On Holographic Non-Local Operators and Multiple M2-Branes Theories

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Physics

Waterloo, Ontario, Canada, 2009

Author's declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners. I understand that my thesis may be made electronically available to the public.

Abstract

Gauge-string duality has provided a powerful framework for the study of strongly coupled gauge theories and non-perturbative string models. This thesis analyzes the holographic description of non-local gauge theory operators and some aspects of the Bagger-Lambert theory. The latter, as a proposal for a multiple M2-branes effective theory, is conjectured to be the holographic dual of a compactification of M-theory.

We show that all half-BPS Wilson loop operators in $\mathcal{N} = 4$ SYM – which are labeled by Young tableaus – have a gravitational dual description in terms of D5-branes or alternatively in terms of D3-branes in $AdS_5 \times S^5$. We prove that the insertion of a half-BPS Wilson loop operator in the $\mathcal{N} = 4$ SYM path integral is achieved by integrating out the degrees of freedom on a configuration of bulk D5-branes or alternatively on a configuration of bulk D3-branes. We construct a new class of supersymmetric surface operators in $\mathcal{N} = 4$ SYM and find the corresponding dual supergravity solutions. Consistency requires constructing $\mathcal{N} =4$ SYM in the D7 supergravity background and not in flat space. This enlarges the class of holographic gauge theories dual to string theory backgrounds to gauge theories in non-trivial supergravity backgrounds. We write down a maximally supersymmetric one parameter deformation of the field theory action of Bagger and Lambert and we show that this theory on $R \times T^2$ is invariant under the superalgebra of the maximally supersymmetric Type IIB plane wave. It is argued that this theory holographically describes the Type IIB plane wave in the discrete light-cone quantization (DLCQ). Finally, we show by explicit computation that the Bagger-Lambert Lagrangian realizes the M2-brane superalgebra, including also two p-form central charges that encode the M-theory intersections involving M2-branes.

Acknowledgements

It is a pleasure to thank my advisor, Jaume Gomis, for his support and encouragement, and for sharing his ideas and knowledge with me.

I would like to thank Sujay Ashok, Dario Benedetti, Paolo Benincasa, Evgeny Buchbinder, Freddy Cachazo, Cohl Furey, Eleonora Dell'Aquila, Xiao Liu, Shunji Matsuura, Rob Myers, Sara Pasquetti, Federico Piazza, Amir Jafari Salim, Simone Severini, Aninda Sinha, Elena Taddei, Samuel Vazquez and Yidun Wan for collaborations and fruitful conversations.

Contents

1. Introduction

Technological development during the past century has led to a radical improvement of our understanding of nature. Many disciplines have been completely renewed and reformulated in terms of new paradigms and principles. Fundamental physics, like many other areas of science, has gone through revolutionary and surprising breakthroughs.

At the beginning of the twentieth century, Einstein realized that in order to explain the independence of the speed of light from the reference frame, the separate concepts of space and time had to be unified in the new concept of continuum spacetime. When the velocities involved are comparable to the speed of light, the continuum spacetime is the arena where physical phenomena take place. The unification of space and time is the main point of the Einstein improvement of the principle of relativity, formulated by Galilei in the seventeenth century. From a mathematical point of view, the Einstein relativity principle implies that the kinematical group of invariance of the theory is the Poincaré group, differently from Galilean relativity that implies that the group of invariance is the Galilei group. The modern terminology refers to Poincaré invariant theories as relativistic theories. Indeed, the Poincaré kinematical group is thought to be at a more fundamental level respect to the Galilei group. This last one is seen as an approximate symmetry when the velocities of the physical system are much smaller than the speed of light.

Newtonian gravity is not a relativistic theory, it is not invariant under the Poincare group. In order to reconcile gravity with the new principle of relativity, Einstein was led to the theory of general relativity where the gravitational field is described by a metric tensor defined on the spacetime continuum. In this way, gravity is encoded as a geometrical property of the spacetime manifold.

Approximately at the same time of Einstein's work, quantum mechanics was introduced as the theory describing the microscopic world. The theory correctly reproduces features that are not described by classical physics, like for instance the discretization of energy levels and the uncertainty principle.

Given the successful results of Einstein relativity and quantum mechanics, it became clear that a good theory for the fundamental constituents of matter has to be formulated in a relativistic quantum framework. Furthermore, the combination of relativity with quantum mechanics gives rise to new features, like for example the concept of antiparticles. The standard approach to describe quantum relativistic systems is given by quantum field theory. Among other things, this theory describes particle-antiparticle creation and annihilation and thus can deal with processes with a variable number of particles. In quantum field theory, the objects that are quantized are fields. The particles are described as quanta of excitation of the relevant field.

Our current description of the fundamental constituents of matter is based on four fundamental interactions: the electromagnetic force, the weak force, the strong force and the gravitational force. The first three can be correctly described in the framework of quantum field theory and the guiding principle to construct the theory is the gauge principle. With this approach, a global symmetry of the theory is promoted to be a local symmetry, i.e. it is "gauged". This procedure leads to the introduction of vector fields that are associated to bosons that mediate the relevant interaction. The standard model of particle physics is a quantum field theory based on the gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$. The strong interaction is associated to $SU(3)_C$ and sometimes this is stated by saying that the strongly interacting particles have three colors. For this reason the theory is called Quantum Chromodynamics (QCD). The electro-weak interaction is associated to $SU(2)_L \times U(1)_Y$. The matter fields are quarks and leptons, spinor fields that transform in the fundamental representation of the gauge group. The vector bosons that mediate the interactions transform in the adjoint of the gauge group. The masses of massive particles in the standard model can be generated through what is known as the Higgs mechanism. This implies the existence of a scalar particle, the Higgs boson, that to date hasn't been detected yet.

The standard model represents the highest achievement of fundamental theoretical physics and its predictions have been tested with great precision. However, it also shows several problems. First of all, it does not describe gravity and thus it cannot be considered as a complete theory for fundamental physics. As we have already mentioned, the standard model is constructed in the framework of quantum field theory and gravity cannot be quantized with this approach; indeed it is a non-renormalizable theory. Furthermore, the latest experimental data seems to suggest that neutrinos are particles with a small mass, instead of strictly massless as implied by the standard model. Another problem has to do with our poor understanding of strongly coupled quantum field theory. The coupling constants of the field theories in the standard model assume different values according to the level of energies. In particular, the strength of the coupling constants for the electromagnetic and weak interactions decrease when the energies decrease. In contrast, the strength of the coupling constant for QCD increases when the energy decreases. This feature is called asymptotic freedom and it makes it possible to study the strong interactions at high energies using perturbative techniques. Qualitatively the asymptotic freedom is the explanation of the quark confinement, the property of quarks to appear always in bounded states called hadrons. However, an analytical proof of the quark confinement is currently missing and this is a serious incompleteness of the theory since the matter of the real world appears to be in the confined phase. The standard model is a theory with many free parameters whose values have to be fixed by hand using experimental data. Also the construction of the model, like the choice of the matter fields or the choice of the gauge groups, needs to be guided by observational facts. All of this makes it plausible to look for a more complete theory of fundamental interactions.

Quantum field theory and thus also the standard model deal with pointlike particles, that is 0-dimensional objects. It seems natural to try to construct quantum theories for higher dimensional objects that propagate in the spacetime. String theory [1][2][3][4][5] arises from the study of 1-dimensional objects, strings. The theory was introduced at the end of the 1960s as a theory for strong interactions. At that time the experiments were showing an increasing number of different hadrons and particle physicists started to suspect that not all of them were fundamental constituents of matter. It was thus proposed that all the different hadrons were different excited states of 1-dimensional strings. The spectrum of string theory include open strings and closed strings. The open strings were associated to mesons. This simple model succeeded in explaining some of the phenomenology of hadrons, like the observed Regge trajectories. These are plots of the maximum spin of a hadron as a function of its mass. However, the improvement of the experimental data showed that the hadrons are made up by pointlike constituents and that led to the success of the parton model first and then to the asymptotically free QCD.

String theory was later reinterpreted as a theory of fundamental constituents of matter. According to the theory, all the fundamental particles of nature are different excitation states of a string. The closed string Hilbert space includes a spin-2 massless particle that can be interpreted as a graviton [6][7], the particle that mediates gravity. Therefore, the presence of this higher spin massless state that was a problem for the study of the strong interactions, now becomes one of the interesting features of the theory because also gravity is described in the same manner as the other interactions. For this reason string theory is considered as one of the best candidates for a theory that unifies all the fundamental forces of nature in a consistent way. Indeed, it is possible to study the interaction between the different string states building a perturbative expansion in terms of a string coupling g_s and there is evidence to conclude that the amplitudes are UV-finite at every order of perturbation.

The action for a 1-dimensional string propagating in spacetime was introduced by Nambu and Goto and it is a straightforward generalization of the action for the relativistic particle. It is proportional to the worldsheet area spanned by the string during the propagation

$$S = -T \int d^2\sigma \sqrt{-\det[(\partial_\alpha x^\mu)(\partial_\beta x_\mu)]} \tag{1.1}$$

where $T = \frac{1}{2\pi\alpha'}$ is the string tension and $\alpha' = \ell_s^2$ where ℓ_s is the string length. Worldsheet coordinates are denoted as σ^α with $\alpha = 0, 1$ and x^μ are spacetime coordinates with $\mu = 0, \ldots d-1$. It is possible to introduce a worldsheet metric $h_{\alpha\beta}$ and to rewrite the action (1.1) as

$$S = -\frac{T}{2} \int d^2\sigma \sqrt{-h} h^{\alpha\beta} (\partial_\alpha x^\mu)(\partial_\beta x_\mu). \tag{1.2}$$

The expression (1.2) is the Polyakov formulation of the string action and it is the action suitable for quantization. It results that the spectrum contains only spacetime bosons and there is a tachyonic state. Furthermore, working in a generic spacetime dimension d, the theory presents a Weyl anomaly that makes the quantum analysis inconsistent. The anomaly disappears when the spacetime dimension assumes the critical value $d = 26$ that is not the dimension of the spacetime we live in. In order to try to solve all of these problems one can consider a generalization of the actions (1.1) or (1.2). In particular, it turns out to be useful to add fermionic degrees of freedom to the theory and require the action to be invariant under supersymmetry. This can be done following the Ramond-Neveu-Schwarz (RNS)[8][9] or the Green-Schwarz (GS)[10] prescriptions. In the RNS formulation, worldsheet fermions are added to the action (1.2) so that each worldsheet scalar x^μ is associated with a superpartner ψ^μ that is a 2-dimensional worldsheet fermion. In the GS prescription the string is embedded in the superspace, an extension of the spacetime that includes also Grassmann-odd coordinates. However, once all the gauge freedom is fixed, the two theories reduce to the same action, the action of the superstring.

With this enhancement the theory acquires spacetime fermions and the tachyon disappears[1]. For the supersymmetric strings, the critical dimension is reduced to $d = 10$.

[1] The theory is tachion free thanks to the truncation of the spectrum proposed by Gliozzi, Scherk and Olive (GSO). The GSO projection is fundamental also to get a spacetime supersymmetric spectrum.

This value again does not match the observed physical dimension of the spacetime and to make contact with phenomenology, it is necessary to invoke other mechanisms such as spacetime compactification. The absence of gravitational and gauge anomalies in the spacetime theory imposes other constraints to the theory. It is possible to construct five different kinds of anomaly free string theories: the Type IIA and Type IIB that have $\mathcal{N}=2$ supersymmetry in the spacetime and differ from each other because of the different chirality of the supercharges; the Type I with $\mathcal{N}=1$ supersymmetry and gauge group $SO(32)$; the heterotic string with gauge group $SO(32)$ and the heterotic with gauge group $E_8 \times E_8$. In the limit where the massive states of the strings can be neglected, the dynamics of the massless modes of the strings can be described by the supergravity effective field theories. Indeed, in 10-dimensional spacetime there are exactly five types of supergravities that correspond to the low energy dynamics approximation of the five superstring theories.

It is possible to construct a supergravity theory also in 11-dimensions. This is the maximum dimension of the spacetime where we can have a supergravity theory, considering higher dimensions the theory would include fields with a spin higher than two that do not have a physical interpretation. Furthermore, there is a unique supergravity in 11-dimensions. In the same way like 10-dimensional supergravities are effective theories for string theories, the 11-dimensional supergravity is interpreted as the low energy description of a quantum theory called M-theory. The full formulation of M-theory is not known yet and it is not clear what the fundamental degrees of freedom are.

Performing a Kaluza-Klein reduction of the 11-dimensional supergravity, the theory becomes Type IIA supergravity. To get the precise 10-dimensional action, one has to consider a reduction over a circle of radius R given by

$$R = g_s^{\frac{2}{3}} \ell_{P11} \tag{1.3}$$

where ℓ_{P11} is the 11-dimensional Planck length and g_s is the string coupling. We see that in the regime where perturbative string theory is applicable, that is when $g_s < 1$, the eleventh dimension is very small and is negligible. This relation is believed to survive at any scale of energies and thus the M-theory is associated to the strong coupling regime of Type IIA strings.

All the five string theories are actually related to each other thanks to three different duality relations: the S,T and U duality. The different string theories and also the M-theory can be thought as different vacua of a unique theory [11][12][13]. However, these are not the complete set of vacua that the theory possesses. Indeed, it is possible to consider non-trivial backgrounds with the presence of fluxes and curved spacetime metric. It is also possible to consider string models in the presence of non-perturbative objects. To date, a dynamical mechanism able to select among all the possible vacua is still missing. That means that there are many possible string models and that string theory cannot be seen as a unique theory of the universe. In this picture, string theory can be thought as a completion of quantum field theory and thus, as quantum field theory, it is not a unique theory but rather a mathematical framework that could be used to describe many physical phenomena.

Even though string theory originates from the study of 1-dimensional strings, it includes also higher dimensional objects. Of particular relevance are the so called Dp-branes [14][15][16] (for a review see for instance [17]). These are p-dimensional, non-perturbative objects that are charged under the RR-modes, i.e. differential form massless modes of the perturbative strings. The presence of D-branes can be seen also in the framework of the effective supergravity approximation. In particular, the D-branes are solutions of the supergravity equations of motion where the spacetime metric is the metric of an extended black hole and the RR-flux assumes a source configuration. These solutions preserve one half of the supersymmetries of the background, they are half-BPS states. In this way, it is possible to describe stacks of multiple D-branes where the number of D-branes is encoded in the amount of RR-charge. The dynamics of this system can be studied considering string theory embedded in this supergravity background. This is one of the possible ways to describe the physics of D-branes and this picture is valid when the radius of curvature of the solution is small compared to the string length and the string loops are negligible. These conditions are satisfied when

$$1 \ll g_s N < N \tag{1.4}$$

where g_s is the string coupling and N is the number of D-branes.

Considering the full string theory formulation, the Dp-branes can be described as p-dimensional hyperplanes where the open strings can end. Indeed, it was shown by Polchinski [17] that a stack of N of these objects carry exactly N units of RR-charge. Furthermore, the supersymmetries preserved are the same as the corresponding supergravity solutions. When there are N D-branes, the open string endpoints are associated with a Chan-Paton factor that indicates which brane of the stack the string is ending on. It results that the effective loop expansion parameter is $g_s N$ instead of g_s. This implies that this picture is a good description for D-branes when

$$1 \gg g_s N. \tag{1.5}$$

D-branes have played a fundamental role in most of the latest achievements of string theory. Their applications range from the dualities between the different kinds of string theories to the physics of the black holes [18]. Furthermore, the two different descriptions of the D-branes that we have introduced above have led to the discoveries of the celebrated duality between string theories and gauge theories [19] (for a review see for instance [20]). The key idea is to study the physics of the brane in a particular low energy limit that is called the decoupling limit. In the gravity picture, this limit implies that only the degrees of freedom near the branes survive, that means that we have to consider string theory defined on the near horizon metric, i.e. the region of the background close to the brane. In the flat space picture, the decoupling limit implies that only the massless modes on the worldvolume of the brane are relevant. These degrees of freedom are described in an effective way by a gauge theory. If we believe that the two theories are actually describing the same physical object, it follows that a string theory defined on a particular background is equivalent to a corresponding gauge theory. These relations are strong-weak dualities, in the sense that in the regime where one of the two theories is weakly coupled, the other one is strongly coupled and vice versa. This makes the string-gauge duality a powerful tool to study a theory in a regime where it is difficult to analyze with a direct approach, but on the other hand it makes the checks of the duality a challenging task.

The first example of string-gauge duality was proposed by Maldacena [19] studying the decoupling limit of a stack of D3-branes. This analysis leads to conjecture that Type IIB string theory defined on $AdS_5 \times S^5$ in the presence of N units of 5-form RR-flux is equivalent to $\mathcal{N} = 4$ Supersymmetric Yang Mills theory (SYM) with $U(N)$ gauge group. The gauge theory is defined on a $(3+1)$-dimensional space that can be interpreted as the boundary of the AdS_5 space. The lagrangian preserves four spinorial charges ($\mathcal{N} = 4$), that gives the maximum amount of supercharges for a gauge theory. The $\mathcal{N} = 4$ SYM is a conformal field theory (CFT) and for this reason this duality is often called AdS/CFT. The number N of units of RR-charge corresponds to the rank of the gauge group on the field theory side. The other parameters in the theory are the radius L of the AdS_5 and S^5 spaces, the string coupling g_s and the SYM coupling g_{YM}. They are related by

$$g_{YM}^2 = 2\pi g_s \qquad L^4 = 4\pi g_s N \ell_s^4. \tag{1.6}$$

In its strongest formulation, the conjecture is assumed to be valid for each value of the parameters N and g_s. However, to make the theories more tractable it is useful to set the parameters to particular values. The 't Hooft limit considers $N \to \infty$ keeping fixed the 't Hooft coupling $\lambda = g_{YM}^2 N$. The 't Hooft coupling is the effective coupling constant for the gauge field theory. Under these conditions, g_s is very small and thus the string theory is not interacting. Furthermore, considering λ very large, the curvature radius becomes very large respect to the string length ℓ_s and thus it is possible to approximate the string theory with a supergravity theory defined on $AdS_5 \times S^5$. Thus, in the large N limit it is possible to describe a strongly coupled gauge theory using a supergravity model!

An immediate check of the AdS/CFT duality is given by the symmetries of the two theories. The bosonic group of invariance of $\mathcal{N} = 4$ SYM in $(3+1)$-dimensions is given by the conformal group $SO(4,2)$ and the R-symmetry group $SO(6)$. These groups are the isometry groups of respectively AdS_5 and S^5. Considering also the spinors it is possible to show that both the theories are invariant under the supergroup $PSU(2,2|4)$.

To use the duality, it is fundamental to understand the dictionary that is relating the two theories. Once the mapping between field theory operators and string theory fields is known, for instance it is possible to compute quantum field theory correlation functions at strong coupling using the gravitational dual [21][22]. This is based on the prescribed equivalence between the partition functions of the two theories

$$Z_{CFT}(\phi_0) = \left\langle \exp\left(\int \phi_0 \mathcal{O}\right)\right\rangle = Z_{string}(\phi|_{\partial AdS} = \phi_0) \tag{1.7}$$

where $\mathcal{O}$ is a CFT operator and ϕ is the associated string theory field. The string theory partition function is evaluated constraining the fields at the boundary and the asymptotic value of a string field works as a source for the corresponding gauge theory operator. In the large N, large λ limit, the string theory can be approximated by supergravity and $Z_{string} \sim \exp(iS_{sugra}|_{\phi_0})$ where $S_{sugra}|_{\phi_0}$ is the supergravity action evaluated on a classical solution with fixed boundary conditions.

The decoupling limit has been applied to many other D-brane systems and this has revealed dual systems where the gauge theory shares some features with theories that are relevant for phenomenology. In this way, it is possible to use string theory or supergravity to study strongly coupled QCD-like theories, the Quark Gluon Plasma and Condensed Matter systems.

On the other hand, the gauge string duality can be used to study the strong coupling regime of string theory using a gauge theory defined on the boundary of the spacetime where the string theory lives. Since string models are believed to be theories of quantum gravity, the duality provides an explicit realization of the holographic principle, that states that the quantum-gravity physics of a region of spacetime is encoded on the lower dimensional boundary of the region.

In a gauge theory it is possible to define various non-local operators. The Wilson loop [23] and 't Hooft loop [24] operators are defined on a curve embedded in the spacetime and describe the interaction between an external particle and the gauge field theory. A Wilson loop $W_R(C)$ is the trace in an arbitrary representation R of the gauge group G of the holonomy matrix associated with parallel transport along a closed curve C, that is

$$W_R(C) = \mathrm{Tr}_R \, P \exp\left(i \int_C ds A_\mu \dot{x}^\mu\right), \tag{1.8}$$

where $x^\mu(s)$ is the parametric description of C, P denotes path-ordering and A_μ is the gauge vector potential. It corresponds to inserting in the theory an external particle that has R charge with respect to the gauge group G. The Wilson loops can be used as a basis of operators and the dynamics can be described by the loop equation. This picture is called the loop-space formulation of gauge theory. When the expectation value of a Wilson loop is proportional to the exponent of the area of the minimal surface enclosed by the loop, it is said to satisfy the area law. This is a characteristic feature of confinement and for this reason the Wilson loops can work as order parameter for the confining-deconfining phase transition. The form of these operators for $\mathcal{N} = 4$ SYM is discussed in chapter 2.

It was shown in [25][26] that when the trace is evaluated in the fundamental representation of the gauge group $U(N)$, the Wilson loops are associated to fundamental classical strings in the string theory dual. In detail, given an operator $W(C)$ defined on a loop C, it is associated to a string embedded in the $AdS_5 \times S^5$ that ends on the boundary of AdS_5 on the path C. Indeed the boundary of AdS_5 is a $(3+1)$-dimensional space where the gauge theory is thought to live. Using the duality, it is possible to compute the expectation value of the Wilson loop when the theory is strongly coupled. It gives

$$\langle W(C) \rangle = \exp\left(i S_{NG}(C)\right) \tag{1.9}$$

where $S_{NG}(C)$ is the classical action for strings embedded in $AdS_5 \times S^5$ evaluated for a solution that ends on C at the boundary of the spacetime. When this prescription is applied to confining gauge theories with a gravity dual, the Wilson loops show the expected area law [27].

The 't Hooft operators are defined requiring that near the loop where the operator lives, the gauge field has a singularity of the Dirac monopole kind. The 't Hooft loop describes the coupling to the theory of an external magnetic monopole. Operators that cannot be expressed in terms of an operator insertion made out by the fields in the theory are usually called disorder operators. They are defined requiring some singular behavior for the fields in the spacetime region near the operator. This is the common way to define surface-operators, i.e. operators that live on a 2-dimensional surface [28][29]. The insertion of a surface operator corresponds to probing the gauge theory with a 1-dimensional string and it might be useful to detect new phase transitions that cannot be seen probing the theory with a point particle. It is not possible to construct surface operators using a trivial generalization of the definition of Wilson loop (1.8). Indeed, it is not possible to define an ordering for the operators that would be invariant under the surface reparameterization [30].

Part of this thesis will be focused on the study of non-local operators in the context of the gauge gravity duality, including Wilson loops in higher representations of the gauge group and a special class of surface operators that do admit a description in terms of an operator insertion made out of the fields appearing in the Lagrangian.

We have seen how the string-gauge correspondences arise from the study of the low energy dynamics of stacks of multiple D-branes and in principle one could apply the same procedure also in the context of M-theory, looking at the low energy physics of a stack of M-branes, higher dimensional objects that share some similarities with the D-branes of string theory (see for instance [31]). There exists only 2-dimensional and 5-dimensional M-branes, they are called respectively M2 and M5-branes. Like the D-branes in string theory, they can be described as classical solutions of the low energy theory, that is the 11-dimensional supergravity. However, there are not open strings ending on the M-branes and thus their worldvolume low energy description is not a SYM theory. Indeed, the thermodynamics of the supergravity solutions seem to suggest that the degrees of freedom of a stack of N M2-branes scale as $N^{3/2}$ and for N M5-branes as N^3. These numbers are not reproduced by a $U(N)$ gauge theory that has N^2 degrees of freedom.

The search for an effective theory of multiple M2-branes has gone through a remarkable development during the past year and a half, thanks to the papers by Bagger and Lambert [32][33][34][35] and ABJM [36] (Aharony, Bergman, Jafferis and Maldacena). Bagger and Lambert have constructed a (2+1)-dimensional field theory that is invariant under $\mathcal{N}=8$ supersymmetries and SO(8) R-symmetry. The theory is a gauge theory based on novel algebraic structures called 3-algebras. These are generalizations of Lie algebras and the main difference is that the Lie commutator is replaced by a three entries operator called 3-product (3-algebras are reviewed in Chapter 4 and 5). Given the amount of symmetry, the theory was interpreted as the worldvolume effective theory of a stack of M2-branes, where the SO(8) R-symmetry is interpreted as the rotational invariance of the spacetime transverse to the stack. However, once the 3-algebras were classified [37][38][39], it was shown that there is only one 3-algebra with Euclidean metric. This is interpreted as the gauge group associated to two M2-branes embedded in a particular orbifold called M-fold[2] [40][41]. It is possible to consider 3-algebras with a Lorentzian metric [42][43][44] but their M-theory interpretation is not clear yet [45][46][47][48]. Another proposal was put forward by ABJM in [36] where an $\mathcal{N} = 6$ Chern-Simons theory with gauge group $U(N) \times U(N)$ was proposed as the world volume theory for a stack of N M2-branes embedded in $\mathbf{R}^{1,2} \times \mathbf{C}^4/\mathbf{Z}_k$. This proposal has passed several checks, for instance the moduli space of the Chern-Simons is the same as the moduli space of N M2-branes probing a $\mathbf{C}^4/\mathbf{Z}_k$ singularity in M-theory. It was later shown by Bagger and Lambert that also this theory can be rewritten in terms of a 3-algebra structure [35]. The 3-algebras defined to construct $\mathcal{N} = 6$ theories [35] are different from the 3-algebras originally defined in [33] to construct $\mathcal{N} = 8$ theories.

The study of multiple M2-branes theories is an important task that might lead to a better understanding of the M-theory and to new gauge-gravity dualities suitable for phenomenological applications. Indeed, the effective field theories of multiple M2-branes include gauge fields described by a Chern-Simons Lagrangian and this represents a novelty for the gauge-gravity duality. This might end up being useful for the study of certain condensed matter systems whose physics is well described by 3-dimensional Chern-Simons theories. The last part of this thesis analyzes few aspects of the Bagger Lambert theory in the $\mathcal{N} = 8$ formulation.

Outline of the thesis

In chapter 2 we show that the Wilson loops in a higher representation of the gauge group correspond to D-branes on the string theory side. In particular, operators in the symmetric representation of the gauge group are associated to D3-branes and operators in the antisymmetric representations are associated to D5-branes. A Wilson loop in a generic representation corresponds to a particularly chosen stack of D3-branes or equivalently to

[2] This orbifold combine a geometrical action with an action on the branes.

a stack of D5-branes. The way to prove this correspondence is to look at a D-brane system that in addition to the stack of N D3-branes giving rise to the standard AdS/CFT, includes also some extra D3-branes or D5-branes. In the decoupling limit the extra D-branes become probes in $AdS_5 \times S^5$, when the backreaction can be neglected. On the field theory side, the extra D-branes introduce degrees of freedom localized on a one dimensional subspace of the spacetime. Integrating out the physics on this subspace introduces in the theory Wilson loops in a representation that is encoded in the physical properties of the branes.

In chapter 3 we study a particular D3-D7 intersection, where the D7-branes intersect the stack of D3-branes along two spacetime coordinates. In the gauge theory description of the system, the D7-branes manifest themselves as a surface operator, i.e. a non-local operator defined on a 2-dimensional subspace of the spacetime. The novelty is that this operator is of order-type, in the sense that it is expressed in terms of the fields in the gauge theory. Previously studied surface operators are of disorder-type, i.e. they cannot be expressed by the fields in the theory but they are defined by imposing a singular behavior to the fields on the surface where the operator lives. Another interesting result of this analysis is that also in the low energy field theory description, the backreaction of the D7-branes cannot be neglected so that in the end, we find a gauge-string duality where the gauge theory is defined in a curved spacetime. This result is important because it enlarges the holographic duality to gauge theories defined in a non-trivial background.

We then study the multiple membranes theory in the last part of the thesis. In chapter 4 we construct a one parameter mass deformation of the Bagger-Lambert Lagrangian that preserves all the supersymmetries. This theory represents a novel example of a maximally supersymmetric 3-dimensional gauge theory. Furthermore, when it is compactified on $R \times T^2$ it possesses the same superisometries of the Type IIB pp-wave background and due to the M2-branes interpretation of the Bagger-Lambert theory, it is interpreted as the Matrix theory for strings on Type IIB pp-wave. In chapter 5 we show by explicit computation that the Bagger-Lambert Lagrangian realizes the full M2-brane superalgebra, including also two central charges related to higher dimensional objects. These charges are associated to the intersections between the M2-branes and other M-branes and they should be realized by a Lagrangian describing the low energy physics of M2-branes. It follows that solitons of the Bagger-Lambert theory that are interpreted as worldvolume realizations of intersecting branes correctly saturate a BPS-bound given in terms of the corresponding charge. In chapter 6 we conclude with a summary of our results and discuss possible future directions.

