Zigzags of Finite, Bounded Posets and Monotone Near-Unanimity Functions and Jónsson Operations

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

We define the notion of monotone operations admitted by partially ordered sets, specifically monotone near-unanimity functions and Jónsson operations. We then prove a result of McKenzie's in [8] which states that if a finite, bounded poset **P** admits a set of monotone Jónsson operations then it admits a set of monotone Jónsson operations for which the operations with even indices do not depend on their second variable. We next define zigzags of posets and prove various useful properties about them. Using these zigzags, we proceed carefully through Zádori's proof from [12] that a finite, bounded poset **P** admits a monotone near-unanimity function if and only if **P** admits monotone Jónsson operations.

Acknowledgements

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Dedication

This thesis is dedicated to my nephews and niece in the hopes that one day one of them will be able to read and understand the mathematics in it.

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Chapter 1

Introduction

In the first section of this chapter, we go over some basic definitions that we will need for the rest of the thesis. We then give some background and motivation for the thesis and prove a basic result about monotone Jónsson operations and near-unanimity functions admitted by posets. Following that, we define zigzags of posets and give some basic examples. In the last section, we give an outline of what lays ahead in the following chapters.

1.1 Preliminary definitions

The following definitions are taken from [12], [8], and [4].

Let P be a set. An *n-ary relation* r on P is any subset $r \subseteq P^n$. A *partial order* on P is a binary relation $\leq_{\mathbf{P}}$ on P that is reflexive, antisymmetric, and transitive. With this relation, the pair $(P, \leq_{\mathbf{P}})$ forms a relational structure called a *partially ordered set* or *poset* which we will usually just denote $\mathbf{P}$. When it is apparent by context, we will often drop the subscript of $\leq_{\mathbf{P}}$.

Now, let a and b be elements in a poset $\mathbf{P}$. If $(a, b) \in \leq_{\mathbf{P}}$, we often denote this $a \leq_{\mathbf{P}} b$, and if $a \leq b$ but $a \neq b$, then we denote this $a < b$. As well, if $a < b$ and for every $c \in \mathbf{P}$ we have that $a \leq c \leq b$ implies either $a = c$ or $b = c$, then we say that b *covers* a or a is *covered by* b. We also call a a *lower cover* of b and b an *upper cover* of a and denote this by writing $a \prec_{\mathbf{P}} b$. In fact, the *covering relation* $\prec_{\mathbf{P}}$ of $\mathbf{P}$ is equal to the set of all covering pairs of $\leq_{\mathbf{P}}$. It is clearly contained in $\leq_{\mathbf{P}}$. Notice that there is a natural partial order on P^n induced by $\leq_{\mathbf{P}}$ where $(x_1, \ldots, x_n) \leq_{\mathbf{P}^n} (y_1, \ldots, y_n)$ if and only if $x_i \leq y_i$ for $1 \leq i \leq n$.

A poset $\mathbf{P}$ is *bounded* if there exists some top element $1 \in \mathbf{P}$ and bottom element $0 \in \mathbf{P}$ such that for all $x \in \mathbf{P}$, we have $0 \leq x \leq 1$. A poset $\mathbf{P}$ is said to be a *finite poset* if the underlying set P has finitely many elements. Also, we say that an element x is *maximal* in $\mathbf{P}$ if for all $y \in \mathbf{P}$, whenever $x \leq y$ we have $x = y$. We define *minimal* elements dually. Finite posets have the nice property that we can often draw them in a diagram using their covering relation. We use a circle $\circ$ to represent each element and edges between the circles to indicate a covering pair with the lesser element being lower on the diagram than the greater element. We then see that for any a and b in our poset, $a \leq b$ if and only if $a = b$ or there exists some set $\{c_1, \ldots, c_n\} \subseteq P$ such that $a \prec c_1 \prec \cdots \prec c_n \prec b$. Consider the example given in Figure 1.1 with $P = \{1, a, b, c, d\}$ and $\prec_{\mathbf{P}} = \{(c, a), (a, 1), (d, a), (d, b), (b, 1)\}$.

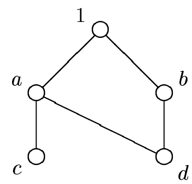


Figure 1.1: A diagram of the poset $\mathbf{P}$

Next, define an *n-ary operation* f on a set P to be function from P^n to P. We say that f is *monotone* for a poset $\mathbf{P}$ if for all $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ in P^n, whenever $(x_1, \ldots, x_n) \leq_{\mathbf{P}^n} (y_1, \ldots, y_n)$ we have $f(x_1, \ldots, x_n) \leq f(y_1, \ldots, y_n)$. If f is monotone for $\mathbf{P}$, we often say that $\mathbf{P}$ *admits* f or f *preserves* $\mathbf{P}$.

A *projection operation* on P^n is an n-ary operation $p_i : P^n \to P$ defined such that $p(x_1, \ldots, x_n) = x_i$ for all $(x_1, \ldots, x_n) \in P^n$ and for some $1 \leq i \leq n$. Also, if $g_1, \ldots, g_n$ are all k-ary operations on P and f is some n-ary operation on P, then the *composition* $f(g_1, \ldots, g_n)$ is the k-ary operation on P defined such that

$$f(g_1, \ldots, g_n)(x_1, \ldots, x_k) = f(g_1(x_1, \ldots, x_k), \ldots, g_n(x_1, \ldots, x_k)).$$

A set of finitary operations on a set P that contains all the projection operations and is closed under the composition just described is called a *clone* for P. It is easy to see that for a poset $\mathbf{P}$, both projection operations and compositions of monotone operations are monotone. Hence we can define the *monotone clone* of a poset $\mathbf{P}$ to be the clone of all monotone operations for $\mathbf{P}$.

Finally, define a *partial n-ary operation* f on a set P to be an n-ary operation from Q to P where $Q \subseteq P^n$. If f is monotone for $\mathbf{P}$ on its domain Q, then we say that f is a *partial n-ary monotone operation.*

1.2 Monotone clones, near-unanimity functions, and Jónsson operations

Many questions have been asked about monotone clones of finite, bounded posets. In particular, what kinds of operations are admitted by a particular finite, bounded poset $\mathbf{P}$ and what are the implications? For $n \geq 3$, we say that n-ary function f is a *n-near unanimity function* (or *n-nuf*) if the identity

$$f(x, \ldots, x, \underset{i}{y}, x, \ldots, x) = x$$

holds for every $1 \leq i \leq n$. It is known thanks to the Baker-Pixley Theorem (see [2]) that if the monotone clone of $\mathbf{P}$ admits a monotone n-nuf, then it is finitely generated but it is still undecided as to whether a finitely generated monotone clone of a finite, bounded poset always admits a n-nuf. If we lose the bounded condition, then there are examples of finite posets whose monotone clones are finitely generated but do not admit an n-nuf [5].

In fact, the study of near-unanimity functions has important consequences in areas besides that of partially ordered sets. One example is that of the area of constraint satisfaction problems (or CSPs) which has seen much new research lately. It has been shown that if a finite relational structure P admits a near-unanimity function, then the problem CSP(P) of determining which finite relational structures Q of the same signature as P admit a homomorphism into P is solvable in polynomial-time [9].

Now, near-unanimity functions are related to sets of operations called Jónsson operations. To motivate this connection, we mention that algebras whose term operations include a near-unanimity function generate a variety $\mathcal{V}$ that is congruence-distributive (every algebra in $\mathcal{V}$ has a distributive congruence lattice). It is well-known in universal algebra that congruence-distributivity of the variety generated by an algebra is equivalent to an algebra including a certain set of operations in its term operations. These operations $d_0, \ldots, d_N$ are *Jónsson operations* and they satisfy the following set of *Jónsson equations*:

$$\begin{aligned} x = d_0(x,y,z) = d_i(x,y,x) = d_N(z,y,x) &\text{ for } 0 \leq i \leq N \\ d_{2i}(x,x,y) = d_{2i+1}(x,x,y) &\text{ for } 0 \leq i \leq (N-1)/2 \\ d_{2i-1}(x,y,y) = d_{2i}(x,y,y) &\text{ for } 1 \leq i \leq N/2 \end{aligned} \tag{1.1}$$

Zádori mentions in [12] the following question, "Does a finite, bounded poset admit an n-nuf if and only if it admits a set of Jónsson operations?" and in the same paper he answers this question in the positive. It is the goal of this thesis to present his proof in a detailed manner. One direction is straightforward as is seen in the next lemma. We note that the lemma actually holds for any relational structure although it is only stated here in terms of posets.

Lemma 1.1. *Let* $\mathbf{P}$ *be a poset. Then* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) *where*

(1) For some $n \geq 3$, $\mathbf{P}$ *admits an n-ary monotone near-unanimity function.*

(2) For some $n \geq 3$, *there exists a partially defined n-ary monotone operation h on* $\mathbf{P}$ *with domain*

$$A_n = \left\{ (x, \ldots, x, \underset{i}{y}, z, \ldots, z) : x, y, z \in P,\ 1 \leq i \leq n \right\}$$

which satisfies $h(a, \ldots, a, \underset{i}{b}, a, \ldots, a) = a$ *for all* $a, b \in P$ *and* $1 \leq i \leq n$.

(3) For some $k \geq 1$, $\mathbf{P}$ admits monotone ternary operations $D_1, \ldots, D_k$ satisfying the following equations:

$$\begin{aligned} D_1(x,x,y) = D_k(y,x,x) = D_i(x,y,x) &= x && \text{for } 1 \leq i \leq k; \\ D_i(x,y,y) &= D_{i+1}(x,x,y) && \text{for } 1 \leq i \leq k-1. \end{aligned} \tag{1.2}$$

(4) For some $k \geq 1$, $\mathbf{P}$ admits monotone Jónsson operations $d_0, \ldots, d_{2k}$ which have the additional property that for $0 \leq i \leq k$, the operation d_{2i} does not depend on its middle variable.

(5) $\mathbf{P}$ admits monotone Jónsson operations.

Proof. (1) $\Rightarrow$ (2): This is immediate once we let h be our monotone n-nuf.

(2) $\Rightarrow$ (3): Let $k = n-2$ and for $1 \leq i \leq k$, define

$$D_i(x,y,z) = h(z, \ldots, z, \underset{i+1}{y}, x, \ldots, x).$$

The first line of (1.2) holds immediately by the property of h and the second line is true since for $1 \leq i \leq k-1$,

$$D_i(x,y,y) = h(y, \ldots, y, \underset{i+1}{y}, x, \ldots, x) = h(y, \ldots, y, \underset{i+2}{x}, x, \ldots, x) = D_{i+1}(x,x,y).$$

(3) $\Rightarrow$ (4): Given $D_1, \ldots, D_k$ satisfying (1.2), let us define $d_0, \ldots, d_{2k}$ by first defining $d_0(x,y,z) = x$. Then for $1 \leq i \leq k$, define $d_{2i-1}(x,y,z) = D_i(x,y,z)$ and $d_{2i}(x,y,z) = D_i(x,z,z)$. It is clear from the definitions that for all $1 \leq i \leq k$ the operation d_{2i} does not depend on its second variable.

Now, clearly by our definitions the first line of (1.1) is satisfied as well as $d_{2i-1}(x,y,y) = d_{2i}(x,y,y)$ for all $1 \leq i \leq k$. Then, for all $0 \leq i \leq (k-1)$ we have

$$d_{2i}(x,x,z) = D_i(x,z,z) = D_{i+1}(x,x,z) = d_{2(i+1)-1}(x,x,z) = d_{2i+1}(x,x,z).$$

Thus $d_0, \ldots, d_{2k}$ are Jónsson operations.

(4) $\Rightarrow$ (5): This is immediate.

■

In order to prove the other direction when $\mathbf{P}$ is finite and bounded, Zádori relies heavily on the idea of zigzags in finite, bounded posets. We introduce these in the next section.

1.3 Zigzags

We begin our discussion of zigzags by defining some important concepts and note that the definitions given in this section come directly from [12].

Let $\mathbf{P}$ be a poset. Now, let $T \subseteq (P \cup \prec_{\mathbf{P}})$ such that $P \nsubseteq T$. We say that T is *cancelled* from $\mathbf{P}$ by denoting the corresponding poset $\mathbf{P}\backslash T = (P\backslash T, (\leq_{\mathbf{P}} |_{P\backslash T})\backslash T)$. As an example, consider the posets $\mathbf{Q}$ and $\mathbf{Q}\backslash T$ with $T = \{a, (d, b)\}$ in Fig. 1.2.

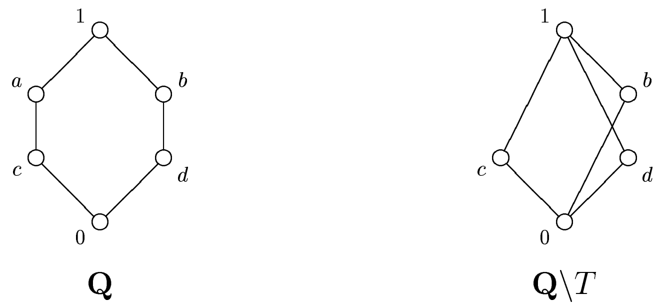


Figure 1.2: T cancelled from $\mathbf{Q}$

Next, let us say a poset $\mathbf{Q}$ is *contained* in a poset $\mathbf{P}$ if $Q \subseteq P$ and $\leq_{\mathbf{Q}} \subseteq \leq_{\mathbf{P}} |_Q$. We write this as $\mathbf{Q} \subseteq \mathbf{P}$. In the case where $\mathbf{Q} \subseteq \mathbf{P}$ but $\mathbf{Q} \neq \mathbf{P}$, we say that $\mathbf{Q}$ is *properly contained* in $\mathbf{P}$. The two posets in Fig. 1.3 illustrate this.

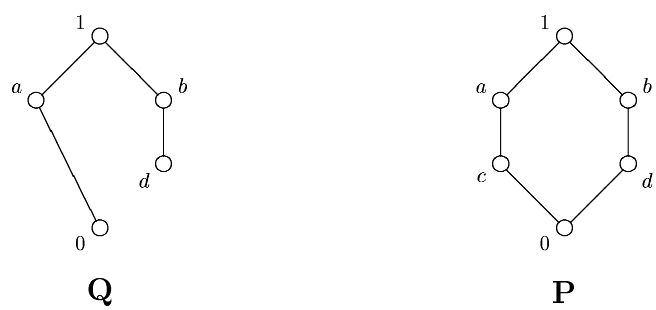


Figure 1.3: $\mathbf{Q} \subseteq \mathbf{P}$

Next, for posets $\mathbf{P}$ and $\mathbf{H}$, call a pair $(\mathbf{H}, f)$ a $\mathbf{P}$-*coloured poset* if f is a partially defined map from H to P. If we can extend f to a fully defined, monotone map f' from $\mathbf{H}$ to $\mathbf{P}$, then we say $(\mathbf{H}, f)$ is $\mathbf{P}$-*extendible*. If not, then we say f and $(\mathbf{H}, f)$ are $\mathbf{P}$-*nonextendible*. Under the partial map f, call the set of elements in the domain of f the *coloured elements* of $(\mathbf{H}, f)$ and denote it $C(\mathbf{H}, f)$. Call the set $N(\mathbf{H}, f) = H\backslash C(\mathbf{H}, f)$ the *noncoloured elements* of $(\mathbf{H}, f)$. In the case where these sets are non-empty, define the posets $\mathbf{C}(\mathbf{H}, f)$ and $\mathbf{N}(\mathbf{H}, f)$ to be given by the restriction of $\leq_{\mathbf{H}}$ to $C(\mathbf{H}, f)$ and $N(\mathbf{H}, f)$, respectively. Let $(\mathbf{Q}, g)$ also be a $\mathbf{P}$-coloured poset. We say that $(\mathbf{H}, f)$ is *contained* in $(\mathbf{Q}, g)$, written $(\mathbf{H}, f) \subseteq (\mathbf{Q}, g)$ if the poset $\mathbf{H} \subseteq \mathbf{Q}$ and the function $f = g|_H$. Let $T \subseteq (H \cup \prec_{\mathbf{H}})$ such that $H \nsubseteq T$. We say that T is *cancelled* from $(\mathbf{H}, f)$ by denoting the corresponding $\mathbf{P}$-coloured poset $(\mathbf{H}, f)\backslash T = (\mathbf{H}\backslash T, f|_{H\backslash T})$.

We are now able to define and give examples of zigzags. A **P**-*zigzag* is a **P**-nonextendible, **P**-coloured poset $(\mathbf{H}, f)$, where H is finite and for every poset **K** properly contained in **H**, the **P**-coloured poset $(\mathbf{K}, f|_K)$ is **P**-extendible. In the following figures, let the black dots indicate the coloured elements of a **P**-coloured poset.

Example 1.2. *Let* **Q** *be the poset in Figure 1.4. The* **Q**-*coloured poset* $(\mathbf{H}, f)$ *in Figure 1.4 is a* **Q**-*zigzag.*

Figure 1.4: **Q** and $(\mathbf{H}, f)$

Proof. First note that $(\mathbf{H}, f)$ is not **Q**-extendible as all the elements of **Q** are incomparable. We need to show that $(\mathbf{H}, f)$ is minimal in terms of **P**-nonextendability. Let n denote the single noncoloured element in $(\mathbf{H}, f)$. If we cancel n from $(\mathbf{H}, f)$, then $(\mathbf{H}, f)\setminus\{n\}$ will have a fully-defined map $f|_{H\setminus\{n\}}$. Hence $(\mathbf{H}, f)\setminus\{n\}$ will be **P**-extendible. If we cancel the element coloured a, then we can just define n to be b under our new map and this will yield a fully defined map. The case for cancelling b follows the same.

Next, consider $(\mathbf{H}, f)\setminus\{(n, a)\}$. This is **Q**-extendible by just mapping n to b. Removal of the other covering pair yields a similar result. Thus $(\mathbf{H}, f)$ is a **Q**-zigzag. ∎

Example 1.3. *Let* **P** *be the poset in Figure 1.5. Then the* **P**-*coloured poset* $(\mathbf{H}, f)$ *in Figure 1.5 is a* **P**-*zigzag.*

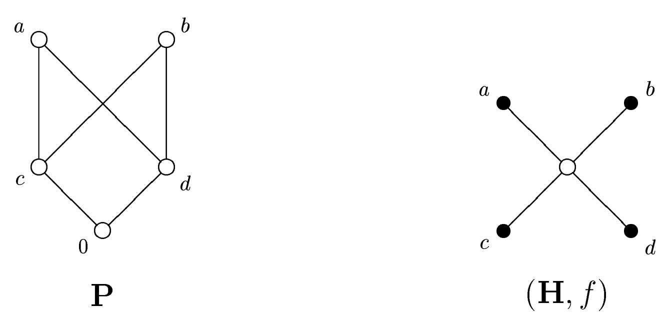


Figure 1.5: **P** and $(\mathbf{H}, f)$

Proof. Clearly $(\mathbf{H}, f)$ is not **P**-extendible as there is no element $x \in \mathbf{P}$ such that $c \le x \le a$ and $d \le x \le b$. Let n denote the single noncoloured element in $(\mathbf{H}, f)$.

If we cancel n from $(\mathbf{H}, f)$, then $(\mathbf{H}, f)\setminus\{n\}$ will have a fully-defined map $f|_{H\setminus\{n\}}$. Hence $(\mathbf{H}, f)\setminus\{n\}$ will be $\mathbf{P}$-extendible. If we remove the element coloured a, then we can just define n to be b under our new map and this will yield a fully defined map. The cases for the other coloured elements follow similarly.

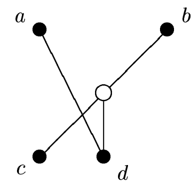


Figure 1.6: $(\mathbf{H}, f)\setminus\{(a, n)\}$

Finally, consider $(\mathbf{H}, f)\setminus\{(n, a)\}$ in Figure 1.6. Notice that $(\mathbf{H}, f)\setminus\{(n, a)\}$ is $\mathbf{P}$-extendible if we just let n map to the element $b \in \mathbf{P}$. Removal of any of the other covering pairs yields a similar result. Thus $(\mathbf{H}, f)$ is a $\mathbf{P}$-zigzag. ■

The natural question to ask is when is the partially defined map f already monotone in a zigzag $(\mathbf{H}, f)$. Call such coloured posets *monotone zigzags* and *nonmonotone zigzags* otherwise. It turns out there is only one possibility for nonmonotone zigzags and we describe it in the next proposition.

Proposition 1.4. *Let* $\mathbf{P}$ *be a poset and* $(\mathbf{H}, f)$ *a nonmonotone* $\mathbf{P}$*-zigzag. Then* $(\mathbf{H}, f)$ *is a two element chain and, for some* $a, b \in \mathbf{P}$ *with* $a \not\leq b$*, the bottom element maps to* a *and the top element maps to* b*.*

Proof. Let $(\mathbf{H}, f)$ be a $\mathbf{P}$-coloured poset such that $(\mathbf{H}, f)$ is a two element chain and, for some $a, b \in \mathbf{P}$ with $a \not\leq b$, the bottom element maps to a and the top element maps to b. Clearly the map f is nonmonotone. The cancellation of either element of H or the single covering pair will yield a $\mathbf{P}$-extendible poset so $(\mathbf{H}, f)$ is a nonmonotone $\mathbf{P}$-zigzag.

