There exist constants $\gamma > 0$ and $\bar{h} > 0$ so that for every $t_0 \geq 0$, $x_0 \in \mathbf{R}^n$, $r \in PC_\infty$, $h \in (0, \bar{h})$, $\bar{u} \in \mathbf{R}$, and $g \in \mathcal{G}(G, T_s, c_g)$, the solutions of (6.2) and (6.3) with*

$$u(t) = \begin{cases} 0, & t \in [t_0, t_0+h_m) \\ \bar{u}, & t \in [t_0+h_m, t_0+2h_m), \end{cases}$$

satisfy the following:

(i) for every $g \in \mathcal{G}(G, T_s, c_g)$

$$\left\| [C_p(A_p)^{m-1}B_p]^{-1} e_{m+1}^T X_m(h)^{-1} S_m^{-1} [\mathcal{E}(t_0) - \mathcal{E}(t_0+h_m)] \right\| \leq$$
$$\gamma|\bar{u}| + \gamma h \left[\|x_p(t_0)\| + \|x_w(t_0)\| + \|r\|_\infty \right],$$

(ii) and if g is continuous on $[t_0, t_0+2h_m)$ then we have:

$$\left\| [C_p(A_p)^{m-1}B_p]^{-1} e_{m+1}^T X_m(h)^{-1} S_m^{-1} [\mathcal{E}(t_0) - \mathcal{E}(t_0+h_m)] - g(t_0)\bar{u} \right\| \leq$$
$$\gamma h \left[\|x_p(t_0)\| + \|x_w(t_0)\| + |\bar{u}| + \|r\|_\infty \right].$$

Proof: Please see Appendix D.

Remark 6.2 *Observe that (6.2) and (6.3) do not contain noise signals, so implicitly, this result holds on intervals* $[t_0, t_0 + 2h_m)$ *where* $w = 0$.

In a major departure from the previous three chapters, here we will *not* take two estimates in series, allowing our controller (and the closed loop system) to be linear. In previous chapters, a main goal was to allow for large controller periods; the introduction of the nonlinearity was critically important to doing so[5]. Since this chapter *always* requires small controller periods, here we would gain very little from making our controller non-linear. Furthermore, using the nonlinearity requires either doubling the duration of the Estimation Phase or halving the sampling period, the former is an issue since we need small periods, and (although we do not prove it here) the latter would degrade the noise performance.

The upshot of this lemma is that the known (measurable) quantity

$$[C_p(A_p)^{m-1}B_p]^{-1}e_{m+1}^T X_m(h)^{-1}S_m^{-1}[\mathcal{E}[kT] - \mathcal{E}(jT + h_m)]$$

provides us with a good estimate of $g[kT]\bar{u}$. The natural next step is to feed a scaled version of that estimate back into the system to obtain an estimate of $g^2[kT]\bar{u}$ and so on. Of course, this only provides us with an estimate of polynomials in g; as in the previous chapter, an additional approximation is required.

6.2.2 Approximation by a Sampled Data Controller

Recall that the desired control signal is

$$u(t) = \frac{1}{g(t)}u^0(t), \quad t \geq 0.$$

Unfortunately, this is not a polynomial in g, so we can not use the Estimation Lemma to estimate it; however, we know from the previous chapter that we can leverage the Stone-Weierstrass Approximation Theorem [32] to obtain a good approximation to the desired control signal which *is* a polynomial in g. Indeed, for every tolerance $\varepsilon > 0$, the Stone-Weierstrass Approximation Theorem guarantees the existence of a (real) polynomial

$$\phi^\varepsilon(f) := \sum_{i=0}^{q} a_i f^i$$

such that

$$|1 - f\phi^\varepsilon(f)| < \varepsilon, \quad f \in G.$$

[5]Recall that, if the estimate was generated over an interval with a switch, then the resulting control signal to be applied over the control phase would also be large, causing havoc with the plant if allowed to persist due to long controller periods.

It follows immediately that

$$\|1 - g\phi^{\varepsilon}(g)\|_{\infty} < \varepsilon, \quad g \in \mathcal{G}(G, T_s, c_g). \tag{6.8}$$

Observe that, for each $\varepsilon > 0$ there are many polynomials ϕ^{ε} that satisfy (6.8); to that end we adopt the following convention: if we fix $\varepsilon > 0$, then we implicitly mean that we fix ϕ^{ε} so that it satisfies (6.8).

If we use x^{ε} to denote the state trajectory when the controller $\phi^{\varepsilon}(g)K_{lti}$ is applied to (6.2), then we can use (6.5) to write the associated closed loop state-space representation by setting

$$\begin{aligned} u(t) &= \phi^{\varepsilon}(g(t))u^0(t) \\ &= \underbrace{\phi^{\varepsilon}(g(t))Fx^{\varepsilon}(t)}_{=:u^{\varepsilon}(t)}, \quad t \geq 0, \end{aligned}$$

yielding

$$\begin{aligned} \dot{x}^{\varepsilon}(t) &= \underbrace{[A + B_u g(t)\phi^{\varepsilon}(g(t))F]}_{=:A^{\varepsilon}_{cl}(g(t))} x^{\varepsilon}(t) + B_r r(t), \quad x^{\varepsilon}(0) := x_0, \\ e^{\varepsilon}(t) &= Cx^{\varepsilon}(t). \end{aligned} \tag{6.9}$$

We use the natural notation of $\Phi_{A^{\varepsilon}_{cl}(g)}(t, t_0)$ to represent the transition matrix associated with $A^{\varepsilon}_{cl}(g(t))$. Clearly, we can obtain a good estimate of the control signal u^{ε} by using the KEL, but before we do so, we first confirm that this provides the desired performance for the plant P_g if ε is small enough.

Proposition 6.1 *For every $K_{lti} \in \mathcal{S}(P_1)$, there exist constants $\bar{\varepsilon} > 0$, $\lambda_0 < 0$ and $\gamma_0 > 0$ such that, for every $\varepsilon \in (0, \bar{\varepsilon})$, $x_0 \in \mathbf{R}^n$, $r \in \mathcal{PC}_{\infty}$, and $g \in \mathcal{G}(G, T_s, c_g)$ the following properties hold:*

(i) $\|\Phi_{A^{\varepsilon}_{cl}(g)}(t, t_0)\| \leq \gamma_0 e^{\lambda_0(t - t_0)}, \quad t \geq t_0,$

(ii) the controller $\phi^{\varepsilon}(g)K_{lti}$ satisfies

$$\|S(g, \phi^{\varepsilon}(g)K_{lti})W - S(1, K_{lti})W\| \leq \gamma\varepsilon.$$

Proof: Please see Appendix D.

At this point, for every $K_{lti} \in \mathcal{S}(P_1)$, we fix the constants $\bar{\varepsilon} > 0$, $\lambda_0 < 0$, and $\gamma_0 > 0$ so that they satisfy Proposition 6.1; clearly these constants will depend on the particular choice of K_{lti}, but we do not make it explicit.

6.2.3 Applying the KEL

From the discussion of Section 6.2.1, if we set

$$u(t) = \begin{cases} 0, & t \in [kT, kT + h_m) \\ u^0[kT], & t \in [kT + h_m, kT + 2h_m), \end{cases}$$

then it is reasonable to define

$$\mathrm{Est}\big\{g[kT]u^0[kT]\big\} := [C_p(A_p)^{m-1}B_p]^{-1}e_{m+1}^T X_m(h)^{-1}S_m^{-1}[\mathcal{E}[kT] - \mathcal{E}(kT + h_m)],$$

which is a good estimate of $g[kT]u^0[kT]$. The natural next step is to feed a scaled version of that estimate back into the system to obtain an estimate of $g^2[kT]u^0[kT]$ and so on, which can be combined to provide a good estimate of control signal

$$u(t) = \phi^\varepsilon g[kT]u^0[kT] \approx \frac{1}{g[kT]}u^0[kT], \quad t \in [kT, (k+1)T);$$

clearly, for small T and ε, this provides a good estimate of the desired optimal control signal.

Observe that this estimate is only good if g is continuous on the interval, so on periods that contain a discontinuity in g, our estimates will (likely) be poor. That being said, the Estimation Lemma says that, even if there is a discontinuity, the size of the estimate is bounded.

6.3 The Controller

In this section, we design the controller $\mathcal{C}$. Our design approach works in the following way: we choose a controller $K_{lti} \in \mathcal{S}(P_1)$ that provides acceptable performance and then design a sampled-data linear periodic compensator κ of period p and sample time h that performs estimation, yielding an output that behaves much like that of the controller $\frac{1}{g}K_{lti}$. From the Estimation Lemma and the structure of ϕ^ε, we know that we need to obtain q estimates, each of which will take $2h_m$ units of time to obtain, so the Estimation Phase is of duration $T' = 2qh_m$. We could choose any duration for the Control Phase; for simplicity, we assume that Control Phase has duration $2h_m$, so the period is

$$T = 2(q+1)h_m = 2(q+1)(m+1)h.$$

Observe that there is a natural relationship between h, h_m, T', and T.

In the following, we use estimation coefficients c_i, which are arbitrary (non-zero) weights that can be chosen to adjust the size of the control signal u. It will be useful to define

$$c_{-1} := 1,$$
$$Est\{u^0(jT)\} := u^0(kT),$$

and

$$Est\{c_i g^{i+1}(kT)u^0(jT)\} := [C_p(A_p)^{m-1}B_p]^{-1}e_{m+1}^T X_m(h)^{-1}S_m^{-1} \times \\ \left[\mathcal{E}(kT+2ih_m) - \mathcal{E}(kT+(2i+1)h_m)\right],\ i = 0, ..., q-1.$$

As usual, we present our controller in open loop form over one period:

THE PROPOSED COMPENSATOR - κ

With $K_{lti} \in \mathcal{S}(P_1)$, $\varepsilon \in (0, \bar{\varepsilon})$, $h \in \mathbf{R}^+$, and $k \in \mathbf{Z}^+$, we define the compensator by

Stage 1 - Estimation Phase: $[kT, kT+T')$

$$u(t) = \begin{cases} 0 & t \in [kT, kT+h_m) \\ c_0 u^0(kT) & t \in [kT+h_m, kT+2h_m) \\ 0 & t \in [kT+2h_m, kT+3h_m) \\ \frac{c_1}{c_0} Est\{c_0 g(kT)u^0(jT)\} & t \in [kT+3h_m, kT+4h_m) \\ \vdots & \\ 0 & t \in [kT+2(q-1)h_m, kT+(2q-1)h_m) \\ \frac{c_{q-1}}{c_{q-2}} Est\{c_{q-2}g^{q-1}(kT)u^0(jT)\} & t \in [kT+(2q-1)h_m, kT+T'), \end{cases} \tag{6.10}$$

Stage 2 - Control Phase: $[kT+T', (k+1)T)$

$$u(t) = \frac{1}{2}\left[\sum_{i=0}^{q} 2(q+1)a_i \frac{1}{c_{i-1}} Est\{c_{i-1}g^i(kT)u^0(jT)\} - \right. \\ \left. \sum_{i=0}^{q-1} \frac{c_i}{c_{i-1}} Est\{c_{i-1}g^i(kT)u^0(jT)\}\right],\ t \in [kT+T', (k+1)T). \tag{6.11}$$

The controller $\mathcal{C}$ consists of K_{lti} and κ, which clearly depends on ε and T; to stress this, we re-write it as $\mathcal{C}(K_{lti}, \varepsilon, T)$. A graphical representation of the resulting control signal is given in Figure 6.4. Observe that κ is a periodic sampled-data controller with period $p = 2(q+1)(m+1)$ and sample time h; we can obtain a state space representation of dimension $q+1$ with each state holding one of the estimates:

$$\begin{aligned} z[k+1] &= A_z[k]z[k] + B_z[k]\begin{bmatrix} u^0 \\ e \end{bmatrix}(kh), \quad z[0] = z_0 \in \mathbf{R}^{q+1}, \\ u(kh+\tau) &= C_z[k]z[k] + D_z[k]\begin{bmatrix} u^0 \\ e \end{bmatrix}(kh), \quad \tau \in [0, h), \end{aligned} \tag{6.12}$$

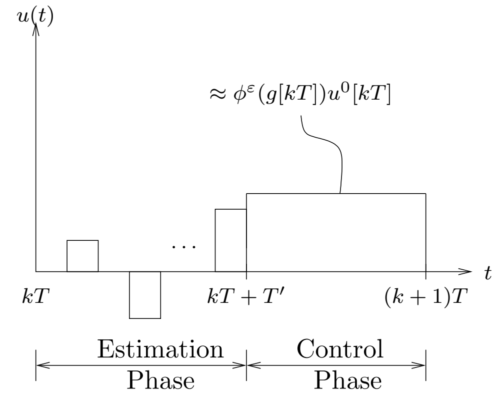

Figure 6.4: Input signal u for one period T - control phase not to scale.

with A_z, B_z, C_z, and D_z periodic of period p. For an example of a construction of such a representation for a special case, please see Appendix D.

We now investigate the closed loop system's behaviour over one period. We will be able to show that, for small T, the effect of the controller $\mathcal{C}(K_{lti}, \varepsilon, T)$ is similar to that of $\phi^\varepsilon(g)K_{lti}$, at least over one period.

Lemma 6.2 *[One Period Lemma] With $K_{lti} \in \mathcal{S}(P_1)$ and $\varepsilon \in (0, \bar{\varepsilon})$, there exist constants $\gamma > 0$ and $\bar{T} > 0$ so that for every $T \in (0, \bar{T})$, $x_0 \in \mathbf{R}^n$, $r \in \mathcal{PC}_\infty$, $g \in \mathcal{G}(G, T_s, c_g)$, and $k \in \mathbf{Z}^+$, when $\mathcal{C}(K_{lti}, \varepsilon, T)$ is applied to P_g the solution to (6.5) satisfies:*

(i) In all cases we have

$$\text{(a)} \quad \left\| x(t) - \Phi_{A_{cl}^\varepsilon(g)}(t, kT)x(kT) - \int_{kT}^{t} \Phi_{A_{cl}^\varepsilon(g)}(t, \tau) B_r r(\tau) d\tau \right\| \leq \gamma T (\|x(kT)\| + \|r\|_\infty),$$

$$\text{(b)} \quad \|u(t)\| \leq \gamma \|x(kT)\| + \gamma T \|r\|_\infty, \quad t \in [kT, (k+1)T).$$

(ii) If g is absolutely continuous on $[kT, (k+1)T)$ then

$$\Big\| x((k+1)T) - \Phi_{A_{cl}^\varepsilon(g)}((k+1)T, kT)x(kT) - \int_{kT}^{(k+1)T} \Phi_{A_{cl}^\varepsilon(g)}((k+1)T, \tau) B_r r(\tau) d\tau \Big\| \leq \gamma T^2 (\|x(kT)\| + \|r\|_\infty).$$

Proof: Please see Appendix D.

We now consider the entire interval $[0, \infty)$ and show that $\mathcal{C}(K_{lti}, \varepsilon, T)$ stabilizes $\mathcal{P}$ and provides weighted sensitivity close to that provided by $\phi^{\varepsilon}(g)K_{lti}$. In a major divergence from prior chapters, observe that stability does **not** require a lower bound on T_s.

Proposition 6.2 *With $K_{lti} \in \mathcal{S}(P_1)$ and $\varepsilon \in (0, \bar{\varepsilon})$, there exists a constant $\gamma > 0$ so that, for every sufficiently small $T > 0$, the controller $\mathcal{C}(K_{lti}, \varepsilon, T)$ stabilizes $\mathcal{P}$ and satisfies*

$$\left\| S\big(g, \mathcal{C}(K_{lti}, \varepsilon, T)\big)W - S\big(g, \phi^{\varepsilon}(g)K_{lti}\big)W \right\| \leq \gamma T, \quad g \in \mathcal{G}(G, T_s, c_g).$$

Proof: Please see Appendix D.

6.4 The Main Result

In our previous chapters, we considered stability and performance separately; however, observe that Proposition 6.2 already provides the desired stability result. Indeed, it says that we can use *any* stabilizing compensator K_{lti} and *any* $\varepsilon \in (0, \bar{\varepsilon})$ to generate a stabilizing controller $\mathcal{C}(K_{lti}, \varepsilon, T)$, provided that T is small enough. We achieve the desired goal of near optimal LTI performance in the face of uncertainty in g by leveraging the results of the previous section. To that end, recall that

$$\alpha_{lti} := \inf_{K \in \mathcal{S}(P_1)} \|S(1, K)W\|$$

and that we would like to design $\mathcal{C}$ so that $\|S(g, \mathcal{C})W\|$ can be made as close as desired to α_{lti}, independent of $g \in \mathcal{G}(G, T_s, c_g)$.

Theorem 6.1 *For every $\delta > 0$ there exists a compensator $K_{lti} \in \mathcal{S}(P_1)$ and a constant $\bar{\varepsilon}_1 \in (0, \bar{\varepsilon})$ so that, for every $\varepsilon \in (0, \bar{\varepsilon}_1)$, if $T > 0$ is sufficiently small, then the controller $\mathcal{C}(K_{lti}, \varepsilon, T)$ stabilizes $\mathcal{P}$ and provides the following performance bound:*

$$\left\| S\left(g, \mathcal{C}(K_{lti}, \varepsilon, T)\right) W \right\| \leq \alpha_{lti} + \delta, \quad g \in \mathcal{G}(G, T_s, c_g).$$

Proof:

Fix $\delta > 0$ and let $g \in \mathcal{G}(G, T_s, c_g)$ be arbitrary. Choose $K_{lti} \in \mathcal{S}(P_1)$ so that it satisfies

$$\|S(1, K_{lti})\| \leq \alpha_{lti} + \frac{\delta}{3}.$$

From Proposition 6.1, for every $\varepsilon \in (0, \bar{\varepsilon})$, it follows directly that

$$\|S(1, K_{lti})W - S(g, \phi^{\varepsilon}(g)K_{lti})W\| \leq \gamma_0 \varepsilon;$$

define

$$\bar{\varepsilon}_1 := \min\left\{\bar{\varepsilon}, \frac{\delta}{3\gamma_0}\right\}$$

so

$$\|S(1, K_{lti})W - S(g, \phi^{\varepsilon}(g)K_{lti})W\| \leq \frac{\delta}{3}, \quad \varepsilon \in (0, \bar{\varepsilon}_1).$$

Fix $\varepsilon \in (0, \bar{\varepsilon}_1)$. From Proposition 6.2 it follows directly that there exist constants $\bar{T}_1 > 0$ and $\gamma_1 > 0$ such that for every $T \in (0, \bar{T}_1)$, the proposed controller $\mathcal{C}(K_{lti}, \varepsilon, T)$ stabilizes $\mathcal{P}$ and satisfies

$$\|S(g, K)W - S(g, \phi^{\varepsilon}(g)K_{lti})W\| \leq \gamma_1 T;$$

define

$$\bar{T}_2 := \min\left\{\bar{T}_1, \frac{\delta}{3\gamma_1}\right\}.$$

If we combine the above inequalities, then it follows that

$$\begin{aligned}
\|S(g, K)W\| &= \|S(g, K)W - S(g, \phi^{\varepsilon}(g)K_{lti})W + \\
&\quad\; S(g, \phi^{\varepsilon}(g)K_{lti})W - S(1, K_{lti})W + S(1, K_{lti})W\| \\
&\leq \|S(g, K)W - S(g, \phi^{\varepsilon}(g)K_{lti})W\| + \\
&\quad\; \|S(g, \phi^{\varepsilon}(g)K_{lti})W - S(1, K_{lti})W\| + \|S(1, K_{lti})W\| \\
&\leq \frac{\delta}{3} + \frac{\delta}{3} + (\alpha_{lti} + \frac{\delta}{3}) \\
&= \alpha_{lti} + \delta, \qquad T \in (0, \bar{T}_2).
\end{aligned}$$

■

Remark 6.3 *From the above proof we see that, for each K_{lti}, the controller $\mathcal{C}(K_{lti}, \varepsilon, T)$ recovers the performance $\|S(1, K_{lti})W\|$ for small values of ε and T.*

6.5 Examples

Here we discuss two examples to illustrate the proposed design. First, we will consider a setup that incorporates our standard two plants $\frac{-1}{s-1}$ and $\frac{1}{s-1}$; since it is minimum-phase, this problem is better solved by using the methods in [22] - we consider it for completeness. Our second example is more complicated and considers a second order, non-minimum phase nominal plant.

6.5.1 Example 1

We consider the nominal plant

$$P_1 : (A_p, B_p, C_p) = (1, 1, 1)$$

the compact set

$$G = [-\sqrt{2}, -1] \cup [1, \sqrt{2}],$$

the minimum time between switches

$$T_s = 2,$$

and the derivative bound

$$c_g = 1.$$

Observe that the relative degree of P_1 is $m = 1$. The class of signals to be tracked is modeled via the filter

$$W : A_w = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \ B_w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ C_w = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

We begin the controller design by selecting the (non-optimal) LTI compensator

$$K_{lti} = 10$$

and choosing the inter-sample time

$$h = 0.001.$$

We use the optimal approach outlined in [35] to find the polynomial

$$\phi^{0.011}(g(t)) = 0.3431g^5(t) - 1.5147g^3(t) + 2.1642g(t).$$

Finally, for simplicity, we choose the arbitrary constants $c_i = 1, \ i = 0, .., q$.

In this simulation we will consider the particular case where all initial conditions are zero, r is a square wave with period 4π:

$$r = \mathrm{sgn}(\cos(\pi/2)),$$

and g is a scaled sinusoid that switches between the positive and negative parts of G:

$$g(t) = \frac{\sqrt{2} - 1}{2} \sin(2t) + \frac{\sqrt{2} + 1}{2} \mathrm{sgn}(\cos(2t/3)).$$

We see from Figure 6.5 that the output follows the desired path quite well; furthermore, if we were to overlay the optimal output y, it would lie on top of our actual output. We suspect that it is possible to optimize the choice of c_i to reduce the size of the control signal, but that is beyond the scope of this thesis. Figure 6.6 provides a zoomed in view of the control signal to better illustrate the behaviour of the controller. Since this example was presented for completeness, we do not consider noise and non-zero initial conditions (we will do so in our next example).

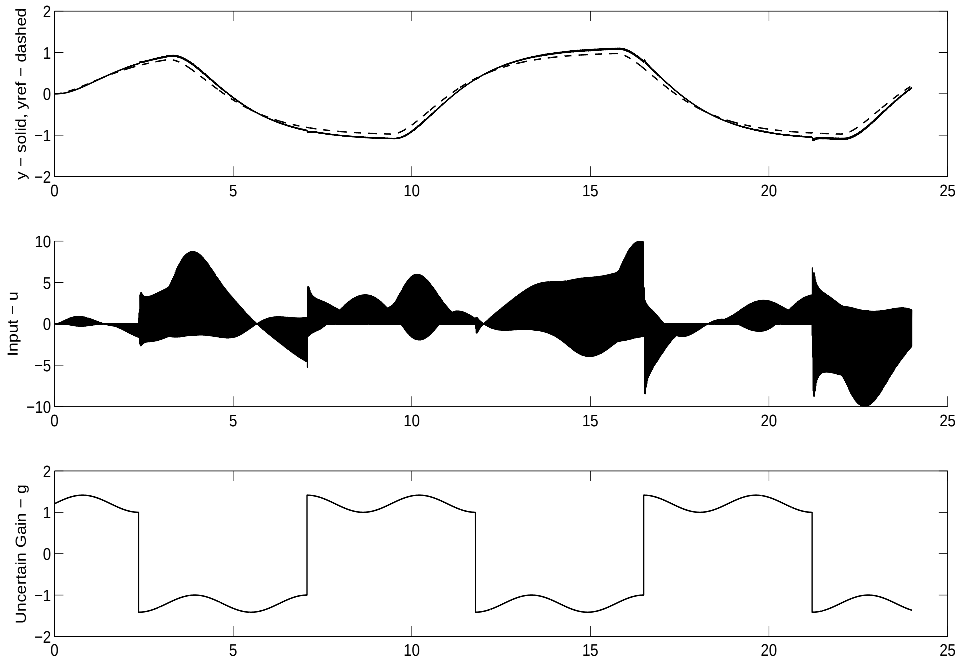

Figure 6.5: Example 1

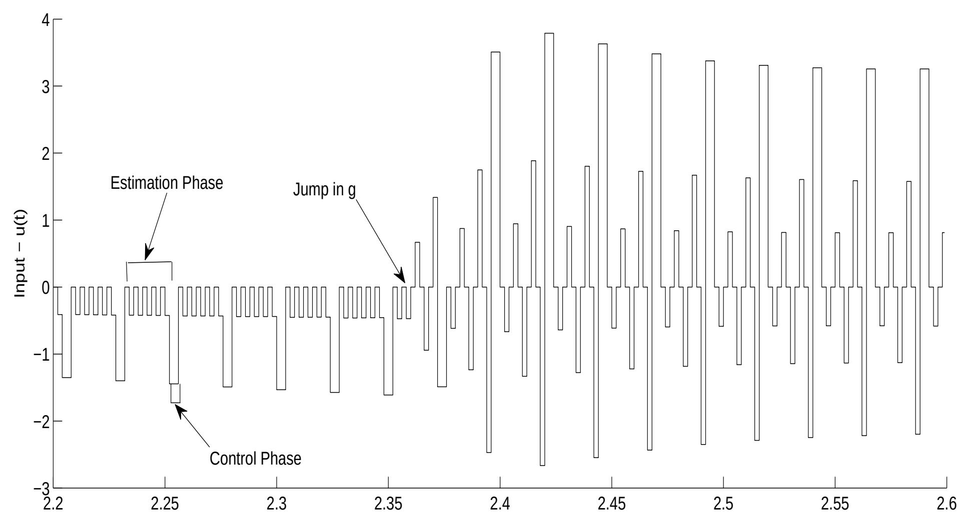

Figure 6.6: Example 1 - zoomed control signal u

6.5.2 Example 2 - A Non-Minimum Phase System

We now move to our second example. The methods in [22] cannot handle non-minimum phase plants, so here we consider the following unstable non-minimum phase nominal plant:

$$P_1 : A_p = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \; B_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C_p = \begin{bmatrix} -1 & 1 \end{bmatrix},$$

whose relative degree is $m = 1$. We consider the same set of admissible gains:

$$\mathcal{G}([-\sqrt{2}, -1] \cup [1, \sqrt{2}], 2, 1)$$

and filter:

$$W : A_w = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \; B_w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C_w = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Using the concepts outlined in [4] we find that the LTI optimal cost is 0.5.

We begin the controller design by selecting the LTI compensator

$$K_{lti} = -0.9\frac{s+1}{0.01s+1},$$

which provides a cost of 0.82. We use the same polynomial ϕ as in our first example. Finally, we again choose the arbitrary constants $c_i = 1$, $i = 0, .., q$.

In the first simulation we will consider the particular case where all initial conditions are zero, r is a square wave with period 8π, and g is the same as in the first example. We look at two different inter-sample times $h = 0.0001s$ and $h = 0.00005s$ and we see from Figures 6.7 and 6.8 that, in both cases, the output follows the desired path reasonably well, considering that our plant is unstable and non-minimum phase; furthermore, for smaller h the output more closely matches the optimal one, as expected (this is further highlighted in Figure 6.9). As in the first example, we suspect that it is possible to optimize the choice of c_i to reduce the size of the control signal, but that is beyond the scope of this thesis.

Finally, we turn off the exogenous input r and add noise on the plant output with size 10^{-4} from $t = 2.4s$ to $t = 10.8s$ and we consider $h = 0.0008s$ and $h = 0.0002s$. Results are shown in Figures 6.10 and 6.11. As usual, a smaller h leads to larger noise gains.

6.6 Controller Redesign - Ideas for Future Work

There are three obvious drawbacks to the above approach:

(i) As usual, the noise gains are large.

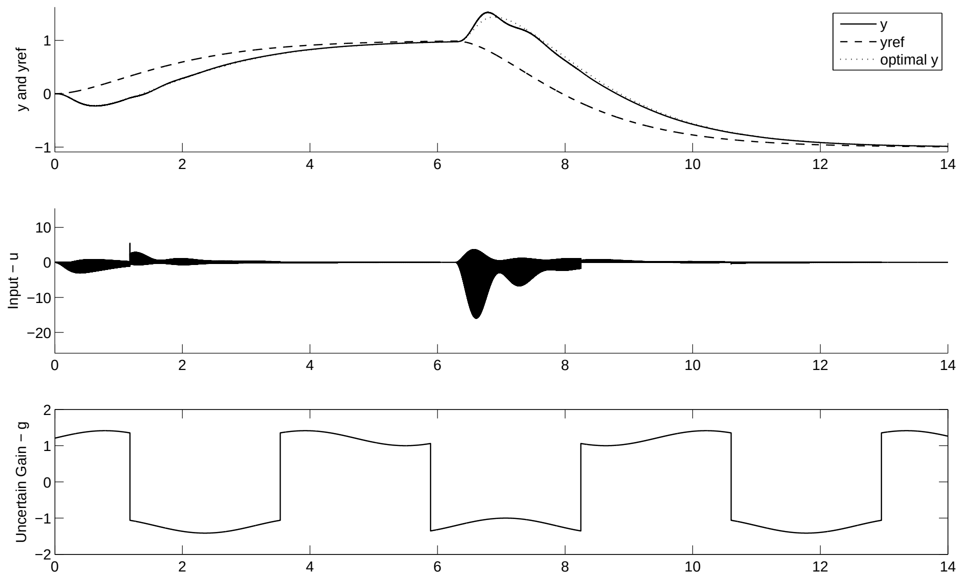

Figure 6.7: Example 2 - $h = 0.00005s$.

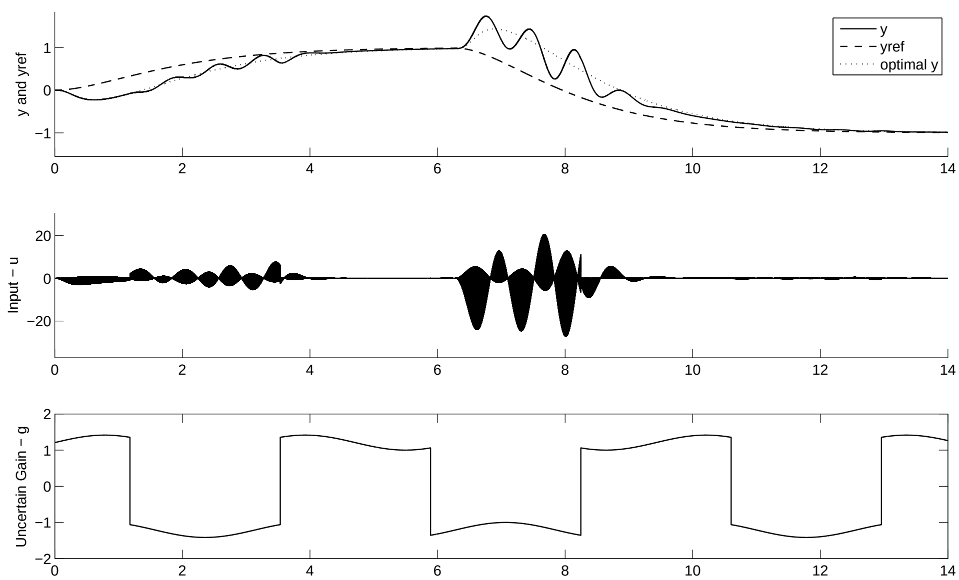

Figure 6.8: Example 2 - $h = 0.0001s$.

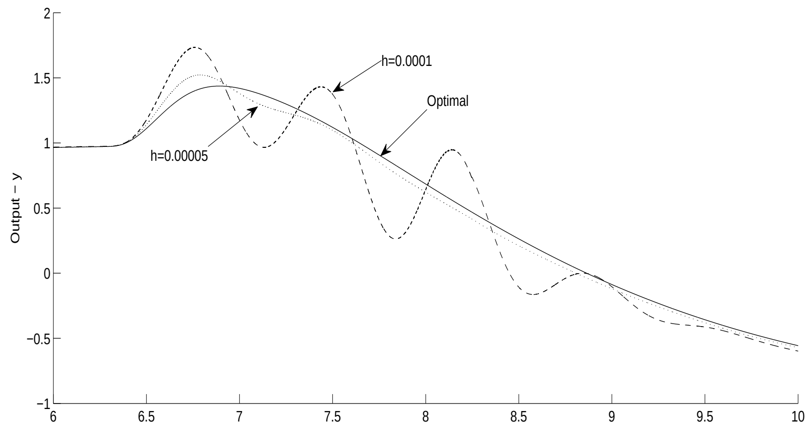

Figure 6.9: Example 2 - overlay of y from both cases of h.

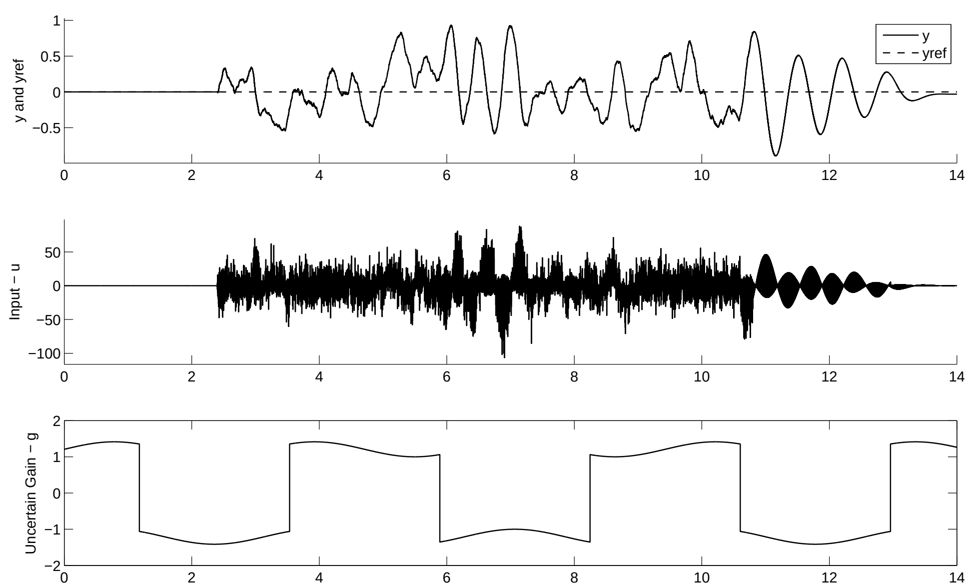

Figure 6.10: Example 2 with noise - $h = 0.00008s$.

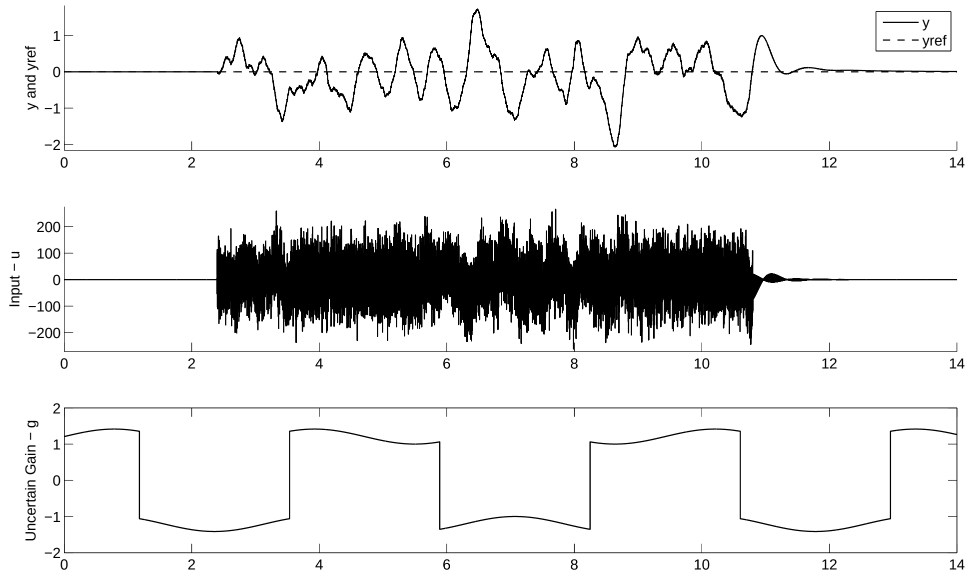

Figure 6.11: Example 2 with noise - $h = 0.00002s$.

(ii) The control signal is very 'jumpy' and requires fast actuators.

(iii) As the desired performance approaches the optimal one, the complexity of the controller increases (as $\varepsilon \to 0$ we typically have $q \to \infty$).

It is not clear how to address (i) or (iii) directly; indeed, (iii) appears to be an artifact of the design approach. To attempt to mitigate (ii), it is natural to try to perform estimation on top of the control signal; it turns out that a new estimation approach based on this idea [23] appears to solve (iii) and provide significant improvements to (i). The idea behind this approach is as follows: rather than estimate the optimal control signal, we attempt to estimate the **error** between our previous estimate and the optimal control signal. This is a promising avenue for future work. Since we do not yet have any details, we will provide a rough outline of the approach in order to illustrate the expected benefits.

It turns out that using a test signal of the form

$$u(t) = \begin{cases} \alpha + \beta & t \in [t_0, t_0 + \tau/2) \\ \alpha - \beta & t \in [t_0 + \tau/2, t_0 + \tau) \end{cases}$$

yields a good estimate of $2\beta g(t_0)$. If we define our estimate of the previous period's optimal control signal via

$$\hat{u}_k := \mathrm{Est}\big\{\phi^\varepsilon(g[(k-1)T])u^0[(k-1)T]\big\} \approx \frac{1}{g[(k-1)T]}u^0[(k-1)T],$$

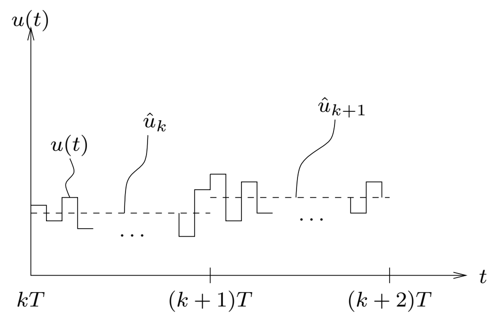

Figure 6.12: New Input Signal for Redesigned Controller.

and assume that it is a reasonably good estimate for not only the previous period, but for the current one as well[6], then we could imagine setting $\beta = \hat{u}_k$ and using a very small α to probe with. For example, with ρ a scaling factor, we could set

$$u(t) = \begin{cases} \hat{u}_k + \rho u^0[kT] & t \in [kT, kT + h_m) \\ \hat{u}_k - \rho u^0[kT] & t \in [kT + h_m, kT + 2h_m), \end{cases}$$

which yields a good estimate of $2\rho g[kT]u^0[kT]$, which we then feed back to obtain a good estimate of $2\rho g^2[kT]u^0[kT]$, and so on, from which we could generate our estimate of

$$\hat{u}_{k+1} = \mathrm{Est}\{\phi^\varepsilon u^0[kT]\}.$$

Figure 6.12 shows an example of this new control signal; observe that the probing takes place around the quantity $\hat{u}_k$ which is updated each period to reflect the new estimate.

Unfortunately, the above approach only has the effect of mitigating (ii). Instead we will try to use an estimate of the *error* between the previous estimate and the current optimal control signal to construct the optimal control signal: define

$$f[kT] := u^0[kT] - g[kT]\hat{u}_k,$$

so

$$\frac{1}{g[kT]} f[kT] = \frac{1}{g[kT]} u^0[kT] - \hat{u}_k$$

is the error signal that we wish to estimate. To do so, we first probe with $\hat{u}_k$ to generate an estimate of $g[kT]\hat{u}_k$, which we use (in conjunction with the known signal u^0) to construct a good estimate of f which we can use to estimate the quantites $g[kT]f[kT]$, $g^2[kT]f[kT]$, etc.. Finally, we combine these estimates to yield

$$\phi^\varepsilon(g[kT])f[kT] \approx \frac{1}{g[kT]} f[kT];$$

therefore, we can construct

$$\hat{u}_{k+1} := \mathrm{Est}\{\phi^\varepsilon(g[kT])f[kT]\} + \hat{u}_k \approx \frac{1}{g[kT]} u^0[kT].$$

[6]The validity of this assumption relies on the period being small.

