The Differential Geometry of Instantons

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The instanton solutions to the Yang-Mills equations have a vast range of practical applications in field theories including gravitation and electro-magnetism. Solutions to Maxwell's equations, for example, are abelian gauge instantons on Minkowski space. Since these discoveries, a generalised theory of instantons has been emerging for manifolds with special holonomy. Beginning with connections and curvature on complex vector bundles, this thesis provides some of the essential background for studying moduli spaces of instantons.

Manifolds with exceptional holonomy are special types of seven and eight dimensional manifolds whose holonomy group is contained in G_2 and $Spin(7)$, respectively. Focusing on the G_2 case, instantons on G_2 manifolds are defined to be solutions to an analogue of the four dimensional anti-self-dual equations. These connections are known as Donaldson-Thomas connections and a couple of examples are noted.

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Dedication

This is dedicated to my friends and family who were extremely supportive during throughout the ups and the downs. Also, to my cats Pepper and Sunny.

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Chapter 1

Introduction

This thesis may be considered as an elementary introduction to the geometry of vector bundles and special connections that live on them. The background material on bundles, connections, and curvature has been developed and explored by many geometers, and some references are Griffiths, Harris, Huybrechts, and Kobayashi in [8, 11, 15]. One of the reasons for developing such objects is to describe special connections that appear in theoretical physics. The instantons are a special type of connection having minimal Yang-Mills energy and have been studied in great depth on four manifolds by Atiyah, Donaldson and Kronheimer in [1, 5]. More recently, instantons have been generalized to special classes of 7 manifolds known as G_2 manifolds. Much work in the area of G_2 manifolds and connections on them is attributed to Donaldson, Karigiannis, Joyce, Leung, and Salamon in [5, 17, 14, 19, 24].

Chapter 2 begins with a motivation and definition of vector bundles along with a recipe for construction of those which are most commonly prescribed in practise. Examples of vector bundles include product, tangent, dual, tensor, exterior, quotient, normal and endomorphism bundles. A distinction is made between trivial and non-trivial vector bundles which is illustrated through classical examples of real line bundles on S^1. Sections of vector bundles are defined and described in order to generalize vector valued function theory over manifolds. Equivalence between frames, which are linearly independent sets of sections, and trivializations is demonstrated resulting in a theorem about trivializability of vector bundles and existence of nowhere vanishing frame fields. The rest of the chapter briefly introduces principal bundles and the associated vector bundles attached to them.

Chapter 3 is intended to serve as an intuitive approach to the ideas and mechanics involved in studying connections as differential operators on sections of vector bundles. The local representation of connections is examined and transformation laws are provided. Using these ideas, one now has means of describing parallel sections with respect to a specified connection. With this, the holonomy group of any connection on a vector bundle is defined by parallel transport of vectors around loops in the base space. The holonomy is used to classify bundles in terms of the possible holonomy groups that connections on them admit. The curvature is then defined as in usual calculus by applying the connection twice and found to be itself a tensor. In particular, the curvature tensor is a section of $\mathrm{End}(E) \otimes \Lambda^2(M)$ fitting nicely into the framework of bundles. The Levi-Civita connection is described as a historical landmark in Riemannian geometry being one of the preferred connections on the tangent bundle of a real Riemannian manifold and explicit calculations on TS^2 are provided. Flat connections and flat vector bundles are completely classified over a fixed base as the quotient $\tilde{M} \times_\rho \mathbb{C}^r$ where $\tilde{M}$ is the universal covering space of the base and ρ is a representation of the fundamental group. Next, moduli spaces are defined as equivalence classes of connections up to gauge symmetry. Explicit calculations of moduli spaces for vector bundles with $U(1)$ gauge group are shown to be the quotient of the first de Rham cohomology group with real coefficients by the first simplicial homology group with integer coefficients. The moduli space of flat connections on a trivial complex line bundle over a torus is found to be again a torus. Finally, Maxwell's electromagnetic field equations are encoded in the language of connections which solidifies the practicality of this theory.

Chapter 4 introduces an L^2 norm on connections known as the Yang-Mills functional and connections that minimize this norm are of interest. Mathematically, these connections are analogous to geodesics and their defining equations are found in a similar fashion. The critical values of the Yang-Mills functional are found to be harmonic connections and the equations describing these are called the Yang-Mills equations. The Hodge star operator, given in the appendix, is used to decompose the space of 2-forms on four manifolds revealing two classes of connections satisfying the Yang-Mills equations known as instantons. The notion of calibrations is introduced to show that the instantons constructed are in fact of minimal Yang-Mills energy.

The final chapter describes a particular class of seven manifolds analogous to

Kähler manifolds whose tangent spaces admit the smoothly varying structure of a two-fold cross product. This structure along with an orientation and Riemannian metric is encoded in a positive three form called a G_2-structure. Using this structure and a slightly varied version of the Hodge star, the 2-forms on M admit a decomposition similar to the case of four manifolds. This decomposition allows for the definition of the Donaldson-Thomas connections which are analogous to the anti-self dual connections and serve as generalized instanton solutions to the Yang-Mills equations. Lastly, a particular Donaldson-Thomas connection on $\mathbb{R}^7$ allowed only to depend on the first four variables is examined and found to necessarily be flat. This type of problem poses as a gateway to further study into the theory of connections on G_2-manifolds.

Chapter 2

Vector bundles

Motivation and definitions

Consider a point mass traveling in a circle of radius 1 whose path is a curve γ in $\mathbb{R}^2$ parameterized by $\gamma(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi]$. The velocity vector field of this mass at any given point is expressed as the vector valued function $v : S^1 \to \mathbb{R}^2$ defined by

$$v(t) = \frac{d\gamma}{dt} = (-\sin t, \cos t).$$

This is an example of a tangent vector field on the one sphere. In Maxwell's theory of electromagnetism and Einstein's gravitational physics, there are many "fields" which play an important role. Often, action at a distance is encoded using "force fields". To date, the only known naturally occurring fields of this type are electromagnetic, gravitational and the strong and weak nuclear forces. In the case of gravitation, the force between a spherically symmetric mass M of radius R and a point mass m at any fixed time is represented in spherical coordinates by

$F(r,\theta,\phi) = -\frac{GMm}{r^2}\hat{r}$, where the origin has been chosen at the center of M. This is a vector valued function $F : \mathbb{R}^3 \to \mathbb{R}^3$ pointing in the direction of gravitational pull whose norm encodes the strength of the pull.

From a theoretical perspective, the notion of a vector bundle is intended to abstract the concept of tangent vector fields on manifolds to arbitrary vector valued functions that may be of interest.

These are just a few examples to motivate the need for further investigation of such geometrical spaces and the functions they admit.

Definition 2.0.1. A *rank-r vector bundle* E over a (smooth) manifold M is a family of isomorphic r-dimensional vector spaces $\{E_p \cong F\}_{p\in M}$ parameterized by M having its own personal (smooth) manifold structure and satisfying:

(i) $\pi : E \to M$, called the *projection map*, is a continuous (smooth) surjection such that $\pi_{-1}(p) = E_p$, for each $p \in M$ and

(ii) for each $p \in M$, there is a neighbourhood $U \subseteq M$ of p and a homeomorphism (diffeomorphism)

$$\varphi_U : E_U := \pi^{-1}(U) \to U \times F$$

that is a point-wise linear isomorphism of vector spaces

$$\varphi_{U|_p} : E_p = \pi^{-1}(p) \cong \{p\} \times F$$

These homeomorphisms are called the *local trivializations* of the *total space* E having M as a *base space*. The family $\{(U_\alpha, \varphi_\alpha)\}$ is an *atlas* for E and the vector spaces F are the *fibres*.

Remark 2.0.2.

- This definition is intentionally stated for the weaker case of continuous maps along side the smooth case in order to reveal the flexibility of such objects. In fact, there is another type of *fibre bundle* having Lie groups for fibres rather than vector spaces. These are called principal bundles and will be discussed briefly in section 2.4.

- Rank-1 vector bundles are known as a *line bundles*.

- It is inherent from the definition of a smooth vector bundle that the total space is again a smooth $(\dim M + \dim F)$-dimensional manifold.

- With the exception of the first few conceptual examples, fibres will be taken to be $\mathbb{C}^r$ and we will work with smooth vector bundles only.

When passing between neighbourhoods on the base, it is necessary to determine the change of coordinates in order to obtain correct calculations. These changes are expressed, like a manifold, by *transition functions*

$$\tau_{\alpha\beta} := \varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(E_{U_\alpha \cap U_\beta}) \to \varphi_\alpha(E_{U_\alpha \cap U_\beta}) \in \mathrm{End}(E_{U_\alpha \cap U_\beta}).$$

These transition functions evaluate pointwise to linear isomorphisms of $\mathbb{C}^r$, so are equivalently realized as smooth maps

$$g_{\alpha\beta} : U_{\alpha\beta} := U_\alpha \cap U_\beta \to \mathrm{GL}_r(\mathbb{C}).$$

The simplest natural example of a smooth vector bundle over a smooth manifold M is the *trivial bundle* $E := M \times V$ having base M, fibres V and natural smooth projection π onto M. A global trivialization is given by the identity map on E. For a visual example, consider the infinite cylinder $S^1 \times \mathbb{R}$ embedded as a two-dimensional sub-manifold of $\mathbb{R}^3$.

The next and potentially most important example is known as the *tangent bundle*. For a smooth real n-dimensional manifold M, the tangent bundle TM is defined as

$$TM := \bigsqcup_{p \in M} T_pM$$

where T_pM is the *tangent space* of M at p spanned by the $n(= \dim M)$ vectors $\frac{\partial}{\partial x^i}|_p := \varphi_*^{-1}(p, e_i)$ where (φ, U) is a coordinate chart for $U \subseteq M$ containing p and e_i is the i^th standard basis vector for the tangent space of $\mathbb{R}^n$ at p. This is continued over the entire manifold to yield trivializations

$$\varphi_\alpha : TU_\alpha \longrightarrow U_\alpha \times \mathbb{R}^n$$

by

$$\sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i}|_p \mapsto (p, \vec{v}).$$

On the overlap $U_{\alpha\beta} := U_\alpha \cap U_\beta$ of the neighbourhoods U_α, U_β in M with coordinate systems x_α, x_β respectively, we have $TU_{\alpha\beta}$ given both in terms of $\frac{\partial}{\partial x^i_\alpha}$ and $\frac{\partial}{\partial x^i_\beta}$. The transition functions are given by the *Jacobian matrix* $\left[\frac{\partial x^j_\beta}{\partial x^i_\beta}\right]$ in the same fashion as multi-variable calculus. This is shown using the chain rule

$$\frac{\partial}{\partial x^i_\alpha} = \sum_{j=1}^{n} \frac{\partial x^j_\beta}{\partial x^i_\alpha} \frac{\partial}{\partial x^j_\beta}$$

or, in familiar notation,

$$\tau_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}$$

maps $(p, \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_\beta^i})$ to $(p, \sum_{i,j=1}^{n} \frac{\partial x_\alpha^j}{\partial x_\beta^i} v_i \frac{\partial}{\partial x_\alpha^j})$ which is captured by the Jacobian. The Jacobian matrix is nowhere singular since it is obtained from a diffeomorphism and represents an invertible change of coordinates at every point.

As with any topological or algebraic space, we define a *bundle morphism* between two vector bundles $\pi_1 : E_1 \to M_1$ and $\pi_2 : E_2 \to M_2$ as a continuous map $\varphi : E_1 \to E_2$ along with a continuous map $f : M_1 \to M_2$ between the base spaces satisfying $f \circ \pi_1 = \pi_2 \circ \varphi$. This type of commutativity is enforced to ensure that φ preserves the fibre structure of the bundles (i.e. in the case where fibres are vector spaces, we want the point-wise evaluation to be a linear isomorphism). The map φ is called a *bundle map covering* f. Two bundles E, F over the same base M are *isomorphic* if there exists a bundle morphism $\varphi : E \to F$ covering the identity map on M that is invertible.

A nice result used for recognizing bundle isomorphisms is

Lemma 2.0.3. *If $h : E_1 \to E_2$ is a continuous map between vector bundles over the same base M covering the identity, then h is an isomorphism of bundles if each fibre $\pi_1^{-1}(p)$ is mapped linear isomorphically to the corresponding fiber $\pi_2^{-1}(p)$ in E_2.*

See [9] Lemma 1.1 for a proof of this result.

A vector bundle $\pi : E \to M$ is called *trivial* if it is isomorphic to the product bundle. For example, this chapter was motivated with the tangent bundle of S^1, which is easily seen to be trivial pending a further result at the end of this chapter.

There is but one non-trivial real line bundle on S^1 known as the *Möbius bundle*. This is defined by the equivalence relation $\ddot{M} := [0, 2\pi] \times \mathbb{R}/\sim$ where $(0, \lambda) \sim (2\pi, -\lambda)$ for each $\lambda \in \mathbb{R}$. This relation identifies opposite endpoints and represents a twist in the topological structure.

It should be mentioned that for a vector bundle E over M, their first fundamental groups coincide because M is a deformation retract of E. That is, $\pi_1(E) = \pi_1(M)$.