Now suppose $(\mathbf{H}, f)$ is some arbitrary nonmonotone $\mathbf{P}$-zigzag. Since f is not monotone, there exist $x, y \in \mathbf{H}$ such that $x \leq y$ but $f(x) \not\leq f(y)$. If we cancel from $(\mathbf{H}, f)$ all but $\{x, y, (x, y)\}$ we are left with a two element chain as described in the previous paragraph. By the minimality of $(\mathbf{H}, f)$, this must be the same as $(\mathbf{H}, f)$. ■

It also easy to see that a monotone zigzag must have at least three elements. We prove this in detail in the Chapter 4. As well, this means we can also assume that every monotone zigzag has at least one noncoloured element and two coloured elements.

As we will see in the upcoming chapters, the structure of $\mathbf{P}$-zigzags of a finite, bounded poset $\mathbf{P}$ yield some very important results concerning the monotone operations admitted by $\mathbf{P}$. The next easy but important lemma is from a remark by Tardos in [10]. It relates monotone near-unanimity functions to the maximum number of coloured elements in any $\mathbf{P}$-zigzag of a poset $\mathbf{P}$.

Lemma 1.5. *Let* **P** *be a poset and* $n \geq 3$. *If for every* **P**-*zigzag* $(\mathbf{H}, f)$, *we have* $|C(\mathbf{H}, f)| \leq n-1$, *then* **P** *admits an* n-*near-unanimity function.*

Proof. We define a new **P**-coloured poset $(\mathbf{P}^n, g)$ as follows. For $1 \leq i \leq n$, consider the set of n-tuples

$$C_i = \left\{ (x, \ldots, x, \underset{i}{y}, x, \ldots, x) : x, y \in P \right\}$$

and let $C = \cup_{i=1}^n C_i$. Clearly $C \subseteq P^n$. Next define $g : C \to P$ such that $g(x, \ldots, x, \underset{i}{y}, x, \ldots, x) = x$; that is, let g be the partially defined n-nuf on P that is fully-defined on C. Now notice that g is monotone for, since $n \geq 3$, we have that

$$(x_1, \ldots, x_1, \underset{i}{y_1}, x_1, \ldots, x_1) \leq (x_2, \ldots, x_2, \underset{j}{y_2}, x_2, \ldots, x_2)$$

implies $x_1 \leq x_2$.

Suppose, for a contradiction, that $(\mathbf{P}^n, g)$ is not **P**-extendible. Then it must contain some **P**-zigzag $(\mathbf{Q}, g|_Q)$. But now notice that by our original assumption, $C(\mathbf{Q}, g|_Q)$ has at most $n-1$ elements. Thus by the pigeonhole principle there is some i such that for each $(x, \ldots, x, y, x, \ldots, x)$ in the domain of $g|_Q$, the projection $p_i(x, \ldots, x, y, x, \ldots, x) = x$. This implies that $g|_Q(x, \ldots, x, y, x, \ldots, x) = p_i(x, \ldots, x, y, x, \ldots, x)$ for all $(x, \ldots, x, y, x, \ldots, x) \in C(\mathbf{Q}, g|_Q)$. Hence $g|_Q$ acts as a coordinate projection on **Q** and this is clearly extendible monotonically to all of **Q**. As this is a contradiction, it must be that our partially defined map n-nuf g can be extended to all of $\mathbf{P}^n$ and we are done. ■

We finish this section by mentioning and proving an interesting fact stated but not proved by Zádori in [12]. It is not particularly useful for the upcoming work but does give us a little more insight into the structure of the posets we are going to be dealing with. Define a *finite lattice* **L** to be a finite, bounded poset for which each pair of elements in **L** has a unique greatest lower bound and unique least upper bound. For any subset S of a lattice **L**, let $S_* = \{x \in \mathbf{L} : x \leq s \text{ for all } s \in S\}$.

Proposition 1.6. *Let* **L** *be a finite lattice. The only* **L**-*zigzags possible are nonmonotone.*

Proof. Suppose otherwise. Then there exists a monotone **L**-zigzag $(\mathbf{H}, f)$ such that $N(\mathbf{H}, f) \neq \emptyset$. Pick some $x \in N(\mathbf{H}, f)$ and consider $\{x\}_*$. Define $f(x)$ to be the least upper bound in **L** of all the elements in $f(C(\mathbf{H}, f) \cap \{x\}_*)$ (if $C(\mathbf{H}, f) \cap \{x\}_* = \emptyset$, then $f(x) = 0$). Now, if $y \geq x$ and $y \in C(\mathbf{H}, f)$, then $y \geq z$ and $f(y) \geq f(z)$ for all $z \in C(\mathbf{H}, f) \cap \{x\}_*$. It follows that, since $f(x)$ is the unique least upper bound, $f(y) \geq f(x)$. This means we have just extended f monotonically to another noncoloured element of $(\mathbf{H}, f)$. If we continue this process with each other noncoloured element of $(\mathbf{H}, f)$, we will get a monotone extension of f and this will be a contradiction. Thus $(\mathbf{H}, f)$ cannot be a monotone **L**-zigzag. ■

We note that, in fact, by the preceding proof **L** need only be a finite joinseimlattice with a bottom element and every **L**-zigzag will be nonmonotone.

1.4 Outline of thesis

In the upcoming chapters, we will work carefully through the proof that a finite, bounded poset which admits monotone Jónsson operations also admits an n-near-unanimity function for some $n \geq 3$. We do this by working backwards in implication from properties (5) down to (1) from Lemma 1.1. Note that the results in sections 2.1 and 2.2 come directly from McKenzie's work in [8] and the results in the following sections and chapters are directly from Zádori in [12]. I have expanded heavily on the details of the proofs and added various examples throughout to help clarify the meaning of many of the lemmas.

In Chapter 2, we prove that (5) implies (4) in Theorem 2.11; that is, we show that a finite, bounded poset which admits monotone Jónsson operations also admits monotone Jónsson operations for which the operations with even indices do not depend on their second variable. We then show in Lemma 2.12 that if a finite bounded poset $\mathbf{P}$ admits operations described in (4) then it also admits operations satisfying (3). The final important result of this chapter uses monotone ternary operations satisfying (1.2) to define an operation satisfying (2). Thus, in Chapter 2, we prove most of the converse of Lemma 1.1: (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2).

In Chapter 3, we prove various properties about zigzags of finite, bounded posets. We define the notion of diameter in a poset and then prove that if a finite, bounded poset $\mathbf{P}$ satisfies (2) of Lemma 1.1, then we can find a finite bound on the diameters of all $\mathbf{P}$-zigzags.

In Chapter 4, we come to the most technical part of Zádori's proof. We state and prove various lemmas and then prove in Theorem 4.9 that if there exist $\mathbf{P}$-zigzags of arbitrarily large size, then we can find $\mathbf{P}$-zigzags of arbitrarily large diameter. As a corollary to this, we show that if $\mathbf{P}$ is a finite, bounded poset for which there exists a finite bound on the diameter of all $\mathbf{P}$-zigzags, then there exist at most finitely many $\mathbf{P}$-zigzags. Finally, we apply Lemma 1.5 which implies that if $\mathbf{P}$ has as at most finitely many $\mathbf{P}$-zigzags, then $\mathbf{P}$ admits a near-unanimity function. This final piece, combined with the main results of Chapters 3 and 4, proves that in a finite, bounded poset (2) implies (1). Hence we will conclude that a finite, bounded poset $\mathbf{P}$ admits a monotone n-near-unanimity function if and only if $\mathbf{P}$ admits a set of monotone Jónsson operations.

Chapter 2

Monotone Jónsson operations

In this chapter, we will consider finite, bounded posets $\mathbf{P}$ that admit monotone Jónsson operations. We will show that from these operations we can derive monotone Jónsson operations $d_0, \dots, d_n$ on $\mathbf{P}$ such that the operations with even indices do not depend on their second variable. With these, we will prove that for some $n \geq 3$ there exists a partially defined near-unanimity function fully defined on a certain subset of $\mathbf{P}^n$.

2.1 Refining monotone Jónsson operations

For the following arguments in this chapter, let $\mathbf{P}$ be a finite, bounded poset which admits monotone Jónsson operations $d_0(x, y, z), \dots, d_n(x, y, z)$ for some positive integer n. In [8], McKenzie defines binary operations $b_0(x, y), \dots, b_{2n-2}(x, y)$ on $\mathbf{P}$ by

$$\begin{aligned} b_0(x, y) &= d_1(x, x, y), && \quad (2.1)\\ b_{2i-1}(x, y) &= d_i(x, 1, y) && \text{for } 1 \leq i \leq n-1,\\ b_{4i+2}(x, y) &= d_{2i+1}(x, y, y) && \text{for } 0 \leq i \leq (n-2)/2, \text{ and}\\ b_{4i}(x, y) &= d_{2i}(x, x, y) && \text{for } 1 \leq i \leq (n-1)/2. \end{aligned}$$

These operations are important to us for the following reason.

Lemma 2.1. *The operations $b_0(x, y), \dots, b_{2n-2}(x, y)$ on $\mathbf{P}$ are monotone and satisfy the following equations:*

$$\begin{aligned} x = b_0(x, y) = b_i(x, x) = b_{2n-2}(y, x) & \quad \textit{for } 0 \leq i \leq 2n-2, && (2.2)\\ b_{2i}(x, y) \leq b_{2i+1}(x, y) & \quad \textit{for } 0 \leq i \leq n-1,\\ b_{2i+1}(x, y) \geq b_{2i+2}(x, y) & \quad \textit{for } 0 \leq i \leq n-2. \end{aligned}$$

Proof. The monotonicity of the b_i follows immediately by the monotonicity of the Jónsson operations. Then, by (1.1), for $1 \leq i \leq n-1$, we have $b_0(x,y) = d_1(x,x,y) = d_0(x,x,y) = x$, $b_{2i-1}(x,x) = d_i(x,1,x) = x$, $b_{2i}(x,x) = d_i(x,x,x) = x$, and either we have $b_{2n-2}(y,x) = d_{n-1}(y,x,x) = d_n(y,x,x) = x$ or $b_{2n-2}(y,x) = d_{n-1}(y,y,x) = d_n(y,y,x) = x$.

Now, by the monotonicity of the Jónsson operations,

$$b_0(x,y) = d_1(x,x,y) \leq d_1(x,1,y) = b_1(x,y).$$

As well, for $0 \leq i \leq (n-1)/2$,

$$\begin{aligned} b_{4i}(x,y) = d_{2i}(x,x,y) &= d_{2i+1}(x,x,y) \\ &\leq d_{2i+1}(x,1,y) = b_{2(2i+1)-1}(x,y) = b_{4i+1}(x,y) \end{aligned}$$

and, for $0 \leq i \leq (n-2)/2)$,

$$\begin{aligned} b_{4i+2}(x,y) = d_{2i+1}(x,y,y) &= d_{2i+2}(x,y,y) \\ &\leq d_{2i+2}(x,1,y) = b_{2(2i+2)-1}(x,y) = b_{4i+3}(x,y). \end{aligned}$$

Hence the second line of (2.2) is satisfied.

Finally, for $0 \leq i \leq n-2$, if i is odd, then

$$b_{2i+1}(x,y) = d_{i+1}(x,1,y) \geq d_{i+1}(x,x,y) = b_{2i+2}(x,y).$$

If i is even,

$$b_{2i+1}(x,y) = d_{i+1}(x,1,y) \geq d_{i+1}(x,y,y) = b_{2i+2}(x,y).$$

Thus the last inequality is also satisfied. ■

In Theorem 2.3 of [8], McKenzie derives from $b_0(x,y), \ldots, b_{2n-2}(x,y)$ monotone Jónsson operations $d_0, \ldots, d_k$ on $\mathbf{P}$ such that the operations with even indices do not depend on their second variable. The majority of this chapter is devoted to a more detailed look at the proof of this theorem. Before the proof is given, however, we provide some preliminary lemmas which will prove useful.

Define a *semigroup* $\mathbf{S}$ to be a set of elements S with an associative binary operation $\cdot$. The following lemma will provide a useful result about the order of elements in a semigroup.

Lemma 2.2. *Let* $\mathbf{S}$ *be a finite semigroup. Then there exists an integer* $N \geq 1$ *such that for all* $a \in \mathbf{S}$, $a^{2N} = a^N$.

Proof. Let $m = |S|$. Fix $a \in \mathbf{S}$ and consider the set $\{a, a^2, \ldots, a^{m+1}\}$. Then $a^u = a^{u+v}$ for some $1 \leq u \leq m$ and $1 \leq v \leq m$. Notice that this implies

$$a^u = a^u a^v = a^{u+v} a^v = a^{u+2v} = a^{u+3v} = \cdots = a^{u+nv}$$

for all positive integers n. Hence $a^u = a^{u+m!} = a^u a^{m!}$ yielding $a^{m!} = a^{2(m!)}$. Thus, since a was arbitrary, the result follows by letting $N = m!$. ■

Let $S = \{f : P \times P \to P\}$ and define the binary operation fg on S such that $fg(x,y) = f(g(x,y),y)$ for all $x, y \in P$. Notice that

$$f(gh)(x,y)) = f((gh)(x,y),y) = f(g(h(x,y),y),y) = (fg)(h(x,y),y) = (fg)h(x,y).$$

Hence the operation is associative so we have the following useful corollary.

Corollary 2.3. *There exists a positive integer N such that $f^{(N)}(x,y) = f^{(2N)}(x,y)$ for all $f : P \times P \to P$ and $x, y \in P$.*

Let $m = 2n-2$. In the following series of lemmas, we iterate the b_i's to obtain a new set of b_i's satisfying various equations. In each instance we replace the original b_i's with our "improved" set of b_i's. Our goal is to show that we can choose the b_i's to satisfy (2.2) and the following equations:

$$\begin{aligned} b_{2j}(x,y) \geq y \Rightarrow b_{2j}(x,y) = b_{2j+1}(x,y) &\qquad \forall j \leq (m-1)/2 \\ b_{2j+1}(x,y) \leq y \Rightarrow b_{2j+1}(x,y) = b_{2j+2}(x,y) &\qquad \forall j \leq (m-2)/2 \end{aligned} \tag{2.3}$$

In order to do so, for $0 \leq i \leq m$, we define $b_i^{(0)}(x,y) = x$ and inductively define $b_i^{(j+1)}(x,y) = b_i(b_i^{(j)}(x,y),y)$.

Lemma 2.4. *If there exist binary operations $b_0(x,y), \ldots, b_m(x,y)$ on $\mathbf{P}$ satisfying (2.2), then for some N the operations $b_0^{(N)}(x,y), \ldots, b_m^{(N)}(x,y)$ satisfy (2.2) and*

$$b_i^{(N)}(b_i^{(N)}(x,y),y) = b_i^{(N)}(x,y) \quad \forall i.$$

Proof. By Corollary 2.3 and the associativity of the operations, there exists a positive integer N such that, for $0 \leq i \leq m$,

$$b_i^{(N)}(x,y) = b_i^{(2N)}(x,y) = b_i^{(N)}(b_i^{(N)}(x,y),y).$$

We claim that the operations $b_0^{(N)}(x,y), \ldots, b_m^{(N)}(x,y)$ satisfy (2.2). For the following assume that $0 \leq k < N$. Suppose $b_0^{(k)}(x,y) = x$. Then $b_0^{(k+1)}(x,y) = b_0(b_0^{(k)}(x,y),y) = b_0^{(k)}(x,y) = x$. Hence by induction $b_0^{(N)}(x,y) = x$. As well, if $b_i^{(k)}(x,x) = x$, then $b_i^{(k+1)}(x,x) = b_i(b_i^{(k)}(x,x),x) = b_i(x,x) = x$. And again by induction $b_i^{(N)}(x,x) = x$ Finally, suppose that $b_m^{(k)}(y,x) = x$. This implies that $b_m^{(k+1)}(y,x) = b_m(b_m^{(k)}(y,x),x) = b_m(x,x) = x$ by our previous argument. Thus $b_m^{(N)}(y,x) = x$ and the first equation of (2.2) holds.

Next, for $0 \leq j \leq (m-1)/2$, notice that if for all $x, y \in P$, we have $b_{2j}^{(k)}(x,y) \leq b_{2j+1}^{(k)}(x,y)$ for some k where $0 \leq k < N$, then

$$\begin{aligned} b_{2j}^{(k+1)}(x,y) &= b_{2j}(b_{2j}^{(k)}(x,y),y) \\ &\leq b_{2j+1}(b_{2j}^{(k)}(x,y),y) \\ &\leq b_{2j+1}(b_{2j+1}^{(k)}(x,y),y) &\qquad \text{(monotonicity of } b_{2j+1}) \\ &= b_{2j+1}^{(k+1)}(x,y). \end{aligned}$$

Hence, by induction $b_{2j}^{(N)}(x,y) \leq b_{2j+1}^{(N)}(x,y)$ for $0 \leq j \leq (m-1)/2$. Similarly we have $b_{2j+1}^{(N)}(x,y) \geq b_{2j+2}^{(N)}(x,y)$ for $0 \leq j \leq (m-2)/2$ and the rest of (2.2) is satisfied. ■

Let us now redefine each $b_i(x,y)$ to be $b_i^{(N)}(x,y)$ so that our b_i's also satisfy

$$b_i(b_i(x,y),y) = b_i(x,y) \quad \forall i. \tag{2.4}$$

Our next step towards satisfying (2.3) will be to show that we can produce b_i's satisfying (2.2) and (2.4) for which

$$b_i(b_j(x,y),y) = b_i(x,y) \text{ whenever } i \geq j. \tag{2.5}$$

We prove this in the following lemma.

Lemma 2.5. *There exist binary operations $b_0(x,y), \ldots, b_m(x,y)$ on $\mathbf{P}$ satisfying* (2.2) *and* (2.5).

Proof. From the previous lemma, we can assume the b_i's already satisfy (2.2) and (2.4). We proceed through the proof using an inductive argument on iterations of our operations.

We start our base case by defining $c_0 = b_0$ and $c_i(x,y) = b_i(b_1(x,y),y)$ for $1 \leq i \leq m$. From (2.4), we have

$$\begin{aligned} c_i(c_1(x,y),y) &= b_i(b_1(b_1(b_1(x,y),y),y),y) \\ &= b_i(b_1(b_1(x,y),y),y) = b_i(b_1(x,y),y) = c_i(x,y) \end{aligned}$$

for all $i \geq 1$. Also notice that since $b_0(x,y) = x$,

$$c_i(c_0(x,y),y) = b_i(b_1(b_0(x,y),y),y) = b_i(b_1(x,y),y) = c_i(x,y).$$

Hence the operations $c_0 \ldots, c_m$ satisfy (2.5) for $j \in \{0,1\}$. We need to show that they still preserve (2.2). The first line is satisfied since

$$c_0(x,y) = b_0(x,y) = x = b_m(b_1(y,x),x) = c_m(y,x)$$

and $c_i(x,x) = b_i(b_1(x,x),x) = b_i(x,x) = x$ for $1 \leq i \leq m$. As well,

$$c_0(x,y) = b_0(x,y) \leq b_1(x,y) = b_1(b_1(x,y),y) = c_1(x,y).$$

The rest of inequalities hold from the definitions of the c_i. Thus the operations $c_0, \ldots, c_m$ satisfy (2.2).

By Lemma 2.4, the iterated operations $c_0^{(N)}, \ldots, c_m^{(N)}$ satisfy (2.2) and (2.4). As well, to show (2.5) still holds for $j \in \{0,1\}$ we need to use induction. For $0 \leq i \leq m$, we have $c_i(c_0^{(N)}(x,y),y) = c_i(x,y)$ by Lemma 2.1. If we assume, for some $1 \leq k < N$, that $c_i^{(k)}(c_0^{(N)}(x,y),y) = c_i^{(k)}(x,y)$, then

$$c_i^{(k+1)}(c_0^{(N)}(x,y),y) = c_i(c_i^{(k)}(c_0^{(N)}(x,y),y),y) = c_i(c_i^{(k)}(x,y),y) = c_i^{(k+1)}(x,y).$$

Hence $c_i^{(N)}(c_0^{(N)}(x,y),y) = c_i^{(N)}(x,y)$. Then, since $c_i(c_1(x,y),y) = c_i(x,y)$, we have

$$c_i(c_1^{(N)}(x,y),y) = c_i(c_1(c_1^{(N-1)}(x,y),y),y) = c_i(c_1^{(N-1)}(x,y),y) = \ldots = c_i(x,y).$$

Thus by a similar argument $c_i^{(N)}(c_1^{(N)}(x,y),y) = c_i^{(N)}(x,y)$. Thus (2.5) holds for $j \in \{0,1\}$ as does the base case of our main induction.

Now, for our inductive hypothesis assume that for $0 < k < m$ there exists a set of operations $f_0(x,y),\ldots,f_m(x,y)$ satisfying (2.2), (2.4), and (2.5) for $j \leq k$. Consider the operations $g_0,\ldots,g_m$ defined such that

$$g_i(x,y) = \begin{cases} f_i(x,y), & \text{if } 0 \leq i \leq k \\ f_i(f_{k+1}(x,y),y), & \text{if } k+1 \leq i. \end{cases}$$

We need to show that our new operations satisfy (2.5) for $j \in \{0,\ldots,k+1\}$. If $i \leq k$, then this is immediate from our inductive hypothesis. If $k+1 \leq i$ and $j = k+1$, then (2.4) gives

$$\begin{aligned} g_i(g_j(x,y),y) &= f_i(f_{k+1}(f_{k+1}(f_{k+1}(x,y),y),y),y) \\ &= f_i(f_{k+1}(f_{k+1}(x,y),y),y) = f_i(f_{k+1}(x,y),y) = g_i(x,y). \end{aligned}$$

If $k+1 \leq i$ and $j < k+1$, then by our hypothesis

$$g_i(g_j(x,y),y) = f_i(f_{k+1}(f_j(x,y),y),y) = f_i(f_{k+1}(x,y),y) = g_i(x,y).$$

Thus (2.5) is satisfied up to $j = k+1$.