At this point, it is unclear what all of this extra work has bought for us; however, observe that

$$\begin{aligned}\hat{u}_{k+1} &= \mathrm{Est}\big\{\phi^{\varepsilon}(g[kT])f[kT]\big\} + \hat{u}_k \\ &\approx \phi^{\varepsilon}(g[kT])f[kT] + \hat{u}_k \\ &= \phi^{\varepsilon}(g[kT])\big[u^0[kT] - g[kT]\hat{u}_k\big] + \hat{u}_k \\ &= \underbrace{\big(1 - \phi^{\varepsilon}(g[kT])g[kT]\big)}_{<\varepsilon}\hat{u}_k + \phi^{\varepsilon}(g[kT])u^0[kT],\end{aligned}$$

which clearly converges for any $\varepsilon < 1$. The upshot of this is that, to obtain near optimal performance, we only need our polynomial estimate to be good enough to ensure that $\varepsilon < 1$; indeed, it turns out that, if the sign of the gain is not known, then a first order polynomial is enough, while if the sign is known, then a constant is sufficient.

This discussion has many ramifications. First, complexity is no longer an issue and, (perhaps more importantly) since the complexity directly contributes to our noise problems, we expect significantly improved noise performance. Additionally, this controller performs estimation and control simultaneously. Finally, the resulting control signal is much more aesthetically pleasing. Of course, proving this result will require totally new tools and will not be straightforward since, unlike all of our previous controllers, this controller has memory from one period to the next.

6.7 Summary and Conclusions

In this chapter we show that we can obtain near LTI optimal weighted sensitivity in the face of time-varying uncertainty in the plant's input gain. To do so, we design a linear periodic RACE controller that requires a short period; this short period requirement leads to one major drawback: we expect poor noise tolerance. At the end of this chapter, we proposed a change to the estimation method which alleviates the noise issue while providing several totally unexpected benefits; for example, we no longer require small ε to obtain near optimal performance, so the polynomial order can be drastically reduced. This new method provides a very promising avenue for future work.

Chapter 7

Summary and Concluding Remarks

In this thesis the goal was to provide stability and near-optimal performance for two classes of time varying uncertain plants. The first class of time-varying plants arises from allowing (sufficiently slow and possibly persistent) switching between elements of a (possibly infinite) set of LTI plants, while the second class arises from allowing a fairly general time-varying gain at the input of an otherwise LTI nominal plant.

The key idea behind our control approach is to periodically estimate and then apply the *optimal control signal*. The resulting controllers adapt to changes in the plant parameters and are able to provide performance that is robust to uncertainty, so we refer to them as Robust Adaptive Control signal Estimation (RACE) Controllers.

For the first class of uncertainty, we start by designing a RACE controller to handle the simple case of a finite set of plants and then show that, with minor modifications, the same approach can be used to provide near-optimal step tracking. The estimation method is very straightforward and does not require probing. These controllers contain a mild nonlinearity to handle plant switches, allowing for larger controller periods and slower sampling than previous RACE approaches, leading to improved noise tolerance. We also provide an easily computable bound on how often switches are allowed.

We complete the investigation of the first class by considering the case of a compact set of LTI plants; estimation is significantly more complex in this case, requiring active probing. We consider only SISO plants and ignore noise in the analysis; furthermore, we only prove the performance result for the time-invariant case (i.e., we do not allow plant switches). As above, this controller also uses a mild nonlinearity to handle plant switches; however, here we require fast sampling (we can still allow large controller periods), so it is unclear how much of an improvement we should expect with respect to noise. That being said, in this context no previous RACE controllers have been shown to stabilize such a system in the face of persistent switching. A major drawback of this approach (shared with that of [25]) is that it is difficult to find the required polynomial approximation and its associated coefficients.

We conclude the thesis by investigating the second class of uncertainty; i.e., the TVGM problem. We allow the uncertain gain to be quite general and design a RACE controller to achieve the objectives of stability and near optimal performance in the face of that uncertainty. Although the problem requires active probing, our estimation method is simpler than that of the Compact Stability problem discussed above. Unlike the previous controllers, here we do not require the nonlinearity, so the closed loop system in linear periodic; however, we do require small controller periods, so we expect larger noise gains.

7.1 Future Work

Regarding the first class of stability, although our first two RACE controllers have smaller gains than their predecessors (and therefore we expect improved noise tolerance), it still remains to perform noise analysis with the goal of finding ways to reduce the noise gain. Additionally, we suspect that the robustness results of [26] could be extended to our setting, which would allow a framework similar to that of [42], in which more general time variations are modeled as piecewise constant ones, with the difference absorbed into unmodeled plant dynamics. Furthermore, in addition to the problem of LQR performance, [25] investigates the problem of pole placement and shows that the optimal control laws have similar structure. We expect that our results can be extended to the pole placement setting. Finally, we indicated that defining optimal performance in the context of plant switches is not straightforward; an alternative definition would be to consider the finite horizon cost, where the horizon is the period of the controller.

In the particular case of the Compact Stability problem, there are significant opportunities for future work beyond those indicated above. The most obvious is to complete the analysis in the same vein as the first two problems; i.e., prove I/O stability (in the face of noise) and near-nominal performance in the face of persistent plant switches. An additional avenue of future work is that of studying efficient ways of finding the polynomial H^{ε} and the coefficients $d_{i,j}$.

Finally, at the end of Chapter 6, we presented a high level outline of a very promising avenue of future work for all of our settings. The idea is to use the previous period's estimate as the baseline control signal and then probe on top of it with the goal of estimating the *error* between the current control signal and the optimal control signal; we then update the control signal for the next period accordingly. This approach was originally proposed as a way to perform estimation and control simultaneously with the goal of making the control signal less 'jumpy'; however, there are several totally unexpected and highly desirable side effects. Two of the most significant are that the noise gain is significantly reduced, as is the complexity of the controller. We expect that this approach can be applied to the TVGM problem as well as the Compact Stability problem; in the latter case this could lead to additional benefits since the complexity of the controller directly relates to the order of the polynomial approximation H^ε.

APPENDICES

Appendix A

Proofs from Chapter 3

Proof of Proposition 3.1:

Fix $T_s > 0$, $T \in (0, T_s/2)$, and $T' \in (0, T/2)$. Let $\sigma \in \Sigma_{T_s}$, $x_0 \in \mathbf{R}^n$, and $w \in \mathcal{PC}_\infty$ be arbitrary.

(i)

Let $k \in \{k_l : l \in \mathbf{N}\}^c$ be arbitrary. Observe that $\sigma(t)$ is constant on $[kT, (k+1)T)$; we denote its value by i, which means that the plant is P_i on this interval. We begin by solving (3.5) over $t \in [kT, kT + 2T')$. Note that, in this interval,

$$\hat{H}(t) = 0,$$

so

$$\nu(t) = 0$$

as well; therefore,

$$e(t + kT) = C_i e^{\hat{A}_i t} x[kT] + \underbrace{C_i \int_0^t e^{\hat{A}_i(t-\tau)} L_i \underline{w}_k(\tau) d\tau + \begin{bmatrix} 0 & I \end{bmatrix} \underline{w}_k(t)}_{=: f_i(\underline{w}_k, t)}, \quad t \in [0, 2T').$$

We substitute this into (3.11) and then use (3.16) to find that

$$v_1[k] = E_i x[kT] + \underbrace{\int_0^{T'} S(t) C_i f_i(\underline{w}_k, t) dt}_{=: \phi_{1,i}(\underline{w}_k)}, \tag{A.1}$$

similarly, for v_2 we find

$$v_2[k] = E_i x[kT] + \underbrace{\int_{T'}^{2T'} S(t) C_i f_i(\underline{w}_k, t) dt}_{=: \phi_{2,i}(\underline{w}_k)}. \tag{A.2}$$

Note that, in (A.1) and (A.2), v_1 and v_2 are implicit functions of i, $\underline{w}_k$, and $x[kT]$; with this in mind, we define χ_i to be the selector function

$$\chi_i(\underline{w}_k, x[kT]) := \begin{cases} 1 & \text{if } \operatorname{argmin}\{\|v_1[k]\|, \|v_2[k]\|\} = v_1[k] \\ 0 & \text{if } \operatorname{argmin}\{\|v_1[k]\|, \|v_2[k]\|\} = v_2[k], \end{cases}$$

so we can write

$$\begin{aligned} &\operatorname{argmin}\{\|v_1[k]\|, \|v_2[k]\|\} = \\ &\qquad E_i x[kT] + \chi_i(\underline{w}_k, x[kT])\phi_{1,i}(\underline{w}_k) + (1 - \chi_i(\underline{w}_k, x[kT]))\phi_{2,i}(\underline{w}_k); \end{aligned} \quad \text{(A.3)}$$

from (3.13) it follows that

$$\begin{aligned} \nu(t) &= \hat{H}(t)\big[E_i x[kT] + \chi_i(\underline{w}_k, x[kT])\phi_{1,i}(\underline{w}_k) + (1 - \chi_i(\underline{w}_k, x[kT]))\phi_{2,i}(\underline{w}_k)\big] \\ &= \hat{H}_i(t)x[kT] + \hat{H}(t)\big[\chi_i(\underline{w}_k, x[kT])\phi_{1,i}(\underline{w}_k) + (1 - \chi_i(\underline{w}_k, x[kT]))\phi_{2,i}(\underline{w}_k)\big], \\ &\qquad\qquad t \in [kT, (k+1)T). \end{aligned}$$

Substituting this into the first equation of (3.5) and using the fact that

$$\sigma(t) = i, \quad t \in [kT, (k+1)T)$$

yields the desired equation:

$$\begin{aligned} \dot{x}(t) &= \hat{A}_{\sigma(t)}x(t) + B_{\sigma(t)}\hat{H}_{\sigma(t)}(t)x[kT] + \begin{bmatrix} B_{\sigma(t)}\hat{H}(t) & L_{\sigma(t)} \end{bmatrix} \times \\ &\quad \begin{bmatrix} \chi_{\sigma(t)}(\underline{w}_k, x[kT])\phi_{1,\sigma(t)}(\underline{w}_k) + [1 - \chi_{\sigma(t)}(\underline{w}_k, x[kT])]\phi_{2,\sigma(t)}(\underline{w}_k) \\ w(t) \end{bmatrix}, \\ &\qquad\qquad t \in [kT, (k+1)T). \end{aligned}$$

Last of all, it is straight-forward, though tedious, to show that $\phi_{1,i}$ and $\phi_{2,i}$ have bounded gain.

(ii)

Let $l \in \mathbf{N}$ be arbitrary. To obtain the desired result we consider three cases.

Case 1: Switch occurs in the Control Phase: $t_l \in [k_lT + 2T', (k_l+1)T)$.

Here the samplers are not affected by the switch, so (A.1), (A.2), and (A.3) hold (with $k = k_l$ and $i = i_{l-1}$); since $\phi_{i,1}$ and $\phi_{i,2}$ have bounded gain for every $i = 1, ..q$, it follows that there exists a constant $\gamma_1(T, T') > 0$ that is independent of i such that

$$\min\{\|v_1[k_l]\|, \|v_2[k_l]\|\} \le \|x[k_lT]\| + \gamma_1(T, T')\|w\|_\infty \quad \text{(A.4)}$$

Case 2: Switch occurs during the second sample: $t_l \in [k_lT + T', k_lT + 2T')$.

Here, the first sampler is not affected; therefore, (A.1) still holds and it follows that we again obtain (A.4).

Case 3: Switch occurs during the first sample: $t_l \in [k_lT, k_lT + T')$.

Here, we have two possibilities: $t_l = k_lT$ and $t_l > k_lT$[1]. We begin by considering the former. Here $\sigma(t)$ is constant on $[k_lT, k_lT + T')$, so it follows that (A.1) and (A.2) hold (this time with $k = k_l$ and $i = i_l$) and we again obtain (A.4).

The other case is that of $t_l \in (k_lT, k_lT + T')$. Unfortunately, the first sampler yields a possibly large output, and, due to the plant switch, (A.2) does not hold. Fortunately, we can still obtain a nice bound on $\|v_2[k_l]\|$. We begin by recognizing that, since $t_l < k_lT + T'$, we have

$$\begin{aligned} e(t) &= C_{i_l} e^{\hat{A}_{i_l}(t-k_lT-T')} x[k_lT+T'] + \\ &\quad \underbrace{C_{i_l} \int_{k_lT+T'}^{t} e^{\hat{A}_{i_l}(t-\tau)} L_{i_l} w(\tau) d\tau + \left[\begin{array}{cc} 0 & I \end{array}\right] w(t),}_{=: g_{0,i_l}(w,T,T',t)} \\ &\qquad\qquad\qquad\qquad t \in [k_lT + T', k_lT + 2T') \end{aligned}$$

so from (3.12), (3.17), and (3.18) we have

$$\begin{aligned} v_2[k_l] &= \int_{k_lT+T'}^{k_lT+2T'} S(t) e(t)\, dt \\ &= E_{i_l} e^{-A_{i_l}T'} x[k_lT+T'] + \underbrace{\int_{k_lT+T'}^{k_lT+2T'} S(t) g_{0,i_l}(w,T,T',t) dt}_{=: g_{1,i_l}(T,T',w)} . \qquad \text{(A.5)} \end{aligned}$$

We would like to write our bound in terms of $x[k_lT]$ instead of $x[k_lT + T']$. To do so, we first observe that

$$x[k_lT+T'] = e^{\hat{A}_{i_l}(k_lT+T'-t_l)} x(t_l) + \underbrace{\int_{t_l}^{k_lT+T'} e^{\hat{A}_{i_l}(k_lT+T'-\tau)} L_{i_l} w(\tau) d\tau}_{=: g_{2,i_l}(T,T',w,t_l)}$$

and

$$x(t_l^)= e^{\hat{A}_{i_{l-1}}(t_l-k_lT)} x[k_lT] + \underbrace{\int_{k_lT}^{t_l} e^{\hat{A}_{i_{l-1}}(t_l-\tau)} L_{i_{l-1}} w(\tau) d\tau}_{=: g_{3,i_l}(T,T',w,t_l)} .$$

If we define

$$\begin{aligned} \hat{\phi}_{i_l}(T,T',w,t_l) &:= g_{1,i_l}(T,T',w) + E_{i_l} e^{-\hat{A}_{i_l}T'} g_{2,i_l}(T,T',w,t_l) + \\ &\quad E_{i_l} e^{-\hat{A}_{i_l}(t_l-k_lT)} g_{3,i_l}(T,T',w,t_l) \end{aligned}$$

[1]The astute reader may be concerned that our result will not hold if σ is such that $t_1 < T$. Recall that there is no discontinuity at t_0, so even if $t_1 < T$, there can be at most one discontinuity on $[0, T)$, occurring at t_1.

then these two equalities combine with (A.5) to provide

$$v_2[k_l] \quad = \quad E_{i_l} e^{-\hat{A}_{i_l}(t_l - k_l T)} e^{\hat{A}_{i_{l-1}}(t_l - k_l T)} x[k_l T] + \hat{\phi}_{i_l}(T, T', w).$$

Since there are a finite number of plants, it is straightforward, though tedius, to show that $\hat{\phi}_{i_l}$ has the property that there exists a constant $\gamma_2(T, T') > 0$ such that

$$\max_{t \in [k_l T, (k_l+1)T]} \|\hat{\phi}_{i_l}(T, T', w, t)\| \leq \gamma_2(T, T')\|w\|_\infty, \quad l \in \mathbf{N},$$

so we have

$$\|v_2[k_l]\| \quad \leq \quad \max_{t \in [0, T')} \left\| e^{-\hat{A}_{i_l} t} e^{\hat{A}_{i_{l-1}} t} \right\| \|x[k_l T]\| + \gamma_2(T, T')\|w\|_\infty;$$

using the definition of $\rho(T')$ we obtain

$$\|v_2[k_l]\| \leq \rho(T')\|x[k_l T]\| + \gamma_2(T, T')\|w\|_\infty. \tag{A.6}$$

Since we clearly have

$$\rho(T') \geq 1$$

we can combine (A.4) and (A.6) to find that, no matter whether or not $t_l = k_l T$ we have

$$\begin{aligned} \min\{\|v_1[k_l]\|, \|v_2[k_l]\|\} \quad \leq \quad & \rho(T')\|x[k_l T]\| + \\ & \underbrace{\max\{\gamma_1(T, T'), \gamma_2(T, T')\}}_{=:\gamma_v(T, T')} \|w\|_\infty. \end{aligned} \tag{A.7}$$

Clearly, (A.7) holds in all three cases, so it provides the desired bound. ∎

Proof of Theorem 3.4:

Fix $\varepsilon > 0$, $T_s > \underline{T_s}$, and $w = 0$; let $\sigma \in \Sigma_{T_s}$ and $T \in (0, \bar{T}(T_s))$ be arbitrary. Stability follows directly from Theorem 3.2. Since T_s is fixed, to reduce clutter we write $\bar{T}$ instead of $\bar{T}(T_s)$. We set

$$\rho := \rho(T_s/4).$$

As in the proof of Theorem 3.3, we define

$$\tilde{x} := x - x^0$$

and

$$\tilde{\nu} := \nu - \nu^0.$$

Finally observe that it is enough to prove that, if T is sufficiently small, then there exists a constant $\bar{T}'(T) \in (0, T/2)$ such that, for every $T' \in (0, \bar{T}'(T))$ we have the desired result.

We first deal with the special case of $l = 0$.

Claim 0: There exists a constant $\bar{T}_0'(T) \in (0, T/2)$ so that, if $T' \in (0, \bar{T}_0(T))$, then

$$|J_{[0,t_1)}(x(t_l)) - J^0_{[0,t_1)}(x(t_l))| \leq \varepsilon \|x_0\|^2, \quad x_0 \in \mathbf{R}^n.$$

Proof:

On the interval $[0, t_1)$, $\sigma(t)$ is constant, so Theorem 3.3 can be applied: it states that, if T' is sufficiently small then, irrespective of the value of $\sigma(t)$ on $[0, t_1)$, we have

$$|J_{[0,t_1)}(x(t_l)) - J^0_{[0,t_1)}(x(t_l))| \leq \varepsilon \|x(t_0)\|^2, \quad x_0 \in \mathbf{R}^n,$$

so the result follows immediately. ∎

Before we move on, recall that, with ε_H given by (3.23), from Lemma 3.1 we have that

$$\|\tilde{H}_i(t)\| \leq \varepsilon_H(T, T'), \quad t \in [2T', T), \quad T' \in (0, T/2) \quad i = 1, .., q,$$

observe that, from the definition of $\hat{H}$ we have that

$$\hat{H}(t) = \begin{cases} 0 & t \in [0, 2T') \\ H(t) + \tilde{H}(t) & t \in [2T', T), \end{cases}$$

so clearly

$$\|\hat{H}\|_\infty \leq f\gamma_0 + \varepsilon_H(T, T'), \quad T' \in (0, T/2).$$

Unlike in the proof of Theorem 3.3, here we will not need to (explicitly) make ε_H small; indeed, it will be enough to simply bound $\|\hat{H}\|_\infty$. To that end, we observe that Lemma 3.1 says that, for every $T \in (0, \bar{T})$,

$$\lim_{T' \to 0} \varepsilon_H(T, T') = 0,$$

so, for every $T \in (0, \bar{T})$, there exists a constant $\bar{T}_1'(T) \in (0, \bar{T}_0'(T))$ so that

$$\|\hat{H}\|_\infty \leq f\gamma_0 + 1, \quad T' \in (0, \bar{T}_1'(T)). \tag{A.8}$$

We now turn to the general case: let $l \in \mathbf{N}$ be arbitrary. It will be useful to partition each interval into two parts; this is best illustrated via Figure A.1. We will be able to leverage Theorem 3.3 to provide a nice result for the second part of the interval shown in Figure A.1, but to do so we must first investigate two important issues that do not arise when there are no switches:

(i) From Proposition 3.1(ii), we know that in intervals with a switch, the size of the controller output depends on $x[k_l T]$; however, we wish to obtain results in terms of $x(t_l)$.

(ii) The control applied during $[t_l, (k_l + 1)T)$ will likely be wrong, so, unlike the case where there are no switches, we will likely **not** have $x[(k_l + 1)T] = x^0[(k_l + 1)T]$.

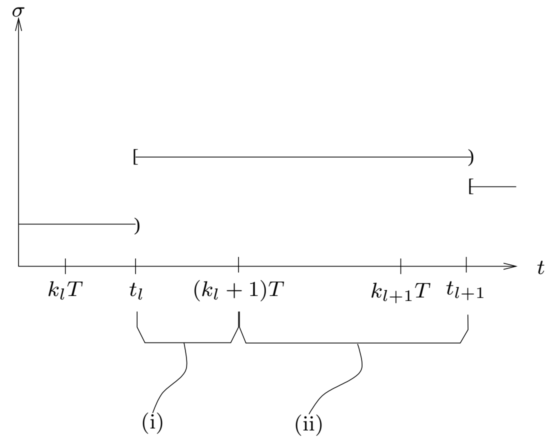

Figure A.1: Partitioning of one interval - time axis is not to scale. Here (i) is the portion of the period immediately following a switch, while (ii) contains the remainder of the period.

We begin by investigating (i).

Claim 1: There exists a constant $\gamma_1 > 0$ such that, if T is sufficiently small, then, for every $T' \in (0, \bar{T}_1'(T))$, we have

$$\|x[k_lT]\| \leq \gamma_1 \|x(t_l)\|.$$

Proof:

Let $T' \in (0, \bar{T}_1'(T))$ be arbitrary. If $t_l = k_lT$, then we conclude that the result is trivially true as long as $\gamma_1 \geq 1$. If $t_l > k_lT$, then we proceed by solving (3.5) backwards in time to yield

$$x[k_lT] = e^{\hat{A}_{i_{(l-1)}}(k_lT - t_l)} x(t_l) + \int_{t_l}^{k_lT} e^{\hat{A}_{i_{(l-1)}}(k_lT-\tau)} B_{i_{(l-1)}} \nu(\tau) d\tau. \tag{A.9}$$

Since the period $[k_lT, (k_l+1)T)$ contains a switch, we use Proposition 3.1(ii) to bound the size of the sampler output, yielding

$$\|\nu(t)\| \leq \|\hat{H}(t)\| \rho \|x[k_lT]\|, \quad t \in [k_lT, (k_l+1)T). \tag{A.10}$$

We can take norms on both sides of (A.9), use (A.8) to bound $\|\hat{H}(t)\|$, and simplify, yielding

$$\|x[k_lT]\| \leq e^{aT} \|x(t_l)\| + \int_{k_lT}^{t_l} e^{aT} b(f\gamma_0 + 1)\rho \|x[k_lT]\| \, d\tau;$$

therefore,

$$\|x[k_lT]\| \leq e^{a\bar{T}} \|x(t_l)\| + T e^{a\bar{T}} b(f\gamma_0 + 1)\rho \|x[k_lT]\|.$$

With

$$\alpha := e^{a\bar{T}} \max\{1, b\rho(f\gamma_0 + 1)\},$$

it follows that

$$\|x[k_lT]\| \le \alpha\big[\|x(t_l)\| + T\|x[k_lT]\|\big]$$

and therefore

$$\big[1-\alpha T\big]\|x[k_lT]\| \le \alpha\|x(t_l)\|$$

Clearly

$$\begin{aligned}\|x[k_lT]\| &\le \frac{\alpha}{1-\alpha T}\|x(t_l)\| \\ &\le 2\alpha\|x(t_l)\|, \quad T \in \left(0, \min\left\{\bar{T}, \frac{1}{2\alpha}\right\}\right).\end{aligned}$$

■

We now investigate the difference between the nominal and the actual state at $(k_l+1)T$.

Claim 2: There exists a constant $\gamma_2 > 0$ such that, if T is sufficiently small, then, for every $T' \in (0, \bar{T}_1'(T))$, we have

$$\|\tilde{x}(t)\| \le \gamma_2 T\|x(t_l)\|, \quad t \in [t_l, (k_l+1)T]$$

and

$$\|x(t)\| \le \gamma_2\|x(t_l)\|, \quad t \in [t_l, (k_l+1)T].$$

Proof:

Let $T' \in (0, \bar{T}_1'(T))$ be arbitrary. Using the definitions of x^0 and ν^0 given by (3.52) and (3.51) respectively, we obtain

$$\dot{x}^0(t) = \hat{A}_{i_l}x^0(t) + B_{i_l}\nu^0(t), \quad t \in [t_l, t_{l+1}]$$

which combines with (3.5) to yield

$$\dot{\tilde{x}}(t) = \hat{A}_{i_l}\tilde{x}(t) + B_{i_l}\tilde{\nu}(t), \quad t \in [t_l, t_{l+1}].$$

Solving this and using the fact that $x(t_l) = x^0(t_l)$, we find that

$$\tilde{x}(t) = \int_{t_l}^{t} e^{\hat{A}_{i_l}(t-\tau)}B_{i_l}\tilde{\nu}(\tau)d\tau, \quad t \in [t_l, t_{l+1}];$$

therefore,

$$\tilde{x}(t) = \int_{t_l}^{t} e^{\hat{A}_{i_l}(t-\tau)}B_{i_l}\left(\nu(\tau) - H_{i_l}(\tau)x(t_l)\right)d\tau, \quad t \in [t_l, (k_l+1)T] \subset [t_l, t_{l+1}].$$

We then take the norm of both sides and use (A.10) and (A.8) to yield

$$\begin{aligned}\|\tilde{x}(t)\| &\le \int_0^T e^{aT}b\left[(f\gamma_0+1)\,\rho\|x[k_lT]\| + f\gamma_0\|x(t_l)\|\right]d\tau \\ &\le Te^{a\bar{T}}b\big[(f\gamma_0+1)\rho\|x[k_lT]\| + f\gamma_0\|x(t_l)\|\big], \quad t \in [t_l, (k_l+1)T]\end{aligned}$$

to which we apply Claim 1 to find that, if T is sufficiently small, then

$$\|\tilde{x}(t)\| \leq \underbrace{e^{a\bar{T}}b\big[(f\gamma_0+1)\rho\gamma_1+f\gamma_0\big]}_{=:\alpha_1}T\|x(t_l)\|, \quad t\in[t_l,(k_l+1)T],$$

which provides the first desired result. To obtain the second desired result, observe that it follows immediately from the definition of x^0 that

$$\|x^0(t)\| \leq \gamma_0\|x^0(t_l)\|, \quad t\in[t_l,(k_l+1)T],$$

so we conclude that, if $T\in(0,\bar{T})$ is sufficiently small, then

$$\begin{aligned}\|x(t)\| &\leq \|\tilde{x}(t)\|+\|x^0(t)\| \\ &\leq \underbrace{(\alpha_1\bar{T}+\gamma_0)}_{=:\alpha_2}\|x(t_l)\|, \qquad t\in[t_l,(k_l+1)T].\end{aligned}$$

■

As indicated earlier, we would like to leverage Theorem 3.3; since the period $[k_lT,(k_l+1)T)$ contains a switch, we will not be able to do so for the interval $[t_l,(k_l+1)T)$. Recall that the proof of Theorem 3.3 was motivated by (3.50) and that we found bounds on $\|\tilde{x}(t)\|$, $\int\|\nu(t)\|dt$, and $\int\|\nu(t)\|^2dt$ to find the desired result; we will do the same for the interval $[t_l,(k_l+1)T)$. Observe that Claim 2 already provides a nice bound on $\|\tilde{x}(t)\|$; we now turn to $\int\|\nu(t)\|dt$ and $\int\|\nu(t)\|^2dt$:

Claim 3: There exists a constant $\gamma_3>0$ such that, if T is sufficiently small, then, for every $T'\in(0,\bar{T}_1'(T))$ we have

$$\int_{t_l}^{(k_l+1)T}\|\tilde{\nu}(t)\|dt \leq \gamma_3T\|x(t_l)\|$$

and

$$\int_{t_l}^{(k_l+1)T}\|\tilde{\nu}(t)\|^2dt \leq \gamma_3T\|x(t_l)\|^2.$$

Proof:

Let $T' \in (0, \bar{T}_1'(T))$ be arbitrary. Observe that

$$\begin{aligned}
\int_{t_l}^{(k_l+1)T} \|\tilde{\nu}(t)\| dt &= \int_{t_l}^{(k_l+1)T} (\|\nu(t) - \nu^0(t)\|) dt \\
&\leq \int_{t_l}^{(k_l+1)T} (\|\nu(t)\| + \|\nu^0(t)\|) dt \\
&\leq \int_{t_l}^{(k_l+1)T} (\|\nu(t)\| + f\gamma_0 \|x(t_l)\|) dt,
\end{aligned}$$

so using (A.10) and (A.8) we find that

$$\begin{aligned}
\int_{t_l}^{(k_l+1)T} \|\tilde{\nu}(t)\| dt &\leq \int_{t_l}^{(k_l+1)T} \left[(f\gamma_0 + 1)\rho \|x[k_l T]\| + f\gamma_0 \|x(t_l)\| \right] dt \\
&\leq T \left[(f\gamma_0 + 1)\rho \|x[k_l T]\| + f\gamma_0 \|x(t_l)\| \right].
\end{aligned}$$

We now apply Claim 1 to find that, if $T > 0$ is sufficiently small, then

$$\int_{t_l}^{(k_l+1)T} \|\tilde{\nu}(t)\| dt \leq T \Big\{ \underbrace{\left[(f\gamma_0 + 1)\rho\gamma_1 + f\gamma_0 \right]}_{=:\alpha_1} \|x(t_l)\| \Big\}.$$

Similarly, we find that, if $T > 0$ is sufficiently small, then

$$\begin{aligned}
\int_{t_l}^{(k_l+1)T} \|\tilde{\nu}(t)\|^2 dt &\leq \int_{t_l}^{(k_l+1)T} (\|\nu(t)\| + \|\nu^0(t)\|)^2 dt \\
&\leq \int_{t_l}^{(k_l+1)T} \left[(f\gamma_0 + 1)\rho\gamma_1 + f\gamma_0 \right]^2 \|x(t_l)\|^2 dt \\
&\leq T\alpha_1^2 \|x(t_l)\|^2.
\end{aligned}$$

■

With $\gamma_{lqr} > 0$ defined in (3.48), for every interval $[\underline{t}, \bar{t}) \subset [t_l, t_{l+1})$, it is routine to confirm that the procedure used to derive (3.49) can be applied here to show that

$$\begin{aligned}
&\left| J_{[\underline{t},\bar{t})}(x(t_l)) - J^0_{[\underline{t},\bar{t})}(x(t_l)) \right| \leq \\
&\qquad \gamma_{lqr} \int_{\underline{t}}^{\bar{t}} \big(\|x^0(t)\| \|\tilde{x}(t)\| + \|\tilde{x}(t)\|^2 + \|\nu^0(t)\| \|\tilde{\nu}(t)\| + \|\tilde{\nu}(t)\|^2 + \\
&\qquad\qquad \|\tilde{\nu}(t)\| \|\tilde{x}(t)\| + \|\nu^0(t)\| \|\tilde{x}(t)\| + \|\tilde{\nu}(t)\| \|x^0(t)\| \big) dt.
\end{aligned}$$

We now apply the definitions of x^0 and ν^0 found in (3.52) and (3.51) yielding

$$\begin{aligned}
&\left| J_{[\underline{t},\bar{t})}(x(t_l)) - J^0_{[\underline{t},\bar{t})}(x(t_l)) \right| \leq \\
&\qquad \gamma_{lqr} \int_{\underline{t}}^{\bar{t}} \big[\gamma_0 (1 + f) e^{\lambda_0 (t - t_l)} \|x(t_l)\| \left(\|\tilde{x}(t)\| + \|\tilde{\nu}(t)\| \right) + \\
&\qquad\qquad \|\tilde{x}(t)\|^2 + \|\tilde{\nu}(t)\|^2 + \|\tilde{\nu}(t)\| \|\tilde{x}(t)\| \big] dt.
\end{aligned}$$

If we apply Claims 2 and 3, then we find that there exists a constant $\alpha_3 > 0$ such that, if $T > 0$ is sufficiently small and $T' \in (0, \bar{T}_1'(T))$, we have

$$\left| J_{[t_l,(k_l+1)T)}(x(t_l)) - J^0_{[t_l,(k_l+1)T)}(x(t_l)) \right| \leq \alpha_3 T \|x(t_l)\|^2. \tag{A.11}$$

It remains to analyze the second part of the interval, namely $[(k_l + 1)T, t_{l+1})$.

Claim 4: If T is sufficiently small, then there exists a constant $\bar{T}_2'(T) \in (0, \bar{T}_1'(T))$ so that, for every $T' \in (0, \bar{T}_2'(T))$, we have that

$$\left| J_{[(k_l+1)T,t_{l+1})}(x(t_l)) - J^0_{[(k_l+1)T,t_{l+1})}(x(t_l)) \right| \leq \frac{\varepsilon}{2} \|x(t_l)\|^2.$$

Proof:

On the interval of interest, namely $[(k_l + 1)T, t_{l+1})$, there are no switches. Recall that, in Theorem 3.3 we showed that we can obtain a nice performance bound when there are no switches; however, there we had the nice property that $x[(k_l + 1)T] = x^0[(k_l + 1)T]$, which is typically not the case here. Nonetheless, we would like to leverage Theorem 3.3, so we define

$$\hat{x}^0(t) := e^{\bar{A}_{i_l}(t-(k_l+1)T)} x[(k_l + 1)T], \quad t \in [(k_l + 1)T, t_{l+1}), \tag{A.12}$$

$$\hat{\nu}^0(t) := F_{i_l} e^{\bar{A}_{i_l}(t-(k_l+1)T)} x[(k_l + 1)T], \quad t \in [(k_l + 1)T, t_{l+1}), \tag{A.13}$$

and

$$\hat{e}^0(t) := C_{i_l} \hat{x}^0(t), \quad t \in [(k_l + 1)T, t_{l+1}),$$

and then define

$$\hat{J}^0_{[(k_l+1)T,t_{l+1})}(x(t_l)) := \int_{(k_l+1)T}^{t_{l+1}} M_{i_l}(\hat{x}^0(t), \hat{\nu}^0(t), \hat{e}^0(t)) dt.$$

Claim 2 says that, if $T > 0$ is sufficiently small, then for every $T' \in (0, \bar{T}_1'(T))$, we have

$$\|x[(k_l + 1)T]\| \leq \gamma_2 \|x(t_l)\|;$$

since our closed loop system is periodic with period T, we can combine this with Theorem 3.3 applied to the interval $[(k_l + 1)T, t_{l+1})$ to find that, if T is sufficiently small, then there exists a constant $\bar{T}_2'(T) \in (0, \bar{T}_1'(T))$ so that, for every $T' \in (0, \bar{T}_2'(T))$, we have

$$\begin{aligned}
&\left| J_{[(k_l+1)T,t_{l+1})}(x(t_l)) - \hat{J}^0_{[(k_l+1)T,t_{l+1})}(x(t_l)) \right| \\
&\qquad \leq \int_{(k_l+1)T}^{t_{l+1}} \|M_{i_l}(x(t), \nu(t), e(t)) - M_{i_l}(\hat{x}^0(t), \hat{\nu}^0(t), \hat{e}^0(t))\| dt \\
&\qquad \leq \frac{\varepsilon}{4\gamma_2^2} \|x[(k_l + 1)T]\|^2 \\
&\qquad \leq \frac{\varepsilon}{4} \|x(t_l)\|^2.
\end{aligned} \tag{A.14}$$

We now find a relationship between $\hat{J}^0(x(t_l))$ and the nominal cost $J^0(x(t_l))$. We define

$$\tilde{x}^0 := \hat{x}^0 - x^0$$

and

$$\tilde{\nu}^0 := \hat{\nu}^0 - \nu^0;$$

as before, it is routine to confirm that the procedure used to derive (3.49) can be applied here to show that

$$\begin{aligned}
&\left|\hat{J}^0_{[(k_l+1)T,t_{l+1})}(x(t_l)) - J^0_{[(k_l+1)T,t_{l+1})}(x(t_l))\right| \leq \\
&\quad \gamma_{lqr}\int_{(k_l+1)T}^{t_{l+1}} \big(\|x^0(t)\|\|\tilde{x}^0(t)\| + \|\tilde{x}^0(t)\|^2 + \|\nu^0(t)\|\|\tilde{\nu}^0(t)\| + \|\tilde{\nu}^0(t)\|^2 + \\
&\qquad \|\tilde{\nu}^0(t)\|\|\tilde{x}^0(t)\| + \|\nu^0(t)\|\|\tilde{x}^0(t)\| + \|\tilde{\nu}^0(t)\|\|x^0(t)\|\big)\, dt.
\end{aligned}$$

From the definitions (3.52), (A.12), (3.51), and (A.13) we have

$$\begin{aligned}
\|x^0(t)\| &\leq \gamma_0 e^{\lambda_0(t-(k_l+1)T)}\|x^0[(k_l+1)T]\|, \\
\|\tilde{x}^0(t)\| &\leq \gamma_0 e^{\lambda_0(t-(k_l+1)T)}\|\tilde{x}^0[(k_l+1)T]\|, \\
\|\nu^0(t)\| &\leq f\gamma_0 e^{\lambda_0(t-(k_l+1)T)}\|x^0[(k_l+1)T]\|, \\
\|\tilde{\nu}^0(t)\| &\leq f\gamma_0 e^{\lambda_0(t-(k_l+1)T)}\|\tilde{x}^0[(k_l+1)T]\|, \quad t \in [(k_l+1)T, t_{l+1}),
\end{aligned}$$

which means that

$$\begin{aligned}
&\left|\hat{J}^0_{[(k_l+1)T,t_{l+1})}(x(t_l)) - J^0_{[(k_l+1)T,t_{l+1})}(x(t_l))\right| \\
&\leq \gamma_{lqr}\gamma_0^2 \left[\int_{(k_l+1)T}^{t_{l+1}} e^{2\lambda_0(t-(k_l+1)T)} dt\right] \Big[(1+f+f^2)\|\tilde{x}^0[(k_l+1)T]\|^2 + \\
&\qquad (1+2f+f^2)\|\tilde{x}^0[(k_l+1)T]\| \times \|x^0[(k_l+1)T]\|\Big] \\
&\leq \gamma_{lqr}\gamma_0^2(1+f)^2 \frac{1}{2|\lambda_0|} \times \\
&\qquad \big[\|\tilde{x}^0[(k_l+1)T]\|^2 + \gamma_0\|\tilde{x}^0[(k_l+1)T]\| \times \|x^0(t_l)\|\big]. \qquad \text{(A.15)}
\end{aligned}$$

By definition

$$\hat{x}^0[(k_l+1)T] = x[(k_l+1)T],$$

so it follows immediately that

$$\tilde{x}^0[(k_l+1)T] = \tilde{x}[(k_l+1)T],$$

so we can apply Claim 2 to obtain a bound on $\|\tilde{x}^0[(k_l+1)T]\|$ in (A.15): it follows that, if T is sufficiently small, then for every $T' \in (0, \bar{T}_2'(T))$ we have

$$\begin{aligned}
&\left|\hat{J}^0_{[(k_l+1)T,t_{l+1})}(x(t_l)) - J^0_{[(k_l+1)T,t_{l+1})}(x(t_l))\right| \leq \\
&\qquad \gamma_{lqr}\gamma_0^2(1+f)^2 \frac{1}{2|\lambda_0|}\Big[\gamma_2^2 T^2\|x(t_l)\|^2 + \gamma_0\gamma_2 T\|x(t_l)\|^2\Big].
\end{aligned}$$

If we define

$$\gamma_4 := \gamma_{lqr}\gamma_0^2(1+f)^2\frac{1}{2|\lambda_0|}\max\{\gamma_2^2\bar{T},\gamma_0\gamma_2\}$$

then

$$\left|\hat{J}^0_{[(k_l+1)T,t_{l+1})}(x(t_l)) - J^0_{[(k_l+1)T,t_{l+1})}(x(t_l))\right| \le \gamma_4 T\|x(t_l)\|^2,$$

so clearly, if T is sufficiently small, then for every $T' \in (0,\bar{T}_2'(T))$ we have

$$\left|\hat{J}^0_{[(k_l+1)T,t_{l+1})}(x(t_l)) - J^0_{[(k_l+1)T,t_{l+1})}(x(t_l))\right| \le \frac{\varepsilon}{4}\|x(t_l)\|^2. \tag{A.16}$$

We now combine (A.14) and (A.16) to find that, if T is sufficiently small, then for every $T' \in (0,\bar{T}_2'(T))$ we have

$$\begin{aligned}
&\left|J_{[(k_l+1)T,t_{l+1})}(x(t_l)) - J^0_{[(k_l+1)T,t_{l+1})}(x(t_l))\right| \\
&\quad\le \left|J_{[(k_l+1)T,t_{l+1})}(x(t_l)) - \hat{J}^0_{[(k_l+1)T,t_{l+1})}(x(t_l))\right| + \\
&\quad\quad \left|\hat{J}^0_{[(k_l+1)T,t_{l+1})}(x(t_l)) - J^0_{[(k_l+1)T,t_{l+1})}(x(t_l))\right| \\
&\quad< \left(\frac{\varepsilon}{4}+\frac{\varepsilon}{4}\right)\|x(t_l)\|^2 \\
&\quad= \frac{\varepsilon}{2}\|x(t_l)\|^2.
\end{aligned}$$

■

It remains to combine the result of Claim 4 with (A.11). Clearly, if T is sufficiently small, then for every $T' \in (0,\bar{T}_2'(T))$ it follows immediately that

$$\begin{aligned}
\left|J_{[t_l,t_{l+1})}(x(t_l)) - J^0_{[t_l,t_{l+1})}(x(t_l))\right| &\le \left|J_{[t_l,(k_l+1)T)}(x(t_l)) - J^0_{[t_l,(k_l+1)T)}(x(t_l))\right| + \\
&\quad \left|J_{[(k_l+1)T,t_{l+1})}(x(t_l)) - J^0_{[(k_l+1)T,t_{l+1})}(x(t_l))\right| \\
&\le \left(\alpha_3 T + \frac{\varepsilon}{2}\right)\|x(t_l)\|^2 \\
&\le \varepsilon\|x(t_l)\|^2.
\end{aligned}$$

■

Appendix B

Proofs from Chapter 4

When there are no plant switches, the proofs here will be virtually identical to those in Chapter 3 (with the exception of some variable name substitutions and some slight changes due to to different structure of the error signal). Even with switches, the proofs will follow the same structure; changes in the details are almost always due to the discontinuity in the state ξ and the inclusion of the reference signal y_{ref}.

