# Qualitative Theory of Impulsive Delay Differential Equations

by

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A thesis
presented to the University of Waterloo
in fulfilment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Applied Mathematics

Waterloo, Ontario, Canada, 1999

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# Abstract

Systems of impulsive delay differential equations are considered and the qualitative theory of these equations are developed. Delay differential equations (without impulses) and impulsive differential equations (without delays) are first discussed and these are then combined to yield impulsive delay differential equations. These more general systems can suitably model evolutionary processes that exhibit both delay and impulse characteristics. After formulating the initial value problem for these systems and defining the notion of a solution, theorems establishing certain fundamental properties of solutions are developed. Specifically, theorems on local and global existence, uniqueness, continuability, and continuous dependence of solutions are presented. Subsequently, a variety of stability and boundedness results are obtained for the case when impulses occur at fixed times. Lyapunov functionals and functions are the main tools used to obtain stability and boundedness results. Finally, a number of examples and applications are presented to help motivate the study of these equations.

# Acknowledgements

I wish to thank my supervisor, Professor Xinzhi Liu, for his guidance and supervision during the development and preparation of this document. The support and input he provided in researching the theory of impulsive delay differential equations was most helpful and is greatly appreciated.

Thanks also go to Professors Sue Ann Campbell, David Siegel, David Wang, and Gail Wolcowicz who read and critiqued this document, and together with Professor Liu, comprised my examining committee. They offered many valuable comments and suggestions for improving the form and content of this thesis.

I am grateful to the Natural Sciences and Engineering Research Council of Canada and to the University of Waterloo for their financial support.

My lovely wife, Marilou, deserves very special thanks. She encouraged and supported me wholeheartedly while I worked on this thesis. Her patience and love are limitless.

I am grateful for the support and friendship that I have received from many special people whom I have met during my studies at the University of Waterloo. In particular, I consider myself very fortunate to have come to know Lise Arseneau, Ada Cheng, Tom Holly, Serge Kruk and Debbie MacLean.

Finally, I am especially thankful for the awesome 1990's dance music, without which, working on this thesis would have been a considerably greater chore.

# Dedication

To my wife, Marilou ♡.

# Contents

# Chapter 1

# Introduction

The development and analysis of mathematical models of dynamic physical processes have been of great importance in bettering our understanding of nature and the world around us. Since the invention of differential and integral calculus by Sir Isaac Newton (1642-1727) and Gottfried Wilhelm Leibnitz (1646-1716), differential equations have aided the investigation of a wide variety of problems in the physical, biological and social sciences. Early mathematicians to have studied differential equations include brothers Jacob (1654-1705) and Johann (1667-1748) Bernoulli, who solved various problems in mechanics [Kli72]. Leonhard Euler (1707-1783), who was a student of Johann Bernoulli and one of the greatest of all mathematicians, contributed a great deal to the advancement of the study of differential equations [Sim91].

Ordinary differential equations involve unknown functions and their derivatives that depend on a single independent variable, often denoted by $t$ and referred to as time. While an ordinary differential equation may not have a physical interpretation at all or it may model some physical system whose independent variable represents some quantity other than time, it is convenient when studying ordinary differential equations as a whole to maintain consistency of notation and label the independent variable by $t$ and refer to it as time. While an ordinary differential equation may involve higher order derivatives of an unknown function, under the usual assumption that the highest derivative can be solved for, it is customary to convert such an equation to a system of first order ordinary differential equations. This leads us to the system of ordinary differential equations,

$$x'(t) = f(t, x(t)), \tag{1.0.1}$$

where the independent variable $t$ is real-valued, the dependent state variable $x$ is an $n$-dimensional real vector, $f$ is some function of both $t$ and $x(t)$, and $x'(t)$ denotes the derivative of $x(t)$ with respect to $t$. Since one could think of (1.0.1) as a system of $n$ individual, albeit coupled, scalar ordinary differential equations or as a single vector ordinary differential equation, then (1.0.1) may be referred to as either a system of ordinary differential equations (or simply a system or a system of equations) or as an ordinary differential equation (or simply an equation).

Over the past 300 years a great deal of work has gone into the study of ordinary differential equations, justified by their wide ranging applications. Because it is often not possible (or even desirable) to solve general equations such as (1.0.1), the analysis of the qualitative properties of solutions of these equations has been of fundamental importance. Texts devoted to differential equations abound (see [Sim91] or [Hal80] for example).

Ordinary differential equations are sometimes inadequate as models of certain physical processes and therefore in such cases one or more generalizations of (1.0.1) is necessary. Two properties inherent in system (1.0.1) are that it satisfies the principle of causality, meaning the future state of the system depends exclusively on the present state and not on past (or future) states, and that the evolution of the state is continuous. Here we use the term "causality" in the sense described by Hale and Lunel [Hal93] in the introductory chapter of their monograph. The term is sometimes defined differently in the literature. In control theory, for instance, it is used interchangeably with the term "nonanticipatory," meaning that there can be dependence on both present and past states but not on future states.

In many processes including physical, chemical, political, economic, biological and control systems, time delays are an important factor. The rate of change of the state, $x'(t)$, may depend on historical values of the state at times $t + s$, where $s \leq 0$, as well as present state values. In other words, it may be unreasonable to assume that the causality principle applies to the system. In fact, strictly speaking, the causality principle does not apply to most real systems. Instead, it is an approximation to how the true system behaves. When time delays are negligible, then they can often be ignored in mathematical models, leading us to simple equations like (1.0.1). When time delays are an important feature of a process, then one is led to consider delay differential equations, also known as retarded functional differential equations.

Various evolutionary processes from fields as diverse as population dynamics, aeronautics, economics and engineering are characterized by the fact that they undergo abrupt changes of state at certain moments of time between intervals of continuous evolution. Because the duration of these changes are often negligible compared to the total duration of the process, such changes can be reasonably well-approximated as being instantaneous changes of state, or impulses. These processes tend to be more suitably modeled by impulsive differential equations, which allow for discontinuities in the evolution of the state. Impulsive differential equations are usually defined by a pair of equations, an ordinary differential equation (1.0.1) to be satisfied during the continuous portion of evolution and a difference equation defining the discrete impulsive actions. The impulses occur when some spatio-temporal relation is satisfied. Impulsive differential equations and delay differential equations will be described more fully later.

Both time delays and impulses are important features commonly encountered in many of the same systems. If a system exhibits both of these properties, then we are led to consider impulsive delay differential equations, which is the subject of this thesis.

Delay differential equations were studied as far back as the eighteenth century by such well-known mathematicians as Euler, Joseph Louis Lagrange (1736-1813), Pierre Simon de Laplace (1749-1827) and others, mostly in relation to various geometrical problems [Ser77]. However, unlike ordinary differential equations, there was comparatively little study of the qualitative theory of delay differential equations until the early twentieth century. Vito Volterra (1860-1940) was among the early mathematicians of this century to have formulated and analyzed delay differential equations, primarily with respect to problems in viscoelasticity [Vol09] and predator-prey models [Vol28, Vol31].

During the 1930's and 1940's a number of applied problems required a more careful examination of delay differential equations. For instance, Minorsky's study of ship stabilization and automatic steering, wherein delay in the feedback mechanism was significant, led him to consider differential equations with delay [Min42]. Myshkis, who was the first to correctly formulate the initial value problem for delay differential equations, was also the first to publish a book [Mys51] devoted exclusively to the theory of delay differential equations.

Within the past forty years or so the study of delay differential equations has matured a great deal and yielded a number of monographs dedicated to the subject. These have included a book on differential difference equations by Bellman and Cooke [Bel63] and later texts by Hale [Hal77] and Driver [Dri77] on more general delay differential equations, the former being revised and updated by Hale and Lunel [Hal93] in 1993. Without being exhaustive, other texts devoted to the subject include those by Oğuztöreli [Ogu66], Burton [Bur85], Kolmanovskii and Nosov [Kol86], and Górecki et al. [Gor89]. In most cases, delay differential equations have been studied on the space of continuous functions with the evolution of the state being absolutely continuous and usually continuously differentiable.

Compared to delay differential equations, impulsive differential equations received scant attention until the 1960's and it wasn't until the 1980's that interest in them began to catch on among mathematicians. Among the earliest articles on impulsive differential equations was the seminal paper by Milman and Myshkis [Mil60]. They considered differential equations with impulses occurring when certain spatio-temporal relations are satisfied. By making use of classical results of ordinary differential equations, they obtained the first results on stability of solutions of impulsive differential equations. Based on this early work, later research into impulsive differential equations culminated in the publishing of several monographs by Samoilenko and Perestyuk [Sam87], Lakshmikantham, Bainov and Simeonov [Lak89], Bainov and Covachev [Bai94a], and Bainov and Simeonov [Bai89, Bai93] along with more recent publications during the 1990's.

These authors considered an impulsive differential equation to be an ordinary differential equation coupled with a difference equation to be satisfied at certain fixed or variable impulse times. The resulting solutions are thereby piecewise continuous with discontinuities occurring at these impulse times. This approach enabled them to apply many well-established results for ordinary differential equations to these systems in order to develop the qualitative theory of the impulsive differential equations. The same approach has been taken in this document with respect to the study of impulsive delay differential equations.

We remark that an alternative approach has been considered by Halanay and Wexler [Hal68] and Pandit and Deo [Pan82]. Their approach involves defining a measure differential equation (incorporating Dirac delta functions for example) where the derivative involved is a distributional derivative. The times at which impulses occur are fixed, generalized functions are considered and the resulting solutions are of bounded variation. Unfortunately, most classical theory cannot be applied to these types of systems, and perhaps as a result, few authors currently study this formulation of impulsive differential equations.

Early work on concrete systems of impulsive differential equations include the investigation of the shock model of a clock mechanism by Bautin [Bau63] and the study of drug distribution in the human body by Kruger-Thiemer [Kru66].

Due to their increased generality, impulsive delay differential equations have the potential for applications to all sorts of physical problems. For instance, in the study of population dynamics, delays in reproduction or in the interactions within or between species may require the adoption of a delay differential equation to best model the system [Gop92]. If one or more of the species regularly undergoes periodic stocking or harvesting, for example, then this behaviour could be modelled using impulsive differential equations [Bal97c, Liu94]. Of course, if both of these characteristics are present in the system, then an impulsive delay differential equation would make for a suitable model.

Despite the apparent abundance of applications, the study of impulsive delay differential equations is in its relative infancy. The first article on the subject was published in 1986 by Anokhin [Ano86]. Recent study of impulsive delay differential equations has tended to focus on special classes of problems such as linear (and often scalar) impulsive delay differential equations [Ano95, Gop89, Zha96] or delay differential difference equations such as $x'(t) = f(t, x(t), x(t-r))$ together with impulses [Bai96, Yu96]. Also, with the odd rare exception such as a paper by Bainov and Covachev [Bai94b], no delays have been assumed in the difference equation governing the impulsive action.

Among the most fundamental of qualitative properties of impulsive delay differential equations are the existence (both local and global), uniqueness and continuability of solutions, the continuous dependence of solutions with respect to initial conditions, and stability and boundedness properties.

For many of the special classes of problems that have recently been studied, properties of solutions such as existence and uniqueness can be obtained directly from the system under fairly weak assumptions. As a result, little mention is ever made about these properties. One exception is a paper by Krishna and Anokhin [Kri94] who recently considered a special case involving a delay differential equation of the form $x'(t) = f(t, x(t - h(t)))$ together with variable time impulses and they established some interesting existence and uniqueness results. Global existence and uniqueness properties of the slightly more general equation $x'(t) = f(t, x(t - h_1(t)), \dots, x(t - h_m(t)))$ together with impulses occurring at fixed times were later obtained by Shen [She97]. The establishment of some existence and uniqueness results for more general impulsive delay differential equations has been done by Shen [She96], Weng and Yang [Wen98] and Ballinger and Liu [Bal99a, Bal99b], with the latter being developed here.

While some stability results have been proven for special classes of impulsive delay differential equations [Ano95, Gop89, Zha96], very little research into stability has been undertaken for more general equations. The only apparent exception to this has been the very recent publication of articles by Weng and Yang [Wen98], Shen and Yan [She98], Luo and Shen [Luo99] and Shen [She99] in late 1998 and 1999. Through the use of Lyapunov functionals and/or Lyapunov functions together with certain Razumikhin techniques they obtained some very interesting stability results.

With the few noted exceptions, the study of general impulsive delay differential equations has been rather minimal to date. This thesis embodies original work into the development of the qualitative theory of impulsive delay differential equations. This of course is a necessary prerequisite to the use of these equations in modelling real physical processes. While some examples are provided to help further motivate the study of these equations, the development of the theory, and not the application to any specific problems, is the primary goal.

We begin by discussing delay differential equations over the space of continuous functions. Throughout this document we restrict our attention to equations that have bounded delays. Moreover, our focus is on studying properties of solutions of the initial value problem associated with our equation. Since we are ultimately interested in incorporating impulses into these equations to yield impulsive delay differential equations, we then discuss delay differential equations over the space of piecewise continuous functions. In particular, the initial condition will involve a piecewise continuous initial function. This seemingly minor change introduces a great deal of complications into the study of these equations.

Meanwhile, after a review of impulsive differential equations wherein we define what we mean by impulses, we introduce the notation for impulsive delay differential equations. We first consider both fixed time and variable time impulses and we develop theorems on existence, uniqueness, continuous dependence and continuability of solutions. In the case where impulses occur at variable times, we impose some additional conditions so as to guarantee that solutions exist locally and are sufficiently well-behaved so as not to exhibit the so-called beating phenomenon.

Once these fundamental results have been established, the remainder of the thesis is devoted to the study and analysis of stability and boundedness of solutions. To simplify things we restrict our attention to impulsive delay differential equations which have impulses at fixed times. Two main approaches are used in the study of stability and boundedness. The first involves the use of Lyapunov functionals. The second involves the use of Lyapunov functions together with a Razumikhin-type condition.

In developing Lyapunov stability and boundedness theorems, we focus on two main approaches. We first consider a system which, in the absence of impulses, is already stable (or bounded). We then develop criteria on the impulses which will guarantee that stability is maintained. The second approach considers a system which, in the absence of impulses, may be highly unstable (or unbounded). In this latter case the impulses are responsible for stabilizing the system and conditions guaranteeing the stabilization of the system are established.

After developing stability and boundedness for a general class of nonlinear systems, special cases of linear impulsive delay differential equations are then considered and the stability theorems of Chapters 4 and 5 are applied to these systems.

Finally, two sources of applications of the stability and boundedness results are presented. The first application considers the problem of permanence of species in population growth models that involve both impulses and delays. The second is a look at the problem of practical stability of control systems that have delays and uncertainties in the feedback mechanism and that experience impulses.

# Chapter 2

# Preliminaries

In this and subsequent chapters we will study delay differential equations with bounded delays. More specifically we will concern ourselves with the initial value problem whereby the state is initially specified over some bounded interval of time. Our goal will then be to examine the qualitative properties of solutions of these systems.

Delay differential equations are most often studied over the space of continuous functions as opposed to the space of piecewise continuous functions or more general spaces of functions [Hal93]. This makes it that much easier to establish qualitative theoretical results.

In the first section we give an introduction to delay differential equations on the space of continuous functions. These we sometimes refer to as continuous delay differential equations. A more thorough discussion can be found in various texts on the subject (see [Hal93] or [Dri77] for instance).

In the next section we briefly describe impulsive differential equations without delay. These are described in greater detail in [Lak89] and [Bai89].

Section 2.3 then goes on to describe delay differential equations on the space of piecewise continuous functions. Comparisons between them and delay differential equations over the space of continuous functions are made.

In the final section of this chapter we introduce impulsive delay differential equations and define the notion of a solution of these equations.

## 2.1 Delay Differential Equations on the Space of Continuous Functions

Delay differential equations are generalizations of ordinary differential equations (1.0.1) in that the derivative of the state, $x'(t)$, is no longer merely a function $t$ and $x(t)$ but may also be a function of $x(t+s)$ for values of $s \leq 0$. By restricting our attention to systems with bounded delays, this means that we assume there exists some $r > 0$, which we call the delay constant, whereby $x'(t)$ depends on only those values of $x(t+s)$ for which $s \in [-r, 0]$.

Ordinary differential equations can be thought of as delay differential equations with an arbitrarily small delay constant $r$. In fact we could think of $r$ as being zero in this case but for simplicity of notation we will always assume $r$ is strictly positive. Note that a delay differential equation with a delay constant $r > 0$ can also be thought of as a delay differential equation having a larger delay constant $\hat{r} \geq r > 0$ although the converse is not generally true.

For simple delay differential equations, $x'(t)$ might depend on $x(t+s)$ for only a finite number of values $s \in [-r, 0]$ that do not depend on $t$. These types of delay differential equations are called delay differential difference equations and have the general form

$$x'(t) = g(t, x(t), x(t-t_1), x(t-t_2), \dots, x(t-t_m)), \tag{2.1.1}$$

where $0 < t_1 < t_2 < \dots < t_m$. For this equation, the delay constant $r = t_m$. Although any larger value for $r$ will suffice, we usually choose the smallest. Equation (2.1.1) may be generalized by replacing the constants $t_k$ by nonnegative bounded functions $h_k(t)$. In this case $r$ is chosen to satisfy $r \geq h_k(t)$ for all $t$ and all $1 \leq k \leq m$. Equations of these types are sometimes referred to as having lumped delays, which are distinct from distributed delays.

In some cases $x'(t)$ may depend on $x(t+s)$ over a whole continuum of points $s \in [-r, 0]$. In such cases the system is said to have distributed delays. Integro-differential equations such as

$$x'(t) = g\left(t, x(t), \int_{t-r}^{t} G(t, s, x(s))ds\right), \tag{2.1.2}$$

are typical examples of these types of equations.

In order to consider all types of delay differential equations (with bounded delays) without having to specify its particular form (such as (2.1.1) or (2.1.2)), we need some standardized notation.

Let $\mathbf{R}$ denote the set of real numbers, $\mathbf{R}_+$ the set of nonnegative real numbers and $\mathbf{R}^n$ the $n$-dimensional Euclidean linear space equipped with the Euclidean norm $\|\cdot\|$. Occasionally we will use the symbol $\mathbb{Z}^+$ to represent the set of positive integers. Let $r > 0$ be the delay constant for the system. For any continuous function $x \in C([t_0 - r, \infty), \mathbf{R}^n)$, where $t_0 \in \mathbf{R}_+$, we define $x_t \in C([-r, 0], \mathbf{R}^n)$ by $x_t(s) = x(t+s)$ for $s \in [-r, 0]$. In other words, $x_t$ is the segment of the function $x$, from $t - r$ up to $t$, that has been shifted to the interval $[-r, 0]$. This notation then allows us to express a general delay differential equation as

$$x'(t) = f(t, x_t), \tag{2.1.3}$$

where $f$ depends on both $t$ and $x_t$. Since $x_t$ is an element of $C([-r, 0], \mathbf{R}^n)$, then $f$ is called a functional instead of simply a function.

Equation (2.1.1) can be written in the form (2.1.3) by letting

$$f(t,\psi)=g(t,\psi(0),\psi(-t_1),\psi(-t_2),\ldots,\psi(-t_m)). \tag{2.1.4}$$

Likewise, equation (2.1.2) can be converted into the form (2.1.3) by defining

$$f(t,\psi)=g\left(t,\psi(0),\int_{-r}^{0}G(t,t+s,\psi(s))ds\right). \tag{2.1.5}$$

In studying delay differential equations, and later impulsive delay differential equations, of particular interest will be the initial value problem. For ordinary differential equations (1.0.1) the initial value problem involves finding a solution (or analyzing the qualitative properties of such a solution) of equation (1.0.1), on some interval, that satisfies

$$x(t_0)=x_0, \tag{2.1.6}$$

where $t_0\in\mathbb{R}_+$ is the initial time and $x_0\in\mathbb{R}^n$ is the initial state. For system (2.1.3) the initial state must be specified over the entire interval $[t_0-r,t_0]$, not just at $t_0$. This leads to the initial condition

$$x_{t_0}=\phi, \tag{2.1.7}$$

where $t_0\in\mathbb{R}_+$ again represents the initial time but $\phi\in C([-r,0],\mathbb{R}^n)$ represents the initial function.

Solutions of (2.1.3) are usually assumed to be continuously differentiable and satisfy the delay differential equation (2.1.3) at all points $t>t_0$ (within their interval of existence). At the initial time $t_0$, (2.1.3) is assumed to be satisfied with $x'(t)$ representing the right-hand derivative. Smoothness assumptions are typically imposed on the functional $f$ to guarantee properties of existence, uniqueness and so on of solutions. A weaker notion of a solution is sometimes considered whereby the solution is absolutely continuous and satisfies (2.1.3) except on a set of Lebesgue measure zero [Hal93, Ogu66]. This allows for weaker assumptions on $f$ but tends to further complicate the study of equation (2.1.3).

Unlike ordinary differential equations, it is generally not possible to extend solutions backwards in time except under very special and unusual circumstances [Dri77]. For this reason the study of delay differential equations usually considers only the forward evolution of the state in time.

Delay differential equations are often difficult to solve explicitly, more so than ordinary differential equations. Our focus will be on the study of qualitative properties of solutions without necessarily having to solve for them. Nevertheless, we mention one frequently used technique for solving special classes of delay differential equations. This method is known as the method of steps or the method of sequential integration. It can be applied only to delay differential equations for which $x'(t)$ depends on $t$, $x(t)$ and $x(t+s)$ for $s\in[-r,-\epsilon]$ where $\epsilon>0$ is some arbitrarily small constant.

Equation (2.1.1) is an example. For values of $t$ in the interval $[t_0, t_0 + \epsilon]$, knowledge of $x(t+s)$ for $s \in [-r, -\epsilon]$ is given by the initial function $\phi$. Therefore on this interval, $x'(t)$ depends only on $t$, $x(t)$ and specified initial data. In other words, the delay differential equation reduces to an ordinary differential equation from which the state $x(t)$ can (hopefully) be solved using techniques of ordinary differential calculus. Once $x(t)$ is known for $t \in [t_0 - r, t_0 + \epsilon]$ it can be solved by using this same technique on the interval $[t_0 + \epsilon, t_0 + 2\epsilon]$. By repeated stepping over intervals of length $\epsilon$, the entire solution curve can be solved. Note that this can be cumbersome if $\epsilon$ is small.

For delay differential equations to which the step method can be applied, theorems on existence, uniqueness and continuous dependence of solutions may be formulated under much weaker assumptions on the functional $f$. This is because at each step the delay differential equation is transformed into an ordinary differential equation without delays and so the well-known theorems on existence, uniqueness and continuous dependence for ordinary differential equations may be applied.

More properties of system (2.1.3) will be described when we generalize the system in Section 2.3 by considering them to be defined over the larger space of piecewise continuous functions.

## 2.2 Impulsive Differential Equations

As noted in Chapter 1, impulsive differential equations are usually defined as an ordinary differential equation coupled with a difference equation, although other formulations do exist. The difference equation is usually given by

$$\Delta x(t) = I(t, x(t)), \tag{2.2.1}$$

where $\Delta x(t) = x(t^+) - x(t^-)$. Here and elsewhere throughout this document we use the abbreviated notation $x(t^+) = \lim_{s \to t^+} x(s)$ and $x(t^-) = \lim_{s \to t^-} x(s)$ to refer to right-hand and left-hand limits, respectively. Equation (2.2.1) is to be satisfied when some spatio-temporal relation $h(t, x(t)) = 0$ is satisfied. In general then, we are led to an impulsive differential equation having the form

$$x'(t) = f(t, x(t)), \quad h(t, x(t)) \neq 0, \tag{2.2.2a}$$

$$\Delta x(t) = I(t, x(t)), \quad h(t, x(t)) = 0. \tag{2.2.2b}$$

Thus for as long as we have $h(t, x(t)) \neq 0$, then the evolution of the state is governed by the ordinary differential equation (2.2.2a). At such time that $h(t, x(t)) = 0$, then the state undergoes an impulse and instantly changes by some amount $I(t, x(t))$ according to (2.2.2b). This causes a jump discontinuity in the solution. Following this impulsive action, and assuming $h(t, x(t))$ is nonzero for some time thereafter, then the solution will continue to evolve according to (2.2.2a) until it again undergoes an impulse.

The arbitrary nature of the relation $h(t, x(t)) = 0$ makes the study of (2.2.2) extremely difficult.

It is therefore common to focus on a particular type of relation. The set of points $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$ for which $h(t, x) = 0$ will be assumed to consist of a sequence of hypersurfaces of the form $t = \tau_k(x)$ where $\tau_k \in C(\mathbb{R}^n, \mathbb{R}_+)$ for $k = 0, 1, 2, \dots$ and $0 = \tau_0(x) < \tau_1(x) < \tau_2(x) < \dots$ and $\lim_{k\to\infty} \tau_k(x) = \infty$ for each $x \in \mathbb{R}^n$. The system is then written as

$$x'(t) = f(t, x(t)), \; t \neq \tau_k(x(t)), \tag{2.2.3a}$$

$$\Delta x(t) = I(t, x(t)), \; t = \tau_k(x(t)). \tag{2.2.3b}$$

When the functions $\tau_k$ depend on the state, then system (2.2.3) is said to have impulses at variable times. This is reflected in the fact that different solutions will tend to undergo impulses at different times. If the functions $\tau_k$ are all constant, then (2.2.3) is said to be a system having impulses at fixed times. In this case all solutions undergo the impulsive action (2.2.3b) at the same times.

The question of existence of solutions of system (2.2.3) is a nontrivial one when impulses occur at variable times. Even the precise notion of what a solution is must be carefully stated. It is fairly clear that solutions should be piecewise continuous and in fact piecewise continuously differentiable (or piecewise absolutely continuous if considering generalized types of solutions). A solution will undergo simple jump discontinuities when it intersects impulse hypersurfaces $t = \tau_k(x(t))$.

It is common practice to assume solutions of (2.2.3) are left-continuous [Lak89]. This is also intuitively appealing since one can think of a solution as approaching and actually reaching a hypersurface before being mapped to some new point. When incorporating impulses into delay differential equations we will deviate from this convention for other reasons and instead assume that solutions are right-continuous.

Under the assumption that solutions should be left-continuous, the initial condition for system (2.2.3) is typically given by

$$x(t_0^+) = x_0, \tag{2.2.4}$$

which is a slight modification of (2.1.6). Both (2.1.6) and (2.2.4) are equivalent when $t_0 \neq \tau_k(x_0)$ for all $k$. Condition (2.2.4) emphasizes the fact that we do not consider a solution to instantly undergo an impulse at the initial time $t_0$.

If instead, one were to consider solutions to be right-continuous, then the corresponding initial condition would be simply (2.1.6) although again we would not expect an impulse to occur at $t = t_0$ even if $t_0 = \tau_k(x_0)$ for some $k$. Under the right-continuous assumption we would also have to modify equation (2.2.3) to read

$$x'(t) = f(t, x(t)), \; t \neq \tau_k(x(t^-)), \tag{2.2.5a}$$

$$\Delta x(t) = I(t, x(t^-)), \; t = \tau_k(x(t^-)). \tag{2.2.5b}$$

Even after focussing on a particular class of relations $h(t, x(t)) = 0$ given by these impulse hypersurfaces, impulsive differential equations still exhibit some unusual behaviour. Solutions, depending on how they are defined, could undergo an infinite number of impulses in a finite amount of time unless some additional restrictions on the functions $\tau_k$ are imposed. In addition, solutions may not exist after reaching one of the impulse hypersurfaces. They may also intercept the same hypersurface $t = \tau_k(x(t))$ more than once or not at all or even intercept it after intercepting a subsequent hypersurface $t = \tau_i(x(t))$ for some $i > k$. The repeated intersection of the same hypersurface by a solution is commonly known as a pulse or beating phenomenon and conditions are often sought to guarantee that it does not happen.

Another problem associated with system (2.2.3) is that the usual notion of continuous dependence and stability of solutions tends to break down since neighbouring solutions tend to undergo jump discontinuities at slightly different times. When impulses occur at fixed times, then much of the theory of ordinary differential equations can be directly carried over to the study of these impulsive differential equations.

Before concluding this section we make one final comment about impulsive differential equations. Suppose for simplicity that the impulses in system (2.2.3) occur at fixed times. If the function $I$ is defined so that $x + I(\tau_k, x)$ is not a one-to-one function in $x$, then it is possible that two different solutions could merge following the impulsive action at time $t = \tau_k$. This merging, also known as confluence, of solutions cannot happen in ordinary differential equations when $f$ is sufficiently smooth. Note also that in this situation if one were to try to extend a solution of (2.2.3) backwards in time, then nonunique solutions would result. In fact, if $x + I(\tau_k, x)$ were not also a surjective function, then backward continuation of solutions would be impossible. It is therefore more fruitful to consider only the forward continuation of solutions.

More will be said about impulsive differential equations when we later discuss their incorporation into delay differential equations.

## 2.3 Delay Differential Equations on the Space of Piecewise Continuous Functions

In describing an impulsive delay differential equation we could either start with an impulsive differential equation and describe how one would incorporate delays into it or we could start with a delay differential equation and then add impulses. We choose the latter approach.

Adding impulses to system (2.1.3) will invariably lead us to the consideration of piecewise continuous functions. Since solutions will be piecewise continuous, then the functional $f$ must be defined on the larger class of piecewise continuous functions. Moreover, the initial condition must be generalized to piecewise continuous initial functions. At first glance one might think we could continue to consider only continuous initial functions. But later in our proofs we will need to make use of the semi-group property of the dynamical system (i.e. $x(t, t_0, \phi) = x(t, t_1, x_{t_1}(t_0, \phi))$ for $t_0 \le t_1 \le t$) and this will require that we consider piecewise continuous initial functions as well.

Before actually adding impulses to system (2.1.3), in this section we will simply examine system (2.1.3) over the larger class of piecewise continuous functions and elaborate on the complications that ensue.

We start by introducing the following notation. For $a, b \in \mathbb{R}$ with $a < b$ and for $S \subset \mathbb{R}^n$ define

$$PC([a,b],S) = \Big\{ \psi : [a,b] \to S \;\Big|\; \psi(t^+) = \psi(t)\ \forall t \in [a,b),\ \psi(t^-) \text{ exists in } S\ \forall t \in (a,b] \text{ and } \psi(t^-) = \psi(t) \text{ for all but at most a finite number of points } t \in (a,b] \Big\},$$

$$PC([a,b),S) = \Big\{ \psi : [a,b) \to S \;\Big|\; \psi(t^+) = \psi(t)\ \forall t \in [a,b),\ \psi(t^-) \text{ exists in } S\ \forall t \in (a,b) \text{ and } \psi(t^-) = \psi(t) \text{ for all but at most a finite number of points } t \in (a,b) \Big\}, \text{ and}$$

$$PC([a,\infty),S) = \Big\{ \psi : [a,\infty) \to S \;\Big|\; \forall c > a,\ \psi|_{[a,c]} \in PC([a,c],S) \Big\}.$$

These function classes describe functions that are right-continuous on their domain and that are left-continuous except at simple jump discontinuities where they have a left-hand limit. This left-hand limit, denoted by $\psi(t^-)$, must be finite-valued and contained in the set $S$. Such functions may have only a finite number of jump discontinuities unless they are defined on a set of the form $[a, \infty)$. In this case they may have simple jump discontinuities at a countably infinite number of points that form an increasing sequence tending to infinity.

Given the delay constant $r > 0$ as in Section 2.2, we equip the linear space $PC([-r,0],\mathbb{R}^n)$ with the norm $\|\cdot\|_r$ defined by $\|\psi\|_r = \sup_{-r \le s \le 0} \|\psi(s)\|$. If $x \in PC([t_0 - r, \infty), \mathbb{R}^n)$ where $t_0 \in \mathbb{R}_+$, then again as in Section 2.2, for each $t \ge t_0$ we define $x_t \in PC([-r,0],\mathbb{R}^n)$ by $x_t(s) = x(t+s)$ for $-r \le s \le 0$.

Let $J \subset \mathbb{R}_+$ be an interval of the form $[a,b)$ where $0 \le a < b \le \infty$ and let $D \subset \mathbb{R}^n$ be an open set. Then we consider the system

$$x'(t) = f(t, x_t), \tag{2.3.1}$$

where $f : J \times PC([-r,0],D) \to \mathbb{R}^n$. Moreover, we impose the initial condition

$$x_{t_0} = \phi, \tag{2.3.2}$$

where $t_0 \in \mathbb{R}_+$ and $\phi \in PC([-r,0],D)$.

Note that one advantage in assuming that solutions are right-continuous can be seen immediately. That is that the entire initial data can be incorporated into the single function $\phi$ and we do not need to separately include an extra initial condition specifying the right-hand limit of the solution at the initial time. Also, it remains reasonable to expect that the solution will be right-differentiable and satisfy (2.3.1) with its right derivative at $t = t_0$.

Without impulses to cause future discontinuities beyond those that may already be present in the initial condition, the solution must evolve continuously for $t \geq t_0$. At the very least it must be absolutely continuous after the initial time. At first glance, a reasonable expectation for solutions would be that they should satisfy (2.3.1) at all times $t > t_0$ (and with their right derivative at $t = t_0$). Then for well-behaved and sufficiently smooth functionals $f$, we would get solutions that are continuously differentiable for $t \geq t_0$.

Unfortunately, restricting one's attention to systems exhibiting this behaviour ends up excluding a whole class of common delay differential equations including delay differential difference equations. The initial discontinuities can cause "solutions" of these systems to fail to be differentiable at future times. For example, consider the scalar delay differential difference equation given by

$$x'(t) = x(t-1), \tag{2.3.3}$$

with $r = 1$, $t_0 = 0$ and a piecewise continuous initial function $\phi$ having a discontinuity at some point $t^* \in (-1, 0]$. This equation can be solved using the method of steps. However, it is clear that any "solution" of (2.3.3) will not have a derivative at the point $t^* + 1$ although it will be continuous and have one-sided derivatives there.

In order to accommodate these important types of systems we must weaken our notion of a solution somewhat. We choose to weaken it by permitting a solution to have a finite number of points on any finite interval of time where the solution may not be differentiable. However, at these points we still expect the solution to be continuous and have a right-hand derivative satisfying (2.3.1).

One could weaken the definition of a solution further, as is occasionally done with delay differential equations, by defining generalized solutions that are solutions of the integral equation corresponding to (2.3.1) and (2.3.2). Such solutions would be absolutely continuous and satisfy (2.3.1) almost everywhere with respect to the Lebesgue measure. Carathéodory-type conditions would have to be imposed on the functional $f$ in order to guarantee existence and uniqueness of these types of solutions.

Nevertheless, we prefer to focus our attention on the more classical types of solutions. The advantages are that the development of the theory is simpler, more closely resembling the classical theory of ordinary differential equations, and that in practice the examples and applications one encounters tend more often than not to be of the form that we first described. Our definition of a solution is the following.

**Definition 2.3.1:** A function $x \in PC([t_0 - r, t_0 + \alpha], D)$ where $\alpha > 0$ and $[t_0, t_0 + \alpha] \subset J$ is said to be a *solution of (2.3.1)* if

(i) $x$ is continuous at each $t \in (t_0, t_0 + \alpha]$;

(ii) the derivative of $x$ exists and is continuous at all but at most a finite number of points $t$ in $(t_0, t_0 + \alpha)$; and

(iii) the right-hand derivative of $x$ exists and satisfies the delay differential equation (2.3.1) for all $t \in [t_0, t_0 + \alpha)$.

If in addition $x$ satisfies the initial condition (2.3.2), then it is said to be a *solution of (the initial value problem) (2.3.1) & (2.3.2)* and we write $x = x(t_0, \phi)$ to emphasize this.

**Definition 2.3.2:** A function $x \in PC([t_0 - r, t_0 + \beta), D)$ where $0 < \beta \leq \infty$ and $[t_0, t_0 + \beta) \subset J$ is said to be a *solution of (2.3.1)* (*solution of (2.3.1) & (2.3.2)*) if for each $0 < \alpha < \beta$ the restriction of $x$ to $[t_0 - r, t_0 + \alpha]$ is a solution of (2.3.1) (solution of (2.3.1) & (2.3.2)) and if $\beta < \infty$, then the derivative of $x$ exists and is continuous at all but at most a finite number of points $t$ in $(t_0, t_0 + \beta)$.

In order to obtain existence and uniqueness results we require that the functional $f$ satisfy various continuity, boundedness and Lipschitz types of conditions. Before elaborating further on these properties we wish to point out some key differences between continuous delay differential equations and delay differential equations on the space of piecewise continuous functions.

One fundamental difference between continuous and piecewise continuous functions with respect to delay differential equations is the following. If $x \in C([t_0 - r, t_0 + \alpha], D)$ for some $\alpha > 0$, then $x_t$ is a continuous function of $t$ (with respect to $\|\cdot\|_r$) for $t \in [t_0, t_0 + \alpha]$, while if $x \in PC([t_0 - r, t_0 + \alpha], D)$, then $x_t$ may be discontinuous at some or all $t \in [t_0, t_0 + \alpha]$. This key difference causes a myriad of problems when analyzing delay differential equations over the space of piecewise continuous functions and we will make note of it time and again. The proof of the first part can be found in [Hal93] and we state it here.

**Lemma 2.3.1:** *Assume $x \in C([t_0 - r, t_0 + \alpha], D)$. Then $x_t$ is a continuous function of $t$ (with respect to $\|\cdot\|_r$) for $t \in [t_0, t_0 + \alpha]$.*

*Proof:* Since $x$ is continuous on $[t_0 - r, t_0 + \alpha]$, then it is uniformly continuous and thus for every $\epsilon > 0$, there exists a $\delta > 0$ such that $\|x(t_1) - x(t_2)\| < \epsilon$ for all $t_1, t_2 \in [t_0 - r, t_0 + \alpha]$ with $|t_1 - t_2| < \delta$. Thus for $t_1, t_2 \in [t_0, t_0 + \alpha]$, $|t_1 - t_2| < \delta$ and $s \in [-r, 0]$ we get $\|x(t_1 + s) - x(t_2 + s)\| < \epsilon$, which implies $\|x_{t_1} - x_{t_2}\|_r \leq \epsilon$, thus proving the lemma. ∎

To show how a single discontinuity can destroy the conclusion of Lemma 2.3.1, consider the function

$$x(t) = \begin{cases} 0, & t \in [-1, 0), \\ 1, & t \in [0, 1], \end{cases} \tag{2.3.4}$$

where $t_0 = 0$, $r = 1$ and $\alpha = 1$. Suppose $t_1, t_2 \in [0, 1]$ and $\delta > 0$ with $0 < t_1 - t_2 < \delta$. Then for $s = -t_1 \in [-r, 0]$, $|x(t_1 + s) - x(t_2 + s)| = |x(0) - x(t_2 - t_1)| = 1$, which implies $\|x_{t_1} - x_{t_2}\|_r = 1$. So clearly $x_t$ is discontinuous at each $t \in [0, 1]$.

While we can say little about the smoothness of $x_t$ as a function of $t$ for a piecewise continuous function $x$, we can show that $\|x_t\|_r$ is a piecewise continuous real-valued function of $t$. We prove this in the following lemma which we will later use.

**Lemma 2.3.2:** *Assume $x \in PC([t_0 - r, t_0 + \alpha], D)$ and let $g(t) = \|x_t\|_r$ for $t \in [t_0, t_0 + \alpha]$. Then $g \in PC([t_0, t_0 + \alpha], \mathbf{R}_+)$ and the only possible points of discontinuity of $g$ are $t^*$ or $t^* + r$, where $t^*$ denotes a point of discontinuity of $x$.*

*Proof:* We will first prove that $g(t_1^+) = g(t_1)$ for all $t_1 \in [t_0, t_0 + \alpha)$. Let $t_1 \in [t_0, t_0 + \alpha)$ and note that $g(t_1) = \sup_{t \in [t_1 - r, t_1]} \|x(t)\|$. Let $\epsilon > 0$. Since $x$ is assumed to be right-continuous at $t_1$ and $t_1 - r$ (and indeed at all points in $[t_0 - r, t_0 + \alpha)$), then there exists a $0 < \delta < t_0 + \alpha - t_1$ such that if $t \in [t_1, t_1 + \delta]$, then $\|x(t) - x(t_1)\| \le \epsilon/2$ and $\|x(t - r) - x(t_1 - r)\| \le \epsilon/2$. Let $t_2 \in (t_1, t_1 + \delta]$. If $t \in [t_1, t_2]$, then $\|x(t)\| \le \|x(t_1)\| + \epsilon/2 \le g(t_1) + \epsilon/2$. Thus $\|x(t)\| \le g(t_1) + \epsilon/2$ for all $t \in [t_1 - r, t_2]$. Since $[t_2 - r, t_2] \subset [t_1 - r, t_2]$, then this in turn implies that $g(t_2) \le g(t_1) + \epsilon/2$. Similarly, if $t \in [t_1 - r, t_2 - r]$, then $\|x(t)\| = \|x(t) - x(t_1 - r) + x(t_1 - r) - x(t_2 - r) + x(t_2 - r)\| \le \|x(t) - x(t_1 - r)\| + \|x(t_1 - r) - x(t_2 - r)\| + \|x(t_2 - r)\| \le \epsilon/2 + \epsilon/2 + g(t_2) = \epsilon + g(t_2)$ and so $\|x(t)\| \le g(t_2) + \epsilon$ for all $t \in [t_1 - r, t_2]$. Since $[t_1 - r, t_1] \subset [t_1 - r, t_2]$, then this implies that $g(t_1) \le g(t_2) + \epsilon$. So combining our results gives us $|g(t_1) - g(t_2)| \le \epsilon$, which proves that $g(t_1^+) = g(t_1)$.

Note that to prove $g(t_1^+) = g(t_1)$ we only required that $x$ be right-continuous at $t_1$ and at $t_1 - r$. By a similar argument we can prove that $g(t_1^-) = g(t_1)$ for all $t_1 \in [t_0, t_0 + \alpha)$ providing $x$ is left-continuous at $t_1$ and at $t_1 - r$. Moreover, if $t_1 \in (t_0, t_0 + \alpha]$ and either $t_1$ or $t_1 - r$ is a point of discontinuity of $x$, then it is easy to verify that $g(t_1^-)$ exists.

Thus $g$ is right-continuous on all of $[t_0, t_0 + \alpha)$ and is left-continuous at each $t_1 \in (t_0, t_0 + \alpha]$ unless $t_1$ or $t_1 - r$ happen to be discontinuities of $x$, in which case $g(t_1^-)$ exists (yet may still equal $g(t_1)$). The only discontinuities of $g$ may occur at times $t^*$ or $t^* + r$ where $t^*$ is a point of discontinuity of the function $x$. ■

Defining $g(t) = \|x_t\|_r$ for the function (2.3.4) gives us $g(t) = 1$ for all $t \in [0, 1]$, which shows that $g \in PC([0, 1], \mathbf{R}_+)$ (in fact $g$ is continuous on $[0, 1]$) as Lemma 2.3.2 predicts.

When dealing with continuous delay differential equations it is common practice to assume that the solution trajectories are of the form $(t, x_t)$ and evolve in the function space $\mathbf{R}_+ \times C([-r, 0], \mathbf{R}^n)$, or some open subset thereof, instead of the more usual integral curve $(t, x(t))$ in $\mathbf{R}_+ \times \mathbf{R}^n$. This is the approach taken by Hale and Lunel [Hal93] where they discuss some of the advantages and disadvantages of the two approaches. Because $x_t$ is generally not even piecewise continuous as a function of $t$ when $x$ is a piecewise continuous function, this approach would not be appropriate for systems involving piecewise continuous functions. Unfortunately, this means that much of the theory developed for continuous delay differential equations cannot be indiscriminately applied to delay differential equations over the space of piecewise continuous functions.

For continuous delay differential equations, the functional $f$ is typically assumed to be jointly continuous in its two variables. This condition is sufficient to guarantee the local existence of solutions of (2.1.3) & (2.1.7) [Hal93]. This turns out to be a poor choice of a condition for functionals defined on piecewise continuous functions. Many simple functionals that are continuous on $\mathbb{R}_+ \times C([-r,0],\mathbb{R}^n)$ cannot be extended continuously to $\mathbb{R}_+ \times PC([-r,0],\mathbb{R}^n)$ for the same reason that $x_t$ is no longer continuous in $t$. An example of such a functional is

$$f(t,\psi) = \psi(-1-e^{-t}), \tag{2.3.5}$$

which corresponds to the equation

$$x'(t) = x(t-1-e^{-t}), \tag{2.3.6}$$

with $r = 2$. Together with a piecewise continuous initial function, this equation can be solved by the method of steps and can be shown to have a solution that is continuous and has a piecewise continuous derivative for $t \geq t_0$ (see Corollary 3.1.1 in the next chapter). This is despite the fact that $f$ is not continuous on $\mathbb{R}_+ \times PC([-2,0],\mathbb{R})$. While $f$ is continuous with respect to $\psi$ for fixed values of $t \in \mathbb{R}_+$, it is not continuous with respect to $t$ for fixed values of $\psi \in PC([-2,0],\mathbb{R})$ that have discontinuities in the interval $(-2,-1)$. While $f$ will be piecewise continuous with respect to $t$ for each fixed $\psi$, the times where it experiences a jump discontinuity will clearly vary depending on the choice of $\psi$.

Instead of the usual continuity assumption on $f$, Driver [Dri77] assumes that the composite function $f(t,x_t)$ is continuous for every continuous function $x$ when he considers continuous delay differential equations. If $f$ were assumed to be continuous in its two variables, then it would satisfy this weaker hypothesis since the composition of continuous functions is continuous (and here $x_t$ would be continuous since $x$ is). We find that an adaptation of this sort of condition is best suited to our functional $f$. For the functional $f$ given in (2.3.5), the composite function $f(t,x_t)$, where $x$ is any piecewise continuous function, is itself piecewise continuous with respect to $t$. This property, which we define more formally later and refer to as $f$ being "composite-PC," will be what we will use in our existence and uniqueness theorems.

When $x$ is piecewise continuous, $x_t$ may be highly discontinuous as a function of $t$. Therefore, even if we were to assume that $f$ was continuous on $\mathbb{R}_+ \times PC([-r,0],\mathbb{R}^n)$, we could not, in general, conclude anything about the continuity of the composite function $f(t,x_t)$. However, we would need to do so in any proof of existence. This further explains the need to consider this alternative continuity assumption for $f$.

As an illustration, let $r = 1$ and let $\{q_n\}$ be some enumeration of the rational numbers in $[-1,0)$.

Next define the functional

$$f(t,\psi) = \sum_{n=1}^{\infty} \frac{\psi(q_n)}{2^n}. \tag{2.3.7}$$

Here $f$ is actually independent of $t$. Since functions $\psi$ in $PC([-1,0],\mathbb{R})$ are bounded, the series in (2.3.7) converges absolutely and so $f$ is well-defined. Moreover, $f$ is clearly continuous on all of $\mathbb{R}_+ \times PC([-1,0],\mathbb{R})$. Let us look at the composite function $f(t,x_t)$ for $t \in [0,1]$ where $x$ is given by the piecewise continuous function (2.3.4). Then

$$f(t,x_t) = \sum_{n=1}^{\infty} \frac{x(t+q_n)}{2^n} = \sum_{\{n:\, q_n \geq -t\}} \frac{1}{2^n} \tag{2.3.8}$$

is discontinuous at each rational number $t$ in $(0,1]$. The functional (2.3.7) is not composite-PC as defined below.

In most practical cases, solutions of (2.3.1), where $f$ is defined on the space of piecewise continuous functions $\mathbb{R}_+ \times PC([-r,0],\mathbb{R}^n)$, will exhibit properties similar to solutions of the same equation but with the domain of $f$ restricted to continuous functions, i.e. $\mathbb{R}_+ \times C([-r,0],\mathbb{R}^n)$. In other words, solutions corresponding to piecewise continuous initial functions $\phi$ will have similar characteristics to those with continuous initial functions. In particular, note that a solution $x = x(t_0,\phi)$ of (2.3.1) satisfying (2.3.2) with $\phi \in PC([-r,0],\mathbb{R}^n)$ will satisfy $x_t \in C([-r,0],\mathbb{R}^n)$ for $t \geq t_0 + r$. Of course if $\phi$ is continuous, then $x_t \in C([-r,0],\mathbb{R}^n)$ for all $t \geq t_0$.

It is possible, however, for $f(t,\psi)$ to be defined in such a way that it depends on the jump discontinuities of $\psi$, if applicable. Consider for example the functional $f : \mathbb{R}_+ \times PC([-1,0],\mathbb{R}) \to \mathbb{R}$ defined by

$$f(t,\psi) = \begin{cases} 0, & \text{if } \psi \text{ is continuous on } [-1,0], \\ \left(\max\limits_{1\leq k\leq m} \{|\psi(s_k) - \psi(s_k^-)|\}\right)\psi^2(0), & \text{where } -1 < s_1 < s_2 < \ldots < s_k < \ldots < s_m \leq 0 \\ & \text{denote the points of discontinuity of } \psi. \end{cases} \tag{2.3.9}$$

This functional $f$ satisfies the conditions of Theorem 3.1.1 on the space $\mathbb{R}_+ \times PC([-1,0],\mathbb{R})$. For continuous initial functions $\phi$, the corresponding solutions are constant and given by $x(t) = \phi(0)$ for $t \geq t_0$. In particular, these solutions exist globally and are uniformly bounded. Next consider the discontinuous initial function

$$\phi(s) = \begin{cases} 0, \; s \in [-1,0), \\ \delta, \; s = 0, \end{cases} \tag{2.3.10}$$

where $\delta > 0$ is arbitrary. For $\delta < 1$ the resulting solution is given by

$$x(t) = \begin{cases} \dfrac{\delta}{1 - \delta^2(t - t_0)}, & t \in [t_0, t_0 + 1), \\ \dfrac{\delta}{1 - \delta^2}, & t \in [t_0 + 1, \infty). \end{cases} \tag{2.3.11}$$

On the other hand, if $\delta \geq 1$, then the solution is given by

$$x(t) = \frac{\delta}{1 - \delta^2(t - t_0)}, \; t \in [t_0, t_0 + 1/\delta^2). \tag{2.3.12}$$

In particular, these solutions become unbounded as $t \to t_0 + 1/\delta^2$ and cannot be extended beyond this maximal interval of existence.

This example illustrates that by enlarging the domain of $f$ to include piecewise continuous functions we may lose desirable properties such as global existence and boundedness of solutions, since the set of solutions is correspondingly larger. Of course, results that establish boundedness of solutions given piecewise continuous initial functions, for example, will also give us boundedness of solutions having continuous initial functions. The functional (2.3.9) is a rather artificial construct and we will not be considering functionals like it in the remainder of this document.

When introducing our existence and uniqueness theorems we will need to impose certain conditions on the functional $f$. We define the following properties which we will frequently refer to.

**Definition 2.3.3:** A functional $f : J \times PC([-r, 0], D) \to \mathbb{R}^n$ is said to be *composite-PC* if for each $t_0 \in J$ and $0 < \alpha \leq \infty$, where $[t_0, t_0 + \alpha) \subset J$, if $x \in PC([t_0 - r, t_0 + \alpha), D)$, then the composite function $g$ defined by $g(t) = f(t, x_t)$ is an element of the function class $PC([t_0, t_0 + \alpha), \mathbb{R}^n)$.

**Definition 2.3.4:** A functional $f : J \times PC([-r, 0], D) \to \mathbb{R}^n$ is said to be *quasi-bounded* if for each $t_0 \in J$ and $\alpha > 0$, where $[t_0, t_0 + \alpha] \subset J$, and for each compact set $F \subset D$, there exists some $M > 0$ such that $\|f(t, \psi)\| \leq M$ for all $(t, \psi) \in [t_0, t_0 + \alpha] \times PC([-r, 0], F)$.

**Definition 2.3.5:** A functional $f : J \times PC([-r, 0], D) \to \mathbb{R}^n$ is said to be *continuous in $\psi$* if for each fixed $t \in J$, $f(t, \psi)$ is a continuous function of $\psi$ on $PC([-r, 0], D)$.

**Definition 2.3.6:** A functional $f : J \times PC([-r, 0], D) \to \mathbb{R}^n$ is said to be *locally Lipschitz in $\psi$* if for each $t_0 \in J$ and $\alpha > 0$, where $[t_0, t_0 + \alpha] \subset J$, and for each compact set $F \subset D$, there exists some $L > 0$ such that $\|f(t, \psi_1) - f(t, \psi_2)\| \leq L\|\psi_1 - \psi_2\|_r$ for all $t \in [t_0, t_0 + \alpha]$ and $\psi_1, \psi_2 \in PC([-r, 0], F)$.

If $f$ is locally Lipschitz in $\psi$, then clearly it is also continuous in $\psi$. If in addition $f$ is composite-PC, then it is also quasi-bounded, since $\|f(t, \psi)\| \leq L\|\psi\|_r + \|f(t, 0)\|$ for $t \in [t_0, t_0 + \alpha]$ where $\|\psi\|_r \leq \sup\{\|x\| \mid x \in F\}$ and where $\|f(t, 0)\|$ is bounded above by some constant since $f(t, 0)$ is a piecewise continuous (and hence bounded) function of $t$. Through slight abuse of notation we use the symbol "0" to represent either a zero scalar, a zero vector or a zero function (or functional). In this case it refers both to the zero element of $PC([-r,0],D)$ and the zero element of $PC([t_0-r,t_0+\alpha],D)$ and in general it will be obvious what it represents from the context. In fact the same goes for the symbol "$x$" which in some instances refers to a piecewise continuous function while in other instances it refers to a point in $\mathbf{R}^n$. Note that while this argument assumes that $D$ contains the origin, it can be easily modified to account for the case when $0 \notin D$ (zero here representing a point in $\mathbf{R}^n$). The Lipschitz condition will be discussed further in Chapter 3 with respect to uniqueness of solutions as well as in Chapters 4 and 5 in the context of Lyapunov functionals and functions.

We conclude this section with a simple lemma that introduces an equivalent integral formulation of system (2.3.1) & (2.3.2). Its proof is straight-forward.

**Lemma 2.3.3:** *Suppose $f$ is composite-PC. Then a function $x \in PC([t_0-r,t_0+\alpha],D)$, where $\alpha > 0$ and $[t_0,t_0+\alpha] \subset J$, is a solution of (2.3.1) & (2.3.2) if and only if $x$ satisfies*

$$x(t) = \begin{cases} \phi(t-t_0), & t \in [t_0-r,t_0], \\ \phi(0) + \int_{t_0}^{t} f(s,x_s)ds, & t \in (t_0,t_0+\alpha], \end{cases} \tag{2.3.13}$$

*for $t \in [t_0-r,t_0+\alpha]$.*

Note that if $x$ is defined on an interval of the form $[t_0-r,t_0+\beta)$ for some $0 < \beta \leq \infty$ where $[t_0,t_0+\beta) \subset J$, then Lemma 2.3.3 will also give the equivalent integral formulation of a solution of (2.3.1) & (2.3.2).

## 2.4 Impulsive Delay Differential Equations

In Section 2.2 we described impulsive differential equations and in Section 2.1 we described delay differential equations that we then expanded upon in Section 2.3 by examining them on the space of piecewise continuous functions. In this section we combine these to give an impulsive delay differential equation.

The impulsive delay differential equation that we will consider will essentially be system (2.2.5), since we are assuming right-continuous solutions, where the ordinary differential equation in (2.2.5a) is replaced by the delay differential equation (2.3.1).

Before constructing an impulsive delay differential equation in this manner, we make one additional generalization. Instead of the impulse operator $I$ depending only on $x(t^-)$, which is essentially the current value of the state, we will allow it to also have delays so that it may depend on past values of the state. So in addition to depending on $x(t^-)$ whenever $t = \tau_k(x(t^-))$, it may also depend on values of $x(t+s)$ for $s \in [-r,0)$. This will make $I$ a functional much like $f$. In most of our examples, however, we will choose $I$ to depend only on $x(t^-)$ since this value usually plays a dominant role.

As we pointed out in Section 2.3, for piecewise continuous functions $x \in PC([t_0 - r, \infty), \mathbf{R}^n)$, the limit $\lim_{s \to t^-} x_s$ does not generally exist with respect to the norm $\|\cdot\|_r$. So we will define $x_{t^-} \in PC([-r, 0], \mathbf{R}^n)$ by $x_{t^-}(s) = x(t+s)$ for $-r \leq s < 0$ and $x_{t^-}(s) = x(t^-)$ for $s = 0$. Note that this does not mean $x_{t^-} = \lim_{s \to t^-} x_s$. Using this notation for $x_{t^-}$ we may express the dependency of $I$ on $x(t^-)$ and past values of the state by $I(t, x_{t^-})$. So just as we have $f : J \times PC([-r, 0], D) \to \mathbf{R}^n$, so too do we assume that $I : J \times PC([-r, 0], D) \to \mathbf{R}^n$. Later we will impose various conditions on the functionals $f$ and $I$.

We now present our impulsive delay differential equation in the following form

$$x'(t) = f(t, x_t), \; t \neq \tau_k(x(t^-)), \tag{2.4.1a}$$

$$\Delta x(t) = I(t, x_{t^-}), \; t = \tau_k(x(t^-)). \tag{2.4.1b}$$

The initial condition for system (2.4.1) will simply be given by the usual

$$x_{t_0} = \phi, \tag{2.4.2}$$

where $t_0 \in \mathbf{R}_+$ and $\phi \in PC([-r, 0], \mathbf{R}^n)$. As in Section 2.2, the functions $\tau_k$ are assumed to satisfy $0 = \tau_0(x) < \tau_1(x) < \tau_2(x) < \ldots$ and $\lim_{k \to \infty} \tau_k(x) = \infty$ for each $x \in \mathbf{R}^n$. However, we only assume they are defined on the domain $D$; in other words $\tau_k \in C(D, \mathbf{R}_+)$ for each $k$.

We next define a solution of (2.4.1). Note that we use the notation $A \backslash B$ to denote the difference of two sets $A$ and $B$ (i.e. $A \backslash B = \{t \mid t \in A \text{ and } t \notin B\}$).

**Definition 2.4.1:** A function $x \in PC([t_0 - r, t_0 + \alpha], D)$ where $\alpha > 0$ and $[t_0, t_0 + \alpha] \subset J$ is said to be a *solution of (2.4.1)* if

(i) the set $T = \{t \in (t_0, t_0 + \alpha] \mid t = \tau_k(x(t^-)) \text{ for some } k\}$ of impulse times is finite (possibly empty);

(ii) $x$ is continuous at each $t \in (t_0, t_0 + \alpha] \backslash T$;

(iii) the derivative of $x$ exists and is continuous at all but at most a finite number of points $t$ in $(t_0, t_0 + \alpha)$;

(iv) the right-hand derivative of $x$ exists and satisfies the delay differential equation (2.4.1a) for all $t \in [t_0, t_0 + \alpha) \backslash T$; and

(v) $x$ satisfies the delay difference equation (2.4.1b) for all $t \in T$.

If in addition, $x$ satisfies the initial condition (2.1.7), then it is said to be a *solution of (the initial value problem) (2.4.1) & (2.4.2)* and we write $x = x(t_0, \phi)$.