We claim that $g_1,\ldots,g_m$ also satisfy (2.2). By our hypothesis,

$$g_0(x,y) = f_0(x,y) = x = f_m(f_{k+1}(y,x),x) = g_m(y,x).$$

As well for $1 \leq i \leq k$, we immediately have $g_i(x,x) = f_i(x,x) = x$. For $k+1 \leq i \leq m$, we get $g_i(x,x) = f_i(f_{k+1}(x,x),x) = f_i(x,x) = x$. To prove the inequalities of (2.2), we only need consider f_k and f_{k+1} as the rest follow by our inductive hypothesis. If k is even, then

$$g_k(x,y) = f_k(f_{k+1}(x,y)) \leq f_{k+1}(f_{k+1}(x,y)) = g_{k+1}(x,y)$$

and the odd case follows dually. Hence (2.2) is satisfied and we can proceed with the iterating the operations.

By Lemma 2.4, the iterated operations $g_0^{(N)},\ldots,g_m^{(N)}$ satisfy (2.2) and (2.4). By associativity, since $g_0,\ldots,g_m$ satisfy (2.5), we can show that $g_0^{(N)},\ldots,g_m^{(N)}$ also satisfy (2.5) for $j \in \{0,\ldots,k+1\}$. Thus our main inductive argument holds.

Therefore we can find operations $b_0(x,y),\ldots,b_m(x,y)$ on $\mathbf{P}$ for which (2.2) holds and (2.5) also holds for all $1 \leq j \leq m$. ■

We end this section by proving there are operations on $\mathbf{P}$ for which both the equations of (2.2) and the implications of (2.3) hold.

Theorem 2.6. *There exist operations $b_0(x,y),\ldots,b_m(x,y)$ on* **P** *satisfying* (2.2) *and* (2.3).

Proof. Firstly, by Lemma 2.5, there exist operations $b_0(x,y),\ldots,b_m(x,y)$ on **P** satisfying (2.2) and (2.5). Suppose that $b_{2j}(x,y) \geq y$ for some $j \leq (m-1)/2$. This implies that

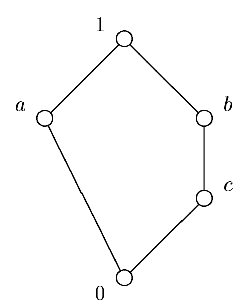
$$\begin{aligned} b_{2j+1}(x,y) &= b_{2j+1}(b_{2j}(x,y),y) && \text{(by (2.5))} \\ &\leq b_{2j+1}(b_{2j}(x,y),b_{2j}(x,y)) \\ &= b_{2j}(x,y) && \text{(by (2.2)).} \end{aligned}$$

On the other hand, by (2.2) we have $b_{2j}(x,y) \leq b_{2j+1}(x,y)$. Thus $b_{2j}(x,y) = b_{2j+1}(x,y)$. The second implication of (2.3) follows by a dual argument. ■

2.2 Modifying the Jónsson operations with even indices

In this section, we derive monotone Jónsson operations on **P** for which the operations with even indices do not depend on their second variable. We create these from the operations given by Theorem 2.6.

First, though, for every element x in a poset $\mathbf{Q}$, define $l_{\mathbf{Q}}(x)$ to be the maximum cardinality of all chains in $\mathbf{Q}$ with top element x. Then, define $m_{\mathbf{Q}}(x)$ to be the maximum cardinality of all chains in $\mathbf{Q}$ with bottom element x. Finally, let $D_k = \{x \in \mathbf{P} : l_{\mathbf{P}}(x) \leq k+1\}$ and $U_k = \{x \in \mathbf{P} : m_{\mathbf{P}}(x) \leq k+1\}$. As an example, consider the following poset:



In this case, $D_0 = \{0\}$, $D_1 = \{0,a,c\}$, $D_2 = \{0,a,c,b\}$, $D_3 = \{0,a,c,b,1\}$, $U_0 = \{1\}$, $U_1 = \{a,b,1\}$, $U_2 = \{c,a,b,1\}$, and $U_3 = \{0,c,a,b,1\}$.

We note some facts about D_k and U_k in the following lemma.

Lemma 2.7. *1. There exists a positive integer m_0 such that $D_{m_0} = P$ and $D_i \subsetneq D_{i+1}$ for $0 \leq i \leq m_0 - 1$.*

2. There exists a positive integer m_1 such that $U_{m_1} = P$ and $U_i \subsetneq U_{i+1}$ for $0 \leq i \leq m_1 - 1$.

3. If $u < v$ in $\mathbf{P}$ and $u \in U_{i+1}$, then $v \in U_i$.

4. If $u < v$ in $\mathbf{P}$ and $v \in D_{i+1}$, then $u \in D_i$.

The general idea behind the proofs in the following lemmas and theorem is that we are building a ladder of d_i's leading from b_0 to b_1 and then from b_1 to b_2 and so on up to b_m. This ladder will allow us to work our way up through the U_k's and D_k's and the first element of this ladder, b_0, and the last element, b_m, will satisfy the first equation of (1.1). At each level, the d_i's will be monotone, satisfy all of (1.1), and for every even index i and any triple $(x, y, z) \in \mathbf{P}^3$, we can find j such that $d_i(x, y, z) = b_j(x, z)$.

For any subset S in a poset $\mathbf{Q}$, let $S^* = \{z \in \mathbf{Q} : z \geq_{\mathbf{Q}} x \text{ for all } x \in S\}$. For the rest of this section, let $b_0(x, y), \dots, b_m(x, y)$ be the operations on $\mathbf{P}$ given by Theorem 2.6. We proceed with the main argument of this section.

Lemma 2.8. *There exist monotone operations $d_0^{(0)}, \dots, d_{2m_1+1}^{(0)}$ satisfying all of (1.1) except possibly $d_N(z, y, x) = x$. Moreover, $d_0^{(0)}(x, y, z) = b_0(x, z)$, $d_{2m_1+1}^{(0)}(x, y, z) = b_1(x, z)$, and for $1 \leq k \leq m_1$ the operation $d_{2k}^{(0)}(x, y, z)$ does not depend on its second variable.*

Proof. First define $d_0^{(0)}(x, y, z) = x$ and define

$$d_1^{(0)}(x, y, z) = \begin{cases} b_1(x, z), & \text{if } \{x, y\}^* \cup \{x, z\}^* \subseteq U_0 \\ x, & \text{otherwise.} \end{cases}$$

Note that $d_0^{(0)}$ is monotone and clearly satisfies the first line of (1.1). We claim that $d_1^{(0)}(x, y, z)$ is monotone and satisfies the necessary Jónsson equations.

Suppose $(x_1, y_1, z_1) \leq (x_2, y_2, z_2)$. If $\{x_1, y_1\}^* \cup \{x_1, z_1\}^* \subseteq U_0$, then $\{x_2, y_2\}^* \cup \{x_2, z_2\}^* \subseteq U_0$. Hence $d_1^{(0)}(x_1, y_1, z_1) = b_1(x_1, z_1) \leq b_1(x_2, z_2) = d_1^{(0)}(x_2, y_2, z_2)$. If not, then either

$$d_1^{(0)}(x_1, y_1, z_1) = x_1 = b_0(x_1, z_1) \leq b_1(x_1, z_1) \leq b_1(x_2, z_2) = d_1^{(0)}(x_2, y_2, z_2)$$

or

$$d_1^{(0)}(x_1, y_1, z_1) = x_1 \leq x_2 = d_1^{(0)}(x_2, y_2, z_2).$$

Thus $d_1^{(0)}(x_1, y_1, z_1)$ is monotone. Moreover, since $b_1(x, x) = x$, then $d_1^{(0)}(x, y, x) = x$ and if $d_0^{(0)}(x, x, y) = x = d_1^{(0)}(x, x, y)$ then (1.1) holds. If $\{x, x\}^* \cup \{x, y\}^* \subseteq U_0$, then notice that x is the maximal element of $\mathbf{P}$. Thus, since $x = b_0(x, z) \leq b_1(x, z)$,

$$d_0^{(0)}(x, x, y) = x = b_1(x, z) = d_1^{(0)}(x, x, y)$$

and (1.1) still holds as does our claim.

Now, let us define the rest of the $d_i^{(0)}$'s. For $1 \leq i \leq m_1$, let $d_{2i}^{(0)}(x,y,z) = d_{2i-1}^{(0)}(x,z,z)$ and let

$$d_{2i+1}^{(0)}(x,y,z) = \begin{cases} b_1(x,z), & \text{if } \{x,z\}^* \subseteq U_{i-1} \text{ or } \{x,y\}^* \cup \{x,z\}^* \subseteq U_i \\ x, & \text{otherwise.} \end{cases}$$

Because of the definition of $d_{2i}^{(0)}$ it is sufficient to just show monotonicity of the operations with odd indices. So let us suppose $(x_1,y_1,z_1) \leq (x_2,y_2,z_2)$. If $\{x_1,z_1\}^* \subseteq U_{i-1}$, then $\{x_2,z_2\}^* \subseteq U_{i-1}$ and we have

$$d_{2i+1}^{(0)}(x_1,y_1,z_1) = b_1(x_1,z_1) \leq b_1(x_2,z_2) = d_{2i+1}^{(0)}(x_2,y_2,z_2).$$

The argument for the other two cases is the same as the argument for $d_1^{(0)}$.

Finally, we show that $d_{2i}^{(0)}$ and $d_{2i+1}^{(0)}$ preserve (1.1). Note first that $d_{2i}^{(0)}(x,y,x) = d_{2i-1}^{(0)}(x,y,x) = b_1(x,x) = x$. The third equation of (1.1) follows immediately from the definition of $d_{2i}^{(0)}$. To show the second equation is satisfied, fix i and suppose first that $\{x,z\}^* \subseteq U_{i-1}$. Then

$$d_{2i}^{(0)}(x,x,z) = d_{2i-1}^{(0)}(x,z,z) = b_1(x,z) = d_{2i+1}^{(0)}(x,x,z).$$

If $\{x,x\}^* \cup \{x,z\}^* \subseteq U_i$ and $\{x,z\}^* \nsubseteq U_{i-1}$, then $\{x\}^* \subseteq U_i$ which implies $x \in U_i$. But there exists some $w \notin U_{i-1}$ such that $w \geq x$. If $w \notin U_i$ then we have a contradiction to Lemma 2.7 (3), so it must be that $w = x$. Hence $z \leq x = b_0(x,z)$ and by equation (2.3) we have $b_1(x,z) = b_0(x,z) = x$. Thus

$$d_{2i+1}^{(0)}(x,x,z) = x = d_{2i-1}^{(0)}(x,z,z) = d_{2i}^{(0)}(x,x,z).$$

The final case occurs when $\{x,x\}^* \cup \{x,z\}^* \nsubseteq U_i$ and $\{x,z\}^* \nsubseteq U_{i-1}$. But in this case $\{x,z\}^* \nsubseteq U_{i-2}$, so we immediately have

$$d_{2i+1}^{(0)}(x,x,z) = x = d_{2i-1}^{(0)}(x,z,z) = d_{2i}^{(0)}(x,x,z).$$

Therefore our $d_i^{(0)}$'s as defined above are all monotone and satisfy the equations of (1.1) except possibly $d_N(z,y,x) = x$. ∎

Let S_* denote the set of all lower bounds of a set S in a poset $\mathbf{Q}$. The next lemma gives the operations leading from $b_{2i-1}(x,y)$ to $b_{2i}(x,y)$ for $1 \leq i \leq n$.

Lemma 2.9. *There exist monotone operations $d_0^{(2i-1)}, \ldots, d_{2m_0+1}^{(2i-1)}$ satisfying*

$$\begin{aligned} d_{2j}^{(2i-1)}(x,x,y) &= d_{2j+1}^{(2i-1)}(x,x,y) \text{ for } 0 \leq j \leq m_0, \\ d_{2j-1}^{(2i-1)}(x,y,y) &= d_{2j}^{(2i-1)}(x,y,y) \text{ for } 1 \leq j \leq m_0, \end{aligned}$$

and $d_j^{(2i-1)}(x,y,x) = x$ for $1 \leq j \leq 2m_0+1$. Moreover, for $1 \leq i \leq m$, we have $d_0^{(2i-1)}(x,y,z) = b_{2i-1}(x,z)$, $d_{2m_0+1}^{(2i-1)}(x,y,z) = b_{2i}(x,z)$, and for $1 \leq k \leq m_0$ the operation $d_{2k}^{(2i-1)}(x,y,z)$ does not depend on its second variable.

Proof. We proceed similarly to the previous lemma. First define $d_0^{(2i-1)}(x,y,z) = b_{2i-1}(x,z)$ and define

$$d_1^{(2i-1)}(x,y,z) = \begin{cases} b_{2i}(x,z), & \text{if } \{b_{2i-1}(x,z),z\}_* \cup \{b_{2i-1}(x,z), b_{2i-1}(y,z)\}_* \subseteq D_0 \\ b_{2i-1}(x,z), & \text{otherwise.} \end{cases}$$

Note that $d_0^{(2i-1)}$ is monotone. We claim that $d_1^{(2i-1)}(x,y,z)$ is monotone and satisfies (1.1).

Suppose $(x_1,y_1,z_1) \le (x_2,y_2,z_2)$. If

$$\{b_{2i-1}(x_2,z_2),z_2\}_* \cup \{b_{2i-1}(x_2,z_2), b_{2i-1}(y_2,z_2)\}_* \subseteq D_0,$$

then

$$\{b_{2i-1}(x_1,z_1),z_1\}_* \cup \{b_{2i-1}(x_1,z_1), b_{2i-1}(y_1,z_1)\}_* \subseteq D_0.$$

Hence

$$d_1^{(2i-1)}(x_1,y_1,z_1) = b_{2i}(x_1,z_1) \le b_{2i}(x_2,z_2) = d_1^{(2i-1)}(x_2,y_2,z_2).$$

If not, then either

$$d_1^{(2i-1)}(x_1,y_1,z_1) = b_{2i-1}(x_1,z_1) \le b_{2i-1}(x_2,z_2) = d_1^{(2i-1)}(x_2,y_2,z_2)$$

or, by (2.2),

$$d_1^{(2i-1)}(x_1,y_1,z_1) = b_{2i}(x_1,z_1) \le b_{2i-1}(x_1,z_1) \le b_{2i-1}(x_2,z_2) = d_1^{(2i-1)}(x_2,y_2,z_2).$$

Thus $d_1^{(2i-1)}(x,y,z)$ is monotone. Now, if $d_0^{(2i-1)}(x,x,y) = b_{2i}(x,z) = d_1^{(2i-1)}(x,x,y)$ then our hypothesis holds. If $\{b_{2i-1}(x,y),y\}_* \cup \{b_{2i-1}(x,y), b_{2i-1}(x,y)\}_* \subseteq D_0$, then notice that $b_{2i-1}(x,y)$ is the minimal element of $\mathbf{P}$. Thus, since $b_{2i}(x,z) \le b_{2i-1}(x,z)$,

$$d_0^{(2i-1)}(x,x,y) = b_{2i-1}(x,z) = b_{2i}(x,z) = d_1^{(2i-1)}(x,x,y)$$

and the hypothesis still holds.

We now define the rest of the $d_j^{(2i-1)}$'s. For $1 \le i \le m_0$, let $d_{2j}^{(2i-1)}(x,y,z) = d_{2j-1}^{(2i-1)}(x,z,z)$ and let

$$d_{2j+1}^{(2i-1)}(x,y,z) = \begin{cases} b_{2i}(x,z), & \text{if } \{b_{2i-1}(x,z),z\}_* \subseteq D_{i-1} \\ & \quad \text{or } \{b_{2i-1}(x,z),z\}_* \cup \{b_{2i-1}(x,z), b_{2i-1}(y,z)\}_* \subseteq D_i \\ b_{2i-1}(x,z), & \text{otherwise.} \end{cases}$$

Because of the definition of $d_{2j}^{(2i-1)}$ it is sufficient to just show monotonicity of the operations with odd indices. So let us suppose $(x_1,y_1,z_1) \le (x_2,y_2,z_2)$. If $\{b_{2i-1}(x_2,z_2),z_2\}_* \subseteq D_{i-1}$, then $\{b_{2i-1}(x_1,z_1),z_1\}_* \subseteq D_{i-1}$ and we have

$$d_{2j+1}^{(2i-1)}(x_1,y_1,z_1) = b_{2i}(x_1,z_1) \le b_{2i}(x_2,z_2) = d_{2j+1}^{(2i-1)}(x_2,y_2,z_2).$$

The argument for the other two cases is the same as the argument for $d_1^{(2i-1)}$.

Next, we show that $d_{2j}^{(2i-1)}$ and $d_{2j+1}^{(2i-1)}$ preserve the equations given. The second equation follows immediately from the definition of $d_{2j}^{(2i-1)}$. To show the first equation is satisfied, fix j and suppose first that $\{b_{2i-1}(x,z),z\}_* \subseteq D_{i-1}$. Then

$$d_{2j}^{(2i-1)}(x,x,z) = d_{2j-1}^{(2i-1)}(x,z,z) = b_{2i}(x,z) = d_{2j+1}^{(2i-1)}(x,x,z).$$

If $\{b_{2i-1}(x,z),z\}_* \cup \{b_{2i-1}(x,z),b_{2i-1}(x,z)\}_* \subseteq D_i$ and $\{b_{2i-1}(x,z),z\}_* \nsubseteq D_{i-1}$, then $\{b_{2i-1}(x,z)\}_* \subseteq D_i$ which implies $b_{2i-1}(x,z) \in D_i$. But there exists some $w \notin D_{i-1}$ such that $w \leq b_{2i-1}(x,z)$. If $w \notin D_i$, then we have a contradiction to Lemma 2.7 (3) so it must be that $w = b_{2i-1}(x,z)$. Hence $z \geq b_{2i-1}(x,z)$ and by equation (2.3) we have $b_{2i}(x,z) = b_{2i-1}(x,z)$. Thus

$$d_{2j+1}^{(2i-1)}(x,x,z) = b_{2i-1}(x,z) = d_{2j-1}^{(2i-1)}(x,z,z) = d_{2j}^{(2i-1)}(x,x,z).$$

The final case occurs when $\{b_{2i-1}(x,z),z\}_* \cup \{b_{2i-1}(x,z),b_{2i-1}(y,z)\}_* \nsubseteq D_i$ and $\{b_{2i-1}(x,z),z\}_* \nsubseteq D_{i-1}$. But in this case $\{b_{2i-1}(x,z),z\}_* \nsubseteq D_{i-2}$, so we immediately have

$$d_{2j+1}^{(2i-1)}(x,x,z) = b_{2i-1}(x,z) = d_{2j-1}^{(2i-1)}(x,z,z) = d_{2j}^{(2i-1)}(x,x,z).$$

Finally, for $0 \leq j \leq 2m_0+1$, we immediately have $d_j^{(2i-1)}(x,y,x) = b_{2i}(x,x) = b_{2i-1}(x,x) = x$ and our proof is complete. ∎

A dual argument to the proof of Lemma 2.9 connects $b_{2i}(x,z)$ to $b_{2i+1}(x,z)$ in the next lemma.

Lemma 2.10. *There exist monotone operations* $d_0^{(2i)}, \ldots, d_{2m_1+1}^{(2i)}$ *satisfying*

$$d_{2j}^{(2i)}(x,y,y) = d_{2j+1}^{(2i)}(x,y,y) \text{ for } 0 \leq j \leq m_1,$$
$$d_{2j-1}^{(2i)}(x,x,y) = d_{2j}^{(2i)}(x,x,y) \text{ for } 1 \leq j \leq m_1,$$

and $d_j^{(2i)}(x,y,x) = x$ *for* $1 \leq j \leq 2m_1+1$. *Moreover, for* $1 \leq i \leq m-1$, *we have* $d_0^{(2i)}(x,y,z) = b_{2i}(x,z)$, $d_{2m_1+1}^{(2i)}(x,y,z) = b_{2i+1}(x,z)$, *and for* $1 \leq k \leq m_1$ *the operation* $d_{2k}^{(2i)}(x,y,z)$ *does not depend on its second variable.*

We are now able to put it all together and produce monotone Jónsson operations that are independent of their second variable.

Theorem 2.11. *There exist monotone Jónsson operations on* $\mathbf{P}$ *for which the operations on the even indices do not depend on their second variable.*

Proof. From Lemmas 2.8, 2.9, and 2.10, we can produce a series of monotone operations

$$d_0^{(0)}, \ldots, d_{2m_1+1}^{(0)} = d_0^{(1)}, \ldots, d_{2m_0+1}^{(1)} = d_0^{(2)}, \ldots, d_0^{(m-1)}, \ldots, d_{2m_0+1}^{(m-1)}$$

when we recall that $m = 2n$. By definition, and since $d_{2m_0+1}^{(m-1)}(z,y,x) = b_m(z,x) = x$, these operations completely satisfy (1.1). More importantly, by definition each d with even indices is independent of its second variable. ∎

2.3 A partially defined n-nuf on $\mathbf{P}^n$

For any positive integer n, define

$$A_n = \left\{ (x, \ldots, x, \underset{i}{y}, z, \ldots, z) : x, y, z \in P,\ 1 \le i \le n \right\}.$$

In this section, we return to Zádori's argument and prove that (4) implies (3) and (3) implies (2) from Lemma 1.1. First, we need to produce a set of operations $D_1, \ldots, D_k$ satisfying the equations of (1.2):

$$\begin{aligned} D_1(x,x,y) = D_k(y,x,x) = D_i(x,y,x) &= x && \text{for } 1 \le i \le k \\ D_i(x,y,y) &= D_{i+1}(x,x,y) && \text{for } 1 \le i \le k-1 \end{aligned}$$

Lemma 2.12. *For some $k \ge 1$, suppose $\mathbf{P}$ admits monotone Jónsson operations $d_0, \ldots, d_{2k-1}$ which have the additional property that for $0 \le i \le k-1$, the operation d_{2i} does not depend on its middle variable. Then $\mathbf{P}$ admits operations $D_1, \ldots, D_k$ satisfying* (1.2).