2. Holographic Wilson Loops

We have already mentioned in the introduction that a necessary step in describing string theory in terms of a holographic dual gauge theory, is to be able to map all gauge invariant operators of the field theory in string theory. Indeed, all physical information is captured by gauge invariant observables.

Gauge theories can be formulated in terms of a non-Abelian vector potential or alternatively in terms of gauge invariant Wilson loop variables. The formulation in terms of non-abelian connections makes locality manifest while it has the disadvantage that the vector potential transforms inhomogeneously under gauge transformation and is therefore not a physical observable. The formulation in terms of Wilson loop variables makes gauge invariance manifest at the expense of a lack of locality. The Wilson loop variables, being non-local, appear to be the natural set of variables in which the bulk string theory formulation should be written down to make holography manifest. It is therefore interesting to consider the string theory realization of Wilson loop operators[3].

Significant progress has been made in mapping local gauge invariant operators in gauge theory in the string theory dual. Local operators in the boundary theory correspond to bulk string fields [19][21][22][20]. Furthermore, the correlation function of local gauge invariant operators is obtained by evaluating the string field theory action in the bulk with prescribed sources at the boundary.

Wilson loop operators are an interesting set of non-local gauge invariant operators in gauge theory in which the theory can be formulated. Mathematically, a Wilson loop is the trace in an arbitrary representation R of the gauge group G of the holonomy matrix associated with parallel transport along a closed curve C in spacetime. Physically, the expectation value of a Wilson loop operator in some particular representation of the gauge group measures the phase associated with moving an external charged particle with charge R around a closed curve C in spacetime.

In this chapter we show that all half-BPS operators in four dimensional $\mathcal{N} = 4$ SYM with gauge group $U(N)$ – which are labeled by an irreducible representation of $U(N)$ – can be realized in the dual gravitational description in terms of $D5$-branes or alternatively in

[3] As described in the introduction, this has been done for Wilson loops in the fundamental representation by [25][26].

terms of $D3$-branes in $\mathrm{AdS}_5 \times \mathrm{S}^5$. We show this by explicitly integrating out the physics on the $D5$-branes or alternatively on the $D3$-branes and proving that this inserts a half-BPS Wilson loop operator in the desired representation in the $\mathcal{N} = 4$ SYM path integral.

The choice of representation of $U(N)$ can be conveniently summarized in a Young tableau. We find that the data of the tableau can be precisely encoded in the AdS bulk description. Consider a Young tableau for a representation of $U(N)$ with n_i boxes in the i-th row and m_j boxes in the j-th column:

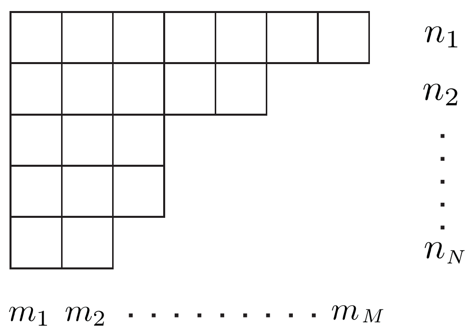

Fig. 1: A Young tableau. For $U(N)$, $i \leq N$ and $m_j \leq N$ while M and n_i are arbitrary.

We show that the Wilson operator labeled by this tableau is generated by integrating out the degrees of freedom on M coincident $D5$-branes in $\mathrm{AdS}_5 \times \mathrm{S}^5$ where the j-th $D5$-brane has m_j units of fundamental string charge dissolved in it. If we label the j-th $D5$-brane carrying m_j units of charge by $D5_{m_j}$, the Young tableau in Fig. 1. has a bulk description in terms of a configuration of $D5$-branes given by $(D5_{m_1}, D5_{m_2}, \ldots, D5_{m_M})$.

We show that the same Wilson loop operator can also be represented in the bulk description in terms of coincident $D3$-branes in $\mathrm{AdS}_5 \times \mathrm{S}^5$ where the i-th $D3$-brane has n_i units of fundamental string charge dissolved in it[4]. If we label the i-th $D3$-brane carrying n_i units of charge by $D3_{n_i}$, the Young tableau in Fig. 1. has a bulk description in terms of a configuration of $D3$-branes[5] given by $(D3_{n_1}, D3_{n_2}, \ldots, D3_{n_N})$.

[4] This D-brane has been previously considered in the study of Wilson loops by Drukker and Fiol [49]. In this chapter we show that these D-branes describe Wilson loops in a representation of the gauge group which we determine.

[5] The number of $D3$-branes depends on the length of the first column, which can be at most N. A $D3$-brane with $\mathrm{AdS}_2 \times \mathrm{S}^2$ worldvolume is a domain wall in AdS_5 and crossing it reduces the amount of five-form flux by one unit. Having such a $D3$-brane solution requires the presence

The way we show that the bulk description of half-BPS Wilson loops is given by D-branes is by studying the effective field theory dynamics on the N $D3$-branes that generate the $\text{AdS}_5 \times \text{S}^5$ background in the presence of bulk $D5$ and $D3$-branes. This effective field theory describing the coupling of the degrees of freedom on the bulk D-branes to the $\mathcal{N}=4$ SYM fields is a defect conformal field theory (see e.g [50][51][52]). It is by integrating out the degrees of freedom associated with the bulk D-branes in the defect conformal field theory that we show the correspondence between bulk branes and Wilson loop operators. We can carry out this procedure exactly and show that this results in the insertion of a half-BPS Wilson loop operator in the $\mathcal{N}=4$ SYM theory and that the mapping between the Young tableau data and the bulk $D5$ and $D3$ brane configuration is the one we described above.

First, we study the defect field theory associated to the bulk $D5$-branes. It results that these branes introduce in the theory fermionic degrees of freedom localized on the codimension three defect which corresponds to the location of the Wilson line. The $D3$-brane description of the Wilson loop is related to the $D5$-brane description by bosonizing the localized degrees of freedom of the defect conformal field theory. Indeed, we find that if we quantize these degrees of freedom as bosons instead, which is allowed in $0+1$ dimensions, that the defect conformal field theory captures correctly the physics of the bulk $D3$-branes.

We then consider the flat space brane configuration which yields in the near horizon limit the $D3$-branes Wilson loop in $\text{AdS}_5 \times \text{S}^5$. It corresponds to separating P D-branes by a distance L from a stack of $N+P$ coincident $D3$-branes and introducing k fundamental strings stretched between the two stacks of branes, in the limit $L \to \infty$. We can exactly integrate out the degrees of freedom introduced by the extra P D-branes from the low energy effective field theory describing this configuration and show that the net effect is to insert into the $U(N)$ $\mathcal{N}=4$ SYM path integral a Wilson loop operator with the expected representation. This explicitly confirms that the defect field theories associated to $D5$-branes and $D3$-branes are related by bosonization.

The outline of the chapter is as follows. In section 1 we identify the Wilson loop operators in $\mathcal{N}=4$ SYM that preserve half of the supersymmetries and study the $\mathcal{N}=4$

of five-form flux in the background to stabilize it. Therefore, we cannot put more that N such $D3$-branes as inside the last one there is no more five-form flux left and the $N+1$-th $D3$-brane cannot be stabilized.

subalgebra preserved by the half-BPS Wilson loops. Section 2 contains the embeddings of the $D5_k$ and $D3_k$ brane in $\mathrm{AdS}_5\times\mathrm{S}^5$ and we show that they preserve the same symmetries as the half-BPS Wilson loop operators. In section 3 we derive the defect conformal field theory produced by the interaction of the bulk $D5_k/D3_k$ branes with the $D3$ branes that generate the $\mathrm{AdS}_5\times\mathrm{S}^5$ background. We also show that a single $D5_k$-brane corresponds to a half-BPS Wilson loop in the k-th antisymmetric product representation of $U(N)$ while the $D3_k$-brane corresponds to the k-th symmetric product representation. In this section, the $D3_k$-brane defect field theory is introduced bosonising the $D5_k$-brane defect field theory. In section 4 we show that a half-BPS Wilson loop in any representation of $U(N)$ is described in terms of the collection of $D5$ or $D3$ branes explained in the introduction. In section 5 we show by first principles that a single $D3$-brane in $\mathrm{AdS}_5\times\mathrm{S}^5$ with k units of fundamental string charge correponds to a half-BPS Wilson loop in the k-th symmetric representation of $U(N)$. This is shown by studying in a certain infinite mass limit the Coulomb branch of $\mathcal{N}=4$ SYM in the presence of k W-bosons. In section 6 this result is generalized to arbitrary representations and confirms the proposal in section 4 that $D5$-branes and $D3$-branes Wilson loops are related by bosonization. Some computations have been relegated to section 7.

2.1. Wilson Loops in $\mathcal{N}=4$ SYM

A Wilson loop operator in $\mathcal{N}=4$ SYM is labeled by a curve C in superspace and by a representation R of the gauge group G. The data that characterizes a Wilson loop, the curve C and the representation R, label the properties of the external particle that is used to probe the theory. The curve C is identified with the worldline of the superparticle propagating in $\mathcal{N}=4$ superspace while the representation R corresponds to the charge carried by the superparticle.

The curve C is parameterized by $(x^\mu(s), y^I(s), \theta^\alpha_A(s))$ and it encodes the coupling of the charged external superparticle to the $\mathcal{N}=4$ SYM multiplet $(A_\mu, \phi^I, \lambda^A_\alpha)$, where μ (α) is a vector(spinor) index of $SO(1,3)$ while I (A) is a vector (spinor) index of the $SO(6)$ R-symmetry group of $\mathcal{N}=4$ SYM. Gauge invariance of the Wilson loop constraints the curve $x^\mu(s)$ to be closed while $(y^I(s), \theta^\alpha_A(s))$ are arbitrary curves.

The other piece of data entering into the definition of a Wilson loop operator is the choice of representation R of the gauge group G. For gauge group $U(N)$, the irreducible representations are conveniently summarized by a Young tableau $R = (n_1, n_2, \ldots, n_N)$, where n_i is the number of boxes in the i-th row of the tableau and $n_1 \geq n_2 \geq \ldots \geq n_N \geq 0$. The corresponding Young diagram is given by:

1	2	·	·	·	·	n_1
1	2	·	·	·	n_2	
1	2	·	·	·	n_3	
·	·	·	·			
1	2	·	n_N			

The main goal of this chapter is to identify all half-BPS Wilson loop operators of $\mathcal{N} = 4$ SYM in the dual asymptotically AdS gravitational description.

In this thesis we consider bosonic Wilson loop operators for which $\theta^\alpha_A(s) = 0$. Wilson loop operators coupling to fermions can be obtained by the action of supersymmetry and are descendant operators. The operators under study are given by

$$W_R(C) = \text{Tr}_R \, P \exp\left(i \int_C ds (A_\mu \dot{x}^\mu + \phi_I \dot{y}^I)\right), \tag{2.1}$$

where C labels the curve $(x^\mu(s), y^I(s))$ and P denotes path-ordering along the curve C.

We now consider the Wilson loop operators in $\mathcal{N} = 4$ SYM which are invariant under one-half of the $\mathcal{N} = 4$ Poincare supersymmetries and also invariant under one-half of the $\mathcal{N} = 4$ superconformal supersymmetries. The sixteen Poincare supersymmetries are generated by a ten dimensional Majorana-Weyl spinor ϵ_1 of negative chirality while the superconformal supersymmetries are generated by a ten dimensional Majorana-Weyl spinor ϵ_2 of positive chirality. The analysis in section 7 shows that supersymmetry restricts the curve C to be a straight time-like line spanned by $x^0 = t$ and $\dot{y}^I = n^I$, where n^I is a unit vector in R^6. The unbroken supersymmetries are generated by $\epsilon_{1,2}$ satisfying

$$\gamma_0 \gamma_I n^I \epsilon_1 = \epsilon_1 \qquad \gamma_0 \gamma_I n^I \epsilon_2 = -\epsilon_2. \tag{2.2}$$

Therefore, the half-BPS Wilson loop operators in $\mathcal{N} = 4$ SYM are given by

$$W_R = W_{(n_1, n_2, \ldots, n_N)} = \text{Tr}_R \, P \exp\left(i \int dt \, (A_0 + \phi)\right), \tag{2.3}$$

where $\phi = \phi_I n^I$. It follows that the half-BPS Wilson loop operators carry only one label: the choice of representation R.

We conclude this section by exhibiting the supersymmetry algebra preserved by the insertion of (2.3) to the $\mathcal{N} = 4$ path integral. This becomes useful when identifying the gravitational dual description of Wilson loops in later sections. In the absence of any operator insertions, $\mathcal{N}=4$ SYM is invariant under the $PSU(2,2|4)$ symmetry group. It is well known [53] that a straight line breaks the four dimensional conformal group $SU(2,2)\simeq SO(2,4)$ down to $SO(4^*)\simeq SU(1,1)\times SU(2)\simeq SL(2,R)\times SU(2)$. Moreover, the choice of a unit vector n^I in (2.3) breaks the $SU(4)\simeq SO(6)$ R-symmetry of $\mathcal{N}=4$ SYM down to $Sp(4)\simeq SO(5)$. The projections (2.2) impose a reality condition on the four dimensional supersymmetry generators, which now transform in the $(4,4)$ representation of $SO(4^*)\times Sp(4)$. Therefore, the supersymmetry algebra preserved[6] by the half-BPS Wilson loops is $Osp(4^*|4)$.

2.2. Giant and Dual Giant Wilson loops

The goal of this section is to put forward plausible candidate D-branes for the bulk description of the half-BPS Wilson loop operators (2.3). In the following sections we show that integrating out the physics on these D-branes results in the insertion of a half-BPS Wilson loop operator to $\mathcal{N}=4$ SYM. This provides the string theory realization of all half-BPS Wilson loops in $\mathcal{N}=4$ SYM.

Given the extended nature of Wilson loop operators in the gauge theory living at the boundary of AdS, it is natural to search for extended objects in $AdS_5\times S^5$ preserving the same symmetries as those preserved by the half-BPS operators (2.3) as candidates for the dual description of Wilson loops. The extended objects that couple to the Wilson loop must be such that they span a time-like line in the boundary of AdS, where the Wilson loop operator (2.3) is defined.

Since we want to identify extended objects with Wilson loops in $\mathcal{N}=4$ SYM on $R^{1,3}$, it is convenient to write the AdS_5 metric in Poincare coordinates

$$ds^2_{AdS}=L^2\left(u^2\eta_{\mu\nu}dx^\mu dx^\nu+\frac{du^2}{u^2}\right), \tag{2.4}$$

where $L=(4\pi g_sN)^{1/4}l_s$ is the radius of AdS_5 and S^5. Furthermore, since the Wilson loop operator (2.3) preserves an $SO(5)$ symmetry, we make this symmetry manifest by foliating the metric on S^5 by a family of S^4's

$$ds^2_{sphere}=L^2\left(d\theta^2+\sin^2\theta\; d\Omega_4^2\right), \tag{2.5}$$

[6] This supergroup has appeared in the past in relation to the baryon vertex [54][55].

where θ measures the latitude angle of the S^4 from the north pole and $d\Omega_4^2$ is the metric on the unit S^4.

In [25][26] the bulk description of a Wilson loop in the fundamental representation of the gauge group associated with a curve C in $R^{1,3}$ was given in terms of a fundamental string propagating in the bulk and ending at the boundary of AdS along the curve C. This case corresponds to the simplest Young tableau $R = (1, 0, \ldots, 0)$, with Young diagram $\square$.

The expectation value of the corresponding Wilson loop operator is identified with the action of the string ending at the boundary along C. This identification was motivated by considering a stack of D3-branes and moving one of them to infinity, leaving behind a massive external particle carrying charge in the fundamental representation of the gauge group.

The embedding corresponding to the half-BPS Wilson loop (2.3) for $R = (1, 0, \ldots, 0)$ is given by[7]

$$\sigma^0 = x^0 \qquad \sigma^1 = u \qquad x^i = 0 \qquad x^I = n^I, \tag{2.6}$$

so that the fundamental string spans an AdS_2 geometry sitting at $x^i = 0$ in AdS_5 and sits at a point on the S^5 labeled by a unit vector n^I, satisfying $n^2 = 1$. Therefore, the fundamental string preserves exactly the same $SU(1,1) \times SU(2) \times SO(5)$ symmetries as the one-half BPS Wilson loop operator (2.3). Moreover the string ends on the time-like line parameretrized by $x^0 = t$, which is the curve corresponding to the half-BPS Wilson loop (2.3).

In section 7 we compute the supersymmetries left unbroken by the fundamental string (2.6). We find that they are generated by two ten dimensional Majorana-Weyl spinors $\epsilon_{1,2}$ of opposite chirality satisfying

$$\gamma_0 \gamma_I n^I \epsilon_1 = \epsilon_1 \qquad \gamma_0 \gamma_I n^I \epsilon_2 = -\epsilon_2, \tag{2.7}$$

which coincides with the unbroken supersymmetries (2.2) of the half-BPS Wilson loop. Therefore, the fundamental string preserves the same $Osp(4^*|4)$ symmetry as the half-BPS Wilson loop (2.3).

The main question in this chapter is, what is the holographic description of half-BPS Wilson loop operators in higher representations of the gauge group?

7 The coordinates $\sigma^0, \ldots \sigma^p$ refer to the worldvolume coordinates on a string/brane.

Intuitively, higher representations correspond to having multiple coincident fundamental strings[8] ending at the boundary of AdS. This description is, however, not very useful as the Nambu-Goto action only describes a single string. A better description of the system is achieved by realizing that coincident fundamental strings in the $AdS_5 \times S^5$ background can polarize [57] into a single D-brane with fundamental strings dissolved in it, thus providing a concrete description of the coincident fundamental strings.

We now describe the way in which a collection of k fundamental strings puff up into a D-brane with k units of fundamental string charge on the D-brane worldvolume.

The guide we use to determine which D-branes are the puffed up description of k-fundamental strings is to consider D-branes in $AdS_5 \times S^5$ which are invariant under the same symmetries as the half-BPS Wilson loops[9], namely we demand invariance under $Osp(4^*|4)$. The branes preserving the $SU(1,1) \times SU(2) \times SO(5)$ symmetries of the Wilson loop are given by:

1) $D5_k$-brane with $AdS_2 \times S^4$ worldvolume.

2) $D3_k$-brane with $AdS_2 \times S^2$ worldvolume.

We now describe the basic properties of these branes that we need for the analysis in upcoming sections.

$D5_k$-brane as a Giant Wilson loop

The classical equations of motion for a $D5$-brane with an $AdS_2 \times S^4$ geometry and with k fundamental strings dissolved in it (which we label by $D5_k$) has been studied in the past in [58][59]. Here we summarize the necessary elements that will allow us to prove in the following section that this D-brane corresponds to a half-BPS Wilson loop operator.

The $D5_k$-brane is described by the following embedding

$$\sigma^0 = x^0 \qquad \sigma^1 = u \qquad \sigma^a = \varphi_a \qquad x^i = 0 \qquad \theta = \theta_k = \text{constant}, \tag{2.8}$$

together with a nontrivial electric field F along the AdS_2 spanned by (x^0, u). Therefore, a $D5_k$-brane spans an $AdS_2 \times S^4$ geometry[10] and sits at a latitude angle $\theta = \theta_k$ on the S^5, which depends on k, the fundamental string charge carried by the $D5_k$-brane:

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[8] Such a proposal was put forward in [56] by drawing lessons from the description of Wilson loops in two dimensonal QCD.

[9] We have already established that the fundamental strings (2.6) have the same symmetries as the half-BPS Wilson loops.

[10] φ_a are the coordinates on the S^4 in (2.5).

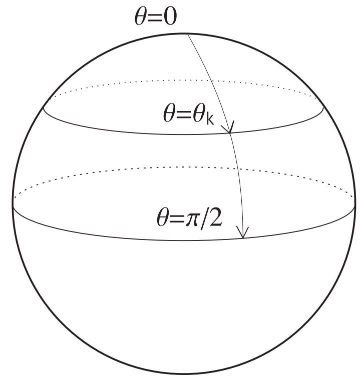

Fig. 2: A $D5_k$-brane sits at a latitude angle θ_k determined by the amount of fundamental string charge it carries.

This brane describes the puffing up of k fundamental strings into a D-brane inside S^5, so in analogy with a similar phenomenon for point-like gravitons [60], such a brane can be called a giant Wilson loop.

It can be shown [59] that θ_k is a monotonically increasing function of k in the domain of θ, that is $[0, \pi]$ and that $\theta_0 = 0$ and $\theta_N = \pi$, where N is the amount of flux in the $AdS_5 \times S^5$ background or equivalently the rank of the gauge group in $\mathcal{N} = 4$ SYM. Therefore, we can dissolve at most N fundamental strings on the $D5$-brane.

The $D5_k$-brane has the same bosonic symmetries as the half-BPS Wilson loop operator and it ends on the boundary of AdS_5 along the time-like line where the half-BPS Wilson loop operator (2.3) is defined. In section 7 we show that it also preserves the same supersymmetries (2.2) as the half-BPS Wilson loop operator (2.3) when $n^I = (1, 0, \ldots, 0)$ and is therefore invariant under the $Osp(4^*|4)$ symmetry group.

$D3_k$-brane as a Dual Giant Wilson loop

The classical equations of motion of a $D3$-brane with an $AdS_2 \times S^2$ geometry and with k fundamental strings dissolved in it (which we label by $D3_k$) has been studied by Drukker and Fiol [49]. We refer the reader to this reference for the details of the solution.

For our purposes we note that unlike for the case of the $D5_k$-brane, an arbitrary amount of fundamental string charge can be dissolved on the $D3_k$-brane. As we shall see in the next section, this has a pleasing interpretation in $\mathcal{N} = 4$.

The geometry spanned by a $D3_k$-brane gives an $\mathrm{AdS}_2\times\mathrm{S}^2$ foliation[11] of AdS_5, the location of the slice being determined by k, the amount of fundamental string charge:

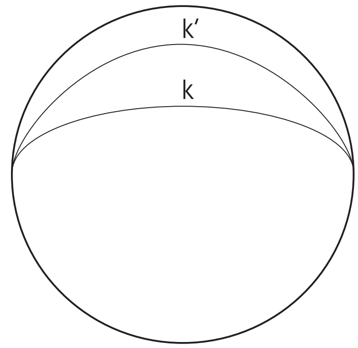

Fig. 3: A $D3_k$-brane gives an $\mathrm{AdS}_2\times\mathrm{S}^2$ slicing of AdS_5.

This brane describes the puffing up of k fundamental strings into a D-brane inside AdS_5, so in analogy with a similar phenomenon for point-like gravitons [64][65], such a brane can be called a dual giant Wilson loop.

By generalizing the supersymmetry analysis in [49] one can show that the $D3_k$-brane preserves precisely the same supersymmetries as the fundamental string (2.7) and therefore the same as the ones preserved by the half-BPS Wilson loop operator.

To summarize, we have seen that k fundamental strings can be described either by a single $D5_k$-brane or by a single $D3_k$-brane. The three objects preserve the same $Osp(4^*|4)$ symmetry if the fundamental string and the $D3_k$-brane sit at the north pole of the S^5, i.e. at $\theta=0$ corresponding to the unit vector $n^I=(1,0,\ldots,0)$. Furthermore, these three objects are invariant under the same $Osp(4^*|4)$ symmetry as the half-BPS Wilson loop operator (2.3).

2.3. Dirichlet Branes as Wilson loops

We show that the half-BPS Wilson loop operators in $\mathcal{N}=4$ SYM are realized by the D-branes in the previous section. We study the modification on the low energy effective field theory on the N $D3$-branes that generate the $\mathrm{AdS}_5\times\mathrm{S}^5$ background due to the

[11] This foliation structure and the relation with $\mathcal{N}=4$ SYM defined on the $\mathrm{AdS}_2\times\mathrm{S}^2$ boundary – which makes manifest the symmetries left unbroken by the insertion of a straight line Wilson loop – has been considered in [61][62][63].

presence of $D5$-brane giants and $D3$-brane dual giants. We can integrate out exactly the degrees of freedom introduced by the Wilson loop D-branes and show that the net effect of these D-branes is to insert into the $\mathcal{N} = 4$ $U(N)$ SYM path integral a Wilson loop operator in the desired representation of the $U(N)$ gauge group.

In order to develop some intuition for how this procedure works, we start by analyzing the case of a single $D5_k$-brane and a single $D3_k$-brane. We now show that a $D5_k$-brane describes a half-BPS Wilson loop operator in the k-th antisymmetric product representation of $U(N)$ while a $D3_k$-brane describes one in the k-th symmetric product representation.

In section 4 we proceed to show that a Wilson loop described by an arbitrary Young tableau corresponds to considering multiple D-branes. We also show that a given Young tableau can be either derived from a collection of $D5_k$-branes or from a collection of $D3_k$-branes and that the two descriptions are related by bosonization.

$D5_k$-brane as a Wilson Loop

We propose to analyze the physical interpretation of a single $D5_k$-brane in the gauge theory by studying the effect it has on four dimensional $\mathcal{N} = 4$ SYM. A $D5_k$-brane with an $\text{AdS}_2 \times \text{S}^4$ worldvolume in $\text{AdS}_5 \times \text{S}^5$ arises in the near horizon limit of a single $D5$-brane probing the N $D3$-branes that generate the $\text{AdS}_5 \times \text{S}^5$ background. The flat space brane configuration is given by:

$$\begin{array}{ccccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ D3 & \text{X} & \text{X} & \text{X} & \text{X} & & & & & & \\ D5 & \text{X} & & & & & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} \end{array} \qquad (2.9)$$

We can now study the effect of the $D5_k$-brane by analyzing the low energy effective field theory on a single $D5$-brane probing N $D3$-branes in flat space.

We note first that the $D5$-brane produces a codimension three defect on the $D3$-branes, since they overlap only in the time direction. In order to derive the decoupled field theory we must analyze the various open string sectors. The 3-3 strings give rise to the familiar four dimensional $\mathcal{N} = 4$ $U(N)$ SYM theory. The sector of 3-5 and 5-3 strings give rise to degrees of freedom that are localized in the defect. There are also the 5-5 strings. The degrees of freedom associated with these strings – a six dimensional vector multiplet on the $D5$-brane – are not dynamical. Nevertheless, as we will see, they play a crucial role in encoding the choice of Young tableau $R = (n_1, \ldots, n_N)$.

This brane configuration gives rise to a defect conformal field theory (see e.g. [50][51]), which describes the coupling of the $\mathcal{N}=4$ SYM to the localized degrees of freedom. The localized degrees of freedom arise from the 3-5 and 5-3 strings and they give rise to fermionic fields χ transforming in the fundamental representation of $U(N)$. We can write the action of this defect conformal field theory by realizing that we can obtain it by performing T-duality on the well studied D0-D8 matrix quantum mechanics (see e.g. [66][67]). Ignoring for the moment the coupling of χ to the non-dynamical 5-5 strings, we obtain that the action of our defect conformal field theory is given by[12]

$$S = S_{\mathcal{N}=4} + \int dt\, i\chi^\dagger \partial_t \chi + \chi^\dagger (A_0 + \phi)\chi, \tag{2.10}$$

where A_0 is the temporal component of the gauge field in $\mathcal{N}=4$ SYM and ϕ is one of the scalars of $\mathcal{N}=4$ SYM describing the position of the $D3$-branes in the direction transverse to both the $D3$ and $D5$ branes; it corresponds to the unit vector $n^I = (1, 0, \ldots, 0)$.

What are the $PSU(2,2|4)$ symmetries that are left unbroken by adding to the $\mathcal{N}=4$ action the localized fields? The supersymmetries of $\mathcal{N}=4$ SYM act trivially on χ. This implies that the computation determining the unbroken supersymmetries is exactly the same as the one we did for the Wilson loop operator (2.3). Likewise for the bosonic symmetries, where we just need to note that the defect fields live on a time-like straight line. Therefore, we conclude that our defect conformal field theory has an $Osp(4^*|4)$ symmetry, just like the half-BPS Wilson loop operator (2.3).

Even though the fields arising from the 5-5 strings are nondynamical, they play a crucial role in the identification of the $D5_k$-brane with a Wilson loop operator in a particular representation of the gauge group. As we discussed in the previous section, a $D5_k$-brane has k fundamental strings ending on it and we must find a way to encode the choice of k in the low energy effective field theory on the D-branes in flat space. This can be accomplished by recalling that a fundamental string ending on a D-brane behaves as an electric charge for the gauge field living on the D-brane. Therefore we must add to (2.10) a term that captures the fact that there are k units of background electric charge localized on the defect. This is accomplished by inserting into our defect conformal field theory path integral the operator:

$$\exp\left(-ik\int dt\, \tilde{A}_0\right). \tag{2.11}$$

[12] We do not write the $U(N)$ indices explicitly. They are contracted in a straighforward manner between the χ_i fields and the $A_{0\,ij}$ gauge field, where $i, j = 1, \ldots, N$.

