Matrix Representations and Extension of the Graph Model for Conflict Resolution

by

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A thesis
presented to the University of Waterloo
in fulfilment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Systems Design Engineering

Waterloo, Ontario, Canada, 2009

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The graph model for conflict resolution (GMCR) provides a convenient and effective means to model and analyze a strategic conflict. Standard practice is to carry out a stability analysis of a graph model, and then to follow up with a post-stability analysis, two critical components of which are status quo analysis and coalition analysis. In stability analysis, an equilibrium is a state that is stable for all decision makers (DMs) under appropriate stability definitions or solution concepts. Status quo analysis aims to determine whether a particular equilibrium is reachable from a status quo (or an initial state) and, if so, how to reach it. A coalition is any subset of a set of DMs. The coalition stability analysis within the graph model is focused on the status quo states that are equilibria and assesses whether states that are stable from individual viewpoints may be unstable for coalitions. Stability analysis began within a simple preference structure which includes a relative preference relationship and an indifference relation. Subsequently, preference uncertainty and strength of preference were introduced into GMCR but not formally integrated.

In this thesis, two new preference frameworks, hybrid preference and multiple-level preference, and an integrated algebraic approach are developed for GMCR. Hybrid preference extends existing preference structures to combine preference uncertainty and strength of preference into GMCR. A multiple-level preference framework expands GMCR to handle a more general and flexible structure than any existing system representing strength of preference. An integrated algebraic approach reveals a link among traditional stability analysis, status quo analysis, and coalition stability analysis by using matrix representation of the graph model for conflict resolution.

To integrate the three existing preference structures into a hybrid system, a new preference framework is proposed for graph models using a quadruple relation to express strong or mild preference of one state or scenario over another,

equal preference, and an uncertain preference. In addition, a multiple-level preference framework is introduced into the graph model methodology to handle multiple-level preference information, which lies between relative and cardinal preferences in information content. The existing structure with strength of preference takes into account that if a state is stable, it may be either strongly stable or weakly stable in the context of three levels of strength. However, the three-level structure is limited in its ability to depict the intensity of relative preference. In this research, four basic solution concepts consisting of Nash stability, general metarationality, symmetric metarationality, and sequential stability, are defined at each level of preference for the graph model with the extended multiple-level preference. The development of the two new preference frameworks expands the realm of applicability of the graph model and provides new insights into strategic conflicts so that more practical and complicated problems can be analyzed at greater depth.

Because a graph model of a conflict consists of several interrelated graphs, it is natural to ask whether well-known results of Algebraic Graph Theory can help analyze a graph model. Analysis of a graph model involves searching paths in a graph but an important restriction of a graph model is that no DM can move twice in succession along any path. (If a DM can move consecutively, then this DM's graph is effectively transitive. Prohibiting consecutive moves thus allows for graph models with intransitive graphs, which are sometimes useful in practice.) Therefore, a graph model must be treated as an edge-weighted, colored multidigraph in which each arc represents a legal unilateral move and distinct colors refer to different DMs. The weight of an arc could represent some preference attribute. Tracing the evolution of a conflict in status quo analysis is converted to searching all colored paths from a status quo to a particular outcome in an edge-weighted, colored multidigraph. Generally, an adjacency matrix can determine a simple digraph and all state-by-state paths between any two vertices. However, if a graph model contains multiple arcs between the same two states controlled by different DMs, the adjacency matrix would be unable to track all aspects of conflict evolution from the status quo. To bridge the gap, a conversion function using the matrix representation is designed to transform the original problem of searching edge-weighted, colored paths in a colored multidigraph to a standard problem of finding paths in a simple digraph with no color constraints. As well, several unexpected and useful links among status quo analysis, stability analysis, and coalition analysis are revealed using the conversion function.

The key input of stability analysis is the reachable list of a DM, or a coalition, by a legal move (in one step) or by a legal sequence of unilateral moves, from a status quo in 2-DM or n-DM ($n > 2$) models. A weighted reachability matrix for a DM or a coalition along weighted colored paths is designed to construct the reachable list using the aforementioned conversion function. The weight of each edge in a graph model is defined according to the preference structure, for example, simple preference, preference with uncertainty, or preference with strength. Furthermore, a graph model and the four basic graph model solution concepts are formulated explicitly using the weighted reachability matrix for the three preference structures. The explicit matrix representation for conflict resolution (MRCR) that facilitates stability calculations in both 2-DM and n-DM ($n > 2$) models for three existing preference structures. In addition, the weighted reachability matrix by a coalition is used to produce matrix representation of coalition stabilities in multiple-decision-maker conflicts for the three preference frameworks.

Previously, solution concepts in the graph model were traditionally defined logically, in terms of the underlying graphs and preference relations. When status quo analysis algorithms were developed, this line of thinking was retained and pseudo-codes were developed following a similar logical structure. However, as was noted in the development of the decision support system (DSS) GMCR II, the nature of logical representations makes coding difficult. The DSS GMCR II, is available for basic stability analysis and status quo analysis within simple preference, but is difficult to modify or adapt to other preference structures. Compared with existing graphical or logical representation, matrix representation for conflict resolution (MRCR) is more effective and convenient for computer implementation and for adapting to new analysis techniques. Moreover, due to an inherent link between stability analysis and post-stability analysis presented, the proposed algebraic approach establishes an integrated paradigm of matrix representation for the graph model for conflict resolution.

Acknowledgments

My first and foremost acknowledgement is to the University of Waterloo, which has not only provided a nurturing environment for my husband and me, but also, is bringing hope and opportunities to my daughter. Two years ago, my husband was awarded his PhD degree by the University of Waterloo. From this fall, my daughter will start her undergraduate study in this amazing university. Today, my journey at the University of Waterloo is coming to an end. This university continues to provide my family with vast knowledge and countless opportunities; it is my sincere hope that my family can make positive contributions to the University of Waterloo in the future.

I next would like to express my genuine gratitude to my supervisors, Professor Keith W. Hipel and Professor D. Marc Kilgour. I am extremely grateful for their guidance for my smooth transition from a background in mathematics to a specialization in engineering. In fact, all my progress in Engineering is owing to their help and support. It is almost impossible to express in words how much help they have provided.

I would also like to convey my thanks to Professor Jian Chen from Tsinghua University, the most famous university in China. I am honored that Professor Chen agreed to be my external examiner. I really appreciate my PhD comprehensive committee members, Professor Xinzhi Liu, Professor Paul Calamai, and Professor Gordon J. Savage, who have provided extensive help in improving my English and refining my research. Their valuable comments and suggestions have improved my work greatly.

I wish to extend my gratitude to my friends in the Faculty of Engineering and Faculty of Mathematics for their sincere friendship and professional support. In addition, from other departments and universities, I am especially thankful for assistance from Joan Kilgour, Mary Mcpherson, Kevin W. Li, and Ye Richard Chen. I have benefited greatly from their patient instructions and constructive advice.

Sincere thanks go to the Natural Science and Engineering Research Council of Canada and Ontario Graduate Scholarship for providing me with full financial support throughout my PhD study. I also wish to express my appreciation to the Department of Systems Design Engineering for providing me with a friendly research environment.

I wish to express my deepest gratitude to my family. I am greatly indebted to my parents for their unchanging love and constant encouragement. Thanks for my husband Ju Jiang for his patience, support, and understanding throughout these years of our marriage. My daughter, Yangzi Jiang, is probably the most important and special person in my life. Thanks to her for bringing me much joy and happiness during these hard years.

Dedication

I dedicate this dissertation to my daughter−Yangzi Jiang.

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List of Acronyms

Acronyms	**Description of acronyms**
DM	Decision Maker
DSS	Decision Support System
GMCR	Graph Model for Conflict Resolution
GMCR II	A Decision Support System for the Implementation of GMCR
GMR	General Metarationality
SMR	Symmetric Metarationality
SEQ	Sequential Stability
UM	Unilateral Move
UI	Unilateral Improvement
UIUUM	Unilateral Improvement or Unilateral Uncertain Move
WI	Unilateral Mild Improvement or Strong Improvement (Weak Improvement)
WIUUM	Weak Improvement or Unilateral Uncertain Move
MRCR	Matrix Representation for Conflict Resolution
MRSC	Matrix Representation of Solution Concepts
MRSCU	Matrix Representation of Solution Concepts with Preference Uncertainty
MRSCS	Matrix Representation of Solution Concepts with Strength of Preference
MRSQA	Matrix Representation of Status Quo Analysis
MRCSA	Matrix Representation of Coalition Stability Analysis
MRCR-DSS	A Decision Support System for the Implementation of MRCR

Chapter 1

Introduction

Strategic conflict arises in diverse contexts, including environmental management and the economic, political, and personal relationships among individuals and organizations. The problem of how to solve strategic conflict has been investigated within many disciplines including international relations, psychology, and law, as well as from mathematical and engineering perspectives [5, 16, 35, 59]. Among the formal methodologies that address strategic conflict, the graph model for conflict resolution (GMCR) [41] provides a remarkable combination of simplicity and flexibility. The graph model provides an effective means to model and analyze stabilities and then encourage follow-up or post-stability analysis–status quo analysis and coalition analysis. To analyze a strategic conflict means to investigate the interaction of two or more decision makers (DMs) to identify possible outcomes.

1.1 Research Motivation

A graph model for a strategic conflict comprises a finite set of DMs, N, a set of feasible states, S, and, for each DM $i \in N$, a preference relation on S and a directed graph $G_i = \{S, A_i\}$. In each directed graph, S is the vertex set, and each oriented arc in $A_i \subseteq S \times S$ indicates that DM i can make a legal move (in one step) from the initial state to the terminal state of the arc. Obviously, preferences play an important role in decision analysis. In the original graph model, only a relative preference relation $\succ$ and an indifference relation $\sim$ are available to represent a particular DM's simple preference for one state over another [16]. The graph model has recently been developed in two new directions—preference uncertainty and preference strength. To enhance GMCR's applicability, more integrated and general preference structures need to be developed. Because a graph model of a conflict consists of several interrelated graphs, it is natural to utilize results of Algebraic Graph Theory to analyze a graph model.

1.1.1 Motivation for New Preference Structures

Preferences that involve incomplete information have been addressed in a significant amount of research such as preference with uncertainty and strength of preference. However, existing structures address preference uncertainty and preference strength separately, so they cannot model complex strategic conflicts arising in practical applications. How to expand the realm of applicability of GMCR and provide more insights into strategic conflicts? In this thesis, a mechanism that is more general and flexible than existing two frameworks of preference with uncertainty and strength of preference is introduced into the paradigm of GMCR to combine together preference uncertainty and preference strength.

The original graph model uses "simple preference $\{\succ, \sim\}$" to represent a DM's relative preference between two states. This model is called a two-level preference structure. Furthermore, a preference framework called "strength of preference" that includes two new binary relations, "greatly preferred $\gg$" and "mildly preferred $>$", expressing a DM's strong or mild preference for one state over another, with the indifference relation $\sim$, is referred to as a three-level preference structure.

As a result of the development of extensive research expressing preference information by degree of strength, existing preference structures in the graph model are limited in their ability to depict the intensity of relative preference. How to handle more specific preference information which lies between relative and cardinal preferences in terms of information content? How to gain better and more realistic insights into strategic conflicts? A multiple-level preference ranking structure is developed to expand earlier 2-level and 3-level structures to an unlimited number of levels of preference. In addition, this new preference structure is incorporated into GMCR for studying multi-objective decision making in conflict situations more realistically.

1.1.2 Motivation for Novel Algebraic Approach

In the graph model, stability analysis (individual stability analysis) is defined using logical structures that refer to the underlying graphs and preference relations [16]. Subsequently, Kilgour et al. [43] developed coalition stability analysis based on Nash stability but pseudo-code was furnished retaining a logical structure. However, as was noted in the development of the DSS GMCR II, the nature of logical representations makes coding difficult. The new preference structure proposed by Li et al. [46] to represent uncertainty in DMs' preferences included some extensions of the four stability definitions, and algorithms were outlined but never developed. Status quo analysis for simple preference and preference with uncertainty were developed by Li et al. [47, 48], but only in the form of pseudo-codes following a similar logical structure, which have never been implemented in a practical decision support system. The work of [27, 28] integrated strength of preference information into the four basic solution concepts consisting of Nash stability, general metarationality (GMR), symmetric metarationality (SMR), and sequential stability (SEQ), but, again, proved difficult to code and was never integrated into GMCR II. Table 1.1 shows the current status of available individual stability and coalition stability analyses and status quo analysis, as well as the development of effective algorithms and codes to implement these stabilities and status quo analysis, which would be essential if they are to be applied to practical problems [44].

How to develop a unique representation of conflict resolution that is easy to code and easy to adapt to new procedures? How to design a comprehensive decision support system for conflict analysis to include individual stability and coalition stability analyses and status quo analysis? These are essential motivations to develop an integrated algebraic approach for the graph model for conflict resolution. An important restriction of a graph model is that no decision maker can move twice in succession along any path. Hence, a graph model can be treated as an edge-weighted, colored multidigraph in which each arc represents a legal unilateral move and distinct colors refer to different DMs. Moreover, arc weights can be used to represent some preference attribute. Thus, tracing the evolution of a conflict in status quo analysis with some preference structure is converted to searching all colored paths assigned specific weights. Generally, the adjacency matrix represents a simple digraph and determines all paths between

Table 1.1: **Current status of the graph model for conflict resolution** (extend from [44])

Preference information	Stability and post-stability analyses	Algorithms?	In GMCR II ?
Simple preference	Individual stability analysis	Yes	Yes
	Status quo analysis	Yes	Yes
	Coalition stability analysis	Yes	Yes
Preference with uncertainty	Individual stability analysis	No	No
	Status quo analysis	Yes	No
	Coalition stability analysis	No	No
Strength of preference	Individual stability	No	No
	Status quo analysis	No	No
	Coalition stability analysis	No	No

any two vertices, but is not readily extendable to colored multidigraphs. How to transform the original problem of searching edge-colored paths in a colored multidigraph to a standard problem of finding paths in a simple digraph?

A conversion function using matrix representation can establish a relationship between a colored multidigraph and a simple digraph with no color constraints. Based on the conversion function, an inherent link among status quo analysis, individual stability analysis, and coalition stability analysis is revealed. Because edge weights in a graph model are used to represent preference attributes, a weight matrix can be designed to represent various preference structures. Therefore, the above analysis provides the possibility of establishing an integrated paradigm using matrix representation for stability analysis and post-stability analysis in a graph model. The explicit matrix representation for conflict resolution (MRCR) is developed to ease the coding of logically-defined individual and coalition stability definitions and status quo analysis. Another benefit of matrix representation is that it facilitates modification and extension of the definitions.

1.2 Objectives

This research has two key objectives: the first is to propose two new preference frameworks to enhance the applicability of GMCR; the second is to develop an integrated algebraic approach for stability analysis, status quo analysis, and coalition stability analysis for three preference structures, simple preference, preference with uncertainty, and strength of preference.

The specific goals are presented as follows:

1. To extend the graph model for conflict resolution including hybrid preference:

   - Propose a new preference structure for the graph model that can represent DMs' preference uncertainty and strength of preference;
   - Extend the four basic solution concepts to models with hybrid preference;
   - Extend status quo analysis from models with simple preference and preference with uncertainty to models with hybrid preference.

2. To extend the graph model for conflict resolution to include multiple levels of preference:

   - Propose a new preference framework for the graph model that can represent multiple levels of preference;
   - Propose appropriate results of the four basic stability definitions for graph models with multiple levels of preference;
   - Investigate the relationships among these new stability definitions;
   - Employ these new stability definitions to analyze a model for presenting the significance of multiple levels of preference.

3. To develop an algebraic approach to searching edge-weighted, colored paths in a weighted colored multidigraph:

   - Propose a procedure (the Rule of Priority) to label colored multidigraphs;
   - Design a conversion function that transforms the problem of searching edge-colored paths in a colored multidigraph to the standard problem of finding paths in a simple digraph;

- Use the conversion function to find all colored paths between any two vertices of a colored multidigraph;
- Develop an algorithm for searching edge-weighted, colored paths between any two vertices in a weighted colored multidigraph;
- Construct a weighted reachability matrix of a coalition by weighted colored paths to reveal the link among individual stability analysis, status quo analysis, and coalition stability analysis.

4. To develop matrix representation of solution concepts (MRSC) in multiple-decision-maker graph models:

- Construct weight matrices to represent preference information for simple preference, preference with uncertainty, strength of preference, and hybrid preference;
- Establish the equivalence of weighted reachability matrices for a DM or a coalition by the weighted colored paths and reachable lists of a DM or a coalition by various legal unilateral moves;
- Develop explicit matrix representations of the four basic solution concepts for graph models with simple preference (MRSC), preference with uncertainty (MRSCU), and strength of preference (MRSCS) based on their weighted reachable matrices.

5. To propose matrix representation for status quo analysis (MRSQA) to track the evolution of a conflict:

- Show how to input efficiently the weight matrices that represent simple preference, preference with uncertainty, and strength of preference;
- Show that weighted edges by 0 or 1 can be used to indicate allowable unilateral moves;
- Show that the algorithm for searching edge-weighted, colored paths can be used to trace the evolution of a conflict under some constraints on unilateral moves.

6. To develop matrix representation of coalition stability analysis (MRCSA):

- Extend coalition stabilities to models including preference uncertainty and strength of preference;
- Construct coalition stability matrices for simple preference, preference with uncertainty, and strength of preference based on the weighted reachability matrix of the coalition;
- Develop an explicit algebraic form conflict model that facilitates coalition stability calculations for the aforementioned three preference structures.

1.3 Outline of the Thesis

The outline of this thesis is presented in Fig. 1.1 to describe the existing research and the main objectives in this work.

This chapter presents the motivation and objectives of this research. Chapter 2 includes some definitions from Algebraic Graph Theory and a brief overview of the graph model for conflict resolution including stability analysis, status quo analysis, and coalition analysis for existing preference structures. In Chapter 3, two new preference frameworks, hybrid preference and multiple-level preference, are proposed for a graph model. The four basic solution concepts and status quo analysis for simple preference are extended to graph models incorporating hybrid preference of uncertainty and strength. To illustrate this method, a model of the conflict over proposed bulk water exports from Lake Gisborne in Newfoundland is extended to hybrid preference. Then the possible resolutions and evolution of this conflict are calculated using the extended stability and status quo analyses. In Chapter 4, the graph model for conflict resolution is extended to multiple-level preference. The redefined solution concepts are then applied to the expanded Garrison Diversion Unit (GDU) conflict to show how the procedure works.

In Chapter 5, a new algebraic approach to constructing the reduced weighted edge consecutive matrix is developed for finding all edge-weighted, colored paths within a weighted colored multidigraph. Then, weight matrices are used to represent simple preference, preference with uncertainty, strength of preference, and hybrid preference. Finally, the reduced weighted edge consecutive matrix is used to obtain weighted reachability matrices that are equivalent to the reachable lists of a coalition by legal unilateral moves within the four preference frameworks, simple preference, preference with uncertainty, strength of

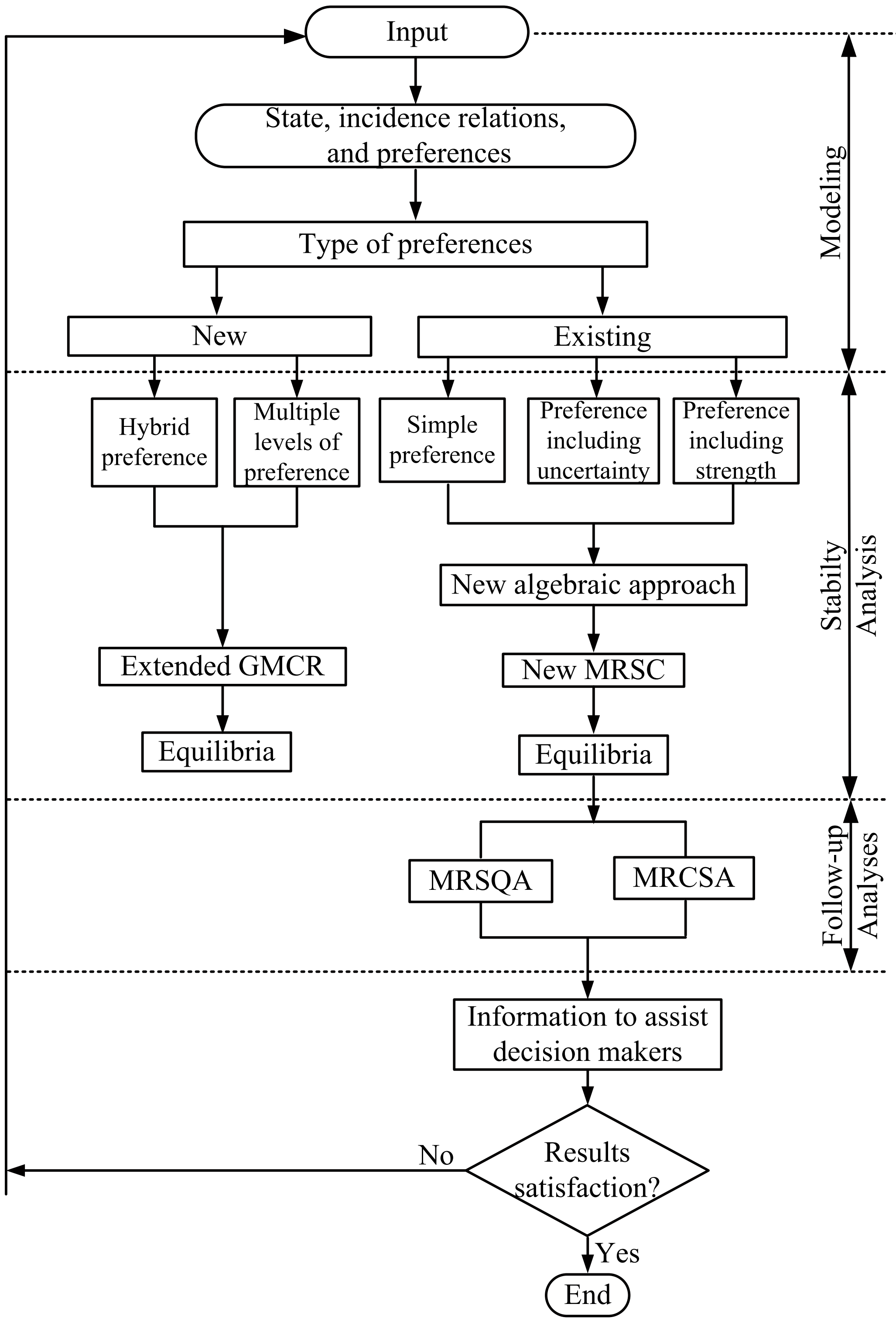


Figure 1.1: **Outline of this thesis.**

preference, and hybrid preference. Furthermore, logical stability definitions are presented using matrix representations for three existing preference structures in Chapter 6. Following is the proposed algebraic approach that is employed to solve real applications for status quo analysis and coalition stability analysis in Chapter 7. Finally, some conclusions and ideas for future work are presented in Chapter 8.

Chapter 2

Background and Literature Review

2.1 Definitions from Algebraic Graph Theory

A graph is a pair (V, E) of sets satisfying $E \subseteq V \times V$. A multidigraph [13] $G = (V, A, \psi)$ is a set of vertices (nodes) V and a set of oriented edges (arcs) A with $\psi : A \to V \times V$. If $a \in A$ satisfies $\psi(a) = (u, v)$, then we say that a has initial vertex u and terminal vertex v. A multidigraph may contain $a, b \in A$ such that $a \neq b$ and $\psi(a) = \psi(b)$, in which case a and b are said to be multiple arcs. If there exists $a \in A$ such that $\psi(a) = (u, v)$, then u is said to be *adjacent* to v and (u, v) is said to be incident from u and incident to v. Hence, (u, v) is called in-incident to v and out-incident to u. When G is drawn, it is common to represent the direction of an edge with an arrowhead. We generally assume loop-free graphs; i.e., for any $a \in A$, if $\psi(a) = (u, v)$, then $u \neq v$.

It should be pointed out that a multidigraph with no multiple edges can be called a simple digraph [13].

Definition 2.1. *For a multidigraph $G = (V, A, \psi)$, edge $a \in A$ and edge $b \in A$* ***are consecutive (in the order** ab**)*** *iff $\psi(a) = (u, v)$ and $\psi(b) = (v, s)$, where $u, v, s \in V$.*

Definition 2.2. *For a multidigraph $G = (V, A, \psi)$, the* ***line digraph*** *$L(G) = (A, LA)$ of G is a simple digraph with vertex set A and edge set LA=$\{d = (a, b) \in A \times A : a$ and b are consecutive (in the order ab)$\}$.*

Definition 2.3. *For a multidigraph $G = (V, A, \psi)$, a* ***path*** *from vertex $u \in V$ to vertex $s \in V$ is a sequence of vertices in G starting with u and ending with s, such that consecutive vertices are adjacent.*

Note that in this thesis a path may contain the same vertex more than once [8]. The length of a path is the number of edges therein.

Important matrices associated with a digraph include the adjacency matrix and the incidence matrix [24]. Let $m = |V|$ denote the number of vertices and $l = |A|$ be the number of edges of the directed graph G. Then,

Definition 2.4. *For a multidigraph $G = (V, A, \psi)$, the* ***adjacency matrix*** *is the $m \times m$ matrix J with (u, v) entry*

$$J(u,v) = \begin{cases} 1 & \textit{if } (u,v) \in A, \\ 0 & \textit{otherwise}, \end{cases}$$

where $u, v \in V$.

Definition 2.5. *For a multidigraph $G = (V, A, \psi)$, the* ***incidence matrix*** *is the $m \times l$ matrix B with (v, a) entry*

$$B(v,a) = \begin{cases} -1 & \textit{if } a = (v,x) \textit{ for some } x \in V, \\ 1 & \textit{if } a = (x,v) \textit{ for some } x \in V, \\ 0 & \textit{otherwise}, \end{cases}$$

where $v \in V$ and $a \in A$.

According to the signed entries, the incidence matrix can be separated into the in-incidence matrix and the out-incidence matrix.

Definition 2.6. *For a multidigraph $G = (V, A, \psi)$, the* ***in-incidence matrix*** *B_{in} and the* ***out-incidence matrix*** *B_{out} are the $m \times l$ matrices with (v, a) entries*

$$B_{in}(v,a) = \begin{cases} 1 & \textit{if } a = (x,v) \textit{ for some } x \in V, \\ 0 & \textit{otherwise}, \end{cases}$$

and

$$B_{out}(v,a) = \begin{cases} 1 & \textit{if } a = (v,x) \textit{ for some } x \in V, \\ 0 & \textit{otherwise}, \end{cases}$$

where $v \in V$ and $a \in A$.

It is obvious that $B_{in} = (B + abs(B))/2$ and $B_{out} = (abs(B) - B)/2$, where $abs(B)$ denotes the matrix in which each entry equals the absolute value of the corresponding entry of B. Definitions 2.2 to 2.6 are adapted from [24].

Definition 2.7. *For two $m \times m$ matrices M and Q, the **Hadamard product** for the two matrices is the $m \times m$ matrix $H = M \circ Q$ with (s, q) entry*

$$H(s, q) = M(s, q) \cdot Q(s, q).$$

Let " $\vee$ " denote the disjunction operator ("or") on two matrices. Assuming that H and G are two $m \times m$ matrices, the disjunction operation on matrices H and G is defined by:

Definition 2.8. *For two $m \times m$ matrices H and G, **disjunction matrix** of H and G is the $m \times m$ matrix $M = H \vee G$ with (u, v) entry*

$$M(u, v) = \begin{cases} 1 & if \ H(u, v) + G(u, v) \neq 0, \\ 0 & otherwise. \end{cases}$$

Definition 2.9. *The **sign function**, $sign(\cdot)$, maps an $m \times m$ matrix with (u, v) entry $M(u, v)$ to the $m \times m$ matrix*

$$sign[M(u, v)] = \begin{cases} 1 & M(u, v) > 0, \\ 0 & M(u, v) = 0, \\ -1 & M(u, v) < 0. \end{cases}$$

2.2 Graph Model for Conflict Resolution: Literature Review

To analyze a strategic conflict means to investigate the interaction of two or more decision makers (DMs) to identify possible outcomes. There are many models available for strategic conflicts, and many ways to analyze a model, including the strategic-form game [53], the option form [34], and the closely-related tabular form [22, 23]. In 1987, the graph model for conflict resolution (GMCR) was proposed by Kilgour et al. [41] to provide a simple, flexible, structure modeling strategic conflicts and insightful methods for analyzing the model. One advantage of the graph model is that it incorporates a range of stability definitions (or solution concepts) that models human behavior in strategic conflicts. See [44] for a recent summary of work on the graph model. Compared with the other ways to represent strategic conflicts, the graph model has several advantages, including its ability to

- handle irreversible moves,
- model common moves easily,
- provide a flexible framework for defining, comparing, and characterizing solution concepts, and
- adapt easily in practice.

This thesis concerns the graph model.

As Fig. 2.1 shows, the graph model provides a methodology for modeling and analyzing strategic conflicts. The modeling stage includes identification of the decision makers (DMs), the states, the state transitions controlled by each DM, and each DM's relative preferences over the states. A DM may be an individual or a group, such as an industrial or governmental organization. Usually, a DM is modeled as having one or more options, each of which may or may not be selected, and a state is defined as a particular selection of options by all DMs. The analysis stage includes the determination of whether a state is stable from each DM's viewpoint for a range of solution concepts. States that are stable for all DMs according to a given solution concept are called equilibria. The analysis stage also includes follow-up analyses such as status quo analysis, coalition stability analysis, and sensitivity analysis [16].

In a graph model, a stability definition (solution concept) is a procedure for determining whether a state is stable for a DM, and represents the situation in which the DM would have no incentive to move away from the state unilaterally. An equilibrium of a graph model, or a possible resolution of the conflict it represents, is a state that all DMs find stable under an appropriate stability definition. To represent various decision styles and contexts, at least seven solution concepts have been formulated for graph models, including Nash stability [51, 52], general metarationality (GMR) [34], symmetric metarationality (SMR) [34], sequential stability (SEQ) [22], limited-move stability (LS) [16, 40, 78], non-myopic stability (NM) [6, 7, 39], and Stackelberg's equilibrium concept [61]. In this thesis, four basic solution concepts consisting of Nash, GMR, SMR, and SEQ are considered because these definitions can be employed with both intransitive and transitive preferences. In 1989, Wang et al. [63] redefined the four basic solution concepts in hypergames. Recently, Zeng et al. [79] suggested more general solution concepts—policy stability—for the

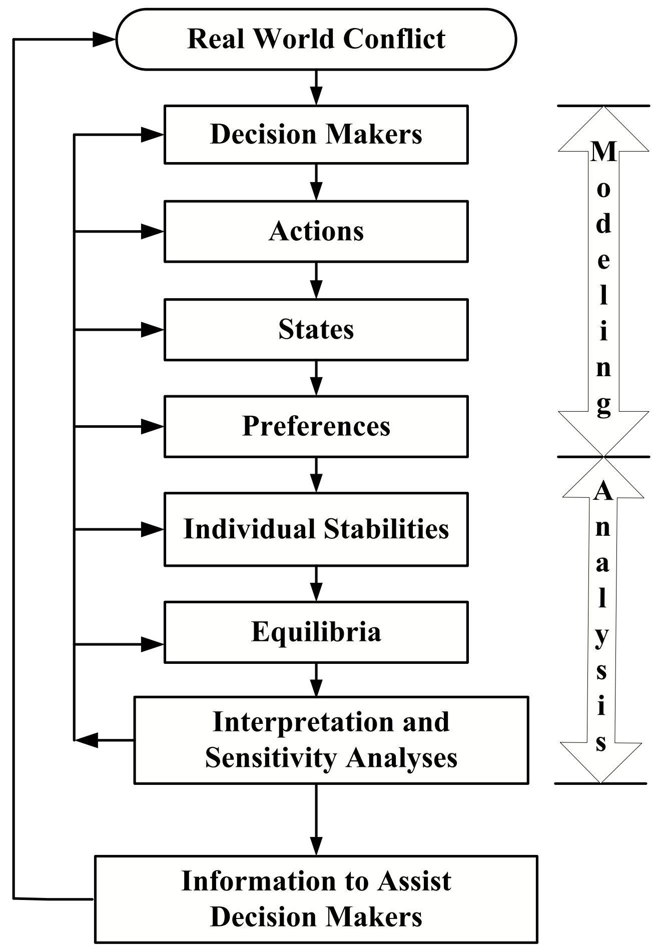


Figure 2.1: **The procedure for applying GMCR [16].**

graph model and Li et al. [46] extended the four basic solution concepts to models having preference uncertainty. Hamouda et al. [27, 28] proposed new solution concepts that take strength of preference (strong or mild) into account. This thesis focuses mainly on the analysis stage: identifying stable states based on the four basic solution concepts and carrying out status quo analysis and coalition stability analysis.

2.2.1 Simple Preference, Uncertain Preference, and Strength of Preference

Obviously, preference information plays an important role in decision analysis. Each DM has preferences among the possible states that can arise. Ordinal preferences, ranking states from most to least preferred (ties allowed), or cardinal preferences using the values of a real-valued preference function on the states are required by some models. The graph model requires only relative preference information for each DM, but can of course use cardinal information; moreover, it can handle both intransitive and transitive preferences. In the original graph model, simple preference [16] of DM i is coded by a pair of relations $\{\succ_i, \sim_i\}$ on S, where $s \succ_i q$ indicates that DM i prefers s to q and $s \sim_i q$ means that DM i is indifferent between s and q (or equally prefers s and q). Note that, for each i, $\succ_i$ is assumed irreflexive and asymmetric, and $\sim_i$ is assumed reflexive and symmetric. Also, $\{\succ_i, \sim_i\}$ is complete, i.e., for any $s, q \in S$, either $s \succ_i q$, $s \sim_i q$, or $q \succ_i s$. The conventions that $s \succeq_i q$ is equivalent to either $s \succ_i q$ or $s \sim_i q$, and that $s \prec_i q$ is equivalent to $q \succ_i s$, are convenient. This completes the definition of the graph model as used until around 2000, and represents the structures encoded in the Decision Support System (DSS) GMCR II [18, 19].

Unfortunately, it is often difficult to obtain accurate preference information in practical cases, so models that allow preference uncertainty can be very useful. Moreover, as pointed out by [20, 21], conflicts among the attributes of alternatives can cause preference uncertainty. To incorporate preference uncertainty into the graph model methodology, Li et al. [46] proposed a new preference structure in which DM i's preferences are expressed by a triple of relations $\{\succ_i, \sim_i, U_i\}$ on S, where $s \succ_i q$ indicates strict preference, $s \sim_i q$ indicates indifference, and sU_iq means DM i may prefer state s to state q, may prefer q to s, or may be indifferent between s and q. If for any relation R and any states k, s, and q, kRs and sRq

imply kRq, then R is transitive. For example, strict preference $\succ$ is transitive in many graph models, though in some cases it is intransitive. In this research, transitivity of preferences is not required, and all results hold whether preferences are transitive or intransitive. For example, the uncertain preference relation, U, is often intransitive.

Another triplet relation $\{\gg_i, >_i, \sim_i\}$ on S that expresses strength of preference (strong or mild preference) was developed by Hamouda et al. [27,28]. For $s, q \in S$, $s \gg_i q$ denotes DM i strongly prefers s to q, $s >_i q$ means DM i mildly prefers s to q, and $s \sim_i q$ indicates that DM i is indifferent between states s and q. Table 2.1 summarizes the three existing types of preferences for DM i.

Table 2.1: **Three types of preferences**

Preference type	Expression of preference	Properties of preference			
		Asymmetric	Symmetric	Reflexive and symmetric	Complete
Simple preference	$\{\succ_i, \sim_i\}$	$\succ_i$		$\sim_i$	$\{\succ_i, \sim_i\}$
Preference with uncertainty	$\{\succ_i, \sim_i, U_i\}$	$\succ_i$	U_i	$\sim_i$	$\{\succ_i, \sim_i, U_i\}$
Preference with strength	$\{\gg_i, >_i, \sim_i\}$	$\gg_i$, $>_i$		$\sim_i$	$\{\gg_i, >_i, \sim_i\}$

Note that $\{\gg_i, >_i, \sim_i\}$ is complete, i.e., if $s, q \in S$, then exactly one of the following relations holds: $s \gg_i q$, $q \gg_i s$, $s >_i q$, $q >_i s$, and $s \sim_i q$.

The state set S can be divided into a set of subsets based on preference relative to a fixed state $s \in S$. These subsets are essential components in stability analysis. The descriptions of these subsets for the three types of preferences are presented in Table 2.2.

Let $s \in S$ and $i \in N$. Based on different structures of preferences, DM i can identify different subsets of S. The details are presented as follows:

- For simple preference, DM i can identify three subsets of S: $\Phi_i^+(s)$, $\Phi_i^=(s)$, and $\Phi_i^-(s)$ [16].

- For preference with uncertainty, DM i can identify four subsets of S: $\Phi_i^+(s)$, $\Phi_i^=(s)$, $\Phi_i^-(s)$, and $\Phi_i^U(s)$ [46].

Table 2.2: **Subsets of S with respect to three structures of preferences for DM i** [16, 27, 28, 46]

Subsets of S	Description
$\Phi_i^{++}(s) = \{q : q \gg_i s\}$	States strongly preferred to state s by DM i
$\Phi_i^{+m}(s) = \{q : q >_i s\}$	States mildly preferred to state s by DM i
$\Phi_i^{+}(s) = \{q : q \succ_i s\}$	States preferred to state s by DM i
$\Phi_i^{=}(s) = \{q : q \sim_i s\}$	States equally preferred to state s by DM i
$\Phi_i^{-}(s) = \{q : s \succ_i q\}$	States less preferred than state s for DM i
$\Phi_i^{-m}(s) = \{q : s >_i q\}$	States mildly less preferred than state s for DM i
$\Phi_i^{--}(s) = \{q : s \gg_i q\}$	States strongly less preferred to state s by DM i
$\Phi_i^{U}(s) = \{q : q\ U_i\ s\}$	States uncertainly preferred to state s by DM i

- For preference with strength, DM i can identify five subsets of S: $\Phi_i^{++}(s)$, $\Phi_i^{+m}(s)$, $\Phi_i^{=}(s)$, $\Phi_i^{-m}(s)$, and $\Phi_i^{--}(s)$ [27, 28].

For ease of use, some additional notation is defined by $\Phi_i^{--,-,=}(s) = \Phi_i^{--}(s) \cup \Phi_i^{-m}(s) \cup \Phi_i^{=}(s)$, where $\cup$ denotes the union operation. Note that in the graph model with strength of preference, $s \succ_i q$ iff either $s >_i q$ or $s \gg_i q$. Therefore, the two preference frameworks of preference with uncertainty and preference with strength expand simple preference.

2.2.2 Reachable Lists for Three Preference Structures

2.2.2.1 Reachable Lists of a DM

Let $i \in N$, $s \in S$, and let $m = |S|$ be the number of the states in S. Notation $\cap$ denotes the intersection operation. Recall that each arc of $A_i \subseteq S \times S$ indicates that DM i can make a unilateral move (in one step) from the initial state to the terminal state of the arc. The reachable lists of DM i's from state $s \in S$ for different preference structures are defined as follows.

- Simple preference [16]:

(i) $R_i(s) = \{q \in S : (s,q) \in A_i\}$ denotes DM i's reachable list from state s by a unilateral move (UM);

(ii) $R_i^+(s) = \{q \in S : (s,q) \in A_i \text{ and } q \succ_i s\}$ denotes DM i's reachable list from state s by a unilateral improvement (UI);

(iii) $R_i^=(s) = \{q \in S : (s,q) \in A_i \text{ and } q \sim_i s\}$ denotes DM i's reachable list from state s by an equally preferred move;

(iv) $R_i^-(s) = \{q \in S : (s,q) \in A_i \text{ and } s \succ_i q\}$ denotes DM i's reachable list from state s by a unilateral disimprovement.

- Preference with uncertainty [46]:

(i) $R_i^U(s) = \{q \in S : (s,q) \in A_i \text{ and } qU_is\}$ denotes DM i's reachable list from state s by a unilateral uncertain move (UUM);

(ii) $R_i^{+,U}(s) = R_i^+(s) \cup R_i^U(s) = \{q \in S : (s,q) \in A_i \text{ and } q \succ_i s \text{ or } qU_is\}$ denotes DM i's reachable list from state s by a unilateral improvement or unilateral uncertain move (UIUUM).

- Strength of preference [27,28]:

(i) $R_i^{+m}(s) = \{q \in S : (s,q) \in A_i \text{ and } q >_i s\}$ denotes DM i's reachable list from state s by a mild unilateral improvement;

(ii) $R_i^{++}(s) = \{q \in S : (s,q) \in A_i \text{ and } q \gg_i s\}$ denotes DM i's reachable list from state s by a strong unilateral improvement;

(iii) $R_i^{-m}(s) = \{q \in S : (s,q) \in A_i \text{ and } s >_i q\}$ denotes DM i's reachable list from state s by a mild unilateral disimprovement;

(iv) $R_i^{--}(s) = \{q \in S : (s,q) \in A_i \text{ and } s \gg_i q\}$ denotes DM i's reachable list from state s by a strong unilateral disimprovement;

(v) $R_i^{+,++}(s) = R_i^{+m}(s) \cup R_i^{++}(s) = \{q \in S : (s,q) \in A_i \text{ and } q >_i s \text{ or } q \gg_i s\}$ denotes DM i's reachable list from state s by a mild unilateral move or strong unilateral move called a weak move (WI).

From the above definitions, these reachable lists from state s by DM i can be summarized as presented in Table 2.3.

The reachable list from state s, $R_i(s)$, represents DM $i's$ unilateral moves (UMs). $R_i(s)$ is partitioned according to the different preference structures as follows [16,27,28,46]:

Table 2.3: **Unilateral movements for DM i in various preference structures** [16, 27, 28, 46]

Type of movements	Description
$R_i^{++}(s) = R_i \cap \Phi_i^{++}(s)$	All strong unilateral improvements from state s for DM i
$R_i^{+m}(s) = R_i \cap \Phi_i^{+m}(s)$	All mild unilateral improvements from state s for DM i
$R_i^{+}(s) = R_i \cap \Phi_i^{+}(s)$	All unilateral improvements (UIs) from state s for DM i
$R_i^{=}(s) = R_i \cap \Phi_i^{=}(s)$	All equally preferred states reachable from state s by DM i
$R_i^{-}(s) = R_i \cap \Phi_i^{-}(s)$	All unilateral disimprovements from state s for DM i
$R_i^{-m}(s) = R_i \cap \Phi_i^{-m}(s)$	All mild unilateral disimprovements from state s for DM i
$R_i^{--}(s) = R_i \cap \Phi_i^{--}(s)$	All strong unilateral disimprovements from state s for DM i
$R_i^{U}(s) = R_i \cap \Phi_i^{U}(s)$	All states reachable by DM i from state s for which DM i's preference relative to s is uncertain

- For simple preference, $R_i(s) = R_i^{+}(s) \cup R_i^{=}(s) \cup R_i^{-}(s)$.
- For preference with uncertainty, $R_i(s) = R_i^{+}(s) \cup R_i^{=}(s) \cup R_i^{-}(s) \cup R_i^{U}(s)$.
- For preference with strength, $R_i(s) = R_i^{++}(s) \cup R_i^{+m}(s) \cup R_i^{=}(s) \cup R_i^{-m}(s) \cup R_i^{--}(s)$.

2.2.2.2 Reachable Lists of a Coalition

Any subset H of DMs in the set N is called a coalition. If $|H| > 0$, then the coalition H is non-empty. If $|H| > 1$, then the coalition H is non-trivial. Below, a coalition $H \subseteq N$ is assumed to be non-trivial. For a two-DM model, DM i's opponent is one DM, j, so DM j's reachable lists from s are the states reachable by one step moves. In an n-DM model ($n > 2$), the opponents of a DM constitute a group of two or more DMs. Therefore, the definition of a legal sequence of UMs is given first.

A legal sequence of UMs for a coalition of DMs is a sequence of states linked by unilateral moves by members of the coalition, in which a DM may move more than once, but not twice consecutively. (If a DM can move consecutively, then this DM's graph is effectively transitive.)

Let the coalition $H \subseteq N$ satisfy $|H| \geq 2$ and let the status quo state be $s \in S$. We now define $R_H(s) \subseteq S$, the reachable list of coalition H from state s (by a legal sequence of UMs). The following definitions are taken from [16]:

Definition 2.10. *A unilateral move by H is a member of $R_H(s) \subseteq S$, defined inductively by*
(1) if $j \in H$ and $s_1 \in R_j(s)$, then $s_1 \in R_H(s)$ and $j \in \Omega_H(s, s_1)$;
(2) if $s_1 \in R_H(s)$, $j \in H$ and $s_2 \in R_j(s_1)$, then, provided $\Omega_H(s, s_1) \neq \{j\}$, $s_2 \in R_H(s)$ and $j \in \Omega_H(s, s_2)$.

Note that this definition is inductive: first, using (1), the states reachable from s are identified and added to $R_H(s)$; then, using (2), all states reachable from those states are identified and added to $R_H(s)$; then the process is repeated until no further states are added to $R_H(s)$ by repeating (2). Because $R_H(s) \subseteq S$, and S is finite, this limit must be reached in finitely many steps.

To interpret Definition 2.10, note that if $s_1 \in R_H(s)$, then $\Omega_H(s, s_1) \subseteq N$ is the set of all last DMs in legal sequences from s to s_1. (If $s_1 \notin R_H(s)$, it is assumed that $\Omega_H(s, s_1) = \emptyset$.) Suppose that $\Omega_H(s, s_1)$ contains only one DM, say $j \in N$. Then any move from s_1 to a subsequent state, say s_2, must be made by a member of H other than j; otherwise DM j would have to move twice in succession. On the other hand, if $|\Omega_H(s, s_1)| \geq 2$, any member of H who has a unilateral move from s_1 to s_2 may exercise it. It should be pointed out that it is possible $s \in R_H(s)$ according to Definition 2.10, but the trivial case will not be discussed in research.

A legal sequence of UIs for a coalition can be defined similarly, leading to the list of coalitional UIs, as follows.

Definition 2.11. *Let $s \in S$, $H \subseteq N$, and $H \neq \emptyset$. A unilateral improvement by H is a member of $R_H^+(s) \subseteq S$, defined inductively by*
(1) if $j \in H$ and $s_1 \in R_j^+(s)$, then $s_1 \in R_H^+(s)$ and $j \in \Omega_H^+(s, s_1)$;
(2) if $s_1 \in R_H^+(s)$, $j \in H$ and $s_2 \in R_j^+(s_1)$, then, provided $\Omega_H^+(s, s_1) \neq \{j\}$, $s_2 \in R_H^+(s)$ and $j \in \Omega_H^+(s, s_2)$.

Definition 2.11 is identical to Definition 2.10 except that all moves are required to be UIs, i.e. each move is to a state strictly preferred by the mover to the current state. Similarly, $\Omega_H^+(s, s_1)$ includes all last movers in a UI by coalition H from state s to state s_1.

The reachable lists of coalition H from state s by the legal sequences of UMs and UIs were defined above for simple preference. Li et al. [46] and Hamouda et al. [28] extended the legal sequences of UMs and UIs and reachable lists of coalition H to preference including possible uncertainty and strength, respectively. To extend the definitions of the reachable lists for a coalition to take preference uncertainty and strength of preference into account, legal sequence of coalitional UIUUMs and legal sequence of coalitional WIs must be defined first, respectively. A legal sequence of UIUUMs is a sequence of allowable unilateral improvements or unilateral uncertain moves by a coalition, with the usual restriction that a member of the coalition may move more than once, but not twice consecutively. Similarly, a legal sequence of WIs is a sequence of allowable mild unilateral improvements or strong unilateral improvements by a coalition, with the same restriction that any member in the coalition may move more than once, but not twice consecutively. The following formal definitions for reachable lists of coalition H by the legal sequence of UIUUMs and by the legal sequence of WIs are respectively taken from [46] and [28]:

Definition 2.12. *Let $s \in S$ and $H \subseteq N$ where $|H| \geq 2$. A unilateral improvement or unilateral uncertain move (UIUUM) by H is a member of $R_H^{+,U}(s) \subseteq S$, defined inductively by*
(1) if $j \in H$ and $s_1 \in R_j^{+,U}(s)$, then $s_1 \in R_H^{+,U}(s)$ and $j \in \Omega_H^{+,U}(s, s_1)$;
(2) if $s_1 \in R_H^{+,U}(s)$, $j \in H$ and $s_2 \in R_j^{+,U}(s_1)$, then, provided $\Omega_H^{+,U}(s, s_1) \neq \{j\}$, $s_2 \in R_H^{+,U}(s)$ and $j \in \Omega_H^{+,U}(s, s_2)$.

Definition 2.13. *Let $s \in S$ and $H \subseteq N$ where $|H| \geq 2$. A weak improvement (WI) by H is a member of $R_H^{+,++}(s) \subseteq S$, defined inductively by:*
(1) if $j \in H$ and $s_1 \in R_j^{+,++}(s)$, then $s_1 \in R_H^{+,++}(s)$ and $j \in \Omega_H^{+,++}(s, s_1)$;
(2) if $s_1 \in R_H^{+,++}(s)$, $j \in H$ and $s_2 \in R_j^{+,++}(s_1)$, then, provided $\Omega_H^{+,++}(s, s_1) \neq \{j\}$, $s_2 \in R_H^{+,++}(s)$ and $j \in \Omega_H^{+,++}(s, s_2)$.

Like Definitions 2.10 and 2.11, Definitions 2.12 and 2.13 are inductive definitions. The roles and interpretations of $R_H^{+,U}(s)$ and $\Omega_H^{+,U}(s, s_1)$, as well as $R_H^{+,++}(s)$ and $\Omega_H^{+,++}(s, s_1)$ are likewise analogous.

Within an n-DM model ($n \geq 2$), DM i's opponents, $N \backslash \{i\}$, where $\backslash$ refers to "set subtraction", consist of a group of one or more DMs. In order to analyze the stability of a state for DM $i \in N$, it is necessary to take into account possible responses by all other DMs $j \in N \backslash \{i\}$. The essential inputs of stability analysis are reachable lists of coalition $N\backslash\{i\}$ from state s, $R_{N\backslash\{i\}}(s)$ and $R^{+}_{N\backslash\{i\}}(s)$ for simple preference, $R_{N\backslash\{i\}}(s)$ and $R^{+,U}_{N\backslash\{i\}}(s)$ for preference with uncertainty, and $R_{N\backslash\{i\}}(s)$ and $R^{+,++}_{N\backslash\{i\}}(s)$ for preference with strength.

2.2.3 Solution Concepts in the Graph Model for Simple Preference

The four basic solution concepts, Nash stability, general metarationality (GMR), symmetric metarationality (SMR), and sequential stability (SEQ) in the graph model for simple preference are taken from [16]. Let $i \in N$ and $s \in S$.

Definition 2.14. *State s is Nash stable for DM i iff $R^{+}_{i}(s) = \emptyset$.*

State $s \in S$ is GMR for DM i iff whenever DM i makes any UI from s, then its opponent can move to hurt i or sanction i in response.

Definition 2.15. *State s is GMR for DM i iff for every $s_1 \in R^{+}_{i}(s)$ there exists at least one $s_2 \in R_{N\backslash\{i\}}(s_1)$ with $s \succeq_i s_2$.*

SMR is a more restrictive stability definition than GMR. SMR is similar to GMR except that DM i expects to have a chance to counterrespond to its opponent's response to i's original move [16].

Definition 2.16. *State s is SMR for DM i iff for every $s_1 \in R^{+}_{i}(s)$ there exists at least one $s_2 \in R_{N\backslash\{i\}}(s_1)$, such that $s \succeq_i s_2$ and $s \succeq_i s_3$ for any $s_3 \in R_i(s_2)$.*

SEQ is similar to GMR, but includes only sanctions that are "credible". A credible action is a unilateral improvement.

Definition 2.17. *State s is SEQ for DM i iff for every $s_1 \in R^{+}_{i}(s)$ there exists at least one $s_2 \in R^{+}_{N\backslash\{i\}}(s_1)$ with $s \succeq_i s_2$.*

When $n = 2$, the DM set N reduces to $\{i, j\}$ in Definitions 2.14 to 2.17. For example, the reachable list $R_{N\backslash\{i\}}(s_1)$ of $N\backslash\{i\}$ from s_1, reduces to reachable list $R_j(s_1)$ of j from s_1.

2.2.4 Solution Concepts in the Graph Model for Preference with Uncertainty

Based on the extended preference structure (including uncertainty), Li et al. [46] defined Nash, GMR, SMR, and SEQ stability to capture a DM's incentives to leave the status quo state and sensitivity to sanctions. Four types of stability definitions were proposed, indexed a, b, c, and d, according to whether the DM would move to a state of uncertain preference and whether the DM would be sanctioned by a responding move to a state of uncertain preference, relative to the status quo. This range of extensions is needed, according to [46], to address the diversity of possible risk profiles in the face of uncertainty. A DM may be conservative or aggressive, avoiding or accepting states of uncertain preference, depending on the level of satisfaction with the current position.

In the definitions indexed a, DM i has an incentive to move to states with uncertain preferences relative to the status quo, but, when assessing possible sanctions, will not consider states with uncertain preferences [46]. Let $i \in N$ and $|N| = n$ in the following definitions taken from [46].

Definition 2.18. *State s is $Nash_a$ stable for DM i iff $R_i^{+,U}(s) = \emptyset$.*

Definition 2.19. *State s is GMR_a for DM i iff for every $s_1 \in R_i^{+,U}(s)$ there exists at least one $s_2 \in R_{N\backslash\{i\}}(s_1)$ with $s \succeq_i s_2$.*

Definition 2.20. *State s is SMR_a for DM i iff for every $s_1 \in R_i^{+,U}(s)$ there exists at least one $s_2 \in R_{N\backslash\{i\}}(s_1)$, such that $s \succeq_i s_2$ and $s \succeq_i s_3$ for any $s_3 \in R_i(s_2)$.*

Definition 2.21. *State s is SEQ_a for DM i iff for every $s_1 \in R_i^{+,U}(s)$ there exists at least one $s_2 \in R_{N\backslash\{i\}}^{+,U}(s_1)$ with $s \succeq_i s_2$.*

For stabilities indexed b, DM i would move only to preferred states from a status quo and would be sanctioned only by less preferred or equally preferred states relative to the status quo. Note that the definitions are different from those discussed in Section 2.2.3 for simple preference, since the current definitions are utilized to analyze conflict models with preference uncertainty.

Definition 2.22. *State s is $Nash_b$ for DM i iff $R_i^{+}(s) = \emptyset$.*

Definition 2.23. *State s is GMR_b for DM i iff for every $s_1 \in R_i^{+}(s)$ there exists at least one $s_2 \in R_{N\backslash\{i\}}(s_1)$ with $s \succeq_i s_2$.*

Definition 2.24. *State s is SMR_b for DM i iff for every $s_1 \in R_i^{+}(s)$ there exists at least one $s_2 \in R_{N\backslash\{i\}}(s_1)$, such that $s \succeq_i s_2$ and $s \succeq_i s_3$ for any $s_3 \in R_i(s_2)$.*

Definition 2.25. *State s is SEQ_b for DM i iff for every $s_1 \in R_i^{+}(s)$ there exists at least one $s_2 \in R_{N\backslash\{i\}}^{+,U}(s_1)$ with $s \succeq_i s_2$.*

For definitions indexed c, DM i would move to preferred states and states having uncertain preference relative to the starting state. With respect to sanctions, DM i does not want to end up at states that are less preferred or equally preferred relative to state s, and states having uncertain preference relative to state s.

Definition 2.26. *State s is $Nash_c$ for DM i iff $R_i^{+,U}(s) = \emptyset$.*

Definition 2.27. *State s is GMR_c for DM i iff for every $s_1 \in R_i^{+,U}(s)$ there exists at least one $s_2 \in R_{N\backslash\{i\}}(s_1)$ with $s \succeq_i s_2$ or sU_is_2.*

Definition 2.28. *State s is SMR_c for DM i iff for every $s_1 \in R_i^{+,U}(s)$ there exists at least one $s_2 \in R_{N\backslash\{i\}}(s_1)$, such that $s \succeq_i s_2$ or sU_is_2 and $s \succeq_i s_3$ or sU_is_3 for any $s_3 \in R_i(s_2)$.*

Definition 2.29. *State s is SEQ_c for DM i iff for every $s_1 \in R_i^{+,U}(s)$ there exists at least one $s_2 \in R_{N\backslash\{i\}}^{+,U}(s_1)$ with $s \succeq_i s_2$ or sU_is_2.*

For the last set of stabilities, indexed by d, a DM is not willing to move to a state with uncertain preference relative to the status quo, but is deterred by sanctions to states that have uncertain preference relative to the status quo.

Definition 2.30. *State s is $Nash_d$ for DM i iff $R_i^{+}(s) = \emptyset$.*

Definition 2.31. *State s is GMR_d for DM i iff for every $s_1 \in R_i^{+}(s)$ there exists at least one $s_2 \in R_{N\backslash\{i\}}(s_1)$ with $s \succeq_i s_2$ or sU_is_2.*

Definition 2.32. *State s is SMR_d for DM i iff for every $s_1 \in R_i^{+}(s)$ there exists at least one $s_2 \in R_{N\backslash\{i\}}(s_1)$, such that $s \succeq_i s_2$ or sU_is_2 and $s \succeq_i s_3$ or sU_is_3 for any $s_3 \in R_i(s_2)$.*

Definition 2.33. *State s is SEQ_d for DM i iff for every $s_1 \in R_i^{+}(s)$ there exists at least one $s_2 \in R_{N\backslash\{i\}}^{+,U}(s_1)$ with $s \succeq_i s_2$ or sU_is_2.*

When $n = 2$, the DM set N reduces to $\{i, j\}$ in Definitions 2.18 to 2.33. For example, the reachable list $R_{N\backslash\{i\}}^{+,U}(s_1)$ of $N\backslash\{i\}$ from s_1 by the legal sequences of UIUUMs reduces to the reachable list $R_j^{+,U}(s_1)$ of j from s_1 by one step UIUUMs.

From the solution concepts indexed a, b, c, and d presented above, it can be seen that a solution concept indexed a represents the stability for the most aggressive DMs. Firstly, the DM is aggressive in deciding whether to move from the status quo, being willing to accept the risk associated with moves to states of uncertain preference. In addition, when evaluating possible moves, the DM is deterred only by sanctions to states that are less preferred than the status quo and does not see states of uncertain preference (relative to the status quo) as sanctions. For the definitions indexed b, uncertainty in preferences is not considered by a DM. The definitions indexed c incorporate a mixed attitude toward the risk associated with states of uncertain preference. Specifically, the DM is aggressive in deciding whether to move from the status quo, but is conservative when evaluating possible moves, being deterred by sanctions to states that are less preferred or have uncertain preference relative to the status quo. Finally, the definition indexed d represents stability for the most conservative DMs, who would move only to preferred states from a status quo, but would be deterred by responses that result in states of uncertain preference [46].

2.2.5 Solution Concepts in the Graph Model with Strength of Preference

Hamouda et al. [27] first integrated strength of preference information into the graph model and extended the four basic solution concepts to handle strength of preference for 2-DM graph models. Lately, they further extended the four solution concepts to multiple-decision-maker graph models [28].

Four standard solution concepts are given below in which strength of preference is not considered in sanctioning. However, the standard stabilities are different from those defined in [16], though they are presented using the same notation, because stability definitions for simple preference cannot analyze conflict models having strength of preference. Let $i \in N$ and $s \in S$ for next definitions taken from [28].

Definition 2.34. *State s is Nash stable for DM i, denoted by $s \in S_i^{Nash}$, iff $R_i^{+,++}(s) = \emptyset$.*

Definition 2.35. *State s is GMR for DM i, denoted by $s \in S_i^{GMR}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$ such that $s_2 \in \Phi_i^{--,-,=}(s)$.*

Definition 2.36. *State s is SMR for DM i, denoted by $s \in S_i^{SMR}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$, such that $s_2 \in \Phi_i^{--,-,=}(s)$ and $s_3 \in \Phi_i^{--,-,=}(s)$ for any $s_3 \in R_i(s_2)$.*

Definition 2.37. *State s is SEQ for DM i, denoted by $s \in S_i^{SEQ}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}^{+,++}(s_1)$ such that $s_2 \in \Phi_i^{--,-,=}(s)$.*

With strength of preference introduced into the graph model, stability definitions can be strong or weak, according to the level of sanctioning. Strong and weak stabilities only include GMR, SMR, and SEQ because Nash stability does not involve sanctions.

