On the Structure of Nonnegative Semigroups of Matrices

by

Peter Williamson

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Mathematics
in
Pure Mathematics

Waterloo, Ontario, Canada, 2009

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

The results presented here are concerned with questions of decomposability of multiplicative semigroups of matrices with nonnegative entries. Chapter 1 covers some preliminary results which become useful in the remainder of the exposition. Chapters 2 and 3 constitute an exposition of some recent known results on special semigroups. Chapter 2 explores conditions for decomposability of semigroups in terms of conditions derived from linear functionals and in Chapter 3, we give a complete proof of an extension of the celebrated Perron-Frobenius Theorem. No originality is claimed for the results in Chapters 2 and 3. In Chapter 4, we present some new results on sufficient conditions for finiteness of semigroups of matrices.

Acknowledgements

I would like to thank Heydar Radjavi for his continual encouragement, and the many patient hours that he spent helping me prepare and proof-read my thesis.

Contents

Chapter 1

Introduction and Preliminaries

This thesis concerns itself with questions of decomposability of multiplicative semigroups of matrices with nonnegative entries. Chapters 2 and 3 constitute an exposition of some recent known results on special semigroups. In particular, we give a complete proof of an extension of the celebrated Perron-Frobenius Theorem. No originality is claimed for the results in Chapters 2 and 3. In Chapter 4, we present some new results on sufficient conditions for finiteness of semigroups of matrices.

Unless otherwise stated, the underlying vector space will be $\mathbb{C}^n$. We will be mostly working with square matrices with nonnegative entries, which we will simply call *nonnegative matrices*. Recall that a *semigroup*, $\mathcal{S}$, is a nonempty set together with an associative binary operation, which in this exposition will be nonnegative matrices with matrix multiplication. It can easily be checked that a collection of nonnegative matrices upon closure under multiplication is in fact a semigroup. A collection of matrices $\mathcal{C}$ is called *reducible* if there exists a proper invariant subspace, common to each member in the collection. This definition is equivalent to the existence of a fixed invertible matrix T such that TAT^{-1} is in block upper triangular form for each $A \in \mathcal{C}$. A collection is *irreducible* if it is not reducible. A *permutation matrix* is defined to have a single 1 in each row and column and all other entries equal to zero. A collection of matrices, $\mathcal{C}$, is *decomposable* if there exists an invariant subspace which is spanned by a proper subset of the basis vectors and is common to each member in the collection. This definition is equivalent to the existence of a fixed permutation matrix P such that PAP^{-1} is in block upper triangular form for each $A \in \mathcal{C}$. A collection is *indecomposable* if it is not decomposable. A collection of matrices is called *completely decomposable* if each member can be simultaneously decomposed into upper-triangular form.

If $\mathcal{V}$ is a vector space and $\mathcal{N}$ is a subspace of $\mathcal{V}$, then the *quotient space* $\mathcal{V}/\mathcal{N}$ is the collection of cosets $\overline{x} = x + \mathcal{N}$. If A is a linear transformation on $\mathcal{V}$ and $\mathcal{N}$ is invariant under A, then the *quotient transformation* $\tilde{A}$ on $\mathcal{V}/\mathcal{N}$ is defined by $\tilde{A}\overline{x} = \overline{Ax}$ for each $x \in \mathcal{V}$. If $\mathcal{S}$ is a collection of linear transformations on $\mathcal{V}$, and if $\mathcal{M}$ and $\mathcal{N}$ are invariant subspaces for $\mathcal{S}$ with $\mathcal{N} \subset \mathcal{M}$ properly, then the *quotients of* $\mathcal{S}$ *on* $\{\mathcal{M}, \mathcal{N}\}$ are the transformations, $\tilde{A}$, on $\mathcal{M}/\mathcal{N}$. A property is *inherited by quotients* if every collection of quotients of a collection satisfying the property also satisfies the property. Although most results presented here will pertain to indecomposable semigroups, we first need a few results on reducibility.

The first result presented is often implicitly used in the proofs of triangularization theorems. However, it appears to have first been formalized in Radjavi-Rosenthal [14].

Theorem 1.1. ***(The Triangularization Lemma)*** *Let $\mathcal{P}$ be a set of properties, each of which is inherited by quotients. If every collection of transformations on a space of dimension greater than 1 that satisfies $\mathcal{P}$ is reducible, then every collection of transformations satisfying $\mathcal{P}$ is triangularizable.*

Proof. Let $\mathcal{S}$ be any collection satisfying $\mathcal{P}$ and let

$$\{0\} = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \ldots \subset \mathcal{M}_m = \mathcal{V}$$

be a maximal chain, $\mathcal{C}$, of invariant subspaces of $\mathcal{S}$. If each quotient $\mathcal{M}_{k+1}/\mathcal{M}_k$ is one-dimensional, then $\mathcal{C}$ will be a triangularizing chain for $\mathcal{S}$. Suppose there exists a k such that the dimension of $\mathcal{M}_{k+1}/\mathcal{M}_k$ is greater than 1. Then, since any quotient transformation of $\mathcal{S}$ on $\mathcal{M}_{k+1}/\mathcal{M}_k$ also satisfies $\mathcal{P}$, the collection of quotients of $\mathcal{S}$ with respect to $\{\mathcal{M}_{k+1}, \mathcal{M}_k\}$ would have a proper invariant subspace $\mathcal{N}$ by hypothesis. Then, $\mathcal{M}_o = \{x \in \mathcal{M}_k : \overline{x} \in \mathcal{N}\}$ would be a nontrivial invariant subspace of $\mathcal{S}$ and thus $\mathcal{M}_k \subset \mathcal{M}_o \subset \mathcal{M}_{k+1}$ where each containment is proper, but this contradicts the maximality of the chain. Therefore, each $\mathcal{M}_{k+1}/\mathcal{M}_k$ is a one dimensional space and so $\mathcal{C}$ is a triangularizing chain.

□

The next result is a famous and extremely powerful result, first established for groups of matrices by Burnside in 1905 [2], and later extended to its present form in Frobenius-Schur [5].

Theorem 1.2. ***(Burnside's Theorem)*** *The only irreducible algebra of linear transformations on the finite-dimensional vector space $\mathcal{V}$ of dimension greater than 1 is the algebra of all linear transformations mapping $\mathcal{V}$ into $\mathcal{V}$.*

Proof. Let $\mathcal{A}$ be an irreducible algebra. Every linear transformation on a finite dimensional vector space can be expressed as a finite sum of rank 1 linear transformations. Thus, in order to prove the theorem, it will suffice to show that $\mathcal{A}$ contains all rank 1 transformations. We will begin by showing that $\mathcal{A}$ contains at least one rank 1 transformation. Let $T_o \in \mathcal{A}$ be a member of minimal nonzero rank. We plan to show that T_o is rank 1 and so suppose for a contradiction that T_o has rank greater than 1. Then, there exists vectors x_1 and x_2 such that $\{T_o x_1, T_o x_2\}$ is a linearly independent set. Note that $\mathcal{J} = \{AT_o x_1 : A \in \mathcal{A}\} = \mathcal{V}$ since otherwise $\mathcal{J}$ would be an invariant subspace of $\mathcal{A}$ which would contradict its irreducibility. Thus, we can find an $A_o \in \mathcal{A}$ such that $A_o T_o x_1 = x_2$. Then, $\{T_o x_1, T_o x_2\} = \{T_o x_1, T_o A_o T_o x_1\}$ is a linearly independent set. Consider $T_o A_o$ restricted to the range of T_o. The spectrum (in this case the set of eigenvalues) of $((T_o A_o)|_{T_o\mathcal{V}})$ is not empty, and so we can find a scalar λ such that $(T_o A_o - \lambda I)|_{T_o\mathcal{V}}$ is not invertible.

Note that $(T_oA_o - \lambda I)T_o$ is not zero since $T_oA_oT_ox_1 - \lambda T_ox_1 \neq 0$ due to the linear independence. The range of $T_oA_oT_o - \lambda T_o$ is contained in that of T_o. We can see this by noting that for each $x \in \mathcal{V}$, $T_oA_oT_ox - \lambda T_ox = T_o(A_oT_ox) - T_o(\lambda x) = T_o(A_oT_ox - \lambda x)$. In fact, the containment is proper since $T_oA_o - \lambda I$ restricted to the range of T_o is not invertible which means that $T_oA_o - \lambda I$ has members of its kernel in the range of T_o. This implies that the rank of $T_oA_oT_o - \lambda T_o$ is less than that of T_o and this contradicts the minimality of the rank of T_o. Therefore, T_o must have rank 1.

Let y_o be a nonzero vector in the range of T_o. Since every vector in the range of T_o is a multiple of y_o, we can find a linear functional ϕ_o on $\mathcal{V}$ such that $T_ox = \phi_o(x)y_o$ for all $x \in \mathcal{V}$. Note that every linear transformation of rank 1 is of the form $x \mapsto \phi(x)y$ for a vector y in $\mathcal{V}$ and a linear functional ϕ. We want to show that $\mathcal{A}$ contains every rank 1 transformation. First note that $T_oA \in \mathcal{A}$ for all $A \in \mathcal{A}$ and that $T_oAx = T_o(Ax) = \phi_o(Ax)y_o$ for all $x \in \mathcal{V}$. Thus, we have all the rank 1 transformations of the form $T_oA = y_o\phi_o \circ A$. We claim that $\Phi = \{\phi_o \circ A : A \in \mathcal{A}\}$ consists of all linear functionals. Suppose for a contradiction that this were not the case. Then, there would be a nonzero $x_o \in \mathcal{V}$ such that $\phi(x_o) = 0$ for all $\phi \in \Phi$. But this means that $\phi_o(Ax_o) = 0$ for all $A \in \mathcal{A}$. Since ϕ_o is nonzero, and $\{Ax_o : A \in \mathcal{A}\} = \mathcal{V}$, this is only possible if $x_o = 0$, a contradiction and so Φ does in fact consist of all linear functionals on $\mathcal{V}$. Thus, we have the transformations of the form $Tx = \phi(x)y_o$ where ϕ can be any linear functional. Now, since $\{Ax : A \in \mathcal{A}\} = \mathcal{V}$, given a $y \in \mathcal{V}$, we can find a $B \in \mathcal{A}$ such that $By_o = y$. Then, $BTx = B(\phi(x)y_o) = \phi(x)By_o = \phi(x)y$. Thus, we have shown that $\mathcal{A}$ contains every rank one linear transformation, which completes the profor sof.

$\square$

The next result has been used ubiquitously in various proofs and as a result it is hard to determine who discovered it first.

Lemma 1.3. *Let $\mathcal{S}$ be a semigroup in $\mathcal{B}(\mathcal{V})$, the set of all linear transfor-*

mations mapping $\mathcal{V}$ to $\mathcal{V}$ where the dimension of $\mathcal{V}$ is greater than 1. If there exists a nonzero linear functional φ on $\mathcal{B}(\mathcal{V})$ with $\varphi|_{\mathcal{S}} = 0$, then $\mathcal{S}$ is reducible.

Proof. Since φ is zero on $\mathcal{S}$, it is also zero on the algebra generated by $\mathcal{S}$. Thus, since the only linear functional which is zero on all of $\mathcal{B}(\mathcal{V})$ is zero, and φ is nonzero, the algebra generated by $\mathcal{S}$ is not equal to $\mathcal{B}(\mathcal{V})$ and thus by Burnside's theorem, the algebra generated by $\mathcal{S}$ is reducible, so $\mathcal{S}$ is reducible too. $\square$

The following result is one of the earliest which uses conditions on the spectra to deduce reducibility and is found in Levitzki [7].

Theorem 1.4. (***Levitzki's Theorem***) *Every semigroup, $\mathcal{S}$, of nilpotent operators is triangularizable.*

Proof. The trace is a linear functional that vanishes on $\mathcal{S}$, so by Lemma 1.3, $\mathcal{S}$ is reducible. Since quotients of nilpotent operators are nilpotent, the Triangularization Lemma yields the result. $\square$

Chapter 2

Conditions for Decomposability of Semigroups

In this chapter, we discuss conditions for decomposability and complete decomposability of semigroups in terms of nonnegative linear functionals, including those that act only on the diagonal entries of the matrices. We explore the structure of idempotents, and special semigroups, including semigroups of idempotents and semigroups of nilpotents.

The first theorem presented is from Radjavi-Rosenthal [15], but is really a special case of known results from Choi-Nordgren-Radjavi-Rosenthal-Zhong [3].

Theorem 2.1. *A semigroup $\mathcal{S}$ of nonnegative nilpotent matrices is completely decomposable.*

Proof. By Levitzki's Theorem, we know that $\mathcal{S}$ is triangularizable (i.e. there exists a fixed invertible matrix U such that every matrix in $U^{-1}\mathcal{S}U$ is in upper-triangular form).

We claim that the product of any n members of $\mathcal{S}$ must be zero, where we assume that the matrices are $n \times n$. Assume that after a similarity, every member is in strictly upper-triangular form (we are no longer assuming nonnegativity). Then the product of any n is clearly zero.

We will now show that $\mathcal{S}$ is decomposable. Assume that $\mathcal{S}$ is back in its nonnegative form. Let k be the smallest integer such that the product of any k members of $\mathcal{S}$ is zero. If $k = 1$, then zero is the only element in $\mathcal{S}$ and we are done. Assume $k > 1$, so that we can find $S_1, S_2, \ldots, S_{k-1}$ such that their product, which we denote T, is not zero. However, $ST = 0$ for all $S \in \mathcal{S}$. Since $T \neq 0$, we know that it has a nonzero column. We can permute the basis to assume that the first entry of that column, say the i-th column, is nonzero. Every entry of ST is zero, so $0 = (ST)_{ji} = \sum_{k=1}^{n} S_{jk}T_{ki} \geq S_{j1}T_{1i}$ for all i and j. Since $T_{1i} \neq 0$, $S_{j1} = 0$ for all j. Thus the first column of S is zero for all $S \in \mathcal{S}$. Therefore, $\mathcal{S}$ is decomposable. We thus have a permutation P such that $P^{-1}\mathcal{S}P$ is block upper triangular. Note that this implies that every block must still be a nonnegative matrix since a permutation does not affect nonnegativity. Also, the diagonal blocks must be nilpotent, which we can see by considering S^n for each $S \in \mathcal{S}$ in block upper-triangular form, under block multiplication. Therefore, we can further decompose each block until the matrices are completely decomposed.

$\square$

The next lemma, and the two ensuing corollaries, are both from [15].

Lemma 2.2. *Let φ_i denote the linear functional on $n \times n$ matrices defined by $\varphi_i(M) = M_{ii}$, the (i,i) entry of M. If there is an i such that φ_i is submultiplicative on a semigroup $\mathcal{S}$ of nonnegative matrices, then $\mathcal{S}$ is decomposable. Furthermore, after a suitable permutation of the basis, every $S \in \mathcal{S}$ has the block form*

$$\begin{pmatrix} R & X & Y \\ 0 & s & Z \\ 0 & 0 & T \end{pmatrix}$$

where s represents a 1×1 block and equals S_{ii}. (If s is in the 1st or nth position, then the corresponding decomposition will be block 2×2)

Proof. First note that the submultiplicativity of φ_i implies multiplicativity, as

$$\varphi_i(A)\varphi_i(B) \geq \varphi_i(AB) = \sum_j A_{ij}B_{ji} \geq A_{ii}B_{ii} = \varphi_i(A)\varphi_i(B)$$

for any A, B in $\mathcal{S}$, where the second inequality is due to the nonnegativity of the entries. Assume, without loss of generality, that $i = 1$ in order to simplify the notation. Then, since $\varphi_1(AB) = \varphi_1(A)\varphi_1(B)$, we have that

$$A_{12}B_{21} + \cdots + A_{1n}B_{n1} = 0,$$

which by nonnegativity implies $A_{1j}B_{j1} = 0\ \forall j = 2, \ldots, n$. If $A_{12} = \cdots = A_{1n} = 0$ for all $A \in \mathcal{S}$, then for $k = 2, \ldots, n$, $Ae_k \in \operatorname{span}\{e_k\}_{k=2}^n$ where $\{e_k\}$ are the standard basis vectors. Thus, we would have decomposability since the proper subspace spanned by $\{e_k\}_{k=2}^n$ would be invariant under all $S \in \mathcal{S}$.

Assume then, that after a permutation, A_{1n} is nonzero for some $A \in \mathcal{S}$, which implies that $B_{n1} = 0$, for all $B \in \mathcal{S}$. Let $\mathcal{J}$ be the maximal subset of $\{2, \ldots, n\}$ such that $B_{i1} = 0$ for all $i \in \mathcal{J}$ and $B \in \mathcal{S}$. After a further permutation, we may assume that $\mathcal{J} = \{l+1, \ldots, n\}$.

Let $m \in \{1, \ldots, l\}$. Since $\mathcal{J}$ is maximal, we know that there exists an element $S \in \mathcal{S}$ such that $S_{m1} \neq 0$. Then,

$$0 = (BS)_{j1} = \sum_{i=1}^n B_{ji}S_{i1} \geq B_{jm}S_{m1} \quad (*)$$

whenever $j \geq l+1$ and for all $B \in \mathcal{S}$, where the inequality in $(*)$ is due to the nonnegativity of the matrices. Thus, $B_{jm} = 0$ whenever $j \geq l+1$ and $1 \leq m \leq l$. This implies that the standard basis vectors, indexed $\{1, \ldots, l\}$ span an invariant subspace of $\mathcal{S}$.