2.1 Basic results and constructions of bundles

We are now ready to make our very own vector bundles using the following recipe:

Lemma 2.1.1. *[Vector Bundle Construction Lemma] Given a smooth manifold M with open cover $\{U_\alpha\}$ indexed by a set A along with a (complex) vector space E_p for each $p \in M$ each of dimension k, let $E := \bigsqcup_{p\in M} E_p$ and $\pi : E \to M$ map E_p to p. If*

- *for each $\alpha \in A$, there exists a bijective map $\Phi_\alpha : E_{U_\alpha} := \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^k$ whose point-wise evaluation is a linear isomorphism of E_p, with $\{p\} \times \mathbb{C}^k \cong \mathbb{C}^k$, and*

- *for each $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$, there exists a smooth map*

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_k(\mathbb{C})$$

such that the composite map

$$\tau_{\alpha\beta} := \Phi_\alpha \circ \Phi_\beta^{-1} : U_\alpha \cap U_\beta \times \mathbb{C}^k \to U_\alpha \cap U_\beta \times \mathbb{C}^k$$

evaluates as

$$\tau_{\alpha\beta}(p, v) = (p, g_{\alpha\beta}(p)v).$$

Then E has a unique smooth manifold structure, making it into a smooth rank-k vector bundle over M having π as projection and the Φ_α's as local trivializations.

The reader is referred to [18] Lemma 5.5. for a proof in the case of real vector bundles and the generalization to complex vector bundles is identical.

Remark 2.1.2. This result is a bit subtle, and may initially appear to be a jumbled-up restatement of the definition of a vector bundle. The difference to notice here is that only the transition functions are required to be smooth. We have reduced the trivializations to being merely bijective stating nothing explicitly about the topological structure of E. Also, the projection no longer depends on the topological structure on the spaces it maps between.

At $p \in U_{\alpha\beta}$ the transition functions are written nicely as

$$\tau_{\alpha\beta}(p, v) = (p, g_{\alpha\beta}(p) \cdot v)$$

where $g_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}_r(\mathbb{C})$ is smooth for each α, β. The maps $g_{\alpha\beta}$ will be referred to as the *gluing functions*. The gluing functions on a vector bundle satisfy a certain cohomological property called the *co-cycle condition*:

$$g_{\alpha\beta} \circ g_{\beta\alpha} = I_r$$

$$g_{\alpha\beta} \circ g_{\beta\gamma} \circ g_{\gamma\alpha} = I_r,$$

where I_r is the identity $r \times r$ matrix. From the simple observation that $g_{\alpha\alpha} = I_r$, these two equations may be reduced to just

$$g_{\alpha\beta} \circ g_{\beta\gamma} = g_{\alpha\gamma} \tag{2.1.1}$$

Intuitively, these equations represent the transitivity between change of coordinates on triply overlapping neighbourhoods. For further reading in this direction see [8] page 34 on Čech cohomology.

As a stronger consequence of Lemma 2.1.1, we find that any family of maps satisfying the co-cycle condition (2.1.1) defines a vector bundle having these maps as gluing functions. More precisely,

Theorem 2.1.3. *Let M be a smooth manifold with an open cover $\{U_\alpha\}_{\alpha\in A}$ and a family of maps $\mathcal{G} = \{g_{\alpha\beta} : U_{\alpha\beta} \to GL_r(\mathbb{C})\}_{\alpha,\beta\in A}$. If $\mathcal{G}$ satisfies(2.1.1), then it defines a smooth rank-r vector bundle $\pi : E \to M$ whose gluing functions are the members of $\mathcal{G}$.*

Proof. Define $E = \bigsqcup_{\alpha\in A}(U_\alpha \times \mathbb{C}^k)/\sim$, given by $(p, v) \sim (p, g_{\alpha\beta}(p)v)$, and a map $\pi : E \to M$ by $\pi([p, v]) = p$ sending each fibre $E_p = \{[p, v] : v \in \mathbb{C}^k\}$ to p. The bijective maps required by the construction lemma are basically identity maps given by

$$\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^k; [p, v] \mapsto (p, v)$$

and evaluate to a linear isomorphisms on the fibres. On overlapping neighbourhoods U_α, U_β, we find

$$\Phi_\alpha \circ \Phi_\beta^{-1}(p, v) = (p, g_{\alpha\beta}(p)v).$$

Hence, by Lemma 2.1.1, E is a smooth rank-k vector bundle over M having transition functions given by the $g_{\alpha\beta}$'s. □

The *dual bundle* is defined using this construction procedure as

$$E^* := \bigsqcup_{p \in M} E_p^*$$

with gluing functions $g_{\alpha\beta}^*$ inherited through E's by

$$(\overline{g}_{\alpha\beta}^T)^{-1} : U_{\alpha\beta} \to \mathrm{GL}_r(\mathbb{C}),$$

where $g_{\alpha\beta}$ is the gluing function of E on $U_{\alpha\beta}$. At first, it seems unclear why the inverse is necessary. With trivializations $\varphi_\alpha, \varphi_\beta$ of E, since point-wise these functions are linear isomorphisms, we are lead to construct trivializations for the dual bundle as the dual-inverse of the initial ones. That is,

$$\psi_\alpha := (\varphi_\alpha^*)^{-1} : E_{U_\alpha}^* \to U_\alpha \times (\mathbb{C}^r)^*$$

is a nice family of local trivializations for the dual bundle. The transition functions

$$\tau_{UV}^* : (U \cap V) \times (\mathbb{C}^r)^* \to (U \cap V) \times (\mathbb{C}^r)^*$$

are now computed as follows:

$$\tau_{UV}^*(p, v) := \psi_U \circ \psi_V^{-1}(p, v) = (p, ((g_{UV})^*)^{-1} v)$$

where $(g_{UV}(x))^* = \bar{g}_{UV}(x)^T$ is the usual adjoint matrix between complex vector spaces. These transition functions are smooth and satisfy equation (2.1.1) which suffices, by Theorem 2.1.3, to say that E^* is a rank-r vector bundle over M.

This type of proof technique will be standard when constructing new vector bundles from old. In a similar fashion, we may define the *conjugate bundle* $\bar{E}$ of E. The fibres here are given as the componentwise conjugates of the fibres of E and transition functions by the conjugates of those for E.

Let's take a look at some classical and practical examples of bundles which are constructed from others that appear frequently in any geometer's personal life.

Example 2.1.4 (Standard constructions)**.**

Let E and F be vector bundles over the same base M having $r = \text{rank}(E), k = \text{rank}(F)$ and gluing functions g^E, g^F respectively.

1. The *Whitney sum bundle* is the vector bundle constructed by taking the direct sum of the two fibres at each point of the given bundles. That is,

$$E \oplus F := \bigsqcup_{x \in M} E_x \oplus F_x.$$

   Using Lemma 2.1.1 we find bijective maps $\Phi_\alpha := \varphi_\alpha \oplus \psi_\alpha : E_{U_\alpha} \oplus F_{U_\alpha} \to U_\alpha \times \mathbb{C}^{r+k}$, where $\varphi_\alpha, \psi_\alpha$ are trivializations of E and F respectively. The point-wise evaluation $\Phi_{\alpha,x} : E_x \oplus F_x \to \{x\} \times \mathbb{C}^{r+k}$ is a linear isomorphism of vector spaces because both $\varphi_{\alpha,x}$ and $\psi_{\alpha,x}$ are.

The gluing functions of such a bundle on overlaps $U_{\alpha\beta}$ are

$$g_{\alpha\beta}(x) = \begin{bmatrix} g^E_{\alpha\beta}(x) & 0 \\ 0 & g^F_{\alpha\beta}(x) \end{bmatrix} \in GL_{r+k}(\mathbb{C}).$$

2. The *tensor bundle* $E \otimes F := \bigsqcup_{x\in M} E_x \otimes F_x$ has gluing functions $g_{\alpha\beta}(x) = g^E_{\alpha\beta}(x) \otimes g^F_{\alpha\beta}(x) \in GL_{r\cdot k}(\mathbb{C})$.

   More generally, the $(k,l)^{th}$ tensor bundle $\tau^k_l(E)$ of E is defined as the fibre-wise $(k,l)^{th}$ tensor power of E_x. That is

$$\tau^k_l(E) = \bigsqcup_{x\in M} E_x^{\otimes k} \otimes E_x^{*\otimes l},$$

   having gluing functions $g^{\otimes k}_{\alpha\beta} \otimes (g^*_{\alpha\beta})^{\otimes l}$ as expected. The rank of of this bundle is $\mathrm{rank}(E)^{kl}$.

3. The *exterior bundle* $\Lambda^k(M) := \bigsqcup_{x\in M} \Lambda^k T^*M$ has gluing functions

$$\tau_{\alpha\beta} = \wedge^k g_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}(\Lambda^k(\mathbb{C}^r)),$$

   where

$$\wedge^k g_{\alpha\beta}(p)(e_1 \wedge \cdots \wedge e_k) := \wedge^k_{i=1} g_{\alpha\beta}(p) e_i|_p.$$

   The entry-wise expansion of this is certainly hideous, but takes almost no time in the special case of *top exterior powers* of M (ie, when $n = \dim(M)$). Indeed, $\Lambda^n M$ has transition functions mapping to $\mathrm{GL}(\wedge^n\mathbb{C}^n) = \mathbb{C}^*$ which makes this is a line bundle and the gluing functions are:

$$\begin{aligned}
\bigwedge^n g_{UV}(e_1 \wedge \cdots \wedge e_n) &= \bigwedge_{i=1}^n g_{UV}(e_i) \\
&= \bigwedge_{i=1}^n (\sum_{j=1}^n g_{ji} e_j) \\
&= \left[\sum_{\sigma \in S_n} (-1)^{|\sigma|} \prod_{i=1}^n g_{i\sigma(i)} \right] \wedge_{i=1}^n e_i \\
&= \det(g_{UV}) e_1 \wedge \cdots \wedge e_n.
\end{aligned}$$

using the relation $\alpha \wedge \beta = -\beta \wedge \alpha$ for any $\alpha, \beta \in \Lambda^1(M)$. This line bundle is known as the *determinant bundle* of M and is denoted $\det(M)$. The co-cycle conditions for these maps are satisfied since determinants are multiplicative.

The *endomorphism bundle* is defined as the tensor bundle $\mathrm{End}(E) := E \otimes E^*$.

A *sub-bundle* $E' \subset E$ is an embedded sub-manifold of E which is also a vector bundle over M having fibres F' that are subspaces of the fibres F of E.

Here is an important chain of sub-bundles that should help with the bigger picture:

$$\Lambda^k(E) \subset \tau_k^0(E) \subset \tau_k^l(E).$$

It is natural to consider complementary/quotient sub-bundles. The *quotient bundle* E/E' over M is defined fibre-wise as the quotient of the vector spaces $E_x/E'_x \cong F/F'$ for each $x \in M$. If $\{(U_\alpha, \varphi_\alpha)\}$ is an atlas for E, then an atlas for E' is obtained by the restriction $\varphi'_U := \varphi_U|_{E'_U} : E'_U \to U \times F'$ and transition functions for E are written

$$g^E_{\alpha\beta}(x) = \begin{pmatrix} g_{\alpha\beta}(x) & * \\ 0 & h_{\alpha\beta}(x) \end{pmatrix} \in GL_{l+(k-l)}(\mathbb{C})$$

where g and h are transition functions for E' and E/E' respectively.

For any embedded sub-manifold $M \hookrightarrow N$, TM is a subbundle of TN and the *normal bundle* is the quotient of tangent bundles

$$NM := TN/TM.$$

This bundle has rank $n - m$ where N, M are of dimension n, m respectively. In particular, when M has codimension 1 in N, the normal bundle is a line bundle. This is for example the case for surfaces in $\mathbb{R}^3$.

For any continuous map $f : M \to N$ between manifolds and any vector bundle $\pi : E \to N$, define the *pullback bundle* as

$$f^*(E) := \{(x, e) \in M \times E : \pi(e) = f(x)\},$$

with projection map π' given by $(x, e) \mapsto x$. This results in the following commutative diagram:

$$\begin{array}{ccc} f^*(E) & \xrightarrow[\tilde{f}]{} & E \\ \downarrow{\scriptstyle \pi'} & & {\scriptstyle \pi}\downarrow \\ M & \xrightarrow{f} & N \end{array} ,$$

where $\pi \circ \tilde{f} = f \circ \pi'$ making $\tilde{f}$ a uniquely defined bundle morphism. Trivializations of $f^*(E)$ are inherited through f by

$$\tilde{\varphi}_\alpha(x, e) = \varphi_\alpha(f(x), e).$$

where φ_α are trivializations of E.

2.2 Sections of vector bundles

A *section* of E_U is a smooth map $\sigma_U : U \subseteq M \to E_U \cong U \times \mathbb{C}^r$ satisfying $\pi \circ \sigma = id_U$ or equivalently, $\sigma(p) \in E_p$ for each $p \in U$. A *frame* for E_U is a collection of r point-wise linearly independent sections $\vec{\sigma} = (\sigma_1, \ldots, \sigma_r)$. Any section or frame defined on all of M is called a *global* section or frame, respectively. Given a local frame field $\vec{\sigma}_U$, on U, one may represent any section τ_U on U with respect to $\vec{\sigma}_U$ by $\tau_U(p) = \sum_{i=1}^r \tau_i(p)\sigma_i(p)$ for each $p \in U$. Local sections will be denoted by $\Gamma(E_U)$ and global sections by $\Gamma(E)$.