Proof. Let $D_i(x,y,z) = d_{2i-1}(x,y,z)$ for $1 \le i \le k$. First, by (1.1), we note that $D_1(x,x,y) = d_1(x,x,y) = d_0(x,x,y) = x$ and $D_k(y,x,x) = d_{2k-1}(y,x,x) = x$. As well, for $1 \le i \le k$, we have $D_i(x,y,x) = d_{2i-1}(x,y,x) = x$ so the first line of (1.2) is satisfied. To satisfy the second line, just use (1.1) to get

$$D_i(x,y,y) = d_{2i-1}(x,y,y) = d_{2i}(x,y,y) = d_{2i}(x,x,y) = d_{2i+1}(x,x,y) = D_{i+1}(x,x,y).$$

■

Now, in order to define our n-nuf properly, we need to first define certain subsets of $\mathbf{P}^n$. For $1 \le i \le n$, let $B_i = \left\{ (x, \ldots, x, \underset{i}{y}, z, \ldots, z) : x, y, z \in P \right\}$ where $B_i \subseteq \mathbf{P}^n$. We can now describe A_n in terms of the B_i's

Lemma 2.13. *For any positive integer $n \ge 2$, the set $A_n = \bigcup_{i=2}^{n-1} B_i$.*

Proof. Just notice that $B_1 \subseteq B_2$ and $B_n \subseteq B_{n-1}$. ■

We are now able to produce, for some n, a partially defined n-near-unanimity function that is fully defined on the set $A_n \subseteq \mathbf{P}^n$.

Theorem 2.14. *If $\mathbf{P}$ admits operations $D_1, \ldots, D_k$ satisfying* (1.2)*, then for some n, there exists a partially defined, monotone n-nuf on $\mathbf{P}^n$ that is fully defined on*

$$A_n = \left\{ (x, \ldots, x, \underset{i}{y}, z, \ldots, z) : x, y, z \in P,\ 1 \le i \le n \right\}.$$

Proof. For $2 \leq i \leq k+1$, partially define a function f on $\mathbf{P}^{k+2}$ such that

$$f(x, \ldots, x, \underset{i}{y}, z, \ldots, z) = D_{i-1}(z, y, x)$$

where $(x, \ldots, x, \underset{i}{y}, z, \ldots, z) \in B_i$ and the D_i's satisfy (1.2).

We start by claiming that f is well-defined. Suppose, for $2 \leq i < j \leq k+1$, that $(a, \ldots, a, \underset{i}{b}, c, \ldots, c) \in B_i \cap B_j$. This means either $a = b = c$, $a = b \neq c$, or $a \neq b = c$. If $a = b = c$, then

$$D_{i-1}(c, b, a) = D_{i-1}(a, a, a) = a = D_{j-1}(a, a, a) = D_{j-1}(c, b, a) \qquad \text{by (1.2).}$$

Otherwise, $a = b \neq c$ implies that $j = i+1$. Hence, by (1.2) again,

$$f(a, \ldots, \underset{i}{a}, c, \ldots, c) = D_{i-1}(c, a, a) = D_{j-1}(c, c, a) = f(a, \ldots, a, \underset{i+1}{c}, \ldots, c).$$

A similar argument holds for $a \neq b = c$ and it follows that f is well-defined. By Lemma 2.13, f has domain equal to A_{k+2}. That f is an $(k+2)$-nuf now falls easily from the first equation of (1.2).

Now we show that f is monotone. Let us suppose that

$$(a, \ldots, a, \underset{i}{b}, c, \ldots, c) \leq (d, \ldots, d, \underset{j}{e}, f, \ldots, f)$$

where $(a, \ldots, a, \underset{i}{b}, c, \ldots, c)$ and $(d, \ldots, d, \underset{j}{e}, f, \ldots, f)$ are both elements of A_{k+2}. We have a few cases to consider. First, if $i = j$, then, by the monotonicity of the D_i's, we immediately have

$$f(a, \ldots, a, \underset{i}{b}, c, \ldots, c) = D_{i-1}(c, b, a) \leq D_{i-1}(f, e, d) = f(d, \ldots, d, \underset{i}{e}, f, \ldots, f).$$

Next, let us assume $i < j$. Then $a \leq d$, $b \leq d$, $c \leq e$, and $c \leq f$. If $j \geq i+2$, then we also get $c \leq d$ which gives us

$$\begin{aligned} f(a, \ldots, a, \underset{i}{b}, c, \ldots, c) &= D_{i-1}(c, b, a) \\ &\leq D_{i-1}(c, d, d) && \text{(by monotonicity)} \\ &= D_i(c, c, d) && \text{(by (1.2))} \\ &\leq D_i(c, d, d) \\ &= \ldots \\ &\leq D_{j-2}(c, d, d) \\ &= D_{j-1}(c, c, d) \\ &\leq D_{j-1}(f, e, d) = f(d, \ldots, d, \underset{j}{e}, f, \ldots, f). \end{aligned}$$

If $j = i+1$, however, we get

$$\begin{aligned} f(a, \ldots, a, \underset{i}{b}, c, \ldots, c) &= D_{i-1}(c, b, a) \leq D_{i-1}(c, d, d) = D_i(c, c, d) \\ &\leq D_i(f, e, d) = D_{j-1}(f, e, d) = f(d, \ldots, d, \underset{j}{e}, f, \ldots, f). \end{aligned}$$

When $j < i$, proceed dually. Thus f is monotone and our proof is complete. ∎

In the next chapter, we will begin to explore in more depth the idea of zigzags in finite, bounded posets. We will then use Theorem 2.14 and various other properties of these zigzags to show that we can find a sort of bound on the size of these subposets.

Chapter 3

Zigzags in finite, bounded posets

In the first section of this chapter we will give some useful properties of the zigzags and show that for every zigzag $(\mathbf{H}, f)$, we can find a standard zigzag which has $(\mathbf{H}, f)$ as a monotone image. In the second section we prove that if a finite, bounded poset $\mathbf{P}$ satisfies property (2) of Lemma 1.1, then we can place a finite bound on the diameter of all $\mathbf{P}$-zigzags.

3.1 Standard P-zigzags

We start with some basic definitions. The *comparability graph* of a poset $\mathbf{P}$ is the undirected graph on the set P such that there is exactly one pair $\{a, b\}$ in the edge set if and only if a is comparable to b in $\mathbf{P}$ for any $a \neq b$ in P. Similarly, the *covering graph* of $\mathbf{P}$ is the undirected graph on the set P such that $\{a, b\}$ is in the edge set if and only if $a \prec_{\mathbf{P}} b$. The poset $\mathbf{P}$ is called *connected* if its comparability graph is a connected graph. Now, if the comparability graph of $\mathbf{P}$ forms sequence of vertices such that from each of its vertices there is an edge to the next vertex in the sequence and in the sequence no vertex is repeated, then $\mathbf{P}$ is called a *fence*.

In a connected poset $\mathbf{P}$, define the *distance* between two elements a and b in $\mathbf{P}$ to be $n - 1$ where n is the smallest integer for which there exists an n-element fence contained in $\mathbf{P}$ connecting a and b. We denote it $d(a, b)$. Next, define the *up distance* from elements a to b in $\mathbf{P}$ to be the least positive integer n such that there is a subset $\{a_0, \ldots, a_n\} \subseteq \mathbf{P}$ with $a = a_0$, $b = a_n$, and $a_0 \leq a_1 \geq a_2 \leq \cdots$. The *down distance* from a to b is defined dually. Finally, the *diameter* of $\mathbf{P}$ is the supremum of the set $\{d(a, b) : a, b \in P\}$. As an example, consider Figure 3.1 and notice that the diameter of $\mathbf{P}$ is 3 and the up distance from a to f is 3 but the down distance from a to f is 2.

For the rest of this chapter, $\mathbf{P}$ will always be a finite, bounded poset. We are now able to restate and prove Proposition 3.1 and Claims 3.2 through 3.4 of [12] in the following lemmas, the first of which gives us an alternate characterization of monotone zigzags.

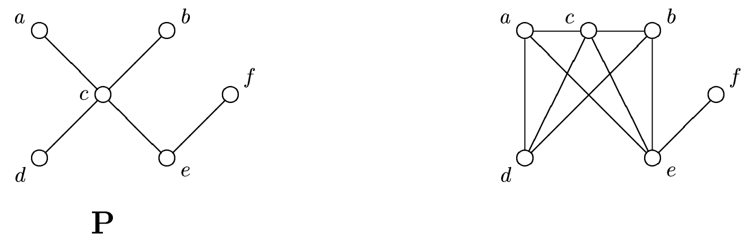


Figure 3.1: A poset $\mathbf{P}$ and its comparability graph

Lemma 3.1. *Let* $(\mathbf{H}, f)$ *be a finite* $\mathbf{P}$*-coloured poset. Then* $(\mathbf{H}, f)$ *is a* $\mathbf{P}$*-zigzag if and only if* $\mathbf{H}$ *is connected,* $(\mathbf{H}, f)$ *is not* $\mathbf{P}$*-extendible, and by cancelling any covering pair in* $(\mathbf{H}, f)$*, the resulting coloured poset is* $\mathbf{P}$*-extendible.*

Proof. The nonmonotone case is immediate from Proposition 1.4. So let us assume that f is monotone on its domain. For the first direction, we only need show that the $\mathbf{P}$-zigzag $(\mathbf{H}, f)$ is connected as the rest follows from the definition. Suppose otherwise. Then the comparability graph of $\mathbf{H}$ contains at least two distinct disconnected components $H_1, \ldots, H_n$. Since $(\mathbf{H}, f)$ is a zigzag, each $\mathbf{P}$-coloured poset $(\mathbf{H}_i, f|_{H_i})$ is $\mathbf{P}$-extendible. Let $f'|_{H_i}$ be the fully-defined monotone map on $\mathbf{H}_i$. Then f' is a fully-defined monotone map on $\mathbf{H}$. This is a contradiction so $(\mathbf{H}, f)$ must be connected.

Now assume $\mathbf{H}$ is connected, $(\mathbf{H}, f)$ is not $\mathbf{P}$-extendible, and by cancelling any covering pair in $(\mathbf{H}, f)$, the resulting coloured poset is $\mathbf{P}$-extendible. Let $(\mathbf{H}', f|_{H'})$ be a $\mathbf{P}$-nonextendible $\mathbf{P}$-coloured poset contained in $(\mathbf{H}, f)$. We show that $(\mathbf{H}', f|_{H'})$ must equal $(\mathbf{H}, f)$. First note that $(\mathbf{H}', f|_{H'})$ must contain all the covering pairs in $(\mathbf{H}, f)$. This implies that $\mathbf{H}'$ contains all elements in the comparability graph of $\mathbf{H}$ that are in components of size greater than or equal to 2. But $\mathbf{H}$ is connected so the comparability graph has only one component and it is of size greater than 1 (otherwise $(\mathbf{H}, f)$ would be $\mathbf{P}$-extendible). Thus $\mathbf{H} \subseteq \mathbf{H}'$ which means $\mathbf{H}' = \mathbf{H}$. Hence $f|_{H'} = f$ and $(\mathbf{H}', f|_{H'}) = (\mathbf{H}, f)$. ■

The next lemma gives us more information about the noncoloured elements of a monotone $\mathbf{P}$-zigzag.

Lemma 3.2. *In any monotone* $\mathbf{P}$*-zigzag* $(\mathbf{H}, f)$*, the subgraph spanned by* $N(\mathbf{H}, f)$ *in the covering graph of* $\mathbf{H}$ *is connected. Moreover, if* $a \in C(\mathbf{H}, f)$ *and there exists a* $b \in \mathbf{H}$ *such that* $a \prec b$ *or* $b \prec a$*, then* $b \in N(\mathbf{H}, f)$*.*

Proof. Suppose otherwise. Then the subgraph spanned by $N(\mathbf{H}, f)$ has at least two components. Pick one component; call it H_1. The $\mathbf{P}$-coloured poset $(\mathbf{H}, f) \backslash H_1$ is $\mathbf{P}$-extendible as is $(\mathbf{H}, f) \backslash (N(\mathbf{H}, f) \backslash H_1)$. Since all the elements in H_1 are incomparable to the elements in $N(\mathbf{H}, f) \backslash H_1$, these two extensions yield a piecewise fully-defined extension of $(\mathbf{H}, f)$. This contradicts $(\mathbf{H}, f)$ being a $\mathbf{P}$-zigzag so $\mathbf{N}(\mathbf{H}, f)$ must be connected.

Now let $a \in C(\mathbf{H}, f)$ and suppose $a \prec b$. If $b \in C(\mathbf{H}, f)$, then cancelling the covering pair (a, b) would yield a $\mathbf{P}$-extendible poset. Since a and b are already coloured in $(\mathbf{H}, f)$, it follows that $f(a) \leq f(b)$ so putting (a, b) back yields a monotone extension of $(\mathbf{H}, f)$. This is a contradiction so it must be that $b \in N(\mathbf{H}, f)$. A similar argument shows that $b \in N(\mathbf{H}, f)$ when $b \prec a$. ■

Lemmas 3.1 and 3.2 begin to give us a good idea as to the general structure of monotone $\mathbf{P}$-zigzags. They are connected posets with a central, connected subposet of noncoloured elements with single coloured elements attached in covering pairs to this central subposet. Consider again the zigzags given in Examples 1.2 and 1.3 and ahead in Figure 3.2.

This added structure comes in very handy in the following chapter but it also allows us to prove the next lemma. Before that, though, we must define a *monotone map* between two $\mathbf{P}$-coloured posets $(\mathbf{H}', f')$ and $(\mathbf{H}, f)$ to be a monotone map $g : \mathbf{H}' \to \mathbf{H}$ such that for all $x \in C(\mathbf{H}', f')$, we have $f'(x) = f(g(x))$ and $g(N(\mathbf{H}', f')) \subseteq N(\mathbf{H}, f)$. If such a monotone map exists and it is onto, then we say $(\mathbf{H}, f)$ is a *monotone image* of $(\mathbf{H}', f')$.

Lemma 3.3. *For every $\mathbf{P}$-zigzag $(\mathbf{H}, f)$, there exists a $\mathbf{P}$-zigzag $(\mathbf{H}', f')$ (possibly the same) such that*

1. $N(\mathbf{H}, f) = N(\mathbf{H}', f')$;
2. *every coloured element of $(\mathbf{H}', f')$ occurs in exactly one covering pair of $\mathbf{H}'$;*
3. *$(\mathbf{H}, f)$ is a monotone image of $(\mathbf{H}', f')$ under a map g which is the identity on $N(\mathbf{H}', f')$ and which sends covering pairs of $\mathbf{H}'$ to covering pairs of $\mathbf{H}$.*

Before we prove this, consider the posets in Figure 3.2. The $\mathbf{P}$-zigzag $(\mathbf{H}, f)$ is a monotone image of $(\mathbf{H}', f')$ under the mapping fixing the non-coloured elements and collapsing the corresponding coloured elements together.

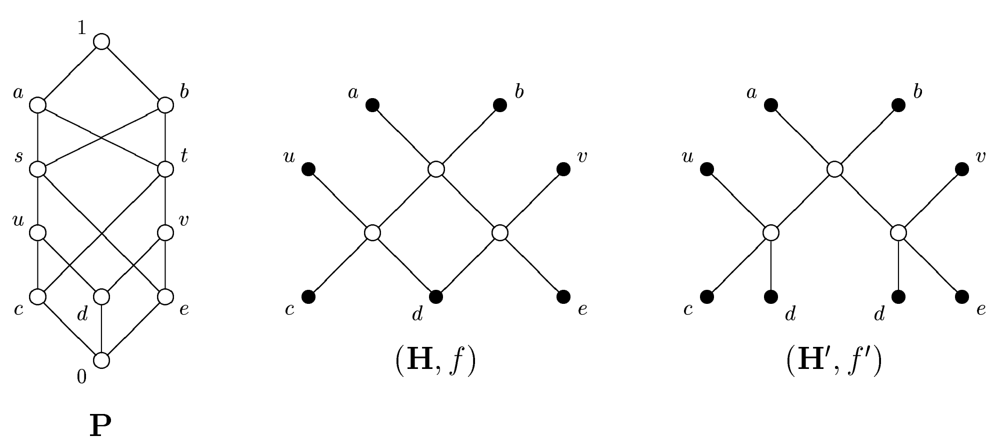


Figure 3.2: $(\mathbf{H}, f)$ as a monotone image of $\mathbf{P}$-zigzag $(\mathbf{H}', f')$

Proof. The nonmonotone case is trivial so assume for the following that $(\mathbf{H}, f)$ is a monotone $\mathbf{P}$-zigzag. We first define a certain $\mathbf{P}$-coloured poset $(\mathbf{H}', f')$, then show that it is a $\mathbf{P}$-zigzag using Lemma 3.1. First, define the following sets

$$\begin{aligned} N &= N(\mathbf{H}, f), \\ C_1 &= \{(x,y) : x \in C(\mathbf{H}, f),\ y \in N(\mathbf{H}, f),\ \text{and } x \prec_{\mathbf{H}} y\}\,,\ \text{and} \\ C_2 &= \{(x,y) : x \in N(\mathbf{H}, f),\ y \in C(\mathbf{H}, f),\ \text{and } x \prec_{\mathbf{H}} y\}\,. \end{aligned}$$

Let $H' = N \cup C_1 \cup C_2$. Next, let us define the following relation on H':

$$\begin{aligned} \prec_{\mathbf{H}'} = &\{(x,y) : x, y \in N,\ x \prec_{\mathbf{H}} y\} \\ &\cup \{((x,y),y) : (x,y) \in C_1\} \cup \{(x,(x,y)) : (x,y) \in C_2\}\,. \end{aligned}$$

Note that elements of C_1 are C_2 are now in only one pair from $\prec_{\mathbf{H}'}$.

We claim that $\prec_{\mathbf{H}'}$ does not contain any cycles. If we suppose otherwise, then there exists an element $a \in H'$ and $\{b_1, \ldots, b_n\} \subseteq H'$ such that $a \prec_{\mathbf{H}'} b_1 \prec_{\mathbf{H}'} \cdots \prec_{\mathbf{H}'} b_n \prec_{\mathbf{H}'} a$. If $a \in N$, then for some $1 \leq i \leq n$, there must be a $b_i \in C_1 \cup C_2$ since $\prec_{\mathbf{H}}$ is a proper covering relation. But if $b_i \in C_1$, then there does not exist an element $z \in H'$ with $z \prec_{\mathbf{H}'} b_i$ while if $b_i \in C_2$, then there does not exist an element $z \in H'$ with $b_i \prec_{\mathbf{H}'} z$. If $a \in C_1$, then $b_1 \in N$ and we eventually run into the same contradiction and if $a \in C_2$, then a is not covered by any elements. Hence $\prec_{\mathbf{H}'}$ does not contain any cycles and generates a partial order H'.

Now we claim that if $a \prec_{\mathbf{H}'} b$, then there does not exist a non-empty set $\{b_1, \ldots, b_n\} \subseteq H'$ such that $a \prec_{\mathbf{H}'} b_1 \prec_{\mathbf{H}'} \cdots \prec_{\mathbf{H}'} b_n \prec_{\mathbf{H}'} b$. The case where $a \in C_2$ does not make sense so suppose $a \in C_1$. Then a is covered by only one element b so $b = b_1$. Thus by our previous cycle argument, we cannot get a chain $a \prec_{\mathbf{H}'} b_1 = b \prec_{\mathbf{H}'} \cdots \prec_{\mathbf{H}'} b_n \prec_{\mathbf{H}'} b$. If $a \in N$, then either $b \in C_2$ or $b \in N$. If $b \in C_2$, then we are done. If $b \in N$, then clearly either $\{b_1, \ldots, b_n\} \subseteq N$ and we are done since $\prec_{\mathbf{H}}$ is a proper covering relation or, for some $1 \leq i \leq n$, the element $b_i \in C_2$ and any covering chain ends. Thus, let $\leq_{\mathbf{H}'}$ be the partial relation on H' generated by $\prec_{\mathbf{H}'}$ (that is, the reflexive, transitive closure of $\prec_{\mathbf{H}'}$ in $(H')^2$.) We have proved that $\prec_{\mathbf{H}'}$ is the covering relation for $\leq_{\mathbf{H}'}$.

To finish our definition, let f' be the partial map on H' fully defined on $C_1 \cup C_2$ such that

$$f'((x,y)) = \begin{cases} f(x), & \text{if } (x,y) \in C_1, \text{ and} \\ f(y), & \text{if } (x,y) \in C_2 \end{cases}$$

to get the $\mathbf{P}$-coloured poset $(\mathbf{H}', f')$. To summarize, we have taken each element $x \in C(\mathbf{H}, f)$ and replaced it with numerous elements, specifically one for each covering pair that x is in. We note that for each copy x' of x corresponding to the covering pair $x \prec_{\mathbf{H}} y$, the covering pair $x' \prec_{\mathbf{H}'} y$ is the unique covering pair to which x' belongs. We then defined the colouring of $\mathbf{H}'$ so that $f'(x') = f(x)$ for each copy x' of x.