Proof of Lemma 4.1:

First note that u and x are continuous, and, by Assumptions 4.1 and 4.3, for every $T_s > 0$, $\sigma \in \Sigma_{T_s}$, and $t \geq 0$, we have that $G^{-1}_{\sigma(t)}$ exists. Let $T_s > 0$, $P_\sigma \in \mathcal{P}_{T_s}$, and $l \in \mathbf{Z}^+$ be arbitrary. From (4.7) and (4.8), we have that

$$\begin{aligned}\begin{bmatrix} x(t_l) \\ u(t_l) \end{bmatrix} &= G^{-1}_{\sigma(t_l^-)}\xi(t_l^-) + G^{-1}_{\sigma(t_l^-)}\begin{bmatrix} 0 \\ I \end{bmatrix} y_{ref} \\ &= G^{-1}_{\sigma(t_l)}\xi(t_l) + G^{-1}_{\sigma(t_l)}\begin{bmatrix} 0 \\ I \end{bmatrix} y_{ref};\end{aligned}$$

these two equations can be readily combined to yield

$$\xi(t_l) = G_{\sigma(t_l)}G^{-1}_{\sigma(t_l^-)}\xi(t_l^-) + \left(G_{\sigma(t_l)}G^{-1}_{\sigma(t_l^-)} - I\right)\begin{bmatrix} 0 \\ I \end{bmatrix} y_{ref} \tag{B.1}$$

and

$$\xi(t_l^-) = G_{\sigma(t_l^-)}G^{-1}_{\sigma(t_l)}\xi(t_l) + \left(G_{\sigma(t_l^-)}G^{-1}_{\sigma(t_l)} - I\right)\begin{bmatrix} 0 \\ I \end{bmatrix} y_{ref}.$$

Hence, by the definition of $\bar{g}$, we obtain our desired result.

∎

Proof of Proposition 4.1:

Fix $T_s > 0$, $T \in (0, T_s/2)$, and $T' \in (0, T/2)$, and let $x_0 \in \mathbf{R}^n$, $u_0 \in \mathbf{R}^m$, $\sigma \in \Sigma_{T_s}$, $y_{ref} \in \mathbf{R}^r$, and $w \in \mathcal{L}_\infty$ be arbitrary.

(i)

Let $k \in \{k_l : l \in \mathbf{N}\}^c$ be arbitrary. Observe that $\sigma(t)$ is constant on $[kT, (k+1)T)$; for notational simplicity, we denote its value by i, which means that the plant is P_i on this interval.

We begin by solving (4.6) for $e(t + kT)$ over $t \in [0, 2T')$. Note that, in this interval,

$$\hat{H}(t) = 0,$$

so

$$\nu(t) = 0$$

as well; therefore,

$$e(t + kT) = \tilde{C}\Big[e^{\tilde{A}_i t}\xi[kT] + \underbrace{\int_0^t e^{\tilde{A}_i(t-\tau)}\tilde{A}_i L_i \underline{w}_k(\tau)d\tau + L_i\underline{w}_k(t)}_{=:f_i(\underline{w}_k,t)}\Big], \quad t \in [0, 2T').$$

We substitute this into (4.13) and then use (4.18) to find that

$$v_1[k] = E_i\xi[kT] + \underbrace{\int_0^{T'} S(t)\tilde{C}f_i(\underline{w}_k, t)dt}_{=:\phi_{1,i}(\underline{w}_k)}. \tag{B.2}$$

Similarly, for v_2 we find

$$v_2[k] = E_i\xi[kT] + \underbrace{\int_{T'}^{2T'} S(t)\tilde{C}f_i(\underline{w}_k, t)dt}_{=:\phi_{2,i}(\underline{w}_k)}. \tag{B.3}$$

Note that, in (B.2) and (B.3), v_1 and v_2 are implicit functions of i, $\underline{w}_k$, and $\xi[kT]$; with this in mind, we define χ_i to be the selector function

$$\chi_i(\underline{w}_k, \xi[kT]) := \begin{cases} 1 & \text{if } \operatorname{argmin}\{\|v_1[k]\|, \|v_2[k]\|\} = v_1[k] \\ 0 & \text{if } \operatorname{argmin}\{\|v_1[k]\|, \|v_2[k]\|\} = v_2[k], \end{cases}$$

so we have

$$\operatorname{argmin}\{\|v_1[k]\|, \|v_2[k]\|\} = \\ E_i\xi[kT] + \chi_i(\underline{w}_k, \xi[kT])\phi_{1,i}(\underline{w}_k) + (1 - \chi_i(\underline{w}_k, \xi[kT]))\phi_{2,i}(\underline{w}_k); \tag{B.4}$$

from (4.15) it follows that

$$\begin{aligned} \nu(t) &= \hat{H}(t)\left[E_i\xi[kT] + \chi_i(\underline{w}_k, \xi[kT])\phi_{1,i}(\underline{w}_k) + (1 - \chi_i(\underline{w}_k, \xi[kT]))\phi_{2,i}(\underline{w}_k)\right] \\ &= \hat{H}_i(t)\xi[kT] + \hat{H}(t)\left[\chi_i(\underline{w}_k, \xi[kT])\phi_{1,i}(\underline{w}_k) + (1 - \chi_i(\underline{w}_k, \xi[kT]))\phi_{2,i}(\underline{w}_k)\right], \\ &\qquad t \in [kT, (k+1)T). \end{aligned}$$

Substituting this into the first equation of (4.6) and using the fact that

$$\sigma(t) = i, \quad t \in [kT, (k+1)T)$$

yields the desired equation:

$$\begin{aligned} \dot{\xi}(t) &= \tilde{A}_{\sigma(t)}\xi(t) + \tilde{B}_{\sigma(t)}\hat{H}_{\sigma(t)}(t)\xi[kT] + \left[\begin{array}{cc} \tilde{B}_{\sigma(t)}\hat{H}(t) & \tilde{A}_{\sigma(t)}L_{\sigma(t)} \end{array} \right] \times \\ & \left[\begin{array}{c} \chi_{\sigma(t)}(\underline{w}_k, \xi[kT])\phi_{1,\sigma(t)}(\underline{w}_k) + [1 - \chi_{\sigma(t)}(\underline{w}_k, \xi[kT])]\phi_{2,\sigma(t)}(\underline{w}_k) \\ w(t) \end{array} \right], \\ & \qquad t \in [kT, (k+1)T). \end{aligned}$$

Last of all, it is straight-forward, though tedious, to show that $\phi_{1,i}$ and $\phi_{2,i}$ have bounded gain.

(ii)

Let $l \in \mathbf{N}$ be arbitrary. To obtain the desired result we consider three cases.

Case 1: Switch occurs in the Control Phase: $t_l \in [k_lT + 2T', (k_l + 1)T)$.

Here the Estimation Phase is not affected by the switch, so (B.2), (B.3), and (B.4) hold (with $k = k_l$ and $i = i_{l-1}$); since $\phi_{i,1}$ and $\phi_{i,2}$ have bounded gain for every $i = 1, ..q$, it follows that there exists a constant $\gamma_1(T, T') > 0$ that is independent of i such that

$$\min\{\|v_1[k_l]\|, \|v_2[k_l]\|\} \leq \|\xi[k_lT]\| + \gamma_1(T, T')\|w\|_\infty;$$

since there is a discontinuity on $[k_lT + 2T', (k_l + 1)T)$, there cannot be one at $t = k_lT$, so $\xi[k_lT] = \xi[k_lT^-]$ and

$$\min\{\|v_1[k_l]\|, \|v_2[k_l]\|\} \leq \|\xi[k_lT^-]\| + \gamma_1(T, T')\|w\|_\infty. \tag{B.5}$$

Case 2: Switch occurs during the second sample: $t_l \in [k_lT + T', k_lT + 2T')$.

Here, the first sampler is not affected; therefore, (B.2) still holds and it follows that we again obtain (B.5).

Case 3: Switch occurs during the first sample: $t_l \in [k_lT, k_lT + T')$.

Here, we have two possibilities: $t_l = k_lT$ and $t_l > k_lT$[1]. We begin by considering the former. Here, in general,

$$\xi[k_lT] \neq \xi[k_lT^-];$$

however, $\sigma(t)$ is constant on $[k_lT, k_lT + T')$, so it follows that (B.2) and (B.3) hold (this time with $k = k_l$ and $i = i_l$) and we obtain

$$\min\{\|v_1[k_l]\|, \|v_2[k_l]\|\} \leq \|\xi[k_lT]\| + \gamma_1(T, T')\|w\|_\infty,$$

[1]The astute reader may be concerned that our result will not hold if σ is such that $t_1 < T$. Recall that there is no discontinuity at t_0, so even if $t_1 < T$, there can be at most one discontinuity on $[0, T)$, occurring at t_1.

to which we apply Lemma 4.1 to yield

$$\min\{\|v_1[k_l]\|, \|v_2[k_l]\|\} \leq \bar{g}\|\xi[k_l T^-]\| + (\bar{g}+1)\|y_{ref}\| + \gamma_1(T, T')\|w\|_\infty. \tag{B.6}$$

The other case is that of $t_l \in (k_l T, k_l T + T')$, so

$$\xi[k_l T] = \xi[k_l T^-];$$

unfortunately, the first sampler yields a possibly large output, and, due to the plant switch, (B.3) does not hold. Fortunately, we can still obtain a nice bound on $\|v_2[k_l]\|$. We begin by recognizing that, since $t_l < k_l T + T'$, we have

$$e(t) = \tilde{C}e^{\tilde{A}_{i_l}(t-k_l T-T')}\xi[k_l T + T'] + \underbrace{\tilde{C}\Big[\int_{k_l T+T'}^{t} e^{\tilde{A}_{i_l}(t-\tau)}\tilde{A}_{i_l}L_{i_l}w(\tau)d\tau + L_{i_l}w(t)\Big]}_{=:g_{0,i_l}(T,T',w,t)},$$
$$t \in [k_l T + T', k_l T + 2T')$$

so from (4.14) and (4.19) we have

$$\begin{aligned} v_2[k_l] &= \int_{k_l T+T'}^{k_l T+2T'} S(t)e(t)\,dt \\ &= E_{i_l}e^{-A_{i_l}T'}\xi[k_l T + T'] + \underbrace{\int_{k_l T+T'}^{k_l T+2T'} S(t)g_{0,i_l}(T, T', w, t)dt}_{=:g_{1,i_l}(T,T',w)}. \end{aligned} \tag{B.7}$$

We would like to write a bound in terms of $\xi[k_l T^-]$ instead of $\xi[k_l T + T']$. To do so, we first observe that

$$\xi[k_l T + T'] = e^{\tilde{A}_{i_l}(k_l T+T'-t_l)}\xi(t_l) + \underbrace{\int_{t_l}^{k_l T+T'} e^{\tilde{A}_{i_l}(k_l T+T'-\tau)}\tilde{A}_{i_l}L_{i_l}w(\tau)d\tau}_{=:g_{2,i_l}(T,T',w,t_l)}$$

and then, as in (B.1) from the proof of Lemma 4.1, we obtain

$$\xi(t_l) = G_{i_l}G_{i_{l-1}}\xi(t_l^-) + \left(G_{i_l}G_{i_{l-1}}^{-1} - I\right)\begin{bmatrix} 0 \\ I \end{bmatrix} y_{ref}$$

and finally

$$\xi(t_l^-) = e^{\tilde{A}_{i_{l-1}}(t_l-k_l T)}\xi[k_l T^-] + \underbrace{\int_{k_l T}^{t_l} e^{\tilde{A}_{i_{l-1}}(t_l-\tau)}\tilde{A}_{i_{l-1}}L_{i_{l-1}}w(\tau)d\tau}_{=:g_{3,i_l}(T,T',w,t_l)}.$$

If we define

$$\begin{aligned} \hat{\phi}_{i_l}(T, T', w, t_l) \;&:=\; g_{1,i_l}(T, T', w) + E_{i_l}e^{-\tilde{A}_{i_l}T'}g_{2,i_l}(T, T', w, t_l) + \\ &\quad\; E_{i_l}e^{-\tilde{A}_{i_l}(t_l-k_l T)}G_{i_l}G_{i_{l-1}}^{-1}g_{3,i_l}(T, T', w, t_l) \end{aligned}$$

then these three equalities combine with (B.7) to provide

$$\begin{aligned} v_2[k_l] &= E_{i_l} e^{-\tilde{A}_{i_l}(t_l - k_l T)} G_{i_l} G_{i_{l-1}}^{-1} e^{\tilde{A}_{i_{l-1}}(t_l - k_l T)} \xi[k_l T^-] + \\ & \quad E_{i_l} e^{-\tilde{A}_{i_l}(t_l - k_l T)} \left(G_{i_l} G_{i_{l-1}}^{-1} - I \right) \begin{bmatrix} 0 \\ I \end{bmatrix} y_{ref} + \hat{\phi}_{i_l}(T, T', w). \end{aligned}$$

Since there are a finite number of plants, $\hat{\phi}_{i_l}$ has the property that there exists a constant $\gamma_2(T, T') > 0$ such that

$$\max_{t \in [k_l T, (k_l+1)T]} \|\hat{\phi}_{i_l}(T, T', w, t)\| \leq \gamma_2(T, T') \|w\|_\infty, \quad l \in \mathbf{N},$$

so we have

$$\begin{aligned} \|v_2[k_l]\| &\leq \max_{t \in [0, T')} \left\| e^{-\tilde{A}_{i_l} t} G_{i_l} G_{i_{l-1}}^{-1} e^{\tilde{A}_{i_{l-1}} t} \right\| \|\xi[k_l T^-]\| + \\ & \quad \max_{t \in [0, T')} \left\| e^{-\tilde{A}_{i_l} t} \left(G_{i_l} G_{i_{l-1}}^{-1} - I \right) \right\| \|y_{ref}\| + \gamma_2(T, T') \|w\|_\infty; \end{aligned}$$

using the definitions of $\rho(T')$ and $\rho_y(T')$ we can write

$$\|v_2[k_l]\| \leq \rho(T') \|\xi[k_l T^-]\| + \rho_y(T') \|y_{ref}\| + \gamma_2(T, T') \|w\|_\infty. \tag{B.8}$$

Since we clearly have both

$$\rho(T') \geq \bar{g}$$

and

$$\rho(T') \geq \bar{g} + 1,$$

we can combine (B.6), and (B.8) to find that, no matter whether or not $t_l = k_l T$ we have

$$\begin{aligned} \min\{\|v_1[k_l]\|, \|v_2[k_l]\|\} \leq \rho(T') \|\xi[k_l T^-]\| + \rho_y(T') \|y_{ref}\| + \\ \underbrace{\max\{\gamma_1(T, T'), \gamma_2(T, T')\}}_{=: \gamma_v(T, T')} \|w\|_\infty. \end{aligned} \tag{B.9}$$

We can now combine the results of our three cases. To do so, observe that

$$\rho(T') \geq 1,$$

and then combine the bounds in (B.5) and (B.9) to find

$$\min\{\|v_1[k_l]\|, \|v_2[k_l]\|\} \leq \rho(T') \|\xi[k_l T^-]\| + \rho_y(T') \|y_{ref}\| + \gamma_v(T, T') \|w\|_\infty, \quad l \in \mathbf{N}.$$

■

Proof of Theorem 4.1:

Fix $T > 0$, $T' \in (0, T/2)$, and $P_i \in \mathcal{P}$. Let $x_0 \in \mathbf{R}^n$, $u_0 \in \mathbf{R}^m$, and $w \in \mathcal{PC}_\infty$ be arbitrary. As discussed in Remark 4.2, we can set $y_{ref} = 0$.

From Corollary 4.1 we have that there exist nonlinear functions

$$\phi_i : \mathcal{L}_\infty[0,T) \times \mathbf{R}^{n+r} \to \mathcal{L}_\infty[0,T)$$

and

$$\theta_i : \mathcal{L}_\infty[0,T) \times \mathbf{R}^{n+r} \to \mathbf{R}^{n+r}$$

that have bounded gain and are such that, for every $k \in \mathbf{Z}^+$, we have

$$\dot{\xi}(t) = \tilde{A}_i\xi(t) + \tilde{B}_i\hat{H}_i(t)\xi[kT] + \phi_i(\underline{w}_k, \xi[kT]), \quad t \in [kT, (k+1)T) \tag{B.10}$$

and

$$\xi[(k+1)T] = e^{\bar{A}_iT}\xi[kT] + \theta_i(\underline{w}_k, \xi[kT]); \tag{B.11}$$

we define

$$\gamma_\phi := \max_{i=1,..,q} \|\phi_i\|$$

and

$$\gamma_\theta := \max_{i=1,..,q} \|\theta_i\|.$$

Since T and T' are fixed, we have that $\|\hat{H}\|_\infty$ is well defined. Finally, observe that, by (4.4), (4.5), and Assumptions 4.1 and 4.3, for $t \geq 0$ we have

$$\left.\begin{array}{rcl} \xi(t) & = & G_i \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \\ \Leftrightarrow \;\; G_i^{-1}\xi(t) & = & \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \end{array}\right\} \tag{B.12}$$

and there exits a constant $g > 0$ so that

$$\|G_i\| \leq g, \quad i = 1, .., q.$$

(Asymptotic Stability)

Let $x_0 \in \mathbf{R}^n$ and $u_0 \in \mathbf{R}^m$ remain arbitrary and set $w = 0$; observe that, since $y_{ref} = 0$, we have

$$\xi_0 = \begin{bmatrix} A_ix_0 + B_iu_0 \\ C_ix_0 \end{bmatrix}.$$

In this context (B.10) and (B.11) reduce to

$$\dot{\xi}(t) = \tilde{A}_i\xi(t) + \tilde{B}_i\hat{H}_i(t)\xi[kT], \quad t \in [kT, (k+1)T) \tag{B.13}$$

and

$$\xi[(k+1)T] = e^{\bar{A}_iT}\xi[kT]. \tag{B.14}$$

If we solve (B.14) and then substitute the result into the solution of (B.13), then we clearly have

$$\xi(t) = \left(e^{\tilde{A}_i(t-kT)} + \int_{kT}^{t} e^{\tilde{A}_i(t-\tau)}\tilde{B}_i\hat{H}_i(\tau)d\tau\right)e^{\bar{A}_ikT}\xi_0, \quad t \in [kT, (k+1)T), \quad k \in \mathbf{Z}^+$$

so

$$\begin{aligned}
\|\xi(t)\| &\leq \left(e^{aT} + Te^{aT} b\|\hat{H}\|_\infty\right)\gamma_0 e^{\lambda_0 kT}\|\xi_0\|, \quad t \in [kT, (k+1)T), \quad k \in \mathbf{Z}^+ \\
&\leq \left(e^{aT} + Te^{aT} b\|\hat{H}\|_\infty\right)\gamma_0 e^{-\lambda_0 T} e^{\lambda_0 t}\|\xi_0\|, \quad t \geq 0;
\end{aligned}$$

clearly

$$\lim_{t\to\infty} \|\xi(t)\| = 0.$$

Since S is admissible and periodic, we have that there exists a constant $\gamma_s > 0$ so that

$$\begin{aligned}
\|v_1[k]\| &= \left\| \int_{kT}^{kT+T'} S(t)e(t)dt \right\| \\
&\leq \gamma_s \max_{t\in[kT,kT+T')} \|e(t)\|, \quad k \in \mathbf{Z}^+
\end{aligned}$$

and

$$\begin{aligned}
\|v_2[k]\| &= \left\| \int_{kT+T'}^{kT+2T'} S(t)e(t)dt \right\| \\
&\leq \gamma_s \max_{t\in[kT+T',kT+2T')} \|e(t)\|, \quad k \in \mathbf{Z}^+.
\end{aligned}$$

Since $e = \tilde{C}\xi$, this clearly yields

$$\begin{aligned}
\lim_{k\to\infty} \|v_1[k]\| &\leq \lim_{t\to\infty} \gamma_s \|\xi(t)\| \\
&= 0,
\end{aligned}$$

and similarly

$$\lim_{k\to\infty} \|v_2[k]\| = 0;$$

finally, by (B.12), we have

$$\begin{aligned}
\lim_{t\to\infty} \left\| \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \right\| &\leq \lim_{t\to\infty} g^{-1} \|\xi(t)\| \\
&= 0.
\end{aligned}$$

■

(I/O Stability)

Let $w \in \mathcal{PC}_\infty$ be arbitrary and set $x_0 = 0$ and $u_0 = 0$. Observe that, by (B.12), we can obtain a bound on $\begin{bmatrix} x \\ u \end{bmatrix}$ in terms of a bound on ξ; additionally, this provides us with

$$\xi_0 = 0.$$

Furthermore, by (4.5)

$$\xi(t) \quad = \quad \begin{bmatrix} \dot{x}(t) \\ e(t) \end{bmatrix} - \begin{bmatrix} B_i & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} w_u(t) \\ w_y(t) \end{bmatrix}, \quad t \geq 0,$$

which we rearrange to find

$$e(t) = \begin{bmatrix} 0 & I \end{bmatrix} \xi(t) + w_y(t),$$

so we can obtain a bound on e in terms of bounds on ξ and w_y. From the structure of the compensator κ given in (4.13)-(4.15), it then follows that we can obtain a bound on ν in terms of ξ and w_y. Hence, to prove I/O stability it is enough to obtain a bound on $\|\xi\|_\infty$ in terms of $\|w\|_\infty$. The remainder of this proof will be devoted to this.

We begin by investigating (B.11). Since $\xi_0 = 0$ we have

$$\xi[kT] = \sum_{j=0}^{k-1} e^{\bar{A}_i(k-1-j)T} \theta_i(\underline{w}_j, \xi[jT]), \quad k \in \mathbf{Z}^+;$$

taking norms and then extending the summation to infinity we find that

$$\begin{aligned} \|\xi[kT]\| \quad &\leq \quad \sum_{j=0}^{\infty} \gamma_0 e^{\lambda_0 jT} \gamma_\theta \|w\|_\infty \\ &= \quad \underbrace{\frac{\gamma_0 \gamma_\theta}{1 - e^{\lambda_0 T}}}_{=:\gamma_1} \|w\|_\infty, \quad k \in \mathbf{Z}^+. \end{aligned} \tag{B.15}$$

We now solve (B.10), to find that, for every $k \in \mathbf{Z}^+$, we have

$$\xi(t) = e^{\tilde{A}_i(t-kT)} \xi[kT] + \int_{kT}^{t} e^{\tilde{A}_i(t-\tau)} \left[\tilde{B}_i \hat{H}_i(t) \xi[kT] + \phi_i(\underline{w}_k, \xi[kT]) \right] d\tau,$$

$$t \in [kT, (k+1)T),$$

so

$$\begin{aligned} \|\xi(t)\| \quad &\leq \quad e^{aT} \|\xi[kT]\| + T e^{aT} \left[b \|\hat{H}\|_\infty \|\xi[kT]\| + \gamma_\phi \|w\|_\infty \right] \\ &\leq \quad \underbrace{e^{aT} \left(1 + Tb \|\hat{H}\|_\infty \right)}_{=:\gamma_2} \|\xi[kT]\| + \underbrace{T e^{aT} \gamma_\phi}_{=:\gamma_3} \|w\|_\infty, \qquad t \in [kT, (k+1)T). \end{aligned}$$

Combining this with (B.15) provides

$$\|\xi(t)\| \leq (\gamma_1 \gamma_2 + \gamma_3) \|w\|_\infty, \quad t \geq 0.$$

∎

Proof of Theorem 4.2:

Fix $T_s > \underline{T_s}$, $T \in (0, \bar{T}(T_s))$, and $\sigma \in \Sigma_{T_s}$, and let $x_0 \in \mathbf{R}^n$, $u_0 \in \mathbf{R}^m$, $T' \in (0, T/2)$, and $w \in \mathcal{L}_\infty$ be arbitrary; as discussed in Remark 4.2, we can set $y_{ref} = 0$. Finally, as in the proof of Theorem 4.1, observe that, by (4.4) and (4.5) and Assumptions 4.1 and 4.3, for $t \geq 0$ we have

$$\left.\begin{array}{rcl} \xi(t) & = & G_{\sigma(t)} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \\ \Leftrightarrow \;\; G_{\sigma(t)}^{-1}\xi(t) & = & \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \end{array}\right\} \tag{B.16}$$

and, since there are a finite number of LTI plants, there exits a constant $g > 0$ so that

$$\|G_{\sigma(t)}\| \leq g, \quad t \geq 0.$$

From Corollary 4.1, there exist $2q$ nonlinear functions

$$\phi_i : \mathcal{L}_\infty[0, T) \times \mathbf{R}^{n+r} \to \mathcal{L}_\infty[0, T), \quad i = 1, .., q$$

and

$$\theta_i : \mathcal{L}_\infty[0, T) \times \mathbf{R}^{n+r} \to \mathbf{R}^{n+r}, \quad i = 1, .., q$$

that have bounded gain with the property that, for every $k \in \{k_l : l \in \mathbf{N}\}^c$, we have

$$\dot{\xi}(t) = \tilde{A}_{\sigma(t)}\xi(t) + \tilde{B}_{\sigma(t)}\hat{H}_{\sigma(t)}(t)\xi[kT] + \phi_{\sigma(t)}(\underline{w}_k, \xi[kT]), \quad t \in [kT, (k+1)T) \tag{B.17}$$

and

$$\xi[(k+1)T^-] = e^{\bar{A}_{\sigma[kT]}T}\xi[kT] + \theta_{\sigma[kT]}(\underline{w}_k, \xi[kT]). \tag{B.18}$$

We define

$$\gamma_\phi(T') := \max_{i=1,..,q} \|\phi_i\|$$

and

$$\gamma_\theta(T') := \max_{i=1,..,q} \|\theta_i\|.$$

Before turning to the particular details of each of our two stability types, it will be useful to perform some preliminary analysis. To proceed, we make two comments. First,

$$k_l T \neq t_l \Rightarrow \xi[k_l T^-] = \xi[k_l T]. \tag{B.19}$$

Second, it is important to note that $\|\hat{H}(t)\|$ and $\|\hat{H}\|_\infty$ are implicit functions of T'; indeed, there is no uniform upper bound on $\|\hat{H}\|_\infty$. However, by definition, for every $i = 1, .., q$ we clearly have that

$$\|\hat{H}_i(t)\| = \begin{cases} 0 & t \in [0, 2T') \\ \|H_i(t) - \tilde{H}_i(t)\| & t \in [2T', T), \end{cases}$$

so

$$\begin{aligned}\|\hat{H}_i\|_\infty &\leq \max_{t\in[2T',T)} \|H_i(t) - \tilde{H}_i(t)\| \\ &\leq f\gamma_0 + \max_{t\in[2T',T)} \|\tilde{H}_i(t)\|;\end{aligned}$$

if we use Lemma 4.2 to bound the rightmost term, then we find

$$\|\hat{H}_i\|_\infty \leq f\gamma_0 + \varepsilon_H(T,T'). \tag{B.20}$$

Since T is fixed, we write $\varepsilon_H(T')$ instead of $\varepsilon_H(T,T')$.

We now address the $l = 0$ case in the sense that we investigate the interval $[0, k_1T)$. Note that, if $k_1 = 0$ then this interval is empty and there is nothing to prove, so we assume that this not the case. We begin by observing that $\sigma(t) = i_0$ over $[0, k_1T)$, so we can solve (B.18) to find

$$\xi[kT^-] = \xi[kT] = e^{\bar{A}_{i_0}kT}\xi_0 + \sum_{j=0}^{k-1} e^{\bar{A}_{i_0}(k-1-j)T}\theta_{i_0}(\underline{w}_j, \xi[jT]), \quad k = 0,..,k_1-1,$$

so

$$\|\xi[kT]\| \leq \gamma_0\|\xi_0\| + \gamma_0\gamma_\theta(T')\frac{1}{1-e^{\lambda_0 T}}\|w\|_\infty, \quad k = 0,..,k_1-1. \tag{B.21}$$

We then solve (B.17), yielding

$$\begin{aligned}\xi(t) &= e^{\tilde{A}_{i_0}(t-kT)}\xi[kT] + \int_{kT}^{t} e^{\tilde{A}_{i_0}(t-\tau)}\left(\tilde{B}_{i_0}\hat{H}_{i_0}(\tau)\xi[kT] + \phi_{i_0}(\underline{w}_k, \xi[kT])\right)d\tau, \\ &\qquad t \in [kT,(k+1)T), \quad k = 0,..,k_1-1,\end{aligned}$$

so, using (B.20) to bound $\|\hat{H}(t)\|$, we find that

$$\begin{aligned}\|\xi(t)\| &\leq \underbrace{\left(e^{aT} + Te^{aT}b(f\gamma_0 + \varepsilon_H(T'))\right)}_{=:\alpha_0(T')}\|\xi[kT]\| + Te^{aT}\gamma_\phi(T')\|w\|_\infty, \\ &\qquad t \in [kT,(k+1)T), \quad k = 0,..,k_1-1.\end{aligned}$$

If we combine this with (B.21), then we obtain

$$\begin{aligned}\|\xi(t)\| &\leq \alpha_0(T')\gamma_0\|\xi_0\| + \underbrace{\left[\alpha_0(T')\left(\gamma_0\gamma_\theta(T')\frac{1}{1-e^{\lambda_0 T}}\right) + Te^{aT}\gamma_\phi(T')\right]}_{=:\bar{\gamma}(T')}\|w\|_\infty, \\ &\qquad t \in [0, k_1T).\end{aligned} \tag{B.22}$$

Clearly

$$\alpha_0(T')\gamma_0 \geq 1,$$

so, for all cases of k_1 we have

$$\|\xi[k_1T^-]\| \leq \alpha_0(T')\gamma_0\|\xi_0\| + \bar{\gamma}(T')\|w\|_\infty. \tag{B.23}$$

We now turn to $t \geq k_1 T$. We will present two claims: the first examines the behavior of ξ on intervals where there *is* a switch and the second uses Corollary 3.1 to examine the behavior of ξ on intervals where there is *no* switch. Together with (B.22), these claims will provide the necessary tools for proving both types of stability. Recall that switches are confined to $\{t_l : l \in \mathbf{N}\}$ and

$$t_l \in [k_l T, (k_l + 1)T).$$

Claim 1: There exist constants $\gamma_1(T') > 0$ and $\bar{\gamma}_1(T') > 0$ such that

$$\|\xi(t)\| \leq \gamma_1(T')\|\xi[k_l T^-]\| + \bar{\gamma}_1(T')\|w\|_\infty, \quad t \in [k_l T, (k_l+1)T), \quad l \in \mathbf{N}.$$

Proof:

Since the interval of interest may contain a plant switch, we use Proposition 4.1(ii) to bound the size of the sampler output: there exists $\gamma_v(T') > 0$ such that

$$\min\{\|v_1[k_l]\|, \|v_2[k_l]\|\} \leq \rho(T')\|\xi[k_l T^-]\| + \gamma_v(T')\|w\|_\infty, \quad l \in \mathbf{N},$$

so, by definition of ν and (B.20) we have

$$\begin{aligned} \|\nu(t)\| \quad &\leq \quad (f\gamma_0 + \varepsilon_H(T'))\left(\rho(T')\|\xi[k_l T^-]\| + \gamma_v(T')\|w\|_\infty\right), \\ &\qquad\qquad t \in [k_l T, (k_l+1)T), \quad l \in \mathbf{N}. \end{aligned} \tag{B.24}$$

A plant switch causes a discontinuity, so we must be careful when solving for ξ in this period. To that end, we split the period into two parts: the time interval before the switching time, namely $[k_l T, t_l)$, and the time interval at and after the switching time, namely $[t_l, (k_l+1)T)$.

We start by investigating the first interval. If $t_l = k_l T$, then this interval is empty. If $t_l > k_l T$, then solving (4.6) yields

$$\xi(t) = e^{\tilde{A}_{i_l}(t-k_l T)}\xi[k_l T] + \int_{k_l T}^{t} e^{\tilde{A}_{i_l}(t-\tau)}\left[\tilde{B}_{i_l}\nu(\tau) + \tilde{A}_{i_l}L_{i_l}w(\tau)\right]d\tau, \quad t \in [k_l T, t_l);$$

taking norms on both sides gives

$$\begin{aligned} \|\xi(t)\| \quad &\leq \quad e^{a(t-k_l T)}\|\xi[k_l T\| + \frac{(\overbrace{e^{a(t-k_l T)}}^{\leq e^{a(t-k_l T)}} - 1)}{a}\left(b \max_{t\in[k_l T, t_l)}\|\nu(\tau)\| + a\ell\|w\|_\infty\right), \\ &\qquad\qquad t \in [k_l T, t_l). \end{aligned}$$

We can now use (B.19) and (B.20) to yield

$$\begin{aligned} \|\xi(t)\| \quad &\leq \quad e^{aT}\|\xi[k_l T^-]\| + \frac{e^{aT}}{a}\Big[b(f\gamma_0 + \varepsilon_H(T')) \times \\ &\qquad \left(\rho(T')\|\xi[k_l T^-]\| + \gamma_v(T')\|w\|_\infty\right) + a\ell\|w\|_\infty\Big], \quad t \in [k_l T, t_l), \end{aligned}$$

which we rearrange to find

$$\begin{aligned}\|\xi(t)\| &\leq \underbrace{e^{aT}\left(1+\frac{b(f\gamma_0+\varepsilon_H(T'))\rho(T')}{a}\right)}_{=:\alpha_1(T')}\|\xi[k_lT^-]\|+\\ &\quad \underbrace{e^{aT}\left(\frac{b(f\gamma_0+\varepsilon_H(T'))}{a}\gamma_v(T')+\ell\right)}_{=:\alpha_2(T')}\|w\|_\infty, \quad t\in[k_lT,t_l). \quad \text{(B.25)}\end{aligned}$$

Now we turn to the second interval. If we solve (4.6) on $[t_l,(k_l+1)T)$ and use (B.24), then, using the same approach as above, we obtain

$$\begin{aligned}\|\xi(t)\| &\leq e^{aT}\|\xi(t_l)\|+\frac{e^{aT}}{a}b(f\gamma_0+\varepsilon_H(T'))\rho(T')\|\xi[k_lT^-]\|+\alpha_2(T')\|w\|_\infty,\\ &\qquad\qquad t\in[t_l,(k_l+1)T),\end{aligned}$$

to which we apply Lemma 4.1 to yield

$$\begin{aligned}\|\xi(t)\| \leq e^{aT}\bar{g}\|\xi(t_l^-)\|+\frac{e^{aT}}{a}b(f\gamma_0+\varepsilon_H(T'))\rho(T')\|\xi[k_lT^-]\|+\alpha_2(T')\|w\|_\infty,\\ t\in[t_l,(k_l+1)T). \quad \text{(B.26)}\end{aligned}$$

Now we use (B.25) and (B.26) to obtain the desired bounds. If $t_l=k_lT$, then from (B.26) we have

$$\begin{aligned}\|\xi(t)\| &\leq \underbrace{e^{aT}\left(\bar{g}+\frac{b(f\gamma_0+\varepsilon_H(T'))\rho(T')}{a}\right)}_{=:\alpha_3(T')}\|\xi[k_lT^-]\|+\alpha_2(T')\|w\|_\infty,\\ &\qquad\qquad t\in[k_lT,(k_l+1)T). \quad \text{(B.27)}\end{aligned}$$

If $t_l>k_lT$, then we use (B.25) to obtain a bound on $\|\xi(t_l^-)\|$ and substitute this into (B.26) to yield

$$\begin{aligned}\|\xi(t)\| &\leq e^{aT}\bar{g}\left(\alpha_1(T')\|\xi[k_lT^-]\|+\alpha_2(T')\|w\|_\infty\right)+\\ &\quad \frac{e^{aT}}{a}b(f\gamma_0+\varepsilon_H(T'))\rho(T')\|\xi[k_lT^-]\|+\alpha_2(T')\|w\|_\infty\\ &= \underbrace{e^{aT}\left(\bar{g}\alpha_1(T')+\frac{b(f\gamma_0+\varepsilon_H(T'))\rho(T')}{a}\right)}_{=:\gamma_1(T')}\|\xi[k_lT^-]\|+\\ &\quad \underbrace{\alpha_2(T')\left(e^{aT}\bar{g}+1\right)}_{=:\bar{\gamma}_1(T')}\|w\|_\infty, \qquad t\in[t_l,(k_l+1)T);\end{aligned}$$

since it is clear that

$$\gamma_1(T')\geq\alpha_1(T')$$

and

$$\bar{\gamma}_1(T') \geq \alpha_2(T'),$$

if we combine this with (B.25), then we have

$$\|\xi(t)\| \leq \gamma_1(T')\|\xi[k_l T^-]\| + \bar{\gamma}_1(T')\|w\|_\infty, \quad t \in [k_l T, (k_l+1)T). \quad \text{(B.28)}$$

Finally, observe that

$$\gamma_1(T') \geq \alpha_3(T'),$$

so we can combine (B.28) with (B.27), so that, regardless of whether or not $t_l = k_l T$, we have

$$\|\xi(t)\| \leq \gamma_1(T')\|\xi[k_l T^-]\| + \bar{\gamma}_1(T')\|w\|_\infty, \quad t \in [k_l T, (k_l+1)T).$$

■

We now turn to the interval $[(k_l+1)T, k_{l+1}T)$.