Example 2.2.1.

1. Any smooth function $f : M \to \mathbb{R}$ is a smooth global section of the trivial bundle $M \times \mathbb{R}$ over M.

2. A *metric tensor* is a symmetric, bilinear section of $\tau_2^0(TM)$ such that, point-wise over M, it is an inner product on each T_xM. The Euclidean metric on $\mathbb{R}^n$, for example, is expressed locally in this language by

$$g = \sum_{i=1}^{n} dx^i \otimes dx^i \in \tau_2^0(TM).$$

   Indeed, if $X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}, Y = \sum_{i=1}^n Y^i \frac{\partial}{\partial x^i} \in T\mathbb{R}^n$ are two vector fields, then

$$g(X,Y) = \sum_{i=1}^{n} dx^i(\sum_{j=1}^{n} X^j \frac{\partial}{\partial x^j}) \cdot dx^i(Y^k \sum_{k=1}^{n} \frac{\partial}{\partial x^k}) = \sum_{i=1}^{n} X^i Y^i.$$

The standard metric for Minkowski space is $\eta = -dt \otimes dt + \sum_{i=1}^{3} dx^i \otimes dx^i$, although this is not positive definite due to the metric's negative signature.

In general, the positive metrics are the smooth sections of the positive cone of $\tau_2^0(M)$ meaning, they are smooth and positive definite at every point. These are commonly known as *Riemannian metrics*.

3. The *differential k-forms* on M are sections of $\Lambda^k(M) := \bigwedge^k(T^*M)$, for $k \geq 1$, and $\Lambda^0(M) := C^\infty(M)$. Locally, these forms are expressed as elements from a left $C^\infty(M)$-module with generating set

$$\{dx^{i_1} \wedge \cdots \wedge dx^{i_k} : 1 \leq i_1 < \cdots < i_k \leq \dim M\}.$$

Generally, differential forms are used to make measurements such as lengths, areas and "volumes" on M. As a familiar example, the infinitesimal arc-length of a curve $\gamma : [0,1] \to M$ on M at time t is given by the differential 1-form

$$\sqrt{g(\dot{\gamma}, \dot{\gamma})}dt$$

where $\dot{\gamma}$ is the velocity vector field and $||\dot{\gamma}||^2 = g(\dot{\gamma}, \dot{\gamma})$ This form may be integrated along $[0,1]$ to obtain the total arc-length of γ.

More generally, when computing areas or volumes, the integrand is a differential form of appropriate *degree* for the measurement at hand. In particular, lengths, areas and volumes on three dimensional manifolds are given by 1, 2 and 3-forms, respectively.

The standard theory of integration on manifolds uses exterior products of T^*M and can be found in any of [18, 21, 22].

A generalization of this idea to *E-valued k-forms* is made by considering the tensor bundle $\Lambda^k(E) := E \otimes \Lambda^k(M)$. This bundle will have sections similar to the differential k-forms with the $C^\infty(M)$ coefficients replaced with $\Gamma(E)$. Hence, the sections $\Lambda^0(E)$ are defined as $\Gamma(E)$.

4. The sections of $\det(M) = \Lambda^n(M)$ are locally equivalent to $\Gamma(E)$ by the correspondence

$$\sigma \leftrightarrow \sigma \cdot dx^1 \wedge \cdots \wedge dx^n$$

2.3 Frames versus trivializations

A local frame $\vec{\sigma}_\alpha$, induces a local trivialization by

$$\varphi_\alpha(\sum a_i \sigma_i(x)) := (x, (a_1, \ldots, a_r)).$$

Conversely, given a local trivialization $\varphi_\alpha : E_{U_\alpha} \to U_\alpha \times \mathbb{C}^r$, we have a corresponding local frame defined by

$$\sigma_i(x) := \varphi_\alpha^{-1}(x, e_i)$$

where e_i is the i^{th} standard basis vector of $\mathbb{C}^r$.

Now, given a trivialization of our bundle, we may represent any section s of E locally as a E_{U_α}-valued function $s_\alpha = (f^1, \ldots, f^r)$ where

$$s_\alpha(x) = \sum_{i=1}^{r} f^i \varphi_\alpha^{-1}(x, e_i),$$

and on overlapping neighbourhoods $U_{\alpha\beta}$, with alternate representation $s_\beta = (g^1, \ldots, g^r)$ that must agree so

$$\sum_{i=1}^{r} f^i \varphi_\alpha^{-1}(x, e_i) = \sigma_\alpha(x) = \sum_{i=1}^{r} g^i \varphi_\beta^{-1}(x, e_i) = \sigma_\beta(x)$$

which is equivalent to

$$\sum_{i=1}^{r} f^i e_i = \sum_{i=1}^{r} g^i \varphi_\alpha \circ \varphi_\beta^{-1}(x, e_i)$$

or in matrix notation as

$$s_\alpha = g_{\alpha\beta} s_\beta. \tag{2.3.1}$$

Now, a section is equivalent to a family of smooth vector valued functions $\{f_\alpha : U_\alpha \to \mathbb{C}^r\}$ satisfying equation (2.3.1). This leads to a nice result demonstrating the equivalence between triviality of a vector bundle and the existence of a nowhere vanishing global frame field on it.

Proposition 2.3.1. *A vector bundle $\pi : E \to M$ is trivial if and only if there exists a nowhere vanishing global frame on E.*

Proof. If $E = M \times V$, define global sections as follows: Pick any point $p \in M$ and any basis $\{v_1, \ldots, v_k\}$ for $E_p = \{p\} \times V$ and the constant sections $\sigma_i(x) := v_i$ form a nowhere vanishing global frame field. Conversely, given a global frame field $\vec{\sigma} = (\sigma_1, \ldots, \sigma_k)$ for E which never vanishes, a global trivialization is given by $\varphi(e) := (\pi(e), \vec{\sigma}(\pi(e)))$. □

This result may be used to prove that the tangent bundles of S^1, S^3, and S^7 are trivial. As mentioned earlier, we can prove triviality of TS^1 by simply noting that the constant tangent vector field $(-\sin\theta, \cos\theta)$ is nowhere vanishing. These trivializations are constructed using the complex, quaternionic and octionion structures existing on the respective spheres. These are the only spheres which admit such a trivialization. The non-triviality of TS^2 is proven in the section on flat connections. The reader is directed to any literature on the parallelizability of spheres for the previous remark such as [9].

The following theorem taken from [21], describes an important algebraic operator between forms

Theorem 2.3.2. *Let $\omega \in \Omega^k(M) = \Gamma(\Lambda^k(M))$, the there exists a unique $(k+1)$-form $d\omega$ which enjoys all of the following properties:*

(i) For local vector fields $X_0, \dots, X_k \in \Gamma(TU)$ we have

$$\begin{aligned} d\omega(X_0, \dots, X_k) &= \sum_{i=0}^{k} (-1)^i X_i \omega(X_0, \dots, \hat{X}_i, \dots, X_k) \\ &\quad + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0 \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k) \end{aligned}$$

(ii) $d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^{\deg(\omega)} \omega \wedge d\tau$

(iii) $d(d\omega) = 0$

(iv) For each 0-form $f \in \Omega^0(M) = C^\infty(M)$, df is the usual differential, given in local coordinates by

$$df = \frac{\partial f}{\partial x^i} dx^i$$

(v) In the case of a smooth map $\psi : M \to N$ between manifolds, $d(\psi^* \omega) = \psi^*(d\omega)$.

dw is called the exterior derivative *of ω and d is a linear map*

$$d : \Omega^k(M) \to \Omega^{k+1}(M).$$

2.4 Principal Bundles

Given a Lie group G and a smooth manifold M, a similar object to a vector bundle is the notion of *principal G-bundles* which are defined to be smooth manifolds P covering M with a (smooth) submersion $\pi : P \to M$ and an action of the group G on P that is free and transitive when restricted to each fibre. One sees immediately from the requirements of the group action that each fibre is in fact diffeomorphic to G itself. Local triviality of the bundle is required here meaning that on certain neighbourhoods of the base, the bundle is diffeomorphic to a product of the neighbourhood with G.

Some examples of principal bundles are as follows.

Example 2.4.1.

1. *Product bundle:* $P = M \times G$.

2. *Frame bundle:* $\mathcal{F}M := \bigsqcup_{x \in M, \vec{\sigma}} \vec{\sigma}(x)$ may be endowed with the structure of a smooth manifold of dimension $n + n^2$ where the first term count for the base and the second is the dimension of GL_n. The Lie group of the frame bundle is GL_n because it acts freely and transitively on the set of frames (bases) at each $x \in M$.

Bundle maps and isomorphisms are defined in the same fashion as before and as always, a principal G-bundle over X is said to be trivial if it is equivalent to the product bundle $X \times G$. In this case, the restriction of the bundle map to the fibres is not required to be a linear isomorphism, since the fibres are not vector spaces. Instead the restriction of the bundle map to the fibres must commute with the G-action.

A simple but important result states that a principle G-bundle P is trivial if and only if it admits a global section. The proof is similar to Proposition 2.3.1, with the exception that groups do not have zeros unlike vector spaces and the non-vanishing does not make sense.

2.5 Associated vector bundles

Given a principal G-bundle over M and a representation $\rho : G \to \mathrm{GL}(\mathbb{C}^k)$, define the *associated* vector bundle with respect to ρ by

$$E_\rho = P \times_\rho \mathbb{C}^k := (P \times \mathbb{C}^k)/\sim$$

where

$$(p, x) \sim (q, y) \qquad \leftrightarrow \qquad p = q \cdot g, x = \rho(g^{-1})y$$

Example 2.5.1. The associated vector bundle of the k^{th} trivial representation $e_k(g) = I_k$ of any principal G-bundle P over M is the product bundle $M \times \mathbb{C}^k$.

Indeed, the equivalence relation now reads $(p, v) \sim (pg, v)$ for each $g \in G$ and $v \in \mathbb{C}^k$. Hence, the fibres will be the entire vector space.

The projection $\tilde{\pi} : E_{e_k} \to M$ is naturally defined by $\tilde{\pi}([p, v]) = \pi(p) \in M$ and is smooth since π is. The fibres are

$$\tilde{\pi}^{-1}(x) = \pi^{-1}(x) \times_{e_k} V = G \times_{e_k} V$$

as described above, and have a nice set of representatives $\{[e, v] : v \in V\}$.

For transition functions, consider local sections $\sigma_\alpha, \sigma_\beta$ on overlapping neighbourhoods $U_\alpha, U_\beta \subseteq M$. Upon evaluation at $x \in M$

$$\sigma_\alpha(x) = [p_\alpha(x), v_\alpha(x)]$$

and

$$\sigma_\beta(x) = [p_\beta(x), v_\beta(x)]$$

where p_α, p_β are local sections of P satisfying $\pi(p_\alpha(x)) = x$ and v_α, v_β are local sections of the trivial bundle $M \times \mathbb{C}^k$.

By the local triviality of P, there are transition maps $g_{\alpha\beta} : U_{\alpha\beta} \to G$ such that $p_\alpha = p_\beta g_{\alpha\beta}$ meaning on overlaps,

$$\sigma_\alpha = [p_\alpha, v_\alpha] = [p_\beta g_{\alpha\beta}, v_\alpha] = [p_\beta, \rho(g_{\alpha\beta}^{-1}) v_\alpha].$$

This shows that gluing functions are inherited through $e_k(g_{\alpha\beta}^{-1})$ and so the associated vector bundle is indeed a vector bundle.

Chapter 3

Connections on bundles

3.1 Connections

With a decent function theory developed for vector bundles, the next step is to perform rate of change calculations on sections. In an attempt to generalize usual differential calculus from an algebraic perspective, operators are defined on $\Gamma(E)$ to act linearly and satisfy a (generalized) product rule. This approach is motivated through Sir Gottfried Wilhelm Leibniz's philosophical teaching that the properties of an object are reflected by its surroundings. One may recall that "you are who you hang with" and, conversely,"if you want be like them you've got to act like them".

It turns out, when defining differentials abstractly in this manner, that there is an infinite dimensional affine space of possibilities! Even in the trivial bundle case, where the usual exterior derivative d is a valid "connection", there are many other choices of differential operators on this bundle. Realizing that the exterior derivative is of this algebraic type on the trivial bundle provides positive indication that everything has been built according to plan. To this end, the following theory reduces to regular everyday normal calculus (when applicable).

In single variable calculus, the differential of a section f of the trivial line bundle $\mathbb{R} \times \mathbb{R}$ over $\mathbb{R}$ is

$$df_p = f'(p)dx = \frac{df}{dx}|_{x=p}dx.$$

Similarly, for a real valued function of several variables, or section f of the trivial real line bundle $\mathbb{R} \times \mathbb{R}^n$ over $\mathbb{R}^n$, the exterior differential evaluates to

$$df_p = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}|_{x=p}dx^i.$$

In both cases, d is a linear map, taking sections to differential 1-forms, which acts linearly on a specified tangent vector field pointing in the direction desired for rate of change. This is known as the *directional derivative* and works well for trivial bundles.