We now show that $(\mathbf{H}', f')$ satisfies our hypothesis. Define $g : H' \to H$ such that

$$g(z) = \begin{cases} z, & \text{if } z \in N, \\ x, & \text{if } z = (x, y) \in C_1, \text{ and} \\ y, & \text{if } z = (x, y) \in C_2. \end{cases}$$

By our definition of H', it should be clear that g is onto. As well, g preserves colourings since for $(x, y) \in C_1$ we have $f(g((x, y))) = f(x) = f'((x, y))$ and similarly for C_2. So we only need show is that g is monotone. It suffices to prove that $x \prec_{\mathbf{H}'} y$ implies that $g(x) \leq_{\mathbf{H}} g(y)$. In fact, we can easily see that $x \prec_{\mathbf{H}'} y$ implies $g(x) \prec_{\mathbf{H}} g(y)$. Thus g is monotone and by our previous observations, $(\mathbf{H}, f)$ is a monotone image of $(\mathbf{H}', f')$.

Now we need to prove that $(\mathbf{H}', f')$ is a $\mathbf{P}$-zigzag. Note that the subgraph spanned by $N(\mathbf{H}', f')$ in the covering graph of $\mathbf{H}'$ is equal to the subgraph spanned by $N(\mathbf{H}, f)$ in the covering graph of $\mathbf{H}$. By Lemma 3.2, the latter is connected so $\mathbf{N}(\mathbf{H}', f')$ must be as well. Then, every other element of $\mathbf{H}'$ is in $C_1 \cup C_2$ and either has an upper or lower cover in N. Hence these elements are comparable to something in N and the comparability graph of $\mathbf{H}'$ is connected. Thus $(\mathbf{H}', f')$ is connected.

Next, suppose for a contradiction that $(\mathbf{H}', f')$ were extendible. Then there would be a monotone extension of f' preserving every covering pair. Without loss of generality, for every $y \in N(\mathbf{H}', f')$, we can then define $f'(y)$ such that $f'(x') \leq f'(y)$ for all $x' \prec_{\mathbf{H}'} y$. But then $f(x) = f'(x') \leq f'(y)$ for all $x \prec_{\mathbf{H}} y$ and this will extend f as well. This is a contradiction so $(\mathbf{H}', f')$ must be non-$\mathbf{P}$-extendible.

Finally, suppose we cancel a covering pair (a, b) in $(\mathbf{H}', f')$. Notice that $g(a) \prec_{\mathbf{H}} g(b)$, so the coloured poset $(\mathbf{H}, f) \setminus \{(g(a), g(b))\}$ is $\mathbf{P}$-extendible since $(\mathbf{H}, f)$ is a $\mathbf{P}$-zigzag. Let h be the extension of f to $N(\mathbf{H}, f)$. Then, since $\mathbf{N}(\mathbf{H}', f') = \mathbf{N}(\mathbf{H}, f)$, the map h is monotone on $\mathbf{N}(\mathbf{H}', f')$ as well. Now without loss of generality, if we consider any cover pair (w, z) with $w \in C_1$ and $z \in N$, we get $f'(w) = f(g(w)) \leq h(z)$. Thus h is a monotone extension of f' to N and $(\mathbf{H}', f') \setminus \{(a, b)\}$ is $\mathbf{P}$-extendible. Therefore, by Lemma 3.1 the $\mathbf{P}$-coloured poset $(\mathbf{H}', f')$ is a $\mathbf{P}$-zigzag. ■

Call a coloured poset for which every coloured element is in exactly one covering pair a *standard coloured poset*.

3.2 Bounding the diameter of a P-zigzag

Before we prove the main result of this chapter, we need a lemma relating the diameter of monotone $\mathbf{P}$-zigzags to the diameter of their monotone images.

Lemma 3.4. *Let $(\mathbf{H}, f)$ be a monotone $\mathbf{P}$-zigzag and let $(\mathbf{H}', f')$ be the $\mathbf{P}$-zigzag given by Lemma 3.3. The diameter of $(\mathbf{H}', f')$ is greater than or equal to the diameter of $(\mathbf{H}, f)$.*

Proof. Let g be the onto, monotone map from $(\mathbf{H}', f')$ to $(\mathbf{H}, f)$. Suppose the diameter of $\mathbf{H}$ is equal to n. Then there exists elements a and b in $\mathbf{H}$ with distance n between them. In $\mathbf{H}'$, pick some $a' \in g^{-1}(a)$ and $b' \in g^{-1}(b)$. Now suppose $d(a', b') = k < n$ in $\mathbf{H}'$. Then there exists a $(k-1)$-element subfence between a' and b'. But since g is onto and monotone, this implies that there exists a $(k-1)$-element subfence in $\mathbf{H}$ between $g(a') = a$ and $g(b') = b$. This contradicts $d(a, b) = n$ so $d(a', b') \geq n$. Hence the diameter of $\mathbf{H}'$ must be greater than or equal to the diameter of $\mathbf{H}$. ■

We now have enough information to put another piece together in the main argument of this thesis. In order to do so, define an *up set* of $\mathbf{P}$ to be a subset S of P such that, for all $x \in S$ and $y \in P$, if $x \leq y$, then $y \in S$. Define a *down set* of $\mathbf{P}$ dually. We now prove in the next theorem that there exists a bound on the diameter of all $\mathbf{P}$-zigzags when $\mathbf{P}$ satisfies (2) of Theorem 1.1.

Theorem 3.5. *If for some n, there exists a partially defined, monotone n-nuf on $\mathbf{P}^n$ that is fully defined on $A_n = \left\{(x, \ldots, x, \underset{i}{y}, z, \ldots, z) : x, y, z \in P, 1 \leq i \leq n\right\}$, then there exists a finite m such that every $\mathbf{P}$-zigzag has a diameter at most m.*

Proof. Suppose that there is a $\mathbf{P}$-zigzag $(\mathbf{Q}, g)$ with a diameter of at least $n+2$. Clearly $(\mathbf{Q}, g)$ is a monotone $\mathbf{P}$-zigzag. We shall prove that this yields a contradiction hence giving a bound on the diameters of all $\mathbf{P}$-zigzags. By Lemma 3.3, we can find a standard monotone $\mathbf{P}$-zigzag $(\mathbf{H}, f)$ with monotone image $(\mathbf{Q}, g)$ such that $N(\mathbf{H}, f) = N(\mathbf{Q}, g)$. Then, by Lemma 3.4, the diameter of $(\mathbf{H}, f)$ is at least $n+2$. Hence, since $(\mathbf{H}, f)$ is standard, the diameter d of $\mathbf{N}(\mathbf{H}, f)$ satisfies $d \geq n$.

Now, this means there exists some a and b in $N(\mathbf{H}, f)$ such that $d(a, b) = d$. If we cancel the element a, then we can find a monotone extension of $(\mathbf{H}, f) \setminus \{a\}$. Call it f_a. Similarly, let f_b be a monotone extension of $(\mathbf{H}, f) \setminus \{b\}$. In addition, we need to define, for $1 \leq i \leq d+1$,

$$B_i = \{h \in N(\mathbf{H}, f) : h \text{ has down distance } i \text{ from } a \in \mathbf{N}(\mathbf{H}, f)\}.$$

Clearly $a \in B_1$. Also, define $d_0 = d$ if $B_{d+1} = \emptyset$ and let $d_0 = d+1$ otherwise. Notice now that, for $1 \leq i \leq d_0$, the sets $B_1, \ldots, B_{d_0}$ are non-empty and form a partition of $N(\mathbf{H}, f)$. As well, we have $b \in B_{d_0}$.

We now make two important observations about $B_1, \ldots, B_{d_0}$. The first observation is that, for $1 \leq i \leq d_0$, when i is odd the set B_i is a down-set and when i is even B_i is an up set. We argue the odd case only as the even case follows dually. Suppose i is odd and $x \leq y$ for some $x \in N(\mathbf{H}, f)$ and $y \in B_i$. Then there exists $\{a_0, \ldots, a_i\} \subseteq N(\mathbf{H}, f)$ such that $a = a_0 \geq a_1 \leq \cdots \leq a_{i-1} \geq a_i = y \geq x$. Hence $a = a_0 \geq a_1 \leq \cdots \leq a_{i-1} \geq x$ and the down distance from a to x is at most i. Suppose the down distance from a to x is some $j < i$. If j is odd, then $j+1 < i$ so there exists $\{b_0, \ldots, b_j\} \subseteq N(\mathbf{H}, f)$ such that $a = b_0 \geq b_1 \leq \cdots \leq b_{j-1} \geq b_j = x \leq y$. This would imply that the down-distance from a to y is less than or equal to $j+1$ which is a contradiction. If we suppose j is even, then there exists $\{c_0, \ldots, c_j\} \subseteq N(\mathbf{H}, f)$ such that $a = c_0 \geq c_1 \leq \cdots \leq c_j = x \leq y$ which yields a similar contradiction. Hence it must be that $x \in B_i$ and it follows that B_i is a down set.

Our second observation is that when $2 \leq i \leq d_0$ the set $N(\mathbf{H}, f) \backslash \bigcup_{j=1}^{i} B_j$ and $\bigcup_{j=1}^{i-1} B_j$ span two subposets that are not connected in $\mathbf{N}(\mathbf{H}, f) \backslash B_i$. If we suppose otherwise, then for some $i < j \leq d_0$ and $x \in B_j$ there would exist a $y \in \bigcup_{j=1}^{i-1} B_j$ with either $x < y$ or $y < x$. This would imply that the down distance from a to x is either less than or equal to $i-1$ or $(i-1)+1 = i$ which is a contradiction. Hence our second observation holds.

We now have enough information to define a monotone map from $\mathbf{H}$ to $\mathbf{P}$ which will extend $(\mathbf{H}, f)$ and give us the contradiction necessary to complete the proof. For $1 \leq i \leq d_0$, define $g_i : \mathbf{H} \to \mathbf{P}$ in the following way. For $x \in N(\mathbf{H}, f)$, and $i \geq 1$, define

$$g_i(x) = \begin{cases} f_b(x), & \text{if } x \in \bigcup_{j=1}^{i-1} B_j \\ f_a(x), & \text{if } x \in N(\mathbf{H}, f) \backslash \bigcup_{j=1}^{i} B_j \\ 0, & \text{if } x \in B_i \text{ and } i \text{ is odd} \\ 1, & \text{if } x \in B_i \text{ and } i \text{ is even.} \end{cases}$$

For each $x \in C(\mathbf{H}, f)$, by Lemma 3.2 there is a unique covering pair with element $y \in N(\mathbf{H}, f)$ so this allows us to define

$$g_i(x) = \begin{cases} 0, & \text{if } x \in C(\mathbf{H}, f),\ y \in B_i, \text{ and } i \text{ is odd} \\ 1, & \text{if } x \in C(\mathbf{H}, f),\ y \in B_i, \text{ and } i \text{ is even} \\ f(x), & \text{if } y \in N(\mathbf{H}, f) \backslash B_i. \end{cases}$$

Since $B_1, \ldots, B_{d_0}$ are mutually disjoint, the functions g_i are well-defined on H.

We claim that they are monotone as well. Suppose $w <_{\mathbf{H}} z$ and i is odd. Then

$$g_i(x) = \begin{cases} f_b(x), & \text{if } x \in \bigcup_{j=1}^{i-1} B_j \\ f_a(x), & \text{if } x \in N(\mathbf{H}, f) \backslash \bigcup_{j=1}^{i} B_j \\ 0, & \text{if } x \in B_i \\ 0, & \text{if } x \in C(\mathbf{H}, f),\ y \in B_i, \text{ and } x \prec_{\mathbf{H}} x' \text{ or } x' \prec_{\mathbf{H}} x \\ f(x), & \text{otherwise.} \end{cases}$$

We have a few cases to consider.

Case 1: Suppose $z \in B_j$. Then either $w \in N(\mathbf{H}, f)$, so since B_i is a down-set, $w \in B_i$ or $w \prec_{\mathbf{H}} w' \in B_i$. Either way, $g_i(w) = 0 \leq_{\mathbf{P}} 0 = g_i(z)$.

Case 2: If $z \in C(\mathbf{H}, f)$ and $z \prec_{\mathbf{H}} z' \in B_i$, then $w <_{\mathbf{H}} z$ doesn't make sense since $(\mathbf{H}, f)$ is standard.

Case 3: If $z \in C(\mathbf{H}, f)$ and $z' \prec_{\mathbf{H}} z$ for some $z' \in B_i$, then since z is in only one covering pair, $w \leq z'$. By a similar argument to Case 1, $g_i(w) = 0$.

We are now able to assume that w and z are in cases 1, 2, or 5 of the definition of g_i. Moreover, we can assume they are in different cases otherwise we can can just use the monotonicity of f_a, f_b, and f. Now by our second observation, it is not possible to have one of w and z in $x \in \bigcup_{j=1}^{i-1} B_j$ and the other in $N(\mathbf{H}, f) \setminus \bigcup_{j=1}^{i} B_j$. Hence one of w and z is in the first two cases of g_i and one is in the fifth. Thus without loss of generality, if w in $\bigcup_{j=1}^{i-1} B_j$, then z is also must be in the domain of f_a and we can just use the monotonicity of f_a. If one of w and z is in $N(\mathbf{H}, f) \setminus \bigcup_{j=1}^{i} B_j$, then we just use a similar argument. This covers all possible cases so we always have $g_i(w) \leq_{\mathbf{P}} g_i(z)$ when $w <_{\mathbf{H}} z$. Therefore g_i is a monotone map from $\mathbf{H}$ to $\mathbf{P}$ when i is odd and the even case follows dually.

The functions g_i are monotone but do not necessarily extend f. We need to use our original assumption to complete the proof. Recall that there exists monotone partial n-nuf that is fully defined on A_n. Call it M_n. Since $d_0 \geq n$, define the map M_{d_0} on A_{d_0} such that $M_{d_0}(\overline{x}_{d_0}) = M_{d_0}(x_1, \ldots, x_{d_0}) = M_n(x_1, \ldots, x_n) = M_n(\overline{x}_n)$. It is easy to see that M_{d_0} is a partial d_0-nuf fully defined on A_{d_0} since, for $n < i \leq d_0$,

$$M_{d_0}(x, \ldots, x, \underset{i}{y}, x, \ldots, x) = M_n(x, \ldots, x) = x.$$

Is it monotone? Yes, for if $\overline{k}_{d_0} = (k, \ldots, k, \underset{i}{l}, m, \ldots, m) \leq (o, \ldots, o, \underset{j}{p}, q, \ldots, q) = \overline{o}_{d_0}$ in $\mathbf{P}^{d_0}$, then $\overline{k}_n \leq \overline{o}_n$ in $\mathbf{P}^n$, and hence $M_{d_0}(\overline{k}_{d_0}) = M_n(\overline{k}_n) \leq M_n(\overline{o}_n) = M_{d_0}(\overline{o}_{d_0})$.

Now, we claim that the monotone map $M_{d_0}(g_1, \ldots, g_{d_0}) : \mathbf{H} \to \mathbf{P}$ extends f. We must first ask if it is actually even a well-defined, total map on $\mathbf{H}$. More specifically, is $(g_1(x), \ldots, g_{d_0}(x)) \in A_{d_0}$ for all $x \in \mathbf{H}$? Suppose $x \in N(\mathbf{H}, f)$. Then there exists an i such that $x \in B_i$. Consequently, we have

$$(g_1(x), \ldots, g_{d_0}(x)) = (f_b(x), \ldots, f_b(x), g_i(x), f_a(x), \ldots, f_a(x)) \in A_{d_0}.$$

Otherwise, if $x \in C(\mathbf{H}, f)$, then for some $1 \leq i \leq d_0$, the element x is in a covering pair with an element in B_i. Hence

$$(g_1(x), \ldots, g_{d_0}(x)) = (f(x), \ldots, f(x), g_i(x), f(x), \ldots, f(x)) \in A_{d_0}.$$

Moreover,

$$M_{d_0}(g_1(x), \ldots, g_{d_0}(x)) = M_{d_0}(f(x), \ldots, f(x), g_i(x), f(x), \ldots, f(x)) = f(x)$$

which shows that our map restricts to f on $C(\mathbf{H}, f)$. Thus we have a monotone extension of f to $\mathbf{P}$ which contradicts the fact that $(\mathbf{H}, f)$ is a $\mathbf{P}$-zigzag. Consequently, our assumption that there exists a $\mathbf{P}$-zigzag with diameter of at least $n+2$ must be false. This completes the proof. ■

Theorem 3.5 gives us a bound on the diameters of $\mathbf{P}$-zigzags and we will use this to show in Chapter 4 that there are actually only a finite number of $\mathbf{P}$-zigzags. It is important to note that in [10] Tardos gives an example of a finite, bounded poset $\mathbf{P}$ with $\mathbf{P}$-zigzags which can be constructed to have diameters of any size. These can be seen in Figure 3.3 on the following page.

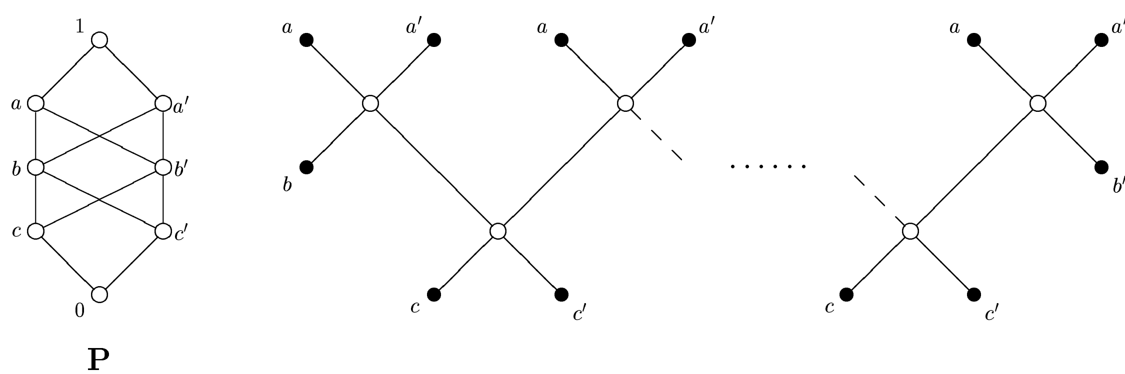


Figure 3.3: A **P**-zigzag with unbounded diameter

Chapter 4

Bounding the number of zigzags of a finite, bounded poset

We begin this chapter with various lemmas that give us a better idea of the shape of zigzags of finite, bounded posets. Using those results, we proceed with a fairly long and technical proof that a finite, bounded poset **P** with arbitrarily large **P**-zigzags must have **P**-zigzags with arbitrarily large diameter. For us, the useful consequence is that **P** can have at most finitely many **P**-zigzags if it has a finite bound on the diameter of its zigzags. Then, in the last section we complete the proof that a finite, bounded poset **P** admits Jónsson operations if and only if it admits a near-unanimity function. Note again that for the following chapter, **P** is always a finite, bounded poset.

4.1 The shape of a zigzag

In Section 1.3, we started to get some idea of the structure of **P**-zigzags. In the following lemmas in this section, we fill in the details. The first lemma of this chapter gives us more information about the colouring of elements in a **P**-zigzag.

Lemma 4.1. *Let* $(\mathbf{H}, f)$ *be a monotone* **P***-zigzag and let* a *and* b *be two distinct elements in* $C(\mathbf{H}, f)$*.*

1. *If* $a < b$*, then* $f(a) \neq f(b)$*.*
2. *If there exists a* $c \in N(\mathbf{H}, f)$ *such that* $c \prec a$ *and* $c \prec b$*, then* $f(a) \nleq f(b)$*.*

Proof. 1. Suppose $a < b$ but $f(a) = f(b)$. Lemma 3.2 tells us that there must be a $c \in N(\mathbf{H}, f)$ such that $a < c < b$. If we consider the **P**-coloured poset given when c is cancelled from $(\mathbf{H}, f)$, we can find a monotone extension f'. Since f' preserves f, we can put c back and colour it such that $f(a) = f(c) = f(b)$. But this will yield a **P**-extension of $(\mathbf{H}, f)$ which cannot happen. Hence $f(a) \neq f(b)$.

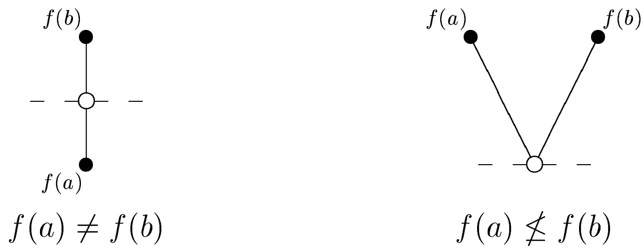


Figure 4.1: Lemma 4.1

2. Let $c \prec a, b$ but suppose $f(a) \leq f(b)$. If we consider $(\mathbf{H}, f) \setminus \{(c, b)\}$, then we can extend f fully to all of $\mathbf{H}$. Since $f(c) \leq f(a) \leq f(b)$, we can put (c, b) back and we will have a $\mathbf{P}$-extension of $(\mathbf{H}, f)$. This is a contradiction so $f(a) \not\leq f(b)$.

■

Define, for any poset $\mathbf{Q}$, an element $a \in \mathbf{Q}$ to be *retractable* if we can find some non-onto monotone map on $\mathbf{Q}$ that is the identity on $Q \setminus \{a\}$. The next lemma tells that only the coloured elements of a monotone zigzag are retractable.

Lemma 4.2. *If* $(\mathbf{H}, f)$ *is a monotone* $\mathbf{P}$*-zigzag, then* $N(\mathbf{H}, f)$ *has no retractable elements of* $\mathbf{H}$.