**Definition 2.4.2:** A function $x \in PC([t_0 - r, t_0 + \beta), D)$ where $0 < \beta \leq \infty$ and $[t_0, t_0 + \beta) \subset J$ is said to be a ***solution of (2.4.1)*** (***solution of (2.4.1) & (2.4.2)***) if for each $0 < \alpha < \beta$ the restriction of $x$ to $[t_0 - r, t_0 + \alpha]$ is a solution of (2.4.1) (solution of (2.4.1) & (2.4.2)) and if $\beta < \infty$, then the derivative of $x$ exists and is continuous at all but at most a finite number of points $t$ in $(t_0, t_0 + \beta)$ and the set $T = \{t \in (t_0, t_0 + \beta) \mid t = \tau_k(x(t^-)) \text{ for some } k\}$ is finite.

Note that the points where a solution fails to have a continuous derivative will generally include but may not be limited to impulse times. We require that there be at most a finite number of such exceptional points on any finite interval of time. In our definition, we do not explicitly require that solutions have a right-hand derivative satisfying (2.4.1a) at the impulse times in $T$ although in practice they will.

A solution $x = x(t_0, \phi)$ of (2.4.1) & (2.4.2) existing on $[t_0 - r, t_0 + \alpha]$ and experiencing impulses at the points $T = \{t_k\}_{k=1}^m$ where $t_0 < t_1 < t_2 < \ldots < t_m \leq t_0 + \alpha$ can be described by

$$x(t, t_0, \phi) = \begin{cases} x(t, t_0, \phi), & t \in [t_0 - r, t_1), \\ x(t, t_k, x_{t_k}), & t \in [t_k, t_{k+1}),\ k = 1, 2, \ldots, m-1, \\ x(t, t_m, x_{t_m}), & t \in [t_m, t_0 + \alpha], \end{cases} \tag{2.4.3}$$

where $x(t_k) = x(t_k^-) + I(t_k, x_{t_k^-})$. If the solution exists on $[t_0 - r, \infty)$, then it may experience impulses at an infinite number of points $T = \{t_k\}_{k=1}^\infty$ where $t_0 < t_1 < t_2 < \ldots < t_k < \ldots$ and $\lim_{k\to\infty} t_k = \infty$. Of course in this case we may express the solution by

$$x(t, t_0, \phi) = \begin{cases} x(t, t_0, \phi), & t \in [t_0 - r, t_1), \\ x(t, t_k, x_{t_k}), & t \in [t_k, t_{k+1}),\ k = 1, 2, \ldots. \end{cases} \tag{2.4.4}$$

For each $k$, $x(t, t_k, x_{t_k})$ for $t$ in the interval $[t_k, t_{k+1})$ represents a solution of (2.3.1) with $t_k$ denoting the initial time and $x_{t_k}$ representing the initial function. Before developing existence and uniqueness results for system (2.4.1) we will first develop them for (2.3.1) and then add additional conditions, as necessary, to establish those same properties for system (2.4.1).

Just as Lemma 2.3.3 gives an alternate integral formulation for problem (2.3.1) & (2.3.2), so too can we do the same for system (2.4.1) & (2.4.2). We state this in the next lemma whose proof is again straight-forward and hence omitted.

**Lemma 2.4.1:** *Suppose $f$ is composite-PC. Then a function $x \in PC([t_0 - r, t_0 + \alpha], D)$, where $\alpha > 0$ and $[t_0, t_0 + \alpha] \subset J$, that experiences the impulsive effect at the points $T = \{t_k\}_{k=1}^m$ where*

$t_0 < t_1 < t_2 < \ldots < t_m \leq t_0 + \alpha$ *is a solution of (2.4.1) & (2.4.2) if and only if* $x$ *satisfies*

$$x(t) = \begin{cases} \phi(t - t_0), & t \in [t_0 - r, t_0], \\ \phi(0) + \int_{t_0}^{t} f(s, x_s) ds, & t \in (t_0, t_1), \\ x(t_k^-) + I(t_k, x_{t_k^-}) + \int_{t_k}^{t} f(s, x_s) ds, & t \in [t_k, t_{k+1}),\ k = 1, 2, \ldots, m-1, \\ x(t_m^-) + I(t_m, x_{t_m^-}) + \int_{t_m}^{t} f(s, x_s) ds, & t \in [t_m, t_0 + \alpha], \end{cases} \tag{2.4.5}$$

*or equivalently,*

$$x(t) = \begin{cases} \phi(t - t_0), & t \in [t_0 - r, t_0], \\ \phi(0) + \int_{t_0}^{t} f(s, x_s) ds + \sum_{\{k:\ t_k \in (t_0, t]\}} I(t_k, x_{t_k^-}), & t \in (t_0, t_0 + \alpha]. \end{cases} \tag{2.4.6}$$

If $x$ is defined on an interval of the form $[t_0 - r, t_0 + \beta)$ for some $0 < \beta \leq \infty$ where $[t_0, t_0 + \beta) \subset J$, then Lemma 2.4.1 also gives us the equivalent integral formulation of a solution of (2.4.1) & (2.4.2).

Occasionally in the study of impulsive differential equations without delay one considers other kinds of "solutions". These more bizarre solutions are characterized by the fact that they undergo an infinite number of impulses in a finite amount of time, often caused by rhythmical beating upon an impulse hypersurface. The simplest of these cases is where the impulse times form an increasing sequence tending to some positive finite value. It is unclear how one could interpret a solution beyond this time, however.

These unusual types of solutions tend to lack a good physical interpretation and little qualitatively or quantitatively can be said about them. The same holds true for impulsive delay differential equations. Krishna and Anokhin [Kri94] discuss these so-called singular solutions in their study of impulsive delay differential equations and they mention the need to isolate circumstances where such solutions can exist or are guaranteed not to exist.

We prefer to restrict our attention only to those solutions that undergo impulses a finite number of times over any finite interval, but that, if defined for all $t \geq t_0$, may undergo impulses at a sequence of times tending to infinity. This preference is reflected in our notion of a solution given in Definitions 2.4.1 and 2.4.2. We will have to take this into account when obtaining conditions for the continuability of solutions of (2.4.1).

A special case of system (2.4.1) that we will focus a great deal of our attention on in later chapters is the system

$$x'(t) = f(t, x_t), \; t \neq \tau_k, \tag{2.4.7a}$$

$$\Delta x(t) = I(t, x_{t^-}), \; t = \tau_k, \tag{2.4.7b}$$

where the $\tau_k$ are constant and satisfy $0 = \tau_0 < \tau_1 < \ldots < \tau_k < \ldots$ and $\lim_{k\to\infty} \tau_k = \infty$. This system lacks many of the undesirable complications that the more general system (2.4.1) has.

# Chapter 3

# Fundamental Properties

In this chapter we will establish existence, uniqueness, continuability and continuous dependence conditions for solutions of (2.3.1) and more generally for the impulsive delay differential equation (2.4.1). Special conditions will have to be imposed to guarantee local existence and continuation results for system (2.4.1) in the case where impulse times are variable. Continuous dependence results will only be developed for the impulsive delay differential equation (2.4.7) having fixed impulse times since it is generally not feasible to expect continuous dependence for the arbitrary system (2.4.1).

## 3.1 Local Existence

We start by considering the delay differential equation (2.3.1), which is defined on the space of piecewise continuous functions but which lacks impulses. Our first theorem gives sufficient conditions for the existence of a local solution of system (2.3.1). The techniques used to prove this theorem are based on a similar existence proof developed for delay differential equations in [Ogu66].

**Theorem 3.1.1:** *Assume $f$ is composite-PC, quasi-bounded and continuous in $\psi$. Then for each $(t_0, \phi) \in J \times PC([-r,0], D)$, there exists a solution $x = x(t_0, \phi)$ of (2.3.1) & (2.3.2) on $[t_0 - r, t_0 + \beta]$ for some $\beta > 0$.*

*Proof:* Let $(t_0, \phi) \in J \times PC([-r,0], D)$ and choose $\alpha > 0$ so that $[t_0, t_0+\alpha] \subset J$. Since $\phi(0) \in D$ and $D$ is open choose $\lambda > 0$ such that $F_1 = \{z \in \mathbf{R}^n \mid \|z - \phi(0)\| \leq \lambda\} \subset D$. Since $\phi \in PC([-r,0], D)$, then the closure of the range of $\phi$, which we denote by $F_2$, is a compact subset of $D$. So define $F = F_1 \cup F_2$. Then $F$ is a compact subset of $D$. Since $f$ is quasi-bounded, then there must exist some $M > 0$ so that $\|f(t, \psi)\| \leq M$ for all $(t, \psi) \in [t_0, t_0 + \alpha] \times PC([-r,0], F)$. Let $\beta = \min\{\alpha, \lambda/M\} > 0$.

For $0 < \beta_1 \le \beta$ define

$$R(t_0, \phi, \lambda, \beta_1) = \Big\{ x \in PC([t_0 - r, t_0 + \beta_1], D) \,\Big|\, x_{t_0} = \phi, \ x \text{ is continuous on } (t_0, t_0 + \beta_1] \text{ and } \|x(t) - \phi(0)\| \le \lambda \ \forall t \in (t_0, t_0 + \beta_1] \Big\}. \tag{3.1.1}$$

If $x \in R(t_0, \phi, \lambda, \beta_1)$, then $x_t \in PC([-r, 0], F)$ for all $t \in [t_0, t_0 + \beta_1]$ from the definition of $F$ and so $\|f(t, x_t)\| \le M$ for $t \in [t_0, t_0 + \beta_1]$. Moreover, if $x \in R(t_0, \phi, \lambda, \beta_1)$, then the composite function $f(t, x_t)$ is in $PC([t_0, t_0 + \beta_1], \mathbb{R}^n)$ since $f$ is composite-PC. Note that when restricted to the domain $[t_0, t_0 + \beta_1]$, functions in $R(t_0, \phi, \lambda, \beta_1)$ are continuous since they are continuous on $(t_0, t_0 + \beta_1]$ and are right-continuous at $t_0$.

For $\mu = 1, 2, 3, \ldots$ define

$$x^{(\mu)}(t) = \begin{cases} \phi(t - t_0), & t \in [t_0 - r, t_0], \\ \phi(0), & t \in (t_0, t_0 + \beta/\mu], \\ \phi(0) + \displaystyle\int_{t_0}^{t - \beta/\mu} f(s, x_s^{(\mu)})ds, & t \in (t_0 + \beta/\mu, t_0 + \beta]. \end{cases} \tag{3.1.2}$$

We will first prove that each function $x^{(\mu)}$ is well-defined and is in $R(t_0, \phi, \lambda, \beta)$. This is obviously true for $x^{(1)}$. For any $\mu \ge 2$ the first two expressions in (3.1.2) define $x^{(\mu)}$ for $t \in [t_0 - r, t_0 + \beta/\mu]$ and, restricted to this interval, $x^{(\mu)} \in R(t_0, \phi, \lambda, \beta/\mu)$. Thus $f(t, x_t^{(\mu)})$ is piecewise continuous and consequently integrable on $[t_0, t_0 + \beta/\mu]$ and $\|f(t, x_t^{(\mu)})\| \le M$ for $t \in [t_0, t_0 + \beta/\mu]$. Therefore the third expression in (3.1.2) defines $x^{(\mu)}$ as a continuous function for $t \in (t_0 + \beta/\mu, t_0 + 2\beta/\mu]$. Moreover, for $t \in (t_0 + \beta/\mu, t_0 + 2\beta/\mu]$ we have

$$\begin{aligned} \|x^{(\mu)}(t) - \phi(0)\| &= \left\| \int_{t_0}^{t - \beta/\mu} f(s, x_s^{(\mu)})ds \right\| \le \int_{t_0}^{t - \beta/\mu} \|f(s, x_s^{(\mu)})\|ds \\ &\le \int_{t_0}^{t_0 + \beta/\mu} M ds = M\beta/\mu \le \lambda. \end{aligned} \tag{3.1.3}$$

This shows that $x^{(\mu)}$ is well-defined on $[t_0 - r, t_0 + 2\beta/\mu]$ and, when restricted to this interval, is in $R(t_0, \phi, \lambda, 2\beta/\mu)$. Now suppose that $x^{(\mu)}$ is well-defined on $[t_0 - r, t_0 + k\beta/\mu]$ for some $1 < k < \mu$ and, when restricted to this interval, is in $R(t_0, \phi, \lambda, k\beta/\mu)$. Then $\|f(t, x_t^{(\mu)})\| \le M$ and $f(t, x_t^{(\mu)})$ is piecewise continuous for $t \in [t_0, t_0 + k\beta/\mu]$. Thus (3.1.2) defines $x^{(\mu)}$ as a continuous function for $t \in (t_0 + k\beta/\mu, t_0 + (k+1)\beta/\mu]$. Also, inequality (3.1.3) holds for $t$ in this interval, which shows that $x^{(\mu)}$ restricted to this time interval is in $R(t_0, \phi, \lambda, (k+1)\beta/\mu)$. So by induction, $x^{(\mu)}$ is a well-defined function in $R(t_0, \phi, \lambda, \beta)$.

For each $\mu$ let $y^{(\mu)}$ denote the restriction of $x^{(\mu)}$ to $[t_0, t_0 + \beta]$. Then $y^{(\mu)}$ is continuous on $[t_0, t_0 + \beta]$. Moreover, for $t \in [t_0, t_0 + \beta]$, $\|y^{(\mu)}(t)\| \le \lambda + \|\phi(0)\|$ and so the functions $y^{(\mu)}$ are uniformly bounded. In addition, for any $t_1, t_2 \in [t_0, t_0+\beta]$ we have

$$\|y^{(\mu)}(t_1) - y^{(\mu)}(t_2)\| \le \left\| \int_{t_2-\beta/\mu}^{t_1-\beta/\mu} f(s, x_s^{(\mu)})ds \right\| \le M|t_1 - t_2|, \tag{3.1.4}$$

for all $\mu$, which implies that the functions $y^{(\mu)}$ are equicontinuous on the interval $[t_0, t_0+\beta]$. Hence, by Ascoli's Theorem, there exists a subsequence $\{y^{(\mu_k)}\}$ of the sequence of functions $\{y^{(\mu)}\}$ that converges uniformly to some continuous function $y$ on $[t_0, t_0+\beta]$ as $k \to \infty$.

Define

$$x(t) = \begin{cases} \phi(t-t_0), & t \in [t_0 - r, t_0], \\ y(t), & t \in (t_0, t_0+\beta]. \end{cases} \tag{3.1.5}$$

For each fixed $t \in [t_0, t_0+\beta]$, $\|x_t^{(\mu_k)} - x_t\|_r \to 0$ as $k \to \infty$ and since $f(t, \psi)$ is assumed to be continuous in $\psi$ for $t$ fixed, then $\lim_{k\to\infty} f(t, x_t^{(\mu_k)}) = f(t, x_t)$. Moreover, since $x^{(\mu_k)} \in R(t_0, \phi, \lambda, \beta)$, then $\|f(t, x_t^{(\mu_k)})\| \le M$ for $t \in [t_0, t_0+\beta]$.

By Lebesgue's Dominated Convergence Theorem we obtain

$$\lim_{k\to\infty} \int_{t_0}^{t} f(s, x_s^{(\mu_k)})ds = \int_{t_0}^{t} f(s, x_s)ds, \tag{3.1.6}$$

for all $t \in [t_0, t_0+\beta]$. From (3.1.2) we get

$$x^{(\mu_k)}(t) = \phi(0) + \int_{t_0}^{t} f(s, x_s^{(\mu_k)})ds - \int_{t-\beta/\mu_k}^{t} f(s, x_s^{(\mu_k)})ds, \; t \in (t_0 + \beta/\mu_k, t_0+\beta], \tag{3.1.7}$$

where the second integral tends to zero as $k \to \infty$. By taking the limit as $k \to \infty$ in (3.1.7) and using (3.1.6) we find

$$x(t) = \begin{cases} \phi(t-t_0), & t \in [t_0 - r, t_0], \\ \phi(0) + \displaystyle\int_{t_0}^{t} f(s, x_s)ds, & t \in (t_0, t_0+\beta], \end{cases} \tag{3.1.8}$$

which in light of Lemma 2.3.3 proves the theorem. ∎

The following examples provide a list of commonly encountered functionals defined on $\mathbf{R}_+ \times PC([-r,0], \mathbf{R}^n)$ that are all composite-PC, quasi-bounded and continuous in their second variable and hence satisfy the conditions of Theorem 3.1.1:

(F1) $f(t, x_t) = g(t, x(t), x(t-t_1), x(t-t_2), \dots, x(t-t_m))$,
where $g \in C(\mathbf{R}_+ \times \mathbf{R}^{n\times(m+1)}, \mathbf{R}^n)$ and the constants $t_k$ satisfy $0 \le t_k \le r$ for $k = 1, 2, \dots, m$;

(F2) $f(t, x_t) = g(t, x(t), x(t-h_1(t)), x(t-h_2(t)), \dots, x(t-h_m(t)))$, where $g \in C(\mathbb{R}_+ \times \mathbb{R}^{n\times(m+1)}, \mathbb{R}^n)$, the functions $h_k$ are continuous and satisfy $0 \le h_k(t) \le r$ for $t \in \mathbb{R}_+$ and the functions $t - h_k(t)$ are nondecreasing on $\mathbb{R}_+$;

(F3) $f(t, x_t) = g\left(t, x(t), \int_{t-r}^{t} G(t, s, x(s))ds\right)$,
where $g \in C(\mathbb{R}_+ \times \mathbb{R}^{n\times 2}, \mathbb{R}^n)$ and $G \in C(\mathbb{R}_+ \times [-r, \infty) \times \mathbb{R}^n, \mathbb{R}^n)$;

(F4) $f(t, x_t) = g\left(t, x(t), x(t - t_1), x(t - t_2), \dots, x(t - t_m), \int_{t-r}^{t} G(t, s, x(s))ds\right)$,
where $g \in C(\mathbb{R}_+ \times \mathbb{R}^{n\times(m+2)}, \mathbb{R}^n)$, $G \in C(\mathbb{R}_+ \times [-r, \infty) \times \mathbb{R}^n, \mathbb{R}^n)$ and the constants $t_k$ satisfy $0 \le t_k \le r$.

We provide a proof that functionals of the form (F2) satisfy the conditions of Theorem 3.1.1 from which we can conclude that local solutions of (2.3.1) & (2.3.2) exist.

**Corollary 3.1.1:** *If (2.3.1) is of the form*

$$x'(t) = g(t, x(t), x(t - h_1(t)), x(t - h_2(t)), \dots, x(t - h_m(t))), \tag{3.1.9}$$

*where* $g \in C(\mathbb{R}_+ \times \mathbb{R}^{n\times(m+1)}, \mathbb{R}^n)$, *the functions* $h_k$ *are continuous and satisfy* $0 \le h_k(t) \le r$ *for* $t \in \mathbb{R}_+$ *and the functions* $t - h_k(t)$ *are nondecreasing on* $\mathbb{R}_+$, *then for each* $(t_0, \phi) \in \mathbb{R}_+ \times PC([-r, 0], \mathbb{R}^n)$, *there exists a solution* $x = x(t_0, \phi)$ *of (2.3.1) & (2.3.2) on* $[t_0 - r, t_0 + \beta]$ *for some* $\beta > 0$.

*Proof:* Here we have $f(t, \psi) = g(t, \psi(0), \psi(h_1(t)), \dots, \psi(h_m(t)))$. We need only show that $f$ is quasi-bounded, composite-PC and continuous in $\psi$ and then we can apply Theorem 3.1.1. If $t_0 \in \mathbb{R}_+$, $\alpha > 0$ and $F \subset \mathbb{R}^n$ is compact, then since $g$ is continuous and $[t_0, t_0 + \alpha] \times F^{m+1}$ is compact, there exists some $M \ge 0$ such that $\|g(t, y_0, y_1, \dots, y_m)\| \le M$ for all $t \in [t_0, t_0 + \alpha]$ and $y_k \in F$. If $\psi \in PC([-r, 0], F)$, then $\psi(0) \in F$ and $\psi(h_k(t)) \in F$ for all $k$ and so $\|f(t, \psi)\| = \|g(t, \psi(0), \psi(h_1(t)), \dots, \psi(h_m(t)))\| \le M$. In other words, $f$ is quasi-bounded.

Suppose $x \in PC([t_0 - r, t_0 + \alpha), \mathbb{R}^n)$ for some $0 < \alpha < \infty$. By our hypotheses, each of the functions $t - h_k(t)$ are continuous and nondecreasing. Thus $x(t - h_k(t))$, which is a piecewise continuous function composed with a nondecreasing continuous function, must be an element of the function class $PC([t_0, t_0 + \alpha), \mathbb{R}^n)$. Note that the assumption that $t - h_k(t)$ be nondecreasing ensures that $x(t - h_k(t))$ will be right-continuous everywhere and have only a finite number of jump discontinuities. This condition is not necessary when dealing with continuous delay differential equations. Since each function $x(t - h_k(t))$ is in $PC([t_0, t_0 + \alpha), \mathbb{R}^n)$, then similarly the composite function $f(t, x_t) = g(t, x(t), x(t - h_1(t)), x(t - h_2(t)), \dots, x(t - h_m(t)))$ is in $PC([t_0, t_0 + \alpha), \mathbb{R}^n)$. Since this is true for all $0 < \alpha < \infty$, then it also holds true for the case $\alpha = \infty$. Hence $f$ is composite-PC.

Let $t \in \mathbb{R}_+$ be fixed. Let $\psi \in PC([-r, 0], \mathbb{R}^n)$ and choose any $\epsilon > 0$. Since $g(t, y_0, y_1, \dots, y_m)$ is continuous with respect to the variables $y_0, y_1, \dots, y_m$, then there exists some $\delta > 0$ such that, given

$z_k \in \mathbf{R}^n$ for $k = 0, 1, \ldots, m$, $\|g(t, z_0, z_1, \ldots, z_m) - g(t, \psi(0), \psi(h_1(t)), \ldots, \psi(h_m(t)))\| < \epsilon$ providing $\|z_0 - \psi(0)\| < \delta$ and $\|z_k - \psi(h_k(t))\| < \delta$ for all $k$. If $\bar{\psi} \in PC([-r, 0], \mathbf{R}^n)$ and $\|\bar{\psi} - \psi\|_r < \delta$, then $\|\bar{\psi}(0) - \psi(0)\| < \delta$ and $\|\bar{\psi}(h_k(t)) - \psi(h_k(t))\| < \delta$ for all $k$. So letting $z_0 = \bar{\psi}(0)$ and $z_k = \bar{\psi}(h_k(t))$ for $k = 1, \ldots, m$ gives us $\|g(t, \bar{\psi}(0), \bar{\psi}(h_1(t)), \ldots, \bar{\psi}(h_m(t))) - g(t, \psi(0), \psi(h_1(t)), \ldots, \psi(h_m(t)))\| < \epsilon$. In other words, $\|f(t, \bar{\psi}) - f(t, \psi)\| < \epsilon$. This proves $f$ is continuous in $\psi$. Finally, by applying Theorem 3.1.1 we obtain our conclusion. ∎

Note that functionals of the form (F1) are special cases of those given by (F2). Similarly, the functionals (F4) include those of (F1) and (F3) as special cases. Of course much more general functionals than those presented here are possible.

If the functionals given by (F2) satisfy the additional condition that there exists some $\epsilon > 0$ for which $\epsilon \le h_k(t) \le r$ for all $t \in \mathbf{R}_+$ and $1 \le k \le m$, then system (2.3.1) & (2.3.2) could theoretically be solved by the method of steps and therefore existence of solutions could be proved directly from the assumptions on $g$ and the functions $h_k$, rather than applying the more general Theorem 3.1.1.

Now we wish to obtain a local existence result for the impulsive delay differential equation (2.4.1). The thing to note here is that a solution $x = x(t_0, \phi)$ of (2.3.1) & (2.3.2) defined on $[t_0 - r, t_0 + \beta]$ for some $\beta > 0$, whose existence is guaranteed by Theorem 3.1.1, will also be a solution of (2.4.1) & (2.4.2) providing $x(t) \neq \tau_k(x(t^-))$ for all $t \in (t_0, t_0 + \beta]$ and all $k$. This is seen in the expansion of a solution of (2.4.1) given by (2.4.3). If we are interested only in local existence, then all we need to do beyond assuming the conditions of Theorem 3.1.1 is to ensure that $\beta > 0$ can be chosen sufficiently small so that the solution curve does not intersect any impulse hypersurface at any time $t \in (t_0, t_0 + \beta]$. Our next theorem gives a result in this direction and is based on some similar ideas developed by Lakshmikantham, Bainov and Simeonov [Lak89] for impulsive differential equations without delays.

**Theorem 3.1.2:** *Assume $f$ is composite-PC, quasi-bounded and continuous in $\psi$ and that $\tau_k \in C^1(D, \mathbf{R}_+)$ for $k = 1, 2, \ldots$. Furthermore, assume that whenever $t^* = \tau_k(x^*)$ for some $(t^*, x^*) \in J \times D$ and some $k$, then there exists a $\delta > 0$, where $[t^*, t^* + \delta] \subset J$, such that*

$$\nabla \tau_k(x(t)) \cdot f(t, x_t) \neq 1, \tag{3.1.10}$$

*for all $t \in (t^*, t^* + \delta]$ and for all functions $x \in PC([t^* - r, t^* + \delta], D)$ that are continuous on $(t^*, t^* + \delta]$ and satisfy $x(t^*) = x^*$ and $\|x(s) - x^*\| < \delta$ for $s \in [t^*, t^* + \delta]$. Then for each $(t_0, \phi) \in J \times PC([-r, 0], D)$, there exists a solution $x = x(t_0, \phi)$ of (2.4.1) & (2.4.2) on $[t_0 - r, t_0 + \beta]$ for some $\beta > 0$.*

*Proof:* The assumptions on $f$ guarantee the local existence of a solution $x = x(t_0, \phi)$ of (2.3.1) & (2.3.2) on some interval $[t_0 - r, t_0 + \beta]$ where $\beta > 0$. This will correspond to a local solution of (2.4.1) & (2.4.2) providing the solution does not intersect any impulse hypersurface in $(t_0, t_0 + \beta]$. If $t_0 \neq \tau_k(\phi(0))$ for any $k$, then the solution must initially lie strictly between some pair of hypersurfaces.

It is therefore obvious that it will evolve for a short amount of time before intersecting its first hypersurface. While this may require a reduction in the value of $\beta$, the local existence of a solution of (2.4.1) & (2.4.2) follows rather trivially.

The more interesting case is what happens when $t_0 = \tau_k(\phi(0))$ for some $k$. The solution of (2.3.1) & (2.3.2) is guaranteed not to intersect any hypersurface $t = \tau_j(x(t))$ for $j \neq k$ for a sufficiently small amount of time beyond $t_0$. However, we must be certain that it cannot continue along the hypersurface $t = \tau_k(x(t))$ after initially beginning on it. Define

$$g(t) = t - \tau_k(x(t)), \tag{3.1.11}$$

for $t \in [t_0, t_0 + \beta]$. Then $g(t_0) = 0$ and

$$g'(t) = 1 - \nabla\tau_k(x(t)) \cdot x'(t) = 1 - \nabla\tau_k(x(t)) \cdot f(t, x_t). \tag{3.1.12}$$

Since $\tau_k$ is assumed to be continuously differentiable and $f$ is composite-PC, then the right-hand side of (3.1.12) is continuous at least in a sufficiently small neighbourhood of $t_0$. Letting $t^* = t_0$ and $x^* = \phi(0)$ and applying (3.1.10) tells us that $g'(t)$ must be either strictly positive or strictly negative on some interval $(t_0, t_0 + \delta)$ for a sufficiently small $\delta > 0$. This of course implies that $g$ is either strictly increasing or strictly decreasing on $(t_0, t_0 + \delta)$ and can therefore not equal zero on this interval. Hence the solution of (2.3.1) & (2.3.2) will be guaranteed not to intersect the hypersurface $t = \tau_k(x(t))$ for some small amount of time beyond $t_0$, which allows us to conclude that it too is a local solution of (2.4.1) & (2.4.2). ∎

While not as general as Theorem 3.1.2, the conditions of the next corollary are easier to check.

**Corollary 3.1.2:** *Assume $f$ is composite-PC, quasi-bounded and continuous in $\psi$ and that $\tau_k \in C^1(D, \mathbf{R}_+)$ for $k = 1, 2, \ldots$. Furthermore, assume that for each $t^* \in J$ there exists some $\delta > 0$, where $[t^*, t^* + \delta] \subset J$, such that*

$$\nabla\tau_k(\psi(0)) \cdot f(t, \psi) \neq 1, \tag{3.1.13}$$

*for all $(t, \psi) \in (t^*, t^* + \delta] \times PC([-r, 0], D)$ and $k = 1, 2, \ldots$. Then for each $(t_0, \phi) \in J \times PC([-r, 0], D)$, there exists a solution $x = x(t_0, \phi)$ of (2.4.1) & (2.4.2) on $[t_0 - r, t_0 + \beta]$ for some $\beta > 0$.*

*Proof:* Inequality (3.1.13) implies (3.1.10) and so Theorem 3.1.2 immediately gives us our conclusion that a local solution of (2.4.1) & (2.4.2) exists. ∎

Note that geometrically condition (3.1.13) ensures that the tangent to the integral curve of the solution at $(t_0, \phi(0)) \in J \times D$ will not be orthogonal to the normal to the hypersurface $t = \tau_k(x)$ at $(t_0, \phi(0))$ (and hence it will not coincide with the tangent to the hypersurface $t = \tau_k(x)$ at this point).

The next corollary tells us that under the same hypotheses on $f$, solutions of the impulsive delay differential equation with fixed impulse times given by (2.4.7) & (2.4.2) exist locally.

**Corollary 3.1.3:** *Assume $f$ is composite-PC, quasi-bounded and continuous in $\psi$. Then for each $(t_0, \phi) \in J \times PC([-r, 0], D)$, there exists a solution $x = x(t_0, \phi)$ of (2.4.7) & (2.4.2) on $[t_0 - r, t_0 + \beta]$ for some $\beta > 0$.*

*Proof:* Since each $\tau_k$ is constant, $\nabla \tau_k(x) = 0$ for all $x \in D$ and so condition (3.1.13) holds trivially. So by Corollary 3.1.2 we obtain local existence. ∎

In the case that the functions $\tau_k(x)$ are not constant, then inequality (3.1.13) tells us that the point $\psi(0)$ must play a dominant role in the definition of the function $f(t, \psi)$.

## 3.2 Continuation

In this section we discuss the property of continuability of solutions of (2.4.1). We first define what is meant by a continuation of a solution and then we establish some associated theorems.

**Definition 3.2.1:** If $x$ and $y$ are solutions of (2.4.1) on the intervals $J_1$ and $J_2$, respectively, where $J_2$ properly contains $J_1$ and both intervals have the same closed left endpoint, and if $x(t) = y(t)$ for $t \in J_1$, then $y$ is said to be a *proper continuation of $x$ to the right*, or simply a *continuation* of $x$. A solution $x$ of (2.4.1) defined on $J_1$ is said to be *continuable* if there exists some continuation $y$ of $x$. Otherwise $x$ is said to be *noncontinuable* and the interval $J_1$ is called a *maximal interval of existence* of $x$.

In Definition 3.2.1 if $J_1$ is of the form $[t_0 - r, t_0 + \beta_1)$, then the interval $J_2$ which we describe may be either $[t_0 - r, t_0 + \beta_2]$ for some $\beta_2 \geq \beta_1$ or $[t_0 - r, t_0 + \beta_2)$ for some $\beta_1 < \beta_2 \leq \infty$. Similarly, if $J_1$ is of the form $[t_0 - r, t_0 + \beta_1]$, then $J_2$ may be either $[t_0 - r, t_0 + \beta_2]$ for some $\beta_2 > \beta_1$ or $[t_0 - r, t_0 + \beta_2)$ for some $\beta_1 < \beta_2 \leq \infty$.

We concern ourselves only with the forward continuation of solutions (i.e. for increasing time) since this is most natural for real physical systems and because backward continuation of solutions for both delay differential equations and impulsive systems is generally difficult if not impossible to achieve as discussed in Chapter 2.

If $x$ is a continuable solution of (2.4.1) on some interval, can it always be continued to a maximal interval of existence? In other words, must there exist a continuation $y$ of $x$ that is noncontinuable? This depends on our definition of a solution of (2.4.1). With solutions of (2.4.1) defined as in Definitions 2.4.1 and 2.4.2 the answer is no, a solution of (2.4.1) cannot necessarily be continued to a maximal interval of existence. The main reason for this is that we have excluded from our notion of a solution functions defined on finite intervals of the form $[t_0 - r, t_0 + \beta)$ where $0 < \beta < \infty$ that experience impulses at an infinite increasing sequence of times $T = \{t_k\}_{k=1}^{\infty}$ where $\lim_{k\to\infty} t_k =$

$t_0 + \beta < \infty$. A similar problem crops up even in system (2.3.1) where there are no impulses. In this case as well as in the more general case of system (2.4.1), as we continue a solution $x$ to the right it might tend to some function defined on a finite interval $[t_0 - r, t_0 + \beta)$ but whose derivative may fail to exist and be continuous at more than just a finite number of times in $(t_0, t_0 + \beta)$ as required by Definition 2.3.2 (or Definition 2.4.2). There could be an infinite sequence of times tending to $t_0 + \beta$ where the derivative fails to exist and be continuous and in this case we would not consider $y$ to be a solution of (2.3.1) (or of (2.4.1)).

Two questions that we would like to address are the following. Under what conditions can solutions of (2.4.1) always be continued to a maximal interval of existence? Secondly, under what circumstances can a solution of (2.4.1) be noncontinuable and, more specifically, what happens to it at the right endpoint of its domain of definition? Our first theorem is concerned with the first of these questions.

Note that in Section 3.1 when we established criteria for local existence of solutions of (2.4.1) we imposed a condition that would guarantee that solutions of (2.3.1) beginning on an impulse hypersurface would not evolve along it for some arbitrarily small amount of time. If this solution later does reach an impulse hypersurface, then we know there must exist a first time $t_1 > t_0$ when this happens. Without a condition such as (3.1.10) of Theorem 3.1.2 there will not necessarily be a first time $t_1 > t_0$ for which a solution of (2.3.1) intersects an impulse hypersurface, since there may be times arbitrarily close to $t_0$ at which the solution intersects the same hypersurface from which it evolves. In this case it is unclear how we could interpret a solution of (2.4.1) and so we would be inclined to say that it does not exist.

Along similar lines, since we only want to consider solutions of (2.4.1) as defined in Definitions 2.4.1 or 2.4.2, then we will need to establish conditions that will ensure that as a solution evolves and experiences impulses, the times at which the impulsive effect occurs do not tend to some finite limiting value. Of course even these "solutions" can be of interest, but we prefer to ignore them for the most part since they cannot be continued beyond this limiting value, much less all the way to infinity, without one's having to completely redefine the notion of a solution to include functions whose points of discontinuity cannot be listed in the form of an increasing sequence.

As a solution of (2.4.1) evolves it could intersect successive impulse hypersurfaces $t = \tau_k(x)$, $t = \tau_{k+1}(x)$, $t = \tau_{k+2}(x), \ldots$ (in the sense that $t_1 = \tau_k(x(t_1^-))$, $t_2 = \tau_{k+1}(x(t_2^-))$, $t_3 = \tau_{k+2}(x(t_3^-)), \ldots$) and yet its impulse times $t_1 < t_2 < t_3 < \ldots$ may be bounded from above. To avoid this situation we can simply require that $\lim_{k\to\infty} \tau_k(x) = \infty$ uniformly in $x$. Then on any finite interval of time, there would be only a finite number of impulse hypersurfaces that any such solution could intersect. The next problem is that we would want to establish conditions to avoid solutions exhibiting the pulse phenomenon. In other words, we would like to ensure that a solution could only reach (in terms of its left-hand limit) a given impulse hypersurface at most once in its evolution. If it intersected the same hypersurface multiple times, then it could potentially intersect it an infinite number of times over a finite interval, which is something we would not want. In Theorem 3.2.1 we give conditions that will ensure that as a solution of (2.4.1) evolves it will intersect each impulse hypersurface at most once.

Before introducing Theorem 3.2.1, we state Zorn's Lemma which we will need in the proof of the theorem. Zorn's Lemma is equivalent to the Axiom of Choice and a proof of this equivalency can be found in many analysis texts including [Lan83].

**Lemma 3.2.1: (Zorn's Lemma)** *Let $X$ be a partially ordered set with partial ordering $\prec$. If every nonempty, totally ordered subset $S$ of $X$ has an upper bound in $X$ (i.e. there is some $z \in X$, depending on $S$, such that $y \prec z$ for all $y \in S$), then $X$ has a maximal element (i.e. there is some $x \in X$ such that if $y \in X$ and $x \prec y$ then $x = y$).*

**Theorem 3.2.1:** *Assume $f$ is composite-PC, quasi-bounded and continuous in $\psi$ and that $\tau_k \in C^1(D, \mathbf{R}_+)$ for $k = 1, 2, \ldots$ and the limit $\lim_{k\to\infty} \tau_k(x) = \infty$ is uniform in $x$. Furthermore, assume that*

$$\nabla\tau_k(\psi(0)) \cdot f(t, \psi) < 1, \tag{3.2.1}$$

*for all $(t, \psi) \in J \times PC([-r, 0], D)$ and $k = 1, 2, \ldots$. Finally, assume that $\psi(0) + I(\tau_k(\psi(0)), \psi) \in D$ and*

$$\tau_k(\psi(0) + I(\tau_k(\psi(0)), \psi)) \leq \tau_k(\psi(0)), \tag{3.2.2}$$

*for all $\psi \in PC([-r, 0], D)$ for which $\psi(0^-) = \psi(0)$ and for all $k = 1, 2, \ldots$. Then for every continuable solution $x$ of (2.4.1), there exists a continuation $y$ of $x$ that is noncontinuable. Moreover, any solution $x$ of (2.4.1) can intersect each impulse hypersurface (in the sense that $t = \tau_k(x(t^-))$) at most once.*

*Proof:* Suppose $x$ is some continuable solution of (2.4.1). Let $[t_0 - r, t_0 + \beta_1)$ or $[t_0 - r, t_0 + \beta_1]$ where $0 < \beta_1 < \infty$ be its domain of definition depending on whether it is defined at its rightmost endpoint or not. Let $X$ be the set consisting of the solution $x$ together with all continuations of $x$. For any pair of elements $y, z \in X$ let us define the partial ordering $\prec$ by $y \prec z$ if either $z = y$ or $z$ is a continuation of $y$. Then Zorn's Lemma tells us that $X$ has a maximal element, (i.e. there is a solution of (2.4.1) that is both a continuation of $x$ and is itself noncontinuable), if for each totally ordered subset $S$ of $X$, there exists an element $z \in X$ such that $y \prec z$ for all $y \in S$.

So suppose $S$ is some totally ordered nonempty subset of $X$. We must show that $S$ has an upper bound in $X$. Associated with each solution $y \in S$, there exists a unique $\beta(y)$ satisfying $\beta_1 \leq \beta(y) \leq \infty$ where $t_0 + \beta(y)$ represents the rightmost endpoint of the interval on which $y$ is defined. In other words, $y$ is defined on either $[t_0 - r, t_0 + \beta(y))$ or on $[t_0 - r, t_0 + \beta(y)]$ if $\beta(y) < \infty$.

If $x \in S$, then of course $\beta(x) = \beta_1$. Now define

$$\beta_2 = \sup\{\beta(y) \mid y \in S\}. \tag{3.2.3}$$

Then $\beta_1 \le \beta_2 \le \infty$ and the domain of every solution $y \in S$ is a subset of $[t_0 - r, t_0 + \beta_2]$ if $\beta_2 < \infty$ or of $[t_0 - r, \infty)$ if $\beta_2 = \infty$. If $\beta_2 < \infty$ and there exists some $y \in S$ that is defined on $[t_0 - r, t_0 + \beta_2]$, then this solution $y$ is clearly an upper bound on $S$. So let us disregard that case. If we can show the existence of a solution $z$ of (2.4.1) that is defined on $[t_0 - r, t_0 + \beta_2)$ and for which $y \prec z$ for all $y \in S$, then $z$ will be an upper bound on $S$ and from Zorn's Lemma we will be able to conclude the existence of a maximal element of $X$.

For $t \in [t_0 - r, t_0 + \beta_2)$ define the function

$$z(t) = y(t), \text{ where } y \text{ is any solution in } S \text{ for which } t < t_0 + \beta(y). \tag{3.2.4}$$

From the definition of $\beta_2$ it follows that whenever $t \in [t_0 - r, t_0 + \beta_2)$ there will always exist a solution $y \in S$ for which $t_0 + \beta(y) > t$ and such a solution $y$ is therefore defined at $t$. Moreover, since the set $S$ is totally ordered, then all solutions $y \in S$ that are defined at $t$ have the same function value at $t$. In other words, the function $z$ is well-defined. The challenge then is to show that $z$ is actually a solution of (2.4.1) according to Definition 2.4.2. If so, then $z \in X$ and it follows from the construction of $z$ that $y \prec z$ for all $y \in S$.

First note that since $y(t) \in D$ and $y(t^+) = y(t)$ for all $y \in S$ and for all $t \in [t_0 - r, t_0 + \beta(y))$ it follows immediately that $z(t) \in D$ and $z(t^+) = z(t)$ for all $t \in [t_0 - r, t_0 + \beta_2)$. Thus $z : [t_0 - r, t_0 + \beta_2) \to D$ and $z$ is right-continuous everywhere. Also since for each $y \in S$, $y(t^-)$ exists in $D$ for all $t \in (t_0 - r, t_0 + \beta(y))$, then $z(t^-)$ exists in $D$ for all $t \in (t_0 - r, t_0 + \beta_2)$. Similarly $z(t^-) = z(t)$ for all but at most a finite number of points $t \in (t_0 - r, t_0]$. We cannot yet claim that $z \in PC([t_0 - r, t_0 + \beta_2), D)$, however, since if $\beta_2 < \infty$, then the function $z$ may experience an infinite number of jump discontinuities.

Choose any $0 < \alpha < \beta_2$. Then there exists some $y \in S$ for which $\beta(y) > \alpha$. The restriction of $y$ to $[t_0 - r, t_0 + \alpha]$ is clearly a solution of (2.4.1) and is an element of $PC([t_0 - r, t_0 + \alpha], D)$, but it is also none other than the restriction of $z$ to $[t_0 - r, t_0 + \alpha]$. Therefore if $\beta_2 = \infty$, then $z \in PC([t_0 - r, \infty), D)$ and by Definition 2.4.2 the function $z$ is a solution of (2.4.1).

On the other hand, if $\beta_2 < \infty$, then define $T = \{t \in (t_0, t_0 + \beta_2) \mid t = \tau_k(z(t^-)) \text{ for some } k\}$. Then $z(t^-) = z(t)$ for all $t \in (t_0, t_0 + \beta_2)\backslash T$. If the set $T$ is finite, then $z \in PC([t_0 - r, t_0 + \beta_2), D)$. Since $f$ is composite-PC, then $f(t, z_t)$ can have only a finite number of discontinuities on $(t_0, t_0 + \beta_2)$ and except at these points or at points in $T$ the function $z$ must have a continuous derivative. This is because if at some $t \in (t_0, t_0 + \beta_2)\backslash T$ the composite function $f(t, z_t)$ were continuous, then the same would be true for $f(t, y_t)$ and any $y \in S$ for which $t_0 + \beta(y) > t$ and from the integral form of the solution $y$ given by (2.4.6) of Lemma 2.4.1 (where $\phi = y_{t_0}$) it is clear that the derivative of $y$

and hence that of $z$ exists and is continuous at $t$. It therefore follows that $z$ is a solution of (2.4.1) according to Definition 2.4.2.

Since $\lim_{k\to\infty} \tau_k(\omega) = \infty$ uniformly in $\omega$, then there exists some positive integer $N$ for which $\tau_k(\omega) \geq t_0 + \beta_2$ for all $\omega \in D$ and for all $k \geq N$. Note that we have replaced the $x$ in the assumption of the theorem by $\omega$ to avoid confusion with the $x$ that we took to initially be a solution of (2.4.1).

We are now left with the remaining case to consider, which is when $\beta_2 < \infty$ and yet the set $T$ is infinite. We will show that the conditions of our theorem will guarantee this cannot happen. In this situation $T$ must consist of an increasing sequence of impulse times $T = \{t_k\}_{k=1}^{\infty}$ where $t_0 < t_1 < t_2 < \ldots < t_k < \ldots < t_0 + \beta_2$ and $\lim_{k\to\infty} t_k = t_0 + \beta_2$. For each $k = 1, 2, \ldots$ let $j_k$ denote the index of the unique hypersurface that $z$ reaches at time $t_k$. In other words, $t_k = \tau_{j_k}(z(t_k^-))$ for $k = 1, 2, \ldots$. Then $j_k < N$ for all $k$, which means that there are only a finite number of impulse hypersurfaces that $z$ can intersect. Thus $z$ must reach at least one hypersurface more than once. In other words, $j_k = j_{k+m}$, and hence $t_k = \tau_{j_k}(z(t_k^-))$ and $t_{k+m} = \tau_{j_k}(z(t_{k+m}^-))$, for some positive integers $k$ and $m$. So let $y \in S$ be any solution where $t_0 + \beta(y) > t_{k+m}$. Then $t_k = \tau_{j_k}(y(t_k^-))$ and $t_{k+m} = \tau_{j_k}(y(t_{k+m}^-))$. We will show, however, that it is in fact impossible for the solution $y$ to reach the same impulse hypersurface two or more times. The same technique can then be used to prove that every solution of (2.4.1) can intersect a given impulse hypersurface at most once.

For $i = 0, 1, 2, \ldots, m$ define

$$h_{k+i}(t) = t - \tau_{j_{k+i}}(y(t)), \tag{3.2.5}$$

for $t \in [t_0 - r, t_{k+m}]$. Then these functions are all continuous on $[t_k, t_{k+m}]$ except possibly at the impulse times where they are right-continuous and have a left-hand limit. Note that $h_{k+i}(t_{k+i}^-) = 0$ for all $i$.

Suppose that for some $0 \leq i \leq m-1$ we have $j_{k+i} \geq j_{k+i+1}$. Then $\tau_{j_{k+i}}(\omega) \geq \tau_{j_{k+i+1}}(\omega)$ for all $\omega \in D$ and so therefore we find that

$$\begin{aligned} h_{k+i+1}(t_{k+i}) &= t_{k+i} - \tau_{j_{k+i+1}}(y(t_{k+i})) \geq t_{k+i} - \tau_{j_{k+i}}(y(t_{k+i})) \\ &= t_{k+i} - \tau_{j_{k+i}}(y(t_{k+i}^-) + I(t_{k+i}, y_{t_{k+i}^-})) \\ &= t_{k+i} - \tau_{j_{k+i}}(y(t_{k+i}^-) + I(\tau_{j_{k+i}}(y(t_{k+i}^-)), y_{t_{k+i}^-})) \\ &\geq t_{k+i} - \tau_{j_{k+i}}(y(t_{k+i}^-)) = h_{k+i}(t_{k+i}^-) = 0, \end{aligned} \tag{3.2.6}$$

where we used (3.2.2) to get the final inequality.

On the other hand if we differentiate $h_{k+i+1}(t)$ with respect to $t$ over the interval $(t_{k+i}, t_{k+i+1})$ we get

$$h'_{k+i+1}(t) = 1 - \nabla\tau_{j_{k+i+1}}(y(t)) \cdot f(t, y_t), \tag{3.2.7}$$

for all $t \in (t_{k+i}, t_{k+i+1})$, which is valid at least in terms of a right-hand derivative (and in terms of its ordinary derivative at all but a finite number of points). From inequality (3.2.1) we conclude that $h'_{k+i+1}(t) > 0$ and therefore that $h_{k+i+1}(t)$ is strictly increasing on this interval. So in particular $h_{k+i+1}(t_{k+i}) < h_{k+i+1}(t^-_{k+i+1})$. However, $h_{k+i+1}(t^-_{k+i+1}) = 0$ and as we showed in (3.2.6) $h_{k+i+1}(t_{k+i}) \geq 0$. This gives a contradiction. Thus it must be true that $j_{k+i} < j_{k+i+1}$ for all $i$. In other words, $j_k < j_{k+1} < j_{k+2} < \ldots < j_{k+m}$. But finally this contradicts the assumption that $j_k = j_{k+m}$. Therefore $y$ and consequently $z$ could not have intersected the same impulse hypersurface twice. This concludes the final case and proves the theorem. ∎

In Theorem 3.2.1 we assumed that $\psi(0) + I(\tau_k(\psi(0)), \psi) \in D$ for all $k$ and for all $\psi \in PC([-r, 0], D)$ for which $\psi(0^-) = \psi(0)$. This will of course be necessary if we want solutions to be defined and continuable beyond impulse times. It ensures that solutions remain in the domain of the functional $f$ following the impulsive action. In most applications $D = \mathbf{R}^n$ and so this condition is trivially satisfied.

The next corollary of Theorem 3.2.1 involves a strengthening of inequality (3.2.2), which occasionally may be easier to check.

**Corollary 3.2.1:** *Assume $f$ is composite-PC, quasi-bounded and continuous in $\psi$ and that $\tau_k \in C^1(D, \mathbf{R}_+)$ for $k = 1, 2, \ldots$ and the limit $\lim_{k\to\infty} \tau_k(x) = \infty$ is uniform in $x$. Furthermore, assume that*

$$\nabla\tau_k(\psi(0)) \cdot f(t, \psi) < 1, \tag{3.2.8}$$

*for all $(t, \psi) \in J \times PC([-r, 0], D)$ and $k = 1, 2, \ldots$. Finally, assume that $\psi(0) + sI(\tau_k(\psi(0)), \psi) \in D$ and*

$$\nabla\tau_k(\psi(0) + sI(\tau_k(\psi(0)), \psi)) \cdot I(\tau_k(\psi(0)), \psi) \leq 0 \tag{3.2.9}$$

*for all $\psi \in PC([-r, 0], D)$ for which $\psi(0^-) = \psi(0)$, for all $0 \leq s \leq 1$ and for all $k = 1, 2, \ldots$. Then for every continuable solution $x$ of (2.4.1), there exists a continuation $y$ of $x$ that is noncontinuable. Moreover, any solution $x$ of (2.4.1) can intersect each impulse hypersurface (in the sense that $t = \tau_k(x(t^-))$) at most once.*

*Proof:* For fixed $\psi \in PC([-r, 0], D)$ for which $\psi(0^-) = \psi(0)$ and for any fixed positive integer $k$, define

$$g(s) = \tau_k(\psi(0) + sI(\tau_k(\psi(0)), \psi)), \tag{3.2.10}$$

for $s \in [0, 1]$. Differentiating $g$ with respect to $s$ gives

$$g'(s) = \nabla\tau_k(\psi(0) + sI(\tau_k(\psi(0)), \psi)) \cdot I(\tau_k(\psi(0)), \psi). \tag{3.2.11}$$

Thus $g'(s) \leq 0$ for $0 \leq s \leq 1$ according to (3.2.9). Therefore $g$ is nonincreasing on $[0, 1]$ and in particular $g(1) \leq g(0)$. In other words, inequality (3.2.2) is satisfied. So by Theorem 3.2.1 the conclusion of the corollary follows. ∎

Note that if the set $D$ is convex, then when checking the requirement in Corollary 3.2.1 that $\psi(0) + sI(\tau_k(\psi(0)), \psi)$ be contained in $D$ for $0 \leq s \leq 1$, one need only check the special case when $s = 1$ (i.e. that $\psi(0) + I(\tau_k(\psi(0)), \psi) \in D$).

The next corollary tells us that solutions of system (2.4.7) where impulses occur at fixed times can always be continued to a maximal interval of existence. The fact that solutions of system (2.4.7) can intersect impulse hyperplanes $t = \tau_k$ at most once (in fact exactly once for each $\tau_k > t_0$ in the domain of the solution) is also obvious.

**Corollary 3.2.2:** *Assume $f$ is composite-PC, quasi-bounded and continuous in $\psi$. Also assume that $\psi(0)+I(\tau_k, \psi) \in D$ for all $\psi \in PC([-r, 0], D)$ for which $\psi(0^-) = \psi(0)$ and for all $k = 1, 2, \ldots$. Then for every continuable solution $x$ of (2.4.7), there exists a continuation $y$ of $x$ that is noncontinuable.*

*Proof:* Since each $\tau_k$ is constant, inequality (3.2.2) holds trivially. Moreover, $\nabla\tau_k(x) = 0$ for all $x \in D$ and so inequality (3.2.1) also holds. So by Theorem 3.2.1 continuable solutions of (2.4.7) can be continued to a maximal interval of existence. ∎

Now that we have established conditions that guarantee that solutions of (2.4.1) can be continued to a maximal interval of existence and at the same time avoid beating upon an impulse hypersurface, let us turn our attention to what happens to noncontinuable solutions.

**Theorem 3.2.2:** *Assume that all of the conditions of Theorem 3.2.1 are satisfied and let $x$ be any solution of (2.4.1). If $x$ is defined on a closed interval of the form $[t_0 - r, t_0 + \alpha]$, where $\alpha > 0$ and $[t_0, t_0 + \alpha] \subset J$, then $x$ is continuable. If $x$ is defined on an interval of the form $[t_0 - r, t_0 + \beta)$, where $0 < \beta < \infty$ and $[t_0, t_0 + \beta] \subset J$, and if $x$ is noncontinuable, then for every compact set $F \subset D$, there exists a sequence of numbers $\{s_k\}_{k=1}^{\infty}$ with $t_0 < s_1 < s_2 < \ldots < s_k < \ldots < t_0 + \beta$ and $\lim_{k\to\infty} s_k = t_0 + \beta$ such that $x(s_k) \notin F$.*

*Proof:* Suppose $x$ is a solution of (2.4.1) defined on $[t_0-r, t_0+\alpha]$ for some $\alpha > 0$ where $[t_0, t_0+\alpha] \subset J$. Define $\bar{t}_0 = t_0+\alpha$ and $\bar{\phi} = x_{t_0+\alpha}$. By Corollary 3.1.2 we know there exists a local solution $y = y(\bar{t}_0, \bar{\phi})$ of (2.4.1) & (2.4.2) on $[\bar{t}_0 - r, \bar{t}_0 + \beta]$ for some $\beta > 0$ where $\bar{t}_0 \in J$ denotes the new initial time and $\bar{\phi} \in PC([-r, 0], D)$ denotes the new initial function. By extending the domain of definition of the function $x$ so that $x(t) = y(t)$ for $t \in (\bar{t}_0, \bar{t}_0 + \beta]$ it is clear that this new function is also a solution of (2.4.1) & (2.4.2) and is a continuation of the original solution $x$.

Next assume $x$ is a noncontinuable solution of (2.3.1) that is defined on $[t_0 - r, t_0 + \beta)$ for some $0 < \beta < \infty$ where $[t_0, t_0+\beta] \subset J$. Suppose, for the sake of contradiction, that there exists a compact set $F_1 \subset D$ and a constant $0 < \beta_1 < \beta$ such that $x(t) \in F_1$ for all $t \in [t_0+\beta_1, t_0+\beta)$. Without loss of generality we may assume that $\beta_1$ is sufficiently small so that the solution $x$ does not experience any impulses on the interval $[t_0+\beta_1, t_0+\beta)$. Define $F_2$ to be the closure of the range of $x$ when restricted to $[t_0 - r, t_0 + \beta_1]$. Then $F_2$ is a compact set properly contained in $D$. The set $F = F_1 \cup F_2 \subset D$ is clearly compact and $x(t) \in F$ for all $t \in [t_0 - r, t_0 + \beta)$.

Since $f$ is quasi-bounded, there exists some $M > 0$ such that $||f(t,\psi)|| \le M$ for all $(t,\psi) \in [t_0, t_0+\beta] \times PC([-r, 0], F)$. So in particular $||f(t, x_t)|| \le M$ for $t \in [t_0, t_0 + \beta)$. Given $t, \bar{t} \in [t_0 + \beta_1, t_0 + \beta)$ we have by (2.4.6) of Lemma 2.4.1 (where $\phi = x_{t_0}$),

$$||x(t) - x(\bar{t})|| = \left\|\int_{\bar{t}}^{t} f(s, x_s)ds\right\| \le \left|\int_{\bar{t}}^{t} ||f(s, x_s)||\, ds\right| \le M|t - \bar{t}|. \tag{3.2.12}$$

Then (3.2.12) implies that $\lim_{t\to(t_0+\beta)^-} x(t)$ exists by the Cauchy criterion and this limit point, call it $\omega$, is in $F$. Thus $x$ may be continued as a solution of (2.4.1) & (2.4.2) to $t_0 + \beta$ (as well as beyond as proven earlier) by defining $x(t_0 + \beta) = \omega$. This contradicts the assumption that $x$ is noncontinuable. Thus, for every compact set $F \in D$, there is a strictly increasing sequence of times $\{s_k\}_{k=1}^{\infty}$ in $(t_0, t_0 + \beta)$ that converges to $t_0 + \beta$ and for which $x(s_k) \notin F$. ■

For completeness we state the following corollary to Theorem 3.2.2, which applies to system (2.4.7) which has impulses occurring at fixed times.

**Corollary 3.2.3:** *Assume $f$ is composite-PC, quasi-bounded and continuous in $\psi$. Also assume that $\psi(0)+I(\tau_k,\psi) \in D$ for all $\psi \in PC([-r, 0], D)$ for which $\psi(0^-) = \psi(0)$ and for all $k = 1, 2, \ldots$. Let $x$ be any solution of (2.4.7). If $x$ is defined on a closed interval of the form $[t_0-r, t_0+\alpha]$, where $\alpha > 0$ and $[t_0, t_0 + \alpha] \subset J$, then $x$ is continuable. If $x$ is defined on an interval of the form $[t_0 - r, t_0 + \beta)$, where $0 < \beta < \infty$ and $[t_0, t_0 + \beta] \subset J$, and if $x$ is noncontinuable, then for every compact set $F \subset D$, there exists a sequence of numbers $\{s_k\}_{k=1}^{\infty}$ with $t_0 < s_1 < s_2 < \ldots < s_k < \ldots < t_0 + \beta$ and $\lim_{k\to\infty} s_k = t_0 + \beta$ such that $x(s_k) \notin F$.*

*Proof:* The conditions of Corollary 3.2.2 and hence those of Theorem 3.2.1 are satisfied. Thus by Theorem 3.2.2 the conclusion of the corollary follows. ■

The first part of Theorem 3.2.2 states that a maximal interval of existence of a solution of (2.4.1) is open on the right. The second part says that a noncontinuable solution $x$ is either defined for all $t \in J$ or it is defined on a bounded proper sub-interval of $J$ and in this latter case the solution either becomes unbounded as $t \to t_0 + \beta$ (i.e. $\limsup_{t\to t_0+\beta} ||x(t)|| = \infty$) or it takes on values arbitrarily close to the boundary of $D$ as $t \to t_0 + \beta$. In the special case where $J = \mathbb{R}_+$ and $D = \mathbb{R}^n$, Theorem 3.2.2 essentially says that bounded solutions are continuable to $t = \infty$.