Definition 2.38. *State s is strongly GMR (SGMR) for DM i, denoted by $s \in S_i^{SGMR}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$ such that $s_2 \in \Phi_i^{--}(s)$.*

Definition 2.39. *State s is strongly SMR (SSMR) for DM i, denoted by $s \in S_i^{SSMR}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$, such that $s_2 \in \Phi_i^{--}(s)$ and $s_3 \in \Phi_i^{--}(s)$ for all $s_3 \in R_i(s_2)$.*

Definition 2.40. *State s is strongly SEQ (SSEQ) for DM i, denoted by $s \in S_i^{SSEQ}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}^{+,++}(s_1)$ such that $s_2 \in \Phi_i^{--}(s)$.*

Definitions 2.38 to 2.40 are adapted from [28] in which Nash stability is excluded from SGMR, SSMR, and SSEQ. The definition of weak stability is presented next.

Definition 2.41. *Let $s \in S$ and $i \in N$. State s is weakly stable for DM i iff s is stable, but not strongly stable for some stability definition.*

Based on the individual stability analysis, DMs can request additional follow up analyses to generate valuable decision guidance. The follow-up analyses include status quo analysis, coalition analysis, and sensitivity analyses.

2.2.6 Status Quo Analysis

When a conflict is modeled as a graph model, a point in time must be selected first; the current (or initial) state of the conflict is then referred to as the status quo [47]. Two fundamental steps are involved in analyzing a graph model, stability analysis and post-stability (or follow-up) analysis. When the stability of a state is assessed at the stability stage, it is not a concern whether this state is actually achievable from the status quo state. As a follow-up analysis, status quo analysis is to determine whether a particular equilibrium is reachable from the status quo and, if so, how to reach it. Thus, in contrast to stability analysis, which identifies states that would be stable if attained, status quo analysis provides a dynamic and forward-looking perspective, identifying states that are attainable, and describing how to reach them [47, 48].

Let $i \in N$ and $H \subseteq N$ and let $k \geq 1$ be an integer. New notation is required, as follows:

- SQ denotes the status quo state;
- The state sets, $S_i^{(k)}(s)$, $S_i^{(k,+)}(s)$, and $S_i^{(k,+U)}(s)$, denote the states reachable from $SQ = s$ in legal sequences of exactly k UMs, UIs, and UIUUMs, respectively, with last mover DM i;
- The state sets, $V_H^{(k)}(s)$, $V_H^{(k,+)}(s)$, and $V_H^{(k,+U)}(s)$, denote the sets of states reachable from $SQ = s$ in legal sequences of at most k UMs, UIs, and UIUUMs by H, respectively; (if $H = N$, then $V_H^{(k)}(s) = V^{(k)}(s)$, $V_H^{(k,+)}(s) = V^{(k,+)}(s)$, and $V_H^{(k,+U)}(s) = V^{(k,+U)}(s)$.)
- The arc sets, $A_i^{(k)}(s)$, $A_i^{(k,+)}(s)$, and $A_i^{(k,+U)}(s)$, denote the arcs controlled by DM i that are final arcs in legal sequences of at most k UMs, UIs, and UIUUMs, respectively, from $SQ = s$.

Recall that A_i is DM i's arc set in a graph model. Let A_i^+ and $A_i^{+,U}$ denote i's UI arc set and UIUUM arc set, respectively. For $s \in S$, let $A_i(s)$, $A_i^+(s)$, and $A_i^{+,U}(s)$ denote the respective subsets of these three arc sets with initial state s. Therefore, these arc sets are expressed by $A_i = \bigcup\limits_{s \in S} A_i(s)$, $A_i^+ = \bigcup\limits_{s \in S} A_i^+(s)$, and $A_i^{+,U} = \bigcup\limits_{s \in S} A_i^{+,U}(s)$.

The following algorithm permits all UMs for simple preference:

Algorithm for status quo analysis in the graph model with legal UMs [47]

1. Let $h = 0$, $S_i^{(0)}(SQ) = \{SQ\}$, $V^{(0)}(SQ) = \{SQ\}$, and $A_i^{(0)}(SQ) = \emptyset$ (for $i \in N$)
2. Let $h = h + 1$, and for each $i \in N$, update $S_i^{(h)}(SQ)$ and $A_i^{(h)}(SQ)$ as:

$$S_i^{(h)}(SQ) = \bigcup\{R_i(s) : s \in \bigcup_{j \in N \setminus i} S_j^{(h-1)}(SQ)\}$$

$$A_i^{(h)}(SQ) = \begin{cases} A_i^{(h-1)}(SQ) & \text{if } S_i^{(h)}(SQ) = \emptyset, \\ A_i^{(h-1)}(SQ) \bigcup \{(s, s') : s \in \bigcup_{j \in N \setminus i} S_i^{(h-1)}(SQ) \text{ } and \text{ } s' \in R_i(s)\} & \text{otherwise.} \end{cases}$$

$$V^{(h)}(SQ) = (\bigcup_{i \in N} S_i^{(h)}(SQ)) \bigcup V^{(h-1)}(SQ)$$

3. If $\bigcup_{i \in N} A_i^{(h)}(SQ) = \bigcup_{i \in N} A_i^{(h-1)}(SQ)$, stop.
   Otherwise, go to 2.

Although [47] indicates that the process must stop in a finite number of iterations, this condition is not explained in detail. If the algorithm stops at step k, the status quo diagram of permitted UMs in the graph model is given by $(V^{(k)}(SQ), \bigcup_{i \in N} A_i^{(k)}(SQ))$. Similarly, an algorithm that permits only UIs can be found in [47].

The following algorithm permits only UIUUMs for preference with uncertainty.

Algorithm for status quo analysis in the graph model with legal UIUUMs [48]

1. Let $h = 0$, $S_i^{(0,+U)}(SQ) = \{SQ\}$, $V^{(0,+U)}(SQ) = \{SQ\}$, and $A_i^{(0,+U)}(SQ) = \emptyset$ (for $i \in N$)
2. Let $h = h + 1$, and for each $i \in N$, update $S_i^{(h,+U)}(SQ)$, $A_i^{(h,+U)}(SQ)$ and $V^{(h,+U)}(SQ)$ as:

$$S_i^{(h,+U)}(SQ) = \bigcup\{R_i^{+,U}(s) : s \in \bigcup_{j \in N \setminus i} S_j^{(h-1,+U)}(SQ)\}$$

$$A_i^{(h,+U)}(SQ) = \begin{cases} A_i^{(h-1,+U)}(SQ) & \text{if } S_i^{(h,+U)}(SQ) = \emptyset, \\ A_i^{(h-1,+U)}(SQ) \bigcup \{(s, s') : s \in \bigcup_{j \in N \setminus i} S_i^{(h-1,+U)}(SQ), s' \in R_i^{+,U}(s)\} & \text{otherwise.} \end{cases}$$

$$V^{(h,+U)}(SQ) = (\bigcup_{i \in N} S_i^{(h,+U)}(SQ)) \bigcup V^{(h-1,+U)}(SQ)$$

3. If $\bigcup_{i \in N} A_i^{(h,+U)}(SQ) = \bigcup_{i \in N} A_i^{(h-1,+U)}(SQ)$, stop.
   Otherwise, go to 2.

Similarly, $(V^{(k,+U)}(SQ), \bigcup_{i \in N} A_i^{(k,+U)}(SQ))$ presents the status quo diagram permitted UIUUMs in the graph model when the above algorithm stops at iteration step k. Although the algorithms were developed for status quo analysis for simple preference and preference with uncertainty but have never been integrated into GMCR II.

Using status quo diagrams, significant information about the conflict under investigation can be obtained. Specifically, if an equilibrium is in the diagram, the analysis provides a path from the status quo to the reachable equilibrium; if not, the DMs have no way to control the conflict to the equilibrium. Status quo analysis can provide guidance for DMs and analysts by identifying how to attain reachable equilibria from a status quo state [47, 48].

2.2.7 Coalition Stability Analysis

Coalition H is a subset of DMs with $|H| \geq 2$. For an equilibrium, no DM has the incentive to move away from it, but a coalition may sometimes be able to move away from the equilibrium to a better state for all members of the coalition. Hence, analysts can detect equilibria that are vulnerable to coalition moves in strategic conflicts [43].

Coalitions and coalition stability have been widely studied in the area of conflict analysis. For example, inspired by Aumann [3], Kilgour et al. [43] proposed coalition stability based on Nash stability within the framework of the GMCR. Then, Inohara and Hipel [36, 37] extended the above Nash coalition stability to GMR, SMR, and SEQ coalition stabilities. However, to make coding easier, these extensions were based on a transitive graph that allows the same DM to move twice in succession, which is inconsistent with the standard restriction in the graph model. For example, in the work of [36, 37, 43], the reachable list of a coalition, $R_H(s)$, may include states reachable only by consecutive moves of the same DM. Additionally, these coalition stabilities were defined logically within a simple preference structure. The Simple preference structure is often inadequate for modeling the complex strategic conflicts that arise in practical applications. The following coalition stabilities based on Nash stability are taken from [43].

Definition 2.42. *For $s_1 \in R_H(s)$, s_1 is a coalition improvement by H from state s iff, for every $i \in H$, satisfies $s_1 \succ_i s$.*

A coalition improvement s_1 by H indicates a threat, or potential threat, to the stability of state s.

Definition 2.43. *State s is unstable for coalition H iff there exists a coalition improvement by H from s.*

Even if state s is stable for each DM $i \in N$, the instability of s for coalition H makes s impossible to become a resolution for a conflict.

Definition 2.44. *State $s \in S$ is stable for coalition $H \subseteq N$ ($\mid H \mid \geq 2$) iff, for every $s_1 \in R_H(s)$, there exists $i \in H$ with $s \succeq_i s_1$.*

Definition 2.45. *State $s \in S$ is coalitionally stable iff s is stable for every coalition $H \subseteq N$ ($\mid H \mid \geq 2$).*

Note that if the reachable list $R_H(s)$ of H from state s in the above definitions is adapted to use the definition in Section 2.2.2, then a transitive graph is extended to a general graph.

2.2.8 The Decision Support System GMCR II

Although the graph model for conflict resolution has many advantages, it is difficult to apply to real problems without computational assistance, even to small models. For this purpose, the basic decision support system (DSS) GMCR I was developed in [42]. However, GMCR I only includes a basic analysis engine, so that a model must be converted to the GMCR I data format first, which is a difficult conversion process. The DSS GMCR II [32, 54], including modeling and analysis procedures, later replaced GMCR I. GMCR II, is written in Visual C++, a computer implementation of the graph model for conflict resolution, and is described by [16, 18, 19].

The DSS GMCR II offers model management and stability analysis and includes some basic coalition analysis and status quo analysis for simple preference. At present, GMCR II allows for status quo analysis, but does not implement it fully. A consistent and effective set of status quo analysis definitions and algorithms was proposed by [47, 48] but has not been included in GMCR II.

Sensitivity analyses in GMCR II are carried out by varying the model input in the following categories: options, state transitions, preferences, DMs (including addition and deletion), and solution concepts, including changing individual stabilities into coalition stabilities. Although sensitivity analyses are a popular technique in solving engineering problems, in GMCR II, few papers focus on sensitivity analyses. If a conflict analytical result is very sensitive to variations of some parameters, the result may not provide useful guidance in real applications, so sensitivity analysis should be an important research topic in conflict analyses.

2.3 Degree of Preference

Obviously, preferences play an important role in decision analysis. How to obtain individual preference information has already been ascertained by extensive research, such as the Analytic Hierarchy Process (AHP) approach [57] and some approaches used by modeling preference relations of consumers in microeconomics [49]. Normally, for the graph model only a relative preference relation, "$\succ$ preferred", and an equal relation, "$\sim$ indifferent" are needed to represent a particular DM i's preference for one state with respect to another to calibrate a specific model [16]. This type of preference is called a two-level preference in this thesis. Different definitions for strength of preference can be found in [4, 15]. Dyer and Sarin [15] indicate the relations between strength of preference and risky behavior. In 2004, Hamouda et al. [27] proposed "strength of preference" that includes two new binary relations, "$\gg$ strongly preferred", and, "$>$ mildly preferred", to express DM i's strong and mild preferences for one state over another, respectively, as well as an equal relation. This is referred to as a three-level preference.

However, the 3-level structure is limited in its ability to depict the intensity of relative preference. For example, in the Analytic Hierarchy Process (AHP) [58], strength of preference is reflected a scale from 1 to 9. Table 2.4 presents an interpretation for strength at levels 1, 3, 5, 7, and 9 in the AHP approach and motivates the extension of the three-level model to a multiple-level model that can capture a range of degrees. In related, but quite different research, significant effort has been devoted to representing preference information by degree or strength. For example, Wang et al. [64] presented a probability method to represent preferences with certain degrees or strength.

2.4 Summary

After reviewing the background of Graph Theory and the Graph Model for Conflict Resolution, we know that a graph model of a conflict consists of several interrelated graphs and preference relations, and three types of preference structures have been developed and introduced into the graph model for conflict resolution. To enhance the applicability of GMCR, in Chapters 3 and 4, the three preference frameworks are extended to a hybrid system in which preference uncertainty and strength

Table 2.4: **Scale of relative preference** [58]

Intensity of relative preference	Definition	Description
1	Equally important	Two events are equally preferred.
3	Moderately important	The first event is slightly preferred to the second.
5	Quite important	The first event is much more preferred than the second.
7	Demonstrably important	The first event is very strongly preferred to the second.
9	Extremely important	The first event is extremely preferable to the second.

of preference are combined together and a system of multiple levels of preference. Previously, individual and coalition stabilities in the graph model were traditionally defined logically, in terms of the underlying graphs and preference relations. Status quo analysis follows a similar logical structure. However, as was noted in the development of the DSS GMCR II, the nature of logical representations makes coding difficult. A new algebraic system based on Algebraic Graph Theory to represent stability analysis and post-stability analysis is proposed in Chapter 5 to Chapter 7.

Chapter 3

Hybrid Preference for the Graph Model for Conflict Resolution

3.1 Combining Preference Uncertainty and Strength of Preference

A hybrid preference framework is proposed for strategic conflict analysis to integrate preference strength and preference uncertainty into the paradigm of the graph model for conflict resolution (GMCR) under multiple decision makers. This structure offers decision makers a more flexible mechanism for preference expression, which can include strong or mild relative relationship of one state over another, an indifference relation, and uncertain preference between two states.

To date, only three types of preference structures–simple preference, preference possibly including uncertainty, and preference having strength–have been integrated into GMCR. To integrate the three existing preference frameworks into a hybrid system, a new preference framework $\{\gg_i, >_i, \sim_i, U_i\}$ is defined using a quadruple relation in a graph model for DM i. The preference structure $\{\gg_i, >_i, \sim_i, U_i\}$ is complete, i.e. if $s, q \in S$, then exactly one of the following relations holds: $s \gg_i q$, $q \gg_i s$, $s >_i q$, $q >_i s$, $s \sim_i q$, and sU_iq. Note that notation, $\Phi_i^{+m}(s)$, $\Phi_i^{-m}(s)$, $R_i^{+m}(s)$, and $R_i^{-m}(s)$, is replaced with $\Phi_i^{+}(s)$, $\Phi_i^{-}(s)$, $R_i^{+}(s)$, and $R_i^{-}(s)$, respectively, in this chapter. For hybrid preference, DM i can identify six subsets of S: $\Phi_i^{++}(s)$, $\Phi_i^{+}(s)$, $\Phi_i^{=}(s)$, $\Phi_i^{U}(s)$, $\Phi_i^{-}(s)$, and $\Phi_i^{--}(s)$, and can control six corresponding reachable lists from state s: $R_i^{++}(s)$,

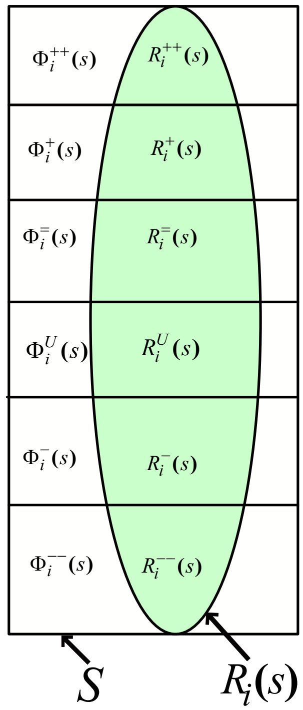


Figure 3.1: **Relations among subsets of S and reachable lists from s.**

$R_i^{+}(s)$, $R_i^{=}(s)$, $R_i^{U}(s)$, $R_i^{-}(s)$, and $R_i^{--}(s)$, where these subsets and reachable lists from state s are defined in Tables 2.2 and 2.3, respectively. The relationships among the subsets of state set S and the reachable lists from state s for DM i are portrayed in Fig. 3.1.

The reachable list from state s for DM i in one step, $R_i(s)$, represents DM $i's$ various unilateral moves (UMs) for hybrid preference, so $R_i(s) = R_i^{++}(s) \cup R_i^{+}(s) \cup R_i^{=}(s) \cup R_i^{U}(s) \cup R_i^{-}(s) \cup R_i^{--}(s)$. For ease of use, the notation with respect to UMs and subsets of the state set S for hybrid preference is presented as follows:

- $R_i^{+,++,U}(s) = R_i^{+}(s) \cup R_i^{++}(s) \cup R_i^{U}(s)$ stands for mild unilateral improvements, strong unilateral improvements, or unilateral uncertain moves called weak improvements or unilateral uncertain moves (WIUUMs) from state s for DM i;

- $\Phi_i^{--,U}(s) = \Phi_i^{--}(s) \cup \Phi_i^{U}(s)$; and

- $\Phi_i^{--,-,=,U}(s) = \Phi_i^{--}(s) \cup \Phi_i^{-}(s) \cup \Phi_i^{=}(s) \cup \Phi_i^{U}(s)$.

Note that the assumption of transitivity of preferences is not required, and thus the results in this research hold for both transitive and intransitive preferences.

3.2 Stability Analysis in the Graph Model for Hybrid Preference

To analyze the stability of a state for DM $i \in N$ for hybrid preference, it is necessary to take into account possible responses by all other DMs $j \in N\backslash\{i\}$. Therefore, the previous definitions for legal sequences of decisions in the graph model with preference uncertainty [46] and with preference of strength [28] must first be extended to take combining preference uncertainty and preference strength into account.

3.2.1 Reachable Lists of Coalition H

The legal sequences of UMs, UIs, and UIUUMs are defined in Subsection 2.2.2. For hybrid preference, a legal sequence of WIUUMs for a coalition of DMs is a sequence of states linked by weak improvements or unilateral uncertain moves by members of the coalition, in which a DM may move more than once, but not twice consecutively.

Let $H \subseteq N$ be any subset of DMs. Within hybrid preference, the definition of the reachable list $R_H(s)$ for coalition H by the legal UMs starting at state s is similar to Definition 2.10 in Subsection 2.2.2. The definition of $R_H^{+,++}(s)$ in hybrid preference is similar to Definition 2.13. Let coalition $H \subseteq N$ satisfy $|H| \geq 2$ and let the status quo state be $s \in S$. We now define reachable list $R_H^{+,++,U}(s)$ for coalition H with the explicit hybrid preference.

Definition 3.1. *Let $R_j^{+,++,U}(s) = R_j^{+}(s) \cup R_j^{++}(s) \cup R_j^{U}(s)$ for any $j \in H$. A weak improvement or unilateral uncertain move by H is a member of $R_H^{+,++,U}(s) \subseteq S$, defined inductively by:*
(1) if $j \in H$ and $s_1 \in R_j^{+,++,U}(s)$, then $s_1 \in R_H^{+,++,U}(s)$ and $j \in \Omega_H^{+,++,U}(s, s_1)$;
(2) if $s_1 \in R_H^{+,++,U}(s)$, $j \in H$ and $s_2 \in R_j^{+,++,U}(s_1)$, then, provided $\Omega_H^{+,++,U}(s, s_1) \neq \{j\}$, $s_2 \in R_H^{+,++,U}(s)$ and $j \in \Omega_H^{+,++,U}(s, s_2)$.

Note that this definition is inductive: first, using (1), the states reachable by a single DM of coalition H in one step WIUUM from s are identified and added to $R_H^{+,++,U}(s)$; then, using (2), all states reachable from those states are identified and added to $R_H^{+,++,U}(s)$; then the process is repeated until no further states are added to $R_H^{+,++,U}(s)$ by repeating (2). Because $R_H^{+,++,U}(s) \subseteq S$, and S is finite, this limit must be reached in a finite number of steps.

To interpret Definition 3.1, note that if $s_1 \in R_H^{+,++,U}(s)$, then $\Omega_H^{+,++,U}(s, s_1) \subseteq N$ is the set of all last DMs in legal sequence of WIUUMs from s to s_1. (If $s_1 \notin R_H^{+,++,U}(s)$, it is assumed that $\Omega_H^{+,++,U}(s, s_1) = \emptyset$.) Suppose that $\Omega_H^{+,++,U}(s, s_1)$ contains only one DM, say $j \in N$. Then any move from s_1 to a subsequent state, say s_2, must be made by a member of H other than j; otherwise DM j would have to move twice in succession. On the other hand, if $|\Omega_H^{+,++,U}(s, s_1)| \geq 2$, any member of H who has a mild unilateral improvement or strong unilateral improvement (weak improvement) or unilateral uncertain move from s_1 to s_2 may exercise it.

For the simple preference structure, a state s is either stable or unstable [16]. For the framework with strength of preference, if a state s is stable, then it is either strongly stable or weakly stable based on sanctioning strength [27, 28]. Li et al. [46] proposed solution concepts with preference uncertainty that are separately classified into four extensions, indexed a, b, c, and d, according to the incentives to leave the status quo state and the motivation to avoid states of uncertain preference relative to the status quo. Since possible uncertainty is included in DMs' preferences, a range of extensions of stability definitions is needed to address DMs' attitudes with distinct risk profiles in face of uncertainty. For example, a DM will make a conservative or aggressive decision depended on the DM's current status "satisfied" or "unsatisfied" [46].

According to the proposed new preference structure, the hybrid versions of solution concepts refer to stabilities, strong stabilities, and weak stabilities indexed a, b, c, and d, respectively. In the following stabilities, strength of preference is not considered in sanctioning.

3.2.2 Stabilities in the Graph Model for Hybrid Preference

The stability definitions in the graph model for two DM conflicts with hybrid preference are special cases of the definitions proposed in the next subsection, the details are not given here.

3.2.2.1 Stabilities, Indexed a, for Hybrid Preference

For stabilities indexed a, DM i is willing to move to states that are mildly preferred or strongly preferred, as well as states having uncertain preference relative to the status quo but does not wish to be sanctioned by a strongly less preferred, mildly less preferred, or equally preferred state relative to the status quo. The definitions given below assume that $s \in S$ and $i \in N$.

Definition 3.2. *State s is $Nash_a$ for DM i, denoted by $s \in S_i^{Nash_a}$, iff $R_i^{+,++,U}(s) = \emptyset$.*

Definition 3.3. *State s is GMR_a for DM i, denoted by $s \in S_i^{GMR_a}$, iff for every $s_1 \in R_i^{+,++,U}(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$ with $s_2 \in \Phi_i^{--,-,=}(s)$.*

Definition 3.4. *State s is SMR_a for DM i, denoted by $s \in S_i^{SMR_a}$, iff for every $s_1 \in R_i^{+,++,U}(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$, such that $s_2 \in \Phi_i^{--,-,=}(s)$ and $s_3 \in \Phi_i^{--,-,=}(s)$ for any $s_3 \in R_i(s_2)$.*

Definition 3.5. *State s is SEQ_a for DM i, denoted by $s \in S_i^{SEQ_a}$, iff for every $s_1 \in R_i^{+,++,U}(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}^{+,++,U}(s_1)$ with $s_2 \in \Phi_i^{--,-,=}(s)$.*

It should be pointed out that the same notation for stabilities indexed a for preference with uncertainty presented in Subsection 2.2.4 is used for hybrid preference. However, they have different meaning, since current definitions can analyze conflict models including hybrid preference. The following definitions are still presented using the same notation as those including preference uncertainty.

3.2.2.2 Stabilities, Indexed b, for Hybrid Preference

For stabilities indexed b, DM i will move only to mildly or strongly preferred states from a status quo, but does not want to be sanctioned by a strongly less preferred, mildly less preferred, or equally preferred state relative to the status quo.

Definition 3.6. *State s is $Nash_b$ for DM i, denoted by $s \in S_i^{Nash_b}$, iff $R_i^{+,++}(s) = \emptyset$.*

Definition 3.7. *State s is GMR_b for DM i, denoted by $s \in S_i^{GMR_b}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$ with $s_2 \in \Phi_i^{--,-,=}(s)$.*

Definition 3.8. *State s is SMR_b for DM i, denoted by $s \in S_i^{SMR_b}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$, such that $s_2 \in \Phi_i^{--,-,=}(s)$ and $s_3 \in \Phi_i^{--,-,=}(s)$ for any $s_3 \in R_i(s_2)$.*

Definition 3.9. *State s is SEQ_b for DM i, denoted by $s \in S_i^{SEQ_b}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}^{+,++,U}(s_1)$ with $s_2 \in \Phi_i^{--,-,=}(s)$.*

The above definitions indexed b which exclude uncertainty in preference are different from those discussed by Hamouda et al. [28], since current definitions are utilized to analyze conflict models under combining preference uncertainty and strength of preference.

3.2.2.3 Stabilities, Indexed c, for Hybrid Preference

For definitions indexed c, DM i can move to mildly preferred, strongly preferred states, as well as states having uncertain preference relative to the starting state. With respect to sanctioning, DM i does not want to be ended up at states that are mildly less preferred, strongly less preferred, or equally preferred, as well as states having uncertain preference relative to state s.

Definition 3.10. *State s is $Nash_c$ for DM i, denoted by $s \in S_i^{Nash_c}$, iff $R_i^{+,++,U}(s) = \emptyset$.*

Definition 3.11. *State s is GMR_c for DM i, denoted by $s \in S_i^{GMR_c}$, iff for every $s_1 \in R_i^{+,++,U}(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$ with $s_2 \in \Phi_i^{--,-,=,U}(s)$.*

Definition 3.12. *State s is SMR_c for DM i, denoted by $s \in S_i^{SMR_c}$, iff for every $s_1 \in R_i^{+,++,U}(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$, such that $s_2 \in \Phi_i^{--,-,=,U}(s)$ and $s_3 \in \Phi_i^{--,-,=,U}(s)$ for any $s_3 \in R_i(s_2)$.*

Definition 3.13. *State s is SEQ_c for DM i, denoted by $s \in S_i^{SEQ_c}$, iff for every $s_1 \in R_i^{+,++,U}(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}^{+,++,U}(s_1)$ with $s_2 \in \Phi_i^{--,-,=,U}(s)$.*

3.2.2.4 Stabilities, Indexed d, for Hybrid Preference

For the last set of stabilities, indexed d, a DM is not willing to move to a state with uncertain preference relative to the status quo, but is deterred by sanctions to states that have uncertain preference relative to the status quo.

Definition 3.14. *State s is $Nash_d$ for DM i, denoted by $s \in S_i^{Nash_d}$, iff $R_i^{+,++}(s) = \emptyset$.*

Definition 3.15. *State s is GMR_d for DM i, denoted by $s \in S_i^{GMR_d}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$ with $s_2 \in \Phi_i^{--,-,=,U}(s)$.*

Definition 3.16. *State s is SMR_d for DM i, denoted by $s \in S_i^{SMR_d}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$, such that $s_2 \in \Phi_i^{--,-,=,U}(s)$ and $s_3 \in \Phi_i^{--,-,=,U}(s)$ for any $s_3 \in R_i(s_2)$.*

Definition 3.17. *State s is SEQ_d for DM i, denoted by $s \in S_i^{SEQ_d}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}^{+,++,U}(s_1)$ with $s_2 \in \Phi_i^{--,-,=,U}(s)$.*

When $n = 2$, the DM set N becomes to $\{i, j\}$ in Definitions 3.2 to 3.17, and the reachable lists for $H = N \setminus \{i\}$ by legal sequences of UMs and WIUUMs from s_1, $R_{N\setminus\{i\}}(s_1)$ and $R_{N\setminus\{i\}}^{+,++,U}(s_1)$, degenerate to $R_j(s_1)$ and $R_j^{+,++,U}(s_1)$, DM j's corresponding reachable lists from s_1.

If the binary relation $\succ$ denotes $>$ or $\gg$ in this research, i.e., $s \succ q$ iff either $s > q$ or $s \gg q$, then Definitions 3.2 to 3.17 are identical with Definitions 2.18 to 2.33 in Chapter 2 proposed by Li et al. [46]. On the other hand, when each DM does not consider including uncertain preference in stability analysis, the above definitions reduce to the standard stability definitions from Definitions 2.34 to 2.37 in Chapter 2 developed by Hamouda et al. [28].

3.2.3 Strong Stabilities under Hybrid Preference for Multiple Decision Makers

With the hybrid preference framework introduced into the graph model, stable states can be classified into strongly stable or weakly stable according to strength of the possible sanctions and indexed a, b, c, or d by a DM's attitudes toward the risk associated with uncertain preferences. Strong and weak stabilities include only GMR, SMR, and SEQ because Nash stability does not involve sanctions.

3.2.3.1 Strong Stabilities, Index a, for Hybrid Preference with Strength of Preference

Definition 3.18. *State s is **strongly** GMR_a ($SGMR_a$) for DM i, denoted by $s \in S_i^{SGMR_a}$, iff for every $s_1 \in R_i^{+,++,U}(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$ such that $s_2 \in \Phi_i^{--}(s)$.*

Definition 3.19. *State s is **strongly** SMR_a ($SSMR_a$) for DM i, denoted by $s \in S_i^{SSMR_a}$, iff for every $s_1 \in R_i^{+,++,U}(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$, such that $s_2 \in \Phi_i^{--}(s)$ and $s_3 \in \Phi_i^{--}(s)$ for all $s_3 \in R_i(s_2)$.*

Definition 3.20. *State s is **strongly** SEQ_a ($SSEQ_a$) for DM i, denoted by $s \in S_i^{SSEQ_a}$, iff for every $s_1 \in R_i^{+,++,U}(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}^{+,++,U}(s_1)$ such that $s_2 \in \Phi_i^{--}(s)$.*

The above definitions indexed a represent strong stabilities for the most aggressive DMs. Firstly, DM i is aggressive in deciding whether to move from the status quo, since the DM considers moving to mildly or strongly preferred states, as well as states having uncertain preference relative to the status quo. This means that DM i is willing to accept the risk associated with moves from the status quo to states of uncertain preferences. In addition, when evaluating possible moves, DM i is strongly deterred by sanctions to states that are strongly less preferred relative to status quo state s.

3.2.3.2 Strong Stabilities, Index b, for Hybrid Preference with Strength of Preference

For the following definitions indexed b, DM i would move only to mildly or strongly preferred states and be deterred by sanctions to strongly less preferred states relative to the status quo.

Definition 3.21. *State s is **strongly** GMR_b ($SGMR_b$) for DM i, denoted by $s \in S_i^{SGMR_b}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$ such that $s_2 \in \Phi_i^{--}(s)$.*

Definition 3.22. *State s is **strongly** SMR_b ($SSMR_b$) for DM i, denoted by $s \in S_i^{SSMR_b}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$, such that $s_2 \in \Phi_i^{--}(s)$ and $s_3 \in \Phi_i^{--}(s)$ for all $s_3 \in R_i(s_2)$.*

Definition 3.23. *State s is **strongly** SEQ_b ($SSEQ_b$) for DM i, denoted by $s \in S_i^{SSEQ_b}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}^{+,++,U}(s_1)$ such that $s_2 \in \Phi_i^{--}(s)$.*

3.2.3.3 Strong Stabilities, Index c, for Hybrid Preference with Strength of Preference

The definitions indexed c refer to a DM's mixed attitudes toward the risk associated with uncertain preferences. Specifically, DM i is aggressive in deciding whether to move from the status quo, but is conservative when evaluating possible moves, because DM i is deterred by sanctions to states that are strongly less preferred and states that have uncertain preference relative to the status quo.

Definition 3.24. *State s is **strongly** GMR_c ($SGMR_c$) for DM i, denoted by $s \in S_i^{SGMR_c}$, iff for every $s_1 \in R_i^{+,++,U}(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$ such that $s_2 \in \Phi_i^{--,U}(s)$.*

Definition 3.25. *State s is **strongly** SMR_c ($SSMR_c$) for DM i, denoted by $s \in S_i^{SSMR_c}$, iff for every $s_1 \in R_i^{+,++,U}(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$, such that $s_2 \in \Phi_i^{--,U}(s)$ and $s_3 \in \Phi_i^{--,U}(s)$ for all $s_3 \in R_i(s_2)$.*

Definition 3.26. *State s is **strongly** SEQ_c ($SSEQ_c$) for DM i, denoted by $s \in S_i^{SSEQ_c}$, iff for every $s_1 \in R_i^{+,++,U}(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}^{+,++,U}(s_1)$ such that $s_2 \in \Phi_i^{--,U}(s)$.*

3.2.3.4 Strong Stabilities, Index d, for Hybrid Preferences with Strength of Preference

Definition 3.27. *State s is **strongly** GMR_d ($SGMR_d$) for DM i, denoted by $s \in S_i^{SGMR_d}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$ such that $s_2 \in \Phi_i^{--,U}(s)$.*

Definition 3.28. *State s is **strongly** SMR_d ($SSMR_d$) for DM i, denoted by $s \in S_i^{SSMR_d}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$, such that $s_2 \in \Phi_i^{--,U}(s)$ and $s_3 \in \Phi_i^{--,U}(s)$ for all $s_3 \in R_i(s_2)$.*

Definition 3.29. *State s is **strongly** SEQ_d ($SSEQ_d$) for DM i, denoted by $s \in S_i^{SSEQ_d}$, iff for every $s_1 \in R_i^{+,++}(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}^{+,++,U}(s_1)$ such that $s_2 \in \Phi_i^{--,U}(s)$.*

The above definitions indexed d indicate that DM i would move only to mildly or strongly preferred states, but is deterred by sanctions that could move i to strongly less preferred states and states that have uncertain preference relative to the status quo. Therefore, definitions indexed d represent strong stabilities for the most conservative DMs.

When $n = 2$ and the DM set N reduces to two DMs $\{i, j\}$, the reachable lists of coalition $N \setminus \{i\}$ by the legal sequences of UMs and WIUUMs from state s_1, $R_{N\setminus\{i\}}(s_1)$ and $R_{N\setminus\{i\}}^{+,++,U}(s_1)$, reduce to the reachable lists from s_1 by DM j, $R_j(s_1)$ and $R_j^{+,++,U}(s_1)$. Thus, Definitions 3.2 to 3.29 reduce to the definitions presented in [68] for two DM conflicts. Therefore, if one considers neither strength nor uncertainty in preferences, the above definitions will reduce to Definitions 2.14

to 2.17 proposed by Fang et al. [16] for simple preference. When DMs' preferences do not include uncertainty, Definitions 3.18 to 3.29 reduce to the strong stability definitions 2.38 to 2.40 defined by Hamouda et al. [28]; when DM i's preferences do not include strength, they reduce to Definitions 2.18 to 2.33 for the graph model with preference uncertainty developed by Li et al. [46].

3.2.4 Weak Stabilities, Index l, for Hybrid Preference with Strength of Preference

Let l denote one of the four extensions indexed a, b, c, and d, i.e., $l = a, b, c$, or d. In the following theorems, the symbol GS denotes a solution concept, GMR, SMR, or SEQ. Then GS_l refers to the GS solution concept indexed l, SGS refers to the strong solution concept of GS, and WGS refers to the weak solution concept of GS (defined below). The symbol $s \in S_i^{GS_l}$ denotes that $s \in S$ is stable for DM i according to stability GS indexed l. Similarly, $s \in S_i^{SGS_l}$ denotes that $s \in S$ is strongly stable for DM i according to strong stability SGS indexed l. A state is weakly stable iff it is stable, but not strongly stable. The formal weak stability concept is defined next.

Definition 3.30. *Let $s \in S$ and $i \in N$. State s is weakly stable WGS_l for DM i according to stability WGS indexed l, denoted by $s \in S_i^{WGS_l}$, iff $s \in S_i^{GS_l}$ and $s \notin S_i^{SGS_l}$.*

3.2.5 Interrelationships among Stabilities for Hybrid Preference

In 1993, Fang et al. [16] determined relationships among Nash, GMR, SMR, and SEQ for the simple preference structure. Following this research direction, Li et al. [46] and Hamouda et al. [28] established interrelationships among stability definitions with preference uncertainty and with strength of preference, respectively.

The following interrelationships among proposed stabilities for hybrid preference are similar to those clarified by Fang et al. [16]. Let $l = a, b, c$, or d. Then, the inclusion relationships among the four stabilities indexed l for hybrid preference are shown in Fig. 3.2.

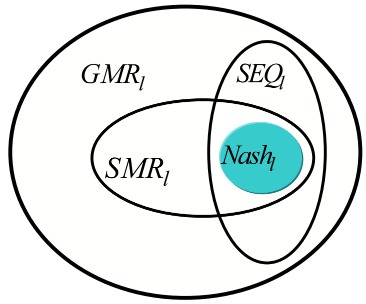


Figure 3.2: **Interrelationships among stabilities indexed l for hybrid preference.**

Under the hybrid preference, the interrelationships of stabilities, strong stabilities, and weak stabilities are as follows:

Theorem 3.1. *Let $l = a, b, c,$ or d and $i \in N$. The interrelationships among stability GS, strong stability SGS, and weak stability WGS indexed l for DM i are*

$$S_i^{WGS_l} = S_i^{GS_l} - S_i^{SGS_l}.$$

This result is obvious from Definition 3.30.

Based on definitions 3.2 to 3.29, the interrelationships among the four stabilities of Nash, GMR, SMR, and SEQ and the three strong stabilities of SGMR, SSMR, and SSEQ, indexed l for hybrid preference are given next.

Theorem 3.2. *Let $l = a, b, c,$ or d and $i \in N$. The interrelationships among the four stabilities and the three strong stabilities indexed l are*

$$S_i^{Nash_l} \subseteq S_i^{SSMR_l} \subseteq S_i^{SMR_l} \subseteq S_i^{GMR_l},$$

$$S_i^{Nash_l} \subseteq S_i^{SSEQ_l} \subseteq S_i^{SEQ_l} \subseteq S_i^{GMR_l},$$

and

$$S_i^{Nash_l} \subseteq S_i^{SGMR_l} \subseteq S_i^{GMR_l}.$$

The proof of Theorem 3.2 easily follows from the above definitions. Note that there is no necessary inclusion relationship between $S_i^{SSMR_l}$ and $S_i^{SSEQ_l}$, i.e., it may or may not be true that $S_i^{SSMR_l} \supseteq S_i^{SSEQ_l}$, or that $S_i^{SSMR_l} \subseteq S_i^{SSEQ_l}$.

Theorem 3.3. *The interrelationships among Nash stabilities indexed a, b, c, and d for DM i are*

$$S_i^{Nash_a} = S_i^{Nash_c}, \; S_i^{Nash_b} = S_i^{Nash_d},$$

and

$$S_i^{Nash_a} \subseteq S_i^{Nash_b}.$$

This result is obvious from the above Nash stability definitions.

Theorem 3.4. *Let $i \in N$. The interrelationships among stabilities GS and SGS indexed a, b, c, and d are*

$$S_i^{GS_a} \subseteq S_i^{GS_b} \subseteq S_i^{GS_d},\ S_i^{GS_a} \subseteq S_i^{GS_c} \subseteq S_i^{GS_d},$$

and

$$S_i^{SGS_a} \subseteq S_i^{SGS_b} \subseteq S_i^{SGS_d},\ S_i^{SGS_a} \subseteq S_i^{SGS_c} \subseteq S_i^{SGS_d}.$$

The interrelationships are shown in Fig. 3.3.

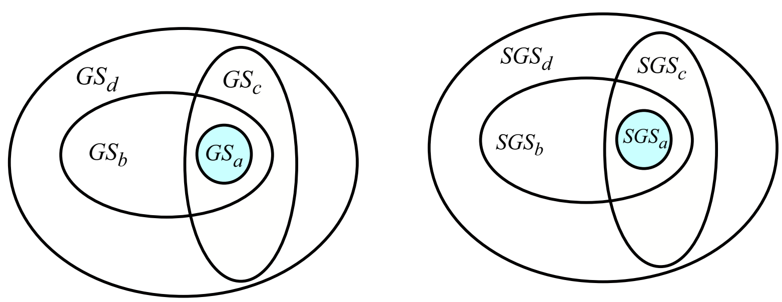


Figure 3.3: **Interrelationships for stability GS and strong stability SGS for all indexes.**

The inclusion relations about GS are similar with those regarding SGS, so we only provide the proofs about SGS. We first prove inclusion relations $S_i^{SSMR_a} \subseteq S_i^{SSMR_c} \subseteq S_i^{SSMR_d}$.

Proof: If state $s \in S_i^{SSMR_a}$, this implies that if $R_i^{+,++,U}(s) \neq \emptyset$ and $s_1 \in R_i^{+,++,U}(s)$, then there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$, such that $s_2 \in \Phi_i^{--}(s)$ and $s_3 \in \Phi_i^{--}(s)$ for all $s_3 \in R_i(s_2)$. Since $\Phi_i^{--}(s) \subseteq \Phi_i^{--,U}(s)$, then $s_2 \in \Phi_i^{--,U}(s)$ and $s_3 \in \Phi_i^{--,U}(s)$ for all $s_3 \in R_i(s_2)$. Therefore, if state $s \in S_i^{SSMR_a}$, then state $s \in S_i^{SSMR_c}$.

If state $s \in S_i^{SSMR_c}$, this implies that if $s_1 \in R_i^{+,++,U}(s)$, then there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$, such that $s_2 \in \Phi_i^{--,U}(s)$ and $s_3 \in \Phi_i^{--,U}(s)$ for all $s_3 \in R_i(s_2)$. Since $R_i^{+,++}(s) \subseteq R_i^{+,++,U}(s)$, then $s \in S_i^{SSMR_c}$ implies that if $s_1 \in R_i^{+,++}(s)$, then there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$, such that $s_2 \in \Phi_i^{--,U}(s)$ and $s_3 \in \Phi_i^{--,U}(s)$ for all $s_3 \in R_i(s_2)$. Therefore, $S_i^{SSMR_c} \subseteq S_i^{SSMR_d}$.

The inclusion relations $S_i^{SSMR_a} \subseteq S_i^{SSMR_c} \subseteq S_i^{SSMR_d}$ is proved. Other inclusion relations about $SGMR$ and $SSEQ$ can be proved, similarly. So

$$S_i^{SGS_a} \subseteq S_i^{SGS_c} \subseteq S_i^{SGS_d}.$$

The proof of the inclusion relations

$$S_i^{SGS_a} \subseteq S_i^{SGS_b} \subseteq S_i^{SGS_d}$$

can be similarly proved. □

3.3 Computational Stability Analysis and Status Quo Analysis

In n-DM models, $R_H(s)$, the reachable list of coalition H by the legal UMs starting at s, and $R_H^{+,++,U}(s)$, the reachable list of coalition H by the legal WIUUMs starting at s, are key inputs to stability analysis in the hybrid preference framework. Although Li et al. [46] and Hamouda et al. [28] proposed definitions for related sets $R_H^{+,U}$ and $R_H^{+,++}$, no algorithms for them have been developed.

As a follow-up analysis, status quo analysis traces conflict evolution from a status quo state to any specific outcome. It usually focuses on whether possible equilibria are reachable from the status quo, and if so, how to reach them. Thus, status quo analysis provides useful forward-looking insights into a strategic conflict, helping DMs and analysts to identify how to attain a reachable equilibria from a status quo state. GMCR II [18, 19] allows for status quo analysis, but does not implement it. Subsequently, [47, 48] developed status quo analysis definitions and the corresponding pseudo codes, but did not induce strength of preference. In this section, the algorithms for the essential inputs of stability analysis and status quo analysis are developed for hybrid preference.

Let $i \in N$ and $H \subseteq N$ and let $k \geq 1$ be an integer. With the notation defined in Section 2.2.6, some new notation for hybrid preference is as follows:

- $S_i^{(k,+,++,U)}(s)$ stands for states reachable from $SQ = s$ in exactly k legal WIUUMs by the DMs in H with last mover DM i;

- $V_H^{(k,+,++,U)}(s)$ denotes states reachable from $SQ = s$ in at most k legal WIUUMs by H;

- $A_i^{(k,+,++,U)}(s)$ indicates arcs with last mover DM i in sequences of at most k ($k>1$) legal WIUUMs by the DMs in H from $SQ=s$. Let $A_i^{+,++,U}(s)$ denote the sets of arcs associated with DM i in one step WIUUM from state s. Therefore, $A_i^{+,++,U}=\bigcup\limits_{s\in S} A_i^{+,++,U}(s)$.

In the status quo analysis, if a DM moves twice in succession, the DM is deemed to make illegal moves. The following Theorem 3.5 asserts that if there does not exist any new appropriate arc in the graph model, then corresponding joint moves will stop.

Theorem 3.5. *For $SQ=s$ and $H\subseteq N$, the following results hold:*

$$(1)\ If\ \bigcup_{i\in H} A_i^{(k+1)}(s)=\bigcup_{i\in H} A_i^{(k)}(s),$$

$$then\ V_H^{(k+1)}(s)=V_H^{(k)}(s)\ and\ R_H(s)=V_H^{(k)}(s);$$

$$(2)\ If\ \bigcup_{i\in H} A_i^{(k+1,+,++)}(s)=\bigcup_{i\in H} A_i^{(k,+,++)}(s),$$

$$then\ V_H^{(k+1,+,++)}(s)=V_H^{(k,+,++)}(s)\ and\ R_H^{+,++}(s)=V_H^{(k,+,++)}(s);$$

$$(3)\ If\ \bigcup_{i\in H} A_i^{(k+1,+,++,U)}(s)=\bigcup_{i\in H} A_i^{(k,+,++,U)}(s),$$

$$then\ V_H^{(k+1,+,++,U)}(s)=V_H^{(k,+,++,U)}(s)\ and\ R_H^{+,++,U}(s)=V_H^{(k,+,++,U)}(s).$$

Proof: The proofs of three statements (1), (2), and (3) are similar. We prove (3) that explicitly shows the hybrid preference structure.

Assume that there exists $q\in V_H^{(k+1,+,++,U)}(s)\backslash V_H^{(k,+,++,U)}(s)$ but $\bigcup\limits_{j\in H} A_j^{(k+1,+,++,U)}(s)=\bigcup\limits_{j\in H} A_j^{(k,+,++,U)}(s)$.

Since $V_H^{(k+1,+,++,U)}(s)=(\bigcup\limits_{j\in H} S_j^{(k+1,+,++,U)}(s))\cup V_H^{(k,+,++,U)}(s)$, then, there exists $i\in H$, such that $q\in S_i^{(k+1,+,++,U)}(s)\setminus V_H^{(k,+,++,U)}(s)$. Hence, there exists $s_1\in\bigcup\limits_{j\in H\backslash\{i\}} S_j^{(k,+,++,U)}(s)$ such that $q\in R_i^{+,++,U}(s_1)$. Clearly, this implies that arc $(s_1,q)\in A_i^{(k+1,+,++,U)}(s)\setminus\bigcup\limits_{j\in H} A_j^{(k,+,++,U)}(s)$ which contradicts with the hypothesis that $\bigcup\limits_{i\in H} A_i^{(k+1,+,++,U)}(s)=\bigcup\limits_{i\in H} A_i^{(k,+,++,U)}(s)$. Thus, $V_H^{(k+1,+,++,U)}(s)=V_H^{(k,+,++,U)}(s)$ when $\bigcup\limits_{i\in H} A_i^{(k+1,+,++,U)}(s)=\bigcup\limits_{i\in H} A_i^{(k,+,++,U)}(s)$. It is clear that if $V_H^{(k+1,+,++,U)}(s)=V_H^{(k,+,++,U)}(s)$, then $\bigcup\limits_{i\in H} S_i^{(k+1,+,++,U)}(s)\subseteq V_H^{(k,+,++,U)}(s)$. Consequently, if there are no new arcs in

$\bigcup_{i\in H} A_i^{(k+1,+,++,U)}(s)$, then the legal WIUUMs will stop after k legal WIUUMs from state s. i.e., $R_H^{+,++,U}(s) = V_H^{(k,+,++,U)}(s)$. (1) and (2) can be similarly verified. $\square$

Fix $H \subseteq N$. Let $|\bigcup_{i\in H} A_i|$, $|\bigcup_{i\in H} A_i^{+,++}|$, and $|\bigcup_{i\in H} A_i^{+,++,U}|$ respectively denote the cardinalities of UM arcs, WI arcs, and WIUUM arcs in a directed graph associated with the DMs in H. Then, the following lemma can be easily derived using Theorem 3.5.

Lemma 3.1. *Let δ_1, δ_2, and δ_3 respectively stand for the number of iteration steps required to construct $R_H(s)$, $R_H^{+,++}(s)$, and $R_H^{+,++,U}(s)$ for any $s \in S$. Then*

(1) $\delta_1 \leq |\bigcup_{i\in H} A_i|$;

(2) $\delta_2 \leq |\bigcup_{i\in H} A_i^{+,++}|$; and

(3) $\delta_3 \leq |\bigcup_{i\in H} A_i^{+,++,U}|$.

Let $l_1 = |\bigcup_{i\in H} A_i|$, $l_2 = |\bigcup_{i\in H} A_i^{+,++}|$, and let $l_3 = |\bigcup_{i\in H} A_i^{+,++,U}|$. By Theorem 3.5 and Lemma 3.1, the following theorem can be proved.

Theorem 3.6. *Let $s \in S$, $H \subseteq N$, and $H \neq \emptyset$. Then the reachable lists of H by the legal sequences of UMs, WIs, and WIUUMs from state s, $R_H(s)$, $R_H^{+,++}(s)$, and $R_H^{+,++,U}(s)$, can be respectively expressed by*

(1) $R_H(s) = V_H^{(l_1)}(s)$;

(2) $R_H^{+,++}(s) = V_H^{(l_2,+,++)}(s)$;

(3) $R_H^{+,++,U}(s) = V_H^{(l_3,+,++,U)}(s)$.

Proof: The proofs of equations (1), (2), and (3) can be carried out similarly. Here, we prove (3) including explicit hybrid preferences. Based on Theorem 3.5, $R_H^{+,++,U}(s) = V_H^{(\delta_3,+,++,U)}(s)$. It is obvious that no new arc is produced by legal WIUUMs in the graph model after δ_3 iteration steps. Since $\delta_3 \leq l_3$ by using Lemma 3.1, then $V_H^{(l_3,+,++,U)}(s)= V_H^{(\delta_3,+,++,U)}(s)$. Therefore, (3) is proved.

(1) and (2) can be similarly proved. $\square$

The following algorithm presented in Table 3.1 implements constructions of the state set and arc set, $V_H^{(k)}(s)$ and $A_i^{(k)}(s)$, which include all states reachable by coalition H in at most k legal UMs starting at state s and all arcs with last mover DM i in sequences of at most k legal UMs from $SQ = s$ for $k = 1, 2, \cdots, \delta_1$. Obviously, Table 3.1 also provides the construction of the reachable list of H from state s, $R_H(s)$, using Theorem 3.6. The arcs, $A_i^{(k)}(s)$ for $k = 1, 2, \cdots$

$\cdot, \delta_1$, sufficiently track the evolution of a conflict permitting all UMs from state s. Similarly, the computational implementation of the state set $V_H^{(k,+,++,U)}(s)$ and the arc set $A_i^{(k,+,++,U)}(s)$ can be accomplished by using the following algorithms described in Table 3.2. Therefore, the algorithms designed in Tables 3.1 and 3.2 operationalize the key inputs of stability analysis, $R_H(s)$ and $R_H^{+,++,U}(s)$, and the evolution paths of status quo analysis for hybrid preference.

If UMs have no strength of preference to be considered, then the state set $R_H^{+,++,U}(s)$ reduces to $R_H^{+,U}(s)$ defined by Li et al. [46]. If no uncertain preference is associated with UMs, $R_H^{+,++,U}(s)$ reduces to $R_H^{+,++}(s)$, introduced by Hamouda et al. [28]. Obviously, the developed results for hybrid preference expand the existing stability analysis [16, 28, 46] and status quo analysis [47, 48].

3.4 Application: Gisborne Conflict with Hybrid Preference

Lake Gisborne is located near the south coast of the Canadian Atlantic province of Newfoundland and Labrador. In June 1995, a local division of the McCurdy Group of Companies, Canada Wet Incorporated, proposed a project to export bulk water from Lake Gisborne to foreign markets. On December 5, 1996, this project was registered with the government of Newfoundland and Labrador. At the time of registration, no policy existed on bulk water exports. However, this proposal immediately aroused considerable opposition from a wide variety of lobby groups. In addition to unpredictable harmful impacts on local environment and First Nations culture, a critical issue is its potential implication of making water a tradeable "commodity" that is thus subject to WTO (World Trade Organization) and NAFTA (North American Free Trade Agreement). Therefore, if the Lake Gisborne bulk water export project was successfully executed, the water policy in Canada might have to undergo a significant shift as any firm would be able to follow the suit. As such, the Federal Government of Canada sided with the opposing groups by introducing a policy to forbid bulk water export from major drainage basins in Canada. The mounting pressure eventually forced the government of Newfoundland and Labrador to introduce a new bill to ban bulk water export from Newfoundland and Labrador, which effectively terminated the Gisborne water export project. (See details in [17, 46]).

Table 3.1: **The pseudocode for constructing $R_H(s)$**

Initialize //initialize the necessary parameters
H: any subset of DMs;
h: the number of H;
m: the number of states;
s: the status quo state;
δ_1: the max step we want to calculate;
$R_i(s)$: reachable list from state s by DM i, $i = 1, \cdots, h$;
$k = 1$
$S_i^{(k)}(s) = R_i(s), i = 1, \cdots, h$
$V_i^{(k)}(s) = S_i^{(k)}(s), i = 1, \cdots, h$
$A_i^{(k)}(s) = \bigcup\limits_{q \in R_i(s)} (s, q)$, (s, q) for $i = 1, \cdots, h$
loop 1
$k = k + 1$
loop 2 i from 1 to h // the last mover is DM i
$S' = \bigcup\limits_{j \in H \setminus \{i\}} S_j^{(k-1)}(s)$
$S_i^{(k)}(s) = \bigcup\limits_{s' \in S'} R_i(s')$
$V_i^{(k)}(s) = V_i^{(k-1)}(s) \bigcup S_i^{(k)}(s)$
$A_i^{(k)}(s) = A_i^{(k-1)}(s) \bigcup \{(s_1, s_2) : s_1 \in \bigcup\limits_{j \in H \setminus \{i\}} S_j^{(k-1)}(s)$, and $s_2 \in R_i(s_1)\}$
return to loop 2
$V_H^{(k)}(s) = \bigcup\limits_{i \in H} V_i^{(k)}(s)$
return to loop 1 if $\bigcup\limits_{i \in H} A_i^{(k)}(s) \neq \bigcup\limits_{i \in H} A_i^{(k-1)}(s)$
$\delta_1 = k$
$R_H(s) = V_H^{(\delta_1)}(s)$.

Table 3.2: **The pseudocode for constructing $R_H^{+,++,U}$**

Initialize //initialize the necessary parameters
H: any set of DMs;
h: the number of H;
m: the number of states;
s: status quo state;
δ_3: the max step we want to calculate;
$R_i^{+,++,U}(s)$: reachable list by a WIUUM from state s by DM i, $i = 1, \cdots, h$;
$k = 1$
$S_i^{(k,+,++,U)}(s) = R_i^{+,++,U}(s), i = 1, \cdots, h$
$V_i^{(k,+,++,U)}(s) = S_i^{(k,+,++,U)}(s), i = 1, \cdots, h$
$A_i^{(k,+,++,U)}(s) = \bigcup_{q \in R_i^{+,++,U}(s)} (s, q)$ for $i = 1, \cdots, h$
loop 1
$k = k + 1$
loop 2 i from 1 to h // the last mover is DM i
$S' = \bigcup_{j \in H \setminus \{i\}} S_j^{(k-1,+,++,U)}(s)$
$S_i^{(k,+,++,U)}(s) = \bigcup_{s' \in S'} R_i^{+,++,U}(s')$
$V_i^{(k,+,++,U)}(s) = V_i^{(k-1,+,++,U)}(s) \bigcup S_i^{(k,+,++,U)}(s)$
$A_i^{(k,+,++,U)}(s) = A_i^{(k-1,+,++,U)}(s) \bigcup S''$
$S'' = \{(s_1, s_2) : s_1 \in \bigcup_{j \in H \setminus \{i\}} S_j^{(k-1,+,++,U)}(s)$, and $s_2 \in R_i^{+,++,U}(s_1)\}$
return to loop 2
$V_H^{(k,+,++,U)}(s) = \bigcup_{i \in H} V_i^{(k,+,++,U)}(s)$
return to loop 1 if $\bigcup_{i \in H} A_i^{(k,+,++,U)}(s) \neq \bigcup_{i \in H} A_i^{(k-1,+,++,U)}(s)$
$\delta_3 = k$
$R_H^{+,++,U}(s) = V_H^{(\delta_3,+,++,U)}(s)$.

Nevertheless, several support groups remain interested in the project, and the provincial government might restart the project at an appropriate time in the future due to its urgent need for cash. This prospect introduces uncertainty into the preferences of the provincial government for the Gisborne conflict model. This conflict is modeled using three DMs: DM 1, **Federal (Fe)**; DM 2, **Provincial (Pr)**; and DM 3, **Support (Su)**; and a total of three options, as shown in Table 3.3. The following is a summary of the three DMs and their options [46]:

- Federal government of Canada (**Federal**): its option is to **continue** a Canada-wide accord on the prohibition of bulk water export or not,
- Provincial government of Newfoundland and Labrador (**Provincial**): its option is to **lift** the ban on bulk water export or not, and
- Support groups (**Support**): its option is to **appeal** for continuing the Gisborne project or not.

Table 3.3: **Options and feasible states for the Gisborne conflict** [46]

Federal								
1. Continue	N	Y	N	Y	N	Y	N	Y
Provincial								
2. Lift	N	N	Y	Y	N	N	Y	Y
Support								
3. Appeal	N	N	N	N	Y	Y	Y	Y
State number	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8

In the Lake Gisborne conflict model, the three options together determine 8 possible states as listed in Table 3.3, where a "Y" indicates that an option is selected by the DM controlling it and an "N" means that the option is not chosen. The graph model of this conflict is depicted based on the 8 feasible states by Fig. 3.4, in which a label on an arc indicates which DM controls the moves between the two states connected by the arc.

In this section, the extended stability definitions with hybrid preference are applied to an extended Lake Gisborne conflict. Li et al. [46] introduced uncertainty into the preferences of the Provincial Government for the Gisborne conflict. We extend the graph model to include the hybrid preference of

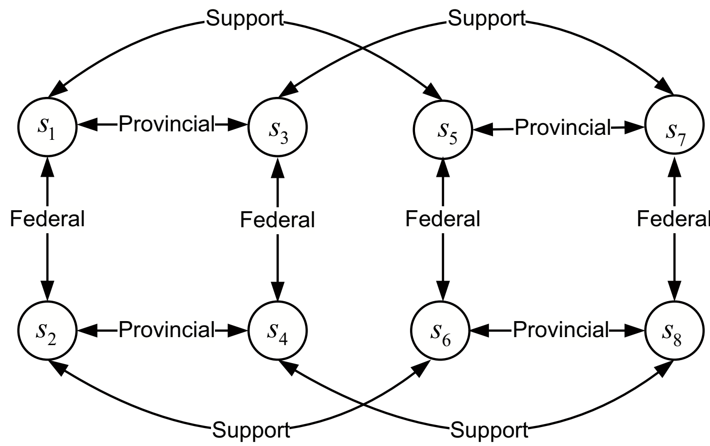


Figure 3.4: **Graph model for the Gisborne conflict [46].**

uncertainty and strength in the Gisborne dispute. The preference information for this conflict over the feasible states is given in Table 3.4. We assume that state s_7 is strongly less preferred to all other states by the Federal Government, the Support Groups consider state s_2 to be strongly less preferred relative to all other states, and the Provincial Government strongly prefers state s_2 to state s_6. Note that DM **Provincial** only knows that it mildly prefers state s_3 to s_7, state s_4 to s_8, state s_1 to s_5, and strongly prefers state s_2 to s_6. This DM is uncertain for preference relations between other any two states. It is obvious that DM **Provincial**'s preference information includes uncertainty and strength. Additionally, this representation of preference information presented in Table 3.4 implies that the preferred relations, $>$ and $\gg$, are transitive. For instance, since $s_5 > s_3$ and $s_3 \gg s_7$, then $s_5 \gg s_7$. However, in general, the preference structure presented in this research does not require the transitivity of preference relations and, hence, the developed results can be used to handle intransitive preferences.