Proof. We claim that every monotone map g from $\mathbf{H}$ to itself that fixes elements of $C(\mathbf{H}, f)$ must be onto. Suppose g is the identity on $C(\mathbf{H}, f)$ but $g(H) \subsetneq H$. Then there exists an extension f' of $g(H)$ to $\mathbf{P}$ since $(\mathbf{H}, f)$ is a $\mathbf{P}$-zigzag. But this means that, since $f' \circ g$ is monotone and restricts to f on $C(\mathbf{H}, f)$, our $\mathbf{P}$-zigzag $(\mathbf{H}, f)$ is $\mathbf{P}$-extendible. This contradiction must mean that g is onto and our claim holds. Consequently, the only possible retractable elements in $(\mathbf{H}, f)$ are the coloured elements. ■

This next lemma gives us more information about the maximal chains in a $\mathbf{P}$-zigzag.

Lemma 4.3. *In any* $\mathbf{P}$*-zigzag, the top and bottom elements of a maximal chain are coloured.*

Proof. The nonmonotone case is trivial. For the monotone case, if we suppose otherwise, then we can remove the offending element to get a $\mathbf{P}$-extendible poset. Then all we need do is put the offending element back and colour it 0 or 1 to extend $(\mathbf{H}, f)$. ■

For posets $\mathbf{Q}$ and $\mathbf{K}$, define a bijective map $f : \mathbf{Q} \to \mathbf{K}$ to be an *order-isomorphism* when, for all a and b in $\mathbf{Q}$, we have $a \leq b$ in $\mathbf{Q}$ if and only if $f(a) \leq f(b)$ in $\mathbf{K}$. The next lemma will give us the complete picture of monotone $\mathbf{P}$-zigzags with only one or two noncoloured elements. It will also provide the base case for the theorem that follows.

Lemma 4.4. *Let* $(\mathbf{H}, f)$ *be a monotone* $\mathbf{P}$*-zigzag.*

1. *If* $|N(\mathbf{H}, f)| = 1$*, then* $(\mathbf{H}, f)$ *is the first poset seen in Figure 4.2 where* m *and* n *are nonnegative integers such that* $m+n > 0$ *and* m *and* n *do not equal* 1*. Moreover,* f *is an order-isomorphism on its domain.*

2. *If* $|N(\mathbf{H}, f)| = 2$*, then* $(\mathbf{H}, f)$ *is the second poset in Figure 4.2 where* $k \geq 1$*,* $l \geq 1$*, and* m *and* n *do not equal* 1*. Moreover, the only comparable pairs of elements in the range of* f *but not seen in the figure are of the form* $d_i < c_j$*,* $c_j < b_s$*, or* $a_t < d_i$ *for* $1 \leq i \leq k$*,* $1 \leq j \leq l$*,* $1 \leq s \leq m$*, and* $1 \leq t \leq n$*.*

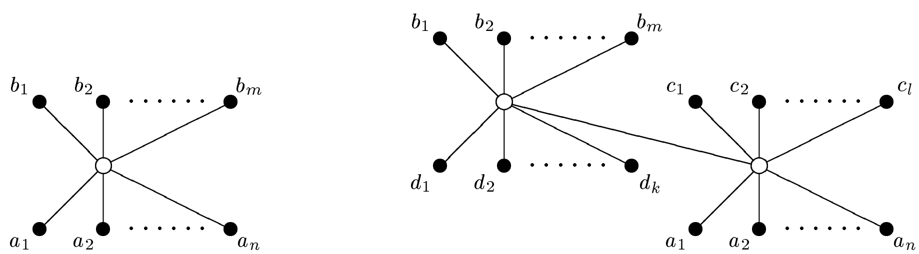


Figure 4.2: Zigzags with one or two noncoloured elements

Proof. 1. First, notice that the picture comes immediately from our assumption and Lemma 3.2. As well, it should be clear that $m + n > 0$. Also, without loss of generality, $m \neq 1$ otherwise the noncoloured element would be retractable by mapping it to b_m, contradicting Lemma 4.2. Finally, f is an order-isomorphism on its domain since, without loss of generality, for all $1 \leq i, j \leq m$ with $i \neq j$, the element $f(b_i)$ is not comparable to $f(b_j)$ in $\mathbf{P}$ by Lemma 4.1.

2. We know from Lemma 3.2 that $N(\mathbf{H}, f)$ is connected and every coloured element is in a covering pair with noncoloured. Moreover, $(\mathbf{H}, f)$ is standard since, without loss of generality, if a coloured element is above the greatest noncoloured element then it is already above the least noncoloured element. Hence it would only ever be connected in the covering graph to one noncoloured element. Thus we get the second figure seen. Now, without loss of generality, if $k = 0$ then the greatest noncoloured element would be retractable by sending it to the least noncoloured element and for similar reasons, $m, n \neq 1$.

   For the final claim, suppose there exists $f(x)$ and $f(y)$ in the range of f such that $f(x) \leq f(y)$ but $x \nleq y$ in $\mathbf{H}$. Clearly, $y \nleq x$ since f is monotone so y and x must be incomparable. By Lemma 4.1(2), there does not exist a $z \in N(\mathbf{H}, f)$ in a covering pair both with x and y. So let us suppose for some $1 \leq j \leq l$ and $1 \leq s \leq m$ that $f(x) = b_s$ and $f(y) = c_j$. Then $c_j \geq b_s$ so in $(\mathbf{H}, f) \backslash \{c_j\}$ any colouring of the least noncoloured element will still have to preserve the same inequalities as $(\mathbf{H}, f)$. Thus $(\mathbf{H}, f) \backslash \{c_j\}$ is not

P-extendible which contradicts the minimality of $(\mathbf{H}, f)$. Hence $c_j \not\geq b_s$ and using similar arguments for the other cases, we are left only with the strict inequalities given in the hypothesis.

■

We now have a complete picture of **P**-zigzags with one or two noncoloured elements. Define the *length* between two elements a and b in a poset $\mathbf{Q}$ to be the maximum cardinality of all chains between a and b and denote it $l_{\mathbf{Q}}(a, b)$. The next theorem lets us produce standard monotone **P**-zigzags whose maximal chains have lengths with bounds related to the lengths of the corresponding chains in **P**.

Theorem 4.5. *For every monotone* **P***-zigzag* $(\mathbf{H}, f)$*, there exists a standard monotone* **P***-zigzag* $(\mathbf{H}', f')$ *such that* $(\mathbf{H}, f)$ *is a monotone image of* $(\mathbf{H}', f')$ *and for every maximal chain* $a = a_1 < \cdots < a_n = b$ *in* $\mathbf{H}'$*, we have* $n \leq l_{\mathbf{P}}(f'(a), f'(b)) + 1$.

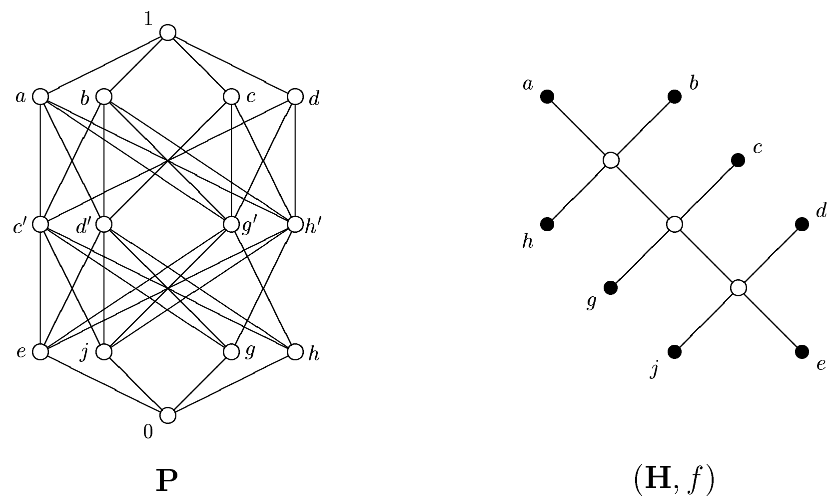


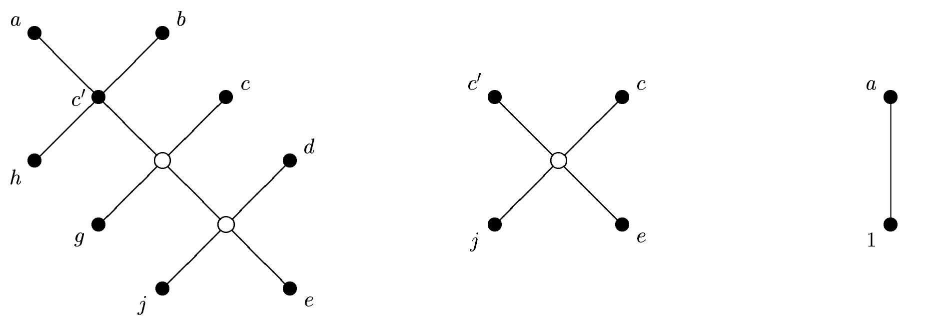
Figure 4.3: The length $l_{\mathbf{H}}(a, e) = 5 \not\leq 4 = l_{\mathbf{P}}(f(a), f(e)) + 1$

Proof. First recall that, by Lemma 4.3, the top and bottom elements of a maximal chain in a **P**-zigzag are coloured. Hence it is reasonable to consider the length between their images in **P**.

Now, the main idea of this proof is an induction on the cardinality of $N(\mathbf{H}, f)$. The base case $|N(\mathbf{H}, f)| = 1$ follows quickly from Lemma 4.4 when we notice that the zigzag is already standard and every maximal chain has at most 3 elements. Hence for any two elements a and b in $C(\mathbf{H}, f)$ we have $f(a) \neq f(b)$ so certainly $3 \leq l_{\mathbf{P}}(f(a), f(b)) + 1$. Thus let $(\mathbf{H}, f)$ be a **P**-zigzag such that $|N(\mathbf{H}, f)| = m \geq 2$. As our inductive hypothesis, suppose that every **P**-zigzag with $m - 1$ noncoloured elements satisfies the theorem. We produce a **P**-zigzag $(\mathbf{H}', f')$ satisfying the claim using the following steps:

Step 1: Fix some maximal element $h \in \mathbf{N}(\mathbf{H}, f)$. For each $p \in \mathbf{P}$, let $f_p^+(h) = p$ and $f_p^+(c) = f(c)$ for $c \in C(\mathbf{H}, f)$ to get a new **P**-coloured poset $(\mathbf{H}, f_p^+)$. Then, since

$(\mathbf{H}, f)$ is nonextendible, $(\mathbf{H}, f_p^+)$ must be nonextendible so we can find a $\mathbf{P}$-zigzag $(\mathbf{H}_p, f_p)$ contained in it. Moreover, we can assume $h \in \mathbf{H}_p$ since otherwise $(\mathbf{H}, f)$ would properly contain a $\mathbf{P}$-zigzag. Now, if there exists a nonmonotone zigzag contained in the poset, then let that be $(\mathbf{H}_p, f_p)$. Let t_p be the monotone map embedding $(\mathbf{H}_p, f_p)\backslash \{h\}$ in $(\mathbf{H}, f)$. (Consider the poset $\mathbf{P}$ and $\mathbf{P}$-zigzag $(\mathbf{H}, f)$ in Figure 4.3 and then consider the corresponding coloured posets and $\mathbf{P}$-zigzags in Figure 4.4.)



$(\mathbf{H}, f)$ with $f(h) = c'$ $\quad$ $(\mathbf{H}_{c'}, f_{c'})$ $\quad$ $(\mathbf{H}_1, f_1)$

Figure 4.4: First step of Theorem 4.5

Step 2: If $(\mathbf{H}_p, f_p)$ is nonmonotone, then let $(\mathbf{Q}_p, g_p)$ be the same coloured poset as $(\mathbf{H}_p, f_p)$. Otherwise, f_p will be monotone and $(\mathbf{H}_p, f_p)$ will satisfy the inductive hypothesis. Hence there exists a standard $\mathbf{P}$-zigzag $(\mathbf{Q}_p, g_p)$ with monotone image $(\mathbf{H}_p, f_p)$ that satisfies our claims about maximal chain lengths. For each $p \in \mathbf{P}$, let s_p be the onto monotone map taking $(\mathbf{Q}_p, g_p)$ to $(\mathbf{H}_p, f_p)$. It is important to note that, in the monotone case, all the elements in $(\mathbf{Q}_p, g_p)$ mapping to $h \in (\mathbf{H}_p, f_p)$ under s_p are maximal in $(\mathbf{Q}_p, g_p)$. (In our example from Figure 4.4, every $(\mathbf{H}_p, f_p) = (\mathbf{Q}_p, g_p)$ for all $p \in \mathbf{P}$ and each one looks like one of the two $\mathbf{P}$-zigzags shown.)

Step 3: Now let us create a new standard $\mathbf{P}$-coloured poset $(\mathbf{Q}, g)$. We do this by gluing all the elements which are preimages of h in each $(\mathbf{Q}_p, g_p)$ together in a noncoloured point h' but preserve all the other colourings and orders of the $\mathbf{P}$-zigzags. In more detail, let $Q = \{h'\} \bigcup_{p \in P} Q_p \backslash \left\{s_p^{-1}(h)\right\}$ with the colourings inherited from each g_p (except on h'). Define the covering relation of $\mathbf{Q}$ to be

$$\begin{aligned}\prec_{\mathbf{Q}} = \bigcup_{p \in P} \left\{\prec_{\mathbf{Q}_p \backslash \left\{s_p^{-1}(h)\right\}}\right\} &\cup \left\{(x, h') : x \prec_{\mathbf{Q}_p} y \text{ for some } p \in P, y \in s_p^{-1}(h)\right\} \\ &\cup \left\{(h', x) : y \prec_{\mathbf{Q}_p} x \text{ for some } p \in P, y \in s_p^{-1}(h)\right\}.\end{aligned}$$

Then, using the fact that cancelling h' from $(\mathbf{Q}, g)$ would only disconnected components with distinct partial relations, it is straightforward to show that $\prec_{\mathbf{Q}}$ generates a partial relation $\leq_{\mathbf{Q}}$ on $\mathbf{Q}$. (See Figure 4.5, a continuation of the example in Figure 4.4.)

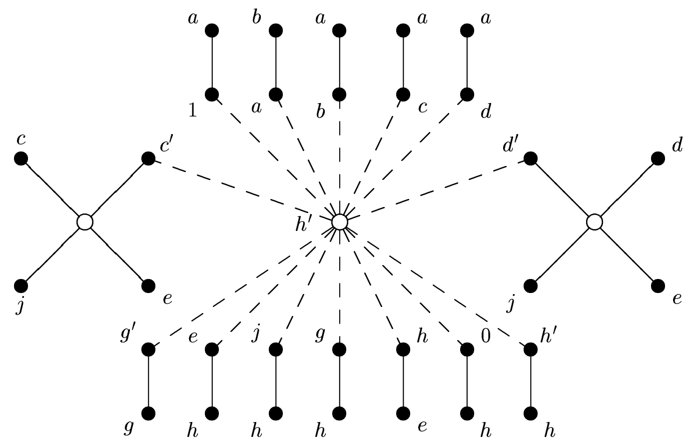
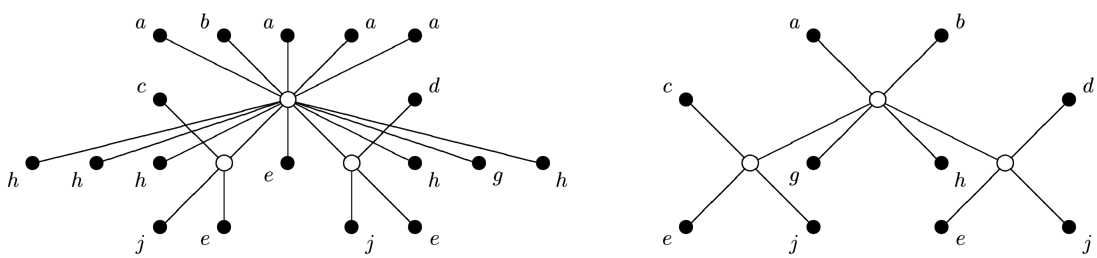


Figure 4.5: Third step of Theorem 4.5

Step 4: Finally, notice that $(\mathbf{Q}, g)$ is not extendible as any colouring of h' by some $p \in \mathbf{P}$ will cause $(\mathbf{Q}_p, g_p) \subseteq (\mathbf{Q}, g)$. Hence $(\mathbf{Q}, g)$ must contain a $\mathbf{P}$-zigzag, call it $(\mathbf{Q}', g')$, and we can standardize $(\mathbf{Q}', g')$ using Lemma 3.3 to get a standard $\mathbf{P}$-zigzag $(\mathbf{H}', f')$. Let q be the monotone map embedding $(\mathbf{Q}', g')$ into $(\mathbf{Q}, g)$ and let r be the monotone map from $(\mathbf{H}', f')$ onto $(\mathbf{Q}', g')$. (See Figure 4.6.)



$(\mathbf{Q}, g)$ $(\mathbf{H}', f') = (\mathbf{Q}', g')$

Figure 4.6: Fourth step of Theorem 4.5

We claim that this new $\mathbf{P}$-zigzag $(\mathbf{H}', f')$ has $(\mathbf{H}, f)$ as a monotone image. Let $m : (\mathbf{H}', f') \to (\mathbf{H}, f)$ be defined such that

$$m(x) = \begin{cases} h, & \text{if } x = h' \\ t_p(s_p(q(r(x)))), & \text{if } x \in (\mathbf{Q}_p, g_p) \setminus \{h'\}. \end{cases}$$

On $C(\mathbf{H}', f')$, the map m is simply a composition of monotone maps so it clearly preserves colourings. To show monotonicity between $\mathbf{H}'$ and $\mathbf{H}$, suppose $a < b$ in $(\mathbf{H}', f')$. If, for some $p \in \mathbf{P}$, the elements $q(r(a))$ and $q(r(b))$ are in the same component $(\mathbf{Q}_p, g_p) \setminus s_p^{-1}(h)$ of $\mathbf{Q}$ or $a = h$ or $b = h$, then $m(a) \leq m(b)$ by our construction of $\mathbf{Q}$. If $q(r(a)) \in (\mathbf{Q}_{p_1}, g_{p_1}) \setminus s_{p_1}^{-1}(h)$ and $q(r(b)) \in (\mathbf{Q}_{p_2}, g_{p_2}) \setminus s_{p_2}^{-1}(h)$, then $m(a) = t_{p_1}(s_{p_1}(q(r(a)))) < h < t_{p_2}(s_{p_2}(q(r(b)))) = m(b)$. To see that it is onto, just notice that if $m(\mathbf{H}', f') \subsetneq (\mathbf{H}, f)$, then $m(\mathbf{H}', f')$ would be extendible causing $(\mathbf{H}', f')$ to be as well. Thus m is an onto, monotone map from $(\mathbf{H}', f')$ to $(\mathbf{H}, f)$.

Finally, to complete our induction we show that $(\mathbf{H}', f')$ satisfies our claims about the lengths of maximal chains. Let $a = a_1 < \cdots < a_n = b$ be a maximal chain in $(\mathbf{H}', f')$. If $a_i \neq h'$ for all i, then it must be that $q(r(\{a_i\}_{i=1}^n)) \subseteq (\mathbf{Q}_p, g_p)$ for some p. Otherwise, for some p_1 and p_2 there exists a j with $q(r(a_j)) \in (\mathbf{Q}_{p_1}, g_{p_1})$ and $q(r(a_{j+1})) \in (\mathbf{Q}_{p_2}, g_{p_2})$. Hence $a_j < h < a_{j+1}$ and our chain would not be maximal. Thus, either $n = 1$ and we are done or by our inductive hypothesis, $n \leq l_{\mathbf{P}}(f'(a), f'(b)) + 1$.

However, if h' is in our maximal chain, then by our construction it must be that $a_{n-1} = h'$ (the preimages of h in $(\mathbf{Q}_p, g_p)$ are maximal elements). This means that if $(\mathbf{Q}_p, g_p)$ is monotone, then $a = a_1 < \cdots < a_{n-1} = h'$ is maximal in the preimage of some $(\mathbf{H}_p, f_p)$ for some p so by our induction applied to $(\mathbf{Q}_p, g_p)$, we have $n - 1 \leq l_{\mathbf{P}}(g_p(a), g_p(h)) + 1 = l_{\mathbf{P}}(g_p(a), f_p(h)) + 1$. Now notice that, since $(\mathbf{H}_p, f_p)$ is monotone, it must be that $f_p(h) \leq f'(b)$. Since h is a maximal noncoloured element in $(\mathbf{H}, f)$, by Lemma 4.1, if there was more than one coloured element above h, then $f_p(h) \lneq f'(b)$. On the other hand, if h has only one coloured cover in $(\mathbf{H}, f)$, then h is retractable which contradicts Lemma 4.2. Thus $f_p(h) < f'(b)$. Since $f'(a) = g_p(a)$, this gives $l_{\mathbf{P}}(g_p(a), f_p(h)) + 1 \leq l_{\mathbf{P}}(f'(a), f'(b))$ and, combining this with the previous inequality, we get $n \leq l_{\mathbf{P}}(f'(a), f'(b)) + 1$. In the case where $(\mathbf{Q}_p, g_p)$ is nonmonotone, we have $n = 3$ and since a and b are distinct it follows immediately. ■

4.2 Finitely many P-zigzags

We now have enough details to proceed with the most technical part of this thesis. The idea of Theorem 4.9 is this: we pick a **P**-zigzag of size at least $\sum_{i=0}^{m+1} k^{ik}$. Using the following series of lemmas, we will recursively define a series of **P**-zigzags of strictly increasing minimum diameter.