## 3.3 Global Existence

In this section we impose further conditions on the functional $f$ to establish global existence results for solutions of system (2.4.1). Before stating our main theorem we need a generalization of

Gronwall's Inequality.

While Gronwall's Inequality applies to continuous functions, generalizations of the lemma to piecewise continuous functions and even Lebesgue integrable functions are available (see [Sam77] and [Hal80], respectively). We state the lemma as it applies to piecewise continuous functions and include a proof.

**Lemma 3.3.1: (Gronwall's Inequality)** *If* $g, h \in PC([a,b], \mathbb{R}_+)$, $c \in \mathbb{R}_+$, *and*

$$g(t) \le c + \int_a^t g(s)h(s)ds, \tag{3.3.1}$$

*for all* $t \in [a,b]$, *then*

$$g(t) \le c \exp\left(\int_a^t h(s)ds\right), \tag{3.3.2}$$

*for all* $t \in [a,b]$.

*Proof:* Let $s_0 = a$, let $s_i$ denote the $i^{th}$ point in $(a,b)$ at which either function $g$ or $h$ has a simple jump discontinuity, and let $s_n = b$, where $n-1$ is the number of such discontinuous points. Then on each interval $[s_{i-1}, s_i]$, for $i = 1, 2, \ldots, n$, the functions $g$ and $h$ are continuous except perhaps at $s_i$ where they have a left-hand limit.

For $t \in [a,b]$ define

$$u(t) = c + \int_a^t g(s)h(s)ds. \tag{3.3.3}$$

Then $u$ is continuous on $[a,b]$ and is continuously differentiable on each interval $(s_{i-1}, s_i)$. Moreover, for $t \in (s_{i-1}, s_i)$, $u'(t) = g(t)h(t)$, and since $g(t) \le u(t)$ on $[a,b]$ by (3.3.1), then $u'(t) \le h(t)u(t)$ and hence $u'(t) - h(t)u(t) \le 0$ for $t \in (s_{i-1}, s_i)$. Multiplying both sides of this inequality by the integrating factor $\exp(-\int_a^t h(s)ds)$, which is also differentiable at each $t \in (s_{i-1}, s_i)$, and applying the product rule identity gives us

$$\frac{d}{dt}\left[\exp\left(-\int_a^t h(s)ds\right)u(t)\right] \le 0, \tag{3.3.4}$$

for all $t \in (s_{i-1}, s_i)$. Thus the function

$$p(t) = \exp\left(-\int_a^t h(s)ds\right)u(t), \tag{3.3.5}$$

is nonincreasing on $(s_{i-1}, s_i)$. Since $p(t)$ is continuous on $[a,b]$ then it is nonincreasing on each closed interval $[s_{i-1}, s_i]$ and hence on all of $[a,b]$. So in particular, for any $t \in [a,b]$ we have

$p(t) \leq p(a) = u(a) = c$ which gives us

$$g(t) \leq u(t) \leq c \exp\left(\int_a^t h(s)ds\right). \tag{3.3.6}$$

■

**Theorem 3.3.1:** *Assume $J = \mathbf{R}_+$, $D = \mathbf{R}^n$, and the conditions of Theorem 3.2.1 are satisfied. Suppose that there exist functions $h_1, h_2 \in PC(\mathbf{R}_+, \mathbf{R}_+)$ such that $\|f(t,\psi)\| \leq h_1(t) + h_2(t)\|\psi\|_r$ for all $(t,\psi) \in \mathbf{R}_+ \times PC([-r,0],\mathbf{R}^n)$. Then for each $(t_0,\phi) \in \mathbf{R}_+ \times PC([-r,0],\mathbf{R}^n)$, there exists a (local) solution $x = x(t_0,\phi)$ of (2.4.1) & (2.4.2) and any such solution can be continued to $[t_0 - r, \infty)$.*

*Proof:* Let $(t_0,\phi) \in \mathbf{R}_+ \times PC([-r,0],\mathbf{R}^n)$ and let $x = x(t_0,\phi)$ be a local solution of (2.4.1) & (2.4.2) (whose existence is guaranteed by Corollary 3.1.2). If $x$ is continuable, then we know it can be extended to a maximal interval of existence of the form $[t_0 - r, t_0 + \beta)$ for some $0 < \beta \leq \infty$ according to Theorems 3.2.1 and 3.2.2. The question then is whether or not $x$ can be noncontinuable on $[t_0 - r, t_0 + \beta)$ for a finite value of $\beta$. So let us suppose for the sake of contradiction that $\beta < \infty$ and $x$ is noncontinuable on $[t_0 - r, t_0 + \beta)$.

Let $\{t_k\}_{k=1}^m$ denote the impulse times corresponding to the solution $x$. By Theorem 3.2.2, the solution becomes unbounded as $t \to t_0 + \beta$. We will obtain a contradiction by showing that $x$ is bounded on $[t_0 - r, t_0 + \beta)$.

Let $M_i = \sup\{h_i(t) \mid t \in [t_0, t_0 + \beta]\}$ for $i = 1, 2$ and note that both $M_1$ and $M_2$ are finite numbers. Let $B_1 = \beta M_1$, $B_2 = \beta M_2$, $B_3 = \|\phi\|_r$ and $B_4 = \|\phi(0)\| + \sum_{k=1}^m \|I(t_k, x_{t_k^-})\|$ (or simply $B_4 = \|\phi(0)\|$ if $x$ experiences no impulses).

Now from (2.4.6) of Lemma 2.4.1 we obtain, for $t \in [t_0, t_0 + \beta)$,

$$\begin{aligned}
\|x(t)\| &\leq \|\phi(0)\| + \left\| \sum_{\{k:\ t_k \in (t_0,t]\}} I(t_k, x_{t_k^-}) \right\| + \left\| \int_{t_0}^t f(s, x_s)ds \right\| \\
&\leq \|\phi(0)\| + \sum_{\{k:\ t_k \in (t_0,t]\}} \left\| I(t_k, x_{t_k^-}) \right\| + \int_{t_0}^t \|f(s,x_s)\|\, ds \\
&\leq B_4 + \int_{t_0}^t (h_1(s) + h_2(s)\|x_s\|_r)\, ds \leq B_4 + B_1 + \int_{t_0}^t h_2(s)\|x_s\|_r ds,
\end{aligned} \tag{3.3.7}$$

which implies that

$$\|x_t\|_r \leq B_4 + B_1 + B_3 + \int_{t_0}^t h_2(s)\|x_s\|_r ds, \tag{3.3.8}$$

for all $t \in [t_0, t_0 + \beta)$. Define $g(t) = \|x_t\|_r$ for $t \in [t_0, t_0 + \beta)$. Let $0 < \beta_1 < \beta$. Then restricted to $[t_0, t_0 + \beta_1]$ we know $g \in PC([t_0, t_0 + \beta_1], \mathbf{R}_+)$ by Lemma 2.3.2. Thus by Gronwall's Inequality we get

$$g(t) \leq (B_4 + B_1 + B_3)\exp\left(\int_{t_0}^{t} h_2(s)ds\right). \tag{3.3.9}$$

If we let $B = (B_4 + B_1 + B_3)\exp(B_2)$, then $g(t) \leq B$ for all $t \in [t_0, t_0 + \beta_1]$, which in turn implies $\|x(t)\| \leq B$ for all $t \in [t_0 - r, t_0 + \beta_1]$. Since this holds for all $\beta_1$ arbitrarily close to $\beta$, then this implies that $\|x(t)\| \leq B$ for all $[t_0 - r, t_0 + \beta)$. Thus we get a contradiction which proves the theorem. ■

Our final corollary of this section essentially restates Theorem 3.3.1 insofar as it applies to system (2.4.7).

**Corollary 3.3.1:** *Assume $J = \mathbf{R}_+$, $D = \mathbf{R}^n$, and that $f$ is composite-PC and continuous in $\psi$. Suppose that there exist functions $h_1, h_2 \in PC(\mathbf{R}_+, \mathbf{R}_+)$ such that $\|f(t,\psi)\| \leq h_1(t) + h_2(t)\|\psi\|_r$ for all $(t,\psi) \in \mathbf{R}_+ \times PC([-r,0], \mathbf{R}^n)$. Then for each $(t_0, \phi) \in \mathbf{R}_+ \times PC([-r,0], \mathbf{R}^n)$, there exists a (local) solution $x = x(t_0, \phi)$ of (2.4.7) & (2.4.2) and any such solution can be continued to $[t_0 - r, \infty)$.*

*Proof:* The conditions of Corollary 3.2.2, which are assumed, imply those of Theorem 3.2.1 and so Theorem 3.3.1 yields the conclusion.

Note, however, that we have dropped the explicit reference to the quasi-boundedness assumption on $f$ as a condition of our corollary. This is because it follows from the assumption that $\|f(t,\psi)\| \leq h_1(t) + h_2(t)\|\psi\|_r$. Suppose $F \subset \mathbf{R}^n$ is compact and $[t_0, t_0 + \alpha] \subset \mathbf{R}_+$ for some $t_0 \in \mathbf{R}_+$ and $\alpha > 0$. Let $M_i = \sup\{h_i(t) \mid t \in [t_0, t_0 + \alpha]\}$ for $i = 1, 2$ and let $M_3 = \sup\{\|z\| \mid z \in F\}$. Clearly $M_1, M_2$ and $M_3$ are all finite. If we define $M = M_1 + M_2 M_3$, then for $(t,\psi) \in [t_0, t_0 + \alpha] \times PC([-r,0], F)$, $\|f(t,\psi)\| \leq h_1(t) + h_2(t)\|\psi\|_r \leq M_1 + M_2 M_3 = M$, and hence $f$ is quasi-bounded. ■

## 3.4 Uniqueness

This section looks at the uniqueness properties of solutions of (2.4.1) & (2.4.2). Rather than first prove uniqueness for system (2.3.1) & (2.3.2) and then apply these results to obtain uniqueness for system (2.4.1) & (2.4.2), we will prove uniqueness for system (2.4.1) & (2.4.2) directly.

**Definition 3.4.1:** A solution $x = x(t_0, \phi)$ of (2.4.1) & (2.4.2) is said to be *unique* if given any other solution $y = y(t_0, \phi)$ of (2.4.1) & (2.4.2), $x(t) = y(t)$ on their common interval of existence.

Note that two distinct solutions $x = x(t_0, \phi)$ and $y = y(t_0, \bar{\phi})$ of (2.4.1) & (2.4.2) having different initial functions $\phi$ and $\bar{\phi}$, respectively, for which $\phi(0) \neq \bar{\phi}(0)$, may intersect or even merge at some time $t > t_0$. This could be a result of the delay differential equation (2.4.1a) even with a sufficiently smooth functional $f$. Winston and Yorke [Win69] give a rather interesting example illustrating this behaviour. This is one feature that distinguishes delay differential equations from ordinary differential equations. On the other hand, merging of solutions can be caused by the impulses as discussed in Section 2.2.

As with ordinary differential equations, additional smoothness assumptions on $f$ are required if we want to expect uniqueness of solutions. In Theorem 3.4.1 we show that the addition of a local Lipschitz condition on $f$ is sufficient to guarantee uniqueness of solutions of (2.4.1) & (2.4.2). This theorem is modified from a similar uniqueness theorem in [Hal93] for delay differential equations, except that we account for the discontinuities caused by impulses.

**Theorem 3.4.1:** *Assume $f$ is composite-PC and locally Lipschitz in $\psi$. Then there exists at most one solution of (2.4.1) & (2.4.2) on $[t_0 - r, t_0 + \beta)$ where $0 < \beta \leq \infty$ and $[t_0, t_0 + \beta) \subset J$.*

*Proof:* Assume $x = x(t_0, \phi)$ and $y = y(t_0, \phi)$ are two solutions of (2.4.1) & (2.4.2) on $[t_0 - r, t_0 + \beta)$ where $0 < \beta \leq \infty$ and $[t_0, t_0 + \beta) \subset J$ and assume $x \not\equiv y$. Since $x(t) = y(t) = \phi(t - t_0)$ for $t \in [t_0 - r, t_0]$, then there exists some $t \in (t_0, t_0 + \beta)$ for which $x(t) \neq y(t)$. Define $t_1 = \inf\{t \in (t_0, t_0 + \beta) \mid x(t) \neq y(t)\}$. Then $t_1 \in [t_0, t_0 + \beta)$ and $x(t) = y(t)$ for $t \in [t_0 - r, t_1)$. In particular $x(t_1^-) = y(t_1^-)$. If $t_1 > t_0$ and $t_1 \neq \tau_k(x(t_1^-))$ (equivalently, $t_1 \neq \tau_k(y(t_1^-))$) for all $k$, then $x(t_1) = x(t_1^-) = y(t_1^-) = y(t_1)$, while if $t_1 > t_0$ and $t_1 = \tau_k(x(t_1^-))$ (equivalently, $t_1 = \tau_k(y(t_1^-))$) for some $k$, then $x(t_1) = x(t_1^-) + I(t_1, x_{t_1^-}) = y(t_1^-) + I(t_1, y_{t_1^-}) = y(t_1)$. Thus $x(t) = y(t)$ for $t \in [t_0 - r, t_1]$. Let $\epsilon > 0$ be sufficiently small so that $t_1 + \epsilon < t_0 + \beta$ and both $t \neq \tau_k(x(t))$ and $t \neq \tau_k(y(t))$ for all $t \in (t_1, t_1 + \epsilon]$ and for all $k$. Using the integral form of the system of equations given by (2.4.6) in Lemma 2.4.1 we get

$$x(t) - y(t) = \int_{t_1}^{t} (f(s, x_s) - f(s, y_s))\, ds, \tag{3.4.1}$$

for $t \in [t_1, t_1 + \epsilon]$. Let $S = \{x(t) \mid t \in [t_0 - r, t_1 + \epsilon]\} \cup \{y(t) \mid t \in [t_0 - r, t_1 + \epsilon]\}$ and $F = \overline{S}$ (i.e. the closure of $S$). Clearly $F$ is a compact subset of $D$. Since $f$ is locally Lipschitz in $\psi$, there exists a constant $L > 0$ such that $\|f(t, \psi_1) - f(t, \psi_2)\| \leq L\|\psi_1 - \psi_2\|_r$ for all $t \in [t_0, t_1 + \epsilon]$ and $\psi_1, \psi_2 \in PC([-r, 0], F)$. Let $\delta > 0$ be sufficiently small so that $\delta < \epsilon$ and $L\delta \leq 1/2$. Then for all $t \in [t_1, t_1 + \delta]$, we have $x_t, y_t \in PC([-r, 0], F)$ and from (3.4.1) we obtain

$$\begin{aligned} \|x(t) - y(t)\| &\leq \int_{t_1}^{t} \|f(s, x_s) - f(s, y_s)\| ds \leq \int_{t_1}^{t} L\|x_s - y_s\|_r ds \\ &\leq L \int_{t_1}^{t} \sup_{u \in [t_1, s]} \|x(u) - y(u)\| ds \leq L \int_{t_1}^{t_1 + \delta} \sup_{u \in [t_1, t_1 + \delta]} \|x(u) - y(u)\| ds \\ &= L\delta \sup_{u \in [t_1, t_1 + \delta]} \|x(u) - y(u)\| \leq \frac{1}{2} \sup_{u \in [t_1, t_1 + \delta]} \|x(u) - y(u)\|. \end{aligned} \tag{3.4.2}$$

Since inequality (3.4.2) is satisfied for all $t \in [t_1, t_1 + \delta]$, then this means we must have $x(t) = y(t)$ for $t \in [t_1, t_1 + \delta]$, which contradicts the definition of $t_1$ and proves the theorem. ∎

When considering the continuous system (2.1.3) without impulses and where $f$ is defined on

$J \times C([-r, 0], D)$, then uniqueness of solutions may be proven by assuming a seemingly weaker form of Lipschitz condition. Lakshmikantham [Lak64] has shown, in a somewhat more general setting, that $f$ does not have to satisfy the Lipschitz inequality for every pair of functions $\psi_1, \psi_2 \in C([-r, 0], F)$ but only for those pairs of functions that satisfy $\|\psi_1 - \psi_2\|_r = \|\psi_1(0) - \psi_2(0)\|$. We could similarly prove uniqueness under this alternative assumption. The technique is a little more complicated since it requires Razumikhin-type arguments similar to those we use in the study of stability in Chapter 5. However, we refrain from presenting a theorem based on this assumption since it does not seem to generalize the theorem significantly. Functionals that satisfy this alternate Lipschitz condition invariably end up also satisfying the standard Lipschitz condition defined in Definition 2.3.6. It is unknown whether a functional exists that satisfies this alternate Lipschitz condition, but that fails to satisfy Definition 2.3.6.

## 3.5 Continuous Dependence

We now wish to examine the property of continuous dependence of solutions with respect to initial conditions. When impulses occur at variable times in system (2.4.1), then different solutions will generally experience the impulsive effect at different times. This means that one cannot ordinarily expect solutions to depend continuously on initial data. For this reason we restrict our attention to impulsive delay differential equations where impulses occur at fixed times (2.4.7). We note, however, that although continuous dependence in the classical sense will not normally exist for the variable impulse system (2.4.1), one could sufficiently redefine and weaken the concept to a point where one might expect system (2.4.1) to satisfy it. Results along this line for impulsive systems without delay can be found in [Lak89], although here we will only consider a predominantly classical notion of continuous dependence and hence focus on system (2.4.7).

Roughly speaking, by continuous dependence we mean that solutions starting sufficiently close to one another in terms of their initial conditions will remain close together on finite intervals of time. Thus small perturbations of the initial time $t_0$ or the initial function $\phi$ of a solution $x = x(t_0, \phi)$ of (2.4.7) & (2.4.2) will not cause a large change to the corresponding solution on any finite interval of time. One could talk about continuous dependence just with respect to the initial function, just with respect to the initial time, or jointly with respect to both the initial function and initial time. In the former case, continuous dependence with respect to the initial function is similar to the concept of stability, which we introduce in Chapter 4. The main difference is that stability applies to infinite intervals of time and is therefore a stronger property.

In this section we develop a theorem which establishes continuous dependence of solutions of (2.4.7) with respect to initial functions. We also show that because of the discontinuities caused by the impulses and because of the delays in system (2.4.7), continuous dependence with respect to the initial time, or more generally with respect to the entire initial data, is unfortunately, and perhaps surprisingly, not generally satisfied even under otherwise strong assumptions on the functionals $f$ and $I$.

Let $(t_0, \phi) \in J \times PC([-r, 0], D)$ and suppose $x = x(t_0, \phi)$ is a solution of (2.4.7) & (2.4.2) on $[t_0 - r, t_0 + \beta]$ for some $\beta > 0$ for which $[t_0, t_0 + \beta] \subset J$. One way to precisely define the notion of continuous dependence is as follows. For every $\epsilon > 0$, there exists some $\delta > 0$ such that if $(t_0^*, \phi^*) \in J \times PC([-r, 0], D)$ where $|t_0 - t_0^*| \leq \delta$ and $\|\phi - \phi^*\|_r \leq \delta$, then if $y = y(t_0^*, \phi^*)$ is a solution of (2.4.7) & (2.4.2), then $y$ exists on (or can be continued to) the interval $[t_0^* - r, t_0 + \beta]$ and $\|x(t) - y(t)\| \leq \epsilon$ for all $t \in [\bar{t} - r, t_0 + \beta]$ where $\bar{t} = \max\{t_0, t_0^*\}$. This is essentially the definition given by Hale and Lunel [Hal93] in their study of continuous delay differential equations. Note that $\delta$ generally depends on $\epsilon$ as well as the particular solution $x$ (which in turn specifies $t_0$, $\phi$ and $\beta$).

A weaker notion of continuous dependence is sometimes considered where $\delta$ also depends on the particular value of $t$ in the interval $[\bar{t} - r, t_0 + \beta]$. In other words, any solution $y = y(t_0^*, \phi^*)$ of (2.4.7) & (2.4.2) satisfying $|t_0 - t_0^*| \leq \delta$ and $\|\phi - \phi^*\|_r \leq \delta$ is continuable at least up to time $t$ and at that particular time (although not necessarily at other times) we have $\|x(t) - y(t)\| \leq \epsilon$.

The first thing to notice about the original definition is that if $t_0^* \neq t_0$, then it is generally unreasonable to expect that $\|x(t) - y(t)\| \leq \epsilon$ for $t \in [\bar{t} - r, \bar{t}]$ (i.e. $\|x_{\bar{t}} - y_{\bar{t}}\|_r \leq \epsilon$) although for $t \in [\bar{t}, t_0 + \beta]$ one might expect the inequality to hold. This has to do with the piecewise continuous space upon which system (2.4.7) is defined. Indeed, if $\phi$ has one or more discontinuities (say at the points $-r < s_1 < s_2 < \ldots < s_m \leq 0$) and we were to choose $\phi^* = \phi$ so that our main focus is on continuous dependence with respect to $t_0$, then as $t_0^* \to t_0$, $\|x_{\bar{t}} - y_{\bar{t}}\|_r$ will not tend to zero. Instead it will approach $\max_{1 \leq k \leq m}\{\|\phi(s_k) - \phi(s_k^-)\|\}$, which would be some positive number.

Of course we do not have this problem if $t_0^* = t_0$ and we are considering only continuous dependence with respect to the initial function $\phi$. Before further discussing continuous dependence with respect to $t_0$ or the more general continuous dependence with respect to both $(t_0, \phi)$, let us introduce a theorem establishing continuous dependence with respect to the initial function only. For delay differential equations this is often the only type of continuous dependence that is considered in the literature. We will assume that the functional $f$ satisfies a local Lipschitz condition in $\psi$ just as we did for our proof of uniqueness in Theorem 3.4.1.

We first formally state the definition of continuous dependence of solutions of (2.4.7) & (2.4.2) with respect to initial conditions. Then we prove a theorem for system (2.3.1) & (2.3.2) and finally apply those results to give us a corresponding theorem for system (2.4.7) & (2.4.2).

**Definition 3.5.1:** Solutions of (2.4.7) & (2.4.2) are said to *depend continuously on initial functions* if given any solution $x = x(t_0, \phi)$ of (2.4.7) & (2.4.2) defined on some interval $[t_0 - r, t_0 + \beta]$, where $(t_0, \phi) \in J \times PC([-r, 0], D)$, $\beta > 0$ and $[t_0, t_0 + \beta] \subset J$, then for every $\epsilon > 0$, there exists some $\delta > 0$ such that if $\phi^* \in PC([-r, 0], D)$ and $\|\phi - \phi^*\|_r \leq \delta$, then if $y = y(t_0, \phi^*)$ is any solution of (2.4.7) & (2.4.2), then it exists on (or can be continued to) $[t_0 - r, t_0 + \beta]$ and will satisfy $\|x(t) - y(t)\| \leq \epsilon$ for all $t \in [t_0 - r, t_0 + \beta]$.

The above definition applies to solutions $x = x(t_0, \phi)$ of (2.4.7) defined on closed intervals of the form $[t_0 - r, t_0 + \beta]$. For solutions defined instead on finite half-open intervals of the form $[t_0 - r, t_0 + \beta)$, we can simply modify the definition to require that continuous dependence, as given by Definition 3.5.1, holds for $x$ restricted to $[t_0 - r, t_0 + \beta_1]$ for each $0 < \beta_1 < \beta$.

We remark that if Definition 3.5.1 is satisfied, then solutions of (2.4.7) & (2.4.2) are unique.

**Theorem 3.5.1:** *Assume $f$ is composite-PC and locally Lipschitz in $\psi$. Then solutions of (2.3.1) & (2.3.2) depend continuously on initial functions.*

*Proof:* Let $(t_0, \phi) \in J \times PC([-r, 0], D)$ and suppose $x = x(t_0, \phi)$ is a solution of (2.3.1) & (2.3.2) defined on some interval $[t_0 - r, t_0 + \beta]$ where $\beta > 0$ and $[t_0, t_0 + \beta] \subset J$.

Let $\epsilon > 0$ and assume, without loss of generality, that $\epsilon$ is chosen sufficiently small so that the set $S = \{z \in \mathbf{R}^n \mid \|x(t) - z\| \leq \epsilon$ for some $t \in [t_0 - r, t_0 + \beta]\}$ is contained in some compact subset $F$ of $D$. Next let $L > 0$ be the Lipschitz constant for $f$ associated with the time interval $[t_0, t_0 + \beta]$ and the compact set $F \subset D$. Define $\delta = e^{-L\beta}\epsilon/(\beta L + 1)$ and note that $0 < \delta < \epsilon$.

Now let $\phi^* \in PC([-r, 0], D)$ satisfy $\|\phi - \phi^*\|_r \leq \delta$ and suppose $y = y(t_0, \phi^*)$ is some solution of (2.3.1) & (2.3.2) defined on $[t_0 - r, t_0 + \alpha]$ for some $\alpha > 0$ for which $[t_0, t_0 + \alpha] \subset J$. According to Corollaries 3.2.2 and 3.2.3 (with the functional $I$ being identically zero so that system (2.4.7) reduces to system (2.3.1)) this solution can be extended to a maximal interval of existence $[t_0 - r, t_0 + \beta_1)$ for some $\alpha < \beta_1 \leq \infty$. We must show that $\beta_1 > \beta$ so that $y$ can be continued to at least $[t_0 - r, t_0 + \beta]$ and we must also show that $\|x(t) - y(t)\| \leq \epsilon$ for all $t \in [t_0 - r, t_0 + \beta]$.

If $\beta_1 \leq \beta$, then by Corollary 3.2.3 we know that there must exist some point $t \in [t_0 - r, t_0 + \beta_1)$ for which $y(t) \notin F$ and in particular $\|x(t) - y(t)\| > \epsilon$. We will show that $\|x(t) - y(t)\| \leq \epsilon$ for all $t \in [t_0 - r, t_0 + \beta_1) \cap [t_0 - r, t_0 + \beta]$, which will in turn imply that $y(t) \in F$ for all such $t$ and thus guarantee that $\beta_1 > \beta$.

Suppose for the sake of contradiction that $\|x(t) - y(t)\| > \epsilon$ for some $t \in [t_0 - r, t_0 + \beta_1) \cap [t_0 - r, t_0 + \beta]$. Define $t^* = \inf\{t \in [t_0 - r, t_0 + \beta_1) \cap [t_0 - r, t_0 + \beta] \mid \|x(t) - y(t)\| > \epsilon\}$. For $t \in [t_0 - r, t_0]$ we have $\|x(t) - y(t)\| = \|\phi(t - t_0) - \phi^*(t - t_0)\| \leq \delta < \epsilon$. Since both solutions are continuous when restricted to the interval $[t_0, t_0 + \beta_1) \cap [t_0, t_0 + \beta]$, then clearly we have $t_0 < t^* < \min\{t_0 + \beta, t_0 + \beta_1\}$, $\|x(t^*) - y(t^*)\| = \epsilon$ and $\|x(t) - y(t)\| \leq \epsilon$ for $t \in [t_0 - r, t^*]$. Thus $x_t, y_t \in PC([-r, 0], F)$ for all $t \in [t_0, t^*]$.

From (2.3.13) of Lemma 2.3.3, for $t \in [t_0, t^*]$ we get

$$\begin{aligned}
\|x(t) - y(t)\| &\leq \|\phi(0) - \phi^*(0)\| + \int_{t_0}^{t} \|f(s, x_s) - f(s, y_s)\| ds \\
&\leq \|\phi(0) - \phi^*(0)\| + \int_{t_0}^{t} L\|x_s - y_s\|_r ds \qquad (3.5.1)\\
&\leq \|\phi(0) - \phi^*(0)\| + L \int_{t_0}^{t} \left( \|\phi - \phi^*\|_r + \sup_{u \in [t_0, s]} \|x(u) - y(u)\| \right) ds.
\end{aligned}$$

Let $g(t_0) = \|\phi(0) - \phi^*(0)\|$ and $g(t) = \sup_{u\in[t_0,t]} \|x(u) - y(u)\|$ for $t \in (t_0, t^*]$. Then clearly $g \in C([t_0, t^*], \mathbb{R}_+)$ and from (3.5.1), noting that the right-hand side of (3.5.1) is nondecreasing in $t$, we get

$$\begin{aligned} g(t) &\le \|\phi(0) - \phi^*(0)\| + L\int_{t_0}^{t}\left(\|\phi - \phi^*\|_r + \sup_{u\in[t_0,s]}\|x(u) - y(u)\|\right)ds \\ &\le \|\phi(0) - \phi^*(0)\| + \beta L\|\phi - \phi^*\|_r + L\int_{t_0}^{t} g(s)ds \le (\beta L + 1)\delta + L\int_{t_0}^{t} g(s)ds, \end{aligned} \tag{3.5.2}$$

for $t \in [t_0, t^*]$. Therefore by Gronwall's Inequality we find that

$$g(t) \le (\beta L + 1)\delta e^{L(t-t_0)} = e^{L(t-\beta-t_0)}\epsilon, \tag{3.5.3}$$

for $t \in [t_0, t^*]$. Since $t^* < t_0 + \beta$, then in particular $g(t^*) < \epsilon$. In other words, $\sup_{u\in[t_0,t^*]} \|x(u) - y(u)\| < \epsilon$. But this contradicts the fact that $\|x(t^*) - y(t^*)\| = \epsilon$. This contradiction therefore completes the proof. ∎

**Theorem 3.5.2:** *Assume $f$ is composite-PC and locally Lipschitz in $\psi$ and that $I$ is continuous in $\psi$. Also assume that $\psi(0) + I(\tau_k, \psi) \in D$ for all $\psi \in PC([-r, 0], D)$ for which $\psi(0^-) = \psi(0)$ and for all $k = 1, 2, \ldots$. Then solutions of (2.4.7) & (2.4.2) depend continuously on initial functions.*

*Proof:* Let $(t_0, \phi) \in J \times PC([-r, 0], D)$ and suppose $x = x(t_0, \phi)$ is a solution of (2.4.7) & (2.4.2) defined on some interval $[t_0 - r, t_0 + \beta]$ where $\beta > 0$ and $[t_0, t_0 + \beta] \subset J$. Then $t_0 \in [\tau_{l-1}, \tau_l)$ for some positive integer $l$. If $t_0 + \beta < \tau_l$, then $x$ is simply a solution of (2.3.1) & (2.3.2) and so Theorem 3.5.1 gives us continuous dependence with respect to $\phi$.

Next suppose $t_0 + \beta = \tau_l$. Let $\epsilon > 0$. Since $I$ is continuous with respect to $\psi$, then there exists some $0 < \bar{\delta} \le \epsilon/2$ such that $\|I(\tau_l, x_{\tau_l^-}) - I(\tau_l, \psi)\| \le \epsilon/2$ if $\psi \in PC([-r, 0], D)$ and $\|x_{\tau_l^-} - \psi\|_r \le \bar{\delta}$. If we were to redefine, for the moment, the value of the solution $x$ at $\tau_l$ to be $x(\tau_l^-)$, and thereby make $x$ continuous on $[t_0, \tau_l]$, then we could apply Theorem 3.5.1 to this function, where we let $\delta > 0$ be given as in Definition 3.5.1 and where the $\epsilon$ in the definition is replaced by $\bar{\delta}$.

Now suppose $\phi^* \in PC([-r, 0], D)$ satisfies $\|\phi - \phi^*\|_r \le \delta$. Then any solution $y = y(t_0, \phi^*)$ of (2.4.7) & (2.4.2) exists on or can be extended (as a solution of (2.3.1) & (2.3.2)) to $[t_0 - r, \tau_l]$, and on the interval $[t_0 - r, \tau_l)$, will satisfy $\|x(t) - y(t)\| \le \bar{\delta}$. Redefining the value of $y$ at $\tau_l$ to be $y(\tau_l^-) + I(\tau_l, y_{\tau_l^-})$ will then make it a solution (which is unique according to Theorem 3.4.1) of (2.4.7) & (2.4.2).

Since for $t \in [t_0 - r, \tau_l)$ we have $\|x(t) - y(t)\| \le \bar{\delta} \le \epsilon$, then in particular it follows that $\|x_{\tau_l^-} - y_{\tau_l^-}\|_r \le \bar{\delta}$. Finally, at $t = \tau_l$ we have

$$\begin{aligned} \|x(\tau_l) - y(\tau_l)\| &= \|x(\tau_l^-) + I(\tau_l, x_{\tau_l^-}) - y(\tau_l^-) - I(\tau_l, y_{\tau_l^-})\| \\ &\le \|x(\tau_l^-) - y(\tau_l^-)\| + \|I(\tau_l, x_{\tau_l^-}) - I(\tau_l, y_{\tau_l^-})\| \le \bar{\delta} + \epsilon/2 \le \epsilon. \end{aligned} \tag{3.5.4}$$

Therefore for all $t \in [t_0 - r, \tau_l]$, it follows that $\|x(t) - y(t)\| \leq \epsilon$ and so this concludes the case when $t_0 + \beta = \tau_l$.

For the final case suppose $\tau_{l+m-1} < t_0 + \beta \leq \tau_{l+m}$ for some positive integer $m$ and let $\epsilon > 0$. For the solution $x$ restricted to $[\tau_{l+m-1} - r, t_0 + \beta]$ and with $\tau_{l+m-1}$ and $x_{\tau_{l+m-1}}$ being thought of as the initial time and initial function, respectively, let $\delta_1 > 0$ be defined so that given any $\phi^* \in PC([-r,0], D)$ with $\|x_{\tau_{l+m-1}} - \phi^*\|_r \leq \delta_1$, then any solution $y = y(\tau_{l+m-1}, \phi^*)$ of (2.4.7) & (2.4.2) either already exists on or can be continued to $[\tau_{l+m-1} - r, t_0 + \beta]$ and will satisfy $\|x(t) - y(t)\| \leq \epsilon$ for all $t \in [\tau_{l+m-1} - r, t_0 + \beta]$. The existence of such a $\delta_1 > 0$ follows from what we showed earlier in this proof.

Similarly, if $m \geq 2$, then for $i = 1, 2, \ldots, m-1$, consider $x$ restricted to $[\tau_{l+m-i-1} - r, \tau_{l+m-i}]$ where $\tau_{l+m-i-1}$ and $x_{\tau_{l+m-i-1}}$ are considered the new initial time and function. Then let $\delta_{i+1} > 0$ be defined inductively so that $\delta_{i+1} < \delta_i$ and if $\phi^* \in PC([-r,0], D)$ and $\|x_{\tau_{l+m-i-1}} - \phi^*\|_r \leq \delta_{i+1}$, then any solution $y = y(\tau_{l+m-i-1}, \phi^*)$ of (2.4.7) & (2.4.2) either already exists on or can be continued to $[\tau_{l+m-i-1} - r, \tau_{l+m-i}]$ and will satisfy $\|x(t) - y(t)\| \leq \delta_i$ for all $t \in [\tau_{l+m-i-1} - r, \tau_{l+m-i}]$.

Finally let $\delta > 0$ be chosen so that given any $\phi^* \in PC([-r,0], D)$ satisfying $\|\phi - \phi^*\|_r \leq \delta$ and given any solution $y = y(t_0, \phi^*)$ of (2.4.7) & (2.4.2), then $y$ exists on or can be continued to $[t_0 - r, \tau_l]$ and $\|x(t) - y(t)\| \leq \delta_m$ for all $t$ in this interval.

With this choice of $\delta$ we can show that any solution $y = y(t_0, \phi^*)$ of (2.4.7) & (2.4.2) for which $\phi^* \in PC([-r,0], D)$ and $\|\phi - \phi^*\|_r \leq \delta$ can be continued all the way to $[t_0 - r, t_0 + \beta]$ and will satisfy $\|x(t) - y(t)\| \leq \epsilon$ for all $t \in [t_0 - r, t_0 + \beta]$. This is because initially we know that $y$ can be continued to $[t_0 - r, \tau_l]$ and on this interval $\|x(t) - y(t)\| \leq \delta_m$. In particular $\|x_{\tau_l} - y_{\tau_l}\|_r \leq \delta_m$. Thus $y$ can be continued further to all of $[t_0 - r, \tau_{l+1}]$ and where $\|x(t) - y(t)\| \leq \delta_{m-1}$ on this interval (or if $m = 1$ and $t_0 + \beta \leq \tau_{l+1}$, then $y$ can be continued to $[t_0 - r, t_0 + \beta]$ and on this interval $\|x(t) - y(t)\| \leq \epsilon$). By repeating this same argument $m-1$ more times we find that $y$ can be continued to the impulse times $\tau_{l+2}, \tau_{l+3}, \ldots, \tau_{l+m-1}$ and then to $t_0 + \beta$. On each interval $[t_0 - r, \tau_{l+i}]$ we find that $\|x(t) - y(t)\| \leq \delta_{m-i}$ and then finally on $[t_0 - r, t_0 + \beta]$ we get $\|x(t) - y(t)\| \leq \epsilon$, which completes the proof. ∎

Let us now return to the discussion of continuous dependence of solutions of (2.4.7) & (2.4.2) with respect to the initial time. For the time being we are treating the initial function as being fixed but allowing for small perturbations of the initial time. We have already mentioned that a solution $y = y(t_0^*, \phi)$ cannot be expected to be close to a given solution $x = x(t_0, \phi)$ on the interval $[\bar{t} - r, \bar{t}]$, where $\bar{t} = \max\{t_0, t_0^*\}$, regardless of how close $t_0^*$ is to $t_0$. Another important observation to make is that one cannot expect solutions of (2.4.7) & (2.4.2) to depend continuously on $t_0$ if $t_0$ happens to be an impulse time. If $t_0 = \tau_k$ for some $k$, then by our definition of a solution of (2.4.7) & (2.4.2) we do not consider a solution $x = x(t_0, \phi)$ to experience an impulse instantly at its initial time $t_0$. However, if $t_0^*$ were chosen so that $0 < t_0 - t_0^* \leq \delta$ for an arbitrarily small $\delta > 0$, then the corresponding solution $y = y(t_0^*, \phi)$ would experience the impulsive effect at time $t = t_0$ and this would likely cause the solution $y$ to deviate a great deal from $x$ at times $t \geq t_0$.

As it turns out, even if we were to modify the definition of continuous dependence of solutions of (2.4.7) & (2.4.2) to account for this problem (by excluding impulse times from the set of initial times), we still cannot generally expect to get continuous dependence with respect to $t_0$ as we will demonstrate by way of examples.

One problem with establishing continuous dependence with respect to $t_0$ stems from the impulse functional $I$. In general one can only expect solutions of (2.4.7) & (2.4.2) to depend continuously on $t_0$ if $I$ does not involve delays. To illustrate the problem that delays in $I$ can cause, consider the simple scalar equation

$$x'(t) = 0, \ t \neq k, \tag{3.5.5a}$$

$$\Delta x(t) = x(t-1), \ t = k, \tag{3.5.5b}$$

where $k = 1, 2, \ldots$ and where delays are present only in the impulse equation. Let the initial function $\phi$ be defined by $\phi(s) = 0$ for $s \in [-1, 0)$ and $\phi(0) = 1$. Letting $t_0 = 0$ and solving for the solution $x = x(t_0, \phi)$ of (3.5.5) we find that $x(t) = 1$ for $t \in [0, 1)$ and $x(t) = 2$ for $t \in [1, 2)$. On the other hand for arbitrarily small $\delta > 0$ if $t_0^* = \delta$ and $y = y(t_0^*, \phi)$, then we find that $y(t) = 1$ for all $t \in [\delta, 2)$. So clearly in this case these solutions are not close to each other. This problem can be avoided if we only consider initial times $t_0$ that are not contained in any set of the form $[\tau_k - r, \tau_k]$. Of course this becomes very restrictive if $r$ is large especially relative to the time differences between successive impulse times.

Solutions of the continuous system (2.1.3) & (2.1.7) can be shown to depend continuously on initial times (in fact they depend continuously on the initial data $(t_0, \phi)$) if $f$ is a continuous functional and if $f$ satisfies a local Lipschitz condition in $\psi$ [Ogu66]. More generally, continuous dependence can be proven under the assumption that $f$ is continuous and that solutions of (2.1.3) & (2.1.7) are unique [Hal93]. Unfortunately, these proofs fall apart when one tries to enlarge the space from the set of continuous functions to the set of piecewise continuous functions. In fact it no longer remains true that such conditions will guarantee continuous dependence with respect to $t_0$. We will illustrate this with an example of a functional $f$ that is composite-PC and locally Lipschitz in $\psi$. It will therefore satisfy the conditions of Theorem 3.4.1, which ensures uniqueness of solutions. Solutions of the corresponding system (2.3.1) & (2.3.2) having no impulses will depend continuously on initial functions according to Theorem 3.5.1. Despite all of this, however, the solutions (having piecewise continuous initial functions) fail to depend continuously on initial times.

Define the functional $f : \mathbf{R}_+ \times PC([-1, 0], \mathbf{R}) \to \mathbf{R}$ as follows,

$$f(t, \psi) = \begin{cases} \psi(-t), & t \in [0, 1], \\ \psi(-1), & t \in (1, \infty). \end{cases} \tag{3.5.6}$$

This corresponds to the equation

$$x'(t) = \begin{cases} x(0), & t \in [0,1], \\ x(t-1), & t \in (1,\infty). \end{cases} \tag{3.5.7}$$

This equation can also be written in the form

$$x'(t) = x(t - h(t)), \tag{3.5.8}$$

where

$$h(t) = \begin{cases} t, & t \in [0,1], \\ 1, & t \in (1,\infty). \end{cases} \tag{3.5.9}$$

Clearly $h$ is continuous, $0 \le h(t) \le 1$ for all $t \in \mathbb{R}_+$ and $t - h(t)$ is nondecreasing on $\mathbb{R}_+$. The functional $f$ is therefore of the type (F2) defined in Section 2.3. Therefore by Corollary 3.1.1, $f$ is composite-PC, quasi-bounded and continuous in $\psi$. It is not hard to see that $f$ is also locally Lipschitz in $\psi$.

Now let the initial function be defined to be $\phi(s) = 1$ for $s \in [-1,-\rho)$ and $\phi(s) = 0$ for $s \in [-\rho, 0]$ where $0 \le \rho < 1$ is some constant. Let us start by considering the initial time $t_0 = \rho$. Solving $x = x(t_0, \phi)$ of equation (3.5.7) gives us $x(t) = 0$ for all $t \ge 0$ and so in particular $x(1) = 0$. Now let $0 < \delta < 1 - \rho$ and consider $t_0^* = \rho + \delta$. If we solve for $y = y(t_0^*, \phi)$, then we find that on the interval $[\rho + \delta, 1]$, $y(t) = t - (\rho + \delta)$ and so in particular at $t = 1$ we get $y(1) = 1 - (\rho + \delta)$. As $\delta \to 0^+$ this value approaches $1 - \rho$, which does not equal $x(1) = 0$. Hence, solutions of (3.5.7) do not depend continuously on $t_0$.

Another thing to note about equation (3.5.7) is that it is a typical example of an equation to which the step method could be applied (on any interval $[\rho, \infty)$ where $\rho > 0$). So here is an example where the theory of continuous dependence of solutions of ordinary differential equations cannot directly be carried over to an impulsive delay differential equation despite the ability to reduce the equation to one of an ordinary differential equation.

The rather simple equation (3.5.7) illustrates that continuous dependence of solutions with respect to initial times (or more generally with respect to initial data) is unfortunately not a property that one can ordinarily expect from the more general equation (2.4.7) & (2.4.2) even under strong smoothness assumptions on $f$.

Some positive results establishing continuous dependence of solutions with respect to initial times could be established for very restrictive classes of functionals. Obviously smooth functionals that don't involve time delays would have this property. Moreover, if one were to restrict the class of initial functions to only continuous functions as well as restrict the set of initial times to non-impulse times and assume $I$ had no delays, then one could fairly easily show that under the same conditions as Theorem 3.5.2, this weaker form of continuous dependence would follow. However, as a general rule, continuous dependence of solutions of (2.4.7) & (2.4.2) with respect to initial times does not hold, although continuous dependence with respect to initial functions holds under fairly standard smoothness assumptions on $f$ and $I$.

# Chapter 4

# Stability and Boundedness Using Lyapunov Functionals

In this chapter and the next, we obtain stability and boundedness conditions for system (2.4.7). In this chapter we make use of Lyapunov functionals in our proofs of stability and boundedness. Their use represents a natural generalization of the direct method (or second method) of Lyapunov for ordinary differential equations. In Chapter 5 we utilize Lyapunov functions, which tend to be simpler to work with but which require the use of Razumikhin-type techniques in order to make the most of these functions.

Lyapunov functions (and functionals) are often found in an ad hoc fashion and suitable functions can sometimes be difficult to obtain. Nevertheless, for some simple equations one can create an appropriate Lyapunov function relatively easily. Other times the physics of the problem might suggest that one consider a particular form of Lyapunov function (such as an energy function). Alexander Lyapunov (1857-1918) pioneered the use of functions which carry his name. The idea is to use an auxiliary function that satisfies any of various conditions (such as, for example, being scalar, nonnegative-valued, positive definite, decrescent, radially unbounded, having a negative definite derivative with respect to the system of equations, etc.). By looking at how such an auxiliary function, or Lyapunov function, evolves along solutions of an ordinary differential equation (or in our case, an impulsive delay differential equation), information about stability or boundedness properties of the system may be obtained.

## 4.1 Properties of Lyapunov Functionals

In order to make use of Lyapunov functionals in our theorems we must first define certain properties that we shall later refer to.

**Definition 4.1.1:** A functional $V : J \times PC([-r,0],D) \to \mathbb{R}_+$ is said to be *consistently composite-PC* if for each $t_0 \in J$ and $\alpha > 0$, where $[t_0, t_0+\alpha] \subset J$, if $x \in PC([t_0 - r, t_0 + \alpha], D)$ and $x$ is continuous at each $t \neq \tau_k$ in $(t_0, t_0+\alpha]$, then the composite function $g$ defined by $g(t) = V(t, x_t)$ is continuous at each $t \neq \tau_k$ in $(t_0, t_0+\alpha]$ and $g \in PC([t_0, t_0+\alpha], \mathbb{R}_+)$.

Note that this definition is similar to the one given for composite-PC functionals $f$ in Section 2.3. One minor difference is that here the functional $V$ maps into $\mathbb{R}_+$ instead of $\mathbb{R}^n$. The main distinguishing feature, however, is that the composite function $V(t, x_t)$ may not have jump discontinuities at non-impulse times. Most useful Lyapunov functionals are of this form and so this does not impose a significant constraint. When looking at how a Lyapunov functional evolves along solutions of (2.4.7) it will be important to know that discontinuities can only occur at the impulse times. For this reason we will normally assume that the Lyapunov functionals are consistently composite-PC instead of just composite-PC.

Recall that in Chapter 2 we defined what it meant for a functional to be locally Lipschitz in $\psi$. We restate the definition here, for convenience, insofar as it applies to a Lyapunov functional $V$.

**Definition 4.1.2:** A functional $V : J \times PC([-r,0],D) \to \mathbb{R}_+$ is said to be *locally Lipschitz in* $\psi$ if for each $t_0 \in J$ and $\alpha > 0$, where $[t_0, t_0+\alpha] \subset J$, and for each compact set $F \subset D$, there exists some $L > 0$ such that $|V(t,\psi_1) - V(t,\psi_2)| \leq L\|\psi_1 - \psi_2\|_r$ for all $t \in [t_0, t_0+\alpha]$ and $\psi_1, \psi_2 \in PC([-r,0],F)$.

Next we define the derivative of Lyapunov functionals with respect to system (2.4.7).

**Definition 4.1.3:** Given a functional $V : J \times PC([-r,0],D) \to \mathbb{R}_+$, the upper right-hand derivative of $V$ with respect to system (2.4.7) is defined by

$$D^+V_{(2.4.7)}(t,\psi) = \limsup_{h \to 0^+} \frac{1}{h}[V(t+h, \psi_{[h]}) - V(t,\psi)], \tag{4.1.1}$$

for $(t,\psi) \in J \times PC([-r,0],D)$, where

$$\psi_{[h]}(s) = \begin{cases} \psi(h+s), & s \in [-r,-h], \\ \psi(0) + (h+s)f(t,\psi), & s \in (-h, 0], \end{cases} \tag{4.1.2}$$

for $0 < h \leq r$.

Note that $D^+V_{(2.4.7)}(t,\psi)$ given by (4.1.1) is well-defined (as an extended real number) and can be computed directly from $f$ without advanced knowledge of the solutions of system (2.4.7). Occasionally, explicit reference to system (2.4.7) in (4.1.1) may be omitted for brevity, whereupon we simply write $D^+V(t,\psi)$. If $V$ is locally Lipschitz in $\psi$, then this is equivalent to the time derivative of $V$ along solutions of (2.4.7). This is the subject of our next theorem.

**Theorem 4.1.1:** *Assume $f$ satisfies the conditions of Corollary 3.1.3. Let* $V : J \times PC([-r,0],D) \to$

*$\mathbb{R}_+$ and assume $V$ is locally Lipschitz in $\psi$. If $(t,\psi) \in J \times PC([-r,0],D)$ and $x = x(t,\psi)$ is any solution of (2.4.7) satisfying the initial condition $x_t = \psi$, then*

$$D^+V_{(2.4.7)}(t,\psi) = \limsup_{h\to 0^+} \frac{1}{h}\left[V(t+h, x_{t+h}(t,\psi)) - V(t,\psi)\right]. \tag{4.1.3}$$

*Proof:* Assume $x$ is defined on $[t-r, t+\beta]$ for some $\beta > 0$ where $[t,t+\beta] \subset J$. Let $h_1 > 0$ be sufficiently small so that $h_1 \le r$, $h_1 \le \beta$, $\psi(0) + sf(t,\psi) \in D$ for $s \in [0,h_1]$, $\tau_k \notin (t, t+h_1]$ for all $k$ and $f(s,x_s)$ is a continuous function of $s$ for $s \in [t, t+h_1]$. This final constraint imposed on $h_1$ is justified by the composite-PC assumption on $f$.

Let $F \subset D$ be a compact set containing $x(s)$ for $s \in [t-r, t+h_1]$ and $\psi(0)+sf(t,\psi)$ for $s \in [0,h_1]$. Then $\psi_{[h]}, x_{t+h} \in PC([-r,0],F)$ for $0 < h \le h_1$ and $x_t = \psi \in PC([-r,0],F)$. Let $L > 0$ be the Lipschitz constant for $V$ associated with the sets $[t,t+h_1] \subset J$ and $F \subset D$.

Since $f(s,x_s)$ is continuous on $[t,t+h_1]$ and $\tau_k \notin (t,t+h_1]$ for all $k$, then the solution $x$ is continuous on $[t,t+h_1]$ and differentiable on $(t,t+h_1)$ with derivative (not just right-hand derivative) equalling $f(s,x_s)$ at each $s \in (t,t+h_1)$. Let $0 < h < h_1$. Then for each $0 < s \le h$, there exists some $\bar{t}(s) \in (t,t+s)$ such that $x(t+s) - x(t) = sf(\bar{t}(s), x_{\bar{t}(s)})$ by the Mean Value Theorem. In other words, $x(t+s,t,\psi) - \psi(0) = sf(\bar{t}(s), x_{\bar{t}(s)})$. Thus we obtain the following inequality

$$\begin{aligned}
\frac{1}{h}|V(t+h, x_{t+h}(t,\psi)) - V(t+h,\psi_{[h]})| &\le \frac{L}{h}\|x_{t+h}(t,\psi) - \psi_{[h]}\|_r \\
&= \frac{L}{h}\sup_{s\in[-r,0]}\|x_{t+h}(s,t,\psi) - \psi_{[h]}(s)\| \\
&= \frac{L}{h}\sup_{s\in(-h,0]}\|x(t+h+s,t,\psi) - (\psi(0) + (h+s)f(t,\psi))\| \\
&= \frac{L}{h}\sup_{s\in(0,h]}\|x(t+s,t,\psi) - (\psi(0) + sf(t,\psi))\| \\
&= \frac{L}{h}\sup_{s\in(0,h]} s\|f(\bar{t}(s), x_{\bar{t}(s)}) - f(t,\psi)\| \\
&\le L\sup_{s\in(0,h]}\|f(\bar{t}(s), x_{\bar{t}(s)}) - f(t,\psi)\| \\
&\le L\sup_{s\in(0,h]}\|f(t+s, x_{t+s}) - f(t,x_t)\|.
\end{aligned} \tag{4.1.4}$$

As $h \to 0^+$ this term clearly tends to 0 since $f(s,x_s)$ is right-continuous at $t$.

Thus we get the following equality, which completes the proof.

$$\begin{aligned} D^+V_{(2.4.7)}(t,\psi) &= \limsup_{h\to 0^+} \frac{1}{h}[V(t+h,\psi_{[h]}) - V(t,\psi)] \\ &= \limsup_{h\to 0^+} \frac{1}{h}[V(t+h,\psi_{[h]}) - V(t,\psi)] + \lim_{h\to 0^+} \frac{1}{h}[V(t+h,x_{t+h}(t,\psi)) - V(t+h,\psi_{[h]})] \\ &= \limsup_{h\to 0^+} \left\{ \frac{1}{h}[V(t+h,\psi_{[h]}) - V(t,\psi)] + \frac{1}{h}[V(t+h,x_{t+h}(t,\psi)) - V(t+h,\psi_{[h]})] \right\} \\ &= \limsup_{h\to 0^+} \frac{1}{h}[V(t+h,x_{t+h}(t,\psi)) - V(t,\psi)]. \end{aligned} \tag{4.1.5}$$

■

**Corollary 4.1.1:** *Assume $f$ and $V$ are given as in Theorem 4.1.1. Suppose $x = x(t_0,\phi)$ is a solution of (2.4.7) & (2.4.2) defined on $[t_0 - r, t_0 + \beta]$ for some $\beta > 0$ where $[t_0, t_0 + \beta] \subset J$. Then*

$$D^+V_{(2.4.7)}(t,x_t) = \limsup_{h\to 0^+} \frac{1}{h}[V(t+h,x_{t+h}(t_0,\phi)) - V(t,x_t(t_0,\phi))], \tag{4.1.6}$$

*for $t \in [t_0, t_0 + \beta)$.*

*Proof:* Fix $t \in [t_0, t_0+\beta)$. Then restricted to $[t, t_0+\beta]$, $x$ is a solution of (2.4.7) satisfying the initial condition $x_t = \psi$ where $\psi = x_t(t_0,\phi)$. For $h > 0$ satisfying $t + h < t_0 + \beta$, $x_{t+h}(t_0,\phi) = x_{t+h}(t,\psi)$. So replacing $\psi$ by $x_t(t_0,\phi)$ and $x_{t+h}(t,\psi)$ by $x_{t+h}(t_0,\phi)$ in (4.1.3) gives us (4.1.6). ■

Corollary 4.1.1 says that under suitable assumptions on $f$ and $V$, if one were to construct the composite function $m(t) = V(t,x_t)$ for $t \in [t_0, t_0 + \beta]$ where $x = x(t_0,\phi)$ is a solution of (2.4.7) & (2.4.2), then the upper right-hand derivative (a generalized Dini derivative) of $m(t)$ defined by

$$D^+m(t) = \limsup_{h\to 0^+} \frac{1}{h}[m(t+h) - m(t)], \tag{4.1.7}$$

is equal, at each $t \in [t_0, t_0+\beta)$, to the upper right-hand derivative of $V$ with respect to system (2.4.7) as defined by (4.1.1) evaluated at $(t,\psi) = (t,x_t)$.

When analyzing stability properties of system (2.4.7), the derivative of a Lyapunov functional (or function) along solutions as given by (4.1.7) will play an integral role. In particular, $D^+m(t)$ will be required to be bounded above at all non-impulse times by some (usually nonpositive) function. This will enable us to obtain an upper bound on the growth rate of $m(t)$. It is customary to consider generalized Dini derivatives since this allows one to apply stability theorems to a wider class of problems. The other three Dini derivatives (lower right, lower left and upper left) would be less suitable in the study of impulsive delay differential equations since solutions are assumed to be right-continuous, their evolution is considered only to the right, and because we are only interested in imposing upper bounds on the derivative of $m(t)$.

In (4.1.1) we have defined an upper right-hand derivative of $V$ with respect to system (2.4.7), which can be evaluated at any $(t, \psi) \in J \times PC([-r, 0], D)$ without advanced knowledge of solutions. In practice one can often form the composite function $V(t, x_t)$ where $x$ is some as yet unknown solution. Then calculating the time derivative of $V(t, x_t)$ becomes a simpler task.

The next few examples demonstrate the calculation of the derivative of a Lyapunov functional along solutions of a delay differential equation by using Definition 4.1.3. These we compare to what one would get by differentiating $V(t, x_t)$ directly with respect to $t$. For simplicity we refer to equation (2.3.3) in each case.

Suppose we consider the functional

$$V(t, \psi) = \int_{-1}^{0} |\psi(s)|ds, \tag{4.1.8}$$

which is clearly locally Lipschitz in $\psi$. From (4.1.1) we find that

$$\begin{aligned} D^+V_{(2.3.3)}(t, \psi) &= \limsup_{h \to 0^+} \frac{1}{h}[V(t+h, \psi_{[h]}) - V(t, \psi)] \\ &= \limsup_{h \to 0^+} \frac{1}{h}\left[\int_{-1}^{0} |\psi_{[h]}(s)|ds - \int_{-1}^{0} |\psi(s)|ds\right] \\ &= \limsup_{h \to 0^+} \frac{1}{h}\left[\int_{-1}^{-h} |\psi(h+s)|ds + \int_{-h}^{0} |\psi(0) + (h+s)f(t, \psi)|ds - \int_{-1}^{0} |\psi(s)|ds\right] \\ &= \limsup_{h \to 0^+} \frac{1}{h}\left[\int_{-h}^{0} |\psi(0) + (h+s)f(t, \psi)|ds - \int_{-1}^{h-1} |\psi(s)|ds\right] \\ &= |\psi(0)| - |\psi(-1)|, \end{aligned} \tag{4.1.9}$$

for $(t, \psi) \in \mathbb{R}_+ \times PC([-1, 0], \mathbb{R})$. According to Corollary 4.1.1 the derivative of $V$ along solutions of (2.3.3) should be

$$D^+V_{(2.3.3)}(t, x_t) = |x(t)| - |x(t-1)|. \tag{4.1.10}$$

This can be verified more directly by looking at the composite function

$$V(t, x_t) = \int_{t-1}^{t} |x(s)|ds, \tag{4.1.11}$$

from which (4.1.10) easily follows.

Next, suppose we consider the Lyapunov functional

$$V(t, \psi) = \psi^2(0). \tag{4.1.12}$$

This is in fact an example of a Lyapunov function, which we will look at in more detail in the next chapter. From (4.1.1) we calculate

$$\begin{aligned} D^+V_{(2.3.3)}(t,\psi) &= \limsup_{h\to 0^+}\frac{1}{h}[V(t+h,\psi_{[h]})-V(t,\psi)] = \limsup_{h\to 0^+}\frac{1}{h}\left[\psi^2_{[h]}(0)-\psi^2(0)\right] \\ &= \limsup_{h\to 0^+}\frac{1}{h}\left[(\psi(0)+hf(t,\psi))^2-\psi^2(0)\right] \\ &= \limsup_{h\to 0^+}\frac{1}{h}\left[2h\psi(0)f(t,\psi)+(hf(t,\psi))^2\right] \\ &= 2\psi(0)f(t,\psi) = 2\psi(0)\psi(-1). \end{aligned} \tag{4.1.13}$$

So along solutions of (2.3.3),

$$D^+V_{(2.3.3)}(t,x_t) = 2x(t)x(t-1), \tag{4.1.14}$$

which may be verified by differentiating

$$V(t,x_t) = x^2(t), \tag{4.1.15}$$

with respect to $t$.