Equivalently, we must add to the action (2.10) the Chern-Simons term:

$$-\int dt\, k\tilde{A}_0. \tag{2.12}$$

The effect of (2.12) on the $\tilde{A}_0$ equation of motion is to insert k units of electric charge at the location of the defect, just as desired.

We must also consider the coupling of the χ fields to the nondynamical gauge field $\tilde{A}$ on the $D5$-brane, as they transform in the fundamental representation of the $D5$-brane gauge field. Summarizing, we must add to (2.10) :

$$S_{extra} = \int dt\, \chi^\dagger \tilde{A}_0 \chi - k\tilde{A}_0. \tag{2.13}$$

The addition of these extra couplings preserves the $Osp(4^*|4)$ symmetry of our defect conformal field theory.

We want to prove that a $D5_k$-brane corresponds to a half-BPS Wilson loop operator in $\mathcal{N} = 4$ SYM in a very specific representation of $U(N)$. The way we show this is by integrating out explicitly the degrees of freedom associated with the $D5_k$-brane. We must calculate the following path integral

$$Z = \int [D\chi][D\chi^\dagger][D\tilde{A}_0]\, e^{i(S+S_{extra})}, \tag{2.14}$$

where S is given in (2.10) and S_{extra} in (2.13).

Let's us ignore the effect of S_{extra} for the time being; we will take it into account later. We first integrate out the χ fields. This can be accomplished the easiest by perfoming a choice of gauge such that the matrix $A_0 + \phi$ has constant eigenvalues[13]:

$$A_0 + \phi = \text{diag}(w_1, \ldots, w_N). \tag{2.15}$$

The equations of motion for the χ fields are then given by:

$$(i\partial_t + w_i)\chi_i = 0 \qquad \text{for} \qquad i = 1, \ldots, N. \tag{2.16}$$

[13] Here there is a subtlety. This gauge choice introduces a Fadeev-Popov determinant which changes the measure of the path-integral over the $\mathcal{N} = 4$ SYM fields. Nevertheless, after we integrate out the degrees of freedom associated with the $D5$-brane, we can write the result in a gauge invariant form, so that the Fadeev-Popov determinant can be reabsorbed to yield the usual measure over the $\mathcal{N} = 4$ SYM fields in the path integral.

Therefore, in this gauge, one has a system of N fermions χ_i with energy w_i.

The path integral can now be conveniently evaluated by going to the Hamiltonian formulation, where integrating out the χ fermions corresponds to evaluating the partition function of the fermions[14]. Therefore, we are left with

$$Z^* = e^{iS_{\mathcal{N}=4}} \cdot \prod_{i=1}^{N}(1 + x_i), \tag{2.17}$$

where $x_i = e^{i\beta w_i}$ and the $*$ in (2.17) is to remind us that we have not yet taken into account the effect of S_{extra} in (2.14). A first glimpse of the connection between a $D5_k$-brane and a half-BPS Wilson loop operator is to recognize that the quantity $x_i = e^{i\beta w_i}$ appearing in (2.17) with w_i given in (2.15), is an eigenvalue of the holonomy matrix appearing in the Wilson loop operator (2.3), that is $\exp i\beta\,(A_0 + \phi)$.

Since our original path integral (2.14) is invariant under $U(N)$ conjugations, it means that Z^* should have an expansion in terms of characters or invariant traces of $U(N)$, which are labeled by a Young tableau $R = (n_1, n_2, \ldots, n_N)$. In order to exhibit which representations R appear in the partition function, we split the computation of the partition function into sectors with a fixed number of fermions in a state. This decomposition allows us to write

$$\prod_{i=1}^{N}(1 + x_i) = \sum_{l=0}^{N} E_l(x_1, \ldots, x_l), \tag{2.18}$$

where $E_l(x_1, \ldots, x_l)$ is the symmetric polynomial:

$$E_l(x_1, \ldots, x_l) = \sum_{i_1 < i_2 \ldots < i_l} x_{i_1} \ldots x_{i_l}. \tag{2.19}$$

Physically, $E_l(x_1, \ldots, x_l)$ is the partition function over the Fock space of N fermions, each with energy w_i, that have l fermions in a state.

We now recognize that the polynomial E_l is the formula (see e.g [68]) for the trace of the half-BPS Wilson loop holonomy matrix in the l-th antisymmetric representation

$$E_l = \mathrm{Tr}_{(\underbrace{1,\ldots,1}_{l},0,\ldots,0)}\, P\exp\left(i\int dt\,(A_0+\phi)\right) = W_{(\underbrace{1,\ldots,1}_{l},0,\ldots,0)}, \tag{2.20}$$

[14] Here we introduce, for convenience an infrared regulator, so that t is compact $0 \le t \le \beta$.

where $W_{(\underbrace{1,\ldots,1}_{l},0,\ldots,0)}$ is the half-BPS Wilson loop operator (2.3) corresponding to the following Young diagram:

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \cdot \\ \hline \cdot \\ \hline \cdot \\ \hline l \\ \hline \end{array}$$

Therefore, integrating out the χ fields has the effect of inserting into the $\mathcal{N} = 4$ path integral a sum over all half-BPS Wilson loops in the l-th antisymmetric representation:

$$Z^* = e^{iS_{\mathcal{N}=4}} \cdot \sum_{l=0}^{N} W_{(\underbrace{1,\ldots,1}_{l},0,\ldots,0)} \cdot \tag{2.21}$$

It is now easy to go back and consider the effect of S_{extra} (2.13) on the path integral (2.14). Integrating over $\tilde{A}_0$ in (2.14) imposes the following constraint:

$$\sum_{i=1}^{N} \chi_i^\dagger \chi_i = k. \tag{2.22}$$

This constraint restrict the sum over states in the partition function to states with precisely k fermionic excitations. These states are of the form:

$$\chi_{i_1}^\dagger \ldots \chi_{i_k}^\dagger |0\rangle. \tag{2.23}$$

This picks out the term with $l = k$ in (2.21).

Therefore, we have shown that a single $D5_k$-brane inserts a half-BPS operator in the k-th antisymmetric representation in the $\mathcal{N} = 4$ path integral

$$D5_k \longleftrightarrow Z = e^{iS_{\mathcal{N}=4}} \cdot W_{(\underbrace{1,\ldots,1}_{k},0,\ldots,0)}, \tag{2.24}$$

where $S_{\mathcal{N}=4}$ is the action of $\mathcal{N} = 4$ SYM. The expectation value of this operator can be computed by evaluating the classical action of the $D5_k$-brane.

$D3_k$-brane as a Wilson Loop

We now consider what a $D3_k$ dual giant brane corresponds to in four dimensional $\mathcal{N} = 4$ SYM. For the time being, we make a very simple proposal for how to study the effect produced by a $D3_k$-brane on $\mathcal{N} = 4$ SYM and show that it leads to a consistent physical picture. In particular, we associate to the $D3_k$-brane a defect field theory that is obtained bosonizing the defect field theory associated to the $D5_k$-brane. In section 5 and section 6 we will give a first principle proof of this proposal, analyzing the flat space brane configuration that reproduces the $D3_k$-brane solution in the near horizon limit. In this section and in the next, the basic observation is that if we quantize the χ fields appearing in (2.10)(2.13) not as fermions but as bosons, which is something that is consistent when quantizing degrees of freedom in $0+1$ dimensions, we can show that the effect of the $D3_k$-brane is to insert a half-BPS Wilson loop operator (2.3) in the k-th symmetric representation of $U(N)$.

This result is in concordance with the basic physics of the probe branes. In the previous section we found that the amount of fundamental string charge k on a $D5_k$-brane can be at most N. On the other hand, we have shown that a $D5_k$-brane corresponds to a Wilson loop in the k-th antisymmetric representation of $U(N)$ so that indeed $k \leq N$, otherwise the operator vanishes. For the $D3_k$-brane, however, the string charge k can be made arbitrarily large. The proposal that the $D3_k$-brane can be studied in the gauge theory by quantizing χ as bosons leads, as we will show, to a Wilson loop in the k-th symmetric representation, for which there is a non-trivial representation of $U(N)$ for all k and fits nicely with the $D3_k$-brane probe expectations.

Formally, going from the $D5_k$ giant to the $D3_k$ dual giant Wilson line picture amounts to performing a bosonization of the defect field χ . It would be very interesting to understand from a more microscopic perspective the origin of this bosonization[15].

Having motivated treating χ as a boson we can now go ahead and integrate out the χ fields in (2.14). As before, we ignore for the time being the effect of S_{extra} in (2.14). We also diagonalize the matrix $A_0 + \phi$ as in (2.15).

The equations of motion are now those for N chiral bosons χ_i with energy w_i

$$(i\partial_t + w_i)\chi_i = 0 \qquad \text{for} \qquad i = 1, \ldots, N, \tag{2.25}$$

[15] A similar type of bosonization seems to be at play in the description of half-BPS local operators in $\mathcal{N} = 4$ SYM in terms of giants and dual giant gravitons [69].

where w_i are the eigenvalues of the matrix $A_0 + \phi$.

The path integral over χ in (2.14) is computed by evaluating the partition function of the chiral bosons, which yield

$$Z^* = e^{iS_{\mathcal{N}=4}} \cdot \prod_{i=1}^{N} \frac{1}{1-x_i}, \tag{2.26}$$

where $x_i = e^{i\beta w_i}$ and the $*$ in (2.17) is to remind us that we have not yet taken into account the effect of S_{extra} in (2.14). x_i are the eigenvalues of the holonomy matrix appearing in the Wilson loop operator (2.3).

In order to connect this computation with Wilson loops in $\mathcal{N} = 4$ SYM it is convenient to decompose the Fock space of the chiral bosons in terms of subspaces with a fixed number of bosons in a state. This decomposition yields

$$\prod_{i=1}^{N} \frac{1}{1-x_i} = \sum_{l=0}^{\infty} H_l(x_1, \ldots, x_l), \tag{2.27}$$

where $H_l(x_1, \ldots, x_l)$ is the symmetric polynomial:

$$H_l(x_1, \ldots, x_l) = \sum_{i_1 \leq i_2 \ldots \leq i_l} x_{i_1} \ldots x_{i_l}. \tag{2.28}$$

Physically, $H_l(x_1, \ldots, x_l)$ is the partition function over the Fock space of N chiral bosons with energy w_i that have l bosons in a state.

We now recognize that the polynomial H_l is the formula (see e.g [68]) for the trace of the half-BPS Wilson loop holonomy matrix in the l-th symmetric representation

$$H_l = \mathrm{Tr}_{(l,0,\ldots,0)}\, P \exp\left(i \int dt\; (A_0 + \phi) \right) = W_{(l,0,\ldots,0)}, \tag{2.29}$$

where $W_{(l,0,\ldots,0)}$ is the half-BPS Wilson loop operator (2.3) corresponding to the following Young diagram:

1	2	·	·	·	·	l

Therefore, integrating out the χ fields has the effect of inserting into the $\mathcal{N} = 4$ path integral a sum over all half-BPS Wilson loops in the l-th symmetric representation:

$$Z^* = e^{iS_{\mathcal{N}=4}} \cdot \sum_{l=0}^{N} W_{(l,0,\ldots,0)}. \tag{2.30}$$

It is now straightforward to take into account the effect of S_{extra} (2.13) in (2.14). Integrating over $\tilde{A}_0$ imposes the constraint (2.22). This constraint picks out states with precisely k bosons (2.23) and therefore selects the term with $l = k$ in (2.27).

Therefore, we have shown that a single $D3_k$-brane inserts a half-BPS operator in the k-th symmetric representation in the $\mathcal{N} = 4$ path integral

$$D3_k \longleftrightarrow Z = e^{iS_{\mathcal{N}=4}} \cdot W_{(k,0,\ldots,0)}, \tag{2.31}$$

where $S_{\mathcal{N}=4}$ is the action of $\mathcal{N} = 4$ SYM. The expectation value of this operator can be computed by evaluating the classical action of the $D3_k$-brane.

2.4. D-brane description of an Arbitrary Wilson loop

In the previous section we have shown that Wilson loops labeled by Young tableaus with a single column are described by a $D5$-brane while a $D3$-brane gives rise to tableaus with a single row. What is the gravitational description of Wilson loops in an arbitrary representation?

We now show that given a Wilson loop operator described by an arbitrary Young tableau, that it can be described either in terms of a collection of giants or alternatively in terms of a collection of dual giants.

Wilson loops as $D5$-branes

In the previous section, we showed that the information about the number of boxes in the Young tableau with one column is determined by the amount of fundamental string charge ending on the $D5$-brane. For the case of a single $D5_k$-brane, this background electric charge is captured by inserting (2.11)

$$\exp\left(-ik\int dt\ \tilde{A}_0\right) \tag{2.32}$$

in the path integral of the defect conformal field theory. Equivalently, we can add the Chern-Simons term:

$$-\int dt\ k\tilde{A}_0. \tag{2.33}$$

to the action (2.10). This injects into the theory a localized external particle of charge k with respect to the $U(1)$ gauge field $\tilde{A}_0$ on the $D5$-brane.

We now show that describing half-BPS Wilson loop operators (2.3) labeled by tableaus with more than one column corresponds to considering the brane configuration in (2.9) with multiple $D5$-branes.

In order to show this, we must consider the low energy effective field theory on M $D5$-branes probing N $D3$-branes. In this case, the $U(1)$ symmetry associated with the $D5$-brane gets now promoted to a $U(M)$ symmetry, where M is the number of $D5$-branes. Therefore, the defect conformal field theory living on this brane configuration is given by[16]

$$S = S_{\mathcal{N}=4} + \int dt \; i\chi_i^{I\dagger}\partial_t\chi_i^I + \chi_i^{I\dagger}(A_{0\,ij} + \phi_{ij})\chi_j^I, \tag{2.34}$$

where i, j is a fundamental index of $U(N)$ while I, J is a fundamental index of $U(M)$.

We need to understand how to realize in our defect conformal field theory that we have M $D5$-branes in $\mathrm{AdS}_5\times\mathrm{S}^5$ with a configuration of fundamental strings dissolved in them. Physically, the string endpoints introduce into the system a background charge for the $U(M)$ gauge field which depends on the distribution of string charge among the M $D5$-branes. The charge is labeled by a representation $\rho = (k_1, \ldots, k_M)$ of $U(M)$, where now $\rho = (k_1, \ldots, k_M)$ is a Young tableau of $U(M)$. A charge in the representation $\rho = (k_1, \ldots, k_M)$ is produced when k_i fundamental strings end on the i-th $D5$-brane. This $D5$-brane configuration can be labeled by the array $(D5_{k_1}, \ldots, D5_{k_M})$:

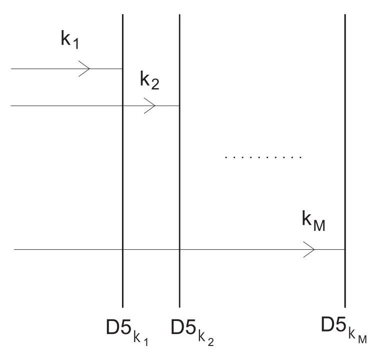

Fig. 4: Array of strings producing a background charge given by the representation $\rho = (k_1, \ldots, k_M)$ of $U(M)$. The $D5$-branes are drawn separated for illustration purposes only, as they sit on top of each other.

[16] For clarity, we write explicitly the indices associated with $U(N)$ and $U(M)$.

We must now add to the defect conformal field theory a term that captures that there is a static background charge $\rho = (k_1, \ldots, k_M)$ induced in the system by the fundamental strings. This is accomplished by inserting into the path integral a Wilson loop operator for the gauge field $\tilde{A}_0$. The operator insertion is given by

$$\mathrm{Tr}_{(k_1,k_2,\ldots,k_M)}\, P\exp\left(-i\int dt\ \tilde{A}_0\right), \tag{2.35}$$

which generalizes (2.32) when there are multiple $D5$-branes. We must also take into account the coupling of the localized fermions χ_i^I to $\tilde{A}_0$:

$$S_{extra} = \int dt\ \chi_i^{I\dagger}\tilde{A}_{0\,IJ}\chi_i^J. \tag{2.36}$$

In order to study what the $(D5_{k_1}, \ldots, D5_{k_M})$ array in $\mathrm{AdS}_5\times\mathrm{S}^5$ corresponds to in $\mathcal{N} = 4$ SYM, we need to calculate the following path integral

$$Z = \int [D\chi][D\chi^\dagger][D\tilde{A}_0]\ e^{i(S+S_{extra})}\cdot \mathrm{Tr}_{(k_1,k_2,\ldots,k_M)}\, P\exp\left(-i\int dt\ \tilde{A}_0\right), \tag{2.37}$$

where S is given in (2.34) and S_{extra} in (2.36).

We proceed by gauge fixing the $U(N)\times U(M)$ symmetry of the theory by diagonalizing $A_0 + \phi$ and $\tilde{A}_0$ to have constant eigenvalues respectively. The eigenvalues are given by:

$$\begin{aligned} A_0 + \phi &= \mathrm{diag}(w_1, \ldots, w_N) \\ \tilde{A}_0 &= \mathrm{diag}(\Omega_1, \ldots, \Omega_M). \end{aligned} \tag{2.38}$$

Since the path integral in (2.37) involves integration over $\tilde{A}_0$ care must be taken in doing the gauge fixing procedure[17]. As shown in section 7, the measure over the Hermitean matrix $\tilde{A}_0$ combines with the Fadeev-Popov determinant Δ_{FP} associated with the gauge choice

$$\tilde{A}_0 = \mathrm{diag}(\Omega_1, \ldots, \Omega_M) \tag{2.39}$$

to yield the measure over a unitary matrix U. That is

$$[D\tilde{A}_0]\cdot\Delta_{FP} = [DU], \tag{2.40}$$

[17] As discussed in footnote 11, the gauge fixing associated with the $U(N)$ symmetry can be undone once one is done integrating out over χ and $\tilde{A}_0$.

with $U = e^{i\beta\tilde{A}_0}$ and

$$[DU] = \prod_{I=1}^{M} d\Omega_I \; \Delta(\Omega)\bar{\Delta}(\Omega), \tag{2.41}$$

where $\Delta(\Omega)$ is the Vandermonde determinant[18]:

$$\Delta(\Omega) = \prod_{I<J}(e^{i\beta\Omega_I} - e^{i\beta\Omega_J}). \tag{2.42}$$

In this gauge, another simplification occurs. The part of the action in (2.37) depending on the χ fields is given by:

$$\int dt\; \chi_i^{I\dagger}(\partial_t + w_i + \Omega_I)\chi_i^I. \tag{2.43}$$

Correspondingly, the equations of motion are:

$$(i\partial_t + w_i + \Omega_I)\chi_i^I = 0 \qquad \text{for} \qquad i = 1, \ldots, N \quad I = 1, \ldots, M. \tag{2.44}$$

Therefore, we have a system of $N \cdot M$ fermions χ_i^I with energy $w_i + \Omega_I$.

We can explicitly integrate out the χ fields in Z (2.37) by going to the Hamiltonian formulation, just as before. The fermion partition function is:

$$\prod_{i=1}^{N}\prod_{J=1}^{M}(1 + x_i e^{i\beta\Omega_J}), \tag{2.45}$$

where as before $x_i = e^{i\beta w_i}$ is an eigenvalue of the holonomy matrix appearing in the Wilson loop operator (2.3) and $e^{i\beta\Omega_J}$ is an eigenvalue of the unitary matrix U.

Combining this with the computation of the measure, the path integral (2.37) can be written as

$$Z = e^{iS_{\mathcal{N}=4}} \cdot \int [DU]\; \chi_{(k_1,\ldots,k_M)}(U^*) \prod_{i=1}^{N}\prod_{J=1}^{M}(1 + x_i e^{i\beta\Omega_J}), \tag{2.46}$$

where we have identified the operator insertion (2.35) with a character in the $\rho = (k_1, \ldots, k_M)$ representation of $U(M)$:

$$\chi_{(k_1,\ldots,k_M)}(U^*) \equiv \mathrm{Tr}_{(k_1,\ldots,k_M)} e^{-i\beta\tilde{A}_0}. \tag{2.47}$$

[18] There is a residual $U(1)^N$ gauge symmetry left over after the gauge fixing (2.39) which turns Ω_I into angular coordinates. We are then left with the proper integration domain over the angles of a unitary matrix.

The partition function of the fermions (2.45) can be expanded either in terms of characters of $U(N)$ or $U(M)$ by using a generalization of the formula we used in (2.18). We find it convenient to write it in terms of characters of $U(M)$

$$\prod_{J=1}^{M}(1+x_i e^{i\Omega_J}) = \sum_{l=0}^{M} x_i^l\, \chi_{(\underbrace{1,\ldots,1}_{l},0,\ldots,0)}(U) = \sum_{l=0}^{M} x_i^l E_l(U_1,\ldots,U_M), \tag{2.48}$$

where

$$E_l(U) = \mathrm{Tr}_{(\underbrace{1,\ldots,1}_{l},0,\ldots,0)}\, e^{i\beta\tilde{A}_0} \tag{2.49}$$

is the character of $U(M)$ in the l-th antisymmetric product representation. We recall that $U = e^{i\beta\tilde{A}_0}$ and that $U_I = e^{i\beta\Omega_I}$ for $I = 1,\ldots,M$ are its eigenvalues.

We now use the following mathematical identity [70]

$$\prod_{i=1}^{N}\sum_{l=0}^{M} x_i^l E_l(U) = \sum_{M\geq n_1\geq n_2\geq\ldots\geq n_N} \det(E_{n_j+i-j}(U))\; \chi_{(n_1,\ldots,n_N)}(x), \tag{2.50}$$

where

$$\chi_{(n_1,\ldots,n_N)}(x) = W_{(n_1,\ldots,n_N)} \tag{2.51}$$

is precisely the half-BPS Wilson loop operator (2.3) in the $R = (n_1,\ldots,n_N)$ representation of $U(N)$. Therefore, the fermion partition function (2.45) can be written in terms of $U(N)$ and $U(M)$ characters as follows

$$\prod_{i=1}^{N}\prod_{J=1}^{M}(1+x_i e^{i\beta\Omega_J}) = \sum_{M\geq n_1\geq n_2\geq\ldots\geq n_N} \det(E_{n_j+i-j}(U)) W_{(n_1,\ldots,n_N)}. \tag{2.52}$$

The determinant $\det(E_{n_j+i-j}(U))$ can be explicitly evaluated by using Giambelli's formula (see e.g [68])

$$\det(E_{n_j+i-j}(U)) = \chi_{(m_1,m_2,\ldots,m_M)}(U), \tag{2.53}$$

where $\chi_{(m_1,m_2,\ldots,m_M)}(U)$ is the character of $U(M)$ associated with the Young tableau $(m_1,m_2,\ldots,m_M)$. This tableau is obtained from $(n_1,n_2,\ldots,n_N)$ by conjugation, which corresponds to transposing the tableau $(n_1,n_2,\ldots,n_N)$:

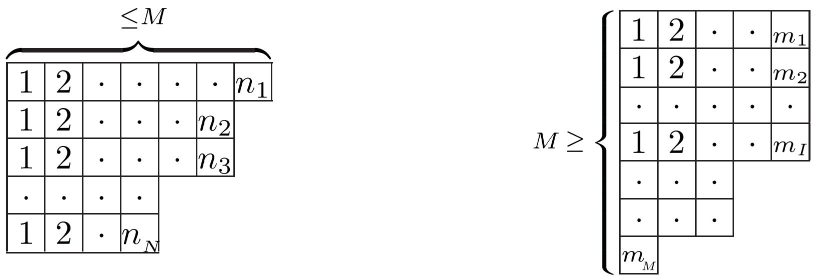

Fig. 5: A Young tableau and its conjugate. In the conjugate tableau the number of boxes in the i-th row is the number of boxes in the i-th column of the original one.

The number of rows in the conjugated tableau $(m_1, m_2, \ldots, m_M)$ is constrained to be at most M due to the $M \geq n_1 \geq n_2 \geq \ldots \geq n_N$ constraint in the sum (2.52).

These computations allow us to write (2.46) in the following way:

$$Z = e^{iS_{\mathcal{N}=4}} \cdot \sum_{M \geq n_1 \geq n_2 \geq \ldots \geq n_N} W_{(n_1,\ldots,n_N)} \cdot \int [DU]\ \chi_{(m_1,m_2,\ldots,m_M)}(U)\chi_{(k_1,\ldots,k_M)}(U^*). \tag{2.54}$$

Now using orthogonality of $U(M)$ characters:

$$\int [DU]\ \chi_{(m_1,m_2,\ldots,m_M)}(U)\chi_{(k_1,\ldots,k_M)}(U^*) = \prod_{I=1}^{M} \delta_{m_I,k_I}, \tag{2.55}$$

we arrive at the final result

$$Z = e^{iS_{\mathcal{N}=4}} \cdot W_{(l_1,\ldots,l_N)}, \tag{2.56}$$

where $(l_1, \ldots, l_N)$ is the tableau conjugate to $(k_1, \ldots, k_M)$.

To summarize, we have shown that a collection of $D5$-branes described by the array $(D5_{k_1}, \ldots, D5_{k_M})$ in $\mathrm{AdS}_5\times\mathrm{S}^5$ corresponds to the half-BPS Wilson loop operator (2.3) in $\mathcal{N} = 4$ SYM in the representation $R = (l_1, \ldots, l_N)$ of $U(N)$

$$(D5_{k_1}, \ldots, D5_{k_M}) \longleftrightarrow Z = e^{iS_{\mathcal{N}=4}} \cdot W_{(l_1,\ldots,l_N)}, \tag{2.57}$$

where $(l_1, \ldots, l_N)$ is the tableau conjugate to $(k_1, \ldots, k_M)$. Thererefore, any half-BPS Wilson loop operator in $\mathcal{N} = 4$ has a bulk realization. We now move on to show that there is an alternative bulk formulation of Wilson loop operators in $\mathcal{N} = 4$, now in terms of an array of $D3$-branes.

Wilson loops as $D3$-branes

Let's now consider the $\mathcal{N}=4$ gauge theory description of a configuration of multiple $D3$-branes in $\mathrm{AdS}_5\times\mathrm{S}^5$. As we have argued in section 3, the only modification in the defect conformal field theory compared to the case with the $D5$-branes is to quantize the χ_i^I fields as chiral bosons as opposed to fermions. Therefore, we consider the defect conformal field theory action (2.34) treating χ_i^I now as bosons.

Similarly to the case with multiple $D5$-branes, we realize the charge induced by the fundamental strings ending on the $D3$-branes by the Wilson loop operator (2.35) in the representation $\rho=(k_1,\ldots,k_M)$ of $U(M)$, where $\rho=(k_1,\ldots,k_M)$ is a Young tableau of $U(M)$. A charge in the representation $\rho=(k_1,\ldots,k_M)$ is produced when k_i fundamental strings end on the i-th $D3$-brane. This $D3$-brane configuration can be labeled by the array $(D3_{k_1},\ldots,D3_{k_M})$:

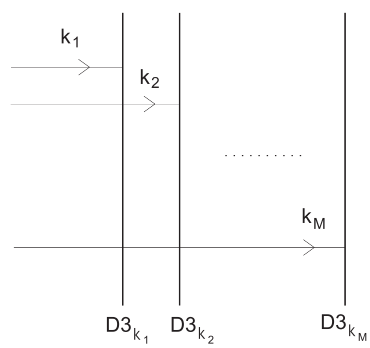

Fig. 6: Array of strings producing a background charge given by the representation $\rho=(k_1,\ldots,k_M)$ of $U(M)$. The $D3$-branes are drawn separated for illustration purposes only, as they sit on top of each other.

Therefore, in order to integrate out the degrees of freedom on the probe $D3$-branes we must calculate the path integral (2.37) treating χ_i^I as bosons.

We gauge fix the $U(N)\times U(M)$ as before. This gives us that χ_i^I are chiral bosons with energy $w_i+\Omega_I$. Their partition function is then given by

$$\prod_{i=1}^{N}\prod_{J=1}^{M}\left(\frac{1}{1-x_i e^{i\beta\Omega_J}}\right), \tag{2.58}$$

where as before $x_i=e^{i\beta w_i}$ is an eigenvalue of the holonomy matrix appearing in the Wilson loop operator.

Taking into account the measure change computed earlier, we have that

$$Z = e^{iS_{\mathcal{N}=4}} \cdot \int [DU]\; \chi_{(k_1,\ldots,k_M)}(U^*) \prod_{i=1}^{N} \prod_{J=1}^{M} \left(\frac{1}{1 - x_i e^{i\beta\Omega_J}} \right), \tag{2.59}$$

where we have identified the operator insertion (2.35) with a character in the $\rho = (k_1, \ldots, k_M)$ representation of $U(M)$:

$$\chi_{(k_1,\ldots,k_M)}(U^*) \equiv \mathrm{Tr}_{(k_1,\ldots,k_M)} e^{-i\beta\tilde{A}_0}. \tag{2.60}$$

Now we use that the partition function of the bosons can be expanded in terms of characters of $U(M)$ by generalizing formula (2.27)

$$\prod_{J=1}^{M} \left(\frac{1}{1 - x_i e^{i\beta\Omega_J}} \right) = \sum_{l=0}^{\infty} x_i^l\; \chi_{\,(l,0\ldots,0)}\,(U) = \sum_{l=0}^{\infty} x_i^l H_l(U_1, \ldots, U_M), \tag{2.61}$$

where

$$H_l(U) = \mathrm{Tr}_{\,(l,0\ldots,0)}\, e^{i\beta\tilde{A}_0} \tag{2.62}$$

is the character of $U(M)$ in the l-th symmetric product representation.