The stable states and equilibria under the hybrid preference structure are summarized in Table 3.5, in which a check mark ($\surd$) opposited a given state and an index means that this state is stable for the indicated DM, solution concept and associated index (a, b, c, or d), "Eq" is an equilibrium for a corresponding solution concept, and 1, 2, and 3 denote three DMs, **Federal**, **Provincial**, and **Support**, respectively. In fact, if analysts are not willing to take the risk to switch the current strategy to another strategy having uncertain preference relative to the initial strategy, and are conservative when considering sanctions,

Table 3.4: **Certain preferences information for the Gisborne model** (extended from [46])

DMs	Certain preferences
Federal	$s_2 > s_6 > s_4 > s_8 > s_1 > s_5 > s_3 \gg s_7$
Provincial	$s_3 > s_7$, $s_4 > s_8$, $s_1 > s_5$, $s_2 \gg s_6$, only
Support	$s_3 > s_4 > s_7 > s_8 > s_5 > s_6 > s_1 \gg s_2$

then they would consider selecting equilibria with index d as resolutions for decision making. On the other hand, if developers are very aggressive, they would like to find the stable states under index a. Table 3.6 compares stability results for preference with uncertainty only and hybrid preference of uncertainty and strength. State s is a strong equilibrium for some stability if s is strongly stable for all DMs under the stability. By Table 3.6, we select states s_4 and s_6 as better choices for making decision, since s_8 is not a strong equilibrium.

The evolution of the Gisborne conflict by WIUUMs from status quo s_1

DMs	Status quo	Transitional states			Equilibrium
Federal	N	N	Y	Y	Y
Provincial	N	N	N	Y	Y
Support	N	Y	Y	Y	N
State number	s_1	s_5	s_6	s_8	s_4

Figure 3.5: **The Gisborne conflict evolution from states s_1 to s_4.**

The aim of stability analysis in this research is to find strong equilibria of a graph model associated with some index according to DM's attitudes toward the risk associated with uncertain preferences. Status quo analysis examines the dynamics of a conflict model and assesses whether predicted equilibria are reachable from the status quo. Therefore, by taking a status quo analysis into account, additional insights are revealed about the attainability of any potential resolution. Fig. 3.5 shows the evolution of the Gisborne conflict by legal WIUUMs from statu quo state state s_1 to the desirable equilibrium s_4.

Table 3.5: **Stability results of the Gisborne conflict with hybrid preference**

State		Nash				GMR				SMR				SEQ				SGMR				SSMR				SSEQ				WGMR				WSMR				WSEQ			
		1	2	3	Eq	1	2	3	Eq	1	2	3	Eq	1	2	3	Eq	1	2	3	Eq	1	2	3	Eq	1	2	3	Eq	1	2	3	Eq	1	2	3	Eq	1	2	3	Eq
s_1	a																																								
	b		√				√				√				√				√				√				√														
	c						√				√				√				√				√				√														
	d		√				√				√				√				√				√				√														
s_2	a	√				√				√				√				√				√				√															
	b	√	√			√	√			√	√			√	√			√	√			√	√			√	√														
	c	√				√	√			√	√			√				√	√			√	√			√															
	d	√	√			√	√			√	√			√	√			√	√			√	√			√	√														
s_3	a			√				√				√				√				√				√				√													
	b		√	√			√	√			√	√			√	√			√	√			√	√			√	√													
	c			√			√	√			√	√			√	√			√	√			√	√			√	√													
	d		√	√			√	√			√	√			√	√			√	√			√	√			√	√													
s_4	a	√		√		√		√		√		√		√		√		√		√		√		√		√		√													
	b	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√												
	c	√		√		√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√												
	d	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√												
s_5	a			√				√				√				√				√				√				√													
	b		√	√			√	√			√	√			√	√			√	√			√	√			√	√													
	c			√			√	√			√	√			√	√			√	√			√	√			√	√													
	d		√	√			√	√			√	√			√	√			√	√			√	√			√	√													
s_6	a	√		√		√		√		√		√		√		√		√		√		√		√		√		√													
	b	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√												
	c	√		√		√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√												
	d	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√												
s_7	a							√				√				√				√								√								√					
	b		√				√	√			√	√			√	√			√	√			√				√	√								√					
	c						√	√			√	√			√	√			√	√			√				√	√								√					
	d		√				√	√			√	√			√	√			√	√			√				√	√								√					
s_8	a	√				√		√		√		√		√		√		√		√		√				√										√				√	
	b	√	√			√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√			√	√									√				√	
	c	√				√	√	√	√	√	√	√	√	√		√		√	√	√	√	√	√			√										√			√	√	
	d	√	√			√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√			√	√									√				√	

Table 3.6: **The comparison of stability results for two types of preference structures**

Preference structure	States	Analysis method	Analysis result
Preference with uncertainty	s_4 and s_6	see Chapter 2	Equilibria under extensions b and d for Nash, GMR, SMR and SEQ
	s_8	see Chapter 2	Equilibrium under extensions b and d for GMR, SMR and SEQ
Hybrid preference	s_4 and s_6	see Chapter 3	Strong equilibria under extensions b and d for Nash, GMR, SMR and SEQ
	s_8	see Chapter 3	Weakly stable under extensions b and d for SMR and SEQ

3.5 Summary

A hybrid preference framework is developed in this chapter for strategic conflict analysis to integrate preference strength and preference uncertainty into GMCR for multiple decision makers [68, 70]. The hybrid system is more general than existing models, which consider preference strength and preference uncertainty separately. Within the hybrid preference structure, the hybrid versions of four basic stabilities are defined and algorithms are developed to calculate efficiently the essential inputs of the stabilities and status quo analysis. The new stability concepts under the hybrid preference structure can be used to model complex strategic conflicts arising in practical applications, and can provide new insights for the conflicts.

Chapter 4

Multiple levels of Preference in the Graph Model for Conflict Resolution

A multiple-level preference ranking structure is developed within the paradigm of the Graph Model for Conflict Resolution to study multi-objective decision making in conflict situations more realistically. In this structure, a decision maker may have multiple levels of preference for one state or scenario over another; for example, if state A is preferred to state B, it may be mildly preferred at level 1, more strongly preferred at level 2, $\cdots$, or maximally preferred at level r, where $r > 0$ is a fixed parameter. The number of levels, r, is unrestricted in this system, thereby extending earlier two-level ($r = 1$) and three-level ($r = 2$) structures. Multilevel versions of four stability definitions, Nash stability, general metarationality, symmetric metarationality, and sequential stability, are defined for the graph model with this extended preference structure and the relationships among them are investigated. A specific case study, including multiple decision makers and multiple levels of preference, is carried out to illustrate how the new solution concepts can be applied in practice.

4.1 Multiple Levels of Preference

The simple preference structure $\{\succ, \sim\}$ [16] and the structure with strength of preference $\{\gg, >, \sim\}$ [27, 28] are referred to as two levels of preference and three levels of preference, respectively. As a result of the development of a significant amount of research expressing preference information by degree of strength [58,64], the existing preference structures in the graph model would be unable to depict the intensity of relative preference. Therefore, it may be worthwhile to extend the existing two levels of preference and three levels of preference in the graph model to an unlimited number of levels of preference, which in this thesis are referred to as degrees of preference.

Table 4.1: **Degree of relative preference**

Degree of strength	Description	Notation
$d = 0$	Equally preferred	$\sim$
$d = 1$	Moderately preferred	$>$
$d = 2$	Strongly preferred	$\gg$
$d = 3$	Very strongly preferred	$\ggg$
$\cdots\cdots$	$\cdots\cdots$	$\cdots\cdots$
$d = r$	Preferred at level r	$\overbrace{> \cdots >}^{r}$

A set of new and more general binary relations $\overbrace{> \cdots >}^{d}$ for $d = 1, 2, \cdots, r$, as listed in Table 4.1, are proposed in this research to represent DM i's preference at each level d. With the introduction of these new binary relations, the three levels of preference in the graph model are extended from a triplet relations, to an $r + 1$-level relations for DM i over the set of states, which is expressed as $\{\sim_i, >_i, \gg_i, \cdots, \overset{\mathrm{r}}{\succ}_i\}$ on S, where $\overset{\mathrm{r}}{\succ}_i$ denotes $\overbrace{> \cdots >}^{r}_i$, i.e., DM i has preference by degree r for comparing states with respect to preference. For instance, $s \ggg_i q$ means that DM i very strongly prefers state s to state q. It is assumed that the preference relations of each DM $i \in N$ have the following properties:

(i) $\overset{\mathrm{d}}{\succ}_i$ for $d = 1, 2, \cdots, r$, is asymmetric;

(ii) $\sim_i$ is reflexive and symmetric; and

(iii) $\{\sim_i, >_i, \gg_i, \cdots, \overset{\text{r}}{\succ}_i\}$ is strongly complete, i.e. if $s, q \in S$, then exactly one of the following relations holds: $s \overset{\text{d}}{\succ}_i q$, $q \overset{\text{d}}{\succ}_i s$, for $d = 1, 2, \cdots, r$, or $s \sim_i q$.

Preference information can be either transitive or intransitive. If $k \overset{\text{d}}{\succ}_i s$ and $s \overset{\text{d}}{\succ}_i q$ imply $k \overset{\text{d}}{\succ}_i q$, then the preference $\overset{\text{d}}{\succ}_i$ is transitive. Otherwise, preferences are called intransitive. Note that the assumption of transitivity of preferences is not required in the following definitions so that the results in this research hold for both transitive and intransitive preferences. When all preferences for a given DM i are transitive, the preferences are said to be ordinal and, hence, the states in a conflict can be ordered or ranked from most to least preferred, where ties are allowed. Sometimes this ranking of states according to preference is referred to as a "preference ranking".

For the new preference structure, DM i can identify $2r+1$ subsets of S: $\Phi_i^{+(r)}(s)$, $\cdots$, $\Phi_i^{+(1)}(s)$, $\Phi_i^{=}(s)$, $\Phi_i^{-(1)}(s)$, $\cdots$, and $\Phi_i^{-(r)}(s)$. Here, $\Phi_i^{+(d)}(s)$ and $\Phi_i^{-(d)}(s)$ for $d = 0, 1, \cdots, r$, are defined and described in Table 4.2. The set $R_i(s)$ denotes the *unilateral moves (UMs)* of DM i from $s \in S$, and is also called i's *reachable list* from s. It contains all states to which DM i can move, unilaterally and in one step, from state s. Similarly, the set $R_i^+(s) = \{q \in S : q \in R_i(s) \; and \; q \overset{\text{d}}{\succ}_i s \; for \; d = 1, 2, \cdots, r\}$ contains DM i's *unilateral improvements (UIs)* from state s at various levels of preference. All reachable lists from state s at each level of preference for DM i are expressed by $R_i^{+(r)}(s)$, $\cdots$, $R_i^{+(1)}(s)$, $R_i^{(0)}(s)$, $R_i^{-(1)}(s)$, $\cdots$, and $R_i^{-(r)}(s)$. Let $R_i(s) = \bigcup_{d=0}^{r} (R_i^{-(d)}(s) \cup R_i^{+(d)}(s))$ and $R_i^+(s) = \bigcup_{d=1}^{r} R_i^{+(d)}(s)$, where $R_i^{+(d)}(s)$ and $R_i^{-(d)}(s)$ for $d = 0, 1, \cdots, r$, are described in Table 4.3. Additionally, the relations among the subsets of S, $\Phi_i^{+(d)}(s)$ and $\Phi_i^{-(d)}(s)$ for $d = 0, 1, \cdots, r$, and the corresponding reachable lists from state s for DM i, $R_i^{+(d)}(s)$ and $R_i^{-(d)}(s)$ for $d = 0, 1, \cdots, r$, are depicted in Fig. 4.1.

4.2 Multiple Levels of Preference in the Graph Model for Conflict Resolution

Incorporating this extended multiple levels of preference into the graph model for conflict resolution results in multilevel versions of the four basic solution concepts, $Nash_k$, GMR_k, SMR_k, and SEQ_k for $k = 0, 1, \cdots, r$. The stability definitions in a 2-DM conflict model are presented next.

Table 4.2: **Subsets of S for DM i with respect to multiple levels of preference**

Degree of strength	Subsets of S	Description
$d = r$	$\Phi_i^{+(r)}(s) = \{q : q \overbrace{> \cdots >_i}^{r} s\}$	States preferred to state s at level r by DM i
	$\Phi_i^{-(r)}(s) = \{q : s \overbrace{> \cdots >_i}^{r} q\}$	States less preferred to state s at level r by DM i
$\cdots\cdots$		$\cdots\cdots$
		$\cdots\cdots$
$d = 3$	$\Phi_i^{+(3)}(s) = \{q : q \ggg_i s\}$	States very strongly preferred to state s by DM i
	$\Phi_i^{-(3)}(s) = \{q : s \ggg_i q\}$	States very strongly less preferred to state s by DM i
$d = 2$	$\Phi_i^{+(2)}(s) = \{q : q \gg_i s\}$	States strongly preferred to state s by DM i
	$\Phi_i^{-(2)}(s) = \{q : s \gg_i q\}$	States strongly less preferred to state s by DM i
$d = 1$	$\Phi_i^{+(1)}(s) = \{q : q >_i s\}$	States moderately preferred to state s by DM i
	$\Phi_i^{-(1)}(s) = \{q : s >_i q\}$	States moderately less preferred to state s by DM i
$d = 0$	$\Phi_i^{(0)}(s) = \Phi_i^{=}(s) = \{q : q \sim_i s\}$	States equally preferred to state s by DM i

Table 4.3: **Reachable lists by DM i at some level of preference**

Type of movement	Description
$R_i^{+(d)}(s) = R_i(s) \cap \Phi_i^{+(d)}(s)$	All unilateral improvements of degree d from state s for DM i
$R_i(s)^{-(d)}(s) = R_i(s) \cap \Phi_i^{-(d)}(s)$	All unilateral disimprovements of degree d from state s for DM i
$R_i^{(0)}(s) = R_i^{=}(s) = R_i(s) \cap \Phi_i^{=}(s)$	All equally preferred states reachable from state s by DM i

4.2.1 Stabilities for Multiple Levels of Preference in Two DM Conflicts

First, in the solution concepts given below, strength of preference is not considered in sanctioning, so the following solution concepts are called general stabilities. This idea is analogous to the concept of standard stability proposed by Hamouda et al. [27]. For all of the definitions given in this section, assume that $N = \{i, j\}$ and $s \in S$.

4.2.1.1 General Stabilities for Multiple Levels of Preference

Definition 4.1. *State s is **general Nash stable** (SNash) for DM i, denoted by $s \in S_i^{GNash}$, iff $R_i^+(s) = \emptyset$.*

Definition 4.2. *State s is **general GMR** (GGMR) for DM i, denoted by $s \in S_i^{GGMR}$, iff for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_j(s_1)$ with $s_2 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$.*

Definition 4.3. *State s is **general SMR** (GSMR) for DM i, denoted by $s \in S_i^{GSMR}$, iff for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_j(s_1)$ with $s_2 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$ and $s_3 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$ for all $s_3 \in R_i(s_2)$.*

Definition 4.4. *State s is **general SEQ** (GSEQ) for DM i, denoted by $s \in S_i^{GSEQ}$, iff for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_j^+(s_1)$ with $s_2 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$.*

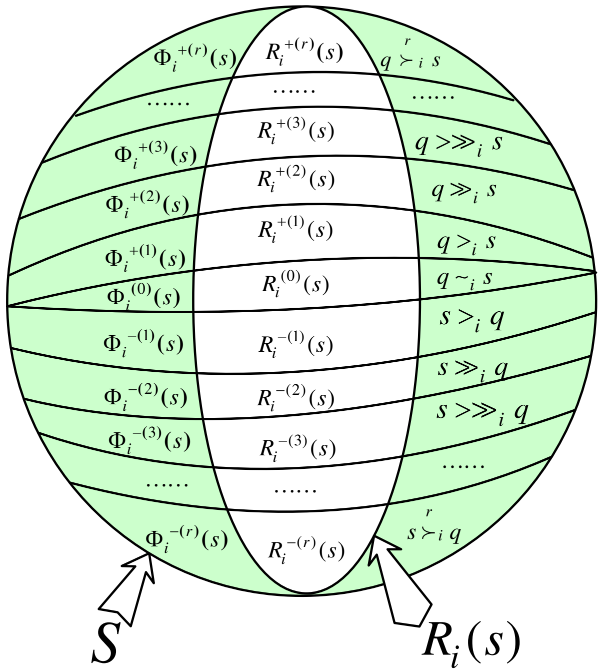


Figure 4.1: **Relations among subsets of S and reachable lists from s.**

Note that, in this research, the meaning of $R_i^+(s)$ differs from that of Fang et al. [16]; there, it denotes all one-level unilateral improvements from s by DM i, whereas here, it includes all unilateral improvements, no matter how many levels. For three levels of preference, stabilities are divided into strongly and weakly stable according to the strength of the possible sanction, i.e., if a particular state s is general stable, then s is either strongly stable or weakly stable [28]. Within multiple levels of preference, the general stabilities are constituted by stabilities at each level of preference.

4.2.1.2 Stabilities at Level k for Multiple Levels of Preference

Firstly, definitions are given in this research for different strengths of Nash stability. Even though unilateral improvements do not exist under Nash stability, the idea of strength of stability can still be captured using the level of preference for the most preferred states to which the DM could unilaterally move. All these states must be less preferred than the initial state. A special connection is required for the case when no movements of any type exist for the DM. If DM i has no any unilateral move at all levels of preference from state s, state s is extremely stable. We proposed the stability next.

Definition 4.5. *If $R_i(s) = \emptyset$, then state s is **super stable** for DM i at any level of preference, denoted by $s \in S_i^{Super}$.*

Definition 4.6. *State s is **Nash stable** ($Nash_0$) at level 0 for DM i, denoted by $s \in S_i^{Nash_0}$, iff $R_i^+(s) = \emptyset$ and $R_i^{(0)}(s) \neq \emptyset$.*

Definition 4.7. *For $1 \leq k \leq r$, state s is **Nash stable** ($Nash_k$) at level k for DM i, denoted by $s \in S_i^{Nash_k}$, iff $R_i^+(s) \cup (\bigcup_{d=0}^{k-1} R_i^{-(d)}(s)) = \emptyset$ and $R_i^{-(k)}(s) \neq \emptyset$.*

The k-th level Nash stability is depicted in Fig. 4.2. The super stability is referred to as Nash stability at the highest level.

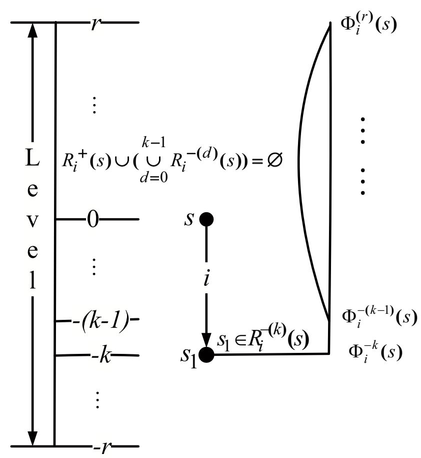


Figure 4.2: **Nash stability at level k**

When multiple-level preference is incorporated into the graph model, GMR, SMR, and SEQ stabilities at different levels can be distinguished according to the strength of the sanction. For DM i, if a UI from state s is sanctioned in exactly k levels below s and all other UIs from state s are sanctioned in at least k levels below s, then the status quo s is called general metarational at level k. Its formal definition is given below.

Definition 4.8. *State s is **general metarational** (GMR_0) at level 0 for DM i, denoted by $s \in S_i^{GMR_0}$, iff either $R_i^+(s) = \emptyset$ and $R_i^{(0)}(s) \neq \emptyset$, or $R_i^+(s) \neq \emptyset$ and for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_j(s_1)$ with $s_2 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$ and there exists at least one $s_1' \in R_i^+(s)$ and $s_2' \in R_j(s_1')$ such that $s_2' \in \Phi_i^{(0)}(s)$ and $R_j(s_1') \bigcap (\bigcup_{d=1}^{r} \Phi_i^{-(d)}(s)) = \emptyset$.*

Definition 4.9. *For $1 \leq k \leq r-1$, state s is **general metarational** (GMR_k) at level k for DM i, denoted by $s \in S_i^{GMR_k}$, iff either $\bigcup_{d=0}^{k-1} R_i^{-(d)}(s) \cup R_i^{+}(s) = \emptyset$ and $R_i^{-(k)}(s) \neq \emptyset$, or $R_i^{+}(s) \neq \emptyset$ and for every $s_1 \in R_i^{+}(s)$ there exists at least one $s_2 \in R_j(s_1)$ with $s_2 \in \bigcup_{d=k}^{r} \Phi_i^{-(d)}(s)$ and there exists at least one $s_1' \in R_i^{+}(s)$ and $s_2' \in R_j(s_1')$ such that $s_2' \in \Phi_i^{-(k)}(s)$ and $R_j(s_1') \bigcap (\bigcup_{d=k+1}^{r} \Phi_i^{-(d)}(s)) = \emptyset$.*

If all of DM i's UIs from a state are sanctioned at the highest level r (exactly r levels below the state), then the state is called general metarational at level r. Its formal definition is given below.

Definition 4.10. *State s is **general metarational** (GMR_r) at level r for DM i, denoted by $s \in S_i^{GMR_r}$, iff either $\bigcup_{d=0}^{r-1} R_i^{-(d)}(s) \cup R_i^{+}(s) = \emptyset$ and $R_i^{-(r)}(s) \neq \emptyset$, or $R_i^{+}(s) \neq \emptyset$ and for every $s_1 \in R_i^{+}(s)$ there exists at least one $s_2 \in R_j(s_1)$ with $s_2 \in \Phi_i^{-(r)}(s)$.*

For DM i, if a UI from a state is sanctioned at level k below the state and all other UIs from the particular state are sanctioned at a level of at least k below the state, and these corresponding sanctions cannot be avoided by any counterresponse, then the state is called SMR stable at level k. Its formal definition is given below.

Definition 4.11. *State s is **symmetric metarational** (SMR_0) at level 0 for DM i, denoted by $s \in S_i^{SMR_0}$, iff either $R_i^{+}(s) = \emptyset$ and $R_i^{(0)}(s) \neq \emptyset$, or $R_i^{+}(s) \neq \emptyset$ and for every $s_1 \in R_i^{+}(s)$ there exists at least one $s_2 \in R_j(s_1)$ with $s_2 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$ and there exists at least one $s_1' \in R_i^{+}(s)$ and $s_2' \in R_j(s_1')$ such that $s_2' \in \Phi_i^{(0)}(s)$ and $R_j(s_1') \bigcap (\bigcup_{d=1}^{r} \Phi_i^{-(d)}(s)) = \emptyset$, as well as $s_3 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$ for any $s_3 \in R_i(s_2) \cup R_i(s_2')$.*

Symmetric metarationality at level k ($0 < k \leq r$) for DM i consists of SMR_{k^+} and SMR_{k^-} that are defined next.

Definition 4.12. *For $1 \leq k \leq r-1$, state s is **symmetric metarational** (SMR_{k^+}) at level k for DM i, denoted by $s \in S_i^{SMR_{k^+}}$, iff either $\bigcup_{d=0}^{k-1} R_i^{-(d)}(s) \cup R_i^{+}(s) = \emptyset$ and $R_i^{-(k)}(s) \neq \emptyset$, or $R_i^{+}(s) \neq \emptyset$ and for every $s_1 \in R_i^{+}(s)$ there exists at least one $s_2 \in R_j(s_1)$ with $s_2 \in \bigcup_{d=k}^{r} \Phi_i^{-(d)}(s)$ and there exists at least one $s_1' \in R_i^{+}(s)$*

and $s_2' \in R_j(s_1')$ *such that* $s_2' \in \Phi_i^{-(k)}(s)$ *and* $R_j(s_1') \bigcap (\bigcup_{d=k+1}^{r} \Phi_i^{-(d)}(s)) = \emptyset$, *as well as* $s_3 \in \bigcup_{d=k}^{r} \Phi_i^{-(d)}(s)$ *for any* $s_3 \in R_i(s_2) \cup R_i(s_2')$.

Stability SMR_{k^-} is defined by $S_i^{SMR_{k^-}} = S_i^{GSMR} \cap S_i^{GMR_k} - S_i^{SMR_{k^+}}$. Equivalently,

Definition 4.13. *For* $1 \leq k \leq r-1$, *state* s *is* ***symmetric metarational*** (SMR_{k^-}) *at level* k *for DM* i, *denoted by* $s \in S_i^{SMR_{k^-}}$, *iff* $s \in S_i^{GMR_k}$ *and* $R_i^+(s) \neq \emptyset$ *and for every* $s_1 \in R_i^+(s)$ *there exists at least one* $s_2 \in R_j(s_1)$ *with* $s_2 \in \bigcup_{d=k}^{r} \Phi_i^{-(d)}(s)$ *and* $s_3 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$ *for all* $s_3 \in R_i(s_2)$, *as well as there exists* $s_1' \in R_i^+(s)$ *and for every* $s_2' \in R_j(s_1') \cap (\bigcup_{d=k}^{r} \Phi_i^{-(d)}(s))$, $R_i(s_2') \cap \Phi_i^{(-d)}(s) \neq \emptyset$ *for at least one* $d \in \{0, \cdots, (k-1)\}$.

Definition 4.14. *State* s *is* ***symmetric metarational*** (SMR_{r^+}) *at level* r *for DM* i, *denoted by* $s \in S_i^{SMR_{r^+}}$, *iff either* $\bigcup_{d=0}^{r-1} R_i^{-(d)}(s) \cup R_i^+(s) = \emptyset$ *and* $R_i^{-(r)}(s) \neq \emptyset$, *or* $R_i^+(s) \neq \emptyset$ *and for every* $s_1 \in R_i^+(s)$ *there exists at least one* $s_2 \in R_j(s_1)$ *with* $s_2 \in \Phi_i^{-(r)}(s)$ *and* $s_3 \in \Phi_i^{-(r)}(s)$ *for any* $s_3 \in R_i(s_2)$.

Definition 4.15. *State* s *is* ***symmetric metarational*** (SMR_{r^-}) *at level* r *for DM* i, *denoted by* $s \in S_i^{SMR_{r^-}}$, *iff* $R_i^+(s) \neq \emptyset$ *and for every* $s_1 \in R_i^+(s)$ *there exists at least one* $s_2 \in R_j(s_1)$ *with* $s_2 \in \Phi_i^{-(r)}(s)$ *and* $s_3 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$ *for all* $s_3 \in R_i(s_2)$, *as well as there exists* $s_1' \in R_i^+(s)$ *and for every* $s_2' \in R_j(s_1) \cap \Phi_i^{-(r)}(s)$, $R_i(s_2') \cap \Phi_i^{(-d)}(s) \neq \emptyset$ *for at least one* $d \in \{0, \cdots, (r-1)\}$.

Sequential stability at level k is similar to the stability of GMR at the same level. The only modification is that all DM i's UIs are subject to credible sanctions by DM i's opponent. Its formal definition is given below.

Definition 4.16. *State* s *is* ***sequential stable*** (SEQ_0) *at level 0 for DM* i, *denoted by* $s \in S_i^{SEQ_0}$, *iff either* $R_i^+(s) = \emptyset$ *and* $R_i^{(0)}(s) \neq \emptyset$, *or* $R_i^+(s) \neq \emptyset$ *and for every* $s_1 \in R_i^+(s)$ *there exists at least one* $s_2 \in R_j^+(s_1)$ *with* $s_2 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$ *and there exists at least one* $s_1' \in R_i^+(s)$ *and* $s_2' \in R_j^+(s_1')$ *such that* $s_2' \in \Phi_i^{(0)}(s)$ *and* $R_j^+(s_1') \bigcap (\bigcup_{d=1}^{r} \Phi_i^{-(d)}(s)) = \emptyset$.

Definition 4.17. *For $1 \leq k \leq r-1$, state s is **sequentially stable** (SEQ_k) at level k for DM i, denoted by $s \in S_i^{SEQ_k}$, iff either $\bigcup_{d=0}^{k-1} R_i^{-(d)}(s) \cup R_i^{+}(s) = \emptyset$ and $R_i^{-(k)}(s) \neq \emptyset$, or $R_i^{+}(s) \neq \emptyset$ and for every $s_1 \in R_i^{+}(s)$ there exists at least one $s_2 \in R_j^{+}(s_1)$ with $s_2 \in \bigcup_{d=k}^{r} \Phi_i^{-(d)}(s)$ and there exists at least one $s_1' \in R_i^{+}(s)$ and $s_2' \in R_j^{+}(s_1')$ such that $s_2' \in \Phi_i^{-(k)}(s)$ and $R_j^{+}(s_1') \bigcap (\bigcup_{d=k+1}^{r} \Phi_i^{-(d)}(s)) = \emptyset$.*

Definition 4.18. *State s is **sequentially stable** (SEQ_r) at level r for DM i, denoted by $s \in S_i^{SEQ_r}$, iff either $\bigcup_{d=0}^{r-1} R_i^{-(d)}(s) \cup R_i^{+}(s) = \emptyset$ and $R_i^{-(r)}(s) \neq \emptyset$, or $R_i^{+}(s) \neq \emptyset$ and for every $s_1 \in R_i^{+}(s)$ there exists at least one $s_2 \in R_j^{+}(s_1)$ with $s_2 \in \Phi_i^{-(r)}(s)$.*

4.2.2 Stabilities for Multiple Levels of Preference in Multiple DM Conflicts

In an n-DM model, where $n \geq 2$, the opponents of a DM can be thought of as a coalition of one or more DMs. To calculate the stability of a state for DM $i \in N$, it is necessary to examine possible responses by all other DMs $j \in N \setminus \{i\}$, which may include sequential responses. To extend the graph model stability definitions to stability definitions in n-DM models with multiple levels of preference, the definition of a legal sequence of decisions for three levels of preference [28] must first be extended to take multiple levels of preference into account.

4.2.2.1 Legal Sequences of Unilateral Moves and Unilateral Improvements

A legal sequence of UMs in a graph model with multiple levels of preference for a coalition of DMs is a sequence of states linked by unilateral moves controlled by members of the coalition, in which a DM may move more than once, but not twice in succession. (If a DM can move in succession, then this DM's graph is effectively transitive. Prohibiting consecutive moves thus allows for graph models with intransitive graphs, which are sometimes useful in practice.) When $H = \{i\}$, a legal sequence of UMs for the coalition H reduces to a unilateral move of DM i.

Let the coalition $H \subseteq N$ satisfy $|H| \geq 2$ and let the status quo state be $s \in S$. We now define $R_H(s) \subseteq S$, the reachable list of coalition H from state s by a legal sequence of UMs in a graph model with multiple levels of preference. The following definitions are adapted from [16, 28]:

Definition 4.19. *Let $s \in S$, $H \subseteq N$, and $H \neq \emptyset$. Here, $R_j(s) = \bigcup_{d=0}^{r}(R_j^{-(d)}(s) \cup R_j^{+(d)}(s))$ for any $j \in H$. A unilateral move by H is a member of $R_H(s) \subseteq S$, defined inductively by:*
(1) if $j \in H$ and $s_1 \in R_j(s)$, then $s_1 \in R_H(s)$ and $j \in \Omega_H(s, s_1)$;
(2) if $s_1 \in R_H(s)$, $j \in H$ and $s_2 \in R_j(s_1)$, then, provided $\Omega_H(s, s_1) \neq \{j\}$, $s_2 \in R_H(s)$ and $j \in \Omega_H(s, s_2)$.

Note that Definition 4.19 is analogous to Definition 2.10, but, here, unilateral moves include the states that are reachable from state s by multiple levels of preference (may more than three levels) listed in Table 4.3.

In a graph model with multiple levels of preference, a legal sequence of UIs for coalition H is a sequence of states linked by unilateral improvements including each-level UIs controlled by members of the coalition H with the usual restriction that a member of the coalition may move more than once, but not twice consecutively. The formal definition is given below.

Definition 4.20. *Let $R_j^+(s) = \bigcup_{d=1}^{r} R_j^{+(d)}(s)$ for any $j \in H$. A unilateral improvement by H is a member of $R_H^+(s) \subseteq S$, defined inductively by:*
(1) if $j \in H$ and $s_1 \in \bigcup_{d=1}^{r} R_j^{+(d)}(s)$, then $s_1 \in R_H^+(s)$ and $j \in \Omega_H^+(s)(s, s_1)$;
(2) if $s_1 \in R_H^+(s)$, $j \in H$ and $s_2 \in \bigcup_{d=1}^{r} R_j^{+(d)}(s_1)$, then, provided $\Omega_H^+(s)(s, s_1) \neq \{j\}$, $s_2 \in R_H^+(s)$ and $j \in \Omega_H^+(s, s_2)$.

Definition 4.20 is identical to Definition 4.19 except that each move is to a state strictly preferred with some degree of preference by the mover to the current state. Similarly, $\Omega_H^+(s, s_1)$ includes all last movers in a legal sequence of UIs by coalition H from state s to state s_1. Specifically, this definition is inductive: first, using (1), the states reachable by a single DM in H from s by one step UIs in multiple levels of preference are identified and added to $R_H^+(s)$; then, using (2), all states reachable from those states are identified and added to $R_H^+(s)$; then the process is repeated until no further states are added to $R_H^+(s)$ by repeating (2). Because $R_H^+(s) \subseteq S$, and S is finite, this limit must be reached in finitely many steps.

4.2.2.2 General Stabilities for Multiple Levels of Preference

Super stability and Nash stability definitions are identical for both the 2-DM and the n-DM models because these stabilities do not consider the opponents' responses. Let $i \in N$ and $s \in S$ for the following Definitions.

Definition 4.21. *State $s \in S$ is GGMR for DM i, denoted by $s \in S_i^{GGMR}$, iff for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_{N\backslash\{i\}}(s_1)$ with $s_2 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$.*

Definition 4.22. *State $s \in S$ is GSMR for DM i, denoted by $s \in S_i^{GSMR}$, iff for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_{N\backslash\{i\}}(s_1)$ with $s_2 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$ and $s_3 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$ for all $s_3 \in R_i(s_2)$.*

Definition 4.23. *State $s \in S$ is GSEQ for DM i, denoted by $s \in S_i^{GSEQ}$, iff for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_{N\backslash\{i\}}^+(s_1)$ with $s_2 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$.*

4.2.2.3 Stabilities at Level k for Multiple Levels of Preference

Similar to 2-DM conflicts, solution concepts for n-DM conflicts can be defined as different-level stabilities, according to degrees of preference. Nash stability definitions in multiple DM conflicts are the same as those in 2-DM cases. Therefore, only the extended GMR, SMR, and SEQ are defined here. For DM i, if a UI from state s is sanctioned by the legal sequence of UMs of i's opponents in exactly k levels below s and all other UIs from state s are sanctioned in at least k levels below s, then the status quo s is called general metarational at level k. The process is portrayed in Fig. 4.3 and the formal definition is given below.

Definition 4.24. *State s is GMR_0 for DM i, denoted by $s \in S_i^{GMR_0}$, iff either $R_i^+(s) = \emptyset$ and $R_i^{(0)}(s) \neq \emptyset$, or $R_i^+(s) \neq \emptyset$ and for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_{N\backslash\{i\}}(s_1)$ with $s_2 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$ and there exists at least one $s_1' \in R_i^+(s)$ and $s_2' \in R_{N\backslash\{i\}}(s_1')$ such that $s_2' \in \Phi_i^{(0)}(s)$ and $R_{N\backslash\{i\}}(s_1') \bigcap (\bigcup_{d=1}^{r} \Phi_i^{-(d)}(s)) = \emptyset$.*

Definition 4.25. *For $1 \leq k \leq r-1$, state s is GMR_k for DM i, denoted by $s \in S_i^{GMR_k}$, iff either $\bigcup_{d=0}^{k-1} R_i^{-(d)}(s) \cup R_i^+(s) = \emptyset$ and $R_i^{-(k)}(s) \neq \emptyset$, or $R_i^+(s) \neq \emptyset$ and for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_{N\backslash\{i\}}(s_1)$ with $s_2 \in \bigcup_{d=k}^{r} \Phi_i^{-(d)}(s)$*

and there exists at least one $s_1' \in R_i^+(s)$ *and* $s_2' \in R_{N\setminus\{i\}}(s_1')$ *such that* $s_2' \in \Phi_i^{-(k)}(s)$ *and* $R_{N\setminus\{i\}}(s_1') \bigcap (\bigcup_{d=k+1}^{r} \Phi_i^{-(d)}(s)) = \emptyset$.

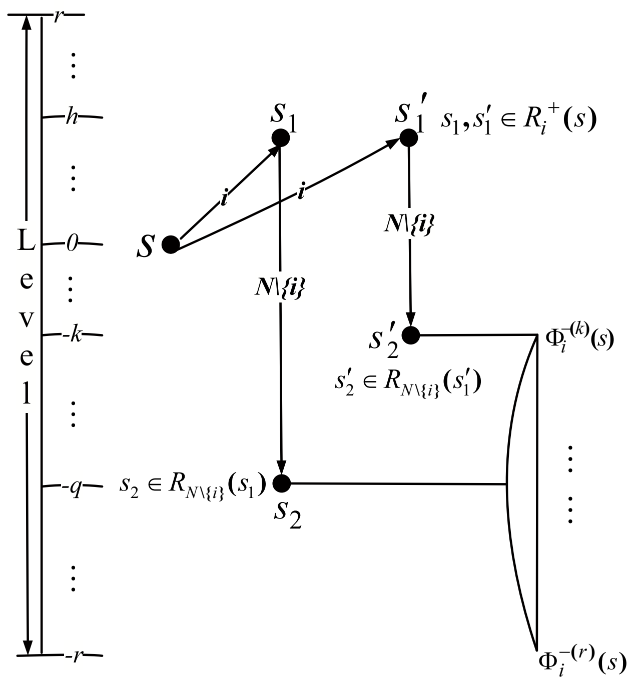


Figure 4.3: **General metarationality at level k.**

If all of DM i's UIs from a state are sanctioned at exactly r levels below the state, then the state is called general metarational at level r. Its formal definition is given below.

Definition 4.26. *State s is GMR_r for DM i, denoted by $s \in S_i^{GMR_r}$, iff either* $\bigcup_{d=0}^{r-1} R_i^{-(d)}(s) \cup R_i^+(s) = \emptyset$ *and* $R_i^{-(r)}(s) \neq \emptyset$, *or* $R_i^+(s) \neq \emptyset$ *and for every* $s_1 \in R_i^+(s)$ *there exists at least one* $s_2 \in R_{N\setminus\{i\}}(s_1)$ *with* $s_2 \in \Phi_i^{-(r)}(s)$.

For DM i, if a UI from a state is sanctioned by the legal sequence of UMs of i's opponents at level k and all other UIs from the particular state are sanctioned at level at least k, and these corresponding sanctions cannot be avoided by any counterresponse, then the state is called symmetric metarational at level k. The stability of SMR at level k is portrayed in Fig. 4.4 and the formal definition is given below.

Definition 4.27. *State s is SMR_0 for DM i, denoted by $s \in S_i^{SMR_0}$, iff either* $R_i^+(s) = \emptyset$ *and* $R_i^{(0)}(s) \neq \emptyset$, *or* $R_i^+(s) \neq \emptyset$ *and for every* $s_1 \in R_i^+(s)$ *there exists at* *least one* $s_2 \in R_{N\backslash\{i\}}(s_1)$ *with* $s_2 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$ *and there exists at least one* $s_1' \in R_i^+(s)$ *and* $s_2' \in R_{N\backslash\{i\}}(s_1')$ *such that* $s_2' \in \Phi_i^{(0)}(s)$ *and* $R_{N\backslash\{i\}}(s_1') \bigcap (\bigcup_{d=1}^{r} \Phi_i^{-(d)}(s)) = \emptyset$*, as well as* $s_3 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$ *for any* $s_3 \in R_i(s_2) \cup R_i(s_2')$*.*

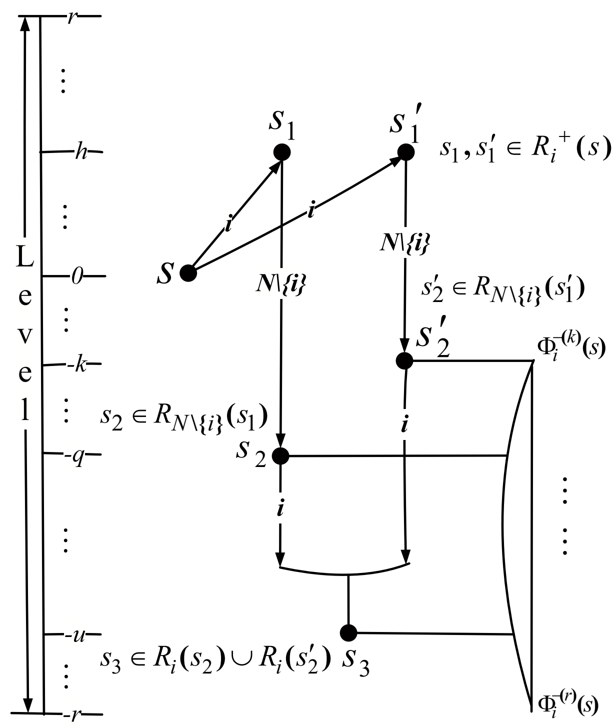


Figure 4.4: **Symmetric metarationality at level k^+.**

Symmetric metarationality at level k $(0 < k \leq r)$ for DM i consists of SMR_{k^+} and SMR_{k^-} that are defined next.

Definition 4.28. *For* $1 \leq k \leq r-1$*, state* s *is* SMR_{k^+} *for DM* i*, denoted by* $s \in S_i^{SMR_{k^+}}$*, iff either* $\bigcup_{d=0}^{k-1} R_i^{-(d)}(s) \cup R_i^+(s) = \emptyset$ *and* $R_i^{-(k)}(s) \neq \emptyset$*, or* $R_i^+(s) \neq \emptyset$ *and for every* $s_1 \in R_i^+(s)$ *there exists at least one* $s_2 \in R_{N\backslash\{i\}}(s_1)$ *with* $s_2 \in \bigcup_{d=k}^{r} \Phi_i^{-(d)}(s)$ *and there exists at least one* $s_1' \in R_i^+(s)$ *and* $s_2' \in R_{N\backslash\{i\}}(s_1')$ *such that* $s_2' \in \Phi_i^{-(k)}(s)$ *and* $R_{N\backslash\{i\}}(s_1') \bigcap (\bigcup_{d=k+1}^{r} \Phi_i^{-(d)}(s)) = \emptyset$*, as well as* $s_3 \in \bigcup_{d=k}^{r} \Phi_i^{-(d)}(s)$ *for any* $s_3 \in R_i(s_2) \cup R_i(s_2')$*.*

Stability SMR_{k^-} is defined by $S_i^{SMR_{k^-}} = S_i^{GSMR} \cap S_i^{GMR_k} - S_i^{SMR_k}$. Equivalently,

Definition 4.29. *For $1 \leq k \leq r-1$, state s is SMR_{k^-} for DM i, denoted by $s \in S_i^{SMR_{k^-}}$, iff $s \in S_i^{GMR_k}$ and $R_i^+(s) \neq \emptyset$, and for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_{N\backslash\{i\}}(s_1)$ with $s_2 \in \bigcup_{d=k}^{r} \Phi_i^{-(d)}(s)$ and $s_3 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$ for all $s_3 \in R_i(s_2)$, as well as there exists $s_1' \in R_i^+(s)$ and for every $s_2' \in R_{N\backslash\{i\}}(s_1') \cap (\bigcup_{d=k}^{r} \Phi_i^{-(d)}(s))$, $R_i(s_2') \cap \Phi_i^{(-d)}(s) \neq \emptyset$ for at least one $d \in \{0, \cdots, (k-1)\}$.*

Definition 4.30. *State s is SMR_{r^+} for DM i, denoted by $s \in S_i^{SMR_{r^+}}$, iff either $\bigcup_{d=0}^{r-1} R_i^{-(d)}(s) \cup R_i^+(s) = \emptyset$ and $R_i^{-(r)}(s) \neq \emptyset$, or $R_i^+(s) \neq \emptyset$ and for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_{N\backslash\{i\}}(s_1)$ with $s_2 \in \Phi_i^{-(r)}(s)$ and $s_3 \in \Phi_i^{-(r)}(s)$ for any $s_3 \in R_i(s_2)$.*

Definition 4.31. *State s is SMR_{r^-} for DM i, denoted by $s \in S_i^{SMR_{r^-}}$, iff $R_i^+(s) \neq \emptyset$ and for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_{N\backslash\{i\}}(s_1)$ with $s_2 \in \Phi_i^{-(r)}(s)$ and $s_3 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$ for all $s_3 \in R_i(s_2)$, as well as there exists $s_1' \in R_i^+(s)$ and for every $s_2' \in R_{N\backslash\{i\}}(s_1) \cap \Phi_i^{-(r)}(s)$, $R_i(s_2') \cap \Phi_i^{(-d)}(s) \neq \emptyset$ for at least one $d \in \{0, \cdots, (r-1)\}$.*

The only modification between GMR_k and SEQ_k is that all DM i's UIs are subject to credible sanctions by the legal sequence of UIs of DM i's opponents. Fig. 4.5 depicts sequential stability at level k. Its formal definition is given below.

Definition 4.32. *State s is* ***sequentially stable*** *(SEQ_0) at level 0 for DM i, denoted by $s \in S_i^{SEQ_0}$, iff either $R_i^+(s) = \emptyset$ and $R_i^{(0)}(s) \neq \emptyset$, or $R_i^+(s) \neq \emptyset$ and for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_{N\backslash\{i\}}^+(s_1)$ with $s_2 \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$ and there exists at least one $s_1' \in R_i^+(s)$ and $s_2' \in R_{N\backslash\{i\}}^+(s_1')$ such that $s_2' \in \Phi_i^{(0)}(s)$ and $R_{N\backslash\{i\}}^+(s_1') \bigcap (\bigcup_{d=1}^{r} \Phi_i^{-(d)}(s)) = \emptyset$.*

Definition 4.33. *For $1 \leq k \leq r-1$, state s is* ***sequentially stable*** *(SEQ_k) at level k for DM i, denoted by $s \in S_i^{SEQ_k}$, iff either $\bigcup_{d=0}^{k-1} R_i^{-(d)}(s) \cup R_i^+(s) = \emptyset$ and $R_i^{-(k)}(s) \neq \emptyset$, or $R_i^+(s) \neq \emptyset$ and for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_{N\backslash\{i\}}^+(s_1)$ with $s_2 \in \bigcup_{d=k}^{r} \Phi_i^{-(d)}(s)$ and there exists at least one $s_1' \in R_i^+(s)$ and $s_2' \in R_{N\backslash\{i\}}^+(s_1')$ such that $s_2' \in \Phi_i^{-(k)}(s)$ and $R_{N\backslash\{i\}}^+(s_1') \bigcap (\bigcup_{d=k+1}^{r} \Phi_i^{-(d)}(s)) = \emptyset$.*

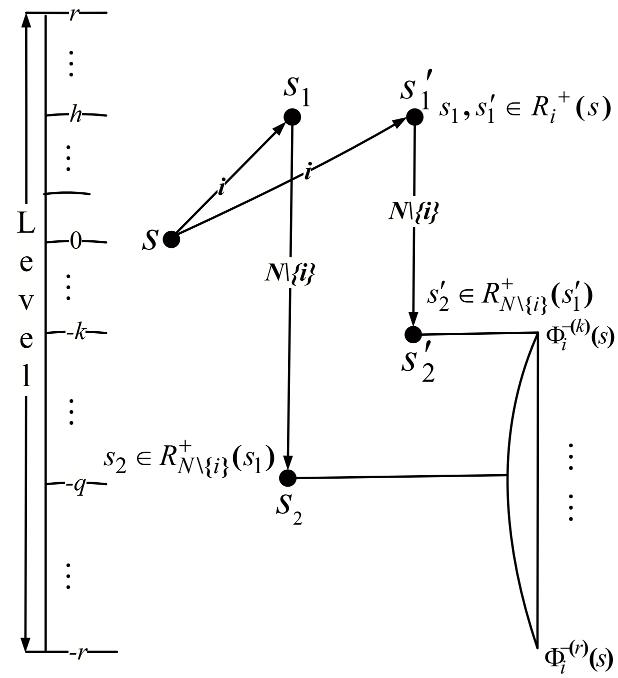


Figure 4.5: **Sequential stability at level k.**

Definition 4.34. *State s is **sequentially stable** (SEQ_r) at level r for DM i, denoted by $s \in S_i^{SEQ_r}$, iff either $\bigcup_{d=0}^{r-1} R_i^{-(d)}(s) \cup R_i^{+}(s) = \emptyset$ and $R_i^{-(r)}(s) \neq \emptyset$, or $R_i^+(s) \neq \emptyset$ and for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R^+_{N\setminus\{i\}}(s_1)$ with $s_2 \in \Phi_i^{-(r)}(s)$.*

When $n = 2$, the DM set N becomes to $\{i, j\}$ in Definitions 4.24 to 4.34, and the reachable lists for $H = N \setminus \{i\}$ by legal sequences of UMs and UIs from s_1, $R_{N\setminus\{i\}}(s_1)$ and $R^+_{N\setminus\{i\}}(s_1)$, degenerate to $R_j(s_1)$ and $R_j^+(s_1)$, DM j's corresponding reachable lists from s_1. Obviously, Definitions 4.8 to 4.18 are special cases of Definition 4.24 to 4.34, so we use the same notation for two DM cases and n-DM situations.

4.3 Interrelationships among the Solution Concepts

In 1993, Fang et al. [16] established relationships among the four basic stabilities of Nash, GMR, SMR, and SEQ for two levels of preference. Then, Hamouda et al. [27, 28] extended these results to a graph model with three levels of preference.

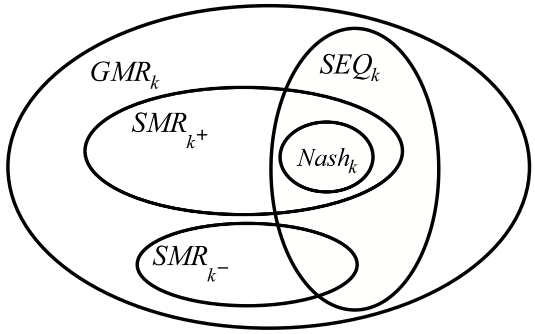


Figure 4.6: **Interrelationships among four stabilities at level k.**

The inclusion relations among the multilevel versions of the four solution concepts are presented as follows.

Theorem 4.1. *The interrelationships among the four basic stabilities at level k are*

$$S_i^{Nash_k} \subseteq S_i^{SMR_{k+}} \subseteq S_i^{GMR_k}, S_i^{SMR_{k-}} \subseteq S_i^{GMR_k}, \text{ and } S_i^{Nash_k} \subseteq S_i^{SEQ_k} \subseteq S_i^{GMR_k},$$

for $0 \le k \le r$.

Proof: When $k = 0$, the results are obvious. Assume that $0 < k \le r$. If $s \in S_i^{Nash_k}$, then $\bigcup_{d=0}^{k-1} R_i^{-(d)}(s) \cup R_i^{+}(s) = \emptyset$ and $R_i^{-(k)}(s) \neq \emptyset$. This implies that state $s \in S_i^{SMR_{k+}}$ using Definitions 4.28 and 4.30. Hence, if $s \in S_i^{Nash_k}$ for $0 \le k \le r$, then $s \in S_i^{SMR_{k+}}$, which implies $S_i^{Nash_k} \subseteq S_i^{SMR_{k+}}$.

Using Definitions 4.24 to 4.30, if $s \in S_i^{SMR_{k+}}$, it is obvious that $s \in S_i^{GMR_k}$ for $0 \le k \le r$. Therefore, inclusion relations $S_i^{Nash_k} \subseteq S_i^{SMR_{k+}} \subseteq S_i^{GMR_k}$ now follow.

Based on Definitions 4.29 and 4.31, the relation $S_i^{SMR_{k-}} \subseteq S_i^{GMR_k}$ is obvious. Relations $S_i^{Nash_k} \subseteq S_i^{SEQ_k} \subseteq S_i^{GMR_k}$ can be similarly verified. □

Let $0 \le k \le r$. The inclusion relationships presented by Theorem 4.1 are depicted in Fig. 4.6.

Theorem 4.2. *Let* $0 \le h, q \le r$. *When* $h \neq q$, *the relationships between stabilities at h level and at q level are*

$$S_i^{Nash_h} \cap S_i^{Nash_q} = \emptyset, \tag{4.1}$$

$$S_i^{GMR_h} \cap S_i^{GMR_q} = \emptyset, \tag{4.2}$$

$$S_i^{SMR_{h+}} \cap S_i^{SMR_{q+}} = \emptyset, S_i^{SMR_{h-}} \cap S_i^{SMR_{q-}} = \emptyset, S_i^{SMR_{h+}} \cap S_i^{SMR_{h-}} = \emptyset, and \quad (4.3)$$

$$S_i^{SEQ_h} \cap S_i^{SEQ_q} = \emptyset. \quad (4.4)$$

Proof: We first prove equation (4.1). Assume that $h > q$. If there exists $s \in S_i^{Nash_h} \cap S_i^{Nash_q}$, then $s \in S_i^{Nash_h}$ and $s \in S_i^{Nash_q}$. Therefore, $R_i^+(s) \cup (\bigcup_{d=0}^{h-1} R_i^{-(d)}(s)) = \emptyset$ and $R_i^{-(h)}(s) \neq \emptyset$ as s is $Nash_h$ stable. Since $h-1 \geq q$, $R_i^{-(q)}(s) = \emptyset$. This contradicts the hypothesis that s is $Nash_q$ stable. Therefore, (4.1) holds.

Now, equation (4.2) is verified. If $s \in (S_i^{Nash_h} \cup S_i^{Nash_q})$, equation (4.2) is obvious. Assume that $h > q$ and $s \notin (S_i^{Nash_h} \cup S_i^{Nash_q})$. If there exists $s \in S_i^{GMR_h} \cap S_i^{GMR_q}$, then $s \in S_i^{GMR_h}$ and $s \in S_i^{GMR_q}$. Since s is GMR_q stable, $R_i^+(s) \neq \emptyset$ and for every $s_1 \in R_i^+(s)$ there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$ with $s_2 \in \bigcup_{d=q}^{r} \Phi_i^{-(d)}(s)$ and there exists at least one $s_1' \in R_i^+(s)$ and $s_2' \in R_{N\setminus\{i\}}(s_1')$ such that $s_2' \in \Phi_i^{-(q)}(s)$ and $R_{N\setminus\{i\}}(s_1') \bigcap (\bigcup_{d=q+1}^{r} \Phi_i^{-(d)}(s)) = \emptyset$. This implies that for all $s_2' \in R_{N\setminus\{i\}}(s_1')$, $s_2' \in \bigcup_{d=0}^{q} \Phi_i^{-(d)}(s)$ i.e., $s_2' \notin \bigcup_{d=h}^{r} \Phi_i^{-(d)}(s)$ as $h > q$. This contradicts with the hypothesis that s is GMR_h stable. Therefore, (4.2) follows now.

The proofs of (4.3) and (4.4) can be similarly carried out. □

The interrelationships among general stabilities, super stability, and stabilities at each level are presented in the following theorem.

Theorem 4.3. *The interrelationships among general stabilities, super stability, and stabilities at each level are*

$$S_i^{GNash} = (S_i^{Super}) \cup (\bigcup_{d=0}^{r} S_i^{Nash_d}), \quad (4.5)$$

$$S_i^{GGMR} = (S_i^{Super}) \cup (\bigcup_{d=0}^{r} S_i^{GMR_d}), \quad (4.6)$$

$$S_i^{GSMR} = (S_i^{Super}) \cup (\bigcup_{d=0}^{r} (S_i^{SMR_{d+}} \cup S_i^{SMR_{d-}})), and \quad (4.7)$$

$$S_i^{GSEQ} = (S_i^{Super}) \cup (\bigcup_{d=0}^{r} S_i^{SEQ_d}). \quad (4.8)$$

Proof: Equation (4.5) is obvious. Equation (4.6) is verified first. The inclusion relation $S_i^{GGMR} \supseteq (S_i^{Super}) \cup (\bigcup_{d=0}^{r} S_i^{GMR_d})$ is obvious. We will prove that the inclusion relation $S_i^{GGMR} \subseteq (S_i^{Super}) \cup (\bigcup_{d=0}^{r} S_i^{GMR_d})$ holds. For any $s \in S_i^{GGMR}$, based on Definition 4.21, if $s \in (S_i^{Super} \cup S_i^{GNash})$, then the above inclusion relation is true.

Let $|R_i^+(s)| = l$ denote the cardinality of $R_i^+(s)$. Assume that $s \notin (S_i^{Super} \cup S_i^{GNash})$. Then, for any $s \in S_i^{GGMR}$, $R_i^+(s) \neq \emptyset$ and for every $s_k \in R_i^+(s)$ ($k = 1, \cdots, l$), there exists at least one $s'_k \in R_{N\backslash\{i\}}(s_k)$ with $s'_k \in \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)$. Let $Q_k = \{q : q \in R_{N\backslash\{i\}}(s_k) \cap \bigcup_{d=0}^{r} \Phi_i^{-(d)}(s)\}$. It is obvious that $s'_k \in Q_k$. Hence, $Q_k \neq \emptyset$. Let $z \in Q_k$ and be DM i's least preferred in the state set Q_k. Since $z \in R_{N\backslash\{i\}}(s_k) \cap (\bigcup_{d=0}^{r} \Phi_i^{-(d)}(s))$, there exists $0 \leq r_k \leq r$ such that $z \in \Phi_i^{-(r_k)}(s)$ for $k = 1, \cdots, l$. Therefore, either $r_k = r$ or $R_{N\backslash\{i\}}(s_k) \cap (\bigcup_{d=r_k+1}^{r} \Phi_i^{-(d)}(s)) = \emptyset$. This process is portrayed in Fig. 4.7.

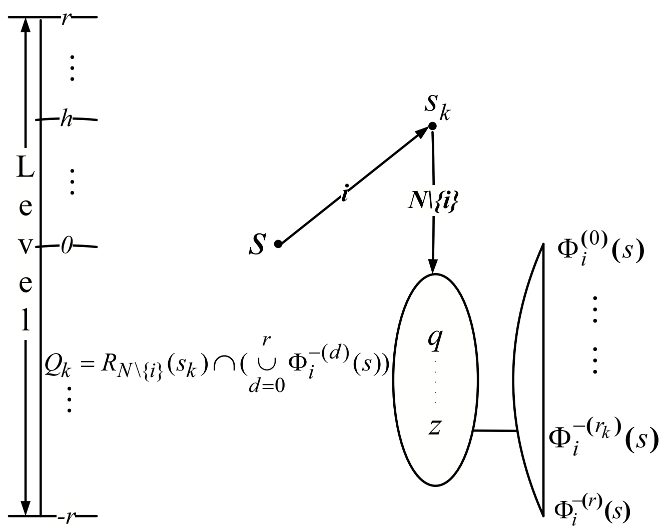


Figure 4.7: **The legal sequence of UM from state s_k.**

Let $r_m = \min\{r_k : k = 1, \cdots, l\}$. Then, $0 \leq r_m \leq r$. It is easy to follow that if $s \in S_i^{GGMR}$ and $R_i^+(s) \neq \emptyset$, then $s \in S_i^{GMR_{r_m}}$. In fact, for every $s_k \in R_i^+(s)$, there exists at least one $s'_k \in R_{N\backslash\{i\}}(s_k)$ with $s'_k \in \Phi_i^{-(r_k)}(s)$. Since $0 \leq r_m \leq r_k$, then $s'_k \in \bigcup_{d=r_m}^{r} \Phi_i^{-(d)}(s)$, and $s'_m \in R_{N\backslash\{i\}}(s_m)$ with $s'_m \in \Phi_i^{-(r_m)}(s)$. Based on the rule of selecting r_m, either $r_m = r$ so that $s \in S_i^{GMR_r}$, or $R_{N\backslash\{i\}}(s_m) \cap (\bigcup_{d=r_m+1}^{r} \Phi_i^{-(d)}(s)) = \emptyset$ so that $s \in S_i^{GMR_{r_m}}$. From the above discussion, equation (4.6) is proved.