As a final example let

$$V(t,\psi) = |\psi(-1)|, \tag{4.1.16}$$

from which we get

$$\begin{aligned} D^+V_{(2.3.3)}(t,\psi) &= \limsup_{h\to 0^+}\frac{1}{h}[V(t+h,\psi_{[h]})-V(t,\psi)] = \limsup_{h\to 0^+}\frac{1}{h}\left[|\psi_{[h]}(-1)|-|\psi(-1)|\right] \\ &= \limsup_{h\to 0^+}\frac{1}{h}\left[|\psi(h-1)|-|\psi(-1)|\right]. \end{aligned} \tag{4.1.17}$$

In particular, if $\psi(-1)\neq 0$, then

$$D^+V_{(2.3.3)}(t,\psi) = \operatorname{sgn}(\psi(-1))D^+\psi(-1), \tag{4.1.18}$$

where $\operatorname{sgn}(x) = 1$ for $x > 0$ and $\operatorname{sgn}(x) = -1$ for $x < 0$. An important thing to note is that the derivative of $V$ depends only on $\psi$ and not on the functional $f(t,\psi) = \psi(-1)$ for system (2.3.3). This is true in general for any system if $V(t,\psi)$ is independent of the values $\psi(s)$ for $s \in [-\epsilon, 0]$ and any $\epsilon > 0$ arbitrarily small. Note that while the derivative of the Lyapunov functional (4.1.8) with respect to equation (2.3.3) seemed initially to depend on $f$, when taking the limit in (4.1.9) we found that the value of $f(t,\psi)$ was actually immaterial.

If from (4.1.16) one forms the composite function

$$V(t, x_t) = |x(t-1)|, \tag{4.1.19}$$

then by Corollary 4.1.1,

$$D^+ V_{(2.3.3)}(t, x_t) = \operatorname{sgn}(x(t-1)) D^+ x(t-1), \tag{4.1.20}$$

for $x(t-1) \neq 0$. We might be tempted to simplify (4.1.20) even further by claiming

$$D^+ V_{(2.3.3)}(t, x_t) = \operatorname{sgn}(x(t-1)) f(t-1, x_{t-1}) = \operatorname{sgn}(x(t-1)) x(t-2), \tag{4.1.21}$$

where we are making use of the relation (2.3.3) with $t$ replaced by $t-1$. There are several problems with this. The first is that while $D^+ x(t-1)$ is well-defined, the ordinary derivative or even right-hand derivative of $x$ at $t-1$ may not exist. Even if it does exist there is no guarantee that at this earlier time the solution must satisfy the delay differential equation (2.3.3). This is particularly true if $t \in [t_0, t_0+1)$ where $x(t-1)$ is then defined in terms of the initial function $\phi$, which need not be right-differentiable anywhere, only piecewise continuous, and which may have no relation to the derivative of $x$.

Nevertheless, in some instances we can use this technique for simplifying the expression for $D^+ V$ for the general system (2.4.7). However, we can only do this if we know that the solution on the interval $[t-r, t]$ also satisfies (2.4.7). In other words, the technique is only useful when considering values of $t \geq t_0 + r$ where $t_0$ is the initial time. One final comment about the functional $V$ given by (4.1.16) is that it is not consistently composite-PC. Our theorems in the next sections will assume, for simplicity, that the Lyapunov functional under consideration be consistently composite-PC. Therefore functionals such as (4.1.16) will not be considered.

We would now like to further discuss why we chose to assume that solutions of our various systems are right-continuous instead of left-continuous. We have already made mention of a few of the advantages and disadvantages of the two alternatives, which we will summarize. The assumption of left-continuity is consistent with current practice in the study of impulsive differential equations without delay. Left-continuity also seems more natural (to this author) when thinking of the evolution of a state in the direction of increasing time and having it actually reach an impulse hypersurface (not merely approach it) before experiencing the impulsive effect. A third advantage to the left-continuity assumption is that it would be simpler to express the delay difference equation as simply $\Delta x(t) = I(t, x_t)$ when $t$ is an impulse time instead of having to define the notation $x_{t^-}$ and then consider $I(t, x_{t^-})$.

One advantage of assuming right-continuity, which has already been mentioned, is that the initial condition need only be represented by a single function $\phi$ and not also by a point in $\mathbf{R}^n$ representing the value of $x(t_0^+)$. In the study of delay differential equations, right-hand derivatives if not actual ordinary derivatives are usually considered. So assuming right-continuity of solutions is consistent with the notion of right-differentiability.

Before pointing out the final key advantage to the right-continuous assumption, let us mention why it is even important to require that solutions always satisfy a one-sided continuity assumption. Of course from a modelling point of view it should make little difference whether or not solutions are left- or right-continuous (or neither) at impulse times unless the system being modelled is highly sensitive to this sort of thing. However, in the theoretical analysis of the equations it helps to make some underlying simplifying assumptions. Suppose we were to simply insist that solutions be piecewise continuous with simple jump discontinuities but not require that the value of the solution at the discontinuities bear any resemblance to the left- or right-hand limiting values. This would not make much difference for impulsive differential equations without delay but when there is delay this could cause two notable problems.

Suppose an initial function $\phi \in PC([-1,0],\mathbb{R})$ were chosen for equation (2.3.3) that had simple jump discontinuities at two points $t_1, t_2 \in (-1,0)$. Suppose $\phi$ were right-continuous at $t_1$ and left-continuous at $t_2$. The resulting "solution" $x = x(0,\phi)$ would satisfy (2.3.3) at $t_1 + 1$ only in terms of its right-hand derivative. Likewise at $t_2 + 1$ it would satisfy (2.3.3) only in terms of its left-hand derivative. Therefore we would not be able to claim that equation (2.3.3) is satisfied in terms of at least a one-sided derivative (either left or right) at all times $t > t_0$ (excluding impulse times). Similarly if we considered a Lyapunov function such as (4.1.12) and looked at the composite function $m(t) = V(t, x_t)$, then the upper right-hand derivative of $m(t)$ as defined by (4.1.7) would satisfy

$$D^+m(t) = 2x(t)x(t-1), \tag{4.1.22}$$

at time $t_1 + 1$ but not at time $t_2 + 1$. If instead of the upper right derivative we were to consider the lower left-hand derivative, which is occasionally done in the study of ordinary differential equations, then it would satisfy

$$D_-m(t) = \liminf_{h\to 0^+} \frac{1}{h}[m(t) - m(t-h)] = 2x(t)x(t-1) \tag{4.1.23}$$

at time $t_2 + 1$ but not at $t_1 + 1$. It will prove useful to know in our theorems and examples that at least one of these Dini derivatives of $m(t)$ exists and satisfies an equation such as (4.1.22) at every non-impulse time $t > t_0$.

The second problem stems from the question of what to do when a solution reaches an impulse time $\tau_k$ (or when it simply has a discontinuity in its initial function). Should we consider it to take on the value $x(\tau_k^-)$, $x(\tau_k^+)$ or maybe something in between, like the average of these two values, at the impulse time $t = \tau_k$. The value of $x(\tau_k)$ could be rather arbitrarily assigned if it weren't for the fact that there are delays. Unfortunately the value of $x(\tau_k)$ could play a crucial role in the evolution of the solution at future times. For instance in equation (2.3.3), the choice of what to define $x(\tau_k)$ will determine what happens at time $\tau_k + 1$ and whether or not equation (2.3.3) will be satisfied at time $\tau_k + 1$ with its left-hand derivative, right-hand derivative or neither. A more drastic illustration comes from looking at equation (3.5.7). The value of $x(0)$ determines how the solution will evolve over the time interval $[0, 1]$. Depending on whether $x(0)$ were taken to be $x(0^-)$ or $x(0^+)$ or something else entirely, the resulting solution could vary a great deal. In the effort to maintain some form of consistency it is therefore important that we impose a condition on our solutions (and the initial functions) that they always be left-continuous or that they always be right-continuous.

If solutions were thought of as being left-continuous and if the space $PC([-r, 0], D)$ were correspondingly redefined to include only left-continuous functions, then as a solution of (2.4.7) evolves it would tend to satisfy (2.4.7a) only in terms of its left-hand derivative at all $t > t_0$ excluding impulse times. This would lead us to have to consider only left-hand (or lower left-hand or upper left-hand) derivatives of Lyapunov functionals. The problem here, however, is that we cannot define the lower (or upper) left-hand derivative of $V$ with respect to system (2.4.7) in a similar fashion to Definition 4.1.3 and expect it to equal $D_-m(t)$ where $m(t) = V(t, x_t)$ and $x$ is any solution of (2.4.7). In other words, one cannot generally calculate a left-hand derivative of $V$ with respect to system (2.4.7) without knowing the solutions of (2.4.7) in advance.

Any definition of $D_-V(t, \psi)$ should satisfy

$$D_-V(t, x_t) = \liminf_{h\to 0^+} \frac{1}{h}[V(t, \psi) - V(t-h, x_{t-h}(t, \psi))], \tag{4.1.24}$$

for any solution $x$ of (2.4.7) much like the equality given by (4.1.3) in Theorem 4.1.1. In (4.1.3) the term $x_{t+h}$ is approximated by using $f$. For $s \in [-r, -h]$ the value of $x_{t+h}(s)$ is known and through the forward continuation of solutions we approximate $x(t + h + s) = x_{t+h}(s)$ for $s \in [-h, 0]$ by $x(t + h + s) \approx x(t) + (h + s)f(t, \psi)$. However, in (4.1.24) the value of $x_{t-h}(s)$ is known for $s \in [-r + h, 0]$. We cannot expect to use $f$ to approximate $x_{t-h}(s)$ for $s \in [-r, -r + h]$. Indeed, the value of $f(t, \psi)$ will have little bearing on these values. In fact these values of $x_{t-h}(s)$ may be part of some initial function and not be related to the delay differential equation (2.4.7a) at all.

To illustrate this let us have another look at the Lyapunov functional given by (4.1.16). We showed that if $\psi(-1) \neq 0$, then

$$D^+V_{(2.3.3)}(t, \psi) = \operatorname{sgn}(\psi(-1))D^+\psi(-1). \tag{4.1.25}$$

By analogy, if we could somehow define $D_-V_{(2.3.3)}(t, \psi)$ without explicit reference to solutions of (2.3.3), then it should be

$$D_-V_{(2.3.3)}(t, \psi) = \operatorname{sgn}(\psi(-1))D_-\psi(-1), \tag{4.1.26}$$

for $\psi(-1) \neq 0$. However, $D_-\psi(-1)$ is not defined since $\psi(s)$ is defined only for $s \in [-1, 0]$ and not at values $s < -1$. We could only calculate $D_-V_{(2.3.3)}(t, x_t)$ for some solution $x$ of (2.3.3) if we knew what values $x$ assumed in some arbitrarily small neighbourhood to the left of time $t-1$.

We will find it convenient to refer to the following function classes in later sections. Therefore we end this section with the following definitions.

$$K_1 = \{g \in C(\mathbf{R}_+, \mathbf{R}_+) \mid g(0) = 0 \text{ and } g(s) > 0 \text{ for } s > 0\},$$

$$K_2 = \{g \in C(\mathbf{R}_+, \mathbf{R}_+) \mid g(0) = 0,\ g(s) > 0 \text{ for } s > 0 \text{ and } \liminf_{s\to\infty} g(s) > 0\},$$

$$K_3 = \{g \in C(\mathbf{R}_+, \mathbf{R}_+) \mid g(0) = 0,\ g(s) > 0 \text{ for } s > 0 \text{ and } g \text{ is nondecreasing in } s\}, \text{ and}$$

$$K_4 = \{g \in C(\mathbf{R}_+, \mathbf{R}_+) \mid g(0) = 0,\ g(s) > 0 \text{ for } s > 0 \text{ and } \lim_{s\to\infty} g(s) = \infty\}.$$

## 4.2 Stability

For our stability analysis we assume that $0 \in D$, $J = \mathbf{R}_+$, $f(t, 0) = 0$ for all $t \in \mathbf{R}_+$ and $I(\tau_k, 0) = 0$ for all $\tau_k \in \mathbf{R}_+$. System (2.4.7) therefore possesses a trivial solution $x(t) = 0$. Through the use of Lyapunov functionals we will obtain conditions under which it satisfies various stability properties. Our study of stability will focus on this solution.

While in general, $x(t) = 0$ need not be a solution of (2.4.7) and as well the stability of the trivial solution generally does not imply stability of other nonzero solutions if system (2.4.7) is nonlinear, we can of course examine separately the stability of an arbitrary solution $z(t)$. By a change of variables $y(t) = x(t) - z(t)$ we can transform system (2.4.7) to

$$y'(t) = f(t, y_t + z_t) - f(t, z_t) = \hat{f}(t, y_t),\ t \neq \tau_k, \tag{4.2.1a}$$

$$\Delta y(t) = I(t, y_{t^-} + z_{t^-}) - I(t, z_{t^-}) = \hat{I}(t, y_t),\ t = \tau_k, \tag{4.2.1b}$$

where $\hat{f}(t, 0) = \hat{I}(t, 0) = 0$, from which we can study the stability of the trivial solution $y(t) = 0$ of this system. Occasionally we may talk simply about the stability of system (2.4.7). But by this we really mean the stability of just the trivial solution of (2.4.7).

For our stability analysis we assume that $0 \in D$, $J = \mathbf{R}_+$, $f(t, 0) = 0$ for all $t \in \mathbf{R}_+$ and $I(\tau_k, 0) = 0$ for all $\tau_k \in \mathbf{R}_+$. System (2.4.7) therefore possesses a trivial solution $x(t) = 0$. Through the use of Lyapunov functionals we will obtain conditions under which it satisfies various stability properties.

Because of the local nature of our stability analysis we need not assume that the Lyapunov functional is defined on the whole space $\mathbf{R}_+ \times PC([-r, 0], D)$. Instead we will assume $V$ is defined on the smaller set $\mathbf{R}_+ \times PC([-r, 0], S(\rho))$ where $S(\rho) = \{z \in \mathbf{R}^n \mid \|z\| \leq \rho\}$ and $\rho > 0$ is sufficiently small to ensure that $S(\rho) \subset D$. In Definitions 4.1.1, 4.1.2, etc. we therefore substitute this smaller domain for $V$. Moreover, Theorem 4.1.1 and Corollary 4.1.1 are valid providing $\|x(t)\| \leq \rho$ for all $t$. In order to consider this smaller domain for $V$ we will make the following additional assumption on the functional $I$ beyond that which is stated in Corollary 3.2.3:

(H1) there exists some $0 < \rho_1 \leq \rho$ such that if $(\tau_k, \psi) \in \mathbb{R}_+ \times PC([-r, 0], S(\rho_1))$ and $\psi(0^-) = \psi(0)$, then $\psi(0) + I(\tau_k, \psi) \in S(\rho)$.

We begin by defining some standard stability concepts.

**Definition 4.2.1:** The solution $x(t) = 0$ of system (2.4.7) is said to be

(S1) *stable* if for every $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists some $\delta = \delta(t_0, \epsilon) > 0$ such that if $\phi \in PC([-r, 0], D)$ with $\|\phi\|_r \leq \delta$ and $x = x(t_0, \phi)$ is any solution of (2.4.7) & (2.4.2), then $x(t, t_0, \phi)$ is defined and $\|x(t, t_0, \phi)\| \leq \epsilon$ for all $t \geq t_0$;

(S2) *uniformly stable* if $\delta$ in (S1) is independent of $t_0$;

(S3) *asymptotically stable* if (S1) holds and for every $t_0 \in \mathbb{R}_+$, there exists some $\eta = \eta(t_0) > 0$ such that if $\phi \in PC([-r, 0], D)$ with $\|\phi\|_r \leq \eta$, then $\lim_{t \to \infty} x(t, t_0, \phi) = 0$;

(S4) *uniformly asymptotically stable* if (S2) holds and there exists some $\eta > 0$ such that for every $\gamma > 0$, there exists some $T = T(\eta, \gamma) > 0$ such that if $\phi \in PC([-r, 0], D)$ with $\|\phi\|_r \leq \eta$, then $\|x(t, t_0, \phi)\| \leq \gamma$ for $t \geq t_0 + T$;

(S5) *unstable* if (S1) fails to hold.

The concept of stability stated in (S1) is usually defined with the two inequalities $\|\phi\|_r \leq \delta$ and $\|x(t, t_0, \phi)\| \leq \epsilon$ being replaced by the strict inequalities $\|\phi\|_r < \delta$ and $\|x(t, t_0, \phi)\| < \epsilon$, respectively. Of course these two ways of defining stability are equivalent. One advantage in using non-strict inequalities when dealing with piecewise continuous functions is the observations that if $\phi \in PC([-r, 0], D)$ and $\|\phi(s)\| < \delta$ for all $s \in [-r, 0]$, then although $\|\phi\|_r \leq \delta$, we cannot conclude that $\|\phi\|_r < \delta$. However, if $\phi$ were continuous, then we could claim $\|\phi\|_r < \delta$. For this same reason we choose to define $S(\rho)$ to be the closed ball of radius $\rho$ about the origin in $\mathbb{R}^n$ instead of the open ball.

Occasionally one may consider a weaker notion of stability than (S1), namely stability at some fixed $t_0 \in \mathbb{R}_+$. The trivial solution of (2.4.7) is said to be stable at $t_0$ if for each $\epsilon > 0$, there exists some $\delta = \delta(t_0, \epsilon) > 0$ such that if $\phi \in PC([-r, 0], D)$ with $\|\phi\|_r \leq \delta$, then $\|x(t, t_0, \phi)\| \leq \epsilon$ for all $t \geq t_0$. Of course if the trivial solution is stable at each $t_0 \in \mathbb{R}_+$, then it is stable as defined by (S1). For ordinary differential equations, stability at a point implies stability. However, for delay differential equations the trivial solution can be stable at $t_0$ but unstable at some $t_1 > t_0$. This effect results from the fact that solutions of delay differential equations are generally not continuable to the left. For an example illustrating this behaviour see [Zve59].

In order to prove asymptotic stability using a traditional approach with Lyapunov functionals, we will need to strengthen our quasi-boundedness assumption on $f$. The need for this assumption is discussed in [Hal93] with regard to continuous delay differential equations. In light of this we introduce the following definition.

**Definition 4.2.2:** A functional $f : \mathbf{R}_+ \times PC([-r,0],D) \to \mathbf{R}^n$ is said to be *strongly quasi-bounded* if for each compact set $F \subset D$, there exists some $M > 0$ such that $\|f(t,\psi)\| \leq M$ for all $(t,\psi) \in \mathbf{R}_+ \times PC([-r,0],F)$.

A strongly quasi-bounded functional is obviously quasi-bounded which we defined in Section 2.3. The main distinction is that the upper bound $M$ for the functional $f$ must be valid for every time interval. If $f$ is independent of $t$, then the two notions are identical.

In the following theorems we will generally be assuming that system (2.4.7) without impulses (i.e. system (2.3.1)) has a stable trivial solution. The impulses will tend to be thought of as small perturbations and we will be interested in determining how much the impulses can disturb the system and yet still preserve the stability properties. We will assume the existence of a Lyapunov functional whose growth along solutions between impulses is nonpositive although at the impulse times it may experience jump increases. The idea will be to ensure that the overall net effect of the impulses is small.

**Theorem 4.2.1:** *Assume the conditions of Corollary 3.2.3 and hypothesis (H1) are satisfied. Suppose there exist functions $a, b \in K_1$ and $c \in C(\mathbf{R}_+,\mathbf{R}_+)$ and there exist constants $d_k \geq 0$ with $d = \sum_{k=1}^{\infty} d_k < \infty$ . Suppose $V : \mathbf{R}_+ \times PC([-r,0],S(\rho)) \to \mathbf{R}_+$ is consistently composite-PC, locally Lipschitz in $\psi$ and satisfies the following conditions:*

*(i) $b(\|\psi(0)\|) \leq V(t,\psi) \leq a(\|\psi\|_r)$ for all $t \in \mathbf{R}_+$ and $\psi \in PC([-r,0],S(\rho))$;*

*(ii) $D^+_{(2.4.7)}V(t,\psi) \leq -c(\|\psi(0)\|)$ for all $t \neq \tau_k$ in $\mathbf{R}_+$ and $\psi \in PC([-r,0],S(\rho))$; and*

*(iii) for all $t_0 \in \mathbf{R}_+$ and $x \in PC([t_0-r,\infty),S(\rho))$ where $x$ is continuous at each $t \neq \tau_k$ in $(t_0,\infty)$, $V(\tau_k, x_{\tau_k}) \leq (1+d_k)\lim_{t\to\tau_k^-} V(t,x_t)$ whenever $(\tau_k, x_{\tau_k^-}) \in (t_0,\infty) \times PC([-r,0],S(\rho_1))$ and $x(\tau_k) = x(\tau_k^-) + I(\tau_k, x_{\tau_k^-})$.*

*Then the solution $x(t) = 0$ of (2.4.7) is uniformly stable.*

*Proof:* Condition (i) implies $b(s) \leq a(s)$ for all $s \in [0,\rho]$. So let $\hat{a}$ and $\hat{b}$ be continuous, strictly increasing functions satisfying $\hat{b}(s) \leq b(s) \leq a(s) \leq \hat{a}(s)$ for all $s \in [0,\rho]$. Then

$$\hat{b}(\|\psi(0)\|) \leq V(t,\psi) \leq \hat{a}(\|\psi\|_r), \tag{4.2.2}$$

for all $t \in \mathbf{R}_+$ and $\psi \in PC([-r,0],S(\rho))$.

Let $\epsilon > 0$ and assume without loss of generality that $\epsilon \leq \rho_1$. Define $\bar{d} = \prod_{k=1}^{\infty}(1+d_k)$. Then $1 \leq \bar{d} < \infty$ since $d < \infty$. Choose $\delta = \delta(\epsilon) > 0$ so that $\delta < \hat{a}^{-1}(\hat{b}(\epsilon)/\bar{d})$ and note that $0 < \delta < \epsilon$.

Let $(t_0,\phi) \in \mathbf{R}_+ \times PC([-r,0],D)$ where $\|\phi\|_r \leq \delta$ and $t_0 \in [\tau_{l-1},\tau_l)$ for some positive integer $l$.

Suppose $x = x(t_0, \phi)$ is a solution of (2.4.7) & (2.4.2) and let $[t_0 - r, t_0 + \beta)$ be its maximal interval of existence. From Corollary 3.2.3 we know that if $\beta < \infty$, then there exists some $t \in (t_0, t_0 + \beta)$ for which $\|x(t)\| > \epsilon$. We will prove that $\|x(t)\| \leq \epsilon$ for $t \in [t_0, t_0 + \beta)$, which in turn will imply that $\beta = \infty$ and that the trivial solution of (2.4.7) is thereby uniformly stable.

Suppose for the sake of contradiction that $\|x(t)\| > \epsilon$ for some $t \in [t_0, t_0 + \beta)$. Then let $\hat{t} = \inf\{t \in [t_0, t_0 + \beta) \mid \|x(t)\| > \epsilon\}$. Note that $\|x(t)\| \leq \|\phi\|_r \leq \delta < \epsilon$ for $t \in [t_0 - r, t_0]$ and in particular $\|x(t_0)\| < \epsilon$.

By the definition of $\hat{t}$ we see that $\hat{t} \in (t_0, t_0 + \beta)$, $\|x(t)\| \leq \epsilon \leq \rho_1$ for $t \in [t_0 - r, \hat{t})$ and either $\|x(\hat{t})\| = \epsilon$ or $\|x(\hat{t})\| > \epsilon$ and $\hat{t} = \tau_k$ for some $k$. In the latter case $\|x(\hat{t})\| \leq \rho$ by hypothesis (H1) since $\|x_{\hat{t}^-}\|_r \leq \epsilon \leq \rho_1$. Thus in either case $V(t, x_t)$ is defined for $t \in [t_0, \hat{t}]$.

For $t \in [t_0, \hat{t}]$ define

$$m(t) = V(t, x_t), \tag{4.2.3}$$

and note that $m(t_0) = V(t_0, \phi)$. Since $V$ is consistently composite-PC we know that $m \in PC([t_0, \hat{t}], \mathbb{R}_+)$ and $m(t)$ is continuous at each $t \neq \tau_k$ in $(t_0, \hat{t}]$. By (4.2.2) we have

$$\hat{b}(\|x(t)\|) \leq m(t) \leq \hat{a}(\|x_t\|_r), \tag{4.2.4}$$

for $t \in [t_0, \hat{t}]$. Thus $m(t_0) \leq \hat{a}(\|\phi\|_r) \leq \hat{a}(\delta)$. By condition (ii) and from Corollary 4.1.1, since $V$ is locally Lipschitz in $\psi$, we have

$$D^+ m(t) = \limsup_{h \to 0^+} \frac{1}{h}[m(t+h) - m(t)] \leq -c(\|x(t)\|), \tag{4.2.5}$$

for all $t \neq \tau_k$ in $(t_0, \hat{t}]$. Also by condition (iii) we have

$$m(\tau_k) \leq (1 + d_k) m(\tau_k^-), \tag{4.2.6}$$

for all $\tau_k \in (t_0, \hat{t}]$.

From (4.2.5) it is clear that $m(t)$ is nonincreasing on $[t_0, \hat{t}]$ between impulses. If $\hat{t} \in (t_0, \tau_l)$, then $m(\hat{t}) \leq m(t_0) \leq \hat{a}(\delta)$. Otherwise $\hat{t} \in [\tau_k, \tau_{k+1})$ for some $k \geq l$. In this case we have $m(\hat{t}) \leq m(\tau_k)$, $m(\tau_l^-) \leq m(t_0) \leq \hat{a}(\delta)$, $m(\tau_i^-) \leq m(\tau_{i-1})$ for $i = l+1, l+2, \ldots, k$ and $m(\tau_i) \leq (1 + d_i) m(\tau_i^-)$ for $i = l, l+1, \ldots, k$. Combining all of these inequalities gives us $m(\hat{t}) \leq \bar{d}\hat{a}(\delta)$. So in either case $m(\hat{t}) \leq \bar{d}\hat{a}(\delta) < \hat{b}(\epsilon)$ and so by (4.2.4) we have $\|x(\hat{t})\| < \epsilon$. But this contradicts the fact that $\|x(\hat{t})\| \geq \epsilon$ and therefore the proof is complete. ■

**Theorem 4.2.2:** *Suppose that in addition to the conditions of Theorem 4.2.1,* $c \in K_2$, $\tau = \inf_{k \in \mathbb{Z}^+}\{\tau_k - \tau_{k-1}\} > 0$ *and* $f$ *is strongly quasi-bounded. Then the solution* $x(t) = 0$ *of (2.4.7) is uniformly asymptotically stable.*

*Proof:* Let $\hat{c}$ be a continuous, strictly increasing function satisfying $\hat{c}(s) \le c(s)$ for all $s \in \mathbf{R}_+$.

Theorem 4.2.1 establishes uniform stability and more specifically it tells us that there exists some $\eta > 0$ such that if $\|\phi\|_r \le \eta$, then $\|x(t, t_0, \phi)\| \le \rho_1$ for all $t \ge t_0 - r$ where $x = x(t_0, \phi)$ is any solution of (2.4.7) & (2.4.2).

Since $f$ is strongly quasi-bounded, there exists some $M > 0$ such that $\|f(t, \psi)\| \le M$ for all $(t, \psi) \in \mathbf{R}_+ \times PC([-r, 0], \mathbf{R}^n)$ with $\|\psi\|_r \le \rho_1$. Without loss of generality we may assume $M > \max\{1/r, 1/\tau\}$.

Let $\gamma > 0$. For $\epsilon = \gamma$ let $\delta = \delta(\gamma) > 0$ satisfy $\delta < \hat{a}^{-1}(\hat{b}(\epsilon)/\bar{d})$ as in Theorem 4.2.1 and moreover assume $\delta < 1$. Let $N = N(\gamma)$ be some positive integer satisfying $N > 2M\bar{d}\hat{a}(\rho_1)/[\delta\hat{c}(\delta/2)]$ and define $T = T(\eta, \gamma) = (2N + 1/2)r$. Let $x = x(t_0, \phi)$ be a solution of (2.4.7) & (2.4.2) with $\|\phi\|_r \le \eta$ and $t_0 \in [\tau_{l-1}, \tau_l)$ for some positive integer $l$. Also let $m(t)$ be defined as in (4.2.3). Then $\|x(t)\| \le \rho_1$ for $t \ge t_0 - r$ and we will show that $\|x(t)\| \le \gamma$ for $t \ge t_0 + T$.

We consider two cases. Either $\|x_{t^*}\|_r \le \delta$ for some $t^* \in [t_0, t_0 + T]$ or $\|x_t\|_r > \delta$ for all $t \in [t_0, t_0 + T]$. In the first case the restriction of $x$ to $[t^* - r, \infty)$ is again a solution of (2.4.7) but with the initial time $t^*$ and initial function $x_{t^*}$. Since $\|x_{t^*}\|_r \le \delta$, then by Theorem 4.2.1 we know $\|x(t)\| \le \gamma$ for $t \ge t^* - r$ and in particular for $t \ge t_0 + T$ as desired.

On the other hand suppose $\|x_t\|_r > \delta$ for every $t \in [t_0, t_0 + T]$. The remainder of this proof will show by way of contradiction that this case cannot happen. This implies that on every interval $[t, t + r] \subset [t_0, t_0 + T]$, there exists some $\bar{t} \in [t, t + r]$ such that $\|x(\bar{t})\| > \delta$ and moreover we may assume, without loss of generality, that $\bar{t} \ne \tau_k$ for any $k$.

Thus for $j = 1, 2, \ldots, N$, there exists some $t_j \in [t_0 + (2j - 1)r, t_0 + 2jr]$ with $t_j \ne \tau_k$ for any $k$ and $\|x(t_j)\| > \delta$. Note that $t_{j+1} - t_j \ge r$ for all $j$. For each $j$ consider the interval $[t_j - \delta/(2M), t_j + \delta/(2M)]$. Since $M > 1/r$ and $\delta < 1$, then these intervals are nonoverlapping and each is contained in $[t_0, t_0 + T]$. Each interval has length $\delta/M < 1/M < \tau$ and so can contain at most one impulse time $\tau_k$. Suppose that there are no impulse times in $[t_j, t_j + \delta/(2M)]$. Since $\|x(t)\| \le \rho_1$ for all $t \ge t_0 - r$, then $\|x_t\|_r \le \rho_1$ for $t \in [t_j, t_j + \delta/(2M)]$, which implies $\|x'(t)\| = \|f(t, x_t)\| \le M$ at all points in this interval. Thus $\|x(t)\| > \delta/2$ on this interval. This in turn implies $D^+m(t) \le -\hat{c}(\delta/2)$ on this interval by (4.2.5) and so $m(t)$ decreases by at least $\delta\hat{c}(\delta/2)/(2M)$. A similar argument shows that $m(t)$ decreases by at least $\delta\hat{c}(\delta/2)/(2M)$ on $[t_j - \delta/(2M), t_j]$ if this interval is free from impulses.

For $t \ge t_0$ define

$$\hat{m}(t) = \begin{cases} m(t), & t \in [t_0, \tau_l), \\ \left[\prod_{k=l}^{i}(1 + d_k)\right]^{-1} m(t), & t \in [\tau_i, \tau_{i+1}), \quad i = l, l+1, l+2, \ldots. \end{cases} \tag{4.2.7}$$

Inequalities (4.2.5) and (4.2.6) imply that this new function $\hat{m}(t)$ is nonincreasing for $t \ge t_0$. Also since $\hat{m}(t) \ge \bar{d}^{-1}m(t)$ for all $t \ge t_0$, then $\hat{m}(t)$ must decrease by at least $\bar{d}^{-1}\delta\hat{c}(\delta/2)/(2M)$ on either $[t_j, t_j + \delta/(2M)]$ or $[t_j - \delta/(2M), t_j]$. So in particular it must be true that $\hat{m}(t_0 + T) \le$

$\hat{m}(t_0) - N\bar{d}^{-1}\delta\hat{c}(\delta/2)/(2M) = m(t_0) - N\bar{d}^{-1}\delta\hat{c}(\delta/2)/(2M)$.

Using (4.2.4) together with our definition for $N$ gives us $\hat{m}(t_0+T) \le \hat{a}(\rho_1) - N\bar{d}^{-1}\delta\hat{c}(\delta/2)/(2M) < 0$, which is a contradiction, leading us to conclude that this second case is impossible. ∎

We remark that if condition (ii) in the previous theorem were replaced by the stronger condition (ii*) $D^+_{(2.4.7)}V(t,\psi) \le -c(V(t,\psi))$ for all $t \ne \tau_k$ in $\mathbf{R}_+$ and $\psi \in PC([-r,0],S(\rho))$, then the strongly quasi-boundedness assumption on $f$ could be dropped. Moreover, condition (iii) could be weakened further (eliminating the requirement that $d = \sum_{k=1}^{\infty} d_k < \infty$ for example). However, relating the derivative of the functional $V$ to the Lyapunov functional itself via (ii*) is generally difficult to do in practice. This is where Lyapunov functions have the advantage.

In the absence of impulses, and so when $d_k = 0$ for all $k$, the proofs of Theorems 4.2.1 and 4.2.2 reduce to proofs given by Hale and Lunel in [Hal93].

We end this section with the following example.

**Example 4.2.1:** Consider the impulsive delay differential equation

$$x'(t) = -p(t)x(t) + \int_{t-r}^{t} g(x(s))ds, \quad t \ne k, \tag{4.2.8a}$$

$$\Delta x(t) = \frac{1}{(t+1)^2}\exp\left(-\int_{t-r}^{t}|x(s)|ds\right)x(t^-), \quad t = k, \tag{4.2.8b}$$

where $k = 0, 1, 2, 3, \ldots$ are the impulse times and $r > 0$ is some delay constant. Suppose $p \in PC(\mathbf{R}_+, \mathbf{R})$ that satisfies $M_1 \le p(t) \le M_2$ for all $t \in \mathbf{R}_+$ for constants $M_1 > 0$ and $M_2 > 0$. Suppose $g \in C(\mathbf{R}, \mathbf{R})$ and there exists some $M_3 > 0$ such that $|g(x)| \le M_3|x|$ for all $x \in \mathbf{R}$. Furthermore, assume that $M_1$ and $M_3$ are related by the following inequality

$$M_3 \le \frac{M_1^2}{4(1+e^{M_1 r/2})}. \tag{4.2.9}$$

Then we claim that the trivial solution of equation (4.2.8) is uniformly asymptotically stable.

Here the functionals $f$ and $I$ are defined by

$$f(t,\psi) = -p(t)\psi(0) + \int_{-r}^{0} g(\psi(s))ds, \tag{4.2.10}$$

and

$$I(t,\psi) = \frac{1}{(t+1)^2}\exp\left(-\int_{-r}^{0}|\psi(s)|ds\right)\psi(0), \tag{4.2.11}$$

for all $(t,\psi) \in \mathbf{R}_+ \times PC([-r,0],\mathbf{R})$.

The functional $f$ is clearly composite-PC, strongly quasi-bounded (since $p$ is assumed to be bounded) and continuous in $\psi$. Given any $\rho > 0$ let $\rho_1 = \rho/2$. If $(k,\psi) \in \mathbb{R}_+ \times PC([-r,0],\mathbb{R})$, $\psi(0^-) = \psi(0)$ and $\|\psi\|_r \le \rho_1$, then

$$|\psi(0) + I(k,\psi)| \le |\psi(0)| + \frac{1}{(k+1)^2}\exp\left(-\int_{-r}^{0}|\psi(s)|ds\right)|\psi(0)| \le 2|\psi(0)| \le 2\rho_1 = \rho. \quad (4.2.12)$$

In other words, hypothesis (H1) is satisfied.

Let $d_k = 1/(k+1)^2$ for $k = 1, 2, \ldots$. Then $d = \sum_{k=1}^{\infty} d_k < \infty$.

Next, for $(t,\psi) \in \mathbb{R}_+ \times PC([-r,0],\mathbb{R})$ define the following Lyapunov functional

$$V(t,\psi) = |\psi(0)| + \int_{-r}^{0}\left[e^{M_1 s/2}\left(\frac{M_1}{2} - \frac{2M_3}{M_1}\right) - \frac{2M_3}{M_1}\right]|\psi(s)|ds. \quad (4.2.13)$$

Inequality (4.2.9) implies

$$M_3 \le \frac{M_1}{2}e^{-M_1 r/2}\left(\frac{M_1}{2} - \frac{2M_3}{M_1}\right), \quad (4.2.14)$$

and so

$$\frac{2M_3}{M_1} \le e^{-M_1 r/2}\left(\frac{M_1}{2} - \frac{2M_3}{M_1}\right) \le e^{M_1 s/2}\left(\frac{M_1}{2} - \frac{2M_3}{M_1}\right). \quad (4.2.15)$$

for all $s \in [-r,0]$. Hence $V(t,\psi)$ is always nonnegative. Moreover, $V$ is also consistently composite-PC and locally Lipschitz in $\psi$.

Condition (i) of Theorem 4.2.1 is satisfied by letting $b(s) = s$ and $a(s) = (1 + r(M_1/2 - 4M_3/M_1))s$.

As for condition (iii) suppose we let $x \in PC([t_0 - r, \infty), \mathbb{R})$ be some function that is continuous at each $t \ne k$ in $(t_0, \infty)$. If $x(k) = x(k^-) + I(k, x_{k^-})$ (i.e. the function $x$ satisfies the delay difference equation (4.2.8b) at $t = k$), then we get

$$\begin{aligned} V(k,x_k) &= |x(k)| + \int_{-r}^{0}\left[e^{M_1 s/2}\left(\frac{M_1}{2} - \frac{2M_3}{M_1}\right) - \frac{2M_3}{M_1}\right]|x_k(s)|ds \\ &= |x(k^-) + I(k,x_{k^-})| + \int_{k-r}^{k}\left[e^{M_1(s-k)/2}\left(\frac{M_1}{2} - \frac{2M_3}{M_1}\right) - \frac{2M_3}{M_1}\right]|x(s)|ds \\ &\le (1+d_k)|x(k^-)| + \int_{k-r}^{k}\left[e^{M_1(s-k)/2}\left(\frac{M_1}{2} - \frac{2M_3}{M_1}\right) - \frac{2M_3}{M_1}\right]|x(s)|ds \\ &\le (1+d_k)\left(|x(k^-)| + \int_{k-r}^{k}\left[e^{M_1(s-k)/2}\left(\frac{M_1}{2} - \frac{2M_3}{M_1}\right) - \frac{2M_3}{M_1}\right]|x(s)|ds\right) \\ &= (1+d_k)\lim_{t\to k^-} V(t,x_t). \end{aligned} \quad (4.2.16)$$

Finally. we must check condition (ii). Calculating the derivative of $V$ with respect to system (4.2.8) gives us

$$\begin{aligned}
D^+V_{(4.2.8)}(t,\psi) &\leq -p(t)|\psi(0)| + \int_{-r}^{0} |g(\psi(s))|ds - \frac{M_1}{2}\left(\frac{M_1}{2} - \frac{2M_3}{M_1}\right)\int_{-r}^{0} e^{M_1 s/2}|\psi(s)|ds \\
&\quad + \left[\left(\frac{M_1}{2} - \frac{2M_3}{M_1}\right) - \frac{2M_3}{M_1}\right]|\psi(0)| - \left[e^{-M_1 r/2}\left(\frac{M_1}{2} - \frac{2M_3}{M_1}\right) - \frac{2M_3}{M_1}\right]|\psi(-r)| \\
&\leq -M_1|\psi(0)| + M_3\int_{-r}^{0} |\psi(s)|ds - \frac{M_1}{2}\left(\frac{M_1}{2} - \frac{2M_3}{M_1}\right)e^{-M_1 r/2}\int_{-r}^{0} |\psi(s)|ds \\
&\quad + \left[\left(\frac{M_1}{2} - \frac{2M_3}{M_1}\right) - \frac{2M_3}{M_1}\right]|\psi(0)| - \left[e^{-M_1 r/2}\left(\frac{M_1}{2} - \frac{2M_3}{M_1}\right) - \frac{2M_3}{M_1}\right]|\psi(-r)| \\
&= -\left(\frac{M_1}{2} + \frac{4M_3}{M_1}\right)|\psi(0)| - \left[e^{-M_1 r/2}\left(\frac{M_1}{2} - \frac{2M_3}{M_1}\right) - \frac{2M_3}{M_1}\right]|\psi(-r)| \\
&\quad - \left[\frac{M_1}{2}e^{-M_1 r/2}\left(\frac{M_1}{2} - \frac{2M_3}{M_1}\right) - M_3\right]\int_{-r}^{0} |\psi(s)|ds \\
&\leq -\left(\frac{M_1}{2} + \frac{4M_3}{M_1}\right)|\psi(0)|.
\end{aligned} \tag{4.2.17}$$

The final inequality follows from (4.2.14) and (4.2.15).

By letting $c(s) = (M_1/2 + 4M_3/M_1)s$, then $c \in K_2$ and condition (ii) is satisfied. Therefore by Theorem 4.2.2 we have proven that the trivial solution of (4.2.8) is uniformly asymptotically stable.

□

## 4.3 Boundedness

Now we introduce some theorems that establish boundedness of solutions of system (2.4.7). As in the previous section we make use of Lyapunov functionals. In fact the theorems presented here are very similar to those given for stability. For simplicity we assume that $D = \mathbb{R}^n$ and $J = \mathbb{R}_+$.

**Definition 4.3.1:** Solutions of (2.4.7) are said to be

(B1) *bounded* if for every $B_1 > 0$ and $t_0 \in \mathbb{R}_+$, there exists some $B_2 = B_2(t_0, B_1) > 0$ such that if $\phi \in PC([-r,0],\mathbb{R}^n)$ with $||\phi||_r \leq B_1$ and $x = x(t_0,\phi)$ is any solution of (2.4.7) & (2.4.2), then $x(t,t_0,\phi)$ is defined and $||x(t,t_0,\phi)|| \leq B_2$ for all $t \geq t_0$;

(B2) *uniformly bounded* if (B1) holds with $B_2$ independent of $t_0$;

(B3) *ultimately bounded with bound B* if (B1) holds and for every $B_3 > 0$ and $t_0 \in \mathbb{R}_+$, there exists some $T = T(t_0, B_3) > 0$ such that if $\phi \in PC([-r,0],\mathbb{R}^n)$ with $||\phi||_r \leq B_3$, then $||x(t,t_0,\phi)|| \leq B$ for $t \geq t_0 + T$;

(B4) *uniformly ultimately bounded with bound B* if (B2) holds and (B3) holds with $T$ independent of $t_0$.

In the definition of boundedness, global existence of solutions is implicitly assumed. The boundedness theorems of this section will establish global existence of solutions as well as boundedness without assuming, *a priori*, that solutions can necessarily be continued to infinity.

When discussing the property of (uniform) ultimate boundedness of solutions of (2.4.7), we will often omit the phrase "with bound $B$". In these cases the existence of such a bound will be understood. We are less interested in establishing a value for $B$ than in proving the existence of such a bound. This is in keeping with our study of stability where, for example, we care mainly that there exists a $\delta > 0$ (no matter how small it may be) corresponding to each $\epsilon > 0$. Only in Chapter 8 where we examine the concept of practical stability as applied to system (2.4.7) will we make some effort in assigning values to the various quantities.

**Theorem 4.3.1:** *Assume $J = \mathbb{R}_+$, $D = \mathbb{R}^n$ and the conditions of Corollary 3.2.3 are satisfied. Suppose there exist functions $a, c \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $b \in K_4$ and there exist constants $d_k, \mu_k \geq 0$ with $d = \sum_{k=1}^{\infty} d_k < \infty$ and $\mu = \sum_{k=1}^{\infty} \mu_k < \infty$. Suppose $V : \mathbb{R}_+ \times PC([-r, 0], \mathbb{R}^n) \to \mathbb{R}_+$ is consistently composite-PC, locally Lipschitz in $\psi$ and satisfies the following conditions:*

*(i) $b(\|\psi(0)\|) \leq V(t, \psi) \leq a(\|\psi\|_r)$ for all $t \in \mathbb{R}_+$ and $\psi \in PC([-r, 0], \mathbb{R}^n)$;*

*(ii) $D^+_{(2.4.7)}V(t, \psi) \leq -c(\|\psi(0)\|)$ for all $t \neq \tau_k$ in $\mathbb{R}_+$ and $\psi \in PC([-r, 0], \mathbb{R}^n)$; and*

*(iii) for all $t_0 \in \mathbb{R}_+$ and $x \in PC([t_0 - r, \infty), \mathbb{R}^n)$ where $x$ is continuous at each $t \neq \tau_k$ in $(t_0, \infty)$, $V(\tau_k, x_{\tau_k}) \leq (1 + d_k) \lim_{t \to \tau_k^-} V(t, x_t) + \mu_k$ for all $\tau_k > t_0$ whenever $x(\tau_k) = x(\tau_k^-) + I(\tau_k, x_{\tau_k^-})$.*

*Then solutions of (2.4.7) are uniformly bounded.*

*Proof:* Let $\hat{b}$ be a continuous, strictly increasing function satisfying $\lim_{s \to \infty} \hat{b}(s) = \infty$ and $\hat{b}(s) \leq b(s)$ for all $s \in \mathbb{R}_+$. Condition (i) implies $b(s) \leq a(s)$ for all $s$, which in turn implies $\lim_{s \to \infty} a(s) = \infty$. So let $\hat{a}$ be a continuous, strictly increasing function satisfying $\hat{a}(s) \geq a(s)$ for all $s \in \mathbb{R}_+$. Note that $\hat{b}$ has an inverse defined on $\mathbb{R}_+$ while $\hat{a}$ has an inverse defined on $[\hat{a}(0), \infty)$. Condition (i) then implies that

$$\hat{b}(\|\psi(0)\|) \leq V(t, \psi) \leq \hat{a}(\|\psi\|_r), \tag{4.3.1}$$

for all $t \in \mathbb{R}_+$ and $\psi \in PC([-r, 0], \mathbb{R}^n)$.

We will prove uniform boundedness of solutions. Define $\bar{d} = \prod_{k=1}^{\infty}(1 + d_k)$ and note that $\bar{d} < \infty$ since $d < \infty$. Let $B_1 > 0$ and define $B_2 = B_2(B_1) = \hat{b}^{-1}(\bar{d}(\hat{a}(B_1) + \mu))$. Note that $B_2 \geq B_1$. Let $(t_0, \phi) \in \mathbb{R}_+ \times PC([-r, 0], \mathbb{R}^n)$ where $\|\phi\|_r \leq B_1$ and $t_0 \in [\tau_{l-1}, \tau_l)$ for some positive integer $l$. Let $x = x(t_0, \phi)$ be a solution of (2.4.7) & (2.4.2) and let $[t_0 - r, t_0 + \beta)$ be its maximal interval of existence. Recall from Corollary 3.2.3 that if $\beta < \infty$, then $\|x(t)\| > B_2$ for some $t \in (t_0, t_0 + \beta)$. We will prove that $\|x(t)\| \leq B_2$ for all $t \in [t_0, t_0 + \beta)$, which will in turn imply that $\beta = \infty$ and that solutions of (2.4.7) are thereby uniformly bounded.

For $t \in [t_0, t_0 + \beta)$ define

$$m(t) = V(t, x_t), \tag{4.3.2}$$

and note that $m(t_0) = V(t_0, \phi)$. Since $V$ is consistently composite-PC, then we know $m \in PC([t_0, t_0 + \beta), \mathbb{R}_+)$ and $m(t)$ is continuous at each $t \neq \tau_k$ in $(t_0, t_0 + \beta)$. By (4.3.1) we have

$$\hat{b}(\|x(t)\|) \leq m(t) \leq \hat{a}(\|x_t\|_r), \tag{4.3.3}$$

for $t \in [t_0, t_0 + \beta)$. By condition (ii) and from Corollary 4.1.1, since $V$ is locally Lipschitz in $\psi$, we have

$$D^+ m(t) = \limsup_{h \to 0^+} \frac{1}{h} [m(t+h) - m(t)] \leq -c(\|x(t)\|), \tag{4.3.4}$$

for all $t \neq \tau_k$ in $(t_0, t_0 + \beta)$. Also by condition (iii) we have

$$m(\tau_k) \leq (1 + d_k) m(\tau_k^-) + \mu_k, \tag{4.3.5}$$

for all $\tau_k \in (t_0, t_0 + \beta)$.

From (4.3.4) it is clear that $m(t)$ is nonincreasing on $(t_0, t_0 + \beta)$ between impulses. Suppose $t \in (t_0, t_0 + \beta)$. If $t \in (t_0, \tau_l)$, then $m(t) \leq m(t_0) \leq \hat{a}(\|x_{t_0}\|_r) = \hat{a}(\|\phi\|_r) \leq \hat{a}(B_1)$. Otherwise $t \in [\tau_k, \tau_{k+1})$ for some $k \geq l$. In this case we have $m(t) \leq m(\tau_k)$, $m(\tau_l^-) \leq m(t_0) \leq \hat{a}(B_1)$, $m(\tau_i^-) \leq m(\tau_{i-1})$ for $i = l+1, l+2, \ldots, k$ and $m(\tau_i) \leq (1 + d_i) m(\tau_i^-) + \mu_i$ for $i = l, l+1, \ldots, k$. Combining all of these inequalities gives us $m(t) \leq \bar{d}(\hat{a}(B_1) + \mu)$. So by (4.3.3) we have $\|x(t)\| \leq \hat{b}^{-1}(\bar{d}(\hat{a}(B_1) + \mu)) = B_2$ for $t \in (t_0, t_0 + \beta)$ and indeed for all $t \in [t_0 - r, t_0 + \beta)$. Thus solutions of (2.4.7) are uniformly bounded. ■

**Theorem 4.3.2:** *Suppose that in addition to the conditions of Theorem 4.3.1,* $c \in K_2$, $\tau = \inf_{k \in \mathbb{Z}^+}\{\tau_k - \tau_{k-1}\} > 0$ *and* $f$ *is strongly quasi-bounded. Then solutions of (2.4.7) are uniformly ultimately bounded.*

*Proof:* Let $\hat{c}$ be a continuous, strictly increasing function satisfying $\hat{c}(s) \leq c(s)$ for all $s \in \mathbb{R}_+$. Let $B_1 = 1$ and $B = B_2(B_1)$ where $B_2$ is defined as in Theorem 4.3.1 by $B_2(B_1) = \hat{b}^{-1}(\bar{d}(\hat{a}(B_1) + \mu))$. Note that $B \geq 1$.

Let $B_3 > 0$. By our assumption on $f$, there exists a constant $M = M(B_3) > 0$ such that $\|f(t, \psi)\| \leq M$ for all $(t, \psi) \in \mathbb{R}_+ \times PC([-r, 0], \mathbb{R}^n)$ with $\|\psi\|_r \leq B_2(B_3)$. Without loss of generality let us assume that $M > \max\{1/r, 1/\tau\}$. Let $N = N(B_3)$ be some positive integer satisfying $N > 2M[2\hat{a}(B_3) + d\bar{d}(\hat{a}(B_3) + \mu) + \mu]/\hat{c}(1/2)$ and define $T = T(B_3) = 2(N+1)r$. Let $x = x(t_0, \phi)$ be a solution of (2.4.1) with $\|\phi\|_r \leq B_3$. We know $\|x(t)\| \leq B_2(B_3)$ for $t \in [t_0 - r, \infty)$ by Theorem 4.3.1.

We will show that $\|x(t)\| \le B$ for $t \ge t_0 + T$. We consider two cases.

In the first case suppose $\|x_{t^*}\|_r \le 1$ for some $t^* \in [t_0, t_0 + T]$. The restriction of $x$ to $[t^*, \infty)$ is a solution of (2.4.7) with initial time $t^*$ and initial function $x_{t^*}$. Therefore by the uniform boundedness results of Theorem 4.3.1 we know that $\|x(t)\| \le B = B_2(1)$ for $t \ge t^* - r$ and in particular for $t \ge t_0 + T$.

For the second case suppose $\|x_t\|_r > 1$ for all $t \in [t_0, t_0 + T]$. We will show this to be impossible. On every interval $[t, t + r] \subset [t_0, t_0 + T]$, there exists some $\bar{t} \in [t, t + r]$ such that $\|x(\bar{t})\| > 1$, and moreover we may assume, without loss of generality, that $\bar{t} \neq \tau_k$ for any $k$.

Thus for $j = 1, 2, \ldots, N$, there exists some $t_j \in [t_0 + (2j - 1)r, t_0 + 2jr]$ with $t_j \neq \tau_k$ for any $k$ and $\|x(t_j)\| > 1$. Note that $t_{j+1} - t_j \ge r$ for all $j$. For each $j$ consider the interval $[t_j - 1/(2M), t_j + 1/(2M)]$. Since $M > 1/r$, then these intervals are nonoverlapping and each is contained in $[t_0, t_0+T]$. Each interval has length $1/M < r$ and so can contain at most one impulse time $\tau_k$. Suppose that there are no impulse times in $[t_j, t_j + 1/(2M)]$. Since $\|x(t)\| \le B_2(B_3)$ for all $t \ge t_0 - r$, then $\|x_t\|_r \le B_2(B_3)$ for $t \in [t_j, t_j + 1/(2M)]$, which implies $\|x'(t)\| = \|f(t, x_t)\| \le M$ at all points in this interval. Thus $\|x(t)\| > 1/2$ on this interval. This in turn implies $D^+ m(t) \le -\hat{c}(1/2)$ on this interval and so $m(t)$ decreases by at least $\hat{c}(1/2)/(2M)$. A similar argument shows that $m(t)$ decreases by at least $\hat{c}(1/2)/(2M)$ on $[t_j - 1/(2M), t_j]$ if this interval is free from impulses.

On $[t_0, t_0 + T]$, $m(t)$ is of bounded variation since it is nonincreasing except possibly at the discrete impulse times $\tau_k$ where it may undergo a jump discontinuity. Since $m(t)$ decreases by at least $\hat{c}(1/2)/(2M)$ on either $[t_j, t_j+1/(2M)]$ or $[t_j-1/(2M), t_j]$ for each $j$, then the negative variation of $m(t)$ on $[t_0, t_0+T]$ must be no less than $N\hat{c}(1/2)/(2M)$. Since $m(t) \le \bar{d}(\hat{a}(B_3) + \mu)$ on $[t_0, t_0+T]$ as shown in the proof of Theorem 4.3.1, then the positive variation of $m(t)$ on $[t_0, t_0+T]$ is bounded above by $m(t_0)+\sum_{k:\ \tau_k \in (t_0, t_0+T]}(d_k m(\tau_k^-)+\mu_k) \le \hat{a}(B_3)+\sum_{k:\ \tau_k \in (t_0, t_0+T]}(d_k(\bar{d}(\hat{a}(B_3)+\mu))+\mu_k) \le \hat{a}(B_3) + d\bar{d}(\hat{a}(B_3) + \mu) + \mu$. Since the difference between the positive and negative variations is $m(t_0 + T) - m(t_0)$, then $m(t_0 + T) \le m(t_0) + \hat{a}(B_3) + d\bar{d}(\hat{a}(B_3) + \mu) + \mu - N\hat{c}(1/2)/(2M) \le 2\hat{a}(B_3) + d\bar{d}(\hat{a}(B_3) + \mu) + \mu - N\hat{c}(1/2)/(2M) < 0$ by our choice of $N$, which is impossible. This contradiction completes the proof of uniform ultimate boundedness. ∎

To illustrate the application of the preceding theorems we conclude by providing the following example.

**Example 4.3.1:** Consider the scalar impulsive delay differential equation

$$x'(t) = -p(t)x(t) + q(t)x(t - r) + w(t),\ t \neq k, \tag{4.3.6a}$$

$$\Delta x(t) = h_k x(t^-),\ t = k, \tag{4.3.6b}$$

where $r > 0$, $p, q, w \in PC(\mathbf{R}_+, \mathbf{R})$, $w$ is bounded and square integrable on $\mathbf{R}_+$ (for example $w$ could be defined to be $w(t) = e^{-t}$), and $h_k > 0$ for $k = 0, 1, 2, \ldots$ and $\sum_{k=1}^{\infty} h_k < \infty$. Assume that for some $M_2 \ge M_1 > 1/2$ and $0 < M_3 < M_1 - 1/2$, $M_1 \le p(t) \le M_2$ and $|q(t)| \le M_3$ for all $t \in \mathbf{R}_+$.

We will show that the conditions of Theorem 4.3.2 are satisfied and thereby conclude that solutions of this impulsive delay differential equation are uniformly ultimately bounded.

The functionals $f$ and $I$ are defined by

$$f(t,\psi) = -p(t)\psi(0) + q(t)\psi(-r) + w(t), \tag{4.3.7}$$

and

$$I(t,\psi) = h_k\psi(0). \tag{4.3.8}$$

The functional $f$ is clearly composite-PC, strongly quasi-bounded (since $p, q$ and $w$ are bounded) and continuous in $\psi$.

Define the Lyapunov functional $V$ by

$$V(t,\psi) = \psi^2(0) + M_3\int_{-r}^{0}\psi^2(s)ds + \int_t^{\infty} w^2(s)ds. \tag{4.3.9}$$

Clearly $V$ satisfies condition (i) of Theorem 4.3.1 with $b(s) = s^2$ and $a(s) = (1+M_3r)s^2 + \int_0^{\infty} w^2(t)dt$. Moreover, $V$ is locally Lipschitz in $\psi$ and consistently composite-PC. Calculating the upper right-hand derivative of $V$ with respect to equation (4.3.6) gives us

$$\begin{aligned}
D^+_{(4.3.6)}V(t,\psi) &= 2\psi(0)(-p(t)\psi(0) + q(t)\psi(-r) + w(t)) + M_3(\psi^2(0) - \psi^2(-r)) - w^2(t)\\
&\leq (-2M_1 + M_3)\psi^2(0) + 2M_3|\psi(0)\psi(-r)| + 2\psi(0)w(t) - M_3\psi^2(-r) - w^2(t)\\
&= (-2M_1 + M_3 + 1)\psi^2(0) + 2M_3|\psi(0)\psi(-r)| - M_3\psi^2(-r) - (\psi(0) - w(t))^2\\
&\leq (-2M_1 + M_3 + 1)\psi^2(0) + 2M_3|\psi(0)\psi(-r)| - M_3\psi^2(-r)\\
&= (-2M_1 + 2M_3 + 1)\psi^2(0) - M_3(|\psi(0)| - |\psi(-r)|)^2\\
&\leq -K\psi^2(0),
\end{aligned} \tag{4.3.10}$$

where $K = 2(M_1 - M_3) - 1 > 0$. Thus condition (ii) of Theorem 4.3.1 is satisfied with $c(s) = Ks^2$.