Claim 2: There exist constants $\gamma_2(T') > 0$ and $\bar{\gamma}_2(T') > 0$ such that, for all $l \in \mathbf{N}$ we have

$$\|\xi(t)\| \leq e^{\lambda_0(t-(k_l+1)T)}\gamma_2(T')\|\xi[k_l T^-]\| + \bar{\gamma}_2(T')\|w\|_\infty, \quad t \in [(k_l+1)T, k_{l+1}T).$$

Proof:

Let $l \in \mathbf{N}$ be arbitrary. To reduce notational clutter we write

$$[\underline{k}T, \bar{k}T) := [(k_l+1)T, k_{l+1}T);$$

on this interval

$$\sigma(t) = i_l.$$

We begin by using the fact that $\xi(t)$ is continuous over $[\underline{k}T, \bar{k}T)$ to solve (B.18), which results in

$$\xi[kT^-] = \xi[kT] = e^{\bar{A}_{i_l}(k-\underline{k})T}\xi[\underline{k}T] + \sum_{j=\underline{k}}^{k-1} e^{\bar{A}_{i_l}(k-1-j)T}\theta_{i_l}(\underline{w}_j, \xi[jT]), \quad \underline{k} \leq k \leq \bar{k}-1;$$

taking norms on both sides yields

$$\begin{aligned}
\|\xi[kT]\| &\leq \gamma_0 e^{\lambda_0(k-\underline{k})T}\|\xi[\underline{k}T]\| + \sum_{j=0}^{\infty}\gamma_0 e^{\lambda_0 jT}\gamma_\theta(T')\|w\|_\infty \\
&\leq \gamma_0 e^{\lambda_0(k-\underline{k})T}\|\xi[\underline{k}T]\| + \frac{\gamma_0\gamma_\theta(T')}{1-e^{\lambda_0 T}}\|w\|_\infty, \quad \underline{k} \leq k \leq \bar{k}-1. \quad \text{(B.29)}
\end{aligned}$$

Next, we solve (B.17), to find that, for every integer k satisfying $\underline{k} \leq k \leq \bar{k}-1$, we have

$$\begin{aligned}
\xi(t) &= e^{\tilde{A}_{i_l}(t-kT)}\xi[kT] + \int_{kT}^{t} e^{\tilde{A}_{i_l}(t-\tau)}\left[\tilde{B}_{i_l}\hat{H}_{i_l}(t)\xi[kT] + \phi_{i_l}(\underline{w}_k, \xi[kT])\right]d\tau, \\
&\qquad t \in [kT, (k+1)T),
\end{aligned}$$

so taking norms on both sides and using (B.20) to bound $\|\hat{H}(t)\|$ yields

$$\begin{aligned}
\|\xi(t)\| &\leq e^{aT}\|\xi[kT]\| + \frac{e^{aT}}{a}\left[b(f\gamma_0 + \varepsilon_H(T'))\|\xi[kT]\| + \gamma_\phi(T')\|w\|_\infty\right] \\
&= e^{aT}\left(1 + \frac{b(f\gamma_0 + \varepsilon_H(T'))}{a}\right)\|\xi[kT]\| + \frac{e^{aT}}{a}\gamma_\phi(T')\|w\|_\infty, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in [kT, (k+1)T),
\end{aligned}$$

which, combined with (B.29), yields

$$\begin{aligned}
\|\xi(t)\| &\leq e^{aT}\left(1 + \frac{b(f\gamma_0 + \varepsilon_H(T'))}{a}\right)\left(\gamma_0 e^{\lambda_0(k-\underline{k})T}\|\xi[\underline{k}T]\| + \frac{\gamma_0\gamma_\theta(T')}{1 - e^{\lambda_0 T}}\|w\|_\infty\right) \\
&\quad + \frac{e^{aT}}{a}\gamma_\phi(T')\|w\|_\infty \\
&= e^{\lambda_0(k-\underline{k})T}\underbrace{\gamma_0 e^{aT}\left(1 + \frac{b(f\gamma_0 + \varepsilon_H(T'))}{a}\right)}_{=:\alpha_4(T')}\|\xi[\underline{k}T]\| + \\
&\quad \underbrace{e^{aT}\left[\left(1 + \frac{b(f\gamma_0 + \varepsilon_H(T'))}{a}\right)\frac{\gamma_0\gamma_\theta(T')}{1 - e^{\lambda_0 T}} + \frac{\gamma_\phi(T')}{a}\right]}_{=:\alpha_5(T')}\|w\|_\infty, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in [kT, (k+1)T).
\end{aligned}$$

We now use Claim 1. Since ξ is continuous at $(k_l+1)T$, it follows immediately that

$$\|\xi[(k_l+1)T]\| = \|\xi[\underline{k}T]\| \leq \gamma_1(T')\|\xi[k_l T^-]\| + \bar{\gamma}_1(T')\|w\|_\infty,$$

so, for every integer k satisfying $\underline{k} \leq k \leq \bar{k} - 1$, we have

$$\begin{aligned}
\|\xi(t)\| &\leq e^{\lambda_0(k-\underline{k})T}\alpha_4(T')\|\xi[\underline{k}T]\| + \alpha_5(T')\|w\|_\infty \\
&\leq e^{\lambda_0(k-\underline{k})T}\alpha_4(T')\left(\gamma_1(T')\|\xi[k_l T^-]\| + \bar{\gamma}_1(T')\|w\|_\infty\right) + \alpha_5(T')\|w\|_\infty, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in [kT, (k+1)T);
\end{aligned}$$

it follows easily that

$$\begin{aligned}
\|\xi(t)\| &\leq e^{\lambda_0(t-\underline{k}T)}\underbrace{e^{-\lambda_0 T}\alpha_4(T')\gamma_1(T')}_{=:\gamma_2(T')}\|\xi[k_l T^-]\| + \\
&\quad \underbrace{[\alpha_4(T')\bar{\gamma}_1(T') + \alpha_5(T')]}_{=:\bar{\gamma}_2(T')}\|w\|_\infty, \qquad t \in [\underline{k}T, \bar{k}T).
\end{aligned}$$

■

We now assemble our results to find a bound over the entire interval $[0, \infty)$. To do so, notice that, in particular, Claim 2 says that

$$\|\xi[k_{l+1}T^-]\| \leq e^{\lambda_0(k_{l+1}-(k_l+1))T}\gamma_2(T')\|\xi[k_l T^-]\| + \bar{\gamma}_2(T')\|w\|_\infty, \quad l \in \mathbf{N},$$

so, since

$$t_{l+1} - t_l \geq T_s, \quad l \in \mathbf{N},$$

we have that

$$\|\xi[k_{l+1}T^-]\| \leq e^{\lambda_0(T_s-2T)}\gamma_2(T')\|\xi[k_lT^-]\| + \bar{\gamma}_2(T')\|w\|_\infty, \quad l \in \mathbf{N}.$$

If

$$e^{\lambda_0(T_s-2T)}\gamma_2(T') < 1^2, \tag{B.30}$$

then it follows immediately that

$$\|\xi[k_lT^-]\| \leq \underbrace{\left(e^{\lambda_0(T_s-2T)}\gamma_2(T')\right)^{(l-1)}}_{\leq 1}\|\xi[k_1T^-]\| + \frac{1}{1-e^{\lambda_0(T_s-2T)}\gamma_2(T')}\bar{\gamma}_2(T')\|w\|_\infty, \quad l \in \mathbf{N}$$

which combines with (B.23) to yield

$$\begin{aligned}\|\xi[k_lT^-]\| \leq\ & \left(e^{\lambda_0(T_s-2T)}\gamma_2(T')\right)^{(l-1)}\alpha_0(T')\gamma_0\|\xi_0\| + \\ & \underbrace{\left[\bar{\gamma}(T') + \frac{1}{1-e^{\lambda_0(T_s-2T)}\gamma_2(T')}\bar{\gamma}_2(T')\right]}_{=:\bar{\gamma}_3(T')}\|w\|_\infty, \quad l \in \mathbf{N}.\end{aligned} \tag{B.31}$$

It will turn out that our hypothesis ensures that (B.30) holds (for sufficiently small T'). To maintain the flow of the proof, we defer showing this until the end; in the meantime, we assume that it holds (and restrict T' accordingly) and proceed. We have from Claims 1 and 2 that

$$\begin{aligned}\|\xi(t)\| \leq\ & \max\left\{\gamma_1(T'), \gamma_2(T')\right\}\|\xi[k_lT^-]\| + \max\left\{\bar{\gamma}_1(T'), \bar{\gamma}_2(T')\right\}\|w\|_\infty, \\ & t \in [k_lT, k_{l+1}T), \quad l \in \mathbf{N};\end{aligned}$$

if we combine this with (B.31), then it follows immediately that

$$\begin{aligned}\|\xi(t)\| \leq\ & \max\left\{\gamma_1(T'), \gamma_2(T')\right\}\left(e^{\lambda_0(T_s-2T)}\gamma_2(T')\right)^{(l-1)}\alpha_0(T')\gamma_0\|\xi_0\| + \\ & \underbrace{\left(\max\left\{\gamma_1(T'), \gamma_2(T')\right\} \times \bar{\gamma}_3(T') + \max\left\{\bar{\gamma}_1(T'), \bar{\gamma}_2(T')\right\}\right)}_{=:\bar{\gamma}_4(T')}\|w\|_\infty, \\ & t \in [k_lT, k_{l+1}T), \quad l \in \mathbf{N}.\end{aligned} \tag{B.32}$$

which we will use, in conjunction with (B.22) to prove our stability results.

(Asymptotic Stability)

If we set $w = 0$ and let $x_0 \in \mathbf{R}^n$ and $u_0 \in \mathbf{R}^m$ remain arbitrary, then by (B.22) we have

$$\|\xi(t)\| \leq \alpha_0(T')\gamma_0\|\xi_0\|, \quad t \in [0, k_1T]$$

[2]We will prove that our hypothesis ensures that this holds for sufficiently small T' shortly.

and, by (B.32), we have

$$\begin{aligned}\|\xi(t)\| &\leq \max\left\{\gamma_1(T'), \gamma_2(T')\right\}\left(e^{\lambda_0(T_s-2T)}\gamma_2(T')\right)^{(l-1)}\alpha_0(T')\gamma_0\|\xi_0\|, \\ &\qquad\qquad\qquad\qquad t\in[k_lT, k_{l+1}T], \quad l\in\mathbf{N};\end{aligned}$$

clearly, for each admissible T',

$$\lim_{t\to\infty}\|\xi(t)\| \leq \lim_{l\to\infty}\left(\max\left\{\gamma_1(T'), \gamma_2(T')\right\}\left(e^{\lambda_0(T_s-2T)}\gamma_2(T')\right)^{(l-1)}\alpha_0(T')\gamma_0\|\xi_0\|\right),$$

so, using (B.30), it follows that

$$\lim_{t\to\infty}\|\xi(t)\| = 0.$$

As in the proof of Theorem 4.1, since S is periodic and admissible and (with the noise turned off) $e = \tilde{C}\xi$, this clearly yields

$$\lim_{k\to\infty}\|v_1[k]\| = 0, \text{ and } \lim_{k\to\infty}\|v_2[k]\| = 0,$$

and, by (B.16), we have

$$\begin{aligned}\lim_{t\to\infty}\left\|\begin{bmatrix} x(t) \\ u(t)\end{bmatrix}\right\| &\leq \lim_{t\to\infty} g^{-1}\|\xi(t)\| \\ &= 0.\end{aligned}$$

■

(I/O Stability)

Set $x_0 = 0$ and $u_0 = 0$ and let $w \in \mathcal{PC}_\infty$ be arbitrary. Observe that, by (B.16) we can obtain a bound on $\begin{bmatrix} x \\ u \end{bmatrix}$ in terms of a bound on ξ; additionally, this provides us with $\xi_0 = 0$. Furthermore, by (4.5)

$$\xi(t) = \begin{bmatrix}\dot{x}(t) \\ e(t)\end{bmatrix} - \begin{bmatrix} B_{\sigma(t)} & 0 \\ 0 & I\end{bmatrix}\begin{bmatrix} w_u(t) \\ w_y(t)\end{bmatrix}, \quad t \geq 0,$$

which we rearrange to find

$$e(t) = \begin{bmatrix} 0 & I \end{bmatrix}\xi(t) + w_y(t),$$

so we can obtain a bound on e in terms of bounds on ξ and w_y. From the structure of the compensator κ given in (4.13)-(4.15), it then follows that we can obtain a bound on ν in terms of ξ and w_y. Hence, to prove I/O stability it is enough to obtain a bound on $\|\xi\|_\infty$ in terms of $\|w\|_\infty$. From (B.32), we have

$$\|\xi(t)\| \leq \bar{\gamma}_4(T')\|w\|_\infty, \quad t \geq k_1T,$$

so using (B.22) to provide a bound on $\|\xi(t)\|$ over the interval $[0, k_1 T]$ yields

$$\|\xi\|_\infty \leq \max\left\{\bar{\gamma}(T'), \bar{\gamma}_4(T')\right\}\|w\|_\infty.$$

$\blacksquare$

It remains to show that our hypothesis ensures that (B.30) holds for small T'. We begin by using the explicit formula for $\gamma_2(T')$ derived in this proof:

$$\gamma_2(T') = e^{-\lambda_0 T}\alpha_4(T')\gamma_1(T'),$$

with

$$\alpha_4(T') = \gamma_0 e^{aT}\left(1 + \frac{b(f\gamma_0 + \varepsilon_H(T'))}{a}\right),$$

$$\gamma_1(T') = e^{aT}\left(\bar{g}\alpha_1(T') + \frac{b(f\gamma_0 + \varepsilon_H(T'))\rho(T')}{a}\right),$$

and

$$\alpha_1(T') = e^{aT}\left(1 + \frac{b(f\gamma_0 + \varepsilon_H(T'))\rho(T')}{a}\right);$$

back-substituting we find

$$\begin{aligned}
\gamma_2(T') &= e^{-\lambda_0 T}\gamma_0 e^{2aT}\left(1 + \frac{b(f\gamma_0 + \varepsilon_H(T'))}{a}\right) \times \\
&\quad \left[\bar{g}e^{aT}\left(1 + \frac{b(f\gamma_0 + \varepsilon_H(T'))\rho(T')}{a}\right) + \frac{b(f\gamma_0 + \varepsilon_H(T'))\rho(T')}{a}\right] \\
&\leq e^{-\lambda_0 T}\gamma_0 e^{3aT}\left(1 + \frac{b(f\gamma_0 + \varepsilon_H(T'))}{a}\right)\left(\bar{g} + \frac{b(f\gamma_0 + \varepsilon_H(T'))\rho(T')}{a}(1+\bar{g})\right).
\end{aligned}$$

But

$$\lim_{T' \to 0} \rho(T') = \bar{g}.$$

and, from Lemma 2,

$$\lim_{T' \to 0} \varepsilon_H(T') = 0,$$

so

$$\begin{aligned}
\lim_{T' \to 0} \gamma_2(T') &\leq e^{-\lambda_0 T}\gamma_0 e^{3aT}\left(1 + \frac{bf\gamma_0}{a}\right)\left(\bar{g} + \frac{bf\gamma_0\bar{g}}{a}(1+\bar{g})\right) \\
&= e^{-\lambda_0 T}e^{3aT}\gamma_0\bar{g}\left(1 + \frac{bf\gamma_0}{a}\right)\left(1 + \frac{bf\gamma_0}{a}(1+\bar{g})\right)
\end{aligned}$$

and therefore

$$\lim_{T' \to 0} e^{\lambda_0(T_s - 2T)}\gamma_2(T') \leq e^{\lambda_0 T_s}e^{3(a-\lambda_0)T}\gamma_0\bar{g}\left(1 + \frac{bf\gamma_0}{a}\right)\left(1 + \frac{bf\gamma_0}{a}(1+\bar{g})\right).$$

To simplify this expression, we will use the hypothesis that $T_s > \underline{T_s}$ and $T < \bar{T}(T_s)$; it follows directly from this and the definition of $\bar{T}(T_s)$ that

$$\begin{array}{rrcl} & T & < & \frac{|\lambda_0|}{3(a-\lambda_0)}(T_s - \underline{T_s}) \\ \Rightarrow & 3(a-\lambda_0)T & < & |\lambda_0|(T_s - \underline{T_s}) \\ \Rightarrow & e^{3(a-\lambda_0)T} & < & e^{|\lambda_0|(T_s - \underline{T_s})} \\ & & = & e^{\lambda_0(\underline{T_s} - T_s)}, \end{array}$$

so

$$\lim_{T' \to 0} e^{\lambda_0(T_s - 2T)}\gamma_2(T') \;\; < \;\; e^{\lambda_0 \underline{T_s}}\gamma_0 \bar{g}\left(1 + \frac{bf\gamma_0}{a}\right)\left(1 + \frac{bf\gamma_0}{a}(1+\bar{g})\right).$$

If we now apply the definition of $\underline{T_s}$, we find that

$$\lim_{T' \to 0} e^{\lambda_0(T_s - 2T)}\gamma_2(T') < 1,$$

and therefore, for sufficiently small T', we have

$$e^{\lambda_0(T_s - 2T)}\gamma_2(T') < 1.$$

■

Proof of Theorem 4.3:

Fix $\varepsilon > 0$ and $T > 0$. Let $T' \in (0, T/2)$ be arbitrary. Stability follows directly from Theorem 4.1. Let $y_{ref} \in \mathbf{R}^r$, $x_0 \in \mathbf{R}^n$, $u_0 \in \mathbf{R}^r$, and $i = 1, .., q$, be arbitrary and assume that

$$\sigma(t) = i, \;\; t \geq 0;$$

observe that this yields a corresponding ξ_0 via (4.24). Since we are not allowing plant switches ξ is continuous; furthermore, since $w = 0$, from Corollary 4.1 we have

$$\dot{\xi}(t) = \tilde{A}_i \xi(t) + \tilde{B}_i \underbrace{\hat{H}_i(t)\xi[kT]}_{=\nu(t)}, \quad t \in [kT, (k+1)T), \quad k \in \mathbf{Z}^+ \tag{B.33}$$

and

$$\xi[(k+1)T] = e^{\bar{A}_i T}\xi[kT] = e^{\bar{A}_i (k+1)T}\xi_0, \quad k \in \mathbf{Z}^+. \tag{B.34}$$

With $\varepsilon_H(T, T')$ given by (4.23), from Lemma 4.2, we have that

$$\|\tilde{H}_i(t)\| \leq \varepsilon_H(T, T'), \quad t \in [2T', T)$$

and, for every $T > 0$,

$$\lim_{T' \to 0} \varepsilon_H(T, T') = 0; \tag{B.35}$$

since T is fixed we will write $\varepsilon_H(T')$ instead of $\varepsilon(T, T')$. To minimize the complexity of the forthcoming algebra, henceforth we restrict our attention to those choices of T' that are sufficiently small to ensure that

$$\varepsilon_H(T') < 1.$$

We also define

$$\tilde{\xi} := \xi - \xi^0$$

and

$$\tilde{\nu} := \nu - \nu^0,$$

so from (B.34) we have that

$$\tilde{\xi}[kT] = 0, \quad k \in \mathbf{Z}^+. \tag{B.36}$$

We will investigate the cost function over one period T, and then extend the result to the entire time range. It follows immediately from the definitions of J_i and J_i^0 that

$$\begin{aligned}
&\left|J_i(\xi_0) - J_i^0(\xi_0)\right| \\
&\qquad \le \int_0^\infty \left\|M_i(\xi(t), \nu(t), e(t)) - M_i(\xi^0(t), \nu^0(t), e^0(t))\right\| dt \\
&\qquad = \sum_{k=0}^{\infty} \left(\int_{kT}^{(k+1)T} \left\|M_i(\xi(t), \nu(t), e(t)) - M_i(\xi^0(t), \nu^0(t), e^0(t))\right\| dt \right).
\end{aligned} \tag{B.37}$$

We now find a relationship between the actual and optimal cost functions over a single period T. With

$$\gamma_{lqr} := 2 \max_{i=1,..,q} \left\{ \max\{\|\bar{Q}_i + \tilde{C}'K'R_iK\tilde{C}\|, \|R_i\|, \|R_iK\tilde{C}\|\} \right\} \tag{B.38}$$

it is straight-forward to check that

$$\begin{aligned}
&\int_{kT}^{(k+1)T} \left\|M_i(\xi(t), \nu(t), e(t)) - M_i(\xi^0(t), \nu^0(t), e^0(t))\right\| dt \le \\
&\qquad \gamma_{lqr} \int_{kT}^{(k+1)T} \Big[\|\xi^0(t)\|\|\tilde{\xi}(t)\| + \|\tilde{\xi}(t)\|^2 + \|\nu^0(t)\|\|\tilde{\nu}(t)\| + \|\tilde{\nu}(t)\|^2 + \\
&\qquad\qquad \|\tilde{\nu}(t)\|\|\tilde{\xi}(t)\| + \|\nu^0(t)\|\|\tilde{\xi}(t)\| + \|\tilde{\nu}(t)\|\|\xi^0(t)\|\Big] dt, \\
&\qquad\qquad\qquad k \in \mathbf{Z}^+.
\end{aligned} \tag{B.39}$$

We now use the definitions of ξ^0 and ν^0 given in (4.25) and (4.26) respectively, then apply (4.11) and simplify, to find

$$\begin{aligned}
&\int_{kT}^{(k+1)T} \left\|M_i(\xi(t), \nu(t), e(t)) - M_i(\xi^0(t), \nu^0(t), e^0(t))\right\| dt \\
&\qquad \le \gamma_{lqr} \int_{kT}^{(k+1)T} \Big[\|e^{\bar{A}_i(t-kT)}\| \times \|\xi[kT]\|\big(\|\tilde{\xi}(t)\| + \|\tilde{\nu}(t)\|\big) + \\
&\qquad\qquad \|F_ie^{\bar{A}_i(t-kT)}\| \times \|\xi[kT]\|\big(\|\tilde{\xi}(t)\| + \|\tilde{\nu}(t)\|\big) + \\
&\qquad\qquad \|\tilde{\nu}(t)\|^2 + \|\tilde{\xi}(t)\|^2 + \|\tilde{\nu}(t)\|\|\tilde{\xi}(t)\|\Big] dt \\
&\qquad \le \gamma_{lqr} \int_{kT}^{(k+1)T} \Big[\gamma_0(1+f)\|\xi[kT]\|\big(\|\tilde{\xi}(t)\| + \|\tilde{\nu}(t)\|\big) + \\
&\qquad\qquad \|\tilde{\nu}(t)\|^2 + \|\tilde{\xi}(t)\|^2 + \|\tilde{\nu}(t)\|\|\tilde{\xi}(t)\|\Big] dt, \quad k \in \mathbf{Z}^+.
\end{aligned} \tag{B.40}$$

The upshot of this is that, if we can bound $\|\tilde{\xi}(t)\|$, $\int_{kT}^{(k+1)T} \|\tilde{\nu}(t)\|dt$, and $\int_{kT}^{(k+1)T} \|\tilde{\nu}(t)\|^2 dt$ by a suitably scaled version of $\|\xi[kT]\|$, then we can leverage (B.34) to obtain the desired result. We begin with $\int_{kT}^{(k+1)T} \|\tilde{\nu}(t)\|dt$ and $\int_{kT}^{(k+1)T} \|\tilde{\nu}(t)\|^2 dt$:

Claim 1: There exists a constant $\gamma_1 > 0$ satisfying

$$\int_{kT}^{(k+1)T} \|\tilde{\nu}(t)\|dt \leq \gamma_1(T' + \varepsilon_H(T'))\|\xi[kT]\|, \quad k \in \mathbf{Z}^+,$$

and

$$\int_{kT}^{(k+1)T} \|\tilde{\nu}(t)\|^2 dt \leq \gamma_1(T' + \varepsilon_H(T'))\|\xi[kT]\|^2, \quad k \in \mathbf{Z}^+.$$

Proof:

By definition and (B.36) we have that

$$\begin{aligned}
\int_{kT}^{(k+1)T} \|\tilde{\nu}(t)\|dt &= \int_{kT}^{(k+1)T} \|\nu(t) - \nu^0(t)\|dt \\
&= \int_0^{2T'} \|-H_i(t)\xi[kT]\|dt + \int_{2T'}^{T} \|\tilde{H}_i(t)\xi[kT]\|dt \\
&\leq \left[\int_0^{2T'} f\gamma_0 \, dt + \int_{2T'}^{T} \varepsilon_H(T')dt\right] \|\xi[kT]\| \\
&\leq [2f\gamma_0 T' + \varepsilon_H(T')T] \, \|\xi[kT]\|, \quad k \in \mathbf{Z}^+.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_{kT}^{(k+1)T} \|\tilde{\nu}(t)\|^2 dt &\leq \left[\int_0^{2T'} (f\gamma_0)^2 dt + \int_{2T'}^{T} \underbrace{\varepsilon_H(T')^2}_{<\varepsilon_H(T')} dt\right] \|\xi[kT]\|^2 \\
&\leq \left[2(f\gamma_0)^2 T' + \varepsilon_H(T')T\right] \|\xi[kT]\|^2,
\end{aligned}$$

Set $\gamma_1 = \max\{2f\gamma_0, 2(f\gamma_0)^2, T\}$ to obtain the desired result. ∎

Now we turn to $\|\tilde{\xi}(t)\|$:

Claim 2: There exists a constant $\gamma_2 > 0$ satisfying

$$\|\tilde{\xi}(t)\| \leq \gamma_2(T' + \varepsilon_H(T'))\|\xi[kT]\|, \quad t \in [kT, (k+1)T), \quad k \in \mathbf{Z}^+.$$

Proof:

Using the definitions of $\bar{A}_i$, ξ^0, and ν^0 we can write

$$\dot{\xi}^0(t) = \tilde{A}_i \xi^0(t) + \tilde{B}_i \nu^0(t), \quad t \geq 0$$

which combines with (B.33) to yield

$$\dot{\tilde{\xi}}(t) = \tilde{A}_i\tilde{\xi}(t) + \tilde{B}_i\tilde{\nu}(t), \quad t \geq 0.$$

Solving this and using (B.36), we find that

$$\tilde{\xi}(t) = \int_{kT}^{t} e^{\tilde{A}_i(t-\tau)}\tilde{B}_i\tilde{\nu}(t)d\tau, \quad t \in [kT, (k+1)T), \quad k \in \mathbf{Z}^+$$

so

$$\begin{aligned}\|\tilde{\xi}(t)\| &\leq \int_{kT}^{t} \|e^{\tilde{A}_i(t-\tau)}\tilde{B}_i\|\|\tilde{\nu}(t)\|d\tau \\ &\leq be^{aT}\int_{kT}^{(k+1)T} \|\tilde{\nu}(t)\|d\tau,\end{aligned}$$

to which we apply Claim 1, to obtain

$$\|\tilde{\xi}(t)\| \leq \underbrace{be^{aT}\gamma_1}_{=:\gamma_2}(T' + \varepsilon_H(T'))\|\xi[kT]\|, \quad t \in [kT, (k+1)T), \quad k \in \mathbf{Z}^+.$$

■

Now we return to (B.40), to which we apply Claims 1 and 2 to yield

$$\begin{aligned}&\int_{kT}^{(k+1)T} \left\|M_i(\xi(t), \nu(t), e(t)) - M_i(\xi^0(t), \nu^0(t), e^0(t))\right\| dt \\ &\leq \gamma_{lqr}\Big\{\gamma_0(1+f)\|\xi[kT]\|\left[T\gamma_2(T' + \varepsilon_H(T'))\|\xi[kT]\| + \gamma_1(T' + \varepsilon_H(T'))\|\xi[kT]\|\right] \\ &\quad +\gamma_1(T' + \varepsilon_H(T'))\|\xi[kT]\|^2 + T\gamma_2^2(T' + \varepsilon_H(T'))^2\|\xi[kT]\|^2 + \\ &\quad \gamma_1(T' + \varepsilon_H(T'))\|\xi[kT]\|\gamma_2(T' + \varepsilon_H(T'))\|\xi[kT]\|\Big\} \\ &= \gamma_{lqr}\Big\{\gamma_0(1+f)\left[T\gamma_2 + \gamma_1\right] + \gamma_1 + T\gamma_2^2(T' + \varepsilon_H(T')) + \gamma_1\gamma_2(T' + \varepsilon_H(T'))\Big\} \times \\ &\quad (T' + \varepsilon_H(T'))\|\xi[kT]\|^2, \quad k \in \mathbf{Z}^+.\end{aligned}$$

Observe that

$$(T' + \varepsilon_H(T')) \leq \frac{T}{2} + 1,$$

so this reduces to

$$\begin{aligned}&\int_{kT}^{(k+1)T} \left\|M_i(\xi(t), \nu(t), e(t)) - M_i(\xi^0(t), \nu^0(t), e^0(t))\right\| dt \\ &\leq \underbrace{\gamma_{lqr}\Big\{\left[\gamma_0(1+f) + \gamma_2(T/2+1)\right]\left[T\gamma_2 + \gamma_1\right] + \gamma_1\Big\}}_{=:\gamma_3}(T' + \varepsilon_H(T'))\|\xi[kT]\|^2, \\ &\qquad k \in \mathbf{Z}^+.\end{aligned}$$

We can now combine this with (B.34) and then (4.11) to yield

$$\begin{aligned}
\sum_{k=0}^{\infty} \int_{kT}^{(k+1)T} & \big\|M_i(\xi(t),\nu(t),e(t)) - M_i(\xi^0(t),\nu^0(t),e^0(t))\big\|\,dt \\
&\leq \gamma_3(T' + \varepsilon_H(T')) \sum_{k=0}^{\infty} \|e^{\bar{A}_i kT}\xi_0\|^2 \\
&\leq \gamma_3(T' + \varepsilon_H(T')) \sum_{k=0}^{\infty} \gamma_0^2 e^{2\lambda_0 kT}\|\xi_0\|^2 \\
&= (T' + \varepsilon_H(T'))\underbrace{\gamma_3\gamma_0^2(1 - e^{2\lambda_0 T})^{-1}}_{=:\gamma_4}\|\xi_0\|^2.
\end{aligned}$$

From (B.35) we have

$$\lim_{T' \to 0} \varepsilon_H(T') = 0,$$

so, for every sufficiently small T', we obtain

$$(T' + \varepsilon_H(T'))\gamma_4 < \varepsilon,$$

and therefore, using (B.37), we obtain

$$\begin{aligned}
\big|J_i(\xi_0) - J_i^0(\xi_0)\big| &\leq \int_0^{\infty} \big\|M_i(\xi(t),\nu(t),e(t)) - M_i(\xi^0(t),\nu^0(t),e^0(t))\big\|\,dt \\
&\leq \varepsilon\|\xi_0\|^2.
\end{aligned}$$

■

Proof of Theorem 4.4

Fix $\varepsilon > 0$, $T_s > \underline{T_s}$, and $w = 0$; let $\sigma \in \Sigma_{T_s}$ and $T \in (0, \bar{T}(T_s))$ be arbitrary. Stability follows directly from Theorem 4.2. Since T_s is fixed, to reduce clutter we write $\bar{T}$ instead of $\bar{T}(T_s)$. We set

$$\rho := \rho(T_s/4)$$

and

$$\rho_y := \rho_y(T_s/4).$$

As in the proof of Theorem 4.3, we define

$$\tilde{\xi} := \xi - \xi^0$$

and

$$\tilde{\nu} := \nu - \nu^0.$$

As in the proof of Theorem 3.4, our proof works by showing that, if T is sufficiently small, then there exists a constant $\bar{T}'(T) \in (0, T/2)$ such that, for every $T' \in (0, \bar{T}'(T))$ we have the desired result.

We first deal with the special case of $l = 0$.

Claim 0: There exists a constant $\bar{T}_0'(T) \in (0, T/2)$ so that, if $T' \in (0, \bar{T}_0(T))$, then

$$|J_{[0,t_1)}(\xi(t_l), y_{ref}) - J^0_{[0,t_1)}(\xi(t_l), y_{ref})| \leq \varepsilon\big[\|\xi(t_0)\| + \|y_{ref}\|\big]^2,$$
$$x_0 \in \mathbf{R}^n, \quad u_0 \in \mathbf{R}^r, \quad y_{ref} \in \mathbf{R}^r.$$

Proof:

On the interval $[0, t_1)$, $\sigma(t)$ is constant, so Theorem 4.3 can be applied: it states that, if T' is sufficiently small then, irrespective of the value of $\sigma(t)$ on $[0, t_1)$, we have

$$|J_{[0,t_1)}(\xi(t_l), y_{ref}) - J^0_{[0,t_1)}(\xi(t_l), y_{ref})| \leq \varepsilon\|\xi(t_0)\|^2, \quad x_0 \in \mathbf{R}^n, \quad u_0 \in \mathbf{R}^r, \quad y_{ref} \in \mathbf{R}^r,$$

so the result follows immediately. ∎

Before we move on, we follow the same process as in the proof of Theorem 3.4 to bound $\hat{H}$. To that end, recall that, with ε_H given by (4.23), from Lemma 4.2 we have that

$$\|\tilde{H}_i(t)\| \leq \varepsilon_H(T, T'), \quad t \in [2T', T), \quad T' \in (0, T/2) \quad i = 1, .., q,$$

observe that, from the definition of $\hat{H}$ we have that

$$\hat{H}(t) = \begin{cases} 0 & t \in [0, 2T') \\ H(t) + \tilde{H}(t) & t \in [2T', T), \end{cases}$$

so clearly

$$\|\hat{H}\|_\infty \leq f\gamma_0 + \varepsilon_H(T, T'), \quad T' \in (0, T/2).$$

Unlike in the proof of Theorem 4.3, here we will not need to (explicitly) make ε_H small; indeed, it will be enough to simply bound $\|\hat{H}\|_\infty$. To that end, we observe that Lemma 4.2 says that, for every $T \in (0, \bar{T})$,

$$\lim_{T' \to 0} \varepsilon_H(T, T') = 0,$$

so, for every $T \in (0, \bar{T})$, there exists a constant $\bar{T}_1'(T) \in (0, \bar{T}_0'(T))$ so that

$$\|\hat{H}\|_\infty \leq f\gamma_0 + 1, \quad T' \in (0, \bar{T}_1'(T)). \tag{B.41}$$

We now turn to the general case: let $l \in \mathbf{N}$ be arbitrary. As in Theorem 3.4, it will be useful to partition each interval into two parts, illustrated via Figure A.1, which we reproduce here (Figure B.1) for convenience. Since we do not know whether or not $t_l = k_l T$, we must take care to differentiate between $\xi[k_l T]$ and $\xi[k_l T^-]$; furthermore, since

$$\bar{g} \geq 1,$$

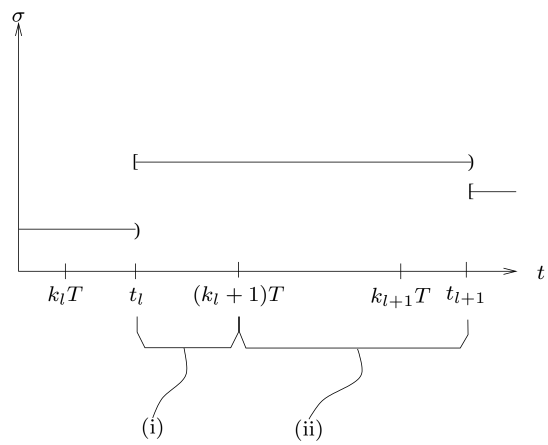

Figure B.1: Partitioning of one interval - time axis is not to scale. Here (i) is the portion of the period immediately following a switch, while (ii) contains the remainder of the period.

whether or not $t_l = k_lT$, from Lemma 4.1 we have

$$\|\xi[k_lT^-]\| \leq \bar{g}\|\xi[k_lT]\| + (\bar{g}+1)\|y_{ref}\|. \tag{B.42}$$

We will be able to leverage Theorem 4.3 to provide a nice result for the second part of the interval shown in Figure B.1, but to do so we must first investigate two important issues that do not arise when there are no switches:

(i) From Proposition 4.1(ii), we know that in intervals with a switch, the size of the controller output depends on $\xi[k_lT^-]$; however, we wish to obtain results in terms of $\xi(t_l)$ and y_{ref}.

(ii) The control applied during $[t_l, (k_l+1)T)$ will likely be wrong, so, unlike the case where there are no switches, we will likely **not** have $\xi[(k_l+1)T] = \xi^0[(k_l+1)T]$.

We begin by investigating (i).

Claim 1: There exists a constant $\gamma_1 > 0$ such that, if T is sufficiently small, then, for every $T' \in (0, \bar{T}_1'(T))$, we have

$$\|\xi[k_lT^-]\| \leq \gamma_1 \left[\|\xi(t_l)\| + \|y_{ref}\|\right].$$

Proof:

Let $T' \in (0, \bar{T}_1'(T))$ be arbitrary. If $t_l = k_lT$, then we apply (B.42) and conclude that the result is trivially true as long as $\gamma_1 \geq \bar{g}+1$. If $t_l > k_lT$, then we proceed by solving (4.6) backwards in time to yield

$$\xi[k_lT] = e^{\tilde{A}_{i_{(l-1)}}(k_lT-t_l)}\xi(t_l^-) + \int_{t_l}^{k_lT} e^{\tilde{A}_{i_{(l-1)}}(k_lT-\tau)}\tilde{B}_{i_{(l-1)}}\nu(\tau)d\tau. \tag{B.43}$$

Since the period $[k_lT, (k_l + 1)T)$ contains a switch, we use Proposition 4.1(ii) to bound the size of the sampler output, yielding

$$\|\nu(t)\| \leq \|\hat{H}(t)\|(\rho\|\xi[k_lT^-]\| + \rho_y\|y_{ref}\|), \quad t \in [k_lT, (k_l + 1)T). \tag{B.44}$$

Here $\xi[k_lT] = \xi[k_lT^-]$, so we can take norms on both sides of (B.43), use (B.41) to bound $\|\hat{H}(t)\|$, and simplify, yielding

$$\|\xi[k_lT^-]\| \leq e^{aT}\|\xi(t_l^-)\| + \int_{k_lT}^{t_l} e^{aT}b(f\gamma_0 + 1)\left(\rho\|\xi[k_lT^-]\| + \rho_y\|y_{ref}\|\right)\, d\tau;$$

therefore,

$$\|\xi[k_lT^-]\| \leq e^{a\bar{T}}\|\xi(t_l^-)\| + Te^{a\bar{T}}b(f\gamma_0 + 1)\left(\rho\|\xi[k_lT^-]\| + \rho_y\|y_{ref}\|\right).$$

With

$$\alpha := e^{a\bar{T}} \max\{1, b(f\gamma_0 + 1)\rho, b(f\gamma_0 + 1)\rho_y\},$$

it follows that

$$\|\xi[k_lT^-]\| \leq \alpha\Big[\|\xi(t_l^-)\| + T\big(\|\xi[k_lT^-]\| + \|y_{ref}\|\big)\Big]$$

and therefore

$$\big[1 - \alpha T\big]\|\xi[k_lT^-]\| \leq \alpha\big[\|\xi(t_l^-)\| + T\|y_{ref}\|\big]. \tag{B.45}$$

Clearly,

$$\begin{aligned}
\|\xi[k_lT^-]\| &\leq \frac{\alpha}{1 - \alpha T}\big[\|\xi(t_l^-)\| + T\|y_{ref}\|\big] \\
&\leq 2\alpha\|x(t_l)\| + \|y_{ref}\|, \quad T \in \left(0, \min\left\{\bar{T}, \frac{1}{2\alpha}\right\}\right),
\end{aligned}$$

so, since $\alpha > 1$, we have

$$\|\xi[k_lT^-]\| \leq 2\alpha\left[\|\xi(t_l^-)\| + \|y_{ref}\|\right];$$

finally, we use Lemma 4.1 to obtain a bound on $\|\xi(t_l^-)\|$:

$$\|\xi[k_lT^-]\| \leq 2\alpha\left[\bar{g}\|\xi(t_l)\| + (\bar{g} + 2)\|y_{ref}\|\right].$$

Since $\alpha > 1$, if we define

$$\gamma_1 := 2\alpha(\bar{g} + 2),$$

then, for every sufficiently small T, it follows that

$$\|\xi[k_lT^-]\| \leq \gamma_1\left[\|\xi(t_l)\| + \|y_{ref}\|\right].$$

■

We now investigate the difference between the nominal and the actual state at $(k_l + 1)T$.