Given a tangent vector field $V \in \Gamma(TM)$ and a local tangent vector field $u = u_i \frac{\partial}{\partial x^i}$ the usual exterior derivative evaluates as

$$dV_p(u) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}|_p dx^i \left(\sum_{j=1}^{n} u_j \frac{\partial}{\partial x^j}|_p \right) = \frac{\partial V}{\partial x_i}|_p u_i(p).$$

However, these coefficients are no longer well-defined. The problem here lies in partial derivatives defined in terms of limits as

$$\frac{\partial V}{\partial x_i}(x) := \lim_{\Delta x_i \to 0} \frac{V(x + \Delta x_i) - V(x)}{\Delta x_i}$$

where it is not necessarily possible to perform subtraction in the numerator. This is because $V(x+\Delta x_i) \in T_{x+\Delta x_i}$ and $V(x) \in T_x(M)$ live in different tangent spaces. This only fails when the fibres of our bundle are "twisted", or the bundle is non-trivial, and hence the fibres are not algebraically comparable. A concept of parallel transport of vectors from one tangent space to another is needed in order to properly subtract them.

Once this has been solidified, a generalized notion of parallel transport on arbitrary vector bundles leads to the generalized theory of geodesics and curvature. All of this and more helps to begin classifying the intrinsic geometry of non-trivial vector bundles.

For the rest of this section let $\pi : E \to M$ be a rank k vector bundle over a complex manifold M. The following definition will make use of the fact that $\Gamma(E)$ and $\Omega^k(E) := \Gamma(\Lambda^k(E))$ are sheaves whose definition can be found in appendix B of [11].

Definition 3.1.1. A *connection* on E is a sheaf homomorphism

$$D : \Gamma(E) \to \Omega^1(E)$$

satisfying the *Leibniz rule*: $D(f\sigma) = df \otimes \sigma + f(D\sigma)$ for each $f \in C^\infty(M)$ and section $\sigma \in \Gamma(E)$.

From our previous discussion, this operator maps sections linearly to an E-valued differential form whose coefficients are sections of E representing the "rate of change" of the original section in the direction corresponding to the coefficient's index with respect to a frame field. Differential 1-forms are, by design, linear operators on vector fields. Hence, given a connection on E, the *covariant derivative* of $\sigma \in \Gamma(E)$ in the direction of a vector field $X \in TM$ is defined by

$$D_X : \Gamma(E) \to \Gamma(E),$$

where $D_X(\cdot) := D(\cdot)(X)$. This is an operator which is linear in X defined on $\Gamma(E)$ satisfying the Leibniz rule (product rule).

Comparing connections with exterior differentiation should, at least locally, reveal similarities. In fact, a connection is locally just an affine perturbation of the exterior derivative. Observe, on a trivialization $E_\alpha \cong U_\alpha \times \mathbb{C}^r$, having frame field $\vec{\sigma} = (\sigma_1, \dots, \sigma_r)$, then $\eta \in \Gamma(E_\alpha)$ may be expressed as $\eta = \sum_{i=1}^r a_i \sigma_i$ with $a_i \in C^\infty(U)$ for each i. One finds, a connection D evaluates on basis vectors as

$$D\sigma_i = \sum_{j=1}^r \sigma_j A_{ji}$$

or in matrix notation

$$D\vec{\sigma} = \vec{\sigma} A$$

where $A := (A_{ij})$ is an $r \times r$ matrix of 1-forms called the *connection 1-form* with respect to the local frame $\vec{\sigma}$. For mathematical accuracy, the connection 1-form is a section of $\mathrm{End}(E_\alpha) \otimes \Lambda^1(U_\alpha)$. Extending this linearly and using the Leibniz rule one finds,

$$D\eta = \sum_i (da_i \otimes \sigma_i + a_i D\sigma_i) = d\eta + A\eta.$$

This means, locally that $D_\alpha = d + A_\alpha$.

The simplest example of a connection is the *trivial connection* on the product bundle. This is defined globally by the usual exterior derivative d. Another classical example known as the Levi-Civita connection will be discussed once the curvature tensor has been introduced.

It is important to keep in mind, just like in linear algebra, that the connection form is only a matrix representation of an operator with respect to a predetermined local frame field for E_α. This is analogous to the fact that the set of linear transformations on $\mathbb{C}^n$ is given as the quotient $M_n(\mathbb{C})/\mathrm{GL}_n(\mathbb{C})$, where GL_n acts by conjugation on M_n, whose orbits represent equivalence classes of similar matrices (equivalent transformations). For connections, these quotients are called *moduli spaces* and turn out to be elegant geometric spaces inheriting many structures from the base space.

A change of frame, also known as a *gauge transformation*, between local frames σ' and σ for E_α is expressed as $\sigma' = \sigma g$ where $g : U_\alpha \to GL_r(\mathbb{C})$ is a family of invertible matrices varying smoothly over the neighbourhood U_α. The connection 1-forms are related by

$$A' = g^{-1}dg + g^{-1}Ag \tag{3.1.1}$$

since

$$\begin{aligned}
\sigma' A' &= D\sigma' = D(\sigma g) \\
&= \sigma(dg) + (D\sigma)g \\
&= \sigma' g^{-1}dg + \sigma A g \\
&= \sigma'(g^{-1}dg + g^{-1}Ag).
\end{aligned}$$

Similarly, on overlapping neighbourhoods U_α, U_β of M, connection forms A_α, A_β are related by equation (3.1.1) with appropriate relabeling $g_{\alpha\beta} : U_{\alpha\beta} \to GL_r(\mathbb{C})$ in place g.

Remark 3.1.2. Connections give rise to a family of 1-forms defined for each trivialization of a bundle that glue up on overlaps according to equation 3.1.1. Conversely, a connection is uniquely specified by any such family of 1-forms $\{A_\alpha\}$, on an atlas $\{(U_\alpha, \varphi_\alpha)\}$ that glues up properly. This type of construction of connections will turn out to be useful when proving flat bundles admit flat connections.

Lemma 3.1.3. *Let $f : M \to N$ be a smooth map between manifolds and D a connection on some vector bundle E over N. Then the pullback bundle $f^*(E)$ over M inherits a connection $f^*(D)$.*

Proof. $f^*(D)$ is defined locally on $f^*(E_{U_\alpha})$ by $f^*(d + A_\alpha) := d + f^*(A_\alpha)$ where A_α is the connection form of D on U_α and $f^*(A_\alpha)$ is an $r \times r$ matrix whose entries are the entry-wise pull-back of forms contained in A_α. □

3.2 Parallel transport and geodesics

For a connection D on E, a section $\sigma \in \Gamma(E)$ is said to be *parallel (with respect to D)*, if for every tangent vector $X \in T_pM$ at $p \in M$, we have

$$D_X(\sigma)(p) = 0.$$

This says that the "rate of change" of σ in the direction of each tangent vector X at p is 0.

For any curve $\gamma : [0,1] \to M$, a section σ is called *parallel* along γ if

$$D_{\dot{\gamma}(t)}\sigma(\gamma(t)) = 0 \tag{3.2.1}$$

for each $t \in [0,1]$. Locally, this equation is a linear system of differential equations having a unique solution upon specification of a single initial condition $\sigma(\gamma(0)) = V_0$. The proof of existence and uniqueness of this solution can be found in [18] Chapter 17.

The parallel transport with respect to D of $V_0 \in E_p$ along γ from p to q is $\sigma(q)$ where σ is the unique solution to 3.2.1 having $\sigma(p) = V_0$. The map

$$\mathcal{P}_D^\gamma : E_p \to E_q; V_0 \mapsto \sigma(q)$$

is a linear isomorphism of vector spaces (because equation 3.2.1 is $\mathbb{R}$-linear in σ and the inverse is given by $\mathcal{P}_D^{\gamma^{-1}}$) which is uniquely defined for any path γ from p to q. $\mathcal{P}_D^\gamma$ is called the *parallel transport along* γ *with respect to* D.

For a closed curve (loop) γ in M and $v \in E_{\gamma(0)}$, let $A \in \mathrm{GL}_n(\mathbb{C})$ be such that $\mathcal{P}^\gamma v = Av$. For most vector bundles this operator will be non-trivial and path dependent. Moreover, these matrices will always form a group called the *holonomy group* of E at $\gamma(0)$.

3.3 Curvature

This is the next most natural operator after connections.

Definition 3.3.1. Let D be a connection on E. The *curvature* tensor F_D with respect to D is given by the bundle morphism

$$F_D := D^2 : \Omega^0(E) \to \Omega^2(E).$$

Currently this is an ill-defined operation because connections have yet to be extended to k-forms. However, since connections are intended to mimic exterior differentiation, it is natural to extend the domain of connections to $\Omega^k(E)$ by the rule

$$D(\sigma \otimes \alpha) := d\alpha \otimes \sigma + (-1)^k \alpha \wedge D\sigma$$

for $\sigma \in \Gamma(E)$, $\alpha \in \Omega^k(M)$.

The curvature of D is $C^\infty(M)$-linear because

$$F_D(f\sigma) = D(df \otimes \sigma + f \cdot D\sigma) = -df \wedge D\sigma + df \wedge D\sigma + fD^2\sigma = f \cdot F_D(\sigma),$$

meaning it is a valid tensor, unlike D.

The local representation of F_D in terms of the connection form is $F_D\vec{\sigma} = \vec{\sigma}F$ where

$$F = dA + A \wedge A. \tag{3.3.1}$$

Indeed, for any local frame $\vec{\sigma}$

$$F_D(\vec{\sigma}) = D^2(\vec{\sigma}) = D(\vec{\sigma}A) = D\vec{\sigma} \wedge A + \vec{\sigma}dA = \vec{\sigma}(A \wedge A + dA) = \vec{\sigma}F.$$

Exterior differentiation of equation 3.3.1 reveals the well known *Bianchi identity*

$$dF = dA \wedge A - A \wedge dA = [F, A], \tag{3.3.2}$$

since

$$\begin{aligned}
[F, A] &= F \wedge A - A \wedge F \\
&= (dA + A \wedge A) \wedge A - A \wedge (dA + A \wedge A) \\
&= dA \wedge A - A \wedge dA.
\end{aligned}$$

Equivalently this reads $dF + [A, F] = 0$. This identity may be viewed in a different light as follows: Thinking of the curvature F of a connection A as a section of $\mathrm{End}(E) \otimes \Lambda^2(M)$, there is a natural connection d_A induced on the endomorphism bundle acting on $\tau \in \Gamma(\mathrm{End}(E))$ as $d_A\tau = d\tau + [A, \tau]$ because when applied to a section $D(\tau(s)) = (D\tau)(s) + \tau Ds$ so that

$$\begin{aligned}
(D\tau)(s) &= D(\tau(s)) - \tau(D(s)) \\
&= (d + A)\tau(s) - \tau(d + A)(s) \\
&= d(\tau(s)) + A\tau(s) - \tau ds - \tau As \\
&= d\tau(s) + \tau ds + [A, \tau](s) - \tau ds \\
&= (d\tau + [A, \tau])(s).
\end{aligned}$$

Hence, the Bianchi identity may be expressed as

$$d_A F = 0.$$

Under local change of gauge, the curvature tensor is transformed by the usual similarity of matrices under conjugation. That is, if σ', σ are local frames on E_U with $\sigma' = \sigma g$ for some $g : U \to GL_r(\mathbb{C})$, then

$$F' = g^{-1} F g$$

which is proven in the same fashion as 3.1.1.

The following proposition, taken from [11] (4.3.7) part (iv), describes the curvature of a pull-back bundle.

Proposition 3.3.2. *Let $f : M \to N$ a smooth map and consider E over N with connection D. Then the curvature of the pull-back connection on $f^*(E)$ is*

$$F_{f^*(D)} = f^*(F_D)$$

Proof. Looking locally, where $D = d + A$, we find

$$F_{f^*(D)} = d(f^*(A)) + f^*(A) \wedge f^*(A) = f^*(dA + A \wedge A) = f^*(F)$$

□

3.4 Levi-Civita connection

An important and well known example of a connection on the tangent bundle of a Riemannian manifold is the Levi-Civita (Riemannian) connection. This connection has two additional geometrically appealing constraints associated to it. Namely, the *Levi-Civita connection*, ∇, of a Riemannian manifold (M, g) is a connection on TM that is

(i) *compatible* with the Riemannian metric, meaning

$$\nabla[g(X, Y)] = g(\nabla X, Y) + g(X, \nabla Y),$$

and

(ii) *torsion free*, meaning

$$T_\nabla(X, Y) = 0$$

for all $X, Y \in \Gamma(TM)$, where

$$T_\nabla(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$$

is called the *torsion tensor*.

Geometrically speaking, the compatibility condition means that ∇ preserves the inner product of parallely transported vectors in TM and vanishing torsion means that ∇ is symmetric. Condition (ii) implies ∇ is symmetric, which is not immediately apparent so justification is provided. Suppose that M has local coordinates $(x_1, \ldots, x_n)$, and consider the corresponding frames $\{\partial_i := \frac{\partial}{\partial x_i}\}_{i=1}^n$ and $\{dx_i\}_{i=1}^n$ of TM and T^*M respectively. Then,

$$[\partial_i, \partial_j] = 0$$

for all i, j so that $T_\nabla = 0$ is equivalent to

$$\nabla_{\partial_i}\partial_j = \nabla_{\partial_j}\partial_i$$

for all i, j. If we set

$$\nabla_{\partial_i}\partial_j = \sum_{k=1}^{n} \Gamma_{ij}^k \partial_k,$$

this means that

$$\Gamma_{ij}^k = \Gamma_{ji}^k, \tag{3.4.1}$$

for all i, j. The Γ_{ij}^k are called the *Christofel symbols of the second kind* and equation 3.4.1 tells us they are symmetric in i and j.