Finally, let us check condition (iii). If $t_0 \in \mathbb{R}_+$ and $x \in PC([t_0 - r, \infty), \mathbb{R})$ with discontinuities occurring only at impulse times in $(t_0, \infty)$ and if $x(k) = x(k^-) + I(k, x_{k^-}) = (1 + h_k)x(k^-)$ for some $k$, then

$$\begin{aligned}
V(k, x_k) &= (1 + h_k)^2x^2(k^-) + M_3\int_{k-r}^{k} x^2(s)ds + \int_k^{\infty} w^2(s)ds\\
&\leq (1 + h_k)^2 \lim_{t\to k^-} V(t, x_t) = (1 + d_k)\lim_{t\to k^-} V(t, x_t),
\end{aligned} \tag{4.3.11}$$

where $d_k = 2h_k + h_k^2 > 0$ (and $\mu_k = 0$). Since $\sum_{k=1}^{\infty} h_k < \infty$, then $\sum_{k=1}^{\infty} d_k < \infty$ also.

We can therefore conclude in light of Theorem 4.3.2 that solutions of system (4.3.6) are uniformly ultimately bounded. □

Note that in this example, the boundedness conclusion is independent of the delay term $r$. Also, what is interesting is that solutions are uniformly ultimately bounded despite the fact that the state $x$ increases in magnitude at each impulse time.

# Chapter 5

# Stability and Boundedness Using Lyapunov Functions

In this chapter we continue to examine the stability and boundedness properties of system (2.4.7), only this time our main tool in developing these results is through the use of Lyapunov functions.

Lyapunov functions $V(t, x)$ can be thought of as special types of Lyapunov functionals $V(t, \psi)$ that depend only on $t$ and $x = \psi(0)$. Lyapunov functions are often simpler to work with. The theorems of the previous chapter can all be applied to Lyapunov functions. Unlike the case of ordinary differential equations, however, the derivative of Lyapunov functions along solutions of delay differential equations are functionals in that they depend on past values of the state as well as the current values. In applying the theorems of the previous chapter it is therefore usually too much to expect that its derivative will be nonpositive along solutions all of the time. Only if $f$ were severely restricted to the point where the dependence of $f(t, \psi)$ on the values $\psi(s)$ for $-r \leq s < 0$ was relatively insignificant, so that the delay differential equation closely resembled an ordinary differential equation, could we expect this. When considering Lyapunov functions and delay differential equations it is standard practice to employ Razumikhin-type arguments, developed by Razumikhin [Raz56], whereby the derivative of the Lyapunov function is no longer assumed to be nonpositive all of the time but only when $V(t, x(t))$ is sufficiently large relative to $V(s, x(s))$ for $t - r \leq s \leq t$.

The proofs developed in this chapter are based, in part, on earlier work by Ballinger and Liu on impulsive differential equations without delays [Bal97a, Bal97b, Bal98].

## 5.1 Properties of Lyapunov Functions

We define the derivative of a Lyapunov function with respect to system (2.4.7) as follows.

**Definition 5.1.1:** Given a function $V : J \times D \to \mathbb{R}_+$, the upper right-hand derivative of $V$ with respect to system (2.4.7) is defined by

$$D^+V_{(2.4.7)}(t, \psi(0)) = \limsup_{h \to 0^+} \frac{1}{h}[V(t+h, \psi(0) + hf(t, \psi)) - V(t, \psi(0))], \tag{5.1.1}$$

for $(t, \psi) \in J \times PC([-r, 0], D)$.

As before we may drop the subscript and simply write $D^+V(t, \psi(0))$ where it is understood which system the derivative of $V$ is with respect to. Note that $D^+V(t, \psi(0))$ is a functional whereas $V$ is a function. Moreover, this definition is consistent with our earlier definition of the derivative of a Lyapunov functional.

We remark that if $V(t, x)$ has continuous partial derivatives with respect $t$ and $x$, then (5.1.1) reduces to

$$D^+V_{(2.4.7)}(t, \psi(0)) = \frac{\partial V(t, \psi(0))}{\partial t} + \nabla_x V(t, \psi(0)) \cdot f(t, \psi). \tag{5.1.2}$$

Next we state for thoroughness what we mean for the function $V$ to be locally Lipschitz in $x$.

**Definition 5.1.2:** A function $V : J \times D \to \mathbb{R}_+$ is said to be *locally Lipschitz in $x$* if for each $t_0 \in J$ and $\alpha > 0$, where $[t_0, t_0 + \alpha] \subset J$, and for each compact set $F \subset D$, there exists some $L > 0$ such that $|V(t, x_1) - V(t, x_2)| \leq L\|x_1 - x_2\|$ for all $t \in [t_0, t_0 + \alpha]$ and $x_1, x_2 \in F$.

This definition is again consistent with our notion of a local Lipschitz condition for Lyapunov functionals. Occasionally a seemingly weaker form of local Lipschitz condition is defined by some authors which says that for each $(t, x) \in J \times D$, there exists a neighbourhood $U$ of $(t, x)$ and a constant $L > 0$ such that $|V(t_1, x_1) - V(t_1, x_2)| \leq L\|x_1 - x_2\|$ for all $(t_1, x_1), (t_1, x_2) \in U$. Because of the compactness of the set $[t_0, t_0 + \alpha] \times F$ in Definition 5.1.2 it can be shown that these two definitions are in fact equivalent.

One might ask whether one could similarly define a local Lipschitz condition for Lyapunov functionals by requiring that for each $(t, \psi) \in J \times PC([-r, 0], D)$, there exists a neighbourhood $U$ of $(t, \psi)$ and a constant $L > 0$ such that $|V(t_1, \psi_1) - V(t_1, \psi_2) \leq L\|\psi_1 - \psi_2\|_r$ for all $(t_1, \psi_1), (t_1, \psi_2) \in U$. Unfortunately, such a definition has two significant shortcomings, which make it substantially different from Definition 4.1.2 and hence not very practical or useful. The first problem is that the set $[t_0, t_0 + \alpha] \times PC([-r, 0], F)$ in Definition 4.1.2 is not compact. In fact even $[t_0, t_0 + \alpha] \times C([-r, 0], F)$ is not compact. The second problem stems from the fact that if $\psi = x_t$ and $U$ is a neighbourhood of $(t, x_t)$, then the point $(t + \epsilon, x_{t+\epsilon})$ will not generally be in $U$ no matter how small $\epsilon$ is chosen to be, since the function $x_t$ may not be continuous in $t$. The Lipschitz constant $L$, which works for $(t, x_t)$, may not work for $(t + \epsilon, x_{t+\epsilon})$. This latter problem occurs because of the piecewise continuous space upon which $V$ is defined.

Here too if the Lyapunov function $V$ is locally Lipschitz in $x$, then (5.1.1) is equivalent to the time derivative of $V$ along solutions of (2.4.7). We state the following theorem and corollary, which follow directly from Theorem 4.1.1 and Corollary 4.1.1, respectively.

**Theorem 5.1.1:** *Assume $f$ satisfies the conditions of Corollary 3.1.3. Let $V : J \times D \to \mathbb{R}_+$ and assume $V$ is locally Lipschitz in $x$. If $(t, \psi) \in J \times PC([-r, 0], D)$ and $x = x(t, \psi)$ is any solution of (2.4.7) satisfying the initial condition $x_t = \psi$, then*

$$D^+ V_{(2.4.7)}(t, \psi(0)) = \limsup_{h \to 0^+} \frac{1}{h} \left[V(t + h, x(t + h, t, \psi)) - V(t, \psi(0))\right]. \tag{5.1.3}$$

**Corollary 5.1.1:** *Assume $f$ and $V$ are given as in Theorem 5.1.1. Suppose $x = x(t_0, \phi)$ is a solution of (2.4.7) & (2.4.2) defined on $[t_0 - r, t_0 + \beta]$ for some $\beta > 0$ where $[t_0, t_0 + \beta] \subset J$. Then*

$$D^+ V_{(2.4.7)}(t, x(t)) = \limsup_{h \to 0^+} \frac{1}{h}[V(t + h, x(t + h, t_0, \phi)) - V(t, x(t, t_0, \phi))], \tag{5.1.4}$$

*for $t \in [t_0, t_0 + \beta)$.*

## 5.2 Stability

Our next two theorems establish conditions for uniform asymptotic stability of the zero solution of system (2.4.7). As in the previous chapter we assume $0 \in D$, $J = \mathbb{R}_+$, $f(t, 0) = 0$ for all $t \in \mathbb{R}_+$ and $I(\tau_k, 0) = 0$ for all $\tau_k \in \mathbb{R}_+$. We also restrict the domain of our Lyapunov function to the set $\mathbb{R}_+ \times S(\rho)$ (or rather $[-r, \infty) \times S(\rho)$) much like we did for Lyapunov functionals in Section 4.2. We will need to assume, as before, that hypothesis (H1) is satisfied.

**Theorem 5.2.1:** *Assume the conditions of Corollary 3.2.3 and hypothesis (H1) are satisfied. Suppose there exist functions $a, b, c \in K_1$, $p \in PC(\mathbb{R}_+, \mathbb{R}_+)$ and $g \in K_3$. Suppose $V : [-r, \infty) \times S(\rho) \to \mathbb{R}_+$ is continuous on $[-r, \tau_0) \times S(\rho)$ and on $[\tau_{k-1}, \tau_k) \times S(\rho)$ for $k = 1, 2, \ldots$ and that for each $x \in S(\rho)$ and $k = 0, 1, 2, \ldots$, $\lim_{(t,y) \to (\tau_k^-, x)} V(t, y) = V(\tau_k^-, x)$ exists. Moreover assume that $V$, restricted to $\mathbb{R}_+ \times S(\rho)$, is locally Lipschitz in $x$ and that the following conditions are satisfied:*

*(i)* $b(\|x\|) \le V(t, x) \le a(\|x\|)$ *for all* $(t, x) \in [-r, \infty) \times S(\rho)$;

*(ii)* $D^+_{(2.4.7)} V(t, \psi(0)) \le p(t)c(V(t, \psi(0)))$ *for all* $t \ne \tau_k$ *in* $\mathbb{R}_+$ *and* $\psi \in PC([-r, 0], S(\rho))$ *whenever* $V(t, \psi(0)) \ge g(V(t + s, \psi(s)))$ *for* $s \in [-r, 0]$;

*(iii)* $V(\tau_k, \psi(0) + I(\tau_k, \psi)) \le g(V(\tau_k^-, \psi(0)))$ *for all* $(\tau_k, \psi) \in \mathbb{R}_+ \times PC([-r, 0], S(\rho_1))$ *for which* $\psi(0^-) = \psi(0)$; *and*

*(iv)* $r = \sup_{k \in \mathbb{Z}^+} \{\tau_k - \tau_{k-1}\} < \infty$,

$$M_1 = \sup_{t\geq 0} \int_t^{t+r} p(s)ds < \infty, \text{ and}$$
$$M_2 = \inf_{q>0} \int_{g(q)}^{q} \frac{ds}{c(s)} > M_1.$$

*Then the solution $x(t) = 0$ of (2.4.7) is uniformly asymptotically stable.*

*Proof:* Condition (i) implies $b(s) \leq a(s)$ for all $s \in [0, \rho]$. So let $\hat{a}$ and $\hat{b}$ be continuous, strictly increasing functions satisfying $\hat{b}(s) \leq b(s) \leq a(s) \leq \hat{a}(s)$ for all $s \in [0, \rho]$. Then

$$\hat{b}(||x||) \leq V(t, x) \leq \hat{a}(||x||), \tag{5.2.1}$$

for all $(t, x) \in [-r, \infty) \times S(\rho)$.

From the definition of $M_2$ we see that $0 < g(q) < q$ for all $q > 0$.

We first show uniform stability. Let $\epsilon > 0$ and assume without loss of generality that $\epsilon \leq \rho_1$. Choose $\delta = \delta(\epsilon) > 0$ so that $\delta < \hat{a}^{-1}(g(\hat{b}(\epsilon)))$ and note that $0 < \delta < \epsilon$.

Let $(t_0, \phi) \in \mathbb{R}_+ \times PC([-r, 0], D)$ where $||\phi||_r \leq \delta$ and $t_0 \in [\tau_{l-1}, \tau_l)$ for some positive integer $l$. Suppose $x = x(t_0, \phi)$ is a solution of (2.4.7) & (2.4.2) and let $[t_0 - r, t_0 + \beta)$ be its maximal interval of existence. From Corollary 3.2.3 we know that if $\beta < \infty$, then there exists some $t \in (t_0, t_0 + \beta)$ for which $||x(t)|| > \epsilon$. We will prove that $||x(t)|| \leq \epsilon$ for $t \in [t_0, t_0 + \beta)$, which in turn will imply that $\beta = \infty$ and that the trivial solution of (2.4.7) is thereby uniformly stable.

Suppose for the sake of contradiction that $||x(t)|| > \epsilon$ for some $t \in [t_0, t_0 + \beta)$. Then let $\hat{t} = \inf\{t \in [t_0, t_0 + \beta) \mid ||x(t)|| > \epsilon\}$. Note that $||x(t)|| \leq ||\phi||_r \leq \delta < \epsilon$ for $t \in [t_0 - r, t_0]$ and in particular $||x(t_0)|| < \epsilon$.

By the definition of $\hat{t}$ we see that $\hat{t} \in (t_0, t_0 + \beta)$, $||x(t)|| \leq \epsilon \leq \rho_1$ for $t \in [t_0 - r, \hat{t})$ and either $||x(\hat{t})|| = \epsilon$ or $||x(\hat{t})|| > \epsilon$ and $\hat{t} = \tau_k$ for some $k$. In the latter case $||x(\hat{t})|| \leq \rho$ since $||x_{\hat{t}-}||_r \leq \epsilon \leq \rho_1$ and by our assumption on the functional $I$. Thus in either case $V(t, x(t))$ is defined for $t \in [t_0 - r, \hat{t}]$.

For $t \in [t_0 - r, \hat{t}]$ define

$$m(t) = V(t, x(t)). \tag{5.2.2}$$

By the piecewise continuity assumption on $V$ it follows that $m \in PC([t_0 - r, \hat{t}], \mathbb{R}_+)$ and $m(t)$ is continuous at each $t \neq \tau_k$ in $(t_0, \hat{t}]$. By (5.2.1) we have

$$\hat{b}(||x(t)||) \leq m(t) \leq \hat{a}(||x(t)||), \tag{5.2.3}$$

for $t \in [t_0 - r, \hat{t}]$. Thus $m(t) \leq \hat{a}(||\phi||_r) \leq \hat{a}(\delta) < g(\hat{b}(\epsilon))$ for $t \in [t_0 - r, t_0]$. By condition (ii) and from Corollary 5.1.1, since $V$ is locally Lipschitz in $x$, we have

$$D^+m(t) = \limsup_{h\to 0^+} \frac{1}{h}\left[m(t+h) - m(t)\right] \leq p(t)c(m(t)), \tag{5.2.4}$$

for all $t \neq \tau_k$ in $(t_0, \hat{t}]$ whenever $m(t) \geq g(\|m_t\|_r)$. Also by condition (iii) we have

$$m(\tau_k) \leq g(m(\tau_k^-)), \tag{5.2.5}$$

for all $\tau_k \in (t_0, \hat{t}]$.

Let $t^* = \inf\{t \in [t_0, \hat{t}] \mid m(t) \geq \hat{b}(\epsilon)\}$. Since $m(t_0) < g(\hat{b}(\epsilon)) < \hat{b}(\epsilon)$ and $m(\hat{t}) \geq \hat{b}(\epsilon)$, then $t^* \in (t_0, \hat{t}]$. Moreover, $m(t) < \hat{b}(\epsilon)$ for $t \in [t_0 - r, t^*)$. We claim that $m(t^*) = \hat{b}(\epsilon)$ and that $t^* \neq \tau_k$ for any $k$. Clearly we must have $m(t^*) \geq \hat{b}(\epsilon) > 0$. If $t^* = \tau_k$ for some $k$, then $0 < \hat{b}(\epsilon) \leq m(t^*) \leq g(m(t^{*-})) < m(t^{*-}) \leq \hat{b}(\epsilon)$ by (5.2.5), which is impossible. Thus $t^* \neq \tau_k$ for any $k$ and that in turn implies that $m(t^*) = \hat{b}(\epsilon)$ since $m(t)$ is continuous at $t^*$.

Now let us first consider the case where $\tau_{l-1} \leq t_0 < t^* < \tau_l$. Let $\bar{t} = \sup\{t \in [t_0, t^*] \mid m(t) \leq g(\hat{b}(\epsilon))\}$. Since $m(t_0) < g(\hat{b}(\epsilon))$, $m(t^*) = \hat{b}(\epsilon) > g(\hat{b}(\epsilon))$ and $m(t)$ is continuous on $[t_0, t^*]$, then $\bar{t} \in (t_0, t^*)$, $m(\bar{t}) = g(\hat{b}(\epsilon))$ and $m(t) \geq g(\hat{b}(\epsilon))$ for $t \in [\bar{t}, t^*]$.

Hence for $t \in [\bar{t}, t^*]$ and $s \in [-r, 0]$, we have $g(m(t+s)) \leq g(\hat{b}(\epsilon)) \leq m(t)$ or in other words $m(t) \geq g(\|m_t\|_r)$. Thus inequality (5.2.4) holds for all $t \in (\bar{t}, t^*]$. Integrating this differential inequality gives us

$$\int_{m(\bar{t})}^{m(t^*)} \frac{ds}{c(s)} \leq \int_{\bar{t}}^{t^*} p(s)ds \leq \int_{\bar{t}}^{\bar{t}+r} p(s)ds \leq M_1. \tag{5.2.6}$$

However, we also have

$$\int_{m(\bar{t})}^{m(t^*)} \frac{ds}{c(s)} = \int_{g(\hat{b}(\epsilon))}^{\hat{b}(\epsilon)} \frac{ds}{c(s)} \geq M_2. \tag{5.2.7}$$

Since we have assumed that $M_2 > M_1$, then we arrive at our desired contradiction.

Alternatively, suppose that $\tau_k < t^* < \tau_{k+1}$ for some $k \geq l$. Then $m(\tau_k) \leq g(m(\tau_k^-)) \leq g(\hat{b}(\epsilon))$ by (5.2.5). Similar to before define $\bar{t} = \sup\{t \in [\tau_k, t^*] \mid m(t) \leq g(\hat{b}(\epsilon))\}$. Then $\bar{t} \in [\tau_k, t^*)$, $m(\bar{t}) = g(\hat{b}(\epsilon))$ and $m(t) \geq g(\hat{b}(\epsilon))$ for $t \in [\bar{t}, t^*]$. Applying exactly the same argument as before yields a contradiction.

So in either case we obtain a contradiction, which proves that the zero solution of (2.4.7) is uniformly stable. Now we show that it is uniformly asymptotically stable.

Since the zero solution of (2.4.7) is uniformly stable, then there exists some $\eta > 0$ such that if $\|\phi\|_r \leq \eta$, then $\|x(t, t_0, \phi)\| \leq \rho_1$ for all $t \geq t_0 - r$ where $x = x(t_0, \phi)$ is any solution of (2.4.7) & (2.4.2). Moreover, $V(t, x(t)) \leq \hat{a}(\|x(t)\|) \leq \hat{a}(\rho_1)$ for $t \geq t_0 - r$.

Now let $\gamma > 0$ and assume without loss of generality that $\gamma < \rho_1$. Define

$$M = M(\gamma) = \sup\left\{\frac{1}{c(s)} \;\middle|\; g(\hat{b}(\gamma)) \leq s \leq \hat{a}(\rho_1)\right\}, \tag{5.2.8}$$

and note that $0 < M < \infty$.

For $\hat{b}(\gamma) \le q \le \hat{a}(\rho_1)$ we have $g(\hat{b}(\gamma)) \le g(q) < q \le \hat{a}(\rho_1)$ and so

$$M_2 \le \int_{g(q)}^{q} \frac{ds}{c(s)} \le M[q - g(q)], \tag{5.2.9}$$

from which we obtain $g(q) \le q - M_2/M < q - d$ where $d = d(\gamma) > 0$ is chosen so that $d < (M_2 - M_1)/M$.

Let $N = N(\gamma)$ be the smallest positive integer for which $\hat{a}(\rho_1) \le \hat{b}(\gamma) + Nd$ and define $T = T(\gamma) = \tau + (r + \tau)(N - 1)$. Given a solution $x = x(t_0, \phi)$ of (2.4.7) & (2.4.2) where $||\phi||_r \le \eta$ and $t_0 \in [\tau_{l-1}, \tau_l)$ for some positive integer $l$, we will show that $||x(t)|| \le \gamma$ for $t \ge t_0 + T$. Let

$$m(t) = V(t, x(t)), \tag{5.2.10}$$

for $t \ge t_0 - r$. Then $m(t) \le \hat{a}(\rho_1)$ for $t \ge t_0 - r$.

Given $0 < A \le \hat{a}(\rho_1)$ and $j \ge l$ we will show that if $m(t) \le A$ for $t \in [\tau_j - r, \tau_j)$, then $m(t) \le A$ for $t \ge \tau_j$ and if in addition $A \ge \hat{b}(\gamma)$, then $m(t) \le A - d$ for $t \ge \tau_j$.

To prove the first part, suppose for the sake of contradiction that there exists some $t \ge \tau_j$ for which $m(t) > A$. Then let $t^* = \inf\{t \ge \tau_j \mid m(t) > A\}$. Thus $t^* \in [\tau_k, \tau_{k+1})$ for some $k \ge j$. Since $m(\tau_k) \le g(m(\tau_k^-)) \le g(A) < A$, then $t^* \in (\tau_k, \tau_{k+1})$. Moreover, $m(t^*) = A$ and $m(t) \le A$ for $t \in [\tau_j - r, t^*]$.

Let $\bar{t} = \sup\{t \in [\tau_k, t^*] \mid m(t) \le g(A)\}$. Since $m(t^*) = A > g(A) \ge m(\tau_k)$, then $\bar{t} \in [\tau_k, t^*)$, $m(\bar{t}) = g(A)$ and $m(t) \ge g(A)$ for $t \in [\bar{t}, t^*]$.

Thus for $t \in [\bar{t}, t^*]$ and $s \in [-r, 0]$, we have $g(m(t + s)) \le g(A) \le m(t)$ implying that inequality (5.2.4) holds for all $t \in (\bar{t}, t^*]$ and thus (5.2.6) holds true. However, since

$$\int_{m(\bar{t})}^{m(t^*)} \frac{ds}{c(s)} = \int_{g(A)}^{A} \frac{ds}{c(s)} \ge M_2, \tag{5.2.11}$$

and $M_2 > M_1$, then we get a contradiction proving the first part.

The proof of the second part is similar. Assume for the sake of contradiction that there exists some $t \ge \tau_j$ for which $m(t) > A - d$. Then define $t^* = \inf\{t \ge \tau_j \mid m(t) > A - d\}$ and let $k \ge j$ be chosen so that $t^* \in [\tau_k, \tau_{k+1})$. Since $\hat{b}(\gamma) \le A \le \hat{a}(\rho_1)$, then $g(A) < A - d$ and so $m(\tau_k) \le g(m(\tau_k^-)) \le g(A) < A - d$. Thus $t^* \in (\tau_k, \tau_{k+1})$. Moreover, $m(t^*) = A - d$ and $m(t) \le A - d$ for $t \in [\tau_k, t^*]$.

Define $\bar{t}$ as before. Since $m(t^*) = A - d > g(A) \ge m(\tau_k)$, then $\bar{t} \in [\tau_k, t^*)$, $m(\bar{t}) = g(A)$ and

$m(t) \geq g(A)$ for $t \in [\bar{t}, t^*]$. Thus we obtain inequality (5.2.6) as before. However,

$$\int_{m(\bar{t})}^{m(t^*)} \frac{ds}{c(s)} = \int_{g(A)}^{A-d} \frac{ds}{c(s)} = \int_{g(A)}^{A} \frac{ds}{c(s)} - \int_{A-d}^{A} \frac{ds}{c(s)}. \tag{5.2.12}$$

Since $\hat{b}(\gamma) \leq A \leq \hat{a}(\rho_1)$, then $g(\hat{b}(\gamma)) \leq g(A) < A - d < A \leq \hat{a}(\rho_1)$ and so $1/c(s) \leq M$ for $A - d \leq s \leq A$. Thus from (5.2.12) we get

$$\int_{m(\bar{t})}^{m(t^*)} \frac{ds}{c(s)} \geq M_2 - \int_{A-d}^{A} M ds = M_2 - dM > M_2 + (M_1 - M_2) = M_1, \tag{5.2.13}$$

which contradicts (5.2.6), establishing the second part.

We define the indices $k^{(i)}$ for $i = 1, 2, \ldots, N$ as follows. Let $k^{(1)} = l$ and for $i = 2, \ldots, N$ let $k^{(i)}$ be chosen so that $\tau_{k^{(i)}-1} < \tau_{k^{(i-1)}} + r \leq \tau_{k^{(i)}}$.

Then $\tau_{k^{(1)}} = \eta \leq t_0 + \tau$ and for $i = 2, \ldots, N$ we have $\tau_{k^{(i)}} \leq \tau_{k^{(i)}-1} + \tau \leq \tau_{k^{(i-1)}} + r + \tau$. Combining these inequalities gives us $\tau_{k^{(N)}} \leq t_0 + \tau + (r + \tau)(N - 1) = t_0 + T$.

We claim that for each $i = 1, 2, \ldots, N$, $m(t) \leq \hat{a}(\rho_1) - id$ for $t \geq \tau_{k^{(i)}}$. Since $m(t) \leq \hat{a}(\rho_1)$ for $t \in [t_0 - r, \tau_{k^{(1)}})$, then by setting $A = \hat{a}(\rho_1)$ in our earlier argument we get $m(t) \leq \hat{a}(\rho_1) - d$ for $t \geq \tau_{k^{(1)}}$, which establishes the base case. We now proceed by induction and assume $m(t) \leq \hat{a}(\rho_1) - jd$ for $t \geq \tau_{k^{(j)}}$ for some $1 \leq j \leq N - 1$. Let $A = \hat{a}(\rho_1) - jd$. Since $\tau_{k^{(j)}} \leq \tau_{k^{(j+1)}} - r$, then $m(t) \leq A$ for $t \in [\tau_{k^{(j+1)}} - r, \tau_{k^{(j+1)}})$ and so $m(t) \leq A - d = \hat{a}(\rho_1) - (j + 1)d$ for $t \geq \tau_{k^{(j+1)}}$. Thus we have proven our claim by induction.

In particular we have $m(t) \leq \hat{a}(\rho_1) - Nd \leq \hat{b}(\gamma)$ for $t \geq t_0 + T \geq \tau_{k^{(N)}}$. Finally, by (5.2.3) we get that $\|x(t)\| \leq \gamma$ for $t \geq t_0 + T$, which completes the proof of this theorem. ∎

In Theorem 5.2.1, condition (iii) ensures that along solutions of (2.4.7) the Lyapunov function must decrease at each impulse time. Condition (ii) effectively imposes a bound on the growth rate of $V$ along solutions through a Razumikhin-type of argument. The definitions of $M_1$ and $M_2$ in (iv) and the requirement that $M_2$ be strictly greater than $M_1$ ensures that any possible growth in $V$ between impulses is more than offset by a reduction in $V$ at impulses. Stability results along this line for impulsive differential equations without delay can be found in [Liu94].

The importance of Theorem 5.2.1 is in its applicability to delay differential equations that are not already stable but that can be stabilized through the incorporation of impulses. Note that Theorem 5.2.1 does not directly impose any bounds on the delay constant $r$. Thus the stability results are delay independent.

**Example 5.2.1:** Consider the nonlinear delay differential equation

$$x'(t) = -ax(t - 1)[1 + x(t)], \tag{5.2.14}$$

where $a > 0$ is constant. This equation has been used in the study of population dynamics and was considered by Wright [Wri55] in 1955. He proved that for $a > \pi/2$, there exists some $\epsilon > 0$ such that $\limsup_{t\to\infty} |x(t, t_0, \phi)| \geq \epsilon$ for all $\phi \in C([-1,0], \mathbb{R})$ for which $\phi(s) \neq 0$ for $s \in (-1, 0)$. In other words, the zero equilibrium solution is unstable.

Suppose we define the Lyapunov function

$$V(x) = |x|. \tag{5.2.15}$$

If $\psi \in PC([-1,0], \mathbb{R})$, then we have, providing $\psi(0) \neq 0$,

$$\begin{aligned} D^+_{(5.2.14)}V(t, \psi(0)) &= \operatorname{sgn}(\psi(0))[-a\psi(-1)[1 + \psi(0)]] \leq a|\psi(-1)|(1 + |\psi(0)|) \\ &= aV(\psi(-1))(1 + V(\psi(0))). \end{aligned} \tag{5.2.16}$$

If $\psi(0) = 0$, then $D^+_{(5.2.14)}V(t, \psi(0))$ also satisfies the inequality derived in (5.2.16).

Suppose $0 < \lambda < 1$ for some constant $\lambda$ and let $g(s) = \lambda s$ for $s \in \mathbb{R}_+$. Then whenever $V(\psi(0)) \geq g(V(\psi(s)))$ for $s \in [-1, 0]$, we have in particular $V(\psi(-1)) \leq (1/\lambda)V(\psi(0))$, and so

$$D^+_{(5.2.14)}V(t, \psi(0)) \leq \frac{a}{\lambda}V(\psi(0))(1 + V(\psi(0))). \tag{5.2.17}$$

Now let $\rho > 0$ be arbitrarily small and define $\rho_1 = \rho$. Larger values of $\rho$ (and correspondingly $\rho_1$) will be associated with larger domains of attraction of the zero equilibrium solution. Then for $||\psi||_r \leq \rho$, we have $V(\psi(0)) = |\psi(0)| \leq \rho$, and so

$$D^+_{(5.2.14)}V(t, \psi(0)) \leq \frac{a(1+\rho)}{\lambda}V(\psi(0)) = p(t)c(V(\psi(0))). \tag{5.2.18}$$

where $p(t) = a(1 + \rho)/\lambda$ for $t \in \mathbb{R}_+$ and $c(s) = s$ for $s \in \mathbb{R}_+$.

We are interested in asymptotically stabilizing (5.2.14) through the use of impulses according to Theorem 5.2.1. For simplicity let us assume that the impulse times are equally spaced. In other words, $\tau_k - \tau_{k-1} = \tau > 0$ for all $k$. Calculating $M_1$ and $M_2$ as defined in Theorem 5.2.1, we get $M_1 = a\tau(1 + \rho)/\lambda$ and $M_2 = -\ln(\lambda)$. In order to satisfy $M_2 > M_1$, $\tau$ must be chosen sufficiently small so that

$$\tau < \frac{1}{a(1+\rho)}[-\lambda \ln(\lambda)]. \tag{5.2.19}$$

This illustrates that, at least insofar as Theorem 5.2.1 is concerned, impulses must occur frequently enough in order to stabilize the system.

In the case of impulsive differential equations the impulses can be widely spaced apart (i.e. $\tau$ can be arbitrarily large) and yet the system can still be impulsively stabilized. This is providing of course that when each impulse does occur, the impulse operator $I$ maps the solutions sufficiently close to zero. The same cannot normally be said for impulsive delay differential equations and this is why Theorem 5.2.1 indirectly imposes a constraint on the parameter $\tau$ (although note that $\tau$ does not depend on the delay constant $r = 1$). To illustrate this latter point one need only look at the delay differential equation

$$x'(t) = -x\left(t - \frac{\pi}{2}\right). \tag{5.2.20}$$

If $t_0 = 0$ and $\phi(s) = a\sin(s)$ for $s \in [-\pi/2, 0]$ where $a > 0$ is some constant, then the unique solution of (5.2.20) is given by $x(t) = a\sin(t)$ for all $t \geq -\pi/2$. This shows that the trivial solution of (5.2.20) is not asymptotically stable. Suppose we were to apply impulses to equation (5.2.20) at the times $\tau_k = k\pi$. Then these particular solutions are already zero at these times and so using impulses to map the solutions all the way to zero would have no effect and could therefore not asymptotically stabilize equation (5.2.20). If $\tau$ were sufficiently small, however, we could impulsively stabilize equation (5.2.20) by way of Theorem 5.2.1.

Returning to equation (5.2.14) again, suppose we wish to stabilize the system but with as large a value of $\tau$ as possible. Looking at inequality (5.2.19) we see that $a$ is fixed for our system, $\rho > 0$ may be made arbitrarily small and $\lambda$ can be chosen strictly between 0 and 1. The optimum choice of $\lambda$ that gives the greatest upper bound is $\lambda = 1/e$. With this choice of $\lambda$, then $\tau$ would have to satisfy $\tau < 1/[ae(1+\rho)]$ or more simply $\tau < 1/(ae)$ if $\rho$ were sufficiently small depending on $\tau$.

In any case, in order to satisfy the conditions of Theorem 5.2.1 we must define the functional $I$ so that it satisfies

$$|\psi(0) + I(\tau_k, \psi)| \leq \lambda|\psi(0)| \tag{5.2.21}$$

for all $(\tau_k, \psi) \in \mathbf{R}_+ \times PC([-1, 0], S(\rho_1))$ for which $\psi(0^-) = \psi(0)$. The simplest choice of $I$ that will work is $I(t, \psi) = (\lambda - 1)\psi(0)$. □

**Theorem 5.2.2:** *Assume $f$ satisfies the conditions of Corollary 3.2.3 and there exist functions $a, b, c \in K_1$, $p \in PC(\mathbf{R}_+, \mathbf{R}_+)$ and $g, \hat{g} \in K_3$ where $s \leq \hat{g}(s) < g(s)$ for $s > 0$. Suppose $V : [-r, \infty) \times S(\rho) \to \mathbf{R}_+$ is continuous on $[-r, \tau_0) \times S(\rho)$ and on $[\tau_{k-1}, \tau_k) \times S(\rho)$ for $k = 1, 2, \ldots$ and that for each $x \in S(\rho)$ and $k = 0, 1, 2, \ldots$, $\lim_{(t,y)\to(\tau_k^-, x)} V(t, y) = V(\tau_k^-, x)$ exists. Moreover assume that $V$, restricted to $\mathbf{R}_+ \times S(\rho)$, is locally Lipschitz in $x$ and that the following conditions are satisfied:*

*(i)* $b(\|x\|) \leq V(t, x) \leq a(\|x\|)$ *for all* $(t, x) \in [-r, \infty) \times S(\rho)$;

*(ii)* $D^+_{(2.4.7)}V(t, \psi(0)) \leq -p(t)c(V(t, \psi(0)))$ *for all* $t \neq \tau_k$ *in* $\mathbf{R}_+$ *and* $\psi \in PC([-r, 0], S(\rho))$ *whenever* $g(V(t, \psi(0))) \geq V(t + s, \psi(s))$ *for* $s \in [-r, 0]$;

*(iii)* $V(\tau_k, \psi(0) + I(\tau_k, \psi)) \leq \hat{g}(V(\tau_k^-, \psi(0)))$ *for all* $(\tau_k, \psi) \in \mathbf{R}_+ \times PC([-r, 0], S(\rho_1))$ *for which* $\psi(0^-) = \psi(0)$; *and*

*(iv)* $\mu = \inf_{k \in \mathbb{Z}^+} \{\tau_k - \tau_{k-1}\} > 0$,

$$M_2 = \sup_{q>0} \int_q^{g(q)} \frac{ds}{c(s)}, \text{ and}$$

$$M_1 = \inf_{t \geq 0} \int_t^{t+\mu} p(s)ds > M_2.$$

*Then the solution $x(t) = 0$ of (2.4.7) is uniformly asymptotically stable.*

*Proof:* As in Theorem 5.2.1 let $\hat{a}$ and $\hat{b}$ be continuous, strictly increasing functions satisfying $\hat{b}(s) \leq b(s) \leq a(s) \leq \hat{a}(s)$ for all $s \in [0, \rho]$. Then

$$\hat{b}(\|x\|) \leq V(t, x) \leq \hat{a}(\|x\|), \tag{5.2.22}$$

for all $(t, x) \in [-r, \infty) \times S(\rho)$.

Note that given how $M_1$ and $M_2$ are defined and related we know that $0 < M_2 < M_1 < \infty$.

We first show uniform stability. Let $\epsilon > 0$ and assume without loss of generality that $\epsilon \leq \rho_1$. Choose $\delta = \delta(\epsilon) > 0$ so that $g(\hat{a}(\delta)) < \hat{b}(\epsilon)$ and note that $0 < \delta < \epsilon$.

Let $(t_0, \phi) \in \mathbb{R}_+ \times PC([-r, 0], D)$ where $\|\phi\|_r \leq \delta$ and $t_0 \in [\tau_{l-1}, \tau_l)$ for some positive integer $l$. Suppose $x = x(t_0, \phi)$ is a solution of (2.4.7) & (2.4.2) and let $[t_0 - r, t_0 + \beta)$ be its maximal interval of existence. From Corollary 3.2.3 we know that if $\beta < \infty$, then there exists some $t \in (t_0, t_0 + \beta)$ for which $\|x(t)\| > \epsilon$. We will prove that $\|x(t)\| \leq \epsilon$ for $t \in [t_0, t_0 + \beta)$, which in turn will imply that $\beta = \infty$ and that the trivial solution of (2.4.7) is thereby uniformly stable.

Suppose for the sake of contradiction that $\|x(t)\| > \epsilon$ for some $t \in [t_0, t_0 + \beta)$. Then let $\hat{t} = \inf\{t \in [t_0, t_0 + \beta) \mid \|x(t)\| > \epsilon\}$. Note that $\|x(t)\| \leq \|\phi\|_r \leq \delta < \epsilon$ for $t \in [t_0 - r, t_0]$ and in particular $\|x(t_0)\| < \epsilon$.

By the definition of $\hat{t}$ we see that $\hat{t} \in (t_0, t_0 + \beta)$, $\|x(t)\| \leq \epsilon \leq \rho_1$ for $t \in [t_0 - r, \hat{t})$ and either $\|x(\hat{t})\| = \epsilon$ or $\|x(\hat{t})\| > \epsilon$ and $\hat{t} = \tau_k$ for some $k$. In the latter case $\|x(\hat{t})\| \leq \rho$ since $\|x_{\hat{t}^-}\|_r \leq \epsilon \leq \rho_1$ and by our assumption on the functional $I$. Thus in either case $V(t, x(t))$ is defined for $t \in [t_0 - r, \hat{t}]$.

For $t \in [t_0 - r, \hat{t}]$ define

$$m(t) = V(t, x(t)). \tag{5.2.23}$$

By the piecewise continuity assumption on $V$ it follows that $m \in PC([t_0 - r, \hat{t}], \mathbb{R}_+)$ and $m(t)$ is continuous at each $t \neq \tau_k$ in $(t_0, \hat{t}]$. By (5.2.22) we have

$$\hat{b}(\|x(t)\|) \leq m(t) \leq \hat{a}(\|x(t)\|), \tag{5.2.24}$$

for $t \in [t_0 - r, \hat{t}]$. Thus $m(t) \leq \hat{a}(\|\phi\|_r) \leq \hat{a}(\delta)$ implying $g(m(t)) \leq g(\hat{a}(\delta)) < \hat{b}(\epsilon)$ for $t \in [t_0 - r, t_0]$.

By condition (ii) and from Corollary 5.1.1, since $V$ is locally Lipschitz in $x$, we have

$$D^+m(t) = \limsup_{h\to 0^+} \frac{1}{h}[m(t+h) - m(t)] \le -p(t)c(m(t)), \tag{5.2.25}$$

for all $t \neq \tau_k$ in $(t_0, \hat{t}]$ whenever $g(m(t)) \ge \|m_t\|_r$. Also by condition (iii) we have

$$m(\tau_k) \le \hat{g}(m(\tau_k^-)), \tag{5.2.26}$$

for all $\tau_k \in (t_0, \hat{t}]$.

Let $t^* = \inf\{t \in [t_0, \hat{t}] \mid m(t) \ge \hat{b}(\epsilon)\}$. Since $m(t_0) < \hat{b}(\epsilon)$ and $m(\hat{t}) \ge \hat{b}(\epsilon)$, then $t^* \in (t_0, \hat{t}]$. Moreover, $m(t) < \hat{b}(\epsilon)$ for $t \in [t_0 - r, t^*)$, $m(t^{*-}) \le \hat{b}(\epsilon)$ and $m(t^*) \ge \hat{b}(\epsilon)$.

First let us consider the case where $\tau_{l-1} \le t_0 < t^* < \tau_l$. Then $m(t^*) = \hat{b}(\epsilon)$ and $g(m(t^*)) = g(\hat{b}(\epsilon)) > \hat{b}(\epsilon)$. Recall that $g(m(t_0)) < \hat{b}(\epsilon)$. So define $\bar{t} = \sup\{t \in [t_0, t^*] \mid g(m(t)) \le \hat{b}(\epsilon)\}$ and note that $\bar{t} \in (t_0, t^*)$, $g(m(\bar{t})) = \hat{b}(\epsilon)$ and $g(m(t)) \ge \hat{b}(\epsilon)$ for $t \in [\bar{t}, t^*]$. Thus for $t \in [\bar{t}, t^*]$ and $s \in [-r, 0]$, we have $m(t+s) \le \hat{b}(\epsilon) \le g(m(t))$ implying $\|m_t\|_r \le g(m(t))$. Thus inequality (5.2.25) holds for $t \in [\bar{t}, t^*]$, which in turn means that $m(t)$ is nonincreasing on this interval. Thus $m(\bar{t}) \ge m(t^*)$ and so $g(m(\bar{t})) \ge g(m(t^*))$. But this contradicts the fact that $g(m(t^*)) > \hat{b}(\epsilon) = g(m(\bar{t}))$.

Alternatively suppose $\tau_k \le t^* < \tau_{k+1}$ for some $k \ge l$. We will eventually get a contradiction for this case as well. First we claim that $g(m(\tau_k^-)) \le \hat{b}(\epsilon)$. To prove this we consider the cases $k = l$ and $k > l$ separately.

Suppose $k = l$. To prove $g(m(\tau_l^-)) \le \hat{b}(\epsilon)$ we suppose, for the sake of contradiction, that $g(m(\tau_l^-)) > \hat{b}(\epsilon)$. Recall that $g(m(t_0)) < \hat{b}(\epsilon)$ and so define $\bar{t} = \sup\{t \in [t_0, \tau_l) \mid g(m(t)) \le \hat{b}(\epsilon)\}$. Then $\bar{t} \in (t_0, \tau_l)$, $g(m(\bar{t})) = \hat{b}(\epsilon)$ and $g(m(t)) \ge \hat{b}(\epsilon) \ge m(t+s)$ for $t \in [\bar{t}, \tau_l)$ and $s \in [-r, 0]$. Thus from inequality (5.2.25) we conclude $m(\bar{t}) \ge m(\tau_l^-)$ from which we get the contradiction $\hat{b}(\epsilon) = g(m(\bar{t})) \ge g(m(\tau_l^-)) > \hat{b}(\epsilon)$, proving this case.

Now consider the case $k > l$ and suppose for the sake of contradiction that $g(m(\tau_k^-)) > \hat{b}(\epsilon)$. Either $g(m(t)) > \hat{b}(\epsilon)$ for all $t \in [\tau_{k-1}, \tau_k)$ or there exists some $t \in [\tau_{k-1}, \tau_k)$ for which $g(m(t)) \le \hat{b}(\epsilon)$. In the former case $g(m(t)) > \hat{b}(\epsilon) \ge m(t+s)$ for all $t \in [\tau_{k-1}, \tau_k)$ and $s \in [-r, 0]$. Therefore we may integrate the differential inequality (5.2.25) over the interval $[\tau_{k-1}, \tau_k)$ to give us

$$\int_{m(\tau_k^-)}^{m(\tau_{k-1})} \frac{ds}{c(s)} \ge \int_{\tau_{k-1}}^{\tau_k} p(s)ds \ge \int_{\tau_{k-1}}^{\tau_{k-1}+\mu} p(s)ds \ge M_1. \tag{5.2.27}$$

However,

$$\int_{m(\tau_k^-)}^{m(\tau_{k-1})} \frac{ds}{c(s)} \le \int_{m(\tau_k^-)}^{\hat{b}(\epsilon)} \frac{ds}{c(s)} \le \int_{m(\tau_k^-)}^{g(m(\tau_k^-))} \frac{ds}{c(s)} \le M_2 < M_1, \tag{5.2.28}$$

giving us our contradiction.

In the latter case let $\bar{t} = \sup\{t \in [\tau_{k-1}, \tau_k) \mid g(m(t)) \leq \hat{b}(\epsilon)\}$. Then $\bar{t} \in [\tau_{k-1}, \tau_k)$, $g(m(\bar{t})) = \hat{b}(\epsilon)$ and $g(m(t)) \geq \hat{b}(\epsilon) \geq m(t+s)$ for all $t \in [\bar{t}, \tau_k)$ and $s \in [-r, 0]$. From (5.2.25) we get $m(\bar{t}) \geq m(\tau_k^-)$ implying $\hat{b}(\epsilon) = g(m(\bar{t})) \geq g(m(\tau_k^-)) > \hat{b}(\epsilon)$ giving us our contradiction. Thus the case $k > l$ is proven.

We have proven that $g(m(\tau_k^-)) \leq \hat{b}(\epsilon)$ when $\tau_k \leq t^* < \tau_{k+1}$ and $k \geq l$. Thus $m(\tau_k) \leq \hat{g}(m(\tau_k^-)) < \hat{b}(\epsilon)$ by (5.2.26) and since $\hat{g}(s) < g(s)$ for $s > 0$. Since $m(t^*) \geq \hat{b}(\epsilon)$, then we cannot have $t^* = \tau_k$. Thus $t^* \in (\tau_k, \tau_{k+1})$ and $m(t^*) = \hat{b}(\epsilon)$. If $g(m(t)) > \hat{b}(\epsilon)$ for $t \in [\tau_k, t^*]$, then let $\bar{t} = \tau_k$ and note that $m(\bar{t}) < \hat{b}(\epsilon)$. Otherwise let $\bar{t} = \sup\{t \in [\tau_k, t^*] \mid g(m(t)) \leq \hat{b}(\epsilon)\}$ and note that $m(\bar{t}) < g(m(\bar{t})) = \hat{b}(\epsilon)$. Since $m(t^*) = \hat{b}(\epsilon)$, then $\bar{t} \in [\tau_k, t^*)$. Moreover, $g(m(t)) \geq \hat{b}(\epsilon) \geq m(t+s)$ for $t \in [\bar{t}, t^*]$ and $s \in [-r, 0]$. Thus from (5.2.25) we know $m(\bar{t}) \geq m(t^*)$ but this contradicts the fact that $m(\bar{t}) < \hat{b}(\epsilon) = m(t^*)$. This final contradiction completes the proof of uniform stability.

Since the trivial solution of (2.4.7) is uniformly stable, then there exists some $\eta > 0$ satisfying $g(\hat{a}(\eta)) < \hat{b}(\rho_1)$ such that if $||\phi||_r \leq \eta$, then $||x(t, t_0, \phi)|| \leq \rho_1$ for all $t \geq t_0 - r$ where $x = x(t_0, \phi)$ is any solution of (2.4.7) & (2.4.2). Moreover, $V(t, x(t)) \leq \hat{a}(||x(t)||) \leq \hat{a}(\rho_1)$ for $t \geq t_0 - r$.

Now let $\gamma > 0$ and assume without loss of generality that $\gamma < \rho_1$. Let $\hat{d} = \hat{d}(\gamma) = \inf\{g(s) - \hat{g}(s) \mid g(s) \geq \hat{b}(\gamma)/2 \text{ and } s \leq \hat{a}(\rho_1)\}$. Note that $\hat{d}$ is well-defined and positive since the set of points $s$ for which $g(s) \geq \hat{b}(\gamma)/2$ and $s \leq \hat{a}(\rho_1)$ is a closed (nonempty) interval excluding zero. Choose $d = d(\gamma) > 0$ such that $d < \min\{\hat{a}(\rho_1) - \hat{b}(\gamma), \hat{d}, \hat{b}(\gamma)/2\}$.

Let $N = N(\gamma)$ be the smallest positive integer for which $\hat{a}(\rho_1) \leq \hat{b}(\gamma) + Nd$. Let $\tau = \sup_{k \in \mathbb{Z}^+}\{\tau_k - \tau_{k-1}\}$. We can assume without loss of generality that $\tau < \infty$. This is because if $\tau$ were infinite, then we could artificially add additional impulse times to system (2.4.7). In particular if consecutive impulse times $\tau_{k-1}$ and $\tau_k$ satisfy $j\mu \leq \tau_k - \tau_{k-1} < (j+1)\mu$ for some integer $j \geq 2$, then we may declare the times $\tau_{k-1} + \tau, \tau_{k-1} + 2\mu, \ldots, \tau_{k-1} + (j-1)\mu$ as new impulse times. For this modified system $\tau \leq 2\mu$. At these new impulse times we simply define the functional $I$ to be identically zero so that there is no effect on the solutions. Condition (iii) will continue to be satisfied. Stability of this system will then imply stability of the original system. In light of this define $T(\eta, \gamma) = 2\tau + (r + 3\tau)(N-1)$.

Let $x = x(t_0, \phi)$ be a solution of (2.4.7) & (2.4.2) where $||\phi||_r \leq \eta$ and $t_0 \in [\tau_{l-1}, \tau_l)$ for some positive integer $l$. We will show that $||x(t)|| \leq \gamma$ for $t \geq t_0 + T$. Define $m(t)$ for $t \geq t_0 - r$ as in (5.2.23). Then $m(t) \leq \hat{a}(\rho_1)$ for $t \geq t_0 - r$.

We first claim that $g(m(\tau_l^-)) \leq \hat{a}(\rho_1)$. Since $||x(t_0)|| \leq ||\phi||_r \leq \eta$, then $m(t_0) \leq \hat{a}(\eta)$ and so $g(m(t_0)) \leq g(\hat{a}(\eta)) < \hat{b}(\rho_1)$. We prove that $g(m(\tau_l^-)) \leq \hat{b}(\rho_1)$ in the same way we proved earlier in the case of uniform stability that $g(m(\tau_l^-)) \leq \hat{b}(\epsilon)$ when $k = l$ only this time $\epsilon$ is replaced by $\rho_1$. Thus we get $g(m(\tau_l^-)) \leq \hat{b}(\rho_1) \leq \hat{a}(\rho_1)$.

Let $0 < A \leq \hat{a}(\rho_1)$ and $j \geq l$. If $m(t) \leq A$ for $t \in [\tau_j - r, \tau_j)$ and $g(m(\tau_j^-)) \leq A$, then we claim that $m(t) \leq A$ for $t \in [\tau_j, \tau_{j+1})$ and $g(m(\tau_{j+1}^-)) \leq A$.

To show $m(t) \leq A$ for $t \in [\tau_j, \tau_{j+1})$ suppose for the sake of contradiction that there exists some $t \in [\tau_j, \tau_{j+1})$ for which $m(t) > A$. Let $t^* = \inf\{t \in [\tau_j, \tau_{j+1}) \mid m(t) \geq A\}$. Since $m(\tau_j) \leq \hat{g}(m(\tau_j^-)) < A$, then $t^* \in (\tau_j, \tau_{j+1})$, $m(t^*) = A$ and $m(t) \leq A$ for $t \in [\tau_j - r, t^*]$. If $g(m(t)) > A$ for $t \in [\tau_j, t^*]$, then let $\bar{t} = \tau_j$ and note that $m(\bar{t}) < A$. Otherwise let $\bar{t} = \sup\{t \in [\tau_j, t^*] \mid g(m(t)) \leq A\}$ and note that $m(\bar{t}) < g(m(\bar{t})) = A$. Since $g(m(t^*)) = g(A) > A$, then $\bar{t} \in [\tau_j, t^*)$. For $t \in [\bar{t}, t^*]$ and $s \in [-r, 0]$ we have $g(m(t)) \geq A \geq m(t+s)$. Thus from (5.2.25) we know that $m(\bar{t}) \geq m(t^*)$, which contradicts the fact that $m(t^*) = A > m(\bar{t})$.

Now that we have proven $m(t) \leq A$ for $t \in [\tau_j, \tau_{j+1})$, we must show that $g(m(\tau_{j+1}^-)) \leq A$. The proof of this follows in an identical fashion to the arguments used earlier in the proof of uniform stability where we showed that $g(m(\tau_k^-)) \leq \hat{b}(\epsilon)$ for the case $k > l$, if we replace $k$ by $j+1$ and $\hat{b}(\epsilon)$ by $A$.

Having proven our claim we may repeatedly apply it to successive intervals of the form $[\tau_k, \tau_{k+1})$ for $k \geq j$ to conclude that in fact $m(t) \leq A$ for all $t \geq \tau_j$ and that $g(m(\tau_k^-)) \leq A$ for all $k \geq j+1$.

Next we claim that if, in addition, $A \geq \hat{b}(\gamma)$, then $m(t) \leq A - d$ for $t \in [\tau_{j+1}, \tau_{j+2})$. To show this we must first show $m(\tau_{j+1}) < A - d$ and to do this we look at two cases. In the first case we suppose $0 \leq g(m(\tau_{j+1}^-)) \leq \hat{b}(\gamma)/2$. Then $m(\tau_{j+1}) \leq \hat{g}(m(\tau_{j+1}^-)) \leq g(m(\tau_{j+1}^-)) \leq \hat{b}(\gamma)/2 \leq A - \hat{b}(\gamma)/2 < A - d$. For the second case we suppose $\hat{b}(\gamma)/2 < g(m(\tau_{j+1}^-)) \leq A$. Since $g(m(\tau_{j+1}^-)) \geq \hat{b}(\gamma)/2$ and $m(\tau_{j+1}^-) \leq g(m(\tau_{j+1}^-)) \leq A \leq \hat{a}(\rho_1)$, then $\hat{g}(m(\tau_{j+1}^-)) \leq g(m(\tau_{j+1}^-)) - \hat{d} < g(m(\tau_{j+1}^-)) - d \leq A - d$ from the definition of $\hat{d}$ and so $m(\tau_{j+1}) \leq \hat{g}(m(\tau_{j+1}^-)) < A - d$. So in either case we have shown $m(\tau_{j+1}) < A - d$.

To prove $m(t) \leq A - d$ for all $t \in [\tau_{j+1}, \tau_{j+2})$ we assume for the sake of contradiction that $m(t) > A - d$ for some $t \in [\tau_{j+1}, \tau_{j+2})$. Let $t^* = \inf\{t \in [\tau_{j+1}, \tau_{j+2}) \mid m(t) \geq A - d\}$. Then $t^* \in (\tau_{j+1}, \tau_{j+2})$, since $m(\tau_{j+1}) < A - d$. Moreover, $m(t^*) = A - d$ and $m(t) \leq A - d$ for $t \in [\tau_{j+1}, t^*]$. One thing to note is that since $g(A-d) \geq A - d \geq \hat{b}(\gamma)/2$ and $A - d \leq A \leq \hat{a}(\rho_1)$, then $g(A-d) \geq \hat{g}(A-d) + \hat{d} > A - d + d = A$. Therefore $g(m(t^*)) = g(A-d) > A$. If $g(m(t)) > A$ for all $t \in [\tau_{j+1}, t^*]$, then let $\bar{t} = \tau_{j+1}$ and note that $m(\bar{t}) < A - d$. Otherwise let $\bar{t} = \sup\{t \in [\tau_{j+1}, t^*] \mid g(m(t)) \leq A\}$ and note that $m(\bar{t}) \leq \hat{g}(m(\bar{t})) \leq g(m(\bar{t})) - \hat{d} = A - \hat{d} < A - d$ since $g(m(\bar{t})) = A \geq \hat{b}(\gamma)/2$ and $m(\bar{t}) \leq A \leq \hat{a}(\rho_1)$. Clearly $\bar{t} \in [\tau_{j+1}, t^*)$ since $g(m(t^*)) > A$. Thus for $t \in [\bar{t}, t^*]$ and $s \in [-r, 0]$, $g(m(t)) \geq A \geq m(t+s)$, which implies by (5.2.25) that $m(\bar{t}) \geq m(t^*)$. This, however, contradicts the fact that $m(t^*) = A - d > m(\bar{t})$.

We have therefore proven our additional claim in the case where $A \geq \hat{b}(\gamma)$. Again by applying this result to successive intervals $[\tau_k, \tau_{k+1})$ for $k \geq j+1$ we can argue that $m(t) \leq A - d$ for all $t \geq \tau_{j+1}$.

To summarize then, we have shown that if $0 < A \leq \hat{a}(\rho_1)$ and $j \geq l$ and if $m(t) \leq A$ for $t \in [\tau_j - r, \tau_j)$ and $g(m(\tau_j^-)) \leq A$, then $m(t) \leq A$ for $t \geq \tau_j$ and if, in addition, $A \geq \hat{b}(\gamma)$, then $m(t) \leq A - d$ for $t \geq \tau_{j+1}$.

We define the indices $k^{(i)}$ for $i = 1, 2, \ldots, N$ recursively as follows. Let $k^{(1)} = l + 1$ and for $i = 2, \ldots, N$ let $k^{(i)}$ satisfy $\tau_{k^{(i)}-3} < \tau_{k^{(i-1)}} + r \leq \tau_{k^{(i)}-2}$. Note that $\tau_{k^{(i-1)}} < \tau_{k^{(i-1)}} + r \leq \tau_{k^{(i)}-2}$

and so $k^{(i-1)} < k^{(i)} - 2$ implying $k^{(i)} \geq 3 + k^{(i-1)}$. Therefore $\tau_{k^{(1)}} = \tau_{l+1} \leq 2\tau + \tau_{l-1} \leq 2\tau + t_0$ and for $i = 1, 2, \ldots, N$, $\tau_{k^{(i)}} \leq 3\tau + \tau_{k^{(i)}-3} \leq 3\tau + r + \tau_{k^{(i-1)}}$. Combining these inequalities gives $\tau_{k^{(N)}} \leq 2\tau + (2\tau + r)(N-1) + t_0 = t_0 + T$.

We claim that for each $i = 1, 2, \ldots, N$, $m(t) \leq \hat{a}(\rho_1) - id$ for $t \geq \tau_{k^{(i)}}$. Since $m(t) \leq \hat{a}(\rho_1)$ for $t \in [t_0 - r, \tau_l)$ and $g(m(\tau_l^-)) \leq \hat{a}(\rho_1)$, then by setting $A = \hat{a}(\rho_1)$ in our earlier argument we get $m(t) \leq \hat{a}(\rho_1) - d$ for $t \geq \tau_{l+1} = \tau_{k^{(1)}}$, which establishes the base case. We now proceed by induction and assume $m(t) \leq \hat{a}(\rho_1) - jd$ for $t \geq \tau_{k^{(j)}}$ for some $1 \leq j \leq N-1$.

Let $A = \hat{a}(\rho_1) - jd$ and note that $A \geq \hat{b}(\gamma)$. Since $\tau_{k^{(j+1)}-2} \geq \tau_{k^{(j)}} + r$, then $m(t) \leq A$ for $t \geq \tau_{k^{(j+1)}-2} - r$. Next we must show that $g(m(\tau^-_{k^{(j+1)}-1})) \leq A$. This is proven the same way we proved $g(m(\tau_k^-)) \leq \hat{b}(\epsilon)$ for the case $k > l$ in the proof of uniform stability only this time $k$ and $\hat{b}(\epsilon)$ are replaced by $k^{(j+1)} - 1$ and $A$, respectively. We may therefore conclude that $m(t) \leq A - d = \hat{a}(\rho_1) - (j+1)d$ for $t \geq \tau_{k^{(j+1)}}$, thus completing the induction step.