We now have that $\mathcal{S}$ is decomposable and so after a permutation of the basis, each $S \in \mathcal{S}$ is of the form:

$$S = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}.$$

The (i,i) entry occurs in either A or C. Without loss of generality assume it occurs in A. Note that the set of all matrices A as S runs through $\mathcal{S}$ forms a semigroup which we denote $\mathcal{F}$, and $\phi_i|_{\mathcal{F}}$ is clearly submultiplicative. Therefore, $\mathcal{F}$ is decomposable. We can repeat this procedure for whichever decomposed block contains the (i,i) entry until that block is 1×1 and the stated block form for $\mathcal{S}$ is obtained. $\square$

Corollary 2.3. *A semigroup $\mathcal{S}$ of nonnegative matrices is completely decomposable if and only if every φ_i is submultiplicative on it.*

Proof. If $\mathcal{S}$ is completely decomposable, then every φ_i is clearly multiplicative and thus submultiplicative on it. If every φ_i is submultiplicative on $\mathcal{S}$, then proceeding by induction on Lemma 2.2 yields the result. $\square$

Corollary 2.4. *Let A be a nonnegative matrix such that every positive power of A has at least one diagonal entry equal to 0. Then A is decomposable and has 0 as an eigenvalue.*

Proof. If we apply the previous lemma (2.2) to the semigroup generated by A, denoted $\mathcal{S}$, then it suffices to show that $\phi_i(A^m) = 0$ for some i and all positive m, since this functional being constantly zero implies its submultiplicativity.

We will proceed by contradiction. Suppose that for each i there is an m_i such that $\phi_i(A^{m_i}) \neq 0$. Recall from the proof of Lemma 2.2 that $\phi_i(AB) \geq \phi_i(A)\phi_i(B)$ for any nonnegative matrices. Thus, for any positive integer, r,

$$\phi_i(A^{rm_i}) \geq (\phi_i(A^{m_i})^r).$$

Let $m = m_1 \ldots m_n$ so that

$$\phi_i(A^m) \geq (\phi_i(A^{m_i}))^{m/m_i} > 0$$

for all i. This contradicts the hypothesis, and so $\phi_i(A^m) = 0$ for some i and all positive m. Now using Lemma 2.2, we can decompose $\mathcal{S}$ into the form stated in the lemma which implies that A is decomposable with zero as an eigenvalue.

$\square$

An *ideal*, $\mathcal{J}$, of a semigroup $\mathcal{S}$ is a subsemigroup, such that SJ and JS are in $\mathcal{J}$ for all $S \in \mathcal{S}$ and $J \in \mathcal{J}$. A *nonnegative linear functional* is a linear functional which when acting on nonnegative elements (such as nonnegative matrices), returns nonnegative real numbers. The next lemma is from Marwaha [10], except for part (vi) which is an obvious equivalence of (iii). This Lemma will be very useful when studying indecomposable semigroups, as we will have the negation of each of these equivalences.

Lemma 2.5. *For a semigroup $\mathcal{S}$ of nonnegative matrices, the following are mutually equivalent:*

(i) $\mathcal{S}$ is decomposable;

(ii) $A\mathcal{S}B = \{0\}$ for some nonzero nonnegative matrices A and B;

(iii) for some fixed i and j, the (i,j) entry of every member of $\mathcal{S}$ is zero;

(iv) every sum of members of $\mathcal{S}$ has a zero entry;

(v) some nonzero ideal of $\mathcal{S}$ is decomposable;

(vi) some nonzero, nonnegative linear functional is zero on $\mathcal{S}$.

Proof. $(i) \Rightarrow (ii)$: Suppose $\mathcal{S}$ is decomposable. Then after a permutation, each matrix in $\mathcal{S}$ has simultaneous 2×2 block upper-triangular form where the $(2,1)$ block is zero. The block matrices

$$A = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \quad B = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

give $ASB = 0$ for each $S \in \mathcal{S}$. Thus, $A\mathcal{S}B = \{0\}$.

$(ii) \Rightarrow (iii)$: Since A and B are nonzero, $A_{h,i} \neq 0$ for some h, i and $B_{j,k} \neq 0$ for some j, k. Since $ASB = 0$ for all $S \in \mathcal{S}$ and A and B are nonnegative, a simple calculation shows that $S_{i,j} = 0$ for all $S \in \mathcal{S}$, implying (iii).

$(iii) \Rightarrow (iv)$: This is immediate.

$(iv) \Rightarrow (iii)$: Let S be a sum of members of $\mathcal{S}$ such that it has a minimal number of zero entries. We know that it must have at least one zero entry by hypothesis, say (i, j). This means that every $S \in \mathcal{S}$ must have a zero in the (i, j) entry since otherwise adding it into the sum would contradict the minimality of S.

$(iii) \Rightarrow (i)$: Assume $S_{i,j} = 0$ for all $S \in \mathcal{S}$. If $i = j$, then we are done by Lemma 2.2, so assume $i \neq j$. We can assume, after a permutation, that it is the $(n, 1)$ entry. Let $\mathcal{J}$ be the maximal subset of $(1, \ldots, n)$ such that $S_{i,1} = 0$ for all $i \in \mathcal{J}$ and that after a permutation, $\mathcal{J} = \{k+1, \ldots, n\}$. By the maximality of $\mathcal{J}$, for each $1 \leq j \leq k$, we can find a $T \in \mathcal{S}$ such that $T_{j,1} \neq 0$. Then, for $l \geq k+1$, and $1 \leq j \leq k$,

$$0 = (ST)_{l1} = \sum_{i=1}^{n} S_{li} T_{i1} \geq S_{lj} T_{j1}$$

and hence $S_{lj} = 0$ for all l with $l \geq k+1$ and all j with $1 \leq j \leq k$. Thus, $\mathcal{S}$ is decomposable.

$(i) \Rightarrow (v)$: This is immediate.

$(v) \Rightarrow (iii)$: Assume (v), and let $\mathcal{J}$ be a decomposable nonzero ideal of $\mathcal{S}$. Then, after a permutation, $\mathcal{J}$ has simultaneous $k \times k$ block upper-triangular form in which the blocks under the diagonal blocks are zero, where we assume that the constantly zero rows appear at the bottom of the matrices and that $\mathcal{J}$ cannot be further decomposed. Suppose the first column of J is zero for all $J \in \mathcal{J}$. Then, $JS \in \mathcal{J}$ has its first column zero for all $S \in \mathcal{S}$. We can assume that some $J \in \mathcal{J}$ has its i-th entry in its first row is nonzero. Then multiplying on the right by $S \in \mathcal{S}$ gives that $0 = (JS)_{11} \geq J_{1i} S_{i1}$. Since

$J_{1i} \neq 0$, S_{i1} must be zero for all $S \in \mathcal{S}$. Now suppose that the first column is not zero. Pick $J \in \mathcal{J}$ such that the $(1,1)$ entry is nonzero. We can find such a J since each block on the diagonal of $\mathcal{J}$ is indecomposable (since by part (iii) of this Lemma, we know that no fixed entry of an indecomposable semigroup can be constantly zero). Since $\mathcal{J}$ is an ideal, we have that $SJ \in \mathcal{J}$ for all $S \in \mathcal{S}$. Note that for all $J \in \mathcal{J}$, $J_{n,1} = 0$. Thus $0 = (SJ)_{n,1} = S_{n,i}J_{i,1} \geq S_{n,1}J_{1,1}$ which means that $S_{n,1} = 0$ for all $S \in \mathcal{S}$. Thus, we have (iii).

$(vi) \Leftrightarrow (iii)$: Assume (vi). Let ϕ be a nonzero nonnegative linear functional that is zero on every member of $\mathcal{S}$. Let $S \in \mathcal{S}$. $\phi(S) = \sum_{i,j} a_{ij}S_{ij}$ where each a_{ij} is nonnegative and at least one is nonzero. Let a_{rs} be nonzero. Since $0 = \phi(S) \geq a_{rs}S_{rs}$, we have that $S_{rs} = 0$. This must hold for all $S \in \mathcal{S}$ and so the (r,s) entry of each $S \in \mathcal{S}$ is zero. Assume (iii) and let the (i,j) entry be the one that is constantly zero. Then the linear functional which returns the value of the (i,j) entry of the matrix it acts upon is a non-zero, nonnegative linear functional which is zero on $\mathcal{S}$.

$\square$

The following lemma can be found in Berman-Plesmmons [1]. This Lemma will be used throughout the remainder of the exposition.

Lemma 2.6. *Let E be a nonnegative idempotent of rank r.*

(i) *If E has no zero rows or columns, then there exists a permutation matrix P such that $P^{-1}EP$ has the block-diagonal form*

$$E_1 \oplus \cdots \oplus E_r,$$

where each E_i is an idempotent of rank one whose entries are all positive.

(ii) *In general, there exists a permutation matrix P such that $P^{-1}EP$ has the block-triangular form*

$$E = \begin{pmatrix} 0 & XF & XFY \\ 0 & F & Y \\ 0 & 0 & 0 \end{pmatrix}$$

where $F = E_1 \oplus \cdots \oplus E_r$ *as in* (i) *and where* X *and* Y *are nonnegative matrices.*

Proof. (i): If E has rank one, then $E = xy^*$ where x and y are nonnegative column vectors with $y^*x = 1$. The last condition is simply the requirement that the $tr(E) = 1$. A simple calculation shows that if x had a zero entry, then E would have a zero row, and if y had a zero entry, then E would have a zero column. Thus x and y must be positive vectors (more precisely, each entry must be nonzero and the $i-th$ entry of x must have the same sign as the $i-th$ entry of y), which means that the entries of E must too be positive. We will prove the result by induction on the rank of E, which we will denote by r.

We claim that if $r \geq 2$, then there is a nonnegative vector in the range of E with at least one zero entry. Since E is at least rank 2, we can pick two linearly independent columns, x and y. If either x or y has a zero entry, we are done, and so assume both are positive vectors. Let $y_j/x_j = \max_i\{y_i/x_i\}$ and let $z = y_j x - x_j y$. The j-th entry of z is clearly zero and z is nonzero since x and y are linearly independent. Note also that z is nonnegative by the way that we picked our y_j and x_j. $Ez = E(y_j x) - E(x_j y) = y_j x - x_j y = z$ and so we have proven our claim.

Now we will show that if $r \geq 2$, then E is decomposable. From above, we know that E has a nonzero vector with a zero entry in its range. Let $z = (z_1, \ldots, z_n)$ be such a column vector with a maximal number of zero entries, and permute the basis so that $z_1 \geq \ldots z_s > z_{s+1} = \cdots = z_n = 0$. Recall that E and z have nonnegative entries. Thus the equation $Ez = z$ implies that the (i, j) entry of E is zero for all $i \geq s+1$ and all $j \leq s$ (if this were not the case, then Ez would be nonzero for some entry which is zero for z) and so the span of the first s basis vectors is an invariant subspace under E. Therefore, E is decomposable.

Since E is decomposable, we can assume after a permutation that

$$E = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

where A and C are idempotents whose rank is less than r. Since E has no zero rows or columns (and permutations do not create or destroy zero rows or columns), A has no zero columns and C has no zero rows. Observe that

$$E^2 = \begin{pmatrix} A^2 & AB + BC \\ 0 & C^2 \end{pmatrix},$$

which implies that $AB + BC = B$ (as well as $A^2 = A$ and $C^2 = C$, which proves the statement above that A and C are idempotents). Multiplying this equation on the left by A and the right by C gives $2ABC = ABC$ which implies $ABC = 0$. Since A has no zero columns, $A(BC) = 0$ implies that $BC = 0$. Now, since $BC = 0$, and C has no zero rows, $B = 0$.

Since $B = 0$, A cannot have any zero rows, and C cannot have any zero columns. Thus, A and C satisfy the hypothesis of (i), and so we can continue to decompose A and C until each block is rank 1. At this point, the nonzero blocks will be on the diagonal, and they will be positive since they are each rank 1. Therefore, after a permutation, E is in the stated form.

(ii): We can assume that, after a permutation

$$E = \begin{pmatrix} 0 & X & Z \\ 0 & F & Y \\ 0 & 0 & 0 \end{pmatrix},$$

where the first m columns are exactly the zero columns, and all the zero rows numbered higher than m are at the end. $E^2 = E$ gives that $F^2 = F$, $XF = X$, $Z = XY$ and $FY = Y$. Combining the last two gives $Z = XFY$. If F had a zero row, then $Y = FY$ would have the same zero row. Likewise, if F had a zero column, then $X = XF$ would have the same zero column. This would mean that E would have another zero row or column, but we have assumed that all zero columns and the rows numbered higher than m have been account for by the first permutation, so this is a contradiction. Thus, the idempotent F has no zero rows or columns, and is thus, after a permutation, the direct sum of positive rank-one idempotents as shown in (i).

□

The next corollary comes from Radjavi [12].

Corollary 2.7. *Let E be a nonnegative idempotent matrix of rank r. Then there exists r columns $x_1, \cdots, x_r$ of E whose nonnegative linear combinations include all columns of E. In the special case where E has no zero columns or rows, every column of E is a positive multiple of some x_i, with $1 \leq i \leq r$.*

Proof. Assume first that E has no zero rows or columns. Then, we can decompose E as in Lemma 2.6 such that $E = E_1 \oplus \cdots \oplus E_r$ where each E_i has rank 1. Then, define x_i as the column of E which corresponds to the first column of E_i. Since each column of E_i is a positive multiple of its first column, the $\{x_i\}$ defined above span the columns of E.

Now consider the case where E has a zero row or column. From Lemma 2.6(ii), we know that

$$E = \begin{pmatrix} 0 & XF & XFY \\ 0 & F & FY \\ 0 & 0 & 0 \end{pmatrix}$$

where $F = E_1 \oplus \cdots \oplus E_r$ and where X and Y are nonnegative matrices. Consider the block decomposition of the block column

$$\begin{pmatrix} XF \\ F \\ 0 \end{pmatrix} = \begin{pmatrix} X_1E_1 & X_2E_2 & \cdots & X_rE_r \\ E_1 & 0 & \cdots & 0 \\ 0 & E_2 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ \vdots & \vdots & 0 & E_r \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\{X_1, \ldots, X_r\}$ is the decomposition of X conforming to the decomposition, $F = E_1 \oplus \cdots \oplus E_r$. Recall that each column of E_i is a positive multiple of the first column of E_i. The rows of $X_i E_i$ are nonnegative linear combinations of the rows of E_i and so each column is a multiple of the first column of E_i as well. Thus, each column of the block column

$$\begin{pmatrix} X_i E_i \\ 0 \\ \vdots \\ E_i \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

is a multiple of the first column of this block. Note that the columns of

$$\begin{pmatrix} XFY \\ FY \\ 0 \end{pmatrix}$$

are all nonnegative linear combinations of the columns of

$$\begin{pmatrix} XF \\ F \\ 0 \end{pmatrix}.$$

Each x_i can therefore be chosen to be the first column of the i-th of these block columns.

$\square$

This next Lemma is important in the proof of Lemma 3.4, and is found in Radjavi [13].

Lemma 2.8. *If $\mathcal{G}$ is a group of invertible nonnegative matrices, then every member of $\mathcal{G}$ has exactly one nonzero entry in each of its rows and columns.*

Furthermore, if $\mathcal{G}$ is bounded, then there is a diagonal matrix D with positive entries such that $D^{-1}\mathcal{G}D$ is a group of permutation matrices.

Proof. Since every member in $\mathcal{G}$ is invertible, each row and column of must contain at least one nonzero entry. Suppose, for a contradiction, that some $G \in \mathcal{G}$ had two nonzero entries in one row. Without loss of generality, we can assume that it occurs in row 1, so $G_{1i} \neq 0 \neq G_{1j}$. Since $\mathcal{G}$ is a group, G^{-1} exists. Note that $(G^{-1})_{ik}$ and $(G^{-1})_{jk}$ must be zero for all $k \geq 2$, since otherwise, $I = GG^{-1}$ would have nonzero entries occurring off of the diagonal. But, then the i-th and j-th row of G^{-1} would be linearly dependent and this implies that the rank of G^{-1} is less than n. This is a contradiction. The proof for columns is similar and so each member of $\mathcal{G}$ must have exactly one nonzero entry in each row and column.

Now for the second part of the proof, assume $\mathcal{G}$ is bounded. We claim that this implies that $\rho(G) = 1$ for all $G \in \mathcal{G}$ ($\rho(G)$ is the spectral radius of G). First, suppose for a contradiction that $\rho(G) > 1$. Then, there is an eigenvalue, λ, of G such that $|\lambda| > 1$. Then, λ^k is an eigenvalue of G^k. Let v_o be the eigenvector corresponding to the eigenvalue λ. $\|G^k\| = \sup_{0 \neq v \in \mathcal{V}}\{\|G^k(v)\|/\|v\|\} \geq \|G^k(v_o)\|/\|v_o\| = |\lambda^k| = |\lambda|^k$. Thus, the norm of G^k is unbounded which is a contradiction. If $\rho(G) < 1$, then $\rho(G^{-1}) > 1$ and so the same argument can be used again to show that this is impossible. Thus, $\rho(G) = \rho(G^{-1}) = 1$ for all $G \in \mathcal{G}$.