Theorem 3.4.1. *For any Riemannian manifold (M, g), there exists a unique Levi-Civita connection ∇. Moreover, ∇ is expressed locally with respect to the coordinate frame $\{\partial_i := \frac{\partial}{\partial x_i}\}_{i=1}^n$ as $\nabla = d + A$, where $A = (A_{ij})$ with*

$$A_{ij} = \sum_{k=1}^{n} \Gamma^i_{kj} dx_k, \tag{3.4.2}$$

and the Christoffel symbols are given by

$$\Gamma^k_{ij} = \frac{1}{2} \sum_{l=1}^{n} g^{kl} \left(\partial_j g_{li} + \partial_i g_{lj} - \partial_l g_{ij} \right) \tag{3.4.3}$$

Proof. See any text on Differential Geometry and/or General Relativity such as [7, 18, 22] for this construction. □

Example 3.4.2. The Levi-Civita connection of S^2. The standard round metric on S^2, which is just the restriction of the Euclidean metric on $\mathbb{R}^3$ to the unit sphere, in spherical coordinates is given locally by

$$g = d\phi^2 + \sin^2(\phi) d\theta^2.$$

The Christoffel symbols with respect to this frame, computed using equation 3.4.3, are $\Gamma^\phi_{\theta\theta} = -\cos(\phi)\sin(\phi)$, $\Gamma^\theta_{\phi\theta} = \Gamma^\theta_{\theta\phi} = \cot(\phi)$ with all others zero.

So, by equation 3.4.2, the connection form is $A = \begin{pmatrix} 0 & -\cos(\phi)\sin(\phi)d\theta \\ \cot(\phi)d\theta & \cot(\phi)d\phi \end{pmatrix}$ making the curvature tensor

$$F = dA + A \wedge A = \begin{pmatrix} 0 & \sin^2(\phi) d\phi \wedge d\theta \\ -d\phi \wedge d\theta & 0 \end{pmatrix}$$

Remark 3.4.3. Metric compatibility of connections is often a constraint of concern. On arbitrary vector bundles $E \to M$, we can have a metric structure analogous to the Riemannian metric which is a smoothly varying field of inner products on the fibres of E. When considering a complex vector bundle, one is concerned with a Hermitian metric h which is a smoothly varying field of Hermitian inner products. If such a structure exists on E, the (E, h) is called a *Hermitian vector bundle*. A *Hermitian connection* on a Hermitian vector bundle E is ∇ satisfying $dh(\alpha, \beta) = h(\nabla\alpha, \beta) + h(\alpha, \nabla\beta)$. One can see from these requirements that the connection form has special restricted values upon such compatibility. Indeed, on a rank r Hermitian bundle $\pi : E \to M$, choose a local orthonormal frame $\sigma_1, \ldots, \sigma_r$ and any Hermitian connection D satisfies

$$\begin{aligned} 0 = d < \sigma_i, \sigma_j > &= < D\sigma_i, \sigma_j > + < \sigma_i, D\sigma_j > \\ &= < A_{ki}\sigma_k, \sigma_j > + < \sigma_i, A_{kj}\sigma_k > \\ &= A_{ji} + \bar{A}_{ij} \end{aligned}$$

This means the Hermitian connections are locally represented by skew-Hermitian valued 1-forms. In general, the connection forms are Lie algebra valued 1-forms, where the Lie group is the structure group of the bundle (i.e., the group where the gluing functions take values).

3.5 Flat bundles, flat connections and some homotopy theory

The purpose here is to examine the types of curvature tensors existing on bundles over a fixed base space in order to begin a classification of bundles. The simplest bundles to classify are the flat ones. These bundles provide a foundation for working on these types of problems and involve links to the algebraic topology of the base space.

The trivial connection is the most natural choice of connection on trivial vector bundles. This connection satisfies $d^2 = 0$ meaning it has zero curvature. This is a mathematical way of saying that trivial bundles are "flat". The question is, for non-trivial vector bundles, is there a connection D on E whose curvature tensor vanishes? Such a connection D on E satisfying $D^2 = 0$ is known as a *flat connection.* Equation 3.3.1, shows that this is not always the case as shown for the Levi Civita connection of S^2.

In a similar, and soon to be equivalent respect, E is called *flat* if it admits an open cover along with local trivializations $\{(U_\alpha, \varphi_\alpha)\}$ whose gluing functions are constant. Such an atlas is called a *flat structure* for E.

It is proven in [5] (Theorem 2.2.1), that if D is a flat connection on E then for every point $p \in M$ there is an open neighbourhood U of p and a trivialization $\varphi_U : E_U \to U \times \mathbb{C}^r$ for which the connection form $A_U = 0$. This means that, on overlapping neighbourhoods, our transition functions are constant because

$$0 = A_U = g_{UV}^{-1} A_V g_{UV} + g_{UV}^{-1} dg_{UV} = g_{UV}^{-1} dg_{UV}$$

meaning $dg_{UV} = 0$. Hence, a bundle admitting a flat connection also admits a flat structure. In fact,

Proposition 3.5.1. *E is flat if and only if E admits a flat connection.*

Proof. We have already described the converse of this result.

Given that E is flat with flat structure $\{(U_\alpha, \varphi_\alpha)\}$, define connection forms $\omega_\alpha := 0$ for each α. To ensure this is a connection, it suffices to show equation 3.1.1 is satisfied. Since transition functions, hence gluing functions $g_{\alpha\beta}$, are constant on $U_\alpha \cap U_\beta$, it follows that

$$\omega_\alpha = 0 = g_{\alpha\beta}^{-1} \cdot 0 \cdot g_{\alpha\beta} + g_{\alpha\beta}^{-1} \cdot 0 = g_{\alpha\beta}^{-1} \omega_\beta g_{\alpha\beta} + g_{\alpha\beta}^{-1} d g_{\alpha\beta}$$

as required. This connection is flat since locally for each α

$$\Omega_\alpha = d\omega_\alpha + \omega_\alpha \wedge \omega_\alpha = 0.$$

□

The two notions of flatness may now be used interchangeably. Both interpretations of flatness here are very differential in nature, meaning they depend heavily on the differential structure of the bundle. A remarkable fact which serves as a classification of all flat bundles is a description in terms of representations of the first fundamental group of the base space. This section is dedicated to developing the theory required to prove and visualize such a classification theorem linking several tools from modern mathematics.

The concepts of connection and curvature are very closely related to the parallel transport of tensor fields on vector bundles. Hence, naturally one encounters the homotopy theory of paths and loops on M.

Theorem 3.5.2. *If D is a flat connection on E, then $\mathcal{P}_D^{\gamma_1} = \mathcal{P}_D^{\gamma_2}$ for any homotopic paths γ_1, γ_2 from p to q in M.*

Proof. Let $H : [0,1]^2 \to M$ be a homotopy from γ_1 to γ_2 and let $\{(U_\alpha, \varphi_\alpha)\}$ be a flat structure for E satisfying that $D_{U_\alpha} = d$ for each U_α. Suppose now that $U_{1,1}, \ldots, U_{1,M}, U_{2,1}, \ldots, U_{N,M}$ is a good covering for the image of H meaning that $U_{i,j} \supset H([(i-1)/N, i/N] \times [(j-1)/M, j/M])$ for each $1 \leq i \leq N$,$1 \leq j \leq M$, where N, M have been chosen large enough so this type of covering is achievable. Without loss of generality assume $p \in U_{11}$. The parallel transport along any of the $U'_{i,j}s$ is constant because $D_{U_{i,j}} = d$. In particular, consider the path γ_{11} in $[0,1]^2$ by connecting (with straight lines) the points $(0,0), (0,1/M), (1/N,1/M), (1/N,0)$ and $(1,0)$. Then $\mathcal{P}^{\gamma_1} = \mathcal{P}^{H(\gamma_{11})}$ because the points in M where these paths differ are contained in U_{11} where parallel transport is constant. Similarly, the path γ_{12} in $[0,1]^2$ connecting $(0,0), (0,2/M), (1/N,2/M), (1/N,0)$ and $(1,0)$ will satisfy $\mathcal{P}^{\gamma_1} = \mathcal{P}^{H(\gamma_{12})}$. This holds because a vector transported parallely along $H(\gamma_{12})$ remain constant within U_{11}, transitions into U_{12}, is constant within U_{12}, transitions back into U_{11} and continues along a path coinciding with γ_1. Since transition functions are constant on $U_{11} \cap U_{12}$, the change of bases experienced upon entering and exiting U_{12} are inverses of each other. Iteratively, we construct paths γ_{1k} for $1 \leq k \leq M$ in $[0,1]^2$ beginning at $(0,0)$ and connecting points $(0,k/M), (1/N,k/M), (1/N,0), (1,0)$ to find that $\mathcal{P}^{\gamma_1} = \mathcal{P}^{H(\gamma_{1k})}$ for each k. We continue to define paths γ_{kl} for $1 < k \leq M$, $1 \leq l \leq N$ by connecting $(0,0), (0,1), ((k-1)/N, 1), ((k-1)/N, l/M), (k/N, l/M), (k/N, 0)$ and $(1,0)$. It follows by the same arguments as above that $\mathcal{P}^{\gamma_1} = \mathcal{P}^{H(\gamma_{kl})}$ for each k, l. In particular, since $\gamma_2 = H(\gamma_{NM})$ we have $\mathcal{P}^{\gamma_1} = \mathcal{P}^{\gamma_2}$ as required.

□

Corollary 3.5.3. *The parallel transport with respect to a flat connection between distinct points on a simply connected M is path independent.*

Proof. Let γ_1, γ_2 be distinct paths from p to q in M. Then, since M is simply connected, γ_1 is homotopic to γ_2 so the above result implies path independence. $\square$

Lemma 3.5.4. *If D is flat on E, there is a uniquely defined representation*

$$\rho : \pi(M, x_0) \to \mathrm{GL}_k(\mathbb{C})$$

of the first fundamental group of M at x_0 defined using parallel transport with respect to D.

Proof. Theorem 3.5.2 implies the representation $\rho([\gamma]) := \mathcal{P}^\gamma$ is well-defined and given by a unique matrix $A \in \mathrm{GL}_k(\mathbb{C})$. It remains to see this is a group homomorphism.

Certainly, $\rho(\gamma^{-1}) = \rho(\gamma)^{-1}$ as traversing the path backwards would yield an inverse parallel transport. Finally, $\rho(\gamma_1 \circ \gamma_2) = A_{\gamma_1 \circ \gamma_2} = A_{\gamma_1} \cdot A_{\gamma_2} = \rho(\gamma_1) \cdot \rho(\gamma_2)$, by the uniqueness of solutions to the ODE's involved in construction of these matrices. This makes ρ a well-defined representation of $\pi_1(M, x_0)$ induced uniquely by D on E. $\square$

Remark 3.5.5. This is called the *holonomy* representation of $\pi_1(M, x_0)$ with respect to D. In the case where E is the tangent bundle of M, this is called the *monodromy* of M.

Donaldson and Kronheimer show, in their section on connections and curvature of [5], that a vector bundle with a flat connection over a hyper-cube is necessarily trivial. This can be easily abstracted to show that any vector bundle admitting a flat connection over a simply connected manifold is trivial.

Proposition 3.5.6. *If E is flat and M is simply connected, then E is equivalent to the trivial bundle $M \times \mathbb{C}^r$.*

Proof. It suffices, by proposition 2.3.1, to find a global frame field for E. Given a flat structure $\{(U_\alpha, \varphi_\alpha)\}$, Proposition 3.5.1 provides a flat connection D on E having local 1-forms $A_\alpha = 0$ on each U_α. A global frame is constructed using parallel transport of a basis $\beta = \{v_1, \ldots, v_r\}$ at an arbitrary fibre E_p of E. This is globally well-defined since M is simply connected, as we now explain.

Indeed, for $p \in M$ and the basis β, path independence from Corollary 3.5.3 is used to define

$$\sigma^i(x) := \mathcal{P}^\gamma v_i$$

for each $i = 1, \ldots, k$ where γ is any path from p to x in M.

To see linear independence of these sections, consider locally beginning at p. For $x \in U_\alpha$ containing p, the parallel transport is given by the differential equation $D_{\dot\gamma}\vec{\sigma}(\gamma(t)) = 0$ with initial conditions $\sigma_i(p) = \sigma_i(\gamma(0)) = v_i$ for each i. Since each $A_\alpha = 0$, $d\vec{\sigma}(\gamma(t)) = 0$ on U_α which implies the unique solution $\vec{\sigma}$ to our differential equation is constant along γ. This extends to the entire neighbourhood U_α showing $\vec{\sigma}_\alpha = (\sigma^1_\alpha, \ldots, \sigma^r_\alpha) = (v_1, \ldots, v_r)$ which is certainly nowhere zero.

For any point y outside of U_α and any path γ from p to y, the vectors $v_1, \ldots, v_k$ will experience a non-zero change of basis on the overlapping neighbourhoods covering γ and remain constant within them, hence remaining linearly independent. It is clear that $\vec{\sigma}$ is a globally defined frame field trivializing E. $\square$

Using this result provides a very broad class of flat bundles.