So in particular we have $m(t) \leq \hat{a}(\rho_1) - Nd \leq \hat{b}(\gamma)$ for $t \geq t_0 + T \geq \tau_{k^{(N)}}$. Finally, by (5.2.24) we get that $\|x(t)\| \leq \gamma$ for $t \geq t_0 + T$, which completes the proof of this theorem. ∎

Theorem 5.2.2 is in some ways the opposite of Theorem 5.2.1. Here the derivative of $V$ is always nonpositive, implying that $V$ is nonincreasing along solutions between impulses. In the absence of impulses the trivial solution of system (2.4.7) is uniformly asymptotically stable. Theorem 5.2.2 allows for significant increases in $V$, and correspondingly the solutions themselves, at impulse times but only as long as these are balanced sufficiently by the decrease of $V$ between impulses. These techniques are based in part on earlier work in the study of boundedness properties of solutions of impulsive differential equations without delay (see [Bal97a, Bal97b, Bal98]).

Shen and Yan [She98] obtained a stability result similar to Theorem 5.2.2, which was recently improved upon by Shen and Luo [She99]. However, Theorem 5.2.2 represents a significant improvement over both of these results. One advantage of Theorem 5.2.2 is that we do not assume *a priori* that solutions of (2.4.7) exist for all $t$. Second, as pointed out in the proof of Theorem 5.2.2, it is not necessary to assume that the duration between consecutive impulse times is bounded above. The main advantage, however, is that there is no restriction on the delay constant $r$ in terms of its relation to $\mu$. In [She98] it is assumed that $\mu > 3r$ (using our notation) while in [She99] this condition was weakened to $\mu \geq r$. In Theorem 5.2.2 the delay constant $r$ can be significantly larger than $\mu$.

We remark that Theorem 5.2.1 could be weakened slightly by redefining $M_1$ to be

$$M_1 = \sup_{k \in \mathbf{Z}^+} \int_{\tau_{k-1}}^{\tau_k} p(s)ds. \tag{5.2.29}$$

Moreover, with this new definition of $M_1$, the requirement that $\tau < \infty$ may be dropped if all we are interested in is uniform stability. As we see from the proof of Theorem 5.2.1, it was only when proving uniform asymptotic stability that we made use of this condition.

If in Theorem 5.2.2 we were to redefine $M_1$ to be

$$M_1 = \inf_{k \in \mathbb{Z}^+} \int_{\tau_{k-1}}^{\tau_k} p(s)ds, \tag{5.2.30}$$

then the condition $\mu > 0$ could likewise be dropped if we only want uniform stability. Moreover, the proof of uniform asymptotic stability in Theorem 5.2.2 carries through with this new definition of $M_1$ providing we assume that $\tau < \infty$. If $\tau = \infty$, then the addition of new artificial impulse times into the system as described in the theorem would cause a reduction in this value of $M_1$, perhaps to the point where it is no longer strictly greater than $M_2$.

**Example 5.2.2:** Consider the two dimensional impulsive delay differential equation

$$\begin{aligned} x'(t) &= -y^3(t)\sin(x(t-1)) - 6x(t) + y^2(t-1),\ t \neq 3k, \\ y'(t) &= x(t)\sin(x(t-1)) - 3y(t) + y(t-1),\ t \neq 3k, \end{aligned} \tag{5.2.31a}$$

$$\begin{aligned} \Delta x(t) &= \frac{1}{2}x(t^-),\ t = 3k, \\ \Delta y(t) &= \frac{1}{2}y(t^-),\ t = 3k, \end{aligned} \tag{5.2.31b}$$

where $k = 0, 1, 2, \ldots$.

The conditions of Corollary 3.2.3 are clearly satisfied. Moreover, given any $\rho > 0$ we may define $\rho_1 = 2\rho/3$ so that hypothesis (H1) is satisfied. We claim that the trivial solution of equation (5.2.31) is uniformly asymptotically stable.

For $(x, y) \in \mathbb{R}^2$ define the following Lyapunov function

$$V(x, y) = \frac{1}{2}x^2 + \frac{1}{4}y^4. \tag{5.2.32}$$

Then condition (i) of Theorem 5.2.2 is satisfied. So let us calculate the upper right-hand derivative of $V$ with respect to system (5.2.31). We get for $(t, \psi) \in \mathbb{R}_+ \times PC([-r, 0], \mathbb{R}^2)$, where $\psi = (\psi_1, \psi_2)$.

$$\begin{aligned} D^+_{(5.2.31)}V(t, \psi_1(0), \psi_2(0)) &= \psi_1(0)(-\psi_2^3(0)\sin(\psi_1(-1)) - 6\psi_1(0) + \psi_2^2(-1)) \\ &\quad + \psi_2^3(0)(\psi_1(0)\sin(\psi_1(-1)) - 3\psi_2(0) + \psi_2(-1)), \\ &= -6\psi_1^2(0) - 3\psi_2^4(0) + \psi_1(0)\psi_2^2(-1) + \psi_2^3(0)\psi_2(-1) \\ &\leq -12V(\psi_1(0), \psi_2(0)) + |\psi_1(0)||\psi_2(-1)|^2 + |\psi_2(0)|^3|\psi_2(-1)| \\ &\leq -12V(\psi_1(0), \psi_2(0)) + (2V(\psi_1(0), \psi_2(0)))^{1/2}(4V(\psi_1(-1), \psi_2(-1)))^{1/2} \\ &\quad + (4V(\psi_1(0), \psi_2(0)))^{3/4}(4V(\psi_1(-1), \psi_2(-1)))^{1/4}. \end{aligned} \tag{5.2.33}$$

If we define $g(s) = 4s$ for all $s \in \mathbb{R}_+$, then whenever $g(V(\psi_1(0), \psi_2(0))) \geq V(\psi_1(s), \psi_2(s))$ for

$s \in [-1, 0]$, we get

$$\begin{aligned} D^+_{(5.2.31)}V(t, \psi_1(0), \psi_2(0)) &\le -12V(\psi_1(0), \psi_2(0)) + (2V(\psi_1(0), \psi_2(0)))^{1/2}(16V(\psi_1(0), \psi_2(0)))^{1/2} \\ &\quad +(4V(\psi_1(0), \psi_2(0)))^{3/4}(16V(\psi_1(0), \psi_2(0)))^{1/4} \\ &= (-12 + 4\sqrt{2} + 4\sqrt{2})V(\psi_1(0), \psi_2(0)) \le -\frac{2}{3}V(\psi_1(0), \psi_2(0)). \end{aligned} \tag{5.2.34}$$

So in Theorem 5.2.2 let $c(s) = s$ for all $s \in \mathbb{R}_+$ and let $p(t) = 2/3$ for all $t \in \mathbb{R}_+$. Then condition (ii) is satisfied.

Calculating $M_1$ and $M_2$ in Theorem 5.2.2 gives us $M_2 = \ln(4)$ and, since $\mu = 3$, $M_1 = 2 > M_2$. This verifies condition (iv).

Finally, to check condition (iii) we note that

$$V\left(\psi_1(0) + \frac{1}{2}\psi_1(0), \psi_2(0) + \frac{1}{2}\psi_2(0)\right) = \frac{1}{2}\left(\frac{3}{2}\psi_1(0)\right)^2 + \frac{1}{4}\left(\frac{3}{2}\psi_2(0)\right)^4 \le \frac{9}{4}V(\psi_1(0), \psi_2(0)), \tag{5.2.35}$$

and hence condition (iii) is satisfied with $\bar{g}(s) = 9s/4$. □

## 5.3 Boundedness

In this section we develop some uniform ultimate boundedness results using the techniques of the previous section. We will be assuming $J = \mathbb{R}_+$ and $D = \mathbb{R}^n$ for simplicity.

**Theorem 5.3.1:** *Assume $J = \mathbb{R}_+$, $D = \mathbb{R}^n$ and the conditions of Corollary 3.2.3 are satisfied. Suppose there exists a constant $\rho > 0$ and functions $a \in C(\mathbb{R}_+, \mathbb{R}_+)$, $b \in K_4$, $c \in K_1$, $p \in PC(\mathbb{R}_+, \mathbb{R}_+)$ and $g \in K_3$ with $g(s) < s$ for all $s > 0$ and $\lim_{s\to\infty} g(s) = \infty$. Suppose $V : [-r, \infty) \times \mathbb{R}^n \to \mathbb{R}_+$ is continuous on $[-r, \tau_0) \times \mathbb{R}^n$ and on $[\tau_{k-1}, \tau_k) \times \mathbb{R}^n$ for $k = 1, 2, \ldots$ and that for each $x \in \mathbb{R}^n$ and $k = 0, 1, 2, \ldots$, $\lim_{(t,y)\to(\tau_k^-, x)} V(t, y) = V(\tau_k^-, x)$ exists. Moreover assume that $V$, restricted to $\mathbb{R}_+ \times \mathbb{R}^n$, is locally Lipschitz in $x$ and that the following conditions are satisfied:*

*(i)* $b(\|x\|) \le V(t, x) \le a(\|x\|)$ *for all* $(t, x) \in [-r, \infty) \times \mathbb{R}^n$;

*(ii)* $D^+_{(2.4.7)}V(t, \psi(0)) \le p(t)c(V(t, \psi(0)))$ *for all* $t \ne \tau_k$ *in* $\mathbb{R}_+$ *and* $\psi \in PC([-r, 0], \mathbb{R}^n)$ *whenever* $\|\psi(0)\| \ge \rho$ *and* $V(t, \psi(0)) \ge g(V(t + s, \psi(s)))$ *for* $s \in [-r, 0]$;

*(iii)* $V(\tau_k, \psi(0) + I(\tau_k, \psi)) \le g(V(\tau_k^-, \psi(0)))$ *for all* $(\tau_k, \psi) \in \mathbb{R}_+ \times PC([-r, 0], \mathbb{R}^n)$ *for which* $\psi(0^-) = \psi(0)$ *and* $\|\psi(0)\| \ge \rho$;

*(iv) there exists some* $\rho_1 \ge \rho$ *such that if* $(\tau_k, \psi) \in \mathbb{R}_+ \times PC([-r, 0], \mathbb{R}^n)$, $\psi(0^-) = \psi(0)$ *and* $\|\psi(0)\| \le \rho$, *then* $\|\psi(0) + I(\tau_k, \psi)\| \le \rho_1$; *and*

*(v)* $\tau = \sup_{k \in \mathbb{Z}^+} \{\tau_k - \tau_{k-1}\} < \infty$,

$M_1 = \sup_{t \geq 0} \int_t^{t+\tau} p(s)ds < \infty$, *and*

$M_2 = \inf_{q>0} \int_{g(q)}^{q} \frac{ds}{c(s)} > M_1$.

*Then solutions of (2.4.7) are uniformly ultimately bounded.*

*Proof:* Let $\hat{b}$ be a continuous, strictly increasing function satisfying $\lim_{s\to\infty} \hat{b}(s) = \infty$ and $\hat{b}(s) \leq b(s)$ for all $s \in \mathbb{R}_+$. Condition (i) implies $b(s) \leq a(s)$ for all $s$, which in turn implies $\lim_{s\to\infty} a(s) = \infty$. So let $\hat{a}$ be a continuous, strictly increasing function satisfying $\hat{a}(s) \geq a(s)$ for all $s \in \mathbb{R}_+$. Condition (i) then implies

$$\hat{b}(\|x\|) \leq V(t,x) \leq \hat{a}(\|x\|), \tag{5.3.1}$$

for all $(t,x) \in [-r,\infty) \times \mathbb{R}^n$.

We first show uniform boundedness. Let $B_1 > 0$ and assume, without loss of generality, that $B_1 \geq \rho_1 \geq \rho > 0$. Choose $B_2 = B_2(B_1)$ so that $\hat{a}(B_1) < g(\hat{b}(B_2))$ and note that $B_2 > B_1$. Here we are making use of the assumptions on $\hat{b}$ and $g$ that $\lim_{s\to\infty} \hat{b}(s) = \lim_{s\to\infty} g(s) = \infty$. Otherwise there may not exist such a $B_2$.

Let $(t_0,\phi) \in \mathbb{R}_+ \times PC([-r,0],\mathbb{R}^n)$ where $\|\phi\|_r \leq B_1$ and $t_0 \in [\tau_{l-1},\tau_l)$ for some positive integer $l$. Suppose $x = x(t_0,\phi)$ is a solution of (2.4.7) & (2.4.2) and let $[t_0 - r, t_0+\beta)$ be its maximal interval of existence. From Corollary 3.2.3 we know that if $\beta < \infty$, then there exists some $t \in (t_0, t_0+\beta)$ for which $\|x(t)\| > B_2$. We will prove that $\|x(t)\| \leq B_2$ for $t \in [t_0, t_0+\beta)$, which in turn will imply that $\beta = \infty$ and that solutions of (2.4.7) are hence uniformly bounded.

Suppose for the sake of contradiction that $\|x(t)\| > B_2$ for some $t \in [t_0, t_0+\beta)$. Then let $\hat{t} = \inf\{t \in [t_0,t_0+\beta) \mid \|x(t)\| > B_2\}$. Note that $\|x(t)\| \leq \|\phi\|_r \leq B_1 < B_2$ for $t \in [t_0 - r, t_0]$ and in particular $\|x(t_0)\| < B_2$. By the definition of $\hat{t}$ we see that $\hat{t} \in (t_0, t_0+\beta)$, $\|x(t)\| \leq B_2$ for $t \in [t_0 - r, \hat{t})$ and $\|x(\hat{t})\| \geq B_2$.

For $t \in [t_0 - r, \hat{t}]$ define

$$m(t) = V(t, x(t)). \tag{5.3.2}$$

By the piecewise continuity assumption on $V$ it follows that $m \in PC([t_0 - r, \hat{t}], \mathbb{R}_+)$ and $m(t)$ is continuous at each $t \neq \tau_k$ in $(t_0, \hat{t}]$. By (5.3.1) we have

$$\hat{b}(\|x(t)\|) \leq m(t) \leq \hat{a}(\|x(t)\|), \tag{5.3.3}$$

for $t \in [t_0 - r, \hat{t}]$. By condition (ii) and from Corollary 5.1.1, since $V$ is locally Lipschitz in $x$, we have

$$D^+m(t) = \limsup_{h\to 0^+} \frac{1}{h}[m(t+h) - m(t)] \le p(t)c(m(t)), \tag{5.3.4}$$

for all $t \ne \tau_k$ in $(t_0, \hat{t}]$ whenever $\|x(t)\| \ge \rho$ and $m(t) \ge g(\|m_t\|_r)$. Also by condition (iii) we have for each $\tau_k \in (t_0, \hat{t}]$,

$$m(\tau_k) \le g(m(\tau_k^-)), \tag{5.3.5}$$

if $\|x(\tau_k^-)\| \ge \rho$.

For $t \in [t_0 - r, t_0]$ we have $m(t) \le \hat{a}(\|x(t)\|) \le \hat{a}(\|\phi\|_r) \le \hat{a}(B_1) < g(\hat{b}(B_2)) < \hat{b}(B_2)$. In particular $m(t_0) < \hat{b}(B_2)$. Moreover, $m(\hat{t}) \ge \hat{b}(\|x(\hat{t})\|) \ge \hat{b}(B_2)$. So define $t^* = \inf\{t \in [t_0, \hat{t}] \mid m(t) \ge \hat{b}(B_2)\}$. Then $t^* \in (t_0, \hat{t}]$ and $m(t) < \hat{b}(B_2)$ for $t \in [t_0 - r, t^*)$.

We claim that $m(t^*) = \hat{b}(B_2)$ and that $t^* \ne \tau_k$ for any $k$. Clearly we must have $m(t^*) \ge \hat{b}(B_2)$. Suppose $t^* = \tau_k$ for some $k$. If $\|x(t^{*-})\| \ge \rho$, then $\hat{b}(B_2) \le m(t^*) \le g(m(t^{*-})) < m(t^{*-}) \le \hat{b}(B_2)$ by (5.3.5) which is impossible. On the other hand if $\|x(t^{*-})\| < \rho$, then $\|x(t^*)\| = \|x(t^{*-}) + I(t^*, x_{t^{*-}})\| \le \rho_1 \le B_1$ by condition (iv) and so $\hat{b}(B_2) \le m(t^*) \le \hat{a}(B_1) < g(\hat{b}(B_2)) < \hat{b}(B_2)$, which is again a contradiction. Therefore we may conclude that $t^* \ne \tau_k$ for all $k$, which in turn implies that $m(t^*) = \hat{b}(B_2)$ since $m(t)$ is continuous at $t^*$.

Now let us first consider the case where $\tau_{l-1} \le t_0 < t^* < \tau_l$. Let $\bar{t} = \sup\{t \in [t_0, t^*] \mid m(t) \le g(\hat{b}(B_2))\}$. Since $m(t_0) < g(\hat{b}(B_2))$, $m(t^*) = \hat{b}(B_2) > g(\hat{b}(B_2))$ and $m(t)$ is continuous on $[t_0, t^*]$, then $\bar{t} \in (t_0, t^*)$, $m(\bar{t}) = g(\hat{b}(B_2))$ and $m(t) \ge g(\hat{b}(B_2))$ for $t \in [\bar{t}, t^*]$.

Hence for $t \in [\bar{t}, t^*]$ and $s \in [-r, 0]$, we have $g(m(t+s)) \le g(\hat{b}(B_2)) \le m(t)$ or in other words $m(t) \ge g(\|m_t\|_r)$. Also since $\hat{a}(B_1) < g(\hat{b}(B_2)) \le m(t) \le \hat{a}(\|x(t)\|)$ for $t \in [\bar{t}, t^*]$, then $\|x(t)\| \ge B_1 \ge \rho$ for $t \in [\bar{t}, t^*]$. Thus inequality (5.3.4) holds for all $t \in (\bar{t}, t^*]$. Integrating this differential inequality gives us

$$\int_{m(\bar{t})}^{m(t^*)} \frac{ds}{c(s)} \le \int_{\bar{t}}^{t^*} p(s)ds \le \int_{\bar{t}}^{\bar{t}+r} p(s)ds \le M_1. \tag{5.3.6}$$

However, we also have

$$\int_{m(\bar{t})}^{m(t^*)} \frac{ds}{c(s)} = \int_{g(\hat{b}(B_2))}^{\hat{b}(B_2)} \frac{ds}{c(s)} \ge M_2. \tag{5.3.7}$$

Since we have assumed that $M_2 > M_1$, then we arrive at our desired contradiction.

Alternatively, suppose that $\tau_k < t^* < \tau_{k+1}$ for some $k \ge l$. If $\|x(\tau_k^-)\| \ge \rho$, then $m(\tau_k) \le g(m(\tau_k^-)) \le g(\hat{b}(B_2))$ by (5.3.5). Otherwise if $\|x(\tau_k^-)\| < \rho$, then $\|x(\tau_k)\| = \|x(\tau_k^-) + I(\tau_k, x_{\tau_k^-})\| \le \rho_1 \le B_1$ by condition (iv) and so $m(\tau_k) \le \hat{a}(B_1) < g(\hat{b}(B_2))$. Thus $m(\tau_k) \le g(\hat{b}(B_2))$ in either case.

Similar to before define $\bar{t} = \sup\{t \in [\tau_k, t^*] \mid m(t) \leq g(\hat{b}(B_2))\}$. Then $\bar{t} \in [\tau_k, t^*)$, $m(\bar{t}) = g(\hat{b}(B_2))$ and $m(t) \geq g(\hat{b}(B_2))$ for $t \in [\bar{t}, t^*]$. Applying exactly the same argument as before yields a contradiction.

So in either case we obtain a contradiction, which proves that solutions of (2.4.7) are uniformly bounded. Now we show that they are uniformly ultimately bounded.

Since solutions of (2.4.7) are uniformly bounded, then there exists some $B > \rho_1$ satisfying $\hat{a}(\rho_1) < g(\hat{b}(B))$ such that if $\|\phi\|_r \leq \rho_1$, then $\|x(t, t_0, \phi)\| \leq B$ for all $t \geq t_0 - r$ where $x = x(t_0, \phi)$ is any solution of (2.4.7) & (2.4.2).

Now let $B_3 > 0$ and assume without loss of generality that $B_3 > B$. Then from the proof of uniform boundedness, there exists some $B_2 = B_2(B_3) > B_3$ for which if $\|\phi\|_r \leq B_3$, then $\|x(t, t_0, \phi)\| \leq B_2$ for all $t \geq t_0 - r$ where $x = x(t_0, \phi)$ is any solution of (2.4.7) & (2.4.2).

Define

$$M = M(B_3) = \sup\left\{\frac{1}{c(s)} \;\middle|\; g(\hat{b}(B)) \leq s \leq \hat{a}(B_2)\right\}, \tag{5.3.8}$$

and note that $0 < M < \infty$.

For $\hat{b}(B) \leq q \leq \hat{a}(B_2)$ we have $g(\hat{b}(B)) \leq g(q) < q \leq \hat{a}(B_2)$ and so

$$M_2 \leq \int_{g(q)}^{q} \frac{ds}{c(s)} \leq M[q - g(q)], \tag{5.3.9}$$

from which we obtain $g(q) \leq q - M_2/M < q - d$ where $d = d(B_3) > 0$ is chosen so that $d < (M_2 - M_1)/M$.

Let $N = N(B_3)$ be the smallest positive integer for which $\hat{a}(B_2) \leq \hat{b}(B) + Nd$ and define $T = T(B_3) = \tau + (r + \tau)(N - 1)$. Given a solution $x = x(t_0, \phi)$ of (2.4.7) & (2.4.2) where $\|\phi\|_r \leq B_3$ and $t_0 \in [\tau_{l-1}, \tau_l)$ for some positive integer $l$, we will show that $\|x(t)\| \leq B$ for $t \geq t_0 + T$. Let

$$m(t) = V(t, x(t)), \tag{5.3.10}$$

for $t \geq t_0 - r$. Then $m(t) \leq \hat{a}(B_2)$ for $t \geq t_0 - r$.

Given $\hat{b}(B) \leq A \leq \hat{a}(B_2)$ and $j \geq l$ we will show that if $m(t) \leq A$ for $t \in [\tau_j - r, \tau_j)$, then $m(t) \leq A - d$ for $t \geq \tau_j$.

Suppose for the sake of contradiction that there exists some $t \geq \tau_j$ for which $m(t) > A - d$ and define $t^* = \inf\{t \geq \tau_j \mid m(t) > A - d\}$. Then $t^* \in [\tau_k, \tau_{k+1})$ for some $k \geq j$, $m(t) \leq A$ for $t \in [\tau_j - r, t^*)$ and $m(t^*) \geq A - d$. If $\|x(\tau_k^-)\| \geq \rho$, then $m(\tau_k) \leq g(m(\tau_k^-)) \leq g(A)$. However, if $\|x(\tau_k^-)\| < \rho$, then $\|x(\tau_k)\| \leq \rho_1$ and so $m(\tau_k) \leq \hat{a}(\rho_1) < g(\hat{b}(B)) \leq g(A)$. Thus in either case $m(\tau_k) \leq g(A)$.

Now since $\hat{b}(B) \leq A \leq \hat{a}(B_2)$, then $g(A) < A - d$ and so $m(\tau_k) < A - d$. This means that $t^* \neq \tau_k$ since $m(t^*) \geq A - d$. Thus $t^* \in (\tau_k, \tau_{k+1})$ and $m(t^*) = A - d$ since $m(t)$ is continuous at $t^*$. Also for $t \in [\tau_k, t^*]$ we have $m(t) \le A - d$.

Define $\bar{t} = \sup\{t \in [\tau_k, t^*] \mid m(t) \le g(A)\}$. Since $m(t^*) = A - d > g(A) \ge m(\tau_k)$, then $\bar{t} \in [\tau_k, t^*)$, $m(\bar{t}) = g(A)$ and $m(t) \ge g(A)$ for $t \in [\bar{t}, t^*]$. Therefore if $t \in [\bar{t}, t^*]$ and $s \in [-r, 0]$, then $g(m(t+s)) \le g(A) \le m(t)$. Also since for all such $t$ we have $\hat{a}(\|x(t)\|) \ge m(t) \ge g(A) \ge g(\hat{b}(B)) > \hat{a}(\rho_1)$, then $\|x(t)\| \ge \rho_1 \ge \rho$ for $t \in [\bar{t}, t^*]$. Thus inequality (5.3.4) holds for all $t \in (\bar{t}, t^*]$, which in turn means that inequality (5.3.6) is satisfied.

However,

$$\int_{m(\bar{t})}^{m(t^*)} \frac{ds}{c(s)} = \int_{g(A)}^{A-d} \frac{ds}{c(s)} = \int_{g(A)}^{A} \frac{ds}{c(s)} - \int_{A-d}^{A} \frac{ds}{c(s)}. \tag{5.3.11}$$

Since $\hat{b}(B) \le A \le \hat{a}(B_2)$, then $g(\hat{b}(B)) \le g(A) < A - d < A \le \hat{a}(B_2)$ and so $1/c(s) \le M$ for $A - d \le s \le A$. Thus from (5.3.11) we get

$$\int_{m(\bar{t})}^{m(t^*)} \frac{ds}{c(s)} \ge M_2 - \int_{A-d}^{A} M\,ds = M_2 - dM > M_2 + (M_1 - M_2) = M_1, \tag{5.3.12}$$

which contradicts (5.3.6) and proves that $m(t) \le A - d$ for all $t \ge \tau_j$.

As in the proof of Theorem 5.2.1 we define the indices $k^{(i)}$ for $i = 1, 2, \ldots, N$ by letting $k^{(1)} = l$ and, for $i = 2, \ldots, N$, we let $k^{(i)}$ be chosen so that $\tau_{k^{(i)}-1} < \tau_{k^{(i-1)}} + r \le \tau_{k^{(i)}}$.

Then $\tau_{k^{(1)}} = \tau_l \le t_0 + \tau$ and for $i = 2, \ldots, N$ we have $\tau_{k^{(i)}} \le \tau_{k^{(i)}-1} + \tau \le \tau_{k^{(i-1)}} + r + \tau$. Combining these inequalities gives us $\tau_{k^{(N)}} \le t_0 + \tau + (r + \tau)(N - 1) = t_0 + T$.

We claim that for each $i = 1, 2, \ldots, N$, $m(t) \le \hat{a}(B_2) - id$ for $t \ge \tau_{k^{(i)}}$. Since $m(t) \le \hat{a}(B_2)$ for $t \in [t_0 - r, \tau_{k^{(1)}})$, then by setting $A = \hat{a}(B_2)$ in our earlier argument we get $m(t) \le \hat{a}(B_2) - d$ for $t \ge \tau_{k^{(1)}}$, which establishes the base case. We now proceed by induction and assume $m(t) \le \hat{a}(B_2) - jd$ for $t \ge \tau_{k^{(j)}}$ for some $1 \le j \le N - 1$. Let $A = \hat{a}(B_2) - jd$. Since $\tau_{k^{(j)}} \le \tau_{k^{(j+1)}} - r$, then $m(t) \le A$ for $t \in [\tau_{k^{(j+1)}} - r, \tau_{k^{(j+1)}})$ and so $m(t) \le A - d = \hat{a}(B_2) - (j + 1)d$ for $t \ge \tau_{k^{(j+1)}}$. Thus we have proven our claim by induction.

In particular we have $m(t) \le \hat{a}(B_2) - Nd \le \hat{b}(B)$ for $t \ge t_0 + T \ge \tau_{k^{(N)}}$. Finally, by (5.3.3) we get that $\|x(t)\| \le B$ for $t \ge t_0 + T$, which completes the proof of this theorem. ∎

In Chapter 8 we will show how Theorem 5.3.1 can be used to prove boundedness and persistence of certain population growth models that involve delays and impulses. Meanwhile, we introduce our second boundedness theorem of this section, which is modelled after the stability Theorem 5.2.2 of the previous section.

**Theorem 5.3.2:** *Assume $J = \mathbf{R}_+$, $D = \mathbf{R}^n$ and the conditions of Corollary 3.2.3 are satisfied. Suppose there exists a constant $\rho > 0$ and functions $a \in C(\mathbf{R}_+, \mathbf{R}_+)$, $b \in K_4$, $c \in K_1$, $p \in PC(\mathbf{R}_+, \mathbf{R}_+)$ and $g, \hat{g} \in K_3$ with $s \le \hat{g}(s) < g(s)$ for all $s > 0$. Suppose $V : [-r, \infty) \times \mathbf{R}^n \to \mathbf{R}_+$ is continuous on $[-r, \tau_0) \times \mathbf{R}^n$ and on $[\tau_{k-1}, \tau_k) \times \mathbf{R}^n$ for $k = 1, 2, \ldots$ and that for each $x \in \mathbf{R}^n$*

*and* $k = 0, 1, 2, \ldots$, $\lim_{(t,y)\to(\tau_k^-,x)} V(t, y) = V(\tau_k^-, x)$ *exists. Moreover assume that* $V$, *restricted to* $\mathbb{R}_+ \times \mathbb{R}^n$, *is locally Lipschitz in* $x$ *and that the following conditions are satisfied:*

*(i)* $b(\|x\|) \le V(t, x) \le a(\|x\|)$ *for all* $(t, x) \in [-r, \infty) \times \mathbb{R}^n$;

*(ii)* $D^+_{(2.4.7)}V(t, \psi(0)) \le -p(t)c(V(t, \psi(0)))$ *for all* $t \ne \tau_k$ *in* $\mathbb{R}_+$ *and* $\psi \in PC([-r, 0], \mathbb{R}^n)$ *whenever* $\|\psi(0)\| \ge \rho$ *and* $g(V(t, \psi(0))) \ge V(t + s, \psi(s))$ *for* $s \in [-r, 0]$;

*(iii)* $V(\tau_k, \psi(0) + I(\tau_k, \psi)) \le \hat{g}(V(\tau_k^-, \psi(0)))$ *for all* $(\tau_k, \psi) \in \mathbb{R}_+ \times PC([-r, 0], \mathbb{R}^n)$ *for which* $\psi(0^-) = \psi(0)$ *and* $\|\psi(0)\| \ge \rho$;

*(iv) there exists some* $\rho_1 \ge \rho$ *such that if* $(\tau_k, \psi) \in \mathbb{R}_+ \times PC([-r, 0], \mathbb{R}^n)$, $\psi(0^-) = \psi(0)$ *and* $\|\psi(0)\| \le \rho$, *then* $\|\psi(0) + I(\tau_k, \psi)\| \le \rho_1$; *and*

*(v)* $\mu = \inf_{k\in\mathbb{Z}^+}\{\tau_k - \tau_{k-1}\} > 0$,

$$M_2 = \sup_{q>0} \int_q^{g(q)} \frac{ds}{c(s)}, \text{ and}$$

$$M_1 = \inf_{t\ge 0} \int_t^{t+\mu} p(s)ds > M_2.$$

*Then solutions of (2.4.7) are uniformly ultimately bounded.*

*Proof:* As in the proof of Theorem 5.3.1, let $\hat{b}$ and $\hat{a}$ be continuous, strictly increasing functions satisfying $\lim_{s\to\infty} \hat{b}(s) = \lim_{s\to\infty} \hat{a}(s) = \infty$ and $\hat{b}(s) \le b(s) \le a(s) \le \hat{a}(s)$ for all $s \in \mathbb{R}_+$. Condition (i) then implies

$$\hat{b}(\|x\|) \le V(t, x) \le \hat{a}(\|x\|), \tag{5.3.13}$$

for all $(t, x) \in [-r, \infty) \times \mathbb{R}^n$.

We first show uniform boundedness. Let $B_1 > 0$ and assume, without loss of generality, that $B_1 \ge \rho_1 \ge \rho > 0$. Choose $B_2 = B_2(B_1)$ so that $g(\hat{a}(B_1)) < \hat{b}(B_2)$. Then $B_2 > B_1$.

Let $(t_0, \phi) \in \mathbb{R}_+ \times PC([-r, 0], \mathbb{R}^n)$ where $\|\phi\|_r \le B_1$ and $t_0 \in [\tau_{l-1}, \tau_l)$ for some positive integer $l$. Suppose $x = x(t_0, \phi)$ is a solution of (2.4.7) & (2.4.2) having a maximal interval of existence $[t_0 - r, t_0 + \beta)$. If $\beta < \infty$, then from Corollary 3.2.3 it follows that $\|x(t)\| > B_2$ for some $t \in (t_0, t_0 + \beta)$. We will prove that $\|x(t)\| \le B_2$ for $t \in [t_0, t_0 + \beta)$. This will show that $\beta = \infty$ and that solutions of (2.4.7) are uniformly bounded.

Suppose for the sake of contradiction that $\|x(t)\| > B_2$ for some $t \in [t_0, t_0 + \beta)$. Then let $\hat{t} = \inf\{t \in [t_0, t_0 + \beta) \mid \|x(t)\| > B_2\}$. Note that $\|x(t)\| \le \|\phi\|_r \le B_1 < B_2$ for $t \in [t_0 - r, t_0]$ and in particular $\|x(t_0)\| < B_2$. Clearly $\hat{t} \in (t_0, t_0 + \beta)$, $\|x(t)\| \le B_2$ for $t \in [t_0 - r, \hat{t})$ and $\|x(\hat{t})\| \ge B_2$.

For $t \in [t_0 - r, \hat{t}]$ define

$$m(t) = V(t, x(t)). \tag{5.3.14}$$

Since $V$ is assumed to be piecewise continuous, then $m \in PC([t_0 - r, \hat{t}], \mathbb{R}_+)$ and $m(t)$ is continuous at each $t \neq \tau_k$ in $(t_0, \hat{t}]$. By (5.3.13) we have

$$\hat{b}(\|x(t)\|) \leq m(t) \leq \hat{a}(\|x(t)\|), \tag{5.3.15}$$

for $t \in [t_0 - r, \hat{t}]$. By condition (ii) and from Corollary 5.1.1, since $V$ is locally Lipschitz in $x$, we have

$$D^+ m(t) = \limsup_{h \to 0^+} \frac{1}{h}\left[m(t+h) - m(t)\right] \leq -p(t)c(m(t)), \tag{5.3.16}$$

for all $t \neq \tau_k$ in $(t_0, \hat{t}]$ whenever $\|x(t)\| \geq \rho$ and $m(t) \geq g(\|m_t\|_r)$. Also by condition (iii) we have for each $\tau_k \in (t_0, \hat{t}]$,

$$m(\tau_k) \leq \hat{g}(m(\tau_k^-)), \tag{5.3.17}$$

if $\|x(\tau_k^-)\| \geq \rho$. Finally, if $\tau_k \in (t_0, \hat{t}]$ and $\|x(\tau_k^-)\| \leq \rho$, then from condition (iv) we get $\|x(\tau_k)\| = \|x(\tau_k^-) + I(\tau_k, x_{\tau_k^-})\| \leq \rho_1$, which implies, by (5.3.15), that

$$m(\tau_k) \leq \hat{a}(\rho_1). \tag{5.3.18}$$

For $t \in [t_0 - r, t_0]$ we have $m(t) \leq \hat{a}(\|x(t)\|) \leq \hat{a}(\|\phi\|_r) \leq \hat{a}(B_1)$, which implies $m(t) \leq g(m(t)) \leq g(\hat{a}(B_1)) < \hat{b}(B_2)$. In particular $m(t_0) \leq g(m(t_0)) < \hat{b}(B_2)$. Also $m(\hat{t}) \geq \hat{b}(\|x(\hat{t})\|) \geq \hat{b}(B_2)$. So define $t^* = \inf\{t \in [t_0, \hat{t}] \mid m(t) \geq \hat{b}(B_2)\}$. Then $t^* \in (t_0, \hat{t}]$, $m(t^*) \geq \hat{b}(B_2)$ and $m(t) < \hat{b}(B_2)$ for $t \in [t_0 - r, t^*)$.

Consider first the case when $\tau_{l-1} \leq t_0 < t^* < \tau_l$. Then $m(t^*) = \hat{b}(B_2)$ and $g(m(t^*)) = g(\hat{b}(B_2)) > \hat{b}(B_2)$. So define $\bar{t} = \sup\{t \in [t_0, t^*] \mid g(m(t)) \leq \hat{b}(B_2)\}$, noting that $\bar{t} \in (t_0, t^*)$, $g(m(\bar{t})) = \hat{b}(B_2)$ and $g(m(t)) \geq \hat{b}(B_2)$ for $t \in [\bar{t}, t^*]$. Therefore for any $t \in [\bar{t}, t^*]$ and $s \in [-r, 0]$ we have $m(t+s) \leq \hat{b}(B_2) \leq g(m(t))$, implying $\|m_t\|_r \leq g(m(t))$ and $g(\hat{a}(B_1)) < \hat{b}(B_2) \leq g(m(t))$. This final inequality implies $m(t) \geq \hat{a}(B_1)$, which in light of (5.3.15), implies $\|x(t)\| \geq B_1 \geq \rho$. Therefore the differential inequality (5.3.16) holds for all $t \in [\bar{t}, t^*]$. But this means that $m(t)$ is nonincreasing on this interval. Thus $m(t^*) \leq m(\bar{t})$ and so $g(\hat{b}(B_2)) = g(m(t^*)) \leq g(m(\bar{t})) = \hat{b}(B_2) < g(\hat{b}(B_2))$ giving us our desired contradiction, which proves this case.

Alternatively suppose $\tau_k \leq t^* < \tau_{k+1}$ for some $k \geq l$. In order to get a contradiction for this case as well we must first show that $g(m(\tau_k^-)) \leq \hat{b}(B_2)$ and to do this we look at the cases $k = l$ and $k > l$ separately.

First suppose $k = l$. To prove $g(m(\tau_l^-)) \leq \hat{b}(B_2)$ we assume, for the sake of contradiction, that $g(m(\tau_l^-)) > \hat{b}(B_2)$. Define $\bar{t} = \sup\{t \in [t_0, \tau_l) \mid g(m(t)) \leq \hat{b}(B_2)\}$. Then $\bar{t} \in (t_0, \tau_l)$, $g(m(\bar{t})) = \hat{b}(B_2)$ and $g(m(t)) \geq \hat{b}(B_2) \geq m(t+s)$ for $t \in [\bar{t}, \tau_l)$ and $s \in [-r, 0]$. As before this also implies $\|x(t)\| \geq B_1 \geq \rho$ for $t \in [\bar{t}, \tau_l)$. Since (5.3.16) holds for all $t \in [\bar{t}, \tau_l)$, then $m(\tau_l^-) \leq m(\bar{t})$, which implies $\hat{b}(B_2) < g(m(\tau_l^-)) \le g(m(\bar{t})) = \hat{b}(B_2)$ giving us our contradiction. This proves then that $g(m(\tau_l^-)) \le \hat{b}(B_2)$.

Now consider the case $k > l$ and suppose $g(m(\tau_k^-)) > \hat{b}(B_2)$. Either $g(m(t)) > \hat{b}(B_2)$ for all $t \in [\tau_{k-1}, \tau_k)$ or there exists some $t \in [\tau_{k-1}, \tau_k)$ for which $g(m(t)) \le \hat{b}(B_2)$. In the first case $g(m(t)) > \hat{b}(B_2) \ge m(t+s)$ and $\|x(t)\| \ge \rho$ for all $t \in [\tau_{k-1}, \tau_k)$ and $s \in [-r, 0]$. Since inequality (5.3.16) is satisfied for all $t \in (\tau_{k-1}, \tau_k)$ we may integrate it to give us

$$\int_{m(\tau_k^-)}^{m(\tau_{k-1})} \frac{ds}{c(s)} \ge \int_{\tau_{k-1}}^{\tau_k} p(s)ds \ge \int_{\tau_{k-1}}^{\tau_{k-1}+\mu} p(s)ds \ge M_1. \tag{5.3.19}$$

However,

$$\int_{m(\tau_k^-)}^{m(\tau_{k-1})} \frac{ds}{c(s)} \le \int_{m(\tau_k^-)}^{\hat{b}(B_2)} \frac{ds}{c(s)} \le \int_{m(\tau_k^-)}^{g(m(\tau_k^-))} \frac{ds}{c(s)} \le M_2 < M_1, \tag{5.3.20}$$

giving us our contradiction.

In the second case let $\bar{t} = \sup\{t \in [\tau_{k-1}, \tau_k) \mid g(m(t)) \le \hat{b}(B_2)\}$. Then $\bar{t} \in [\tau_{k-1}, \tau_k)$, $g(m(\bar{t})) = \hat{b}(B_2)$ and for $t \in [\bar{t}, \tau_k)$ and $s \in [-r, 0]$, $g(m(t)) \ge \hat{b}(B_2) \ge m(t+s)$ and $\|x(t)\| \ge \rho$. Therefore from (5.3.16) we get $m(\bar{t}) \ge m(\tau_k^-)$ implying $\hat{b}(B_2) < g(m(\tau_k^-)) \le g(m(\bar{t})) = \hat{b}(B_2)$ giving us our contradiction, which verifies the second case. This in turn completes the proof of the case $k > l$.

We have proven that $g(m(\tau_k^-)) \le \hat{b}(B_2)$ when $\tau_k \le t^* < \tau_{k+1}$ and $k \ge l$. If $\|x(\tau_k^-)\| \ge \rho$, then $m(\tau_k) \le \hat{g}(m(\tau_k^-)) < \hat{b}(B_2)$ by (5.3.17) since $\hat{g}(s) < g(s)$ for $s > 0$. On the other hand, if $\|x(\tau_k^-)\| < \rho$, then $m(\tau_k) \le \hat{a}(\rho_1) \le \hat{a}(B_1) < g(\hat{a}(B_1)) < \hat{b}(B_2)$ by (5.3.18). So in either case $m(\tau_k) < \hat{b}(B_2)$. Since $m(t^*) \ge \hat{b}(B_2)$, then $t^* \in (\tau_k, \tau_{k+1})$ and $m(t^*) = \hat{b}(B_2)$. If $g(m(t)) > \hat{b}(B_2)$ for $t \in [\tau_k, t^*]$, then let $\bar{t} = \tau_k$ and note that $m(\bar{t}) < \hat{b}(B_2)$. Otherwise let $\bar{t} = \sup\{t \in [\tau_k, t^*] \mid g(m(t)) \le \hat{b}(B_2)\}$ and note that $m(\bar{t}) < g(m(\bar{t})) = \hat{b}(B_2)$. Since $m(t^*) = \hat{b}(B_2)$, then $\bar{t} \in [\tau_k, t^*)$. Moreover, $g(m(t)) \ge \hat{b}(B_2) \ge m(t+s)$ and $\|x(t)\| \ge \rho$ for $t \in [\bar{t}, t^*]$ and $s \in [-r, 0]$. Thus by (5.3.16) we know $m(\bar{t}) \ge m(t^*)$, but this contradicts the fact that $m(\bar{t}) < \hat{b}(B_2) = m(t^*)$. This final contradiction completes the proof of uniform boundedness.

Since solutions of (2.4.7) are uniformly bounded, then there exists some $B > \rho_1$ satisfying $g(\hat{a}(\rho_1)) < \hat{b}(B)$ such that if $\|\phi\|_r \le \rho_1$, then $\|x(t, t_0, \phi)\| \le B$ for all $t \ge t_0 - r$ where $x = x(t_0, \phi)$ is any solution of (2.4.7) & (2.4.2).

Now let $B_3 > 0$ and assume without loss of generality that $B_3 > B$. Then from the proof of uniform boundedness, there exists some $B_2 = B_2(B_3) > B_3$ satisfying $g(\hat{a}(B_3)) < \hat{b}(B_2)$ for which if $\|\phi\|_r \le B_3$, then $\|x(t, t_0, \phi)\| \le B_2$ for all $t \ge t_0 - r$ where $x = x(t_0, \phi)$ is any solution of (2.4.7) & (2.4.2).

Let $\hat{d} = \hat{d}(B_3) = \inf\{g(s) - \hat{g}(s) \mid g(s) \ge \hat{b}(B)/2 \text{ and } s \le \hat{a}(B_2)\}$ and choose $d = d(B_3) > 0$ such that $d < \min\{\hat{a}(B_2) - \hat{b}(B), \hat{d}, \hat{b}(B)/2, \hat{b}(B) - \hat{a}(\rho_1)\}$.

Let $N = N(B_3)$ be the smallest positive integer for which $\hat{a}(B_2) \le \hat{b}(B) + Nd$. As in the proof of Theorem 5.2.2 we may assume, without loss of generality, that $\tau = \sup_{k\in\mathbb{Z}^+}\{\tau_k - \tau_{k-1}\} < \infty$. Define $T(B_3) = 2\tau + (r+3\tau)(N-1)$.

Let $x = x(t_0, \phi)$ be a solution of (2.4.7) & (2.4.2) where $\|\phi\|_r \le B_3$ and $t_0 \in [\tau_{l-1}, \tau_l)$ for some positive integer $l$. We will show that $\|x(t)\| \le B$ for $t \ge t_0 + T$ using essentially the same arguments as was used in the proof of Theorem 5.2.2. Define $m(t)$ for $t \ge t_0 - r$ as in (5.3.14). Then $m(t) \le \hat{a}(B_2)$ for $t \ge t_0 - r$. In fact $m(t) \le \hat{b}(B_2)$ for $t \ge t_0 - r$, lest we get a contradiction as we did in the proof of uniform boundedness.

We prove that $g(m(\tau_l^-)) \le \hat{b}(B_2)$ in the same way we proved it earlier in the case of uniform boundedness, only this time $B_1$ is replaced by $B_3$. From this we get $g(m(\tau_l^-)) \le \hat{a}(B_2)$.

Given $\hat{b}(B) \le A \le \hat{a}(B_2)$ and $j \ge l$ we will show that if $m(t) \le A$ for $t \in [\tau_j - r, \tau_j)$ and $g(m(\tau_j^-)) \le A$, then $m(t) \le A$ for $t \in [\tau_j, \tau_{j+1})$ and $g(m(\tau_{j+1}^-)) \le A$.

Suppose for the sake of contradiction that there exists some $t \in [\tau_j, \tau_{j+1})$ for which $m(t) > A$. Let $t^* = \inf\{t \in [\tau_j, \tau_{j+1}) \mid m(t) \ge A\}$. If $\|x(\tau_j^-)\| \ge \rho$, then $m(\tau_j) \le \hat{g}(m(\tau_j^-)) < A$ by (5.3.17), while if $\|x(\tau_j^-)\| < \rho$, then $m(\tau_j) \le \hat{a}(\rho_1) < g(\hat{a}(\rho_1)) < \hat{b}(B) \le A$ by (5.3.18). So in either case $m(\tau_j) < A$. Thus $t^* \in (\tau_j, \tau_{j+1})$, $m(t^*) = A$ and $m(t) \le A$ for $t \in [\tau_j - r, t^*]$. If $g(m(t)) > A$ for $t \in [\tau_j, t^*]$, then let $\bar{t} = \tau_j$ and note that $m(\bar{t}) < A$. Otherwise let $\bar{t} = \sup\{t \in [\tau_j, t^*] \mid g(m(t)) \le A\}$ and note that $m(\bar{t}) < g(m(\bar{t})) = A$. Since $g(m(t^*)) = g(A) > A$, then $\bar{t} \in [\tau_j, t^*)$. For $t \in [\bar{t}, t^*]$ and $s \in [-r, 0]$ we have $g(m(t)) \ge A \ge m(t+s)$. Moreover for $t \in [\bar{t}, t^*]$ we have $g(\hat{a}(\|x(t)\|)) \ge g(m(t)) \ge A \ge \hat{b}(B) > g(\hat{a}(\rho_1))$ implying $\|x(t)\| \ge \rho_1 \ge \rho$. Thus from (5.3.16) we know that $m(\bar{t}) \ge m(t^*)$, which contradicts the fact that $m(t^*) = A > m(\bar{t})$.

Now that we have proven $m(t) \le A$ for $t \in [\tau_j, \tau_{j+1})$ we must show that $g(m(\tau_{j+1}^-)) \le A$. The proof of this follows in an identical fashion to the arguments used earlier in the proof of uniform boundedness where we showed that $g(m(\tau_k^-)) \le \hat{b}(B_2)$ for the case $k > l$, if we replace $k$ by $j+1$ and $\hat{b}(B_2)$ by $A$.

Having proven our claim we may argue by induction that in fact $m(t) \le A$ for all $t \ge \tau_j$ and that $g(m(\tau_k^-)) \le A$ for all $k \ge j+1$. Next we claim that $m(t) \le A - d$ for $t \in [\tau_{j+1}, \tau_{j+2})$ but to show this we must first show $m(\tau_{j+1}) < A - d$.

If $\|x(\tau_{j+1}^-)\| \le \rho$, then $m(\tau_{j+1}) \le \hat{a}(\rho_1) < \hat{b}(B) - d \le A - d$. On the other hand, if $\|x(\tau_{j+1}^-)\| > \rho$, then we look at two cases. If $0 \le g(m(\tau_{j+1}^-)) \le \hat{b}(B)/2$, then $m(\tau_{j+1}) \le \hat{g}(m(\tau_{j+1}^-)) \le g(m(\tau_{j+1}^-)) \le \hat{b}(B)/2 \le A - \hat{b}(B)/2 < A - d$ since $A \ge \hat{b}(B)$ and $d < \hat{b}(B)/2$. Alternatively, suppose $g(m(\tau_{j+1}^-)) > \hat{b}(B)/2$. Recall that $g(m(\tau_{j+1}^-)) \le A$. Since $g(m(\tau_{j+1}^-)) \ge \hat{b}(B)/2$ and $m(\tau_{j+1}^-) \le g(m(\tau_{j+1}^-)) \le A \le \hat{a}(B_2)$, then $\hat{g}(m(\tau_{j+1}^-)) \le g(m(\tau_{j+1}^-)) - \hat{d} < g(m(\tau_{j+1}^-)) - d \le A - d$ from the definition of $\hat{d}$ and so $m(\tau_{j+1}) \le \hat{g}(m(\tau_{j+1}^-)) < A - d$. Therefore in all cases we have shown $m(\tau_{j+1}) < A - d$.

In order to prove $m(t) \le A - d$ for all $t \in [\tau_{j+1}, \tau_{j+2})$ we assume for the sake of contradiction that $m(t) > A - d$ for some $t \in [\tau_{j+1}, \tau_{j+2})$. Let $t^* = \inf\{t \in [\tau_{j+1}, \tau_{j+2}) \mid m(t) \ge A - d\}$. Then $t^* \in (\tau_{j+1}, \tau_{j+2})$, since $m(\tau_{j+1}) < A - d$. Moreover, $m(t^*) = A - d$ and $m(t) \le A - d$ for $t \in [\tau_{j+1}, t^*]$. Note that since $g(A-d) \ge A - d \ge \hat{b}(B)/2$ and $A - d \le A \le \hat{a}(B_2)$, then $g(A-d) \ge \hat{g}(A-d) + \hat{d} >$

$A - d + d = A$. Therefore $g(m(t^*)) = g(A - d) > A$. If $g(m(t)) > A$ for all $t \in [\tau_{j+1}, t^*]$, then let $\bar{t} = \tau_{j+1}$ and note that $m(\bar{t}) < A - d$. Otherwise let $\bar{t} = \sup\{t \in [\tau_{j+1}, t^*] \mid g(m(t)) \leq A\}$ and note that $m(\bar{t}) \leq \hat{g}(m(\bar{t})) \leq g(m(\bar{t})) - \hat{d} = A - \hat{d} < A - d$ since $g(m(\bar{t})) = A \geq \hat{b}(B)/2$ and $m(\bar{t}) \leq A \leq \hat{a}(B_2)$. Clearly $\bar{t} \in [\tau_{j+1}, t^*)$ since $g(m(t^*)) > A$. Thus for $t \in [\bar{t}, t^*]$ and $s \in [-r, 0]$, $g(m(t)) \geq A \geq m(t+s)$, and consequently $\|x(t)\| \geq \rho$, which implies by (5.3.16) that $m(\bar{t}) \geq m(t^*)$. This, however, contradicts the fact that $m(t^*) = A - d > m(\bar{t})$.

Having proven that $m(t) \leq A - d$ for $t \in [\tau_{j+1}, \tau_{j+2})$ we can then claim that in fact $m(t) \leq A - d$ for all $t \geq \tau_{j+1}$ by applying our results to successive intervals of the form $[\tau_k, \tau_{k+1})$ for $k \geq j+1$.

As in Theorem 5.2.2 we define the indices $k^{(i)}$ for $i = 1, 2, \ldots, N$ recursively as follows. Let $k^{(1)} = l + 1$ and for $i = 2, \ldots, N$ let $k^{(i)}$ satisfy $\tau_{k^{(i)}-3} < \tau_{k^{(i-1)}} + r \leq \tau_{k^{(i)}-2}$. Note that $\tau_{k^{(i-1)}} < \tau_{k^{(i-1)}} + r \leq \tau_{k^{(i)}-2}$ and so $k^{(i-1)} < k^{(i)} - 2$ implying $k^{(i)} \geq 3 + k^{(i-1)}$. Thus $\tau_{k^{(1)}} = \tau_{l+1} \leq 2\tau + \tau_{l-1} \leq 2\tau + t_0$ and for $i = 1, 2, \ldots, N$, $\tau_{k^{(i)}} \leq 3\tau + \tau_{k^{(i)}-3} \leq 3\tau + r + \tau_{k^{(i-1)}}$. Combining these inequalities gives $\tau_{k^{(N)}} \leq 2\tau + (3\tau + r)(N-1) + t_0 = t_0 + T$.

We claim that for each $i = 1, 2, \ldots, N$, $m(t) \leq \hat{a}(B_2) - id$ for $t \geq \tau_{k^{(i)}}$. Since $m(t) \leq \hat{a}(B_2)$ for $t \in [t_0 - r, \tau_l)$ and $g(m(\tau_l^-)) \leq \hat{a}(B_2)$, then by setting $A = \hat{a}(B_2)$ in our earlier argument we get $m(t) \leq \hat{a}(B_2) - d$ for $t \geq \tau_{l+1} = \tau_{k^{(1)}}$, which establishes the base case. We now proceed by induction and assume $m(t) \leq \hat{a}(B_2) - jd$ for $t \geq \tau_{k^{(j)}}$ for some $1 \leq j \leq N - 1$.

Let $A = \hat{a}(B_2) - jd$ and note that $A \geq \hat{b}(B)$. Since $\tau_{k^{(j+1)}-2} \geq \tau_{k^{(j)}} + r$, then $m(t) \leq A$ for $t \geq \tau_{k^{(j+1)}-2} - r$. Next we must show that $g(m(\tau^-_{k^{(j+1)}-1})) \leq A$. This is proven the same way we proved $g(m(\tau_k^-)) \leq \hat{b}(B_2)$ for the case $k > l$ in the proof of uniform boundedness only this time $k$ and $\hat{b}(B_2)$ are replaced by $k^{(j+1)} - 1$ and $A$, respectively. We may therefore conclude that $m(t) \leq A - d = \hat{a}(B_2) - (j+1)d$ for $t \geq \tau_{k^{(j+1)}}$, thus completing the induction step.

So in particular we have $m(t) \leq \hat{a}(B_2) - Nd \leq \hat{b}(B)$ for $t \geq t_0 + T \geq \tau_{k^{(N)}}$. Finally, by (5.3.15) we get that $\|x(t)\| \leq B$ for $t \geq t_0 + T$, which completes the proof of this theorem. ∎

The comments following Theorems 5.2.1 and 5.2.2 about how $M_1$ may be similarly redefined hold for Theorems 5.3.1 and 5.3.2.

We conclude this section with an example showing an application of Theorem 5.3.2.

**Example 5.3.1:** Consider the following impulsive delay differential equation

$$x'(t) = -4x^3(t) + e^{-t}x(t-1) + x^{1/3}(t-2) + \cos(t),\ t \neq \tau_k, \tag{5.3.21a}$$

$$\Delta x(t) = \frac{1}{10},\ t = \tau_k, \tag{5.3.21b}$$

where $\tau_k = k^2$ for $k = 0, 1, 2, \ldots$. The conditions of Corollary 3.2.3 are obviously satisfied.

Define

$$V(x) = x^4, \tag{5.3.22}$$

for $x \in \mathbb{R}$. This Lyapunov function clearly satisfies the smoothness assumptions of Theorem 5.3.2 as well as condition (i). For $(t, \psi) \in \mathbb{R}_+ \times PC([-2, 0], \mathbb{R})$ we have

$$\begin{aligned} D^+_{(5.3.21)}V(t, \psi) &= 4\psi^3(0)\left(-4\psi^3(0) + e^{-t}\psi(-1) + \psi^{1/3}(-2) + \cos(t)\right) \\ &\leq -16|\psi(0)|^6 + 4|\psi(0)|^3|\psi(-1)| + 4|\psi(0)|^3|\psi(-2)|^{1/3} + 4|\psi(0)|^3. \end{aligned} \tag{5.3.23}$$

Now let $\rho = 1$ and $g(s) = 3s$. Then whenever $|\psi(0)| \geq \rho$ and $V(\psi(s)) \leq g(V(\psi(0)))$ for $s \in [-2, 0]$, we have $|\psi(0)| \geq 1$ and $|\psi(s)| \leq 3^{1/4}|\psi(0)|$ for $s \in [-2, 0]$, and so from (5.3.23) we get

$$\begin{aligned} D^+_{(5.3.21)}V(t, \psi(0)) &\leq -16|\psi(0)|^4 + 4 \cdot 3^{1/4}|\psi(0)|^4 + 4 \cdot 3^{1/12}|\psi(0)|^3|\psi(0)|^{1/3} + 4|\psi(0)|^4 \\ &\leq -(16 - 4 \cdot 3^{1/4} - 4 \cdot 3^{1/12} - 4)|\psi(0)|^4 \leq -2V(\psi(0)). \end{aligned} \tag{5.3.24}$$

So let $p(t) = 2$ for all $t \in \mathbb{R}_+$ and $c(s) = s$ for all $s \in \mathbb{R}_+$. Then condition (ii) of Theorem 5.3.2 is verified.

To check condition (iv) we calculate $\mu = 1$, $M_2 = \ln(3)$ and $M_1 = 2$ from which we verify that $M_1 > M_2$.

Next let $\tilde{g}(s) = 2s$. Whenever $|\psi(0)| \geq 1$ we have $|\psi(0) + I(\tau_k, \psi)| \leq |\psi(0)| + |I(\tau_k, \psi)| = |\psi(0)| + 1/10 \leq (11/10)|\psi(0)|$ from which we obtain

$$V(\psi(0) + I(\tau_k, \psi)) \leq \left(\frac{11}{10}\right)^4 V(\psi(0)) \leq 2V(\psi(0)) = \tilde{g}(V(\psi(0))), \tag{5.3.25}$$

which verifies condition (iii). Finally, condition (iv) is satisfied with $\rho_1 = 11/10$.

Therefore, according to Theorem 5.3.2, solutions of equation (5.3.21) are uniformly ultimately bounded. □

# Chapter 6

# Linear Systems

Linear systems are important because they tend to be simpler to study and analyze both qualitatively and quantitatively than more complicated nonlinear systems and also because they often make good approximations to nonlinear systems.