The only diagonal element in $\mathcal{G}$ is I, since otherwise if $D \neq I$ were diagonal, then $\rho(D)$ or $\rho(D^{-1})$ would be greater than 1. Since each member $G \in \mathcal{G}$ has only one nonzero entry in each row and column, some power of G must be diagonal which means that any diagonal entry of G must be 1. Thus, for each $G \in \mathcal{G}$, there exists a k such that $G^k = I$. After a permutation, members of $\mathcal{G}$ thus have the form,

$$G = G_1 \oplus \cdots \oplus G_m,$$

where $G_i \in \mathcal{G}_i$ and $\mathcal{G}_i$ is an indecomposable group. If we now prove the lemma for an indecomposable group, we will be done, since if D_i is a diagonal matrix such that $D_i^{-1}\mathcal{G}_i D_i$ is a permutation group for each i, then $D = D_1 \oplus \cdots \oplus D_m$ is a diagonal matrix with the desired property for the group $\mathcal{G}$.

Assume $\mathcal{G}$ is indecomposable. First, note that each (i, j) entry of G can take only one nonzero value as G is varied over $\mathcal{G}$. To show this, suppose G_1 and G_2 have different nonzero values in their (i, j) entry. This means that G_1^{-1} and G_2^{-1} will have different (j, i) entries. Then, the i-th diagonal entry of $G_1G_2^{-1}$ will be $(G_1)_{ij}(G_2^{-1})_{ji}$, which is neither 0 nor 1, and this is a contradiction. By Lemma 2.5(iii), we know that there is no entry that is zero for all $G \in \mathcal{G}$, and so for each i, there is a $G_i \in \mathcal{G}$ such that the i-th coordinate, p_i, in the first column is nonzero. Let $D = diag(p_1, \ldots, p_n)$. Note that $p_1 = 1$ since p_1 is on the diagonal of G_1 and any element on the diagonal must be 1 as was shown above. Now, the nonzero element, in the i-th position of the first column of $D^{-1}GD$ is $1/p_i \cdot p_i \cdot 1 = 1$ for each G. Therefore, the nonzero element in the first row of each $D^{-1}GD$ must also be 1, since otherwise, we would have an element on the diagonal that is neither 0 nor 1. Let $G_{ij} = c$ be nonzero for some G. Then, there exists S and T such that $S_{1i} \neq 0$ and $T_{j1} \neq 0$ and so

$$1 = (SGT)_{11} = S_{1i}G_{ij}T_{j1} = 1 \cdot G_{ij} \cdot 1 = c.$$

This implies that every entry of $D^{-1}GD$ is either 0 or 1 for all $G \in \mathcal{G}$.

□

The next result first appeared in Radjavi [15].

Lemma 2.9. *Let A be a nonzero nonnegative matrix and assume that A has at least one zero column or row. Let $\mathcal{S}$ be the semigroup of nonnegative matrices that commute with A. Then $\mathcal{S}$ is decomposable.*

Proof. If A has a zero row, then A^* has a zero column and commutes with the members in semigroup $\mathcal{S}^*$ of nonnegative matrices. Thus, without loss of generality, we can assume that A has a zero column. Now, permute the basis so that

$$A = \begin{pmatrix} 0 & B \\ 0 & C \end{pmatrix}$$

where the first m columns displayed above are the only zero columns of A. Let $S \in \mathcal{S}$, where we express S in the same block decomposition as A. The equation $AS = SA$ for all $S \in \mathcal{S}$ implies that

$$BS_{21} = CS_{21} = 0, \quad (*)$$

where S_{21} is the $(2,1)$ block of S. If S_{21} is zero for all S we are done, and so assume $S_{21} \neq 0$. Then, S_{21} has a nonzero entry, say the (r,k) entry of this block. Then, the nonnegativity of B and C and equation $(*)$ imply that

$$0 = (BS_{21})_{ik} = \sum_{j=1} B_{ij}(S_{21})_{jk} \geq B_{i,r}(S_{21})_{r,k},$$

which means that $B_{i,r} = 0$ for all i, and so the r-th column of B is zero. Likewise, the r-th column of C is zero. However, this is a contradiction, because it implies that A has a zero column other than those exhibited above. Therefore, $S_{21} = 0$ for all $S \in \mathcal{S}$, and so $\mathcal{S}$ is decomposable. $\square$

Definition 2.10. *A band is a semigroup that consists only of idempotents.*

Example 2.11. *A good example of a band is the semigroup consisting of all the matrices which have a single column of 1's and all other entries zero. Each element is a rank one idempotent, and the product of any two gives one back.*

This next theorem on bands is another result from Marwaha [10].

Theorem 2.12. *Let $\mathcal{S}$ be a band of nonnegative matrices, and denote the minimal rank of members of $\mathcal{S}$ by r. If $r > 1$, then $\mathcal{S}$ is decomposable. In fact, there exists a permutation P such that $P^{-1}\mathcal{S}P$ has an $r \times r$ block-upper-triangular form.*

Proof. Suppose $r > 1$ and let Q be a rank-r member of $\mathcal{S}$. By Lemma 2.6, there is a permutation of the basis such that

$$Q = \begin{pmatrix} Q_1 & X \\ 0 & Q_2 \end{pmatrix},$$

where Q_1 and Q_2 are idempotents of positive rank less than r. Now, for any $S \in \mathcal{S}$, QSQ is a rank r idempotent with the same range and kernel as Q. This implies that $QSQ = Q$. Consider the block form of S that corresponds to the block form of Q given above,

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix},$$

Then, $QSQ = Q$ implies that $Q_2 S_{21} Q_1 = 0$. Then,

$$\begin{pmatrix} 0 & 0 \\ 0 & Q_2 \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} Q_1 & 0 \\ 0 & 0 \end{pmatrix} = 0,$$

for all $S \in \mathcal{S}$ which implies that $\mathcal{S}$ is decomposable by Lemma 2.5.

We now want to show that under a suitable permutation, $\mathcal{S}$ exhibits a simultaneous $r \times r$ block-upper-triangular form. We have shown $\mathcal{S}$ is decomposable, and so we can assume that every $S \in \mathcal{S}$ is of the form,

$$S = \begin{pmatrix} S_1 & X \\ 0 & S_2 \end{pmatrix},$$

where $S^2 = S$ tells us that $S_1^2 = S_1$ and $S_2^2 = S_2$. Let S and T be members of $\mathcal{S}$. Each has the form of S above, and since ST is also an idempotent, we can see that S_1T_1 and S_2T_2 are idempotents. Therefore, the set of matrices S_1 for $S \in \mathcal{S}$ is a band which we will denote $\mathcal{S}_1$. Similarly, the set of matrices S_2 for $S \in \mathcal{S}$ is a band which we will denote $\mathcal{S}_2$. We will next show that $r = r_1 + r_2$, where r_1 and r_2 are the minimal ranks of $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively. Once we have shown this, then by induction we can continue to decompose whichever of the $\mathcal{S}_i$ has rank minimal rank greater than 1, until the desired block decomposition is achieved. That $r \geq r_1 + r_2$ is clear. To show the reverse inequality, let S and T be members of $\mathcal{S}$, such that the rank of S_1 is r_1 and the rank of T_2 is r_2. $S_2 \neq 0 \neq T_1$, since otherwise, SQ or TQ would have rank less than r. Note that an idempotent's rank is equal to its trace, and so we have

$$r \leq rank(ST) = tr(ST) = tr(S_1T_1) + tr(S_2T_2)$$

$$= rank(S_1T_1) + rank(S_2T_2) = rank(S_1) + rank(T_2) = r_1 + r_2,$$

which completes the proof.

$\square$

Chapter 3

The Structure of Indecomposable Semigroups

In this chapter, we extend the Perron-Frobenius Theorem to the general case of indecomposable semigroups of nonnegative matrices with a certain condition on the minimal idempotents. If the Perron-Frobenius Theorem is viewed as a result about singly generated semigroups, it turns out that the uniqueness of the minimal idempotent is what makes it work. The uniqueness of the range of the minimal idempotents turns out to be sufficient for many of the theorem's results, which is the main result of the section. The results in the section are an expansion of the material in Radjavi [13]. The beautiful Perron-Frobenius Theorem was first proven in Perron [11] for matrices with positive entries, and generalized by Frobenius [4] to nonnegative matrices.

Here is a statement of the Perron-Frobenius Theorem, which will be restated and proven later. Some expressions may not be defined yet, but will be shortly.

Corollary 3.1. ***The Perron-Frobenius Theorem*** *Let A be an indecomposable nonnegative matrix with $\rho(A) = 1$. Denote by r the minimal rank of nonzero members of $\overline{\mathbb{R}^+\mathcal{S}}$, where $\mathcal{S}$ is the semigroup generated by A. Then the following hold:*

(i) *The sequence* $\{A^{rj}\}_{j=1}^{\infty}$ *converges to an idempotent* E *of rank* r*;*

(ii) *if* $r > 1$*, there is a permutation matrix* P *such that* $P^{-1}AP$ *has the block form*

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & A_r \\ A_1 & 0 & \cdots & 0 & 0 \\ 0 & A_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & A_{r-1} & 0 \end{pmatrix}$$

(with square diagonal blocks);

(iii) *there is a positive column vector* x*, unique up to scalar multiple, such that* $Ax = x$*;*

(iv) *the set* $\{\lambda \in \rho(A) : |\lambda| = 1\}$ *consists precisely of all the* r*-th roots of unity; each member of the set is a simple eigenvalue;*

(v) $\sigma(A)$ *is invariant under the rotation about the origin by the angle* $2\pi/r$*;*

(vi) *1 is dominant in* $\sigma(A)$ *if and only if some power of* A *has all its entries positive. This occurs precisely when* $\{A^j\}$ *is convergent.*

We will now present a few lemmas which will be useful in the generalizations.

Lemma 3.2. *Let* $\mathcal{S}$ *be a semigroup of nonnegative matrices and let* E *be a nonnegative idempotent of rank* r *(not necessarily in* $\mathcal{S}$*). Then*

(i) *relative to some basis* $x_1, \cdots, x_r$ *of* $E\mathcal{V}$*, every operator in the collection* $E\mathcal{S}E|_{E\mathcal{V}}$ *has a nonnegative matrix, and*

(ii) *this representation of* $E\mathcal{S}E|_{E\mathcal{V}}$ *is indecomposable if* $\mathcal{S}$ *is indecomposable.*

Proof. (i): Since E has rank r, by Corollary 2.6, there exists r linearly independent columns, $x_1, \ldots, x_r$, such that every column of E is a nonnegative linear combination of these $\{x_i\}$. This is a basis of $E\mathcal{V}$, since the range of E is equal to the column space of E. Let $ETE \in E\mathcal{S}E$. Then $ETEx_i = ETx_i$ and ETx_i is a nonnegative linear combination of the columns of E. Thus $ETEx_i = a_1x_1 + \cdots + a_rx_r$ where $a_j \geq 0$. This is true for all i and so $ETE|_{E\mathcal{V}}$ is nonnegative with respect to the basis $\{x_i\}$.

(ii): Since $\mathcal{S}$ is indecomposable, Lemma 2.5 implies that there exists $S_1, \ldots, S_k$ in $\mathcal{S}$ such that $T = S_1 + \cdots + S_k$ has all positive entries. Consider $ETE|_{E\mathcal{V}} = (ES_1E + \cdots + ES_kE)|_{E\mathcal{V}}$. $ETEx_i = ETx_i$ and since T is positive and $x_i \neq 0$, $Tx_i = y_i$ is a positive vector. Thus, $ETEx_i = Ey_i$ is a positive linear combination of the columns of E. This is true for all i so $ETE|_{E\mathcal{V}}$ in the basis $\{x_i\}$ has all positive entries. $(ES_1E + \cdots + ES_kE)|_{E\mathcal{V}}$ is thus a sum of matrices in $E\mathcal{S}E|_{E\mathcal{V}}$ that has no zero entry and so by Lemma 2.5, $E\mathcal{S}E|_{E\mathcal{V}}$ is indecomposable. $\square$

Definition 3.3. *A nonzero idempotent E in a semigroup $\mathcal{S}$ is called minimal if $EF = FE = F$ for any idempotent $F \in \mathcal{S}$ implies that either $F = E$ or $F = 0$.*

Before we proceed with the next Lemma, we need one more definition. If $\mathcal{S}$ is a semigroup of matrices, it will be useful to consider all positive scalar multiples of members in $\mathcal{S}$, since then we can make assumptions such as $\rho(S) = 1$ for any S with a desired property. Also, it is useful to be able to assume that our semigroup is closed, so that we may assume in convergent sequences of powers that the limit is in our semigroup. Thus, let $\mathbb{R}^+\mathcal{S} = \{cS : c \in \mathbb{R}^+, S \in \mathcal{S}\}$ so that $\overline{\mathbb{R}^+\mathcal{S}}$ is the closure of the preceding set in the norm topology. The next Lemma is extremely useful in the rest of this section, and also in the next chapter, where part (iv) will be used to help prove that certain semigroups are finite.

Lemma 3.4. *Let $\mathcal{S}$ be an indecomposable semigroup of nonnegative matrices* *such that $\mathcal{S} = \overline{\mathbb{R}^+\mathcal{S}}$ and let r be the minimal rank of nonzero members of $\mathcal{S}$. Then the following hold:*

(i) an idempotent $E \in \mathcal{S}$ is of rank r if and only if it is minimal;

(ii) for each A of rank r in $\mathcal{S}$ there is a minimal idempotent $F \in \mathcal{S}$ with $FA = A$;

(iii) for each i there exists a minimal idempotent in $\mathcal{S}$ whose i-th row is nonzero, and the same assertion is true for columns;

(iv) if E is a minimal idempotent in $\mathcal{S}$, then $E\mathcal{S}E \setminus \{0\}$ is a group with identity E. The set

$$\{ESE : S \in \mathcal{S}, \rho(ESE) = 1\}$$

is a subgroup whose restriction $\mathcal{G}$ to the range of E is simultaneously similar, via a diagonal matrix, to a transitive group of permutation matrices.

Proof. (i): Let E be an idempotent of rank r and let F be an idempotent such that $EF = FE = F$. Then the range of F is contained in that of E and the kernel of E is contained in that of F. If there were an element x_o in the kernel of F which is not in the kernel of E, then the rank of F would be strictly less than that of E which implies that $F = 0$ since E has minimal nonzero rank. Likewise, if there were an element y_o in the range of E that is not in the range of F, then the rank of E would be strictly greater than that of F which implies $F = 0$. Thus, if $F \neq 0$, then E and F have the same kernel and range and are thus equal. Hence, E is minimal.

Now suppose E is minimal. Let the rank of E be $s \geq r$. Let $\mathcal{S}_r$ be the ideal of all elements with rank r or 0. $\mathcal{S}_r$ is indecomposable by Lemma 2.5 and so $E\mathcal{S}_rE|_{E\mathcal{V}}$ is nonnegative and indecomposable with respect to the basis $x_1 \ldots x_r$ (where $\{x_i\}$ are the vectors as in Lemma 3.2). Now, we only need to show that there exists a nonzero idempotent $F \in E\mathcal{S}_rE$. Then, we will have that $EF = FE = F$, which will imply that $E = F$ since E is minimal (and so E is rank r).

Since $E\mathcal{S}_r E$ is indecomposable, Lemma 2.1 implies that it must contain a non-nilpotent element, A. We can assume $\rho(A) = 1$ since $\mathcal{S} = \overline{\mathbb{R}^+\mathcal{S}}$. Now, since r is the minimal rank in $\mathcal{S}$, we know that every element in $\{A^k : k \in \mathbb{N}\}$ has rank r. The proof of the existence of this idempotent does not require nonnegativity, and so we will assume that A is in its Jordan form, and that

$$A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$$

where $\sigma(B)$ lies on the unit circle and $\sigma(C)$ lies inside the unit circle. Note that since $\rho(C) < 1$, $\lim_{n\to\infty} \|C^n\| = 0$. Now, since we have assumed that A is in Jordan form, $B = U + N$ where $U|_{E\mathcal{V}}$ is unitary, $N|_{E\mathcal{V}}$ is nilpotent and $UN = NU$. Note that the rank of N must be strictly less than that of B which can be seen easily by visualizing the Jordan form of B.

Claim: $N = 0$ and C acts on the zero dimensional space. To show this, let $k \in \mathbb{N}$ such that $N^k \neq 0$ and $N^{k+1} = 0$. Now for $n > k$, the binomial expansion yields

$$(U + N)^n = U^n + \begin{pmatrix} n \\ 1 \end{pmatrix} U^{n-1}N + \cdots + \begin{pmatrix} n \\ k \end{pmatrix} U^{n-k}N^k$$

since $N^{k+j} = 0$ for all $j \geq 1$. Since U is unitary, some subsequence of U^n converges to I. Pick the subsequence n_j such that $U^{n_j - k} \to I$. If $k \neq 0$, divide both sides by $\begin{pmatrix} n \\ k \end{pmatrix}$ to get

$$\lim_{j\to\infty} \frac{(U + N)^{n_j}}{\begin{pmatrix} n_j \\ k \end{pmatrix}} = \lim_{j\to\infty} U^{n_j - k}N^k = N^k.$$

Since $\lim_{n\to\infty}\|C^n\| = 0$,

$$\lim_{j\to\infty}\frac{A^{n_j}}{\begin{pmatrix} n_j \\ k \end{pmatrix}} = \begin{pmatrix} N^k & 0 \\ 0 & 0 \end{pmatrix}$$

which is a contradiction since N^k has rank strictly less than r (even if k=1). Thus, $k = 0$ which implies $N = 0$. Thus, C must act on the zero dimensional space, since otherwise $\lim_{j\to\infty} A^{n_j}$ would have nonzero rank strictly less than r. Thus, $A|_{A\mathcal{V}}$ is in fact unitary and so there is a subsequence n_k such that A^{n_k} converges to an idempotent, F, completing the proof.