Hence, equations (4.7) and (4.8) can be similarly proved. $\square$

Let S_i^{Nash}, S_i^{GMR}, S_i^{SMR}, and S_i^{SEQ} denote all stable states for Nash, GMR, SMR, and SEQ, respectively, in the graph model for simple preference [16]. When $r = 1$, stabilities having multiple-level preference degenerate to the stabilities presented in [16], including two levels of preference. Specifically,

Theorem 4.4. *For the multiple levels of preference, when $r = 1$, $S_i^{Super} \cup S_i^{Nash_0} \cup S_i^{Nash_1} = S_i^{Nash}$, $S_i^{Super} \cup S_i^{GMR_0} \cup S_i^{GMR_1} = S_i^{GMR}$, $S_i^{Super} \cup S_i^{SMR_0} \cup S_i^{SMR_{1+}} \cup S_i^{SMR_{1-}} = S_i^{SMR}$, and $S_i^{Super} \cup S_i^{SEQ_0} \cup S_i^{SEQ_1} = S_i^{SEQ}$.*

Let S_i^{SGMR}, S_i^{SSMR}, and S_i^{SSEQ} denote all strongly stable states for strongly GMR, SMR , and SEQ, respectively, in the graph model with strength of preference [27, 28]. When $r = 2$, stabilities having multiple levels of preference degenerate to the stabilities presented in [27, 28]. Specifically,

Theorem 4.5. *For the multiple levels of preference, when $r = 2$, $S_i^{GMR_2} \backslash S_i^{Nash_2} = S_i^{SGMR}$, $S_i^{SMR_{2+}} \backslash S_i^{Nash_2} = S_i^{SSMR}$, and $S_i^{SEQ_2} \backslash S_i^{Nash_2} = S_i^{SSEQ}$.*

The stabilities at level 2 in the graph model with three levels of preference degenerate to the corresponding strong stabilities presented in [27, 28], except for the states that are Nash stable, because Hamouda et al. [27, 28] have not included Nash stable states into strongly GMR, SMR, and SEQ.

The above two theorems can be easily proved using the corresponding definitions.

4.4 Application: GDU Conflict

In this section, the four-level versions of stability definitions are applied to the Garrison Diversion Unit (GDU) conflict to illustrate how the procedure works. The history of this conflict dates back to the nineteenth century. In order to irrigate land in the northeastern section of North Dakota, an irrigation project was proposed by the **United States Support (USS)** regarding construction of a crucial canal and holding reservoir to transfer water from the Missouri River Basin to the Hudson Bay Basin [16]. Because the irrigation runoff finally flow into the Canadian province of Manitoba via the Red and Souris rivers, which will cause environmental damage, this proposal immediately aroused the **Canadian Opposition (CDO)**. In order to resolve this conflict, the **International Joint Commission (IJC)** consisting of representatives from the governments of USA and Canada plays an important role for taking an unbiased attitude and making recommendations on this project [16, 28]. This irrigation project for the water diversion is called the Garrison Diversion Unit (GDU) project. A conflict arose among US, Canada and IJC for the GDU project (see the book [16] and the paper [28] for more details).

Table 4.4: **Feasible states for the GDU model** [28]

USS									
1. Proceed	Y	Y	N	Y	N	Y	N	Y	N
2. Modify	N	N	Y	N	Y	N	Y	N	Y
CDO									
3. Legal	N	N	N	Y	Y	N	N	Y	Y
IJC									
4. Completion	N	Y	Y	Y	Y	N	N	N	N
5. Modification	N	N	N	N	N	Y	Y	Y	Y
State number	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9

Fang et al. [16] analyzed the environmental dispute over the GDU project and established a graph model with two levels of preference for this conflict. Recently, Hamouda et al. [28] carried out a strategic study of this conflict using an extended graph model which includes three levels of preference. The graph model for the GDU conflict is comprised of three DMs: 1. **USS**, 2. **CDO**, and 3. **IJC**; and five options: 1. **Proceed**−Proceed with the project regardless of Canada's concerns; 2. **Modify**−Modify the project to reduce impacts on Canada; 3. **Legal**−Legal action based on Boundary Waters Treaty; 4. **Completion**−Recommend completion of the project as originally planned; and 5. **Modification**−Recommend modification of the project to reduce impacts on Canada [28]. A state is defined as a selection of options for each DM using some principle. In the GDU conflict, five options are combined to form 2^5 possible states. Usually, however, not all option combinations are feasible or logical. After all infeasible states are eliminated, only nine states are identified as being feasible and listed in Table 4.4 in which a "Y" indicates that an option is selected by the DM controlling it and an "N" means that the option is not chosen.

The graph model of the GDU conflict is shown in Fig. 4.8, in which labels on the arcs indicate each DM who controls the move. All that is required for a graph

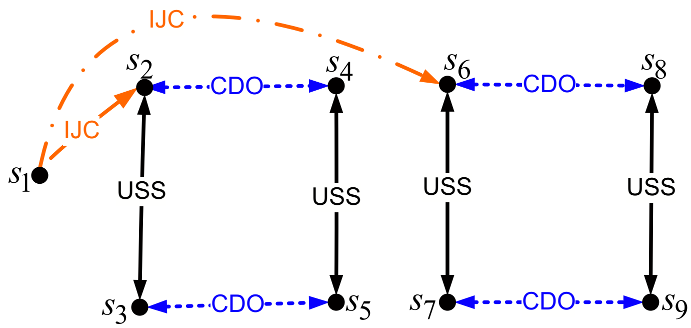


Figure 4.8: **The graph model for the GDU conflict [28].**

model is knowledge of each DM's preference ranking of the feasible states. We extend the graph model introduced in [28] to have four levels of preference in the GDU conflict. The preference information for this conflict over the feasible states is given in Table 4.5. We assume that state s_8 is very strongly less preferred to all other states for USS, and the DM, CDO considers states s_1, s_2, and s_6 to be equally preferred and very strongly less preferred relative to all other states. Note that this representation of preference information presented in Table 4.5 implies that the preferred relations, $>$, $\gg$, and $\ggg$ are transitive. For instance, since $s_9 > s_7$ and $s_7 \ggg s_8$, then $s_9 \ggg s_8$. However, in general, the preference structure presented in this research does not require the transitivity of preference relations, and hence can handle intransitive preferences.

Table 4.5: **Four levels of preferences for DMs in the GDU conflict** (extended from [28])

DM	Preference
USS	$s_2 > s_4 > s_3 > s_5 > s_1 > s_6 > s_9 > s_7 \ggg s_8$
CDO	$\{s_3 \sim s_7\} > \{s_5 \sim s_9\} > \{s_4 \sim s_8\} \ggg \{s_1 \sim s_2 \sim s_6\}$
IJC	$\{s_2 \sim s_3 \sim s_4 \sim s_5 \sim s_6 \sim s_7 \sim s_8 \sim s_9\} \gg s_1$

Formally, stability analysis determines the stability of each state for each DM according to some solution concept. Here, four-level versions of five stability definitions of super stability, Nash stability, $Nash_k$, GMR_k, SMR_k, and sequential stability, SEQ_k, for $k = 0, 1, 2, 3$ are employed to analyze the GDU conflict. An equilibrium indexed k, which represents a likely resolution to the conflict, is a state that is stable for every DM according to some stability definition at level k. Note that the super stable states are treated as Nash stable at the highest level when determining an equilibrium in the graph model with multiple levels of preference. Here, we analyze DM 2's SMR_k stability at state s_5 for $k = 0, 1, 2, 3$ as an example. Since $R_2^+(s_5) = \{s_3\}$ and $R_{N\backslash\{2\}}(s_3) = \{s_2\}$ with $s_5 \ggg_2 s_2$ and $s_5 >_2 s_4$ for $R_2(s_2) = \{s_4\}$, state s_5 is stable for SMR_{3^-} using Definition 4.31. Other cases can be analyzed similarly. The stability results for the GDU conflict are summarized in Table 4.6, in which "$\surd$" for a given state under a DM means that this state is stable at a given level for the given DM; "$\surd^{k^+}$" and "$\surd^{k^-}$" for a given state under a DM means that this state is SMR_{k^+} or SMR_{k^-} stable for the given DM; and "$\surd^{k}$" for a state under "Eq" signifies that this state is an equilibrium for a corresponding solution concept at level k. Note that U, C, and I displayed in Table 4.6 denote the three DMs, USS, CDO, and IJC, respectively.

Table 4.7 provides stability results for different structures of preference. When stabilities are analyzed using two levels of preference, states s_4, s_7, and s_9 are equilibria [16]; if preference information is provided using three levels of preference, then states s_7 and s_9 are equilibria [28]; there is only one equilibrium state s_9 for four levels of preference. If state s_4 is selected as a resolution for the GDU conflict, this means that IJC recommends completing the GDU project regardless of Canada's concerns, so USS proceeds with this project. It is obvious that this resolution cannot really resolve this conflict. State s_7 means that the USS follows the IJC recommendation to modify this project, but Canada does not take legal action based on the Boundary Waters Treaty. The strategy of state s_9 is the same as that of state s_7 except that Canada chooses legal procedures. Compared with states s_7 and s_9, equilibrium s_9 is a more reasonable resolution for resolving this conflict. Therefore, the multilevel versions of stability analysis provide new insights and valuable guidance for decision analysts.

Although the example of the GDU conflict shown in Table 4.4 and Fig. 4.8 is a small model with three DMs, five options, and nine feasible states, a graph model structure can handle any finite number of states and DMs, each of whom can control any finite number of options [18]. As pointed out by Fang et al. [19], an available decision support system (DSS) for stability analysis of a graph model with two levels of preference can work well. Theorem 4.4 reveals the relation of stabilities between two levels of preference [16] and multiple levels of preference. This theorem

Table 4.6: **Stability results of the GDU conflict for the graph model with four levels of preference**

State	Super				Level(k)	Nash				GMR				SMR				SEQ			
	U	C	I	Eq		U	C	I	Eq	U	C	I	Eq	U	C	I	Eq	U	C	I	Eq
					0																
					1																
s_1	√	√			2																
					3	√	√			√	√			$\surd^{3+}$	$\surd^{3+}$			√	√		
					0																
					1	√				√				$\surd^{1+}$				√			
s_2			√		2																
					3			√				√				$\surd^{3+}$				√	
					0																
					1		√				√				$\surd^{1+}$				√		
s_3			√		2																
					3			√				√				$\surd^{3+}$				√	
					0																
					1	√				√				$\surd^{1+}$				√			
s_4			√		2																
					3		√	√			√	√			$\surd^{3+}$	$\surd^{3+}$			√	√	
					0																
					1																
s_5			√		2																
					3			√			√	√			$\surd^{3-}$	$\surd^{3+}$			√	√	
					0																
					1	√				√				$\surd^{1+}$				√			
s_6			√		2																
					3			√				√				$\surd^{3+}$				√	
					0																
					1		√				√				$\surd^{1+}$				√		
s_7			√		2																
					3			√		√		√				$\surd^{3+}$		√		√	
					0																
					1																
s_8			√		2																
					3		√	√			√	√			$\surd^{3+}$	$\surd^{3+}$			√	√	
					0																
					1																
s_9			√		2																
					3	√		√		√	√	√	$\surd^{3}$	$\surd^{3+}$	$\surd^{3-}$	$\surd^{3+}$		√	√	√	$\surd^{3}$

Table 4.7: **The comparison of stability results for three versions of preference**

Version of preference	Equilibria	Analysis method
Two levels of preference	s_4, s_7, s_9	see [16]
Three levels of preference	s_4, s_9	see [28]
Four levels of preference	s_9	this paper

indicates the possibility of developing an effective algorithm to implement the multilevel versions of the four stabilities within a DSS, which would be essential if the proposed stability analysis is applied to larger practical problems.

4.5 Summary

In this chapter, a multiple-level preference framework is developed for the graph model methodology to handle multiple levels of preference, which lie between relative and cardinal preferences in terms of information content [74]. Multilevel versions of four solution concepts consisting of Nash, GMR, SMR, and SEQ are defined in the graph model for multiple levels of preference. Specifically, solution concepts at level k are defined as $Nash_k$, GMR_k, SMR_k, and SEQ_k for $k = 1, \cdots, r$, where r is the maximum number of levels of preference between two states. The proposed stability definitions extend existing definitions based on two levels and three levels of preference, so that more practical and complicated problems can be analyzed at greater depth. To date, new stability definitions are defined by logical representation, so algorithms to implement these new stabilities are difficult to develop. A new algebraic system to ease the coding of logically-defined stability definitions is proposed in the following chapters.

Chapter 5

Novel Algebraic Approach to Searching Weighted Colored Paths

An algebraic approach to finding all edge-weighted, colored paths within a weighted colored multidigraph is developed in this chapter. Generally, an adjacency matrix can determine a simple digraph and all paths between any two vertices. However, the adjacency matrix is not readily extendable to the context of a colored multidigraph. To bridge the gap, a conversion function is proposed to transform the original problem of searching edge-colored paths in a colored multidigraph to a standard problem of finding paths in a simple digraph with no color constraints. To date, for general graph classes, searching for particular paths, such as Hamilton paths [2, 56], Euler paths, and shortest path routing between two vertices, can be solved efficiently. Some algorithms to search colored paths for colored simple graphs are available [1], but there exist very limited algorithms to search colored paths for colored multidigraph classes.

5.1 Extended Definitions in a Weighted Colored Multidigraph

A multidigraph $G = (V, A, \psi)$ defined in Section 2.1 is a set of vertices (nodes) V and a multiset of oriented edges (arcs) A with $\psi : A \to V \times V$. Let $m = |V|$ denote the number of vertices and $l = |A|$ be the number of edges in a multidigraph G.

Definition 5.1. *A **colored multidigraph** (V, A, N, ψ, c) is a multidigraph (V, A, ψ) and a set of colors N, and a function $c : A \to N$ such that $c(a) \in N$ is the color of $a \in A$, provided that multiple edges of (V, A, ψ) are assigned different colors , i.e., if $a \neq b$, but $\psi(a) = \psi(b)$, then $c(a) \neq c(b)$.*

If $a \in A$ such that $\psi(a) = (u, v)$ and $c(a) = i$ for $i \in N$, then a can be written as $a = d_i(u, v)$. The line digraph of $G = (V, A, N, \psi, c)$, $L(G)$, is a simple digraph and each vertex in $L(G)$ corresponds to an edge in the multidigraph G. Hence, coloring edges in G is equivalent to assigning colors to vertices in $L(G)$.

Definition 5.2. *For a colored multidigraph $G = (V, A, N, \psi, c)$, the **reduced line digraph** $L_r(G) = (A, LA_r)$ of G is a simple vertex-colored digraph with vertex set A and edge set LA_r=$\{d = (a, b) \in A \times A$: a and b are consecutive (in the order ab) and $c(a) \neq c(b)\}$.*

Definition 5.3. *A **weighted colored multidigraph** (V, A, N, ψ, c, w) is a colored multidigraph (V, A, N, ψ, c) together with a map $w : A \to \mathbb{R}_0^+$ (the set of non-negative real numbers).*

Thus an arc $a \in A$, $a = d_i(u, v)$, carries a weight $w(a)$, representing some attribute of the move from node u to node v along the arc a, which is assigned color i. A network, for instance, is a multidigraph with weighted edges. Let $H \subseteq N$ be a subset of the color set N in the following definitions. An edge-weighted, colored path is defined as follows:

Definition 5.4. *Let $H \subseteq N$. For a weighted colored multidigraph (V, A, N, ψ, c, w), an **edge-weighted, colored path by** H **from vertex** $u \in V$ **to vertex** $v \in V$, $PA_H^{(W)}(u, v)$, is a path from u to v in the multidigraph (V, A, ψ) in which any two consecutive edges have different colors and each edge a on the path carries a weight $w(a) \geq 0$ and $c(a) = i \in H$.*

Definition 5.5. *For a weighted colored multidigraph (V, A, N, ψ, c, w), the **shortest colored path between two vertices** is the colored path that minimizes the sum of the weights of its constituent edges.*

Definition 5.6. *Let $H \subseteq N$. For a weighted colored multidigraph (V, A, N, ψ, c, w), the **weighted arc set for** H denotes $A_H^{(W)} = \{a \in A : w(a) > 0$ and $c(a) = i \in H.\}$.*

Note that a colored multidigraph (V, A, N, ψ, c) is a unit weighted colored multidigraph if $w(u, v) = 1$ for any $a \in A$ such that $\psi(a) = (u, v)$.

Let $l = |A|$ denote the cardinality of A in G. The weight matrix of a weighted colored multidigraph (V, A, N, ψ, c, w) is defined as follows:

Definition 5.7. *For a weighted colored multidigraph (V, A, N, ψ, c, w), let $H \subseteq N$ and w_k denote the weight of arc $a_k \in A$. The **weight matrix for** H is an $l \times l$ diagonal matrix W_H with (k, k) entry*

$$W_H(k,k) = \begin{cases} w_k & \text{if } c(a_k) = i \in H, \\ 0 & \text{otherwise.} \end{cases}$$

It should be pointed out that if $H = N$, then W_N is expressed as W; if $H = \{i\}$, then $W_H = W_i$. A weighted line digraph $L^{(W)}(G) = (A, LA, w)$ is a set of vertices A together with a set of oriented edges LA, and a map $w : A \to \mathbb{R}_0^+$. In traditional graph coloring problems, such as vertex coloring and edge coloring, colors are assigned to vertices or edges such that adjacent vertices or consecutive edges have different colors, and the number of colors needed is minimized [13]. In this research, the edge-weighted, colored graph problem is not concerned with coloring edges, but aims at searching edge-weighted, colored paths in a given weighted colored multidigraph.

Important matrices associated with a digraph include the adjacency matrix J and the incidence matrix B [24]. J and B can be extended to the weighted adjacency and incidence matrices. Let $m = |V|$ denote the cardinality of V in G.

Definition 5.8. *Let $H \subseteq N$. For a weighted colored multidigraph (V, A, N, ψ, c, w), the weighted adjacency matrix for H is the $m \times m$ matrix $J_H^{(W)}$ with (s, q) entry*

$$J_H^{(W)}(s,q) = \begin{cases} 1 & \text{if there exists } a \in A_H^{(W)} \text{ such that } \psi(a) = (s,q) \text{ for } s, q \in V, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 5.9. *For a weighted colored multidigraph (V, A, N, ψ, c, w), w_a denotes the weight of arc $a \in A$. The **weighted incidence matrix for** H is the $m \times l$ matrix $B^{(W_H)}$ with (v, a) entry*

$$B^{(W_H)}(v,a) = \begin{cases} -w_a & \text{if } a = (v,x) \text{ for some } x \in V \text{ and } c(a) = i \in H, \\ w_a & \text{if } a = (x,v) \text{ for some } x \in V \text{ and } c(a) = i \in H, \\ 0 & \text{otherwise,} \end{cases}$$

where $v \in V$.

According to the signed entries, the weighted incidence matrix can be separated into the weighted in-incidence matrix and the weighted out-incidence matrix.

Definition 5.10. *For a weighted colored multidigraph (V, A, N, ψ, c, w), let $H \subseteq N$ and w_a denote the weight of arc $a \in A$. The **weighted in-incidence matrix for** H and the **weighted out-incidence matrix for** H are two $m \times l$ matrices $B_{in}^{(W_H)}$ and $B_{out}^{(W_H)}$ with (v, a) entries*

$$B_{in}^{(W_H)}(v,a) = \begin{cases} w_a & \text{if } a = (x, v) \text{ for some } x \in V \text{ and } c(a) = i \in H, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$B_{out}^{(W_H)}(v,a) = \begin{cases} w_a & \text{if } a = (v, x) \text{ for some } x \in V \text{ and } c(a) = i \in H, \\ 0 & \text{otherwise,} \end{cases}$$

where $v \in V$.

It is obvious that

$$B_{in}^{(W_H)} = (B^{(W_H)} + abs(B^{(W_H)}))/2 \ \ and \ \ B_{out}^{(W_H)} = (abs(B^{(W_H)}) - B^{(W_H)})/2,$$

where $abs(B^{(W_H)})$ denotes the matrix in which each entry equals the absolute value of the corresponding entry of $B^{(W_H)}$. Let I denote the identity matrix. If $W_H = I$, then $B^{(W_H)} = B$, $B_{in}^{(W_H)} = B_{in}$, and $B_{out}^{(W_H)} = B_{out}$.

A reachability by the weighted colored paths for H matrix is called a reachability matrix by H in this research. Its formal definition is given as follows.

Definition 5.11. *Let $H \subseteq N$. For a weighted colored multidigraph (V, A, N, ψ, c, w), the weighted reachability matrix by H is the $m \times m$ matrix $M_H^{(W)}$ with (s, q) entry*

$$M_H^{(W)}(s,q) = \begin{cases} 1 & \text{if } q \text{ is reachable from vertex } s \text{ by a weighted} \\ & \text{colored path } PA_H^{(W)}(s,q), \text{ for } s, q \in V, \\ 0 & \text{otherwise.} \end{cases}$$

Let $l_H^{(W)} = |A_H^{(W)}|$ denote the number of arcs in $A_H^{(W)}$. Since all arcs are distinct on a path, the length of any path in $PA_H^{(W)}$ is less than $l_H^{(W)}$.

The following result can be obtained by Definition 2.2, on the line digraph $L(G)$, and Definition 2.4, on the adjacency matrix J.

For a weighted colored multidigraph $G = (V, A, N, \psi, c, w)$, the **adjacency matrix of the line graph** of G is the $l \times l$ matrix LJ with (a, b) entry

$$LJ(a,b) = \begin{cases} 1 & \text{if edges } a \text{ and } b \text{ are consecutive in order } ab \text{ in the graph } G, \\ 0 & \textit{otherwise.} \end{cases}$$

In this research, LJ matrix is called an edge consecutive matrix.

Definition 5.12. *For a weighted colored multidigraph $G = (V, A, N, \psi, c, w)$, let $H \subseteq N$ and w_a and w_b denote the weights of arcs $a, b \in A$. The **weighted edge consecutive matrix for** H is the $l \times l$ matrix $LJ^{(W_H)}$ with (a, b) entry*

$$LJ^{(W_H)}(a,b) = \begin{cases} w_a \cdot w_b & \textit{if edges } a \textit{ and } b \textit{ are consecutive in order } ab \\ & \textit{and } c(a) = i \textit{ and } c(b) = j \textit{ for } i, j \in H, \\ 0 & \textit{otherwise.} \end{cases}$$

Definition 5.13. *For a weighted colored multidigraph $G = (V, A, N, \psi, c, w)$, the **reduced weighted edge consecutive matrix for** H is the $l \times l$ matrix $LJ_r^{(W_H)}$ with (a, b) entry*

$$LJ_r^{(W_H)}(a,b) = \begin{cases} w_a \cdot w_b & \textit{if edges } a \textit{ and } b \textit{ are consecutive in order } ab \textit{ and} \\ & c(a) = i \textit{ and } c(b) = j \textit{ such that } i, j \in H \textit{ and } i \neq j, \\ 0 & \textit{otherwise.} \end{cases}$$

Let c_i denote the cardinality of the arc set in color i. I_{c_i} is defined as a $c_i \times c_i$ identity matrix with each diagonal entry being set to 1 for $i = 1, 2, \cdots, n$. Let I_i denote an $l \times l$ diagonal matrix for which

$$I_i = \begin{pmatrix} 0 & & \cdots\cdots & & 0 \\ \vdots & \ddots & & & \vdots \\ 0 & \cdots & I_{c_i} & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & & \cdots\cdots & & 0 \end{pmatrix}.$$

For $H \subseteq N$, $H \neq \emptyset$, and $I_H = \bigvee_{i \in H} I_i$, $W_H = W \circ I_H$. (" $\circ$ " denotes the Hadamard product.)

5.2 The Proposed Rule of Priority to Label Colored Arcs

An incidence matrix can represent a multidigraph if all edges are labeled. The proposed algebraic approach for colored multidigraphs starts with a unique edge-labeling rule.

A colored multidigraph may contain several arcs with the same initial and terminal vertices, but each arc in this case must be assigned a different color. To work with the set of all arcs, we must label them carefully. Assuming that all colors and nodes are pre-numbered. Therefore, the vertex set V and the color set N in $G = (V, A, N, \psi, c)$ are numbered as $V = \{1, 2, \cdots, m\}$ and $N = \{1, 2, \cdots, n\}$, respectively. Let c_i denote the cardinality of arc set assigned color i, i.e., $c_i = |A_i|$, where $A_i = \{x \in A : c(x) = i\}$ for each $i \in N$.

To label the arcs in a colored multidigraph $G = (V, A, N, \psi, c)$, set $\varepsilon_0 = 0$ and $\varepsilon_i = \sum_{j=1}^{i} c_j$ for $i \in N$, and note that $l = \varepsilon_n = \sum_{i=1}^{n} c_i$ is the cardinality of A in G. The arcs, $a_1, a_2, \ldots, a_l$, will be labeled according to the color order; within each color, according to the sequence of initial nodes; and within each color and initial node, according to the sequence of terminal nodes. The ordering, referred to as the *Rule of Priority*, has the following properties:

1. If $\varepsilon_{i-1} < k \leq \varepsilon_i$, then $c(a_k) = i$, i.e., a_k has color i;
2. For $k < h$, if a_k and a_h both have color i for some $i \in N$, and if $\psi(a_k) = (v_x, v_y)$ and $\psi(a_h) = (v_z, v_w)$, then $x \leq z$ and, if $x = z$, then $y < w$.

If all arcs in a colored multidigraph have been labeled according to the Rule of Priority, then the index of an arc uniquely determines its color. Therefore, $A_i = \{a_{\varepsilon_{i-1}+1}, \ldots, a_{\varepsilon_i}\}$, where A_i denotes the set of arcs with color i.

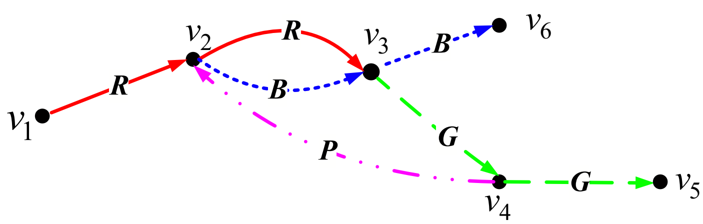


Figure 5.1: **The colored multidigraph G.**

Example 1. *Fig 5.1 shows a colored multidigraph $G = (V, A, N, \psi, c)$. The labels on the arcs of the graph indicate that the corresponding arcs are colored in red (R), blue (B), green (G), and pink (P), respectively. Assume that the vertex set $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. According to the Rule of Priority, label all edges to determine the edge-labeled graph.*

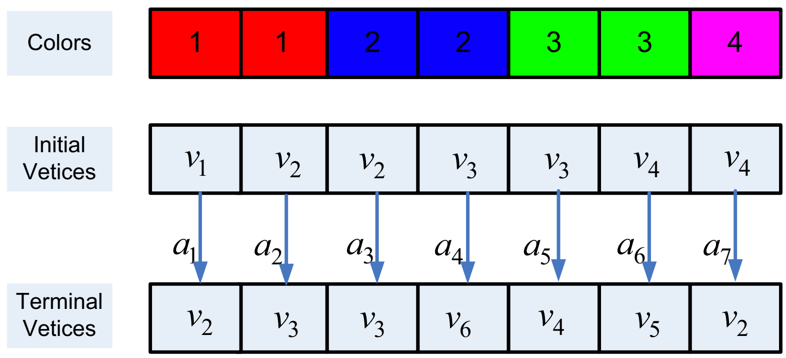


Figure 5.2: **Labeling edges for the graph G.**

First number red 1, blue 2, green 3, and pink 4 so that $N = \{1, 2, 3, 4\}$. The cardinalities of the arc sets A_1, A_2, A_3, and A_4 are 2, 2, 2, and 1, respectively. Then, according to the Rule of Priority, the process to label all colored edges is presented in Fig. 5.2. Recall that $a_k = d_i(u, v)$ for $i \in N$ and $\psi(a_k) = (u, v)$. Obviously, $a_1 = d_1(v_1, v_2); a_2 = d_1(v_2, v_3); a_3 = d_2(v_2, v_3); a_4 = d_2(v_3, v_6); a_5 = d_3(v_3, v_4)$; $a_6 = d_3(v_4, v_5)$; and $a_7 = d_4(v_4, v_2)$. Therefore, the edge labeled graph is expressed as $\langle V, \{A_i, i \in N\}\rangle$, where $A_1 = \{a_1, a_2\}$, $A_2 = \{a_3, a_4\}$, $A_3 = \{a_5, a_6\}$, and $A_4 = \{a_7\}$.

5.3 New Algebraic Approach

5.3.1 A Conversion Function for Finding Colored Paths

Lemma 5.1. *For a weighted colored multidigraph (V, A, N, ψ, c, w), the weighted incidence matrix $B^{(W_H)}$ for H and the incidence matrix B have the following relation*

$$B^{(W_H)} = B \cdot W_H = B \cdot (W \circ I_H).$$

Lemma 5.1 shows a conversion function to transform an original colored multidigraph in the color set N to a reduced weighted colored multidigraph in the color set $H \subseteq N$.

Now let W be a weight matrix and let $L^{(W)}(G)$ denote the weighted line digraph of G. The following theorem is obtained based on Definition 5.10, on the weighted in-incidence and out-incidence matrices $B_{in}^{(W)}$ and $B_{out}^{(W)}$, and Definition 5.12, on the weighted adjacency matrix $LJ^{(W)}$ of the digraph $L^{(W)}(G)$.

Theorem 5.1. *For a weighted colored multidigraph $G = (V, A, N, \psi, c, w)$, W is the weight matrix, $B_{in}^{(W)}$ is the weighted in-incidence matrix, and $B_{out}^{(W)}$ is the weighted out-incidence matrix of the graph G. Then, the weighted edge consecutive matrix $LJ^{(W)}$ satisfies $LJ^{(W)} = (B_{in}^{(W)})^T \cdot (B_{out}^{(W)})$.*

Proof: Let $M = (B_{in}^{(W)})^T \cdot (B_{out}^{(W)})$. Any (k, h) entry of matrix M can be expressed as $M(k,h) = e_k^T \cdot M \cdot e_h = [(B_{in}^{(W)}) \cdot e_k]^T \cdot [(B_{out}^{(W)}) \cdot e_h]$, where e_k^T denotes the transpose of the k^{th} standard basis vector of the l-dimensional Euclidean space.

The q^{th} nonzero element of the row vector $e_k^T \cdot (B_{in}^{(W)})^T$ is equal to the weight w_k of edge $a_k = d_i(s,q)$ for some $s \in V$. Similarly, the q^{th} nonzero element of the column vector $(B_{out}^{(W)}) \cdot e_h$ is equal to the weight w_h of edge $a_h = d_j(q,r)$ for some $r \in V$. Hence, $M(k,h) = w_k \cdot w_h \neq 0$ iff a_k and a_h are consecutive from a_k to a_h (See Fig. 5.3). Then, by Definition 5.12, $B_{in}^{(W)} \cdot B_{out}^{(W)} = LJ^{(W)}$.

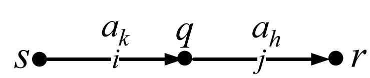


Figure 5.3: a_k **and** a_h **are consecutive in order** $a_k a_h$**.**

□

Obviously, when W is reduced to W_H, $LJ^{(W_H)} = (B_{in}^{(W_H)})^T \cdot (B_{out}^{(W_H)})$.

Let $T_1(B^{(W)}) = (B_{in}^{(W)})^T \cdot (B_{out}^{(W)}) = LJ^{(W)}$ denote a conversion function. The conversion function, $T_1(B^{(W)})$, maps the weighted incidence matrix $B^{(W)}$ to the weighted edge consecutive matrix $LJ^{(W)}$ of the graph G. It shows that this conversion function transforms the original edge-weighted, colored multidigraph G to a simple vertex-weighted-colored line digraph $L(G)$. When $W = I$, $LJ = (B_{in})^T \cdot (B_{out})$. This matrix captures the adjacency relation between pairs of consecutive edges without considering the color(s) of the consecutive edges. Another conversion function is thus presented next to transform the original problem of searching edge-colored paths in a colored multidigraph to the standard problem of finding paths in a simple digraph without color constraints.

Recall that c_i denotes the cardinality of the arc set in color i and let E_{c_i} denote a $c_i \times c_i$ matrix with each entry being set to 1 for $i = 1, 2, \cdots, n$. Then, D is defined as the following block diagonal matrix

$$D = \begin{pmatrix} E_{c_1} & 0 & \cdots & 0 \\ 0 & E_{c_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E_{c_n} \end{pmatrix}. \tag{5.1}$$

It is obvious that this matrix D encodes the color scheme in the graph G, where the dimension of each diagonal block E_{c_i} depends on the number of edges in color i. More specifically, recall that $\varepsilon_i = \sum_{j=1}^{i} c_j$ for $1 \leq i \leq n$. According to the Rule of Priority for labeling edges, for any $a_k \in A$ and $\varepsilon_{i-1} < k \leq \varepsilon_i$, the edge a_k has color i. Hence, for any $a_k, a_h \in A$, if there exists $1 \leq i \leq n$ such that $k, h \in (\varepsilon_{i-1}, \varepsilon_i]$, then edges a_k and a_h have the same color i, and $D(k,h) = 1$. Also, $D(k,h) = 0$ iff edges a_k and a_h have different colors.

The conversion function can now be obtained in matrix form by the following theorem.

Theorem 5.2. *For the weighted colored multidigraph $G = (V, A, N, \psi, c, w)$, let E_l be the $l \times l$ matrix with each entry equal to 1. Then the reduced matrix $LJ_r^{(W)}$ satisfies $LJ_r^{(W)} = LJ^{(W)} \circ (E_l - D)$, where "$\circ$" denotes the Hadamard product.*

Proof: Let $LJ^{(W)}(k,h)$ and $(E_l - D)(k,h)$ denote the (k,h) entries of matrices $LJ^{(W)}$ and $E_l - D$, respectively. Then, $LJ^{(W)}(k,h) \cdot (E_l - D)(k,h) = w_k \cdot w_h \neq 0$ iff $LJ^{(W)}(k,h) = w_k \cdot w_h \neq 0$ and $D(k,h) = 0$. Based on the definitions of matrices $LJ^{(W)}$ and D, $LJ^{(W)}(k,h) \neq 0$ iff edges a_k and a_h are consecutive in order $a_k a_h$. $D(k,h) = 0$ iff edges a_k and a_h have different colors. Obviously, based on the definition of matrix $LJ_r^{(W)}$, $LJ_r^{(W)} = LJ^{(W)} \circ (E_l - D)$. $\square$

Obviously, when W is reduced to W_H, $LJ_r^{(W_H)} = LJ^{(W_H)} \circ (E_l - D)$ satisfies that

$$LJ_r^{(W_H)}(a,b) = \begin{cases} w_a \cdot w_b & \text{if edges } a \text{ and } b \text{ are consecutive in order } ab \text{ and} \\ & c(a) = i \text{ and } c(b) = j \text{ such that } i \neq j \text{ for } i, j \in H, \\ 0 & \textit{otherwise.} \end{cases} \tag{5.2}$$

From Theorem 5.2, $T_2(LJ^{(W)}) = LJ^{(W)} \circ (E_l - D) = LJ_r^{(W)}$. The conversion function, $T_2(LJ^{(W)})$, maps the weighted adjacency matrix $LJ^{(W)}$ of the weighted line digraph $L^{(W)}(G)$ to its reduced matrix $LJ_r^{(W)}$. It reveals that this conversion function T_2 converts the simple vertex-weighted, colored line digraph $L^{(W)}(G)$ to its reduced subgraph $L_r^{(W)}(G)$, called reduced weighted line digraph, which is a simple digraph with no color constraints.

Theorems 5.1 and 5.2 together present a conversion function $F(B^{(W)})$ such that

$$F(B^{(W)}) = [(B_{in}^{(W)})^T \cdot B_{out}^{(W)}] \circ (E_l - D), \tag{5.3}$$

where $B_{in}^{(W)} = (B^{(W)} + abs(B^{(W)}))/2$ and $B_{out}^{(W)} = (abs(B^{(W)}) - B^{(W)})/2$. Therefore, $F(B^{(W)})$ transforms a problem of searching weighted colored paths in an edge-weighted, colored multidigraph to a standard problem of finding paths in a simple digraph with no color constraints. Note that the incident relations between vertices and edges of a graph can uniquely characterize the graph. Therefore, the incidence matrix is treated as the original graph and used for computer implementation.

Example 2. *Fig. 5.1 shows a colored multidigraph $G = (V, A, N, \psi, c)$. If G is associated with a map $w : A \to \mathbb{R}_0^+$, then $G = (V, A, N, \psi, c, w)$ is a weighted colored multidigraph. Construct conversion functions to determine the vertex labeled weighted line digraph $L^{(W)}(G)$ and its reduced line digraph $L_r^{(W)}(G)$.*

By Example 1, the colored multidigraph is labeled using the Rule of Priority. It is easy to obtain incident relations between vertices and edges from the graph. Thus, matrices $B_{in}^{(W)}$ and $B_{out}^{(W)}$ are constructed by Definition 5.10 as follows:

$$B_{in}^{(W)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ w_1 & 0 & 0 & 0 & 0 & 0 & w_7 \\ 0 & w_2 & w_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & w_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & w_6 & 0 \\ 0 & 0 & 0 & w_4 & 0 & 0 & 0 \end{pmatrix},$$

and

$$B_{out}^{(W)} = \begin{pmatrix} w_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & w_2 & w_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & w_4 & w_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & w_6 & w_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

From Theorems 5.1 and 5.2, we obtain that

$$T_1(B^{(W)}) = \begin{pmatrix} 0 & w_1w_2 & w_1w_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & w_2w_4 & w_2w_5 & 0 & 0 \\ 0 & 0 & 0 & w_3w_4 & w_3w_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & w_5w_6 & w_5w_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & w_7w_2 & w_7w_3 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$T_2(LJ^{(W)}) = \begin{pmatrix} 0 & 0 & w_1w_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & w_2w_4 & w_2w_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & w_3w_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & w_5w_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & w_7w_2 & w_7w_3 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The weight matrix designed here is convenient, since edge-weighted (0 or 1) can be used to flexibly control any move between any two vertices in G. For instance, if $w_4 = 0$, then the original graph will be reduced to a new graph with no edge a_4. If $W = I$, then the conversion function T_1 transforms the edge-labeled multidigraph G portrayed in Fig. 5.4 (1) to the vertex-labeled line digraph $L(G)$ shown in Fig. 5.4 (2). Then, the reduced line digraph $L_r(G)$ presented in Fig. 5.4 (3) for finding colored paths is obtained by using the conversion function T_2. The conversion process is illustrated in Fig. 5.4.

5.3.2 Computer Implementation

Many well-known algorithms have been developed to solve the shortest path problems in digraphs, such as Dijkstra's algorithm [14] and Johnson's algorithm [38]. Some other algorithms are available for searching for all paths in undirected graphs, such as the algorithm presented by Migliore et al [50]. Although finding path problems in general graph classes has been extensively investigated, searching colored paths in weighted colored multidigraphs is still a novel topic.

Let $A_S = \{a \in A : B^{(W)}_{out}(s, a) \neq 0\}$ and $A_E = \{b \in A : B^{(W)}_{in}(q, b) \neq 0\}$ for $s, q \in V$. Here, matrices W, $B^{(W)}_{out}$, and $B^{(W)}_{in}$ have been introduced by Definitions 5.7 and 5.10. A_S is the set of arcs starting from vertex s and A_E is the arc set ending at vertex q. The matrix $LJ^{(W)}_r$ provided by Theorem 5.2 is used to search the edge-weighted, colored paths between any two arcs in a weighted colored multidigraph. Let $PA^{(W)}(a, b)$ for $a, b \in A$ denote the weighted colored paths between two edges

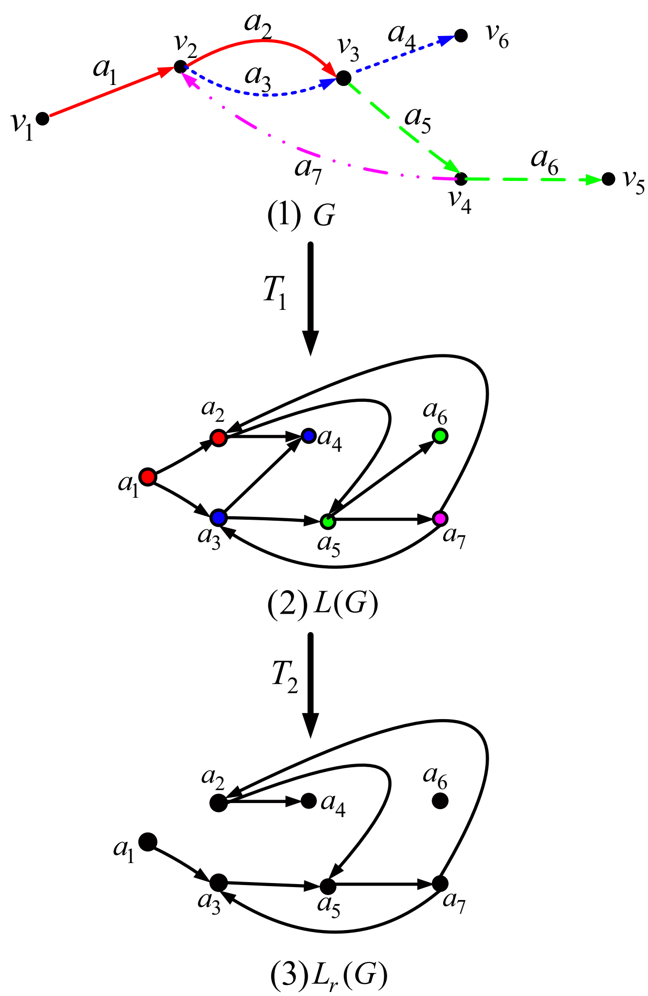


Figure 5.4: **Transformed graphs of G.**

a and b. The weighted colored paths between two vertices s and q for $s, q \in V$ are expressed as $PA^{(W)}(s, q)$. A vertex-by-vertex path between any two vertices in the graph G can be obtained by tracing arc-by-arc paths between two appropriate arcs in the line graph $L(G)$. Specifically, the paths between s and q can be expressed as $PA^{(W)}(s, q) = \{PA^{(W)}(a, b) : a \in A_S,\ b \in A_E\}$.

The proposed algebraic method is convenient for computer implementation. A pseudo code for the proposed algorithm is presented in Table 5.1.

Table 5.1: **Pseudo code of the proposed algorithm for finding colored paths**

Step 0: Input the starting arc set A_S, the ending arc set A_E, and the reduced weighted edge consecutive matrix $LJ_r^{(W)}$.

Step 1: For each arc $a_s \in A_S$ and each arc $a_e \in A_E$, set a_s as the starting arc and a_e as the ending arc. For each pair of a_s and a_e, repeat the steps from Step 2 to Step 5.

Step 2: Put a_s into Path-Recorder as the last arc $a_l(1)$ of the first path.

Step 3: In Path-Recorder, for each path i, e.g., $PA^{(W)}(i)$, check its last arc $a_l(i)$. Obtain all the new arcs starting from $a_l(i)$ based on matrix $LJ_r^{(W)}$.

Case 1: If there is no arc starting from $a_l(i)$, path $PA^{(W)}(i)$ ends.
Eliminate $PA^{(W)}(i)$ from Path-Recorder;

Case 2: If a new arc has appeared in the path, which means that the path forms a cycle, do not record the new path. If all the new arcs have appeared, eliminate $PA^{(W)}(i)$ from Path-Recorder;

Case 3: If the new arc is the end arc a_e, add a_e to the path $PA^{(W)}(i)$ to form a new path. Reserve the path into Path-Recorder and set an *end-mark* at the end of the path;
Otherwise: Add each new arc to path $PA^{(W)}(i)$, respectively, to form several new paths.
Reserve these paths into Path-Recorder, and eliminate the original path $PA^{(W)}(i)$ from Path-Recorder.

Step 4: Repeat Step 3 until all the paths in Path-Recorder have the *end-mark* at the end.

Step 5: Output Path-Recorder, which records all paths starting from a_s and ending at a_e.

Because the algebraic expressions are explicitly given, the proposed method facilitates the development of improved algorithms to search colored paths and is easy to adapt to new path searching problems. For instance, a transportation network problem of finding the shortest path with specific constraints can be solved by using the conversion function $F(B^{(W)}) = [(B_{in}^{(W)})^T \cdot B_{out}^{(W)}] \circ M$, where $B^{(W)}$ denotes the original network and matrix M is designed to capture constraint requirements, to transform the original problem to a general shortest path searching problem without the constraints.

Note that in this research all arcs are distinct on a path but the restriction that all nodes be distinct on a path is relaxed. The process that converts an edge-colored multidigraph to a simple digraph with no color constraints is presented in Fig. 5.5.

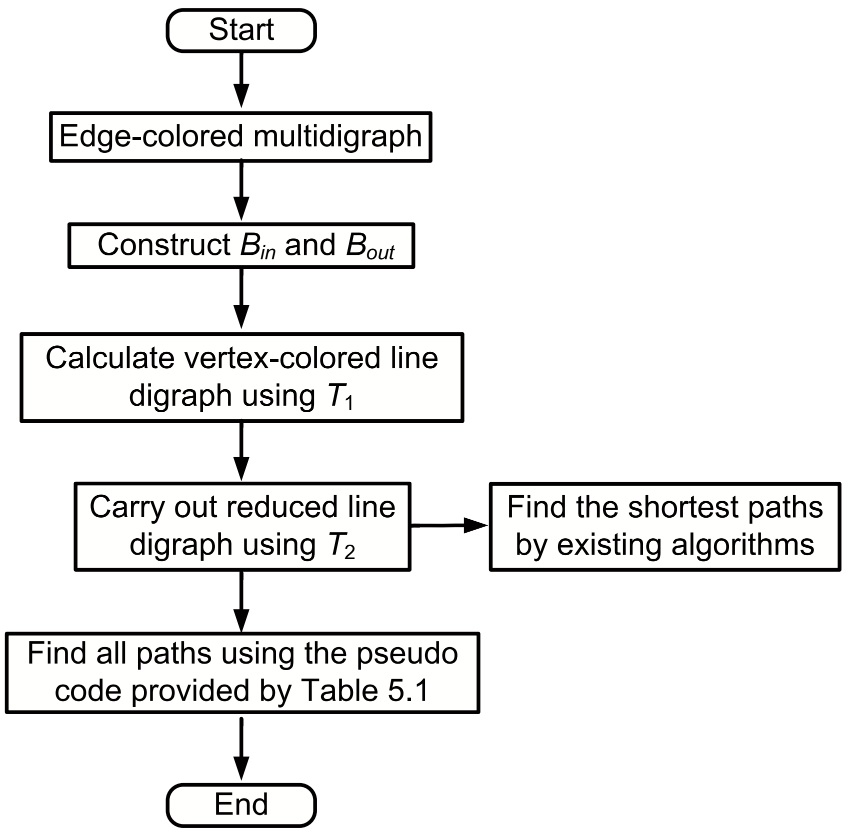


Figure 5.5: **The process of finding all colored paths or the shortest colored path**

5.3.3 Constructing Weighted Reachability Matrix using Weighted Colored Paths

Theorem 5.3. *For a weighted colored multidigraph (V, A, N, ψ, c, w), $B_{in}^{(W_H)}$ and $B_{out}^{(W_H)}$ denote the weighted in-incidence and out-incidence matrices for H. The* *weighted adjacency matrix by H is expressed as*

$$J_H^{(W)} = sign[(B_{out}^{(W_H)}) \cdot (B_{in}^{(W_H)})^T]. \tag{5.4}$$

From algebraic graph theory [24], Theorem 5.3 can easily follow.

Lemma 5.2. *For a weighted colored multidigraph $G = (V, A, N, \psi, c, w)$, let t be an integer, $H \subseteq N$, and $(LJ_r^{(W_H)})^t(a, b)$ be the (a, b) entry of matrix $(LJ_r^{(W_H)})^t$. Then, $(LJ_r^{(W_H)})^t(a, b)$ denotes the number of weighted colored paths by H in the G from edge a to edge b with length t for $a, b \in A$. Moreover, if $\psi(a) = (u, s)$ and $\psi(b) = (q, v)$ for $u, s, q, v \in V$, then, the number of the weighted colored paths by H from vertex u to vertex v with length $t + 1$ is at least $(LJ_r^{(W_H)})^t(a, b)$.*

Proof: This Lemma is proved using induction on t.

When $t = 1$, the result is obvious.

Assume that when $t = k$, the result holds. Then, when $t = k + 1$, $(LJ_r^{(W_H)})^{k+1}(a, b) = \sum_{h=1}^{l}[(LJ_r^{(W_H)})^k(a, h) \cdot LJ_r^{(W_H)}(h, b)]$.

By the induction hypothesis, $(LJ_r^{(W_H)})^k(a, h)$ denotes the number of the weighted colored paths by H from a to h with length k, and $LJ_r^{(W_H)}(h, b)$ indicates the number of weighted colored paths by H from h to b with length 1. Thus, $(LJ_r^{(W_H)})^k(a, h) \cdot LJ_r^{(W_H)}(h, b)$ denotes the number of weighted colored paths by H from a to b through h with length $k + 1$. Therefore, $\sum_{h=1}^{l}[(LJ_r^{(W_H)})^k(a, h) \cdot LJ_r^{(W_H)}(h, b)]$ is the total number of weighted colored paths by H from a to b with length $k + 1$. Thus, $(LJ_r^{(W_H)})^t(a, b)$ denotes the number of weighted colored paths by H in the G from edge a to edge b with length t for $a, b \in A$.

Obviously, if $\psi(a) = (u, s)$ and $\psi(b) = (q, v)$ for $u, s, q, v \in V$, then, the number of the weighted colored paths by H from vertex u to vertex v with length $t + 1$ is at least $(LJ_r^{(W_H)})^t(a, b)$. □

Note that, in Lemma 5.2, when calculating the length of an edge-by-edge path, the edges in the path should be treated as vertices. i.e., edge-by-edge paths are treated as state-by-state paths in the line graph $L(G)$.

Theorem 5.4. *Let $l_H^{(W)}$ denote the number of arcs in $A_H^{(W)}$. For a weighted colored multidigraph (V, A, N, ψ, c, w), the weighted reachability matrix M_H by H can be obtained by*

$$M_H^{(W)} = sign[(B_{out}^{(W_H)}) \cdot (LJ_r^{(W_H)} + I)^{l_H^{(W)}-1} \cdot (B_{in}^{(W_H)})^T], \tag{5.5}$$

where I is the identity matrix.

Proof: Let $L = l_H^{(W)}$ and $C_{L-1}^t = \left(\begin{array}{c} L-1 \\ t \end{array} \right) = \frac{(L-1)\cdot(L-2)\cdots(L-t)}{t!}$ and $(LJ_r^{(W_H)})^0 = I$. Using matrix theory, $(LJ_r^{(W_H)} + I)^{L-1} = \sum_{t=0}^{L-1} C_{L-1}^t \cdot (LJ_r^{(W_H)})^t$.

Let $Q = sign[B_{out}^{(W_H)} \cdot (LJ_r^{(W_H)} + I)^{L-1} \cdot (B_{in}^{(W_H)})^T]$. Since $C_{L-1}^t > 0$, then

$$Q = sign[\sum_{t=0}^{L-1} C_{L-1}^t \cdot B_{out}^{(W_H)} \cdot (LJ_r^{(W_H)})^t \cdot (B_{in}^{(W_H)})^T]$$

$$= (B_{out}^{(W_H)} \cdot I \cdot (B_{in}^{(W_H)})^T) \vee [\bigvee_{t=1}^{L-1} (B_{out}^{(W_H)} \cdot (LJ_r^{(W_H)})^t \cdot (B_{in}^{(W_H)})^T)].$$

Based on Theorem 5.3, $Q = J_H^{(W)} \vee [\bigvee_{t=1}^{L-1} (B_{out}^{(W_H)} \cdot (LJ_r^{(W_H)})^t \cdot (B_{in}^{(W_H)})^T)]$.

Then, $Q(s,q) \neq 0$ iff $J_H^{(W)}(s,q) \neq 0$ or for $1 \leq t \leq L-1$, there exist $(LJ_r^{(W_H)})^t(a,b) \neq 0$ such that $a, b \in A_H^{(W)}$, $\psi(a) = (s,u)$, and $\psi(b) = (v,q)$ for $s,q,u,v \in V$. $J_H^{(W)}(s,q) \neq 0$ implies that vertex q is reachable from vertex s by paths $PA_H^{(W)}(s,q)$ with length 1. By Lemma 5.2, $(LJ_r^{(W_H)})^t(a,b) \neq 0$ iff vertex q is reachable from vertex s by the weighted colored paths $PA_H^{(W)}(s,q)$ with length $t+1$. Therefore, $Q(s,q) \neq 0$ iff vertex q is reachable from vertex s by the weighted colored paths $PA_H^{(W)}(s,q)$ with length 1 or $t+1$ for $0 \leq t \leq L-1$.

By Definition 5.11, $M_H^{(W)}(s,q) \neq 0$ iff vertex q is reachable from vertex s by the weighted colored paths $PA_H^{(W)}(s,q)$ with length $k \leq L$. Then $M_H^{(W)}(s,q) \neq 0$ implies that $Q(s,q) \neq 0$. $Q(s,q) \neq 0$ implies that there exists an edge weighted colored path $PA_H^{(W)}(s,q)$ with length $1 \leq t \leq L$, then $M_H^{(W)}(s,q) \neq 0$. Since $M_H^{(W)}$ and Q are 0-1 matrices, $M_H^{(W)} = sign[(B_{out}^{(W_H)}) \cdot (LJ_r^{(W_H)} + I)^{L-1} \cdot (B_{in}^{(W_H)})^T]$. $\square$

The algebraic method to search edge-weighted, colored paths in a colored multidigraph can have many benefits presented as follows.

5.4 Applications

5.4.1 Application 1: Transportation Network

Because of the accelerating globalization trend, a major logistic challenge is to design a reliable, efficient, and economical system for moving merchandise within a multi-modal transportation network. Due to diverse geography and weather conditions, cost and time constraints, as well as other factors, chartered companies may have to switch their transport mode when passing through a transfer station. In order to design a competitive transportation system, one must analyze all possible paths from any initial station to a destination to make the best choice. This transportation problem can be conveniently modeled as a problem of finding colored paths and the shortest colored path in a weighted colored multidigraph.

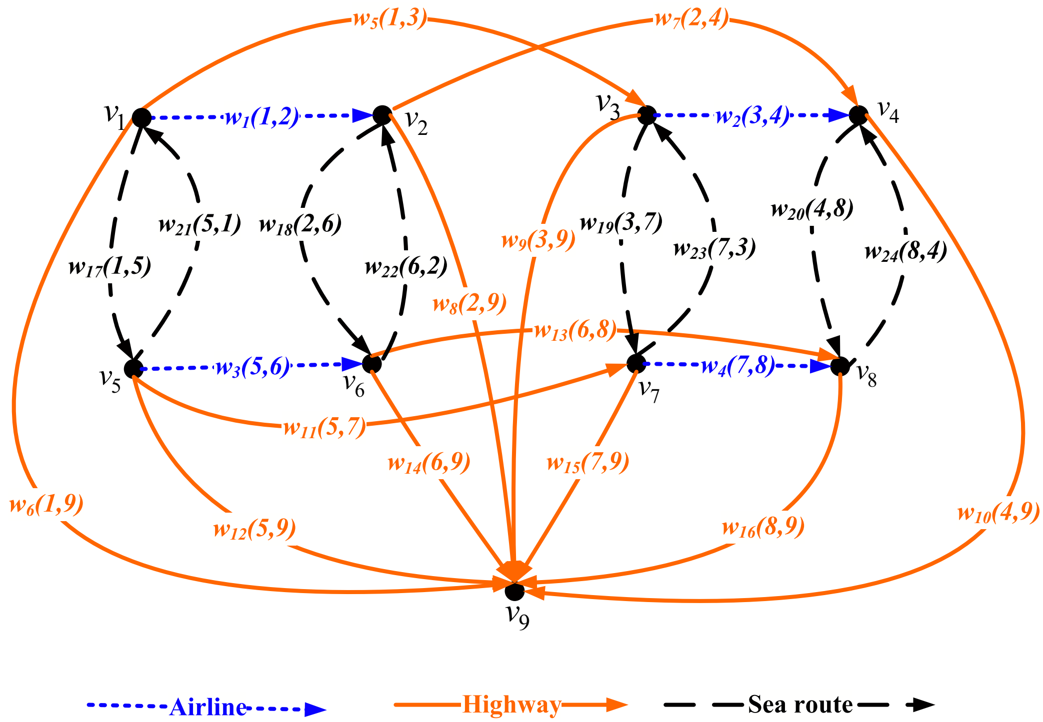


Figure 5.6: **A transportation network.**

A hypothetical transportation network is shown in Fig. 5.6. The label on each arc indicates its weight. Three different line styles, encoded in three colors, denote three transportation modes: Color 1, **Airline**; Color 2, **Highway**; and Color 3, **Sea route**, respectively. The numbers of airlines, highways, and sea routes are $c_1 = 4$, $c_2 = 12$, and $c_3 = 8$, respectively. Nine transfer stations are expressed using vertices v_1 to v_9 as shown in the graph. According to the Rule of Priority, each edge is labeled as shown in Fig. 5.6. Charter companies will move merchandise from a starting station to some destinations. Assume also that this network is consolidated in such a way that merchandise will have to be switched from one transportation mode to another at any transfer station. In order to design a competitive transportation system, one needs to search all possible colored paths between any two vertices in the transportation network. Using Theorem 5.2, the reduced weighted edge consecutive matrix $LJ_r^{(W)}$ is calculated and its nonzero entries are listed in Table 5.2. Using the algorithm presented in Table 5.1, all colored paths in the network can be found based on the information in Table 5.2.

Table 5.2: **The nonzero entries of matrix LJ_r for the transportation network**

Status quo	Nonzero entries of the reduced edge consecutive matrix LJ_r
v_1	(a_1, a_7), (a_1, a_8), (a_1, a_{18}),(a_5, a_2) , (a_5, a_{19}), (a_{17}, a_3), (a_{17}, a_{11}), (a_{17}, a_{12})
v_2	(a_7, a_{20}), (a_{18}, a_{13}), (a_{18}, a_{14})
v_3	(a_2, a_{10}), (a_2, a_{20}), (a_{19}, a_4), (a_{19}, a_{15})
v_4	(a_{20}, a_{16})
v_5	(a_3, a_{13}), (a_3, a_{14}), (a_3, a_{22}), (a_{11}, a_4), (a_{11}, a_{23}), (a_{21}, a_1), (a_{21}, a_5), (a_{21}, a_6)
v_6	(a_{13}, a_{24}), (a_{22}, a_7), (a_{22}, a_8)
v_7	(a_4, a_{16}), (a_4, a_{24}), (a_{23}, a_2), (a_{23}, a_9)
v_8	(a_{24}, a_{10})

Fig. 5.7 shows that the colored multidigraph is mapped by the conversion function $F(\cdot)$ designed by equation (5.3) to a simple digraph with no color constraints. Note that the numbers labeled in circles shown in Fig. 5.7 denote edge numbers. For the standard digraph, several well-known algorithms, such as depth-first search algorithm [25] and Dijkstra algorithm [14], are available for searching the shortest path on the reduced digraph.