We need some preliminary lemmas which will help us prove Theorem 4.9. Before we get to the first lemma, for a finite poset $\mathbf{Q}$ let us define $l(\mathbf{Q})$ to be the number of elements of a chain in $\mathbf{Q}$ with maximum cardinality. As well, for any $a \in \mathbf{Q}$, let $l_{\mathbf{Q}}(a)$ be the number of elements in a chain with top element a of maximum cardinality and note that, for all $a \in \mathbf{Q}$, we have $l_{\mathbf{Q}}(a) \leq l(\mathbf{Q})$. As well, for any subset $S \subseteq Q$, the set S_* is the set of all lower bounds of S in $\mathbf{Q}$.

This first lemma tells us that for every down-set D of a **P**-zigzag $(\mathbf{H}, f)$, we can find another **P**-zigzag $(\mathbf{H}', f')$ which preserves most of $(\mathbf{H}, f)$ but now has the added property that in the corresponding down-set D', every element has at most k lower covers.

Lemma 4.6. *Let $|\mathbf{P}| = k$, let $(\mathbf{H}, f)$ be a **P**-zigzag, and let D be a down-set of $\mathbf{H}$. There exist a **P**-zigzag $(\mathbf{H}', f')$, down-set D' in $\mathbf{H}'$, and onto, monotone map g such that $(\mathbf{H}, f)$ is the monotone image of $(\mathbf{H}', f')$ under g and the following hold:*

(a) $\mathbf{H}' \backslash D' = \mathbf{H} \backslash D$,

(b) $g(u) = u$ *for all* $u \in H' \backslash D'$,

(c) $g(D') = D$,

(d) $|\{d'\}_*| < k^{l_{\mathbf{H}}(g(d'))}$ *for all* $d' \in D'$,

(e) $l(\mathbf{H}') \leq l(\mathbf{H})$.

Proof. We proceed with the proof by using induction on the size of the down-set D. If $|D| = 0$, then just let g be the identity map and we are done. So let $|D| \geq 1$ and suppose that the induction hypothesis holds for $(\mathbf{H}, f)$ and all down-sets D' of $\mathbf{H}$ where $|D'| < |D|$. Pick some maximal element d in the poset $\mathbf{D}$ spanned by D in $\mathbf{H}$. Now, our induction hypothesis applies to $D \backslash \{d\}$ so we can find a $\mathbf{P}$-zigzag $(\mathbf{H}_0, f_0)$, a down-set D_0 in $\mathbf{H}_0$, and an onto, monotone map g_0 such that $(\mathbf{H}, f)$ is the monotone image of $(\mathbf{H}_0, f_0)$ under g_0 and the following hold:

(a′) $\mathbf{H}_0 \backslash D_0 = \mathbf{H} \backslash (D \backslash \{d\})$,

(b′) $g_0(u) = u$ for all $u \in H_0 \backslash D_0$,

(c′) $g_0(D_0) = D \backslash \{d\}$,

(d′) $|\{d_0\}_*| < k^{l_{\mathbf{H}}(g(d_0))}$ for all $d_0 \in D_0$,

(e′) $l(\mathbf{H}_0) \leq l(\mathbf{H})$.

We claim that in $\mathbf{H}_0$, the set $\{d\}_* \backslash \{d\} \subseteq D_0$. By (a′), the element $d \in \mathbf{H}_0 \backslash D_0$ and by (b′), we get $g_0(d) = d$. Now for any $x \in \{d\}_* \backslash \{d\}$, since $x \leq d$, by the monotonicity of g_0, we have $g_0(x) \leq d$. Then, since D is a down-set, it must be that $g_0(\{d\}_* \backslash \{d\}) \subseteq D$. Hence by (b′) and (c′), it follows that $\{d\}_* \backslash \{d\} \subseteq D_0$.

We have two cases.

Case 1: If d already has at most k lower covers in $\mathbf{H}_0$, then just let $(\mathbf{H}', f') = (\mathbf{H}_0, f_0)$. Thanks to our hypothesis, we only need to check that (d′) holds for $d_0 = d$. Notice that, for $d_0 < d$ in D_0 we have $g(d_0) < g(d) = d$ which implies $l_{\mathbf{H}}(g(d_0)) \leq l_{\mathbf{H}}(g(d)) - 1$ (this follows since $g(d) = d$ by (b′) and $g(d_0) \in D \backslash \{d\}$). Thus

$$|\{d\}_*| \leq \sum_{d_0 \prec d} |\{d_0\}_*| < k k^{l_{\mathbf{H}}(g(d_0))} \leq k k^{l_{\mathbf{H}}(g(d))-1} = k^{l_{\mathbf{H}}(g(d))}.$$

Therefore, in this case, $(\mathbf{H}', f')$ already satisfies (a) through (e) and we are done.

Case 2: Things are a lot trickier if d has more than k lower covers in $\mathbf{H}_0$. If so, then we must define a new $\mathbf{P}$-coloured poset $(\mathbf{H}_1, f_1)$ in the following way. Let $A_1, \ldots, A_t$ range over the k-element subsets of the set of lower covers of d. For each such set, create a new element d_i. Let $C = \{c \in H_0 : d \prec_{\mathbf{H}_0} c\}$. Define $H_1 = (H_0 \backslash \{d\}) \cup \{d_1, \ldots, d_t\}$ and define the covering relation for $\mathbf{H}_1$ to be

$$\prec_{\mathbf{H}_1} = (\prec_{\mathbf{H}_0} |_{H_0 \backslash \{d\}}) \cup \{(d_i, c) : 1 \leq i \leq t, c \in C\} \cup_{i=1}^{t} \{(a, d_i) : a \in A_i\}.$$

It is easy to see that this relation contains no cycles and will generate a proper partial relation on $\mathbf{H}_1$ whose covering relation is $\prec_{\mathbf{H}_1}$. To complete the construction, for all i, leave the d_i noncoloured if $d \in N(\mathbf{H}_0, f_0)$ and let $f_1(d_i) = f_0(d)$ otherwise.

We claim that $(\mathbf{H}_1, f_1)$ is not $\mathbf{P}$-extendible. Suppose otherwise. Then there exists some extension f_1' of f_1 to all of $\mathbf{H}_1$. Now this means $f_1'|_{\mathbf{H}_0 \setminus \{d\}}$ is an extension of f_0 to $\mathbf{H}_0 \setminus \{d\}$, since $\mathbf{H}_0 \setminus \{d\} = \mathbf{H}_1 \setminus \{d_1, \dots, d_t\}$. However, if we look at $(\mathbf{H}_0, f_1'|_{\mathbf{H}_0 \setminus \{d\}})$, we find that, since $(\mathbf{H}_0, f_0)$ is not extendible, it is also not extendible. So this means there exists some $\mathbf{P}$-zigzag $(\mathbf{Q}, g) \subseteq (\mathbf{H}_0, f_1'|_{\mathbf{H}_0 \setminus \{d\}})$ and d must be in $\mathbf{Q}$ since otherwise $f_1'|_Q$ would extend it. Now, because $(\mathbf{Q}, g) \subseteq (\mathbf{H}_0, f_1'|_{\mathbf{H}_0 \setminus \{d\}})$, we know that either $(\mathbf{Q}, g)$ is nonmonotone if $d \in C(\mathbf{Q}, g)$ or $N(\mathbf{Q}, g) = \{d\}$. In either case, by Lemma 4.4 this implies that $C(\mathbf{Q}, g) \subseteq \{d\}^* \cup \{d\}_*$. Moreover, $(\mathbf{Q}, g) \cap \{d\}_* \setminus \{d\}$ forms an antichain and, by Lemma 4.1, it has no more than k elements. But this means that d covers an antichain of no more than k elements so the $\mathbf{P}$-zigzag $(\mathbf{Q}, g)$ can be embedded by a monotone map into $(\mathbf{H}_1, f_1'|_{\mathbf{H}_1 \setminus \{d_1, \dots, d_t\}})$. This is a contradiction since $(\mathbf{H}_1, f_1'|_{\mathbf{H}_1 \setminus \{d_1, \dots, d_t\}})$ is $\mathbf{P}$-extendible but $(\mathbf{Q}, g)$ is not. Hence our original claim - that $(\mathbf{H}_1, f_1)$ is not $\mathbf{P}$-extendible - must be true. Moreover this means that there exists a $\mathbf{P}$-zigzag $(\mathbf{H}', f') \subseteq (\mathbf{H}_1, f_1)$.

Let $g_1 : \mathbf{H}_1 \to \mathbf{H}_0$ be defined such that

$$g_1(x) = \begin{cases} x, & \text{if } x \in \mathbf{H}_1 \setminus \{d_1, \dots, d_t\}, \\ d, & \text{if } x \in \{d_1, \dots, d_t\}. \end{cases}$$

It is clear from our construction that g_1 is an onto, monotone map. Now we claim that there does not exist a nonempty set T such that $(\mathbf{H}', f')$ has $(\mathbf{H}_0, f_0) \setminus T$ as a monotone image. Otherwise, if we let h be a monotone extension of f_0 to $(\mathbf{H}_0, f_0) \setminus T$, we could compose $h \circ g'|_{\mathbf{H}'}$, a monotone extension of $(\mathbf{H}', f')$. Thus $g_1(\mathbf{H}') = \mathbf{H}_0$ which implies that $\mathbf{H}' \setminus \{d_1, \dots, d_t\} = \mathbf{H}_1 \setminus \{d_1, \dots, d_t\}$ and for some i, the element $d_i \in \mathbf{H}'$. We are now able to define $g = g_0 \circ g_1|_{H'}$ and note that it is an onto, monotone map from $(\mathbf{H}', f')$ to $(\mathbf{H}, f)$. Consequently, $D' = g^{-1}(D)$ will be a down-set in $\mathbf{H}'$.

The final step to complete this proof is to show that (a) through (e) are all satisfied.

(a) Since $\mathbf{H}' \setminus \{d_1, \dots, d_t\} = \mathbf{H}_1 \setminus \{d_1, \dots, d_t\}$, it follows from (a$'$) that

$$\mathbf{H}' \setminus D' = \mathbf{H}' \setminus g^{-1}(D) = \mathbf{H}_1 \setminus g_0^{-1}(D) = \mathbf{H}_0 \setminus g_0^{-1}(D) = \mathbf{H} \setminus D.$$

(b) For all $u \in H' \setminus D'$, by (b$'$) we have $g(u) = g_0 \circ g_1|_{H'}(u) = g_0(u) = u$.

(c) This follows immediately from our definition of D'.

(d) Let $x \in D'$. Firstly, if $x \notin \{d_1, \dots, d_t\}$, then $x \in \mathbf{H}_1 \setminus \{d_1, \dots, d_t\} = \mathbf{H}_0 \setminus \{d\}$. Even better, since $x < d_i$ for some i, it must be that $x \in D_0$. Hence (d) holds thanks to (d$'$).

Now suppose $x \in \{d_1, \ldots, d_t\}$. If x is minimal in $\mathbf{H}_1$, then (d) holds since $|\{x\}_*| = 1$. Otherwise, x is not minimal and by our construction, the number of elements covered by x is between 1 and k. Now our inductive hypothesis becomes very useful when we notice that by the definition of $\mathbf{H}_1$, all elements covered by x must be in D_0. Hence each element covered by x satisfies (d′) and so, for some $x_0 \in D_0$, it must be that $|\{x\}_*| < k \cdot k^{l_{\mathbf{H}}(g(x_0))}$. Moreover, since g is monotone and $g(D' \backslash \{d_1, \ldots, d_t\}) = D \backslash \{d\}$, the element $g(x_0) <_{\mathbf{H}} g(x)$. Thus $l_{\mathbf{H}}(g(x_0)) \leq l_{\mathbf{H}}(g(x)) - 1$ and we have $|\{x\}_*| < k \cdot k^{l_{\mathbf{H}}(g(x))-1} = k^{l_{\mathbf{H}}(g(x))}$.

(e) Notice that $(\mathbf{H}', f') \subseteq (\mathbf{H}_1, f_1)$ so $l(\mathbf{H}') \leq l(\mathbf{H}_1)$. Now, from our definition of $\mathbf{H}_1$ it should be clear that $l(\mathbf{H}_1) = l(\mathbf{H}_0)$. Then by applying (e′) we are able to get $l(\mathbf{H}') \leq l(\mathbf{H}_1) = l(\mathbf{H}_0) \leq l(\mathbf{H})$.

Therefore, the $\mathbf{P}$-zigzag $(\mathbf{H}', f')$, monotone map g, and down-set D' satisfy the necessary claims and our induction holds. ■

For the next small but useful lemma, we need another definition. Let $\mathbf{Q}$ be a connected poset and let $a \in \mathbf{Q}$ and $B \subseteq \mathbf{Q}$. Define $d_{\mathbf{Q}}(a, B) = \min_{b \in B} \{d_{\mathbf{Q}}(a, b)\}$ where $d_{\mathbf{Q}}(a, b)$ is the distance between a and b in $\mathbf{Q}$.

Lemma 4.7. *Suppose $(\mathbf{H}, f)$ is a monotone image of $(\mathbf{H}', f')$ via the map g, $B' \subseteq H'$, and $a' \in H'$. Then $d_{\mathbf{H}'}(a', B') \geq d_{\mathbf{H}}(g(a'), g(B'))$.*

Proof. Pick $b' \in B'$ such that $d_{\mathbf{H}'}(a', b') = d_{\mathbf{H}'}(a', B') = n$. Then there exists an $(n+1)$-element fence in $\mathbf{H}'$ from a' to b'. By the monotonicity of g, there exists an $(n+1)$-element fence in $\mathbf{H}$ from $g(a')$ to $g(b')$. But $g(b') \in g(B')$ so $d_{\mathbf{H}}(g(a'), g(B')) \leq d_{\mathbf{H}}(g(a'), g(b')) \leq n$ and the result follows. ■

This is the last lemma we will need before we prove the main theorem of this chapter.

Lemma 4.8. *Let $|\mathbf{P}| = k$ and let $(\mathbf{H}, f)$ be a $\mathbf{P}$-zigzag with $w \in \mathbf{H}$. Suppose that*

(A) $H = A \cup B \cup C$ where A, B, and C are all pairwise disjoint,

(B) B and C are nonempty and B is an up-set of $\mathbf{H}$,

(C) For all $a \in A$ and $c \in C$, a is not comparable to c in $\mathbf{H}$, and

(D) $w \in A \cup B$.

Then there exist a $\mathbf{P}$-zigzag $(\mathbf{H}', f')$ and $w' \in \mathbf{H}'$ such that

(a) $H' = A' \cup B' \cup C'$ where A', B', and C' are all pairwise disjoint,

(b) B' is a nonempty up-set of $\mathbf{H}'$, $|B'| \leq |B|$ and $|C| \leq |C'|$,

(c) For all $a' \in A'$ and $c' \in C'$, a' is not comparable to c' in $\mathbf{H}'$,

(d) $w' \in A' \cup B'$,

(e) $d_{\mathbf{H}'}(w', B') \geq d_{\mathbf{H}}(w, B)$,

(f) $l(\mathbf{H}') \leq l(\mathbf{H})$, and

(g) If $d = \max_{c \in C} |\{c\}_*|$, then the number of elements $c' \in C'$ with $c' < b'$ for some $b' \in B'$ is at most $kd|B|$.

This is probably the most technical result of this thesis and the most opaque as far as meaning goes. Because of this, before we begin the proof we will motivate it with a relatively simple example. Let us suppose $(\mathbf{H}, f)$ is a $\mathbf{P}$-zigzag with some maximal element w. If for some positive integer n, we define

$$\begin{aligned} A_n &= \{x \in H : d_{\mathbf{H}}(x, w) < n\}, \\ B_n &= \{x \in H : d_{\mathbf{H}}(x, w) = n\}, \text{ and} \\ C_n &= \{x \in H : d_{\mathbf{H}}(x, w) > n\}, \end{aligned}$$

and assume that $C_n \neq \emptyset$, then $(\mathbf{H}, f)$, w, A_n, B_n and C_n satisfy (A) through (D). Notice in this case that A_n and C_n are down-sets, $d_{\mathbf{H}}(w, B) = n$, and the set

$$\{c \in C_n : c < b \text{ for some } b \in B_n\} = \{x \in H : d_{\mathbf{H}}(x, w) = n + 1\} = B_{n+1}.$$

Lemma 4.8 tells us that, speaking somewhat inaccurately, we can modify $(\mathbf{H}, f)$ so that B_{n+1} is made fairly small; thus if the original C_n were large enough, then the new $C_n \backslash B_{n+1} = C_{n+1}$ will be non-empty. By iterating this process in the main theorem after this lemma, we will be able to produce $\mathbf{P}$-zigzags of strictly increasing diameter. We now prove the lemma.

Proof. Before we begin, observe from assumptions (B) and (C) that in addition to B being an up-set, the sets A and C are both down-sets in $\mathbf{H}$.

There is no induction in this proof. We just construct from $(\mathbf{H}, f)$ a nonextendible coloured poset $(\mathbf{Q}, g)$. We then let $(\mathbf{H}', f')$ be any $\mathbf{P}$-zigzag in $(\mathbf{Q}, g)$, define w' and sets A', B', and C', and show that they satisfy the necessary properties. So first let us define the coloured poset $(\mathbf{Q}, g)$ and then we will show that it is not $\mathbf{P}$-extendible. We break it down into steps.

Step 1: Given properties (A) and (B), we have $H \neq A \cup B$ so it is reasonable to define the poset $\mathbf{C} = \mathbf{H} \backslash (A \cup B)$. Then, since $(\mathbf{H}, f)$ is a $\mathbf{P}$-zigzag, there exist a set of extensions $\{t_1, \ldots, t_n\}$ of $(\mathbf{C}, f|_C)$ to all of $\mathbf{C}$. Now, for $1 \leq i \leq n$, the map f restricted to $C \cap \text{dom} f$ is equal to t_i so for each i, let us consider the coloured poset $(\mathbf{H}, f \cup t_i)$ where $f \cup t_i$ is the partially defined map from H to P given by

$$(f \cup t_i)(x) = \begin{cases} f(x), & \text{if } x \in \text{dom } f, \\ t_i(x), & \text{if } x \in C. \end{cases}$$

Since $(\mathbf{H}, f)$ is nonextendible, we can find some $\mathbf{P}$-zigzag $(\mathbf{Q}_{t_i}, g_{t_i}) \subseteq (\mathbf{H}, f \cup t_i)$.

Step 2: Fix i. We make some observations about $(\mathbf{Q}_{t_i}, g_{t_i})$. First, $Q_{t_i} \cap C$ must be nonempty; otherwise, $(\mathbf{Q}_{t_i}, g_{t_i}) \subseteq (\mathbf{H} \backslash C, f|_{H \backslash C})$ which is extendible since C is nonempty. Second, note that $Q_{t_i} \nsubseteq C$ since t_i fully extends C and $t_i|_{\mathrm{dom} g_{t_i}} = g_{t_i}$. Thirdly, $Q_{t_i} \cap C$ consists of coloured elements of $(\mathbf{Q}_{t_i}, g_{t_i})$ since g_{t_i} extends $f|_C$ to all of $\mathbf{C}$. Moreover, the elements in $Q_{t_i} \cap C$ must be minimal in $\mathbf{Q}_{t_i}$ since C is a down-set. Fourth and finally, we claim that every upper cover of an element in $Q_{t_i} \cap C$ is in $Q_{t_i} \cap B$. If $(\mathbf{Q}_{t_i}, g_{t_i})$ is a monotone $\mathbf{P}$-zigzag, by Lemma 3.2 all the elements in $\mathbf{Q}_{t_i}$ covering elements in $Q_{t_i} \cap C$ are noncoloured and so must be from $Q_{t_i} \cap (A \cup B)$. Then by property (C), our claim follows. In the nonmonotone case, we also have the bottom element of $Q_{t_i} \cap C$ covered by an element of $Q_{t_i} \cap B$ since the top element of $\mathbf{Q}_{t_i}$ is in B ($Q_{t_i} \cap C \neq \emptyset$ but t_i is a monotone extension of $(\mathbf{C}, f|_C)$ so this follows from (B) and (C)).

Step 3: Let $\mathbf{C}_0$ be a disjoint copy of $\mathbf{C}$ and for $1 \leq i \leq n$ and let $\mathbf{Q}_{t_i,0}$ be the same for $\mathbf{Q}_{t_i}$. Let $p_0 : \mathbf{C}_0 \to \mathbf{C}$ and $p_i : \mathbf{Q}_{t_i,0} \to \mathbf{Q}_{t_i}$ be the corresponding order-isomorphisms. Now, define the set $Q = C_0 \cup_{i=1}^n (Q_{t_i,0} \backslash p_i^{-1}(Q_{t_i} \cap C))$ and define the covering relation for $\mathbf{Q}$ to be

$$\begin{aligned} \prec_{\mathbf{Q}} = \prec_{\mathbf{C}_0} \cup_{i=1}^n \prec_{\mathbf{Q}_{t_i,0}} |_{Q_{t_i,0} \backslash p_i^{-1}(Q_{t_i} \cap C)} & \\ \cup_{i=1}^n \big\{ (x, y) : x \in \mathbf{C}_0 \cap p_0^{-1}(Q_{t_i} \cap C), & \\ y \in Q_{t_i,0} \cap p_i^{-1}(Q_{t_i} \cap B), \text{ and } p_0(x) \prec_{\mathbf{H}} p_i(y) \big\}. \end{aligned}$$

The basic idea is that we are gluing the disjoint copies of $(\mathbf{Q}_{t_i}, g_{t_i})$ together with the copy of $\mathbf{C}$ at the points in $Q_{t_i} \cap C$ that they originally shared. Notice that, since for all i every upper cover of an element in $p_i^{-1}(Q_{t_i} \cap C)$ is in $p_i^{-1}(Q_{t_i} \cap B)$ and $p_i^{-1}(B)$ is an up-set, this covering graph does not contain any cycles and generates a partial relation $\leq_{\mathbf{Q}}$ on Q. (Consider the example given in Figure 4.7 where $C = \{c_1, c_2, c_3\}$.)