(ii): It suffices to show that given $A \in \mathcal{S}$, there exists $B \in \mathcal{S}$ such that A and B have the same range and B is not nilpotent. Once we have such a B, we can assume that $\rho(B) = 1$ since $\mathcal{S} = \overline{\mathbb{R}^+\mathcal{S}}$, and then apply the method in part (i) to obtain an idempotent with the same range as B.

Consider the ideal $\mathcal{S}A\mathcal{S}$. We know that $A\mathcal{S} \neq \{0\}$ since otherwise $\mathcal{S}$ would be decomposable by Lemma 2.5. Thus, pick $T \in \mathcal{S}$ such that $AT \neq 0$ and note that the same lemma implies that $\mathcal{S}AT \neq 0$ and so $\mathcal{S}A\mathcal{S} \neq \{0\}$. By Lemma 2.5, this ideal is indecomposable. Thus, by Lemma 2.1, every element in $\mathcal{S}A\mathcal{S}$ cannot be nilpotent and so there exists S_1 and S_2 in $\mathcal{S}$ such that S_1AS_2 is not nilpotent.

Claim: $\sigma(AS_2S_1) = \sigma(S_1AS_2)$. Suppose a nonzero λ is not an eigenvalue of AS_2S_1. Then $\lambda I - AS_2S_1$ is invertible. A simple calculation shows that $\frac{1}{\lambda}S_1(\lambda I - AS_2S_1)^{-1}AS_2 + \frac{1}{\lambda}$ is the inverse of $\lambda I - S_1AS_2$ and so if $\lambda \neq 0$ is not an eigenvalue of one then it is not an eigenvalue of the other. Now, S_1AS_2 is invertible if and only if both S_1 and AS_2 are, so 0 is an eigenvalue of S_1AS_2 if and only if it is an eigenvalue of AS_2S_1 too. Thus, the claim holds.

Therefore, we have that $\sigma(AS_2S_1) = \sigma(S_1AS_2) \neq \{0\}$. Letting $B = AS_2S_1$ and using (i) gives us an E such that $EB = B$, and so $EA = A$ too.

(iii): Recall the indecomposable ideal $\mathcal{S}_r$ from (i). There exists an $A \in \mathcal{S}_r$ such that the i-th row is nonzero by Lemma 2.5 and there exists an idempotent, E, such that $EA = A$ from (ii). The i-th row of E must be nonzero, since otherwise the range of E would not be the same as the range of A. Consider $\mathcal{S}^*$, which is clearly an indecomposable semigroup. Define similarly the indecomposable ideal, $\mathcal{S}_r^*$. Then pick $B^* \in \mathcal{S}_r^*$ such that the i-th row is nonzero. Then, again we have an idempotent $F^* \in \mathcal{S}_r^*$ such that $F^*B^* = B^*$. Then, the i-th row of F^* must be nonzero. Note that F^* is an idempotent if and only if F is. Thus, $F = (F^*)^* \in \mathcal{S}_r$ has its i-th column nonzero.

(iv): E is clearly an identity on $E\mathcal{S}E$. We will prove later that $\{ESE|_{E\mathcal{V}} : S \in \mathcal{S}, \rho(ESE) = 1\}$ is a group and so for now, we will assume it. If $ETE \in E\mathcal{S}E \setminus \{0\}$ such that $0 \neq \rho(ETE) = c$, then $\frac{1}{c}ETE \in \{ESE : S \in \mathcal{S}, \rho(ESE) = 1\}$ and so it has an inverse, ERE. Then,

$$(\frac{1}{c}ETE)(ERE) = (ETE)(\frac{1}{c}ERE) = E$$

and so $(\frac{1}{c}ERE)$ is the inverse of ETE. We claim that for all A and B in $E\mathcal{S}E \backslash \{0\}$, $AB \neq 0$. Let $x_1, \ldots, x_r$ be the r linearly independent columns of E obtained in Corollary 2.6. From Lemma 3.2, $E\mathcal{S}E|_{E\mathcal{V}}$ with respect to the basis, $x_1, \ldots, x_r$, is still nonnegative and indecomposable. $E\mathcal{V}$ is an r-dimensional subspace and every element in $E\mathcal{S}E|_{E\mathcal{V}}$ is rank r. Thus, each member must have r nonzero eigenvalues, and so the product of any two will also. Therefore, the rank of AB is r for any A and B in $E\mathcal{S}E$. Thus, $\rho(S) \neq 0$ for all $S \in E\mathcal{S}E$. Thus, if $\{ESE|_{E\mathcal{V}} : S \in \mathcal{S}, \rho(ESE) = 1\}$ is a group, then so is $E\mathcal{S}E \setminus \{0\}$.

We will now show that $\{ESE|_{E\mathcal{V}} : S \in \mathcal{S}, \rho(ESE) = 1\}$ is a semigroup. Let $S \in \mathcal{S}$ with $\rho(ESE) = 1$. Since ESE has minimal rank and is not nilpotent, we can apply the same argument as applied to the matrix A in the proof of (i), to deduce that $ESE|_{E\mathcal{V}}$ is in fact similar to a unitary operator. Let $A, B \in \mathcal{G}$ which means that $\rho(A) = 1 = \rho(B)$. Clearly, $\rho(AB) \geq 1$ since otherwise $1 > |det(AB)| = |det(A)||det(B)| = 1$, a contradiction. Suppose $\rho(AB) > 1$. Then, since $|det(AB)| = 1$, $\sigma(AB)$ must contain elements which have modulus less than 1. Using the argument applied to the matrix A in (i) to the operator $AB/\rho(AB)$ will result in a nonzero element of rank strictly less than r which is a contradiction. Thus $\rho(AB) = 1$ and so $\mathcal{G}$ is a semigroup.

We now need to show that $ESE \in \mathcal{G}$ has an inverse. Since $\rho(ESE) = 1$, there is a sequence $\{m_i\}$ such that

$$E = \lim_{i\to\infty}(ESE)^{m_i} = \lim_{i\to\infty}(ESE)(ESE)^{m_i-1} = (ESE)\lim_{i\to\infty}(ESE)^{m_i-1}.$$

Thus, (ESE) thus has an inverse in $\mathcal{G}$ which is $\lim_{i\to\infty}(ESE)^{m_i-1}$, and so $\mathcal{G}$ is a group.

To complete (iv), we must show that $\mathcal{G}$ is simultaneously similar, via a diagonal matrix, to a transitive group of permutations matrices. As in Lemma 2.7, let $x_1, \ldots, x_r$ be the r independent columns of E whose span contains all of the columns of E. From Lemma 3.2, $\mathcal{G}$ is nonnegative and indecomposable relative to this basis. We claim that $\mathcal{G}$ is bounded. Suppose for a contradiction, that there exists a sequence, $\{A_n\}$ in $\mathcal{G}$ such that $\lim_{n\to\infty}\|A_n\| = \infty$. Note that $\|(A_n)/\|A_n\|\| = 1$ for all n, and so the bounded sequence $\lim_{n\to\infty}\|(A_n)/\|A_n\|\|$ has a subsequence which converges to some element, A. Since $\rho(A_n) = 1$ for all n, $\rho(A) = 0$ by the continuity of the spectral radius. Thus A is nilpotent, which is a contradiction and so $\mathcal{G}$ is bounded. All of the conditions of Lemma 2.8 are satisfied, and we may apply it to complete the proof.

$\square$

Definition 3.5. *A right ideal of a semigroup $\mathcal{S}$ is a subset $\mathcal{J}$ such that $JS \in \mathcal{J}$ for all $J \in \mathcal{J}$ and $S \in \mathcal{S}$. A minimal right ideal is a nonzero right ideal that contains no other nonzero right ideal.*

Lemma 3.6. *Let $\mathcal{S} = \overline{\mathbb{R}^+\mathcal{S}}$ be an indecomposable semigroup of nonnegative matrices. The following hold:*

(i) Every nonzero right ideal of $\mathcal{S}$ contains a minimal right ideal;

(ii) Every minimal right ideal is of the form $E\mathcal{S}$ for some minimal idempotent $E \in \mathcal{S}$.

Proof. Let $\mathcal{J}$ be a right ideal of $\mathcal{S}$. Let r be the minimal nonzero rank of elements in $\mathcal{S}$ and let $\mathcal{S}_r$ be the ideal of all elements of rank r or 0. Pick a nonzero $S \in \mathcal{S}_r$ and note that $\mathcal{J}S \neq \{0\}$ by Lemma 2.5 parts (v) and (ii), and so we can assume that there is an element, S_o in $\mathcal{J}$ of rank r. By Lemma 3.4(ii), we know there exists a minimal idempotent E such that $ES_o = S_o$. Furthermore, from part (iv) of that same lemma, $E\mathcal{S}E \setminus \{0\}$ is a group and so there exists a $ET_oE \in E\mathcal{S}E$ such that $(ES_oE)(ET_oE) = E$. Thus, $E = (ES_oE)(ET_oE) = (ES_oET_oE) = (S_o)(ET_oE) \in \mathcal{J}$ and so $E\mathcal{S} \subset \mathcal{J}$.

To complete the proof of (i), we need to show that if E is a minimal idempotent, then $E\mathcal{S}$ is a minimal right ideal. Suppose $\mathcal{K}$ is a nonzero right ideal contained in $E\mathcal{S}$. From above, we know that there exists a minimal idempotent F such that $F \in F\mathcal{S} \subset \mathcal{K} \subset E\mathcal{S}$. Thus, $F = EA$ for some $A \in \mathcal{S}$ which means that E and F have the same range. Since these idempotents have the same range, and each acts like the identity on the other's range, they satisfy $FE = E$ (and $EF = F$), and so $E\mathcal{S} = FE\mathcal{S} \subset F\mathcal{S} \subset \mathcal{K} \subset E\mathcal{S}$ and so $E\mathcal{S} = \mathcal{K}$, meaning $E\mathcal{S}$ is minimal. This also proves (ii) since every minimal right ideal will contain (and thus be equal to) an ideal of the form $E\mathcal{S}$ for some minimal idempotent.

□

Lemma 3.7. *Let $\mathcal{S}$ be an indecomposable semigroup of nonnegative matrices and let $\mathcal{E}$ denote the set of all minimal idempotents in $\overline{\mathbb{R}^+\mathcal{S}}$. The following conditions are mutually equivalent:*

(i) $EF = F$ *for every* E *and* F *in* $\mathcal{E}$ *(i.e., all minimal idempotents have the same range);*

(ii) *all nonzero minimal-rank members of* $\mathcal{S}$ *have the same range;*

(iii) $\mathcal{S}E = E\mathcal{S}E$ *for some* $E \in \mathcal{E}$*;*

(iv) $\mathcal{S}E = E\mathcal{S}E$ *for all* $E \in \mathcal{E}$*;*

(v) $\mathcal{S}E \subset E\mathcal{S}$ *for some* $E \in \mathcal{E}$*;*

(vi) $\mathcal{S}E \subset E\mathcal{S}$ *for all* $E \in \mathcal{E}$*;*

(vii) $\mathcal{S}$ *leaves the range of some* $E \in \mathcal{E}$ *invariant;*

(viii) $\mathcal{S}$ *leaves the range of every* $E \in \mathcal{E}$ *invariant;*

(ix) some minimal right ideal is an ideal;

(x) every minimal right ideal is an ideal;

(xi) $\mathcal{S}$ *has a unique minimal right ideal.*

Proof. $(i) \Leftrightarrow (xi)$: Assume (i). Let $\mathcal{J}$ be a minimal right ideal. Then, from Lemma 3.6, $\mathcal{J} = E\mathcal{S}$ for some minimal idempotent E. But, $FE = E$ for any minimal idempotent $F \in \mathcal{S}$ and so $E\mathcal{S} = FE\mathcal{S} \subset F\mathcal{S}$ but $F\mathcal{S}$ is also a minimal right ideal and so $E\mathcal{S} = F\mathcal{S}$. Thus (i) implies (xi). Assume (xi). Then $E\mathcal{S} = F\mathcal{S}$ for all minimal idempotents E and F in $\mathcal{S}$, and so $E\mathcal{S} = EF\mathcal{S} = F\mathcal{S}$ and so $EF = F$. Thus, (i) and (xi) are equivalent.

$(i) \Leftrightarrow (ii)$: Assume (i). Let S and T be nonzero, minimal rank members of $\mathcal{S}$. Then, $ES = S$ and $FT = T$ for some minimal idempotent E and F in $\mathcal{S}$ by Lemma 3.4(ii). Since $EF = F$ by hypothesis, $T = FT = EFT = ET$ and so S and T have the same range. (ii) implies (i) immediately.

$(ii) \Rightarrow (iv)$: If $E \in \mathcal{E}$, then since every nonzero member of $\mathcal{S}E$ is minimal rank, they all have the same range as E. Thus $\mathcal{S}E = E\mathcal{S}E$.

$(iv) \Rightarrow (vi)$: This is immediate.

$(vi) \Rightarrow (viii)$: This is also immediate.

$(viii) \Rightarrow (x)$: Let $E\mathcal{S}$ be a minimal right ideal. Since $\mathcal{S}$ leaves the range of E invariant, we have that $SE = ESE$ for all $S \in \mathcal{S}$ and so $\mathcal{S}E\mathcal{S} = E\mathcal{S}E\mathcal{S} \subset E\mathcal{S} \subset \mathcal{S}E\mathcal{S}$. Thus $\mathcal{S}E\mathcal{S} = E\mathcal{S}$.

$(x) \Rightarrow (xi)$: Let E and F be two minimal idempotents. $E\mathcal{S}F \neq 0$ by Lemma 2.5(ii) and so we can pick an $A \in \mathcal{S}$ such that $EAF \neq 0$. $EAF\mathcal{S}$ is a right ideal and $EAF\mathcal{S} \subset E\mathcal{S}$. Since $E\mathcal{S}$ is minimal, $EAF\mathcal{S} = E\mathcal{S}$.

By hypothesis, $F\mathcal{S}$ is not only a right ideal, but an ideal, so that $E\mathcal{S} = (EA)F\mathcal{S} \subset F\mathcal{S}$. Since $F\mathcal{S}$ is minimal, $E\mathcal{S} = F\mathcal{S}$, and so $\mathcal{S}$ has a unique minimal (right) ideal.

$(ix) \Rightarrow (v)$: Suppose $E\mathcal{S}$ is an ideal. Then, $\mathcal{S}E\mathcal{S} = E\mathcal{S}$ and so $\mathcal{S}E \subset \mathcal{S}E\mathcal{S} = E\mathcal{S}$.

$(v) \Rightarrow (vii)$: This is immediate.

$(vii) \Rightarrow (iii)$: This is immediate.

$(iii) \Rightarrow (ix)$: This is immediate.

$(x) \Rightarrow (ix)$: This is immediate.

$(ix) \Rightarrow (i)$: Let $E\mathcal{S}$ be the ideal and let $F \in \mathcal{E}$. We know there exists a $T \in \mathcal{S}$ such that $FTE \neq 0$ by Lemma 2.5, and then FTE has the same range as F since both are the same rank. Then $FTE \in \mathcal{S}E \subset \mathcal{S}E\mathcal{S} = E\mathcal{S}$, and so the ranges of E and F are the same.

□

This next result will be very useful in the proof of the main result of this section, the generalization of the Perron-Frobenius Theorem.

Lemma 3.8. *Let $\mathcal{S} = \overline{\mathbb{R}^+\mathcal{S}}$ be an indecomposable semigroup of nonnegative matrices satisfying one, and thus all of the conditions in Lemma 3.7. If $\mathcal{R}$ be the common range of the minimal idempotents in $\mathcal{S}$, then*

(i) there exists a nonnegative idempotent P with range $\mathcal{R}$ in the closed convex hull of $\mathcal{S}$ that has no zero rows or columns and satisfies

$$\mathcal{S}|_{\mathcal{R}} = P\mathcal{S}P|_{\mathcal{R}}$$

;

(ii) the only nilpotent element of $\mathcal{S}$ is zero, and

(iii) the spectral radius is multiplicative on $\mathcal{S}$, so that

$$\mathcal{S}_o = \{S/\rho(S) : 0 \neq S \in \mathcal{S}\}$$

is a sub-semigroup on which ρ is identically one.

Proof. (i): We know from Lemma 3.4 that there exists minimal idempotents $E_1, \ldots, E_n$ such that the i-th column of E_i is nonzero for each i. If any row of an E_i were zero, then it would be zero for all minimal idempotents since all minimal idempotents have the same range. But, by Lemma 3.4, we know that for each j, there exists a minimal idempotent whose j-th row is nonzero. Thus, no minimal idempotent has a zero row.