Using an identity from [70]

$$\prod_{i=1}^{N} \sum_{l=0}^{\infty} x_i^l H_l(U) = \sum_{n_1 \geq n_2 \geq \ldots \geq n_N} \det(H_{n_j+i-j}(U))\; \chi_{(n_1,\ldots,n_N)}(x), \tag{2.63}$$

where

$$\chi_{(n_1,\ldots,n_N)}(x) = W_{(n_1,\ldots,n_N)} \tag{2.64}$$

is the half-BPS Wilson loop operator corresponding to the Young tableau $R = (n_1, \ldots, n_N)$ of $U(N)$.

The Jacobi-Trudy identity (see e.g [68]) implies that

$$\det(H_{n_j+i-j}(U)) = \chi_{(n_1,n_2,\ldots,n_N)}(U), \tag{2.65}$$

where $\chi_{(n_1,n_2,\ldots,n_N)}(U)$ is the character of $U(M)$ associated with the Young tableau $(n_1, n_2, \ldots, n_N)$. Considering the antisymmetry of the elements in the same column, we get the constraint that $n_{M+1} = \ldots = n_N = 0$, otherwise the character vanishes.

These computations allow us to write (2.59) as:

$$Z = e^{iS_{\mathcal{N}=4}} \cdot \sum_{n_1 \geq n_2 \geq \ldots \geq n_N} W_{(n_1,\ldots,n_N)} \cdot \int [DU]\, \chi_{(n_1,\ldots,n_N)}(U)\, \chi_{(k_1,\ldots,k_M)}(U^*). \qquad (2.66)$$

Using

$$\int [DU]\, \chi_{(n_1,\ldots,n_N)}(U)\, \chi_{(k_1,\ldots,k_M)}(U^*) = \prod_{I=1}^{M} \delta_{n_I,k_I} \prod_{i=M+1}^{N} \delta_{n_i,0}, \qquad (2.67)$$

we get that:

$$Z = e^{iS_{\mathcal{N}=4}} \cdot W_{(k_1,\ldots,k_M,\ldots,0)}. \qquad (2.68)$$

We have shown that a collection of $D3$-branes described by the array $(D3_{k_1}, \ldots, D3_{k_M})$ in $\mathrm{AdS}_5 \times \mathrm{S}^5$ corresponds to the half-BPS Wilson loop operator (2.3) in $\mathcal{N} = 4$ SYM in the representation $R = (k_1, \ldots, k_N)$ of $U(N)$

$$(D3_{k_1}, \ldots, D3_{k_M}) \longleftrightarrow Z = e^{iS_{\mathcal{N}=4}} \cdot W_{(k_1,\ldots,k_M,0,\ldots,0)}. \qquad (2.69)$$

Therefore, any half-BPS Wilson loop operator in $\mathcal{N} = 4$ has a bulk realization in terms of $D3$-branes.

To summarize, we have shown that a half-BPS Wilson loop described by an arbitrary Young tableau can be described in terms of a collection of $D5$-branes or $D3$-branes. We have shown that indeed the relation between a Wilson loop in an arbitrary representation and a D-brane configuration is precisely the one described at the beginning of this chapter.

2.5. A $D3_k$-brane as a Wilson loop in the k-th symmetric representation

We have argued that the $D3_k$-brane solution in $\mathrm{AdS}_5 \times \mathrm{S}^5$ of [25][71] corresponds to a half-BPS Wilson loop operator labeled by the following Young tableau:

1	2	·	·	·	·	k

.

This solution [25][71] has an $\mathrm{AdS}_2 \times \mathrm{S}^2$ worldvolume geometry and carries k units of fundamental string charge. The fact that k is arbitrary, that there can be at most N such $D3$-branes in $\mathrm{AdS}_5 \times \mathrm{S}^5$, and its proposed relation through bosonization to the defect conformal field theory derived for the $D5_k$-brane[19] led us to the abovementioned proposal.

[19] This $D5$-brane, which has an $\mathrm{AdS}_2 \times \mathrm{S}^4$ worldvolume geometry and $k \leq N$ units of fundamental string charge, was shown to correspond to a Wilson loop in the k-th antisymmetric representation – a Young tableau with k boxes in one column – by integrating out the degrees of freedom on the $D5$-brane.

In this section we show that this proposal is indeed correct by studying a brane configuration in flat space. We integrate out the physics on the brane and show that the D-brane inserts the desired Wilson loop into the $\mathcal{N}=4$ SYM path integral. This brane configuration can also be studied in the near horizon limit and indeed reproduces the $D3_k$-brane solution of [25][71].

A half-BPS Wilson loop of $\mathcal{N}=4$ SYM in a representation[20] R

$$W_R = \text{Tr}_R P \exp\left(i\int dt\,(A_0+\phi)\right), \tag{2.70}$$

is obtained by adding a static, infinitely massive charged probe to $\mathcal{N}=4$ SYM. As already shown in [25][26](see also [72]), one way of introducing external charges in $U(N)$ $\mathcal{N}=4$ SYM is to consider a stack of $N+1$ $D3$-branes and going along the Coulomb brach of the gauge theory.

Let's consider the gauge theory on $N+1$ $D3$-branes and break the gauge symmetry down to $U(N)\times U(1)$ by separating one of the branes. In the gauge theory description this corresponds to turning on the following expectation value

$$\langle\phi\rangle = \begin{pmatrix} 0 & 0 \\ 0 & L \end{pmatrix}, \tag{2.71}$$

where ϕ is one of the scalar fields of $\mathcal{N}=4$ SYM, thus breaking the $SO(6)$ R-symmetry of $\mathcal{N}=4$ SYM down to $SO(5)$.

We are interested in studying the low energy physics of this D-brane configuration in a background where k static fundamental strings are stretched between the two stacks of $D3$-branes:

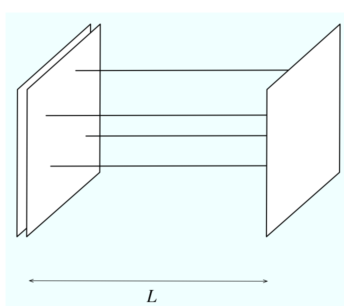

Fig. 7: Two separated stacks of $D3$-branes with k fundamental strings stretched between them.

[20] $R=(R_1,R_2,\ldots,R_N)$, with $R_i\geq R_{i+1}$ labels a representation of $U(N)$ given by a Young tableau with R_i boxes in the i-th row.

In the gauge theory description, we must study the low energy effective field theory of $U(N+1)$ $\mathcal{N}=4$ SYM when spontaneously broken to $U(N)\times U(1)$. The presence of k stretched static fundamental strings corresponds to inserting at $t\to-\infty$ k W-boson creation operators $w^\dagger$ and k W-boson annihilation operators w at $t\to\infty$. Since we are interested in the limit when the charges are infinitely massive probes, we must study this field theory vacuum in the limit $L\to\infty$. In this limit the $U(1)$ theory completely decouples from the $U(N)$ theory.

Physically, the $L\to\infty$ limit can be thought of as a non-relativistic limit. The dynamics can be conveniently extracted by defining

$$w=\frac{1}{\sqrt{L}}e^{-itL}\chi, \tag{2.72}$$

making the kinetic term for the W-bosons non-relativistic. As shown in section 7, the terms in the effective action surviving the limit are given by

$$S=S_{\mathcal{N}=4}+S_\chi, \tag{2.73}$$

where:

$$S_\chi=\int dt\, i\chi^\dagger\partial_t\chi+\chi^\dagger(A_0+\phi)\chi. \tag{2.74}$$

Therefore, the path integral describing k fundamental strings stretching between the two stacks of D-branes in the $L\to\infty$ limit is given by[21]

$$Z\equiv e^{iS_{\mathcal{N}=4}}\int[D\chi][D\chi^\dagger]\; e^{iS_\chi}\frac{1}{k!}\sum_{i_1,\ldots i_k}\chi_{i_1}(\infty)\chi_{i_2}(\infty)\ldots\chi_{i_k}(\infty)\chi^\dagger_{i_1}(-\infty)\chi^\dagger_{i_2}(-\infty)\ldots\chi^\dagger_{i_k}(-\infty), \tag{2.75}$$

where $i_l=1,\ldots N$ is a fundamental index of $U(N)$.

From the formula for the W-boson propagator that follows from (2.74)[22]

$$\langle\chi_i(t_1)\chi^\dagger_j(t_2)\rangle=\theta(t_1-t_2)\delta_{ij}, \tag{2.76}$$

one can derive the following "effective" propagator

$$\langle\chi_i(\infty)\chi^\dagger_j(-\infty)\rangle_{eff}\equiv\langle\exp\left(i\int dt\; i\chi^\dagger\partial_t\chi+\chi^\dagger(A_0+\phi)\chi\right)\chi_i(\infty)\chi^\dagger_j(-\infty)\rangle=U_{ij}, \tag{2.77}$$

[21] The path integral over the $U(N)$ $\mathcal{N}=4$ SYM is to be performed at the end.

[22] $\theta(t)$ is the Heaviside step function.

where U is the holonomy matrix appearing in the half-BPS Wilson loop operator (2.70):

$$U = P \exp\left(i \int dt \, (A_0 + \phi) \right) \in U(N). \tag{2.78}$$

Using this "effective" propagator we can now evaluate (2.75). We must sum over all Wick contractions between the W-bosons. Contractions are labeled by a permutation ω of the symmetric group S_k. The path integral (2.75) is then given by:

$$Z = e^{iS_{\mathcal{N}=4}} \cdot \frac{1}{k!} \sum_{\omega \in S_k} U^{i_1}_{i_{\omega(1)}} \dots U^{i_k}_{i_{\omega(k)}}. \tag{2.79}$$

Permutations having the same cycle structure upon decomposing a permutation into the product of disjoint cycles give identical contributions in (2.79). Since all elements in a given conjugacy class of S_k have the same cycle structure, we can replace the sum over permutations ω in (2.79) by a sum over conjugacy classes $C(\vec{k})$ of S_k. Conjugacy classes of S_k are labeled by partitions of k, denoted by $\vec{k}$, so that

$$k = \sum_{l=1}^{k} l k_l, \tag{2.80}$$

and each permutation in the conjugacy class has k_l cycles of length l.

Therefore, (2.79) can be written as

$$Z = e^{iS_{\mathcal{N}=4}} \cdot \frac{1}{k!} \sum_{C(\vec{k})} N_{C(\vec{k})} \gamma_{\vec{k}}(U), \tag{2.81}$$

where

$$\gamma_{\vec{k}}(U) = \prod_{l=1}^{k} (\text{Tr}U^l)^{k_l}, \tag{2.82}$$

and $N_{C(\vec{k})}$ is the number of permutations in the conjugacy class $C(\vec{k})$, which is given by

$$N_{C(\vec{k})} = \frac{k!}{z_{\vec{k}}}, \tag{2.83}$$

with:

$$z_{\vec{k}} = \prod_{l=1}^{k} k_l! l^{k_l}. \tag{2.84}$$

Therefore, we are led to

$$Z = e^{iS_{\mathcal{N}=4}} \cdot \sum_{C(\vec{k})} \frac{1}{z_{\vec{k}}} \gamma_{\vec{k}}(U), \tag{2.85}$$

which can also be written (see e.g. [73]) as

$$Z = e^{iS_{\mathcal{N}=4}} \cdot \mathrm{Tr}_{(k,0,\ldots,0)} U, \tag{2.86}$$

as we wanted to show.

To summarize, we have shown that integrating out the degrees of freedom associated to the single separated $D3$-brane – when k fundamental strings are stretching between the $D3$-brane and a stack of N $D3$-branes – inserts a half-BPS Wilson loop operator into the $\mathcal{N}=4$ SYM path integral in the k-th symmetric representation of $U(N)$.

We can now make contact with the $D3_k$-brane solution [25][71] in $\mathrm{AdS}_5\times\mathrm{S}^5$. The solution of the Born-Infeld equations of motion for a single $D3$ brane with k fundamental strings stretched between that brane and a stack of N $D3$-branes was already found in [25]. In this solution, the N $D3$-branes are replaced by their supergravity background and the other $D3$-brane with the attached strings as a BION solution [74][75]. In the near horizon limit, the $D3$-brane solution in [25] indeed becomes the $D3_k$-brane solution in $\mathrm{AdS}_5\times\mathrm{S}^5$.

Therefore, we have given a microscopic explanation of the identification

$$D3_k \longleftrightarrow Z = e^{iS_{\mathcal{N}=4}} \cdot W_{(k,0,\ldots,0)}, \tag{2.87}$$

proposed in the previous sections [76].

2.6. Multiple $D3_k$-branes as Wilson loop in arbitrary representation

In the previous section, we have shown that a single $D3_k$-brane corresponds to a Wilson loop in the k-th symmetric representation. We now show that an arbitrary representation R with P rows in a Young tableau can be realized by considering P $D3$-branes.

We consider a stack of $N+P$ $D3$-branes and break the gauge symmetry down to $U(N)\times U(P)$ by separating P of the branes a distance L. In the gauge theory description this corresponds to turning on a scalar expectation value as in (2.71). We also consider a background of k fundamental strings stretched between the two stacks of branes.

Therefore, we must study the low energy effective field theory of $U(N+P)$ $\mathcal{N}=4$ SYM when spontaneously broken to $U(N)\times U(P)$ and in the limit $L\to\infty$, where the charges become infinitely massive probes[23]. The presence of k fundamental strings is realized in

[23] Just as before, the $U(P)$ gauge dynamics completely decouples from the $U(N)$ gauge theory in the $L\to\infty$ limit.

the gauge theory by inserting the creation operator of a k W-boson state at $t \to -\infty$ and the annihilation operator of a k W-boson state at $t \to \infty$. The k W-boson annihilation operator is given by

$$\Psi(t) = \chi_{i_1}^{I_1}(t)\chi_{i_2}^{I_2}(t)\dots\chi_{i_k}^{I_k}(t) \tag{2.88}$$

and the k W-boson creation operator by $\Psi^\dagger(t)$, where[24] $i_l = 1, \dots, N$ and $I_l = 1, \dots, P$.

Such a k W-boson state transforms under $U(N)$ and $U(P)$ as a sum over representations with k boxes in a Young tableau. In order to project to a specific representation R we can apply to the k W-boson annihilation operator (2.88) the following projection operator

$$P_\alpha^R = \frac{d_R}{k!} \sum_{\sigma \in S_k} D_{\alpha\alpha}^R(\sigma)\sigma \tag{2.89}$$

where $R = (n_1, n_2, \dots, n_P)$, with $k = \sum_i n_i$, labels an irreducible representation of both[25] S_k, $U(N)$ and $U(P)$. $D_{\alpha\beta}^R(\sigma)$ is the representation matrix for the permutation σ in the representation R, d_R is the dimension of the representation R of S_k and $\alpha, \beta = 1, \dots, d_R$. Therefore, the operator

$$\Psi_\alpha^R(t) = P_\alpha^R \Psi = \frac{d_R}{k!} \sum_{\sigma \in S_k} D_{\alpha\alpha}^R(\sigma)\chi_{i_1}^{I_{\sigma(1)}}(t)\chi_{i_2}^{I_{\sigma(2)}}(t)\dots\chi_{i_k}^{I_{\sigma(k)}}(t) \tag{2.90}$$

describes a k W-boson state transforming in the irreducible representation R of S_k, $U(N)$ and $U(P)$.

The path integral to perform, representing our brane configuration with k fundamental strings stretching between the two stacks of D-branes, in the $L \to \infty$ limit is given by[26]

$$Z = e^{iS_{\mathcal{N}=4}} \int [D\chi][D\chi^\dagger]\ e^{iS_\chi} \sum_{\alpha=1}^{d_R} \Psi_\alpha^R(\infty)\Psi_\alpha^{\dagger R}(-\infty), \tag{2.91}$$

where S_χ is the straightforward generalization of (2.74) when the gauge group is $U(N) \times U(P)$.

[24] The W-bosons transform in the $(N, \bar{P})$ representation of the $U(N) \times U(P)$ gauge group, see the last section for details.

[25] There is a natural action of S_k, $U(N)$ and $U(P)$ on $\Psi(t)$. The projected operator in fact transforms in the same representation R for both S_k, $U(N)$ and $U(P)$ groups (see e.g. [77]). The representations of the unitary and symmetric groups are both labeled by the same Young tableau $R = (n_1, n_2, \dots, n_P)$.

[26] To avoid cluttering the formulas, the sum over $U(N)$ and $U(P)$ indices is not explicitly written throughout the rest of this chapter.

The "effective" propagator for the W-bosons is now

$$\langle \chi_i^I(\infty)\chi_j^{J\dagger}(-\infty)\rangle_{eff} \equiv \langle \exp\left(i\int dt\ i\chi^\dagger\partial_t\chi + \chi^\dagger(A_0+\phi)\chi\right)\chi_i^I(\infty)\chi_j^{J\dagger}(-\infty)\rangle = U_{ij}\delta^{IJ}, \tag{2.92}$$

with U given in (2.78). The sum over all Wick contractions in (2.91) gives:

$$Z = e^{iS_{\mathcal{N}=4}}\left(\frac{d_R}{k!}\right)^2 \sum_{\alpha=1}^{d_R}\sum_{\sigma,\tau,\omega\in S_k} D^R_{\alpha\alpha}(\sigma)D^R_{\alpha\alpha}(\tau)U^{i_1}_{i_{\omega(1)}}\dots U^{i_k}_{i_{\omega(k)}}\delta^{I_{\sigma(1)}}_{I_{\omega\tau(1)}}\dots\delta^{I_{\sigma(k)}}_{I_{\omega\tau(k)}}. \tag{2.93}$$

By appropriate change of variables, this can be simplified to

$$Z = e^{iS_{\mathcal{N}=4}}\left(\frac{d_R}{k!}\right)^2 \sum_{\alpha=1}^{d_R}\sum_{\sigma,\tau,\omega\in S_k} D^R_{\alpha\alpha}(\sigma)D^R_{\alpha\alpha}(\tau)U^{i_1}_{i_{\omega(1)}}\dots U^{i_k}_{i_{\omega(k)}}P^{C(\sigma^{-1}\omega\tau)}, \tag{2.94}$$

where $C(\sigma)$ is the number of disjoint cycles in the permutation σ and:

$$P^{C(\sigma^{-1}\omega\tau)} = \sum_{I_1,\dots,I_k}\delta^{I_1}_{I_{\sigma^{-1}\omega\tau(1)}}\dots\delta^{I_k}_{I_{\sigma^{-1}\omega\tau(k)}}. \tag{2.95}$$

We proceed[27] by introducing $\delta(\rho)$, an element in the group algebra, which takes the value 1 when the argument is the identity permutation and 0 when the argument is any other permutation. This allows (2.94) to be written as:

$$e^{iS_{\mathcal{N}=4}}\left(\frac{d_R}{k!}\right)^2 \sum_{\alpha=1}^{d_R}\sum_{\sigma,\tau,\omega,\rho\in S_k} D^R_{\alpha\alpha}(\sigma)D^R_{\alpha\alpha}(\tau)U^{i_1}_{i_{\omega(1)}}\dots U^{i_k}_{i_{\omega(k)}}P^{C(\rho)}\delta(\rho^{-1}\sigma^{-1}\omega\tau). \tag{2.96}$$

Summing over τ yields

$$e^{iS_{\mathcal{N}=4}}\left(\frac{d_R}{k!}\right)^2 \sum_{\alpha=1}^{d_R}\sum_{\sigma,\omega\in S_k} D^R_{\alpha\alpha}(\sigma)D^R_{\alpha\alpha}(\omega^{-1}\sigma\sum_{\rho\in S_k}\rho P^{C(\rho)})U^{i_1}_{i_{\omega(1)}}\dots U^{i_k}_{i_{\omega(k)}}. \tag{2.97}$$

Since $\mathcal{C} = \sum_{\rho\in S_k}\rho P^{C(\rho)}$ commutes with all elements in the group algebra, we can use the identity $D^R_{\alpha\alpha}(\mathcal{C}\sigma) = \frac{1}{d_R}D^R_{\alpha\alpha}(\sigma)\chi_R(\mathcal{C})$, where $\chi_R(\mathcal{C}) = \sum_{\alpha=1}^{d_R}D^R_{\alpha\alpha}(\mathcal{C})$ is the character of S_k in the representation R for $\mathcal{C}$. Therefore, (2.97) reduces to

$$e^{iS_{\mathcal{N}=4}}\frac{d_R}{k!}Dim_P(R)\sum_{\alpha=1}^{d_R}\sum_{\sigma,\omega\in S_k} D^R_{\alpha\alpha}(\sigma)D^R_{\alpha\alpha}(\omega^{-1}\sigma)U^{i_1}_{i_{\omega(1)}}\dots U^{i_k}_{i_{\omega(k)}}, \tag{2.98}$$

[27] The paper [78] has a useful compilation of useful formulas relevant for this paper.

where

$$Dim_P(R) = \frac{1}{k!} \sum_{\sigma \in S_k} \chi_R(\sigma) P^{C(\sigma)} \tag{2.99}$$

is the dimension of the irreducible representation R of $U(P)$. By using the relation satisfied by the fusion of representation matrices

$$\sum_{\sigma \in S_k} D^R_{\alpha\alpha}(\sigma) D^R_{\alpha\alpha}(\omega^{-1}\sigma) = \frac{k!}{d_R} D^R_{\alpha\alpha}(\omega^{-1}) \tag{2.100}$$

we are arrive at:

$$Z = e^{iS_{\mathcal{N}=4}} Dim_P(R) \sum_{\omega \in S_k} \chi_R(\omega) U^{i_1}_{i_{\omega(1)}} \dots U^{i_k}_{i_{\omega(k)}}. \tag{2.101}$$

Finally, we use the Frobenius character formula (see e.g. [73]), which relates the trace of a matrix U in an arbitrary representation $R = (n_1, n_2, \dots, n_P)$ of $U(N)$ to the trace in the fundamental representation

$$\mathrm{Tr}_R(U) = \frac{1}{k!} \sum_{\omega \in S_k} \sum_{i_1, \dots i_k} \chi_R(\omega) U^{i_1}_{i_{\omega(1)}} \dots U^{i_k}_{i_{\omega(k)}} \tag{2.102}$$

to show that the final result of the path integral is

$$Z = e^{iS_{\mathcal{N}=4}} \cdot k!\; Dim_R(M) \mathrm{Tr}_R(U), \tag{2.103}$$

the insertion of a half-BPS Wilson loop in the representation R.

In the near horizon limit, when the N D3-branes are replaced by their near horizon geometry, the P D3-branes with the array of stretched fundamental strings labeled by $R = (n_1, n_2, \dots, n_P)$ become the brane configuration $(D3_{n_1}, D3_{n_2}, \dots, D3_{n_P})$ in $\mathrm{AdS}_5 \times \mathrm{S}^5$, thus arriving at the identification[28]

$$(D3_{n_1}, \dots, D3_{n_P}) \longleftrightarrow Z = e^{iS_{\mathcal{N}=4}} \cdot W_{(n_1, \dots, n_P, 0, \dots, 0)} \tag{2.104}$$

in section 4.

[28] We can trivially reabsorb the overall constant in (2.103) in the normalization of Ψ.

Supersymmetry of Wilson loops in $\mathcal{N}=4$ SYM

In this Appendix we study the constraints imposed by unbroken supersymmetry on the Wilson loop operators (2.1) of $\mathcal{N}=4$ SYM. Previous studies of supersymmetry of Wilson loops in $\mathcal{N}=4$ SYM include [79][80][81].

We want to impose that the Wilson loop operator (2.1) is invariant under one-half of the $\mathcal{N}=4$ Poincare supersymmetries and also invariant under one-half of the conformal supersymmetries. The Poincare supersymmetry transformations are given by

$$\begin{aligned}\delta_{\epsilon_1} A_\mu &= i\bar{\epsilon}_1\gamma_\mu\lambda \\ \delta_{\epsilon_1}\phi_I &= i\bar{\epsilon}_1\gamma_I\lambda,\end{aligned} \tag{2.105}$$

while the superconformal supersymmetry transformations are given

$$\begin{aligned}\delta_{\epsilon_2} A_\mu &= i\bar{\epsilon}_2 x^\nu\gamma_\nu\gamma_\mu\lambda \\ \delta_{\epsilon_2}\phi_I &= i\bar{\epsilon}_2 x^\nu\gamma_\nu\gamma_I\lambda,\end{aligned} \tag{2.106}$$

where $\epsilon_{1,2}$ are ten dimensional Majorana-Weyl spinors of opposite chirality. The use of ten dimensional spinors is useful when comparing with string theory computations.

Preservation of one-half of the Poincare supersymmetries locally at each point in the loop where the operator is defined yields:

$$P\epsilon_1 = (\gamma_\mu\dot{x}^\mu + \gamma_I\dot{y}^I)\epsilon_1 = 0. \tag{2.107}$$

Therefore, there are invariant spinors at each point in the loop if and only if $\dot{x}^2+\dot{y}^2=0$. This requires that $x^\mu(s)$ is a time-like curve and that $\dot{y}^I = n^I(s)\sqrt{-\dot{x}^2}$, where $n^I(s)$ is a unit vector in R^6, satisfying $n^2(s)=1$. Without loss of generality we can perform a boost and put the external particle labeling the loop at rest so that the curve along $R^{1,3}$ is given by $(x^0(s), x^i(s)=0)$ and we can also choose an affine parameter s on the curve such that $\sqrt{-\dot{x}^2}=1$.

In order for the Wilson loop to be supersymmetric, each point in the loop must preserve the same spinor. Therefore, we must impose that

$$\frac{dP(s)}{ds} = 0, \tag{2.108}$$

which implies that $\ddot{x}^0 = 0$ and that $n^I(s) = n^I$. Therefore, supersymmetry selects a preferred curve in superspace, the straight line Wilson loop operator, given by

$$W_R(C) = \mathrm{Tr}_R\, P \exp\left(i \int dt\, (A_0 + \phi)\right), \tag{2.109}$$

where $\phi = n^I \phi_I$. The operators are now just labelled by a choice of Young tableau R. For future reference, we write explicitly the 8 unbroken Poincare supersymmetries. They must satisfy

$$i\bar{\epsilon}_1 \gamma_0 \lambda + i n^I \bar{\epsilon}_1 \gamma_I \lambda = 0. \tag{2.110}$$

Using relations for conjugation of spinor with the conventions used here

$$\bar{\chi}\zeta = \bar{\zeta}\chi, \qquad \chi = \gamma^I \zeta \to \bar{\chi} = -\bar{\zeta}\gamma^I \tag{2.111}$$

we arrive at

$$\gamma_0 \gamma_I n^I \epsilon_1 = \epsilon_1. \tag{2.112}$$

In a similar manner it is possible to prove that the straight line Wilson loop operator (2.109) also preserves one-half of the superconformal supersymmetries. The 8 unbroken superconformal supersymmetries are given by:

$$\gamma_0 \gamma_I n^I \epsilon_2 = -\epsilon_2. \tag{2.113}$$

Supersymmetry of Fundamental String and of $D5_k$-brane

In this Appendix we show that the particular embeddings considered for the fundamental string and the $D5_k$-brane in section 2 preserve half of the supersymmetries of the background. We will use conventions similar to those in [82].