Claim 2: There exists a constant $\gamma_2 > 0$ such that, if T is sufficiently small, then, for every $T' \in (0, \bar{T}_1'(T))$, we have

$$\|\tilde{\xi}(t)\| \leq \gamma_2 T(\|\xi(t_l)\| + \|y_{ref}\|), \quad t \in [t_l, (k_l+1)T]$$

and

$$\|\xi(t)\| \leq \gamma_2(\|\xi(t_l)\| + \|y_{ref}\|), \quad t \in [t_l, (k_l+1)T].$$

Proof:

Let $T' \in (0, \bar{T}_1'(T))$ be arbitrary. Using the definitions of ξ^0 and ν^0 given by (4.28) and (4.27) respectively, we obtain

$$\dot{\xi}^0(t) = \tilde{A}_{i_l}\xi^0(t) + \tilde{B}_{i_l}\nu^0(t), \quad t \in [t_l, t_{l+1})$$

which combines with (4.6) to yield

$$\dot{\tilde{\xi}}(t) = \tilde{A}_{i_l}\tilde{\xi}(t) + \tilde{B}_{i_l}\tilde{\nu}(t), \quad t \in [t_l, t_{l+1}).$$

Solving this and using the fact that $\xi(t_l) = \xi^0(t_l)$, we find that

$$\tilde{\xi}(t) = \int_{t_l}^{t} e^{\tilde{A}_{i_l}(t-\tau)}\tilde{B}_{i_l}\tilde{\nu}(\tau)d\tau, \quad t \in [t_l, t_{l+1});$$

since $t_{l+1} - t_l > 2T$, we know that $\tilde{\xi}(t)$ is continuous at $k_{l+1}T$, so

$$\begin{aligned}
\tilde{\xi}(t) &= \int_{t_l}^{t} e^{\tilde{A}_{i_l}(t-\tau)}\tilde{B}_{i_l}\tilde{\nu}(\tau)d\tau \\
&= \int_{t_l}^{t} e^{\tilde{A}_{i_l}(t-\tau)}\tilde{B}_{i_l}\left(\nu(\tau) - H_{i_l}(\tau)\xi(t_l)\right)d\tau, \quad t \in [t_l, (k_l+1)T] \subset [t_l, t_{l+1}).
\end{aligned}$$

We then take the norm of both sides and use (B.44) and (B.41) to yield

$$\begin{aligned}
\|\tilde{\xi}(t)\| &\leq \int_0^T e^{aT} b\left[(f\gamma_0 + 1)\left(\rho\|\xi[k_l T^-]\| + \rho_y\|y_{ref}\|\right) + f\gamma_0\|\xi(t_l)\|\right]d\tau \\
&\leq Te^{a\bar{T}} b\left[(f\gamma_0+1)(\rho\|\xi[k_l T^-]\| + \rho_y\|y_{ref}\|) + f\gamma_0\|\xi(t_l)\|\right], \\
&\qquad\qquad t \in [t_l, (k_l+1)T]
\end{aligned}$$

to which we apply Claim 1 to find that, if T is sufficiently small, then we have

$$\begin{aligned}
\|\tilde{\xi}(t)\| &\leq Te^{a\bar{T}} b\Big\{(f\gamma_0+1)\left[\rho\gamma_1(\|\xi(t_l)\| + \|y_{ref}\|) + \rho_y\|y_{ref}\|\right] + f\gamma_0\|\xi(t_l)\|\Big\}, \\
&\leq \underbrace{e^{a\bar{T}} b \max\left\{(f\gamma_0+1)\rho\gamma_1 + f\gamma_0, (f\gamma_0+1)(\rho\gamma_1 + \rho_y)\right\}}_{=:\alpha_1} \times \\
&\quad T\left[\|\xi(t_l)\| + \|y_{ref}\|\right], \quad t \in [t_l, (k_l+1)T],
\end{aligned}$$

which provides the first desired result. To find the second desired result, observe that it follows immediately from the definition of ξ^0 that

$$\|\xi^0(t)\| \leq \gamma_0 \|\xi^0(t_l)\|, \quad t \in [t_l, (k_l+1)T],$$

so we conclude that, if $T > 0$ is sufficiently small, then

$$\begin{aligned} \|\xi(t)\| &\leq \|\tilde{\xi}(t)\| + \|\xi^0(t)\| \\ &\leq \underbrace{(\alpha_1 \bar{T} + \gamma_0)}_{=:\alpha_2} \left[\|\xi(t_l)\| + \|y_{ref}\|\right], \qquad t \in [t_l, (k_l+1)T]. \end{aligned}$$

■

As indicated earlier, we would like to leverage Theorem 4.3; since the period $[k_l T, (k_l+1)T)$ contains a switch, we will not be able to do so for the interval $[t_l, (k_l+1)T)$. Recall that the proof of Theorem 4.3 was motivated by (B.40) and that we found bounds on $\|\tilde{\xi}(t)\|$, $\int \|\nu(t)\| dt$, and $\int \|\nu(t)\|^2 dt$ to find the desired result; we will do the same for the interval $[t_l, (k_l+1)T)$. Observe that Claim 2 already provides a nice bound on $\|\tilde{\xi}(t)\|$; we now turn to $\int \|\nu(t)\| dt$ and $\int \|\nu(t)\|^2 dt$:

Claim 3: There exists a constant $\gamma_3 > 0$ such that, if T is sufficiently small, then, for every $T' \in (0, \bar{T}_1'(T))$ we have

$$\int_{t_l}^{(k_l+1)T} \|\tilde{\nu}(t)\| dt \leq \gamma_3 T \left[\|\xi(t_l)\| + \|y_{ref}\|\right]$$

and

$$\int_{t_l}^{(k_l+1)T} \|\tilde{\nu}(t)\|^2 dt \leq \gamma_3 T \left[\|\xi(t_l)\| + \|y_{ref}\|\right]^2.$$

Proof:

Let $T' \in (0, \bar{T}_1'(T))$ be arbitrary. Observe that

$$\begin{aligned} \int_{t_l}^{(k_l+1)T} \|\tilde{\nu}(t)\| dt &= \int_{t_l}^{(k_l+1)T} (\|\nu(t) - \nu^0(t)\|) dt \\ &\leq \int_{t_l}^{(k_l+1)T} (\|\nu(t)\| + \|\nu^0(t)\|) dt \\ &\leq \int_{t_l}^{(k_l+1)T} (\|\nu(t)\| + f\gamma_0 \|\xi(t_l)\|) dt, \end{aligned}$$

so using (B.44) and (B.41) we find that

$$\begin{aligned} \int_{t_l}^{(k_l+1)T} \|\tilde{\nu}(t)\| dt &\leq \int_{t_l}^{(k_l+1)T} \left[(f\gamma_0 + 1)\left(\rho \|\xi[k_l T^-]\| + \rho_y \|y_{ref}\|\right) + \right. \\ &\qquad \left. f\gamma_0 \|\xi(t_l)\|\right] dt \\ &\leq T\left[(f\gamma_0 + 1)\left(\rho \|\xi[k_l T^-]\| + \rho_y \|y_{ref}\|\right) + f\gamma_0 \|\xi(t_l)\|\right]. \end{aligned}$$

We now apply Claim 1 to find that, if $T > 0$ is sufficiently small, then

$$\begin{aligned}\int_{t_l}^{(k_l+1)T} \|\tilde{\nu}(t)\| dt \;\; &\leq \;\; T\Big\{ \underbrace{[(f\gamma_0+1)\rho\gamma_1 + f\gamma_0]}_{=:\alpha_1} \|\xi(t_l)\| + \\ &\qquad\qquad \underbrace{(f\gamma_0+1)(\rho\gamma_1+\rho_y)}_{=:\alpha_2} \|y_{ref}\| \Big\}.\end{aligned}$$

Similarly, we find that, if T is sufficiently small, then

$$\begin{aligned}\int_{t_l}^{(k_l+1)T} \|\tilde{\nu}(t)\|^2 dt \;\; &\leq \;\; \int_{t_l}^{(k_l+1)T} (\|\nu(t)\| + \|\nu^0(t)\|)^2 dt \\ &\leq \;\; \int_{t_l}^{(k_l+1)T} \big(\|\nu(t)\| + f\gamma_0\|\xi(t_l)\|\big)^2 dt \\ &\leq \;\; T\left[\alpha_1\|\xi(t_l)\| + \alpha_2\|y_{ref}\|\right]^2 .\end{aligned}$$

If we set

$$\gamma_3 := \max\{\alpha_1+\alpha_2, (\alpha_1+\alpha_2)^2\}$$

then the result follows. ∎

With $\gamma_{lqr} > 0$ defined in (B.38), for every interval $[\underline{t}, \bar{t}) \subset [t_l, t_{l+1})$, it is routine to confirm that the procedure used to derive (B.39) can be applied here to show that

$$\begin{aligned}&\left| J_{[\underline{t},\bar{t})}(\xi(t_l), y_{ref}) - J^0_{[\underline{t},\bar{t})}(\xi(t_l), y_{ref})\right| \leq \\ &\qquad \gamma_{lqr}\int_{\underline{t}}^{\bar{t}} \Big(\|\xi^0(t)\|\|\tilde{\xi}(t)\| + \|\tilde{\xi}(t)\|^2 + \|\nu^0(t)\|\|\tilde{\nu}(t)\| + \|\tilde{\nu}(t)\|^2 + \\ &\qquad\qquad \|\tilde{\nu}(t)\|\|\tilde{\xi}(t)\| + \|\nu^0(t)\|\|\tilde{\xi}(t)\| + \|\tilde{\nu}(t)\|\|\xi^0(t)\|\Big) dt.\end{aligned}$$

We now apply the definitions of ξ^0 and ν^0 found in (4.28) and (4.27) yielding

$$\begin{aligned}&\left| J_{[\underline{t},\bar{t})}(\xi(t_l), y_{ref}) - J^0_{[\underline{t},\bar{t})}(\xi(t_l), y_{ref})\right| \leq \\ &\qquad \gamma_{lqr}\int_{\underline{t}}^{\bar{t}} \Big[\gamma_0(1+f)e^{\lambda_0(t-t_l)}\|\xi(t_l)\| \Big(\|\tilde{\xi}(t)\| + \|\tilde{\nu}(t)\|\Big) + \\ &\qquad\qquad \|\tilde{\xi}(t)\|^2 + \|\tilde{\nu}(t)\|^2 + \|\tilde{\nu}(t)\|\|\tilde{\xi}(t)\|\Big] dt.\end{aligned}$$

If we apply Claims 2 and 3, then we find that there exists a constant $\alpha_3 > 0$ such that for every $T \in (0, \bar{T}_1)$, if $T' > 0$ is sufficiently small, then we have

$$\left| J_{[t_l,(k_l+1)T)}(\xi(t_l), y_{ref}) - J^0_{[t_l,(k_l+1)T)}(\xi(t_l), y_{ref})\right| \leq \alpha_3 T\left[\|\xi(t_l)\| + \|y_{ref}\|\right]^2 . \quad \text{(B.46)}$$

It remains to analyze the second part of the interval, namely $[(k_l+1)T, t_{l+1})$.

Claim 4: If T is sufficiently small, then there exists a constant $\bar{T}_2'(T) \in (0, \bar{T}_1'(T))$ so that, for every $T' \in (0, \bar{T}_2'(T))$, we have

$$\left| J_{[(k_l+1)T, t_{l+1})}(\xi(t_l), y_{ref}) - J^0_{[(k_l+1)T, t_{l+1})}(\xi(t_l), y_{ref}) \right| \leq \frac{\varepsilon}{2} \left[\|\xi(t_l)\| + \|y_{ref}\| \right]^2.$$

Proof:

On the interval of interest, namely $[(k_l + 1)T, t_{l+1})$, there are no switches. Recall that, in Theorem 4.3 we showed that we can obtain a nice performance bound when there are no switches; however, there we had the nice property that $\xi[(k_l + 1)T] = \xi^0[(k_l + 1)T]$, which is typically not the case here. Nonetheless, we would like to leverage Theorem 4.3, so we define

$$\hat{\xi}^0(t) := e^{\bar{A}_{i_l}(t-(k_l+1)T)} \xi[(k_l + 1)T], \quad t \in [(k_l + 1)T, t_{l+1}), \tag{B.47}$$

$$\hat{\nu}^0(t) := F_{i_l} e^{\bar{A}_{i_l}(t-(k_l+1)T)} \xi[(k_l + 1)T], \quad t \in [(k_l + 1)T, t_{l+1}), \tag{B.48}$$

and

$$\hat{e}^0(t) := C_{i_l} \hat{\xi}^0(t), \quad t \in [(k_l + 1)T, t_{l+1}),$$

and then define

$$\hat{J}^0_{[(k_l+1)T, t_{l+1})}(\xi(t_l), y_{ref}) := \int_{(k_l+1)T}^{t_{l+1}} M_{i_l}(\hat{\xi}^0(t), \hat{\nu}^0(t), \hat{e}^0(t)) dt.$$

Claim 2 says that, if $T > 0$ is sufficiently small, then for every $T' \in (0, \bar{T}_1'(T))$, we have

$$\|\xi[(k_l + 1)T]\| \leq \gamma_2(\|\xi(t_l)\| + \|y_{ref}\|);$$

since our closed loop system is periodic with period T, we can combine this with Theorem 4.3 applied to the interval $[(k_l + 1)T, t_{l+1})$ to find that, if $T > 0$ is sufficiently small, then there exists a constant $\bar{T}_2'(T) \in (0, \bar{T}_1'(T))$ so that, for every $T' \in (0, \bar{T}_2'(T))$, we have

$$\begin{aligned}
&\left| J_{[(k_l+1)T, t_{l+1})}(\xi(t_l), y_{ref}) - \hat{J}^0_{[(k_l+1)T, t_{l+1})}(\xi(t_l), y_{ref}) \right| \\
&\qquad \leq \int_{(k_l+1)T}^{t_{l+1}} \|M_{i_l}(\xi(t), \nu(t), e(t)) - M_{i_l}(\hat{\xi}^0(t), \hat{\nu}^0(t), \hat{e}^0(t))\| dt \\
&\qquad \leq \frac{\varepsilon}{4\gamma_2^2} \|\xi[(k_l + 1)T]\|^2 \\
&\qquad \leq \frac{\varepsilon}{4} (\|\xi(t_l)\| + \|y_{ref}\|)^2.
\end{aligned} \tag{B.49}$$

We now find a relationship between $\hat{J}^0(\xi(t_l), y_{ref})$ and $J^0(\xi(t_l), y_{ref})$. We define

$$\tilde{\xi}^0 := \hat{\xi}^0 - \xi^0$$

and

$$\tilde{\nu}^0 := \hat{\nu}^0 - \nu^0;$$

as before, it is routine to confirm that the procedure used to derive (B.39) can be applied here to show that

$$\left|\hat{J}^0_{[(k_l+1)T,t_{l+1})}(\xi(t_l),y_{ref}) - J^0_{[(k_l+1)T,t_{l+1})}(\xi(t_l),y_{ref})\right| \leq$$
$$\gamma_{lqr}\int_{(k_l+1)T}^{t_{l+1}} \Big(\|\xi^0(t)\|\|\tilde{\xi}^0(t)\| + \|\tilde{\xi}^0(t)\|^2 + \|\nu^0(t)\|\|\tilde{\nu}^0(t)\| + \|\tilde{\nu}^0(t)\|^2 +$$
$$\|\tilde{\nu}^0(t)\|\|\tilde{\xi}^0(t)\| + \|\nu^0(t)\|\|\tilde{\xi}^0(t)\| + \|\tilde{\nu}^0(t)\|\|\xi^0(t)\|\Big)\,dt.$$

From the definitions (4.28), (B.47), (4.27), and (B.48) we have

$$\begin{aligned}
\|\xi^0(t)\| &\leq \gamma_0 e^{\lambda_0(t-(k_l+1)T)}\|\xi^0[(k_l+1)T]\|,\\
\|\tilde{\xi}^0(t)\| &\leq \gamma_0 e^{\lambda_0(t-(k_l+1)T)}\|\tilde{\xi}^0[(k_l+1)T]\|,\\
\|\nu^0(t)\| &\leq f\gamma_0 e^{\lambda_0(t-(k_l+1)T)}\|\xi^0[(k_l+1)T]\|,\\
\|\tilde{\nu}^0(t)\| &\leq f\gamma_0 e^{\lambda_0(t-(k_l+1)T)}\|\tilde{\xi}^0[(k_l+1)T]\|, \quad t\in[(k_l+1)T,t_{l+1}),
\end{aligned}$$

which means that

$$\begin{aligned}
&\left|\hat{J}^0_{[(k_l+1)T,t_{l+1})}(\xi(t_l),y_{ref}) - J^0_{[(k_l+1)T,t_{l+1})}(\xi(t_l),y_{ref})\right|\\
&\quad\leq \gamma_{lqr}\gamma_0^2\left[\int_{(k_l+1)T}^{t_{l+1}} e^{2\lambda_0(t-(k_l+1)T)}dt\right]\times\\
&\qquad\Big[(1+f+f^2)\|\tilde{\xi}^0[(k_l+1)T]\|^2 +\\
&\qquad(1+2f+f^2)\|\tilde{\xi}^0[(k_l+1)T]\|\times\|\xi^0[(k_l+1)T]\|\Big]\\
&\quad\leq \gamma_{lqr}\gamma_0^2(1+f)^2\frac{1}{2|\lambda_0|}\times\\
&\qquad\big[\|\tilde{\xi}^0[(k_l+1)T]\|^2 + \gamma_0\|\tilde{\xi}^0[(k_l+1)T]\|\times\|\xi^0(t_l)\|\big]. \qquad \text{(B.50)}
\end{aligned}$$

By definition

$$\hat{\xi}^0[(k_l+1)T] = \xi[(k_l+1)T],$$

so it follows immediately that

$$\tilde{\xi}^0[(k_l+1)T] = \tilde{\xi}[(k_l+1)T],$$

so we can apply Claim 2 to obtain a bound on $\|\tilde{\xi}^0[(k_l+1)T]\|$ in (B.50): it follows that, if T is sufficiently small, then, for every $T' \in (0,\bar{T}_2'(T))$, we have

$$\left|\hat{J}^0_{[(k_l+1)T,t_{l+1})}(\xi(t_l),y_{ref}) - J^0_{[(k_l+1)T,t_{l+1})}(\xi(t_l),y_{ref})\right| \leq$$
$$\gamma_{lqr}\gamma_0^2(1+f)^2\frac{1}{2|\lambda_0|}\Big[\gamma_2^2T^2\big(\|\xi(t_l)\|+\|y_{ref}\|\big)^2 + \gamma_0\gamma_2T\big(\|\xi(t_l)\|+\|y_{ref}\|\big)\|\xi^0(t_l)\|\Big].$$

If we define

$$\gamma_4 := \gamma_{lqr}\gamma_0^2(1+f)^2\frac{1}{2|\lambda_0|}\max\{\gamma_2^2\bar{T},\gamma_0\gamma_2\}$$

then

$$\left|\hat{J}^0_{[(k_l+1)T,t_{l+1})}(\xi(t_l),y_{ref})-J^0_{[(k_l+1)T,t_{l+1})}(\xi(t_l),y_{ref})\right|\leq\gamma_4T\big(\|\xi(t_l)\|+\|y_{ref}\|\big)^2,$$

so clearly, if T is sufficiently small, then, for every $T'\in(0,\bar{T}'_2(T))$, we have

$$\left|\hat{J}^0_{[(k_l+1)T,t_{l+1})}(\xi(t_l),y_{ref})-J^0_{[(k_l+1)T,t_{l+1})}(\xi(t_l),y_{ref})\right|\leq \frac{\varepsilon}{4}\big(\|\xi(t_l)\|+\|y_{ref}\|\big)^2. \tag{B.51}$$

We now combine (B.49) and (B.51) to find that, if T is sufficiently small, then, for every $T'\in(0,\bar{T}'_2(T))$, we have

$$\begin{aligned}
&\left|J_{[(k_l+1)T,t_{l+1})}(\xi(t_l),y_{ref})-J^0_{[(k_l+1)T,t_{l+1})}(\xi(t_l),y_{ref})\right|\\
&\quad\leq \left|J_{[(k_l+1)T,t_{l+1})}(\xi(t_l),y_{ref})-\hat{J}^0_{[(k_l+1)T,t_{l+1})}(\xi(t_l),y_{ref})\right|+\\
&\qquad \left|\hat{J}^0_{[(k_l+1)T,t_{l+1})}(\xi(t_l),y_{ref})-J^0_{[(k_l+1)T,t_{l+1})}(\xi(t_l),y_{ref})\right|\\
&\quad< \left(\frac{\varepsilon}{4}+\frac{\varepsilon}{4}\right)\big[\|\xi(t_l)\|+\|y_{ref}\|\big]^2\\
&\quad= \frac{\varepsilon}{2}\big[\|\xi(t_l)\|+\|y_{ref}\|\big]^2.
\end{aligned}$$

■

It remains to combine the result of Claim 4 with (B.46). Clearly, if T is sufficiently small, then, for every $T'\in(0,\bar{T}'_2(T))$, it follows immediately that

$$\begin{aligned}
&\left|J_{[t_l,t_{l+1})}(\xi(t_l),y_{ref})-J^0_{[t_l,t_{l+1})}(\xi(t_l),y_{ref})\right|\\
&\quad\leq \left|J_{[t_l,(k_l+1)T)}(\xi(t_l),y_{ref})-J^0_{[t_l,(k_l+1)T)}(\xi(t_l),y_{ref})\right|+\\
&\qquad \left|J_{[(k_l+1)T,t_{l+1})}(\xi(t_l),y_{ref})-J^0_{[(k_l+1)T,t_{l+1})}(\xi(t_l),y_{ref})\right|\\
&\quad\leq \left(\alpha_3T+\frac{\varepsilon}{2}\right)\big[\|\xi(t_l)\|+\|y_{ref}\|\big]^2\\
&\quad\leq \varepsilon\big[\|\xi(t_l)\|+\|y_{ref}\|\big]^2.
\end{aligned}$$

■

Appendix C

Proofs from Chapter 5

We will require the following two technical lemmas.

Lemma C.1 *If all eigenvalues of $A \in \mathbf{R}^{n\times n}$ are less that one in magnitude, then there exist constants $\delta > 0$, $\gamma > 0$, and $\lambda \in (0,1)$ so that, for every $\Delta \in \mathcal{PC}_\infty$ satisfying $\|\Delta\|_\infty < \delta$ we have*

$$\left\|(A + \Delta[T])^k\right\| \leq \gamma(\lambda)^k, \quad k \in \mathbf{Z}^+, \quad T \geq 0.$$

Proof:

Let $k \in \mathbf{Z}^+$, $T \geq 0$, and $\Delta \in \mathcal{PC}_\infty$ be arbitrary. We obtain this result by using (discrete time) Lyaponov energy methods. Since all eigenvalues of $A \in \mathbf{R}^{n\times n}$ are less that one in magnitude, there exists a positive definite symmetric matrix P that satisfies

$$A'PA - P = -I. \tag{C.1}$$

We define x via

$$x[k+1] = (A + \Delta[T])\; x[k], \quad x[0] = x_0. \tag{C.2}$$

Additionally, we define

$$V(x[k]) := x'[k]Px[k],$$

$$\bar{\lambda} := \max\{1, \text{maximum eigenvalue of } P\},$$

$$\underline{\lambda} := \text{minimum eigenvalue of } P,$$

and

$$a := \|A\|;$$

observe that, since P is positive definite and symmetric, by the definition of V, we have

$$\underline{\lambda}\|x[k]\|^2 \leq V(x[k]) \leq \bar{\lambda}\|x[k]\|^2. \tag{C.3}$$

Using the definition of V combined with (C.2) and then (C.1) yields

$$\begin{aligned}
V(x[k+1]) &= x'[k]\big\{(A+\Delta[T])'P(A+\Delta[T])\big\}x[k] \\
&= x'[k]\big\{A'PA + 2A'P\Delta[T] + \Delta'[T]P\Delta[T] - P + P\big\}x[k] \\
&= -x'[k]x'[k] + x'[k]\big\{2A'P\Delta[T] + \Delta'[T]P\Delta[T]\big\}x[k] + x'[k]Px'[k] \\
&\le -\|x[k]\|^2 + \big\|2A'P\Delta[T] + \Delta'[T]P\Delta[T]\big\| \times \|x[k]\|^2 + V(x[k]) \\
&\le (-1 + 2a\bar{\lambda}\|\Delta\|_\infty + \bar{\lambda}\|\Delta\|_\infty^2)\|x[k]\|^2 + V(x[k]). \qquad \text{(C.4)}
\end{aligned}$$

If we fix $\delta \in \left(0, -a + \sqrt{a^2 + \frac{1}{\bar{\lambda}}}\right)$, then

$$\begin{aligned}
-1 &\le -1 + 2a\bar{\lambda}\|\Delta\|_\infty + \bar{\lambda}\|\Delta\|_\infty^2 \\
&< -1 + 2a\bar{\lambda}\delta + \bar{\lambda}\delta^2 \\
&< 0, \qquad \|\Delta\|_\infty < \delta,
\end{aligned}$$

and there exists a constant $\alpha \in (-1, 0)$ satisfying

$$-1 + 2a\bar{\lambda}\|\Delta\|_\infty + \bar{\lambda}\|\Delta\|_\infty^2 < \alpha, \quad \|\Delta\|_\infty < \delta;$$

we now restrict ourselves to only those $\Delta \in \mathcal{PC}_\infty$ that satisfy $\|\Delta\|_\infty < \delta$. Together with (C.4), (C.3), and the fact that $\bar{\lambda} \ge 1$, this yields

$$\begin{aligned}
V(x[k+1]) &< \alpha\|x[k]\|^2 + V(x[k]) \\
&\le \alpha\frac{1}{\bar{\lambda}}V(x[k]) + V(x[k]), \\
&\le (\alpha+1)\,V(x[k]); \qquad \text{(C.5)}
\end{aligned}$$

clearly

$$\alpha + 1 \in (0, 1),$$

so, since $V > 0$, we can solve (C.5) iteratively to yield

$$V(x[k]) \le (\alpha+1)^k\, V(x[0]).$$

Using (C.3), this becomes

$$\underline{\lambda}\|x[k]\| \le (\alpha+1)^k\, \bar{\lambda}\|x_0\|,$$

so

$$\|x[k]\| \le \underbrace{\frac{\bar{\lambda}}{\underline{\lambda}}}_{=:\gamma}\underbrace{(\alpha+1)^k}_{=:\lambda^k}\|x_0\|,$$

which implies that

$$\big\|(A+\Delta[T])^k\big\| \le \gamma(\lambda)^k.$$

■

Lemma C.2 *With $\bar{T} > 0$ and $\Delta \in \mathcal{PC}_\infty$, consider the difference equation*

$$z[(k+1)T] = A(T)z[kT] + \Delta[k], \quad k \in \mathbf{Z}^+$$

for which there exist constants $\gamma_1 > 1$, $\gamma_2 > 0$, and $\lambda_1 < 0$ so that

$$\|\big(A(T)\big)^k\| \leq \gamma_1 e^{\lambda_1 kT}, \quad k \in \mathbf{Z}^+, \quad T \in (0, \bar{T})$$

and

$$\|\Delta[k]\| \leq \gamma_2 T \|z[kT]\|, \quad k \in \mathbf{Z}^+, \quad T \in (0, \bar{T}).$$

For every $T \in (0, \bar{T})$, there exists a constant $\varepsilon_\lambda \in (\lambda_1, 0)$ so that, for every $\lambda_2 \in (\lambda_1, \varepsilon_\lambda)$ satisfying

$$\frac{2\gamma_1\gamma_2 e^{-\lambda_2 \bar{T}}}{|\lambda_1 - \lambda_2|} < 1,$$

there exists a constant $\bar{\gamma}(\gamma_2, \lambda_2) > 1$ so that

$$\|z[kT]\| \leq \bar{\gamma}(\gamma_2, \lambda_2) e^{\lambda_2 (k - \underline{k})T} \|z[\underline{k}T]\|, \quad k \geq \underline{k}, \quad \underline{k} \in \mathbf{Z}^+$$

and

$$\lim_{\gamma_2 \to 0} \bar{\gamma}(\gamma_2, \lambda_2) = \gamma_1.$$

Proof:

Fix $\bar{T} > 0$ and $T \in (0, \bar{T})$ and let $\underline{k} \in \mathbf{Z}^+$ and $\lambda_2 \in (\lambda_1, 0)$ be arbitrary. Define

$$\bar{z}[kT] = e^{-\lambda_2 (k - \underline{k})T} z[kT], \quad k \geq \underline{k};$$

using this definition in the difference equation under study, we obtain

$$\begin{aligned} \bar{z}[(k+1)T] &= e^{-\lambda_2 T (k - \underline{k} + 1)T}\big(A(T)z[kT] + \Delta[k]\big) \\ &= \underbrace{e^{-\lambda_2 T} A(T)}_{=:\bar{A}(T)} \bar{z}[kT] + e^{-\lambda_2 T} \underbrace{e^{-\lambda_2 (k - \underline{k})T} \Delta[k]}_{=:\bar{\Delta}[k]}, \quad k \geq \underline{k} \end{aligned}$$

whose solution satisfies

$$\bar{z}[kT] = \big(\bar{A}(T)\big)^{k - \underline{k}} \bar{z}[\underline{k}T] + \sum_{i=\underline{k}}^{k-1} \big(\bar{A}(T)\big)^{k-1-i} e^{-\lambda_2 T} \bar{\Delta}[i], \quad k \geq \underline{k}.$$

We adopt the notation

$$\|\bar{z}\|_\infty := \sup_{k \geq \underline{k}} \|\bar{z}[kT]\|.$$

For $k \in \mathbf{Z}^+$, clearly,

$$\begin{aligned} \|\big(\bar{A}(T)\big)^k\| &\leq e^{-\lambda_2 T} \gamma_1 e^{\lambda_1 kT} \\ &= \gamma_1 e^{(\lambda_1 - \lambda_2)kT} \end{aligned}$$

and

$$\begin{aligned}\|\bar{\Delta}[k]\| &\leq e^{-\lambda_2(k-\underline{k})T}\gamma_2 T\|z[kT]\| \\ &= \gamma_2 T\|\bar{z}[kT]\|,\end{aligned}$$

so, since $\lambda_2 \in (\lambda_1, 0)$, we have

$$\begin{aligned}\|\bar{z}\|_\infty &\leq \gamma_1\|\bar{z}[\underline{k}T]\| + \sum_{i=\underline{k}}^{\infty}\gamma_1 e^{(\lambda_1-\lambda_2)iT}e^{-\lambda_2 T}\gamma_2 T\|\bar{z}\|_\infty \\ &\leq \gamma_1\|\bar{z}[\underline{k}T]\| + \frac{\gamma_1\gamma_2 e^{-\lambda_2\bar{T}}}{1-e^{(\lambda_1-\lambda_2)T}}T\|\bar{z}\|_\infty\end{aligned}$$

and therefore

$$\left(1 - \frac{\gamma_1\gamma_2 e^{-\lambda_2\bar{T}}}{1-e^{(\lambda_1-\lambda_2)T}}T\right)\|\bar{z}\|_\infty \leq \gamma_1\|\bar{z}[\underline{k}T]\|. \tag{C.6}$$

For $\|\bar{z}\|_\infty$ to be well defined, it is enough for the coefficient on the left be positive. To that end, we observe that for sufficiently small $|\gamma_1 - \gamma_2|$, we have

$$\frac{\gamma_1\gamma_2 e^{-\lambda_2\bar{T}}}{1-e^{(\lambda_1-\lambda_2)T}}T \leq \frac{2\gamma_1\gamma_2 e^{-\lambda_2\bar{T}}}{|\lambda_1-\lambda_2|}$$

which, by assumption, is less than one; therefore, for sufficiently small $|\lambda_1 - \lambda_2|$, we obtain

$$1 - \frac{\gamma_1\gamma_2 e^{-\lambda_2\bar{T}}}{1-e^{(\lambda_1-\lambda_2)T}}T > 0.$$

Returning to (C.6), this means that, if $|\lambda_1 - \lambda_2|$ is sufficiently small, then we have

$$\|\bar{z}\|_\infty \leq \underbrace{\frac{\gamma_1}{1-\frac{2\gamma_1\gamma_2 e^{-\lambda_2\bar{T}}}{|\lambda_1-\lambda_2|}}}_{=:\bar{\gamma}(\gamma_2,\lambda_2)}\|\bar{z}[\underline{k}T]\|,$$

so, since $\bar{z}[\underline{k}T] = z[\underline{k}T]$, we have

$$\|\bar{z}[kT]\| \leq \bar{\gamma}(\gamma_2,\lambda_2)\|z[\underline{k}T]\|, \quad k \geq \underline{k},$$

and therefore

$$\begin{aligned}\|z[kT]\| &= \|e^{\lambda_2(k-\underline{k})T}\bar{z}[kT]\| \\ &\leq \bar{\gamma}(\gamma_2,\lambda_2)e^{\lambda_2(k-\underline{k})T}\|z[\underline{k}T]\|, \quad k \geq \underline{k}.\end{aligned}$$

Finally, when λ_2 is fixed, clearly

$$\lim_{\gamma_2\to 0}\bar{\gamma}(\gamma_2,\lambda_2) = \gamma_1.$$

■

Proof of Proposition 5.1:

Let $\xi_0 \in \mathbf{R}^n$, $\varepsilon \geq 0$, and $T \in (0, T_{max})$ be arbitrary. It will turn out that, once we have (iii), the remaining items follow almost directly, so we will start with it.

(iii)

For each $p \in \mathcal{M}$, Lemma C.1 provides a bound on $\|\Phi_p^\varepsilon(T,0)\|$; however, we wish to find a uniform bound, which is easier said than done. To obtain this result we will need to leverage the properties of compact sets and analytic functions.

For any two (not necessarily distinct) sets of Markov Parameters $p, \bar{p} \in \mathcal{M}$ we have

$$\Phi_p^\varepsilon(T,0) = e^{\bar{A}_{\bar{p}}T} + \underbrace{\left(\Phi_p^\varepsilon(T,0) - e^{\bar{A}_{\bar{p}}T}\right)}_{=:\Delta_{p,\bar{p}}[T]}.^{1}$$

Furthermore, since $\bar{A}_p$ is Hurwitz, for every $T > 0$ and $p \in \mathcal{M}$, from Lemma C.1, there exists constants $\delta_p > 0$, $\gamma_p > 0$, and $\lambda_p \in (0,1)$ so that, for every function $\Delta \in \mathcal{PC}_\infty$ which is such that $\|\Delta\|_\infty \leq \delta_p$, we have

$$\left\|\left(e^{\bar{A}_pT} + \Delta(T)\right)^k\right\| \leq \gamma_p(\lambda_p)^k, \quad k \in \mathbf{Z}^+. \tag{C.7}$$

This bound does not have the desired structure, but that is a secondary concern; more importantly the bound depends on the choice of p and hence is not uniform. To make this bound independent of p, we will cover the set of all of our admissible plants with a finite number of open sets, each of which admits a uniform bound, and then leverage the finiteness to find a uniform bound for the entire set $\mathcal{M}$, which we will then convert to the desired structure. To do so, we must first show that $\|\Delta_{p,\bar{p}}\|$ can be made small.

Recall that, by the definition of $\bar{A}_p$ and $H(p,t)$ we have that

$$e^{\bar{A}_{\bar{p}}T} = e^{A_{\bar{p}}T} + \int_0^T e^{A_{\bar{p}}(T-\tau)}B_{\bar{p}}H(\bar{p},\tau)d\tau$$

which combines with the definition of Φ_p^ε to yield

$$\begin{aligned}
\|\Delta_{p,\bar{p}}[T]\| &= \|\Phi_p^\varepsilon(T,0) - e^{\bar{A}_{\bar{p}}T}\| \\
&\leq \|e^{A_{\bar{p}}T} - e^{A_pT}\| + \\
&\quad \int_0^T \|e^{A_{\bar{p}}(T-\tau)}B_{\bar{p}} - e^{A_p(T-\tau)}B_p\|\|H(\bar{p},\tau)\|d\tau + \\
&\quad \int_0^T \|e^{A_p(T-\tau)}B_p\|\Big[\|H(\bar{p},\tau) - H(p,\tau)\| + \\
&\qquad\qquad \|H(p,\tau) - H^\varepsilon(p,\tau)\|\Big]d\tau, \quad p, \bar{p} \in \mathcal{M}.
\end{aligned} \tag{C.8}$$

[1]Observe that, with our definition of Φ^0, if $\varepsilon = 0$ then $\Delta_{p,\bar{p}}[T] = 0$.