Consider $P = (E_1 + \cdots + E_n)/n$. Since every E_i has the same range, we have that $E_iE_j = E_j$ and so $P^2 = (nE_1 + \cdots + nE_n)/n = P$. Thus P is an idempotent with range $\mathcal{R}$ which satisfies $PE = E$ and $EP = P$. By Lemma 3.7, $SP = PSP$ as well as $SE = ESE$ for all $S \in \mathcal{S}$ and every minimal idempotent E. Since P clearly does not have any zero rows are columns and $Ex = Px = x$ for any $x \in \mathcal{R}$, we have that

$$\mathcal{S}|_{\mathcal{R}} = E\mathcal{S}E|_{\mathcal{R}} = P\mathcal{S}P|_{\mathcal{R}}.$$

(ii): Suppose $0 \neq N \in \mathcal{S}$ is nilpotent of order k and let $E \in \mathcal{S}$ be a minimal idempotent. We claim that $N|_{\mathcal{R}}$ is also nilpotent. Since the range of P is invariant under $\mathcal{S}$ by Lemma 3.7, we have that $NP \cdots NP = N^kP = 0$ and this implies $N^k|_{\mathcal{R}} = 0$. Since E has no zero rows, $NE \neq 0$. Now, since $E\mathcal{S}E$ is a group, by Lemma 3.4(iv), we have that $E = (ENE)(EME)$ and since the range of E is invariant under $\mathcal{S}$, we have that $E = (ENE)(EME) = NME$ which means that the restriction of N to the range of E is invertible, and this is a contradiction. Therefore, $N = 0$.

(iii): We will now show that $\rho(S) = \rho(SP)$, i.e., an eigenvalue which is largest in modulus corresponds to an eigenvector in $\mathcal{R}$. We can see that $\rho(SP) \leq \rho(S)$, since $\sigma(SP) = \sigma(S|_{P\mathcal{V}}) \cup \{0\}$ which is the set of those eigenvalues of S which have eigenvectors in the range of P and so this is clearly a subset of $\sigma(S)$. Suppose, for a contradiction, that $\rho(SP) < \rho(S)$. We can scale S so that $\rho(S) = 1$ since $\mathcal{S} = \overline{\mathbb{R}^+\mathcal{S}}$. We know that $\{S^m\}$ must be a bounded sequence, since otherwise a subsequence of $\{S^m/\|S^m\|\}$ would converge to a nilpotent, by continuity of the spectrum, and this nilpotent would be nonzero since $\|S^m/\|S^m\|\| = 1$ for all m. Thus, $S^{m_i} \to A$ with $\rho(A) = 1$ for some subsequence, $\{m_i\}$. Now, $AP = \lim_{i\to\infty} S^{m_i}P$. Note that

$$S^{m_i}P = S^{m_i-1}SP = S^{m_i-1}PSP$$

since $PSP = SP$ by Lemma 3.7 and so $S^{m_i}P = (SP)^{m_i}$ for all i. Thus

$$AP = \lim_{i\to\infty}(SP)^{m_i} = 0$$

where the equality to zero is due to the fact that $\rho(SP) < 1$. Since P has no zero rows or columns by construction, $AP = 0$ implies that $A = 0$, which is a contradiction.

Let T and S be in $\mathcal{S}$. We have that $\rho(ST) = \rho(STP)$ from the above argument. $\rho(STP) = \rho(PSPPTP)$ since the range of P is invariant under $\mathcal{S}$. Since ρ is multiplicative on $\mathcal{S}|_{\mathcal{R}}$ by Lemma 3.4(iv), it is also multiplicative on $\mathcal{S}|_{\mathcal{R}} \oplus 0$ (where 0 is the appropriately sized zero matrix) and so also on $P\mathcal{S}P$ meaning $\rho(PSP \cdot PTP) = \rho(PSP)\rho(PTP)$. Thus

$$\rho(ST) = \rho(PSP)\rho(PTP) = \rho(SP)\rho(TP) = \rho(S)\rho(T).$$

□

Here is the main result of the section.

Theorem 3.9. *Let $\mathcal{S}$ be an indecomposable semigroup of nonnegative matrices and denote the minimal positive rank in $\overline{\mathbb{R}^+\mathcal{S}}$ by r. If $\overline{\mathbb{R}^+\mathcal{S}}$ has a unique minimal right ideal (or satisfies any of the other conditions of Lemma 3.7), then the following hold:*

(i) there is a vector x with positive entries, unique up to scalar multiples, such that

$$Sx = \rho(S)x$$

for all $S \in \mathcal{S}$;

(ii) after a permutation of the basis, $\mathcal{S}$ has an $r \times r$ block partition such that the block matrix $(S_{ij})_{i,j=1}^{r}$ of each nonzero $S \in \mathcal{S}$ has exactly one nonzero block in each block row and in each block column;

(iii) every $S \in \mathcal{S}$ has at least r eigenvalues of modulus $\rho(S)$, counting multiplicities; these are all of the form $\rho(S)\theta$ with $\theta^{r!} = 1$;

(iv) if, for any $S \in \mathcal{S}$, the block matrix (S_{ij}) has a cyclic pattern (i.e., there exists a permutation $\{i_1, \cdots, i_r\}$ of $\{1, \cdots, r\}$ such that the nonzero blocks of S are precisely $S_{i_1,i_2}, S_{i_2,i_3}, \cdots, S_{i_r,i_1}$), then $\sigma(S)$ is invariant under the rotation about the origin by the angle $2\pi/r$;

(v) $r = 1$ if and only if some member of $\mathcal{S}$ has at least one positive column.

Proof. Since multiplication by a constant will not effect the claims of (i) through (iv), apart from in (v) we can assume that $\mathcal{S} = \overline{\mathbb{R}^{+}\mathcal{S}}$.

(i): Let $\mathcal{R}$ again be the common range of all minimal idempotents, and P be the positive idempotent defined in 3.8. By Lemma 2.6, after a permutation of the basis, P is of the form

$$P = P_1 \oplus \cdots \oplus P_r$$

where each P_i is a rank one positive idempotent. Since each P_i is positive, there is a positive vector, v_i, in the range of each. Note that the span of v_i is the range of P_i. Thus, the vectors

$$x_1 = \begin{pmatrix} v_1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \cdots, x_r = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ v_r \end{pmatrix}$$

form a basis for the range of P which is $\mathcal{R}$. After a diagonal similarity, the group

$$\mathcal{G} = \{S|_{\mathcal{R}} : S \in \mathcal{S}, \rho(S) = 1\}$$

is a transitive group of permutations on the $\{x_i\}$. The vector $x = x_1 + \cdots + x_r$ satisfies $Gx = x$ for all $G \in \mathcal{G}$, since $Gx = G(x_1 + \cdots + x_r) = x_{\tau(1)} + \cdots + x_{\tau(r)}$ where τ is a permutation on $\{1, \ldots, r\}$. From Lemma 3.8, we have that for $S \in \mathcal{S}$, $(PSP)_{|}\mathcal{R} = S_{|}\mathcal{R}$ and so $x = PSPx = SPx = Sx$. This means that $Sx = \rho(S)x$ for all $S \in \mathcal{S}$.

We will now show that this vector x is unique up to scalar multiplies. Let y be a vector such that $Sy = \rho(S)y$ for all $S \in \mathcal{S}$. Then, $Py = (1/n)(E_1 y + \cdots + E_r y) = y$ since each $E_i \in \mathcal{S}$, which means y is a linear combination of the columns of P; i.e. $y = a_1 x_1 + \cdots + a_r x_r$. $\mathcal{G}$ acts transitively on the x_i, and so for any i and j, we can find a G such that $Gx_i = x_j$. Since $Gx = x$, this means that $a_i = a_j$ for all i and j. Thus, y is a multiple of x.

(ii): Let $S \in \mathcal{S}$ and consider the block decomposition $S = (S_{ij})_{i,j=1}^{r}$ which corresponds to the decomposition of $P = P_1 \oplus \cdots \oplus P_r$. Let $x_1, \ldots, x_r$ be the r vectors from above which span the range of P, where for each i, x_i spans the range of P_i. Note that the nonzero blocks of these vectors are mutually disjoint. Recall that $PSP/\rho(S)$ in the group $\mathcal{G}$ is a permutation on these x_i. Also, from Lemma 3.8, $PSP|_{\mathcal{R}} = S|_{\mathcal{R}}$. Thus, $Sx_i = PSPx_i = \rho(S)x_j$ for some j.

$$Sx_i = \begin{pmatrix} S_{11} & \ldots & S_{1r} \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ S_{r1} & \ldots & S_{rr} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ v_i \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} S_{1i}v_i \\ \vdots \\ S_{1j}v_j \\ \vdots \\ S_{ir}v_r \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ v_j \\ 0 \\ \vdots \end{pmatrix}$$

where v_i and v_j are in the i-th and j-th block row respectively. This means that the i-th column of S must have all zero blocks except for the j-th row block. This argument can be repeated with each x_i, giving that each column of S has only one nonzero block. Since S is a permutation on $\{x_i\}$, each row must also have exactly one nonzero block. Clearly S must have at least one nonzero block in each row, since S maps to each x_i, and if the i-th row of S were zero, then the i-th block of Sx would be zero for all vectors x. If S had two nonzero blocks, then S would map distinct x_i and x_j to vectors with the same nonzero block row. Since the x_i have mutually disjoint block rows, this implies that $Sx_i = Sx_j$ for some $i \neq j$, but this is a contradiction since S is a permutation on $\{x_i\}$.

(iii): The order of a permutation on r elements is the product of the orders of its disjoint sub-cycles. The orders of the disjoint sub-cycles must sum to r, and thus, the order of the permutation must be less than $r!$. Therefore, from (ii), $S^{r!}$ is block diagonal. Also from (ii), $S^{r!}x_i = \rho(S)^{r!}x_j$, but since $S^{r!}$ is block diagonal, $i = j$. This means that $S^{r!}|_{\mathcal{R}} = \rho(S)^{r!}P$ and so $\sigma(PSP/\rho(S))$ contains the set, $\{\theta_i\}_{i=1}^{r}$ with $\theta_i^{r!} = 1$ for all i. In general, $\sigma(\eta S) = \eta\sigma(S)$ for any scalar η and so $\sigma(S) \supset \sigma(PSP) = \rho(S)\sigma(PSP/\rho(S)) = \{\rho(S)\theta_i\}_{i=1}^{r}$.

(iv): We can assume that the nonzero blocks of S are precisely $S_{i_1,i_2}, S_{i_2,i_3}, \cdots, S_{i_r,i_1}$. Let $\theta = e^{2\pi i/r}$ and

$$D = \theta I_1 \oplus \theta^2 I_2 \oplus \cdots \oplus \theta^r I_r,$$

where for each i, I_i is the identity matrix of the same size as S_{ii}. A simple calculation shows that $\theta DS = SD$. Thus,

$$\sigma(S) = \sigma(SDD^{-1}) = \sigma(\theta DSD^{-1}) = \theta\sigma(DSD^{-1}) = \theta\sigma(S).$$

Multiplying each member of the spectrum by $e^{2\pi i/r}$ is equivalent to rotating the spectrum by $2\pi/r$ and so the proof of (iv) is complete.

(v): ($\Leftarrow$) Suppose that $S \in \mathcal{S}$ has a positive column. Then in the decomposition of (ii), there must be only 1 block, since every row and column has only 1 nonzero block. Thus, $r = 1$.

($\Rightarrow$) Suppose $r = 1$. Then, we know that there is a rank one idempotent $E \in \overline{\mathbb{R}^+\mathcal{S}}$. From (i), we know that there is a positive vector x such that

$Ex = x$, but this implies that x is a multiple of some column of E. Therefore, E has a positive column and is the limit of a sequence in $\mathbb{R}^+\mathcal{S}$. If no member of $\mathcal{S}$ had a positive column, then there could not be such a sequence in $\mathbb{R}^+\mathcal{S}$, since every column in every S would have a zero, a property which would thus remain in the limit.

$\square$

It turns out that imposing a unique minimal idempotent on the semigroup is a stronger condition than a unique minimal ideal. The next Lemma gives useful equivalent conditions of a unique minimal idempotent.

Lemma 3.10. *Let $\mathcal{S} = \overline{\mathbb{R}^+\mathcal{S}}$ be an indecomposable semigroup of nonnegative matrices. Denote its center by $\mathcal{Z}$ and its subset of minimal idempotents by $\mathcal{E}$. The following are mutually equivalent:*

(i) $\mathcal{E}$ *is a singleton;*

(ii) $\mathcal{E} \subset \mathcal{Z}$*;*

(iii) $\mathcal{E} \cap \mathcal{Z} \neq \varnothing$*;*

(iv) $\mathcal{S}E = E\mathcal{S}$ *for some* $E \in \mathcal{E}$

(v) $\mathcal{S}E = E\mathcal{S}$ *for every* $E \in \mathcal{E}$

Proof. $(i) \Rightarrow (ii)$: Let E be the unique minimal idempotent in $\mathcal{S}$. Since E is an idempotent in $\mathcal{S}$ if and only if E^* is an idempotent in $\mathcal{S}^*$, E^* is the unique minimal idempotent in $\mathcal{S}^*$. The range of E is invariant under $\mathcal{S}$ by Lemma 3.7 and so $SE = ESE$ for all $S \in \mathcal{S}$. Likewise, the range of E^* is invariant under $\mathcal{S}^*$ and so $S^*E^* = E^*S^*E^*$ for all $S^* \in \mathcal{S}^*$. Thus, $ES = (S^*E^*)^* = (E^*S^*E^*)^* = ESE = SE$ for all $S \in \mathcal{S}$ and so $\mathcal{E} \in \mathcal{Z}$.

$(ii) \Rightarrow (iii)$: This is immediate.

$(iii) \Rightarrow (i)$: Let $E \in \mathcal{E} \cap \mathcal{Z}$. This clearly implies that $\mathcal{S}E = E\mathcal{S}$ and condition (v) of Lemma 3.7 is satisfied. If $F \in \mathcal{E}$, then by Lemma 3.7(i),

$EF = F$ and $FE = E$. Since E commutes with all elements of $\mathcal{S}$, $F = EF = FE = E$ and so E is the unique minimal idempotent.

We have shown the equivalence of (i), (ii) and (iii). (iii) clearly implies (v) and (v) clearly implies (iv). Thus, we only have to show that (iv) implies (i), (ii) or (iii).

$(iv) \Rightarrow (ii)$: Let $E \in \mathcal{E}$ such that $\mathcal{S}E = E\mathcal{S}$. This means that the range of E is invariant under $\mathcal{S}$ and so $SE = ESE$ for all $S \in \mathcal{S}$. Suppose $Ex = 0$ for some vector x. Then, since $\mathcal{S}E = E\mathcal{S}$, there exists a $T \in \mathcal{S}$ such that $ESx = TEx$ and since $Ex = 0$, $0 = TEx = ESx$ and so the kernel of E is also invariant under $\mathcal{S}$, giving $ESE = ES$. Thus, $SE = ESE = ES$ for all $S \in \mathcal{S}$ and so $\mathcal{E} \cap \mathcal{Z} \neq \emptyset$.

$\square$

Corollary 3.11. *Let $\mathcal{S}$ be an indecomposable semigroup of nonnegative matrices such that $\overline{\mathbb{R}^+\mathcal{S}}$ has a unique minimal idempotent E. Then all the conclusions of Theorem 3.9 hold. Moreover,*

(i) no nonzero member of $\mathcal{S}$ has a zero column or row;

(ii) the adjoint semigroup $\mathcal{S}^*$ also has a common positive eigenvector y, unique up to scalar multiples, such that $S^*y = \rho(S)y$ for all $S \in \mathcal{S}$, and;

(iii) the rank of the minimal idempotent is 1 if and only if some member of $\mathcal{S}$ is positive.

Proof. Lemma 3.7 is trivially met, so the conditions for Theorem 3.9 are satisfied.

(i): First note that E has no zero rows or columns, since by Lemma 3.4(iii), there is a minimal idempotent with nonzero i-th row (column) for all i and E is the only minimal idempotent.

Suppose for a contradiction that $S \in \mathcal{S}$ has a zero row. After a permutation of the basis, we can assume it is the last row of S. We can also assume that S has minimal rank, since SE is minimal rank, and still has the same zero row. If $SE = 0$, then since $S\mathcal{S}E \neq \{0\}$ by Lemma 2.5, we can find an S_1 such that $SS_1E \neq 0$ which will be rank r and have the same zero row. Thus assume that S is nonzero and has minimal rank. We can also assume that S is not nilpotent. Suppose it were. Then, we can find A_1 and A_2 such that $\{0\} \neq \sigma(A_1SA_2) = \sigma(SA_2A_1)$ where we note that SA_2A_1 still has the same zero row. We can also assume that $\rho(S) = 1$ since $\mathcal{S} = \overline{\mathbb{R}^+\mathcal{S}}$. As in the proof of Lemma 3.4(i), we know that some subsequence of $\{S^k\}$ converges to a minimal idempotent, which must be the unique minimal idempotent E. However, S^k has the same zero row for all k and so E must also have it which is a contradiction.

(ii): Since $S^* \in \mathcal{S}^*$ is an idempotent if and only if S is, and the rank of S^* is equal to the rank of S, the indecomposable semigroup $\mathcal{S}^*$ has a unique minimal idempotent E^* and thus as above, the conditions for Theorem 3.9 are satisfied.