For instance, if a firm wants to find the shortest path to move merchandise from station v_1 to station v_8. Fig. 5.7 shows that there exist six colored paths between vertexes v_1 and v_8 in terms of arcs:

$$a_{17} \longrightarrow a_3 \longrightarrow a_{13}$$

$$a_{17} \longrightarrow a_{11} \longrightarrow a_4$$

$$a_5 \longrightarrow a_2 \longrightarrow a_{20}$$

$$a_1 \longrightarrow a_7 \longrightarrow a_{20}$$

$$a_5 \longrightarrow a_{19} \longrightarrow a_4$$

$$a_1 \longrightarrow a_{18} \longrightarrow a_{13}$$

Based on the Rule of Priority and the relation between state-by-state paths and arc-by-arc paths, $PA^{(W)}(s, q) = \{PA^{(W)}(a, b) : a \in A_S,\ b \in A_E\}$, the above arc-by-arc paths can be easily expressed in terms of nodes as follows:

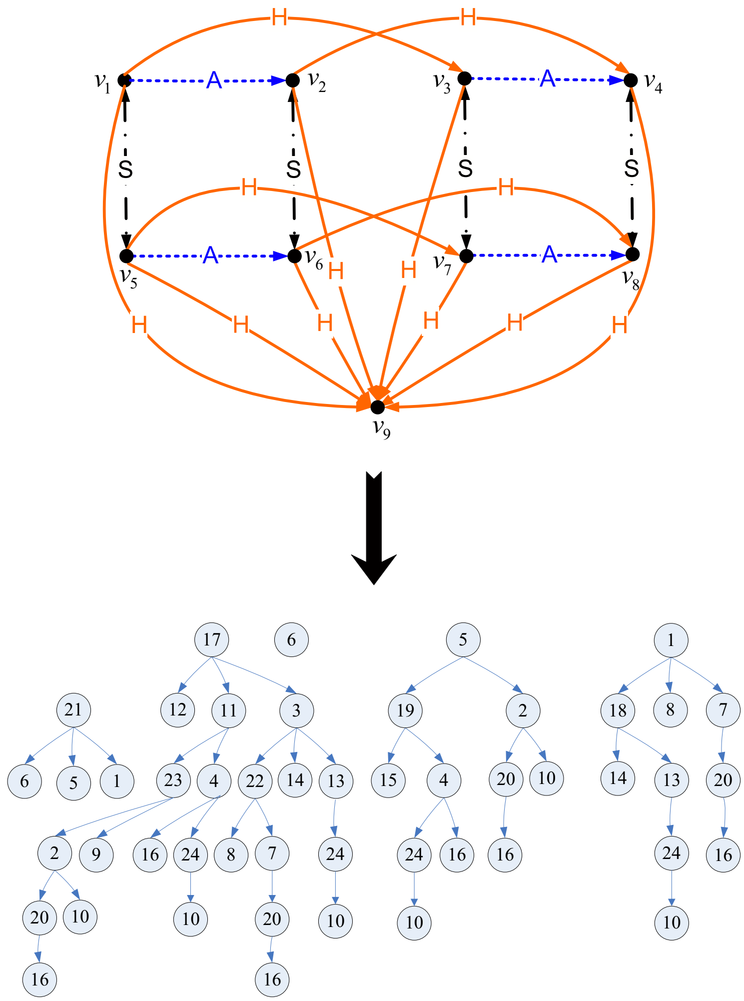


Figure 5.7: **Graph transformation.**

$$v_1 \longrightarrow v_5 \longrightarrow v_6 \longrightarrow v_8,$$

$$v_1 \longrightarrow v_5 \longrightarrow v_7 \longrightarrow v_8,$$

$$v_1 \longrightarrow v_3 \longrightarrow v_4 \longrightarrow v_8,$$

$$v_1 \longrightarrow v_2 \longrightarrow v_4 \longrightarrow v_8,$$

$$v_1 \longrightarrow v_3 \longrightarrow v_7 \longrightarrow v_8,$$

$$v_1 \longrightarrow v_2 \longrightarrow v_6 \longrightarrow v_8.$$

If the following weights are assigned, $w_1 = 13, w_2 = 24, w_3 = 10, w_4 = 17, w_5 = 14, w_7 = 26, w_{11} = 15, w_{13} = 19, w_{17} = 20, w_{18} = 19, w_{19} = 18$, and $w_{20} = 17$, then the shortest colored path between vertices v_1 and v_8 is the path consisting of edges a_{17}, a_3, and a_{13}, or equivalently in terms of nodes, $v_1 \longrightarrow v_5 \longrightarrow v_6 \longrightarrow v_8$.

5.4.2 Application 2: Graph Model for Conflict Resolution

This proposed algebraic approach can also be conveniently applied to solve problems of stability and status quo analyses in the graph model for conflict resolution. If the state set S is treated as a vertex set and DM i's oriented arcs are coded in color i, then a graph model of a conflict is equivalent to a colored multidigraph with appropriate preference relations. Hence, a graph model can be conveniently treated as an edge-weighted, colored multidigraph in which each arc represents a legal unilateral move, distinct colors refer to different DMs, and the weight along the arc identifies some preference attribute.

As a post-stability analysis in the graph model, status quo analysis examines whether predicted equilibria (or potential resolutions) are reachable from a status quo or an initial state by tracing the moves and countermoves among DMs. An important restriction of a graph model is that no DM can move twice in succession along any path [16]. Thus, tracing the evolution of a conflict in status quo analysis is converted to searching all colored paths with some preference structure such as simple preference [16], uncertain preference [46], or strength of preference [28]. The proposed algebraic approach also highlights a link between status quo analysis and traditional stability analysis, thereby suggesting the possibility of an integrated approach to stability and status quo analyses.

5.4.2.1 Weight Matrix for GMCR under Simple Preference

In the original information, the preference of DM i is coded by a pair of relations $\{\succ_i, \sim_i\}$ on S. This preference structure is called simple preference.

Definition 5.7 presents a weight matrix W_H for a weighted colored multidigraph $G = (V, A, N, \psi, c, w)$. In a graph model $G = (S, A)$, let $H \subseteq N$. By the proposed Rule of Priority, the oriented arcs in the graph model are labeled according to the DM order; within each DM, according to the sequence of initial states; and within each DM and initial state, according to the sequence of terminal states. When an edge $a_k = d_i(u, v)$ for $u, v \in S$ and $i \in H \subseteq N$, then its weight w_k can be defined by

$$w_k = \begin{cases} P_w & \text{if } v \succ_i u \text{ and } i \in H, \\ E_w & \text{if } u \sim_i v \text{ and } i \in H, \\ N_w & \text{if } u \succ_i v \text{ and } i \in H, \\ 0 & \text{otherwise.} \end{cases} \tag{5.6}$$

The weight matrix W_H represents preference information of each edge in the graph model for simple preference. Recall that notation UMs and UIs denote unilateral movers and unilateral improvements, respectively. Based on the statement (5.6), the UM weight matrix and the UI weight matrix for H are defined as follows.

Definition 5.14. *For the graph model $G = (S, A)$, let $H \subseteq N$.*

- *when $P_w = E_w = N_w = 1$, the weight matrix W_H is called the UM weight matrix by H, denoted by $W_H^{(UM)}$;*

- *when $P_w = 1$ and $E_w = N_w = 0$, the weight matrix W_H is called the UI weight matrix by H, denoted by $W_H^{(UI)}$ or W_H^+.*

Recall that each arc of A_i and A_i^+ denotes that DM i can make a UM and a UI (in one step) from the initial state to the terminal state of the arc, respectively. Therefore, $A_H = \bigcup\limits_{i \in H} A_i$ and $A_H^+ = \bigcup\limits_{i \in H} A_i^+$ denote the UM and the UI arcs associated with any DM in H. Based on Definition 5.6, on the weighted arc set for H, the following result relative to the UM arc set and the UI arc set is obvious for the graph model with simple preference.

Corollary 5.1. *For the graph model $G = (S, A)$, let $H \subseteq N$.*

- *If $W_H = W_H^{(UM)}$, then the arc set $A_H^{(W)} = A_H$;*

- *If $W_H = W_H^+$, the arc set $A_H^{(W)} = A_H^+$.*

Note that when $H = N$, A_H and A_H^+ are denoted by A and A^+, respectively.

In a weighted colored multidigraph, the edge-weighted, colored paths by H between two vertices u and v are described in Definition 5.4 which can represent conflict evolution by the legal UMs and the legal UIs in a graph model for simple preference.

Corollary 5.2. *For the graph model $G = (S, A)$, let $u, v \in S$ and $H \subseteq N$.*

- *If $W_H = W_H^{(UM)}$, the weighted colored paths between states u and v, $PA_H^{(W)}(u, v)$, give all paths from u to v where all legal UMs are allowed. Then $PA_H^{(W)}(u, v)$ are called legal UM paths from u to v by coalition H, denoted by $PA_H(u, v)$;*

- *If $W_H = W_H^+$, the weighted colored paths between states u and v, $PA_H^{(W)}(u, v)$, give all paths from u to v where only legal UIs are allowed. Then $PA_H^{(W)}(u, v)$ are called legal UI paths from u to v by coalition H, denoted by $PA_H^+(u, v)$.*

The weighted colored paths $PA_H^{(W)}$ can be used to trace conflict evolution of status quo analysis for simple preference. When u is selected as a status quo and v is an equilibrium for some stability in a graph model, $PA_H(u, v)$ and $PA_H^+(u, v)$ trace conflict evolution to confirm that the equilibrium is in fact reachable from the status quo and reveal how to reach it.

Definition 5.15. *In the graph model $G = (S, A)$, the legal UM and the legal UI edge consecutive matrices are two $l \times l$ matrices $LJ_r^{(UM)}$ and LJ_r^{+} with (a, b) entries*

$$LJ_r^{(UM)}(a,b) = \begin{cases} 1 & \text{if edges } a \text{ and } b \text{ are consecutive in order } ab \text{ and} \\ & \text{are controlled by difference DMs} \ \text{ for } a,b \in A, \\ 0 & \text{otherwise,} \end{cases}$$

$$LJ_r^{+}(a,b) = \begin{cases} 1 & \text{if edges } a \text{ and } b \text{ are consecutive in order } ab \text{ and} \\ & \text{are controlled by difference DMs} \ \text{ for } a,b \in A^{+}, \\ 0 & \text{otherwise.} \end{cases}$$

Let LJ_{H_r} and $LJ_{H_r}^{+}$ denote the legal UM and the legal UI edge consecutive matrices in the graph model (S, A_H). Based on Definition 5.13, on the reduced weighted edge consecutive matrix by H, and Definition 5.15, the following result is obvious.

Corollary 5.3. *For the graph model $G = (S, A)$, let $W^{(UM)}$ and W^{+} denote the UM and the UI weight matrices, and $W_H^{(UM)}$ and W_H^{+} be the UM and the UI weight matrices for H. Then*

$$LJ_r^{(W^{(UM)})} = LJ_r^{(UM)} = LJ_r, \ LJ_r^{(W^{+})} = LJ_r^{+},$$

and

$$LJ_r^{(W_H^{(UM)})} = LJ_{H_r}, \ LJ_r^{(W_H^{+})} = LJ_{H_r}^{+}.$$

As the proposed algorithm presented in Table 5.1 for searching weighted colored paths in a weighted colored multidigraph, the legal UM and UI edge consecutive matrices LJ_{H_r} and $LJ_{H_r}^{+}$ are applied to find paths PA_H and PA_H^{+} between any two states for status quo analysis in a graph model. Specific applications for status quo analysis using the algebraic approach are presented in Chapter 7.

For simple preference, the key inputs of stability analysis, $R_H(s)$ and $R_H^{+}(s)$, are the reachable lists by coalition H from state $s \in S$ by the legal UMs and the legal UIs. Algorithms are complicated to implement the key inputs of stability analysis [16]. This research provides an algebraic approach to construct $R_H(s)$ and $R_H^{+}(s)$ using the weighted reachability matrix $M_H^{(W)}$ shown by Definition 5.11. The details are discussed in Chapter 6.

5.4.2.2 Weight Matrix for GMCR under Preference with Uncertainty

Preference information plays an important role in the decision analysis. To incorporate preference uncertainty into the graph model methodology, Li et al. [46] proposed a new preference structure in which DM i's preferences are expressed by a triple of relations $\{\succ_i, \sim_i, U_i\}$ on S, where $s \succ_i q$ indicates strict preference, $s \sim_i q$ indicates indifference, and sU_iq means DM i may prefer state s to state q, may prefer q to s, or may be indifferent between s and q.

The weight matrix W_H can be employed to represent preference with uncertainty. When an edge $a_k = d_i(u,v)$ for $u, v \in S$ and $i \in H \subseteq N$, then its weight w_k can be defined by

$$w_k = \begin{cases} P_w & \text{if } v \succ_i u \text{ and } i \in H, \\ N_w & \text{if } u \succ_i v \text{ and } i \in H, \\ E_w & \text{if } u \sim_i v \text{ and } i \in H, \\ U_w & \text{if } uU_iv \text{ and } i \in H, \\ 0 & \text{otherwise.} \end{cases} \tag{5.7}$$

Recall that notation UIUUMs denotes unilateral improvements or unilateral uncertain moves. Based on the statement (5.7), the UIUUM weight matrix for H is defined as follows.

Definition 5.16. *For the graph model $G = (S, A)$, let $H \subseteq N$. When $P_w = U_w = 1$ and $E_w = N_w = 0$, the weight matrix W_H is called the UIUUM weight matrix for H, denoted by $W_H^{(UIUUM)}$ or $W_H^{+,U}$.*

Each arc of arc set $A_i^{+,U}$ denotes that DM i can make a UIUUM from the initial state to the terminal state of the arc. Therefore, $A_H^{+,U} = \bigcup_{i \in H} A_i^{+,U}$ indicates the UIUUM arcs associated with any DM in H. By Definition 5.6 for the weighted arc set $A_H^{(W)}$, the UIUUM arc set is obtained for a graph model with preference uncertainty by the following Corollary.

Corollary 5.4. *For the graph model $G = (S, A)$, let $H \subseteq N$. If $W_H = W_H^{+,U}$, then the arc set $A_H^{(W)} = A_H^{+,U}$.*

Note that when $H = N$, $A_H^{+,U}$ is expressed by $A^{+,U}$.

The weighted colored paths $PA_H^{(W)}$ can be applied to trace conflict evolution by the legal UIUUMs for the graph model with preference uncertainty.

Corollary 5.5. *For the graph model $G = (S, A)$, let $u, v \in S$ and $H \subseteq N$. If $W_H = W_H^{+,U}$, the weighted colored paths between states u and v, $PA_H^{(W)}(u,v)$, give all paths from u to v where only the legal UIUUMs are allowed. Then $PA_H^{(W)}(u,v)$ are called the legal UIUUM paths from u to v by coalition H, denoted by $PA_H^{+,U}(u,v)$.*

The conflict evolution by the legal UIUUMs can be tracked using the reduced weighted edge consecutive matrix. The legal UIUUM edge consecutive matrix is defined first.

Definition 5.17. *In the graph model $G = (S, A)$, the legal UIUUM edge consecutive matrix is an $l \times l$ matrix $LJ_r^{+,U}$ with (a, b) entry*

$$LJ_r^{+,U}(a,b) = \begin{cases} 1 & \textit{if edges } a \textit{ and } b \textit{ are consecutive in order } ab \textit{ and} \\ & \textit{are controlled by difference DMs} \ \ \textit{for } a, b \in A^{+,U}, \\ 0 & \textit{otherwise.} \end{cases}$$

Let $LJ_{H_r}^{+,U}$ denote the legal UIUUM edge consecutive matrix for the graph model (V, A_H). Based on Definitions 5.13 and 5.17, the following result is obtained.

Corollary 5.6. *For the graph model $G = (S, A)$, let $W^{+,U}$ denote the UIUUM weight matrix and $W_H^{+,U}$ be the UIUUM weight matrix for H. Then*

$$LJ_r^{(W^{+,U})} = LJ_r^{+,U},$$

and

$$LJ_r^{(W_H^{+,U})} = LJ_{H_r}^{+,U}.$$

The key input of stability analysis for the graph model with preference uncertainty is the reachable list $R_H^{+,U}(s)$ of coalition $H \subseteq N$ from state $s \in S$ by the legal UIUUMs. The algebraic approach to searching weighted colored paths can also be used to construct $R_H^{+,U}(s)$. The details are discussed in Chapter 6.

5.4.2.3 Weight Matrix for GMCR under Strength of Preference

Another triplet relation $\{\gg_i, >_i, \sim_i\}$ on S that expresses strength of preference (strong or mild preference) was developed by Hamouda et al. [27,28]. For $s, q \in S$, $s \gg_i q$ denotes DM i strongly prefers s to q, $s >_i q$ means DM i mildly prefers s to q, and $s \sim_i q$ indicates that DM i is indifferent between states s and q. The weight matrix W_H can represent strength of preference. When an edge $a_k = d_i(u, v)$ for $u, v \in S$ and $i \in H \subseteq N$, then its weight w_k is defined by

$$w_k = \begin{cases} P_s & \text{if } v \gg_i u \text{ and } i \in H, \\ P_m & \text{if } v >_i u \text{ and } i \in H, \\ E_w & \text{if } u \sim_i v \text{ and } i \in H, \\ N_w & \text{if } u \gg_i v \text{ or } u >_i v \text{ and } i \in H, \\ 0 & \text{otherwise.} \end{cases} \tag{5.8}$$

Recall that notation WIs denotes strong unilateral improvements or mild unilateral improvements called weak improvements. Based on the statement (5.8), the WI weight matrix for H is defined as follows.

Definition 5.18. *For the graph model $G = (S, A)$, let $H \subseteq N$. When $P_s = P_m = 1$ and $E_w = N_w = 0$, the weight matrix W_H is called the WI weight matrix for H, denoted for $W_H^{(WI)}$ or $W_H^{+,++}$.*

Each arc of the arc set $A_i^{+,++}$ denotes that DM i can make a WI from the initial state to the terminal state of the arc. Therefore, $A_H^{+,++} = \bigcup_{i \in H} A_i^{+,++}$ denotes the WI arcs associated with any DM in H. By Definition 5.6 for the weighted arc set $A_H^{(W)}$, the WI arc set is obtained for a graph model with strength of preference by the following Corollary.

Corollary 5.7. *For the graph model $G = (S, A)$, let $H \subseteq N$. If $W_H = W_H^{+,++}$, then the arc set $A_H^{(W)} = A_H^{+,++}$.*

Note that when $H = N$, $A_H^{+,++}$ is expressed by $A^{+,++}$.

The weighted colored paths $PA_H^{(W)}$ can be applied to trace conflict evolution by the legal WIs for the graph model with strength of preference.

Corollary 5.8. *For the graph model $G = (S, A)$, let $u, v \in S$ and $H \subseteq N$. If $W_H = W_H^{+,++}$, the weighted colored paths between states u and v, $PA_H^{(W)}(u, v)$, give all paths from u to v where only the legal WIs are allowed. Then $PA_H^{(W)}(u, v)$ are called the legal WI paths from u to v by coalition H, denoted by $PA_H^{+,++}(u, v)$.*

Definition 5.19. *In the graph model $G = (S, A)$, the legal WI edge consecutive matrix is an $l \times l$ matrix $LJ_r^{+,++}$ with (a, b) entry*

$$LJ_r^{+,++}(a,b) = \begin{cases} 1 & \text{if edges } a \text{ and } b \text{ are consecutive in order } ab \text{ and} \\ & \text{are controlled by difference DMs for } a, b \in A^{+,++}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $LJ_{H_r}^{+,++}$ denote the legal WI edge consecutive matrix for the graph model (V, A_H). Based on Definition 5.13, on the reduced weighted edge consecutive matrix by H, and Definition 5.19, the following result can be easily obtained.

Corollary 5.9. *For the graph model $G = (S, A)$, let $W^{+,++}$ denote the WI weight matrix and $W_H^{+,++}$ be the WI weight matrix for H. Then*

$$LJ_r^{(W^{+,++})} = LJ_r^{+,++},$$

and

$$LJ_r^{(W_H^{+,++})} = LJ_{H_r}^{+,++}.$$

The key input of stability analysis in the graph model with strength of preference is state set $R_H^{+,++}(s)$, the reachable list of coalition $H \subseteq N$ from state $s \in S$ by the legal WIs. The algebraic approach provides a new method to construct $R_H^{+,++}(s)$. The details are discussed in Chapter 6.

5.4.2.4 Weight Matrix for GMCR under Hybrid Preference

A hybrid preference framework is presented in Chapter 3 to combine preference uncertainty and strength of preference using a quadruple relation $\{\gg_i, >_i, \sim_i, U_i\}$ in a graph model for DM i. The weight matrix W_H can also represent the combining preference of uncertainty and strength. When an edge $a_k = d_i(u, v)$ for $u, v \in S$ and $i \in H \subseteq N$, then its weight w_k is defined by

$$w_{a_k} = \begin{cases} P_s & \text{if } v \gg_i u \text{ and } i \in H, \\ P_m & \text{if } v >_i u \text{ and } i \in H, \\ E_w & \text{if } u \sim_i v \text{ and } i \in H, \\ U_w & \text{if } uU_iv \text{ and } i \in H, \\ N_w & \text{if } u \gg_i v \text{ or } u >_i v \text{ and } i \in H, \\ 0 & \text{otherwise.} \end{cases} \tag{5.9}$$

Recall that notation WIUUMs denotes strong unilateral improvements, mild unilateral improvements, or unilateral uncertain moves. By the statement (5.9), the WIUUM weight matrix for H is defined as follows.

Definition 5.20. *For the graph model $G = (S, A)$, let $H \subseteq N$. When $P_s = P_m = U_w = 1$ and $E_w = N_w = 0$, the weight matrix W_H is called the WIUUM weight matrix for H, denoted by $W_H^{(WIUUM)}$ or $W_H^{+,++,U}$.*

Each arc of the arc set $A_i^{+,++,U}$ denotes that DM i can make a WIUUM from the initial state to the terminal state of the arc. Therefore, $A_H^{+,++,U} = \bigcup_{i \in H} A_i^{+,++,U}$ denotes the WI arcs associated with any DM in H. By Definition 5.6 for the weighted arc set, the WIUUM arc set is obtained for a graph model with hybrid preference by the following Corollary.

Corollary 5.10. *For the graph model $G = (S, A)$, let $H \subseteq N$. If $W_H = W_H^{+,++,U}$, then the arc set $A_H^{(W)} = A_H^{+,++,U}$.*

Note that when $H = N$, $A_H^{+,++,U}$ is expressed by $A^{+,++,U}$.

The weighted colored paths $PA_H^{(W)}$ can be applied to trace conflict evolution by the legal WIUUMs for the graph model with strength of preference.

Corollary 5.11. *For the graph model $G = (S, A)$, let $u, v \in S$ and $H \subseteq N$. If $W_H = W_H^{+,++,U}$, the weighted colored paths between states u and v, $PA_H^{(W)}(u, v)$, give all paths from u to v where only the legal WIUUMs are allowed. Then $PA_H^{(W)}(u, v)$ are called the legal WI paths from u to v by coalition H, denoted by $PA_H^{+,++,U}(u, v)$.*

Definition 5.21. *In the graph model $G = (S, A)$, the legal WIUUM edge consecutive matrix is an $l \times l$ matrix $LJ_r^{+,++,U}$ with (a, b) entry*

$$LJ_r^{+,++,U}(a,b) = \begin{cases} 1 & \text{if edges } a \text{ and } b \text{ are consecutive in order } ab \text{ and} \\ & \text{are controlled by difference DMs for } a, b \in A^{+,++,U}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $LJ_{H_r}^{+,++,U}$ denote the legal WIUUM edge consecutive matrix for the graph model (V, A_H). Based on Definition 5.13, on the reduced weighted edge consecutive matrix for H, and Definition 5.21, the following result is obtained.

Corollary 5.12. *For the graph model $G = (S, A)$, let $W^{+,++,U}$ denote the WIUUM weight matrix and $W_H^{+,++,U}$ be the WIUUM weight matrix for $H \subseteq N$. Then*

$$LJ_r^{(W^{+,++,U})} = LJ_r^{+,++,U},$$

and

$$LJ_r^{(W_H^{+,++,U})} = LJ_{H_r}^{+,++,U}.$$

The key input of stability analysis in the graph model with hybrid preference is state set $R_H^{+,++,U}(s)$, the reachable list of coalition $H \subseteq N$ from state $s \in S$ by the legal WIUUMs. A logical method is presented in Chapter 3 to construct $R_H^{+,++,U}(s)$. An algebraic approach to obtain the state set will be addressed in future research as mentioned in Section 8.2.

5.5 Summary

From the above discussions, we find that although many approaches and algorithms for coloring vertices and edges have been developed in graph theory and computer science [9], the edge-weighted, colored graph research here differs from previous work in that it is not concerned with how to color edges. Instead, the fundamental problem is to search edge-weighted, colored paths in a given colored multidigraph. This research is also different from the well-known network analysis problem of finding paths between two vertices due to the additional color restriction feature that is not present in these problems. Therefore, it is difficult to use existing methods or algorithms directly, including genetic algorithms [12], neural networks [65], and reinforcement learning algorithms [45], to find the shortest colored path. In this research, an adjacency matrix of an undirected line graph is extended to a reduced weighted edge consecutive matrix to search all weighted colored paths, thereby providing new insights into Algebraic Graph Theory [24]. Based on the matrix thus designed, a conversion function is proposed to transform a colored multidigraph to a simple digraph so that the original complex problem of searching edge-colored paths in a colored multidigraph is converted to a standard problem of finding paths in a simple digraph with no color constraints [72]. In Chapters 6 and 7, the capability of the developed algebraic approach will further be investigated.

Chapter 6

Matrix Representation for Stability Analysis in the Graph Model

Stability definitions in the graph model are traditionally defined logically, in terms of the underlying graphs and preference relations. However, as was noted in the development of the DSS GMCR II, the nature of logical representations makes coding difficult. The new preference structures proposed by Li et al. [46] , Hamouda et al. [28] and Xu et al. [70] to represent uncertainty, strength, and combining uncertainty and strength in DMs' preferences included some extensions of the four basic stability definitions, but algorithms have not been developed for the three structures. Table 1.1 shows the current state of development of effective algorithms and codes to implement these solution concepts, which would be essential if they are to be applied to practical problems [44].

In this chapter, matrix expressions are used to capture relative preferences, reachable lists by a coalition from a status quo by legal sequences of UMs and UIs for simple preference, legal sequence of UIUUMs for preference with uncertainty, and legal sequence of WIs for preference with strength. An explicit algebraic form conflict model is developed to facilitate stability calculations in two-DM and n-DM ($n > 2$) models for simple preference, preference with uncertainty, and preference with strength.

Note that if the state set S is treated as a vertex set and DM i's oriented arcs are coded in color i, then a graph model of a conflict is equivalent to a colored multidigraph with appropriate preference relations. As shown in Chapter 5, the weight matrix is convenient and flexible to represent preference information in the graph model. Therefore, the graph model is converted to a weighted colored multidigraph. It is natural to use the results of Graph Theory to assist in analyzing of a graph model. Hence, we will hereafter use the same notation as Chapter 5 to represent a graph model for conflicts.

6.1 Matrix Representation of Solution Concepts for Simple Preference

In this section, a graph model and four graph model solution concepts are formulated explicitly using matrices. More specifically, matrix expressions are given for relative preferences and the reachable lists of a coalition from a status quo state by the legal sequences of UMs and UIs in a multiple-decision-maker model. Then it is shown how to calculate stability under each of the four solution concepts using the matrix representation.

6.1.1 Matrix Representation of Essential Components for Stabilities for Simple Preference

Important matrices associated with a digraph include the adjacency matrix and the incidence matrix [24]. These matrices are extended to the graph model for conflict resolution. Let $i \in N$ and $m = |S|$. Recall that UMs and UIs represent unilateral moves and unilateral improvements, respectively.

Definition 6.1. *For the graph model $G = (S, A)$, the* ***UM adjacency matrix*** *J_i and* ***UI adjacency matrix*** *J_i^+ for DM i are two $m \times m$ matrices with (s, q) entries*

$$J_i(s,q) = \begin{cases} 1 & \textit{if } (s,q) \in A_i, \\ 0 & \textit{otherwise}, \end{cases} \quad \textit{and } J_i^+(s,q) = \begin{cases} 1 & \textit{if } (s,q) \in A_i^+, \\ 0 & \textit{otherwise}, \end{cases}$$

where $s, q \in S$ and $A_i^+ = \{(s,q) \in A_i : q \succ_i s\}$.

The reachable lists by DM i from state s defined in Section 2.2.2, $R_i(s)$ and $R_i^+(s)$, are expressed as $R_i(s) = \{q : J_i(s,q) = 1\}$ and $R_i^+(s) = \{q : J_i^+(s,q) = 1\}$. The following result is obtained based on Definition 5.8, on the weighted adjacency matrix by H, $J_H^{(W)}$, Theorem 5.3 for constructing matrix $J_H^{(W)}$, and Definition 6.1.

Corollary 6.1. *For the graph model $G = (S, A)$, the UM and the UI adjacency matrices of DM i can be expressed as*

$$J_i = J_i^{(W^{(UM)})} = sign[(B_{out}^{(W_i^{(UM)})}) \cdot (B_{in}^{(W_i^{(UM)})})^T]$$

and

$$J_i^+ = J_i^{(W^+)} = sign[(B_{out}^{(W_i^+)}) \cdot (B_{in}^{(W_i^+)})^T].$$

Recall that $R_H(s)$ and $R_H^+(s)$ are the reachable lists of coalition H from state s by the legal sequences of UMs and UIs. Two essential matrices for stability analysis are defined as follows.

Definition 6.2. *Let $H \subseteq N$. For the graph model $G = (S, A)$, the UM reachability matrix and the UI reachability matrix of coalition H are two $m \times m$ matrices M_H and M_H^+ with (s, q) entries*

$$M_H(s,q) = \begin{cases} 1 & \text{if } q \in R_H(s) \text{ for } q \in S, \\ 0 & \text{otherwise,} \end{cases}$$

$$M_H^+(s,q) = \begin{cases} 1 & \text{if } q \in R_H^+(s) \text{ for } q \in S, \\ 0 & \text{otherwise.} \end{cases}$$

The following result is obtained based on Definition 5.11, on the weighted reachability matrix by H, $M_H^{(W)}$, Theorem 5.4 for constructing the weighted reachability matrix and Corollary 5.3 for constructing the legal UM and UI edge consecutive matrices LJ_r and LJ_r^+, and Definition 6.2.

Corollary 6.2. *For the graph model $G = (S, A)$, the UM reachability and the UI reachability matrices of coalition H can be expressed as*

$$M_H = M_H^{(W^{(UM)})} = sign[(B_{out}^{(W_H^{(UM)})}) \cdot (LJ_{H_r} + I)^{l_1 - 1} \cdot (B_{in}^{(W_H^{(UM)})})^T]$$

and

$$M_H^+ = M_H^{(W^+)} = sign[(B_{out}^{(W_H^+)}) \cdot (LJ_{H_r}^+ + I)^{l_2 - 1} \cdot (B_{in}^{(W_H^+)})^T],$$

where $l_1 = |A_H|$ and $l_2 = |A_H^+|$.

Below, several $m \times m$ preference matrices, P_i^+, P_i^-, and $P_i^=$ for DM i, are respectively defined as

$$P_i^+(s,q) = \begin{cases} 1 & if\ q \succ_i s, \\ 0 & otherwise, \end{cases} \quad P_i^-(s,q) = \begin{cases} 1 & if\ s \succ_i q, \\ 0 & otherwise, \end{cases}$$

and

$$P_i^{=}(s,q) = \begin{cases} 1 & if\ s \sim_i q, \\ 0 & otherwise, \end{cases} \quad P_i^{-,=} = P_i^{-} \vee P_i^{=}.$$

It follows that

$$P_i^{-,=}(s,q) = \begin{cases} 1 - P_i^{+}(s,q) & if\ s \neq q, \\ 0 & otherwise. \end{cases}$$

Based on the definitions of the UM adjacency matrix, J_i, the UI adjacency matrix, J_i^{+}, and preference matrix, P_i^{+}, for DM i, the relationship among them is

$$J_i^{+} = J_i \circ P_i^{+}.$$

6.1.2 Matrix Representation of Solution Concepts for Two-DMs under Simple Preference

Matrix representation of Nash stability, GMR, SMR, and SEQ in two-DM conflict models for simple preference is developed in this chapter. The system, called the MRSC method, incorporated a set of $m \times m$ matrices, M_i^{GMR}, M_i^{SMR}, and M_i^{SEQ}, to capture GMR, SMR, and SEQ for DM $i \in N$, where $|N| = 2$ and $m = |S|$.

Since the following results are special cases of those developed in the next subsection, the details are not given here. Let $N = \{i, j\}$. Then

Theorem 6.1. *State $s \in S$ is Nash stable for DM i iff $e_s^T \cdot J_i^{+} = \overrightarrow{0}^T$. ($T$ denotes matrix transpose and e_s^T is the transpose of the s^{th} standard basis vector of the m-dimensional Euclidean space.)*

A state $s \in S$ is general metarational for DM i iff whenever DM i makes any UI from s, then its opponent can hurt i in response. Define the $m \times m$ matrix M_i^{GMR}

$$M_i^{GMR} = J_i^{+} \cdot [E - sign\left(J_j \cdot (P_i^{-,=})^T\right)],\ for\ j \in N \backslash \{i\}.$$

Theorem 6.2. *State $s \in S$ is GMR for DM i iff $M_i^{GMR}(s,s) = 0$.*

Define the $m \times m$ matrix $M_i^{SMR} = J_i^{+} \cdot [E - sign(G)]$ in which

$$G = J_j \cdot [(P_i^{-,=})^T \circ \left(E - sign\left(J_i \cdot (P_i^{+})^T\right)\right)],\ for\ j \in N \backslash \{i\}.$$

Theorem 6.3. *State $s \in S$ is SMR for DM i iff $M_i^{SMR}(s,s) = 0$.*

Define the $m \times m$ matrix $M_i^{SEQ} = J_i^{+} \cdot [E - sign\left(J_j^{+} \cdot (P_i^{-,=})^T\right)],\ for\ j \in N \backslash \{i\}$.

Theorem 6.4. *State $s \in S$ is SEQ for DM i iff $M_i^{SEQ}(s,s) = 0$.*

These theorems prove that the proposed matrix representation of solution concepts are equivalent to the solution concepts for two DM conflicts defined by Fang et al. [16]. The matrix representation can be extended to models including more than two DMs, which is the objective of the next subsection.

6.1.3 Matrix Representation of Solution Concepts for n-DMs under Simple Preference

Equivalent matrix representations of the logical definitions for Nash stability, GMR, SMR, and SEQ can be determined directly by using the relationship that has been established between matrix elements and the state set of a graph model, and by using preference relation matrices among the states.

Let $i \in N$ and $|N| = n$ for the following theorems.

Theorem 6.5. *State $s \in S$ is Nash stable for DM i, denoted by $s \in S_i^{Nash}$, iff $\langle e_s, J_i^+ e\rangle = 0$, where $<,>$ denotes the inner product.*

Theorem 6.1 and Theorem 6.5 are identical because Nash stability does not consider opponents' responses.

It should be pointed out that the following stability matrices for n-DMs use the same notation as that presented in Subsection 6.1.2 for two-DMs. For general metarationality, DM i will take into account the opponents' possible responses, which are the legal sequence of UMs by members of $N \backslash \{i\}$. For $i \in N$, find the UI adjacency matrix J_i^+ and the UM reachability matrix $M_{N \backslash \{i\}}$ using Corollary 6.1 and Corollary 6.2, for which $H = N \backslash \{i\}$. Define the $m \times m$ matrix M_i^{GMR} by

$$M_i^{GMR} = J_i^+ \cdot [E - sign\big(M_{N \backslash \{i\}} \cdot (P_i^{-,=})^T\big)].$$

Theorem 6.6. *State $s \in S$ is GMR for DM i, denoted by $s \in S_i^{GMR}$, iff $M_i^{GMR}(s, s) = 0$.*

Proof: Since the diagonal element of matrix M_i^{GMR}

$$\begin{aligned} M_i^{GMR}(s,s) &= \langle (J_i^+)^T e_s, \big(E - sign\big(M_{N-\{i\}} \cdot (P_i^{-,=})^T\big)\big) e_s \rangle \\ &= \sum_{s_1=1}^{m} J_i^+(s, s_1)[1 - sign\big(\langle (M_{N \backslash \{i\}})^T e_{s_1}, (P_i^{-,=})^T e_s \rangle\big)], \end{aligned}$$

then $M_i^{GMR}(s,s) = 0$ iff

$$J_i^+(s, s_1)[1 - sign\big(\langle (M_{N \backslash \{i\}})^T e_{s_1}, (P_i^{-,=})^T e_s \rangle\big)] = 0, \forall s_1 \in S.$$

This implies that $M_i^{GMR}(s,s)=0$ iff

$$(e_{s_1}^T M_{N\backslash\{i\}})\cdot(e_s^T P_i^{-,=})^T \neq 0, \forall s_1 \in R_i^+(s). \tag{6.1}$$

Statement (6.1) means that, for any $s_1 \in R_i^+(s)$, there exists $s_2 \in S$, such that the m-dimensional row vector, $e_{s_1}^T \cdot M_{N\backslash\{i\}}$, with s_2^{th} element 1 and the m-dimensional column vector, $(P_i^{-,=})^T \cdot e_s$, with s_2^{th} element 1.

Therefore, $M_i^{GMR}(s,s)=0$ iff for any $s_1 \in R_i^+(s)$, there exists at least one $s_2 \in R_{N\backslash\{i\}}(s_1)$ with $s \succeq_i s_2$. □

For symmetric metarationality, the n-DM model is similar to the two-DM model. The only modification is that responses come from DM i's opponents instead of from a single DM. Let

$$G=(P_i^{-,=})^T \circ [E - sign\big(J_i\cdot(P_i^+)^T\big)],$$

then define the $m\times m$ matrix M_i^{SMR} by

$$M_i^{SMR}=J_i^+\cdot[E-sign(M_{N\backslash\{i\}}\cdot G)].$$

Theorem 6.7. *State $s\in S$ is SMR for DM i, denoted by $s\in S_i^{SMR}$, iff $M_i^{SMR}(s,s)=0$.*

Proof: Since the diagonal element of matrix M_i^{SMR}

$$M_i^{SMR}(s,s)=\langle (J_i^+)^T\cdot e_s, \big(E-sign(M_{N\backslash\{i\}}\cdot W)\big)e_s\rangle$$

$$=\sum_{s_1=1}^{m} J_i^+(s,s_1)[1-sign\big(\langle (M_{N\backslash\{i\}})^T\cdot e_{s_1}, G\cdot e_s\rangle\big)],$$

then $M_i^{SMR}(s,s)=0$ iff

$$J_i^+(s,s_1)[1-sign\big(\langle (M_{N\backslash\{i\}})^T\cdot e_{s_1}, G\cdot e_s\rangle\big)]=0, \forall s_1\in S.$$

This means that $M_i^{SMR}(s,s)=0$ iff

$$(e_{s_1}^T\cdot M_{N\backslash\{i\}})\cdot(G\cdot e_s)\neq 0, \forall s_1\in R_i^+(s). \tag{6.2}$$

Let $G(s_2,s)$ denote the (s_2,s) entry of matrix G. Since

$$(e_{s_1}^T M_{N\backslash\{i\}})\cdot(G\cdot e_s)=\sum_{s_2=1}^{m} M_{N\backslash\{i\}}(s_1,s_2)\cdot G(s_2,s),$$

then (6.2) holds iff for any $s_1 \in R_i^+(s)$, there exists $s_2 \in R_{N\backslash\{i\}}(s_1)$ such that $G(s_2, s) \neq 0$.

Because $G(s_2, s) = P_i^{-,=}(s, s_2)[1 - sign(\sum_{s_3=1}^{m} J_i(s_2, s_3) P_i^+(s, s_3))]$, then $G(s_2, s) \neq 0$ implies that for $s_2 \in R_{N\backslash\{i\}}(s_1)$,

$$P_i^{-,=}(s, s_2) \neq 0 \tag{6.3}$$

and

$$\sum_{s_3=1}^{m} J_i(s_2, s_3) P_i^+(s, s_3) = 0. \tag{6.4}$$

(6.3) is equivalent to the statement that, $\forall s_1 \in R_i^+(s), \exists s_2 \in R_{N\backslash\{i\}}(s_1)$ such that $s \succeq_i s_2$. (6.4) is the same as the statement that, $\forall s_1 \in R_i^+(s), \exists s_2 \in R_{N\backslash\{i\}}(s_1)$ such that $P_i^+(s, s_3) = 0$ for $\forall s_3 \in R_i(s_2)$. Based on the definition of $m \times m$ matrix P_i^+, one knows that $P_i^+(s, s_3) = 0 \Longleftrightarrow s \succeq_i s_3$.

Therefore, we conclude the above discussion that $M_i^{SMR}(s, s) = 0$ iff for any $s_1 \in R_i^+(s)$, there exists at least one $s_2 \in R_{N\backslash\{i\}}(s_1)$ with $s \succeq_i s_2$ and $s \succeq_i s_3$ for all $s_3 \in R_i(s_2)$. $\square$

Sequential stability examines the credibility of the sanctions by DM i's opponents. For $i \in N$, find the UI reachability matrix $M_{N\backslash\{i\}}^+$ using Corollary 6.2. Define the $m \times m$ matrix M_i^{SEQ} by

$$M_i^{SEQ} = J_i^+ \cdot [E - sign\big(M_{N\backslash\{i\}}^+ \cdot (P_i^{-,=})^T\big)].$$

Theorem 6.8. *State $s \in S$ is SEQ for DM i, denoted by $s \in S_i^{SEQ}$, iff $M_i^{SEQ}(s, s) = 0$.*

Proof: Since the diagonal element of matrix M_i^{SEQ}

$$\begin{aligned} M_i^{SEQ}(s, s) &= \langle (J_i^+)^T \cdot e_s, \big(E - sign\big(M_{N\backslash\{i\}}^+ \cdot (P_i^{-,=})^T\big)\big) e_s \rangle \\ &= \sum_{s_1=1}^{m} J_i^+(s, s_1)[1 - sign\big(\langle (M_{N\backslash\{i\}}^+)^T \cdot e_{s_1}, (P_i^{-,=})^T \cdot e_s \rangle\big)], \end{aligned}$$

then $M_i^{SEQ} \cdot (s, s) = 0$ iff $J_i^+(s, s_1)[1 - sign\big(\langle (M_{N\backslash\{i\}}^+)^T \cdot e_{s_1}, (P_i^{-,=})^T \cdot e_s \rangle\big)] = 0, \forall s_1 \in S$. This implies that $M_i^{SEQ}(s, s) = 0$ iff

$$(e_{s_1}^T M_{N\backslash\{i\}}^+) \cdot (e_s^T \cdot P_i^{-,=})^T \neq 0, \forall s_1 \in R_i^+(s). \tag{6.5}$$

Statement (6.5) means that, for any $s_1 \in R_i^+(s)$, there exists $s_2 \in S$, such that the m-dimensional row vector, $e_{s_1}^T \cdot M_{N-\{i\}}^+$, with s_2^{th} element 1 and the m-dimensional column vector, $(P_i^{-,=})^T \cdot e_s$, with s_2^{th} element 1.

Therefore, $M_i^{SEQ}(s,s) = 0$ iff for any $s_1 \in R_i^+(s)$, there exists at least one $s_2 \in R_{N\setminus\{i\}}^+(s_1)$ with $s \succeq_i s_2$. $\square$

When $n = 2$, the DM set N becomes to $\{i, j\}$ in Theorems 6.5 to 6.8, and the reachable lists for $H = N \setminus \{i\}$ by legal sequences of UMs and UIs from s_1, $R_{N\setminus\{i\}}(s_1)$ and $R_{N\setminus\{i\}}^+(s_1)$, degenerate to $R_j(s_1)$ and $R_j^+(s_1)$, DM j's corresponding reachable lists from s_1. Thus, Theorems 6.5 to 6.8 are reduced to those Theorems 6.1 to 6.4.

So far, the matrix representation of solution concepts has been established in multiple decision maker graph models for simple preference. As shown below, the matrix method for calculating the individual stability and equilibria is also attractive from a computational point of view. Many researchers are now attempting to develop faster algorithms for matrix operations. For example, for the multiplication of two $m \times m$ matrices, the standard method requires $O(m^3)$ arithmetic operations, but the Strassen algorithm [62] requires only $O(m^{2.807})$ operations. Coppersmith and Winograd's work [11] shows that the computational complexity of matrix multiplication was decreased to $O(m^{2.376})$. In fact, some researchers believe that an optimal algorithm for multiplying $m \times m$ matrices will reduce the complexity to $O(m^2)$ [10]. Therefore, the proposed matrix method not only is propitious for theoretical analysis, but also has the potential to deal with large and complicated conflict problems.

6.1.4 Interrelationships among the Solution Concepts

In 1993, Fang et al. [16] established general relationships among Nash stability, GMR, SMR, and SEQ (See Fig. 6.1) in the following theorem.

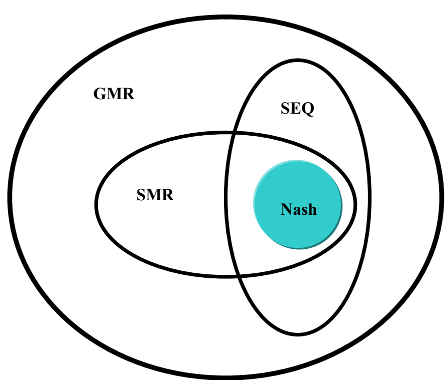


Figure 6.1: **Interrelationships among the four solution concepts [16].**

Theorem 6.9. *Let $i \in N, \mid N \mid = n$, and $n \geq 2$. Then interrelationships among the four solution concepts are*

$$S_i^{Nash} \subseteq S_i^{SMR} \subseteq S_i^{GMR} \tag{6.6}$$

and

$$S_i^{Nash} \subseteq S_i^{SEQ} \subseteq S_i^{GMR}. \tag{6.7}$$

As shown below, the interrelationships among four solution concepts formulated explicitly using matrices are easy to verify.

Proof: If $s \in S_i^{Nash}$, then $e_s^T \cdot J_i^+ = \overrightarrow{0}^T$. Let

$$B = E - sign(M_{N\setminus\{i\}} \cdot G)$$

and let $G = (P_i^{-,=})^T \circ [E - sign\big(J_i \cdot (P_i^+)^T\big)]$. Since $M_i^{SMR}(s,s) = (e_s^T \cdot J_i^+) \cdot (B \cdot e_s)$, it follows that $M_i^{SMR}(s,s) = 0$, when $e_s^T \cdot J_i^+ = \overrightarrow{0}^T$. Hence, if $s \in S_i^{Nash}$, then $s \in S_i^{SMR}$, which implies $S_i^{Nash} \subseteq S_i^{SMR}$.

Because $G = (P_i^{-,=})^T \circ [E - sign\big(J_i \cdot (P_i^+)^T\big)]$, it follows that for $\forall s \in S$,

$$e_s^T \cdot \big(M_{N\setminus\{i\}} \cdot (P_i^{-,=})^T\big) \cdot e_s \neq 0,$$

when $e_s^T \cdot (M_{N\setminus\{i\}} \cdot G) \cdot e_s \neq 0$, this implies that $e_s^T \cdot [E - sign\big(M_{N\setminus\{i\}} \cdot (P_i^{-,=})^T\big)] \cdot e_s = 0$, if $e_s^T \cdot [E - sign(M_{N\setminus\{i\}} \cdot G)] \cdot e_s = 0$. Therefore, if $s \in S_i^{SMR}$, then

$$M_i^{SMR}(s,s) = (e_s^T \cdot J_i^+) \cdot [\big(E - sign(M_{N\setminus\{i\}} \cdot G)\big) \cdot e_s] = 0,$$

which implies that

$$M_i^{GMR}(s,s) = (e_s^T \cdot J_i^+) \cdot [\big(E - sign\big(M_{N\setminus\{i\}} \cdot (P_i^{-,=})^T\big)\big) e_s] = 0.$$

Hence, $S_i^{SMR} \subseteq S_i^{GMR}$. Thus, relation (6.6) now follows. Relation (6.7) can be verified, similarly.

There is no necessary inclusion relation between S_i^{SMR} and S_i^{SEQ}, i. e., it may or may not be true that $S_i^{SMR} \supseteq S_i^{SEQ}$, or that $S_i^{SMR} \subseteq S_i^{SEQ}$. However, we can take advantage of the algebraic characterization of MRSC to establish some facts about their interrelationship.

Theorem 6.10. *Let $i \in N, \mid N \mid = n$, and $n \geq 2$. Let $G = (P_i^{-,=})^T \circ [E - sign\big(J_i \cdot (P_i^+)^T\big)]$. Then, when $(M_{N\setminus\{i\}} \cdot G) \bigvee [M_{N\setminus\{i\}}^+ \cdot (P_i^{-,=})^T] = sign(M_{N\setminus\{i\}} \cdot G)$,*

$$S_i^{SMR} \supseteq S_i^{SEQ}; \text{ and} \tag{6.8}$$

when $(M_{N\setminus\{i\}} \cdot G) \bigvee [M_{N\setminus\{i\}}^+ \cdot (P_i^{-,=})^T] = sign[M_{N\setminus\{i\}}^+ \cdot (P_i^{-,=})^T]$,

$$S_i^{SMR} \subseteq S_i^{SEQ}. \tag{6.9}$$

Proof: If $s \in S_i^{SEQ}$, then $M_i^{SEQ}(s,s) = 0$, which is equivalent to

$$(e_s^T \cdot J_i^+) \cdot [\big(E - sign\big(M_{N\setminus\{i\}}^+ \cdot (P_i^{-,=})^T\big)\big) \cdot e_s] = 0,$$

so that $e_s^T \cdot J_i^+ = (e_s^T \cdot J_i^+) \circ [sign\big(M_{N\setminus\{i\}}^+ \cdot (P_i^{-,=})^T\big) \cdot e_s]^T$. Since

$$(M_{N\setminus\{i\}} \cdot W) \bigvee [M_{N\setminus\{i\}}^+ \cdot (P_i^{-,=})^T] = sign(M_{N\setminus\{i\}} \cdot G),$$

it follows that $e_s^T \cdot J_i^+ = (e_s^T \cdot J_i^+) \circ [sign(M_{N\setminus\{i\}} \cdot G) \cdot e_s]^T$, and therefore

$$(e_s^T \cdot J_i^+) \cdot [\big(E - sign(M_{N\setminus\{i\}} \cdot G)\big) \cdot e_s] = 0,$$

which implies that $M_i^{SMR}(s,s) = 0$. Relation (6.8) now follows. Relation (6.9) can be proved, similarly. $\square$

6.1.5 Applications for Simple Preference

6.1.5.1 Superpower Nuclear Confrontation

In two-DM conflicts, a simplified model of a superpower nuclear confrontation, including the "nuclear winter" possibility [16], is used to illustrate how stability analysis is carried out using MRSC. This conflict is modeled using two DMs and a total of six options. In the superpower nuclear confrontation conflict, the six options together determine five feasible states as listed in Table 6.1, where a "Y" indicates that an option is selected by the DM controlling it and an "N" means that the option is not chosen. The graph model of the superpower nuclear confrontation conflict is shown in Fig. 6.2. Note that state W is assumed to trigger a nuclear winter. Given that the preferences are ordinal for DM 1 and DM 2 [16],

$$PP \succ_1 CP \succ_1 CC \succ_1 PC \succ_1 W,$$

and

$$PP \succ_2 PC \succ_2 CC \succ_2 CP \succ_2 W.$$

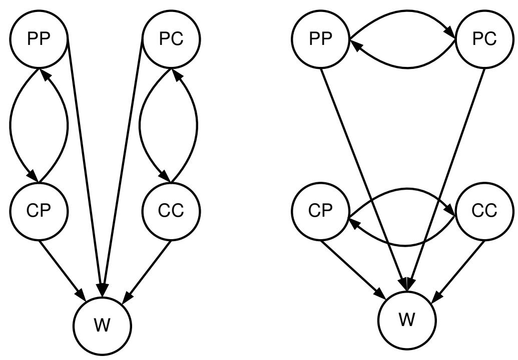
In order to carry out a stability analysis for each of the five states and each of the two DMs, the MRSC method is used for the superpower nuclear confrontation model.

Let the five states, PP, PC, CP, CC, and W, be numbered from 1 to 5, respectively. From the graph model, we have

$$J_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, J_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

Table 6.1: **Options and feasible states for the superpower nuclear confrontation conflict** [16]

DM 1					
1. Peace (labeled P)	Y	Y	N	N	N
2. Conventional attack (labeled C)	N	N	Y	Y	N
3. Full nuclear attack (labeled W)	N	N	N	N	Y
DM 2					
1. Peace (labeled P)	Y	N	Y	N	N
2. Conventional attack (labeled C)	N	Y	N	Y	N
3. Full nuclear attack (labeled W)	N	N	N	N	Y
States	**PP**	**PC**	**CP**	**CC**	**W**



(a) Graph model for DM 1 (b) Graph model for DM 2

Figure 6.2: **The graph model of the superpower nuclear confrontation conflict [16].**

$$P_1^+ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}, P_2^+ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Then

$$J_i^+ = J_i \circ P_i^+, for\ i = 1, 2,$$

and

$$P_i^{-,=} = E - I - P_i^+, for\ i = 1, 2,$$

where I is a 5×5 identity matrix. Next, we can calculate the stabilities of Nash, GMR, SMR, and SEQ, respectively, for the superpower nuclear confrontation conflict, using MRSC for two-DM cases introduced by Theorems 6.1 to 6.4. The stability results using MRSC are provided in Table 6.2 in which "$\surd$" denotes that this state is stable for DM 1 or DM 2 under the appropriate stability definitions, and "Eq" means an equilibrium that is stable for the two DMs. States PP, CC, and W are equilibria for four basic solution concepts.

Table 6.2: **Stability results of the superpower nuclear confrontation**

State Number	Nash			GMR			SMR			SEQ		
	DM 1	DM 2	Eq	DM 1	DM 2	Eq	DM 1	DM 2	Eq	DM 1	DM 2	Eq
PP	√	√	√	√	√	√	√	√	√	√	√	√
PC				√	√	√	√	√	√			
CP				√	√	√	√	√	√			
CC	√	√	√	√	√	√	√	√	√	√	√	√
W	√	√	√	√	√	√	√	√	√	√	√	√

6.1.5.2 Rafferty-Alameda Dams Conflict

The Rafferty-Alameda dam, in the Souris River basin in southern Saskatchewan, was planned for flood control, recreation and cooling the Shand generating plant [55]. The **province of Saskatchewan** wanted to finish the project promptly, seeking a license from the Environment Minister of the Federal government. An **environment group**, the Canadian Wildlife Federation, quickly petitioned against the license and argued that the provincial government had not respected regulations. The **federal court** sided with the environment group and ordered the suspension of the license, but later the license was reissued by a new federal environment minister. The environment group petitioned again, and this time the federal court ordered the suspension of the license and the creation of a **review panel** to reevaluate the project. However, construction of the dam continued during the review period, and the federal and provincial governments even reached an agreement that the project would continue while ten million dollars are set aside to alleviate any future environmental impacts. As the province had hoped, the project moved ahead at full speed, and the review panel resigned in protest. (See Hipel et al. [29] for details.)

This conflict is modeled using four DMs: DM 1, **Federal (F)**; DM 2, **Saskatchewan (S)**; DM 3, **Groups (G)**; and DM 4, **Panel (P)**, each having some options. The following is a summary of the four DMs and their options [29]:

- Federal Court (**Federal**): its options are to create a federal government review panel (Court Order) or to lift the license **(Lift)**,

- Province of Saskatchewan (**Saskatchewan**): its option is to go ahead at full speed **(Full speed)**,

- Environmental Group (**Group**): its option is to threaten court action to halt the project **(Court action)**, and

- Federal Environmental Review Panel (**Panel**): its option is to resign **(Resign)**.

The five options are combined to form 32 possible states in this conflict. Only a part of the combinations of the options create feasible states listed in Table 6.3, where a "Y" indicates that an option is selected by the DM controlling it, an "N" means that the option is not chosen, and a dash " − ", means that the entry may be "Y" or may be "N". The graph model of the Rafferty-Alameda dams conflict is shown in Fig. 6.3 (1), where the labels on the arcs identify the DMs who control the relevant moves. If DM i's oriented arcs are coded in color i, then, according to the *Rule of Priority* presented in Subsection 5.2, Fig. 6.3 (1) is converted to an edge labeled multidigraph as shown in Fig. 6.3 (2).

Table 6.3: **Feasible states for the Rafferty-Alameda dams Model** [29]

Federal										
1. Court Order	-	N	Y	N	Y	N	Y	N	Y	N
2. Lift	-	N	N	N	N	N	N	N	N	Y
Saskatchewan										
3. Full speed	N	Y	Y	Y	Y	Y	Y	Y	Y	-
Groups										
4. Court action	-	N	N	Y	Y	N	N	Y	Y	-
Panel										
5. Resign	-	N	N	N	N	Y	Y	Y	Y	-
State number	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}

Unilateral moves and preference information over the states are given in Table 6.4, where $R_i(s)$, defined in Chapter 2, is DM $i's$ reachable list from state s, and p_i denotes DM $i's$ preference function. For this function, DM i prefers a state with a greater function value than a low one. For example, the Federal Government most prefers state s_1 and least prefers state s_{10}. We calculate the stabilities of Nash, GMR, SMR, and SEQ for the conflict with four DMs, using the proposed MRSC method.

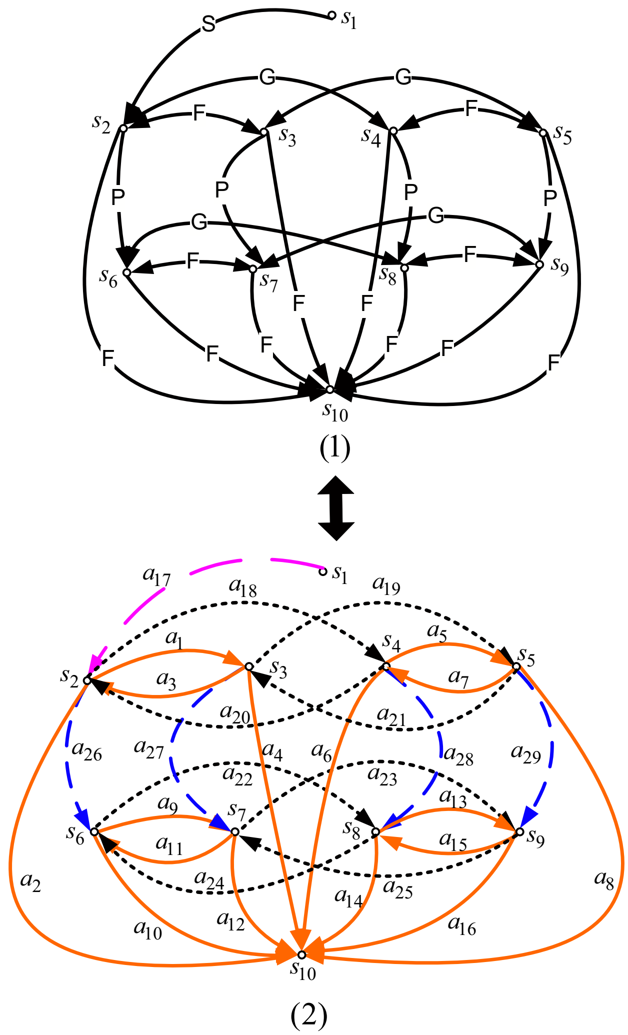


Figure 6.3: **The graph model of the Rafferty-Alameda dams conflict.**

Let $N = \{1, 2, 3, 4\}$ denote the set of four DMs. We use the Rafferty-Alameda dams model as an example to show the procedures to carry out matrix representation of the four solution concepts in the graph model.