Recall that the set

$$\tilde{\mathcal{P}} = \{(A_p, B_p, C) : p \in \mathcal{M}\}$$

contains only minimal representations, so if we define $\tilde{\Gamma}$ to be the set of all (not necessarily canonical form) state-space triples

$$(A, B, C) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times 1} \times \mathbf{R}^{1\times n}$$

which are such that the pair (A, B) is controllable and the pair (C, A) is observable, then $\tilde{\mathcal{P}}$ is a compact subset of $\tilde{\Gamma}$. Furthermore, since $H(p, \tau)$ is an analytic function of p and τ, and p is an analytic function of (A_p, B_p, C), clearly $H(p, \tau)$ is bounded on the compact set $\tilde{\mathcal{P}} \times [0, T_{max}]$, say by $\bar{h}$. Finally, since $\tilde{\Gamma}$ is open, there exists an open ball $\mathcal{B}_{\bar{p}} \subset \tilde{\Gamma}$ centered at $(A_{\bar{p}}, B_{\bar{p}}, C)$ whose closure lies in $\tilde{\Gamma}$, so there exists a constant $\tilde{\gamma}_{\bar{p}} > 0$ such that

$$\|e^{A_{\bar{p}}T} - e^{A_pT}\| \leq \tilde{\gamma}_{\bar{p}}\|A_{\bar{p}} - A_p\|, \quad (A_p, B_p, C) \in \mathcal{B}_{\bar{p}},$$

$$\|e^{A_{\bar{p}}T}B_{\bar{p}} - e^{A_pT}B_p\| \leq \tilde{\gamma}_{\bar{p}}\Big[\|A_{\bar{p}} - A_p\| + \|B_{\bar{p}} - B_p\|\Big], \quad (A_p, B_p, C) \in \mathcal{B}_{\bar{p}},$$

and

$$\|H(\bar{p}, \tau) - H(p, \tau)\| \leq \tilde{\gamma}_{\bar{p}}\Big[\|A_{\bar{p}} - A_p\| + \|B_{\bar{p}} - B_p\|\Big], \quad (A_p, B_p, C) \in \mathcal{B}_{\bar{p}}.$$

Combining these and the fact that $\|H - H^{\varepsilon}\| < \varepsilon$ with (C.8) we see that

$$\begin{aligned}\|\Delta_{p,\bar{p}}[T]\| \;\leq\; & \tilde{\gamma}_{\bar{p}}\|A_{\bar{p}} - A_p\| + \\ & T_{max}\tilde{\gamma}_{\bar{p}}\Big[\|A_{\bar{p}} - A_p\| + \|B_{\bar{p}} - B_p\|\Big]\bar{h} + \\ & T_{max}e^{aT_{max}}b\Big[\tilde{\gamma}_{\bar{p}}\Big[\|A_{\bar{p}} - A_p\| + \|B_{\bar{p}} - B_p\|\Big] + \varepsilon\Big], \quad (A_p, B_p, C) \in \mathcal{B}_{\bar{p}},\end{aligned}$$

so clearly, there exists a constant $\bar{\varepsilon}_{\bar{p}} > 0$ and a ball $\tilde{\mathcal{B}}_{\bar{p}} \subset \mathcal{B}_{\bar{p}}$ with a sufficiently small radius such that

$$\|\Delta_{p,\bar{p}}[T]\| \leq \delta_{\bar{p}}, \quad (A_p, B_p, C) \in \tilde{\mathcal{B}}_{\bar{p}}, \quad \varepsilon \in [0, \bar{\varepsilon}_{\bar{p}}).$$

We can now apply (C.7) to find that

$$\begin{aligned}\Big\|\big(\Phi_p^{\varepsilon}(T, 0)\big)^k\Big\| \;=\; & \Big\|\Big(\Delta_{p,\bar{p}}[T] + e^{\bar{A}_{\bar{p}}T}\Big)^k\Big\| \\ \leq\; & \gamma_{\bar{p}}(\lambda_{\bar{p}})^k, \quad k \in \mathbf{Z}^+, \quad (A_p, B_p, C) \in \tilde{\mathcal{B}}_{\bar{p}}, \quad \varepsilon \in [0, \bar{\varepsilon}_{\bar{p}}).\end{aligned}$$

At this point, we have found a uniform bound over (sufficiently small) balls. We would now like to leverage these to find a uniform bound over the entire set $\tilde{\mathcal{P}}$. To do so, we observe that the open sets $\tilde{\mathcal{B}}_{\bar{p}}$ form an open cover of $\tilde{\mathcal{P}}$, so, since $\tilde{\mathcal{P}}$ is compact, there exists a finite sub-cover in the sense that there exists

$$\{\bar{p}_i : i = 1, .., l\} \subset \mathcal{M}$$

so that

$$\tilde{\mathcal{P}} \subset \bigcup_{i=1}^{l} \tilde{\mathcal{B}}_{\bar{p}_i}$$

and

$$\left\| \left(\Phi_p^\varepsilon(T,0)\right)^k \right\| \le \gamma_{\bar{p}_i} (\lambda_{\bar{p}_i})^k, \quad k \in \mathbf{Z}^+, \quad (A_p, B_p, C) \in \tilde{\mathcal{B}}_{\bar{p}_i}, \quad \varepsilon \in [0, \bar{\varepsilon}_{\bar{p}_i}).$$

If we set

$$\gamma_1 := \max_{i=1,..,l} \gamma_{\bar{p}_i},$$

$$\bar{\lambda}_1 := \max_{i=1,..,l} \lambda_{\bar{p}_i},$$

and

$$\bar{\varepsilon}_1 := \min_{i=1,..,l} \bar{\varepsilon}_{\bar{p}_i},$$

then

$$\left\| \left(\Phi_p^\varepsilon(T,0)\right)^k \right\| \le \gamma_1 (\bar{\lambda}_1)^k, \quad k \in \mathbf{Z}^+, \quad p \in \mathcal{M}, \quad \varepsilon \in [0, \bar{\varepsilon}_1).$$

It remains to put this bound in the desired form. Observe that, since $\bar{\lambda}_1 \in (0,1)$, there exists $\lambda_1 < 0$ such that

$$(\bar{\lambda}_1)^k \le e^{\lambda_1 kT}, \quad T \in [0, T_{max}], \quad k \in \mathbf{Z}^+;$$

in fact, any $\lambda_1 < 0$ satisfying

$$\lambda_1 \in \left(\frac{\ln(\bar{\lambda}_1)}{T_{max}}, 0 \right)$$

will work. We choose any such λ_1 to find

$$\left\| \left(\Phi_p^\varepsilon(T,0)\right)^k \right\| \le \gamma_1 e^{\lambda_1 kT}, \quad k \in \mathbf{Z}^+, \quad p \in \mathcal{M}, \quad \varepsilon \in [0, \bar{\varepsilon}_1). \tag{C.9}$$

■

Henceforth, we restrict $\varepsilon \in [0, \bar{\varepsilon}_1)$ and let $p \in \mathcal{M}$ and $k \in \mathbf{Z}^+$ be arbitrary.

(i)

Define

$$\tilde{\xi} := \xi^0 - \xi^\varepsilon$$

and recall that, by definition,

$$\xi^0[kT] = e^{\bar{A}_p kT} \xi_0.$$

Clearly,

$$\begin{aligned} \dot{\tilde{\xi}}(t) &= A_p \tilde{\xi}(t) + B_p H(p,t) \xi^0[kT] - B_p H^\varepsilon(p,t) \xi^\varepsilon[kT] \\ &= A_p \tilde{\xi}(t) + B_p H^\varepsilon(p,t) \tilde{\xi}[kT] + \\ &\quad B_p \big(H(p,t) - H^\varepsilon(p,t) \big) \xi^0[kT], \qquad t \in [kT, (k+1)T), \end{aligned}$$

whose solution satisfies

$$\tilde{\xi}(t) = \Phi_p^{\varepsilon}(t-kT,0)\tilde{\xi}[kT] + \int_{kT}^{t} \left[e^{A_p(t-\tau)} B_p \big(H(p,\tau) - H^{\varepsilon}(p,\tau) \big) e^{\bar{A}_p kT} \xi_0 \right] d\tau, \quad t \in [kT,(k+1)T). \quad \text{(C.10)}$$

As usual, we will begin by analyzing the period endpoints. To that end, observe that, in particular, (C.10) says that

$$\tilde{\xi}[(k+1)T] = \Phi_p^{\varepsilon}(T,0)\tilde{\xi}[kT] + \underbrace{\left[\int_0^T e^{A_p(T-\tau)} B_p \big(H(p,\tau) - H^{\varepsilon}(p,\tau) \big) d\tau \right]}_{=:\Psi_p^{\varepsilon}(T)} e^{\bar{A}_p kT} \xi_0;$$

solving iteratively and using the fact that $\tilde{\xi}(0) = 0$ yields

$$\tilde{\xi}[kT] = \sum_{i=0}^{k-1} \big(\Phi_p^{\varepsilon}(T,0) \big)^{k-1-i} \Psi_p^{\varepsilon}(T) e^{\bar{A}_p iT} \xi_0.$$

Now,

$$\|\Psi_p^{\varepsilon}(T)\| \le T e^{aT_{max}} b\, \varepsilon,$$

and both $e^{\bar{A}_p iT}$ and $\big(\Phi_p^{\varepsilon}(T,0) \big)^{k-1-i}$ can be bounded by (C.9), so

$$\begin{aligned} \|\tilde{\xi}[kT]\| &\le \sum_{i=0}^{k-1} \gamma_1 e^{\lambda_1(k-1-i)T} T e^{aT_{max}} b\, \varepsilon \gamma_1 e^{\lambda_1 iT} \|\xi_0\| \\ &\le \underbrace{\gamma_1^2 e^{aT_{max}} b}_{=:\gamma_2} T\varepsilon \sum_{i=0}^{k-1} e^{\frac{\lambda_1}{2}(k-1+i)T} \|\xi_0\| \\ &\le e^{\frac{\lambda_1}{2}(k+1)T} \left(\frac{T\gamma_2 e^{-\lambda_1 T}}{1 - e^{\frac{\lambda_1}{2}T}} \right) \varepsilon\, \|\xi_0\|. \end{aligned}$$

Since T is bounded above, there exists a constant $\gamma_3 > 0$ so that

$$\frac{T\gamma_2 e^{-\lambda_1 T}}{1 - e^{\frac{\lambda_1}{2}T}} < \gamma_3, \quad T \in (0, T_{max});$$

therefore,

$$\|\tilde{\xi}[kT]\| \le \gamma_3 e^{\frac{\lambda_1}{2}(k+1)T} \varepsilon\, \|\xi_0\|.$$

Finally, from the design of H^{ε} and (C.9), we have that

$$\begin{aligned} \|H^{\varepsilon}(p,\tau)\| &< \varepsilon + \|H(p,\tau)\| \\ &\le \bar{\varepsilon}_1 + f\gamma_1 e^{\lambda_1 T_{max}}, \quad \tau \in (0,T), \end{aligned}$$

so, from the definition of Φ_p^ε, observe that,

$$\|\Phi_p^\varepsilon(t-kT,0)\| \leq \underbrace{e^{aT_{max}} + T_{max}e^{aT_{max}}b\big(\bar{\varepsilon}_1 + f\gamma_1 e^{\lambda_1 T_{max}}\big)}_{=:\gamma_\phi}.$$

Applying these to (C.10) yields

$$\begin{aligned}
\|\tilde{\xi}(t)\| &\leq \gamma_\phi\big(\gamma_3 e^{\frac{\lambda_1}{2}(k+1)T}\varepsilon\|\xi_0\|\big) + T_{max}e^{aT_{max}}b\varepsilon\big(\gamma_1 e^{\lambda_1 kT}\|\xi_0\|\big) \\
&\leq \underbrace{\big(\gamma_\phi\gamma_3 + T_{max}e^{aT_{max}}b\gamma_1 e^{-\lambda_1 T_{max}}\big)}_{=:\gamma_4} e^{\frac{\lambda_1}{2}(k+1)T}\varepsilon\|\xi_0\|, \quad t\in[kT,(k+1)T) \\
&\leq \gamma_4 e^{\frac{\lambda_1}{2}t}\,\varepsilon\|\xi_0\|, \quad t\geq 0. \qquad \text{(C.11)}
\end{aligned}$$

■

(ii)

Define

$$\tilde{u} := u^0 - u^\varepsilon.$$

Clearly,

$$\begin{aligned}
\tilde{u}(t) &= H(p,t)\xi^0[kT] - H^\varepsilon(p,t)\xi^\varepsilon[kT] \\
&= H(p,t)\tilde{\xi}[kT] - \big(H(p,t) - H^\varepsilon(p,t)\big)\xi^\varepsilon[kT] \\
&= H(p,t)\tilde{\xi}[kT] - \big(H(p,t) - H^\varepsilon(p,t)\big)\left(\Phi_p^\varepsilon(T,0)\right)^k\xi_0;
\end{aligned}$$

therefore, using (C.9) and (C.11), we obtain

$$\|\tilde{u}(t)\| \leq \underbrace{\big(\bar{h}\gamma_4 e^{\frac{\lambda_1}{2}T_{max}} + \gamma_1 e^{-\lambda_1 T_{max}}\big)}_{=:\gamma_5} e^{\frac{\lambda_1}{2}t}\,\varepsilon\|\xi_0\|. \qquad \text{(C.12)}$$

■

(iv)

Observe that

$$\begin{aligned}
\left\|\begin{bmatrix} y^\varepsilon \\ r^{1/2}u^\varepsilon \end{bmatrix} - \begin{bmatrix} y^0 \\ r^{1/2}u^0 \end{bmatrix}\right\|_2^2 &\leq \|y^\varepsilon - y^0\|_2^2 + r\|u^\varepsilon - u^0\|_2^2 \\
&\leq \|\xi^\varepsilon - \xi^0\|_2^2 + r\|u^\varepsilon - u^0\|_2^2
\end{aligned}$$

so by (C.11) and (C.12), we have

$$\begin{aligned}
\left\|\begin{bmatrix} y^\varepsilon \\ r^{1/2}u^\varepsilon \end{bmatrix} - \begin{bmatrix} y^0 \\ r^{1/2}u^0 \end{bmatrix}\right\|_2^2 &\leq \underbrace{(\gamma_4^2 + r\gamma_5^2)}_{=:\gamma_6}\varepsilon^2\|\xi_0\|^2\int_0^\infty \left(e^{\frac{\lambda_1}{2}t}\right)^2 dt \\
&= \frac{\gamma_6}{|\lambda_1|}\varepsilon^2\|\xi_0\|^2.
\end{aligned}$$

■

If we set

$$\gamma_0 := \max\left\{\gamma_1, \gamma_4, \gamma_5, \sqrt{\frac{\gamma_6}{|\lambda_1|}}\right\}$$

and

$$\lambda_0 := \frac{\lambda_1}{2},$$

then the result follows. ■

Proof of Proposition 5.2:

Fix $T_s > 0$ and let $\xi_0 \in \mathbf{R}^n$, $\varepsilon \in (0, \bar{\varepsilon})$, $T \in (0, \bar{T}(T_s))$, $\sigma \in \Sigma_{T_s}$, and $k \in \mathbf{Z}^+$ be arbitrary. Observe that forcing T' to be small is equivalent to forcing h and thereby h_m to be small. Furthermore, it will be convenient to define

$$T_{i,j} := T_i + 2(j-1)h_m$$

and

$$\begin{aligned}
&\operatorname{Est}\big\{\operatorname{Est}\big\{\xi[kT] \otimes^i \sigma[kT]\big\}_j \otimes \sigma[kT]\big\} := \\
&\quad \frac{1}{\rho}\operatorname{argmin}\Big\{ \left\| \begin{bmatrix} I_n & 0 \end{bmatrix} H(h)^{-1} S^{-1} \big[\mathcal{Y}(kT + T_{i,j}) - \mathcal{Y}(kT)\big] \right\|, \\
&\qquad\qquad \left\| \begin{bmatrix} I_n & 0 \end{bmatrix} H(h)^{-1} S^{-1} \big[\mathcal{Y}(kT + T_{i,j} + h_m) - \mathcal{Y}(kT + h_m)\big] \right\| \Big\}.
\end{aligned}$$

Finally, throughout this proof it will be important to keep in mind that ξ is continuous, which provides many nice properties; for example, for any $[\underline{t}, \bar{t}] \subset \mathbf{R}^+$, we have that $\max_{t \in [\underline{t}, \bar{t}]} \|\xi(t)\|$ is well defined.

This proof works by using the KEL to find bounds on the size of our estimates and thereby the desired variables. Since the KEL does not hold on subintervals in which there is a switch, we must be careful; we now obtain a bound on the control signal over the Estimation Phase which holds whether or not there is a switch.

Claim 1: There exists a constant $\gamma_1 > 0$ so that, **in all cases**, for every sufficiently small T' we have that

$$|u(t)| \leq \gamma_1 \max_{\tau \in [kT, kT + 2T']} \|\xi(\tau)\|, \quad t \in [kT, (k+1)T).$$

Proof:

The key to this proof is the following. Since $T < T_s/2$ there can be *at most* one switch on any period. Recall that all of our estimates are obtained via an argmin of two sets of samples obtained on disjoint intervals. Since we know that there is no switch during the acquisition of at least one of the samples, then that sample can be bounded by the KEL and so can the associated estimates.

Using the KEL to bound at least one of the samples, we find that there exists a constant $\gamma_{kel} > 0$ so that, for every sufficiently small $T' > 0$, we have that

$$\begin{aligned} \|\mathrm{Est}[\xi[kT] \otimes^0 \sigma[kT]]\| &\leq \gamma_{kel} \max\{\xi[kT], \xi[kT + h_m]\} \\ &\leq \gamma_{kel} \max_{t\in[kT,kT+h_m]} \{\xi(t)\} \end{aligned} \tag{C.13}$$

and similarly, with $i = 1, .., q$ and $j = 1, .., n_{i-1}$,

$$\mathrm{Est}\big\{\mathrm{Est}\big\{\xi[kT] \otimes^{i-1} \sigma[kT]\big\}_j \otimes \sigma[kT]\big\} \leq$$
$$\left(\sup_{p\in\mathcal{M}} \|p\| + \gamma_{kel}T'\right) \left\|\mathrm{Est}\big\{\xi[kT] \otimes^{i-1} \sigma[kT]\big\}_j\right\| + \gamma_{kel}T' \max_{t\in[kT,kT+T_{i,j}]} \|\xi(t)\|. \tag{C.14}$$

By definition,

$$\mathrm{Est}\big\{\xi[kT] \otimes^i \sigma[kT]\big\} = \begin{bmatrix} \mathrm{Est}\big\{\mathrm{Est}\big\{\xi[kT] \otimes^{i-1} \sigma[kT]\big\}_1 \otimes \sigma[kT]\big\} \\ \cdots \\ \mathrm{Est}\big\{\mathrm{Est}\big\{\xi[kT] \otimes^{i-1} \sigma[kT]\big\}_{n_{i-1}} \otimes \sigma[kT]\big\} \end{bmatrix};$$

therefore, using (C.14), we find that there exists a constant $\alpha_1 > 0$ so that, for every sufficiently small T' we have that

$$\left\|\mathrm{Est}\big\{\xi[kT] \otimes^i \sigma[kT]\big\}\right\| \leq \alpha_1 \left\|\mathrm{Est}\big\{\xi[kT] \otimes^{i-1} \sigma[kT]\big\}\right\| + \alpha_1 T' \max_{t\in[kT,kT+2T']} \|\xi(t)\|,$$

which we can solve iteratively to find that

$$\left\|\mathrm{Est}\big\{\xi[kT] \otimes^i \sigma[kT]\big\}\right\| \leq \alpha_1^i \left\|\mathrm{Est}\big\{\xi[kT] \otimes^0 \sigma[kT]\big\}\right\| + \sum_{j=0}^{i-1} \alpha_1^j T' \max_{t\in[kT,kT+2T']} \|\xi(t)\|.$$

With (C.13), this becomes

$$\left\|\mathrm{Est}\big\{\xi[kT] \otimes^i \sigma[kT]\big\}\right\| \leq \left(\alpha_1^i \gamma_{kel} + \sum_{j=0}^{i-1} \alpha_1^j T'\right) \max_{t\in[kT,kT+2T']} \|\xi(t)\|,$$

so, since i is bounded by q, there exists a constant $\alpha_2 > 0$ so that, for every sufficiently small T' we have

$$\left\|\mathrm{Est}\big\{\xi[kT] \otimes^i \sigma[kT]\big\}\right\| \leq \alpha_2 \max_{t\in[kT,kT+2T']} \|\xi(t)\|, \quad i = 1, .., q$$

and therefore, from the structure of u, if we define

$$\gamma_1 := \max\{\gamma_{kel}, \alpha_2\} \times \max\left\{\rho, \sum_{j=0}^{\bar{q}} (T_{max})^j \sum_{i=0}^{q} \|d_{i,j}\|\right\},$$

then

$$|u(t)| \leq \gamma_1 \max_{\tau\in[kT,kT+2T']} \|\xi(\tau)\|, \quad t \in [kT, (k + 1)T].$$

$\blacksquare$

(i)

Assume that there are **no** switches on the interval $[kT,(k+1)T)$; to simplify notation, with $p \in \mathcal{M}$ arbitrary, we assume that $\sigma(t) = p$ over this interval. We begin by refining the bound on u that was obtained in Claim 1. To do so, we first find a bound on ξ over the Estimation Phase.

Claim 2: There exists a constant $\gamma_2 > 0$ so that, for every sufficiently small T' we have that

$$\|\xi(t)\| \leq \gamma_2 \|\xi[kT]\|, \quad t \in [kT, kT+2T').$$

Proof:

To proceed, observe that, solving (5.3) and using $\sigma(t) = p$ provides

$$\xi(t) = e^{A_p(t-kT)}\xi[kT] + \int_{kT}^{t} e^{A_p(t-\tau)} B_p u(\tau) d\tau, \quad t \in [kT, kT+2T'),$$

so

$$\|\xi(t)\| \leq e^{aT_{max}} \|\xi[kT]\| + 2T' e^{aT_{max}} b \max_{\tau \in [kT, kT+2T']} |u(\tau)|, \quad t \in [kT, kT+2T').$$

If we apply Claim 1, then for every sufficiently small $T' > 0$, we have that

$$\|\xi(t)\| \leq e^{aT_{max}} \|\xi[kT]\| + 2T' e^{aT_{max}} b \left(\gamma_1 \max_{\tau \in [kT, kT+2T']} \|\xi(\tau)\| \right), \quad t \in [kT, kT+2T'),$$

so clearly

$$\max_{t \in [kT, kT+2T']} \|\xi(t)\| \leq e^{aT_{max}} \|\xi[kT]\| + 2T' e^{aT_{max}} b \gamma_1 \max_{\tau \in [kT, kT+2T']} \|\xi(\tau)\|,$$

and, if

$$T' \in \left(0, \frac{1}{4e^{aT_{max}} b \gamma_1}\right),$$

then

$$\begin{aligned} \max_{t \in [kT, kT+2T']} \|\xi(t)\| &\leq \frac{e^{aT_{max}}}{1 - 2T' e^{aT_{max}} b \gamma_1} \|\xi[kT]\| \\ &\leq \underbrace{2e^{aT_{max}}}_{=:\gamma_2} \|\xi[kT]\|; \end{aligned}$$

therefore

$$\|\xi(t)\| \leq \gamma_2 \|\xi[kT]\|, \quad t \in [kT, kT+2T'].$$

$\blacksquare$

Clearly, Claims 1 and 2 combine to provide (i)(c): for every sufficiently small T' we have that

$$\begin{aligned} |u(t)| &\leq \gamma_1 \max_{\tau \in [kT, kT+2T']} \|\xi(\tau)\| \\ &\leq \gamma_1 \gamma_2 \|\xi[kT]\|, \quad t \in [kT, kT+2T']. \end{aligned} \tag{C.15}$$

To proceed, we first need to leverage the KEL to find a nice result regarding our estimates when there is no switch.

Claim 3: There exists a constant $\gamma_3 > 0$ so that, if T' is sufficiently small, then

$$\|\mathrm{Est}\big\{\xi[kT] \otimes^i p\big\} - \xi[kT] \otimes^i p\| \leq \gamma_3 T' \|\xi[kT]\|, \quad i = 0, .., q$$

Proof:

The case of $i = 0$ is trivial: the KEL and Claim 2, together, say that

$$\mathrm{Est}\big\{\xi[kT] \otimes^0 p\big\} = \xi[kT] + \mathcal{O}(T')\|\xi[kT]\|.$$

By the definition of our controller, the KEL, and Claim 2 we have that

$$\begin{aligned} \mathrm{Est}\Big\{\mathrm{Est}\big\{\xi[kT] \otimes^0 p\big\}_j \otimes p\Big\} &= p\mathrm{Est}\big\{\xi[kT] \otimes^0 p\big\}_j + \\ &\quad \mathcal{O}(T') \left(\gamma_2 \|\xi[kT]\| + \|\mathrm{Est}\big\{\xi[kT] \otimes^0 p\big\}_j\|\right), \quad j = 1, .., n. \end{aligned}$$

therefore, back substituting yields

$$\mathrm{Est}\Big\{\mathrm{Est}\big\{\xi[kT] \otimes^0 p\big\}_j \otimes p\Big\} = p\xi[kT]_j + \mathcal{O}(T')\|\xi[kT]\|, \quad j = 1, .., n$$

and therefore

$$\begin{aligned} \mathrm{Est}\big\{\xi[kT] \otimes^1 p\big\} &= \begin{bmatrix} \mathrm{Est}\Big\{\mathrm{Est}\big\{\xi[kT] \otimes^0 p\big\}_1 \otimes p\Big\} \\ \dots \\ \mathrm{Est}\Big\{\mathrm{Est}\big\{\xi[kT] \otimes^0 p\big\}_n \otimes p\Big\} \end{bmatrix} \\ &= \xi[kT] \otimes^1 p + \mathcal{O}(T')\|\xi[kT]\|. \end{aligned}$$

We can continue this process iteratively to find the desired result:

$$\mathrm{Est}\big\{\xi[kT] \otimes^i p\big\} = \xi[kT] \otimes^i p + \mathcal{O}(T')\|\xi[kT]\|, \quad i = 1, .., q.$$

■

By the definition of the controller and Claim 3, we have (i)(b): for $t \in [kT + 2T', (k+1)T)$, for every sufficiently small $T' > 0$, using the fact that, on this interval

$\sigma[kT] = p$, we have that

$$\begin{aligned}
&|u(t) - H^{\varepsilon}(\sigma[kT], t)| \\
&\quad = \left|\sum_{j=0}^{\bar{q}}(t-kT)^j\sum_{i=0}^{q} d_{i,j}\left(\mathrm{Est}\big\{\xi[kT]\otimes^i\sigma[kT]\big\} - \xi[kT]\otimes^i\sigma[kT]\right)\right| \\
&\quad \leq \sum_{j=0}^{\bar{q}}(t-kT)^j\sum_{i=0}^{q}\|d_{i,j}\|\left\|\mathrm{Est}\big\{\xi[kT]\otimes^i\sigma[kT]\big\} - \xi[kT]\otimes^i\sigma[kT]\right\| \\
&\quad \leq \underbrace{\left(\sum_{j=0}^{\bar{q}}T_{max}^j\sum_{i=0}^{q}\|d_{i,j}\|\right)\gamma_3}_{=:\alpha_3} T'\|\xi[kT]\|. \qquad \text{(C.16)}
\end{aligned}$$

With this in hand, we can now obtain (i)(a): observe that (again using $\sigma[kT] = p$)

$$\begin{aligned}
&\|\xi(t) - \Phi^{\varepsilon}_{\sigma[kT]}(t-kT,0)\xi[kT]\| \\
&\quad = \left\|\int_0^t e^{A_{\sigma[kT]}(t-\tau)}B_{\sigma[kT]}\big(u(\tau) - H^{\varepsilon}(\sigma[kT],\tau)\xi[kT]\big)d\tau\right\| \\
&\quad \leq \int_0^{2T'}\|e^{A_{\sigma[kT]}(t-\tau)}B_{\sigma[kT]}\big(u(\tau) - H^{\varepsilon}(\sigma[kT],\tau)\xi[kT]\big)\|d\tau + \\
&\qquad\quad \int_{2T'}^{T}\|e^{A_{\sigma[kT]}(t-\tau)}B_{\sigma[kT]}\big(u(\tau) - H^{\varepsilon}(\sigma[kT],\tau)\xi[kT]\big)\|d\tau, \quad t\in[kT,(k+1)T).
\end{aligned}$$

Since ε is fixed, we have that $\|H^{\varepsilon}\|_{\infty}$ is well defined, so we can use Claim 2, (C.15) and (C.16) to find that

$$\begin{aligned}
&\|\xi(t) - \Phi^{\varepsilon}_{\sigma[kT]}(t-kT,0)\xi[kT]\| \\
&\quad \leq 2T'e^{aT_{max}}b\left(\gamma_1\gamma_2 + \|H^{\varepsilon}\|_{\infty}\gamma_2\right)\|\xi[kT]\| + (T-2T')e^{aT_{max}}b\alpha_3T'\|\xi[kT]\| \\
&\quad \leq e^{aT_{max}}b\Big[2\left(\gamma_1\gamma_2 + \|H^{\varepsilon}\|_{\infty}\gamma_2\right) + T_{max}\alpha_3\Big]T'\|\xi[kT]\|
\end{aligned}$$

∎

(ii)

Since we do not know whether or not there is a switch in this period, rather than use the sequence $\{t_l\}$, we introduce some new notation. We assume that if there is a switch in the period $[kT,(k+1)T)$, then it occurs at $\hat{t}$, i.e., there is a time $\hat{t}\in[kT,(k+1)T)$ and two (not necessarily distinct) sets of Markov parameters $p_1, p_2 \in \mathcal{M}$ satisfying[2]

$$\sigma(t) = \begin{cases} p_1, & t\in[kT,\hat{t}) \\ p_2, & t\in[\hat{t},(k+1)T); \end{cases}$$

[2]Recall that we assume that σ is continuous from the right for convenience.

if there is no switch in the interval, then $p_1 = p_2$.

To proceed, we re-investigate the solution to (5.3) in the context of switches. Observe that

$$\xi(t) = e^{A_{p_1}(t-kT)}\xi[kT] + \int_{kT}^{t} e^{A_{p_1}(t-\tau)} B_{p_1} u(\tau) d\tau, \quad t \in [kT, \hat{t}),$$

and

$$\begin{aligned} \xi(t) &= e^{A_{p_2}(t-\hat{t})}\xi(\hat{t}) + \int_{\hat{t}}^{t} e^{A_{p_2}(t-\tau)} B_{p_2} u(\tau) d\tau, \\ &= e^{A_{p_2}(t-\hat{t})} \left[e^{A_{p_1}(\hat{t}-kT)}\xi[kT] + \int_{kT}^{\hat{t}} e^{A_{p_1}(t-\tau)} B_{p_1} u(\tau) d\tau \right] + \\ & \int_{\hat{t}}^{t} e^{A_{p_2}(t-\tau)} B_{p_2} u(\tau) d\tau, \qquad t \in [\hat{t}, (k+1)T); \end{aligned}$$

if we combine these with Claim 1, then we conclude that, for every sufficiently small T', we have

$$\begin{aligned} \|\xi(t) - \xi[kT]\| \;\; \leq \;\; & \max \left\{ \left\| I - e^{A_{p_1} t} \right\|, \left\| I - e^{A_{p_2}(t-\hat{t})} e^{A_{p_1}(\hat{t}-kT)} \right\| \right\} \|\xi[kT]\| + \\ & T e^{aT_{max}} b \gamma_1 \max_{\tau \in [kT, (k+1)T]} \|\xi(\tau)\|, \quad t \in [kT, (k+1)T), \end{aligned} \tag{C.17}$$

and the simpler bound

$$\|\xi(t)\| \leq e^{aT_{max}} \|\xi[kT]\| + T e^{aT_{max}} b \gamma_1 \max_{\tau \in [kT, (k+1)T]} \|\xi(\tau)\|, \quad t \in [kT, (k+1)T).$$

We analyze this simpler bound in the same way as in the proof of Claim 2, to find that, with

$$\bar{T}_0 := \min \left\{ \bar{T}(T_s), \frac{1}{2e^{aT_{max}} b \gamma_1} \right\},$$

for every $T \in (0, \bar{T}_0)$, if $T' > 0$ is sufficiently small, then

$$\|\xi(t)\| \leq \frac{e^{aT_{max}}}{2} \|\xi[kT]\|, \quad t \in [kT, (k+1)T]; \tag{C.18}$$

when combined with Claim 1, this yields (ii)(b):

$$|u(t)| \leq \frac{\gamma_1 e^{aT_{max}}}{2} \|\xi[kT]\|, \quad t \in [kT, (k+1)T].$$

Furthermore, when (C.18) is applied to (C.17), we obtain (ii)(a); to see how, first observe that, for small T,

$$\begin{aligned} \left\| I - e^{A_{p_1}(t-kT)} \right\| \;\; &\approx \;\; \|A_{p_1}(t - kT)\| \\ &\leq \;\; aT, \quad t \in [kT, (k+1)T], \end{aligned}$$

and

$$\left\| I - e^{A_{p_1}(\hat{t}-kT)} e^{A_{p_2}(t-\hat{t})} \right\| \approx \left\| A_{p_1}(\hat{t}-kT) + A_{p_2}(t-\hat{t}) \right\| \leq aT, \quad t \in [kT,(k+1)T],$$

so there exists a constant $\bar{T}_1 \in (0, \bar{T}_0)$ so that, for every $T \in (0, \bar{T}_1)$, if $T' > 0$ is sufficiently small, then we have

$$\begin{aligned} \|\xi(t) - \xi[kT]\| &\leq \max\left\{ \left\| I - e^{A_{p_1}T} \right\|, \left\| I - e^{A_{p_1}A_{p_2}T} \right\| \right\} \|\xi[kT]\| + \\ & \quad T e^{2aT_{max}} b\gamma_1 \frac{1}{2} \|\xi[kT]\| \\ &\leq \left(2a + \frac{e^{2aT_{max}} b\gamma_1}{2} \right) T\|\xi[kT]\|, \quad t \in [kT,(k+1)T]. \end{aligned}$$

■

■

Proof of Lemma 5.3:

Fix $T_s > 0$, $T \in (0, \bar{T}(T_s))$, and $\varepsilon \in (0, \bar{\varepsilon})$ and let $T' \in (0, T/2)$ be arbitrary. To reduce clutter, with $p \in \mathcal{M}$ arbitrary, assume that $\sigma(t) = p$ over the interval $[\underline{k}T, \bar{k}T)$. From Proposition 5.2(i)(a), if T' is sufficiently small, then there exists a function $\Delta_p[k] \in \mathcal{PC}_\infty$ and a constant $\delta > 0$ satisfying

$$\|\Delta_p[k]\| \leq \delta T' \|\xi[kT]\|, \quad p \in \mathcal{M} \tag{C.19}$$

and

$$\xi[(k+1)T] = \Phi_p^\varepsilon(T,0)\xi[kT] + \Delta_p[k], \quad k = \underline{k}, .., \bar{k} - 1. \tag{C.20}$$

From Proposition 5.1(iii) we have that

$$\| \left(\Phi_p^\varepsilon(T,0) \right)^k \| \leq \gamma_0 e^{\lambda_0 kT}, \quad p \in \mathcal{M},$$

so the difference equation (C.20) is of the form in Lemma C.2, which we would like to apply; to that end, we let $\lambda_1 \in (\lambda_0, 0)$ be arbitrary. We would like our result to hold for all admissible λ_1; to do so, for each $\lambda_1 \in (\lambda_0, 0)$, we must be able to ensure the existence of a constant $\gamma_1 > 0$ that satisfies

$$\delta T' < \gamma_1 T$$

and

$$\frac{2\gamma_0 \gamma_1 e^{-\lambda_1 T_{max}}}{|\lambda_1 - \lambda_0|} < 1.$$

Together, these are equivalent to

$$\delta T' < \gamma_1 T < \frac{|\lambda_0 - \lambda_1|}{2\gamma_0 e^{-\lambda_1 T_{max}}} T.$$

Clearly

$$\begin{aligned}\delta T' &< \frac{|\lambda_0 - \lambda_1|}{2\gamma_0 e^{-\lambda_1 T_{max}}} T \\ \Leftrightarrow T' &< \frac{|\lambda_0 - \lambda_1|}{2\delta\gamma_0 e^{-\lambda_1 T_{max}}} T =: \bar{T}'(\lambda_1, T),\end{aligned}$$

so for every $T' \in (0, \bar{T}'(\lambda_1, T)/2)$ we have that such a γ_1 exists. Observe that γ_1 can be made independent of T' but that it depends on T and λ_1; the dependence on T will be useful, but the dependence on λ_1 will not, so we re-write it as $\gamma_1(T)$. Furthermore, if we freeze λ_1, then such a γ_1 has the property that

$$\lim_{T \to 0} \gamma_1(T) = 0.$$

For the remainder of this proof, we let $T' \in (0, \bar{T}'(\lambda_1, T)/2)$ be arbitrary.

We now apply Lemma C.2 to (C.20) to find that there exist constants $\varepsilon_\lambda \in (\lambda_0, 0)$ and $\bar{\gamma}(\gamma_1, \lambda_1) > 1$ such that, for every $\lambda_1 \in (\lambda_0, \varepsilon_\lambda)$ we have

$$\|\xi[kT]\| \leq \bar{\gamma}(\gamma_1, \lambda_1) e^{\lambda_1 (k - \underline{k})T} \|\xi[\underline{k}T]\|, \quad k = \underline{k}, .., \bar{k}, \tag{C.21}$$

and, if λ_1 is fixed, then

$$\lim_{\gamma_1 \to 0} \bar{\gamma}(\gamma_1, \lambda_1) = \gamma_0.$$

Observe that this implies (with λ_1 fixed)

$$\lim_{T \to 0} \bar{\gamma}(\gamma_1(T), \lambda_1) = \gamma_0;$$

It serves our purposes to stress the dependence on T, so, since γ_1 is a function of both T and λ_1 we will write $\bar{\gamma}(T, \lambda_1)$ instead of $\bar{\gamma}(\gamma_1, \lambda_1)$.

We now have the desired structure at the sample points; it remains to show that our state is well behaved everywhere. To that end, we observe that, since Φ_p^ε is clearly bounded, Proposition 5.2(i)(a) says that there exists a constant $\gamma_3(T) > 0$ such that, for every sufficiently small $T' > 0$, we have

$$\|\xi(t)\| \leq \gamma_3(T) \|\xi[kT]\|, \quad t \in [kT, (k+1)T), \quad k \in \mathbf{Z}^+;$$

since ξ is continuous, we have that

$$\lim_{T \to 0} \gamma_3(T) = 1.$$

Using this bound together with (C.21), if T' is sufficiently small, then we obtain

$$\begin{aligned}\|\xi(t)\| &\leq \gamma_3(T)\bar{\gamma}(T, \lambda_1) e^{\lambda_1 (k - \underline{k})T} \|\xi[\underline{k}T]\|, \quad k = \underline{k}, .., \bar{k}, \quad t \in [kT, (k+1)T), \\ &\leq \underbrace{\gamma_3(T)\bar{\gamma}(T, \lambda_1) e^{-\lambda_1 T}}_{=:\gamma(T, \lambda_1)} e^{\lambda_1 (t - \underline{k}T)} \|\xi[\underline{k}T]\|, \quad t \in [\underline{k}T, \bar{k}T).\end{aligned}$$

Finally, with λ_1 fixed we have

$$\begin{aligned}\lim_{T\to 0}\gamma(T,\lambda_1) &= \lim_{T\to 0}\left(\gamma_3(T)\bar{\gamma}(T,\lambda_1)e^{-\lambda_1 T}\right)\\ &= 1\times\gamma_0\times 1\\ &= \gamma_0.\end{aligned}$$

■

Proof of Lemma 5.4:

Note that we do not need a minimum realization. We will use the controller state z to hold all of the samples of y, so we will need z to have dimension

$$n_z := 2\left(m+\sum_{i=1} qn^i\right).$$

We assume that the period of the controller is ℓ, so

$$T = \ell h$$

and therefore,

$$kT = k\ell h, \quad k\in\mathbf{Z}^+.$$

With e_i the standard basis vector, we then define the periodic state update function

$$G(j,z[j],y(jh)) = \begin{cases} e_1^T y(jh) & j=0\\ z[j]+e_{j+1}^T y(jh) & j=1,..,\ell-2;\end{cases}$$

observe that z_i will hold $y[kT+(i-1)h]$ and that, at the beginning of each period we clear the state. Clearly, this provides the desired property

$$G(0,z[j],y(jh)) = G(0,0,y(jh)).$$

It remains to carefully design the output function J. We will do so in a series of phases that mirror how the controller is defined, but first we introduce some notation. The discrete time analogue to T_i is defined by

$$\ell_1 := 2(m+1) \text{ and } \ell_i := \ell_{i-1}+2n_{i-1}(m+1);$$

we also define the discrete time analogue to $T_{i,l}$:

$$\ell_{i,l} := \ell_i + 2(l-1)(m+1).$$

Phase 1: State Estimation Phase

Here we want $u=0$, so we set

$$J(j,z[j],\tau) = 0, \quad j=0,..,\ell_1-1.$$

Phase 2: Control Estimation Phase

Here we need u to be the sequence of test signals:

$$u(t) = \rho \text{Est}\big\{\xi[kT] \otimes^{i-1} \sigma[kT]\big\}_l, \quad t \in [kT + \ell_{i,l}h, kT + \ell_{i,l+1}h).$$

Recall that we use $m+1$ instead of m to avoid initialization issues; however, this means that we are obtaining one more sample than we require; i.e. we want to throw away the $m+1^{th}$ sample. To that end, we will now abuse the notion of the basis matrix E_i introduced elsewhere in this document. We define

$$\bar{I} := \begin{bmatrix} I_m \\ 0 \end{bmatrix} \in \mathbf{R}^{(m+1)\times m}$$

and then, with $E_i \in \mathbf{R}^{n_z \times m}$ consisting of block elements of size $(m+1)\times m$, we set

$$E_i := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \bar{I} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow i^{th} \text{ block element,} \\ \\ \\ \\ \end{matrix}$$

so $E_i^T z[j]$ is a vector consisting of the $(i-1)(m+1)$ to $i(m+1)-2^{th}$ elements of $z[j]$.