(iii): Suppose the minimal rank, denoted by r is 1. Note that E has no zero rows or columns, and so by Lemma 2.6, $E = E_1 \oplus \cdots \oplus E_r$, where each E is positive and r is the rank. Thus, $E = E_1$ is positive.

Suppose that $S \in \mathcal{S}$ is positive. Then clearly S has a positive column, and so we can apply Theorem 3.9(v) to conclude that some member $\mathcal{S}$, S_o is rank 1. Then, there exists an idempotent, E, with the same range as S_o. Thus E has rank 1, and since 1 is the minimal possible nonzero rank, by Lemma 3.4(i), E is the minimal idempotent.

$\square$

Corollary 3.12. *Let $\mathcal{S}$ be an indecomposable semigroup of nonnegative matrices. If $\mathcal{S}$ is normal (i.e. $A\mathcal{S} = \mathcal{S}A$ for every $A \in \mathcal{S}$), then all the conclusions of Corollary 3.11 hold.*

Proof. In order to apply Corollary 3.11, we must show that $\overline{\mathbb{R}^+\mathcal{S}}$ has a unique minimal idempotent. We know that $\overline{\mathbb{R}^+\mathcal{S}}$ has idempotents, since by Lemma 3.4(ii), for each $A \in \overline{\mathbb{R}^+\mathcal{S}}$, there exists a minimal idempotent, $E \in \overline{\mathbb{R}^+\mathcal{S}}$ such that $EA = A$. Since $\mathcal{S}$ is normal, we have that $E\mathcal{S} = \mathcal{S}E$ and so condition (iv) of Lemma 3.10 holds and so E is the unique minimal idempotent of $E \in \overline{\mathbb{R}^+\mathcal{S}}$.

□

We need the following definition for the next Lemma on groups of permutation matrices.

Definition 3.13. *A transitive group of permutations, $\mathcal{G}$, on $x_1, \ldots, x_n$ is one in which given any x_i and x_j, we can find $G \in \mathcal{G}$ such that $Gx_i = x_j$.*

Lemma 3.14. *Let $\mathcal{G}$ be an abelian, transitive group of $n \times n$ permutation matrices. For each $G \in \mathcal{G}$ there exists a positive integer m dividing n such that, after a permutation of the basis, G is the direct sum of m copies of the cyclic permutation*

$$G_o = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

of size n/m.

Proof. If G is cyclic, then clearly a permutation will put G in the desired form. Thus assume G is not cyclic. Then, after a permutation of the basis, we can assume that G is the direct sum of m cyclic matrices; i.e. $G = G_1 \oplus \cdots \oplus G_m$ where each G_i is of the form of G_o. We want to show now that each G_i is the same size. Fix G_j and G_k. We can permute the basis to assume that $j = 1$ and $k = 2$, in order to simplify the notation. We will proceed by contradiction, and assume that G_2 has larger size than G_1. We will let S_{ij} denote the (i, j) block according to the decomposition of G above, for each $S \in \mathcal{G}$.

Since $\mathcal{G}$ is transitive, we have that for some element $S \in \mathcal{G}$, $S_{12} \neq 0$. Since S_{12} is not square, it has a nontrivial kernel, $\mathcal{K}$. Let $\mathcal{B}_i$ be the set of basis vectors which spans G_i. Note that S_{12} is an operator which maps elements in the span of $\mathcal{B}_2$ to elements in the span of $\mathcal{B}_1$. Since $SG = GS$ for all S and G in $\mathcal{S}$, we have that $S_{12}G_2 = G_1S_{12}$ and so $S_{12}G_2u = G_1S_{12}u = 0$ for each $u \in \mathcal{K}$. This implies that $\mathcal{K}$ is a proper invariant subspace of G_2 and this contradicts the cyclicity of $\mathcal{G}_2$. Thus, every block must be the same size.

□

Corollary 3.15. *Let $\mathcal{S}$ be a commutative indecomposable semigroup of non-negative matrices. Then $\overline{\mathbb{R}^+\mathcal{S}}$ has a unique minimal idempotent P, whose rank we denote by r, and all the conclusions of Theorem 3.9 and Corollary 3.11 hold. Furthermore, the following assertions are true:*

(i) *$S^rP = \rho(S)^rP$ for every $S \in \mathcal{S}$, so that the spectrum of $S/\rho(S)$ contains r-th roots of unity whenever $S \neq 0$;*

(ii) *for each nonzero $S \in \mathcal{S}$, if k=k(S) denotes the least positive integer such that $S^kP = \rho(S)^kP$, then k divides r;*

(iii) *the multiplicity of each k-th root of unity in the spectrum of $S/\rho(S)$ is at least r/k;*

(iv) *$\sigma(S)$ is invariant under rotation about the origin by the angle $2\pi/k(S)$, for each $S \in \mathcal{S}$.*

Proof. By Lemma 3.10(v), we have immediately that $\overline{\mathbb{R}^+\mathcal{S}}$ has a unique minimal idempotent, and so the conclusions of Theorem 3.9 and Corollary 3.11 hold.

Since $\mathcal{S}$ is a commutative, indecomposable semigroup, so is $P\mathcal{S}P|_{P\mathcal{V}}$ with respect to the basis $\{x_1, \ldots, x_r\}$ by Lemma 3.2. By Theorem 3.8, $P\mathcal{S}P|_{P\mathcal{V}} =$

$\mathcal{S}|_{P\mathcal{V}}$. Recall the group $\mathcal{G} = \{\mathcal{S}|_{P\mathcal{V}} : S \in \mathcal{S}, \rho(S) = 1\}$ from the proof of Theorem 3.9(i). Note that $\mathcal{S}|_{P\mathcal{V}}$ has been represented with respect to the basis $\{x_1, \dots, x_r\}$. As is shown in the proof of the theorem, $\mathcal{G}$ is a transitive subgroup of permutations on $\{x_1, \dots, x_r\}$. Moreover, we now have that $\mathcal{G}$ is an abelian permutation group, and so by Lemma 3.14, the r-th power of every element in it must be the identity, which in $\mathcal{G}$ is $P|_{P\mathcal{V}} = I|_{P\mathcal{V}}$. Considering $G \in \mathcal{G}$ as $G = (S/\rho(S))|_{P\mathcal{V}}$ for some nonzero $S \in \mathcal{S}$ gives $S^r|_{P\mathcal{V}} = \rho(S)^r P|_{P\mathcal{V}}$ and so $S^r P = \rho(S)^r P$.

The eigenvalues of a cyclic permutation on k elements are the k-th roots of unity. By Lemma 3.14, every element $G \in \mathcal{G}$ is the direct sum of cyclic permutations, $G = G_1 \oplus \cdots \oplus G_m$, where each G_i is the same size and so the order of G_i divides the order of G which is r. Thus, each G_i has the k-th roots of unity as its eigenvalues, and since k divides r, they are also r-th roots of unity. The case for general $S \in \mathcal{S}$ follows by noting that if $G = (S/\rho(S))|_{P\mathcal{V}}$, then $\sigma(G) \subset \sigma(S/\rho(S))$. This proves ($i$), ($ii$) and ($iii$).

The form given for $\mathcal{G}$ in Lemma 3.14 implies that after a permutation, there is a block decomposition of each PSP, $(PSP)_{ij}$, such that

$$PSP = T_1 \oplus \cdots \oplus T_m,$$

where each T_i is a $k \times k$ block matrix. Each $T_i = P_i^{-1} S_i P_i$ where P_i is an idempotent with no zero rows or columns with $P = P_1 \oplus \cdots \oplus P_m$ and $S_1 \oplus \cdots \oplus S_m = S$ for some $S \in \mathcal{S}$. Recall from the proof of 3.9 that the nonzero blocks of $T_i = P_i^{-1} S_i P_i$ correspond to the nonzero blocks of S_i. Thus, each of the block matrices $\{S_i\}$ conforms to the cyclic pattern of G_o in Lemma 3.14, but in place of each 1 is a block matrix. In the proof of Theorem 3.9, we showed that the spectrum of a matrix of the form of S_i is invariant under rotation by $2\pi/k$. Since this holds for each i, $\sigma(S)$ is also invariant under rotation by $2\pi/k$.

□

Corollary 3.16. ***The Perron-Frobenius Theorem*** *Let A be an indecomposable nonnegative matrix with $\rho(A) = 1$. Denote by r the minimal rank of* *nonzero members of* $\overline{\mathbb{R}^+\mathcal{S}}$*, where* $\mathcal{S}$ *is the semigroup generated by* A*. Then the following hold:*

(i) The sequence $\{A^{rj}\}_{j=1}^{\infty}$ *converges to an idempotent* E *of rank* r*;*

(ii) if $r > 1$*, there is a permutation matrix* P *such that* $P^{-1}AP$ *has the block form*

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & A_r \\ A_1 & 0 & \cdots & 0 & 0 \\ 0 & A_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & A_{r-1} & 0 \end{pmatrix}$$

(with square diagonal blocks);

(iii) there is a positive column vector x*, unique up to scalar multiple, such that* $Ax = x$*;*

(iv) the set $\{\lambda \in \sigma(A) : |\lambda| = 1\}$ *consists precisely of all the* r*-th roots of unity; each member of the set is a simple eigenvalue;*

(v) $\sigma(A)$ *is invariant under the rotation about the origin by the angle* $2\pi/r$*;*

(vi) 1 is dominant in $\sigma(A)$ *(the only eigenvalue of modulus* $\rho(A)$*) if and only if some power of* A *has all its entries positive. This occurs precisely when* $\{A^j\}$ *is convergent.*

Proof. Since every member in $\overline{\mathbb{R}^+\mathcal{S}}$ is a scalar multiple of a power of A, $\overline{\mathbb{R}^+\mathcal{S}}$ is commutative and so the conclusions of Corollary 3.15 hold. Recall from the proof of part (iv) of that corollary, that for each $S \in \mathcal{S}$ there is a permutation of the basis such that $S = S_1 \oplus \cdots \oplus S_m$ where each S_i is block cyclic. A is indecomposable, so in its corresponding decomposition, $m = 1$, and so A has the form claimed in (ii). Also, this form for A implies that $k(A) = r$ (recall k from the previous Corollary) and so part (iv) of the corollary implies that $\sigma(A)$ is invariant under rotation about the origin by the angle $2\pi/r$, proving part (v) of this Theorem. Part (iii) of this Theorem is just (i) in Theorem 3.9.

By part (iii) of Corollary 3.15, we know that each r-th root of unity is in $\sigma(A)$. We know that $\mathcal{S}$ has a unique minimal idempotent, E, also by Corollary 3.15, which is the limit of some subsequence of $\{A^j\}$ as in the proof of Lemma 3.4. Therefore, the rank of E is precisely the number of modulus one eigenvalues of A. Thus $\sigma(A)$ can only contain r elements of modulus 1 including multiplicities and so those are precisely the r-th roots of unity, proving (iv).

By what we have proven above, each eigenvector of A, which corresponds to a modulus one eigenvalue of A, lies in the range of E. Thus, all the elements of $\sigma(A(I-E))$ lie inside the unit circle. Note that $(I-E)$ is an idempotent and elements of $\overline{\mathbb{R}^{+}\mathcal{S}}$ commute. Thus, $(A(I-E))^j = A^j(I-E)^j = A^j(I-E)$ and so

$$\lim_{j\to\infty} A^j(I-E) = 0.$$

We also have that $A^rE = E$ from part (i) of Corollary 3.15, which gives

$$\lim_{j\to\infty} A^{rj} = \lim_{j\to\infty} A^{rj}E + \lim_{j\to\infty} A^{rj}(I-E) = E,$$

proving (i).

The dominance of 1 in $\sigma(A)$ means, by definition, that 1 is the only modulus 1 eigenvalue of A. This is equivalent by (iv) to $r = 1$ which occurs if and only if $\lim_{j\to\infty} A^j = E$. Also, $r = 1$ is equivalent to the positivity of all the entries of E which occurs if and only if a sufficiently large power of A has all positive entries. This proves (vi).

$\square$

Chapter 4

Finiteness Conditions

In this section we impose the condition that the range of certain linear functionals acting on a semigroup is a finite set and conclude, under certain conditions, that the semigroup itself is finite. It has been proven [8] that if the set diagonal entries of all members of an indecomposable semigroup of nonnegative matrices consists exactly of zeros and ones, then after a simultaneous positive diagonal similarity, all entries are either 1 or 0. We would like to replace the condition on the diagonal entries with the more general requirement that the set of diagonal entries of members in the semigroup be finite. It turns out that the diagonal entries taking finitely many values does not ensure that the semigroup is itself finite. However, an affirmative result is achieved if we restrict ourselves to a semigroup of constant rank. We begin this section with several lemmas.

This lemma comes from Kaplansky [6].

Lemma 4.1. *Let $\{a_1, \cdots, a_n\}$ and $\{b_1, \cdots, b_n\}$ be n-tuples of elements of any field whose characteristic is either zero or larger than n.*

(i) If $\sum_{i=1}^{n} a_i^k = \sum_{i=1}^{n} b_i^k$ for $k = 1, \cdots, n$, then there is a permutation τ on n letters such that $b_i = a_{\tau(i)}$ for all i.

(ii) If $\sum_{i=1}^n a_i^k = 0$ *for* $k = 1, \cdots, n$*, then* $a_i = 0$ *for all* i*.*

Proof. (i): Let $T_k = T_k(x_1, \cdots, x_n) = \sum_{i=1}^n x_i^k$ be the symmetric polynomials in n variables. For each k, let S_k denote the elementary symmetric polynomial in $x_1, \cdots, x_n$ of degree k; i.e., S_k is the sum of all products of k distinct variables:

$$\begin{gathered} S_1 = x_1 + \cdots + x_n \\ S_2 = x_1x_2 + \cdots + x_{n-1}x_n \\ \vdots \\ S_n = x_1x_2 \cdots x_n. \end{gathered}$$

A calculation shows that

$$T_k - T_{k-1}S_1 + T_{k-2}S_2 - \cdots + (-1)^{k-1}T_1S_{k-1} + (-1)^k S_k = 0$$

for $k = 1, \cdots, n$. This formula enables us to determine uniquely each S_k inductively in terms of the T_j.

Now, the hypothesis that $T_k(a_1, \cdots, a_n) = T_k(b_1, \cdots, b_n)$ for $1 \leq k \leq n$ implies that $S_k(a_1, \cdots, a_n) = S_k(b_1, \cdots, b_n)$ for $k = 1, \cdots, n$. Consider the following polynomial:

$$\begin{gathered} P = (x - a_1) \cdots (x - a_n) = x^n - S_1(a_1, \ldots, a_n)x^{n-1} + \\ S_2(a_1, \ldots, a_n)x^{n-2} - \cdots + (-1)^{n-1}S_{n-1}(a_1, \ldots, a_n)x + \\ (-1)^n S_n(a_1, \ldots, a_n). \end{gathered}$$

This polynomial has zeros, $a_1, \ldots, a_n$. Since $S_j(a_1, \ldots, a_n) = S_j(b_1, \ldots, b_n)$ for all j, we have that

$$\begin{gathered} P = x^n - S_1(b_1, \ldots, b_n)x^{n-1} + S_2(b_1, \ldots, b_n)x^{n-2} - \ldots \\ +(-1)^{n-1}S_{n-1}(b_1, \ldots, b_n)x + (-1)^n S_n(b_1, \ldots, b_n) = (x - b_1) \cdots (x - b_n). \end{gathered}$$

Thus the two n-tuples are the same except for a permutation.

(ii) follows from (i) by taking $b_i = 0$ for all i.

$\square$

This next two results are my own and are useful when imposing the condition that the range of the trace operator is finite on a given semigroup.

Lemma 4.2. *Let $\{\lambda_i\}_{i=1}^n$ be a set of complex numbers such that the following hold:*

(i) $|\lambda_i| \leq 1 \ \forall i$*;*

(ii) Each modulus-one member of the set is a root of unity;

(iii) $\mathcal{J} = \{\Sigma_{i=1}^n \lambda_i^k : k \in \mathbb{N}\}$ *is a finite set.*

Then, elements in $\{\lambda_i\}_{i=1}^n$ which are nonzero consist precisely of roots of unity.

Proof. We will show that there are no nonzero elements λ such that $|\lambda| < 1$. First, remove any zero elements from the set and let

$$1 = |\lambda_1| = ... = |\lambda_{l-1}| > |\lambda_l| \geq ... \geq |\lambda_n| > 0$$

Since $\{\lambda_i\}_{i=1}^{l-1}$ are all roots of unity, ∃r such that $\lambda_i^r = 1 \ \forall i \in \{1, ..., l-1\}$ and so $\Sigma_{i=1}^{l-1} \lambda_i^k$ is r periodic in k. Now for $\mathcal{J}$ to be finite, $\Sigma_{i=l}^n \lambda_i^k$ must too be periodic in k. Given $\varepsilon > 0$, $\exists N$ such that $\forall k \geq N$

$$\varepsilon > \Sigma_{i=l}^n |\lambda_i|^k \geq |\Sigma_{i=l}^n \lambda_i^k| \geq 0.$$

Thus, $\{|\Sigma_{i=l}^n \lambda_i^k|\}_{k=1}^\infty$ either has a strictly decreasing subsequence, meaning it is not a periodic sequence, or it is constantly zero. If it has strictly decreasing subsequence, then $\mathcal{J}$ is infinite which cannot be, and so it must be constantly zero. By Lemma 4.1(ii), each λ_i in the sum is zero. Thus, there are no elements, $\lambda \neq 0$ such that $|\lambda| < 1$.