For convinience we write again the metric we are interested in (we set $L = 1$)

$$ds^2_{AdS\times S} = u^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{du^2}{u^2} + d\theta^2 + \sin^2\theta\, d\Omega_4^2, \tag{2.114}$$

where the metric on S^4 is given by:

$$d\Omega_4 = d\varphi_1^2 + \sin\varphi_1^2 d\varphi_2^2 + \sin\varphi_1^2 \sin\varphi_2^2 d\varphi_3^2 + \sin\varphi_1^2 \sin\varphi_2^2 \sin\varphi_3^2 d\varphi_4^2. \tag{2.115}$$

It is useful to introduce tangent space gamma matrices, i.e. $\gamma_{\underline{m}} = e^m_{\underline{m}}\Gamma_m$ $(m, \underline{m} = 0, \ldots, 9)$ where $e^m_{\underline{m}}$ is the inverse vielbein and Γ_m are the target space matrices:

$$\begin{aligned} \gamma_\mu = \frac{1}{u}\Gamma_\mu \qquad (\mu = 0,1,2,3), \qquad \gamma_4 = u\Gamma_u, \qquad \gamma_5 = \Gamma_\theta, \\ \gamma_{a+5} = \frac{1}{\sin\theta}\left(\prod_{j=1}^{a-1}\frac{1}{\sin\varphi_j}\right)\Gamma_{\varphi_a} \qquad (a = 1,2,3,4) \end{aligned} \tag{2.116}$$

The Killing spinor of $\mathrm{AdS}_5\times\mathrm{S}^5$ in the coordinates (2.114) is given by [82]

$$\epsilon = \left[-u^{-\frac{1}{2}}\gamma_4 h(\theta,\varphi_a) + u^{\frac{1}{2}}h(\theta,\varphi_a)(\eta_{\mu\nu}x^\mu\gamma^\nu)\right]\eta_2 + u^{\frac{1}{2}}h(\theta,\varphi_a)\eta_1 \tag{2.117}$$

where

$$h(\theta,\varphi_a) = e^{\frac{1}{2}\theta\gamma_{45}}e^{\frac{1}{2}\varphi_1\gamma_{56}}e^{\frac{1}{2}\varphi_2\gamma_{67}}e^{\frac{1}{2}\varphi_3\gamma_{78}}e^{\frac{1}{2}\varphi_4\gamma_{89}} \tag{2.118}$$

η_1 and η_2 are constant ten dimensional complex spinors with negative and positive ten dimensional chirality, i.e.

$$\gamma_{11}\eta_1 = -\eta_1 \qquad \gamma_{11}\eta_2 = \eta_2. \tag{2.119}$$

They also satisfy:

$$P_-\eta_1 = \eta_1 \qquad P_+\eta_2 = \eta_2 \tag{2.120}$$

where $P_\pm = \frac{1}{2}(1 \pm i\gamma^{0123})$. Thus, each spinor $\eta_{1,2}$ has 16 independent real components. These can be written in terms of ten dimensional Majorana-Weyl spinors ϵ_1 and ϵ_2 of negative and positive chirality respectively:

$$\begin{aligned} \eta_1 &= \epsilon_1 - i\gamma^{0123}\epsilon_1 \\ \eta_2 &= \epsilon_2 + i\gamma^{0123}\epsilon_2. \end{aligned} \tag{2.121}$$

By going to the boundary of AdS at $u \to \infty$, we can identify from (2.117) ϵ_1 as the Poincare supersymmetry parameter while ϵ_2 is the superconformal supersymmetry parameter of $\mathcal{N} = 4$ SYM.

The supersymmetries preserved by the embedding of a probe, are those that satisfy

$$\Gamma_\kappa\epsilon = \epsilon \tag{2.122}$$

where Γ_κ is the κ symmetry transformation matrix in the probe worldvolume theory and ϵ is the Killing spinor of the $AdS_5 \times S_5$ background (2.117). Both Γ_κ and ϵ have to be evaluated at the location of the probe.

Let's now consider a fundamental string with an AdS_2 worldvolume geometry with embedding:

$$\sigma^0 = x^0 \qquad \sigma^1 = u \qquad x^i = 0 \qquad x^I = n^I. \tag{2.123}$$

The position of the string on the S^5 is parametrized by the five constant angles $(\theta, \varphi_1, \varphi_2, \varphi_3, \varphi_4)$ or alternatively by a unit vector n^I in R^6. The matrix Γ_κ for a fundamental string with this embedding reduces to

$$\Gamma_{F1} = \gamma_{04} K \tag{2.124}$$

where K acts on a spinor ψ by $K\psi = \psi^*$. For later convenience we define also the operator I such that $I\psi = -i\psi$.

The equation (2.122) has to be satisfied at every point on the string. Thus, the term proportional to $u^{\frac{1}{2}}$ gives:

$$\Gamma_{F1} h(\theta, \varphi_a)\eta_1 = h(\theta, \varphi_a)\eta_1. \tag{2.125}$$

The terms proportional to $u^{-\frac{1}{2}}$ and $u^{-\frac{1}{2}}x_0$ both give:

$$\Gamma_{F1} h(\theta, \varphi_a)\eta_2 = -h(\theta, \varphi_a)\eta_2. \tag{2.126}$$

These can be rewritten as

$$n^I \gamma_{0I}\eta_1 = \eta_1^* \qquad n^I \gamma_{0I}\eta_2 = -\eta_2^* \qquad I = 4, 5, 6, 7, 8, 9 \tag{2.127}$$

where

$$n^I(\theta, \varphi_1, \varphi_2, \varphi_3, \varphi_4) = \begin{pmatrix} \cos\theta \\ \sin\theta\cos\varphi_1 \\ \sin\theta\sin\varphi_1\cos\varphi_2 \\ \sin\theta\sin\varphi_1\sin\varphi_2\cos\varphi_3 \\ \sin\theta\sin\varphi_1\sin\varphi_2\sin\varphi_3\cos\varphi_4 \\ \sin\theta\sin\varphi_1\sin\varphi_2\sin\varphi_3\sin\varphi_4 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta\, l^\alpha \end{pmatrix}, \tag{2.128}$$

where $\alpha = (5, 6, 7, 8, 9)$ and these vectors satisfy $n^2 = 1$ and $l^2 = 1$. Considering the parametrization (2.121), the projection (2.127) becomes:

$$\gamma_{0I} n^I \epsilon_1 = \epsilon_1 \qquad \gamma_{0I} n^I \epsilon_2 = -\epsilon_2. \tag{2.129}$$

We note that n^I define the position of the string in the S_5, so it characterizes the unbroken rotational symmetry of the system. Therefore, the fundamental string preserves exactly the same supersymmetries as the Wilson loop operator (2.109).

We now study the $D5_k$-brane embedding considered first by [58][59]:

$$\sigma^0 = x^0 \qquad \sigma^1 = u \qquad \sigma^a = \varphi_a \qquad x^i = 0 \qquad \theta = \theta_k = \text{constant}. \tag{2.130}$$

There is an electric flux on the brane given by

$$F_{04} = F = \cos\theta_k, \tag{2.131}$$

where k is the amount of fundamental string charge on the $D5_k$-brane.

For this configuration, Γ_κ is

$$\begin{aligned}\Gamma_{D5} &= \frac{1}{\sqrt{1-F^2}}\gamma_{046789}KI + \frac{F}{\sqrt{1-F^2}}\gamma_{6789}I \\ &= \frac{1}{\sin\theta_k}\gamma_{046789}KI + \frac{\cos\theta_k}{\sin\theta_k}\gamma_{6789}I\end{aligned} \tag{2.132}$$

Following similar steps as for the fundamental string, we arrive at

$$\Gamma_{D5}h(\theta_k,\varphi_a)\epsilon_1 = h(\theta_k,\varphi_a)\epsilon_1 \qquad \bar{\Gamma}_{D5}h(\theta_k,\varphi_a)\epsilon_2 = h(\theta_k,\varphi_a)\epsilon_2, \tag{2.133}$$

where

$$\bar{\Gamma}_{D5} = -\frac{1}{\sin\theta_k}\gamma_{046789}KI + \frac{\cos\theta_k}{\sin\theta_k}\gamma_{6789}I. \tag{2.134}$$

Using that $h^{-1}\gamma_{04}h = n^I\gamma_{0I}$ and that $h^{-1}\gamma_{6789}h = l^\alpha\gamma_{\alpha56789}$ we have that the supersymmetry left unbroken by a $D5_k$-brane is given by:

$$\gamma_{04}\epsilon_1 = \epsilon_1 \qquad \gamma_{04}\epsilon_2 = -\epsilon_2. \tag{2.135}$$

Therefore it preserves the same supersymmetries as a fundamental string sitting at the north pole (i.e $\theta = 0$), labeled by the vector $n^I = (1,0,0,0,0,0)$. This vector selects the unbroken rotational symmetry.

Gauge Fixing and the Unitary Matrix Measure

In section 4 we have gauge fixed the $U(M)$ symmetry by imposing the diagonal, constant gauge:

$$\tilde{A}_0 = \text{diag}(\Omega_1,\ldots,\Omega_M). \tag{2.136}$$

There is an associated Fadeev-Popov determinant Δ_{FP} corresponding to this gauge choice. This modifies the measure to

$$[D\tilde{A}_0] \cdot \Delta_{FP}, \tag{2.137}$$

where now $[D\tilde{A}_0]$ involves integration only over the constant mode of the hermitean matrix $\tilde{A}_0$. Under an infinitessimal gauge transformation labelled by α, $\tilde{A}_0$ transforms by

$$\delta\tilde{A}_0 = \partial_t\alpha + i[\tilde{A}_0, \alpha], \tag{2.138}$$

so that:

$$\Delta_{FP} = \det\left(\partial_t \ + i[\tilde{A}_0, \ ]\right). \tag{2.139}$$

An elementary computation yields

$$\Delta_{FP} = \prod_{l\neq 0}^{\infty} \frac{2\pi i l}{\beta} \prod_{I<J} \prod_{k=1}^{\infty} \left(1 - \frac{\beta^2(\Omega_I - \Omega_J)^2}{4\pi^2 k^2}\right), \tag{2.140}$$

where we have introduced β as an infrared regulator. Now, using the product representation of the *sin* function we have that up to an irrelevant constant:

$$\Delta_{FP} = \prod_{I<J} 4\, \frac{\sin^2\left(\beta\left(\frac{\Omega_I - \Omega_J}{2}\right)\right)}{(\Omega_I - \Omega_J)^2}. \tag{2.141}$$

This together with the formula for the measure of the Hermitean matrix $\tilde{A}_0$

$$[D\tilde{A}_0] = \prod_{I<J} d\Omega_I (\Omega_I - \Omega_J)^2 \tag{2.142}$$

proves our claim that the gauge fixing effectively replaces the measure over the Hermitean matrix $\tilde{A}_0$ by the measure over the unitary $U = e^{i\beta\tilde{A}_0}$

$$[D\tilde{A}_0] \cdot \Delta_{FP} = [DU] = \prod_{I<J} d\Omega_I \Delta(\Omega)\bar{\Delta}(\Omega), \tag{2.143}$$

where

$$\Delta(\Omega) = \prod_{I<J} (e^{i\beta\Omega_I} - e^{i\beta\Omega_J}). \tag{2.144}$$

Gauge Theory Along Coulomb Branch

The low energy dynamics of a stack of $N + P$ coincident $D3$-branes is described by four dimensional $\mathcal{N} = 4$ SYM with $U(N + P)$ gauge group. The spectrum of the theory includes a vector field $\hat{A}_\mu$, six scalar fields $\hat{\Phi}_i$ and a ten dimensional Majorana-Weyl spinor $\hat{\Psi}$. The action is given by

$$\hat{S}_{\mathcal{N}=4} = \frac{1}{2g_{YM}^2}\int \mathrm{Tr}\left(-\frac{1}{2}\hat{F}_{\mu\nu}^2 - (\hat{D}_\mu\hat{\Phi}_i)^2 + \frac{1}{2}[\hat{\Phi}_i,\hat{\Phi}_j]^2 - i\hat{\bar{\Psi}}\Gamma^\mu\hat{D}_\mu\hat{\Psi} - \hat{\bar{\Psi}}\Gamma^i[\hat{\Phi}_i,\hat{\Psi}]\right), \tag{2.145}$$

where each field is in the adjoint of the gauge group $U(N+P)$. We use real ten dimensional gamma matrices Γ^i and Γ^μ and we choose Γ^0 as charge conjugation matrix. Thus, the Majorana-Weyl spinor λ has 16 real components and $\bar{\lambda} = \lambda^T\Gamma^0$.

Now we separate a stack of P branes from the remaining stack of N branes, i.e. we give a non trivial vacuum expectation value to the scalar fields. Without lost of generality, we take

$$< \hat{\Phi}_9 >= \begin{pmatrix} 0 & 0 \\ 0 & LI_P \end{pmatrix}, \tag{2.146}$$

where I_P is the $P \times P$ unit matrix and L is a constant with the dimensions of mass. To expand the action around this vacuum, we first define the fields as

$$\hat{A}_\mu = \begin{pmatrix} A_\mu & \omega_\mu \\ \omega_\mu^\dagger & \tilde{A}_\mu \end{pmatrix} \qquad \hat{\Phi}_i = \begin{pmatrix} \Phi_i & \omega_i \\ \omega_i^\dagger & \delta_{i9}LI_P + \tilde{\Phi}_i \end{pmatrix} \qquad \hat{\Psi} = \begin{pmatrix} \Psi & \theta \\ \theta^\dagger & \tilde{\Psi} \end{pmatrix}, \tag{2.147}$$

where A_μ, Φ_i and Ψ transform in the adjoint representation of $U(N)$ and $\tilde{A}_\mu$, $\tilde{\Phi}_i$ and $\tilde{\Psi}$ transform in the adjoint representation of $U(P)$. ω_μ, ω_i and θ are W-bosons fields and transform in the $(N,\bar{P})$ representation of the gauge group $U(N)\times U(P)$.

The action becomes:

$$\hat{S}^L_{\mathcal{N}=4} = S_{\mathcal{N}=4} + \tilde{S}_{\mathcal{N}=4} + S_W + S_{interactions}. \tag{2.148}$$

$S_{\mathcal{N}=4}$ and $\tilde{S}_{\mathcal{N}=4}$ are the actions for the effective field theories living on the two stacks of branes, i.e. four dimensional $\mathcal{N}=4$ SYM with gauge group respectively $U(N)$ and $U(P)$. S_W is the action for the W-bosons and their superpartners

$$S_W = \int \mathrm{Tr}\Big(-\frac{1}{2}f^\dagger_{\mu\nu}f^{\mu\nu} - L^2\omega_\mu^\dagger\omega^\mu - \sum_{i=4}^{9}\partial^\mu\omega_i^\dagger\partial_\mu\omega_i - L^2\sum_{k=4}^{8}\omega_k^\dagger\omega_k - i\bar{\theta}^\dagger\Gamma^\mu\partial_\mu\theta + L\bar{\theta}^\dagger\Gamma^9\theta + \dots\Big), \tag{2.149}$$

where $f_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu$ and $\dots$ denote terms fourth order in the W-boson fields. $S_{interactions}$ is the action describing the interactions between the W-bosons and the fields living on the two stacks of branes. It includes terms of the third and fourth order in the fields.

We are interested in the limit where the two stacks of branes are infinitely separated, i.e. in the limit where $L \to \infty$. From the quadratic action (2.149) we see that the W-bosons ω_m with $m = 1, \dots 8$ and the fermions become infinitely massive. Taking the infinite mass limit of a relativistic massive field, corresponds to considering the non-relativistic limit. The surviving dynamics in the limit can be explicitly extracted by making the following redefinition:

$$\omega_m = \frac{1}{\sqrt{L}} e^{-itL} \chi_m \qquad \text{where} \qquad m = 1, \dots, 8. \tag{2.150}$$

For the W-bosons superpartners, which also become infinitely massive, we first define

$$\Gamma_{09} \theta_\pm = \pm \theta_\pm, \tag{2.151}$$

where this projection must be understood in the spinors space. To extract the physics in the limit we make the following rescaling:

$$\theta = \theta_+ + \theta_- = e^{-iLt} \xi_+ + e^{-iLt} \xi_-. \tag{2.152}$$

Considering (2.150) and (2.152) and then taking the infinite mass limit $L \to \infty$ the W-boson action (2.149) reduces to

$$S_W^{NR} = \int \mathrm{Tr} \left(i \sum_{m=1}^{8} \chi_m^\dagger \partial_t \chi_m + i (\xi_+^T)^\dagger \partial_t \xi_+ \right) \tag{2.153}$$

where the transposition is in the space of fermions and the hermitian conjugation is in the matrix space. The ξ_- fermions become infinitely massive and decouple from the theory, as expected, since there are no antiparticles in the non-relativistic limit.

The interaction action $S_{interaction}$ in (2.148) is now given by

$$\begin{aligned} S_{interactions}^{NR} = \int \sum_{m=1}^{8} \mathrm{Tr} \Big(& \chi_m^\dagger (A_0 + \Phi_9) \chi_m - \chi_m^\dagger \chi_m (\tilde{A}_0 + \tilde{\Phi}_9) + \\ & + \xi_m^\dagger (A_0 + \Phi_9) \xi_m - \xi_m^\dagger \xi_m (\tilde{A}_0 + \tilde{\Phi}_9) \Big) \end{aligned} \tag{2.154}$$

where ξ_m ($m = 1, \dots 8$) are the spinor components of ξ_+. All higher order terms in (2.148) vanish in the $L \to \infty$ limit. Note that, in this limit the dynamics of the $U(P)$ gauge theory effectively decouples from the $U(N)$ gauge theory.

Therefore, we can then write the action describing the coupling of the W-bosons to $U(N)$ $\mathcal{N}=4$ SYM as[29]

$$S = S_{\mathcal{N}=4} + \sum_{m=1}^{8} S_m , \tag{2.155}$$

where $S_{\mathcal{N}=4}$ is the action of $\mathcal{N}=4$ SYM with gauge group $U(N)$ while S_m with $m=1,\ldots 8$ is the action for one of the eight non-relativistic supersymmetric W-bosons

$$S_m = \int [(\chi_m^\dagger)^I_i \Delta^{IJ}_{ij} (\chi_m)^J_j + (\xi_m^\dagger)^I_i \Delta^{IJ}_{ij} (\xi_m)^J_j] , \tag{2.156}$$

where

$$\Delta^{IJ}_{ij} = i\delta_{ij}\delta^{IJ}\partial_t + (A_0 + \Phi_9)_{ij}\delta^{IJ} , \tag{2.157}$$

which is what we have used in the main text.

We note that integrating out the degrees of freedom associated to the W-bosons, without any insertions, we get

$$\begin{aligned} Z = & \int \prod_{i=1}^{8} ([D\chi_m][D\chi_m^\dagger][D\xi_m][D\xi_m^\dagger]) e^{iS} \\ = & e^{iS_{\mathcal{N}=4}} \frac{(\det\Delta)^{n_F}}{(\det\Delta)^{n_B}} \\ = & e^{iS_{\mathcal{N}=4}} , \end{aligned} \tag{2.158}$$

where in the last step we used that $n_F = n_B$. Note that we recover the expected result that the metric in the Coulomb branch of $\mathcal{N}=4$ gets no corrections upon integrating out the massive modes.

[29] There is decoupled contribution for the $U(P)$ gauge theory which does not talk to $U(N)$ SYM.

3. Holographic Gauge Theories in Background Fields and Surface Operators

The phase structure of a gauge theory can be probed by studying the behaviour of the order parameters of the theory as we change external parameters, such as the temperature. In order to characterize the possible phases, one may insert an infinitely heavy probe charged particle, and study its response, as it will depend on the phase the gauge theory is in. Known examples of operators inserting such probes are Wilson, Polyakov and 't Hooft operators, which distinguish between the confined, deconfined and the Higgs phase.

It is a natural question to ask whether one can construct an operator which inserts a probe string instead of a probe particle. If so, we can then study the response of the string and analyze whether new phases of gauge theory can be found that are not discriminated by particle probes. Candidate probe strings range from cosmic strings to the wrapped D-branes of string theory.

Geometrically, an operator inserting a probe string is characterized by a surface Σ in space-time, which corresponds to the worldsheet spanned by the string. One may refer to such operators as surface operators and will label them by $\mathcal{O}_\Sigma$. Such operators are nonlocal in nature and the challenge is to construct them and to understand their physical meaning. For early studies of these operators see for instance [28].

Recently, a class of supersymmetric surface operators in $\mathcal{N} = 4$ SYM have been constructed by Gukov and Witten [29][30], while the corresponding gravitational description in terms of smooth solutions of Type IIB supergravity which are asymptotically $AdS_5 \times S^5$ has been identified in [84]. These operators are defined by a path integral with a codimension two singularity near Σ for the $\mathcal{N} = 4$ SYM fields. Therefore, these operators are of disorder type as they do not admit a description in terms of an operator insertion which can be written in terms of the classical fields appearing in the Lagrangian.

In this chapter we construct a family of surface operators in four dimensional $\mathcal{N} = 4$ SYM that do admit a description in terms of an operator insertion made out of the $\mathcal{N} = 4$ SYM fields. In the standard nomenclature, they are order operators. The surface operator is obtained by inserting into the $\mathcal{N} = 4$ SYM path integral the WZW action supported on the surface Σ

$$\exp\left[iM\Gamma_{WZW}(A)\right], \tag{3.1}$$

[30] These operators play an important role in enriching the gauge theory approach [83] to the geometric Langlands program to the case with ramification.

where:[31]

$$\Gamma_{WZW}(A) = -\frac{1}{8\pi}\int_\Sigma dx^+dx^-\mathrm{Tr}\left[\left(U^{-1}\partial_+U\right)\left(U^{-1}\partial_-U\right) - \left(U^{-1}\partial_+U\right)\left(V^{-1}\partial_-V\right)\right]$$
$$- \frac{1}{24\pi}\int d^3x\epsilon^{ijk}\mathrm{Tr}\left[\left(U^{-1}\partial_iU\right)\left(U^{-1}\partial_jU\right)\left(U^{-1}\partial_kU\right)\right]. \tag{3.2}$$

The $U(N)$ group elements U and V are nonlocally related to the $\mathcal{N}=4$ SYM gauge field A_μ along Σ by:

$$A_+ = U^{-1}\partial_+U \qquad\qquad A_- = V^{-1}\partial_-V. \tag{3.3}$$

M is an arbitrary positive integer which labels the level of the WZW model[32].

We construct these operators by considering the field theory limit of a supersymmetric $D3/D7$ brane intersection along a two dimensional surface Σ. We find that a consistent description of the low energy dynamics of this brane intersection requires that the gauge theory on the $D3$-branes is written down not in flat space *but* in the non-trivial supergravity background created by the $D7$-branes.

In this chapter we construct this supersymmetric field theory in the $D7$-brane supergravity background and show that if we integrate out the degrees of freedom introduced by the $D7$-branes that the net effect is to insert the operator (3.2) into the gauge theory action. The same strategy of integrating out the new degrees of freedom introduced on a brane intersection was used in chapter 2 [76] to construct the Wilson loop operators in $\mathcal{N}=4$ SYM and to find the bulk AdS description of a Wilson loop in an arbitrary representation of the gauge group.

The physics responsible for having to consider the gauge theory on the non-trivial supergravity background is that there are chiral fermions localized on Σ arising from the open strings stretching between the $D3$ and $D7$ branes. It is well known that the gauge anomalies introduced by these chiral degrees of freedom are cancelled only after the appropriate Chern-Simons terms on the D-brane worldvolume are included [85]. The Chern-Simons terms needed to cancel the anomalies become non-trivial due to the presence of the RR one-form flux produced by the $D7$-branes. We show, however, that it is inconsistent to

[31] We note that $\Gamma_{WZW}(A)$ differs from the conventional WZW model action by the addition of a local counterterm which is needed to guarantee that the operator has all the appropriate symmetries.

[32] In the string construction of this operator N denotes the number of $D3$-branes while M is the number of $D7$-branes.

consider only the RR background produced by the $D7$-branes. One must also take into account the non-trivial background geometry and dilaton produced by the $D7$-branes as they are of the same order as the effect produced by the RR flux. This can be seen by showing that the gauge theory in flat space in the presence of the Chern-Simon terms does not capture the supersymmetries of the brane intersection. Therefore, we are led to consider the low energy action of N D3-branes in the supergravity background produced by the intersecting $D7$-branes. The gauge theory describing the low energy dynamics preserves eight supersymmetries and is $ISO(1,1) \times SU(4)$ invariant.

Given the construction of the surface operator in term of D-branes we proceed to study the bulk Type IIB supergravity description of these surface operators. We start by showing that there is a regime in the bulk description where the $D7$-branes can be treated as probe branes in $AdS_5 \times S^5$. We show that this corresponds to the regime where the gauge anomaly is suppressed, the Chern-Simons term can be ignored and the gauge theory lives in flat space. This corresponds to considering the limit where $g^2 M << 1$, where g is the gauge theory coupling constant. In this limit the symmetries of the gauge theory are enhanced to the $SU(1,1|4)$ supergroup.

We go beyond the probe approximation and construct the exact Type IIB supergravity solutions that are dual to the surface operators we have constructed[33]. These solutions can be found by taking the near horizon limit of the supergravity solution describing the localized $D3/D7$ brane intersection from which the surface operator is constructed. The dual supergravity solutions take the form of a warped $AdS_3 \times S^5 \times \mathcal{M}$ metric, where $\mathcal{M}$ is a two dimensional complex manifold. These solutions also shed light on the geometry where the holographic field theory lives. One can infer that the gauge theory lives on the curved background produced by the $D7$-branes by analyzing the dual supergravity geometry near the conformal boundary, thus showing that holography requires putting the gauge theory in a curved space-time. The explicit construction of the supergravity solutions also gives us information about the quantum properties of our surface operators. To leading order in the $g^2 M$ expansion, the surface operator preserves an $SO(2,2) \subset SU(1,1|4)$ symmetry, which is associated with conformal transformations on the surface $\Sigma = \mathbf{R}^{1,1}$. In the probe brane description – where $g^2 M$ effects are suppressed – we also have the $SO(2,2)$ symmetry, while the explicit supergravity solution shows that the $SO(2,2)$ symmetry is

[33] The supergravity solution dual to other (defect) operators in $\mathcal{N} = 4$ have appeared in [86],[87],[88],[89],[90],[84],[91],[92],[93].

broken by g^2M corrections. This shows that g^2M corrections in the field theory break conformal invariance, which can be seen explicitly by analyzing the gauge theory on the $D7$-brane background. This field theory statement is reminiscent[34] to the breaking of conformal invariance by g^2M effects that occurs when considering $\mathcal{N}=4$ SYM coupled to M hypermultiplets, whose β-function is proportional to g^2M.

The plan of the chapter is as follows. In section 1 we introduce the $D3/D7$ brane intersection, the corresponding low energy spectrum and discuss the cancellation of the gauge anomalies via anomaly inflow. We show that the gauge theory on the $D3$-branes has to be placed in the supergravity background produced by the $D7$-branes and construct explicitly the relevant gauge theory action, derive the appropriate supersymmetry transformations and show that the action has all the required symmetries. We integrate out all the degrees of freedom introduced by the $D7$-branes and show that the net effect is to insert the WZW action (3.1) into the $\mathcal{N}=4$ SYM path integral. In section 2 we give the bulk description of the surface operators. We show that there is a regime where the $D7$-branes can be treated as probe branes in $AdS_5\times S^5$ and identify this with the regime in the field theory where the anomaly is suppressed, the Chern-Simons term can be ignored and the gauge theory lives in flat space. We find the explicit exact supergravity solution describing the supergravity background produced by the localized $D3/D7$ brane intersection and show that in the near horizon limit it is described by an $AdS_3\times S^5$ warped metric over a two dimensional manifold. We show that the metric on the boundary, where the gauge theory lives, is precisely the $D7$-brane metric on which we constructed the field theory in section 1. Some of technical details and computations are relegated in section 3.

3.1. Gauge Theory and Surface Operators

Brane Intersection and Anomalies

The surface operators in this chapter are constructed from the low energy field theory on a $D3/D7$ brane configuration that intersects along a surface $\Sigma=\mathbf{R}^{1,1}$. More precisely, we consider the effective description on N $D3$-branes with worldvolume coordinates $x^\mu=$

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[34] Such models have been realized in string theory using brane intersections in e.g. [94], [95]. For attempts at computing the supergravity description of this system see e.g [96], [97], [98].

(x^0, x^1, x^2, x^3) and M $D7$-branes whose worldvolume is parameterized by (x^0, x^1) and $x^I = (x^4, x^5, x^6, x^7, x^8, x^9)$. The coordinates that parametrize the surface Σ are x^0 and x^1:

$$\begin{array}{lcccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ N\,D3 & \mathrm{X} & \mathrm{X} & \mathrm{X} & \mathrm{X} & & & & & & \\ M\,D7 & \mathrm{X} & \mathrm{X} & & & \mathrm{X} & \mathrm{X} & \mathrm{X} & \mathrm{X} & \mathrm{X} & \mathrm{X} \end{array} \tag{3.4}$$

The supersymmetries preserved by the $D3$-branes are the following[35]

$$i\gamma^{0123}\epsilon = \epsilon, \tag{3.5}$$

where ϵ is a ten dimensional complex Weyl spinor satisfying $\gamma^{01\ldots 89}\epsilon = \epsilon$, which labels the thirty-two supersymmetries of Type IIB supergravity. The supersymmetries preserved by the $D7$-branes are given by:

$$i\gamma^{01456789}\epsilon = \epsilon. \tag{3.6}$$

Therefore, in total there are eight supersymmetries preserved by the brane intersection, which can be shown to be chiral in the two dimensional intersection. If we introduce coordinates

$$x^\pm = x^0 \pm x^1 \qquad z = x^2 + ix^3, \tag{3.7}$$

then the unbroken supersymmetries satisfy

$$\gamma_+\epsilon = 0, \tag{3.8}$$

or can alternatively be written as

$$\gamma_{\bar{z}}\epsilon = 0, \tag{3.9}$$

where:

$$\gamma_+ = \frac{1}{2}(\gamma_0 + \gamma_1), \qquad \gamma_{\bar{z}} = \frac{1}{2}(\gamma_2 + i\gamma_3). \tag{3.10}$$

In constructing the supersymmetry transformations of the gauge theory living on the brane intersection we will use four dimensional Weyl spinors. In the four dimensional notation, the sixteen supersymmetries preserved by the $D3$-branes (3.5) are generated by $(\epsilon_\alpha{}^i, \bar{\epsilon}_{\dot{\alpha}i})$, where $\epsilon_\alpha{}^i$ is a four dimensional Weyl spinor of positive chirality transforming in the $(\mathbf{2}, \mathbf{4})$ representation of $SL(2, \mathbf{C}) \times SU(4)$ and $\bar{\epsilon}_{\dot{\alpha}i} = (\epsilon_\alpha{}^i)^*$. These spinors generate

[35] In this chapter we denote the γ-matrices in flat space by γ. The curved space γ-matrices are denoted by Γ. They satisfy $\{\Gamma^M, \Gamma^N\} = 2g^{MN}$, where g_{MN} is the space-time metric.

the usual Poincare supersymmetry transformations of $\mathcal{N}=4$ Yang-Mills theory. In this notation, the projectors (3.8) and (3.9) can be written as:[36]

$$\tilde{\sigma}_{+}{}^{\dot{\alpha}\alpha}\epsilon_{\alpha}{}^{i}=0, \qquad \tilde{\sigma}_{\bar{z}}{}^{\dot{\alpha}\alpha}\epsilon_{\alpha}{}^{i}=0. \tag{3.11}$$

Therefore, the projections (3.11) imply that $\epsilon_1{}^i=\epsilon_2{}^i$, which parametrize the eight real supersymmetries preserved by the brane intersection.