Lemma 3.5.7. *Let ρ be the holonomy, $\tilde{M}$ the universal covering space of M and*

$$E_\rho := \tilde{M} \times_\rho \mathbb{C}^k \tag{3.5.1}$$

with the equivalence relation $(x, v) \sim_\rho (\gamma(x), \rho(\gamma)v)$ for each $[\gamma] \in \pi_1$. Then E_ρ is a flat bundle on M.

Proof. A flat connection d_ρ is inherited on E_ρ descending from the trivial connection d on $\tilde{M} \times \mathbb{C}^r$ as follows: Let $\sigma \in \Gamma(E_\rho)$. Then, using the notation from section 2.4, $\sigma = [\tilde{\sigma}]$ for some section $\tilde{\sigma}$ of $\tilde{M} \times \mathbb{C}^k$ such that if, locally, $\tilde{\sigma}(x) = (x, v(x))$ for some vector-valued function v, then

$$\tilde{\sigma}(\gamma(x)) = (\gamma(x), \rho(\gamma)v(x)).$$

The exterior derivative applied to such sections will thus evaluate as

$$d\tilde{\sigma}(\gamma(x)) = (\gamma(x), \rho(\gamma)dv(x)),$$

which descends to an E_ρ-valued 1-form on M. Defining

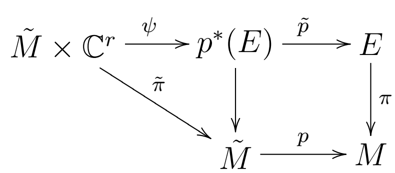
$$d_\rho(\sigma) := [d(\tilde{\sigma})]$$

gives a connection on E_ρ. Since $d_\rho^2(\sigma) = [d^2(\tilde{\sigma})] = 0$, we find d_ρ is flat implying that E_ρ is flat. $\square$

The previous lemma has provided a class of flat bundles so large, that it is in fact all of them, as the following theorem demonstrates.

Theorem 3.5.8. *E is flat if and only if $E = E_\rho$ for some representation ρ of $\pi_1(M, x_0)$.*

Proof. The previous lemma provides the "only if" direction of this result. Given E with flat D, consider the smooth projection $p : \tilde{M} \to M$ and the pullback bundle $p^*(E)$ over $\tilde{M}$. Proposition 3.3.2 provides $p^*(E)$ with a flat connection $\tilde{D} := p^*(D)$ and since $\tilde{M}$ is simply connected, Proposition 3.5.6 implies the existence of a bundle isomorphism $\psi : \tilde{M} \times \mathbb{C}^k \cong p^*(E)$. This gives the commutative diagram:



where $\pi \circ \tilde{p} \circ \psi = p \circ \tilde{\pi}$. Now, for an element $e_x = (x, v) \in E$ we find the pre-image under $\tilde{p} \circ \psi$ to be

$$(\tilde{p} \circ \psi)^{-1}(e_x) = \{(\gamma(x), \rho(\gamma)v) : \gamma \in \pi_1(M)\}$$

making this into a fibration over E which is invariant under the free transitive action of $\pi_1(M) \cong \pi_1(E)$. In fact, $\tilde{M} \times \mathbb{C}^r$ is a $\pi_1(E)$-principal bundle over E so that $\tilde{M} \times \mathbb{C}^k$ is the universal covering space of E. Hence,

$$E \cong (\tilde{M} \times \mathbb{C}^k)/\pi_1(E) = (\tilde{M} \times_\rho \mathbb{C}^r) = E_\rho$$

as required. □

Remark 3.5.9. If ρ is reducible, one finds E_ρ as the Whitney sum of lower rank bundles corresponding to the decomposition of ρ. Thus, for any theoretical purposes the only ones of concern are irreducible ones.

3.6 Moduli Spaces of connections

Generally speaking, a *moduli space* is a geometric space whose points consist of equivalence classes. For example, the class of all linear transformations on a vector space is represented as the quotient space of M_n/GL_k where GL_k acts on M_n by conjugation to represent change of basis. In the case of vector bundles, one is concerned with the moduli space of all connections on a fixed vector bundle E up to gauge symmetry.

In this section, for simplicity E is a trivial complex line bundle over a smooth manifold M. The space of connections is parameterized by

$$\Omega^1(\mathrm{End}(E)) = \mathbb{C} \otimes \Omega^1(M)$$

However, when working with Hermitian metric compatible connections, $\Omega^1(\mathrm{End}(E))$ refers to skew-Hermitian valued 1-forms. In the case of a complex line bundle, these are $i\mathbb{R}$-valued 1-forms.

Since we are working on a complex line bundle, we can find local trivializations whose transition functions take values in $U(1)$ as follows: A local frame consisting of a nowhere vanishing section can be normalized by the metric to be $U(1)$-valued. If this is performed on all frames, then the transition maps between $U(1)$-valued sections must again be $U(1)$-valued. This means gauge transformations may be expressed locally as $g : U \to U(1) = S^1$. This simplifies to be $g(p) = e^{i\chi(p)}$ where $\chi : U \to \mathbb{R}$ is a smooth function (if U_α is simply connected). Under this change of gauge, the connection forms transform as

$$A' = e^{-i\chi} A e^{i\chi} + e^{-i\chi} d e^{i\chi} = A + i d\chi.$$

This means that the space of connections $\mathcal{A}(E)$ is, up to local gauge equivalence, in correspondence with $i\mathbb{R}$-valued one forms $\Omega^1(M)$ modulo the exact one forms $d\Omega^0(M)$. That is,

$$\mathcal{A}(E) = \Omega^1(M)/d\Omega^0(M).$$

A special sub-class of these connections are those which are flat. These are found locally by the differential equation 3.3.1. Indeed,

$$0 = F = dA + A \wedge A = dA$$

where $A \wedge A = 0$ because $\text{rank}(E) = 1$. So, the flat connections up to gauge equivalence are

$$\mathcal{F}(E) = \mathcal{Z}^1(M)/d\Omega^0(M).$$

This is the first deRham cohomology group of M denoted $H^1_{dR}(M, \mathbb{R})$.

Considering equivalence up to bundle automorphism allows for further reduction of the space of connections via global gauge symmetry. Since E is trivial, the bundle automorphisms are simply $\varphi = 1_M \times \psi$ where $\psi : M \to U(1) = S^1$ is a smoothly varying map of fibre automorphisms. The homotopy classes of maps from M to S^1

can be identified with $H^1(M,\mathbb{Z})$. Thus, the moduli spaces of connections and flat connections on the trivial complex line bundle E are

$$\mathcal{M}_{\mathcal{A}(E)} = \mathcal{A}(E)/H^1(M,\mathbb{Z})$$

and

$$\mathcal{M}_{\mathcal{F}(E)} = H^1_{dR}(M)/H^1(M,\mathbb{Z}) = H^1(M,\mathbb{R})/H^1(M,\mathbb{Z})$$

respectively, where the second equality is is a result of deRham's isomorphism theorem, discussed in [8].

Example 3.6.1. The moduli space of all flat connections on the trivial complex line bundle over the 2-torus is again a 2-torus. Indeed, from the calculations above

$$\mathcal{M}_{\mathcal{F}(\mathbb{T}\times\mathbb{C})} = H^1(\mathbb{T},\mathbb{R})/H^1(\mathbb{T},\mathbb{Z}) = \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}.$$

See Allan Hatcher's book [10] for details on (co)homology computations.

Remark 3.6.2. This observation is one of the ingredients of *mirror symmetry:* where the moduli space is of the same type as the base.

3.7 Harmonic forms

A differential k-form α is *harmonic* if it is both closed and co-closed (i.e. $d\alpha = d^*\alpha = 0$ where d^* is the formal adjoint from appendix C). These forms are interesting because they are minima for a natural metric on $\Omega^k(M)$ defined using a common "averaging" technique and the Hodge star from appendix C by

$$< \alpha, \beta >:= \int_M \alpha \wedge *\beta,$$

with the property that $< d\alpha, \beta >=< \alpha, d^*\beta >$. This metric allows for an L^2-norm on the exterior forms given as usual by $||\phi||^2 =< \phi, \phi >$.

Proposition 3.7.1. *The class $[\alpha] \in H^k_{dR}(M)$ has a unique representative $\beta = \alpha + da$, for some $a \in \Omega^{k-1}M$, where β is harmonic and of minimal norm within $[\alpha]$.*

Proof. By the Hodge Theorem, there exists a unique representative $\gamma \in [\alpha]$ which minimizes $|| \cdot ||$ within $[\alpha]$. The critical points of $S(\phi) := ||\phi||^2$ are found by the variation

$$\begin{aligned} L(\epsilon) &:= S(\gamma + \epsilon da) = \int_M (\gamma + \epsilon da) \wedge *(\gamma + \epsilon da) \\ &= \int_M \gamma \wedge *\gamma + \epsilon \int_M \gamma \wedge da + \epsilon \int_M da \wedge *\gamma + \epsilon^2 \int_M da \wedge *da \\ &= ||\gamma||^2 + \epsilon < \gamma, da > + \epsilon < da, \gamma > + \epsilon^2 < da, da > \\ &= ||\gamma||^2 + 2\epsilon < \gamma, da > + \epsilon^2 < da, da > . \end{aligned}$$

Hence,

$$0 = \frac{d}{d\epsilon} L(\epsilon)_{|\epsilon=0} = 2 < \gamma, da >= 2 < d^*\gamma, a >,$$

and this holds for every $a \in \Omega^{k-1}M$, implying that $d^*\gamma$ must vanish. Knowing already that γ is closed as a member of the cohomology class $[\alpha]$ means that γ is harmonic. □

A second interesting point about harmonic forms is they yield the source free solutions to Maxwell's equations which is the topic of the next section. The leads us to believe that harmonic forms are natural solutions to systems exhibiting only the forces of nature.

3.8 Application to electromagnetisim

A fundamental fact in the theory of electricity and magnetism is that a time varying magnetic field gives rise to the presence of an electrical field. A set of four differential equations in Minkowski space $E^{1,3} = (\mathbb{R}^4, \eta)$, where $\eta = -dt^2 + dx^2 + dy^2 + dz^2$, governing these fields is given by:

1. $curlE - \frac{\partial B}{\partial t} = 0$ (Farady's law of induction);

2. $divB = 0$ (Gauss' magnetic law);

3. $divE = \rho$ (Gauss' law);

4. $curlB - \frac{\partial E}{\partial t} = j$ (Ampère's current law),

where E, B are the electric and magnetic fields determined by ρ, the *electric charge density*, and $j = (j_x, j_y, j_z)$ the *electric current density* of space.

These equations are classically known as *Maxwell's equations*, although it was Oliver Heaviside who first expressed them in this manner. It turns out that these electromagnetic fields may be encoded as self dual connections on the trivial complex line bundle $E^{1,3} \times \mathbb{C}$. Indeed, consider the 2-form

$$F = dt \wedge (E \cdot d\vec{r}) + B \cdot d\mathbf{S}$$

where

$$E \cdot d\vec{r} = < (E^1, E^2, E^3), (dx, dy, dz) >$$

and

$$B \cdot d\mathbf{S} = < (B^1, B^2, B^3), (dydz, dzdx, dxdy) >$$

whose exterior derivative is

$$dF = dt \wedge \left(\frac{\partial B}{\partial t} - curl\, E \right) \cdot d\mathbf{S} + div\, B \cdot dV.$$

The vanishing of dF is exactly the result of equations (1) and (2) (called the *homogeneous parts* of Maxwell's equations).

Now, the second pair of equations appear to be dual to the first with a non-homogeneous twist. Using the Hodge star defined in Appendix C, one finds

$$*F = dt \wedge B \cdot d\vec{r} + E \cdot d\mathbf{S}$$

and that equations (3) and (4) are satisfied precisely when $d * F = -J$ with $J = dt \wedge (-j \cdot dS) + \rho \cdot dV$.

Therefore, Maxwell's equations, when expressed in this fashion, become

$$(1) \quad dF = 0 \qquad (2) \quad d * F = -J. \tag{3.8.1}$$

Since F is closed in $\Omega^2(E^{1,3})$, the Poincaré Lemma implies it is locally exact, meaning that $F = dA$ for some $A \in \Omega^1(M)$. Now, Maxwell's equations have been encoded as a connection on a complex line bundle over $E^{1,3}$ with curvature $F = dA$.

Interestingly enough and not surprisingly, one finds that the curvature of the connections satisfying the *source-free* equations 3.8.1 (i.e. where $J = 0$), are precisely the harmonic 2-forms $\mathcal{H}^2(E^{1,3})$ and hence minimize the L^2 norm defined on 2-forms.

Remark 3.8.1. This functional is analogous to the one used for geodesics, where the domain is now the moduli space of connections instead of paths between points. The notion of length here is intended to represent a total energy which is observed in nature to be minimal.