□

Lemma 4.3. *Let $\mathcal{S}$ be a semigroup such that $\mathcal{J} = \{\text{tr}\,(A) : A \in \mathcal{S}\}$ is a finite set. If $\rho(A) < 1$ for some $A \in \mathcal{S}$ then A is nilpotent.*

Proof. The spectrum of A, $\sigma(A) = \{\lambda_1, \cdots, \lambda_n\}$, satisfies all the conditions of Lemma 4.2, since the trace condition requires $\{\sum_{i=1}^{n} \lambda_i^k : k \in \mathbb{N}\}$ to be a finite set and each $\lambda_i < 1$. Thus, every $\lambda_i = 0$.

□

The next lemma comes from Longstaff-Radjavi [9].

Lemma 4.4. *If the sequence $\{tr(A^m)\}$ is bounded for a given operator A, then $\rho(A) \leq 1$.*

Proof. If $\rho(A) = 0$, there is nothing to prove. Suppose $a = \rho(A) \neq 0$; we must show that $a \leq 1$. Enumerate the eigenvalues $\{\lambda_i\}$ of A, counting multiplicities, so that

$$a = |\lambda_1| = \cdots = |\lambda_r| > |\lambda_{r+1} \geq \cdots \geq |\lambda_n|.$$

There is an increasing sequence $\{m_i\}$ of integers such that

$$\lim_{i\to\infty} (\frac{\lambda_j}{a})^{m_i} = 1$$

for all $j \leq r$, since each λ_j/a is modulus-one. Since $\lim_{i\to\infty}(\lambda_j/a)^{m_i} = 0$ for $j > r$, this yields

$$\lim_{i\to\infty} tr(\frac{A}{a})^{m_i} = \lim_{i\to\infty} \sum_{j=1}^{n} (\frac{\lambda_j}{a})^{m_i} = r.$$

Thus, $\lim_{i\to\infty} tr(A^{m_i}) = \infty$ if $a > 1$. This implies that $a \leq 1$.

□

The remainder of the results presented here are my own.

Theorem 4.5. *Let $\mathcal{S}$ be the semigroup generated by the indecomposable non-negative matrix A. If $\mathcal{J}$=$\{\text{tr } (A^k)\}_{k=1}^{\infty}$ is finite, then $\mathcal{S}$ is finite.*

Proof. Denote the minimal rank of nonzero members of $\overline{\mathbb{R}^+\mathcal{S}}$ by r and the eigenvalues of A by $\{\lambda_i\}_{i=1}^n$ where $|\lambda_1| \geq ... \geq |\lambda_n|$. Since $\mathcal{J}$ is finite, it is bounded. Thus, by Lemma 4.4, $\rho(A) \leq 1$ giving two cases: $\rho(A) < 1$ and $\rho(A) = 1$.

First assume $\rho(A) < 1$. Then, by Lemma 4.3, A is nilpotent and so $\mathcal{S}$ is finite.

Now assume $\rho(A) = 1$. Then

$$1 = |\lambda_1| = ... = |\lambda_{l-1}| > |\lambda_l| \geq ... \geq |\lambda_n|$$

Since A is nonnegative and indecomposable, the set $\{\lambda \in \sigma(A) : |\lambda| = 1\}$ consists precisely of the r^{th} roots of unity by the Perron-Frobenius Theorem, and by Lemma 4.2, this set possibly together with zero is in fact $\sigma(A)$. Now, $\overline{\mathbb{R}^+\mathcal{S}}$ has a unique minimal idempotent, E, which is in fact in $\overline{\mathcal{S}}$ since $\rho(S) = 1$ for all $S \in \mathcal{S}$. $A^r E = E$ by Corollary 3.15 and the minimal rank of $\mathcal{S}$ is the rank of A, since all of the nonzero eigenvalues of A are modulus 1. Thus, $EA^r = A^r$ by Lemma 3.4. This implies $A^r = E$ since A and E commute and so $\{A^k\}_{k=1}^{\infty}$ is r-periodic. Therefore $\mathcal{S}$ is finite.

$\square$

Lemma 4.6. *Let $\mathcal{S}$ be a semigroup of nonnegative matrices such that $\mathcal{J} = \{\text{tr } (S) : S \in \mathcal{S}\}$ is a finite set. Then, the nonzero eigenvalues of each $S \in \overline{\mathcal{S}}$ are roots of unity.*

Proof. Since the trace is a continuous function, we can assume without loss of generality that $\mathcal{S} = \overline{\mathcal{S}}$. Let $S \in \mathcal{S}$. Since $\mathcal{J}$ is finite, it is also bounded. Thus, by Lemma 4.4, $\rho(S) \leq 1$. By Lemma 4.3, if $\rho(S) < 1$ then S is nilpotent. Assume $\rho(S) = 1$. Then, by a permutation of the basis, S can be put in the block form (with possibly only one block if S is indecomposable):

$$\begin{pmatrix} S_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & S_m \end{pmatrix}$$

where each S_i is an indecomposable $k_i \times k_i$ matrix of rank r_i. By the Perron-Frobenius Theorem, the modulus-one eigenvalues of each S_i are the r-th roots of unity and so the eigenvalues of S meet the conditions in Lemma 4.2, giving that the nonzero elements in $\sigma(S)$ consists entirely of roots of unity.

$\square$

Corollary 4.7. *Let $\mathcal{S}$ be a commutative semigroup generated by finitely many nonnegative indecomposable matrices. Suppose $\mathcal{J} = \{\text{tr}\ (S) : S \in \mathcal{S}\}$ is a finite set. Then $\mathcal{S}$ is finite.*

Proof. Let $\{A_i\}_{i=1}^m$ be the generators of $\mathcal{S}$. Let $S_o \in \mathcal{S}$. Then $S_o = \prod_{i=1}^m A_i^{k_i}$. Let $\mathcal{S}_i$ be the semigroup generated by A_i. By Theorem 4.5, this semigroup is finite and $A_i^{r_i} = E_i$ where E_i is the unique idempotent of $\mathcal{S}_i$ satisfying $E_i A_i = A_i$, and r_i is the rank of A_i. Thus, elements in $\mathcal{S}$ are of the form $\prod_{i=1}^m A_i^{k_i}$ where each $k_i \in \{0, 1, ..., r_i\}$ and so there are at most $\prod_{i=1}^m (r_i + 1)$ elements in $\mathcal{S}$.

$\square$

Theorem 4.8. *Let $\mathcal{S}$ be an indecomposable semigroup of nonnegative invertible matrices such that $\mathcal{J} = \{\text{tr}\ (S) : S \in \mathcal{S}\}$ is a finite set. Then $\mathcal{S}$ is finite.*

Proof. Since the trace is a continuous function, $\overline{\mathcal{J}} = \{\text{tr}\ (S) : S \in \overline{\mathcal{S}}\}$ is also a finite set. Since each member in $\mathcal{S}$ is invertible, we can apply Lemma 4.6 and deduce that the eigenvalues of each $S \in \overline{\mathcal{S}}$ are roots of unity. Also, every $S \in \overline{\mathcal{S}}$ is invertible, and so the only minimal idempotent in $\overline{\mathcal{S}}$ is the identity. Note that since every member in $\mathcal{S}$ has $\rho(S) = 1$ and so $\{S \in \mathcal{S} : \rho(S) = 1\} = \mathcal{S}$. Furthermore, observe that $\{S \in \overline{\mathbb{R}^+\mathcal{S}} : \rho(S) = 1\} = \overline{\mathcal{S}}$. Thus, we can apply part (iv) of Lemma 3.4, and conclude that $I\overline{\mathcal{S}}I = \overline{\mathcal{S}}$ is simultaneously similar via a diagonal similarity (when restricted to the range of I) to a transitive group of permutation matrices. Then, since $\mathcal{S} \subset \overline{\mathcal{S}}$, $\mathcal{S}$ is such a finite too.

$\square$

Corollary 4.9. *Let $\mathcal{S}$ be an indecomposable semigroup of nonnegative matrices of constant rank, r or 0 such that $\mathcal{S} = \overline{\mathcal{S}}$. Suppose that $\overline{\mathbb{R}^+\mathcal{S}}$ has a unique minimal idempotent, E, and that $\mathcal{J} = \{\text{tr } (S) : S \in \mathcal{S}\}$ is a finite set. Then $\mathcal{S}$ is finite.*

Proof. By Lemma 4.6, the nonzero eigenvalues of each $S \in \mathcal{S}$ are roots of unity. By Lemma 3.8, the only nilpotent element in $\mathcal{S}$ is zero, so $\rho(S) = 1$ for all nonzero S. Observe that $\{S \in \overline{\mathbb{R}^+\mathcal{S}} : \rho(S) = 1 \text{ or } S = 0\} = \overline{\mathcal{S}}$. Since $\overline{\mathbb{R}^+\mathcal{S}}$ has a unique minimal idempotent (and $\rho(E) = 1$, meaning $E \in \overline{\mathcal{S}}$), $ES = S$ for each $S \in \mathcal{S}$ by Lemma 3.4. Furthermore, E commutes with all members in $\mathcal{S}$ by Lemma 3.10, so $ES = S = SE$ and thus $E\mathcal{S}E = \mathcal{S}$. Therefore, by Lemma 3.4, $\mathcal{S} \setminus \{0\}$ is simultaneously similar to a transitive group of permutation matrices, and so $\mathcal{S}$ is finite.

□

Corollary 4.10. *Let $\mathcal{S}$ be a finitely generated indecomposable semigroup of nonnegative matrices of constant rank, r or 0. Suppose that the idempotents of $\mathcal{S}$ have a common range $\mathcal{R}$ and $\mathcal{J} = \{\text{tr } (S) : S \in \mathcal{S}\}$ is a finite set. Then $\mathcal{S}$ is finite.*

Proof. Since all of the idempotents in $\overline{\mathbb{R}^+\mathcal{S}}$ have a common range $\mathcal{R}$ and all $S \in \mathcal{S}$ have the same rank, the range of every $S \in \mathcal{S}$ is $\mathcal{R}$ by Lemma 3.7. Thus, $ES = S$ for every idempotent $E \in \overline{\mathbb{R}^+\mathcal{S}}$. Since $\mathcal{J}$ is finite, Lemma 4.6 implies that the nonzero eigenvalues of each $S \in \overline{\mathcal{S}}$ are roots of unity. Combined with the fact that the only nilpotent element in $\mathcal{S}$ is zero by Lemma 3.8, gives that $\rho(S) = 1$ for all nonzero S. As a result, we have that $\{S \in \overline{\mathbb{R}^+\mathcal{S}} : \rho(S) = 1 \text{ or } S = 0\} = \overline{\mathcal{S}}$. Thus, we can apply Lemma 3.4 which gives that $\mathcal{S}E \setminus \{0\}$ is a permutation group on $\mathcal{R}$, with identity E. Let $\{S_1, ..., S_n\}$ be the generators of $\mathcal{S}$. Then, after a simultaneous similarity (which does not necessarily preserve nonnegativity), each nonzero S_i is of the form:

$$S_i = \begin{pmatrix} P_i & A_i \\ 0 & 0 \end{pmatrix}$$

where P_i is a permutation matrix on $\mathcal{R}$. Consider the product of k, not necessarily distinct generators.

$$\prod_{j=1}^{k} S_{i_j} = \begin{pmatrix} \prod_{j=1}^{k} P_{i_j} & \prod_{j=1}^{k} P_{i_j} A_k \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} P_o & P_o A_k \\ 0 & 0 \end{pmatrix}$$

Thus, since the permutation group is finite and there are finitely many A_k, this semigroup is finite.

□

The next result is a corollary of Lemma 2.8.

Corollary 4.11. *Let $\mathcal{G}$ be a group of invertible nonnegative matrices. Define the linear functional $\phi : \mathcal{M}_n(\mathbb{C}) \to \mathbb{C}$ as $\phi(X) = \sum_{i,j} X_{ij}$. If ϕ is constant on the group, then $\phi(A) = n\ \forall A \in \mathcal{G}$ and $\mathcal{G}$ is a group of permutations on the standard basis vectors.*

Proof. Since $I \in \mathcal{G}$ and $\phi(I) = n$, the constant must be n. From Lemma 2.8, we know that each row and column of $A \in \mathcal{G}$ has exactly one nonzero element. Fix A and denote its nonzero entries by $\{a_i\}_{i=1}^{n}$. Since $\mathcal{G}$ is a group, $A^{-1} \in \mathcal{G}$. The nonzero entries of A^{-1} are $\{a_i^{-1}\}_{i=1}^{n}$. The constancy of the functional ϕ yields the following two equations:

$$\sum_{i=1}^{n} a_i = n \qquad \sum_{i=1}^{n} a_i^{-1} = n$$

We will prove that each entry is 1 by contradiction. Suppose that there are entries not equal to 1. Without loss of generality, we can assume that every entry is not equal to 1, since by removing the m entries that are and then equating our new sums to $n - m$, we obtain an equivalent problem. Assume only the first l are less than 1. Let $a_i = 1 - \delta_i$ for $i \leq l$ and $a_i = 1 + \epsilon_i$ for $i \geq l + 1$ where $1 > \delta_i > 0$ and $\epsilon_i > 0$. The sums now become

$$\sum_{i=1}^{l} 1 - \delta_i + \sum_{i=l+1}^{n} 1 + \epsilon_i = n \quad \textit{and} \quad \sum_{i=1}^{l} (1 - \delta_i)^{-1} + \sum_{i=l+1}^{n} (1 + \epsilon_i)^{-1} = n. \quad (*).$$

The first equation implies that

$$\sum_{i=1}^{l} \delta_i = \sum_{i=l+1}^{n} \epsilon_i.$$

Since $1 > 1 - \delta_i^2 = (1 - \delta_i)(1 + \delta_i)$, we have that $(1 - \delta_i)^{-1} > 1 + \delta_i$ (since $0 < \delta_i < 1$) and $(1 + \epsilon_i)^{-1} > 1 - \epsilon_i$ $\forall i$. Combining these with the second equation in (*) gives

$$n > \sum_{i=1}^{l} 1 + \delta_i + \sum_{i=l+1}^{n} 1 - \epsilon_i = \sum_{i=1}^{n} 1 + \sum_{i=1}^{l} \delta_i - \sum_{i=l+1}^{n} \epsilon_i = n$$

and this is a contradiction.

□

Theorem 4.12. *If $\mathcal{S}$ is a semigroup of nonnegative matrices such that the diagonal entries take only finitely many values, then the set of idempotents in $\mathcal{S}$ is finite.*

Proof. We will prove this Theorem in two parts. First we will show that the set of idempotents with no zero rows or columns is a finite set. Then, we will use an induction argument to show that the set of idempotents with zero rows or columns is also a finite set.

Let $\Lambda = \{1, \ldots, k\}$ be an index for the (finite) collection of all possible ordered sets of diagonal entries of the $S \in \mathcal{S}$. For each $i \in \Lambda$, let $\mathcal{F}_i \subset \mathcal{S}$ be the collection of idempotents with no zero rows of columns that have diagonal entries which correspond to the set indexed by that i; i.e. $A_{ii} = B_{ii}$ for every idempotent A and B in $\mathcal{F}_i$. To prove that there are finitely many idempotents, we must show that each collection, $\mathcal{F}_i$, is finite. We can assume that there is some $\mathcal{F}_i$ that is not empty since otherwise there are no (i.e. finitely many) idempotents with no zero rows or columns. Fix $\mathcal{F}_i$. By Lemma 2.6, we know that given $F \in \mathcal{F}_i$, there exists a permutation, P, such that $P^{-1}FP$ is block diagonal, where each diagonal block is a positive rank-one idempotent. There are finitely many permutations, and finitely many block decompositions and thus each $F \in \mathcal{F}$ must conform to one of each of these. Suppose that F_1 and F_2 have the same block decomposition by the same permutation. Then, block $(F_1)_i$ and block $(F_2)_i$ are the same size and have equal diagonal entries. To prove the Theorem, it suffices to show that these blocks, $(F_1)_i$ and $(F_2)_i$ are equal. Relabel these blocks as E and E'. Since they are both rank-one matrices with equal diagonals, there exists a positive diagonal matrix, $D = \text{diag}(d_1, \cdots, d_n)$ such that $E' = DED^{-1}$. If the size is 1, we are done, so assume the size is larger. The rank of $DED^{-1}E \geq 1$ since E is a positive matrix and D is diagonal meaning that $DED^{-1}E$ is nonzero. Also, $DED^{-1}E \leq 1$ since E has rank 1 and in general rank$(AB) \leq$min{rank(A),rank(B)}. Observe that $\{tr(DED^{-1}E), 0\} = \sigma(DED^{-1}E)$. Let $E = (e_{ij})$.

$$tr(DED^{-1}E) - 1 = tr(DED^{-1}E) - tr(E^2) = \sum_{i,j} d_i d_j^{-1} e_{ij} e_{ji} - \sum_{i,j} e_{ij} e_{ji}$$

$$= \sum_{i \neq j} (d_i d_j^{-1} - 1) e_{ij} e_{ji} = \sum_{i<j} (d_i d_j^{-1} + d_i^{-1} d_j - 2) e_{ij} e_{ji}.$$

To finish the proof that there are finitely many idempotents with no zero rows or columns, we shall show by a contradiction that $d_i = 1$ for all i, meaning our diagonal similarity is the identity and so $E = E'$. This will mean that for A and B in $\mathcal{F}_i$ and P a permutation, if PAP^{-1} and PBP^{-1} are both block diagonal with corresponding equally sized diagonal blocks, that $A = B$. Now, each $E \in \mathcal{F}_i$ must uniquely correspond to one of finitely many permutations, and one of finitely many block decompositions and so there can only be finitely many members of each $\mathcal{F}_i$ and thus finitely many idempotents with no zero rows or columns. We will now proceed with the contradiction argument.