- Construct preference matrices, P_i^+, and $P_i^{-,=}$, for $i = 1, 2, 3, 4$, using information provided by Table 6.4;

- Calculate the UM weight matrix and the UI weight matrix of coalition H based on Definition 5.14, preference information presented in Table 6.4, and

Table 6.4: **Unilateral moves and preference functions for the Rafferty-Alameda dams model** [29]

State Number	Federal		Saskatchewan		Groups		Panel	
	R_1	p_1	R_2	p_2	R_3	p_3	R_4	p_4
s_1		10	s_2	1		9		10
s_2	s_3,s_{10}	7		10	s_4	1	s_6	1
s_3	s_2,s_{10}	9		6	s_5	3	s_7	3
s_4	s_5,s_{10}	6		9	s_2	5	s_8	2
s_5	s_4,s_{10}	8		5	s_3	7	s_9	4
s_6	s_7,s_{10}	3		8	s_8	2		6
s_7	s_6,s_{10}	5		4	s_9	4		8
s_8	s_9,s_{10}	2		7	s_6	6		7
s_9	s_8,s_{10}	4		3	s_7	8		9
s_{10}		1		2		10		5

statement (5.6), i.e., if $a_k = d_i(u, v)$, then

$$w_k = \begin{cases} P_w & \text{if } v \succ_i u \text{ and } i \in H, \\ E_w & \text{if } u \sim_i v \text{ and } i \in H, \\ N_w & \text{if } u \succ_i v \text{ and } i \in H, \\ 0 & \text{otherwise.} \end{cases}$$

Table 6.5 shows the process how to calculate matrices $W_H^{(UM)}$ and W_H^+ for $H = N\backslash\{1\}$;

Table 6.5: **Weights of edges by $N\backslash\{1\}$ for Rafferty-Alameda dams conflict**

Arc number	a_{17}	a_{18}	a_{19}	a_{20}	a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	a_{26}	a_{27}	a_{28}	a_{29}
$W_{N\backslash\{1\}}$	P_w	P_w	P_w	N_w	N_w	P_w	P_w	N_w	N_w	P_w	P_w	P_w	P_w
$W_{N\backslash\{1\}}^{(UM)}$	1	1	1	1	1	1	1	1	1	1	1	1	1
$W_{N\backslash\{1\}}^{+}$	1	1	1	0	0	1	1	0	0	1	1	1	1

- Construct weighted in-incidence matrix $B_{in}^{(W_H)}$ and out-incidence matrix $B_{out}^{(W_H)}$ for coalition H, based on the graph model Fig. 6.3, Definition 5.10, and Lemma 5.1;

- Calculate DM i's UM adjacency matrix and UI adjacency matrix

$$J_i = J_i^{(W^{(UM)})} = sign[(B_{out}^{(W_i^{(UM)})}) \cdot (B_{in}^{(W_i^{(UM)})})^T]$$

and

$$J_i^+ = J_i^{(W^+)} = sign[(B_{out}^{(W_i^+)}) \cdot (B_{in}^{(W_i^+)})^T]$$

for $i = 1, 2, 3, 4$, by Corollary 6.1;

- Calculate the UM reachability matrix and the UI reachability matrix by H

$$M_H = M_H^{(W^{(UM)})} = sign[(B_{out}^{(W_H^{(UM)})}) \cdot (LJ_{H_r} + I)^{l_1 - 1} \cdot (B_{in}^{(W_H^{(UM)})})^T]$$

and

$$M_H^+ = M_H^{(W^+)} = sign[(B_{out}^{(W_H^+)}) \cdot (LJ_{H_r}^+ + I)^{l_2 - 1} \cdot (B_{in}^{(W_H^+)})^T]$$

using Corollary 5.3, Corollary 6.2, and Theorem 5.2 for $l_1 = |A_H|$ and $l_2 = |A_H^+|$, where

$$LJ_{H_r} = [(B_{in}^{(W_H^{(UM)})})^T \cdot (B_{out}^{(W_H^{(UM)})})] \circ (E_l - D)$$

and

$$LJ_{H_r}^+ = [(B_{in}^{(W_H^+)})^T \cdot (B_{out}^{(W_H^+)})] \circ (E_l - D);$$

- Analyze the stabilities of Nash, GMR, SMR, and SEQ by Theorems 6.5 to 6.8 for the Rafferty-Alameda dams conflict.

In n-DM models, the UM and the UI reachability matrices of coalition H are important components in MRSC. We have shown the construction of the reachability matrices by a coalition. Next, we analyze the reachability matrices by H presented in Table 6.6.

Let us use an example to analyze the UM reachability matrix by H, M_H, and the UI reachability matrix by H, M_H^+. Using Table 6.6 with $H = N \backslash \{1\}$, we have

$$e_4^T \cdot M_H = (0, 1, 0, 0, 0, 1, 0, 1, 0, 0).$$

This means that the set of states, $R_H(s_4) = \{s_2, s_6, s_8\}$, can be reached by any legal UM sequence, by DMs in $H = \{2, 3, 4\}$, from the status quo $s = s_4$. Similarly,

$$e_4^T \cdot M_H^+ = (0, 0, 0, 0, 0, 0, 0, 1, 0, 0),$$

which denotes that $R_H^+(s_4) = \{s_8\}$ can be reached using the legal UI sequences, by $H = \{2, 3, 4\}$, from status quo $s = s_4$. It is obvious that if $R_H(s)$ and $R_H^+(s)$ are written as 0-1 row vectors, respectively, then

$$R_H(s) = e_s^T \cdot M_H, \quad R_H^+(s) = e_s^T \cdot M_H^+.$$

After determining all reachable matrices $M_{N\backslash\{i\}}$ and $M^+_{N\backslash\{i\}}$ for $i = 1, 2, 3, 4$, stability results of the Rafferty-Alameda dams conflict can be obtained using Theorems 6.5 to 6.8. The stable states and equilibria under the four solution concepts are summarized in Table 6.7, in which "$\surd$" for a given state means that this state is stable for a DM, F, S, G, or P; and "Eq" is an equilibrium for a appropriate solution concept. Additionally, Table 6.7 indicates that states s_9 and s_{10} are equilibria for the four basic solution concepts, so they are called ideal equilibria and are better choices for decision analysis.

6.2 Matrix Representation of Solution Concepts for Preference with Uncertainty

Explicit matrix representations of solution concepts in a graph model of a multiple-decision-maker conflict with preference uncertainty are developed in this section. In a graph model, the relative preferences of each DM over the available states are crucial in determining which states are stable according to any stability definition (solution concept). Unfortunately, it is often difficult to obtain accurate preference information in practical cases, so models that allow preference uncertainty can be very useful. In this work, stability definitions are extended to apply to graph models with this feature. The extension is easiest to implement using the matrix representation of a conflict model, which was developed to ease the coding of logically-defined stability definitions. Another benefit of matrix representation is that it facilitates modification and extension of the definitions.

6.2.1 Matrix Representation of Essential Components for Stabilities under Uncertain Preference

Recall that notation UUMs denotes unilateral uncertain moves and UIUUMs means unilateral improvements or unilateral uncertain moves.

Table 6.6: **UM and UI reachability matrices by $H = N\backslash\{i\}$ for the Rafferty-Alameda dams conflict**

Matrix	State	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	Matrix	State	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}
$M_{N\backslash\{1\}}$	s_1	0	1	0	1	0	1	0	1	0	0	$M^+_{N\backslash\{1\}}$	s_1	0	1	0	1	0	1	0	1	0	0
	s_2	0	0	0	1	0	1	0	1	0	0		s_2	0	0	0	1	0	1	0	1	0	0
	s_3	0	0	0	0	1	0	1	0	1	0		s_3	0	0	0	0	1	0	1	0	1	0
	s_4	0	1	0	0	0	1	0	1	0	0		s_4	0	0	0	0	0	0	0	1	0	0
	s_5	0	0	1	0	0	0	1	0	1	0		s_5	0	0	0	0	0	0	0	0	1	0
	s_6	0	0	0	0	0	0	0	1	0	0		s_6	0	0	0	0	0	0	0	1	0	0
	s_7	0	0	0	0	0	0	0	0	1	0		s_7	0	0	0	0	0	0	0	0	1	0
	s_8	0	0	0	0	0	1	0	0	0	0		s_8	0	0	0	0	0	0	0	0	0	0
	s_9	0	0	0	0	0	0	1	0	0	0		s_9	0	0	0	0	0	0	0	0	0	0
	s_{10}	0	0	0	0	0	0	0	0	0	0		s_{10}	0	0	0	0	0	0	0	0	0	0
$M_{N\backslash\{2\}}$	s_1	0	0	0	0	0	0	0	0	0	0	$M^+_{N\backslash\{2\}}$	s_1	0	0	0	0	0	0	0	0	0	0
	s_2	0	1	1	1	1	1	1	1	1	1		s_2	0	0	1	1	1	1	1	1	1	0
	s_3	0	1	1	1	1	1	1	1	1	1		s_3	0	0	0	0	1	0	1	0	1	0
	s_4	0	1	1	1	1	1	1	1	1	1		s_4	0	0	0	0	1	0	0	1	1	0
	s_5	0	1	1	1	1	1	1	1	1	1		s_5	0	0	0	0	0	0	0	0	1	0
	s_6	0	0	0	0	0	1	1	1	1	1		s_6	0	0	0	0	0	0	1	1	1	0
	s_7	0	0	0	0	0	1	1	1	1	1		s_7	0	0	0	0	0	0	0	0	1	0
	s_8	0	0	0	0	0	1	1	1	1	1		s_8	0	0	0	0	0	0	0	0	1	0
	s_9	0	0	0	0	0	1	1	1	1	1		s_9	0	0	0	0	0	0	0	0	0	0
	s_{10}	0	0	0	0	0	0	0	0	0	0		s_{10}	0	0	0	0	0	0	0	0	0	0
$M_{N\backslash\{3\}}$	s_1	0	1	1	0	0	1	1	0	0	1	$M^+_{N\backslash\{3\}}$	s_1	0	1	1	0	0	1	1	0	0	0
	s_2	0	0	1	0	0	1	1	0	0	1		s_2	0	0	1	0	0	1	1	0	0	0
	s_3	0	1	0	0	0	1	1	0	0	1		s_3	0	0	0	0	0	0	1	0	0	0
	s_4	0	0	0	0	1	0	0	1	1	1		s_4	0	0	0	0	1	0	0	1	1	0
	s_5	0	0	0	1	0	0	0	1	1	1		s_5	0	0	0	0	0	0	0	0	1	0
	s_6	0	0	0	0	0	0	1	0	0	1		s_6	0	0	0	0	0	0	1	0	0	0
	s_7	0	0	0	0	0	1	0	0	0	1		s_7	0	0	0	0	0	0	0	0	0	0
	s_8	0	0	0	0	0	0	0	0	1	1		s_8	0	0	0	0	0	0	0	0	1	0
	s_9	0	0	0	0	0	0	0	1	0	1		s_9	0	0	0	0	0	0	0	0	0	0
	s_{10}	0	0	0	0	0	0	0	0	0	0		s_{10}	0	0	0	0	0	0	0	0	0	0
$M_{N\backslash\{4\}}$	s_1	0	1	1	1	1	0	0	0	0	1	$M^+_{N\backslash\{4\}}$	s_1	0	1	1	1	1	0	0	0	0	0
	s_2	0	1	1	1	1	0	0	0	0	1		s_2	0	0	1	1	1	0	0	0	0	0
	s_3	0	1	1	1	1	0	0	0	0	1		s_3	0	0	0	0	1	0	0	0	0	0
	s_4	0	1	1	1	1	0	0	0	0	1		s_4	0	0	0	0	1	0	0	0	0	0
	s_5	0	1	1	1	1	0	0	0	0	1		s_5	0	0	0	0	0	0	0	0	0	0
	s_6	0	0	0	0	0	1	1	1	1	1		s_6	0	0	0	0	0	0	1	1	1	0
	s_7	0	0	0	0	0	1	1	1	1	1		s_7	0	0	0	0	0	0	0	0	1	0
	s_8	0	0	0	0	0	1	1	1	1	1		s_8	0	0	0	0	0	0	0	0	1	0
	s_9	0	0	0	0	0	1	1	1	1	1		s_9	0	0	0	0	0	0	0	0	0	0
	s_{10}	0	0	0	0	0	0	0	0	0	0		s_{10}	0	0	0	0	0	0	0	0	0	0

Table 6.7: **Stability results of the Rafferty-Alameda dams conflict**

State Number	Nash					GMR					SMR					SEQ				
	F	S	G	P	Eq	F	S	G	P	Eq	F	S	G	P	Eq	F	S	G	P	Eq
s_1	√		√	√		√		√	√		√		√	√		√		√	√	
s_2		√				√	√				√	√				√	√			
s_3	√	√				√	√				√	√				√	√			
s_4		√	√			√	√	√			√	√	√			√	√	√		
s_5	√	√	√			√	√	√			√	√	√			√	√	√		
s_6		√		√			√		√			√		√			√		√	
s_7	√	√		√		√	√		√		√	√		√		√	√		√	
s_8		√	√	√			√	√	√			√	√	√			√	√	√	
s_9	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√
s_{10}	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√

Definition 6.3. *For the graph model $G = (S, A)$, the **UUM adjacency matrix** $J_i^{(U)}$ for DM i is an $m \times m$ matrix with (s, q) entry*

$$J_i^{(U)}(s,q) = \begin{cases} 1 & \text{if } (s,q) \in A_i^U, \\ 0 & \text{otherwise,} \end{cases}$$

where $s, q \in S$ and $A_i^U = \{(s,q) \in A_i : sU_iq\}$.

Note that $J_i^{+,U} = J_i^+ \vee J_i^U$. Then, $J_i^{+,U}$ is called UIUUM adjacency matrix for DM i. Recall that DM i's reachable list $R_i^{+,U}(s)$ from state s by a UIUUM is expressed as $R_i^{+,U}(s) = \{q : J_i^{+,U}(s,q) = 1\}$. From Theorem 5.3 and Definition 6.3, the following result is obtained.

Corollary 6.3. *For the graph model $G = (S, A)$, the UIUUM adjacency matrix of DM i can be expressed as*

$$J_i^{+,U} = J_i^{(W^{(UIUUM)})} = sign[(B_{out}^{(W_i^{+,U})}) \cdot (B_{in}^{(W_i^{+,U})})^T].$$

Recall that $R_H^{+,U}(s)$ denotes the reachable list of H from state s by the legal sequence of UIUUMs.

Definition 6.4. *For the graph model $G = (S, A)$, the UIUUM reachability matrix of coalition $H \subseteq N$ is an $m \times m$ matrix $M_H^{+,U}$ with (s, q) entry*

$$M_H^{+,U}(s,q) = \begin{cases} 1 & \text{if } q \in R_H^{+,U}(s) \text{ for } q \in S, \\ 0 & \text{otherwise.} \end{cases}$$

From Theorem 5.4, Corollary 5.6, and Definition 6.4, the following result is obvious.

Corollary 6.4. *For the graph model $G = (S, A)$, the UIUUM reachability matrix of coalition $H \subseteq N$ can be expressed by*

$$M_H^{+,U} = M_H^{(W^{+,U})} = sign[(B_{out}^{(W_H^{+,U})}) \cdot (LJ_{H_r}^{+,U} + I)^{l_3 - 1} \cdot (B_{in}^{(W_H^{+,U})})^T],$$

where $l_3 = |A_H^{+,U}|$.

The preference matrices P_i^+, P_i^-, and $P_i^=$, for DM i are defined in Subsection 6.1.1. Matrices P_i^U and $P_i^{+,U}$ are defined next.

$$P_i^U(s, q) = \begin{cases} 1 & if\ sU_iq, \\ 0 & otherwise, \end{cases} \quad and\ P_i^{+,U} = P_i^+ \vee P_i^U.$$

Then matrices $P_i^{-,=}$, $P_i^{+,U}$, and $P_i^{-,=,U}$ respectively denote

$$P_i^{-,=} = P_i^- \vee P_i^=,\ P_i^{+,U}(s, q) = E - I - P_i^{-,=},\ and\ P_i^{-,=,U}(s, q) = E - I - P_i^+.$$

Consequently, the relations among the UM adjacency matrix, UI adjacency matrix, UIUUM adjacency matrix, and preference matrices including uncertainty, are established as follows:

$$J_i^+ = J_i \circ P_i^+, \quad and \quad J_i^{+,U} = J_i \circ P_i^{+,U}.$$

Based on the extended preference structure (including uncertainty), Li et al. [46] defined Nash stability, GMR, SMR, and SEQ to capture a DM's incentives to leave the status quo state and sensitivity to sanctions. Four types of stability definition were proposed, indexed a, b, c, and d, according to whether the DM would move to a state of uncertain preference and whether the DM would be sanctioned by a responding move to a state of uncertain preference, relative to the status quo. This range of extensions is needed, according to [46], to address the diversity of possible risk profiles in face of uncertainty. A DM may be conservative or aggressive, avoiding or accepting states of uncertain preference, depending on the level of satisfaction with the current position.

Like all previous stability definitions in the graph model, the four extensions were defined logically, in terms of the underlying graphs. Thus, as has been observed previously, procedures to identify stable states based on these definitions are difficult to code because of the nature of the logical representations. To overcome this limitation, the four stability definitions in multiple-decision-maker graph models with preference uncertainty are formulated explicitly in terms of matrices in the next section.

6.2.2 Matrix Representation of Solution Concepts for Two-DMs with Preference Uncertainty

Matrix representation of the four extensions of Nash, GMR, SMR, and SEQ stability definitions with preference uncertainty (MRSCU) in 2-DM conflict models is developed in this section. The system, called the MRSCU method, incorporated a set of $m \times m$ stability matrices, $M_i^{GMR_l}$, $M_i^{SMR_l}$, and $M_i^{SEQ_l}$, for $l \in Q = \{a, b, c, d\}$, to capture GMR$_l$, SMR$_l$, and SEQ$_l$ stability for DM $i \in N$, where $|N| = 2$, $m = |S|$, and DMs' preferences may include uncertainty. Theses stability matrices for two-DM models with preference uncertainty are summarized in Table 6.8.

Since the following four theorems are special cases of the theorems developed in the next subsection, the details are not given here. However, we note the following theorems, proven in [66]. Let $N = \{i, j\}$ and $l \in Q$. Then

Theorem 6.11. *State $s \in S$ is $Nash_a$ or $Nash_c$ stable for DM i iff $e_s \cdot J_i^{+,U} \cdot e = 0$; and state $s \in S$ is $Nash_b$ or $Nash_d$ stable for DM i iff $e_s \cdot J_i^{+} \cdot e = 0$.*

Theorem 6.12. *State $s \in S$ is GMR_l for DM i iff $M_i^{GMR_l}(s,s) = 0$, $l \in Q$.*

Theorem 6.13. *State $s \in S$ is SMR_l for DM i iff $M_i^{SMR_l}(s,s) = 0$, $l \in Q$.*

Theorem 6.14. *State $s \in S$ is SEQ_l for DM i iff $M_i^{SEQ_l}(s,s) = 0$, $l \in Q$.*

These theorems prove that the proposed matrix representation of solution concepts are equivalent to the solution concepts for two DM conflicts defined by Li et al. [46]. The matrix representation can be extended to models including more than two DMs, which is the objective of the next subsection.

6.2.3 Matrix Representation of Solution Concepts for n-DMs with Preference Uncertainty

In an n-DM model, where $n > 2$, the opponents of a DM can be thought of as a coalition of two or more DMs. To calculate the stability of a state for DM $i \in N$, it is necessary to examine possible responses by all other DMs $j \in N\backslash\{i\}$, which may include sequential responses. To extend the graph model stability definitions to stability definitions in n-DM models with preference uncertainty, the definitions of a legal sequence of decisions [16] must first be extended to take preference uncertainty into account [46].

Table 6.8: **The construction of stability matrices for two-DMs with preference uncertainty**

<table>
<tr><th>Preference</th><th>Sets of definitions</th><th>Stability matrices</th></tr>
<tr><td rowspan="16">Including uncertainty</td><td rowspan="4">a</td><td>$M_i^{Nash_a} = J_i^{+,U}$</td></tr>
<tr><td>$M_i^{GMR_a} = J_i^{+,U} \cdot [E - sign\left(J_j \cdot (P_i^{-,=})^T\right)]$</td></tr>
<tr><td>$M_i^{SMR_a} = J_i^{+,U} \cdot [E - sign(J_j \cdot G)]$, with
$G = (P_i^{-,=})^T \circ [E - sign\big(J_i \cdot (P_i^{+,U})^T\big)]$</td></tr>
<tr><td>$M_i^{SEQ_a} = J_i^{+,U} \cdot [E - sign\left(J_j^{+,U} \cdot (P_i^{-,=})^T\right)]$</td></tr>
<tr><td rowspan="4">b</td><td>$M_i^{Nash_b} = J_i^{+}$</td></tr>
<tr><td>$M_i^{GMR_b} = J_i^{+} \cdot [E - sign\left(J_j \cdot (P_i^{-,=})^T\right)]$</td></tr>
<tr><td>$M_i^{SMR_b} = J_i^{+} \cdot [E - sign(J_j \cdot G)]$, with
$G = (P_i^{-,=})^T \circ [E - sign\big(J_i \cdot (P_i^{+})^T\big)]$</td></tr>
<tr><td>$M_i^{SEQ_b} = J_i^{+} \cdot [E - sign\left(J_j^{+,U} \cdot (P_i^{-,=})^T\right)]$</td></tr>
<tr><td rowspan="4">c</td><td>$M_i^{Nash_c} = J_i^{+,U}$</td></tr>
<tr><td>$M_i^{GMR_c} = J_i^{+,U} \cdot [E - sign\left(J_j \cdot (P_i^{-,=,U})^T\right)]$</td></tr>
<tr><td>$M_i^{SMR_c} = J_i^{+,U} \cdot [E - sign(J_j \cdot G)]$, with
$G = (P_i^{-,=,U})^T \circ [E - sign\left(J_i \cdot (P_i^{+})^T\right)]$</td></tr>
<tr><td>$M_i^{SEQ_c} = J_i^{+,U} \cdot [E - sign\left(J_j^{+,U} \cdot (P_i^{-,=,U})^T\right)]$</td></tr>
<tr><td rowspan="4">d</td><td>$M_i^{Nash_d} = J_i^{+}$</td></tr>
<tr><td>$M_i^{GMR_d} = J_i^{+} \cdot [E - sign\left(J_j \cdot (P_i^{-,=,U})^T\right)]$</td></tr>
<tr><td>$M_i^{SMR_d} = J_i^{+} \cdot [E - sign(J_j \cdot G)]$, with
$G = (P_i^{-,=,U})^T \circ [E - sign\left(J_i \cdot (P_i^{+})^T\right)]$</td></tr>
<tr><td>$M_i^{SEQ_d} = J_i^{+} \cdot [E - sign\left(J_j^{+,U} \cdot (P_i^{-,=,U})^T\right)]$</td></tr>
</table>

The definitions of Nash stability, GMR, SMR, and SEQ in the graph model for multiple-decision-maker conflict models with preference uncertainty are described in [46]. They retain most features of the stability definitions in the 2-DM case, except that DM i's opponents are a subset of N, $N\backslash\{i\}$, instead of a single opponent, j. It is obvious that in the n-DM case, the algebraic characterizations of stabilities are similar to those presented in Section 6.2.2. Consequently, matrix representation of solution concepts with preference uncertainty for 2-DM cases is easy to extend to that for n-DM situations.

6.2.3.1 Matrix Representation of Stabilities, Index a, for Preference with Uncertainty

In the definitions indexed a, DM i has an incentive to move to states with uncertain preferences relative to the status quo, but, when assessing possible sanctions, will not consider states with uncertain preferences [46]. Let $i \in N$ and $|N| = n$ in the following theorems.

Theorem 6.15. *State $s \in S$ is $Nash_a$ stable for DM i iff $e_s \cdot J_i^{+,U} \cdot e = 0$.*

Theorem 6.15 implies that Nash stability definitions are identical for both 2-DM and n-DM models with preference uncertainty because Nash stability does not consider opponents' responses.

For GMR, DM i considers the opponents' responses, which are reachable states $R_{N\backslash\{i\}}$ of coalition $H = N\backslash\{i\}$ by the legal UM sequences. First, we find matrix $M_{N\backslash\{i\}}$ using Corollary 6.2 with $H = N\backslash\{i\}$. Define the $m \times m$ GMR_a stability matrix by

$$M_i^{GMR_a} = J_i^{+,U} \cdot [E - sign\left(M_{N\backslash\{i\}} \cdot (P_i^{-,=})^T\right)].$$

Then the following theorem provides a matrix method to calculate GMR_a stability.

Theorem 6.16. *State $s \in S$ is GMR_a for DM i, denoted by $s \in S_i^{GMR_a}$, iff $M_i^{GMR_a}(s,s) = 0$.*

Proof: Since the diagonal entry of matrix $M_i^{GMR_a}$

$$M_i^{GMR_a}(s,s) = (e_s^T \cdot J_i^{+,U}) \cdot [\left(E - sign\left(M_{N\backslash\{i\}} \cdot (P_i^{-,=})^T\right)\right) \cdot e_s]$$

$$= \sum_{s_1=1}^{m} J_i^{+,U}(s,s_1)[1 - sign\left((e_{s_1}^T \cdot M_{N\backslash\{i\}}) \cdot (e_s^T \cdot P_i^{-,=})^T\right)],$$

then $M_i^{GMR_a}(s,s) = 0$ iff $J_i^{+,U}(s,s_1)[1 - sign\left((e_{s_1}^T \cdot M_{N\backslash\{i\}}) \cdot (e_s^T \cdot P_i^{-,=})^T\right)] = 0$ for any $s_1 \in S$. This implies that $M_i^{GMR_a}(s,s) = 0$ iff

$$(e_{s_1}^T \cdot M_{N\backslash\{i\}}) \cdot (e_s^T \cdot P_i^{-,=})^T \neq 0 \; for \; any \; s_1 \in R_i^{+,U}(s). \tag{6.10}$$

By statement (6.10), for any $s_1 \in R_i^{+,U}(s)$, there exists $s_2 \in S$, such that the m-dimensional row vector, $e_{s_1}^T \cdot M_{N\backslash\{i\}}$, has s_2^{th} element 1 and the m-dimensional column vector, $(P_i^{-,=})^T \cdot e_s$, has s_2^{th} element 1.

Therefore, $M_i^{GMR_a}(s,s) = 0$ iff for any $s_1 \in R_i^{+,U}(s)$, there exists at least one $s_2 \in R_{N\backslash\{i\}}(s_1)$ with $s \succeq_i s_2$. □

Symmetric metarationality in the n-DM model is similar to in the 2-DM model. The only modification is that responses from DM i's opponents instead of a single DM. Let $D = (P_i^{-,=})^T \circ [E - sign\left(J_i \cdot (P_i^{+,U})^T\right)]$, then define the $m \times m$ SMR_a stability matrix by

$$M_i^{SMR_a} = J_i^{+,U} \cdot [E - sign(M_{N\backslash\{i\}} \cdot D)].$$

Thus, the following theorem provides a matrix method to calculate SMR_a stability.

Theorem 6.17. *State $s \in S$ is SMR_a for DM i, denoted by $s \in S_i^{SMR_a}$, iff $M_i^{SMR_a}(s,s) = 0$.*

Proof: Let $G = M_{N\backslash\{i\}} \cdot D$. Since the diagonal element of matrix $M_i^{SMR_a}$

$$\begin{aligned} M_i^{SMR_a}(s,s) &= (e_s^T \cdot J_i^{+,U}) \cdot [(E - sign(G)) \cdot e_s] \\ &= \sum_{s_1=1}^{m} J_i^{+,U}(s,s_1)[1 - sign\,(G(s_1,s))] \end{aligned}$$

with

$$G(s_1,s) = \sum_{s_2=1}^{m} M_{N\backslash\{i\}}(s_1,s_2) \cdot P_i^{-,=}(s,s_2)[1 - sign\left(\sum_{s_3=1}^{m}\left(J_i(s_2,s_3)P_i^{+,U}(s,s_3)\right)\right)],$$

thus, $M_i^{SMR_a}(s,s) = 0$ holds iff $G(s_1,s) \neq 0$ for any $s_1 \in R_i^{+,U}(s)$, which is equivalent to the statement that, for any $s_1 \in R_i^{+,U}(s)$ there exists $s_2 \in R_{N\backslash\{i\}}(s_1)$ such that

$$P_i^{-,=}(s,s_2) \neq 0, \quad and \quad \sum_{s_3=1}^{m}\left(J_i(s_2,s_3)P_i^{+,U}(s,s_3)\right) = 0. \tag{6.11}$$

Obviously, for any $s_1 \in R_i^{+,U}(s)$ there exists $s_2 \in R_{N\backslash\{i\}}(s_1)$ such that statement (6.11) holds iff for every $s_1 \in R_i^{+,U}(s)$ there exists $s_2 \in R_{N\backslash\{i\}}(s_1)$ such that $s \succeq_i s_2$ and $s \succeq_i s_3$ for all $s_3 \in R_i(s_2)$. $\square$

SEQ is similar to GMR, but includes only those sanctions that are "credible" (unilaterally improved) or uncertain moves, i.e., SEQ examines the credibility and uncertainty of the sanctions by DM i's opponents. First, we find matrix $M_{N\backslash\{i\}}^{+,U}$ using Corollary 6.4. Define the $m \times m$ SEQ_a stability matrix $M_i^{SEQ_a}$ by

$$M_i^{SEQ_a} = J_i^{+,U} \cdot [E - sign\left(M_{N\backslash\{i\}}^{+,U} \cdot (P_i^{-,=})^T\right)].$$

Thus the following theorem provides a matrix method to calculate SEQ_a stability.

Theorem 6.18. *State $s \in S$ is SEQ_a for DM i, denoted by $s \in S_i^{SEQ_a}$, iff $M_i^{SEQ_a}(s,s) = 0$.*

Proof: Since the diagonal element of matrix $M_i^{SEQ_a}$

$$M_i^{SEQ_a}(s,s) = (e_s^T \cdot J_i^{+,U}) \cdot [\left(E - sign\left(M_{N\backslash\{i\}}^{+,U} \cdot (P_i^{-,=})^T\right)\right) \cdot e_s]$$

$$= \sum_{s_1=1}^{m} J_i^{+,U}(s,s_1)[1 - sign\left((e_{s_1}^T \cdot M_{N\backslash\{i\}}^{+,U}) \cdot (e_s^T \cdot P_i^{-,=})^T\right)],$$

then $M_i^{SEQ_a}(s,s) = 0$ iff $J_i^{+,U}(s,s_1)[1 - sign\left((e_{s_1}^T \cdot M_{N\backslash\{i\}}^{+,U}) \cdot (e_s^T \cdot P_i^{-,=})^T\right)] = 0$ for any $s_1 \in S$. This implies that $M_i^{SEQ_a}(s,s) = 0$ iff

$$(e_{s_1}^T \cdot M_{N\backslash\{i\}}^{+,U}) \cdot (e_s^T \cdot P_i^{-,=})^T \neq 0 \ for\ any\ s_1 \in R_i^{+,U}(s). \tag{6.12}$$

By statement (6.12), for any $s_1 \in R_i^{+,U}(s)$, there exists $s_2 \in S$ such that the m-dimensional row vector, $e_{s_1}^T \cdot M_{N\backslash\{i\}}^{+,U}$, has s_2^{th} element 1 and the m-dimensional column vector, $(P_i^{-,=})^T \cdot e_s$, has s_2^{th} element 1.

Therefore, $M_i^{SEQ_a}(s,s) = 0$ iff for any $s_1 \in R_i^{+,U}(s)$, there exists at least one $s_2 \in R_{N\backslash\{i\}}^{+,U}(s_1)$ with $s \succeq_i s_2$. $\square$

$Nahs_a$ stability means that the focal DM has no unilateral improvements or unilateral uncertain moves (UIUUMs). GMR_a denotes that the UIUUMs of the focal DM are sanctioned by subsequent unilateral moves by the opponents of the focal DM. SMR_s is similar to GMR_a, but the focal DM considers not only the responses from his opponents but also his own counterresponses. SEQ_a indicts that UIUUMs of the focal DM are sanctioned by subsequent unilateral improvements or unilateral uncertain moves by the opponents of the focal DM.

6.2.3.2 Matrix Representation of Stabilities, Index b, for Preference with Uncertainty

For the next set of definitions indexed b, DM i considers leaving a state or assessing sanctions, excluding uncertain preferences [46]. However, the definitions are different from those for simple preference [16], since the current definitions are utilized to analyze conflict models including preference uncertainty. The following theorems can be similarly verified as the above theorems.

Theorem 6.19. *State $s \in S$ is $Nash_b$ stable for DM i iff $e_s \cdot J_i^{+} \cdot e = 0$.*

Define the $m \times m$ stability matrix $M_i^{GMR_b}$ by

$$M_i^{GMR_b} = J_i^{+} \cdot [E - sign\left(M_{N\backslash\{i\}} \cdot (P_i^{-,=})^T\right)].$$

Theorem 6.20. *State $s \in S$ is GMR_b for DM i iff $M_i^{GMR_b}(s,s) = 0$.*

Define the $m \times m$ stability matrix $M_i^{SMR_b} = J_i^{+} \cdot [E - sign(G)]$, with

$$G = M_{N\backslash\{i\}} \cdot [(P_i^{-,=})^T \circ \left(E - sign\left(J_i \cdot (P_i^{+,U})^T\right)\right)].$$

Theorem 6.21. *State $s \in S$ is SMR_b for DM i iff $M_i^{SMR_b}(s,s) = 0$.*

Define the $m \times m$ stability matrix $M_i^{SEQ_b}$ by

$$M_i^{SEQ_b} = J_i^{+} \cdot [E - sign\left(M_{N\backslash\{i\}}^{+,U} \cdot (P_i^{-,=})^T\right)].$$

Theorem 6.22. *State $s \in S$ is SEQ_b for DM i iff $M_i^{SEQ_b}(s,s) = 0$.*

6.2.3.3 Matrix Representation of Stabilities, Index c, for Preference with Uncertainty

For the extended definitions indexed c, DM i considers moving from a status quo state or evaluating sanctions including uncertain preferences.

Theorem 6.23. *State $s \in S$ is $Nash_c$ stable for DM i iff $e_s \cdot J_i^{+,U} \cdot e = 0$.*

Define the $m \times m$ stability matrix $M_i^{GMR_c}$ by

$$M_i^{GMR_c} = J_i^{+,U} \cdot [E - sign\left(M_{N\backslash\{i\}} \cdot (P_i^{-,=,U})^T\right)].$$

Theorem 6.24. *State $s \in S$ is GMR_c for DM i iff $M_i^{GMR_c}(s,s) = 0$.*

Define the $m \times m$ stability matrix $M_i^{SMR_c}$ by

$$M_i^{SMR_c} = J_i^{+,U} \cdot [E - sign(M_{N\backslash\{i\}} \cdot D)],$$

in which

$$D = (P_i^{-,=,U})^T \circ [E - sign\left(J_i \cdot (P_i^+)^T\right)].$$

Theorem 6.25. *State $s \in S$ is SMR_c for DM i iff $M_i^{SMR_c}(s,s) = 0$.*

Define the $m \times m$ stability matrix $M_i^{SEQ_c}$ by

$$M_i^{SEQ_c} = J_i^{+,U} \cdot [E - sign\left(M_{N\backslash\{i\}}^{+,U} \cdot (P_i^{-,=,U})^T\right)].$$

Theorem 6.26. *State $s \in S$ is SEQ_c for DM i iff $M_i^{SEQ_c}(s,s) = 0$.*

6.2.3.4 Matrix Representation of Stabilities, Index d, for Preference with Uncertainty

For the last definitions, indexed d, a DM is not motivated to leave the status quo to move to states with uncertain preference, but will consider moving to states with uncertain preference to be a sanction.

Theorem 6.27. *State $s \in S$ is $Nash_d$ stable for DM i iff $e_s \cdot J_i^+ \cdot e = 0$.*

Define the $m \times m$ stability matrix

$$M_i^{GMR_d} = J_i^+ \cdot [E - sign\left(M_{N\backslash\{i\}} \cdot (P_i^{-,=,U})^T\right)].$$

Theorem 6.28. *State $s \in S$ is GMR_d for DM i iff $M_i^{GMR_d}(s,s) = 0$.*

Define the $m \times m$ stability matrix $M_i^{SMR_d} = J_i^+ \cdot [E - sign(M_{N\backslash\{i\}} \cdot D)]$, in which $D = (P_i^{-,=,U})^T \circ [E - sign\left(J_i \cdot (P_i^+)^T\right)]$.

Theorem 6.29. *State $s \in S$ is SMR_d for DM i iff $M_i^{SMR_d}(s,s) = 0$.*

Define the $m \times m$ stability matrix $M_i^{SEQ_d} = J_i^+ \cdot [E - sign\left(M_{N\backslash\{i\}}^{+,U} \cdot (P_i^{-,=,U})^T\right)]$.

Theorem 6.30. *State $s \in S$ is SEQ_d for DM i iff $M_i^{SEQ_d}(s,s) = 0$.*

When $n = 2$, the DM set N becomes to $\{i, j\}$ and Theorems 6.15 to 6.30 are reduced to Theorems 6.11 to 6.14.

From the matrix representation of solution concepts indexed a, b, c, and d presented above, it can be seen that a solution concept indexed a represents the stability for the most aggressive DMs. Firstly, the DM is aggressive in deciding whether to move from the status quo, being willing to accept the risk associated with moves to states of uncertain preference. In addition, when evaluating possible moves, the DM is deterred only by sanctions to states that are less preferred than the status quo and does not see states of uncertain preference (relative to the status quo) as sanctions. For the definitions indexed b, uncertainty in preferences is not considered by a DM. The definitions indexed c incorporate a mixed attitude toward the risk associated with states of uncertain preference. Specifically, the DM is aggressive in deciding whether to move from the status quo, but is conservative when evaluating possible moves, being deterred by sanctions to states that are less preferred or have uncertain preference relative to the status quo. Finally, the definition indexed d represents stability for the most conservative DMs, who would move only to preferred states from a status quo, but would be deterred by responses that result in states of uncertain preference.

6.2.4 Applications including Preference Uncertainty

6.2.4.1 Sustainable Development Game

Table 6.9: **Options and feasible states for the sustainable development conflict** [31]

E: environmentalists				
1. Proactive (labeled P)	Y	Y	N	Y
2. Reactive (labeled R)	N	N	Y	Y
D: developers				
3. Sustainable development (labeled S)	Y	N	Y	N
4. Unsustainable development (labeled U)	N	Y	N	Y
State number	s_1	s_2	s_3	s_4

A two-DM conflict model with preference uncertainty is used to illustrate how stability analysis is carried out by MRSCU. Hipel [31] developed a model for a conflict over sustainable development game that was also studied by Li et al. [46]. Specifically, the conflict consists of two DMs: **environmental agencies** (DM 1: E) and **developers** (DM 2: D); and a total of four options: DM 1 controls the two options of being proactive (labeled P) and being reactive (labeled R) in monitoring developers' activities and their impacts on the environment, and DM 2 has the two options of practicing sustainable development (labeled S) and practicing unsustainable development (labeled U) for properly treating the environment. These options are combined to form four feasible states: s_1: PS, s_2: PU, s_3: RS, and s_4: RU. The four feasible states are listed in Table 6.9, where a "Y" indicates that an option is selected by the DM controlling it and an "N" means that the option is not chosen [31, 46].

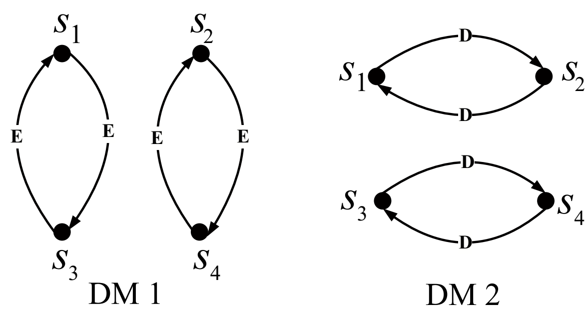


Figure 6.4: **Graph model for the sustainable development conflict [46].**

The graph model of the conflict is shown in Fig. 6.4. DM 1's preference information is provided by the cardinal preference function: $P_1 = (4, 2, 3, 1)$, but DM 2's preference includes uncertainty by $s_1U_2s_2$, $s_1U_2s_4$, $s_2U_2s_3$, $s_3U_2s_4$, $s_3 \succ_2 s_1$, and $s_4 \succ_2 s_2$.

The UM adjacency matrices for DM 1 and DM 2 are

$$J_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \textit{ and } J_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Preference matrices for the DM 1 and DM 2 are

$$P_1^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}, P_1^{-,=} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$P_2^+ = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \textit{ and } P_2^{+,U} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Additionally, P_1^U is a zero matrix, $P_2^{+,U} = P_2^+ \vee P_2^U$, $P_2^{-,=} = E - I - P_2^{+,U}$, $P_2^{-,=,U} = P_2^{-,=} \vee P_2^U$.

Hence, we can calculate the extended stabilities of Nash, GMR, SMR, and SEQ using Theorems 6.11 to 6.14 for the sustainable development conflict including preference uncertainty.

Table 6.10: **Stability results of the sustainable development game with uncertain preference**

State		Nash			GMR			SMR			SEQ		
		1	2	Eq	1	2	Eq	1	2	Eq	1	2	Eq
a	s_1	√			√			√			√		
	s_2	√			√			√			√		
	s_3				√			√			√		
	s_4												
b	s_1	√	√	√	√	√	√	√	√	√	√	√	√
	s_2	√	√	√	√	√	√	√	√	√	√	√	√
	s_3		√		√	√	√	√	√	√	√	√	√
	s_4		√			√			√			√	
c	s_1	√			√	√	√	√			√		
	s_2	√			√	√	√	√			√		
	s_3				√	√	√	√	√	√	√	√	√
	s_4					√			√			√	
d	s_1	√	√	√	√	√	√	√	√	√	√	√	√
	s_2	√	√	√	√	√	√	√	√	√	√	√	√
	s_3		√		√	√	√	√	√	√	√	√	√
	s_4		√			√			√			√	

Table 6.10 provides the stability results for the the sustainable development game calculated by MRSCU method for two-DM situations. They are precisely the same as the results presented in [46]. Obviously, states s_1 and s_2 are equilibria for the four stabilities indexed b and indexed d in the sustainable development conflict.

6.2.4.2 Lake Gisborne Conflict

The background of the Lake Gisborne conflict is described in Section 3.4. This conflict is modeled using three DMs: DM 1, **Federal (Fe)**; DM 2, **Provincial (Pr)**; and DM 3, **Support (Su)**. The graph model of the Lake Gisborne conflict is shown in Fig. 6.5 (1), where the labels on the arcs identify the DMs who control the relevant moves. If DM i's oriented arcs are coded in color i, then, according to the *Rule of Priority* introduced in Section 5.2, Fig. 6.5 (1) is converted to an edge labeled multidigraph as shown in Fig. 6.5 (2).

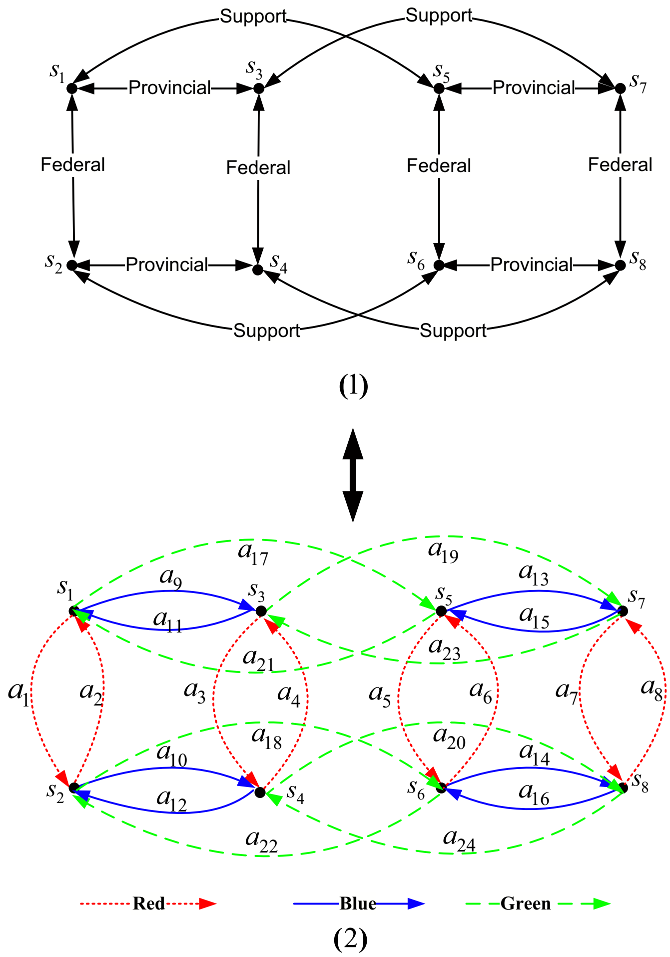


Figure 6.5: **Graph model for the Gisborne conflict.**

Since several groups support the project, the economics-oriented provincial government might consider supporting the project for the urgent need for cash. However, the environment-oriented provincial government might oppose this project because of the devastating consequences to the environment. The two different attitudes of the provincial government result in uncertainty in preferences for the Gisborne conflict model. The details can be found in [46]. Preference information over the states are given in Table 6.11, where $\succ$ represents the strict preference and is transitive. As shown in Table 6.11, DM **Federal**'s and DM **Support**'s preference information is modeled to be known completely without any uncertainty, but DM **Provincial**'s preference includes uncertainty. What is known is that it prefers state s_3 to state s_7, state s_4 to state

Table 6.11: **Preference information for the Gisborne conflict** [46]

Colors	DMs	Certain preferences
Red	Federal	$s_2 \succ s_6 \succ s_4 \succ s_8 \succ s_1 \succ s_5 \succ s_3 \succ s_7$
Blue	Provincial	$s_3 \succ s_7$, $s_4 \succ s_8$, $s_1 \succ s_5$, $s_2 \succ s_6$, only
Green	Support	$s_3 \succ s_4 \succ s_7 \succ s_8 \succ s_5 \succ s_6 \succ s_1 \succ s_2$

s_8, state s_1 to state s_5, and state s_2 to state s_6, but the relative preference across these four groups is uncertain.

Let $N = \{1, 2, 3\}$ denote the set of three DMs. We use the Gisborne conflict as an example to show the procedures using the MRSCU method.

- Construct preference matrices, P_i^+, $P_i^{+,U}$, and $P_i^{-,=}$, for $i = 1, 2, 3$, using information provided by Table 6.11, as well as $P^{-,=,U} = E - I - P_i^+$;

- Calculate the UM weight matrix and the UIUUM weight matrix of coalition H based on Definition 5.16, preference information presented in Table 6.11, and statement (5.7), i.e., if $a_k = d_i(u, v)$, then

$$w_k = \begin{cases} P_w & \text{if } v \succ_i u \text{ and } i \in H, \\ N_w & \text{if } u \succ_i v \text{ and } i \in H, \\ E_w & \text{if } u \sim_i v \text{ and } i \in H, \\ U_w & \text{if } uU_iv \text{ and } i \in H, \\ 0 & \text{otherwise.} \end{cases}$$

  Table 6.12 shows the process how to calculate matrices $W_H^{(UM)}$ and $W_H^{(+,U)}$ for $H = N\backslash\{1\}$;

- Construct weighted in-incidence matrix $B_{in}^{(W_H)}$ and out-incidence matrix $B_{out}^{(W_H)}$ for coalition H, based on the graph model and Definition 5.9;

- Calculate DM i's UM adjacency matrix and UIUUM adjacency matrix,

$$J_i = J_i^{(W^{(UM)})} = sign[(B_{out}^{(W_i^{(UM)})}) \cdot (B_{in}^{(W_i^{(UM)})})^T]$$

  and

$$J_i^{+,U} = J_i^{(W^{+,U})} = sign[(B_{out}^{(W_i^{+,U})}) \cdot (B_{in}^{(W_i^{+,U})})^T]$$

Table 6.12: **Weight matrices for $H = N\backslash\{1\}$ for the Gisborne conflict**

Arc number	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}	a_{18}	a_{19}	a_{20}	a_{21}	a_{22}	a_{23}	a_{24}
W	P_w	N_w	P_w	N_w	P_w	N_w	P_w	N_w	U_w	U_w	U_w	U_w	U_w	U_w	U_w	U_w	P_w	P_w	N_w	N_w	N_w	N_w	P_w	P_w
$W_H^{(UM)}$	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$W_H^{+,U}$	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	0	0	0	0	1	1

for $i = 1, 2, 3$, by Corollary 6.3;

- Calculate the UM reachability matrix and the UIUUM reachability matrix by H,

$$M_H = M_H^{(W^{(UM)})} = sign[(B_{out}^{(W_H^{(UM)})}) \cdot (LJ_{H_r} + I)^{l_1-1} \cdot (B_{in}^{(W_H^{(UM)})})^T]$$

and

$$M_H^{+,U} = M_H^{(W^{+,U})} = sign[(B_{out}^{(W_H^{+,U})}) \cdot (LJ_{H_r}^{+,U} + I)^{l_3-1} \cdot (B_{in}^{(W_H^{+,U})})^T],$$

using Corollary 5.6, Corollary 6.4, and Theorem 5.2 for $l_3 = |A_H^{+,U}|$, where

$$LJ_{H_r} = LJ_r^{(W_H^{(UM)})} = [(B_{in}^{(W_H^{(UM)})})^T \cdot (B_{out}^{(W_H^{(UM)})})] \circ (E_l - D)$$

and

$$LJ_{H_r}^{+,U} = LJ_r^{(W_H^{+,U})} = [(B_{in}^{(W_H^{+,U})})^T \cdot (B_{out}^{(W_H^{+,U})})] \circ (E_l - D);$$

- Analyze the stabilities of Nash, GMR, SMR, and SEQ by Theorems 6.15 to 6.30 for the Gisborne conflict.

Let the state set $S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$. Tables 6.13 and 6.14 show the results for the construction of the reachability matrices by $H = N\backslash\{i\}$ for $i = 1, 2, 3$. It is clear that if $R_H(s)$ and $R_H^{+,U}(s)$ are written as 0-1 row vectors, then

$$R_H(s) = e_s^T \cdot M_H, \text{ and } R_H^{+,U}(s) = e_s^T \cdot M_H^{+,U} \text{ for any } s \in S.$$

For example, using Table 6.14, we have $e_2^T \cdot M_{N\backslash\{1\}}^{+,U} = (0,1,0,1,0,1,0,1)$, which indicates that the reachable list of $N\backslash\{1\}$ by the legal UIUUMs from state s_2, $R_{N\backslash\{1\}}^{+,U}(s_2) = \{s_2, s_4, s_6, s_8\}$, i.e., states s_2, s_4, s_6, and s_8 can be reached by any legal UIUUM sequences, by the coalition consisting of DM **Provincial** and DM **Support**, from the status quo $s = s_2$. Consequently, the reachability matrices of coalition H provides an algebraic method for constructing the reachable lists of H by the legal UM and legal UIUUM sequences, $R_H(s)$ and $R_H^{+,U}(s)$ for all $s \in S$.

Table 6.13: **UM reachability matrices by $N\backslash\{i\}$ for $i = 1, 2, 3$ for the Gisborne conflict**

Matrix	$M_{N\backslash\{1\}}$								$M_{N\backslash\{2\}}$								$M_{N\backslash\{3\}}$							
State	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
s_1	1	0	1	0	1	0	1	0	1	1	0	0	1	1	0	0	1	1	1	1	0	0	0	0
s_2	0	1	0	1	0	1	0	1	1	1	0	0	1	1	0	0	1	1	1	1	0	0	0	0
s_3	1	0	1	0	1	0	1	0	0	0	1	1	0	0	1	1	1	1	1	1	0	0	0	0
s_4	0	1	0	1	0	1	0	1	0	0	1	1	0	0	1	1	1	1	1	1	0	0	0	0
s_5	1	0	1	0	1	0	1	0	1	1	0	0	1	1	0	0	0	0	0	0	1	1	1	1
s_6	0	1	0	1	0	1	0	1	1	1	0	0	1	1	0	0	0	0	0	0	1	1	1	1
s_7	1	0	1	0	1	0	1	0	0	0	1	1	0	0	1	1	0	0	0	0	1	1	1	1
s_8	0	1	0	1	0	1	0	1	0	0	1	1	0	0	1	1	0	0	0	0	1	1	1	1

Table 6.14: **UIUUM reachability matrices by $N\backslash\{i\}$ for $i = 1, 2, 3$ for the Gisborne model**

Matrix	$M_{N\backslash\{1\}}^{+,U}$								$M_{N\backslash\{2\}}^{+,U}$								$M_{N\backslash\{3\}}^{+,U}$							
State	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
s_1	1	0	1	0	1	0	1	0	0	1	0	0	1	1	0	0	0	1	1	1	0	0	0	0
s_2	0	1	0	1	0	1	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0
s_3	1	0	1	0	1	0	1	0	0	0	0	1	0	0	0	0	1	1	0	1	0	0	0	0
s_4	0	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
s_5	1	0	1	0	1	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1
s_6	0	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
s_7	1	0	1	0	1	0	1	0	0	0	1	1	0	0	0	1	0	0	0	0	1	1	0	1
s_8	0	1	0	1	0	1	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0

Next, Theorems 6.15 to 6.30 are used to calculate the stabilities of the Gisborne conflict. The stable states and equilibria under four distinct sets of definitions (indexed a, b, c, and d) and four solution concepts, Nash, GMR, SMR and SEQ, are summarized in Table 6.15, in which "$\surd$" for a given state means that this state is stable for a DM−Fe, Pr, or Su; and "Eq" is an equilibrium for a corresponding solution concept. Additionally, Table 6.15 indicates that states s_4 and s_6 are equilibria for the four solution concepts indexed by b and d. The stability results confirm the calculations of [46]. If the provincial government is economics-oriented and has complete preference information:

$$s_3 \succ s_7 \succ s_4 \succ s_8 \succ s_1 \succ s_5 \succ s_2 \succ s_6,$$

then the likely resolution is state s_4 by using DSS GMCR II [18, 19]. For an environment-oriented provincial government, with preferences

$$s_2 \succ s_6 \succ s_1 \succ s_5 \succ s_4 \succ s_8 \succ s_3 \succ s_7,$$

then state s_6 is the likely resolution. From Table 3.3, we can analyze the two likely resolutions. If the attitude of the provincial government is economics-oriented, then the provincial government will lift the ban on bulk water export. On the other hand, if the provincial government is strongly influenced by the federal government, then it will not lift the ban.

6.3 Matrix Representation of Solution Concepts for Strength of Preference

In this section, an algebraic approach is developed to calculate stabilities in two-DM and n-DM graph models with strength of preference [76, 77]. The original graph model uses "simple preference" to represent a DM's relative preference between two states. This preference structure includes only a relative preference relation and an indifference relation. Basic stability definitions, and algorithms to calculate them, assume simple preference. Due to difficulties in coding the algorithms, mainly because of their logical formulation, led to the development of matrix representations of preference and explicit matrix algorithms to calculate stability. Here, the algebraic approach is extended to representation of a strength-of-preference graph models, which feature multiple levels of preference, and stability analysis for such models.

Table 6.15: **Stability results of the Gisborne model**

State		Nash				GMR				SMR				SEQ			
		Fe	Pr	Su	Eq	Fe	Pr	Su	Eq	Fe	Pr	Su	Eq	Fe	Pr	Su	Eq
s_1	a																
	b		√				√				√				√		
	c						√				√				√		
	d		√				√				√				√		
s_2	a	√				√				√				√			
	b	√	√			√	√			√	√			√	√		
	c	√				√	√			√	√			√			
	d	√	√			√	√			√	√			√	√		
s_3	a			√				√				√				√	
	b		√	√			√	√			√	√			√	√	
	c			√			√	√			√	√			√	√	
	d		√	√			√	√			√	√			√	√	
s_4	a	√		√		√		√		√		√		√		√	
	b	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√
	c	√		√		√	√	√	√	√	√	√	√	√	√	√	√
	d	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√
s_5	a			√				√				√				√	
	b		√	√			√	√			√	√			√	√	
	c			√			√	√			√	√			√	√	
	d		√	√			√	√			√	√			√	√	
s_6	a	√		√		√		√		√		√		√		√	
	b	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√
	c	√		√		√	√	√	√	√	√	√	√	√	√	√	√
	d	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√
s_7	a							√				√				√	
	b		√				√	√			√	√			√	√	
	c						√	√			√	√			√	√	
	d		√				√	√			√	√			√	√	
s_8	a	√				√		√		√		√		√		√	
	b	√	√			√	√	√	√	√	√	√	√	√	√	√	√
	c	√				√	√	√	√	√	√	√	√	√		√	
	d	√	√			√	√	√	√	√	√	√	√	√	√	√	√

6.3.1 Matrix Representation of Essential Components for Stabilities with Strength of Preference

Recall that WIs denotes strong improvements or mild improvements called weak improvements.

Definition 6.5. *For the graph model $G = (S, A)$, the* ***WI adjacency matrix for DM*** *i* ***is an*** *$m \times m$* ***matrix*** *$J_i^{+,++}$ with (s, q) entry*

$$J_i^{+,++}(s,q) = \begin{cases} 1 & \text{if } (s,q) \in A_i^{+,++}, \\ 0 & \text{otherwise,} \end{cases}$$

where $s, q \in S$ and $A_i^{+,++} = \{(s,q) \in A_i : q >_i s \text{ or } q \gg_i s\}$.

DM i's reachable list $R_i^{+,++}(s)$ from state s by a WI can be expressed as $R_i^{+,++}(s) = \{q : J_i^{+,++}(s,q) = 1\}$. From Theorem 5.3 and Definition 6.5, the following result is obvious.

Corollary 6.5. *For the graph model $G = (S, A)$, the WI adjacency matrix of DM i satisfies that*

$$J_i^{+,++} = J_i^{(W^{(WI)})} = sign[(B_{out}^{(W_i^{+,++})}) \cdot (B_{in}^{(W_i^{+,++})})^T].$$

Recall that $R_H^{+,++}(s)$ denotes the reachable list of coalition H from state s by the legal sequence of WIs.

Definition 6.6. *Let $H \subseteq N$. For the graph model $G = (S, A)$, the WI reachability matrix by H is an $m \times m$ matrix $M_H^{+,++}$ with (s, q) entry*

$$M_H^{+,++}(s,q) = \begin{cases} 1 & \text{if } q \in R_H^{+,++}(s) \text{ for } q \in S, \\ 0 & \text{otherwise.} \end{cases}$$

The WI reachability matrix by H can be obtained from the following corollary based on Theorem 5.4, Corollary 5.9, and Definition 6.6.

Corollary 6.6. *For the graph model $G = (S, A)$, the WI reachable matrix by coalition H satisfies that*

$$M_H^{+,++} = M_H^{(W^{(WI)})} = sign[(B_{out}^{(W_H^{+,++})}) \cdot (LJ_{H_r}^{+,++} + I)^{l_4 - 1} \cdot (B_{in}^{(W_H^{+,++})})^T],$$

where $l_4 = |A_H^{+,++}|$.

To carry out stability analysis, a set of matrices corresponding to strength of preference is constructed next. Below, several matrices representing strength of preference for DM i are defined.

$$P_i^{++}(s,q) = \begin{cases} 1 & if\ q \gg_i s, \\ 0 & otherwise, \end{cases}$$

and

$$P_i^{--}(s,q) = \begin{cases} 1 & if\ s \gg_i q, \\ 0 & otherwise. \end{cases}$$

Therefore, $(P_i^{++})^T = P_i^{--}$, where T denotes the transpose of a matrix.

$$P_i^{--,-,=}(s,q) = \begin{cases} 1 & if\ q \in \Phi_i^{--,-,=}(s), \\ 0 & otherwise, \end{cases}$$

and

$$P_i^{+,++}(s,q) = \begin{cases} 1 & if\ q \gg_i s\ or\ q >_i s, \\ 0 & otherwise. \end{cases}$$

It follows that $P_i^{--,-,=}(s,q) = 1 - P_i^{+,++}(s,q)$ for $s,q \in S$ and $s \neq q$.

Based on the above definitions, for DM i, the UM matrix J_i, the WI matrix $J_i^{+,++}$, and the preference matrix $P_i^{+,++}$ have the relationship among them:

$$J_i^{+,++} = J_i \circ P_i^{+,++}.$$

6.3.2 Matrix Representation of Solution Concepts for Two-DMs with Strength of Preference

Stability definitions in the graph model are traditionally defined logically, in terms of the underlying graphs and preference relations. However, as was noted in the development of the DSS GMCR II, the nature of logical representations makes coding difficult. The work of [27, 28] integrated strength of preference information into these four solution concepts but, again, proved difficult to code and was never integrated into GMCR II.

Matrix representation of stabilities of Nash, GMR, SMR, and SEQ with strength of preference in two-DM conflicts is developed in this section. The system, called the MRSCS method, incorporated a set of $m \times m$ stability matrices, M_i^{GMR}, M_i^{SMR}, and M_i^{SEQ}, as well as strong stability matrices, M_i^{SGMR}, M_i^{SSMR}, and M_i^{SSEQ}, to capture GMR, SMR, and SEQ stabilities, as well as strong GMR, strong SMR, and strong SEQ stabilities for DM $i \in N$, where $|N| = 2$, $m = |S|$, and DMs' preferences may include strength.