We begin with $i = 1$. Observe that

$$\begin{aligned} \text{Est}\big\{\xi[kT] \otimes^0 \sigma[kT]\big\}_l &= \text{Est}\big\{\xi[kT]\big\}_l \\ &= \frac{1}{\rho} e_l^T \text{argmin}\{\| \underbrace{\begin{bmatrix} I_n & 0 \end{bmatrix} H(h)^{-1} S^{-1}}_{=:V_0(h)} \mathcal{Y}(kT)\|, \\ &\qquad\qquad \| \begin{bmatrix} I_n & 0 \end{bmatrix} H(h)^{-1} S^{-1} \mathcal{Y}(kT + h_m)\|\} \\ &= \frac{1}{\rho} e_l^T \text{argmin}\{\|V_0(h)E_1^T z[j]\|, \|V_0(h)E_2^T z[j]\|\}, \\ &\qquad\qquad j = \ell_{1,l}, .., \ell - 1. \end{aligned} \quad \text{(C.22)}$$

so we can set

$$\begin{aligned} J(j, z[j], \tau) &= e_l^T \text{argmin}\{\|V_0(h)E_1^T z[j]\|, \|V_0(h)E_2^T z[j]\|\}, \\ &\qquad j = \ell_{1,l}, .., \ell_{1,l+1} - 1, \quad l = 1, .., n. \end{aligned}$$

Via a similar argument, with

$$\bar{n}_1 := 2 \text{ and } \bar{n}_i := \bar{n}_{i-1} + 2n_i + 1$$

observe that, for $i = 2, .., q$ we have

$$\text{Est}\big\{\xi[kT] \otimes^{i-1} \sigma[kT]\big\}_l =$$
$$\frac{1}{\rho} e_l^T \text{argmin} \left\{ \left\| V_i(h) \begin{bmatrix} \left(E_{\bar{n}_i+1}^T - E_1^T\right) z[j] \\ \vdots \\ \left(E_{\bar{n}_{i+1}-2}^T - E_1^T\right) z[j] \end{bmatrix} \right\|, \left\| V_i(h) \begin{bmatrix} \left(E_{\bar{n}_i+2}^T - E_2^T\right) z[j] \\ \vdots \\ \left(E_{\bar{n}_{i+1}-1}^T - E_2^T\right) z[j] \end{bmatrix} \right\| \right\}$$
$$j = \ell_{i,l}, .., \ell - 1, \quad l = 1, .., n_i, \tag{C.23}$$

so, with $*$ the content of the argmin above, we can set

$$J(j, z[j], \tau) = e_l^T \text{argmin}\{*\} \quad j = \ell_{i,l}, .., \ell_{i,l+1} - 1, \quad l = 1, .., n_i.$$

Phase 2: Control Phase

Here we wish to apply

$$u(\tau + kT) = \sum_{j=0}^{\bar{q}} \tau^j \sum_{i=0}^{q} d_{i,j} \text{Est}\big\{\xi[kT] \otimes^i \sigma[kT]\big\}, \quad \tau \in [T_{q+1}, T);$$

if we use the state based estimate realizations (C.22) and (C.23), then it is enough to set

$$J(j, z[j], \tau) = \sum_{j=0}^{\bar{q}} \tau^j \sum_{i=0}^{q} d_{i,j} \text{Est}\big\{\xi[kT] \otimes^i \sigma[kT]\big\}, \quad j = \ell_q, .., \ell - 1.$$

$\blacksquare$

Appendix D

Proofs from Chapter 6

We will require the following technical lemma:

Lemma D.1 *If $A \in \mathbf{R}^{n \times n}$ is Hurwitz, then there exist constants $\delta > 0$, $\lambda < 0$, and $\gamma > 0$ such that, for all $\Delta \in PC_\infty$ satisfying $\|\Delta\|_\infty < \delta$, we have*

(i) $\|\Phi_{A+\Delta}(t, t_0) - \Phi_A(t, t_0)\| \le \gamma \|\Delta\|_\infty e^{\lambda(t-t_0)}, \quad t \ge t_0,$

(ii) $\|\Phi_{A+\Delta}(t, t_0)\| \le \gamma e^{\lambda(t-t_0)}, \quad t \ge t_0.$

Proof:

Let $t_0 \in \mathbf{R}^+$, $t \ge t_0$, and $\Delta \in \mathcal{PC}_\infty$ be arbitrary. Observe that (ii) is the continuous time equivalent to Lemma C.1. Since A is Hurwitz, there exists a positive definite symmetric matrix P that satisfies

$$PA + A'P = -I. \tag{D.1}$$

We define x and x_δ via

$$\dot{x}(t) = A\, x(t), \quad x(t_0) = x_0, \tag{D.2}$$

$$\dot{x}_\delta(t) = (A + \Delta(t))\, x_\delta(t), \quad x_\delta(t_0) = x(t_0) = x_0. \tag{D.3}$$

(ii)

We prove this result by obtaining a bound on $\|x_\delta(t)\|$ in terms of $\|x_\delta(t_0)\|$. To do so we will solve some differential inequalities, constructed through manipulation of energy functions. To that end, set

$$V(x_\delta(t)) = x_\delta'(t) P x_\delta(t),$$

which combines this with (D.1) and (D.3) to yield

$$\begin{aligned}\dot{V}(x_\delta(t)) &= -x_\delta'(t)x_\delta(t) + x_\delta'(t)\left[P\Delta(t) + \Delta'(t)P\right]x_\delta(t)\\ &\leq -\|x_\delta(t)\|^2 + 2\|P\|\|\Delta\|_\infty\|x_\delta(t)\|^2,\end{aligned}$$

If we fix

$$\delta = \frac{1}{4\|P\|},$$

then this becomes

$$\dot{V}(x_\delta(t)) \leq -\frac{1}{2}\|x_\delta(t)\|^2, \quad \|\Delta\|_\infty < \delta. \tag{D.4}$$

To that end, we restrict ourselves to only those $\Delta \in \mathcal{PC}_\infty$ that satisfy $\|\Delta\|_\infty < \delta$. Furthermore, define

$$\bar{\lambda} := \text{maximum eigenvalue of } P,$$

and

$$\underline{\lambda} := \text{minimum eigenvalue of } P$$

and observe that, since P is positive definite and symmetric, by the definition of V, we have

$$\underline{\lambda}\|x(t)\|^2 \leq V(x_\delta(t)) \leq \bar{\lambda}\|x(t)\|^2, \quad t \geq t_0 \tag{D.5}$$

which, when combined with (D.4), yields

$$\dot{V}(x_\delta(t)) \leq -\frac{1}{2}\|x_\delta(t)\|^2 \leq -\frac{1}{2\bar{\lambda}}V(x_\delta(t)),$$

whose solution satisfies

$$\begin{aligned}V(x_\delta(t)) &\leq e^{-\frac{1}{2\bar{\lambda}}(t-t_0)}V(x_\delta(t_0)), \quad t \geq t_0\\ &\leq \bar{\lambda}e^{-\frac{1}{2\bar{\lambda}}(t-t_0)}\|x_\delta(t_0)\|^2, \quad t \geq t_0.\end{aligned}$$

Finally, using (D.5) yields

$$\begin{aligned}\underline{\lambda}\|x_\delta(t)\|^2 &\leq \bar{\lambda}e^{-\frac{1}{2\bar{\lambda}}(t-t_0)}\|x_\delta(t_0)\|^2\\ \Rightarrow \qquad \|x_\delta(t)\| &\leq \underbrace{\left(\frac{\bar{\lambda}}{\underline{\lambda}}\right)^{\frac{1}{2}}}_{=:\gamma_1}\underbrace{e^{-\frac{1}{4\bar{\lambda}}(t-t_0)}}_{=:e^{\lambda_1(t-t_0)}}\|x_\delta(t_0)\|, \quad t \geq t_0,\end{aligned} \tag{D.6}$$

so clearly we have the desired result

$$\|\Phi_{A+\Delta}(t, t_0)\| \leq \gamma_1 e^{\lambda_1(t-t_0)}, \quad t \geq t_0;$$

furthermore, observe that we can set $\Delta = 0$ to obtain

$$\|\Phi_A(t, t_0)\| \leq \gamma_1 e^{\lambda_1(t-t_0)}, \quad t \geq t_0. \tag{D.7}$$

■

(i)

Since $x_\delta(t_0) = x(t_0) = x_0$, we can write

$$(x_\delta - x)(t) = [\Phi_{A+\Delta}(t, t_0) - \Phi_A(t, t_0)]x_0;$$

therefore, we can find our desired bound by solving the differential equation given by

$$(\dot{x}_\delta - \dot{x})(t) = A(x_\delta(t) - x(t)) + \Delta(t)x_\delta(t), \quad x_\delta(t_0) - x(t_0) = 0,$$

whose solution is

$$(x_\delta - x)(t) = \int_{t_0}^{t} \Phi_A(t, \tau)\Delta(\tau)x_\delta(\tau)d\tau,$$

so clearly

$$\|(x_\delta - x)(t)\| \le \int_{t_0}^{t} \big\|\Phi_A(t, \tau)\Delta(\tau)x_\delta(\tau)\big\| d\tau.$$

Using (D.7) to bound $\|\Phi_A\|$ and (D.6) to bound $\|x_\delta\|$ yields

$$\begin{aligned}
\|(x_\delta - x)(t)\| &\le \int_{t_0}^{t} \gamma_1 e^{\lambda_1(t-\tau)}\|\Delta\|_\infty \gamma_1 e^{\lambda_1(\tau - t_0)}\|x_0\| d\tau \\
&\le \gamma_1^2 \|\Delta\|_\infty \|x_0\| \int_{t_0}^{t} e^{\lambda_1(t-\tau)} e^{\frac{\lambda_1}{2}(\tau - t_0)} d\tau \\
&\le \frac{2\gamma_1^2}{|\lambda_1|}\|\Delta\|_\infty e^{\frac{\lambda_1}{2}(t-t_0)}\|x_0\|.
\end{aligned}$$

■

Proof of Lemma 6.1 (The Estimation Lemma):

Let $x_0 \in \mathbf{R}^n$, $r \in PC_\infty$, $t_0 \in \mathbf{R}^+$, $h \in (0, \bar{h})$, $\bar{u} \in \mathbf{R}$, and $g \in \mathcal{G}(G, T_s, c_g)$ be arbitrary. Recall that we are analyzing two models (6.2) and (6.3) that do *not* contain noise signals.

Here we wish to prove a result pertaining to stacked error signals $\mathcal{E}$. To make the proof more tractable, we will split our analysis into two parts: we first investigate y and then y_{ref}. Recall from standard linear system theory that the Zero Input Response (ZIR) is the system response when the initial condition is nonzero and input is zero; conversely, setting the initial condition to zero and introducing a nonzero input produces the Zero State Response (ZSR). The input under consideration in this proof is zero and then constant, and we would like to investigate the system responses to each of these segments independently and then combine the result. To that end, it will be convenient to split the various system responses into two parts in the following way[1]:

$$y(t) = \underbrace{C_p e^{A_p(t-t_0)} x_p(t_0)}_{=:y_{zir}(t,t_0)} + \underbrace{\int_{t_0}^{t} C_p e^{A_p(t-\tau)} B_p g(\tau) u(\tau)\, d\tau}_{=:y_{zsr}(t,t_0)}, \quad t \in [t_0, t_0 + 2h_m).$$

[1]Our use of ZIR and ZSR are slightly non-standard here, but the nomenclature serves the purpose of helping to keep track of which component we are discussing.

We will require the following notation:

$$O_p := \begin{bmatrix} C_p \\ C_p A_p \\ \vdots \\ C_p A_p^m \end{bmatrix} \quad \text{and} \quad O_w := \begin{bmatrix} C_w \\ C_w A_w \\ \vdots \\ C_w A_w^m \end{bmatrix},$$

and we define $\mathcal{Y}(t_0)$ and $\mathcal{Y}_{ref}(t_0)$ so that they are consistent with the definition of $\mathcal{E}(t_0)$ and define $\mathcal{Y}_{zir}(t_0, t_0)$ so that it is consistent with the definition of $\mathcal{Y}(t_0)$ and $y_{zir}(t, t_0)$.

We begin by investigating the plant output y.

Claim 1: There exist constants $\gamma > 0$ and $\bar{h} > 0$ so that for every u of the form

$$u(t) = \begin{cases} 0, & t \in [t_0, t_0 + h_m) \\ \bar{u}, & t \in [t_0 + h_m, t_0 + 2h_m), \end{cases}$$

the solution to (6.1) has the following two properties:

(i) $\|X_m(h)^{-1} S_m^{-1}(\mathcal{Y}(t_0 + h_m) - \mathcal{Y}(t_0))\| \le \gamma\|\bar{u}\| + ch\|x_p(t_0)\|$.

(ii) Furthermore, if $g \in \mathcal{G}(G, T_s, c_g)$ is continuous on $[t_0, t_0 + 2h_m)$, then

$$\|X_m(h)^{-1} S_m^{-1}(\mathcal{Y}(t_0 + h_m) - \mathcal{Y}(t_0)) - C_p(A_p)^{m-1} B_p e_{m+1} g(t_0)\bar{u}\| \le \gamma h(\|x_p(t_0)\| + \|\bar{u}\|).$$

Proof:

We begin by performing some preliminary analysis, starting with the time interval $[t_0, t_0 + h_m)$, in which $u = 0$. For every $l = 0, .., m$ we have

$$\begin{aligned} y(t_0 + lh) &= C_p e^{A_p lh} x_p(t_0), \\ &= \sum_{i=0}^{m} \frac{C_p (A_p lh)^i}{i!} x_p(t_0) + \mathcal{O}(h^{m+1}) x_p(t_0) \\ &= \begin{bmatrix} 1 & l & \dots & l^m \end{bmatrix} X_m(h) O_p x_p(t_0) + \mathcal{O}(h^{m+1}) x_p(t_0). \end{aligned}$$

We then stack all of the signals of the form $y(t_0 + lh)$ to yield

$$\mathcal{Y}(t_0) = S_m X_m(h) O_p x_p(t_0) + \mathcal{O}(h^{m+1}) x_p(t_0). \tag{D.8}$$

Now we turn to the time interval $[t_0 + h_m, t_0 + 2h_m)$, in which $u = \bar{u}$. To simplify notation we define

$$t_1 := t_0 + h_m.$$

Using the same method as above, we obtain the (stacked) response due to the initial conditions:

$$\mathcal{Y}_{zir}(t_1, t_0) = S_m X_m(h) O_p e^{A_p h_m} x_p(t_0) + \mathcal{O}(h^{m+1}) e^{A_p h_m} x_p(t_0). \tag{D.9}$$

(i)

To find a loose bound on the forced response, we first note that the elements of $\mathcal{G}(G, T_s, c_g)$ are uniformly bounded and P_1 has relative degree m; as such, for $t \in [t_1, t_1 + h_m)$ we have

$$\begin{aligned} y_{zsr}(t, t_1) &= \int_{t_1}^{t} C_p e^{A_p(t-\tau)} B_p g(\tau) \bar{u}\, d\tau \\ &= \int_{t_1}^{t} \left[\frac{C_p (A_p(t-\tau))^{m-1} B_p}{(m-1)!} + \mathcal{O}((t-\tau)^m) \right] \mathcal{O}(1) \bar{u}\, d\tau. \\ &= \frac{C_p (A_p)^{m-1} B_p (t-t_1)^m}{m!} \mathcal{O}(1)\bar{u} + \mathcal{O}((t-\tau)^{m+1})\bar{u}, \\ & \qquad\qquad t \in [t_1, t_1 + h_m). \end{aligned} \tag{D.10}$$

In particular, this yields

$$y_{zsr}(t_1 + lh, t_1) = \mathcal{O}(1) \frac{C_p (A_p)^{m-1} B_p (lh)^m}{m!} \bar{u} + \mathcal{O}(h^{m+1})\bar{u}, \quad l = 0, .., m.$$

Combining this with (D.9) and subtracting (D.8) yields

$$\begin{aligned} \mathcal{Y}(t_0 + h_m) - \mathcal{Y}(t_0) &= S_m X_m(h) [O_p (e^{A_p h_m} - I) x_p(t_0) + \mathcal{O}(1)\bar{u}] + \\ &\quad \mathcal{O}(h^{m+1}) x_p(t_0) + \mathcal{O}(h^{m+1})\bar{u}. \end{aligned}$$

Finally, we simplify and invert $S_m X_m(h)$ to obtain

$$\begin{aligned} X_m(h)^{-1} S_m^{-1} (\mathcal{Y}(t_0 + h_m) - \mathcal{Y}(t_0)) &= \mathcal{O}(h) x_p(t_0) + \mathcal{O}(1)\bar{u} + \mathcal{O}(h) x_p(t_0) + \mathcal{O}(h)\bar{u} \\ &= \mathcal{O}(1)\bar{u} + \mathcal{O}(h) x_p(t_0). \end{aligned}$$

■

(ii)

Here we have the additional assumption that g is continuous on $[t_0, t_0 + 2h_m)$, so it follows that for every such $g \in \mathcal{G}(G, T_s, c_g)$

$$|g(t) - g(t_0)| \leq c_g |t - kT|, \quad t \in [t_0, t_0 + 2h_m),$$

which means that we can write

$$g(t) = g(t_0) + \mathcal{O}(h), \quad t \in [t_0, t_0 + 2h_m). \tag{D.11}$$

We can use (D.11) to find a tighter version of (D.10):

$$\begin{aligned} y_{zsr}(t, t_1) &= \int_{t_1}^{t} C_p e^{A_p(t-\tau)} B_p g(\tau) \bar{u}\, d\tau \\ &= \int_{t_1}^{t} \left[\frac{C_p (A_p(t-\tau))^{m-1} B_p}{(m-1)!} + \mathcal{O}((t-\tau)^m) \right] [g(t_1) + \mathcal{O}(h)]\, \bar{u}\, d\tau \\ &= \int_{t_1}^{t} \left[\frac{C_p (A_p(t-\tau))^{m-1} B_p}{(m-1)!} g(t_1) + \right. \\ &\qquad \left. \mathcal{O}((t-\tau)^m) + \mathcal{O}((t-\tau)^{m-1}) \mathcal{O}(h) \right] \bar{u}\, d\tau, \quad t \in [t_1, t_1 + h_m). \end{aligned}$$

so we have

$$y_{zsr}(t+lh,t_1) = \frac{C_p A_p^{m-1} B_p (lh)^m}{m!} g(t_1)\bar{u} + \mathcal{O}(h^{m+1})\bar{u}, \quad l = 0,..,m.$$

Combining this with (D.9) and subtracting (D.8) yields

$$\begin{aligned} \mathcal{Y}(t_0+h_m) - \mathcal{Y}(t_0) &= S_m X_m(h)[O_p e^{A_p h_m}]x_p(t_0) + \mathcal{O}(h^{m+1})x_p(t_0) + \\ & \quad C_p(A_p)^{m-1} B_p g(t_0) S_m X_m(h) e_{m+1}\bar{u} + \mathcal{O}(h^{m+1})\bar{u}, \end{aligned} \tag{D.12}$$

finally, we invert $S_m X_m(h)$ to obtain

$$\begin{aligned} X_m(h)^{-1} S_m^{-1}(\mathcal{Y}(t_0+h_m) - \mathcal{Y}(t_0)) &= \mathcal{O}(h) O_p x_p(t_0) + C_p(A_p)^{m-1} B_p g(t_0) e_{m+1}\bar{u} + \\ & \quad \mathcal{O}(h) x_p(t_0) + \mathcal{O}(h)\bar{u} \\ &= e_{m+1} C_p (A_p)^{m-1} B_p g(t_0)\bar{u} + \\ & \quad \mathcal{O}(h)[\|x_p(t_0)\| + |\bar{u}|]. \end{aligned}$$

■

■

We now turn to the filter output y_{ref}.

Claim 2: There exist constants $\gamma > 0$ and $\bar{h} > 0$ so that the solution to (6.3) satisfies

$$\|X_m(h)^{-1} S_m^{-1}(\mathcal{Y}_{ref}(t_0) - \mathcal{Y}_{ref}(t_0+h_m))\| \leq \gamma h(\|x_w(t_0)\| + \|r\|_\infty).$$

Proof:

The solution to (6.3) satisfies

$$y_{ref}(t) = C_w e^{A_w(t-t_0)} x_w(t_0) + \int_{t_0}^{t} C_w e^{A_w(t-\tau)} B_w r(\tau)\, d\tau.$$

Using Assumption 6.1, and the same methods as in the proof of Claim 1, we obtain

$$\mathcal{Y}_{ref}(t_0) = S_m X_m(h) O_w x_w(t_0) + \mathcal{O}(h^{m+1}) x_w(t_0) + \mathcal{O}(h^{m+1})\|r\|_\infty \tag{D.13}$$

and

$$\mathcal{Y}_{ref}(t_0+h_m) = S_m X_m(h) O_w e^{A_w h_m} x_w(t_0) + \mathcal{O}(h^{m+1}) x_w(t_0) + \mathcal{O}(h^{m+1})\|r\|_\infty, \tag{D.14}$$

which combine to yield

$$\begin{aligned} \mathcal{Y}_{ref}(t_0) - \mathcal{Y}_{ref}(t_0+h_m) &= S_m X_m(h) O_w (e^{A_w h_m} - I) x_w(t_0) + \\ & \quad \mathcal{O}(h^{m+1}) x_w(t_0) + \mathcal{O}(h^{m+1})\|r\|_\infty; \end{aligned}$$

we then multiply by $X_m(h)^{-1} S_m^{-1}$ and simplify to obtain

$$X_m(h)^{-1} S_m^{-1}(\mathcal{Y}_{ref}(t_0) - \mathcal{Y}_{ref}(t_0+h_m)) = \mathcal{O}(h) x_w(t_0) + \mathcal{O}(h)\|r\|_\infty.$$

■

We obtain the desired results via a direct application of Claims 1 and 2. Combining the inequalities in Claim 1(i) and Claim 2 yields

$$X_m(h)^{-1}S_m^{-1}(\mathcal{E}(t_0) - \mathcal{E}(t_0 + h_m)) = \mathcal{O}(1)\bar{u} + \mathcal{O}(h)(\|x_p(t_0)\| + \|x_w(t_0)\| + \|r\|_\infty),$$

which implies

$$\begin{aligned}\left[C_p(A_p)^{m-1}B_p\right]^{-1} e_{m+1}^T X_m(h)^{-1}S_m^{-1}(\mathcal{E}(t_0) - \mathcal{E}(t_0 + h_m)) = \\ \mathcal{O}(1)\bar{u} + \mathcal{O}(h)(\|x_p(t_0)\| + \|x_w(t_0)\| + \|r\|_\infty),\end{aligned}$$

while combining the inequalities in Claim 1(ii) and Claim 2 yields

$$\begin{aligned}X_m(h)^{-1}S_m^{-1}(\mathcal{E}(t_0) - \mathcal{E}(t_0 + h_m)) - e_{m+1}C_p(A_p)^{m-1}B_p g(t_0)\bar{u} = \\ \mathcal{O}(h)(|\bar{u}| + \|x_p(t_0)\| + \|x_w(t_0)\| + \|r\|_\infty),\end{aligned}$$

which implies

$$\begin{aligned}\left[C_p(A_p)^{m-1}B_p\right]^{-1} e_{m+1}^T X_m(h)^{-1}S_m^{-1}(\mathcal{E}(t_0) - \mathcal{E}(t_0 + h_m)) - g(t_0)\bar{u} = \\ \mathcal{O}(h)(|\bar{u}| + \|x_p(t_0)\| + \|x_w(t_0)\| + \|r\|_\infty).\end{aligned}$$

■

Proof of Proposition 6.1:

Fix $K_{lti} \in \mathcal{S}(P_1)$ and let $x_0 \in \mathbf{R}^n$, $r \in \mathcal{PC}_\infty$, and $g \in \mathcal{G}(G, T_s, c_g)$ be arbitrary. Here we will require a state space representation of the ideal system (i.e. $g \equiv 1$ and $\mathcal{C} = K_{lti}$) in closed loop:

$$\begin{aligned}\dot{x}^0(t) &= \underbrace{(A - B_u F)}_{=:A_{cl}^0} x^0(t) + B_r r(t) \\ e^0(t) &= \underbrace{\begin{bmatrix} -C_p & 0 & C_w \end{bmatrix}}_{=:C_{cl}} x^0(t).\end{aligned} \tag{D.15}$$

Define $\Delta(g(t))$ to be the difference between $A_{cl}^\varepsilon(g(t))$ and A_{cl}^0:

$$\begin{aligned}\Delta(g(t)) &:= A_{cl}^\varepsilon(g(t)) - A_{cl}^0 \\ &= B_u\big(g(t)\phi^\varepsilon(g(t)) - 1\big)F\end{aligned}$$

Finally, since A_{cl}^0 is Hurwitz by design of K_{lti} we can apply Lemma D.1: there exist constants $\delta > 0$, $\lambda_1 < 0$, and $\gamma_1 > 0$ so that, for all $\Delta(g)$ satisfying $\|\Delta(g)\|_\infty < \delta$, we have

$$\|\Phi_{A_{cl}^\varepsilon(g)}(t, t_0) - \Phi_{A_{cl}^0}(t, t_0)\| \le \gamma_1 \varepsilon e^{\lambda_1(t - t_0)}, \quad t \ge t_0, \tag{D.16}$$

$$\|\Phi_{A_{cl}^\varepsilon(g)}(t, t_0)\| \le \gamma_1 e^{\lambda_1(t - t_0)}, \quad t \ge t_0. \tag{D.17}$$

(i)

From the definition of Δ we have that

$$\begin{aligned} \|\Delta(g(t))\| &= \|B_u\big(g(t)\phi^\varepsilon(g(t))-1\big)F\| \\ &\leq \|B_u\| \times \|F\| \times \varepsilon, \quad t \geq 0, \end{aligned}$$

so, if we choose $\bar{\varepsilon}$ such that

$$\bar{\varepsilon} \leq \frac{\delta}{\|B_u\| \times \|F\|},$$

then it follows directly from (D.17) that

$$\|\Phi_{A^\varepsilon_{cl}(g)}(t,t_0)\| \leq \gamma_1 e^{\lambda_1(t-t_0)}, \quad t \geq t_0, \quad \varepsilon \in (0,\bar{\varepsilon}).$$

■

(ii)

Since we are interested in weighted sensitivity, we investigate $(e^\varepsilon - e^0)(t)$ with zero initial conditions:

$$\begin{aligned} e^\varepsilon(t) - e^0(t) &= C_{cl}(x^\varepsilon(t) - x^0(t)) \\ &= C_{cl}\int_0^t [\Phi_{A^\varepsilon_{cl}(g)}(t,\tau) - \Phi_{A^0_{cl}}(t,\tau)]B_r r(\tau)d\tau. \end{aligned}$$

We take norms on both sides and use (D.16) to conclude that

$$\begin{aligned} \|(e^\varepsilon - e^0)(t)\| &\leq \|C_{cl}\|\int_0^t \gamma_1\varepsilon e^{\lambda_1(t-\tau)}\|B_r\|\|r\|_\infty d\tau \\ &= \underbrace{\frac{\gamma_1}{|\lambda_1|}\|C_{cl}\|\|B_r\|}_{=:\gamma_2}\varepsilon(1-e^{\lambda_1 t})\|r\|_\infty \\ &\leq \gamma_2\varepsilon\|r\|_\infty, \qquad t \geq t_0, \quad \varepsilon \in (0,\bar{\varepsilon}). \end{aligned}$$

■

Construction of Suitable State Matrices for κ:

For simplicity, in this construction we look at the special case of $q=1$ and $c_1 = 1$; also, recall that $p = 2(q+1)(m+1) = 4(m+1)$. Define:

$$\begin{bmatrix} \gamma_0 & \dots & \gamma_m \end{bmatrix} := (C_p(A_p)^{m+1}B_p)^{-1}e^T_{m+1}X_m(h)^{-1}S_m^{-1}.$$

The controller is periodic of period p, so we choose periodic matrices: for $i \in \mathbf{Z}^+$ we set

$$A_z[ip+k] = \begin{cases} 0 & k=0 \\ I & k=1,..,p-1, \end{cases}$$

$$B_z[ip+k] = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & \gamma_0 \end{bmatrix} & k=0 \\ \begin{bmatrix} 0 & 0 \\ 0 & \gamma_k \end{bmatrix} & k=1,..,m \\ \begin{bmatrix} 0 & 0 \\ 0 & -\gamma_{k-(m+1)} \end{bmatrix} & k=m+1,..,2(m+1)-1 \\ 0 & k=2(m+1),..,p-1, \end{cases}$$

$$C_z[ip+k] = \begin{cases} 0 & k=0,..,m \\ \begin{bmatrix} 1 & 0 \end{bmatrix} & k=m+1,..,2(m+1)-1 \\ \frac{1}{2}\begin{bmatrix} 2(q+1)a_0-1 & 2(q+1)a_1 \end{bmatrix} & k=2(m+1),..,p-1, \end{cases}$$

and

$$D_z[ip+k] = 0.$$

Proof of Lemma 6.2 (One Period Lemma):

Fix $K_{lti} \in \mathcal{S}(P_1)$ and $\varepsilon \in (0, \bar{\varepsilon})$ and let $x_0 \in \mathbf{R}^n$, $r \in PC_\infty$, $k \in \mathbf{Z}^+$, $g \in \mathcal{G}(G, T_s, c_g)$ be arbitrary. Assume that $\mathcal{C}(K_{lti}, \varepsilon, T)$ is attached to the plant P_g.

Before proceeding, we give an overview of the proof. First we use the Estimation Lemma to obtain rough bounds on $|u|$ and $\|x(t)-x[kT]\|$. We then use these bounds to find tighter bounds for the case where g is continuous on the interval. Since $T := 2(q+1)h_m$ and $h_m = (m+1)h$, we have that T, h, and h_m are interchangeable with respect to order notation. We begin by using the Estimation Lemma to obtain bounds on the estimates (and hence on the input):

Claim 1: There exists a constants $\gamma_1 > 0$ so that, if $h > 0$ is sufficiently small, then

$$\|u(t)\| \leq \gamma_1 \sup_{\tau \in [kT,(k+1)T)} \|x(\tau)\| + \gamma_1 h \|r\|_\infty, \quad t \in [kT, (k+1)T).$$

Proof:

From the definition of u and the Estimation Lemma we have

$$\begin{aligned} Est\{c_i g^{i+1}[kT]u^0(jT)\} &= (C_p(A_p)^{m-1}B_p)^{-1} e_{m+1}^T X_m(h)^{-1} S_m^{-1} \times \\ &\quad [\mathcal{E}(kT+(2i+1)h_m) - \mathcal{E}(kT+2ih_m)] \\ &= \mathcal{O}(1) Est\{c_{i-1} g^i[kT]u^0(jT)\} + \\ &\quad \mathcal{O}(h)[x_p(kT+2ih_m) + x_w(kT+2ih_m) + \|r\|_\infty], \\ &\qquad\qquad i = 0,..,q-1; \end{aligned}$$

solving iteratively and using the fact that $u^0[kT] = Fx[kT]$, we obtain

$$\begin{aligned} Est\{c_i g^{i+1}[kT]u^0(jT)\} &= \mathcal{O}(1)x[kT] + \mathcal{O}(h)\left[\max_{\tau \in [kT,(k+1)T)} \|x(\tau)\| + \|r\|_\infty\right] \\ &= \mathcal{O}(1) \max_{\tau \in [kT,(k+1)T)} \|x(\tau)\| + \mathcal{O}(h)\|r\|_\infty, \\ &\qquad\qquad i = 0,..,q-1. \end{aligned}$$

Our desired result follows directly from the special structure of u. ∎

We now turn to an analysis of the system equation (6.5) and derive a bound on the state x over the interval $[kT,(k+1)T)$.

Claim 2 There exists a constant $\gamma_2 > 0$ so that, if $T > 0$ is sufficiently small, then

$$\|x(t)\| \leq \gamma_2 \|x[kT]\| + \gamma_2 T \|r\|_\infty, \quad t \in [kT,(k+1)T).$$

Proof:

Solving (6.5) and then applying Claim 1 yields

$$\begin{aligned}
x(t) &= e^{A(t-kT)}x[kT] + \int_{kT}^{t} e^{A(t-\tau)} \left[B_u g(\tau)u(\tau) + B_r r(\tau)\right] d\tau \\
&= \mathcal{O}(1)x[kT] + \mathcal{O}(T)\left[\sup_{\tau\in[kT,(k+1)T)} \|u(\tau)\| + \|r\|_\infty\right] \\
&= \mathcal{O}(1)x[kT] + \mathcal{O}(T)\left[\|r\|_\infty + \max_{\tau\in[kT,(k+1)T)} \|x(\tau)\|\right], \\
&\qquad\qquad\qquad\qquad\qquad t \in [kT,(k+1)T);
\end{aligned}$$

it follows immediately that

$$\max_{\tau\in[kT,(k+1)T)} \|x(\tau)\| = \mathcal{O}(1)x[kT] + \mathcal{O}(T)[\|r\|_\infty + \max_{\tau\in[kT,(k+1)T)} \|x(\tau)\|],$$

which means that

$$\max_{\tau\in[kT,(k+1)T)} \|x(\tau)\| = \mathcal{O}(1)x[kT] + \mathcal{O}(T)\|r\|_\infty,$$

so clearly

$$x(t) = \mathcal{O}(1)x[kT] + \mathcal{O}(T)\|r\|_\infty, \quad t \in [kT,(k+1)T).$$

∎

We now use Claims 1 and 2 to find a bound on $x(t) - x[kT]$.

Claim 3: There exists a constant $\gamma_3 > 0$ so that, if $T > 0$ is sufficiently small, then

$$\|x(t) - x[kT]\| \leq \gamma_3 T(\|x[kT]\| + \|r\|_\infty), \quad t \in [kT,(k+1)T).$$

Proof:

Solving (6.5) yields

$$\begin{aligned}
x(t) - x[kT] &= \left(e^{A(t-kT)} - I\right)x[kT] + \\
&\quad \int_{kT}^{t} e^{A(t-\tau)} \left[B_u g(\tau)u(\tau) + B_r r(\tau)\right] d\tau \\
&= \mathcal{O}(T)\left[x[kT] + \sup_{\tau\in[kT,(k+1)T)} \|u(\tau)\| + \|r\|_\infty\right], \; t \in [kT,(k+1)T);
\end{aligned}$$

if we simplify and apply Claim 1 followed by Claim 2 then we find

$$x(t) - x[kT] = \mathcal{O}(T)(\|x[kT]\| + \|r\|_\infty).$$

■

(i)

We can use Claims 1 and 2 to obtain (b): from Claim 1 we have

$$u(t) = \mathcal{O}(1) \max_{\tau\in[kT,(k+1)T)} \|x(\tau)\| + \mathcal{O}(h)\|r\|_\infty, \quad t \in [kT, (k+1)T),$$

to which we apply Claim 2, yielding

$$u(t) = \mathcal{O}(1)x[kT] + \mathcal{O}(T)\|r\|_\infty, \quad t \in [kT, (k+1)T). \tag{D.18}$$

To find (a), we observe that, for $t \in [kT, (k+1)T)$, we have

$$\begin{aligned} x(t) - \Phi_{A_{cl}^{\varepsilon}(g)}(t, kT)x[kT] \quad &- \quad \int_{kT}^{t} \Phi_{A_{cl}^{\varepsilon}(g)}(t,\tau)B_r r(\tau)d\tau \\ &= \quad x(t) - x[kT] + (I - \Phi_{A_{cl}^{\varepsilon}(g)}(t,kT))x[kT] - \\ &\quad\quad \int_{kT}^{t} \Phi_{A_{cl}^{\varepsilon}(g)}(t,\tau)B_r r(\tau)d\tau, \end{aligned}$$

to which we apply Claim 3, yielding

$$\begin{aligned} x(t) - \Phi_{A_{cl}^{\varepsilon}(g)}(t, kT)x[kT] - \int_{kT}^{t} \Phi_{A_{cl}^{\varepsilon}(g)}(t,\tau)B_r r(\tau)d\tau = \\ \mathcal{O}(T)(x[kT] + \|r\|_\infty) + \mathcal{O}(T)x[kT] + \mathcal{O}(T)\|r\|_\infty. \end{aligned}$$

■

(ii)

We must now investigate the case in which g is continuous on $[kT, (k+1)T)$; to that end, we restrict ourselves to such pairs g and k.

Observe that we can rewrite the system equation (6.5) in the following way:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_u g(t)u(t) + B_r r(t) \\ &= A_{cl}^{\varepsilon}(g(t))x(t) + \big[A - A_{cl}^{\varepsilon}(g(t))\big]x(t) + B_r r(t) + B_u g(t)u(t), \end{aligned}$$

whose solution satisfies

$$\begin{aligned} x(t) &= \Phi_{A_{cl}^{\varepsilon}(g)}(t, kT)x[kT] + \int_{kT}^{t} \Phi_{A_{cl}^{\varepsilon}(g)}(t,\tau)B_r r(\tau)d\tau + \\ &\quad \int_{kT}^{t} \Phi_{A_{cl}^{\varepsilon}(g)}(t,\tau)\Big\{\big[A - A_{cl}^{\varepsilon}(g(\tau))\big]x(\tau) + g(\tau)B_u u(\tau)\Big\}d\tau. \end{aligned}$$

From the definition of A_{cl}^{ε} we have

$$A - A_{cl}^{\varepsilon}(g(\tau)) = -B_u g(\tau)\phi^{\varepsilon}(g(\tau))L,$$

so

$$x(t) - \Phi_{A_{cl}^{\varepsilon}(g)}(t,kT)x[kT] - \int_{kT}^{t} \Phi_{A_{cl}^{\varepsilon}(g)}(t,\tau)B_r r(\tau)d\tau = \underbrace{\int_{kT}^{t} \Phi_{A_{cl}^{\varepsilon}(g)}(t,\tau)B_u g(\tau)\big[-\phi^{\varepsilon}(g(\tau))u^0(\tau) + u(\tau)\big]d\tau}_{=:\eta(kT,t)}.$$

We proceed by using the Estimation Lemma to show that the effect of u is similar to that of $\phi^{\varepsilon}(g)u^0$ and then use that result to find a nice bound on η.