Observe that

$$0 \leq (a-1)^2 = 1 - 2a + a^2 \Rightarrow 2 \leq a + a^{-1} \quad (*)$$

for any positive a, where equality holds only if and only if $a = 1$. Assume $d_i \neq 1$ for some fixed i. Then, $d_i d_j^{-1} + d_i^{-1} d_j - 2 > 0$ for some j where the inequality is achieved by inserting $d_i d_j^{-1}$ for a in $(*)$. Thus,

$$\sum_{i<j} (d_i d_j^{-1} + d_i^{-1} d_j - 2) e_{ij} e_{ji} > 0$$

by the positivity of the e_{ij}. But then, $1 < tr(DED^{-1}E) \in \sigma(DED^{-1}E)$ and so $\rho(DED^{-1}E) > 1$ and this is a contradiction since the hypothesis implies that $\{tr(S) : S \in \mathcal{S}\}$ is a finite set, so by Lemma 4.6, $\rho(S) = 1 \; \forall S \in \mathcal{S}$. Thus we have shown that the set of idempotents with no zero rows or columns is a finite set.

We now will show by induction that the set of idempotents with zero rows or columns is also a finite set. If the matrices are 1×1, then the result holds trivially. Assume that the result holds for all $k \times k$ semigroups of matrices with $k < n$. Suppose that there exists an idempotent E_o that has a zero row or column. Then, by Lemma 2.6(ii), we have that after a permutation of the basis E_o has the block-triangular form

$$E_o = \begin{pmatrix} 0 & X_o F_o & X_o F_o Y_o \\ 0 & F_o & F_o Y_o \\ 0 & 0 & 0 \end{pmatrix}$$

where $F_o = E_1 \oplus \cdots \oplus E_r$ as in Lemma 2.6(i) and X_o and Y_o are nonnegative matrices where we will assume that our permutation has been chosen such that X_o has no zero rows and that Y_o has no zero columns. Suppose that exactly the first $k < n$ rows of E_o are nonzero. Let $\mathcal{J}_k = \{S \in \mathcal{S} : \text{rows } k+1$ through n of S are zero$\}$. Since $E_o \in \mathcal{J}_k$, we know that $\mathcal{J}_k$ is not empty. It is easily seen that $\mathcal{J}_k$ is a semigroup. Also note that $\mathcal{J}_k \mathcal{S} = \mathcal{J}_k$. To show this, take $J \in \mathcal{J}_k$ and $S \in \mathcal{S}$. A simple calculation shows that JS will have the same zero rows as J. Thus $JS \in \mathcal{J}_k$ for all $J \in \mathcal{J}_k$ and $S \in \mathcal{S}$.

Now we will show that the upper left $k \times k$ block of $\mathcal{J}_k$ imbedded canonically in the $k \times k$ matrices, which we will denote $\mathcal{K}$, is a semigroup. Let $\tilde{A}$

and $\tilde{B}$ be elements in $\mathcal{K}$. Then, there exists A and B in $\mathcal{J}_k$ whose images under the imbedding are $\tilde{A}$ and $\tilde{B}$ respectively. A simple calculation shows that the product of $\tilde{A}$ and $\tilde{B}$ is the image of AB under the imbedding, and so $\mathcal{K}$ is indeed a semigroup. Thus, by the inductive hypothesis, $\mathcal{K}$ is finite. Consider now idempotents of the form of E_o (from above) imbedded in $\mathcal{K}$ (where by form, we mean those that achieve the same block structure under the same basis representation). After the imbedding, they are of the form

$$\begin{pmatrix} 0 & X_oF_o \\ 0 & F_o \end{pmatrix}.$$

Since $\mathcal{K}$ is finite, that implies there are finitely many idempotents of this form, meaning X_oF_o and F_o come from a finite set. A permutation on the transpose of $\mathcal{S}$ allows us to deduce in a similar way that F_oY_o comes from a finite set, which means that so does $(X_oF_o)(F_oY_o) = X_oF_oY_o$. This means that there are finitely many idempotents with $n-k$ zero rows, and this is true for each $k > 0$. Thus, there are finitely many idempotents with at least one zero row. $\square$

Recall that a *band* is a semigroup that consists only of idempotents. In Chapter 2, we showed that the condition of indecomposability in a band is quite strong as it implies that the minimal nonzero rank of members must be 1. This next result, where indecomposability is not assumed, is a simple corollary of Theorem 4.12.

Corollary 4.13. *If $\mathcal{S}$ is a band of nonnegative matrices such that the diagonal entries take only finitely many values, then $\mathcal{S}$ is finite.*

Proof. This result follows immediately from Theorem 4.12. $\square$

This following theorem is the main theorem of the section.

Theorem 4.14. *If $\mathcal{S} = \overline{\mathcal{S}}$ is an indecomposable semigroup of nonnegative matrices of rank r or 0 such that the diagonal entries take only finitely many values, then $\mathcal{S}$ is finite.*

Proof. The result trivially holds for $n = 1$ and so we will proceed by induction. Suppose the Theorem holds for all $k \times k$ semigroups satisfying the hypothesis, where $k < n$. By Theorem 4.12, we know that there are finitely many idempotents in $\mathcal{S}$.

We will first show that there are finitely many nilpotents. Suppose there exists a nonzero nilpotent N. Then, $N^2 = 0$ since otherwise N^2 would be an element with rank strictly between 0 and r. Then, after a permutation, we can assume that

$$N = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}.$$

Again we have that $\{S \in \overline{\mathbb{R}^+\mathcal{S}} : \rho(S) = 1\} = \mathcal{S}$. Thus, by Lemma 3.4, we know there exists idempotents

$$\tilde{E} = \begin{pmatrix} E & EX \\ 0 & 0 \end{pmatrix}$$

and

$$\tilde{F} = \begin{pmatrix} 0 & XF \\ 0 & F \end{pmatrix}$$

such that $\tilde{E}N = N = N\tilde{F}$. This gives us that $EA = A = AF$. Pick a $Z \in \mathcal{S}$,

$$Z = \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix}$$

such that $AZ_3 \neq 0$. We know such a Z exists since $\mathcal{S}$ is indecomposable and nonnegative. Then

$$\tilde{F}Z\tilde{E} = \begin{pmatrix} YFZ_3E & YFZ_3EX \\ FZ_3 & FZ_3EX \end{pmatrix}$$

and so

$$\tilde{E}N\tilde{F} \cdot \tilde{F}Z\tilde{E} = \begin{pmatrix} (EAF)(FZ_3E) & (EAF)(FZ_3E)X \\ 0 & 0 \end{pmatrix}.$$

Note that the columns of $(EAF)(FZ_3E)X$ are just positive linear combinations of the columns of $(EAF)(FZ_3E)$ and so the rank of $(EAF)(FZ_3E)$ must be r. By the inductive hypothesis, there are finitely many possible $(EAF)(FZ_3E)$, as the nilpotents whose upper triangular block forms correspond to a single permutation are varied. Denote the set of all possible $(EAF)(FZ_3E)$ by $\{B_i\}_{i=1}^{m}$. Now, since $EAF = A$ is rank r, FZ_3E must too be rank r. Thus, FZ_3E is an operator from the range of E to the range of F. Both of these spaces are r-dimensional, and FZ_3E is rank r so FZ_3E is one to one and onto when restricted to the range of E. Thus, there is an inverse operator R from the range of F onto the range of E and so $(FZ_3E)R = F$. Then, $A = EAF = EAF \cdot FZ_3E \cdot R = B_i \cdot R$ for some i. Thus, there are finitely many A and so there are finitely many nilpotents corresponding to each permutation.

We now have that the number of idempotents and nilpotents in $\mathcal{S}$ is finite. Suppose S is a nonnilpotent member of $\mathcal{S}$. By Lemma 3.4(ii), there exists an idempotent E such that $ES = S$. This E is constructed as a sequence of powers of S and so E actually commutes with S and any other non-nilpotent $S \in \mathcal{S}$ that generates it, giving, $ESE = S$.

There are finitely many idempotents, $\{E_1, \cdots, E_m\}$ and so for each E_i define $\mathcal{S}_i = \{S \in \mathcal{S} : E_iSE_i = S\}$. By definition, $\mathcal{S}_i \subset E_i\mathcal{S}E_i$. Note that $\bigcup_{i=1}^{m} \mathcal{S}_i = \mathcal{S}$ except possibly for a finite set of nilpotents, since for each non-nilpotent $S \in \mathcal{S}$ there is an E_i such that $E_iSE_i = S$. Note that by Lemma 4.6, $\rho(S) = 1$ for each non-nilpotent $S \in \mathcal{S}$. Now, by Lemma 3.4(iv), for each minimal idempotent E_i, the set $\{E_iSE_i : S \in \mathcal{S}, \rho(E_iSE_i) = 1\}$ is a subsemigroup whose restriction to the range of E_i is simultaneously similar to a (finite) group of permutation matrices and so each $\mathcal{S}_i$ is finite. Let $\mathcal{N}$ be the finite collection of nilpotents in $\mathcal{S}$. Then, $\{\bigcup_{i=1}^{m} \mathcal{S}_i\} \bigcup \mathcal{N} = \mathcal{S}$ is finite.

$\square$

That the semigroup is indecomposable is clearly necessary, which we can observe by considering the infinite semigroup of all upper-triangular matrices which have ones on the diagonal. Adding in the constancy of the rank in the above Theorem is necessary as is demonstrated by the following example.

Example 4.15. *Let $\mathcal{S}$ be the semigroup of nonnegative 4×4 matrices consisting of*

$$\{\begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}, \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}, \begin{pmatrix} 0 & S \\ E & 0 \end{pmatrix}\}$$

with

$$E = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}, S = \begin{pmatrix} p & q \\ q & p \end{pmatrix}$$

and where $p + q = 1$. Note first that $SE = ES = E$ for all S of this form and so every element in $\mathcal{S}$ is of the form of one of the three elements listed above. We can see that $\mathcal{S}$ is indecomposable by Lemma 2.5, since no entry is zero for all $A \in \mathcal{S}$ and the diagonal takes only the values 0 and $1/2$ but there are infinitely many S satisfying $p + q = 1$ and so this semigroup is infinite. The element with the S block has rank 3 except when $p = q = 1/2$ and the other elements have rank 2.

The conditions of nonnegativity and the constancy of the rank, however, are not required if we consider an indecomposable semigroup of 2×2 or 3×3 matrices.

Theorem 4.16. *Let $\mathcal{S}$ be an indecomposable semigroup of 2×2 or 3×3 matrices such that the diagonal entries take only finitely many values. Then $\mathcal{S}$ is finite.*

Proof. We first prove the case for 2×2 matrices. Since $\mathcal{S}$ is indecomposable, there exists an $A \in \mathcal{S}$ such that $A_{12} \neq 0$. Define $\{\mathcal{S}_{ij}\} = \{S_{ij} : S \in \mathcal{S}\}$. Then, $\{(A\mathcal{S})_{11}\} = \{A_{11}S_{11} - A_{12}S_{21} : S \in \mathcal{S}\}$ is a finite set. Thus, since $\{A_{11}S_{11} : S \in \mathcal{S}\}$ is a finite set, $\{A_{12}S_{21} : S \in \mathcal{S}\}$ is a finite set too. Now, since A_{12} is fixed and nonzero, $\{\mathcal{S}_{21}\}$ must be a finite set. Similarly, $\{\mathcal{S}_{12}\}$ can be shown to be a finite set completing the proof. Now we will prove the case for 3×3 matrices.

It suffices to show that the $(2,1)$ entry takes finitely many values in $\mathcal{S}$ since any off diagonal entry can be permuted to appear there. Let $S \in \mathcal{S}$. Then $(S^2)_{11}, (S^2)_{22}$ and $(S^2)_{33}$ each make up a finite as S is varied over $\mathcal{S}$ by hypothesis which implies that

$$\{S_{12}S_{21} + S_{13}S_{31} : S \in \mathcal{S}\},$$

$$\{S_{21}S_{12} + S_{23}S_{32} : S \in \mathcal{S}\}$$

and

$$\{S_{31}S_{13} + S_{32}S_{23} : S \in \mathcal{S}\}$$

are each finite sets. Combining these gives us that $\{S_{21}S_{12} : S \in \mathcal{S}\}$ is a finite set which we denote $\{y_k\}_{k=1}^{p}$.

Since $\mathcal{S}$ is indecomposable, $\exists A \in \mathcal{S}$ such that $A_{12} \neq 0$ by Lemma 2.5. Each of

$$(A\mathcal{S})_{11} = \{A_{11}S_{11} + A_{12}S_{21} + A_{13}S_{31} : S \in \mathcal{S}\}$$

$$(A\mathcal{S})_{22} = \{A_{21}S_{12} + A_{22}S_{22} + A_{23}S_{32} : S \in \mathcal{S}\}$$

$$(\mathcal{S}A)_{33} = \{S_{31}A_{13} + S_{32}A_{23} + S_{33}A_{33} : S \in \mathcal{S}\}$$

is finite. Then, since $\{S_{ii}A_{ii} : S \in \mathcal{S}\}$ is a finite set for $i = 1, 2, 3$,

$$\{S_{21}A_{12} + S_{31}A_{13} : S \in \mathcal{S}\}$$

$$\{S_{12}A_{21} + S_{32}A_{23} : S \in \mathcal{S}\}$$

$$\{S_{31}A_{13} + S_{32}A_{23} : S \in \mathcal{S}\}$$

are finite sets too. Combining these, we get that

$$\{(S_{21}A_{12} + S_{31}A_{13}) - (S_{31}A_{13} + S_{32}A_{23}) + (S_{12}A_{21} + S_{32}A_{23}) : S \in \mathcal{S}\}$$

$$= \{S_{21}A_{12} + S_{12}A_{21} : S \in \mathcal{S}\}$$

is a finite set which we will denote $\{x_k\}_{k=1}^{m}$. Multiplying on the left by S_{21} on both sides gives

$$\{S_{21}^2 A_{12} + S_{21}S_{12}A_{21} - S_{21}x_k = 0 : S \in \mathcal{S}, k = 1, \cdots, m\}$$

and so we have $A_{12}(S_{21})^2 - x_k S_{21} + y_j A_{21} = 0$, $k = 1, \cdots, m$, $j = 1, \cdots, p$ and $S \in \mathcal{S}$. Thus, each S_{21} is a solution to one of finitely many quadratic equations, and thus there can only be finitely many values for S_{21}.

$\square$

References

[1] A. Berman and R. Plesmmons. *Nonnegative Matrices in Mathematical Sciences*. Academic Press, New York, 1979.

[2] W. Burnside. On the condition of reducibility of any group of linear substitutions. *Proceedings of the London Mathematical Society*, s2-3(1):430–434, 1905.

[3] M.D. Choi, E.A. Nordgren, H. Radjavi, P. Rosenthal, and Y. Zhong. Triangularizing semigroups of quasinilpotent operators with nonnegative entries. *Indiana University Mathematics Journal*, 42:15–25, 1993.

[4] G. Frobenius. Über matrizen aus nicht negativen elementen. *Sitzungsberichte der Preussischen Akademie Der Wissenschaften zu Berlin*, pages 456–477, 1912.

[5] G. Frobenius and I. Schur. Über die Äquivalenz der gruppen linearer substitutionen. *Sitzungstber. Preuss. Akad. Wiss.*, pages 209–217, 1906.

[6] I. Kaplansky. *The* E*ngel-*K*olchin Theorem Revisited*. Contributions to Algebra. Academic Press, New York, 1977.

[7] J. Levitzki. Über nilpotente unterrringe. *Mathematische Annalen*, 105:620–627, 1931.

[8] L. Livshits, G. MacDonald, and H. Radjavi. Semigroups of zero-one matrices, reprint.

[9] W.E. Longstaff and H. Radjavi. On permutability and submultiplicativity of spectral radius. *Canadian Journal of Mathematics*, 47:1007–1022, 1995.

[10] A. Marwaha. Decomposability and structure of nonnegative bands. *Linear Algebra Appl.*, 291:63–82, 1999.

[11] O. Perron. Zur theorie der matrizen. *Math. Ann.*, 64:248–263, 1907.

[12] H. Radjavi. On reducibility of semigroups of compact operators. *Indiana University Mathematics Journal*, 39:499–515, 1990.

[13] H. Radjavi. The Perron-Frobenius Theorem revisited. *Journal of Positivity*, 3(4):317–332, 1999.

[14] H. Radjavi and P. Rosenthal. From local to global triangularization. *Journal of Functional Analysis*, 147(2):443–456, 1997.

[15] H. Radjavi and P. Rosenthal. *Simultaneous Triangularization*. Springer, New York, 2000.