Chapter 4

Yang-Mills Functional

Consider a connection form A on a vector bundle E over an orientable Riemannian manifold (M, g) of dimension n. Write $d_A = d + A$ for the connection corresponding to such a form. As in the case of differential forms on M, there is a natural L^2 metric defined on $\Omega^k(\text{End}(E))$ by

$$< \alpha, \beta >:= -\int_M \text{Tr}\ (\alpha \wedge *\beta).$$

This is usually given by $< \alpha, \beta >= \int_M \text{Tr}\ (\bar{\alpha}^T \wedge *\beta)$, but considering unitary connections $\bar{A}^T = -A$. This metric has the property that $< d_A\alpha, \beta >=< \alpha, d_A^*\beta >$, where $d_A^* = (-1)^{nk-n-1} * d_A *$ is the formal adjoint of d_A computed as follows: Let

$\alpha \in \Omega^{k-1}(M)$, $\beta \in \Omega^k(M)$ then by Stokes' theorem

$$\begin{aligned}
0 &= \int_M d_A(\alpha \wedge *\beta) \\
&= \int_M d_A\alpha \wedge *\beta + (-1)^{k-1} \int_M \alpha \wedge d_A * \beta \\
&= < d_A\alpha, \beta > +(-1)^{k-1+(n-k+1)(n-(n-k+1))} \int_M \alpha \wedge * * d_A * \beta \\
&= < d_A\alpha, \beta > +(-1)^{nk-k^2+k+k-n+k-2} < \alpha, *d_A * \beta > \\
&= < d_A\alpha, \beta > +(-1)^{nk-n-2} < \alpha, *d_A * \beta > .
\end{aligned}$$

This is similar to the computation of d^* in Appendix C.

The *Yang-Mills functional* is defined on $\Omega^1(\mathrm{End}(E))$ by

$$YM(A) := \frac{1}{2}||F_A||^2 = -\frac{1}{2}\int_M \mathrm{Tr}\ (F_A \wedge *F_A). \tag{4.0.1}$$

The critical values of YM are the solutions to the *Yang-Mills equations* expressed most simply by

(1) $d_AF_A = 0$,

(2) $d_A * F_A = 0$.

Equation (1) is the Bianchi Identity which is vacuously true. Equation (2) is found using a similar variation as in 3.7.1, where

$$\begin{aligned}
F_{A+\epsilon a} &= d(A + \epsilon a) + (A + \epsilon a) \wedge (A + \epsilon a) \\
&= dA + A \wedge A + \epsilon(da + a \wedge A + A \wedge a) + \epsilon^2 a \wedge a \\
&= F_A + \epsilon d_A(a) + \epsilon^2 a \wedge a,
\end{aligned}$$

so that

$$< F_{A+\epsilon a}, F_{A+\epsilon a} >=< F_A, F_A > +2\epsilon < F_A, d_A(a) > +\mathcal{O}(\epsilon^2).$$

Now, the critical points of YM are found by

$$0 = \frac{d}{d\epsilon} YM(A+\epsilon a)|_{\epsilon=0} = 2 < F_A, d_A(a) >= 2 < d_A^*(F_A), a > .$$

Since this equation holds for any $a \in \Omega^1(\text{End}(E))$, this forces $d_A^*(F_A) = 0$. This is equivalent to equation (2) by applying the Hodge star to both sides of the equation. So the critical points of the Yang-Mills functional are the connections with harmonic curvature with respect to the operation d_A. A connection satisfying the Yang-Mills equations is called a *Yang-Mills connection.*

Example 4.0.2. 1. The flat connections are always Yang-Mills.

2. When M is an orientable 4-dimensional Riemannian manifold, there is an orthogonal decomposition of $\Lambda^2(M)$ into eigen-spaces of the Hodge-star operator $*$. Luckily, in this setting $*$ becomes a linear idempotent operator upon restriction to Λ^2. This leads to a decomposition of the curvature tensors. Indeed, sections of 2-forms are locally expressed as smooth combinations of $\{dx^{ij} | 0 \leq i < j \leq 3\}$ where $dx^{ij} := dx^i \wedge dx^j$. Upon restriction to Λ^2, $*$ satisfies

$$*_{|\Lambda^2} : \Lambda^2(M) \to \Lambda^{4-2}(M) = \Lambda^2(M)$$

and

$$*^2 = (-1)^{2(4-2)}\mathbf{1} = \mathbf{1}.$$

This implies * has eigenvalues ± 1, with eigen-spaces

$$\Lambda^2_{\pm}(M) = \{dx^{01} \pm dx^{23}, dx^{02} \pm dx^{31}, dx^{03} \pm dx^{12}\},$$

and gives the decomposition

$$\Lambda^2(M) = \Lambda^2_{+}(M) \oplus \Lambda^2_{-}(M).$$

These pieces are known as *self dual (SD)* and *anti-self dual (ASD)* 2-forms, respectively.

From the perspective of connections and curvature, a connection D on E over M is *(anti-)self dual* if its curvature lies in one of the $\mathrm{End}(E) \otimes \Lambda^2_{\pm}(M)$.

The self-dual and anti-self dual connections are Yang-Mills since $*F_A = \pm F_A$ so that equation (2) reduces to (1) which always holds. These solutions are called the *instantons.*

3. Suppose there exists a closed (n-4)-form φ on M and $*F_A = F_A \wedge \varphi$, then F_A is Yang-Mills. This holds because

$$d_A * F_A = d_A(F_A \wedge \varphi) = d_A F_A \wedge \varphi + (-1)^2 F_A \wedge d\varphi = 0.$$

The next chapter is dedicated to discussing special 7-manifolds on which such a form exists. These connections will be called *generalized instantons.*

With critical values of YM shown to be solutions of $d_A^* F_A = 0$, it remains to show that, in important cases (such as the examples above), they are in fact minima. In order to do this, the notion of calibrations must be introduced.

Using a fixed form $\Phi \in \Omega^{n-4}(M)$, one may define a quadratic form $q_\Phi : \Omega^2(M) \to C^\infty(M)$ by $q_\Phi(\phi) := *(\phi \wedge \phi \wedge \Phi)$.

Note that q_Φ extends naturally to $\Lambda^2(\mathrm{End}(E))$ by composing with the trace operator. For example $q_\Phi(F_A) = \mathrm{Tr}\ (F_A \wedge F_A) \wedge \Phi$.

A *Yang-Mills calibrating form* is defined to be a closed $\Phi \in \Omega^{n-4}(M)$ satisfying $q_\Phi(\phi) \leq |\phi|^2$ for each $\phi \in \Lambda^2(M)$ where $|\phi|^2 = *(\phi \wedge *\phi)$. Given a Yang-Mills calibrating form Φ, a connection A on a vector bundle E over M is called Φ-*calibrated* if $q_\Phi(F_A) = |F_A|^2$.

The following result, extracted from [19] implies the above examples in fact minimize YM.

Lemma 4.0.3. *If A is a Φ-calibrated connection on E and A' is any other connection then*

$$YM(A') \geq YM(A)$$

Moreover, if $YM(A') = YM(A)$ then A' is also Φ-calibrated.

Proof. Referring to Lemma 4.4.6 from [11], if $\tilde{P}$ is any homogeneous polynomial of degree k, then $\tilde{P}(F_{A'}) = \tilde{P}(F_A) + d\alpha$. $\mathrm{Tr}\ (F_A^2)$ is a homogeneous polynomial of degree 2, so we get that $\mathrm{Tr}\ (F_{A'}^2) = \mathrm{Tr}\ (F_A^2) + d\alpha$. Now,

$$YM(A') = \int_M |F_{A'}|^2 dx^{1\cdots n} \geq \int_M \mathrm{Tr}\ (F_{A'}^2) \wedge \Phi$$

by the calibration property of Φ. Using the above identity, this evaluates to

$$\begin{aligned}\int_M \text{Tr}\,(F_A^2)\wedge\Phi + \int_M d\alpha\wedge\Phi &= \int_M \text{Tr}\,(F_A^2)\wedge\Phi + \underbrace{\int_M d(\alpha\wedge\Phi)}_{=0} + \int_M \alpha\wedge\underbrace{d\Phi}_{=0} \\ &= \int_M \text{Tr}\,(F_A^2)\wedge\Phi\end{aligned}$$

where the first vanishing term is due to Stokes' Theorem and the second because Φ is closed. Finally, since A is calibrated, this is equivalent to

$$\int_M |F_A|^2 dx^{1\cdots n} = YM(A)$$

as claimed. □

Example 4.0.4.

1. The flat connections are precisely the 0-calibrated connections. This holds because $0 \in \Lambda^{n-4}(M)$ is closed, $q_0 = 0 \leq |\phi|^2$ and certainly $|F_A|^2 = 0$ implies $F_A = 0$.

2. The ASD connections on a four manifold are the 1-calibrated connections. Indeed, 1 is closed in $\Lambda^0(E) = \Lambda^{4-4}(E)$ and $q_1(\phi) = \phi\wedge\phi/\mathbf{vol}_M \leq |\phi|^2$ is proven using the decomposition of two forms as follows:

   $$|\phi|^2 =< \phi^+ + \phi^-, \phi^+ + \phi^- >=< \phi^+, \phi^+ > + < \phi^-, \phi^- >= |\phi^+|^2 + |\phi^-|^2$$

   and

   $$\phi\wedge\phi = (\phi^+ + \phi^-)\wedge(*\phi^+ - *\phi^-) = (|\phi^+|^2 - |\phi^-|^2)\mathbf{vol}_M,$$

which certainly satisfies the above inequality upon division by the volume form.

Now, the 1-calibrated connections satisfy $\mathrm{Tr}\,(F_A \wedge F_A) = -\mathrm{Tr}\,(F_A \wedge *F_A)$ forcing $*F_A = -F_A$, meaning A is ASD.

Similarly, the SD connections are -1-calibrated.

3. Donaldson-Thomas connections on G_2 manifolds which will be introduced in the next chapter. They are φ-calibrated connections where φ will be defined in equation B.0.1. This is shown using a decomposition of 2-forms similar to example 2.

Chapter 5

Higher-Dimensional generalizations

An interesting class of connection over orientable Riemannian 7-manifolds, $(M^7, g, \mathbf{vol})$, are those which preserve an octonion cross product structure described in appendix A. These manifolds are known as G_2-manifolds and admit the exact structure necessary for generalized solutions to the Yang-Mills equations. Moreover, these connections may be defined on arbitrary vector bundles where the base is a G_2-manifold.

5.1 G_2-structures and manifolds with G_2 holonomy

Essentially, a G_2-structure on a smooth 7-manifold M is a differential 3-form φ expressed locally by equation B.0.1, encoding the octonionic cross product described in appendix A on the tangent bundle TM. More precisely,

Definition 5.1.1. M has a *G_2-structure* if there exists a smooth metric g, orientation **vol**, cross product $\times$ and 3-form $\varphi \in \Lambda^3(M)$ such that $\varphi(u, v, w) = g(u \times v, w)$ and, for every $p \in M$, we have

$$(T_pM, g_p, \times_p, \varphi_p) \cong (\mathbb{R}^7, g_0, \times_0, \varphi_0).$$

This is analogous to the almost complex structure J which exists on almost complex manifolds. It is not true that an almost complex structure always exists on even dimensional manifolds, for example Steenrod showed in 1951 that S^4 does not admit an almost complex structure.

From here it is natural to consider manifolds which admit G_2-structures that are parallel with respect to a special connection on M.

Lemma 5.1.2. *A G_2-structure on M exists if and only if M is orientable and spin. In other words, the existence of a G_2-structure on M is equivalent to the vanishing of the first two Steifel-Whitney classes of TM.*

Remark 5.1.3. The first Stiefel-Whitney class $\omega_1(TM) \in H^1(M, \mathbb{Z}_2)$ determines the orientability and the second Stiefel-Whitney class determines the existence of a spin structure on M. See [23] for complete exposition on these $\mathbb{Z}_2$ cohomology classes associated to real vector bundles.

This is analogous to the almost complex structure for even dimensional manifolds in the sense that one constructs a tensor $J \in \Gamma(T^*M \otimes TM)$ in the case of complex and $\varphi \in \Gamma(\Lambda^3_+(M))$ for G_2. One further defines an almost complex manifold to be complex if the Nijenhuis tensor N_J vanishes. Further, a complex manifold M is Kähler if and only if the almost complex structure J is parallel with respect to the Levi-Civita connection on M.

Definition 5.1.4. Let (M, φ) be manifold with G_2-structure. Then M is a G_2-*manifold* if φ is parallel with respect to the Levi-Civita connection ∇_φ corresponding to the metric g_φ.

Having a parallel G_2-structure means the holonomy of M is contained within G_2. Some of the few known examples of G_2-manifolds are $X \times S^1$, where X is a Calabi-Yau 3-fold (i.e., a 3-dimensional complex manifold with holonomy contained in $SU(3)$) or $Y \times \mathbb{T}^3$ with Y a $K3$ surface (i.e., a simply connected, compact, complex surface with trivial canonical bundle). These product manifolds are examples of *reducible* G_2-manifolds because their holonomy is properly contained in G_2. Examples of *irreducible* G_2-manifolds (i.e., having holonomy exactly equal to G_2) are few and far between. The first complete non-compact examples were constructed by Bryant and Salamon in [4]. These are $\Lambda^2_-(\mathbb{C}P^2), \Lambda^2_-(S^4)$ and $S^3 \times \mathbb{R}^4$. The first compact examples were constructed by Joyce in [12, 13] and are summarized into four steps in section 11.3 of [14]. The construction is based on the Kummer construction for Calabi-Yau metrics on $K3$ surfaces.