Table 6.16: **The construction of stability matrices for two-DM models with strength of preference**

Types of stabilities	Stability matrices
Stabilities	$M_i^{Nash} = J_i^{+,++}$
	$M_i^{GMR} = J_i^{+,++} \cdot [E - sign\left(J_j \cdot (P_i^{--,-,=})^T\right)]$
	$M_i^{SMR} = J_i^{+,++} \cdot [E - sign(J_j \cdot G)]$, with $G = (P_i^{--,-,=})^T \circ [E - sign\left(J_i \cdot (P_i^{+,++})^T\right)]$
	$M_i^{SEQ} = J_i^{+,++} \cdot [E - sign\left(J_j^{+,++} \cdot (P_i^{--,-,=})^T\right)]$
Strong stabilities	$M_i^{SGMR} = J_i^{+,++} \cdot [E - sign\left(J_j \cdot (P_i^{--})^T\right)]$
	$M_i^{SSMR} = J_i^{+,++} \cdot [E - sign(J_j \cdot G)]$, with $G = (P_i^{++}) \circ [E - sign\left(J_i \cdot (E - P_i^{++})\right)]$
	$M_i^{SSEQ} = J_i^{+,++} \cdot [E - sign\left(J_j^{+,++} \cdot (P_i^{--})^T\right)]$

Let $N = \{i, j\}$. The stability matrices used by Theorems 6.31 to 6.37 are summarized in Table 6.16, which are utilized to calculate the extended stabilities of Nash, GMR, SMR, and SEQ, as well as the strong stabilities of SGMR, SSMR, and SSEQ, in two-DM conflicts for strength of preference, respectively.

It should be pointed out that the stability matrices for strength of preference use the same notation as the stability matrices for simple preference.

6.3.2.1 Matrix Representation of Stabilities

Theorem 6.31. *State s is Nash stable for DM i iff $e_s \cdot J_i^{+,++} \cdot e = 0$, where e denote the m-dimensional column vector with each element being set to 1.*

Theorem 6.32. *State s is GMR for DM i iff $M_i^{GMR}(s,s) = 0$.*

Theorem 6.33. *State s is SMR for DM i iff $M_i^{SMR}(s,s) = 0$.*

Theorem 6.34. *State s is SEQ for DM i iff $M_i^{SEQ}(s,s) = 0$.*

These theorems prove that the proposed matrix representation of solution concepts are equivalent to the standard solution concepts for two DM conflicts defined by Hamouda et al. [27].

6.3.2.2 Matrix Representation of Strong Stabilities

Theorem 6.35. *State $s \in S$ is strong GMR ($SGMR$) for DM i iff $M_i^{SGMR}(s,s) =$* 0.

Theorem 6.36. *State $s \in S$ is strongly SMR ($SSMR$) for DM i iff $M_i^{SSMR}(s,s) = 0$.*

Theorem 6.37. *State $s \in S$ is strongly SEQ ($SSEQ$) for DM i iff $M_i^{SSEQ}(s,s) = 0$.*

Since the seven theorems are special cases of the theorems for n-DM models developed in the next subsection, the details are not given here. These theorems prove that the proposed matrix representation of solution concepts are equivalent to logical representation of the strong stabilities for two DM conflicts [27]. The matrix representation can be extended to models including more than two DMs, which is the objective of the next subsection.

6.3.3 Matrix Representation of Solution Concepts for n-DMs with Strength of Preference

In an n-DM model, where $n > 2$, the opponents of a DM can be thought of as a coalition of two or more DMs. To calculate the stability of a state for DM $i \in N$, it is necessary to examine possible responses by all other DMs $j \in N \backslash \{i\}$, which may include responses by the legal sequences of UMs and WIs.

6.3.3.1 Matrix Representation of Stabilities

Four solution concepts are given below in which strength of preference is not considered in sanctioning. However, they are different from stabilities defined by Fang et al. [16], because the following stabilities can analyze conflict models having strength of preference. Let $i \in N$ and $s \in S$ for next theorems. A coalition is any subset H in DM set N. Let $i \in N$ and $|N| = n$.

Theorem 6.38. *State $s \in S$ is Nash stable for DM i, denoted by $s \in S_i^{Nash}$, iff $\langle e_s, J_i^{+,++} e \rangle = 0$, where $<,>$ denotes the inner product.*

Theorem 6.38 implies that Nash stability definitions are identical for both two-DM and n-DM models because Nash stability does not consider opponents' responses.

Define the $m \times m$ matrix M_i^{GMR} by

$$M_i^{GMR} = J_i^{+,++} \cdot [E - sign\left(M_{N \backslash \{i\}} \cdot (P_i^{--,-,=})^T\right)].$$

Theorem 6.39. *State s is GMR for DM i iff $M_i^{GMR}(s,s) = 0$.*

Define the $m \times m$ matrix M_i^{SMR} by $M_i^{SMR} = J_i^{+,++} \cdot [E - sign(G)]$, with

$$G = M_{N \backslash \{i\}} \cdot [(P_i^{--,-,=})^T \circ \left(E - sign\left(J_i \cdot (P_i^{+,++})^T\right)\right)].$$

Theorem 6.40. *State s is SMR for DM i iff $M_i^{SMR}(s,s) = 0$.*

Define the $m \times m$ matrix M_i^{SEQ} by

$$M_i^{SEQ} = J_i^{+,++} \cdot [E - sign\left(M_{N\backslash\{i\}}^{+,++} \cdot (P_i^{--,-,=})^T\right)].$$

Theorem 6.41. *State s is SEQ for DM i iff $M_i^{SEQ}(s,s) = 0$.*

The proofs of these theorems are similar to those for Theorems 6.5 to 6.8. Theorems 6.38 to 6.41 prove that the proposed matrix representation of solution concepts are equivalent to the standard stabilities for n-DM conflicts [28].

6.3.3.2 Matrix Representation of Strong Stabilities

With strength of preference introduced into the graph model, stability definitions can be strong or weak, according to the level of sanctioning. Strong and weak stabilities only include GMR, SMR, and SEQ because Nash stability does not involve sanctions. Let $i \in N$ and $|N| = n$ in this subsection. First, find matrix $J_i^{+,++}$ by Corollary 6.5 and matrix M_H using Corollary 6.2, for which $H = N\backslash\{i\}$. Define the $m \times m$ strong stability matrix M_i^{SGMR} for DM i by

$$M_i^{SGMR} = J_i^{+,++} \cdot [E - sign\big(M_{N\backslash\{i\}} \cdot (P_i^{--})^T\big)]$$

.

Theorem 6.42. *State $s \in S$ is strong GMR (SGMR) for DM i, denoted by $s \in S_i^{SGMR}$, iff $M_i^{SGMR}(s,s) = 0$.*

Proof: Since $M_i^{SGMR}(s,s) = (e_s^T \cdot J_i^{+,++}) \cdot [\big(E - sign\left(M_{N\backslash\{i\}} \cdot (P_i^{--})^T\right)\big) \cdot e_s]$

$$= \sum_{s_1=1}^{m} J_i^{+,++}(s,s_1)[1 - sign\left((e_{s_1}^T \cdot M_{N\backslash\{i\}}) \cdot (e_s^T \cdot P_i^{--})^T\right)],$$

then

$$M_i^{SGMR}(s,s) = 0 \Leftrightarrow J_i^{+,++}(s,s_1)[1 - sign\left((e_{s_1}^T \cdot M_{N\backslash\{i\}}) \cdot (e_s^T \cdot P_i^{--})^T\right)] = 0, \forall s_1 \in S.$$

This implies that $M_i^{SSGM}(s,s) = 0$ iff

$$(e_{s_1}^T \cdot M_{N\backslash\{i\}}) \cdot (e_s^T \cdot P_i^{--})^T \neq 0, \forall s_1 \in R_i^{+,++}(s). \tag{6.13}$$

By (6.13), for any $s_1 \in R_i^{+,++}(s)$, there exists $s_2 \in S$ such that the m-dimensional row vector, $e_{s_1}^T \cdot M_{N\backslash\{i\}}$, has s_2^{th} element 1 and the m-dimensional column vector, $(P_i^{--})^T \cdot e_s$, has s_2^{th} element 1.

Therefore, $M_i^{SGMR}(s,s) = 0$ iff for any $s_1 \in R_i^{+,++}(s)$, there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$ with $s \gg_i s_2$. □

For strong SMR, the n-DM model is similar to the two-DM model. The only modification is that responses come from DM i's opponents instead of from a single DM. If $D = (P_i^{++}) \circ [E - sign\big(J_i \cdot (E - P_i^{++})\big)]$, then define the $m \times m$ strong stability matrix M_i^{SSMR} for DM i by

$$M_i^{SSMR} = J_i^{+,++} \cdot [E - sign(M_{N\setminus\{i\}} \cdot D)].$$

Theorem 6.43. *State $s \in S$ is strongly SMR (SSMR) for DM i, denoted by $s \in S_i^{SSMR}$, iff $M_i^{SSMR}(s,s) = 0$.*

Proof: Let $G = M_{N\setminus\{i\}} \cdot D$. Since $M_i^{SSMR}(s,s) = (e_s^T \cdot J_i^{+,++}) \cdot [(E - sign(G)) \cdot e_s]$

$$= \sum_{s_1=1}^{m} J_i^{+,++}(s,s_1)[1 - sign(G(s_1,s))]$$

then $M_i^{SSMR}(s,s) = 0$ iff $J_i^{+,++}(s,s_1)[1 - sign(G(s_1,s))] = 0$, for any $s_1 \in S$. This means that $M_i^{SSMR}(s,s) = 0$ iff

$$(e_{s_1}^T \cdot M_{N\setminus\{i\}}) \cdot (D \cdot e_s) \neq 0, \forall s_1 \in R_i^{+,++}(s). \tag{6.14}$$

Since $(e_{s_1}^T \cdot M_{N\setminus\{i\}}) \cdot (D \cdot e_s) = \sum_{s_2=1}^{m} M_{N\setminus\{i\}}(s_1,s_2) \cdot D(s_2,s)$, then (6.14) holds iff for any $s_1 \in R_i^{+,++}(s)$, there exists $s_2 \in R_{N\setminus\{i\}}(s_1)$ such that $D(s_2,s) \neq 0$.

Because $D(s_2,s) = P_i^{++}(s_2,s) \cdot [1 - sign(\sum_{s_3=1}^{m} J_i(s_2,s_3)(1 - P_i^{++}(s_3,s))]$, $D(s_2,s) \neq 0$ implies that for $s_2 \in R_{N\setminus\{i\}}(s_1)$,

$$P_i^{++}(s_2,s) \neq 0 \tag{6.15}$$

and

$$\sum_{s_3=1}^{m} J_i(s_2,s_3)(1 - P_i^{++}(s_3,s)) = 0. \tag{6.16}$$

(6.15) is equivalent to the statement that, $\forall s_1 \in R_i^{+,++}(s), \exists s_2 \in R_{N\setminus\{i\}}(s_1)$ such that $s \gg_i s_2$. (6.16) is the same as the statement that, $\forall s_1 \in R_i^{+,++}(s), \exists s_2 \in R_{N\setminus\{i\}}(s_1)$ such that $P_i^{++}(s_3,s) \neq 0$ for $\forall s_3 \in R_i(s_2)$. Based on the definition of $m \times m$ matrix P_i^{++}, one knows that $P_i^{++}(s_3,s) \neq 0 \Leftrightarrow s \gg_i s_3$.

Therefore, we conclude the above discussion that $M_i^{SMR}(s,s) = 0$ iff for any $s_1 \in R_i^{+,++}(s)$, there exists at least one $s_2 \in R_{N\setminus\{i\}}(s_1)$ with $s \gg_i s_2$ and $s \gg_i s_3$ for all $s_3 \in R_i(s_2)$. □

Strongly sequential stability examines the credibility of the sanctions by DM i's opponents. First, find matrix $M^{+,++}_{N\backslash\{i\}}$ using Corollary 6.6 for $H = N\backslash\{i\}$. Define the $m \times m$ strong stability matrix M_i^{SSEQ} for DM i by

$$M_i^{SSEQ} = J_i^{+,++} \cdot [E - sign\big(M^{+,++}_{N\backslash\{i\}} \cdot (P_i^{--})^T\big)].$$

Theorem 6.44. *State $s \in S$ is strongly SEQ (SSEQ) for DM i, denoted by $s \in S_i^{SSEQ}$, iff $M_i^{SSEQ}(s,s) = 0$.*

Proof: Since $M_i^{SSEQ}(s,s) = (e_s^T \cdot J_i^{+,++}) \cdot [\left(E - sign\left(M^{+,++}_{N\backslash\{i\}} \cdot (P_i^{--})^T\right)\right) \cdot e_s]$

$$= \sum_{s_1=1}^{m} J_i^{+,++}(s,s_1)[1 - sign\left((e_{s_1}^T \cdot M^{+,++}_{N\backslash\{i\}} \cdot (e_s^T \cdot P_i^{--})^T\right)],$$

then

$$M_i^{SSEQ}(s,s) = 0 \Leftrightarrow J_i^{+,++}(s,s_1)[1 - sign\left((e_{s_1}^T \cdot M^{+,++}_{N\backslash\{i\}}) \cdot (e_s^T \cdot P_i^{--})^T\right)] = 0, \forall s_1 \in S.$$

This implies that $M_i^{SSEQ}(s,s) = 0$ iff

$$(e_{s_1}^T \cdot M^{+,++}_{N\backslash\{i\}}) \cdot (e_s^T \cdot P_i^{--})^T \neq 0, \forall s_1 \in R_i^{+,++}(s). \tag{6.17}$$

By (6.17), for any $s_1 \in R_i^{+,++}(s)$, there exists $s_2 \in S$ such that the m-dimensional row vector, $e_{s_1}^T \cdot M^{+,++}_{N\backslash\{i\}}$, has s_2^{th} element 1 and the m-dimensional column vector, $(P_i^{--})^T \cdot e_s$, has s_2^{th} element 1.

Therefore, $M_i^{SSEQ}(s,s) = 0$ iff for any $s_1 \in R_i^{+,++}(s)$, there exists at least one $s_2 \in R^{+,++}_{N\backslash\{i\}}(s_1)$ with $s \gg_i s_2$. □

In the $n = 2$ cases, Theorems 6.42 to 6.44 are reduced to those Theorems 6.35 to 6.37, so we use the same notation for two-DM and n-DM cases.

6.3.4 Weak Stabilities for Strength of Preference

Recall that GS denotes a solution concept, GMR, SMR, or SEQ. Then SGS refers to the strong solution concept of GS, and WGS refers to the weak solution concept of GS (defined below). The symbol $s \in S_i^{GS}$ denotes that $s \in S$ is stable for DM i according to stability GS. Similarly, $s \in S_i^{SGS}$ denotes that $s \in S$ is strongly stable for DM i according to strong stability SGS. A state is weakly stable iff it is stable, but not strongly stable. The formal weak stability concept is defined next.

Definition 6.7. *State s is weakly stable WGS for DM i according to stability GS, denoted by $s \in S_i^{WGS}$, iff $s \in S_i^{GS}$ and $s \notin S_i^{SGS}$.*

6.3.5 Applications including Strength of Preference

6.3.5.1 Sustainable Development Conflict

The description of the sustainable development conflict is presented in Subsection 6.2.4. The graph model for each DM in this conflict is depicted in Fig. 6.4, where vertices designate states and arcs represent movement between states. The letter on a given arc indicates which DM controls the movement while the arrowhead shows the direction of movement. The preference information for each DM is:

DM 1: $s_1 >_1 s_3 \gg_1 s_2 \sim_1 s_4$;

DM 2: $s_3 >_2 s_1 \gg_2 s_4 \sim_2 s_2$.

DM 1 and DM 2's preference information includes strength. From the graph model, the UM adjacency matrices for each DM are constructed by

$$J_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \ and \ J_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The preference matrices for the DMs 1 and 2 are given by

$$P_1^{++} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}, P_1^{+,++} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

,

$$P_2^{++} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ and \ P_2^{+,++} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Therefore, $J_i^{+,++} = J_i \circ P_i^{+,++}$ $P_i^{--,-,=} = E - I - P_i^{+,++}$, $P_i^{--} = (P_i^{++})^T$ for $i = 1, 2$.

The stability matrices used by Theorems 6.31 to 6.37 are included in Table 6.16, which are employed to calculate the extended stabilities of Nash, GMR, SMR, and SEQ, as well as the strong stabilities of SGMR, SSMR, and SSEQ for two-DM conflicts, respectively.

The stable states and equilibria for the sustainable development conflict are summarized in Table 6.17, in which "$\surd$" for a given state means that this state is stable for DM 1 or DM 2 and "Eq" is an equilibrium for an appropriate solution concept.

The results provided by Table 6.17 shows that state s_1 is strong equilibrium for the four basic stabilities. State s_3 is strongly stable for GMR and SMR. Hence, s_1 and s_3 are better choices for decision makers.

Table 6.17: **Stability results of the sustainable development conflict with strength of preference**

State	Nash			GMR			SMR			SEQ			SGMR			SSMR			SSEQ			WGMR			WSMR			WSEQ		
	1	2	Eq	1	2	Eq	1	2	Eq	1	2	Eq	1	2	Eq	1	2	Eq	1	2	Eq	1	2	Eq	1	2	Eq	1	2	Eq
s_1	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓									
s_2	✓			✓			✓			✓			✓			✓			✓											
s_3		✓		✓	✓	✓	✓	✓	✓		✓		✓	✓	✓	✓	✓	✓		✓										
s_4	✓			✓			✓			✓			✓			✓			✓											

6.3.5.2 Garrison Diversion Unit (GDU) Conflict

In this section, the proposed MRSCS method is employed to the Garrison Diversion Unit (GDU) conflict to illustrate how the procedure works. The history and background of this conflict are introduced in Section 4.3. The details of the GDU conflict are described in the book [16]. Recall that the irrigation initiative for the diversion called the Garrison Diversion Unit project concerns three DMs, the **United States Support (USS)**, the **Canadian Opposition (CDO)**, and the **International Joint Commission (IJC)**. The graph model of the GDU conflict is shown in Fig. 6.6 (1), where the labels on the arcs identify the DMs who control the relevant moves. If DM i's oriented arcs are coded in color i, then, according to the *Rule of Priority* introduced in Section 5.2, Fig. 6.6 (1) is converted to an edge labeled multidigraph as shown in Fig. 6.6 (2).

Table 6.18: **Preferences for DMs in the GDU conflict [28]**

DM	Preference
USS	$s_2 > s_4 > s_3 > s_5 > s_1 > s_6 > s_9 > s_7 \gg s_8$
CDO	$\{s_3 \sim s_7\} > \{s_5 \sim s_9\} > \{s_4 \sim s_8\} \gg \{s_1 \sim s_2 \sim s_6\}$
IJC	$\{s_2 \sim s_3 \sim s_4 \sim s_5 \sim s_6 \sim s_7 \sim s_8 \sim s_9\} \gg s_1$

The graph model introduced by Hamouda et al. [28] to have strength of preference in the GDU conflict is used in this section. The preference information for this conflict over the feasible states is given in Table 6.18 in which s_8 is strongly less preferred to all other states for USS, the DM, CDO considers states s_1, s_2, and state s_6 to be equally preferred and strongly less preferred relative to all other states, and s_1 is strongly less preferred to all other equally preferred

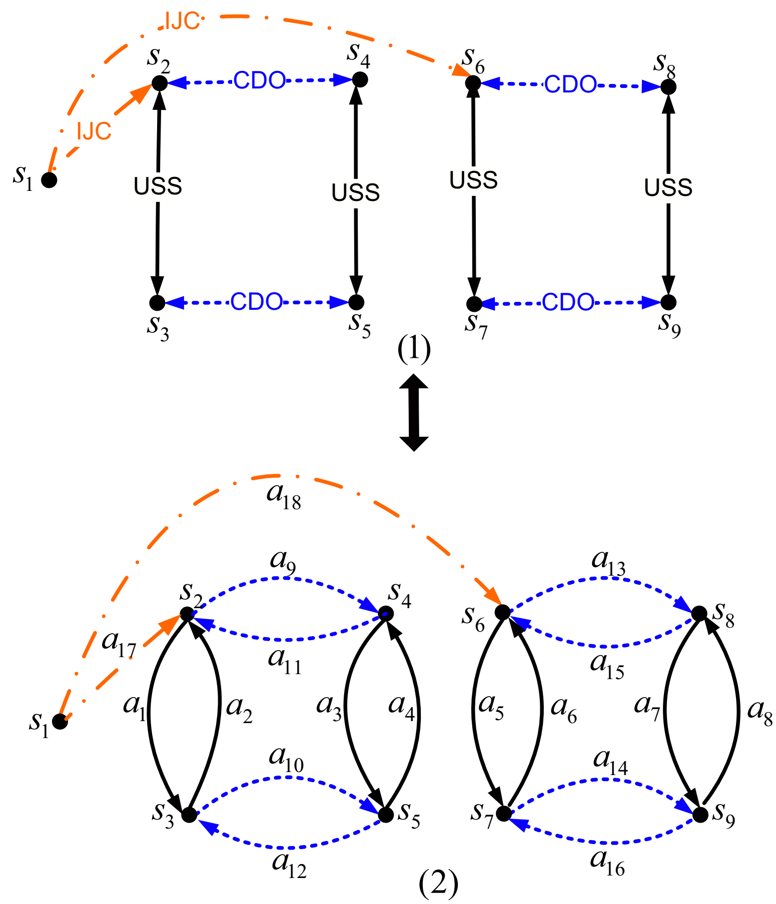


Figure 6.6: **The labeled graph model for the GDU conflict.**

states for IJC. Note that this representation of preference information presented in Table 6.18 implies that the preferred relations, $>$ and $\gg$ are transitive. For instance, since $s_9 > s_7$ and $s_7 \gg s_8$, then $s_9 \gg s_8$. However, in general, the preference structure presented in this research does not require the transitivity of preference, and hence can handle intransitive preferences.

The GDU conflict is used as an example to show the matrix representation of solution concepts with strength of preference obtained by carrying out the following steps. Let $N = \{1, 2, 3\}$ denote the set of the three DMs in the GDU conflict, $i \in N$, and $H = N \backslash \{i\}$.

- Construct preference matrices, P_i^{++} and $P_i^{+,++}$, for $i = 1, 2, 3$, using information provided by Table 6.18, then $P_i^{--} = (P_i^{++})^T$ and $P_i^{--,-,=} = E - I - P_i^{+,++}$;

- Calculate the UM weight matrix and the WI weight matrix of coalition H based on Definition 5.18, preference information presented in Table 6.18, and statement (5.8). Table 6.19 shows the process how to calculate matrices $W_H^{(UM)}$ and $W_H^{(WI)}$ for $H = N\backslash\{1\}$;

Table 6.19: **Weight matrix of coalition $H = N\backslash\{1\}$ by the legal UM and WI sequences for the GDU conflict**

Arc number	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}	a_{18}
Weight matrix W	N_w	P_m	N_w	P_m	N_w	P_m	P_s	N_w	P_s	N_w	N_w	P_m	P_s	N_w	N_w	P_m	P_s	P_s
Weight matrix W_H	0	0	0	0	0	0	0	0	P_s	N_w	N_w	P_m	P_s	N_w	N_w	P_m	P_s	P_s
$W_H^{(UM)}$	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1
$W_H^{(WI)} = W_H^{+,++}$	0	0	0	0	0	0	0	0	1	0	0	1	1	0	0	1	1	1

- Construct weighted in-incidence matrix $B_{in}^{(W_H)}$ and out-incidence matrix $B_{out}^{(W_H)}$ for coalition H, based on the graph model and Definition 5.9;

- Calculate the UM adjacency matrix $J_i = sign[(B_{out}^{(W_i^{(UM)})}) \cdot (B_{in}^{(W_i^{(UM)})})^T]$ and the WI adjacency matrix $J_i^{+,++} = sign[(B_{out}^{(W_i^{+,++})}) \cdot (B_{in}^{(W_i^{+,++})})^T]$ for $i = 1, 2, 3$, by Corollary 6.5 and Theorem 5.3;

- Calculate DM i's UM reachability matrix of coalition H

$$M_H = sign[(B_{out}^{(W_H^{(UM)})}) \cdot (LJ_{H_r} + I)^{l_1 - 1} \cdot (B_{in}^{(W_H^{(UM)})})^T]$$

and WI reachability matrix of coalition H

$$M_H^{+,++} = sign[(B_{out}^{(W_H^{+,++})}) \cdot (LJ_{H_r}^{+,++} + I)^{l_4 - 1} \cdot (B_{in}^{(W_H^{+,++})})^T],$$

using Corollaries 5.9, 6.2, and 6.6, and Theorem 5.2 for $l_1 = |A_H|$ and $l_4 = |A_H^{+,++}|$, where

$$LJ_{H_r} = LJ_r^{(W_H^{(UM)})} = [(B_{in}^{(W_H^{(UM)})})^T \cdot (B_{out}^{(W_H^{(UM)})})] \circ (E_l - D)$$

and

$$LJ_{H_r}^{+,++} = LJ_r^{(W_H^{+,++})} = [(B_{in}^{(W_H^{+,++})})^T \cdot (B_{out}^{(W_H^{+,++})})] \circ (E_l - D);$$

- Analyze the stabilities of Nash, GMR, SMR, and SEQ, as well as the strong stabilities, SGMR, SSMR, and SSEQ, by Theorems 6.38 to 6.44 for the GDU conflict using the stability matrices summarized in Table 6.20.

Table 6.20: **Stability matrices for n-DM conflicts with strength of preference**

Types of stabilities	Stability matrices
Standard stabilities	$M_i^{Nash} = J_i^{+,++}$
	$M_i^{GMR} = J_i^{+,++} \cdot [E - sign\left(M_{N\backslash\{i\}} \cdot (P_i^{--,-,=})^T\right)]$
	$M_i^{SMR} = J_i^{+,++} \cdot [E - sign(M_{N\backslash\{i\}} \cdot G)]$, with $G = (P_i^{--,-,=})^T \circ [E - sign\left(J_i \cdot (P_i^{+,++})^T\right)]$
	$M_i^{SEQ} = J_i^{+,++} \cdot [E - sign\left(M_{N\backslash\{i\}}^{+,++} \cdot (P_i^{--,-,=})^T\right)]$
Strong stabilities	$M_i^{SGMR} = J_i^{+,++} \cdot [E - sign\left(M_{N\backslash\{i\}} \cdot (P_i^{--})^T\right)]$
	$M_i^{SSMR} = J_i^{+,++} \cdot [E - sign(M_{N\backslash\{i\}} \cdot D)]$, with $D = (P_i^{++}) \circ [E - sign\left(J_i \cdot (E - P_i^{++})\right)]$
	$M_i^{SSEQ} = J_i^{+,++} \cdot [E - sign\left(M_{N\backslash\{i\}}^{+,++} \cdot (P_i^{--})^T\right)]$

The stability results for the GDU conflict with strength of preference are summarized in Table 6.21, in which "$\surd$" for a given state under a DM means that this state is stable at a given level for the given DM; Note that U, C, and I displayed in Table 6.21 denote the three DMs, USS, CDO, and IJC, respectively. Obviously, state s_4 is an equilibrium for Nash stability, and is strong GMR, strong SMR, and strong SEQ. State s_9 is a strong equilibrium for GMR and SEQ.

6.4 Summary

An integrated algebraic method is developed to represent several graph model stability definitions for various preference structures using explicit matrix formulations instead of graphical or logical representations. Matrix representations of solution concepts for simple preference (MRSC) [67, 69], for preference with uncertainty (MRSCU) [66, 73], and for preference with strength (MRSCS) [76, 77] are provided for the four basic graph model stability definitions. These explicit algebraic formulations allow algorithms to assess rapidly the

Table 6.21: **Stability results of the GDU conflict with strength of preference**

State	Nash				GMR				SMR				SEQ				SGMR				SSMR				SSEQ				WGMR				WSMR				WSEQ			
	U	C	I	Eq	U	C	I	Eq	U	C	I	Eq	U	C	I	Eq	U	C	I	Eq	U	C	I	Eq	U	C	I	Eq	U	C	I	Eq	U	C	I	Eq	U	C	I	Eq
s_1	✓	✓			✓	✓			✓	✓			✓	✓			✓	✓			✓	✓			✓	✓														
s_2	✓		✓		✓		✓		✓		✓		✓		✓		✓		✓		✓		✓		✓		✓													
s_3		✓	✓			✓	✓			✓	✓			✓	✓			✓	✓			✓	✓			✓	✓													
s_4	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓												
s_5			✓			✓	✓			✓	✓			✓	✓			✓	✓				✓				✓							✓				✓		
s_6	✓		✓		✓		✓		✓		✓		✓		✓		✓		✓		✓		✓		✓		✓													
s_7		✓	✓		✓	✓	✓	✓		✓	✓		✓	✓	✓	✓	✓	✓	✓	✓		✓	✓			✓	✓										✓			
s_8		✓	✓			✓	✓			✓	✓			✓	✓			✓	✓			✓	✓			✓	✓													
s_9	✓		✓		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓		✓		✓	✓	✓	✓						✓						

stabilities of states, and to be applied to large and complicated conflict models. Because of the nature of these explicit expressions, the matrix representations discussed here can be adapted to new solution concepts and contexts.

Chapter 7

Matrix Representations for Status Quo Analysis and Coalition Analysis

The analysis system of GMCR consists of stability analysis and post-stability analysis including status quo analysis and coalition analysis in a graph model. Although pseudo-codes for status quo analysis [47, 48] and basic coalition stability analysis [43] have been developed, they are not yet implemented into a decision support system for use in practical applications. An innovative matrix system to represent various preference structures and calculate corresponding stabilities in a graph model has been presented in Chapter 6. The matrix representation effectively converts the stability analysis from a logical structure to an algebraic system. Due to the difficulty in integrating status quo analysis and coalition stability analysis into the DSS GMCR II and the ease of implementing the matrix representation of stability analysis, it is natural to exploit the matrix approach to perform status quo analysis and coalition analysis.

7.1 Matrix Representation for Status Quo Analysis

It is well-known that matrices can efficiently describe adjacency of vertices, and incidence of arcs and vertices in a graph, thereby permitting tracking of paths between any two vertices [24]. Matrices possess useful algebraic properties that can be exploited to produce improved algorithms for solving graph problems. For instance, extensive research has been conducted to design effective algorithms and efficient search procedures by using relationships between matrices and paths [26, 33, 60].

In a graph model of a conflict, status quo analysis is a form of follow-up analysis designed to trace the evolution of the conflict from a status quo state to any stable state. A legal path in the graph model has the usual restriction that any DM may move more than once, but not twice consecutively. Moreover, Chapter 5 has shown that edge weights can be used to represent preference attributes. The fundamental problem of status quo analysis is thus equivalent to search all weighted colored paths from a given initial state to a desirable state within an edge-weighted, colored multidigraph.

The traditional use of adjacency matrix to search paths is applicable in a simple digraph. The proposed method based on the adjacency matrix will be presented in Subsection 7.1.1 to show its advantages in tracking conflict evolution. However, this method is based on searching state-by-state paths. If a graph model contains two or more arcs between the same two states controlled by different DMs, the adjacency matrix is unable to track all aspects of conflict evolution from a status quo state. An incidence matrix can represent multidigraphs if all edges are labeled. The proposed algebraic approach to searching for the colored paths in a colored multidigraph presented in Chapter 5 starts with a unique edge-labeling rule and then devises a conversion function based on the incidence matrix to transform the colored multidigraph to a simple digraph. The proposed algebraic approach to searching for edge-weighted, colored paths can have many applications, one of which is a main theme of this section.

7.1.1 Status Quo Analysis: Adjacency Matrix

In this subsection, matrix representation is developed for conducting status quo analysis in the graph model for conflict resolution. We now demonstrate how to find matrices to trace conflict evolution by the legal sequences of UMs or UIs from a status quo with the last mover DM i. First, define two $m \times m$ matrices $M_i^{(t)}$ and $M_i^{(t,+)}$ with their (s, q) entries as follows:

Definition 7.1. *In the graph model $G = (S, A)$, let $H \subseteq N$. For $i \in H$ and*

$t = 1, 2, 3, \cdots$,

$$M_i^{(t)}(s,q) = \begin{cases} 1 & \textit{if } q \in S \textit{ is reachable by } H \textit{ from } s \in S \textit{ in exactly } t \textit{ legal} \\ & \textit{UMs with last mover } i, \\ 0 & \textit{otherwise,} \end{cases}$$

and

$$M_i^{(t,+)}(s,q) \;=\; \begin{cases} 1 & \textit{if } q \in S \textit{ is reachable by } H \textit{ from } s \in S \textit{ in exactly } t \textit{ legal} \\ & \textit{UIs with last mover } i, \\ 0 & \textit{otherwise.} \end{cases}$$

Based on Definition 7.1, we have

Lemma 7.1. *In the graph model $G = (S, A)$, let $H \subseteq N$. Then the two $m \times m$ matrices $M_i^{(t)}$ and $M_i^{(t,+)}$ can be expressed inductively by*

$$M_i^{(1)}(s,q) = J_i(s,q) \textit{ and, for } t = 2, 3, \ldots, M_i^{(t)} = sign[(\bigvee_{j \in H \backslash \{i\}} M_j^{(t-1)}) \cdot J_i], \tag{7.1}$$

and

$$M_i^{(1,+)}(s,q) = J_i^+(s,q) \textit{ and, for } t = 2, 3, \ldots, M_i^{(t,+)} = sign[(\bigvee_{j \in H \backslash \{i\}} M_j^{(t-1,+)}) \cdot J_i^+]. \tag{7.2}$$

Proof: The verifications of (7.1) and (7.2) are similar. Now we verify the statement (7.2). For $t = 2$, the definition of matrix multiplication shows that $G(s,q)$, the (s,q) entry of the matrix $G = (\bigvee_{j \in H \backslash \{i\}} J_j^+) \cdot J_i^+$, is nonzero iff state q is reachable from state s by H in exactly two UIs, with last mover DM i. The condition $j \in H \backslash \{i\}$ implies that DM i does not make two moves consecutively. Hence, $G(s,q) \neq 0$ iff state q is reachable by H from state s in exactly two legal UIs. Then

$$sign[(\bigvee_{j \in H \backslash \{i\}} J_j^+) \cdot J_i^+] = sign[(\bigvee_{j \in H \backslash \{i\}} M_j^{(1,+)}) \cdot J_i^+] = M_i^{(2,+)}.$$

Now suppose that $t > 2$. Since

$$M_j^{(t-1,+)}(s,q) = \begin{cases} 1 & \text{if } q \in S \text{ is reachable by } H \text{ from } s \in S \text{ in exactly } t-1 \text{ legal} \\ & \text{UIs with last mover } j, \\ 0 & \text{otherwise,} \end{cases}$$

the definition of matrix multiplication implies that the (s,q) entry of matrix

$$B = sign[(\bigvee_{j \in H \backslash \{i\}} M_j^{(t-1,+)}) \cdot J_i^+]$$

indicates

$$B(s,q) = \begin{cases} 1 & \text{if } q \in S \text{ is reachable by } H \text{ from } s \in S \text{ in exactly } t \text{ legal} \\ & \text{UIs with last mover } i, \\ 0 & \text{otherwise,} \end{cases}$$

which confirms (7.2). □

Next we define two status quo matrices $M_i^{SQ^{(t)}}$ and $M_i^{SQ^{(t,+)}}$ to trace conflict evolution from a status quo to any equilibrium by the legal sequences of UMs and UIs as follows:

Definition 7.2. *In the graph model $G = (S, A)$, let $H \subseteq N$. For $i \in H$ and $t = 1, 2, 3, \cdots$, the UM status quo matrix and the UI status quo matrix are two $m \times m$ matrices with (s, q) entries*

$$M_i^{SQ^{(t)}}(s,q) = \begin{cases} 1 & \textit{if } q \in S \textit{ is reachable by } H \textit{ from } s \in S \textit{ in at most } t \textit{ legal UMs} \\ & \textit{with last mover DM } i, \\ 0 & \textit{otherwise,} \end{cases}$$

$$M_i^{SQ^{(t,+)}}(s,q) = \begin{cases} 1 & \textit{if } q \in S \textit{ is reachable by } H \textit{ from } s \in S \textit{ in at most } t \textit{ legal UIs} \\ & \textit{with last mover DM } i, \\ 0 & \textit{otherwise.} \end{cases}$$

Specifically, $M_i^{SQ^{(t)}}(s,q) = 1$ and $M_i^{SQ^{(t,+)}}(s,q) = 1$ denote that state q is reachable from status quo state s in at most t legal UMs and legal UIs by H, respectively, with last mover i. Based on Definitions 7.1 and 7.2, Theorem 7.1 can be derived.

Theorem 7.1. *In the graph model $G = (S, A)$, let $H \subseteq N$, $i \in H$ and $k \geq 1$ be an integer. Then status quo matrices $M_i^{SQ^{(k)}}$ and $M_i^{SQ^{(k,+)}}$ satisfy that*

$$M_i^{SQ^{(k)}} = \bigvee_{t=1}^{k} M_i^{(t)}, \tag{7.3}$$

$$M_i^{SQ^{(k,+)}} = \bigvee_{t=1}^{k} M_i^{(t,+)}. \tag{7.4}$$

Proof: The proofs of (7.3) and (7.4) are similar. We prove equation (7.4). Let $M_i^{SQ^{(k,+)}}(s,q)$ denote the (s,q) entry of the matrix $M_i^{SQ^{(k,+)}}$. Based on Definition 7.2, $M_i^{SQ^{(k,+)}}(s,q) = 1$ iff q is reachable by H from $SQ = s$ in at most k legal UIs, with last mover $i \in H$.

Let $(\bigvee_{t=1}^{k} M_i^{(t,+)})(s,q)$ denote the (s,q) entry of the matrix $\bigvee_{t=1}^{k} M_i^{(t,+)}$. By Definition 7.1, $(\bigvee_{t=1}^{k} M_i^{(t,+)})(s,q) = 1$ iff there exists $1 \leq t \leq k$, such that

$M_i^{(t,+)}(s,q) = 1$. i.e., q is reachable by H from $SQ = s$ in exactly t legal UIs, with last mover i. It implies that q is reachable from $SQ = s$ in at most k legal UIs, with last mover i. Consequently, $(\bigvee_{t=1}^{k} M_i^{(t,+)})(s,q) = 1$ iff $M_i^{SQ^{(k,+)}}(s,q) = 1$. Since $M_i^{SQ^{(k,+)}}$ and $\bigvee_{t=1}^{k} M_i^{(t,+)}$ are $m \times m$ 0-1 matrices, it follows that $M_i^{SQ^{(k,+)}} = \bigvee_{t=1}^{k} M_i^{(t,+)}$. □

Any nonzero entry (s,q) of the two status quo matrices $M_i^{SQ^{(t)}}$ and $M_i^{SQ^{(t,+)}}$ shows that the desired outcome state q is reachable from the status quo state s in at most t legal UMs and t legal UIs, respectively, with last mover i.

7.1.2 Application: Status Quo Analysis using State-by-State Approach to the Elmira Conflict

In this subsection, the proposed matrix approach to status quo analysis is applied to the Elmira conflict to illustrate how the procedure works. As a small agricultural town renowned for its annual maple syrup festival, Elmira is located in southwestern Ontario, Canada. In 1989, the Ontario **Ministry of Environment (MoE)** detected that the underground aquifer supplying water for Elmira was polluted by N-nitroso demethylamine (NDMA). A local pesticide and rubber manufacturer, **Uniroyal Chemical Ltd. (UR)**, was identified, since the prime suspect as NDMA is a by-product of its production line. Hence, a Control Order was issued by MoE, requiring that UR take expensive measures to remedy the contamination. UR immediately appealed to repeal this control order. The **Local Government (LG)** consisting of the Regional Municipality of Waterloo and the Township of Woolwich, sided with MoE and sought legal advice from independent consultants on its role to resolve this conflict (see [29, 43] for more details).

Hipel et al. [29] established a graph model for this conflict, comprised of three DMs: 1.**MoE**, 2.**UR**, and 3.**LG**; and five options: 1.**Modify**−Modify the Control Order to make it more acceptable to UR; 2.**Delay**−Lengthen the appeal process; 3. **Accept**−Accept the current Control Order; 4.**Abandon**−Abandon its Elmira operation; and 5.**Insist**−Insist that the original Control Order be applied. Although there exist 32 mathematically possible states, given the five options in this model, many of them are infeasible for a variety of reasons and

Table 7.1: **Options and feasible states for the Elmira model** [29, 43]

MoE									
1. Modify	N	Y	N	Y	N	Y	N	Y	–
UR									
2. Delay	Y	Y	N	N	Y	Y	N	N	–
3. Accept	N	N	Y	Y	N	N	Y	Y	–
4. Abandon	N	N	N	N	N	N	N	N	Y
LG									
5. Insist	N	N	N	N	Y	Y	Y	Y	–
State number	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9

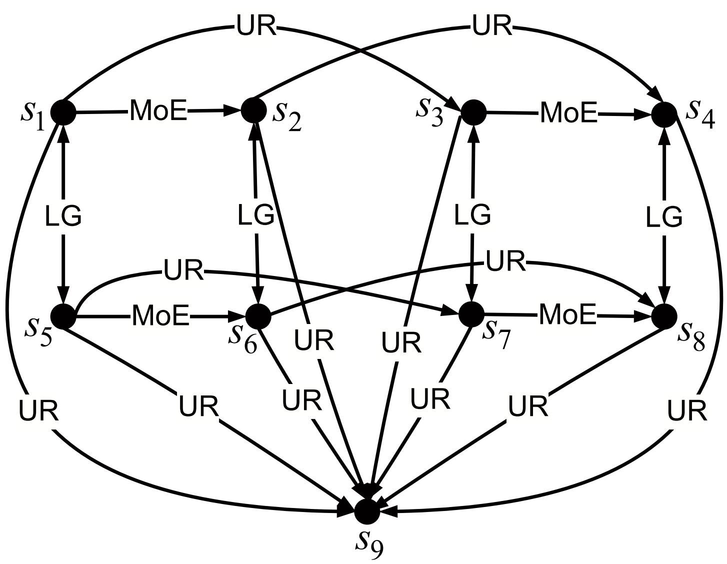


Figure 7.1: **A graph model for the Elmira conflict [29, 43].**

only 9 states are identified as being feasible and listed in Table 7.1 (where a "Y" indicates that an option is selected by the DM controlling it, an "N" means that the option is not chosen, and a dash "–" denotes that the entry may be "Y" or "N"). The graph model of the Elmira conflict is shown in Fig. 7.1, in which labels on the arcs indicate the DM who controls the move; and preference information over the states is given in Table 7.2.

Let $N = \{1, 2, 3\}$ be the set of three DMs (1=MoE, 2=UR, and 3=LG). To carry out status quo analysis for the Elmira model by using the proposed matrix approach, the following steps are needed:

- Construct matrices J_i and P_i^+ for $i = 1, 2, 3$, using information provided by Fig. 7.1 and Table 7.2;

Table 7.2: **Preferences for DMs in the Elmira conflict** [29]

DM	Preference
MoE	$s_7 \succ s_3 \succ s_4 \succ s_8 \succ s_5 \succ s_1 \succ s_2 \succ s_6 \succ s_9$
UR	$s_1 \succ s_4 \succ s_8 \succ s_5 \succ s_9 \succ s_3 \succ s_7 \succ s_2 \succ s_6$
LG	$s_7 \succ s_3 \succ s_5 \succ s_1 \succ s_8 \succ s_6 \succ s_4 \succ s_2 \succ s_9$

- Calculate the UI adjacency matrices $J_i^+ = J_i \circ P_i^+$ for $i = 1, 2, 3,$;
- Determine the matrices $M_i^{(t)}$ and $M_i^{(t,+)}$ for $i = 1, 2, 3$, using inductive formulations provided by Lemma 7.1; and
- Calculate the status quo analysis matrices $M_i^{SQ^{(k)}}$ and $M_i^{SQ^{(k,+)}}$ for $i = 1, 2, 3$, using Theorem 7.1.

Status quo analysis is mainly concerned with the attainability of predicted equilibria. Therefore, stability analysis is usually conducted first. Traditionally, stability analysis is performed by using the DSS GMCR II. Here, to demonstrate the effectiveness of the matrix approach, stability analyses are carried out by using the matrix method developed in Section 6.1 for four basic solution concepts consisting of Nash stability, general metarationality (GMR), symmetric metarationality (SMR), and sequential stability (SEQ). The results are summarized in Table 7.3, in which "$\surd$" for a given state under a DM means that this state is stable for a given DM; and "$\surd$" for a state under Eq signifies that this state is an equilibrium for a corresponding solution concept. It is trivial to verify that the stability results for the four solution concepts are identical to the findings generated by GMCR II. Table 7.3 identifies three states s_5, s_8, and s_9 as ideal equilibria because they are stable for all DMs and for the four solution concepts.

Matrix manipulations generate the status quo analysis matrices given in Table 7.4 (with all UMs) and Table 7.5 (with UIs only). As the status quo state is s_1, we can assess the attainability of any state from the status quo by examining its corresponding entry in the first row for each DM, where a value of 1 indicates that the associated state is reachable from s_1 and a value of 0 means that the corresponding state is not reachable. Given the three matrices in Table 7.4, it is obvious that the three ideal equilibria, s_5, s_8, and s_9, are all attainable. For instance, $M_{MoE}^{SQ^{(3)}}(1,8) = 1$, $M_{UR}^{SQ^{(3)}}(1,8) = 1$, and $M_{LG}^{SQ^{(3)}}(1,8) = 1$ demonstrate

Table 7.3: **Stability results of the Elmira conflict**

State Number	Nash				GMR				SMR				SEQ			
	MoE	UR	LG	Eq	MoE	UR	LG	Eq	MoE	UR	LG	Eq	MoE	UR	LG	Eq
s_1	$\surd$	$\surd$			$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$		
s_2	$\surd$				$\surd$		$\surd$		$\surd$		$\surd$		$\surd$		$\surd$	
s_3	$\surd$				$\surd$		$\surd$		$\surd$		$\surd$		$\surd$		$\surd$	
s_4	$\surd$	$\surd$			$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$		
s_5	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
s_6	$\surd$		$\surd$		$\surd$		$\surd$		$\surd$		$\surd$		$\surd$		$\surd$	
s_7	$\surd$		$\surd$		$\surd$		$\surd$		$\surd$		$\surd$		$\surd$		$\surd$	
s_8	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
s_9	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$

that the ideal equilibrium state s_8 is reachable from s_1 in at most three UMs with the last mover being any of the three DMs, MoE, UR or LG. On the other hand, as the only non-zero $(1,5)$ entry of the three matrices is $M_{LG}^{SQ^{(3)}}(1,5)$, equilibrium s_5 can be reached from the status quo in at most three UMs with LG being the unique last mover. Similarly, the ideal equilibrium s_9 is reachable from s_1 in at most three UMs with a unique last mover UR.

When only UIs are allowed as shown in Table 7.5, only the ideal equilibrium s_5 can be reached from state s_1 in at most three UIs with last mover LG, because the unique non-zero entry in the first row of the three matrices is $M_{LG}^{SQ^{(3,+)}}(1,5)$.

If a different state is selected as the status quo state, one can conveniently examine the elements of the corresponding row in the relevant status quo analysis matrices to evaluate the attainability of any state that is of interest.

By using the proposed inductive formulations in Theorem 7.1, the status quo analysis result can also be presented in a tableau form as shown in Table 7.6 in which number 1, 2, or 3 denotes DM 1, DM 2, or DM 3, as well as $\Omega^{(k)}$ and $\Omega^{(k,+)}$ are the set of all last DMs in at most k legal sequences of UMs and UIs from some status quo, respectively. Note that in Table 7.6, state s_1 ($\surd$) and state s_2 ($\surd$) are sequentially selected as the status quo by the legal sequence of UMs and UIs, respectively. It is easy to verify the equivalence of these results here and those given by Li et al. [47], except for the difference in recording the last mover information. This table offers a wealth of information, such as the specific DM(s) as the last mover(s) and the shortest path(s) to reach a state. For example, the

Table 7.4: **UM status quo matrices for the Elmira conflict**

Matrix	$M_{MoE}^{SQ^{(3)}}$									$M_{UR}^{SQ^{(3)}}$									$M_{LG}^{SQ^{(3)}}$								
State	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9
s_1	0	1	0	1	0	1	0	1	0	0	0	1	1	0	0	1	1	1	0	1	1	0	1	1	1	1	0
s_2	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	1	0	0	0	1	0	1	0	1	0
s_3	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	1	1	0
s_4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0
s_5	0	1	0	1	0	1	0	1	0	0	0	1	1	0	0	1	1	1	1	1	1	1	0	1	1	0	0
s_6	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	1	0	1	0	1	0	0	0	1	0
s_7	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	1	1	0	0	0	1	0
s_8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0
s_9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

shortest path to the ideal equilibrium s_8 from s_1 requires three legal UMs with any of the three DMs being the last mover.

By taking status quo analysis into account, additional insights are revealed about the attainability of any potential resolution and, if attainable, the dynamics of conflict evolution from the status quo state is demonstrated. The results offered by Table 7.6 are identical to those provided by Li et al. [47].

The novel matrix approach to status quo analysis designed here is convenient for computer implementation and easy to employ, as is illustrated by an application to a real-world conflict case: the Elmira conflict. However, the proposed approach is based on the adjacency matrix to search state-by-state paths. If a graph model contains multiple arcs between the same two states controlled by different DMs, the state-by-state paths will not be able to track all aspects of the evolution of a conflict from the status quo state, and an expanded model will be needed to allow for searching arc-by-arc paths. The algebraic approach to searching the edge-weighted, colored paths is developed in Chapter 5. The wide realm of applicabilities is illustrated by a set of real-world conflict cases, which is the objective of the next subsection.

7.1.3 Status Quo Analysis: Edge Consecutive Matrix

Usually, the status quo, or initial state, is specified when a graph model is developed—the conflict is viewed as starting from the status quo and then

Table 7.5: **UI status quo matrices for the Elmira conflict**

Matrix	$M_{MoE}^{SQ^{(3,+)}}$									$M_{UR}^{SQ^{(3,+)}}$									$M_{LG}^{SQ^{(3,+)}}$								
State	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9
s_1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
s_2	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	1	0	0	0	0	0	1	0	1	0
s_3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0
s_4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
s_5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
s_6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0
s_7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
s_8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
s_9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

passing from state to state, according to moves and countermoves controlled by individual DMs, until it stops, eventually, at some equilibrium. A graph model may have many equilibria; some equilibria may be reachable from the status quo by multiple paths, while others may not be reachable at all. Status quo analysis aims to determine whether a particular equilibrium is reachable from the status quo and, if so, how to reach it [47]. The proposed algebraic approach uses the results of Graph Theory to assist in analyzing a graph model and conflict evolution in the graph model by carrying out the following steps:

- The state set S is treated as a vertex set V and DM i's oriented arcs $A_i \subseteq A$ are coded in color $i \in N$, then a graph model (S, A) of a conflict is equivalent to a colored multidigraph (V, A, N, ψ, c) with appropriate preference relations, where ψ and c are two functions with $\psi : A \to V \times V$ such that $\psi(a) = (u, v)$ for $a \in A$ and $u, v \in V$, and $c : A \to N$ such that $c(a) \in N$ is the color of $a \in A$;

- By the proposed *Rule of Priority*, the oriented arcs in the colored multidigraph are labeled according to the color order; within each color, according to the sequence of initial nodes; and within each color and initial node, according to the sequence of terminal nodes;

- The incidence matrix B can represent the colored multidigraph after all edges are labeled;

Table 7.6: **The results of status quo analysis for the Elmira conflict**

State	$\Omega^{(0)}$	$\Omega^{(1)}$	$\Omega^{(2)}$	$\Omega^{(3)}$	$\Omega^{(4)}$	State	$\Omega^{(0,+)}$	$\Omega^{(1,+)}$	$\Omega^{(2,+)}$	$\Omega^{(3,+)}$	$\Omega^{(4,+)}$
s_1	$\surd$					s_2	$\surd$				
s_2		1	1	1,3	1,3	s_4		2	2	2	2
s_3		2	2	2,3	2,3	s_6		3	3	3	3
s_5		3	3	3	3	s_9		2	2,3	2,3	2,3
s_9		2	2	2	2	s_8			2	2	2
s_4			1,2	1,2	1,2	s_1					
s_6			1,3	1,3	1,3	s_3					
s_7			2,3	2,3	2,3	s_5					
s_8				1,2,3	1,2,3	s_7					

- Based on preference structures such as simple preference, preference with uncertainty, and strength of preference, weight matrix W is designed to represent preference information for some preference framework (details presented in equations (5.6), (5.7), and (5.8));

- A graph model is thus conveniently treated as an edge-weighted, colored multidigraph (V, A, N, ψ, c, w) in which each arc represents a legal unilateral move, distinct colors refer to different DMs, and the weight along the arc identifies some preference attribute;

- Tracing the evolution of a conflict in status quo analysis is converted to searching all weighted colored paths between a status quo and a possible equilibrium for some preference structure;

- Let the weighted incidence matrix $B^{(W)}$ represent an original edge-weighted, colored multidigraph (V, A, N, ψ, c, w). Then the conversion function

$$F(B^{(W)}) = [(B_{in}^{(W)})^T \cdot B_{out}^{(W)}] \circ (E_l - D)$$

transforms the problem of searching edge-weighted, colored paths in a weighted colored multidigraph to a standard problem of finding paths in a simple digraph with no color constraints;

- Using existing algorithms or the proposed algorithm presented in Table 5.1, the paths between any two edges can be found in a simple digraph;

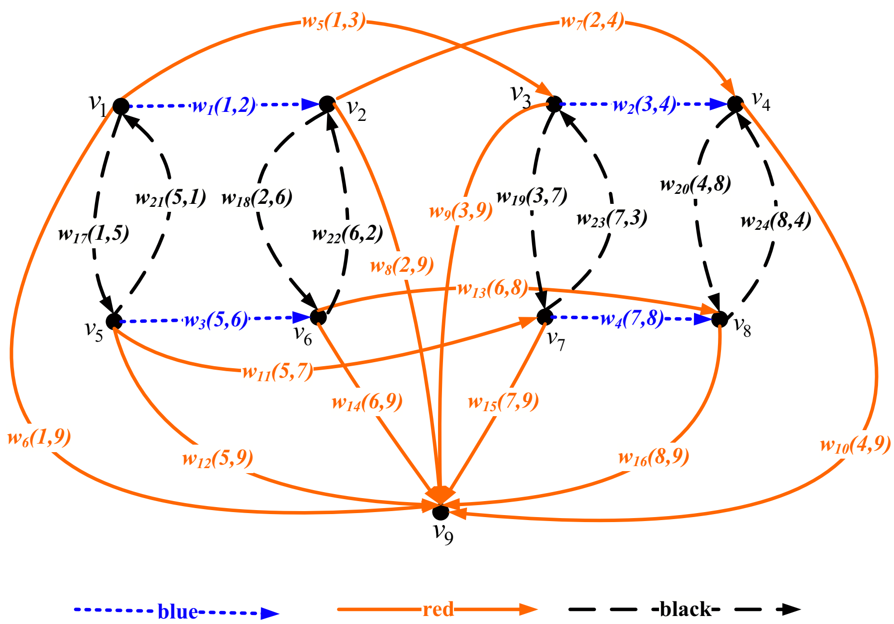


Figure 7.2: **The weighted colored graph for the Elmira conflict.**

- If A_S and A_E are the two sets of arcs starting from vertex s and arcs ending at vertex q with

$$A_S = \{a \in A : B_{out}^{(W)}(s,a) \neq 0\} \ and \ A_E = \{b \in A : B_{in}^{(W)}(q,b) \neq 0\},$$

then paths between any two vertices, $PA^{(W)}(s,q)$ for $s, q \in V$, can be obtained by the paths between two appropriate arcs by

$$PA^{(W)}(s,q) = \{PA^{(W)}(a,b) : a \in A_S, \ b \in A_E\}.$$

7.1.3.1 Application: Status Quo Analysis of the Elmira Conflict for Simple Preference

The background of the Elmira conflict is introduced in Subsection 7.1.2. If the state set $S = \{s_1, s_2, \cdots, s_9\}$ is treated as a vertex set $V = \{v_1, v_2, \cdots, v_9\}$ and DM i's oriented arcs are coded in colors blue, red, and black for $i = 1, 2, 3$, respectively, then the graph model of the Elmira conflict shown in Fig. 7.1 with preference information is equivalent to a weighted colored multidigraph presented in Fig. 7.2, in which $w_k(u,v)$ denotes the weight of arc $a_k = (u,v)$. Although no DM is explicitly shown in the labeled graph, the index number of an arc uniquely determines the DM who controls it when all arcs have been numbered according to the Rule of Priority. Recall that c_i denotes the cardinality of arc set assigned color

i, i.e., $c_i = |A_i|$, where $A_i = \{x \in A : c(x) = i\}$ for each $i \in N$. Specifically, based on the number of arcs in i's graph G_i for $i = 1, 2, 3$, $c_1 = |A_1| = 4, c_2 = |A_2| = 12$, and $c_3 = |A_3| = 8$ provided by Fig. 7.1 for the graph model of the Elmira conflict, arcs a_1 to a_4 are controlled by DM 1 or MoE, arcs a_5 to a_{16} by DM 2 or UR, and arcs a_{17} to a_{24} by DM 3 or LG. The weight of each arc in Fig. 7.2 is assigned based on preference information

$$s_7 \succ_1 s_3 \succ_1 s_4 \succ_1 s_8 \succ_1 s_5 \succ_1 s_1 \succ_1 s_2 \succ_1 s_6 \succ_1 s_9;$$

$$s_1 \succ_2 s_4 \succ_2 s_8 \succ_2 s_5 \succ_2 s_9 \succ_2 s_3 \succ_2 s_7 \succ_2 s_2 \succ_2 s_6;$$

$$s_7 \succ_3 s_3 \succ_3 s_5 \succ_3 s_1 \succ_3 s_8 \succ_3 s_6 \succ_3 s_4 \succ_3 s_2 \succ_3 s_9.$$

Therefore, the diagonal weight matrix, the UM weight matrix, and the UI weight matrix of the Elmira conflict are constructed in Table 7.7.

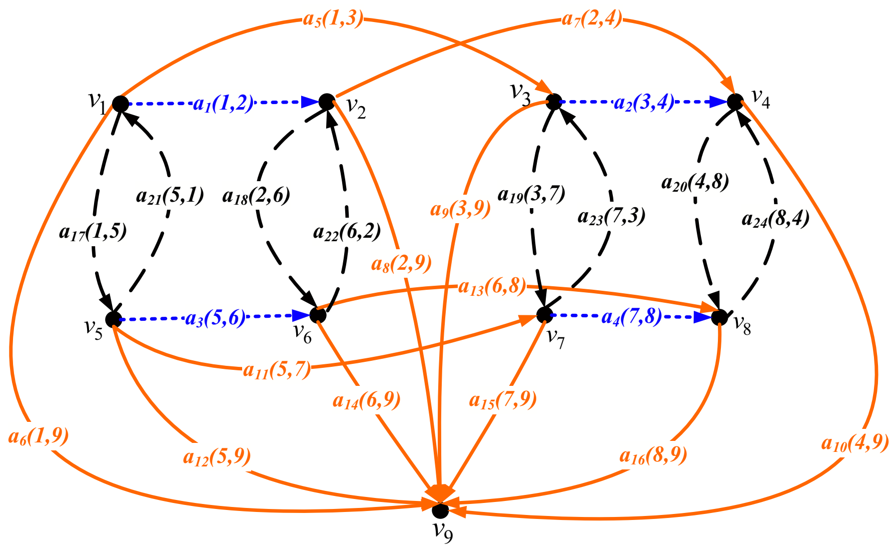
Table 7.7: **UM and UI weight matrices for the Elmira conflict**

Arc number	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}	a_{18}	a_{19}	a_{20}	a_{21}	a_{22}	a_{23}	a_{24}
W	N_w	N_w	N_w	N_w	N_w	N_w	P_w	P_w	P_w	N_w	N_w	N_w	P_w	P_w	P_w	N_w	P_w	P_w	P_w	P_w	N_w	N_w	N_w	N_w
$W^{(UM)}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
W^+	0	0	0	0	0	0	1	1	1	0	0	0	1	1	1	0	1	1	1	1	0	0	0	0

Let

$$F(B^{(W^{(UM)})}) = [(B_{in}^{(W^{(UM)})})^T \cdot (B_{out}^{(W^{(UM)})})] \circ (E_l - D)$$

denote a conversion function. It transforms the labeled multidigraph by node-by-node to the reduced weighted line digraph by arc-by-arc that is a simple digraph with no color constraints to find all evolution of the Elmira conflict by allowing all UMs. The conversion process is depicted in Fig. 7.3 in which each hexagon denotes an arc. Status quo analysis is mainly concerned with the attainability of predicted equilibria. Therefore, stability analysis is usually conducted first. Table 7.3 provides states s_5, s_8, and s_9 are likely resolutions for the Elmira conflict. The three ideal equilibria are reachable from status quo $s = s_1$ by the legal UM paths $PA(s_1, s)$ for $s = s_5, s_8$, and s_9. (See details in Subsection 5.4.1).