Claim 4: There exists a constant $\gamma_4 > 0$ so that, if $T > 0$ is sufficiently small, then

$$\left\| \int_{kT}^{(k+1)T} \big(u(\tau) - \phi^{\varepsilon}(g[kT])u^0[kT]\big)d\tau \right\| \leq \gamma_4 T^2(\|x[kT]\| + \|r\|_{\infty}).$$

Proof:

From the Estimation Lemma (with $t_0 = kT + 2ih_m$ and $\bar{u} = u(kT + (2i+1)h_m)$) we have

$$\begin{aligned} Est\{c_i g^{i+1}[kT]u^0(jT)\} \;&=\; g(kT + 2ih_m)u(kT + (2i+1)h_m) + \\ &\quad \mathcal{O}(T)[\|x_p(kT + 2ih_m)\| + \|x_w(kT + 2ih_m)\| + \\ &\quad |u(kT + (2i+1)h_m)| + \|r\|_{\infty}]. \end{aligned}$$

As in (D.11), since g is continuous on $[kT,(k+1)T]$ we have that

$$g(t) = g[kT] + \mathcal{O}(T), \quad t \in [kT,(k+1)T),$$

so

$$\begin{aligned} Est\{c_i g^{i+1}[kT]u^0(jT)\} \;&=\; [g[kT] + \mathcal{O}(T)]u(kT + (2i+1)h_m) + \\ &\quad \mathcal{O}(T)[\|x_p(kT + 2ih_m)\| + \|x_w(kT + 2ih_m)\| + \\ &\quad |u(kT + (2i+1)h_m)| + \|r\|_{\infty}] \\ &=\; g[kT]u(kT + (2i+1)h_m) + \mathcal{O}(T)[\|x_p(kT + 2ih_m)\| + \\ &\quad \|x_w(kT + 2ih_m)\| + |u(kT + (2i+1)h_m)| + \|r\|_{\infty}]. \end{aligned}$$

Now, from (6.10) we have

$$u(kT + (2i+1)h_m) = \frac{c_i}{c_{i-1}} Est\{c_{i-1}g^i[kT]u^0(jT)\}, \quad i = 0,1,...,q-1,$$

so

$$\begin{aligned} Est\{c_i g^{i+1}[kT]u^0(jT)\} &= g[kT]\frac{c_i}{c_{i-1}}Est\{c_{i-1}g^i[kT]u^0(jT)\} + \\ &\quad \mathcal{O}(T)[\|x_p(kT+2ih_m)\| + \|x_w(kT+2ih_m)\| + \\ &\quad |u(kT+(2i+1)h_m)| + \|r\|_\infty]. \end{aligned}$$

Finally, using Claim 1 to provide a bound on $|u(kT+(2i+1)h_m)|$ and Claim 2 to provide a bound on $\|x(t)\|$ we obtain the update equation

$$Est\{c_i g^{i+1}[kT]u^0(jT)\} = g[kT]\frac{c_i}{c_{i-1}}Est\{c_{i-1}g^i[kT]u^0(jT)\} + \mathcal{O}(T)(\|x[kT]\| + \|r\|_\infty),$$

which, solved iteratively using the 'initial condition' of $Est\{u^0(jT)\} = u^0[kT]$ yields

$$Est\{c_i g^{i+1}[kT]u^0(jT)\} = c_i g^{i+1}[kT]u^0[kT] + \mathcal{O}(T)(\|x[kT]\| + \|r\|_\infty), \;\; i = 0, 1, .., q-1.$$

Since

$$\begin{aligned} \int_{kT}^{(k+1)T} [u(\tau) &- \phi^\varepsilon(g[kT])u^0[kT]]d\tau \\ &= h_m\left[\sum_{i=0}^{q-1} u(kT+(2i+1)h_m)\right] + \\ &\quad 2h_m u(kT+2qh_m) - T\phi^\varepsilon(g[kT])u^0[kT] \\ &= h_m\left[\sum_{i=0}^{q-1} \frac{c_i}{c_{i-1}}Est\{c_{i-1}g^i[kT]u^0(jT)\}\right] + \\ &\quad 2h_m\frac{1}{2}\left[\sum_{i=0}^{q} 2(q+1)a_i\frac{1}{c_{i-1}}Est\{c_{i-1}g^i[kT]u^0(jT)\} - \right. \\ &\quad \left. \sum_{i=0}^{q-1}\frac{c_i}{c_{i-1}}Est\{c_{i-1}g^i[kT]u^0(jT)\}\right] - T\phi^\varepsilon(g[kT])u^0[kT] \end{aligned}$$

this result together with and the fact that $T := 2(q+1)h_m$ yields

$$\begin{aligned} \int_{kT}^{(k+1)T} [u(\tau) &- \phi^\varepsilon(g[kT])u^0[kT]]d\tau \\ &= 2(q+1)h_m\left[\sum_{i=0}^{q} a_i g^i[kT]u^0[kT]\right] - T\phi^\varepsilon(g[kT])u^0[kT] + \\ &\quad \mathcal{O}(T)h_m[x[kT] + \|r\|_\infty] \\ &= \mathcal{O}(T^2)(x[kT] + \|r\|_\infty). \end{aligned}$$

■

We now turn to η. We begin by finding a bound on $\phi^\varepsilon(g(t))u^0(t)$. Since

$$u^0(t) = Fx(t), \tag{D.19}$$

we have

$$\begin{aligned}\phi^{\varepsilon}(g(t))u^0(t) - \phi^{\varepsilon}(g[kT])u^0[kT] &= [\phi^{\varepsilon}(g(t)) - \phi^{\varepsilon}(g[kT])]u^0[kT] + \\ &\quad L[x(t) - x[kT]]\phi^{\varepsilon}(g(t));\end{aligned}$$

to which we apply Claim 3 and use the fact that $g(t) = g[kT] + \mathcal{O}(T)$, yielding

$$\begin{aligned}\phi^{\varepsilon}(g(t))u^0(t) &= \phi^{\varepsilon}(g[kT])u^0[kT] + \\ &\quad \mathcal{O}(T)(|u^0[kT]| + \|x[kT]\| + \|r\|_\infty), \quad t \in [kT, (k+1)T).\end{aligned}$$

We can use this in the definition of η to obtain

$$\begin{aligned}\|\eta(kT,t)\| &\leq \int_{kT}^{(k+1)T} \mathcal{O}(1)\Big[- \phi^{\varepsilon}(g[kT])u^0[kT] + u(\tau) + \\ &\qquad\qquad \mathcal{O}(T)(|u^0[kT]| + \|x[kT]\| + \|r\|_\infty)\Big] d\tau \\ &= \mathcal{O}(1)\left[\int_{kT}^{(k+1)T} \left[-\phi^{\varepsilon}(g[kT])u^0[kT] + u(\tau)\right] d\tau\right] + \\ &\quad \mathcal{O}(T^2)(|u^0[kT]| + \|x[kT]\| + \|r\|_\infty), \quad t \in [kT, (k+1)T).\end{aligned}$$

We then use Claim 4 to bound the first term and (D.19) in the second term, yielding

$$\|\eta(kT,t)\| \leq \mathcal{O}(T^2)(\|x[kT]\| + \|r\|_\infty), \quad t \in [kT, (k+1)T),$$

so, for every $t \in [kT, (k+1)T)$ we have

$$\begin{aligned}x(t) - \Phi_{A^{\varepsilon}_{cl}(g)}(t, kT)x[kT] - \int_{kT}^{t} \Phi_{A^{\varepsilon}_{cl}(g)}(t,\tau)B_r r(\tau)d\tau & \\ &= \eta(kT,t) \\ &= \mathcal{O}(T^2)(\|x[kT]\| + \|r\|_\infty).\end{aligned}$$

■

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Proof of Proposition 6.2:

Fix $x_0 = 0$, $K_{lti} \in \mathcal{S}(P_1)$, and $\varepsilon \in (0, \bar{\varepsilon})$ and let $g \in \mathcal{G}(G, T_s, c_g)$ be arbitrary. Both the stability and the sensitivity parts of this proof work by gluing together the single period results obtained in the Control Lemma. In the previous three chapters, we used the switching sequence $\{t_l\}$ to state and to prove our results since it provided a very natural way to define nominal performance in the face of plant switches. In this chapter, the weighted sensitivity function is well defined, even in the face of discontinuities in g, so instead of using the switching sequence, here we proceed by observing that, as $T \to 0$, the percentage of intervals of length T which contain a discontinuity in g tends to zero, so that the stronger conclusion of Lemma 6.2 is applicable. To that end we define the following integer function:

$$\rho(T) := \left\lfloor \frac{T_s}{T} \right\rfloor.$$

It follows that

$$\rho(T)T \leq T_s$$

and that $\rho(T)T \to T_s$ as $T \to 0$; hence on any interval of $\rho(T)T$ time units long, there is at most one interval of length T which has a discontinuity.

Before proceeding, we invoke two of our previous results. First, from Proposition 6.1 we have that

$$\|\Phi_{A^{\varepsilon}_{cl}(g)}(t, t_0)\| \leq \gamma_0 e^{\lambda_0(t-t_0)}, \quad t \geq t_0. \tag{D.20}$$

Second, from the Control Lemma, there exists a constant $\delta > 0$ and functions Δ_1 and Δ_2 satisfying

$$\|\Delta_1[kT]\| \leq \begin{cases} 0 & \text{if } g \text{ is continuous on } [kT, (k+1)T) \\ \delta T & \text{otherwise} \end{cases}, \quad k \in \mathbf{Z}^+$$

and

$$\|\Delta_2[kT]\| \leq \delta T^2, \quad k \in \mathbf{Z}^+$$

so that, for every sufficiently small $T > 0$, we have

$$x(t) - \Phi_{A^{\varepsilon}_{cl}(g)}(t, kT)x[kT] - \int_{kT}^{t} \Phi_{A^{\varepsilon}_{cl}(g)}(t, \tau)B_r r(\tau)d\tau =$$
$$[\Delta_1[kT] + \Delta_2[kT]] \left[\|x[kT]\| + \|r\|_\infty\right], \; t \in [kT, (k+1)T), \; k \in \mathbf{Z}^+. \tag{D.21}$$

We let $T > 0$ to be arbitrary and sufficiently small to ensure that the above holds.

Before moving to the details of each of our two problems (i.e., stability and weighted sensitivity), we present the following technical result.

Claim 1: There exist constants $\gamma_1 > 0$ and $\lambda_1 \in (\lambda_0, 0)$ so that, for every $r \in PC_\infty$, if $k_0 \in \mathbf{Z}^+$ and $w \in \mathcal{PC}_\infty$ satisfy

$$w(t) = 0, \quad t \geq k_0 T$$

and $T > 0$ is sufficiently small, then

$$\|x[kT]\| \leq \gamma_1 \left(e^{\lambda_1 (k-k_0)T} \|x[k_0 T]\| + \|r\|_\infty \right), \quad k \geq k_0.$$

Proof:

Let $k_0 \in \mathbf{Z}^+$ and $w \in \mathcal{PC}_\infty$ be arbitrary such that they satisfy

$$w(t) = 0, \quad t \geq k_0 T$$

and let $k \geq k_0$ and $r \in \mathcal{PC}_\infty$ be arbitrary.

Using (D.20) to bound $\|\Phi_{A_{cl}^\varepsilon(g)}\|$ in the integrand, from (D.21), in particular, we have that

$$\begin{aligned} x[(k+1)T] &= \Phi_{A_{cl}^\varepsilon(g)}((k+1)T, kT)x[kT] + \\ &\quad \mathcal{O}(1)\int_{kT}^{(k+1)T} \gamma_0 e^{\lambda_0[(k+1)T-\tau]}\|B_r\|\|r\|_\infty d\tau + \\ &\quad \mathcal{O}(1)[\|\Delta_1[kT]\| + \delta T^2][\|x[kT]\| + \|r\|_\infty] \\ &= \Phi_{A_{cl}^\varepsilon(g)}((k+1)T, kT)x[kT] + \\ &\quad \mathcal{O}(T)\|r\|_\infty + \mathcal{O}(1)\left[\|\Delta_1[kT]\| + \delta T^2\right]\|x[kT]\|. \end{aligned}$$

Observe that, from the definition of a state transition matrix, we have that

$$\Phi_{A_{cl}^\varepsilon(g)}(kT, k_0T) = \prod_{i=k_0}^{k-1} \Phi_{A_{cl}^\varepsilon(g)}((i+1)T, iT), \quad k \geq k_0, \quad k_0 \in \mathbf{Z}^+,$$

so we can iteratively solve the above difference equation to obtain

$$\begin{aligned} x[kT] &= \Phi_{A_{cl}^\varepsilon(g)}(kT, k_0T)x[k_0T] + \sum_{i=0}^{j-1}\Big(\Phi_{A_{cl}^\varepsilon(g)}((j-1)T, iT) \times \\ &\qquad \left[\mathcal{O}(T)\|r\|_\infty + \mathcal{O}(1)\left(\|\Delta_1[kT]\| + \delta T^2\right)\|x(iT)\|\right]\Big). \end{aligned}$$

Using (D.20), we find that there exists a constant $\alpha_1 > 0$ such that

$$\begin{aligned} \|x[kT]\| &\leq \gamma_0 e^{\lambda_0(k-k_0)T)}\|x[k_0T]\| + \alpha_1\sum_{i=0}^{j-1}\Big(\gamma_0 e^{\lambda_0(j-1-i)T)} \times \\ &\qquad \left[T\|r\|_\infty + \left(\|\Delta_1[kT]\| + \delta T^2\right)\|x(iT)\|\right]\Big). \end{aligned}$$

If we define the RHS of this inequality to be $\psi[kT]$, then we obtain the more easily handled equation

$$\begin{aligned} \psi[(k+1)T] &= e^{\lambda_0 T}\psi[kT] + \alpha_1\gamma_0\left[T\|r\|_\infty + (\|\Delta_1[kT]\| + \delta T^2)\|x[kT]\|\right], \\ &\qquad \gamma_0\|x[k_0T]\| = \psi[k_0T]; \quad \text{(D.22)} \end{aligned}$$

since

$$\|x[kT]\| \leq \psi[kT]$$

to obtain the desired result, it is sufficient to find a bound on ψ for all k. We begin with (D.22), which implies

$$\psi[(k+1)T] \leq (e^{\lambda_0 T} + \gamma_0\alpha_1\|\Delta_1[kT]\| + \gamma_0\delta T^2)\psi[kT] + \gamma_0\alpha_1 T\|r\|_\infty. \qquad \text{(D.23)}$$

Observe that there exists a $\bar{\lambda}_1 \in (\lambda_0, 0)$ such that, for small T,

$$e^{\lambda_0 T} + \gamma_0\alpha_1\delta T^2 \leq e^{\bar{\lambda}_1 T};$$

if we apply this to (D.23) and use the definition of Δ_1, then we obtain

$$\psi[(k+1)T] \leq \begin{cases} e^{\bar{\lambda}_1 T}\psi[kT] + \gamma_0\alpha_1 T\|r\|_\infty & \text{if } g \text{ is continuous on } [kT,(k+1)T) \\ (1+\gamma_0\alpha_1\delta T)\psi[kT] + \gamma_0\alpha_1 T\|r\|_\infty & \text{otherwise.} \end{cases} \tag{D.24}$$

Recall that, in every $\rho(T)$ periods, we have at least $\rho(T)-1$ periods where g is continuous. As such, if $T>0$ is sufficiently small, then we can use (D.24) and $e^{\bar{\lambda}_1 T} \leq 1$ to write

$$\begin{aligned} \psi[(k+\rho(T))T] &\leq e^{\bar{\lambda}_1 T(\rho(T)-1)}(1+\gamma_0\delta T)\psi[kT] + \\ &\quad \sum_{i=0}^{\rho(T)-1} (1+\gamma_0\alpha_1\delta T)^i\,(\gamma_0\alpha_1 T\|r\|_\infty) \\ &\leq e^{\bar{\lambda}_1 T(\rho(T)-1)}(1+\gamma_0\alpha_1\delta T)\psi[kT] + \\ &\quad \gamma_0\alpha_1 T\rho(T)(1+\gamma_0\alpha_1\delta T)^{\rho(T)}\|r\|_\infty, \end{aligned}$$

but

$$(1+\gamma_0\delta T) \leq e^{\gamma_0\delta T}, \quad T \geq 0$$

and by definition

$$\rho(T)T \leq T_s,$$

so

$$\psi[(k+\rho(T))T] \leq e^{\bar{\lambda}_1 T(\rho(T)-1)+\gamma_0\delta T}\psi[kT] + \gamma_0\alpha_1 T_s e^{\gamma_0\delta T_s}\|r\|_\infty.$$

Finally, observe that there exists a $\bar{\lambda}_2 \in (\bar{\lambda}_1, 0)$ such that, for small T, we have

$$\bar{\lambda}_1 T(\rho(T)-1) + \gamma_0\delta T \leq \bar{\lambda}_2 T\rho(T),$$

so we have

$$\psi[(k+\rho(T))T] \leq e^{\bar{\lambda}_2 T\rho(T)}\psi[kT] + \gamma_0\alpha_1 T_s e^{\gamma_0\delta T_s}\|r\|_\infty;$$

indeed, we can solve this iteratively to find that

$$\begin{aligned} \psi[(k_0+j\rho(T))T] &\leq (e^{\bar{\lambda}_2 T})^{j\rho(T)}\psi[k_0T] + \frac{e^{\gamma_0\delta T_s}}{1-e^{\bar{\lambda}_2 T\rho(T)}}\gamma_0\alpha_1 T_s\|r\|_\infty \\ &\leq (e^{\bar{\lambda}_2 T})^{j\rho(T)}\psi[k_0T] + \underbrace{\frac{e^{\gamma_0\delta T_s}}{1-e^{\bar{\lambda}_2 T_s/2}}\gamma_0\alpha_1 T_s}_{=:\alpha_2}\|r\|_\infty, \\ &\qquad j \in \mathbf{Z}^+. \end{aligned} \tag{D.25}$$

To complete the proof of the claim we must show that ψ is well behaved at the sample points which lie between $j\rho(T)T$ and $(j+1)\rho(T)T$. We consider (D.23) and use a weak bound on Δ_1: for small T

$$\begin{aligned} \psi[(k+1)T] &\leq (e^{\lambda_0 T} + 2\gamma_0\delta T)\psi[kT] + \gamma_0\alpha_1 T\|r\|_\infty \\ &\leq (1+2\gamma_0\delta T)\psi[kT] + \gamma_0\alpha_1 T\|r\|_\infty \end{aligned}$$

which, solved iteratively to obtain a crude bound on $\psi[kT]$, provides the following: for every $\bar{k}_0 \geq k_0$ and small T we have

$$\begin{aligned}\psi[kT] &\leq (1+2\gamma_0\delta T)^{k-\bar{k}_0}\psi[\bar{k}_0 T] + \gamma_0\alpha_1 T\frac{(1+2\gamma_0\delta T)^{k-\bar{k}_0}-1}{2\gamma_0\delta T}\|r\|_\infty \\ &\leq e^{2\gamma_0\delta T(k-\bar{k}_0)}\psi[\bar{k}_0 T] + \frac{\alpha_1}{2\delta}[e^{2\gamma_0\delta T(k-\bar{k}_0)}-1]\|r\|_\infty, \quad k \geq \bar{k}_0 \qquad \text{(D.26)}\end{aligned}$$

so for $l = 0, ..., \rho(T)-1$, set $\bar{k}_0 = k_0 + j\rho(T)$, $k = (k_0 + j\rho(T) + l)T$, and use $l < \rho(T)$ to find

$$\begin{aligned}\psi[(k_0+j\rho(T)+l)T] &\leq e^{2\gamma_0\delta T\rho(T)}\psi[k_0+j\rho(T)T] + \frac{\alpha_1}{2\delta}[e^{2\gamma_0\delta T\rho(T)}-1]\|r\|_\infty \\ &\leq e^{2\gamma_0\delta T_s}\psi[k_0+j\rho(T)T] + \underbrace{\frac{\alpha_1}{2\delta}[e^{2\gamma_0\delta T_s}-1]}_{=:\alpha_3}\|r\|_\infty, \quad j \in \mathbf{Z}^+\end{aligned}$$

which combines with (D.25) to yield

$$\begin{aligned}\psi[(k_0+j\rho(T)+l)T] &\leq e^{2\gamma_0\delta T_s}\left[(e^{\bar{\lambda}_2 T})^{j\rho(T)}\psi[k_0T] + \alpha_2\|r\|_\infty\right] + \alpha_3\|r\|_\infty \\ &\leq e^{2\gamma_0\delta T_s}e^{-\bar{\lambda}_2 T_s}(e^{\bar{\lambda}_2 T})^{j\rho(T)+l}\psi[k_0T] + \\ &\quad \underbrace{\left[e^{2\gamma_0\delta T_s}\alpha_2+\alpha_3\right]}_{=:\alpha_4}\|r\|_\infty, \qquad j \in \mathbf{Z}^+\end{aligned}$$

which implies that

$$\psi[kT] \leq \underbrace{e^{(2\gamma_0\delta-\bar{\lambda}_2)T_s}}_{=:\alpha_5}(e^{\bar{\lambda}_2 T})^{k-k_0}\psi[k_0T] + \alpha_4\|r\|_\infty, \quad k \geq k_0.$$

Using the definition of ψ it follows that

$$\|x[kT]\| \leq \gamma_0\alpha_5 e^{\bar{\lambda}_2 T(k-k_0)}x[k_0T] + \alpha_4\|r\|_\infty, \quad k \geq k_0.$$

■

Sensitivity:

Fix $w \equiv 0$ and let $r \in \mathcal{PC}_\infty$ be arbitrary. By the definition of weighted sensitivity, it is enough to bound $(e - e^\varepsilon)$ by r. Since

$$e = C_p x - C_w r,$$

it clearly sufficient to bound $(x - x^\varepsilon)$ by r; indeed, it is sufficient to show

$$\|(x^\varepsilon - x)(t)\| = \mathcal{O}(T)\|r\|_\infty.$$

To do so, we will begin by showing that $x[kT]$ is bounded, independent of the choice of k and then use this to show that $x^\varepsilon - x$ is well behaved at the sample points; finally, we show that nothing untoward happens between samples. To that end, we now look for a bound on the difference between x and x^ε at the sampling times. Observe that, since $w \equiv 0$, Claim 1 says that, if $T > 0$ is sufficiently small, then

$$\|x[kT]\| \le \gamma_1 \left(e^{\lambda_1 kT} \|x_0\| + \|r\|_\infty \right), \quad k \in \mathbf{Z}^+.$$

It will be convenient to define

$$\tilde{x}^\varepsilon := x - x^\varepsilon.$$

Claim 2: There exists a constant $\gamma_2 > 0$ so that, if $T > 0$ is sufficiently small then

$$\|\tilde{x}^\varepsilon[kT]\| \le \gamma_2 T \|r\|_\infty, \quad k \in \mathbf{Z}^+.$$

Proof:

Let $k \in \mathbf{Z}^+$ be arbitrary. From (6.9) we have

$$x^\varepsilon[(k+1)T] = \Phi_{A_{cl}^\varepsilon(g)}((k+1)T, kT) x^\varepsilon[kT] + \int_{kT}^{(k+1)T} \Phi_{A_{cl}^\varepsilon(g)}((k+1)T, \tau) B_r r(\tau) d\tau,$$

so, using (D.21), we have

$$\begin{aligned} \tilde{x}^\varepsilon[(k+1)T] &= \Phi_{A_{cl}^\varepsilon(g)}((k+1)T, kT) \tilde{x}^\varepsilon[kT] + \\ & \quad [\Delta_1[kT] + \Delta_2[kT]] \times [\|x[kT]\| + \|r\|_\infty], \end{aligned}$$

whose solution satisfies

$$\tilde{x}^\varepsilon[kT] = \sum_{j=0}^{k-1} \left(\Phi_{A_{cl}^\varepsilon(g)}((k-1)T, 0) \right) [\Delta_1[jT] + \Delta_2[jT]] \times [\|x[jT]\| + \|r\|_\infty] \right).$$

If we use Claim 1 to bound $\|x[kT]\|$, the definition of Δ_2, and the bound on $\|\Phi_{A_{cl}^\varepsilon(g)}\|$ then, for sufficiently small T, we have

$$\|\tilde{x}^\varepsilon[kT]\| \le \sum_{j=0}^{k-1} \gamma_0 e^{\lambda_0 (k-1-j)T} \left[\|\Delta_1[jT]\| + \delta T^2 \right] (\gamma_1 + 1) \|r\|_\infty. \tag{D.27}$$

The only problematic term is Δ_1. In the worst case, g has discontinuities every $\rho(T)$ samples, that is, $\Delta(iT) = 0$ for every i that is not an integer multiple of $\rho(T)$. Therefore, we have

$$\begin{aligned} \sum_{i=0}^{k-1} e^{\lambda_0 T(k-1-i)} \|\Delta_1[kT]\| &\le \sum_{i=1}^{\infty} e^{\lambda_0 \rho(T) T i} \delta T \\ &\le \sum_{i=0}^{\infty} e^{\lambda_0 T_s i/2} \delta T \quad \text{for small } T \\ &= \frac{1}{1 - e^{\lambda_0 T_s/2}} \delta\, T. \end{aligned}$$

Furthermore,

$$\begin{aligned}\sum_{i=0}^{k-1} e^{\lambda_0 T(k-1-i)} &\leq \sum_{i=0}^{\infty} e^{\lambda_0 T i}\\ &= \frac{1}{1-e^{\lambda_0 T}}\\ &\leq \frac{2}{|\lambda_0| T} \quad \text{for small } T,\end{aligned}$$

so (D.27) becomes

$$\begin{aligned}\|\tilde{x}^{\varepsilon}[kT]\| &\leq \left[\gamma_0 \mathcal{O}(T) + \gamma_0 \mathcal{O}(T^{-1})\delta T^2\right] \|r\|_\infty\\ &= \mathcal{O}(T)\|r\|_\infty.\end{aligned}$$

■

We can now prove the sensitivity bound; as stated above, it is clearly enough to prove that

$$\tilde{x}^{\varepsilon} = \mathcal{O}(T)\|r\|_\infty.$$

Claim 2 says that $\tilde{x}^{\varepsilon}$ is of the right form at integer multiples of T; it remains to show that $\tilde{x}^{\varepsilon}(t)$ is well behaved between these points. It is straightforward to show that

$$x^{\varepsilon}(t) - x^{\varepsilon}[kT] = \mathcal{O}(T)[\|x^{\varepsilon}[kT]\| + \|r\|_\infty];$$

using Proposition 6.1(i) to find a crude bound on $\|x^{\varepsilon}[kT]\|$ yields

$$x^{\varepsilon}(t) - x^{\varepsilon}[kT] = \mathcal{O}(T)\|r\|_\infty. \tag{D.28}$$

It remains to find a bound on $x(t) - x[kT]$; to do so, we return to (D.21), use the bound on $\Phi_{A^{\varepsilon}_{cl}(g)}$, and assume the worst case (i.e. for every $k \in \mathbf{Z}^+$, $\Delta_1[kT] = \mathcal{O}(T)$) to find that

$$x(t) - x[kT] = \mathcal{O}(T)[\|x[kT]\| + \|r\|_\infty], \quad t \in [kT, (k+1)T),$$

and then we use Claim 1 to provide a crude bound on $\|x[kT]\|$, yielding

$$x(t) - x[kT] = \mathcal{O}(T)\|r\|_\infty. \tag{D.29}$$

Combining (D.28) and (D.29) yields

$$\tilde{x}^{\varepsilon}(t) - \tilde{x}^{\varepsilon}[kT] = \mathcal{O}(T)\|r\|_\infty,$$

which we use with Claim 2 to find the desired result:

$$\begin{aligned}\tilde{x}^{\varepsilon}(t) &= [\tilde{x}^{\varepsilon}(t) - \tilde{x}^{\varepsilon}[kT]] + \tilde{x}^{\varepsilon}[kT]\\ &= \mathcal{O}(T)\|r\|_\infty.\end{aligned}$$

■

Stability:

Fix $r \equiv 0$ and let $w \in \mathcal{PC}_\infty$ be arbitrary. Observe that, when investigating performance we set $w \equiv 0$ but that we can not do so when investigating stability. However, since W is stable, our system is linear, and x incorporates the state of K_{lti} and P_g, and $e = Cx - w_y$, to prove stability it is sufficient to show that, with $r \equiv 0$, both x and u are uniformly bounded functions of w.

This proof works by investigating the response to noise when it is present over only one interval and then stitching together those results to obtain a bound over the entire interval; to this end, with $j \in \mathbf{Z}^+$, define

$$w_j(t) := \begin{cases} w(t) & t \in [jT, (j+1)T) \\ 0 & \text{else}, \end{cases}$$

and let x_j and z_j be the responses due to w_j (we define e_j and u_j^0 in an analogous way), with

$$x_j(0) = x_0 = 0.$$

Observe that, since $r \equiv 0$, Claim 1 says that, if $T > 0$ is sufficiently small, then for every $j \in \mathbf{Z}^+$ we have

$$\|x_j[kT]\| \leq \gamma_1 \left(e^{\lambda_1 (k-j-1)T} \|x[(j+1)T]\| + \|r\|_\infty \right), \quad k \geq j+1. \tag{D.30}$$

Before proceeding, we sketch the proof. Now, observe that (D.30) indicates that we have a bound on x_j as in Figure D.1. Furthermore, since the system is linear and causal, we have that

$$x(t) = \sum_{j=0}^{\infty} x_j(t), \quad t \geq 0,$$

so if we can show that the exponential bound on x_j is uniform over j, then x will also be bounded. The bound on u will follow directly. To proceed, we must first nail down $\bar{T}$ so that we can fix T in an appropriate range. To do so, we require the following claim, which will also provide the exponential part of our bound.

Claim 3: There exists a constant $\gamma_3 > 0$ so that, for every sufficiently small $T > 0$ we have

$$\|x_j(t)\| \leq \gamma_3 e^{\lambda_1 (t-(j+1)T)} \|x_j[(j+1)T]\|, \quad t \geq (j+1)T, \quad j \in \mathbf{Z}^+.$$

Proof:

Let $j \in \mathbf{Z}^+$ be arbitrary. Using (D.30) in conjunction with the One Period Lemma we find that, if $T > 0$ is sufficiently small, then

$$\begin{aligned} x_j(t) &= \mathcal{O}(1)\Phi_{A_{cl}^{\varepsilon}(g)}(t, kT) x_j[kT] + \mathcal{O}(T) x_j[kT], \\ &= \left(\mathcal{O}(1) + \mathcal{O}(T)\right) e^{\lambda_1 (k-j-1)T} \|x_j[(j+1)T]\|, \\ &\qquad\qquad t \in [kT, (k+1)T), \quad k \geq j+1. \end{aligned}$$

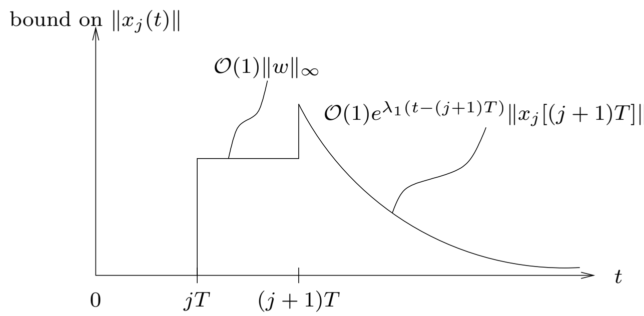

Figure D.1: Bound on x_j

■

At this point we fix $T > 0$ so that it is sufficiently small to satisfy Claims 1 and 3. It remains to investigate the interval $t \in [jT, (j+1)T)$, in which we apply noise. As such, we now rewrite the system equations to reflect the inclusion of noise and $r \equiv 0$:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_u g(t)(u(t) + w_u(t)) + \begin{bmatrix} 0 \\ -B_u \\ 0 \end{bmatrix} w_y(t) \\ u^0(t) &= Fx(t) - D_k w_y(t), \\ e(t) &= Cx(t) - w_y(t). \end{aligned} \tag{D.31}$$

There is no change to κ:

$$\begin{aligned} z[k+1] &= A_z[k]z[k] + B_z[k] \begin{bmatrix} u^0 \\ e \end{bmatrix} (kh) \\ \psi[k] &= C_z[k]z[k] + D_z[k] \begin{bmatrix} u^0 \\ e \end{bmatrix} (kh), \\ u(kh+\tau) &= \psi[k], \qquad \tau \in [0, h). \end{aligned} \tag{D.32}$$

Recall that κ is periodic of period p and that

$$T = ph,$$

so, since the system is linear and causal, we have

$$x_j[jT] = 0,$$

$$z_j[jp] = 0,$$

$$e_j[jT] = -w_y[jT],$$

and

$$u_j^0[jT] = -D_k w_y[jT].$$

If we solve (D.31), then it is easy to show that there exists a constant α_6 so that for all $j \in \mathbf{Z}^+$ and $k = jp, .., (j+1)p-1$ we have

$$\|x_j(t)\| \leq \alpha_6(\|x_j(kh)\| + |u_j(kh)| + \|w\|_\infty), \quad t \in [kh, (k+1)h); \tag{D.33}$$

if we combine this with (D.32), then we see that there exists a constant $\alpha_7 > 1$ so that, for all $j \in \mathbf{Z}^+$ and $k = jp, .., (j+1)p-1$, we have

$$\left\| \begin{bmatrix} x_j((k+1)h) \\ z_j[k+1] \\ \psi_j[k+1] \end{bmatrix} \right\| \leq \alpha_7 \left\| \begin{bmatrix} x_j(kh) \\ z_j[k] \\ \psi_j[k] \end{bmatrix} \right\| + \alpha_7 \|w\|_\infty;$$

solving iteratively yields

$$\begin{aligned} \max_{k=jp,..,(j+1)p-1} \left\| \begin{bmatrix} x_j(kh) \\ z_j[k] \\ \psi_j[k] \end{bmatrix} \right\| &\leq \alpha_7^p \left\| \begin{bmatrix} 0 \\ 0 \\ -D_k w_y(0) \end{bmatrix} \right\| + \frac{\alpha_7^p - 1}{\alpha_7 - 1} \|w\|_\infty \\ &\leq \left(\underbrace{\alpha_7^p \|D_k\| + \frac{\alpha_7^p - 1}{\alpha_7 - 1}}_{=:\alpha_8} \right) \|w\|_\infty. \end{aligned}$$

Using this in (D.33) yields

$$\max_{t \in [jT, (j+1)T)} \|x_j(t)\| \leq \underbrace{\alpha_6(2\alpha_8 + 1)}_{=:\alpha_9} \|w\|_\infty,$$

so, with Claim 3, we have

$$\|x_j(t)\| \leq \gamma_3 \alpha_9 e^{\lambda_1(t-(j+1)T)} \|w\|_\infty, \quad t \geq jT.$$

Finally, we find the full response to w by stitching together the x_js. Recall that

$$x(t) = \sum_{j=0}^{\infty} x_j(t),$$

so for each $k \in \mathbf{Z}^+$ we have

$$\begin{aligned} \|x(t)\| &\leq \sum_{j=0}^{k} \|x_j(t)\| \\ &\leq \gamma_3 \alpha_9 \sum_{j=0}^{k} \underbrace{e^{\lambda_1(t-(j+1)T)}}_{\leq e^{\lambda_1(k-j-1)T}} \|w\|_\infty, \quad t \in [kT, (k+1)T), \quad k \in \mathbf{Z}^+ \\ &\leq \gamma_3 \alpha_9 e^{-\lambda_1 T} \sum_{j=0}^{\infty} \left(e^{\lambda_1 T}\right)^j \|w\|_\infty \\ &= \frac{\gamma_3 \alpha_9 e^{-\lambda_1 T}}{1 - e^{\lambda_1 T}} \|w\|_\infty, \quad t \geq 0. \end{aligned}$$

Hence we have the desired bound on x, independent of g. It remains to show that u is bounded. This follows directly from the special structure of the controller. From (6.10) - (6.11) we see that, since T is fixed, there exists a $\alpha_{10} > 0$ such that

$$\|u(t)\| \le \alpha_{10} \left(\max_{t \in [kT,(k+1)T)} \|e(t)\| + \|u^0[kT]\| \right), \quad t \in [kT, (k+1)T).$$

Since $e(t)$ and $u^0(t)$ are linear functions of $x(t)$ and $u(t)$, it follows that there exists a constant α_{11} so that

$$\|u(t)\| \le \alpha_{11} \|w\|_\infty, \quad t \ge 0.$$

■

Appendix E

List of Notation

Since this thesis contains a significant amount of notation, here we compile the more important terms that were not introduced in Chapter 2. Although it may sometimes be repetitive, to improve accessibility, we will provide four lists of notation, one for each of the chapters 3 to 6. We refer to equation numbers where possible, in the absence of which we refer to page numbers.

Table E.1: Notation for Chapter 3

(A_i, B_i, C_i)	(3.1)	The state-space matrices corresponding to the (MIMO) plant P_i
$(\hat{A}_{\sigma(t)}, B_{\sigma(t)}, C_{\sigma(t)})$	(3.1)	The state-space matrices corresponding to the regularized time-varying plant P_σ at time t
$\hat{A}_i$	(3.4)	The regularized A matrix for the plant P_i
$\bar{A}_i$	p.22	$\bar{A}_i := \hat{A}_i + B_i F_i$, the optimal closed loop A matrix for the plant P_i
a, b, c, ℓ, f	p.23	Uniform bounds on the matrices $\hat{A}_i$, B_i, C_i, L_i, and F_i over every i
F_i	p.22	The optimal state feedback gain for the plant P_i
H_i	(3.10)	The optimal hold gain for the plant P_i
$\hat{H}_i$	p.24	The adjusted version of the hold gain for the plant P_i
$\tilde{H}_i$	(3.20)	The difference between H_i and $\hat{H}_i$
K	(3.3)	The regularization gain
L_i	(3.4)	The noise input state-space matrix for the plant P_i
P_σ	p.20	The time varying plant corresponding to the switching signal σ
S	p.24	The sampler gain
v_1	(3.11)	Output of the first sampler
v_2	(3.12)	Output of the second sampler
γ_0, λ_0	(3.9)	Provides uniform bound on optimal closed loop modes: $$\|e^{\bar{A}_i t}\| \leq \gamma_0 e^{\lambda_0 t},\ i = 1, .., q,\ t \geq 0.$$ Note: $\gamma_0 > 0$ and $\lambda_0 < 0$
σ	p.20	$\sigma : \mathbf{R}^+ \to \{P_1, .., P_q\}$, the signal that specifies the index of the time-varying plant at every time t
ν	(3.3)	The regularized plant input

Table E.2: Notation for Chapter 4

$(\tilde{A}_i, \tilde{B}_i, \tilde{C}_i)$	(4.3)	The state-space matrices corresponding to the regularized, augmented plant P_i
$\bar{A}_i$	p.62	$\bar{A}_i := \tilde{A}_i + \tilde{B}_i F_i$, the optimal closed loop A matrix for the plant P_i
a, b, ℓ, f	p.62	Uniform bounds on the matrices $\tilde{A}_i$, $\tilde{B}_i$, L_i, and F_i over every i
F_i	p.61	The optimal state feedback gain for the augmented, regularized plant P_i
G_i	(4.4)	The gain relating η to x and u for the plant P_i
$\bar{g}$	p.60	A uniform bound on the size of discontinuities in ξ
H_i	(4.12)	The optimal hold gain for the plant P_i
$\hat{H}_i$	p.62	The adjusted version of the hold gain for the plant P_i
$\tilde{H}_i$	(4.20)	The difference between H_i and $\hat{H}_i$
K	(4.3)	The regularization gain
L_i	(4.4)	The noise input state-space matrix for the plant P_i
$\bar{Q}$	p.61	An augmented LQR gain
S	p.63	The sampler gain
v_1	(4.13)	Output of the first sampler
v_2	(4.14)	Output of the second sampler
η	(4.1)	The augmented plant state
γ_0, λ_0	(4.11)	Provides uniform bound on optimal closed loop modes: $$\lVert e^{\bar{A}_i t} \rVert \leq \gamma_0 e^{\lambda_0 t},\ i = 1, .., q,\ t \geq 0.$$ Note: $\gamma_0 > 0$ and $\lambda_0 < 0$
ν	p.58	The regularized augmented plant input
ξ	(4.5)	A change of variables from η to allow noise in the model of the augmented plant

Table E.3: Notation for Chapter 5

(A_p, B_p, C)	(5.2)	The state-space matrices corresponding to the observable canonical form of the plant P_p and the Markov Parameters p
$\bar{A}_p$	p.79	$\bar{A}_p := A_p + B_p F_p$, the optimal closed loop A matrix for the plant P_p
a, b, f	p.82	Uniform bounds on the matrices $\tilde{A}_p$, $\tilde{B}_p$, and F_p over every i
F_p	p.79	The optimal state feedback gain for the plant P_p
H	p.85	The optimal function
H^{ε}	(5.5)	A polynomial approximation to the optimal function
h	various	The controller's sampling rate - an arbitrary integer
$\bar{h}_m$	p.84	$\bar{h}_m := (m+1)h$. An inter-sample rate used in the Estimation Phase
m	p.80	An integer in $\{n, .., 2n\}$, chosen to ensure that p uniquely identifies the plant
p	p.80	The first m Markov parameters of the plant P_p
T_{max}	p.85	An arbitrary upper bound on the length of the controller period T
γ_0, λ_0	p.86	Provides uniform bound on the transmission matrix Φ Note: $\gamma_0 > 0$ and $\lambda_0 < 0$
σ	p.80	$\sigma : \mathbf{R}^+ \to \mathcal{M}$, the signal that specifies the Markov parameters of the time-varying plant at every time t
Φ	p.86	A state transition matrix
ξ	(5.2)	The state of the observer canonical form
ξ^0	p.86	The optimal state trajectory
ξ^{ε}	p.86	The state trajectory when the function H^{ε} is applied

Table E.4: Notation for Chapter 6

(A, B_u, B_r, F, C)	(6.5)	The state-space matrices corresponding to augmented open loop system
A^{ε}_{cl}	(6.9)	The closed loop A matrix when the polynomial approximation ϕ is applied to the time varying plant P_g
h	various	The estimator's sampling rate - an arbitrary integer
$\bar{h}_m$	p.119	$h_m := (m+1)h$. An inter-sample rate used in the Estimation Phase
m	p.113	The relative degree of the nominal plant P_1
x	(6.5)	The state matrices corresponding to augmented open loop system
x^{ε}	p.121	The state trajectory when the polynomial approximation ϕ^{ε} is used
γ_0, λ_0	p.121	Provides uniform bound on the transmission matrix Φ Note: $\gamma_0 > 0$ and $\lambda_0 < 0$
ϕ	p.120	A polynomial approximation to $1/g$

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