An important result taken from [14] (Proposition 11.1.3) regarding the parallelizability of a G_2-structure on M is

Proposition 5.1.5. *Let φ be a G_2 structure on M. Then $\nabla_\varphi \varphi = 0$ if an only if φ satisfies both $d\varphi = 0$ and $d^*\varphi = 0$.*

5.2 Donaldson-Thomas connections

Having this parallel positive three form φ is the key to define the proper connections necessary to discuss solutions to the Yang-Mills functional. This parallel G_2-structure may now be wedged with the usual Hodge star to construct a diagonalizable operator on $\Lambda^2(M)$ as follows:

$$*_\varphi := *(\cdot \wedge \varphi) : \Lambda^2(M) \to \Lambda^5(M) \to \Lambda^2(M)$$

by $*_\varphi(\alpha) := *(\alpha \wedge \varphi)$ where $*$ is the usual Hodge-star.

The 21×21 matrix representing $*_\varphi$ is found to have eigen values 2 and -1 with eigen spaces of dimension 7 and 14 respectively. Thus, the 2-forms on any G_2-manifold decompose as

$$\Lambda^2(M) = \Lambda^2_{14} \oplus \Lambda^2_7,$$

where Λ^2_7 is locally spanned by

$$\{dx^{12} - dx^{47} - dx^{56}, dx^{46} - dx^{13} - dx^{57}, dx^{37} - dx^{15} - dx^{26},$$
$$dx^{16} - dx^{25} - dx^{34}, dx^{24} - dx^{17} - dx^{35}, dx^{23} + dx^{45} + dx^{67}, dx^{14} - dx^{27} - dx^{36}\}$$

and Λ^2_{14} by

$$\{dx^{47} + dx^{12}, dx^{47} - dx^{56}, dx^{13} + dx^{46}, dx^{13} - dx^{57}, dx^{15} + dx^{37},$$
$$dx^{15} - dx^{26}, dx^{25} + dx^{16}, dx^{25} - dx^{34}, dx^{17} + dx^{24}, dx^{17} - dx^{35},$$
$$dx^{23} - dx^{45}, dx^{23} - dx^{67}, dx^{14} - dx^{27}, dx^{14} - dx^{36}\}.$$

Definition 5.2.1. Let E be a vector bundle over a G_2 manifold (M, φ). A connection A on E is a *Donaldson-Thomas connection* if its curvature tensor lies entirely within $\mathrm{End}(E) \otimes \Lambda^2_{14}$.

It will be important to keep in mind

Lemma 5.2.2. *A connection A on a vector bundle E over a G_2-manifold is Donaldson-Thomas if and only if $*F_A = -F_A \wedge \varphi$.*

Proof. $F_A \in \mathrm{End}(E) \otimes \Lambda^2_{14}(M)$ if and only if $*(F_A \wedge \varphi) = -F_A$. This is equivalent to $*F_A = -F_A \wedge \varphi$ by applying $*$ to both sides and multiplying by -1. □

The statement, from example 4.0.4, that Donaldson-Thomas connections are $-\varphi$-calibrated is now simple to see since

$$\begin{aligned} q_{-\varphi}(\phi \wedge \phi) &= (\phi_7 + \phi_{14}) \wedge (-\phi_7 \wedge \varphi - \phi_{14} \wedge \varphi) \\ &= (\phi_7 + \phi_{14}) \wedge (-2 * \phi_7 + *\phi_{14}) \\ &= |\phi_{14}|^2 - 2|\phi_7|^2 \leq |\phi_{14}|^2 + |\phi_7|^2 = |\phi|^2. \end{aligned}$$

Thus, a $-\varphi$-calibrated connection must satisfy $\mathrm{Tr}\,(F_A \wedge F_A) \wedge \varphi = -\mathrm{Tr}\,(F_A \wedge *F_A)$ forcing $*F_A = -F_A \wedge \varphi$. Equivalently, A is a Donaldson-Thomas connection.

Few explicit examples of Donaldson-Thomas connections are known but here are a few.

Example 5.2.3.

1. Flat connections are trivially Donaldson-Thomas.

2. The Levi-Civita connection on a G_2 manifold is Donaldson-Thomas. The result of Theorem 3.1.7 from [14] states the Riemannian curvature (i.e. the curvature of the Levi-Civita connection) lies in $Sym^2(Lie(hol(M)))$. When M is a G_2-manifold, the Lie algebra of G_2 is $\Lambda^2_{14}(M)$. Hence, the Riemannian curvature lies in $Sym^2(\Lambda^2_{14}(M)) \subseteq End(TM) \otimes \Lambda^2_{14}(M)$, implying our claim. This is computed explicitly in Corollary 4.7 of [16].

3. An interesting problem is: do Donaldson-Thomas connections on $\mathbb{R}^7 = \mathbb{R}^4 \times \mathbb{R}^3$ which only depend on the first four variables reduce to instantons on $\mathbb{R}^4$? A connection of this type may be expressed as $A = \sum_{i=1}^4 A^i dx^i + \sum_{i=5}^7 A^i dx^i$ where each A^i is a function of x_1, x_2, x_3, x_4 subject to the constraint $F_A \wedge \varphi = *F_A$. It is straight forward but tedious to extract the 21 coefficients of this equation using

$$\begin{aligned} F_A &= dA + A \wedge A \\ &= \sum_{1 \le i < j \le 4} \left(\frac{\partial A^i}{\partial x^j} - \frac{\partial A^j}{\partial x^i} \right) dx^{ij} + \sum_{j=5}^{7} \sum_{i=1}^{4} \frac{\partial A^j}{\partial x^i} dx^{ij} + \sum_{1 \le i < j \le 7} [A^i, A^j] dx^{ij}. \end{aligned}$$

In the case $A^5 = A^6 = A^7 = 0$, this reduces to

$$F_A = \sum_{1 \le i < j \le 4} \left(\frac{\partial A^i}{\partial x^j} - \frac{\partial A^j}{\partial x^i} + [A^i, A^j] \right) dx^{ij}$$

and our constraints reveal the six independent equations

$$\frac{\partial A^i}{\partial x^j} - \frac{\partial A^j}{\partial x^i} + [A^i, A^j] = 0$$

for each $1 \leq i < j \leq 4$ implying the connection must be flat. This means that the only instantons when pulled back to connections on $\mathbb{R}^7$ that are Donaldson-Thomas are the flat ones. Further investigation when the last three terms are non-zero should lead to higher dimensional reduction for the $N = 4$ instanton equations.

Appendices

Appendix A

Cross products in $\mathbb{R}^7$

Using the standard inner product $g =< \cdot, \cdot >$ and orientation $dx^n = dx^1 \wedge \cdots \wedge dx^n$, one may construct a *2-fold cross product* $\times$, that 'plays nicely' with the multiplicative structure of the octonions. Beginning in the same fashion as $\mathbb{R}^3$ by the constraints that

$$\times : \mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}^7$$

be an alternating, bilinear map satisfying

$$g(u \times v, u) = g(u \times v, v) = 0$$

and

$$|u \times v|^2 + g(u, v)^2 = |u|^2 |v|^2.$$

It is simple to check that two vectors $a, b \in Im(\mathbb{O})$ satisfy

$$a \cdot b = -g(a, b) \cdot \mathbf{1} + a \times b$$

meaning the cross product is precisely the imaginary component of the product $a \cdot b$ as octonions. This is analogous to the cross product on $\mathbb{R}^3$ induced through the multiplicative structure of the quaternions.

Appendix B

G_2 the group

Consider the action of $\mathrm{GL}_7(\mathbb{R})$ on $\Lambda^3(\mathbb{R}^7)$ and define G_2 as the stabilizer of the element

$$\varphi_0 = dx^{123} + dx^{145} + dx^{167} + dx^{246} - dx^{257} - dx^{347} - dx^{356} \tag{B.0.1}$$

where $dx^{ijk} := dx^i \wedge dx^j \wedge dx^k$.

This form, when examined explicitly evaluates as $\varphi_0(a, b, c) = g(a \times b, c)$ where g is an inner product and $\times$ is the 2-fold cross product on $\mathbb{R}^7$ defined in A.

One may recover the metric g_ψ and volume form $dV \in \Lambda^7(\mathbb{R}^7)$ by

$$-6g_\psi(X, Y)dV := (X \lrcorner \psi) \wedge (Y \lrcorner \psi) \wedge \psi. \tag{B.0.2}$$

Also, the cross product is recovered by

$$(x \times y)^\flat = y \lrcorner x \lrcorner \varphi$$

where the flat represents the metric dual tangent vector.

This shows that g_ψ is defined non-linearly in terms of our φ_0 with entries

$$[g_{\varphi_0}]_{ij} = \frac{1}{6^{2/9}} \frac{B_{ij}}{\det(B)^{1/9}}$$

where

$$B_{ij} := \left((\frac{\partial}{\partial x^i} \lrcorner \varphi) \wedge (\frac{\partial}{\partial x^j} \lrcorner \varphi) \wedge \varphi \right) (\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^7}).$$

For computational details of this fact, see [16].

Proposition B.0.4. *G_2 is a compact, connected, simply connected, 14 dimensional, simple real Lie subgroup of $SO(7)$.*

Proof. This is just a rough sketch of some of the ideas involved in working with G_2. A complete in depth proof of this result is found in [2].

Since g_{φ_0} and dV are expressed as a function of φ_0 using equation B.0.2, it follows that G_2 fixes the metric and orientation. This means G_2 is a real subgroup of $SO(7)$. For proper inclusion, it suffices to mention that G_2 fixes the cross product as well which is not true of all members of $SO(7)$.

To see $\dim(G_2) = 14$ is a constructive argument similar to that of the orthogonal group. Indeed, if $A \in G_2$ then the first column a_1 is is only required to be unit length and hence a member of S^6. To choose a_2 the only new requirement is that a_2 be unit length and orthogonal to a_1 meaning $a_2 \in S^5$. Now, $a_3 = a_1 \times a_2$ to ensure preservation of $\times$. a_4 is now chosen from the unit length vectors in the orthogonal compliment of a_1, a_2, a_3 meaning freedom of choice from S^3. Finally, the last three vectors are uniquely determined as the cross products $a_5 = a_1 \times a_4, a_6 = a_2 \times a_4, a_7 = a_3 \times a_4$ hence the dimension of G_2 is $\dim S^6 + \dim S^5 + \dim S^3 = 6 + 5 + 3 = 14$. □

Appendix C

Hodge star on orientable manifolds

On any orientable Riemannian manifold M, there is a nice linear operator defined on $\Lambda(M)$. Using the natural metric induced on $\Lambda^k(M)$, the *Hodge star*

$$* : \Lambda^k(E) \to \Lambda^{n-k}(E)$$

is defined uniquely by the requirement that

$$\omega \wedge *\tau = < \omega, \tau > dV$$

for all $\omega, \tau \in \Lambda^k M$, where $dV = \sqrt{|g|} dx^{1 \cdots n}$ is the *volume form* of M and $< \cdot, \cdot >$ on $\Lambda^k M$ is the natural induced metric described in [21].

Lemma C.0.5. *[Fundamental properties of *] For* $\alpha, \beta \in \Lambda^k(M)$

(i) $*1 = dV$ *and* $*dV = 1$

(ii) $< \alpha, \beta > = < *\alpha, *\beta >$

(iii) $*^2 = (-1)^{k(n-k)}$ *on* $\Lambda^k(M)$

(iv) $\alpha \wedge *\beta = \beta \wedge *\alpha$

Proof.

(i) $*1 \in \Lambda^n(M)$ which means $*1 = f \cdot dV$ for some $f \in \Gamma(E)$. Then,

$$f \cdot dV = 1 \wedge *1 = < 1, 1 > dV = dV \Rightarrow f \equiv 1.$$

Similarly, $*dV = 1$.

(ii) By linearity of $*$, it suffices to prove for basis vectors $dx^{i_1 \cdots i_k}$ where $1 \leq i_1 < \cdots < i_k \leq n$. Now, $< dx^{i_1 \cdots i_k}, dx^{j_1 \cdots j_k} > = \delta_{i_1, \ldots, i_k, j_1, \ldots, j_k}$ and similarly $< *dx^{i_1 \cdots i_k}, *dx^{j_1 \cdots j_k} > = < dx^{i'_1 \cdots i'_{n-k}}, dx^{j'_1 \cdots j'_{n-k}} > = \delta_{i'_1 \cdots i'_{n-k}, j'_1 \cdots j'_{n-k}}$ where $\{i'_1, \ldots, i'_{n-k}\} = \{1, \ldots, n\} \backslash \{i_1, \ldots, i_k\}$. It is simple to see that these evaluations are equivalent since $i_1, \ldots, i_k = j_1, \ldots, j_k$ if and only if their compliments are equal.

(iii) For each $\omega, \tau \in \Lambda^k M$ we have

$$\begin{aligned} *\omega \wedge * * \tau &= < *\omega, *\tau > dV \\ &= < \omega, \tau > dV \\ &= < \tau, \omega > dV \\ &= \tau \wedge *\omega \\ &= (-1)^{k(n-k)} * \omega \wedge \tau \end{aligned}$$

Where the last equality comes from $k(n-k)$ transpositions in swapping a k-form with an $n-k$ form. Thus, $*^2 = (-1)^{k(n-k)}$.

(iv) because $< \alpha, \beta >=< \beta, \alpha >$

$\square$

The *formal adjoint* of d is defined and calculated in chapter 14 of [21] to be $d^* = (-1)^{nk+n+1} * d*$ on $\Lambda^k(M)$ by simple properties of metric and the above lemma.

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