In the low energy limit – where $\alpha'\to 0$ – massive open strings and closed string excitations decouple and only the massless open strings are relevant. The 3-3 strings yield the spectrum of four dimensional $\mathcal{N}=4$ SYM while the quantization of the 3-7 open strings results in two dimensional chiral fermions χ localized on the intersection, and transform in the $(N,\bar{M})$ representation of $U(N)\times U(M)$. The massless 7-7 strings give rise to a SYM multiplet in eight dimensions, but these degrees of freedom are non-dynamical in the decoupling limit and appear in the effective action only as Lagrange multipliers.

The action for the localized chiral fermions is given by

$$S_{defect}=\int dx^{+}dx^{-}\ \bar{\chi}(\partial_{+}+A_{+}+\tilde{A}_{+})\chi, \tag{3.12}$$

where A and $\tilde{A}$ denote the $D3$ and $D7$-brane gauge fields respectively and we have used the coordinates introduced in (3.7). Of the usual Poincare supersymmetries of $\mathcal{N}=4$ SYM, whose relevant transformations are given by

$$\delta A_{\mu}=-i\bar{\lambda}_{\dot{\alpha}i}\tilde{\sigma}_{\mu}{}^{\dot{\alpha}\alpha}\epsilon_{\alpha}{}^{i}+\text{c.c.}, \qquad \delta\chi=0, \qquad \delta\tilde{A}_{\mu}=0, \tag{3.13}$$

the defect term (3.12) is invariant under those supersymmetries for which $\delta A_{+}=0$, which are precisely the ones that satisfy the projections in (3.11) arising from the $D3/D7$ brane intersection.

Quantum mechanically, the path integral over the localized chiral fermions χ is not well defined due to the presence of gauge anomalies in the intersection. In order to see how to cure this problem, it is convenient to split the $U(N)$ and $U(M)$ gauge fields into $SU(N)\times U(1)$ and $SU(M)\times U(1)$ gauge fields . With some abuse of notation, we denote the $SU(N)$ and $SU(M)$ parts of the gauge field by A and $\tilde{A}$ respectively, while the corresponding $U(1)$

[36] Our conventions on σ-matrices are summarized in section 3. They are essentially the same as those in the book [99].

parts of the gauge field are denoted by a and $\tilde{a}$. Then, the variation of the quantum effective action under an $SU(N) \times SU(M)$ gauge transformation

$$\delta A_\mu = \partial_\mu L + [A_\mu, L], \qquad \delta \tilde{A}_\mu = \partial_\mu \tilde{L} + [\tilde{A}_\mu, \tilde{L}] \tag{3.14}$$

is given by

$$\delta_{L,\tilde{L}} S = \frac{1}{8\pi} \int dx^+ dx^- \left[M \mathrm{Tr}_{SU(N)}(L dA) + N \mathrm{Tr}_{SU(M)}(\tilde{L} d\tilde{A}) \right], \tag{3.15}$$

so that the theory is anomalous under $SU(N) \times SU(M)$ gauge transformations. Likewise, $U(1) \times U(1)$ gauge transformations

$$\delta A_\mu = \partial_\mu l, \qquad \delta \tilde{A}_\mu = \partial_\mu \tilde{l}, \tag{3.16}$$

on the quantum effective action yield

$$\delta_{l,\tilde{l}} S = \frac{1}{8\pi} \int dx^+ dx^- NM(l - \tilde{l})(f_{+-} - \tilde{f}_{+-}), \tag{3.17}$$

so that the theory is anomalous under the $U(1)$ gauge transformations generated by $l - \tilde{l}$, and where:

$$f = da, \qquad \tilde{f} = d\tilde{a}. \tag{3.18}$$

Anomalies supported on D-brane intersections are cancelled by the anomaly inflow mechanism [85], which relies on the presence of Chern-Simons couplings in the D-brane worldvolume. The Chern-Simons terms that couple to the $SU(N)$ and $SU(M)$ gauge fields are given by

$$S_{CS}(A) = -\frac{(2\pi\alpha')^2 \tau_3}{2} \int G_1 \wedge \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \tag{3.19}$$

and

$$S_{CS}(\tilde{A}) = -\frac{(2\pi\alpha')^2 \tau_7}{2} \int G_5 \wedge \mathrm{Tr}\left(\tilde{A} \wedge d\tilde{A} + \frac{2}{3} \tilde{A} \wedge \tilde{A} \wedge \tilde{A} \right), \tag{3.20}$$

where g_s is the string coupling constant and τ_3 and τ_7 is the $D3$ and $D7$-brane tension respectively:

$$\tau_3 = \frac{1}{g_s (2\pi)^3 \alpha'^2}, \qquad \tau_7 = \frac{1}{g_s (2\pi)^7 \alpha'^4}. \tag{3.21}$$

G_1 is the RR one-form flux produced by the stack of $D7$-branes and G_5 is the self-dual RR five-form flux produced by the stack of $D3$-branes.

In the presence of localized D-brane sources, the Bianchi identities for the RR fields are modified in a way that the Chern-Simons terms become non-trivial. In our case, the modified Bianchi identities are given by

$$dG_1 = MG_{10}\tau_7\delta^2(z\bar{z}) = g_s M\delta^2(z\bar{z}) \tag{3.22}$$

and

$$dG_5 = NG_{10}\tau_3\delta(x^4)\delta(x^5)\ldots\delta(x^9), \tag{3.23}$$

where G_{10} is the ten-dimensional Newton's constant which is given by:

$$G_{10} = g_s^2(2\pi)^7\alpha'^4. \tag{3.24}$$

Therefore, under an $SU(N)\times SU(M)$ gauge transformation (3.14), the Chern-Simons terms (3.19)and (3.20) are not invariant, and reproduce the two-dimensional anomaly

$$\delta S_{CS}(A) + \delta S_{CS}(\tilde{A}) = -\frac{1}{8\pi}\int dx^+dx^-\left[M\mathrm{Tr}_{SU(N)}(LdA) + N\mathrm{Tr}_{SU(M)}(\tilde{L}d\tilde{A})\right], \tag{3.25}$$

where L and $\tilde{L}$ are taken to vanish at infinity. This mechanism provides a cancellation of the $SU(N)$ and $SU(M)$ gauge anomalies [85].

The Chern-Simons terms containing the $U(1)$ gauge fields a and $\tilde{a}$ are more involved. They have been studied in [100], where the anomalies of a closely related $D5/D5$ brane intersection along a two dimensional defect were studied[37]. The analogous terms for the $D3/D7$ system are given by:

$$\begin{aligned} S_{CS}(a,\tilde{a}) = &-\frac{(2\pi\alpha')^2\tau_3}{2}N\int G_1\wedge a\wedge f - \frac{(2\pi\alpha')^2\tau_7}{2}M\int G_5\wedge\tilde{a}\wedge\tilde{f} \\ &+\frac{(2\pi\alpha')^2\tau_3}{2}N\int G_1\wedge a\wedge\tilde{f} + \frac{(2\pi\alpha')^2\tau_7}{2}M\int G_5\wedge\tilde{a}\wedge f. \end{aligned} \tag{3.26}$$

[37] The physics of that system is quite different from the $D3/D7$ system studied in this chapter. In [100] it was argued that the dynamics of the gauge fields pushes the fermions away from the intersection by a distance determined by the (dimensionful) gauge theory coupling constant. In our system the fermions are stuck at the intersection since the $U(N)$ coupling constant is dimensionless unlike the one on the $D5$-branes which is dimensionful while the $U(M)$ gauge coupling constant vanishes in the decoupling limit, pinning down the fermions at the intersection. Moreover, in [100] the symmetry is enhanced from $ISO(1,1)$ to $ISO(1,2)$ while in our system the symmetry is enhanced from $ISO(1,1)$ to $SO(2,2)$, but only to leading order in the g^2M expansion. Here we also resolve a puzzle left over in their paper, which is to construct the gauge theory action with all the expected supersymmetries.

The first two terms are the usual Chern-Simons couplings analogous to (3.19) and (3.20). The third term arises from the familiar coupling on the $D3$-brane worldvolume of the form

$$\int a \wedge F_3, \tag{3.27}$$

where F_3 is the RR three-form flux, which as argued in [100] is given by $F_3 = G_1 \wedge \tilde{f}$ in the presence of G_1 and $\tilde{f}$ background fields. Note that $\tilde{f}$ in the third term is to be evaluated at $x^I = 0$. Similarly, the last term arises from the Chern-Simons coupling on the $D7$-brane

$$\int \tilde{a} \wedge F_7, \tag{3.28}$$

where the RR seven-form flux is now given by $G_5 \wedge f$, where f is to be evaluated at $z = 0$. If we now perform a $U(1) \times U(1)$ gauge transformation, the variation of (3.26) is given by

$$\delta S_{CS}(a, \bar{a}) = -\frac{1}{8\pi} \int dx^+ dx^- NM(l - \tilde{l})(f_{+-} - \tilde{f}_{+-}), \tag{3.29}$$

where we have used the modified Bianchi identities (3.22) and (3.23). Therefore, by including all the Chern-Simons couplings all anomalies cancel.

Field Theory Construction of Gauge Theories with Anomaly Inflow

Turning on the RR fluxes (3.22) and (3.23) produced by the $D3$ and $D7$ branes is crucial in obtaining an effective theory which is anomaly free. Usually, in analyzing the low energy gauge theory on a D-brane intersection in flat space we can ignore the RR flux produced by the branes. However, whenever there are localized gauge anomalies the RR flux cannot be neglected as it generates the required Chern-Simons needed to cancel the anomaly. But D-branes also source other supergravity fields, such as the metric and the dilaton. It is therefore inconsistent to study the low energy gauge theory in flat space with only the addition of the RR-induced Chern-Simons terms. Physically, one must consider the gauge theory in the *full* supergravity background produced by the other D-brane, as the effect of the metric and dilaton is of the same order as the effect of the RR flux.

One way to see that it is inconsistent to consider the gauge theory on the $D3$-branes in flat space and in the presence of only the RR-flux produced by the $D7$-branes is to note that the naive action of the system

$$S = S_{\mathcal{N}=4} + S_{defect} + S_{CS}(A) + S_{CS}(\tilde{A}) + S_{CS}(a, \tilde{a}), \tag{3.30}$$

is not supersymmetric, where $S_{\mathcal{N}=4}$ is the usual flat space action of $\mathcal{N}=4$ SYM and the other terms appear in (3.12), (3.19), (3.20) and (3.26) respectively. In particular, this low-energy gauge theory does not capture the supersymmetries of the brane intersection (3.11), and therefore is not a faithful description of the low energy dynamics.

In the rest of this section we construct the low energy gauge theory living on the $D3$-branes when embedded in the full supergravity background of the $D7$-branes – which includes the appropriate Chern-Simons terms – and show that the field theory has all the required symmetries.

The $D7$-Brane Background

As just argued, we must construct the low energy gauge theory on the $D3$-branes when placed in the full supergravity background of the $D7$-branes. We will devote this subsection to reviewing the salient features of the $D7$-brane background.

The metric produced by the $D7$-branes in the brane array (3.4) is given by

$$ds^2 = g_{MN}dx^M dx^N = H_7^{-1/2}(-(dx^0)^2 + (dx^1)^2 + dx^I dx^I) + H_7^{1/2} dz d\bar{z}, \tag{3.31}$$

where the coordinates are defined in (3.4). The RR axion C and the dilaton Φ can be combined into a complex field τ with is holomorphic in z, so that the axion and the dilaton produced by the $D7$-branes is given by:

$$\begin{aligned} &\partial_{\bar{z}}\tau = 0 \qquad \text{where} \qquad \tau = C + ie^{-\Phi} \\ &e^{-\Phi} = H_7. \end{aligned} \tag{3.32}$$

This background solves the Killing spinor equations of Type IIB supergravity

$$\begin{aligned} &\delta\Psi_M = \partial_M \epsilon + \frac{1}{4}\omega_M^{AB}\Gamma_{AB}\epsilon - \frac{i}{8}e^{\Phi}\partial_N C \Gamma^N \Gamma_M \epsilon = 0, \\ &\delta\psi = (\Gamma^M \partial_M \Phi)\epsilon + ie^{\Phi}\partial_M C \Gamma^M \epsilon = 0, \end{aligned} \tag{3.33}$$

and preserves the sixteen supersymmetries satisfying

$$\epsilon = H_7^{-1/8}\epsilon_0, \quad \gamma_{\bar{z}}\epsilon_0 = 0, \tag{3.34}$$

where Ψ_M and ψ are the ten-dimensional gravitino and dilatino respectively.

The simplest solution describes the local fields around a coincident stack of $D7$-branes. This local solution has a $U(1)$ symmetry, which acts by rotations in the space transverse to the $D7$-branes, which is parametrized by the coordinate z. It is given by

$$\tau = i\tau_0 + \frac{g_s M}{2\pi i} \ln z, \tag{3.35}$$

so that

$$e^{-\Phi} = H_7 = \tau_0 - \frac{g_s M}{2\pi} \ln r, \quad C = \frac{g_s M}{2\pi} \theta, \tag{3.36}$$

where $z = re^{i\theta}$ and τ_0 is an arbitrary real constant. This solution, however, is only valid very near the branes – for small r – as $e^{-\Phi}$ becomes negative at a finite distance and we encounter a singularity. The local solution for separated branes corresponds to

$$\tau = i\tau_0 + \frac{g_s}{2\pi i} \sum_{l=1}^{M} \ln(z - z_l), \tag{3.37}$$

where z_l is location of the l-th $D7$-brane.

As shown in [101] (see [102], [103] for more recent discussions), the local solution can be patched into global solutions that avoid the pathologies of the local one. The global solutions break the $U(1)$ symmetry present in the local solution of coincident $D7$-branes. In order to describe them it is convenient to switch to the Einstein frame, where the $SL(2,\mathbf{Z})$ invariance of Type IIB string theory is manifest. In this frame, the local metric is given by

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + dx^I dx^I + H_7 dz d\bar{z}. \tag{3.38}$$

Since τ is defined up to the action of $SL(2,\mathbf{Z})$ and $Im\ \tau > 0$, it follows that τ takes values in the fundamental domain $\mathcal{F} = \mathcal{H}^+/SL(2,\mathbf{Z})$, where $\mathcal{H}^+$ is the upper half plane. In order to find a global solution for τ one has to consider the one-to-one map $j : \mathcal{F} \to \mathbf{C}$ from the fundamental domain $\mathcal{F}$ to the complex plane $\mathbf{C}$. This map j is well-known and given by

$$j(\tau) = \frac{(\theta_2(\tau)^8 + \theta_3(\tau)^8 + \theta_4(\tau)^8)^3}{\eta(\tau)^{24}}, \tag{3.39}$$

where the θ's are the usual theta-functions while η is the Dedekind η-function

$$\eta(\tau) = q^{1/24} \prod_n (1 - q^n), \tag{3.40}$$

where $q = e^{2\pi i\tau}$. Then the various solutions for τ are given by

$$j(\tau(z)) = g(z), \tag{3.41}$$

where $g(z)$ is an arbitrary meromorphic function in the complex plane. For a stack of M coincident $D7$-branes we have

$$g(z) = a + \frac{b}{z^{g_s M}}, \tag{3.42}$$

where a sets the value of the dilaton at infinity and b is related to τ_0 in (3.36). Indeed, for $Im\ \tau >> 1$, $j(\tau) \simeq e^{-2\pi i \tau}$ which implies the local behavior (3.35) near $z = 0$.

In general, different choices of $g(z)$ correspond to different types of $D7$-brane solutions. The metric can be written in the following form

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + dx^I dx^I + H_7 f \bar{f} dz d\bar{z}, \tag{3.43}$$

where as in the local case $H_7 = e^{-\Phi}$ and where f is a holomorphic function of z. Locally, one can always choose a coordinate system where $f\bar{f}dzd\bar{z} = dz'd\bar{z}'$ for some local coordinates z' and $\bar{z}'$. This brings the metric (3.43) to the usual local form (3.38). However, globally this cannot be done as discussed above. For the metric to be globally defined, $H_7 f\bar{f}$ has to be $SL(2,\mathbf{Z})$ invariant. The solution studied in [101] is given by

$$H_7 f \bar{f} = e^{-\Phi} \eta^2 \bar{\eta}^2 |\prod_{i=1}^{M} (z - z_i)^{-1/12}|^2, \tag{3.44}$$

where z_i's are the location of the poles of $g(z)$, which correspond to the position of the various $D7$-branes in the z-plane[38].

The metric (3.43) is smooth everywhere except 1) at $z = z_i$ where it behaves as $\ln|z - z_i|$ due to the presence of a $D7$-brane source there and 2) at infinity, where it has a conical singularity with deficit angle $\delta = \frac{\pi M}{6}$. In this thesis, we will mostly be using the $D7$-brane background in the local form (3.31), (3.32). However, as we explained the generalization to the global case is straightforward.

We finish this subsection by constructing the Killing spinors of the gauge theory on the $D3$-branes when placed in the background of the $D7$-branes. If we consider the $D3/D7$ intersection in (3.4), we need the restriction of the $D7$-brane background to the worldvolume of the $D3$-branes. Then the induced metric on the $D3$-branes is given by:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -H_7^{-1/2} dx^+ dx^- + H_7^{1/2} dz d\bar{z}. \tag{3.45}$$

[38] There are restrictions on the range of M coming from the fact that for M large enough the space becomes compact. This was studied in detail in [101]. We will not discuss this point in this chapter.

The Killing spinor equation satisfied by the four dimensional spinors ${\epsilon_\alpha}^i$ that generate the worldvolume supersymmetry transformations on the $D3$-branes is given by[39]

$$D_\mu {\epsilon_\alpha}^i = -\frac{i}{8} e^\Phi \partial_\nu C {\sigma^\nu}_{\alpha\dot\alpha} {\tilde\sigma_\mu}^{\dot\alpha\beta} {\epsilon_\beta}^i \,, \tag{3.47}$$

where D_μ is the covariant derivative in the background metric (3.45). Therefore

$${\epsilon_\alpha}^i = H_7^{-1/8} {\epsilon_{0\alpha}}^i, \tag{3.48}$$

where

$$\tilde\sigma_{\bar z}^{\dot\alpha\alpha} {\epsilon_\alpha}^i = 0, \qquad \tilde\sigma_{+}^{\dot\alpha\alpha} {\epsilon_\alpha}^i = 0, \tag{3.49}$$

thus reproducing the supersymmetry conditions derived for the brane intersection (3.11). In the next subsection we write down the action and supersymmetry transformations of the $D3/D7$ low energy gauge theory and show that the preserved Killing spinors satisfy (3.47) subject to the constraints (3.49).

Holographic Gauge Theory in Background Fields

In this subsection we construct the low energy gauge theory on the $D3$-branes when placed in the full supergravity background of the $D7$-branes. This is the appropriate decoupled field theory that holographically describes the physics of the dual closed string background, which we obtain in section 2 by finding the supergravity solution of the $D3/D7$ intersection. We also construct the corresponding supersymmetry transformations and show that the action is invariant under the subset of $\mathcal{N} = 4$ supersymmetry transformations satisfying the restrictions (3.47) and (3.49), which are precisely the supersymmetries preserved by the D-brane intersection in flat space (3.4).

There is a systematic way of constructing the action and supersymmetry transformations on a single D-brane in an arbitrary supergravity background. The starting point is to consider the covariant D-brane action in an arbitrary curved superspace background [104],

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[39] To go to the four dimensional notation we have used:

$$\Gamma^\mu = i \begin{pmatrix} 0 & \sigma^\mu \\ \tilde\sigma^\mu & 0 \end{pmatrix}. \tag{3.46}$$

[105] (which generalizes the flat space construction in [106].) These actions can in principle be expanded to all orders in the fermions around a given background, even though explicit formulas are not easy to obtain. The covariant action has κ-symmetry and diffeomorphism invariance. By fixing κ-symmetry we can gauge away sixteen of the thirty two fermions of Type IIB supergravity superspace. The remaining sixteen fermions are then identified with the gauginos filling up the SYM multiplet living on the D-brane. Likewise, worldvolume diffeomorphisms can be fixed by specifying how the brane is embedded in the background, which allows for the identification of the scalars of the SYM multiplet parametrizing the position of the D-brane.

In order to construct the explicit supersymmetry transformations of the gauge fixed action one must combine the superspace supersymmetry transformations on the physical fields together with a compensating κ and diffeomorphism transformation to preserve the gauge fixing condition.

Since we are interested in considering a decoupling limit, where $\alpha' \to 0$, this procedure simplifies considerably. In this limit the only terms in the action that survive are quadratic in the fields. Fortunately, the explicit expression for the D-brane action to quadratic order in the fermions in an arbitrary supergravity background can be found in [107][40] (see also [108],[109],[110]). This approach gives the brane action quadratic in fermions with fixed κ-supersymmetry and diffeomorphisms in an arbitrary supersymmetric background. Therefore, we start by finding the action for a single $D3$-brane in the $D7$-brane background following [107]. Later we will show how to extend this analysis to the case when the gauge group is non-Abelian.

Let us start with the bosonic action in the $D7$-brane background. The action for the gauge field A_μ is straightforward to write down. It is given by

$$S_V = -\frac{T_3}{4}\int d^4x\sqrt{-g}e^{-\Phi}F_{\mu\nu}F^{\mu\nu} - \frac{T_3}{4}\int d^4x\sqrt{-g}\partial_\mu C\epsilon^{\mu\nu\rho\sigma}A_\nu F_{\rho\sigma}, \qquad (3.50)$$

where

$$T_3 = (2\pi\alpha')^2\tau_3 = \frac{1}{2\pi g_s} = \frac{1}{g^2}, \qquad (3.51)$$

[40] In that paper the action is written to quadratic order in the fermionic fields and to all order in the bosonic fields. In the decoupling limit, we will only need to extract the action to quadratic order in the bosonic fields.

where g is the SYM coupling constant. The coordinates $x^\mu = (x^+, x^-, z, \bar{z})$ describe the coordinates along the $D3$-brane worldvolume as defined in (3.4). The metric used on the $D3$-brane worldvolume is the induced metric (3.45) from the $D7$-brane background.

In order to obtain the action for the scalar fields on the $D3$-brane it is important to properly identify which are the fields describing the D-brane fluctuations. We introduce vielbeins which are adapted to the symmetries preserved by the $D3$-brane $(e^{\hat{\mu}}, e^{\hat{I}})$, where $\hat{\mu}$ and $\hat{I}$ denote the flat indices along and transverse to the $D3$-brane respectively. The static gauge is fixed by the requirement that the pullback of the vielbein $e_I^{\hat{I}}$ on the $D3$-brane vanishes and the pullback of the vielbein $e_\mu^{\hat{\mu}}$ forms a $D3$-brane worldvolume vielbein. The physical scalar fields are parametrized by

$$\varphi^{\hat{I}} = e_I^{\hat{I}} \delta x^I \tag{3.52}$$

rather than by the fluctuations in the transverse coordinates δx^I. The scalar fields $\varphi^{\hat{I}}$ transform under the local tangent space $SU(4) \simeq SO(6)$ symmetries while the fluctuations δx^I transform under diffeomorphisms in the transverse space. This choice of the static gauge manifestly has the $SO(6)$ R-symmetry since the index $\hat{I}$ is flat.

The low energy action for the scalar fields $\varphi^{\hat{I}}$ can be obtained by expanding the bosonic part of the DBI action:

$$S_{DBI} = -\tau_3 \int d^4x e^{-\Phi} \sqrt{-G}. \tag{3.53}$$

G is the determinant of the metric

$$G_{\mu\nu} = g_{\mu\nu} + G_{IJ} \partial_\mu \delta x^I \partial_\nu \delta x^J, \tag{3.54}$$

where $g_{\mu\nu}$ is the induced metric (3.45) and G_{IJ} is the metric in the transverse space (3.38)

$$G_{IJ} = H^{-1/2} \delta_{IJ} = e^{\Phi/2} \delta_{IJ}, \tag{3.55}$$

where the last equality is a property of the $D7$-brane background.

Therefore, we find that the quadratic action for the scalar fields in the SYM multiplet is given by

$$S_{Sc} = -\frac{T_3}{2} \int d^4x \sqrt{-g} e^{-\Phi} G_{IJ} \partial_\mu \delta x^I \partial^\mu \delta x^J = -\frac{T_3}{2} \int d^4x \sqrt{-g} e^{-\Phi} G_{IJ} \partial_\mu (e^I_{\hat{I}} \varphi^{\hat{I}}) \partial^\mu (e^J_{\hat{J}} \varphi^{\hat{J}}), \tag{3.56}$$

where the worldvolume indices μ are contracted with the induced metric (3.45) and we have used (3.52) to eliminate δx^I in terms of $\varphi^{\hat{I}}$. We note that ∂_μ in (3.56) acts not only on $\varphi^{\hat{I}}$ but also on the vielbein's $e^{I}_{\hat{I}}$. This fact is responsible for giving a mass to the scalar fields $\varphi^{\hat{I}}$. More precisely, evaluating (3.56) gives

$$S_{Sc} = -\frac{T_3}{2}\int d^4x\sqrt{-g}e^{-\Phi}(\partial_\mu\varphi^{\hat{I}}\partial^\mu\varphi^{\hat{I}} + \frac{1}{2}(\mathcal{R} + \partial^\mu\partial_\mu\Phi)\varphi^{\hat{I}}\varphi^{\hat{I}}), \tag{3.57}$$

where $\mathcal{R}$ is the scalar curvature of the induced metric (3.45), which in terms of the dilaton field Φ is given by:

$$\mathcal{R} = -\frac{3}{8}\partial^\mu\Phi\partial_\mu\Phi - \frac{1}{2}\partial^\mu\partial_\mu\Phi. \tag{3.58}$$

A similar mass term proportional to the curvature appears in the action of $\mathcal{N}=4$ Yang-Mills theory in $\mathbf{R}\times S^3$ [111] (for a recent discussion see [112]).