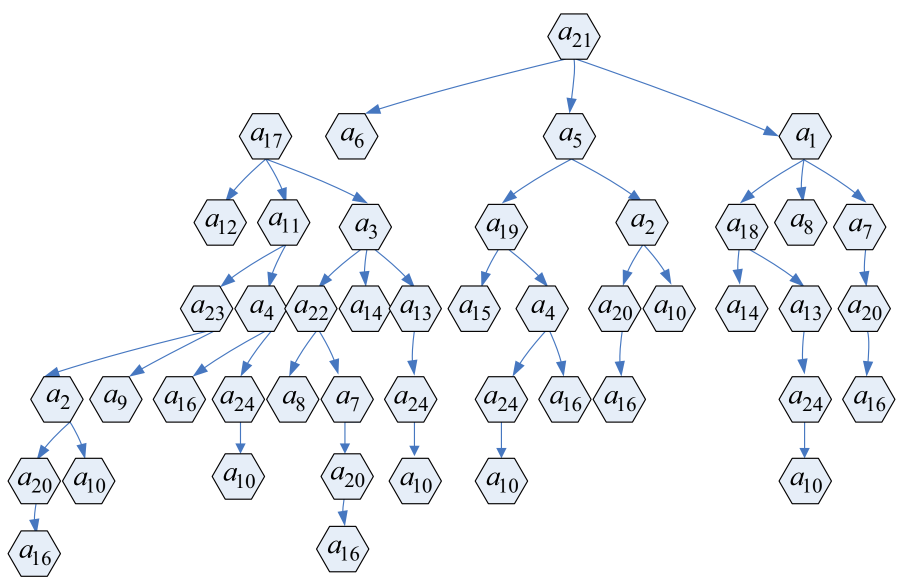


Figure 7.3: **Conversion graph for finding evolutionary UM paths for the Elmira conflict.**

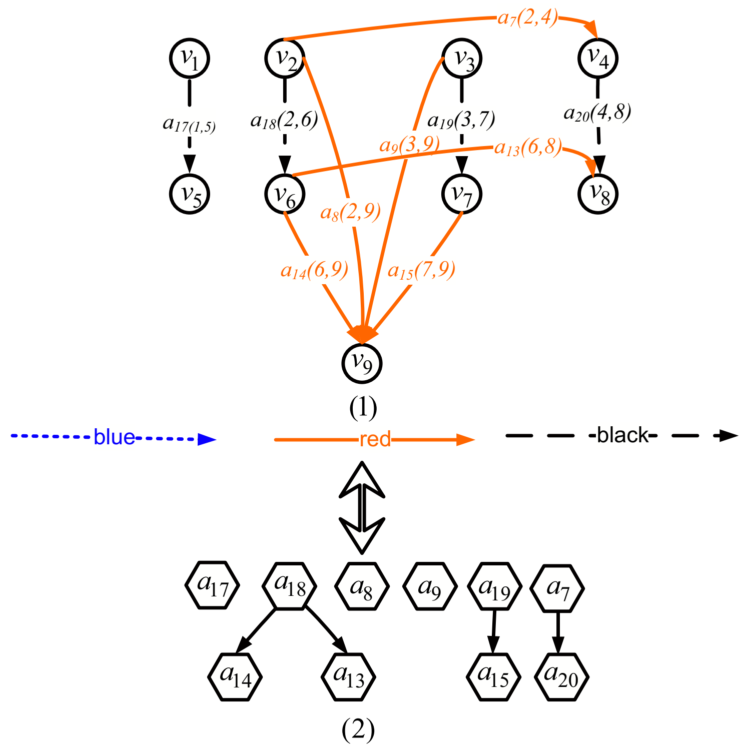
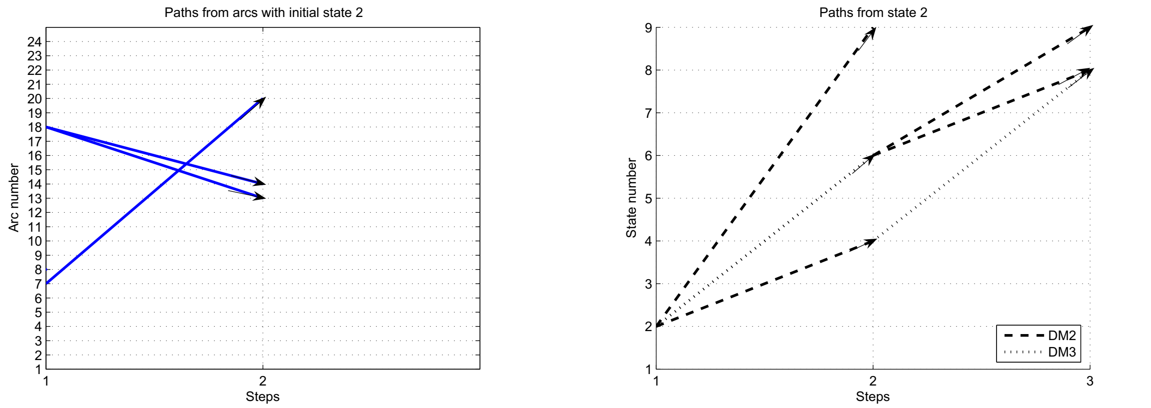


Figure 7.4: **Graph conversion for finding evolutionary UI paths for the Elmira conflict.**

Let $B \Longrightarrow B^{(W^+)}$, then the labeled graph is converted to the reduced colored multidigraph Fig. 7.4(1) including UI arcs only. Let

$$F(B^{(W^+)}) = [(B_{in}^{(W^+)})^T \cdot (B_{out}^{(W^+)})] \circ (E_l - D).$$

The conversion function transforms the original problem of searching the legal UI paths in an edge-colored graph with no repeated colors to the standard problem finding the UI paths on a graph with no color constraints (See Fig. 7.4(2)). For example, if status quo is selected as s_2, then Fig. 7.5(a) shows the UI conflict evolution by arc-by-arc from s_2 for the Elmira conflict. Note that the single arc a_8 does not appear in Fig. 7.5(a) though it is a UI arc and states are denoted by their indexes to make figures clear. Fig. 7.5(b) depicts all possible UI paths from state s_2 by state-by-state and includes the paths of length 1. Obviously, the ideal equilibrium state s_5 cannot be reachable by UIs from status quo state s_2.



(a)Arc-by-arc evolutionary paths by UIs with the initial state s_2

(b)State-by-state evolutionary paths by UIs from status quo state s_2

Figure 7.5: **Evolutionary paths by UIs with status quo state s_2.**

7.1.3.2 Application: Status Quo Analysis of the Gisborne Conflict for Preference with Uncertainty

In this subsection, the proposed matrix method is applied to a case study — status quo analysis of the Gisborne conflict including preference uncertainty. The history and background of the Gisborne conflict is introduced in Subsection 3.4. The edge labeled multidigraph is portrayed in Fig. 7.6 (1) equivalent to the graph model shown in Fig. 6.5. The weight of each arc in Fig. 7.6 (1) is assigned based on preference information

$$s_2 \succ_1 s_6 \succ_1 s_4 \succ_1 s_8 \succ_1 s_1 \succ_1 s_5 \succ_1 s_3 \succ_1 s_7;$$

$$s_3 \succ_2 s_7, s_4 \succ_2 s_8, s_1 \succ_2 s_5, s_2 \succ_2 s_6, \ \mathit{only};$$

$$s_3 \succ_3 s_4 \succ_3 s_7 \succ_3 s_8 \succ_3 s_5 \succ_3 s_6 \succ_3 s_1 \succ_3 s_2.$$

Therefore, the diagonal weight matrix, the diagonal UM weight matrix, the diagonal UI weight matrix, and the diagonal UIUUM weight matrix of the Gisborne conflict are constructed in Table 7.8.

Based on the extended preference structure with uncertainty, Li et al. [46] redefine Nash stability, general metarationality, symmetric metarationality, and sequential stability for graph models with preference uncertainty. According to whether uncertain preferences are deemed as sufficient incentives to motivate the focal DM leaving the current state and credible sanctions to deter the focal DM

Table 7.8: **UM, UI, and UIUUM Weight matrices for the Gisborne conflict**

Arc number	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}	a_{18}	a_{19}	a_{20}	a_{21}	a_{22}	a_{23}	a_{24}
W	P_w	N_w	P_w	N_w	P_w	N_w	P_w	N_w	U_w	U_w	U_w	U_w	U_w	U_w	U_w	U_w	P_w	P_w	N_w	N_w	N_w	N_w	P_w	P_w
$W^{(UM)}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
W^{+}	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	1	1
$W^{+,U}$	1	0	1	0	1	0	1	0	1	1	1	1	1	1	1	1	1	1	0	0	0	0	1	1

from doing so, the aforesaid four types of stability are redefined in four different manners and indexed a, b, c, and d. These four extensions are conceived to depict DMs with distinct risk profiles in face of uncertainty. Li et al. [46] identify states s_4, s_6 and s_8 as equilibria under extension b and d for the Gisborne conflict using logical stability definitions and the proposed algebraic approach presented in Subsection 6.2.4.2 obtain the same results by matrix representation of stabilities. Note that for the stability definitions under extensions b and d, the focal DM is conservative in deciding whether to move away from the current state, since it would only move to preferred states (UIs). For details, one can refer to Subsection 6.2.4.

In parallel to extensions b and d that predict the three equilibria s_4, s_6, and s_8, we examine the evolution paths PA^+ (allowing UIs only) from a status quo to the three equilibria. Based on the UI weight matrix W^+ constructed in Table 7.8, let

$$F(B^{(W^+)}) = [(B_{in}^{(W^+)})^T \cdot (B_{out}^{(W^+)})] \circ (E_l - D)$$

denote a conversion function that transforms the labeled multidigraph Fig. 7.6 (1) to the reduced line digraph Fig. 7.6 (2) including UI arcs only that is a simple digraph with no color constraints. Therefore, finding colored UI paths in Fig. 7.6 (1) is equivalent to searching paths in Fig. 7.6 (2) without constraints. If the status quo is s_1, it is obvious that the equilibria s_4 and s_8 can not be reached by legal UIs and the equilibrium s_6 is the only equilibrium that is attainable from the status quo. Specifically, the evolutionary paths $PA^+(s_1, s_6)$ can be described below:

$$a_1 \longrightarrow a_{18} \Longleftrightarrow s_1 \longrightarrow s_2 \longrightarrow s_6,$$

$$a_{17} \longrightarrow a_5 \Longleftrightarrow s_1 \longrightarrow s_5 \longrightarrow s_6.$$

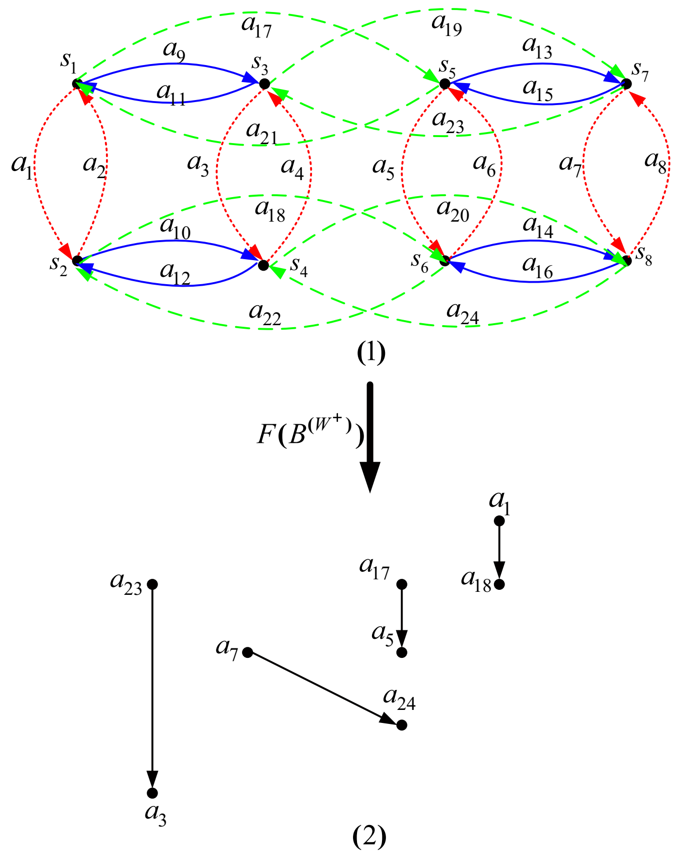


Figure 7.6: **The conversion graphs for finding the evolutionary UI paths for the Gisborne conflict.**

However, if UIUUMs are allowed, equilibrium s_8 is attainable from the status quo s_1. The UIUUM weight matrix $W^{+,U}$ is defined in Table 7.8. Using conversion matrix $B^{(W^{+,U})}$, the labeled graph in Fig. 7.6 (1) is reduced to Fig. 7.7 (1) that illustrates the evolution of the graph model for the Gisborne conflict with allowing UIUUMs only. By the conversion function $F(\cdot)$, the colored multidigraph in Fig. 7.7 (1) is transformed to the reduced line digraph in Fig. 7.7 (2). Searching colored paths $PA^{+,U}(s_1, s_8)$ in Fig. 7.7 (1) is equivalent to finding paths $PA^{+,U}(a_1, a_{14})$, $PA^{+,U}(a_1, a_7)$, $PA^{+,U}(a_9, a_{14})$, $PA^{+,U}(a_9, a_7)$, $PA^{+,U}(a_{17}, a_{14})$, and $PA^{+,U}(a_{17}, a_7)$ in Fig. 7.7 (2). Therefore, the evolution of the Gisborne conflict by the legal $UIUUM$s from status quo state s_1 to equilibrium s_8 is illustrated as follows:

$$a_1 \longrightarrow a_{18} \longrightarrow a_{14},$$

$$a_9 \longrightarrow a_3 \longrightarrow a_{12} \longrightarrow a_{18} \longrightarrow a_{14},$$

$$a_{17} \longrightarrow a_5 \longrightarrow a_{14},$$

$$a_{17} \longrightarrow a_{13} \longrightarrow a_{23} \longrightarrow a_3 \longrightarrow a_{12} \longrightarrow a_{18} \longrightarrow a_{14},$$

$$a_{17} \longrightarrow a_{13} \longrightarrow a_{23} \longrightarrow a_{11} \longrightarrow a_1 \longrightarrow a_{18} \longrightarrow a_{14},$$

$$a_{17} \longrightarrow a_{13} \longrightarrow a_7.$$

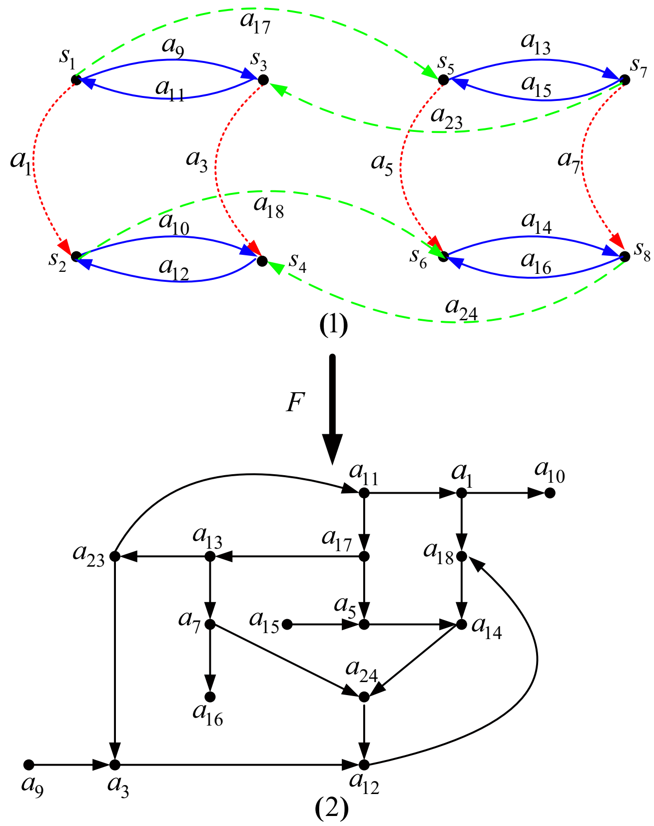


Figure 7.7: **The conversion graphs for finding the evolutionary UIUUM paths for the Gisborne conflict.**

After transforming a colored multidigraph to a simple digraph under conversion functions, existing algorithms such as those reported in [50] and [65] can be used to find all paths or search for the shortest path.

7.1.3.3 Application: Status Quo Analysis of the GDU Conflict for Strength of Preference

As post-stability analysis, the status quo analysis aims at assessing whether predicted equilibria are reachable from the status quo or any other initial state.

Hence, after the stability analysis for the GDU conflict is carried out in the graph model with strength of preference in Subsection 6.3.5, status quo analysis as a post-stability analysis is discussed in this subsection. The history and background of the GDU conflict are introduced in Subsections 4.4 and 6.3.5. The graph model for the GDU conflict Fig. 4.8 is equivalent to the labeled graph Fig. 7.8 (1). Based on preference information

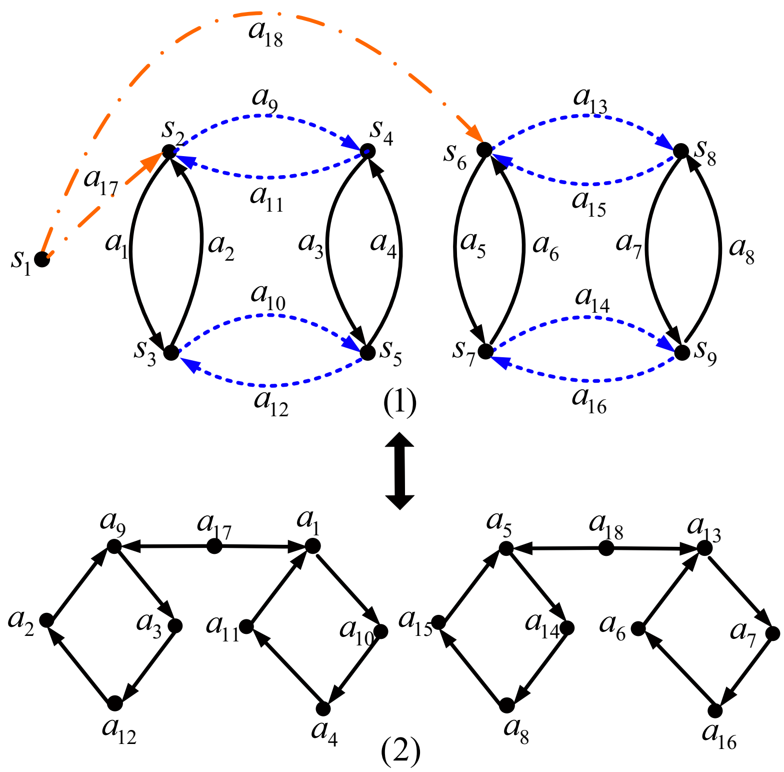


Figure 7.8: **Transformation of the graph model for the GDU conflict.**

$$s_2 >_1 s_4 >_1 s_3 >_1 s_5 >_1 s_1 >_1 s_6 >_1 s_9 >_1 s_7 \gg_1 s_8,$$

$$\{s_3 \sim_2 s_7\} >_2 \{s_5 \sim_2 s_9\} >_2 \{s_4 \sim_2 s_8\} \gg_2 \{s_1 \sim_2 s_2 \sim_2 s_6\},$$

$$\{s_2 \sim_3 s_3 \sim_3 s_4 \sim_3 s_5 \sim_3 s_6 \sim_3 s_7 \sim_3 s_8 \sim_3 s_9\} \gg_3 s_1,$$

the $l \times l$ diagonal weight matrix, the UM weight matrix, and the WI weight matrix are constructed in Table 7.9.

By taking status quo analysis into account, additional insights are revealed about the attainability of any potential resolution. Table 6.21 indicates that state

Table 7.9: **Weight matrix, UM weight matrix, and WI weight matrix for the GDU conflict**

Arc number	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}	a_{18}
Weight matrix W	N_w	P_m	N_w	P_m	N_w	P_m	P_s	N_w	P_s	N_w	N_w	P_m	P_s	N_w	N_w	P_m	P_s	P_s
UM weight matrix $W^{(UM)}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
WI weight matrix $W^{+,++}$	0	1	0	1	0	1	1	0	1	0	0	1	1	0	0	1	1	1

s_4 is a strong equilibrium for Nash stability, GMR, SMR, and SEQ. State s_9 is a strong equilibrium for GMR and SEQ. When state s_1 is selected as a statu quo, all possible UM evolutionary paths of the GDU conflict from s_1 to the equilibrium s_4 are obtained using the following steps:

- Using the UM weight matrix provided by Table 7.9, construct the conversion function

$$F(B^{(W^{(UM)})}) = [(B_{in}^{(W^{(UM)})})^T \cdot (B_{out}^{(W^{(UM)})})] \circ (E_l - D);$$

- This conversion function transforms the labeled multidigraph Fig. 7.8 (1) to the reduced line digraph Fig. 7.8 (2) including all UM arcs that is a simple digraph with no color constraints;

- Searching the colored paths $PA(s_1, s_4)$ between two vertices s_1 and s_4 in Fig. 7.8 (1) is equivalent to finding all paths $PA(a, b)$ for $a \in A_S, b \in A_E$ in Fig. 7.8 (2), where A_S and A_E are the two sets of arcs starting from vertex s_1 and arcs ending at vertex s_4;

- $A_S = \{a_{17}, a_{18}\}$ and $A_E = \{a_4, a_9\}$;

- Finding the legal UM paths $PA(a_{17}, a_4)$, $PA(a_{17}, a_9)$, $PA(a_{18}, a_4)$, and $PA(a_{18}, a_9)$ in the simple digraph Fig. 7.8 (2);

- $PA(a_{17}, a_4) : a_{17} \to a_1 \to a_{10} \to a_4$; $PA(a_{17}, a_9) : a_{17} \to a_9$;

- Find paths between two vertices, s_1 and s_4, using the paths between corresponding two arcs:

$$a_{17} \to a_1 \to a_{10} \to a_4 \Leftrightarrow s_1 \to s_2 \to s_3 \to s_5 \to s_4,$$

$$a_{17} \to a_9 \Leftrightarrow s_1 \to s_2 \to s_4.$$

If a conversion function is designed by $F(B) = B \cdot W^{+,++}$, then the original graph Fig. 7.8 (1) is reduced to the graph shown in Fig. 7.9 including WIs only. The dynamics of the GDU conflict evolution from the status quo state s_1 to the desirable equilibrium state s_9 by the legal WIs is portrayed in Fig. 7.10. Specifically, the evolution path $PA^{+,++}(s_1, s_9)$ of the GDU conflict from state s_1 to state s_9 is

$$s_1 \longrightarrow s_6 \longrightarrow s_8 \longrightarrow s_9.$$

Status quo analysis for the multiple levels of preference will be carried out in future research that is listed in Subsection 8.2.

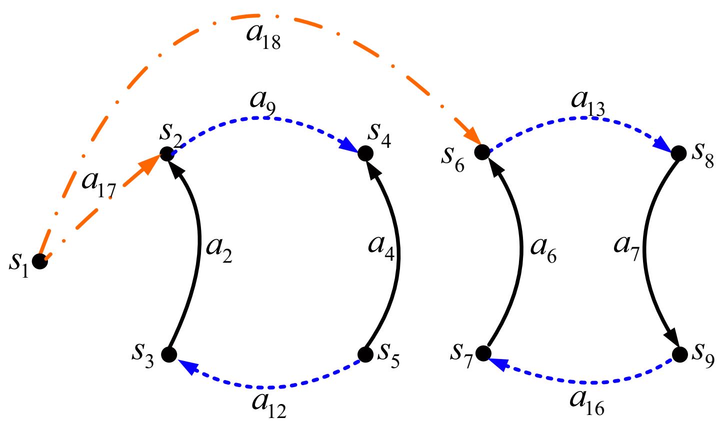


Figure 7.9: **The reduced graph allowing WIs only for the GDU conflict.**

7.2 Matrix Representations for Coalition Stability Analysis

Any subset H of DMs in the set N is called a coalition. If $|H| > 0$, then the coalition H is non-empty. If $|H| > 1$, then the coalition H is non-trivial. If $H = \{i\}$ is trivial, the DM i's reachable lists from a state $s \in S$ by various moves for appropriate preference structures are as follows [16, 27, 46]:

- $R_i(s)$: DM i's reachable list from state s by unilateral moves (UMs) in one step;

- $R_i^+(s)$: DM i's reachable list from state s by unilateral improvements (UIs) in one step;

Options	Status quo	Transitional states		Equilibrium
USS				
1.Proceed	Y	Y	Y →	N
2.Modify	N	N	N →	Y
CDO				
3.Legal	N	N →	Y	Y
IJC				
4.Completion	N	N	N	N
5.Modification	N →	Y	Y	Y
State number	s_1	s_6	s_8	s_9

Figure 7.10: **The GDU conflict evolution from the status quo s_1 to state s_9.**

- $R_i^{+,U}(s)$: DM i's reachable list from state s by unilateral improvements or unilateral uncertain moves (UIUUMs) in one step;

- $R_i^{+,++}(s)$: DM i's reachable list from state s by mild unilateral improvements or strong unilateral improvements (WIs) in one step.

If $|H| > 1$ is non-trivial, the reachable lists of coalition H from state $s \in S$ by various moves for appropriate preference structures are as follows [16, 28, 46]:

- $R_H(s)$: the reachable list of coalition H from state s by the legal UMs;

- $R_H^+(s)$: the reachable list of coalition H from state s by the legal UIs;

- $R_H^{+,U}(s)$: the reachable list of coalition H from state s by the legal UIUUMs;

- $R_H^{+,++}(s)$: the reachable list of coalition H from state s by the legal WIs.

Therefore, for $|H| > 0$, the reachable lists of coalition H from state s are the sets of states attainable by adding states that are one-step moves from state s by some DM in H or adding states that are group moves from status quo state s by some or all DMs in H. Note that the DMs in H do not cooperate. They are assumed to move in some order according to the usual restriction for the legal moves that no decision maker can move twice in succession along any path.

A state that is not an equilibrium has no long-term stability because there is at least one individual DM who has the incentive to move away to upset the temporarily stable state [43]. Therefore, a non-equilibrium state is not expected to persist in any case, including coalition stability. The following discussions of coalition stability are focused on the status quo states that are equilibria. The coalition stability analysis within the graph model assesses whether states that are stable from individual viewpoints may be unstable for coalitions. Therefore, coalition analysis provides valuable guidance for decision analysts.

To date, coalition analysis is based on Nash stability [43], GMR, SMR, and SEQ [36, 37] for simple preference. However, to make coding easier, these stabilities are based on a transitive graph that allows the same DM to move twice in succession, which is inconsistent with the standard restriction in the graph model. The condition of a transitive graph for coalition analysis is relaxed and coalition stabilities based on Nash stability are extended to models including preference uncertainty and strength of preference, which are the objectives of the next subsection. Additionally, the existing coalition stabilities are given in terms of logical representations, which make coding and calculation difficult. To implement coalition analysis in an algebraic system, matrix representation of coalition stability analysis (MRCSA) is developed next.

7.2.1 Extension of Coalition Stability in the Graph Model

The original coalition analysis uses simple preference. To enhance GMCR applicability, the graph model has recently been developed in two new directions—preference uncertainty and preference strength. Therefore, coalition stability analysis is expanded to models including preference uncertainty and strength of preference in this research.

7.2.1.1 Coalition Stability in the Graph Model with Preference Uncertainty

A DM may be conservative or aggressive, avoiding or accepting states of uncertain preference, depending on the level of satisfaction with the current position. Coalition stability is extended to consider conservative coalition stability and aggressive coalition stability for the graph model with preference uncertainty.

Definition 7.3. *For $s_1 \in R_H(s)$, s_1 is* ***a coalition improvement (CI)*** *by H from state s iff, for every $i \in H$, $s_1 \succ_i s$.*

Although this definition has not concerned uncertain preference, it is different from Definition 2.42 of a coalition improvement by H for simple preference, because Definition 2.42 cannot be employed to analyze models with uncertain preference.

Definition 7.4. *For $s_1 \in R_H(s)$, s_1 is **a coalition improvement or uncertain move (CIUM)** by H from state s iff, for every $i \in H$, $s_1 \succ_i s$ or $s_1 U_i s$.*

A coalition improvement or uncertain move s_1 by H is a threat, or potential threat, to the stability of state s. A coalition improvement or uncertain move for H from state s is a state s_1 that is reachable by H from s and preferred or having uncertain preference relative to s by every DM in H.

Definition 7.5. *State s is **unstable for coalition** H iff there exists a coalition improvement or uncertain move by H from s.*

Note that even if s is stable for each DM $i \in N$, the instability of state s for a coalition H makes it unlikely to survive as a resolution. We now define some forms of stability for coalitions in a graph model including preference uncertainty.

Definition 7.6. *Let $H \subseteq N$. State $s \in S$ is **conservatively stable for coalition** H iff, for every $s_1 \in R_H(s)$, there exists $i \in H$ with $s \succeq_i s_1$ or $sU_i s_1$.*

Coalition H is said to be conservative in deciding whether to move from the status quo, because the coalition H is not willing to accept the risk associated with moves from the status quo to states of uncertain preference.

Definition 7.7. *Let $H \subseteq N$. State $s \in S$ is **conservatively coalitionally stable** iff s is conservatively stable for every coalition H.*

Similarly, coalition H may be aggressive when considering whether to move from a status quo, in that the coalition is deterred only by states that are strictly less preferred than the status quo.

Definition 7.8. *Let $H \subseteq N$. State $s \in S$ is **aggressively stable for coalition** H iff, for every $s_1 \in R_H(s)$, there exists $i \in H$ with $s \succeq_i s_1$.*

Although Definition 7.8 excludes uncertainty in preferences when the focal DM considers incentives to leave a state, they are different from Definitions 2.44 since stability Definitions 2.44 cannot analyze conflicts including uncertain preferences.

Definition 7.9. *State $s \in S$ is **aggressively coalitionally stable** iff s is aggressively stable for every coalition $H \subseteq N$.*

When coalition H is trivial with $H = \{i\}$, Definition 7.6 is reduced to the following stability definition.

Definition 7.10. *State $s \in S$ is **conservatively stable for DM** i iff for every $s_1 \in R_i(s)$, $s \succeq_i s_1$ or sU_is_1.*

Obviously, Definition 7.10 is equivalent to $Nash_b$ and $Nash_d$ stabilities. Let us recall $Nash_b$ and $Nash_d$ stabilities.

Definition 7.11. *State $s \in S$ is $Nash_b$ **stable or** $Nash_d$ **stable for DM** i iff $R_i^+(s) = \emptyset$.*

When coalition $H = \{i\}$, Definition 7.8 is reduced to the following stability definition.

Definition 7.12. *State $s \in S$ is **aggressively stable for DM** i iff for every $s_1 \in R_i(s)$, $s \succeq_i s_1$.*

Definition 7.12 is equivalent to $Nash_a$ and $Nash_c$ stabilities. Recall them as follows.

Definition 7.13. *State $s \in S$ is $Nash_a$ **stable or** $Nash_c$ **stable for DM** i iff $R_i^{+,U}(s) = \emptyset$.*

7.2.1.2 Coalition Stability in the Graph Model with Strength of Preference

Definition 7.14. *For $s_1 \in R_H(s)$, s_1 is a **coalition strong or mild improvement** by H from state s iff, for every $i \in H$, $s_1 \gg_i s$ or $s_1 >_i s$.*

Definition 7.15. *State s is unstable for coalition H iff there exists a coalition strong or mild improvement by H from s.*

It should be pointed out even if s is an equilibrium for DM set N, it is possible s is unstable for a coalition $H \subseteq N$. If so, s cannot be selected as a resolution for a conflict.

Definition 7.16. *Let $H \subseteq N$. State $s \in S$ is **stable for coalition** H iff, for every $s_1 \in R_H(s)$, there exists $i \in H$ with $s \geq_i s_1$ or $s \gg_i s_1$.*

Note that if $H = \{i\}$ is trivial, then the stability for coalition $H = \{i\}$ is identical with Nash stability for DM i in the graph model with strength of preference. Recall that Nash stability for strength of preference.

Definition 7.17. *State $s \in S$ is* ***Nash stable for DM*** *$i \in N$ iff $R_H^{+,++}(s) = \emptyset$.*

Definition 7.18. *State $s \in S$ is coalitionally stable iff s is stable for every coalition $H \subseteq N$.*

7.2.2 Matrix Representation of Coalition Stabilities

The explicit algebraic expressions are advantageous for calculating potential resolutions and tracking conflict evolution. It is natural to exploit the matrix approach to perform coalition stability analysis for simple preference, preference with uncertainty, and strength of preference.

7.2.2.1 Matrix Representation of Coalition Stabilities for Simple Preference

Recall that E is the $m \times m$ matrix with each entry equal to 1 and e_s^T denotes the transpose of the s^{th} standard basis vector of the m-dimensional Euclidean space. Let M_H denote the UM reachability matrix by $H \subseteq N$ and find it using Corollary 6.2. The preference matrix $P_i^{-,=}$ denotes the $m \times m$ matrix with (s, q) entry

$$P_i^{-,=}(s,q) = \begin{cases} 1 & \text{if } s \succ_i q \text{ or } s \sim_i q, \\ 0 & \text{otherwise.} \end{cases}$$

Define the $m \times m$ coalition stability matrix by

$$M_H^C = M_H \cdot [E - (P_H^{-,=})^T], \; where \; P_H^{-,=} = \bigvee_{i \in H} P_i^{-,=}.$$

Theorem 7.2. *Let $H \subseteq N$ and $| H | \geq 2$. State $s \in S$ is stable for coalition H iff $e_s^T \cdot M_H^C \cdot e_s = 0$.*

Proof: Since

$$e_s^T \cdot M_H^C \cdot e_s = (e_s^T \cdot M_H) \cdot [\left(E - (P_H^{-,=})^T\right) \cdot e_s]$$

$$= \sum_{s_1=1}^{m} M_H(s, s_1)[1 - P_H^{-,=}(s, s_1)],$$

then $e_s^T \cdot M_H^C \cdot e_s = 0$ iff $P_H^{-,=}(s, s_1) = 1$ for any $s_1 \in R_H(s)$. Clearly,

$$P_H^{-,=}(s, s_1) = (\bigvee_{i \in H} P_i^{-,=})(s, s_1) = 1$$

iff there exists $i \in H$ such that $P_i^{-,=}(s, s_1) = 1$, i.e., $s \succeq_i s_1$. Consequently, the proof of the theorem follows by Definition 2.44. □

Theorem 7.3. *State $s \in S$ is coalitionally stable iff $\sum_{\forall H \subseteq N, |H| \geq 2} e_s^T \cdot M_H^C \cdot e_s = 0$.*

Proof: Since $\sum_{\forall H \subseteq N, |H| \geq 2} e_s^T \cdot M_H^C \cdot e_s = 0$ iff for any $H \subseteq N$ with $|H| \geq 2$, $e_s^T \cdot M_H^C \cdot e_s = 0$. By Theorem 7.2, $e_s^T \cdot M_H^C \cdot e_s = 0$ iff $s \in S$ is stable for coalition H. Consequently, $\sum_{H \subseteq N, |H| \geq 2} e_s^T \cdot M_H^C \cdot e_s = 0$ iff $s \in S$ is stable for any coalition $H \subseteq N$ with $|H| \geq 2$. The proof is completed by Definition 2.45. □

Theorems 7.2 and 7.3 prove the proposed matrix representation of coalition stability analysis (MRCSA) equivalent to logical representations of coalition stabilities (see Subsection 2.2.7) proposed by Kilgour et al. [43]. The matrix representation can be extended to models including uncertain preference, which is the objective of the next subsection.

7.2.2.2 Matrix Representation of Coalition Stabilities for Preference with Uncertainty

Let $P_i^{-,=,U}$ denote the $m \times m$ preference matrix with (s, q) entry

$$P_i^{-,=,U}(s, q) = \begin{cases} 1 & \text{if } s \succ_i q, s \sim_i q \text{ or } sU_iq, \\ 0 & \text{otherwise.} \end{cases}$$

Define the $m \times m$ conservative coalition stability matrix by

$$M_H^{CUc} = M_H \cdot [E - (P_H^{-,=,U})^T], \ \textit{where } P_H^{-,=,U} = \bigvee_{i \in H} P_i^{-,=,U}.$$

Theorem 7.4. *Let $H \subseteq N$ and $|H| \geq 2$. State $s \in S$ is conservatively stable for coalition H iff $e_s^T \cdot M_H^{CUc} \cdot e_s = 0$.*

The proof of this theorem is similar to that for Theorem 7.2.

Let $P_i^{-,=}$ denote the $m \times m$ preference matrix with (s, q) entry

$$P_i^{-,=}(s, q) = \begin{cases} 1 & \text{if } s \succ_i q \textit{ or } s \sim_i q, \\ 0 & \text{otherwise.} \end{cases}$$

Define the $m \times m$ aggressive coalition stability matrix by

$$M_H^{CUa} = M_H \cdot [E - (P_H^{-,=})^T], \ \textit{where } P_H^{-,=} = \bigvee_{i \in H} P_i^{-,=}.$$

Theorem 7.5. *Let $H \subseteq N$ and $\mid H \mid \geq 2$. State $s \in S$ is aggressively stable for coalition H iff $e_s^T \cdot M_H^{CUa} \cdot e_s = 0$.*

The proof of this theorem is similar to that for Theorem 7.2.

Theorems 7.4 and 7.5 prove the proposed matrix representation of coalition stability analysis (MRCSA) for preference with uncertainty equivalent to logical representations of coalition stabilities proposed by Definitions 7.6 and 7.8. The matrix representation can also be extended to models including strength of preference, which is the objective of the next subsection.

7.2.2.3 Matrix Representation of Coalition Stabilities for Preference with Strength

Let $P_i^{+,++}$ denote the $m \times m$ preference matrix with (s, q) entry

$$P_i^{+,++}(s,q) = \begin{cases} 1 & \text{if } q >_i s \text{ or } q \gg_i s, \\ 0 & \text{otherwise.} \end{cases}$$

Define the $m \times m$ coalition stability matrix in the graph model with strength of preference by

$$M_H^{CS} = M_H \cdot (P_H^{+,++})^T, \ where \ E - P_H^{+,++} = \bigvee_{i \in H} (E - P_i^{+,++}).$$

Theorem 7.6. *Let $H \subseteq N$ and $\mid H \mid \geq 2$. State $s \in S$ is stable for coalition H in the graph model with strength of preference iff $e_s^T \cdot M_H^{CS} \cdot e_s = 0$.*

The proof of this theorem is similar to that for Theorem 7.2. Based on Theorem 7.6, the following result can be easily obtained.

Theorem 7.7. *State $s \in S$ is coalitionally stable in the graph model with strength of preference iff $\sum_{\forall H \subseteq N, |H| \geq 2} e_s^T \cdot M_H^{CS} \cdot e_s = 0$.*

Theorems 7.6 and 7.7 prove the proposed matrix representation of coalition stability analysis (MRCSA) for strength of preference equivalent to the logical representations of coalition stabilities proposed by Definitions 7.16 and 7.18. The novel matrix approach to coalition stability analysis designed here is convenient for computer implementation and easy to employ, as is illustrated by the following applications to real-world conflict cases.

7.2.3 Applications

7.2.3.1 Coalition Stability Analysis for the Elmira Conflict including Simple Preference

The proposed algebraic method in this thesis has been employed to carry out stability analysis and status quo analysis for the Elmira conflict (see Subsections 7.1.2 and 7.1.3.1). Stability results presented in Table 7.3 indicates that states s_5, s_8 and s_9 are ideal equilibria for the Elmira conflict because they are stable for all DMs and the four basic solution concepts.

However, the story of the Elmira conflict does not end here. If DMs cooperate to make some agreements, resolution selections for the Elmira conflict will be impacted due to the cooperations among three DMs. For example, if MoE and UR form a coalition, then the coalition prefers state s_8 to state s_5, though neither MoE nor UR can make unilateral move from state s_5 to state s_8. In fact, state $s_8 \in R_H(s_5), s_8 \succ_1 s_5$, and $s_8 \succ_2 s_5$. Therefore, s_8 is a coalition improvement for coalition $H = \{MoE, UR\}$ from state s_5. Coalition stability matrices provided by Table 7.10 using Theorem 7.2 demonstrate that the individual ideal equilibrium s_5 is unstable for the coalition $H = \{MoE, UR\} = \{1, 2\}$, because the diagonal element at entry $(5, 5)$ of the coalition stability matrix M_H^C is nonzero, i.e., $M_H^C(5, 5) = e_5^T \cdot M_H^C \cdot e_5 \neq 0$. Therefore, under communication and cooperation, the individual ideal equilibrium state s_5 is vulnerable to coalition moves and the instability of state s_5 for the coalition H makes it unlikely to survive as a resolution for the Elmira conflict. Since $M_{N\setminus\{i\}}^C(8, 8) = M_{N\setminus\{i\}}^C(9, 9) = 0$, for $i = 1, 2, 3$, then states s_8 and s_9 are not only highly stable individually but also coalitionally stable.

7.2.3.2 Coalition Stability Analysis for the Gisborne Conflict with Preference Uncertainty

The history and background of the Gisborne conflict is introduced in Subsection 3.4. This conflict is modeled using three DMs: DM 1, **Federal (Fe)**; DM 2, **Provincial (Pr)**; and DM 3, **Support (Su)**. The Federal Government of Canada sided with the opposing groups by introducing a policy to forbid bulk water export from major drainage basins in Canada. The provincial government might restart the project at an appropriate time in the future due to its urgent need for cash and several support groups remain interested in the project [46]. Stability analysis and

Table 7.10: **Coalition stability matrices for $H = N\backslash\{i\}$ for the Elmira conflict**

Matrix	$M^C_{N\backslash\{1\}}$									$M^C_{N\backslash\{2\}}$									$M^C_{N\backslash\{3\}}$								
State	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9
s_1	0	1	1	0	1	1	1	0	1	1	1	0	0	1	1	0	0	1	0	1	1	1	1	1	0	1	1
s_2	0	1	0	1	0	1	0	1	1	0	0	0	0	0	1	0	0	1	0	1	0	1	1	1	0	1	1
s_3	0	1	0	0	0	1	1	0	1	1	1	1	1	1	1	1	1	1	0	1	0	1	1	1	0	1	1
s_4	0	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	1	1	0	0	0	0	0	0	0	0	1
s_5	1	1	1	1	0	1	1	1	1	1	1	0	0	0	1	0	0	1	0	1	0	0	1	1	1	1	1
s_6	0	1	0	1	0	1	0	1	1	0	1	0	0	0	0	0	0	1	0	1	0	0	1	1	0	1	1
s_7	0	1	1	0	0	1	0	0	1	1	1	1	1	1	1	0	1	1	0	1	0	0	1	1	0	1	1
s_8	0	1	0	1	0	0	0	0	1	0	1	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	1
s_9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

status quo analysis for the Gisborne conflict have been performed in Subsections 6.2.4.2 and 7.1.3.2 using the proposed algebraic method. Using the results provided by Table 6.15, states s_4 and s_6 are equilibria for the four basic solution concepts indexed b and d, so s_4 and s_6 are likely resolutions for the Gisborne conflict with preference uncertainty.

If the provincial government prefers the suggestion of the support group, then is called an economics-oriented provincial government, which implies that **Provincial** and **Support** cooperate to form a coalition $H = \{Pr, Su\}$. The graph model of the Gisborne conflict shows that neither **Provincial** nor **Support** can make a unilateral move from state s_6 to state s_4, but $s_4 \in R_H(s_6)$. We first use the logical coalition stability presented by Definition 7.8 to analyze the coalition stabilities of states s_4 and s_6. Since $s_4 U_2 s_6$ and $s_4 \succ_3 s_6$, state s_4 is a coalition improvement or uncertain move from s_6 for H. Hence, s_6 is unstable for the aggressive stability of coalition $H = \{Pr, Su\}$. Similarly, we can analyze the aggressive stability of coalition $H = \{Pr, Su\}$ for state s_4.

Using the proposed matrix representation for coalition stability analysis, the conservative and aggressive stability matrices of coalition $H = N\backslash\{i\}$ for $i = 1, 2, 3$, for the Gisborne conflict are presented in Table 7.11 and Table 7.12. Using the information provided by Table 7.11, states s_4 and s_6 are conservatively stable for the three coalitions, because conservative stability matrices of the coalitions have entries $(4, 4)$ and $(6, 6)$ zeros, i.e., $M^{CUc}_{N\backslash\{i\}}(4, 4) = M^{CUc}_{N\backslash\{i\}}(6, 6) = 0$

for $i = 1, 2, 3$. Therefore, states s_4 and s_6 are conservatively coalitionally stable. However, from Table 7.12, the aggressive stability matrix of coalition $H = \{Pr, Su\}$ has $(4,4)$ entry 0, i.e., $M^{CUa}_{N\backslash\{1\}}(4,4) = 0$, but $M^{CUa}_{N\backslash\{1\}}(6,6) \neq 0$. This means that s_4 is a resolution for the Gisborne conflict when the provincial government is economics-oriented. Similarly, since $M^{CUa}_{N\backslash\{3\}}(4,4) \neq 0$, but $M^{CUa}_{N\backslash\{3\}}(6,6) = 0$, state s_6 is a resolution for the Gisborne conflict when the provincial government is environment-oriented to accept the federal government's suggestion. From the above discussions, we find that the selection of the conflict resolution depends on the provincial government's attitude. If the support group convinces the provincial government of the urgent need for cash, state s_4 is selected as a resolution for resolving the Gisborne conflict. It means that the economics-oriented provincial government will lift the ban on bulk water export. On the other hand, for the environment-oriented provincial government, the resolution for the Gisborne conflict is selected as state s_6, which means that the provincial government will not lift the ban.

Table 7.11: **Conservative stability matrices of coalition $H = N\backslash\{i\}$ for $i = 1, 2, 3$ for the Gisborne model**

Matrix	$M^{CUc}_{N\backslash\{1\}}$								$M^{CUc}_{N\backslash\{2\}}$								$M^{CUc}_{N\backslash\{3\}}$							
State	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
s_1	1	0	1	0	1	0	1	0	1	1	0	0	1	1	0	0	0	1	1	1	0	1	1	1
s_2	0	1	0	1	0	1	0	1	1	0	0	0	1	1	0	0	1	0	1	1	1	0	1	1
s_3	1	0	0	0	1	0	1	0	1	0	0	1	1	0	1	1	1	1	0	1	1	1	0	1
s_4	0	1	0	0	0	1	0	1	1	0	1	0	1	0	1	1	1	1	1	0	1	1	1	0
s_5	1	0	1	0	0	0	1	0	1	1	0	0	0	1	0	0	0	0	0	0	0	1	1	1
s_6	0	1	0	1	0	0	0	1	1	1	0	0	1	0	0	0	0	0	0	0	1	0	1	1
s_7	1	0	1	0	1	0	1	0	1	0	1	1	1	0	1	1	0	0	0	0	1	1	0	1
s_8	0	1	0	1	0	1	0	1	1	0	1	1	1	0	1	1	0	0	0	0	1	1	1	0

7.3 Summary

In the original graph model, stability analysis and status quo analysis are carried out within a well-designed logical structure [16, 47, 48]. Nonetheless, the nature of the logical representations makes coding difficult and reduces adaptability. The algorithms for status quo analysis in the graph model for simple preference and

Table 7.12: **Aggressive stability matrices of coalition $H = N\backslash\{i\}$ for $i = 1, 2, 3$ for the Gisborne model**

Matrix	$M^{CUa}_{N\backslash\{1\}}$								$M^{CUa}_{N\backslash\{2\}}$								$M^{CUa}_{N\backslash\{3\}}$							
State	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
s_1	1	1	1	1	1	1	1	1	1	1	0	0	1	1	0	0	1	1	1	1	1	1	1	1
s_2	1	1	0	1	1	1	1	1	1	0	0	0	1	1	0	0	1	0	1	1	1	0	1	1
s_3	1	1	0	0	1	1	1	1	1	0	0	1	1	0	1	1	1	1	1	1	1	1	1	1
s_4	1	1	0	0	1	1	0	1	1	0	1	0	1	0	1	1	1	1	1	1	1	1	1	1
s_5	1	1	1	1	1	1	1	1	1	1	0	0	0	1	0	0	1	0	1	1	1	1	1	1
s_6	1	1	0	1	1	1	1	1	1	1	0	0	1	0	0	0	1	0	1	0	1	0	1	1
s_7	1	1	1	1	1	1	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	1	1	1
s_8	1	1	0	1	1	1	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	1	1	1

preference with uncertainty [47, 48], as well as coalition stability analysis in the graph model for simple preference [43] have been outlined, but were never integrated into GMCR II. Strength of preference was introduced into the graph model for stability analysis [27, 28], but was never integrated into status quo analysis.

To overcome these challenges and keep consistency with matrix representations of stability analysis, the proposed algebraic approach is employed with status quo analysis [71, 75, 80] and coalition analysis in this chapter. Additionally, the algebraic approach also reveals a relationship between stability analysis and post-stability analysis. This algebraic method facilitates the development of improved algorithms to incorporate status quo analysis and coalition stability analysis into a DSS.

Chapter 8

Conclusions and Future Work

8.1 Summary of Contributions

To enhance the applicability for the graph model for conflict resolution, GMCR has been developed in this thesis in two new directions—hybrid preference and multiple levels of preference. The hybrid preference framework is proposed to integrate preference strength and preference uncertainty into the paradigm of GMCR for multiple decision makers. This structure offers decision makers a more flexible mechanism for preference expression, which can include not only strong or mild preference of one state or scenario over another and equal preference, but also uncertain preference between two states. The preference framework is more general than existing models, which consider preference strength and preference uncertainty separately. The new stability concepts for hybrid preference expand the realm of applicability of GMCR and provide new insights for strategic conflicts. Particular advantages along this direction of research described in Chapter 3 are as follows:

- A new hybrid preference system combining strength and uncertainty for preferences, $\{\gg, >, \sim, U\}$, is proposed in Section 3.1 to include simple preference $\{\succ, \sim\}$, preference with uncertainty $\{\succ, \sim, U\}$, and strength of preference $\{\gg, >, \sim\}$ as special cases. Therefore, the new structure can be used to model complex strategic conflicts arising in practical applications.

- Four solution concepts, Nash, GMR, SMR, and SEQ, are expanded in Section 3.2 to take into account a wide range of preference frameworks. The redefined solution concepts handle hybrid preference and provide new insights for conflict studies.

- The algorithms are developed in Section 3.3 to accommodate the essential inputs of stability and status quo analyses efficiently. Specifically, the algorithms to find the reachable lists $R_H^{+,++}(s)$ and $R_H^{+,++,U}(s)$ for coalition H from any status quo s by legal sequences of WIs and WIUUMs are developed. All aspects of conflict evolution from a status quo are tracked, whether states changes occur by WIs or WIUUMs.

A multiple-level preference framework is developed and incorporated into GMCR. In this structure, a decision maker may have multiple levels of preference for one state over another; for example, if state s is preferred to state q, it may be mildly preferred at level 1 or preferred at level r for any positive parameter r. The multiple levels of preference relax the limitation of the current strength of the preference which can only handle two or three levels to an unrestricted degree. Then the extended definitions include extra degrees of stability, thereby improving practicability and gaining better insights into strategic conflicts. Specifically,

- A new multiple-level preference framework is devised in Section 4.1 to expand two-level preference $\{\succ, \sim\}$ and three-level preference $\{\gg, >, \sim\}$ to a more general multiple-level preference $\{>, \gg, \cdots, \overbrace{> \cdots >}^{d}, \sim\}$ for $d = 1, 2, \cdots, r$, where the number of levels, r, is unrestricted.
- Four solution concepts are extended in Section 4.2 to handle multiple levels of preference. Specifically, solution concepts at each level k are defined as $Nash_k$, GMR_k, SMR_k, and SEQ_k for $k = 1, \cdots, r$, where r is the maximum number of levels of preference between two states.

Another contribution is to use Algebraic Graph Theory to analyze a graph model. In this thesis, a graph model is treated as an edge-weighted, colored multidigraph in which each arc represents a legal unilateral move, distinct colors refer to different decision-makers, and the weight along the arc identifies some preference attribute. An important restriction of a graph model is that no decision maker can move twice in succession along any path. An algebraic approach to finding all edge-weighted, colored paths within a weighted colored multidigraph is developed in Chapter 5. The algebraic approach relieves the restriction imposed by the current graph model methodology on the behaviors of the decision maker and establishes an integrated paradigm for stability analysis and post-stability analysis, such as status quo analysis and coalition stability analysis, by revealing the inherent links not only between status quo analysis and the traditional stability analysis, but also among different preference structures for GMCR. It is obvious that this algebraic structure is flexible and can be easily modified to handle large-scale graph models. Specifically

- A reduced weighted edge consecutive matrix $LJ_r^{(W)}$ is designed in Subsection 5.3.1 as a conversion function to transform a weighted colored multidigraph to a simple digraph with no color constraints.

- This conversion function is used to transform the original problem of searching edge-weighted, colored paths in a weighted colored multidigraph to a standard problem of finding paths in a simple digraph.

- Using this conversion function, the weighted reachability matrix is developed to bridge the gap between status quo analysis and stability analysis.

- Utilizing the weight matrix to integrate all of the graph model preference structures.

Useful links between matrix theory and GMCR are revealed in this thesis. Previous stability definitions in the graph model were defined logically, in terms of the underlying graphs and preference relations. Thus, as has been observed previously, procedures to identify stable states based on these definitions are difficult to code because of the nature of the logical representations. To overcome this limitation, stability definitions in multiple-decision-maker graph models for simple preference, preference with uncertainty, and strength of preference are formulated explicitly in terms of matrices in Chapter 6. Specifically,

- Matrix representation of four basic solution concepts (MRSC) for simple preference is developed and its potentially wide realm of applicability is illustrated by two case studies: the Superpower Nuclear Confrontation conflict and the Rafferty-Alameda dams conflict.

- The MRSC method is expanded to models with preference uncertainty (MRSCU) for multiple decision makers and the two case studies of

Sustainable Development game and the Lake Gisborne conflict are used to show the applicability of this proposed matrix method.

- Strength of preference is proposed into the algebraic system to address matrix representation of four basic solution concepts for strength of preference (MRSCS). The developed MRSCS method is carried out using two case studies: the Sustainable Development conflict and the Garrison Diversion Unit (GDU) conflict.

The proposed matrix method is used for follow-up analyses such as status quo analysis and coalition stability analysis in a graph model, as presented in Chapter 7. Specifically,

- Matrix representation of status quo analysis (MRSQA) by tracking state-by-state conflict evolution for simple preference is developed and its applicability is illustrated by the Elmira conflict, in Subsection 7.1.2.

- Matrix representations of status quo analysis are addressed by tracking arc-by-arc conflict evolution for simple preference, preference with uncertainty, and strength of preference. The applications of these methods are illustrated by the Elmira conflict for simple preference, the Gisborne conflict for preference with uncertainty, and the GDU conflict for strength of preference, in Subsection 7.1.3.

- Coalition stability analysis based on Nash stability for simple preference is expanded to models including preference with uncertainty and strength of preference, in Subsection 7.2.1.

- Matrix representations of coalition stability analysis (MRCSA) are explored for simple preference, preference with uncertainty, and strength of preference in Subsection 7.2.2 and their potentials are revealed using two case studies: the Elmira conflict for simple preference and the Gisborne conflict for preference with uncertainty, in Subsection 7.2.3.

8.2 Future Work

To expand the realm of applicability of the algebraic approach, a conversion function will be designed for searching more general paths. To apply the proposed matrix methods to large conflict models, a decision support system MRCRDSS for carrying out individual stability analysis, status quo analysis, and coalition stability analysis would be very useful. The system based on algebraic characterization of MRCR can facilitate the development of a software package for conflict analysis. The following steps will be completed in the future.

- Inohara and Hipel's work [36,37] for coalition stabilities of Nash, GMR, SMR, and SEQ will be improved and expanded to generalized metarationalities in the graph model for conflict resolution;
- Matrix representations of solution concepts for hybrid preference and for multiple levels of preference will be explored;
- Matrix representations of status quo analysis for hybrid preference and for multiple levels of preference will be developed;
- A computer implementation of MRSC, MRSQA, and MRCSA in the graph model with various preference structures will be completed; and
- An integrated decision support system MRCR-DSS with the algebraic characterization will be developed to achieve the objectives presented in Fig. 8.1.

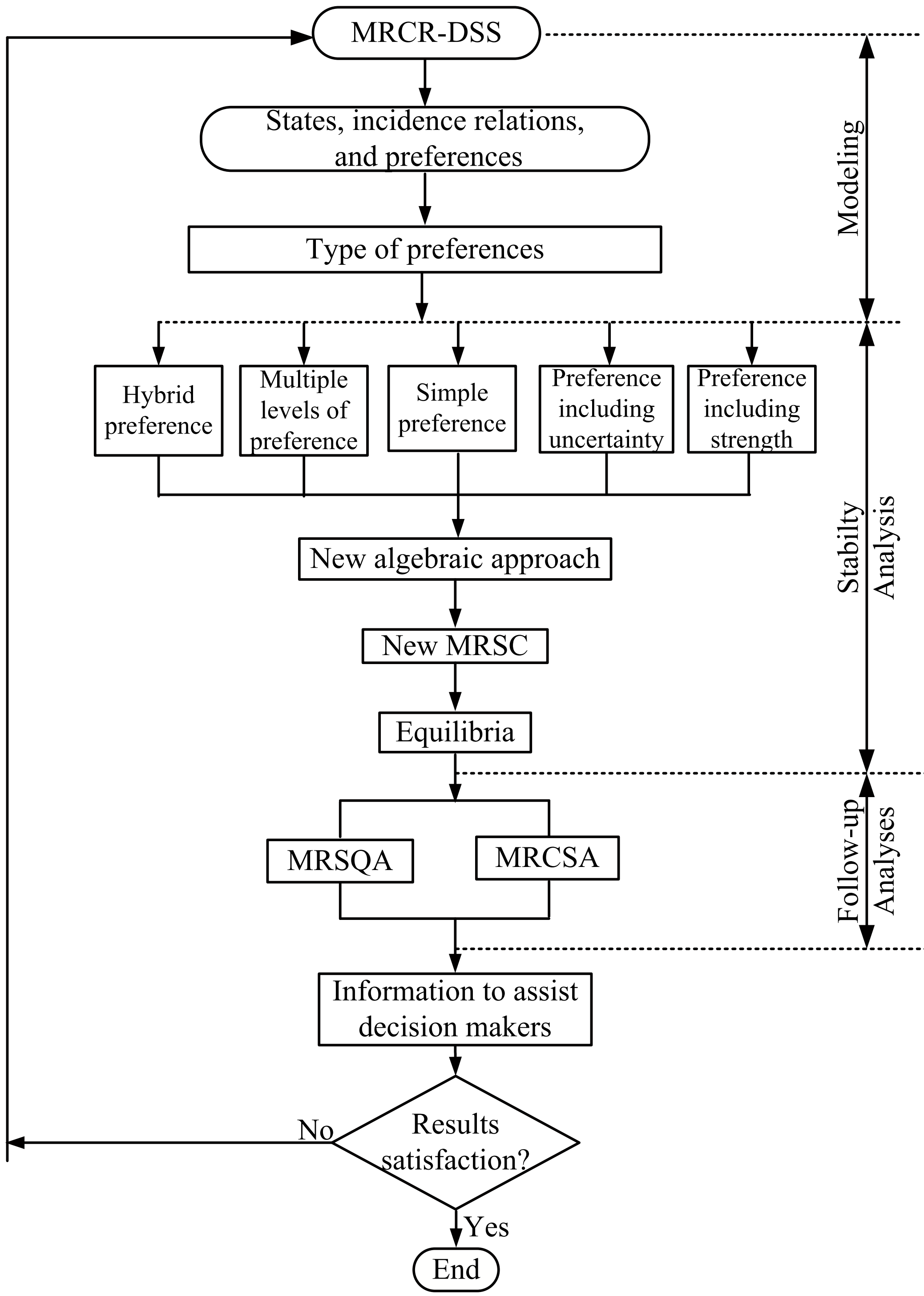


Figure 8.1: **Future objectives.**

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