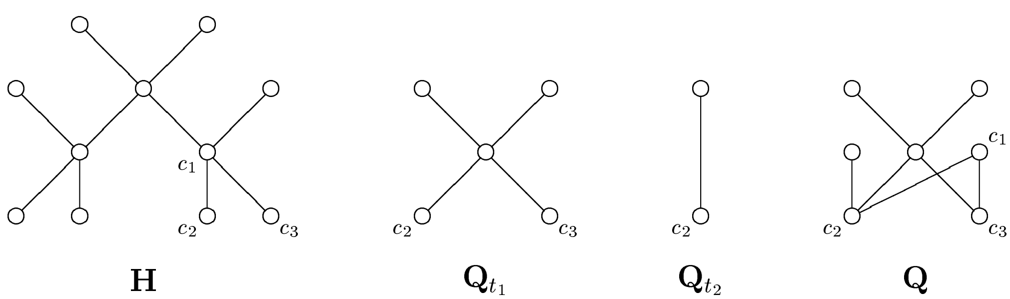


Figure 4.7: Third step of Lemma 4.8

Finally, define the partial colouring $g : \mathbf{Q} \to \mathbf{P}$ such that

$$g(x) = \begin{cases} f(p_0(x)), & \text{if } x \in C_0 \text{ and } p_0(x) \in C(\mathbf{H}, f), \\ f(p_i(x)), & \text{if } x \in Q_{t_i,0} \text{ and } p_i(x) \in C(\mathbf{H}, f) \end{cases}$$

to get the **P**-coloured poset $(\mathbf{Q}, g)$.

Step 4: The last step is to show that $(\mathbf{Q}, g)$ is not **P**-extendible to get our **P**-zigzag $(\mathbf{H}', f')$. But this is easy thanks to our construction, since any extension t of $(\mathbf{Q}, g)$ will restrict to an extension $t|_{C_0}$ of $(\mathbf{C}_0, g|_{C_0})$. For some i, this will correspond to the extension t_i of $\mathbf{C}$ which would mean the exact copy of $(\mathbf{Q}_{t_i}, g_{t_i})$ in $(\mathbf{Q}, g)$ would also be extended by t. But this is a contradiction as $(\mathbf{Q}_{t_i}, g_{t_i})$ is a **P**-zigzag. Thus $(\mathbf{Q}, g)$ is not extendible and must contain a **P**-zigzag $(\mathbf{H}', f')$.

Now, let us define $h : \mathbf{H}' \to \mathbf{H}$ such that

$$h(x) = \begin{cases} p_0(x), & \text{if } x \in C_0, \\ p_i(x), & \text{if } x \in Q_{t_i,0} \backslash C_0 \text{ for some } 1 \leq i \leq n. \end{cases}$$

Since the $Q_{t_i,0}$'s are disjoint outside of C_0, the map is well-defined. It is easy to see it is monotone, for if we suppose that $x \leq y$ we only need check the case where $x \in C_0$ and $y \in Q_{t_i,0} \backslash C_0$ for some i. In this case, there must be some $w \in \mathbf{C}_0 \cap p_0^{-1}(Q_{t_i} \cap C)$ such that $x \leq w \leq y$. Then $h(x) = p_0(x) \leq p_0(w) \leq p_i(y) = h(y)$. As well, h preserves the colourings of elements since $f' = g|_{H'}$. Finally, notice that h must be an onto map; otherwise, $h(\mathbf{H}', f') \subsetneq (\mathbf{H}, f)$ and hence would be **P**-extendible, yielding an extension of $(\mathbf{H}', f')$.

The last thing to do before we verify properties (a) through (g) is to define w', A', B' and C'. Since h is onto, let w' be some element of H' such that $h(w') = w$. Given this w' and property (D), there must exist some j such that $w' \in Q_{t_j,0}$, the complete copy of $\mathbf{Q}_{t_j}$ in $\mathbf{Q}$. Let us define $A' = h^{-1}(A) \cap Q_{t_j,0}$ and $B' = h^{-1}(B) \cap Q_{t_j,0}$. Then the only logical choice to satisfy (a) is to let $C' = H' \backslash (A' \cup B')$. We now check to see if (a) through (g) are satisfied.

(a) Since $h^{-1}(A) \cap h^{-1}(B) = \emptyset$, this is immediate.

(b) Note first that $C_0 \subseteq H'$ since h is an onto map. Moreover, $C_0 \subseteq C'$ which implies that $|C| \leq |C_0| \leq |C'|$. Next note that B' is an up-set for if we suppose otherwise then there is an $x \in \mathbf{H}'$ and $y \in B'$ such that $x \geq y$ but $x \notin B'$. But then $h(x) \in A$ or $h(x) \in C$ and this contradicts (A) as $h(x) \geq h(y)$ implies $h(x) \in B$. Hence B' is an up-set and we also have

$$|B'| = |h^{-1}(B) \cap Q_{t_j,0}| \leq |B \cap Q_{t_j}| \leq |B|.$$

Finally, since $\mathbf{H}'$ is connected, there must be a fence connecting w' to some $x \in C_0$. Now this fence is preserved by h so consider the corresponding fence connecting $h(w') = w$ to $h(x) \in C$. By (D), the element w is either in A or B. If $w \in B$, we are done as $w' \in B'$. If $w \in A$, then by (C) there must be an element b of B in the fence connecting w and $h(x)$. Hence $h^{-1}(b) \in B$ is in our original fence between w' and x and $B \neq \emptyset$.

(c) Let $a' \in A' = h^{-1}(A) \cap Q_{t_j,0}$ and $c' \in C'$. If $c' \in C_0$, then $h(c') \in C$. Hence the result follows from (C) since $h(a') \in A$ and h is monotone. Otherwise, for some

s, we have $c' \in h^{-1}(A) \cap Q_{t_s,0}$ or $c' \in h^{-1}(A) \cap Q_{t_s,0}$. Then a' is not comparable to any elements in $Q_{t_s,0} \cap C_0$ so a' is not comparable to c'.

(d) From (D) we see that $w' \in h^{-1}(A) \cup h^{-1}(B)$. Then, since $w' \in \mathbf{Q}_{t_j,0}$, it must be that $w' \in A'$ or $w' \in B'$. Hence $w' \in A' \cup B'$.

(e) This follows directly from Lemma 4.7.

(f) This is immediate as $l(\mathbf{H}') \leq l(\mathbf{Q}) \leq l(\mathbf{H})$.

(g) Notice first that every coloured element of $(\mathbf{Q}_{t_j}, g_j)$ covers at most one coloured element in $(\mathbf{Q}_{t_j}, g_j)$ (there is one in the nonmonotone case and zero otherwise by Lemma 3.2). As well, every noncoloured element of $(\mathbf{Q}_{t_j}, g_j)$ covers at most k coloured elements by Lemma 4.1. Consequently, the set of coloured elements in $Q_{t_j} \cap C$ covered by elements in $Q_{t_j} \cap B$ is at most $k|Q_{t_j} \cap B| \leq k|B|$. Since $Q_{t_j} \cap C$ consists of all coloured elements, we have $|Q_{t_j} \cap C| \leq k|B|$.

Now consider $\{c' \in C' : c' < b' \text{ for some } b' \in B'\}$. Clearly this set contains no elements of B' or A' so we can break it into two subsets

$$D_1 = \{c' \in H' \cap C_0 : c' < b' \text{ for some } b' \in B'\}$$

and

$$D_2 = \left\{ c' \in H' \cap \bigcup_{i \neq j} (Q_{t_i,0} \backslash C_0) : c' < b' \text{ for some } b' \in B' \right\}.$$

If $c' \in D_1$, then there exists a $b' \in B'$ and $c'' \in Q_{t_j,0} \cap p_0^{-1}(C)$ with $c' \leq c'' < b'$. Hence $c' \in \{c''\}_*$ and we have

$$\begin{aligned} |D_1| &\leq \max_{c'' \in Q_{t_j,0} \cap p_0^{-1}(C)} \{c''\}_* \, |Q_{t_j,0} \cap p_0^{-1}(C)| \\ &\leq \max_{c \in Q_{t_j} \cap C} \{c\}_* \, |Q_{t_j} \cap C| \\ &\leq \max_{c \in C} \{c\}_* \, k|B|. \end{aligned}$$

In the case where $c' \in D_2$, just recall that for all i, the set $Q_{t_i,0} \cap C_0$ consists of minimal elements of $Q_{t_i,0}$. Thus D_2 is empty by the definition of $\prec_{\mathbf{Q}}$.

Thus (a) through (g) are satisfied and our proof is complete. ■

Before we proceed with the main theorem of this chapter, just note that we are able to state and prove duals of both Lemma 4.6 and Lemma 4.8. We leave the details out. In the next theorem, we prove that if we can find $\mathbf{P}$-zigzags of large enough size, then we can also find $\mathbf{P}$-zigzags of correspondingly large diameter.

Theorem 4.9. *Let $|P| = k$. If there exists a $\mathbf{P}$-zigzag $(\mathbf{H}, f)$ such that $|H| \leq \sum_{i=0}^{m+1} k^{ik}$, then there exists a $\mathbf{P}$-zigzag of diameter at least $m+1$.*

Proof. Assume that there exists a $\mathbf{P}$-zigzag $(\mathbf{H}, f)$ such that $|H| \geq \sum_{i=0}^{m+1} k^{ik}$. By Theorem 4.5, we can choose $(\mathbf{H}, f)$ such that $l(\mathbf{H}) \leq k-1$.

We are going to construct a series of $\mathbf{P}$-zigzags $(\mathbf{H}_i, f_i)$, the first being $(\mathbf{H}, f)$, such that each $(\mathbf{H}_i, f_i)$ has a diameter of at least i. The $\mathbf{P}$-zigzag $(\mathbf{H}_{m+1}, f_{m+1})$ will give us our result. For $0 \leq i \leq m+1$, consider the following set of properties:

(a$_i$) $H_i = A_i \cup B_i \cup C_i$ where A_i, B_i, and C_i are all pairwise disjoint.

(b$_i$) If i is even, then B_i is a nonempty up-set of $\mathbf{H}_i$ and if i is odd, then B_i is a nonempty down-set of $\mathbf{H}_i$. Either way, $|B_i| \leq k^{ik}$ and $|C_i| \geq |H| - \sum_{j=0}^{i} k^{jk}$.

(c$_i$) For all $a \in A_i$ and $c \in C_i$, a is not comparable to c in $\mathbf{H}_i$.

(d$_i$) The element $a_i \in A_i \cup B_i$.

(e$_i$) $d_{\mathbf{H}_i}(a_i, B_i) \geq i$.

(f$_i$) $l(\mathbf{H}_i) \leq k-1$.

For each i, we are going to recursively define $(\mathbf{H}_i, f_i)$ and element a_i and sets A_i, B_i, and C_i contained in H_i that satisfy (a$_i$) through (f$_i$).

To start, let $(\mathbf{H}_0, f_0) = (\mathbf{H}, f)$ and let a_0 be a maximal element in $\mathbf{H}$. Now, define $A_0 = \emptyset$, $B_0 = \{a_0\}$, and $C_0 = H \backslash \{a_0\}$. Hence it follows immediately from our definitions that $(\mathbf{H}_0, f_0)$ satisfies properties (a$_0$) through (f$_0$).

Now, for $i \geq 1$ where i is odd, assume that we have a $\mathbf{P}$-zigzag $(\mathbf{H}_{i-1}, f_{i-1})$ with subsets A_{i-1}, B_{i-1}, and C_{i-1} and element a_{i-1} all satisfying properties (a$_{i-1}$) through (f$_{i-1}$). We will define $(\mathbf{H}_i, f_i)$ in the following steps using Lemmas 4.6 and 4.8.

Step 1: First, notice that B_{i-1} is an up-set since $i-1$ is even and hence C_{i-1} is a down-set by property (c$_{i-1}$). Now, if we define $D = C_{i-1}$, then by Lemma 4.6 we get a new $\mathbf{P}$-zigzag $(\mathbf{H}'_{i-1}, f'_{i-1})$ with subsets $A'_{i-1} = A_{i-1}$, $B'_{i-1} = B_{i-1}$, and $C'_{i-1} = D'$ and element $a'_{i-1} = a_{i-1}$. As well, there is an onto, monotone map g from $(\mathbf{H}'_{i-1}, f'_{i-1})$ to $(\mathbf{H}_i, f_i)$. We show that this new zigzag still satisfies (a$_{i-1}$) through (f$_{i-1}$) using (a) through (e) of Lemma 4.6.

(a$_{i-1}$) This follows from (a) since $A'_{i-1} \cap B'_{i-1} = \emptyset$.

(b$_{i-1}$) Since g is monotone and by (b) we have $g(u) = u$, it follows that B'_{i-1} is a nonempty up-set of $\mathbf{H}_i$. We also have $|B'_{i-1}| = |B_{i-1}| \leq k^{(i-1)k}$ and, since g is onto, $|C'_{i-1}| \geq |C_{i-1}| \geq |H| - \sum_{j=0}^{i-1} k^{jk}$.

(c$_{i-1}$) If we suppose otherwise, then there exists an $a \in A'_{i-1} = A_{i-1}$ and $c \in C'_{i-1}$ with $a \leq c$. But $a = g(a) \leq g(c) \in C_{i-1}$ and this contradicts our original (c$_{i-1}$).

(d$_{i-1}$) This is immediate.

(e_{i-1}) From Lemma 4.7, we have

$$d_{\mathbf{H}'_{i-1}}(a'_{i-1}, B'_{i-1}) = d_{\mathbf{H}'_{i-1}}(a_{i-1}, B_{i-1}) \geq d_{\mathbf{H}_{i-1}}(a_{i-1}, B_{i-1}) \geq i-1.$$

(f_{i-1}) Immediately from (c).

We note that, in addition to still satisfying these properties, since $l(\mathbf{H}_{i-1}) \leq k-1$, from (d) we get that $|\{c\}_*| \leq k^{k-1}$ for all $c \in C'_{i-1}$.

Step 2: We now apply Lemma 4.8. Let $A = A'_{i-1}$, $B = B'_{i-1}$, and $C = C'_{i-1}$ and element $w = a'_{i-1}$. By Lemma 4.8, there exists a $\mathbf{P}$-zigzag $(\mathbf{H}_i, f_i)$ with subsets A', B', and C' and element $w' \in H_i$ such that (a) through (g) are satisfied. From these, let us define $A_i = A' \cup B'$ and $B_i = \{c \in C' : c < b \text{ for some } b \in B'\}$. Then, let $C_i = H_i \backslash (A_i \cup B_i)$ and set $a_i = w'$. We show, using properties from Lemma 4.8 and 4.6, that $(\mathbf{H}_i, f_i)$ satisfies (a_i) through (f_i).

(a_i) This follows directly from our definitions and (a) from Lemma 4.8.

(b_i) It follows from the definition that B_i is a down-set. Moreover, as B' and C' are nonempty, by property (c) of 4.8 and the fact that $\mathbf{H}_i$ is connected it must be that B_i is nonempty.

Now, from (g) of 4.8 we get $|B_i| \leq kd|B| = kd|B'_{i-1}| = kd|B_{i-1}|$, where $d = \max_{c\in C} |\{c\}_*|$. In addition, since $C = C'_{i-1}$ we have $d \leq k^{k-1}$ and from (b_{i-1}) we get $|B_{i-1}| \leq k^{(i-1)k}$. Thus $|B_i| \leq kk^{k-1}k^{(i-1)k} = k^{ik}$.

Finally, by (b) of 4.8 we have $|C'| \geq |C'_{i-1}|$ and by (c) of 4.6 we have $|C'_{i-1}| \geq |C_{i-1}|$. Now, once we notice that $C_i = C' \backslash B_i$ and $B_i \subseteq C'$, thanks to (b_{i-1}) we get

$$|C_i| = |C'| - |B_i| \geq |C_{i-1}| - |B_i| \geq |H| - \sum_{j=0}^{i-1} k^{jk} - k^{ik} = |H| - \sum_{j=0}^{i} k^{jk}.$$

(c_i) Let $a \in A_i = A' \cup B'$ and $c \in C_i \subseteq C'$. If $a \in A'$, then we are done by (c) of 4.8. If $a \in B'$, then $c \not< a$ since $c \notin B_i$. But B' is an up-set so $c \ngtr a$ either. Hence a is not comparable to c.

(d_i) This follows from (d) of Lemma 4.8 as $a_i = w' \in A' \cup B' = A_i \subseteq A_i \cup B_i$.

(e_i) From (b_i) we have B_i nonempty. We claim that $d_{\mathbf{H}_i}(a_i, B_i) > d_{\mathbf{H}_i}(a_i, B')$. If $a_i \in B'$, then we are done. Otherwise, suppose $a_i \in A'$ and let $b \in B_i$ be some element such that $d = d_{\mathbf{H}_i}(a_i, b) = d_{\mathbf{H}_i}(a_i, B_i)$. Hence there exists a $(d+1)$-element subfence of elements $\{b_0, \ldots, b_d\}$ where $b_0 = a_i$ and $b_d = b$. Since $b_0 \in A'$, $b_d \in C'$, and b_i is comparable to b_{i+1} for all $i < d$, property (c_i) implies that there must be some $1 \leq j < d$ such that $b_j \in B'$ and this implies $d_{\mathbf{H}_i}(a_i, B') < d = d_{\mathbf{H}_i}(a_i, B_i)$.

Thus by (e) of 4.8 and (e_{i-1}),

$$d_{\mathbf{H}_i}(a_i, B_i) > d_{\mathbf{H}_i}(a_i, B') = d_{\mathbf{H}_i}(w', B') \geq d_{\mathbf{H}'_{i-1}}(w, B) = d_{\mathbf{H}'_{i-1}}(a_{i-1}, B_{i-1}) \geq i-1.$$

(f_i) From (f) of 4.8 and (e) of 4.6, we have $l(\mathbf{H}_i) \leq l(\mathbf{H}'_{i-1}) \leq l(\mathbf{H}_{i-1}) \leq k-1$.

In the case where $i \geq 1$ and i is even, we proceed dually using dual statements of Lemma 4.6 and Lemma 4.8. Thus we are able to define a $\mathbf{P}$-zigzag $(\mathbf{H}_{m+1}, f_{m+1})$ with element a_{m+1} and subset B_{m+1} that satisfies (e_{m+1}) which says $d_{\mathbf{H}_{m+1}}(a_{m+1}, B_{m+1}) \geq m+1$. Therefore the diameter of $(\mathbf{H}_{m+1}, f_{m+1})$ is also greater than or equal to $m+1$. ■

The following corollary gives us the biggest piece of Zádori's proof.

Corollary 4.10. *Let* $\mathbf{P}$ *is a finite, bounded poset. If there exists a finite* m *such that every* $\mathbf{P}$*-zigzag has a diameter at most* m*, then there exist a finite number of* $\mathbf{P}$*-zigzags.*

Proof. If we assume otherwise, then there must be a $\mathbf{P}$-zigzag of size at least $\sum_{i=0}^{m+1} k^{ik}$. By Theorem 4.9, there exists a $\mathbf{P}$-zigzag of diameter $m+1$. ■

We have now completed the toughest part of this thesis. In the next section, we conclude by showing that finite, bounded posets admit Jónsson operations if and only if they admit a near-unanimity function.

4.3 Conclusion

We now complete the last piece of the main idea of this thesis. This final theorem synthesizes the main results of the last four chapters and answers the main question posed in Chapter 1.

Theorem 4.11. *Let* $\mathbf{P}$ *be a finite, bounded poset. Then* $\mathbf{P}$ *admits a monotone* n*-near-unanimity function for some* $n \geq 3$ *if and only if* $\mathbf{P}$ *admits a set of monotone Jónsson operations.*

Proof. The forward direction was proved in Lemma 1.1.

If $\mathbf{P}$ admits monotone Jónsson operations, then by Theorem 2.14, there exists for some n a partially defined, monotone n-nuf on $\mathbf{P}^n$ that is fully defined on

$$A_n = \left\{ (x, \ldots, x, \underset{i}{y}, z, \ldots, z) : x, y, z \in P, 1 \leq i \leq n \right\}.$$

By Theorem 3.5, we can then place a finite bound on the diameters of all $\mathbf{P}$-zigzags. Consequently, there can be at most finitely many $\mathbf{P}$-zigzags by Corollary 4.10. Hence we can find some n such that the number of coloured elements in each $\mathbf{P}$-zigzag is at most $n-1$. Therefore, by Lemma 1.5, the poset $\mathbf{P}$ admits an n-near-unanimity function for some $n \geq 3$. ■

In [12], Zádori provides an example of an infinite, bounded poset which admits Jónsson operations but does not admit a near-unanimity function. It is important to mention also that in a later paper, Larose and Zádori extend the main result of this thesis by proving that an arbitrary finite poset **P** admits a near-unanimity function if and only of **P** admits Jónsson operations [7].

Since Lemma 1.1 is actually true for relational structures in general, the big question that this result pushes us towards is whether all finite relational structures with a finite set of relations admit near-unanimity functions if and only if they admit Jónsson operations. This conjecture is now being referred to as Zádori's conjecture [11] and up to this point there is no published answer. However, there is active work in the area and the conjecture has been discussed and worked on at recent workshops (see [1]). This has led to at least one mathematician claiming to have proved this conjecture [3] and the validity of this proof is still being verified at the time that this thesis was completed.

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