In this chapter we take a brief look at linear systems of impulsive delay differential equations. We first describe linear systems. Then we obtain some stability results for a special class of linear systems by making use of the Lyapunov function theory of the previous chapters.

## 6.1 Description of Linear Systems

In this section we describe linear impulsive delay differential equations. We begin by stating the usual definition of what it means for a functional to be linear. Then we define what it means for system (2.4.7) to be linear.

**Definition 6.1.1:** A functional $L : \mathbb{R}_+ \times PC([-r,0],\mathbb{R}^n) \to \mathbb{R}^n$ is said to be *linear* (in $\psi$) if for any constants $c_1, c_2 \in \mathbb{R}$ and functions $\psi_1, \psi_2 \in PC([-r,0],\mathbb{R}^n)$, $L(t, c_1\psi_1 + c_2\psi_2) = c_1 L(t,\psi_1) + c_2 L(t,\psi_2)$ for all $t \in \mathbb{R}_+$.

**Definition 6.1.2:** System (2.4.7) is said to be *linear* if $f(t,\psi) = L_1(t,\psi) + h_1(t)$ and $I(t,\psi) = L_2(t,\psi) + h_2(t)$, for some linear functionals $L_1$ and $L_2$ and for some functions $h_1, h_2 : \mathbb{R}_+ \to \mathbb{R}^n$. If both $h_1$ and $h_2$ are identically zero, then system (2.4.7) is said to be *homogeneous*; otherwise it is said to be *nonhomogeneous*.

The most common type of linear functionals known to be useful in applications involving delay with a finite delay constant $r$, are of the form [Hal93]

$$L(t,\psi) = \sum_{k=1}^{m} A_k\psi(-t_k) + \int_{-r}^{0} A(t,s)\psi(s)ds + h(t), \tag{6.1.1}$$

where $0 \le t_k \le r$, the matrices $A_k \in \mathbb{R}^{n\times n}$ are constant (alternatively, they could be functions of $t$), $h : \mathbb{R}_+ \to \mathbb{R}^n$, and $A : \mathbb{R}_+ \times [-r, 0] \to \mathbb{R}^{n\times n}$.

Homogeneous linear systems have the solution $x(t) = 0$ and the stability properties of this trivial solution will be of interest. In the next section we will develop stability results for a special class of linear impulsive delay differential equation where the functional $f$ is a special case of the more general linear functional (6.1.1).

## 6.2 Stability of Linear Systems

In this section we consider a simple $n$-dimensional, linear, impulsive delay differential difference equation of the form

$$x'(t) = Ax(t) + Bx(t-r),\ t \ne \tau_k, \tag{6.2.1a}$$

$$\Delta x(t) = C_k x(t^-),\ t = \tau_k, \tag{6.2.1b}$$

where $A, B$ and $C_k$, for $k = 1, 2, \ldots$, are constant matrices in $\mathbb{R}^{n\times n}$. As usual $r > 0$ represents the delay constant and the impulsive times $\tau_k$ are assumed to be fixed.

Global existence and uniqueness of solutions of (6.2.1) follow from Corollary 3.3.1 and Theorem 3.4.1, respectively.

By making use of the stability theorems of Chapters 4 and 5, we obtain conditions on the parameters of system (6.2.1) that will ensure uniform asymptotic stability of the trivial solution.

Let us first introduce some matrix notation. For any matrix $A \in \mathbb{R}^{m\times n}$, $\|A\|$ denotes the matrix norm of $A$ induced by the Euclidean vector norms, $\|\cdot\|$, on $\mathbb{R}^m$ and $\mathbb{R}^n$. In this section all of our matrices will be assumed to be square. However, in Chapter 8 we will also consider non-square matrices and yet still be interested in the norm of such matrices. For a square, symmetric, real matrix $A$, $\lambda_{\max}(A)$ denotes the largest eigenvalue of $A$ and similarly $\lambda_{\min}(A)$ denotes the smallest eigenvalue of $A$. For a positive definite (symmetric) matrix $A \in \mathbb{R}^{n\times n}$, $A^{1/2}$ denotes the square root of $A$, which is defined to be the unique positive definite matrix satisfying $A^{1/2} \cdot A^{1/2} = A$ (see [Hor85]).

Recall that if the eigenvalues of a matrix $A \in \mathbb{R}^{n\times n}$ all have negative real parts, then the matrix is stable (or Hurwitz) and there is a unique positive definite matrix $P \in \mathbb{R}^{n\times n}$ that solves the

Lyapunov equation [Hal80]

$$A^T P + PA = -I, \tag{6.2.2}$$

where, in this section, we use $I$ to denote the $n \times n$ identity matrix. Similarly, if the eigenvalues of $A$ all have positive real parts, then there is a unique positive definite matrix $P$ for which

$$A^T P + PA = I. \tag{6.2.3}$$

Our first theorem makes use of Theorem 4.2.2 and is proven by way of a Lyapunov functional.

**Theorem 6.2.1:** *Assume that the eigenvalues of $A$ have negative real parts and let $P$ be the solution of (6.2.2). Suppose that $\|PB\| < 1/2$, $\sum_{k=1}^{\infty} \|C_k\| < \infty$, and $r = \inf_{k\in\mathbb{Z}^+}\{\tau_k - \tau_{k-1}\} > 0$. Then the solution $x(t) = 0$ of (6.2.1) is uniformly asymptotically stable.*

*Proof:* We first note that the conditions of Corollary 3.2.3 are satisfied and that the functional $f(t,\psi) = A\psi(0) + B\psi(-r)$ is also strongly quasi-bounded.

Let $\rho > 0$ be arbitrarily chosen. Since $\sum_{k=1}^{\infty} \|C_k\| < \infty$, then let $M = \sup_{k\in\mathbb{Z}^+}\{\|C_k\|\} < \infty$ and define $\rho_1 = \rho/(1+M)$. Then $0 < \rho_1 \le \rho$, and if $\|x\| \le \rho_1$, then $\|x + C_k x\| \le \|x\| + \|C_k\| \cdot \|x\| \le (1+M)\rho_1 = \rho$, which verifies that hypothesis (H1) of Chapter 4 is satisfied.

Define the Lyapunov functional

$$V(t,\psi) = \psi(0)^T P\psi(0) + \frac{1}{2}\int_{-r}^{0} \psi(s)^T \psi(s)ds. \tag{6.2.4}$$

Clearly $V$ satisfies the smoothness assumptions of Theorem 4.2.1.

Since, for all $(t,\psi) \in \mathbb{R}_+ \times PC([-r,0],\mathbb{R}^n)$,

$$\lambda_{\min}(P)\|\psi(0)\|^2 \le V(t,\psi) \le \lambda_{\max}(P)\|\psi(0)\|^2 + \frac{1}{2}r\|\psi\|_r^2, \tag{6.2.5}$$

then condition (i) of Theorem 4.2.1 is satisfied with $b(s) = \lambda_{\min}(P)s^2$ and $a(s) = (\lambda_{\max}(P) + r/2)s^2$.

As for condition (iii), suppose we let $x \in PC([t_0 - r, \infty), \mathbb{R}^n)$ be some function that is continuous at each $t \ne \tau_k$ in $(t_0, \infty)$. If $x(\tau_k) = x(\tau_k^-) + C_k x(\tau_k^-)$ (i.e. the function $x$ satisfies the delay difference equation (6.2.1b) at $t = \tau_k$), then we get

$$\begin{aligned}
x(\tau_k)^T P x(\tau_k) &= \left[(I + C_k)x(\tau_k^-)\right]^T P\left[(I + C_k)x(\tau_k^-)\right] \\
&= x(\tau_k^-)^T P x(\tau_k^-) + x(\tau_k^-)^T C_k^T P C_k x(\tau_k^-) + 2x(\tau_k^-)^T P C_k x(\tau_k^-) \\
&\le x(\tau_k^-)^T P x(\tau_k^-) + \|C_k\|^2\|P\| \cdot \|x(\tau_k^-)\|^2 + 2\|C_k\| \cdot \|P\| \cdot \|x(\tau_k^-)\|^2 \\
&\le x(\tau_k^-)^T P x(\tau_k^-) + \frac{1}{\lambda_{\min}(P)}\|P\|(\|C_k\|^2 + 2\|C_k\|)x(\tau_k^-)^T P x(\tau_k^-) \\
&= \left(1 + \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}(\|C_k\|^2 + 2\|C_k\|)\right) x(\tau_k^-)^T P x(\tau_k^-).
\end{aligned} \tag{6.2.6}$$

Therefore, if we let $d_k = (\lambda_{\max}(P)/\lambda_{\min}(P))(||C_k||^2 + 2||C_k||) \geq 0$, then

$$\begin{aligned} V(\tau_k, x_{\tau_k}) &= x(\tau_k)^T P x(\tau_k) + \frac{1}{2}\int_{-r}^{0} x(\tau_k+s)^T x(\tau_k+s)ds \\ &\leq (1+d_k)x(\tau_k^-)^T P x(\tau_k^-) + \frac{1}{2}\int_{-r}^{0} x(\tau_k+s)^T x(\tau_k+s)ds \\ &\leq (1+d_k)\left(x(\tau_k^-)^T P x(\tau_k^-) + \frac{1}{2}\int_{-r}^{0} x(\tau_k+s)^T x(\tau_k+s)ds\right) \\ &= (1+d_k)\lim_{t\to\tau_k^-} V(t, x_t). \end{aligned} \quad (6.2.7)$$

Moreover, since $\sum_{k=1}^{\infty} ||C_k|| < \infty$ then clearly $d = \sum_{k=1}^{\infty} d_k < \infty$.

Finally, we must check condition (ii). Calculating the derivative of $V$ with respect to system (6.2.1) gives us

$$\begin{aligned} D^+V_{(6.2.1)}(t,\psi) &= \psi(0)^T P(A\psi(0) + B\psi(-r)) + (A\psi(0) + B\psi(-r))^T P\psi(0) \\ &\quad + \frac{1}{2}\psi(0)^T\psi(0) - \frac{1}{2}\psi(-r)^T\psi(-r) \\ &= \psi(0)^T(A^TP + PA)\psi(0) + 2\psi(0)^T PB\psi(-r) + \frac{1}{2}\psi(0)^T\psi(0) - \frac{1}{2}\psi(-r)^T\psi(-r) \\ &= -\frac{1}{2}\psi(0)^T\psi(0) + 2\psi(0)^T PB\psi(-r) - \frac{1}{2}\psi(-r)^T\psi(-r) \\ &\leq -\frac{1}{2}||\psi(0)||^2 + 2||PB||\cdot||\psi(0)||\cdot||\psi(-r)|| - \frac{1}{2}||\psi(-r)||^2 \\ &\leq -\frac{1}{2}||\psi(0)||^2 + ||PB||\cdot||\psi(0)||^2 + ||PB||\cdot||\psi(-r)||^2 - \frac{1}{2}||\psi(-r)||^2 \\ &= -\left(\frac{1}{2} - ||PB||\right)(||\psi(0)||^2 + ||\psi(-r)||^2) \\ &\leq -\left(\frac{1}{2} - ||PB||\right)||\psi(0)||^2, \end{aligned} \quad (6.2.8)$$

where the final inequality follows since $||PB|| < 1/2$. Thus condition (ii) is satisfied with $c \in K_2$ defined by $c(s) = (1/2 - ||PB||)s^2$.

Therefore the conditions of Theorem 4.2.2 are satisfied in addition to the conditions of Theorem 4.2.1. So by Theorem 4.2.2 we conclude that the trivial solution of (6.2.1) is uniformly asymptotically stable. ■

**Example 6.2.1:** Consider system (6.2.1) with

$$A = \begin{bmatrix} -7 & 2 & 1 \\ 0 & -5 & 1 \\ 6 & 3 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -3 & -1 \\ -2 & -4 & 3 \\ 1 & 3 & 0 \end{bmatrix}, \text{ and } C_k = 2^{-k}\begin{bmatrix} 3 & 1 & 4 \\ 1 & 3 & -1 \\ 4 & -1 & 3 \end{bmatrix}. \quad (6.2.9)$$

The matrix $A$ is stable since its eigenvalues are -2, -7 and -8. Solving for $P$ in the Lyapunov equation (6.2.2) gives us

$$P = \begin{bmatrix} 0.0939 & 0.0336 & 0.0902 \\ 0.0336 & 0.1161 & 0.0807 \\ 0.0902 & 0.0807 & 0.2566 \end{bmatrix}. \tag{6.2.10}$$

Calculating the norm of $PB$ gives us $||PB|| = 0.4807 < 1/2$. Meanwhile the norm of $C_k$ is $||C_k|| = 7 \cdot 2^{-k}$ and so $\sum_{k=1}^{\infty} ||C_k|| = 7 < \infty$.

Thus if the time between consecutive impulses is bounded below by some positive quantity $\tau$, then Theorem 6.2.1 tells us that the trivial solution of system (6.2.1) will be uniformly asymptotically stable. □

In Theorem 6.2.1, the condition $\sum_{k=1}^{\infty} ||C_k|| < \infty$ implies that the cumulative effect of all of the impulses is finite and that as time evolves the effect of later impulses becomes increasingly negligible since $\lim_{k\to\infty} ||C_k|| = 0$. Since the system without impulses is already stable then the diminishing effect of impulses is not enough to destabilize the system. In some situations the impulses could even accelerate the convergence of solutions to the origin. This could happen if, instead, the magnitude of $||I + C_k||$ became increasingly smaller as $k$ tended to infinity. A simple modification of the proof of Theorem 6.2.1 gives us this alternative result.

If we replace the condition $\sum_{k=1}^{\infty} ||C_k|| < \infty$ by the condition

$$\sum_{k=1}^{\infty} \max\left\{0, \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}||I + C_k||^2 - 1\right\} < \infty, \tag{6.2.11}$$

then the conclusion of Theorem 6.2.1 still holds. The proof of Theorem 6.2.1 remains largely the same except that inequality (6.2.6) now becomes

$$\begin{aligned} x(\tau_k)^T P x(\tau_k) &= \left[(I + C_k)x(\tau_k^-)\right]^T P \left[(I + C_k)x(\tau_k^-)\right] \\ &\le \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}||I + C_k||^2 x(\tau_k^-)^T P x(\tau_k^-) \\ &\le (1 + d_k) x(\tau_k^-)^T P x(\tau_k^-), \end{aligned} \tag{6.2.12}$$

where $d_k = \max\{0, (\lambda_{\max}(P)/\lambda_{\min}(P))||I + C_k||^2 - 1\}$. Thus (6.2.11) is the same as requiring $\sum_{k=1}^{\infty} d_k < \infty$.

Note that (6.2.11) is implied by the following simpler condition

$$\limsup_{k\to\infty} ||I + C_k|| < \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}. \tag{6.2.13}$$

Our next two theorems obtain a different set of conditions under which the trivial solution of (6.2.1) is uniformly asymptotically stable. They make use of Lyapunov functions and apply Theorems 5.2.1 and 5.2.2, respectively.

**Theorem 6.2.2:** *Assume that the eigenvalues of $A$ have positive real parts and let $P$ be the solution of (6.2.3). Suppose that $\tau = \sup_{k\in\mathbb{Z}^+}\{\tau_k - \tau_{k-1}\} < \infty$. If there exists some $0 < \alpha < 1$ which satisfies the following inequalities*

$$\alpha \geq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\|I + C_k\|, \text{ for all } k, \tag{6.2.14}$$

*and*

$$2\lambda_{\min}(P)\alpha\ln(\alpha) + \tau(\alpha + 2\|PB\|) < 0, \tag{6.2.15}$$

*then the solution $x(t) = 0$ of (6.2.1) is uniformly asymptotically stable.*

*Proof:* As in Theorem 6.2.1, the conditions of Corollary 3.2.3 and hypothesis (H1) are satisfied.

Define the Lyapunov function

$$V(x) = x^T Px. \tag{6.2.16}$$

Then $V$ satisfies the smoothness assumptions of Theorem 5.2.1. Since

$$\lambda_{\min}(P)\|x\|^2 \leq V(x) \leq \lambda_{\max}(P)\|x\|^2, \tag{6.2.17}$$

for all $x \in \mathbb{R}^n$, then condition (i) of Theorem 5.2.1 is satisfied with $b(s) = \lambda_{\min}(P)s^2$ and $a(s) = \lambda_{\max}(P)s^2$.

Define $g(s) = \alpha^2 s$. Then

$$\begin{aligned} V(\psi(0) + C_k\psi(0)) &= [(I + C_k)\psi(0)]^T P[(I + C_k)\psi(0)] \\ &\leq \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\|I + C_k\|^2\psi(0)^T P\psi(0) \\ &\leq \alpha^2 V(\psi(0)) = g(V(\psi(0))). \end{aligned} \tag{6.2.18}$$

This verifies condition (iii) of Theorem 5.2.1.

Calculating the derivative of $V$ along solutions of (6.2.1) gives us

$$\begin{aligned} D^+V_{(6.2.1)}(t,\psi) &= \psi(0)^T P(A\psi(0)+B\psi(-r)) + (A\psi(0)+B\psi(-r))^T P\psi(0) \\ &= \psi(0)^T(A^TP+PA)\psi(0) + 2\psi(0)^T PB\psi(-r) \\ &\le \|\psi(0)\|^2 + 2\|PB\|\cdot\|\psi(0)\|\cdot\|\psi(-r)\| \\ &\le \frac{1}{\lambda_{\min}(P)}\psi(0)^T P\psi(0) + \frac{2}{\lambda_{\min}(P)}\|PB\|\sqrt{\psi(0)^T P\psi(0)\cdot\psi(-r)^T P\psi(-r)} \\ &\le \frac{1}{\lambda_{\min}(P)}\left(V(\psi(0)) + 2\|PB\|\sqrt{V(\psi(0))V(\psi(-r))}\right). \end{aligned} \tag{6.2.19}$$

Thus when $V(\psi(0)) \ge g(V(\psi(s)))$ for $s \in [-r,0]$, we have in particular $V(\psi(0)) \ge \alpha^2 V(\psi(-r))$ and so $V(\psi(-r)) \le (1/\alpha^2)V(\psi(0))$. Substituting this into (6.2.19) gives us

$$\begin{aligned} D^+V_{(6.2.1)}(t,\psi) &\le \frac{1}{\lambda_{\min}(P)}\left(V(\psi(0)) + 2\|PB\|\frac{1}{\alpha}V(\psi(0))\right) \\ &= \frac{1}{\lambda_{\min}(P)}\left(1+2\|PB\|\frac{1}{\alpha}\right)V(\psi(0)). \end{aligned} \tag{6.2.20}$$

Therefore, condition (ii) of Theorem 5.2.1 is satisfied if we let $p(t) = (1+2\|PB\|/\alpha)/\lambda_{\min}(P)$ and $c(s) = s$.

Calculating $M_1$ and $M_2$ in condition (iv) of the theorem gives us

$$M_1 = \frac{\tau}{\lambda_{\min}(P)}\left(1+\frac{2}{\alpha}\|PB\|\right), \tag{6.2.21}$$

and

$$M_2 = -2\ln(\alpha). \tag{6.2.22}$$

Finally, we have $M_2 > M_1$, as required by Theorem 5.2.1, by assumption (6.2.15) on $\alpha$. Thus by Theorem 5.2.1 it follows that the solution $x(t) = 0$ of (6.2.1) is uniformly asymptotically stable. ■

**Example 6.2.2:** Consider system (6.2.1) with

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 3 & -1 \\ 2 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 & 2 \\ -2 & -1 & -1 \\ 5 & -1 & 1 \end{bmatrix}, \text{ and } C_k = \begin{bmatrix} -1.1 & 0.1 & 0 \\ 0 & -1.1 & 0.1 \\ 0.2 & 0 & -1.2 \end{bmatrix}. \tag{6.2.23}$$

Suppose in addition that the impulse times occur with a frequency of at least $\tau = 0.03$.

Checking the eigenvalues of $A$, we find that they are 2, 3 and 5. Solving for $P$ in the Lyapunov equation (6.2.3) gives us

$$P = \begin{bmatrix} 0.1929 & -0.0893 & -0.0786 \\ -0.0893 & 0.2500 & 0.0714 \\ -0.0786 & 0.0714 & 0.1643 \end{bmatrix}. \tag{6.2.24}$$

Next we evaluate $||I+C_k|| = 0.3$ for all $k$, $\lambda_{\min}(P) = 0.0986$, $\lambda_{\max}(P) = 0.3686$, and $||PB|| = 0.6517$.
Condition (6.2.14) then reduces to

$$0.5800 \leq \alpha < 1. \tag{6.2.25}$$

For $\alpha$ to satisfy (6.2.15) we must have

$$0.1973\alpha \ln(\alpha) + 0.03\alpha + 0.0391 < 0. \tag{6.2.26}$$

A simple check shows that both of these conditions are satisfied by choosing $\alpha = 0.6$. Therefore, uniform asymptotic stability of the trivial solution of (6.2.1) follows by Theorem 6.2.2. □

Our final theorem makes use of Theorem 5.2.2 to establish conditions for uniform asymptotic stability of the trivial solution of (6.2.1).

**Theorem 6.2.3:** *Assume that the eigenvalues of A have negative real parts and let P be the solution of (6.2.2). Suppose that* $\tau = \inf_{k\in\mathbb{Z}^+}\{\tau_k - \tau_{k-1}\} > 0$. *If there exists some* $\alpha \geq 1$ *which satisfies the following inequalities*

$$\alpha \geq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}||I + C_k||, \text{ for all } k, \tag{6.2.27}$$

$$2||PB||\alpha < \frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}, \tag{6.2.28}$$

*and*

$$2\ln(\alpha) - \frac{\tau}{\lambda_{\max}(P)} + \frac{2\tau\alpha}{\lambda_{\min}(P)}||PB|| < 0, \tag{6.2.29}$$

*then the solution* $x(t) = 0$ *of (6.2.1) is uniformly asymptotically stable.*

*Proof:* As in the previous two theorems, the conditions of Corollary 3.2.3 and hypothesis (H1) are satisfied.

Define the Lyapunov function

$$V(x) = x^T P x. \tag{6.2.30}$$

Then $V$ satisfies the smoothness assumptions of Theorem 5.2.2 as well as condition (i) of Theorem 5.2.2 with $b(s) = \lambda_{\min}(P)s^2$ and $a(s) = \lambda_{\max}(P)s^2$.

Define $\hat{g}(s) = \alpha^2 s$. Then

$$\begin{aligned} V(\psi(0) + C_k\psi(0)) &= [(I + C_k)\psi(0)]^T P [(I + C_k)\psi(0)] \\ &\leq \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \|I + C_k\|^2 \psi(0)^T P \psi(0) \\ &\leq \alpha^2 V(\psi(0)) = \hat{g}(V(\psi(0))). \end{aligned} \tag{6.2.31}$$

This verifies condition (iii) of Theorem 5.2.2.

Calculating the derivative of $V$ along solutions of (6.2.1) gives us

$$\begin{aligned} D^+V_{(6.2.1)}(t, \psi) &= \psi(0)^T P(A\psi(0) + B\psi(-r)) + (A\psi(0) + B\psi(-r))^T P\psi(0) \\ &= \psi(0)^T (A^T P + PA)\psi(0) + 2\psi(0)^T PB\psi(-r) \\ &\leq -\|\psi(0)\|^2 + 2\|PB\| \cdot \|\psi(0)\| \cdot \|\psi(-r)\| \\ &\leq -\frac{1}{\lambda_{\max}(P)}\psi(0)^T P\psi(0) + \frac{2}{\lambda_{\min}(P)}\|PB\|\sqrt{\psi(0)^T P\psi(0) \cdot \psi(-r)^T P\psi(-r)} \\ &\leq -\frac{1}{\lambda_{\max}(P)} V(\psi(0)) + \frac{2}{\lambda_{\min}(P)}\|PB\|\sqrt{V(\psi(0))V(\psi(-r))}. \end{aligned} \tag{6.2.32}$$

Since $\alpha$ satisfies the strict inequalities (6.2.28) and (6.2.29), then both inequalities are satisfied by any $\beta$ chosen sufficiently close to $\alpha$. So let $\beta > \alpha$ satisfy

$$2\|PB\|\beta < \frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}, \tag{6.2.33}$$

and

$$2\ln(\beta) - \frac{r}{\lambda_{\max}(P)} + \frac{2r\beta}{\lambda_{\min}(P)}\|PB\| < 0, \tag{6.2.34}$$

and define $g(s) = \beta^2 s$. Then when $g(V(\psi(0))) \geq V(\psi(s))$ for $s \in [-r, 0]$, we have in particular $\beta^2 V(\psi(0)) \geq V(\psi(-r))$. Substituting this into (6.2.32) gives us

$$\begin{aligned} D^+V_{(6.2.1)}(t, \psi) &\leq -\frac{1}{\lambda_{\max}(P)} V(\psi(0)) + \frac{2}{\lambda_{\min}(P)}\|PB\|\beta V(\psi(0)) \\ &= -\left(\frac{1}{\lambda_{\max}(P)} - \frac{2\beta}{\lambda_{\min}(P)}\|PB\|\right) V(\psi(0)). \end{aligned} \tag{6.2.35}$$

Therefore, condition (ii) of Theorem 5.2.2 is satisfied if we let $p(t) = 1/\lambda_{max}(P) - 2\beta||PB||/\lambda_{min}(P)$ and $c(s) = s$. Note that inequality (6.2.34) implies $p(t) > 0$.

Next, if we calculate $M_1$ and $M_2$ in condition (iv) of the theorem, where what we called $\mu$ in that theorem we now call $\tau$, we get

$$M_1 = \frac{\tau}{\lambda_{max}(P)} - \frac{2\tau\beta}{\lambda_{min}(P)}||PB||, \tag{6.2.36}$$

and

$$M_2 = 2\ln(\beta). \tag{6.2.37}$$

Finally, we have $M_1 > M_2$, as required by Theorem 5.2.2 by inequality (6.2.34). Thus by Theorem 5.2.2 it follows that the solution $x(t) = 0$ of (6.2.1) is uniformly asymptotically stable. ∎

**Example 6.2.3:** Consider system (6.2.1) with

$$A = \begin{bmatrix} -6 & 0 & 3 \\ 0 & -4 & 2 \\ 1 & 0 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 0.11 & 0 & -0.04 \\ -0.05 & 0.03 & -0.07 \\ 0 & -0.03 & 0.03 \end{bmatrix}, \text{ and } C_k = \begin{bmatrix} -0.9 & 0.9 & 0.2 \\ 0 & -1.3 & 0.3 \\ -0.1 & 0.5 & -1.7 \end{bmatrix}. \tag{6.2.38}$$

Suppose in addition that the impulse times occur with a frequency of no more than $\tau = 1$.

The eigenvalues of $A$ are -3, -4 and -7, indicating that $A$ is a stable matrix. Solving for $P$ in the Lyapunov equation (6.2.2) gives us

$$P = \begin{bmatrix} 0.1095 & 0.0216 & 0.0524 \\ 0.0216 & 0.1436 & 0.0372 \\ 0.0524 & 0.0372 & 0.1381 \end{bmatrix}. \tag{6.2.39}$$

The largest and smallest eigenvalues of $P$ are $\lambda_{min}(P) = 0.0690$ and $\lambda_{max}(P) = 0.2070$, respectively. Since $||I + C_k|| = 1.1208$ for all $k$, then condition (6.2.27) reduces to

$$\alpha \geq 1.9412. \tag{6.2.40}$$

Meanwhile, condition (6.2.28) becomes

$$\alpha < 12.8544. \tag{6.2.41}$$

With $\tau = 1$ then condition (6.2.29) becomes

$$2\ln(\alpha) - 4.8309 + 0.3758\alpha < 0. \tag{6.2.42}$$

Choosing $\alpha = 3$, for example, will satisfy all three of these conditions and therefore imply that the trivial solution of (6.2.1) is uniformly asymptotically stable. □

Note that Theorems 6.2.1 and 6.2.3 impose similar conditions on system (6.2.1). Both require that the matrix $A$ have eigenvalues with negative real parts and the matrix $P$ described in each theorem is the same. Also, both theorems require that there be a positive lower bound, $\tau$, on the time between consecutive impulses. This latter condition ensures that the impulses, which might otherwise destabilize the system, do not occur too frequently. In both theorems the conclusion is the same, namely that the trivial solution of (6.2.1) is uniformly asymptotically stable. Theorem 6.2.1 is proven by way of a Lyapunov functional while Theorem 6.2.2 is proven using a Lyapunov function. The conditions on the matrices $B$, $C_k$ and on the parameter $\tau$ differ between the two theorems and this is what distinguishes one from the other.

Theorem 6.2.1 imposes a condition on the magnitude of the norm of the product of the matrices $P$ and $B$, namely that $||PB|| < 1/2$. Theorem 6.2.2 requires an even tighter bound on $||PB||$, namely that

$$||PB|| < \frac{\lambda_{\min}(P)}{2\alpha\lambda_{\max}(P)} \le \frac{\lambda_{\min}(P)}{2\lambda_{\max}(P)} < \frac{1}{2}. \tag{6.2.43}$$

Note that for the matrices $A$ and $B$ defined as in Example 6.2.1, this inequality is not satisfied for any $\alpha \geq 1$ since $||PB|| = 0.4807$ while $\lambda_{\min}(P)/(2\lambda_{\max}(P)) = 0.0808$.

Theorem 6.2.1 imposes no condition on $\tau$ other than to assume that it is positive. Meanwhile, Theorem 6.2.3 requires, in addition, that $\tau$ be sufficiently small in relation to other parameters of the system, that it satisfy the inequality (6.2.29).

While up to this point we have mentioned those assumptions of Theorem 6.2.1 that are the same as or that are weaker than those of Theorem 6.2.3, the one place where Theorem 6.2.3 imposes a generally weaker condition is on the matrices $C_k$. Theorem 6.2.1 requires that $\sum_{k=1}^{\infty} ||C_k|| < \infty$, which in turn requires that $\lim_{k\to\infty} ||C_k|| = 0$. Alternatively, this condition on the matrices $C_k$ could be replaced by (6.2.11). Theorem 6.2.3 does not impose such a stringent condition on the matrices $C_k$. In Example 6.2.3 the matrices $C_k$ are all identical and $||I + C_k|| = 1.1208 > 1$. So clearly neither of the two alternative conditions on the matrices $C_k$ given in Theorem 6.2.1 could be applied in this case.

Theorem 6.2.3 allows for impulses to occur without requiring that they diminish in magnitude as time evolves. This makes the theorem more useful and powerful.

# Chapter 7

# Applications to Population Growth Models

Ordinary differential equations are frequently used to model population dynamics. The Lotka-Volterra equation for predator-prey problems or that of competing species is often considered. The simple logistic growth equation is another classic example. In some instances delays in one or more components of the reaction require the adoption of a delay differential equation to more suitably model a particular system of interacting species. For instance the presence of and the population density level of one species may inhibit or enhance the growth of another but there may be some delay before this effect is felt by the second species. For a thorough discussion of delay differential equations as they apply to the study of population dynamics see Gopalsamy's monograph [Gop92] and the references cited therein.

When population levels repeatedly undergo changes of relatively short duration (due, for instance, to stocking or harvesting of species), then these events may be more suitably modelled by an impulsive differential equation [Bal97c, Liu94, Ang94].

In this chapter we look at some population growth models which consist of an impulsive delay differential equation. In the first section of this chapter we discuss the concept of permanence of populations. By permanence we roughly mean boundedness and persistence (or non-extinction) of all species.

In the second section we look at two examples of population growth models. In the first example we consider a Lotka-Volterra type of equation involving two competing species where there is a fixed, single delay in the intra-specific and inter-specific negative feedbacks. The second example considers one of many possible models for the growth of a single species. Conditions on the impulses as well as the governing delay differential equation will be imposed in order to establish the property of permanence of populations. In order to prove permanence of populations modelled by a system of equations, we perform a change of variables on our system and then apply one of the boundedness theorems of Chapter 5 to this transformed system.

## 7.1 Permanence

We begin by assuming we have $n$ species, where $N_i(t) > 0$ represents the population density (i.e. the number of individuals per unit area) at time $t$ for the $i^{th}$ species. Let $N = (N_1, N_2, \dots, N_n) \in \mathbb{R}^n$. We consider the following impulsive delay differential equation

$$N'(t) = g(t, N_t),\ t \neq \tau_k, \tag{7.1.1a}$$

$$\Delta N(t) = J(t, N(t^-)),\ t = \tau_k, \tag{7.1.1b}$$

where $k = 0, 1, 2, \dots$. For simplicity we have assumed that there are no delays in the difference equation (7.1.1b).

Note that one important property of system (7.1.1) from a modelling point of view is that the interior of the positive cone in $\mathbb{R}^n$ given by $S = \{N \in \mathbb{R}^n \mid N_i > 0 \text{ for } i = 1, 2, \dots, n\}$ should be positively invariant in the sense that given any initial condition specifying $N_i(t) > 0$ for $t \in [t_0 - r, t_0]$ and $i = 1, 2, \dots, n$, we should get that $N_i(t) > 0$ for all $t \geq t_0$ and $i = 1, 2, \dots, n$. One immediate restriction on the vector function $J$ is that we require $N_i + J_i(\tau_k, N) > 0$ for all $N_i > 0$, $k = 0, 1, 2, \dots$, and for $i = 1, 2, \dots, n$.

As usual we let $\phi = (\phi_1, \phi_2, \dots, \phi_n) \in PC([-r, 0], S)$ denote the initial function and we let $t_0$ denote the initial time.

The notion of permanence of solutions of a system of equations which models population dynamics has been considered a great deal in the literature, particularly for models consisting of ordinary differential equations. The extension of the concept of permanence to impulsive delay differential equations is simple and straightforward.

There are numerous mathematical concepts and terminology associated with the ideas of populations avoiding extinction and growing in a bounded fashion. These include: persistence, uniform persistence, weak persistence, strong persistence, dissipativity, uniform boundedness, asymptotic uniform boundedness, permanence, permanent coexistence, cooperativeness, and ecological stability, to name a few.

Hofbauer and Sigmund [Hof88] define permanence for an autonomous ordinary differential equation to mean that there exists some compact subset $F$ of $S$ such that the orbit of every solution starting in $S$ will eventually end up in $F$. Applying this idea to system (7.1.1) would mean that there exist some constants $B_3 > 0$ and $B_4 > 0$ such that if $(t_0, \phi) \in \mathbb{R}_+ \times PC([-r, 0], S)$, then $B_3 \leq \liminf_{t\to\infty} N_i(t, t_0, \phi) \leq \limsup_{t\to\infty} N_i(t, t_0, \phi) \leq B_4$ for $i = 1, 2, \dots, n$.

We choose to strengthen the notion of permanence by defining it as follows.

**Definition 7.1.1:** Solutions of system (7.1.1) are said to be

(P1) *quasi-permanent* if for each $(t_0, \phi) \in \mathbb{R}_+ \times PC([-r, 0], S)$, there exist constants $B_1 = B_1(t_0, \phi) > 0$ and $B_2 = B_2(t_0, \phi) > 0$ such that if $N = N(t_0, \phi)$ is any solution of (7.1.1), then $N(t, t_0, \phi)$ is defined for all $t \geq t_0$ and $B_1 \leq N_i(t, t_0, \phi) \leq B_2$ for $i = 1, 2, \ldots, n$ and for $t \geq t_0$;

(P2) *uniformly quasi-permanent* if (P1) holds with $B_1$ and $B_2$ independent of $t_0$;

(P3) *permanent* if (P1) holds and if there exist constants $B_3 > 0$ and $B_4 > 0$ such that for each $(t_0, \phi) \in \mathbb{R}_+ \times PC([-r, 0], S)$, there exists a constant $T = T(t_0, \phi) > 0$ such that $B_3 \leq N_i(t, t_0, \phi) \leq B_4$ for $i = 1, 2, \ldots, n$ and for $t \geq t_0 + T$;

(P4) *uniformly permanent* if (P2) holds and (P3) holds with $T$ independent of $t_0$.

Note that permanence includes the concepts of boundedness and persistence. Boundedness here is a little different than that which was defined in Definition 4.3.1 in that the bound $B_2$ need not be common to a whole set of functions $\phi$ but could differ depending on which initial function is chosen. The same is true of $T$ in the definition of permanence. Note that $B_4$ is similar to what we called $B$ in the definition of uniform ultimate boundedness. The notion of persistence given in the definition of quasi-permanence is that each individual solution remains bounded away from zero, while in the case of permanence it also means that there is a common lower bound above which solutions will eventually or ultimately remain.

Permanence is often a desirable property to have in models describing population dynamics. It ensures that no single species will either tend to extinction or grow without bound.

## 7.2 Population Growth Models

For our first example, we begin by assuming we have two competing species $N_1$ and $N_2$. We assume that their evolutionary behaviour is governed by the following impulsive delay differential equation

$$\begin{aligned} N_1'(t) &= N_1(t)\left[r_1 - a_{11}N_1(t-r) - a_{12}N_2(t-r)\right], \quad t \neq \tau_k, \\ N_2'(t) &= N_2(t)\left[r_2 - a_{21}N_1(t-r) - a_{22}N_2(t-r)\right], \quad t \neq \tau_k, \end{aligned} \tag{7.2.1a}$$

$$\begin{aligned} \Delta N_1(t) &= J_1(t, N_1(t^-), N_2(t^-)), \quad t = \tau_k, \\ \Delta N_2(t) &= J_2(t, N_1(t^-), N_2(t^-)), \quad t = \tau_k, \end{aligned} \tag{7.2.1b}$$

where $k = 0, 1, 2, \ldots$. The constants $r_i$ for $i = 1, 2$ are assumed to be positive and $r_i$ represents the natural intrinsic growth rate of the $i^{th}$ species if resources were unlimited and inter-species and intra-species effects were neglected. The constants $a_{ij}$ for $i = 1, 2$ and $j = 1, 2$ are assumed to be nonnegative and $a_{ij}$ represents the growth inhibiting effect that species $j$ has on species $i$, after some fixed time delay $r > 0$, as a result of the competition for resources. Conditions on the impulse times $\tau_k$ as well as the functions $J_1, J_2 : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ will be imposed in order to guarantee permanence of solutions of (7.2.1). Equation (7.2.1) without impulses (i.e. $J_1 = J_2 = 0$) has been considered in [Gop92] while equation (7.2.1) without delays (i.e. $r = 0$) has been considered in [Bal97c].

Equation (7.2.1) is a reasonable candidate for a model of a pair of competing species in which there is delay in the effects of species interaction and in which there are impulses causing periodic abrupt changes to the population density levels. We refrain from delving into a deep investigation into just how suitable such a model might be for a particular real-life situation. Instead we prefer to examine equation (7.2.1) from an analytical point of view while keeping in mind the interpretation of the variables as species population densities.

Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}. \tag{7.2.2}$$

and let us assume that $A$ is a nonsingular matrix. Let $N^* = (N_1^*, N_2^*)^T = A^{-1}R$ so that $r_1 = a_{11}N_1^* + a_{12}N_2^*$ and $r_2 = a_{21}N_1^* + a_{22}N_2^*$ and assume further that $N_i^* > 0$ for $i = 1, 2$. Then in the absence of impulses, $N^*$ is the unique positive equilibrium point of equation (7.2.1).

In the absence of both delays and impulses, this Lotka-Volterra model for two competing species has been well-studied. Phase portraits can be easily plotted and it can be determined (see [Hof88]) that the equilibrium point $N^*$ is an unstable saddle point of the nonlinear system if the system parameters happen to satisfy

$$\frac{a_{11}}{a_{21}} < \frac{r_1}{r_2} < \frac{a_{12}}{a_{22}}. \tag{7.2.3}$$

Regardless of the initial condition, both populations remain bounded for all time. However, if (7.2.3) is satisfied then unless the initial population densities (specified only at time $t_0$ when there is no delay) happen to lie on the curve forming the stable manifold of $N^*$, then both populations cannot persist simultaneously. One species will invariably tend to extinction while the other will tend to some positive limiting value. If delays are present then numerical simulations suggest a similar behaviour with one species dying out and the other tending to either a positive equilibrium or, for large values of the delay constant, to a periodic steady state. It has been shown (in the absence of delays) that impulses can be used to ensure the permanence of both species for system (7.2.1) [Bal97c]. We will use similar techniques to develop some permanence results for the impulsive system (7.2.1) when there is delay. The result will be independent of the value of the delay constant $r$.

Rather than study equation (7.2.1) directly we perform a change of variables and then examine the new resulting system. In particular, for $i = 1, 2$ let

$$x_i = \ln\left(\frac{N_i}{N_i^*}\right), \tag{7.2.4}$$

and let $x = (x_1, x_2)$. Note that this transformation is well-defined since we are restricting the domain of $N = (N_1, N_2)$ to $S$ and we have assumed that $N_i^* > 0$ for $i = 1, 2$. Under this transformation the set $S$ gets mapped to all of $\mathbb{R}^2$ and the point $N^*$ gets mapped to the origin.

Under the transformation (7.2.4), system (7.2.1) becomes

$$\begin{aligned} x_1'(t) &= a_{11}N_1^*(1 - e^{x_1(t-r)}) + a_{12}N_2^*(1 - e^{x_2(t-r)}), \ t \neq \tau_k \\ x_2'(t) &= a_{21}N_1^*(1 - e^{x_1(t-r)}) + a_{22}N_2^*(1 - e^{x_2(t-r)}), \ t \neq \tau_k, \end{aligned} \tag{7.2.5a}$$

$$\begin{aligned} \Delta x_1(t) &= I_1(t, x_1(t^-), x_2(t^-)), \ t = \tau_k, \\ \Delta x_2(t) &= I_2(t, x_1(t^-), x_2(t^-)), \ t = \tau_k, \end{aligned} \tag{7.2.5b}$$

where $I_i(t, x_1, x_2) = \ln(1 + J_i(t, N_1^* e^{x_1}, N_2^* e^{x_2})/(N_i^* e^{x_i}))$ for $i = 1, 2$.

The key observation that we make is that if solutions of system (7.2.5) are uniformly ultimately bounded, then solutions of system (7.2.1) are uniformly permanent. Indeed, suppose $(t_0, \phi) \in \mathbb{R}_+ \times PC([-r, 0], S)$ and let $\hat{\phi} \in PC([-r, 0], \mathbb{R}^2)$ be defined according to the transformation (7.2.4) by $\hat{\phi}_i(s) = \ln(\phi_i(s)/N_i^*)$ for $s \in [-r, 0]$ and $i = 1, 2$. If solutions of (7.2.5) are uniformly ultimately bounded, then there exist $B_2^* > 0$ and $T > 0$, which are generally dependent on $\hat{\phi}$, and a bound $B > 0$ independent of $\hat{\phi}$ for which $\|x(t)\| \leq B_2^*$ for $t \geq t_0$ and $\|x(t)\| \leq B$ for $t \geq t_0 + T$. Since $|x_i| \to \infty$ if either $N_i \to 0^+$ or $N_i \to \infty$, then the fact that $\|x(t)\|$ is bounded implies that there must exist $B_1, B_2, B_3, B_4 > 0$ for which $B_1 \leq N_i(t) \leq B_2$ for $t \geq t_0$ and $B_3 \leq N_i(t) \leq B_4$ for $t \geq t_0 + T$ and $i = 1, 2$. Both $B_1$ and $B_2$ can be derived from $B_2^*$ and hence will depend only on $\hat{\phi}$, which is in turn defined by $\phi$, while $B_3$ and $B_4$ can be determined from $B$, which depends only on the system (7.2.5) and not on the initial conditions.

We will define a Lyapunov function for system (7.2.5) and using Theorem 5.3.1 we will obtain conditions under which system (7.2.5) is uniformly ultimately bounded. Specifically we will obtain conditions on the impulse times $\tau_k$ as well as conditions on $I_1$ and $I_2$ that will guarantee uniform ultimate boundedness. Through the relation of $I_i$ to $J_i$ this will therefore indirectly impose conditions on the original impulse functions $J_1$ and $J_2$ of system (7.2.1). Note that to use Theorem 5.3.1 we will require the Lyapunov function to decrease at impulse times for all state values outside of some neighbourhood of the origin. In terms of the original system (7.2.1), one can interpret this as requiring that the solutions, or population density levels, be adjusted at each impulse time so that they become closer to the equilibrium point $N^*$ at each impulse time unless they are already in some predefined fixed compact subset of $S$ containing $N^*$.

Define

$$W(s) = e^s - 1 - s, \tag{7.2.6}$$

for $s \in \mathbb{R}_+$. Note that $W(0) = 0$, $W(s) > 0$ for $s \neq 0$, $W'(s) = e^s - 1$, $W(s)$ is strictly increasing for $s > 0$ and strictly decreasing for $s < 0$ and $W(s) \to \infty$ as $s \to \pm\infty$. Another interesting property of the function $W$ is that given any $a \in [0,1]$ we have $W(as) \le aW(s)$ for all $s \in \mathbb{R}$. We will find the following inequality to be quite useful

$$-(e^{s_1}-1)(e^{s_2}-1) \le (e^{s_1}-1-s_1)+(e^{s_2}-1-s_2) = W(s_1)+W(s_2), \tag{7.2.7}$$

for all $s_1, s_2 \in \mathbb{R}$. This is easily proven by noting that

$$\begin{aligned}(e^{s_1}-1)(e^{s_2}-1)&+(e^{s_1}-1-s_1)+(e^{s_2}-1-s_2)\\ &= e^{s_1+s_2}-e^{s_1}-e^{s_2}+1+e^{s_1}-1-s_1+e^{s_2}-1-s_2\\ &= e^{s_1+s_2}-1-s_1-s_2 = W(s_1+s_2) \ge 0.\end{aligned} \tag{7.2.8}$$

We now define the following Lyapunov function

$$V(x) = d_1W(x_1)+d_2W(x_2), \tag{7.2.9}$$

for all $x \in \mathbb{R}^2$ where $d_1 > 0$ and $d_2 > 0$ are constant. From the properties of the function $W$ we may conclude that $V$ satisfies the smoothness assumptions as well as condition (i) of Theorem 5.3.1.

Define

$$\sigma = \max\left\{N_1^*\left(a_{11}+\frac{d_2}{d_1}a_{21}\right), N_2^*\left(\frac{d_1}{d_2}a_{12}+a_{22}\right)\right\}, \tag{7.2.10}$$

and let $0 < \lambda < 1$ be some constant. Let $g(s) = \lambda s$ and $c(s) = s$ for $s \in \mathbb{R}_+$ and let $p(t) = p = \max\{r_1, r_2\} + \sigma/\lambda$ for all $t \in \mathbb{R}_+$. We will show that condition (ii) of Theorem 5.3.1 is satisfied where $\rho > 0$ is any constant.

For any $\psi \in PC([-r,0],\mathbb{R}^2)$ we have by using inequality (7.2.7)

$$\begin{aligned}D^+_{(7.2.5)}V(t,\psi) &= d_1(e^{\psi_1(0)}-1)\left[a_{11}N_1^*(1-e^{\psi_1(-r)})+a_{12}N_2^*(1-e^{\psi_2(-r)})\right]\\ &\quad+d_2(e^{\psi_2(0)}-1)\left[a_{21}N_1^*(1-e^{\psi_1(-r)})+a_{22}N_2^*(1-e^{\psi_2(-r)})\right]\\ &\le d_1a_{11}N_1^*(W(\psi_1(0))+W(\psi_1(-r)))+d_1a_{12}N_2^*(W(\psi_1(0))+W(\psi_2(-r)))\\ &\quad+d_2a_{21}N_1^*(W(\psi_2(0))+W(\psi_1(-r)))+d_2a_{22}N_2^*(W(\psi_2(0))+W(\psi_2(-r)))\\ &= d_1r_1W(\psi_1(0))+d_2r_2W(\psi_2(0))+d_1N_1^*\left(a_{11}+\frac{d_2}{d_1}a_{21}\right)W(\psi_1(-r))\\ &\quad+d_2N_2^*\left(\frac{d_1}{d_2}a_{12}+a_{22}\right)W(\psi_2(-r))\\ &\le \max\{r_1,r_2\}V(\psi(0))+\sigma V(\psi(-r)).\end{aligned} \tag{7.2.11}$$

Therefore if $V(\psi(0)) \ge g(V(\psi(s)))$ for $s \in [-r,0]$, then in particular $V(\psi(-r)) \le (1/\lambda)V(\psi(0))$

from which we get by (7.2.11) that

$$D^{+}_{(7.2.5)}V(t,\psi) \leq \left(\max\{r_1, r_2\} + \frac{\sigma}{\lambda}\right) V(\psi(0)) = p(t)c(V(\psi(0))), \tag{7.2.12}$$

which verifies condition (ii).

Condition (v) imposes a constraint on $\tau$. Not only must it be finite-valued, but since $M_1 = p\tau$, $M_2 = -\ln(\lambda)$ and we require $M_2 > M_1$, then $\tau$ must be sufficiently small so as to satisfy $\tau < -\ln(\lambda)/p$. Since $p$ depends on $\lambda$, then this actually imposes an absolute upper bound on $\tau$ for the system (7.2.5) and hence system (7.2.1). The upper bound $-\ln(\lambda)/p$ is maximized over all $0 < \lambda < 1$ by the unique solution $\lambda^*$ of

$$\left(\frac{\max\{r_1, r_2\}}{\sigma}\right)\lambda + 1 + \ln(\lambda) = 0. \tag{7.2.13}$$

Having chosen a suitable $\lambda$ and defined $\sigma$, $p$, $\tau$ and so on accordingly, all that we must then establish is that conditions (iii) and (iv) of Theorem 5.3.1 are satisfied for some $\rho_1 \geq \rho > 0$. These two conditions put restrictions on the functions $I_1$ and $I_2$, which in turn impose conditions on the original functions $J_1$ and $J_2$. In the most extreme case, and to show that the set of such functions $J_i$ for which the corresponding functions $I_i$ do satisfy these two conditions is not vacuous, we see that the particular choice of $J_i(t, N_1, N_2) = -N_i + N_i^*$ yields $I_i(t, x_1, x_2) = -x_i$, which does verify both conditions for every $\rho_1 \geq \rho \geq 0$. In this special case the impulses cause the population density levels to get mapped to the equilibrium point $N^*$ at each impulse time. In terms of system (7.2.5) this means that the state $x$ gets mapped to the origin at impulses.

Note that in our analysis the value of the delay constant $r$ was irrelevant in terms of establishing permanence of the two species. In fact we could have derived a similar result even had the four occurrences of $r$ in the original equation (7.2.1) been different from one another. In the special case when $r = 0$ and there is no delay, then one can obtain even better results that impose fewer constraints on the impulse times and the functions $J_1$ and $J_2$ [Bal97c].

We also mention that the previous analysis can be generalized to a similar impulsive delay differential equation involving three or more species without too much difficulty although calculating the analogues of the parameters such as $\sigma$, $p$, etc. becomes a little more unwieldy.

We next look at an actual numerical example that makes use of the previous analysis.

**Example 7.2.1:** We will show that solutions of the following system are uniformly permanent.

$$N_1'(t) = N_1(t)\left[1 - \frac{1}{2}N_1(t-r) - \frac{1}{2}N_2(t-r)\right], \ t \neq \tau_k,$$
$$N_2'(t) = N_2(t)\left[\frac{1}{2} - \frac{3}{8}N_1(t-r) - \frac{1}{8}N_2(t-r)\right], \ t \neq \tau_k, \tag{7.2.14a}$$

$$\Delta N_1(t) = -N_1(t^-) + \sqrt{N_1(t^-)}, \ t = \tau_k,$$
$$\Delta N_2(t) = -N_2(t^-) + \sqrt{N_2(t^-)}, \ t = \tau_k, \tag{7.2.14b}$$

where $r > 0$ can be any delay constant and $\tau_k = k/4$ for $k = 0, 1, 2, \ldots$.

Clearly $\tau = 1/4$. Solving for $N^*$ gives us $N^* = (1,1)$. Let $d_1 = d_2 = 1$ and $\lambda = 1/2$. Then calculating $\sigma$ from (7.2.10) gives us $\sigma = 7/8$ and so $p = 11/4$. Since $M_1 = p\tau = 11/16 = 0.6875$ and $M_2 = -\ln(1/2) = 0.6931$, then $M_2 > M_1$ as required.

Since $J_i(t, N_1, N_2) = -N_i + \sqrt{N_i}$ for $i = 1, 2$, then $I_i(t, x_1, x_2) = -x_i/2$. Therefore $W(x_i + I_i(\tau_k, x_1, x_2)) = W(x_i/2) \leq W(x_i)/2$, which implies

$$V(x_1 + I_1(\tau_k, x_1, x_2), x_2 + I_2(\tau_k, x_1, x_2)) \leq \frac{1}{2}W(x_1) + \frac{1}{2}W(x_2) = \frac{1}{2}V(x_1, x_2) = g(V(x_1, x_2)). \tag{7.2.15}$$

In other words, condition (iii) of Theorem 5.3.1 is satisfied for any choice of $\rho > 0$. Moreover, condition (iv) is satisfied if we simply let $\rho_1 = \rho$.

Therefore according to Theorem 5.3.1, solutions of the transformed system (i.e. in terms of $x_1$ and $x_2$) are uniformly ultimately bounded, which in turn implies that solutions of system (7.2.14) are uniformly permanent. □

Having shown that impulses can ensure permanence of populations in the previous model of two competing species, we now turn our attention to a different equation involving just a single species $N$. The following is a delay differential equation suggested by Gopalsamy in [Gop92] as one possibility for a model of the growth of a single species. The equation is of the form

$$N'(t) = N(t)\left[a - b\ln(N(t)) - c\ln(N(t-r))\right], \tag{7.2.16}$$

where $a, b$ and $c$ are positive constants. This equation has a unique positive equilibrium point $N^* = e^{a/(b+c)}$. We will therefore supplement this equation with impulses, giving us

$$N'(t) = N(t)\left[a - b\ln(N(t)) - c\ln(N(t-r))\right], \ t \neq \tau_k, \tag{7.2.17a}$$

$$\Delta N(t) = J(t, N(t^-)), \ t = \tau_k, \tag{7.2.17b}$$

where $k = 0, 1, 2 \ldots$.

In order to study the permanence of this system we perform a change of variables just as we did in the Lotka-Volterra model. Letting

$$x = \ln\left(\frac{N}{N^*}\right) = \ln(N) - \frac{a}{b+c}, \tag{7.2.18}$$

and transforming equation (7.2.17) gives us

$$x'(t) = -bx(t) - cx(t-r),\ t \neq \tau_k, \tag{7.2.19a}$$

$$\Delta x(t) = I(t, N(t^-)),\ t = \tau_k, \tag{7.2.19b}$$

where $I(t,x) = \ln(1 + J(t, N^*e^x)/(N^*e^x))$.

As before, showing uniform ultimate boundedness of (7.2.19) will in turn prove uniform permanence of (7.2.17).

This time we use a different Lyapunov function and we apply Theorem 5.3.2. We define

$$V(x) = x^2, \tag{7.2.20}$$

for $x \in \mathbb{R}$. Clearly $V$ satisfies the smoothness conditions as well as condition (i) of Theorem 5.3.2.

Let $\lambda > 1$ be some constant and suppose $b > c\lambda$. Let $g(s) = \lambda^2 s$ and $c(s) = s$ for $s \in \mathbb{R}_+$ and let $p(t) = p = 2(b - c\lambda)$ for all $t \in \mathbb{R}_+$. Then for $\psi \in PC([-r,0],\mathbb{R})$ and $g(V(\psi(0))) \geq V(\psi(s))$ for $s \in [-r,0]$, we have in particular $V(\psi(-r)) \leq \lambda^2 V(\psi(0))$ and so

$$\begin{aligned} D^+_{(7.2.19)}V(t,\psi) &= 2\psi(0)[-b\psi(0) - c\psi(-r)] \\ &\leq -2bV(\psi(0)) + 2c\sqrt{V(\psi(0))V(\psi(-r))} \\ &\leq -2bV(\psi(0)) + 2c\lambda V(\psi(0)) \\ &= -2(b - c\lambda)V(\psi(0)) = -p(t)c(V(\psi(0))). \end{aligned} \tag{7.2.21}$$

Thus condition (ii) of Theorem 5.3.2 is satisfied for any $\rho > 0$. Interestingly, this shows that in the absence of impulses, solutions of (7.2.19) are already uniformly ultimately bounded and hence solutions of (7.2.17) are uniformly permanent. Note that this result is independent of the value of $a$.

Solving for $M_1$ and $M_2$ in condition (v) gives us $M_2 = 2\ln(\lambda)$ and $M_1 = 2\mu(b - c\lambda)$ where $\mu = \inf_{k \in \mathbb{Z}^+}\{\tau_k - \tau_{k-1}\} > 0$. Therefore $\mu$ must be sufficiently large to ensure that $M_1 > M_2$.

If we let $1 \leq \hat{\lambda} < \lambda$ and define $\hat{g}(s) = \hat{\lambda}^2 s$, then $s \leq \hat{g}(s) \leq g(s)$ for all $s > 0$. Finally, we need only check that the function $I$ (which depends on $J$) is defined so that conditions (iii) and (iv) of Theorem 5.3.2 are satisfied. If so then Theorem 5.3.2 tells us that solutions of system (7.2.19) are uniformly ultimately bounded and hence solutions of (7.2.17) are uniformly permanent.

Let us conclude by examining a numerical example.

**Example 7.2.2:** Consider the impulsive delay differential equation

$$N'(t) = N(t)[2 - 4\ln(N(t)) - \ln(N(t-r))],\ t \neq \tau_k, \tag{7.2.22a}$$

$$\Delta N(t) = -N(t) + e^{-2/5}N^2(t),\ t = \tau_k, \tag{7.2.22b}$$

where $r > 0$ is arbitrary and $\tau_k = 2k$ for $k = 0, 1, 2, \ldots$.

In this example $a = 2$, $b = 4$ and $c = 1$. Let $\lambda = 3$ and $\hat{\lambda} = 2$. Then $g(s) = 9s$, $\hat{g}(s) = 4s$ and $p = 2$. Calculating $\mu$, $M_1$ and $M_2$ for this equation gives us $\mu = 2$, $M_2 = 2\ln(3) = 2.1972$ and $M_1 = 4$. Therefore the inequality $M_1 > M_2$ is established. The positive equilibrium point of (7.2.22) is $N^* = e^{2/5} = 1.4918$.

Since $J(t, N) = -N + e^{-2/5}N^2$ then $I(t, x) = x$. Therefore

$$V(x + I(\tau_k, x)) = V(2x) = 4x^2 = \hat{g}(V(x)). \tag{7.2.23}$$

In other words, condition (iii) of Theorem 5.3.2 is satisfied for any choice of $\rho > 0$. Moreover, condition (iv) is satisfied if we simply let $\rho_1 = 2\rho$.

Therefore according to Theorem 5.3.2, solutions of the transformed system are uniformly ultimately bounded, implying that solutions of system (7.2.22) are uniformly permanent. □

# Chapter 8

# Applications to Control Systems

From a practical point of view, a physical system is considered to be stable if its state remains within certain bounds of the equilibrium position for all time. Such bounds will depend on the particular physical system and will generally depend on the constraints imposed on the initial values and the system disturbances. The state of a system may be mathematically unstable and yet the system may oscillate sufficiently near an equilibrium that its performance is considered to be acceptable. Many problems fall into this category including the travel of a space vehicle between two points and the problem, in a chemical process, of keeping the temperature within certain bounds. To deal with such situations, the notion of practical stability is very useful [Lak93, Liu92]. Recently, some practical stability results were obtained for linear-quadratic (LQ) regulators with delayed and bounded disturbances in [Hou98].