For later convenience, we parametrize the six scalars $\varphi^{\hat{I}}$ by a two-index antisymmetric tensor φ^{ij} of $SU(4)$ via

$$\varphi^{\hat{I}} = \frac{1}{2}\gamma^{\hat{I}}_{ij}\varphi^{ij}, \quad \varphi^{ij} = \frac{1}{2}\tilde{\gamma}^{\hat{I}ij}\varphi^{\hat{I}}, \quad \varphi_{ij} = \frac{1}{2}\epsilon_{ijkl}\varphi^{kl}, \tag{3.59}$$

where $\gamma^{\hat{I}}_{ij}$ are the Clebsch-Gordan coefficients that couple the **6** representation of $SO(6)$ to the **4**'s of $SU(4)$ labeled by the i,j indices. The Clebsch-Gordan coefficients satisfy a Clifford algebra:

$$\{\gamma^{\hat{I}}, \tilde{\gamma}^{\hat{J}}\} = 2\delta^{\hat{I}\hat{J}}. \tag{3.60}$$

In this parametrization the action of the scalar fields in the SYM multiplet is given by:

$$S_{Sc} = -\frac{T_3}{2}\int d^4x\sqrt{-g}e^{-\Phi}(\partial_\mu\varphi^{ij}\partial^\mu\varphi_{ij} + \frac{1}{2}(\mathcal{R} + \partial^\mu\partial_\mu\Phi)\varphi^{ij}\varphi_{ij}). \tag{3.61}$$

Now we move on to the action for the fermions in the SYM multiplet. As indicated earlier, the κ-supersymmetric DBI action depends on thirty two spinors, which can be parametrized by two ten dimensional Majorana spinors of positive chirality, denoted by θ_1 and θ_2. Fixing κ-supersymmetry is equivalent to setting one of them, say θ_2 to zero. Hence, the fermionic action can be written in terms of θ_1, which is identified with the gaugino in the SYM multiplet. The quadratic fermionic action with fixed κ-supersymmetry was found in [107]. Adopting their answer to our present case we obtain:

$$S_F = \frac{T_3}{2}\int d^4x\sqrt{-g}e^{-\Phi}(\bar{\theta}_1\Gamma^\mu D_\mu\theta_1 - \bar{\theta}_1\hat{\Gamma}^{-1}_{D_3}(\Gamma^\mu W_\mu - \Delta)\theta_1). \tag{3.62}$$

In this expression we have used:

$$\begin{aligned}
\bar{\theta}_1 &= i\theta_1^T\gamma^0,\\
\hat{\Gamma}_{D_3} &= \gamma_0\gamma_1\gamma_2\gamma_3,\\
W_\mu &= \frac{1}{8}e^{-\Phi}\partial_\nu C\Gamma^\nu\Gamma_\mu,\\
\Delta &= -\frac{1}{2}e^{-\Phi}\partial_\mu C\Gamma^\mu.
\end{aligned} \tag{3.63}$$

In order to write the action in terms of four dimensional spinors we use the basis of Γ matrices in (3.46) and decompose

$$\theta_1 = \begin{pmatrix} \lambda_\alpha{}^i \\ \bar{\lambda}_{\dot{\alpha}i} \end{pmatrix}, \tag{3.64}$$

where $\lambda_\alpha{}^i$ is the four dimensional gaugino. We then obtain the action for the fermionic components of the SYM multiplet:

$$S_F = T_3\int d^4x\sqrt{-g}e^{-\Phi}(\frac{i}{2}\bar{\lambda}_i\tilde{\sigma}^\mu D_\mu\lambda^i - \frac{i}{2}D_\mu\bar{\lambda}_i\tilde{\sigma}^\mu\lambda^i) - \frac{T_3}{4}\int d^4x\sqrt{-g}\partial_\mu C\bar{\lambda}_i\tilde{\sigma}^\mu\lambda^i. \tag{3.65}$$

In summary, the total action for the SYM multiplet in the Abelian case is then given by:

$$S_{abel} = S_V + S_{Sc} + S_F, \tag{3.66}$$

where S_V, S_{Sc} and S_F are given by (3.50), (3.61) and (3.65) respectively.

The supersymmetry transformations can be obtained from the superspace supersymmetry transformations on the physical fields with a compensating κ and diffeomorphism transformation to preserve the gauge fixing condition [107]. For the case under consideration we find that the action (3.66) is supersymmetric under the following transformations

$$\begin{aligned}
\delta A_\mu &= -i\bar{\lambda}_i\tilde{\sigma}_\mu\epsilon^i + \text{c.c.}\\
\delta\varphi^{ij} &= (\lambda^{\alpha i}\epsilon_\alpha{}^j - \lambda^{\alpha j}\epsilon_\alpha{}^i) + \epsilon^{ijkl}\bar{\epsilon}_{\dot{\alpha}k}\bar{\lambda}^{\dot{\alpha}}{}_l\\
\delta\lambda_\alpha{}^i &= -\frac{1}{2}F_{\mu\nu}(\sigma^\mu\tilde{\sigma}^\nu)_\alpha{}^\beta\epsilon_\beta{}^i - 2i\sigma^\mu{}_{\alpha\dot{\alpha}}(\partial_\mu\varphi^{ij})\bar{\epsilon}^{\dot{\alpha}}{}_j + \frac{i}{2}\sigma^\mu{}_{\alpha\dot{\alpha}}(\partial_\mu\Phi)\varphi^{ij}\bar{\epsilon}^{\dot{\alpha}}{}_j,
\end{aligned} \tag{3.67}$$

where $\epsilon_\alpha{}^i$ is a Killing spinor satisfying (3.47) and subject to the constraints

$$\tilde{\sigma}_{\bar{z}}{}^{\dot{\alpha}\alpha}\epsilon_\alpha{}^i = 0, \qquad \tilde{\sigma}_+{}^{\dot{\alpha}\alpha}\epsilon_\alpha{}^i = 0, \tag{3.68}$$

so that the action is invariant under eight real supersymmetries.

We note that the variation of the gaugino contains a term proportional to the derivative of the dilaton which is absent in the usual $\mathcal{N} = 4$ SYM theory in flat space. The appearance of this term is consistent with the presence of a scalar "mass term" in the action (3.61). The existence of the mass term in the action indicates that a non-vanishing constant values of φ^{ij} does not solve equations of motion. On the other hand, the set of supersymmetric solutions can be obtained by setting the variations of the fermions to zero. Therefore, the absence of the last term in $\delta\lambda_\alpha{}^i$ would indicate that any constant φ^{ij} was a supersymmetric solution, in direct contradiction with the equations of motion[41].

The formalism using the covariant D-brane action allowed us to write a supersymmetric gauge theory action when the gauge group is Abelian. We now extend the analysis of the action and the supersymmetry transformations to the case when the gauge group is non-Abelian. The extension is relatively straightforward. In the action (3.66) we replace all derivatives D_μ by the gauge covariant derivatives $\mathcal{D}_\mu$, where

$$\mathcal{D}_\mu \cdot = D_\mu \cdot + [A_\mu, \cdot], \tag{3.70}$$

replace the Chern-Simons term in (3.50) by its non-Abelian analog

$$-\frac{T_3}{4} \int d^4x \sqrt{-g} \partial_\mu C \epsilon^{\mu\nu\rho\sigma} \mathrm{Tr}(A_\nu F_{\rho\sigma} - \frac{2}{3} A_\nu A_\rho A_\sigma), \tag{3.71}$$

and add the familiar non-Abelian couplings of $\mathcal{N} = 4$ SYM in flat space:

$$S_{nabe} = T_3 \int d^4x \sqrt{-g} e^{-\Phi} \mathrm{Tr}(\bar{\lambda}_{\dot{\alpha}i} [\bar{\lambda}^{\dot{\alpha}}{}_j, \varphi^{ij}] + \lambda^{\alpha i} [\lambda_\alpha{}^j, \varphi_{ij}] - \frac{1}{2} [\varphi^{ij}, \varphi^{kl}][\varphi_{ij}, \varphi_{kl}]). \tag{3.72}$$

In the supersymmetry transformations (3.67) we replace also all covariant derivatives D_μ with $\mathcal{D}_\mu$, and add to $\delta\lambda_\alpha{}^i$ the usual flat space $\mathcal{N} = 4$SYM commutator term $-2[\varphi_{jk}, \varphi^{ki}]\epsilon_\alpha{}^j$.

---

[41] One can perform a field redefinition and get rid of the "mass term" for the scalar fields in (3.60). To do this, one simply goes from $\varphi^{\hat{I}}$ to δx^I

$$\delta x^I = e^I{}_{\hat{I}} \varphi^{\hat{I}} = e^{-\frac{\Phi}{4}} \varphi^{\hat{I}} \delta^I_{\hat{I}}. \tag{3.69}$$

This transformation eliminates the "mass term" for the scalar fields as well as the term $\frac{i}{2}\sigma^\mu{}_{\alpha\dot{\alpha}}(\partial_\mu\Phi)\varphi^{ij}\bar{\epsilon}^{\dot{\alpha}}{}_j$ in the supersymmetry transformations for the gauginos.

We have found the complete non-Abelian action on N $D3$-branes when embedded in the $D7$-brane background. The full action is given by:

$$
\begin{aligned}
S = & -\frac{T_3}{4}\int d^4x\sqrt{-g}e^{-\Phi}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} - \frac{T_3}{4}\int d^4x\sqrt{-g}\partial_\mu C\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\left(A_\nu F_{\rho\sigma} - \frac{2}{3}A_\nu A_\rho A_\sigma\right) \\
& + T_3\int d^4x\sqrt{-g}e^{-\Phi}\mathrm{Tr}\left(\frac{i}{2}\bar{\lambda}_i\tilde{\sigma}^\mu\mathcal{D}_\mu\lambda^i - \frac{i}{2}\mathcal{D}_\mu\bar{\lambda}_i\tilde{\sigma}^\mu\lambda^i\right) - \frac{T_3}{4}\int d^4x\sqrt{-g}\partial_\mu C\mathrm{Tr}(\bar{\lambda}_i\tilde{\sigma}^\mu\lambda^i) \\
& - \frac{T_3}{2}\int d^4x\sqrt{-g}e^{-\Phi}\mathrm{Tr}\left(\mathcal{D}_\mu\varphi_{ij}\mathcal{D}^\mu\varphi^{ij} + \frac{1}{2}(\mathcal{R} + \partial^\mu\partial_\mu\Phi)\varphi^{ij}\varphi_{ij}\right) \\
& + T_3\int d^4x\sqrt{-g}e^{-\Phi}\mathrm{Tr}\left(\bar{\lambda}_{\dot{\alpha}i}[\bar{\lambda}^{\dot{\alpha}}{}_j, \varphi^{ij}] + \lambda^{\alpha i}[\lambda_\alpha{}^j, \varphi_{ij}] - \frac{1}{2}[\varphi^{ij}, \varphi^{kl}][\varphi_{ij}, \varphi_{kl}]\right).
\end{aligned}
\tag{3.73}
$$

The action on the $D3$-branes (3.73) is invariant under the following explicit supersymmetry transformations

$$
\begin{aligned}
\delta A_\mu = & -i\bar{\lambda}_i\tilde{\sigma}_\mu\epsilon^i + \text{c.c.} \\
\delta\varphi^{ij} = & (\lambda^{\alpha i}\epsilon_\alpha{}^j - \lambda^{\alpha j}\epsilon_\alpha{}^i) + \epsilon^{ijkl}\bar{\epsilon}_{\dot{\alpha}k}\bar{\lambda}^{\dot{\alpha}}{}_l \\
\delta\lambda^i_\alpha = & -\frac{1}{2}F_{\mu\nu}(\sigma^\mu\tilde{\sigma}^\nu)_\alpha{}^\beta\epsilon_\beta{}^i - 2i\sigma^\mu{}_{\alpha\dot{\alpha}}(\partial_\mu\varphi^{ij})\bar{\epsilon}^{\dot{\alpha}}{}_j + \frac{i}{2}\sigma^\mu{}_{\alpha\dot{\alpha}}(\partial_\mu\Phi)\varphi^{ij}\bar{\epsilon}^{\dot{\alpha}}{}_j \\
& - 2[\varphi_{jk}, \varphi^{ki}]\epsilon_\alpha{}^j,
\end{aligned}
\tag{3.74}
$$

where $\epsilon_\alpha{}^i$ is a Killing spinor satisfying (3.47) and subject to the constraints (3.68). It, thus, preserves the same eight supersymmetries preserved by the D-brane intersection.

The detailed check of the invariance of the action (3.73) under the supersymmetry transformations (3.74) is summarized in section 3.

We finish this subsection by stating the symmetries of this field theory. The bosonic symmetry is $ISO(1,1)\times SO(6)$. Furthermore, the field theory is invariant under eight real supercharges. Note that the theory is not conformally invariant. The dilatations and special conformal transformations are broken by z-dependent warp-factors $H_7(z,\bar{z})$ in (3.45).

The WZW Surface Operator

In this final subsection we show that the field theory on the $D3/D7$ intersection describes a surface operator of $\mathcal{N}=4$ SYM in the $D7$-brane background. This surface operator, unlike the one in [29], has a classical expression that can be written down in terms of the classical fields that appear in the Lagrangian of $\mathcal{N}=4$ SYM.

The strategy that we follow for determining the expression for the surface operator is to integrate out explicitly the fermions $\chi, \bar{\chi}$ that are localized on the surface. The effect of the non-dynamical $D7$-brane gauge field is trivial and we suppress it in this section. In section 3 we show that integrating over this gauge field reproduces the same answer as when we suppress it. This same strategy was used in chapter 2 [76] to derive the Wilson loop operators in $\mathcal{N}=4$, which were obtained by integrating out the localized degrees of freedom living on the loop arising from a brane intersection.

We want to perform the following path integral[42]

$$Z = e^{iS} \cdot \int [D\chi][D\bar{\chi}] \exp\left(iS_{defect}\right), \tag{3.75}$$

where:

$$S_{defect} = \int dx^+ dx^- \bar{\chi} \left(\partial_+ + A_+\right) \chi. \tag{3.76}$$

S is the $\mathcal{N}=4$ SYM action in the $D7$-brane background (3.73).

We proceed to integrating out the chiral fermions localized on the surface. This is well known to produce a WZW model, which precisely captures the anomaly of the chiral fermions via the identity

$$\text{Det}(\partial_+ + A_+) = \exp\left(ic_R \Gamma_{WZW}(A)\right), \tag{3.77}$$

where c_R is the index of the representation R under which the fermions transform. The explicit expression for the WZW action one gets is

$$\begin{aligned}\Gamma_{WZW}(A) = &-\frac{1}{8\pi}\int dx^+ dx^- \text{Tr}\left[\left(U^{-1}\partial_+ U\right)\left(U^{-1}\partial_- U\right) - \left(U^{-1}\partial_+ U\right)\left(V^{-1}\partial_- V\right)\right] \\ &- \frac{1}{24\pi}\int d^3x \epsilon^{ijk} \text{Tr}\left[\left(U^{-1}\partial_i U\right)\left(U^{-1}\partial_j U\right)\left(U^{-1}\partial_k U\right)\right],\end{aligned} \tag{3.78}$$

where U and V are $U(N)$ group elements nonlocally related to the gauge field of $\mathcal{N}=4$ SYM:

$$A_+ = U^{-1}\partial_+ U \qquad\qquad A_- = V^{-1}\partial_- V. \tag{3.79}$$

We note that $\Gamma_{WZW}(A)$ differs from the conventional WZW model action by the addition of a local counterterm:

$$\frac{1}{8\pi}\int dx^+ dx^- \text{Tr}\left[\left(U^{-1}\partial_+ U\right)\left(V^{-1}\partial_- V\right)\right] = \frac{1}{8}\int dx^+ dx^- \text{Tr} A_+ A_-. \tag{3.80}$$

[42] After this integral is performed, we must still integrate over the $\mathcal{N}=4$ SYM fields.

The addition of this term is needed to guarantee that (3.78) reproduces the correct chiral anomaly. Indeed, it is straightforward to show that under a $U(N)$ gauge transformation $\delta A_\mu = \partial_\mu + [A_\mu, L]$ we have that the WZW action (3.78) is not invariant:

$$\delta\Gamma_{WZW}(A) = \frac{1}{8\pi}\int dx^+ dx^- \mathrm{Tr}\left[L\left(\partial_+ A_- - \partial_- A_+\right)\right]. \tag{3.81}$$

This gives the same anomalous variation as the usual anomaly in two dimensions (3.15). We recall that our complete action, which combines the $\mathcal{N}=4$ SYM action on the $D7$-brane background (3.73) with the defect term in (3.76) is not anomalous. The anomaly produced by the WZW action is precisely cancelled by a Chern-Simons term.

We also note that $\Gamma_{WZW}(A)$ is not invariant under the supersymmetry transformations (3.74), unlike the original action S_{defect}. But we recall that the Chern-Simons terms always has a boundary term under any variation of the gauge field and that this boundary contribution cancels the variation of $\Gamma_{WZW}(A)$ proportional to δA_-. For this cancellation to occur, it is also crucial to add the local counterterm (3.80).

Therefore, integrating out the localized fields has the effect of inserting the following surface operator into the gauge theory action (3.73):

$$\mathcal{O}_\Sigma = \exp\left(iM\Gamma_{WZW}(A)\right). \tag{3.82}$$

The surface operator is described by a $U(N)$ WZW model at level M. The explicit form of the action is (3.78), where U and V are $U(N)$ valued group elements that are nonlocally related to the $\mathcal{N}=4$ SYM gauge fields via:

$$A_+ = U^{-1}\partial_+ U \qquad A_- = V^{-1}\partial_- V. \tag{3.83}$$

The surface operator (3.82) is supersymmetric under the transformations (3.74) and $U(N)$ invariant when combined with the gauge theory action in the $D7$-brane background (3.73).

Using the explicit expression for the surface operator one can study its properties in perturbation theory. For the case when $\Sigma = \mathbf{R}^{1,1}$ we expect that supersymmetry requires $\langle\mathcal{O}_\Sigma\rangle = 1$, just like in the case of the Wilson line. Another interesting case to consider – which is related by a conformal transformation to the Euclidean version of the previous case – is when $\Sigma = S^2$. In this case Σ is curved and we expect that there is a conformal anomaly associated with the surface which would be interesting to compute explicitly. The bulk description discussed in the next section supports these expectations, as we find that at least in the probe approximation that $\langle \mathcal{O}_\Sigma \rangle = 1$ and that there is a conformal anomaly for the cases $\Sigma = \mathbf{R}^{1,1}$ and S^2 respectively.

Given that these operators are supersymmetric one may expect that the computation of their expectation value is captured by a simpler model, similar to what happens for circular Wilson loops [113], [114]. One may be able to derive the reduced model by topologically twisting the gauge theory by the supercharges preserved by the surface operator.

3.2. The Bulk Description

In this section we study the physics of the surface operator from a dual gravitational point of view. We find that there is a regime in which the $D7$-branes can be treated as a probe brane in $AdS_5 \times S^5$ and identify the corresponding regime in the gauge theory. We also find the exact solutions of the Type IIB supergravity equations of motion – which take the backreaction of the $D7$-branes into account – which are dual to the surface operators in the gauge theory we have constructed in this chapter.

The Probe Approximation and Anomaly Suppression

In the previous section we have constructed the decoupled low energy effective field theory living on the $D3/D7$ intersection (3.4). Following [19] our aim in this section is to provide the bulk gravitational description of this field theory. This requires finding the supergravity solution describing the brane intersection (3.4) [19].

In the absence of the $D7$-branes, the gauge theory on N $D3$-branes is dual to string theory in $AdS_5 \times S^5$ [19]. We are interested in understanding what the effect of introducing the $D7$-branes is in the bulk description.

One may try to first consider the $D7$-branes as a small perturbation around the $AdS_5 \times S^5$ background. The parameter that controls the gravitational backreaction due to the M $D7$-branes can be extracted from the supergravity equations of motion. It is governed by

$$\epsilon = M \cdot G_{10} \tau_7 = g_s M = \frac{g^2}{2\pi} M. \tag{3.84}$$

In the last step we have written the parameter using gauge theory variables, where g is the gauge theory coupling constant. Therefore, we can treat the $D7$-branes as probes in $AdS_5 \times S^5$ as long as $g^2 M$ is small.

In the regime where g^2M is small we can consistently treat the $D7$-branes in the probe approximation. It is straightforward to show that the $D7$-brane equations of motion are solved by the embedding (3.4) even when we place the $D7$-branes in the non-trivial supergravity background produced by the $D3$-branes. Upon taking the $D3$-brane near horizon limit, the brane embedding geometry is that of $AdS_3 \times S^5$ [115].

We are now in a position to determine what is the field theory counterpart of the bulk probe approximation. We recall that the gauge theory we constructed in the previous section is defined on the $D7$-brane background. In the probe regime, where $g^2M << 1$, the background produced by the $D7$-branes becomes trivial, as the metric becomes flat, the dilaton goes to a constant and the RR flux vanishes. Hence, in this limit we get the following gauge theory

$$S = S_{\mathcal{N}=4} + \int dx^+ dx^- \; \bar{\chi}(\partial_+ + A_+)\chi, \tag{3.85}$$

where $S_{\mathcal{N}=4}$ is the standard action of $\mathcal{N} = 4$ SYM in flat space.

However, we have argued that it was crucial to consider the gauge theory on the full $D7$-brane geometry, so as to get an anomaly free and supersymmetric theory. The resolution lies in the observation that the gauge anomaly is suppressed in this limit. In order to better understand the parameter controlling the anomaly, it is convenient to rescale the gauge fields in the action as follows $A_\mu \to gA_\mu$. In this presentation it becomes clear what the effect of the coupling constant is on physical quantities. The quantum effective action obtained by integrating the fermions is anomalous, the obstruction to gauge invariance being measured by[43]

$$\delta_L S = \frac{g^2 M}{8\pi} \int dx^+ dx^- \mathrm{Tr}_{U(N)}(LdA), \tag{3.86}$$

so that the anomaly is controlled by the same parameter that controls the backreaction of the $D7$-branes in the bulk (3.84), and is therefore suppressed in the probe limit $g^2M \to 0$.

Note that to leading order in the g^2M expansion the two dimensional Poincare symmetry of the gauge theory is enlarged to $SO(2,2) \simeq SL(2,\mathbf{R}) \times SL(2,\mathbf{R})$, as long as the $D7$-branes are coincident. This can be understood from the point of view of the symmetries of $\mathcal{N} = 4$ SYM in flat space. A surface $\Sigma = \mathbf{R}^{1,1} \subset \mathbf{R}^{1,3}$ is invariant under an

[43] In the frame where the coupling constant controls the interaction vertices in gauge theory, the gauge parameter must also be rescaled $L \to gL$.

$SO(2,2)$ subgroup of the $SO(2,4)$ four dimensional conformal group. The symmetries are generated by $P_\mu, M_{\mu\nu}, K_\mu$ and D, with $\mu = 0,1$, where K_μ and D generate the special conformal and dilatation transformations respectively. In this case – where the $D7$-branes are coincident – the theory acquires eight extra supersymmetries, which correspond to conformal supersymmetries. Indeed, the$S_{\mathcal{N}=4}$ term in (3.85) is invariant under sixteen superconformal supersymmetries generated by $\varepsilon_\alpha{}^i$. The second term in (3.85), given by S_{defect}, is invariant under the conformal supersymmetries generated by:

$$\tilde{\sigma}_-{}^{\dot{\alpha}\alpha}\varepsilon_\alpha{}^i = 0. \tag{3.87}$$

To see this consider the relevant superconformal transformations

$$\delta A_\mu = -ix^\nu \lambda^{\beta i}\sigma_{\mu\beta\dot{\alpha}}\tilde{\sigma}_\nu{}^{\dot{\alpha}\alpha}\varepsilon_\alpha{}^i + \text{c.c.}, \qquad \delta\chi = 0. \tag{3.88}$$

It is straightforward to show that $\delta A_+ = 0$ if (3.87) is fulfilled and the defect action is localized at $z = 0$. All these symmetries combine into the $SU(1,1|4)\times SL(2,\mathbf{R})$ supergroup [116].

Once g^2M corrections are taken into account, so that the anomaly, the Chern-Simons terms and the $D7$-brane background cannot be neglected, the symmetries are broken down[44] to $ISO(1,1) \times SO(6)$ and the theory is invariant under eight supersymmetries. Even if we start with coincident $D7$-branes, once one takes into account the proper global solution (3.44), the $U(1)$ symmetry is broken.

Let's now consider the symmetries of the bulk theory in the probe approximation. When the M $D7$-branes are coincident the $D7$-branes are invariant under $SO(2,2) \times SO(2) \times SO(6)$. The $SO(2,2)$ and $SO(6)$ symmetries act by isometries on the AdS_3 and S^5 worldvolume geometry respectively. The $U(1)$ symmetry rotates the z-plane in (3.4). We show (see section 3) that the coincident $D7$-branes also preserve half of the Type IIB supersymmetries, which coincide precisely with the Poincare and special conformal supersymmetries preserved in the gauge theory, which are given by

$$\tilde{\sigma}_+{}^{\dot{\alpha}\alpha}\epsilon_\alpha{}^i = 0 \tag{3.89}$$

[44] This is similar to the breaking of conformal invariance by g^2M effects that occurs when considering $\mathcal{N} = 4$ SYM coupled to M hypermultiplets, where the β-function is proportional to g^2M, so that g^2M effects break conformal symmetry.

and

$$\tilde{\sigma}_{-}{}^{\dot{\alpha}\alpha}\varepsilon_{\alpha}{}^{i} = 0 \tag{3.90}$$

respectively. The unbroken symmetries combine to form a chiral superconformal group, which is an $SU(1,1|4) \times SL(2,\mathbf{R})$ supergroup, as thus coincides with the gauge theory symmetries discussed above. If the $D7$-branes are not coincident, in both field and gravity theory the symmetry is broken down to $ISO(1,1) \times SO(6)$ and only eight supersymmetries survive.

The $AdS_3 \times S^5$ $D7$-brane ends on the surface Σ on the boundary of $AdS_5 \times S^5$, thus providing boundary conditions for the surface operator. One can use the probe $D7$-brane to calculate the expectation value of the surface operator in the probe regime. In the semiclassical approximation it is given by [117][118]

$$\langle \mathcal{O}_\Sigma \rangle = \exp(-S_{D7}^{on-shell}). \tag{3.91}$$

For the brane embedding at hand the $D7$-brane on-shell action is given by

$$S_{D7}^{on-shell} = \tau_7 L^8 \mathrm{vol}(S^5)\mathrm{vol}_{ren}(AdS_3), \tag{3.92}$$

where L is the AdS_5/S^5 radius, $\mathrm{vol}(S^5)$ is the volume of the S^5 and $\mathrm{vol}_{ren}(AdS_3)$ is the renormalized volume of AdS_3. As usual the bulk action is infrared divergent and requires renormalization. This is accomplished by adding covariant counterterms. It is easy to show that the renormalized volume of AdS_3 vanishes so we find that $\langle \mathcal{O}_\Sigma \rangle = 1$ in the probe approximation. The same answer is obtained for the gauge theory in the probe approximation (3.85), as one just gets the partition function over free fermions.

One may consider surface operators defined on surfaces Σ other than $\mathbf{R}^{1,1}$ in the probe approximation. In the bulk, this corresponds to considering $D7$-brane solutions of the DBI equations of motion that end on the boundary of $AdS_5 \times S^5$ on Σ. The case when $\Sigma = S^2$ can be obtained easily from the euclidean solution with $\Sigma = \mathbf{R}^2$ by acting with a broken special conformal transformation. In this case the bulk $D7$ brane is still $AdS_3 \times S^5$, but now AdS_3 is in global coordinates and the brane ends on the boundary of AdS_5 on an S^2. In this case, the calculation of D-brane action is non-trivial as the renormalized volume of global AdS_3 is non-trivial. In this case, one finds that the $D7$-brane has a conformal anomaly, similar to the one discussed in [119], [120] in the context of $M2$-branes ending on an S^2 in $AdS_7 \times S^4$. This is encoded in the coefficient of the logarithmic divergence of the on-shell action, which for M $D7$-branes is controlled by $\tau_7 L^8 = g^2 M N^2$. It would be interesting to calculate the corresponding conformal anomaly in the gauge theory in perturbation theory.

The Supergravity Solution

In this final subsection we find the bulk description of a surface operator in terms of solution of the Type IIB supergravity. According to AdS/CFT duality, this solution is obtained by taking the near-horizon limit of the supergravity solution of the brane intersection (3.4). The explicit form of the solution corresponding to (3.4) is given by:

$$\begin{aligned} ds^2 &= -H_3^{-1/2}H_7^{-1/2}dx^+dx^- + H_3^{-1/2}H_7^{1/2}dzd\bar{z} + H_3^{1/2}H_7^{-1/2}dx^Idx^I \\ e^{-\Phi} &= H_7 \\ F_{0123I} &= H_7\partial_I H_3^{-1} \\ \partial_{\bar{z}}\tau &= 0 \qquad \text{where} \qquad \tau = C + ie^{-\Phi}. \end{aligned} \tag{3.93}$$

$H_3 = H_3(x^I)$ is an arbitrary harmonic function in the space transverse to the $D3$-branes while $H_7 = H_7(z, \bar{z})$ determines the $D7$-brane contribution and it is of the same form as in section 1. It is straightforward to show that this supergravity background solves the Type IIB supergravity Killing spinor equations and that moreover the space of solutions is eight real dimensional and can be parametrized by four dimensional spinors satisfying the constraints (3.8) and (3.9).

Here we are interested in the supergravity solution describing the decoupled gauge theory constructed earlier and that lives on the $D3/D7$ intersection. This corresponds to taking the near horizon limit of the supergravity solution corresponding to the case when the N $D3$-branes are coincident – so that $H = 1 + L^4/\rho^4$ – where $dx^Idx^I = d\rho^2 + \rho^2 d\Omega_5$. In this limit the metric can be written in terms of an $AdS_3 \times S^5$ factor. The geometry describing the surface operator is given by

$$ds^2 = H_7^{-1/2}\left(ds^2_{AdS_3} + L^2 d\Omega_5\right) + \frac{\rho^2}{L^2}H_7^{1/2}dzd\bar{z}, \tag{3.94}$$

where:

$$ds^2_{AdS_3} = -\frac{\rho^2}{L^2}dx^+dx^- + L^2\frac{d\rho^2}{\rho^2}. \tag{3.95}$$

This metric reveals several interesting features of the holographic correspondence. We have argued that it is inconsistent to describe a low energy field theory with anomaly inflow by treating the gauge theory in flat space. We have argued that the proper description of the system is in terms of the gauge theory in the supergravity background produced by the other brane. In particular, for our intersection, we have constructed the gauge theory on the $D3$-branes in the background of the $D7$-branes and found that this field theory has all the expected properties. We can now use the dual supergravity solution (3.95) to indeed infer that the holographic dual gauge theory lives in the background geometry of the $D7$-branes and not in flat space. Indeed if we analyze the metric living on the conformal boundary – where $\rho \to \infty$ – we precisely get the metric on which the gauge theory lives (3.45).