We shall investigate practical stability properties of impulsive delay differential equations of the form (2.4.7). Sufficient conditions for strong practical stability are obtained by using the method of Lyapunov functions and Razumikhin-type techniques similar to those introduced in Chapters 4 and 5 where we established boundedness and stability conditions for system (2.4.7). Then as an application we shall examine the practical stability properties of the LQ regulator problem considered in [Hou97, Hou98] with a delayed and disturbed control signal but also with impulsive perturbations of the state.

## 8.1 Practical Stability

In this section we will concern ourselves with the general system (2.4.7) while in the next section we will focus on an even more specific type of equation. For simplicity we will assume throughout this chapter that solutions of system (2.4.7) exist globally. Of course this will be true if the conditions of Corollary 3.3.1 are satisfied, which indeed they will be for system (8.2.8) of Section 8.2. In fact we could deduce global existence from Theorem 8.1.1 if we were to modify its proof. However, we choose to keep the proof relatively simple and so we therefore assume global existence. Let us first introduce the concepts of practical stability and strong practical stability.

**Definition 8.1.1:** System (2.4.7) is said to be

(PS1) practically stable with respect to $(\lambda, \alpha)$ at $t_0$, where $0 < \lambda \leq \alpha$ and $t_0 \in \mathbb{R}_+$, if whenever $\phi \in PC([-r, 0], \mathbb{R}^n)$ with $\|\phi\|_r \leq \lambda$ and $x = x(t_0, \phi)$ is any solution of (2.4.7) & (2.4.2), then $\|x(t, t_0, \phi)\| \leq \alpha$ for all $t \geq t_0$;

(PS2) strongly practically stable with respect to $(\lambda, \alpha, \beta, T)$ at $t_0$, where $0 < \beta < \lambda \leq \alpha$, $T > 0$ and $t_0 \in \mathbb{R}_+$, if whenever $\phi \in PC([-r, 0], \mathbb{R}^n)$ with $\|\phi\|_r \leq \lambda$, then $\|x(t, t_0, \phi)\| \leq \alpha$ for all $t \geq t_0$ and $\|x(t, t_0, \phi)\| \leq \beta$ for all $t \geq t_0 + T$;

As with stability and boundedness one could also define practical stability concepts that are uniform in $t_0$. Specifically, if system (2.4.7) is practically stable with respect to $(\lambda, \alpha)$ (strongly practically stable with respect to $(\lambda, \alpha, \beta, T)$) at every $t_0 \in \mathbb{R}_+$, then we may refer to it as being uniformly practically stable (uniformly strongly practically stable). However, in this chapter we will just focus on the problem of strong practical stability as defined by (PS2).

Our definition of practical stability is taken from [Lak93], although suitably modified for impulses and delays, and is defined fairly consistently throughout the literature. Occasionally, the term "contractively practically stable" is used in place of "strongly practically stable" [Hou98].

While sharing some similarities to the concepts of stability and boundedness, the notion of practical stability is nevertheless distinct from both concepts. Next we introduce our main theorem on practical stability for solutions of system (2.4.7). This theorem is based on work by Hou and Qian [Hou98] who proved a similar theorem for delay differential equations having bounded disturbances but no impulses. Our theorem incorporates impulses and simplifies Hou and Qian's theorem considerably by eliminating some superfluous conditions.

**Theorem 8.1.1:** *Assume $J = [t_0, \infty)$, for some $t_0 \in \mathbb{R}_+$, and $D = \mathbb{R}^n$ and assume that the conditions of Corollary 3.2.3 are satisfied and that solutions of (2.4.7) exist globally. Suppose there exist strictly increasing functions $\nu_1, \nu_2 \in K_4$ and a positive function $p \in PC([t_0 - r, \infty), (0, \infty))$ that is continuous at each $t \neq \tau_k$ in $[t_0 - r, \infty)$ and satisfies $\int_{t_0}^{\infty} p(s)ds = \infty$. Suppose also that there exists a constant $u_0 \geq 0$ and constants $d_k \geq 0$ with $\sum_{k=1}^{\infty} d_k < \infty$ and let $L = \prod_{\{k:\ \tau_k > t_0\}}(1 + d_k) < \infty$. Suppose $V \in C([t_0 - r, \infty) \times \mathbb{R}^n, \mathbb{R}_+)$ and when restricted to $[t_0, \infty) \times \mathbb{R}^n$ assume $V$ is locally Lipschitz in $x$. Assume that the following conditions are satisfied:*

*(i)* $\nu_1(\|x\|) \leq V(t, x) \leq \nu_2(\|x\|)$ *for all* $(t, x) \in [t_0 - r, \infty) \times \mathbb{R}^n$;

*(ii)* $D^+_{(2.4.7)}V(t, \psi(0)) \leq -p(t)(V(t, \psi(0)) - u_0)$ *for all* $t \neq \tau_k$ *in* $(t_0, \infty)$ *and* $\psi \in PC([-r, 0], \mathbb{R}^n)$ *whenever* $V(t, \psi(0)) > u_0$ *and* $V(t, \psi(0)) \geq V(t + s, \psi(s)) \exp\left(-\int_{t-r}^{t} p(\tau)d\tau\right)$ *for* $s \in [-r, 0]$; *and*

*(iii)* $V(\tau_k, \psi(0) + I(\tau_k, \psi)) \le (1 + d_k)V(\tau_k, \psi(0))$ *for all* $(\tau_k, \psi) \in (t_0, \infty) \times PC([-r, 0], \mathbb{R}^n)$ *for which* $\psi(0^-) = \psi(0)$.

*Then for every* $(\lambda, \alpha, \beta)$ *satisfying*

$$0 \le \nu_1^{-1}(Lu_0) < \beta < \lambda \le \nu_1^{-1}(L\nu_2(\lambda)) \le \alpha, \tag{8.1.1}$$

*there exists a* $T > 0$ *such that system (2.4.7) is strongly practically stable with respect to* $(\lambda, \alpha, \beta, T)$ *at* $t_0$. *More specifically,*

$$\|x(t)\| \le \nu_1^{-1}\left(Lu_0 + L(\nu_2(\lambda) - u_0)\exp\left(-\int_{t_0}^{t} p(s)ds\right)\right), \tag{8.1.2}$$

*for* $t \in [t_0 - r, \infty)$.

*Proof:* Suppose $x = x(t_0, \phi)$ is a solution of (2.4.7) & (2.4.2) for some $\phi \in PC([-r, 0], \mathbb{R}^n)$ satisfying $\|\phi\|_r \le \lambda$. By assumption, $x(t)$ exists for $t \in [t_0 - r, \infty)$ and so define

$$m(t) = V(t, x(t)), \tag{8.1.3}$$

for $t \in [t_0 - r, \infty)$. Since $V$ is assumed to be continuous, then $m \in PC([t_0 - r, \infty), \mathbb{R}_+)$ and $m(t)$ is continuous at each $t \ne \tau_k$ in $(t_0, \infty)$. From condition (i) we get

$$\nu_1(\|x(t)\|) \le m(t) \le \nu_2(\|x(t)\|), \tag{8.1.4}$$

for $t \in [t_0 - r, \infty)$.

From condition (ii) and from Corollary 5.1.1, since $V$ is locally Lipschitz in $x$, we get

$$D^+m(t) = \limsup_{h \to 0^+} \frac{1}{h}[m(t + h) - m(t)] \le -p(t)(m(t) - u_0), \tag{8.1.5}$$

for all $t \ne \tau_k$ in $(t_0, \infty)$ whenever $m(t) > u_0$ and $m(t) \ge \|m_t\|_r \exp\left(-\int_{t-r}^{t} p(s)ds\right)$. Also from condition (iii) we have

$$m(\tau_k) \le (1 + d_k)m(\tau_k^-), \tag{8.1.6}$$

for all $\tau_k \in (t_0, \infty)$.

Define

$$w(t) = \begin{cases} 1, & t \in [t_0 - r, t_0], \\ \prod_{\{k:\ \tau_k \in (t_0, t]\}} (1 + d_k), & t \in (t_0, \infty), \end{cases} \tag{8.1.7}$$

where the product is understood to be 1 if the set $\{k : \tau_k \in (t_0, t]\}$ is empty.

Next let $\hat{u} = \max\{u_0, \|m_{t_0}\|_r\}$. For $t \in [t_0 - r, t_0]$ we have $m(t) \leq \nu_2(\|x(t)\|) = \nu_2(\|\phi(t - t_0)\|) \leq \nu_2(\|\phi\|_r) \leq \nu_2(\lambda)$. Also since $\nu_1^{-1}(Lu_0) < \nu_1^{-1}(L\nu_2(\lambda))$ by (8.1.1), then $u_0 < \nu_2(\lambda)$. Therefore $\hat{u} \leq \nu_2(\lambda)$.

Define $l(t) = m(t) - w(t)\left[u_0 + (\hat{u} - u_0)\exp\left(-\int_{t_0}^{t} p(s)ds\right)\right]$. We will show that $l(t) \leq 0$ for $t \in [t_0 - r, \infty)$. For $t \in [t_0 - r, t_0]$, $l(t) = m(t) - \left[u_0 + (\hat{u} - u_0)\exp\left(-\int_{t_0}^{t} p(s)ds\right)\right] \leq \|m_{t_0}\|_r - u_0 - (\hat{u} - u_0) \leq 0$.

Assume $\tau_{i-1} \leq t_0 < \tau_i$ for some $i$. We first show $l(t) \leq 0$ for $t \in (t_0, \tau_i)$. In order to do this we let $\delta > 0$ be arbitrary and show that $l(t) \leq \delta$ for $t \in (t_0, \tau_i)$. Suppose for the sake of contradiction that $l(t) > \delta$ for some $t \in (t_0, \tau_i)$. Then let $t^* = \inf\{t \in (t_0, \tau_i) : l(t) > \delta\}$. Then $t^* > t_0$ since $l(t_0) \leq 0 < \delta$ and $t^* < \tau_i$. Note that $l(t)$ is continuous on $[t_0, \tau_i)$. For $t \in [t_0, t^*]$, $l(t) \leq \delta$ and $l(t^*) = \delta$.

Now $m(t^*) = l(t^*) + \left[u_0 + (\hat{u} - u_0)\exp\left(-\int_{t_0}^{t^*} p(s)ds\right)\right] \geq \delta + u_0 > u_0$. For $s \in [-r, 0]$,

$$\begin{aligned} m(t^* + s) &\leq \delta + \left[u_0 + (\hat{u} - u_0)\exp\left(-\int_{t_0}^{t^*+s} p(\tau)d\tau\right)\right] \\ &\leq \delta + \left[u_0 + (\hat{u} - u_0)\exp\left(-\int_{t_0}^{t^*-r} p(\tau)d\tau\right)\right] \\ &\leq \left[\delta + u_0 + (\hat{u} - u_0)\exp\left(-\int_{t_0}^{t^*} p(\tau)d\tau\right)\right]\exp\left(\int_{t^*-r}^{t^*} p(\tau)d\tau\right) \\ &= m(t^*)\exp\left(\int_{t^*-r}^{t^*} p(\tau)d\tau\right). \end{aligned} \tag{8.1.8}$$

which implies $\|m_{t^*}\|_r \leq m(t^*)\exp\left(\int_{t^*-r}^{t^*} p(s)ds\right)$. So by (8.1.5) we have $D^+m(t^*) \leq -p(t^*)(m(t^*) - u_0)$. Using this inequality, we find that

$$\begin{aligned} D^+l(t^*) &= D^+m(t^*) + (\hat{u} - u_0)\exp\left(-\int_{t_0}^{t^*} p(s)ds\right)p(t^*) \\ &\leq -p(t^*)\left[m(t^*) - u_0 - (\hat{u} - u_0)\exp\left(-\int_{t_0}^{t^*} p(s)ds\right)\right] \\ &= -p(t^*)l(t^*) = -p(t^*)\delta < 0. \end{aligned} \tag{8.1.9}$$

This contradicts the definition of $t^*$. Thus it must be true that $l(t) \leq \delta$ for all $t \in (t_0, \tau_i)$. Letting $\delta \to 0^+$ gives us $l(t) \leq 0$ for $t \in (t_0, \tau_i)$. Consequently $l(\tau_i^-) \leq 0$.

So far we have proven that $l(t) \leq 0$ for $t \in [t_0 - r, \tau_i)$. To prove $l(t) \leq 0$ for all $t \in [t_0 - r, \infty)$ we use an induction argument. As our induction hypothesis, suppose $l(t) \leq 0$ for $t \in [t_0 - r, \tau_j)$ for some $j \geq i$ and let us prove that $l(t) \leq 0$ for $t \in [t_0 - r, \tau_{j+1})$.

To begin with

$$\begin{aligned} l(\tau_j) &= m(\tau_j) - w(\tau_j)\left[u_0 + (\hat{u} - u_0)\exp\left(-\int_{t_0}^{\tau_j} p(s)ds\right)\right] \\ &\le (1+d_j)m(\tau_j^-) - (1+d_j)w(\tau_j^-)\left[u_0 + (\hat{u} - u_0)\exp\left(-\int_{t_0}^{\tau_j} p(s)ds\right)\right] \\ &= (1+d_j)l(\tau_j^-) \le 0. \end{aligned} \tag{8.1.10}$$

by (8.1.6). Let $\delta > 0$. We want to show $l(t) \le \delta$ for $t \in (\tau_j, \tau_{j+1})$. Suppose otherwise and define $t^* = \inf\{t \in (\tau_j, \tau_{j+1}) : l(t) > \delta\}$. Then $\tau_j < t^* < \tau_{j+1}$, $l(t^*) = \delta$ and $l(t) \le \delta$ for $t \in [\tau_j, t^*]$.

Now $m(t^*) = \delta + w(t^*)\left[u_0 + (\hat{u} - u_0)\exp\left(-\int_{t_0}^{t^*} p(s)ds\right)\right] \ge \delta + u_0 > u_0$. For $s \in [-r, 0]$,

$$\begin{aligned} m(t^* + s) &\le \delta + w(t^* + s)\left[u_0 + (\hat{u} - u_0)\exp\left(-\int_{t_0}^{t^*+s} p(\tau)d\tau\right)\right] \\ &\le \delta + w(t^*)\left[u_0 + (\hat{u} - u_0)\exp\left(-\int_{t_0}^{t^*-r} p(\tau)d\tau\right)\right] \\ &\le \left[\delta + w(t^*)\left(u_0 + (\hat{u} - u_0)\exp\left(-\int_{t_0}^{t^*} p(\tau)d\tau\right)\right)\right]\exp\left(\int_{t^*-r}^{t^*} p(\tau)d\tau\right) \\ &= m(t^*)\exp\left(\int_{t^*-r}^{t^*} p(\tau)d\tau\right). \end{aligned} \tag{8.1.11}$$

Thus $\|m_{t^*}\|_r \le m(t^*)\exp\left(\int_{t^*-r}^{t^*} p(s)ds\right)$ and hence by (8.1.5) we have $D^+m(t^*) \le -p(t^*)(m(t^*) - u_0)$.

As before we use this inequality to show $D^+l(t^*) < 0$ as follows.

$$\begin{aligned} D^+l(t^*) &= D^+m(t^*) + w(t^*)(\hat{u} - u_0)\exp\left(-\int_{t_0}^{t^*} p(s)ds\right)p(t^*) \\ &\le -p(t^*)\left[m(t^*) - u_0 - w(t^*)(\hat{u} - u_0)\exp\left(-\int_{t_0}^{t^*} p(s)ds\right)\right] \\ &\le -p(t^*)\left[m(t^*) - w(t^*)u_0 - w(t^*)(\hat{u} - u_0)\exp\left(-\int_{t_0}^{t^*} p(s)ds\right)\right] \\ &= -p(t^*)l(t^*) = -p(t^*)\delta < 0. \end{aligned} \tag{8.1.12}$$

Again this contradicts the definition of $t^*$, which leads us to conclude that $l(t) \le \delta$ for all $t \in (\tau_j, \tau_{j+1})$. Letting $\delta \to 0^+$ gives us $l(t) \le 0$ for $t \in (\tau_j, \tau_{j+1})$. Thus $l(t) \le 0$ for all $t \in [t_0 - r, \tau_{j+1})$, which proves the induction step.

The above induction argument proves that $l(t) \leq 0$ for all $t \in [t_0 - r, \infty)$. Since $w(t) \leq L < \infty$ for all such $t$, then this implies

$$m(t) \leq L\left[u_0 + (\hat{u} - u_0)\exp\left(-\int_{t_0}^{t} p(s)ds\right)\right], \tag{8.1.13}$$

for $t \in [t_0 - r, \infty)$. In light of (8.1.4) and since $\hat{u} \leq \nu_2(\lambda)$ this gives us

$$\|x(t)\| \leq \nu_1^{-1}\left(Lu_0 + L(\nu_2(\lambda) - u_0)\exp\left(-\int_{t_0}^{t} p(s)ds\right)\right), \tag{8.1.14}$$

for $t \in [t_0 - r, \infty)$.

From (8.1.14) we get that $\|x(t)\| \leq \nu_1^{-1}(L\nu_2(\lambda)) \leq \alpha$ for $t \in [t_0, \infty)$. Thus $\|x(t)\| \leq \alpha$ for all $t \in [t_0 - r, \infty)$. Since $\beta > \nu_1^{-1}(Lu_0)$ and $\int_{t_0}^{\infty} p(s)ds = \infty$, then (8.1.14) implies the existence of a $T > 0$ such that $\|x(t)\| \leq \beta$ for all $t \geq t_0 + T$. This completes the proof. ■

We remark that if $u_0 = 0$ in Theorem 8.1.1, then system (2.4.7) will be asymptotically stable. Moreover, weaker assumptions on the constants $d_k$ may be imposed that would not require that $\sum_{k=1}^{\infty} d_k$ be finite. Disturbances in the system are what typically cause $u_0$ to become positive, which in turn increases the practical stability parameter $\beta$.

## 8.2 Control Systems

In this section we will apply the theorem on practical stability proved in Section 8.1 to a control problem. We use similar matrix notation that was used in Chapter 6.

To demonstrate the application of the theorem presented in the previous section we examine the LQ regulator problem with a delayed and disturbed control signal and with impulsive perturbations of the state. As in the previous section, our results in this section are based on the results by Hou and Qian [Hou98] except that our systems also involve impulses and we have corrected some significant flaws in the proof given by Hou and Qian.

First consider the linear system

$$x'(t) = Ax(t) + Bu(t), \; x(t_0) = x_0, \tag{8.2.1}$$

where $x \in \mathbb{R}^n$ denotes the state of the system, $u \in \mathbb{R}^m$ denotes the control signal and $x_0 \in \mathbb{R}^n$ is the state at the initial time $t_0 \in \mathbb{R}_+$. The matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are assumed to be constant with $B \neq 0$ and the pair $(A, B)$ is assumed to be stabilizable (i.e. there exists a matrix $K \in \mathbb{R}^{m\times n}$ such that $A - BK$ is Hurwitz).

Under these assumptions, system (8.2.1) can be stabilized by finding an appropriate matrix $K$

and then using a linear state feedback controller of the form

$$u(t) = -Kx(t). \tag{8.2.2}$$

The gain matrix $K$ for which this feedback control will stabilize system (8.2.1) is not unique. However, a systematic way of finding a unique $K$ is by minimizing the quadratic cost functional defined by

$$v(x_0) = \int_{t_0}^{\infty} \left(x(t)^T Qx(t) + u(t)^T Ru(t)\right) dt, \tag{8.2.3}$$

where $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ are constant positive definite matrices. It can be shown (see [Ath66]) that the input $u(t)$ that minimizes $v(x_0)$ is given by (8.2.2) where

$$K = R^{-1}B^T P, \tag{8.2.4}$$

and $P$ is the unique (constant) positive definite matrix solution of the nonlinear algebraic Riccati equation

$$A^T P + PA - PBR^{-1}B^T P + Q = 0. \tag{8.2.5}$$

In practice, closed loop systems may experience uncertain delayed perturbations, uncertain bounded disturbances and impulsive perturbations. To take these into account we consider the following modified model

$$x'(t) = Ax(t) + B(u(t - h(t)) + \eta(t)), \ t \neq \tau_k, \tag{8.2.6a}$$

$$\Delta x(t) = C_k x(t^-), \ t = \tau_k. \tag{8.2.6b}$$

This is a special case of system (2.4.7). The function $\eta(t)$ represents an uncertain, bounded disturbance. We assume $\|\eta(t)\| \leq \epsilon$ for all $t \geq t_0$ where $\epsilon > 0$ is constant. We assume there exists a constant $\bar{h} > 0$ such that the delay perturbation function $h(t)$ is bounded by $0 \leq h(t) \leq \bar{h}$ for $t \geq t_0$. The functional defined in the delay differential equation (8.2.6a) is of the form (F2) of Section 3.1 if we further assume that $\eta \in C(\mathbf{R}_+, \mathbf{R}^m)$, $h \in C(\mathbf{R}_+, \mathbf{R}_+)$, $t - h(t)$ is nondecreasing in $t$ and the input $u$ is sufficiently smooth. In fact system (8.2.6) then satisfies the conditions of Corollaries 3.1.1 and 3.3.1 (if the input $u$ depends linearly on the state), ensuring global existence of solutions.

The stability properties of linear time invariant control systems such as (8.2.6) without disturbances or impulses (i.e. $h(t) =$ constant, $\eta(t) = 0$ and $C_k = 0$) have been studied using a variety of techniques. These include the use of Lyapunov functions, which we use here, as well as such other techniques as the application of Nyquist stability theorems which can be applied equally well to both delay systems and non-delay systems [Shi92].

As usual the impulse times are assumed to satisfy $0 = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_k < \ldots$ and $\lim_{k\to\infty} \tau_k = \infty$ and we assume the matrices $C_k \in \mathbb{R}^{n\times n}$ are constant for $k = 1, 2, \ldots$. The initial condition for system (8.2.6) will of course be given by

$$x_{t_0} = \phi, \tag{8.2.7}$$

where $\phi \in PC([-\bar{h}, 0], \mathbb{R}^n)$.

We wish to investigate the practical stability of the closed loop system (8.2.6) when subject to the control strategy (8.2.2)-(8.2.5). In other words, we consider the system

$$x'(t) = Ax(t) - BKx(t - h(t)) + B\eta(t), \ t \neq \tau_k, \tag{8.2.8a}$$

$$\Delta x(t) = C_k x(t^-), \ t = \tau_k, \tag{8.2.8b}$$

where $K$ is defined in (8.2.4).

We define the following parameters for future use. Let

$$a = \lambda_{\min}\left(P^{-1/2}\left(Q + K^T RK\right) P^{-1/2}\right), \tag{8.2.9}$$

$$d = \lambda_{\max}\left(P^{-1/2}(A^T P + PA)P^{-1/2}\right), \tag{8.2.10}$$

$$c_1 = \epsilon \left\|P^{1/2}B\right\|, \tag{8.2.11}$$

$$c_2 = \left\|P^{-1/2}K^T RKP^{-1/2}\right\|, \tag{8.2.12}$$

$$c_3 = \max\{0, d + c_1\} + 2c_2, \tag{8.2.13}$$

$$b = 2\bar{h}\left(\left\|P^{-1/2}K^T RKAP^{-1/2}\right\| + c_2^2\right), \tag{8.2.14}$$

$$q = 2\epsilon\left(\left\|P^{1/2}B\right\| + \bar{h}\left\|P^{-1/2}K^T RKB\right\|\right), \tag{8.2.15}$$

$$d_k = \max\left\{0, \lambda_{\max}\left(P^{-1/2}\left(C_k^T PC_k + PC_k + C_k^T P\right) P^{-1/2}\right)\right\}, \ k = 1, 2, \ldots, \tag{8.2.16}$$

$$L = \prod_{\{k:\ \tau_k > t_0\}} (1 + d_k), \tag{8.2.17}$$

$$\hat{L} = \prod_{\{k:\ \tau_k > t_0 + \bar{h}\}} (1 + d_k), \tag{8.2.18}$$

$$\sigma = \sup_{t > t_0 + \bar{h}} \sum_{\{k:\ \tau_k \in (t - \bar{h}, t)\}} \left\|P^{-1/2}K^T RKC_k P^{-1/2}\right\|, \tag{8.2.19}$$

and, if $\sigma < \infty$, we let

$$\hat{b} = b + 2\sigma, \tag{8.2.20}$$

and we define the function

$$g(\theta) = \theta(a - \hat{b}e^{\bar{h}\theta} - \theta), \tag{8.2.21}$$

for $\theta \in \mathbb{R}$. Furthermore, if $a > \hat{b} > 0$, then we define $\theta_0$ to be the unique solution of the equation

$$g'(\theta_0) = a - \hat{b}(1 + \bar{h}\theta_0)e^{\bar{h}\theta_0} - 2\theta_0 = 0, \tag{8.2.22}$$

and we let

$$u_0 = \frac{q^2}{4g(\theta_0)}. \tag{8.2.23}$$

The critical point $\theta_0$ of $g(\theta)$ is where $g(\theta)$ attains an absolute maximum. Note that $a, c_1, c_2, c_3, b$ and $q$ are all positive constants.

**Theorem 8.2.1:** *Suppose that*

*(i)* $\sum_{k=1}^{\infty} d_k < \infty$ *(hence* $\hat{L}, L < \infty$*);*

*(ii)* $\sigma < \infty$*;*

*(iii)* $a > \hat{b} > 0$*;*

*(iv)* $(\lambda, \alpha, \beta)$ *satisfy*

$$0 < \sqrt{\frac{\hat{L}u_0}{\lambda_{\min}(P)}} < \beta \leq \lambda \leq \hat{\lambda}\sqrt{\frac{\hat{L}\lambda_{\max}(P)}{\lambda_{\min}(P)}} \leq \alpha, \tag{8.2.24}$$

*where*

$$\hat{\lambda} = \left\{ \frac{1}{\lambda_{\min}(P)} \prod_{\{k:\ \tau_k \in (t_0, t_0+\bar{h}]\}} (1 + d_k) \left( e^{c_3\bar{h}} \lambda_{\max}(P)\lambda^2 + \frac{c_1}{c_3}\left(e^{c_3\bar{h}} - 1\right)\right)\right\}^{1/2}. \tag{8.2.25}$$

*Then there exists a $T > 0$ such that system (8.2.8) is strongly practically stable with respect to $(\lambda, \alpha, \beta, T)$ at $t_0$. More specifically,*

$$\|x(t)\| \leq \left\{ \frac{1}{\lambda_{\min}(P)} \left[\hat{L}u_0 + \hat{L}(\lambda_{\max}(P)\hat{\lambda}^2 - u_0)e^{-(t-t_0-\bar{h})\theta_0}\right]\right\}^{1/2}, \tag{8.2.26}$$

*for $t \in [t_0 - \bar{h}, \infty)$.*

*Proof:* Define $V(x) = x^T P x$ for $x \in \mathbb{R}^n$. Then

$$\lambda_{\min}(P)\|x\|^2 \leq V(x) \leq \lambda_{\max}(P)\|x\|^2, \tag{8.2.27}$$

for $x \in \mathbb{R}^n$. Let $x = x(t_0, \phi)$ be any solution of (8.2.8) & (8.2.7) with $\|\phi\|_{\bar{h}} \le \lambda$.

If $\tau_k > t_0$ and $\psi \in PC([-\bar{h}, 0], \mathbb{R}^n)$ with $\psi(0) = \psi(0^-)$, then we have

$$\begin{aligned} V(\psi(0) + I(\tau_k, \psi)) &= V(\psi(0) + C_k\psi(0)) = (\psi(0) + C_k\psi(0))^T P(\psi(0) + C_k\psi(0)) \\ &= \psi(0)^T P\psi(0) + \psi(0)^T (C_k^T PC_k + PC_k + C_k^T P)\psi(0) \\ &= V(\psi(0)) + \left(P^{1/2}\psi(0)\right)^T P^{-1/2}\left(C_k^T PC_k + PC_k + C_k^T P\right) P^{-1/2}\left(P^{1/2}\psi(0)\right) \\ &\le V(\psi(0)) + \lambda_{\max}\left(P^{-1/2}\left(C_k^T PC_k + PC_k + C_k^T P\right) P^{-1/2}\right)\psi(0)^T P\psi(0) \\ &\le (1 + d_k)V(\psi(0)). \end{aligned} \tag{8.2.28}$$

For $t \ge t_0 - r$ define

$$m(t) = V(t, x(t)) = x(t)^T Px(t), \tag{8.2.29}$$

to be the Lyapunov function evaluated along the particular solution. Then, in particular, we have

$$m(\tau_k) \le (1 + d_k)m(\tau_k^-), \tag{8.2.30}$$

for each $\tau_k > t_0$.

Also for $t \ne \tau_k$ in $(t_0, \infty)$ we have

$$\begin{aligned} D^+m(t) &= 2x(t)^T Px'(t) = 2x(t)^T P\left(Ax(t) - BKx(t - h(t)) + B\eta(t)\right) \\ &= 2x(t)^T PAx(t) - 2x(t)^T PBKx(t - h(t)) + 2x(t)^T PB\eta(t) \\ &= x(t)^T (A^T P + PA)x(t) - 2x(t)^T K^T RKx(t - h(t)) + 2x(t)^T PB\eta(t). \end{aligned} \tag{8.2.31}$$

We will first show that $\|x(t)\| \le \hat{\lambda}$ for $t \in [t_0 - \bar{h}, t_0 + \bar{h}]$. To do this, we define $W \in PC([t_0 - \bar{h}, t_0 + \bar{h}], \mathbb{R}_+)$ by

$$W(t) = \sup_{s \in [t_0 - \bar{h}, t]} m(s), \tag{8.2.32}$$

for $t \in [t_0 - \bar{h}, t_0 + \bar{h}]$. Note that $W$ is piecewise continuous, with discontinuities occurring only at discontinuous points of $x(t)$, and $W$ is nondecreasing. Therefore $D^+W(t) \ge 0$ for $t \in (t_0, t_0 + \bar{h})$.

If $D^+W(t) > 0$ for some $t \in (t_0, t_0 + \bar{h})$ with $t \ne \tau_k$, then we claim $D^+W(t) = D^+m(t)$. To see this first consider that if $m(t) < W(t)$, then by continuity, $m(s) < W(t)$ for all $s \in [t, t + \delta]$ where $\delta > 0$ is sufficiently small. This would imply $W(s) = W(t)$ for $s \in [t, t + \delta]$, which would in turn imply $D^+W(t) = 0$. But since we are assuming that $D^+W(t) > 0$, we must have $m(t) = W(t)$.

Now,

$$\begin{aligned} D^+W(t) &= \limsup_{\rho\to 0^+} \frac{1}{\rho}[W(t+\rho) - W(t)] = \inf_{\delta>0} \sup_{0<\rho<\delta} \frac{1}{\rho}\left[\sup_{s\in[t_0-h,t+\rho]} m(s) - m(t)\right] \\ &= \inf_{\delta>0} \sup_{0<\rho<\delta} \frac{1}{\rho}\left[\sup_{s\in[t,t+\rho]} m(s) - m(t)\right]. \end{aligned} \tag{8.2.33}$$

Let $\delta > 0$ be sufficiently small so that the interval $[t, t+\delta]$ is free of impulses and let $0 < \rho < \delta$. Let $\rho_1 \in [0, \rho]$ satisfy $m(t+\rho_1) = \sup_{s\in[t,t+\rho]} m(s)$. Since $D^+W(t) > 0$, then $0 < \rho_1 \leq \rho$ and so $1/\rho_1 \geq 1/\rho$, which implies

$$\begin{aligned} \frac{1}{\rho}\left[\sup_{s\in[t,t+\rho]} m(s) - m(t)\right] &\leq \frac{1}{\rho_1}[m(t+\rho_1) - m(t)] \\ &\leq \sup_{0<\rho_2<\delta} \frac{1}{\rho_2}[m(t+\rho_2) - m(t)]. \end{aligned} \tag{8.2.34}$$

Since this is true for all $0 < \rho < \delta$, taking the supremum over all such $\rho$ gives

$$\begin{aligned} \sup_{0<\rho<\delta} \frac{1}{\rho}\left[\sup_{s\in[t,t+\rho]} m(s) - m(t)\right] &\leq \sup_{0<\rho_2<\delta} \frac{1}{\rho_2}[m(t+\rho_2) - m(t)] \\ &\leq \sup_{0<\rho_2<\delta} \frac{1}{\rho_2}\left[\sup_{s\in[t,t+\rho_2]} m(s) - m(t)\right], \end{aligned} \tag{8.2.35}$$

and taking the infimum over $\delta > 0$ reduces these inequalities to

$$D^+W(t) \leq D^+m(t) \leq D^+W(t), \tag{8.2.36}$$

from which we conclude $D^+W(t) = D^+m(t)$.

Therefore if $D^+W(t) > 0$ for some $t \in (t_0, t_0 + \bar{h})$ with $t \neq \tau_k$, then by (8.2.31)

$$\begin{aligned}
D^+W(t) &= D^+m(t) \\
&= x(t)^T(A^TP + PA)x(t) - 2x(t)^TK^TRKx(t-h(t)) + 2x(t)^TPB\eta(t) \\
&= \left(P^{1/2}x(t)\right)^T \left(P^{-1/2}\left(A^TP + PA\right)P^{-1/2}\right)\left(P^{1/2}x(t)\right) \\
&\quad - 2\left(P^{1/2}x(t)\right)^T\left(P^{-1/2}K^TRKP^{-1/2}\right)\left(P^{1/2}x(t-h(t))\right) \\
&\quad + 2\left(P^{1/2}x(t)\right)^T P^{1/2}B\eta(t) \\
&\le \lambda_{\max}\left(P^{-1/2}\left(A^TP + PA\right)P^{-1/2}\right)x(t)^TPx(t) \\
&\quad + 2\left\|P^{-1/2}K^TRKP^{-1/2}\right\| \cdot \left\|P^{1/2}x(t)\right\| \cdot \left\|P^{1/2}x(t-h(t))\right\| \\
&\quad + 2\epsilon\left\|P^{1/2}B\right\| \cdot \left\|P^{1/2}x(t)\right\| \\
&= dm(t) + 2c_2\sqrt{m(t)}\sqrt{m(t-h(t))} + 2c_1\sqrt{m(t)} \\
&\le (d + c_1)m(t) + 2c_2\sqrt{m(t)}\sqrt{m(t-h(t))} + c_1 \\
&\le \max\{0, d + c_1\}W(t) + 2c_2\sqrt{W(t)}\sqrt{W(t)} + c_1 = c_3W(t) + c_1.
\end{aligned} \tag{8.2.37}$$

Thus

$$0 \le D^+W(t) \le c_3W(t) + c_1, \tag{8.2.38}$$

for all $t \in (t_0, t_0 + \bar{h})$ with $t \neq \tau_k$. Given any $t_1, t_2 \in [t_0, t_0 + \bar{h}]$ with $\tau_k \le t_1 < t_2 < \tau_{k+1}$ for some $k$, we can solve this differential inequality to get

$$W(t_2) \le e^{c_3(t_2 - t_1)}W(t_1) + \frac{c_1}{c_3}\left(e^{c_3(t_2-t_1)} - 1\right). \tag{8.2.39}$$

By (8.2.30) we have

$$\begin{aligned}
W(\tau_k) &= \max\left\{\sup_{s\in[t_0-\bar{h},\tau_k)} m(s), m(\tau_k)\right\} = \max\{W(\tau_k^-), m(\tau_k)\} \\
&\le \max\{W(\tau_k^-), (1 + d_k)m(\tau_k^-)\} \le \max\{W(\tau_k^-), (1 + d_k)W(\tau_k^-)\} \\
&= (1 + d_k)W(\tau_k^-),
\end{aligned} \tag{8.2.40}$$

for all $\tau_k \in (t_0, t_0 + \bar{h}]$. Also from (8.2.27) we have, for $t \in [t_0 - \bar{h}, t_0]$,

$$m(t) = V(x(t)) \le \lambda_{\max}(P)\|x(t)\|^2 = \lambda_{\max}(P)\|\phi(t - t_0)\|^2 \le \lambda_{\max}(P)\lambda^2, \tag{8.2.41}$$

and so we get

$$W(t_0) \le \lambda_{\max}(P)\lambda^2, \tag{8.2.42}$$

by the definition of $W$.

By combining inequalities (8.2.39), (8.2.40) and (8.2.42), we obtain the following upper bound for $W(t_0+\bar{h})$

$$W(t_0+\bar{h}) \leq \prod_{\{k:\ \tau_k\in(t_0,t_0+\bar{h}]\}} (1+d_k)\left(e^{c_3\bar{h}}\lambda_{\max}(P)\lambda^2+\frac{c_1}{c_3}\left(e^{c_3\bar{h}}-1\right)\right). \tag{8.2.43}$$

Since

$$\|x(t)\| \leq \left(\frac{1}{\lambda_{\min}(P)}m(t)\right)^{1/2} \leq \left(\frac{1}{\lambda_{\min}(P)}W(t_0+\bar{h})\right)^{1/2}, \tag{8.2.44}$$

for $t\in[t_0-\bar{h},t_0+\bar{h}]$ by (8.2.27), we get

$$\|x(t)\| \leq \hat{\lambda}, \tag{8.2.45}$$

for $t\in[t_0-\bar{h},t_0+\bar{h}]$ where $\hat{\lambda}$ is defined by (8.2.25).

Next we show that the conditions of Theorem 8.1.1 are satisfied where the initial time is now taken to be $t_0+\bar{h}$ and the delay constant $r$ is given by $r=2\bar{h}$. Conditions (i) and (iii) of Theorem 8.1.1 are clearly satisfied by (8.2.27) and (8.2.28), respectively. We define

$$p(t)=\theta_0, \tag{8.2.46}$$

for all $t\in[t_0-\bar{h},\infty)$ where $\theta_0>0$ is defined to be the unique solution of (8.2.22). Thus $p$ satisfies the properties given in Theorem 8.1.1.

As for checking condition (ii) we deviate slightly in terms of what we shall show. From the proof of Theorem 8.1.1 we see that it suffices to show only that

$$D^+m(t) \leq -p(t)(m(t)-u_0), \tag{8.2.47}$$

for all $t\neq\tau_k$ in $(t_0+\bar{h},\infty)$ (where $m(t)$ is defined by (8.2.29)) whenever $m(t)>u_0$ and $m(t)\geq \|m_t\|_r\exp\left(-\int_{t-r}^{t}p(s)ds\right)$. In showing (8.2.47) we will make use of ideas similar to those described in Section 4.1 following equation (4.1.21). We will start with equation (8.2.31) and perform some simplifying inequalities. Then we will express the term $x(t)-x(t-h(t))$ as the integral of the derivative of the state from $t-h(t)$ to $t$ plus the sum of all impulse jumps according to Lemma 2.4.1. We can then replace the derivative of the state inside the integral by equation (8.2.8a), since we know the equation must be satisfied for all such $t$.

Note that $\exp\left(-\int_{t-r}^{t} p(s)ds\right) = e^{-2\bar{h}\theta_0}$. For $t \neq \tau_k$ in $(t_0 + \bar{h}, \infty)$, we have

$$\begin{aligned}
D^+m(t) &= x(t)^T(A^TP + PA)x(t) - 2x(t)^TK^TRKx(t-h(t)) + 2x(t)^TPB\eta(t) \\
&= x(t)^T(K^TRK - Q)x(t) - 2x(t)^TK^TRKx(t-h(t)) + 2x(t)^TPB\eta(t) \\
&= -x(t)^T(K^TRK + Q)x(t) + 2x(t)^TK^TRK(x(t) - x(t-h(t))) \\
&\quad + 2x(t)^TPB\eta(t) \\
&= -\left(P^{1/2}x(t)\right)^T\left(P^{-1/2}\left(K^TRK + Q\right)P^{-1/2}\right)\left(P^{1/2}x(t)\right) \\
&\quad + 2x(t)^TK^TRK(x(t) - x(t-h(t))) + 2\left(P^{1/2}x(t)\right)^TP^{1/2}B\eta(t) \\
&\leq -\lambda_{\min}\left(P^{-1/2}\left(K^TRK + Q\right)P^{-1/2}\right)x(t)^TPx(t) \\
&\quad + 2x(t)^TK^TRK(x(t) - x(t-h(t))) + 2\epsilon\left\|P^{1/2}B\right\|\cdot\left\|P^{1/2}x(t)\right\| \\
&= -am(t) + 2x(t)^TK^TRK(x(t) - x(t-h(t))) + 2c_1\sqrt{m(t)}.
\end{aligned} \tag{8.2.48}$$

Moreover, if $t \in (t_0 + \bar{h}, \infty)$, $t \neq \tau_k$ and $m(t) \geq e^{-2\bar{h}\theta_0}m(s)$ for all $s \in [t - 2\bar{h}, t]$, then

$$\begin{aligned}
&x(t)^TK^TRK(x(t) - x(t-h(t))) \\
&= x(t)^TK^TRK\left(\int_{t-h(t)}^{t} x'(s)ds + \sum_{\{k:\ \tau_k \in (t-h(t),t)\}} \Delta x(\tau_k)\right) \\
&= \left(P^{1/2}x(t)\right)^T\left(\int_{t-h(t)}^{t} P^{-1/2}K^TRKx'(s)ds + \sum_{\{k:\ \tau_k \in (t-h(t),t)\}} P^{-1/2}K^TRKC_kx(\tau_k^-)\right) \\
&\leq \left\|P^{1/2}x(t)\right\|\left(\int_{t-h(t)}^{t}\left\|P^{-1/2}K^TRKx'(s)\right\|ds\right. \\
&\quad \left. + \sum_{\{k:\ \tau_k \in (t-h(t),t)\}}\left\|P^{-1/2}K^TRKC_kP^{-1/2}P^{1/2}x(\tau_k^-)\right\|\right) \\
&\leq \left\|P^{1/2}x(t)\right\|\left(\int_{t-\bar{h}}^{t}\left\|P^{-1/2}K^TRK(Ax(s) - BKx(s-h(s)) + B\eta(s))\right\|ds\right. \\
&\quad \left. + \sum_{\{k:\ \tau_k \in (t-\bar{h},t)\}}\left\|P^{-1/2}K^TRKC_kP^{-1/2}\right\|\cdot\left\|P^{1/2}x(\tau_k^-)\right\|\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \left\|P^{1/2}x(t)\right\| \cdot \Bigg( \int_{t-\bar{h}}^{t} \bigg( \left\|P^{-1/2}K^T RKAP^{-1/2}\right\| \cdot \left\|P^{1/2}x(s)\right\| \\
&\quad + \left\|P^{-1/2}K^T RKBKP^{-1/2}\right\| \cdot \left\|P^{1/2}x(s-h(s))\right\| + \left\|P^{-1/2}K^T RKB\right\| \cdot \|\eta(s)\| \bigg) ds \\
&\quad + \sum_{\{k:\ \tau_k \in (t-\bar{h},t)\}} \left\|P^{-1/2}K^T RKC_k P^{-1/2}\right\| \cdot \left\|P^{1/2}x(\tau_k^-)\right\| \Bigg) \\
&\leq \sqrt{m(t)} \Bigg( \int_{t-\bar{h}}^{t} \bigg( \left\|P^{-1/2}K^T RKAP^{-1/2}\right\| \sqrt{m(s)} \\
&\quad + \left\|P^{-1/2}K^T RKP^{-1/2}\right\|^2 \sqrt{m(s-h(s))} + \epsilon \left\|P^{-1/2}K^T RKB\right\| \bigg) ds \\
&\quad + \sum_{\{k:\ \tau_k \in (t-\bar{h},t)\}} \left\|P^{-1/2}K^T RKC_k P^{-1/2}\right\| \sqrt{m(\tau_k^-)} \Bigg) \qquad (8.2.49) \\
&\leq \sqrt{m(t)} \Bigg( \int_{t-\bar{h}}^{t} \bigg( e^{\bar{h}\theta_0} \left\|P^{-1/2}K^T RKAP^{-1/2}\right\| \sqrt{m(t)} \\
&\quad + e^{\bar{h}\theta_0} \left\|P^{-1/2}K^T RKP^{-1/2}\right\|^2 \sqrt{m(t)} + \epsilon \left\|P^{-1/2}K^T RKB\right\| \bigg) ds \\
&\quad + e^{\bar{h}\theta_0} \sum_{\{k:\ \tau_k \in (t-\bar{h},t)\}} \left\|P^{-1/2}K^T RKC_k P^{-1/2}\right\| \sqrt{m(t)} \Bigg) \\
&= \bar{h}e^{\bar{h}\theta_0} \left\|P^{-1/2}K^T RKAP^{-1/2}\right\| m(t) + \bar{h}e^{\bar{h}\theta_0} \left\|P^{-1/2}K^T RKP^{-1/2}\right\|^2 m(t) \\
&\quad + \epsilon\bar{h} \left\|P^{-1/2}K^T RKB\right\| \sqrt{m(t)} + e^{\bar{h}\theta_0} \sum_{\{k:\ \tau_k \in (t-\bar{h},t)\}} \left\|P^{-1/2}K^T RKC_k P^{-1/2}\right\| m(t) \\
&\leq \frac{b}{2} e^{\bar{h}\theta_0} m(t) + \epsilon\bar{h} \left\|P^{-1/2}K^T RKB\right\| \sqrt{m(t)} + \sigma e^{\bar{h}\theta_0} m(t).
\end{aligned}
$$

By combining this inequality with (8.2.48) we get

$$
\begin{aligned}
D^+m(t) &\leq -\left(a - e^{\bar{h}\theta_0}(b+2\sigma)\right) m(t) + q\sqrt{m(t)} \\
&= -\left(a - \hat{b}e^{\bar{h}\theta_0}\right) m(t) + q\sqrt{m(t)},
\end{aligned} \qquad (8.2.50)
$$

whenever $t \in (t_0 + \bar{h}, \infty)$, $t \neq \tau_k$ and $m(t) \geq e^{-2\bar{h}\theta_0} m(s)$ for all $s \in [t - 2\bar{h}, t]$.

Next we would like to get an inequality for $D^+m(x(t))$ in the form (8.2.47) where, ideally, the term $u_0$ is as small as possible. We made the simplifying assumption that $p(t)$ be constant. As we will see, $\theta_0$ and $u_0$ defined by (8.2.22) and (8.2.23) will allow us to derive the inequality

$$D^+m(t) \le -\theta_0(m(t) - u_0), \tag{8.2.51}$$

from (8.2.50) with $u_0$ minimized.

If we let $z = \sqrt{m(t)}$, then (8.2.50) will imply (8.2.51) if

$$-\left(a - \hat{b}e^{\hat{h}\theta_0}\right) z^2 + qz \le -\theta_0(z^2 - u_0), \tag{8.2.52}$$

or equivalently,

$$-\left(a - \hat{b}e^{\hat{h}\theta_0} - \theta_0\right) z^2 + qz - \theta_0 u_0 \le 0, \tag{8.2.53}$$

for all $z \ge 0$ where $\theta_0 > 0$ and $u_0 \ge 0$. For this to be true we need

$$a - \hat{b}e^{\hat{h}\theta_0} - \theta_0 > 0, \tag{8.2.54}$$

and

$$u_0 \ge \frac{q^2}{4\theta_0\left(a - \hat{b}e^{\hat{h}\theta_0} - \theta_0\right)}. \tag{8.2.55}$$

Now, condition (iii) ensures that (8.2.54) is satisfied providing $\theta_0 > 0$ is sufficiently small. By defining $g(\theta)$ as we did in (8.2.21) we can find the point that maximizes it. This point is $\theta_0$, which is the solution of $g'(\theta) = 0$. The smallest $u_0$ can be and yet still satisfy (8.2.54) and (8.2.55) is to equal the right hand side of inequality (8.2.55) with the denominator of that fraction maximized by $\theta_0$.

To verify that (8.2.50) does indeed imply (8.2.51) one can start with the fact that

$$\left(2g(\theta_0)\sqrt{m(t)} - \theta_0 q\right)^2 \ge 0. \tag{8.2.56}$$

Expanding and dividing through by $4\theta_0 g(\theta_0)$ gives

$$\frac{g(\theta_0)}{\theta_0}m(t) - q\sqrt{m(t)} + \frac{\theta_0 q^2}{4g(\theta_0)} \ge 0, \tag{8.2.57}$$

which implies

$$\left(a - \hat{b}e^{\hat{h}\theta_0} - \theta_0\right) m(t) + \theta_0 u_0 \ge q\sqrt{m(t)}, \tag{8.2.58}$$

which in turn reduces to

$$-\left(a - \hat{b}e^{\bar{h}\theta_0}\right) m(t) + q\sqrt{m(t)} \leq -\theta_0(m(t) - u_0). \tag{8.2.59}$$

We have thus verified that condition (ii) of Theorem 8.1.1 (or a sufficiently similar variation of it) is satisfied. Inequality (8.2.24) gives us

$$0 < \nu_1^{-1}(\hat{L}u_0) < \beta \leq \lambda \leq \hat{\lambda} \leq \nu_1^{-1}(\hat{L}\nu_2(\hat{\lambda})) \leq \alpha, \tag{8.2.60}$$

where $\nu_1(s) = \lambda_{\min}(P)s^2$ and $\nu_2(s) = \lambda_{\max}(P)s^2$. Since $||x(t)|| \leq \hat{\lambda}$ for $t \in [t_0 - \bar{h}, t_0 + \bar{h}]$, Theorem 3.1.1 tells us that

$$\begin{aligned} ||x(t)|| &\leq \nu_1^{-1}\left(\hat{L}u_0 + \hat{L}(\nu_2(\hat{\lambda}) - u_0)\exp\left(-\int_{t_0+\bar{h}}^{t} \theta_0 ds\right)\right) \\ &= \left\{\frac{1}{\lambda_{\min}(P)}\left[\hat{L}u_0 + \hat{L}(\lambda_{\max}(P)\hat{\lambda}^2 - u_0)e^{-(t-t_0-\bar{h})\theta_0}\right]\right\}^{1/2}. \end{aligned} \tag{8.2.61}$$

for $t \in [t_0 - \bar{h}, \infty)$. Moreover, system (8.2.8) is strongly practically stable with respect to $(\lambda, \alpha, \beta, T)$ at $t_0$ for some $T > 0$. The proof is therefore complete. ■

In the previous theorem conditions for practical stability were obtained that were dependent on the delay constant $\bar{h}$ as well as the bound $\epsilon$ on the disturbance function and the magnitudes of the impulse operators $C_k$. In general as $\bar{h}, \epsilon$ and the matrices $C_k$ tend to zero, then conditions (i), (ii) and (iii) of Theorem 8.2.1 become satisfied and so the system becomes strongly practically stable with respect to suitable parameters. Theorem 8.2.1 gives robustness bounds for the feedback controller given in (8.2.2)-(8.2.5).

We conclude by examining a particular numerical example.

**Example 8.2.1:** Consider the control system

$$x'(t) = Ax(t) + Bu(t), \tag{8.2.62}$$

where

$$A = \begin{bmatrix} -4 & 3 & 6 & -5 \\ -1 & -3 & 2 & 5 \\ -2 & 2 & 5 & 3 \\ 1 & 1 & -2 & -7 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -4 & 5 \\ 0 & 0 \\ 1 & 2 \\ 0 & 2 \end{bmatrix}. \tag{8.2.63}$$

System (8.2.62) with zero input (i.e. $u(t) = 0$) is not stable since the matrix $A$ has a positive real eigenvalue of 2.9680. Nevertheless, system (8.2.62) is stabilizable. If we let

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \tag{8.2.64}$$

and then solve for $P$ in the Riccati equation (8.2.5) and calculate $K$ using the formula (8.2.4), we get

$$P = \begin{bmatrix} 0.2217 & -0.0954 & -0.4402 & -0.2556 \\ -0.0954 & 0.2702 & 0.3824 & 0.2673 \\ -0.4402 & 0.3824 & 2.0941 & 0.6978 \\ -0.2556 & 0.2673 & 0.6978 & 0.6151 \end{bmatrix} \quad \text{and} \tag{8.2.65}$$

$$K = \begin{bmatrix} -0.4423 & 0.2547 & 1.2849 & 0.5734 \\ -0.2831 & 0.8223 & 3.3830 & 1.3480 \end{bmatrix}. \tag{8.2.66}$$

Thus using the feedback $u(t) = -Kx(t)$ with this particular choice of $K$ will stabilize system (8.2.62) and minimize the quadratic performance index (8.2.3). The eigenvalues of the matrix $A - BK$ are $-3.1167$, $-9.3516$ and $-3.8160 \pm 2.6731i$, all of which have negative real parts implying $A - BK$ is Hurwitz.

Suppose that in applying the control law $u(t) = -Kx(t)$ we find that the input signal $u(t)$ is delayed by an amount $h(t) \geq 0$ that can never exceed some small fixed amount $\bar{h} > 0$ and suppose the feedback is disturbed by an amount $\eta(t)$ that is bounded by $\epsilon = 0.01$. Finally, suppose the state $x(t)$ is subjected to small and diminishing impulsive perturbations of the form

$$\Delta x(t) = 10^{-k} x(t^-), \tag{8.2.67}$$

at integer times $t = k$ for $k = 1, 2, \ldots$. Then we consider the modified system

$$x'(t) = Ax(t) - BKx(t - h(t)) + B\eta(t), \ t \neq k, \tag{8.2.68a}$$

$$\Delta x(t) = C_k x(t^-), t = k, \tag{8.2.68b}$$

where $0 \leq h(t) \leq \bar{h}$ and $\|\eta(t)\| \leq 0.01$ for all $t \geq t_0 = 0$ and $C_k = 10^{-k} I$ for $k = 1, 2, \ldots$.

Calculating the parameters for our system in order to check the conditions of Theorem 8.2.1, we get $a = 4.0707$, $d = 7.5233$, $c_1 = 0.0363$, $c_2 = 9.1595$, $c_3 = 25.8786$ and $b = 244.2764\bar{h}$. For

$k = 1, 2, \ldots$ we have

$$d_k = \frac{1}{10^{2k}} + \frac{2}{10^k}, \tag{8.2.69}$$

whose sum $\sum_{k=1}^{\infty} d_k = 0.2323$ is finite. Assuming $\bar{h} < 1$ we have $\hat{L} = L = 1.2371$ and $\sigma = 0.9160$ all of which are finite. In order to satisfy condition (iii) of Theorem 8.2.1 we need $a > b + 2\sigma$ or $\bar{h} < (4.0707 - 2 \times 0.9160)/244.2764 = 0.0092$. In other words, Theorem 8.2.1 gives us strongly practical stability results in this example if the delay in feedback never exceeds 0.0092. Let us suppose that $\bar{h} = 0.005$. Then $b = 1.2214$, $\hat{b} = 3.0533$ and $q = 0.0757$.

The maximum of $g(\theta)$ occurs at $\theta = 0.5011$ where $g'(\theta) = 0$. Thus $\theta_0 = 0.5011$ and at this point $g(\theta_0) = 0.2549$ from which we obtain $u_0 = 0.0056$. Since $\bar{h} = 0.005$ is small, the graph of $g(\theta)$ is approximately given by the equation of the parabola $y = \theta(a - \hat{b} - \theta)$ as long as $|\theta|$ is not too large.

We therefore need $0.3103 < \beta \leq \lambda \leq 24.7231(2.9334\lambda^2 + 0.0001936)^{1/2} \leq \alpha$ in order for condition (iv) of Theorem 8.2.1 to be satisfied. If these parameters $(\lambda, \alpha, \beta)$ are so chosen, then according to Theorem 8.2.1, system (8.2.62) will be strongly practically stable with respect to $(\lambda, \alpha, \beta, T)$ at $t_0 = 0$ for some $T > 0$. □

# Chapter 9

# Conclusions and Future Research

Impulsive delay differential equations are useful in modelling physical processes that experience both time delays in their evolutionary behaviour as well as impulsive jump discontinuities. As shown in the early chapters of this thesis, these equations share many similarities, but also a number of important differences, with delay differential equations (without impulses) and with impulsive differential equations (without delays).

In Chapter 2 we carefully defined an impulsive delay differential equation as well as what we considered to be a reasonable definition for a solution of such an equation. Chapter 3 was dedicated to establishing some fundamental theory of these equations. In many cases the theorems on existence, uniqueness, etc. were developed along the lines of similar theorems for non-impulsive systems or non-delay systems.

Chapters 4 and 5 established some new and important stability and boundedness results for impulsive delay differential equations by making use of the theory of Lyapunov functions and functionals. Impulses either maintained the stability (or boundedness) properties of an otherwise stable non-impulsive delay system or else they were used to stabilize the system in the event that it was not already stable.

We applied the results and techniques developed in Chapters 4 and 5 to some linear systems in Chapter 6 as well as to some problems in population dynamics in Chapter 7 and control systems in Chapter 8.

The development of basic definitions and fundamental theorems of impulsive delay differential equations, as given in Chapters 2 and 3, is a necessary part of any thorough study of such equations. While the Lyapunov stability theorems of later chapters and the applications of these theorems represent a significant contribution to the theory of these equations, there are nevertheless many interesting possibilities for future research.

The analysis of impulsive delay differential equations from a more quantitative point of view, by way of simulations, could be very beneficial. Solving such equations numerically could lead to a better understanding of the behaviour of solutions. If through numerical simulations one were to discover, for instance, that certain solutions blew up, then it would be pointless to try to show stability or boundedness by searching for a suitable Lyapunov function or by some other means. The use of numerical techniques could also give some indication as to the rate of convergence of solutions to a steady state or equilibrium.

While stability is usually a desirable property of systems, many systems are unstable, and so having theorems that would analytically verify the instability of such systems would prove useful. Such Lyapunov instability theorems could be developed along lines similar to those used in this thesis and those developed for ordinary differential equations.

With the exception of some of the fundamental theory developed in Chapter 3, most of the results were proven for impulsive delay differential equations where impulses occurred at fixed times. One could establish stability and boundedness results for different types of impulsive delay systems. This may require some form of modification to the classical concepts of stability and boundedness. Not all systems behave in such a way that impulses always occur at the same times. Impulses may occur at variable times as described in Chapters 2 and 3. Alternatively, one could consider an autonomous system where impulses occur when the state reaches some boundary set of predefined values, irrespective of the time. For simplicity we have assumed that impulse times form an increasing sequence tending to infinity. This disallows any clustering of impulse times. Further research into what may happen if this assumption were dropped could be interesting. This may require redefining what is meant by a solution of an impulsive delay differential equation.

Most of the examples and theorems applied to systems having little or no delay in the difference equation defining the impulsive action. It would be interesting to consider how solutions might evolve if delay in the difference equation were more significant.

Other areas of future research could include a more in-depth look at linear systems as well as linearization techniques, the study of periodic solutions and their orbits, and an exploration of other applications beyond those cited here. An analysis of neutral or advanced type equations with impulses, in addition to retarded equations, might also be worthy of consideration. Finally, stability analysis of impulsive delay differential equations using techniques other than Lyapunov functions or functionals would be particularly useful.

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