The solution also gives information about the non-perturbative behavior of the symmetries of the gauge theory. As we discussed earlier in this section, the gauge theory has $SO(2,2)$ symmetry to leading order in a g^2M expansion. This symmetry is intimately related to the geometrical surface on which the fermions live. However, once the g^2M corrections are turned on and the $D7$-brane backreaction cannot neglected, the conformal symmetry is broken. The dual geometry (3.94) has the same symmetries. In particular, the $SO(2,2)$ symmetry is broken down to $ISO(1,1)$. First, the warp-factor H_7 is not invariant under dilatations and special conformal transformations just like in field theory. Second, we see that the AdS_3 radial coordinate ρ does not decouple from the transverse space and appears explicitly in the transverse metric. As usual, the $SO(2,2)$ conformal transformations correspond to AdS_3 isometries. However, since AdS_3 isometries act non-trivially on ρ and z we find that the $SO(2,2)$ conformal symmetry of the surface operator is broken down to $ISO(1,1)$. The supersymmetries are also reduced with respect to the probe approximation. This can be shown (see next section) by explicitly solving the Type IIB Killing spinor equations in the background (3.94). The explicit Killing spinor is given by

$$\epsilon = h(\theta, \varphi_a))H_7^{-1/2}\rho^{1/2}\epsilon_0, \tag{3.96}$$

where $h(\theta, \varphi_a)$ is the standard contribution from S^5 [121], [122] (see last section for the explicit expression). In addition, ϵ (as well as ϵ_0) is subject to the constraints (3.8) and (3.9), which give rise to eight real supersymmetries. Thus, we obtain the same symmetries as those preserved by the gauge theory.

3.3. Supplementary material for chapter 3

The σ-Matrix Conventions

The σ-matrices ${\sigma_\mu}^{\alpha\dot\alpha}$ are defined in the usual way:[45]

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.97}$$

In addition, we define:

$$\begin{aligned} {\tilde\sigma_\mu}^{\dot\alpha\alpha} &= \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}\sigma_{\mu\beta\dot\beta}, \\ \tilde\sigma_\mu &= (\sigma_0, -\sigma_1, -\sigma_2, -\sigma_3). \end{aligned} \tag{3.98}$$

These matrices satisfy the following properties:

$$\begin{aligned} (\sigma_\mu\tilde\sigma_\nu + \sigma_\nu\tilde\sigma_\mu)_\alpha^{\ \beta} &= -2\eta_{\mu\nu}\delta_\alpha^{\ \beta}, \\ (\tilde\sigma_\mu\sigma_\nu + \tilde\sigma_\nu\sigma_\mu)^{\dot\alpha}_{\ \dot\beta} &= -2\eta_{\mu\nu}\delta^{\dot\alpha}_{\ \dot\beta}, \\ \mathrm{tr}(\sigma_\mu\tilde\sigma_\nu) &= -2\eta_{\mu\nu}, \\ {\sigma^\mu}_{\alpha\dot\alpha}{\tilde\sigma_\mu}^{\dot\beta\beta} &= -2\delta^\beta_\alpha\delta^{\dot\beta}_{\dot\alpha}, \\ \sigma_\mu\tilde\sigma_\nu\sigma_\rho &= (\eta_{\mu\rho}\sigma_\nu - \eta_{\nu\rho}\sigma_\mu - \eta_{\mu\nu}\sigma_\rho) + i\epsilon_{\mu\nu\rho\sigma}\sigma^\sigma, \\ \tilde\sigma_\mu\sigma_\nu\tilde\sigma_\rho &= (\eta_{\mu\rho}\tilde\sigma_\nu - \eta_{\nu\rho}\tilde\sigma_\mu - \eta_{\mu\nu}\tilde\sigma_\rho) - i\epsilon_{\mu\nu\rho\sigma}\tilde\sigma^\sigma. \end{aligned} \tag{3.99}$$

In the chapter we go from coordinates x^μ to:

$$x^\pm = x^0 \pm x^1, \quad z = x^2 + ix^3. \tag{3.100}$$

In this basis, we obtain:

$$\begin{aligned} \sigma^- &= \sigma^0 - \sigma^1 = -\sigma_0 - \sigma_1 = -\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \\ \tilde\sigma^- &= \tilde\sigma^0 - \tilde\sigma^1 = -\tilde\sigma_0 - \tilde\sigma_1 = -\sigma_0 + \sigma_1 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \end{aligned} \tag{3.101}$$

etc.

In particular, we have

$$\tilde\sigma_+ = \eta_{+-}\tilde\sigma^- = -\frac{1}{2}\tilde\sigma^- = \frac{1}{2}(\tilde\sigma_0 + \tilde\sigma_1) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{3.102}$$

[45] In this Appendix the index μ is assumed to be flat. In curved space-time we will have to replace in all expressions $\eta_{\mu\nu}$ by the space-time metric.

and:

$$\tilde{\sigma}_{\bar{z}} = \eta_{\bar{z}z}\tilde{\sigma}^{z} = \frac{1}{2}(\tilde{\sigma}_2 + i\tilde{\sigma}_3) = \frac{1}{2}\begin{pmatrix} -i & i \\ -i & i \end{pmatrix}. \tag{3.103}$$

The restriction on the supersymmetry parameter (3.11) found in the chapter can be written as:

$$\tilde{\sigma}_{+}^{\dot{\alpha}\alpha}\epsilon_{\alpha}^{\ i} = 0 \qquad \text{or} \qquad \tilde{\sigma}_{\bar{z}}^{\dot{\alpha}\alpha}\epsilon_{\alpha}^{\ i} = 0. \tag{3.104}$$

Both equations in (3.104) imply that ${\epsilon_1}^i = {\epsilon_2}^i$.

Explicit Check of the Supersymmetry of the Action

In this Appendix, we explicitly show that the non-Abelian $D3$-brane action in the $D7$-brane background given in (3.73) is invariant under the supersymmetry transformations in (3.74). The action has the following structure

$$S = S_V + S_{Sc} + S_F + S_{nab}, \tag{3.105}$$

where:

$$\begin{aligned}
S_V =& -\frac{T_3}{4}\int d^4x\sqrt{-g}e^{-\Phi}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} - \frac{T_3}{4}\int d^4x\sqrt{-g}\partial_\mu C\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\left(A_\nu F_{\rho\sigma} - \frac{2}{3}A_\nu A_\rho A_\sigma\right),\\
S_{Sc} =& -\frac{T_3}{2}\int d^4\sqrt{-g}e^{-\Phi}\mathrm{Tr}\left(\mathcal{D}_\mu\varphi^{ij}\mathcal{D}^\mu\varphi_{ij} + \frac{1}{2}(\mathcal{R} + \partial^\mu\partial_\mu\Phi)\varphi^{ij}\varphi_{ij}\right),\\
S_F =& T_3\int d^4x\sqrt{-g}e^{-\Phi}\mathrm{Tr}\left(\frac{i}{2}\bar{\lambda}_i\tilde{\sigma}^\mu\mathcal{D}_\mu\lambda^i - \frac{i}{2}\mathcal{D}_\mu\bar{\lambda}_i\tilde{\sigma}^\mu\lambda^i\right) - \frac{T_3}{4}\int d^4x\sqrt{-g}\partial_\mu C\bar{\lambda}_i\tilde{\sigma}^\mu\lambda^i,\\
S_{nab} =& T_3\int d^4x\sqrt{-g}e^{-\Phi}\mathrm{Tr}\left(\bar{\lambda}_{\dot{\alpha}i}[\bar{\lambda}^{\dot{\alpha}}_{\ j},\varphi^{ij}] + \lambda^{\alpha i}[\lambda_{\alpha}^{\ j},\varphi_{ij}] - \frac{1}{2}[\varphi^{ij},\varphi^{kl}][\varphi_{ij},\varphi_{kl}]\right).
\end{aligned} \tag{3.106}$$

In looking at the supersymmetry variation of the action we do not write the terms that cancel exactly in the same way as they cancel in $\mathcal{N} = 4$ SYM theory in flat space. That is, we only keep the terms which contain derivatives of the background supergravity fields and ϵ^i and discuss how they cancel. Let us first look at the variation of the terms in the action involving the gauge fields S_V. We obtain:

$$\begin{aligned}
\delta S_V =& -\frac{T_3}{2}\int d^4x\sqrt{-g}\,\partial_\mu\tau\mathrm{Tr}\left(F^{\mu\nu} - \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\right)(\bar{\lambda}_i\tilde{\sigma}_\nu\epsilon^i)\\
&+\frac{T_3}{2}\int d^4x\sqrt{-g}\,\partial_\mu\bar{\tau}\mathrm{Tr}\left(F^{\mu\nu} + \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\right)(\bar{\lambda}_i\tilde{\sigma}_\nu\epsilon^i) + \text{c.c.}
\end{aligned} \tag{3.107}$$

Using the fact that τ is holomorphic and that ϵ^i satisfies equations (3.104), it is easy to see that the first term in the above expression vanishes and only the second term containing $\partial_\mu\bar{\tau}$ survives. Now we vary the fermionic terms in the action in S_F under:

$$\delta\lambda_\alpha^{\ i} = -\frac{1}{2}F_{\mu\nu}(\sigma^\mu\tilde{\sigma}^\nu)_\alpha^{\ \beta}\epsilon_\beta^{\ i}. \tag{3.108}$$

By using the background Killing spinor equation (3.47), we find that the terms in δS_F with $\partial_\mu\tau$ cancel and terms with $\partial_\mu\bar{\tau}$ produce exactly the same expression as in (3.107) but with the opposite sign. This provides the cancellation of terms involving the vectors fields and the fermions.

Now we consider the variation in the action S_{Sc} involving the scalars. It is straightforward to obtain that:

$$\begin{aligned}\delta S_{Sc} =& 2T_3\int d^4x\sqrt{-g}\,(\partial^\mu e^{-\Phi})\mathrm{Tr}\,(\mathcal{D}_\mu\varphi_{ij})\,(\lambda^i\epsilon^j)\\ &- T_3\int d^4x\sqrt{-g}\,e^{-\Phi}\,(\mathcal{R}+\partial^\mu\partial_\mu\Phi)\,\mathrm{Tr}\left(\varphi_{ij}(\lambda^i\epsilon^j)\right)+\text{c.c.}\end{aligned} \tag{3.109}$$

These terms cancel against the variation of S_F under:

$$\delta\bar{\lambda}_{\dot{\alpha}i} = 2i\epsilon^{\alpha j}\sigma^\mu_{\ \alpha\dot{\alpha}}\mathcal{D}_\mu\varphi_{ij} - \frac{i}{2}\epsilon^{\alpha j}\sigma^\mu_{\ \alpha\dot{\alpha}}(\partial_\mu\Phi)\varphi_{ij}. \tag{3.110}$$

Let us make some remarks on how the terms containing derivatives of C cancel when we vary S_F (such terms are not present in the variation of S_{Sc} in (3.109)). Consider the variation of the second term in S_F under (3.110). We get:

$$\begin{aligned}&-\frac{i}{2}\int d^4x\sqrt{-g}\,\partial_\mu C\,\mathrm{Tr}\left(\mathcal{D}_\nu\varphi_{ij}(\lambda^i\sigma^\mu\tilde{\sigma}^\nu\epsilon^j)\right)\\ &+\frac{i}{8}\int d^4x\sqrt{-g}\,\partial_\mu C\,\partial_\nu\Phi\,\mathrm{Tr}\left(\varphi_{ij}(\lambda^i\sigma^\mu\tilde{\sigma}^\nu\epsilon^j)\right)+\text{c.c.}\end{aligned} \tag{3.111}$$

In both terms we anticommute σ^μ and $\tilde{\sigma}^\nu$ using (3.99). Then each term in (3.111) will split into two terms. The first two terms yield:

$$i\int d^4x\sqrt{-g}\,\partial_\mu C\,\mathrm{Tr}\left(\mathcal{D}^\mu\varphi_{ij}(\lambda^i\epsilon^j)\right) - \frac{i}{4}\int d^4x\sqrt{-g}\,\partial_\mu C\,\partial^\mu\Phi\mathrm{Tr}\left(\varphi_{ij}(\lambda^i\epsilon^j)\right)+\text{c.c.} \tag{3.112}$$

They cancel against the variation of the fermion kinetic term when we rewrite $D\epsilon$ in terms of the derivative of the axion by using (3.47). The remaining two terms are:

$$\begin{aligned}&\frac{i}{2}\int d^4x\sqrt{-g}\,\partial_\mu C\,\mathrm{Tr}\left(\mathcal{D}_\nu\varphi_{ij}(\lambda^i\sigma^\nu\tilde{\sigma}^\mu\epsilon^j)\right)\\ &-\frac{i}{8}\int d^4x\sqrt{-g}\,\partial_\mu C\,\partial_\nu\Phi\mathrm{Tr}\left(\varphi_{ij}(\lambda^i\sigma^\nu\tilde{\sigma}^\mu\epsilon^j)\right)+\text{c.c.}\end{aligned} \tag{3.113}$$

We now use the condition that τ is a holomorphic function together with the projection satisfied by the Killing spinor $\tilde{\sigma}_{\bar{z}}\epsilon^i = 0$. This can be summarized by:

$$\partial_\mu \tau \, \tilde{\sigma}^\mu \epsilon^i = 0. \tag{3.114}$$

Using this equation, we can get rid of the terms with derivatives of C in (3.113) and write them using derivatives of $e^{-\Phi}$. The cancellation of such terms arising in δS_F and δS_{Sc} is already straightforward.

In the last step, we vary S_F under the remaining term in the variation of λ:

$$\delta\lambda_\alpha^{\ i} = -2\,[\varphi_{jk}, \varphi ki]\epsilon_\alpha^{\ j}. \tag{3.115}$$

The terms containing ∂C cancel (after we use the Killing spinor equation $D\epsilon \sim \partial C$ as in (3.47)) and we obtain:

$$-i\int d^4x\sqrt{-g}\,(\partial_\mu e^{-\Phi})\mathrm{Tr}\left([\varphi_{jk}, \varphi^{ki}](\bar{\lambda}_i\tilde{\sigma}^\mu\epsilon^j)\right) + \text{c.c.} \tag{3.116}$$

This term cancels against the variation of S_{nab}. In varying S_{nab} we only have to consider:

$$\delta\bar{\lambda}_{\dot{\alpha}i} = -\frac{i}{2}\epsilon^{\alpha j}\sigma^\mu_{\ \alpha\dot{\alpha}}(\partial_\mu\Phi)\varphi_{kj}. \tag{3.117}$$

Anything else gives terms which cancel just like in flat background. It is straightforward to see that the variation of S_{nab} under (3.117) indeed cancels (3.116). This finishes our proof of the supersymmetry of the action.

Integrating Out the Defect Fields

In this Appendix, we perform the explicit integration over the defect fields. We split the $U(N)$ gauge field into an $SU(N)$ gauge field which we denote by A and a $U(1)$ gauge field which we denote by a. Similarly the $U(M)$ gauge field is decomposed into an $SU(M)$ gauge field $\tilde{A}$ and a $U(1)$ gauge field $\tilde{a}$. Therefore, we want to perform the following path integral

$$Z = \int [D\chi][D\bar{\chi}][D\tilde{A}][D\tilde{a}]\exp\left[\left((S_{defect} + S_{CS}(\tilde{A}) + S_{CS}(\tilde{a}) + S_{CS}(a,\tilde{a})\right)\right], \tag{3.118}$$

where:

$$S_{defect} = \int dx^+ dx^- \bar{\chi}\left(\partial_+ + A_+ + \tilde{A}_+ + a_+ - \tilde{a}_+\right)\chi. \tag{3.119}$$

Here we took into account that χ carries the opposite $U(1)$ charges under $U(N)$ and $U(M)$ action. The non-Abelian Chern-Simons term $S_{CS}(\tilde{A})$ is given by

$$S_{CS}(\tilde{A}) = -\frac{(2\pi\alpha')^2\tau_7}{2}\int G_5 \wedge \mathrm{Tr}\left(\tilde{A}\wedge d\tilde{A} + \frac{2}{3}\tilde{A}\wedge\tilde{A}\wedge\tilde{A}\right). \tag{3.120}$$

Similarly:

$$S_{CS}(\tilde{a}) = -\frac{(2\pi\alpha')^2\tau_7}{2}\int G_5 \wedge \tilde{a}\wedge d\tilde{a}. \tag{3.121}$$

Finally, the mixed Chern-Simons terms are given by

$$S_{CS}(a,\tilde{a}) = -\frac{(2\pi\alpha')^2\tau_3}{2}N\int G_1\wedge a\wedge\tilde{f} + \frac{(2\pi\alpha')^2}{2}M\int G_5\wedge\tilde{a}\wedge f, \tag{3.122}$$

where $f = da$ and $\tilde{f} = d\tilde{a}$.

Integrating the fermions in (3.118) yields

$$\int[D\chi][D\bar{\chi}]\exp\left(iS_{defect}\right) = \exp\left[i\left(M\Gamma_{WZW}(A) + N\Gamma_{WZW}(\tilde{A}) + NM\Gamma_{WZW}(a,\tilde{a})\right)\right]. \tag{3.123}$$

We must now integrate the D7-brane gauge fields $\tilde{A}$ and $\tilde{a}$ in (3.118). The gauge field $\tilde{A}$ is completely decoupled from the $\mathcal{N}=4$ SYM gauge fields A and a. Therefore the integral over $\tilde{A}$, which appears in the action through the terms $N\Gamma_{WZW}(\tilde{A}) + S_{CS}(\tilde{A})$ just gives a constant.

Now we have to perform the integral over $\tilde{a}$. In order to simplify the formulas, we consider the case of M coincident $D7$-branes with the local $U(1)$ symmetry. In this case the RR one-form flux is given by:

$$G_1 = \frac{g_s M}{2\pi}d\theta. \tag{3.124}$$

A similar analysis can be easily generalized for the global solutions, as all we require is that G_1 satisfies the Bianchi identities. The path integral we have to study is

$$\int[D\tilde{a}]\exp\left(i\Gamma(a,\tilde{a})\right), \tag{3.125}$$

where

$$\Gamma(a,\tilde{a}) = NM\Gamma_{WZW}(a-\tilde{a}) + S_{CS}(\tilde{a}) + S_{CS}(a,\tilde{a}). \tag{3.126}$$

The explicit expressions are given by

$$\Gamma_{WZW}(a-\tilde{a}) = -\frac{1}{8\pi}\int dx^+dx^-\left[\partial_+\left(u-\tilde{u}\right)\partial_-\left(u-\tilde{u}\right) - \partial_+\left(u-\tilde{u}\right)\partial_-\left(v-\tilde{v}\right)\right], \tag{3.127}$$

where:

$$a_+ = \partial_+ u, \qquad a_- = \partial_+ v, \qquad \tilde{a}_+ = \partial_+ \tilde{u}, \qquad \tilde{a}_- = \partial_+ \tilde{v}. \tag{3.128}$$

The Chern-Simons action $S_{CS}(\tilde{a})$ can be simplified to

$$S_{CS}(\tilde{a}) = -\frac{1}{8\pi} NM \int dx^+ dx^- d\rho \left(\tilde{a}_+ \tilde{f}_{-\rho} + \tilde{a}_- \tilde{f}_{\rho +} + \tilde{a}_\rho \tilde{f}_{+-} \right), \tag{3.129}$$

where ρ is the radial direction away from the N D3-branes and we have restricted the RR flux to s-waves on the S^5. Likewise

$$S_{CS}(a, \tilde{a}) = \frac{1}{8\pi} NM \int dx^+ dx^- \left(\tilde{f}_{+-}(0) \int dr\, a_r + f_{+-}(0) \int d\rho\, \tilde{a}_\rho \right), \tag{3.130}$$

where $f_{+-}(0)$ and $\tilde{f}_{+-}(0)$ are the boundary values of f_{+-} and $\tilde{f}_{+-}$ respectively and r is the radial coordinate away from the $D7$-branes. Note that the path integral is Gaussian and it is enough to evaluate the action on the equations of motion. Since we have both bulk and boundary contributions to the action we need to solve the equations of motion separately on the bulk and on the boundary.

The the bulk equations of motion yield:

$$\tilde{f}_{-\rho} = 0, \qquad \tilde{f}_{+\rho} = 0, \qquad 2\tilde{f}_{+-} = \tilde{f}_{+-}(0). \tag{3.131}$$

Furthermore, the boundary equations of motion give:

$$\int dr\, a_r = -u, \qquad 2\tilde{f}_{+-}(0) = f_{+-}(0). \tag{3.132}$$

Evaluating the action on this solution gives:

$$\Gamma(a, \tilde{a})|_{solution} = \Gamma_{WZW}(a). \tag{3.133}$$

Therefore, the final result of performing the path integral (3.118) is:

$$Z = \exp \left[i (M \Gamma_{WZW}(A) + MN \Gamma_{WZW}(a)) \right]. \tag{3.134}$$

We can now combine the $SU(N)$ connection A with the $U(1)$ connection a into a $U(N)$ gauge field, which with some abuse of notation we will also denote by A.

Therefore, integrating out the localized fields together with the non-dynamical gauge fields on the $D7$-branes has the effect of inserting the following surface operator into the $\mathcal{N} = 4$ SYM path integral in the $D7$-brane background:

$$Z = \exp \left(i M \Gamma_{WZW}(A) \right). \tag{3.135}$$

A Probe D7-Brane in $AdS_5 \times S^5$

In this Appendix we study the sypersymmetries preserved by the $D7$-brane in $AdS_5 \times S^5$ which represents a surface operator in the probe approximation.

We consider the following parametrization for $AdS_5 \times S_5$ (we fix the radius $L = 1$)

$$ds^2_{AdS\times S} = \rho^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{d\rho^2}{\rho^2} + d\theta^2 + \sin^2\theta \, d\Omega_4^2, \tag{3.136}$$

where the metric on S^4 is given by:

$$d\Omega_4^2 = d\varphi_1^2 + \sin\varphi_1^2 d\varphi_2^2 + \sin\varphi_1^2 \sin\varphi_2^2 d\varphi_3^2 + \sin\varphi_1^2 \sin\varphi_2^2 \sin\varphi_3^2 d\varphi_4^2. \tag{3.137}$$

It is useful to introduce tangent space gamma matrices, i.e. $\gamma_{\underline{m}} = e^m_{\underline{m}} \Gamma_m$ $(m, \underline{m} = 0, \ldots, 9)$ where $e^m_{\underline{m}}$ is the inverse vielbein and Γ_m are the target space matrices:

$$\begin{gathered}\gamma_\mu = \frac{1}{\rho}\Gamma_\mu \qquad (\mu = 0,1,2,3), \qquad \gamma_4 = \rho\Gamma_\rho, \qquad \gamma_5 = \Gamma_\theta, \\ \gamma_{a+5} = \frac{1}{\sin\theta}\left(\prod_{j=1}^{a-1} \frac{1}{\sin\varphi_j}\right)\Gamma_{\varphi_a} \qquad (a = 1,2,3,4)\end{gathered} \tag{3.138}$$

The Killing spinor of $AdS_5 \times S^5$ in the coordinates (3.136) is given by [115]

$$\epsilon = \left[-\rho^{-\frac{1}{2}}\gamma_4 h(\theta,\varphi_a) + \rho^{\frac{1}{2}} h(\theta,\varphi_a)(\eta_{\mu\nu}x^\mu\gamma^\nu)\right]\eta_2 + \rho^{\frac{1}{2}} h(\theta,\varphi_a)\eta_1 \tag{3.139}$$

where:

$$h(\theta,\varphi_a) = e^{\frac{1}{2}\theta\gamma_{45}} e^{\frac{1}{2}\varphi_1\gamma_{56}} e^{\frac{1}{2}\varphi_2\gamma_{67}} e^{\frac{1}{2}\varphi_3\gamma_{78}} e^{\frac{1}{2}\varphi_4\gamma_{89}}. \tag{3.140}$$

η_1 and η_2 are constant ten dimensional complex spinors satisfying

$$\gamma_{11}\eta_1 = -\eta_1 \qquad \gamma_{11}\eta_2 = \eta_2 \tag{3.141}$$

with $\gamma_{11} = \gamma_0\gamma_1 \ldots \gamma_9$. They also satisfy

$$\tilde{\gamma}\,\eta_1 = \eta_1 \qquad \tilde{\gamma}\,\eta_2 = -\eta_2, \tag{3.142}$$

where $\tilde{\gamma} = i\gamma^{0123}$ is the four dimensional chirality matrix. Thus, each spinor $\eta_{1,2}$ has 16 independent real components.

The supersymmetries preserved by the embedding of a D-brane probe, are those that satisfy

$$\Gamma_\kappa \epsilon = \epsilon, \tag{3.143}$$

where Γ_κ is the κ-symmetry transformation matrix of the probe worldvolume theory and ϵ is the Killing spinor of the $AdS_5 \times S_5$ background (3.139). Both Γ_κ and ϵ have to be evaluated at the location of the probe.

Let's now consider a $D7$-brane embedding with an $AdS_3 \times S^5$ worldvolume geometry, with embedding:

$$\begin{array}{llllll} \sigma^0 = x^0 & \sigma^1 = x^1 & \sigma^2 = \rho & \sigma^3 = \theta & \sigma^{3+a} = \varphi^a & (a = 1,2,3,4) \\ x^2 = 0 & x^3 = 0 & & & & \end{array} \tag{3.144}$$

and with the worldvolume gauge field set to zero. The matrix Γ_κ for a $D7$-brane in a background with zero B-field and dilaton is given by

$$d^8\sigma\, \Gamma_{D7} = \frac{1}{\sqrt{-\det(g_{ij})}} \Gamma_{(8)} I \tag{3.145}$$

where $\Gamma_{(8)} = \frac{1}{8!}\Gamma_{i_1 \ldots i_8} d\sigma^{i_1} \wedge \ldots \wedge d\sigma^{i_8}$ and I acts on a spinor ψ by $I\psi = -i\psi$. Considering the embedding in (3.144), the matrix in (3.145) reduces to:

$$\Gamma_{D7} = \gamma_{01456789} I. \tag{3.146}$$

The equation (3.143) has to be satisfied at every point on the worldvolume. Thus, the term proportional to $\rho^{\frac{1}{2}}$ gives:

$$\Gamma_{D7} h(\theta, \varphi_a)\eta_1 = h(\theta, \varphi_a)\eta_1. \tag{3.147}$$

The terms proportional to $\rho^{-\frac{1}{2}}$, $\rho^{\frac{1}{2}} x_0$ and $\rho^{\frac{1}{2}} x_1$ give:

$$\Gamma_{D7} h(\theta, \varphi_a)\eta_2 = -h(\theta, \varphi_a)\eta_2. \tag{3.148}$$

Using

$$h^{-1}\gamma_{014} h = n^I \gamma_{01I} \qquad h^{-1}\gamma_{56789} h = n^I \gamma_{I456789} \qquad I = 4,5,6,7,8,9 \tag{3.149}$$

where

$$n^I(\theta, \varphi_1, \varphi_2, \varphi_3, \varphi_4) = \begin{pmatrix} \cos\theta \\ \sin\theta\cos\varphi_1 \\ \sin\theta\sin\varphi_1\cos\varphi_2 \\ \sin\theta\sin\varphi_1\sin\varphi_2\cos\varphi_3 \\ \sin\theta\sin\varphi_1\sin\varphi_2\sin\varphi_3\cos\varphi_4 \\ \sin\theta\sin\varphi_1\sin\varphi_2\sin\varphi_3\sin\varphi_4 \end{pmatrix} \tag{3.150}$$

is a unit vector in $\mathbf{R}^6$ (that is $n^I n^I = 1$) we get:

$$\begin{aligned} h^{-1}\Gamma_{D7}h &= n^I n^J \gamma_{01I}\gamma_{J456789I} \\ &= -i\gamma_{01456789} \\ &= \gamma^{01}\,\tilde{\gamma}\,\gamma_{11}. \end{aligned} \tag{3.151}$$

Thus, the equations (3.147) (3.148) reduce to

$$\begin{aligned} \gamma^{01}\eta_1 &= -\eta_1 \\ \gamma^{01}\eta_2 &= \eta_2 \end{aligned} \tag{3.152}$$

Since η_1 and η_2 satisfy (3.141) and (3.142), they can be written in terms of ten dimensional Majorana-Weyl spinors ϵ and ε of negative and positive chirality respectively:

$$\begin{aligned} \eta_1 &= \epsilon + i\gamma^{0123}\epsilon \\ \eta_2 &= \varepsilon - i\gamma^{0123}\varepsilon. \end{aligned} \tag{3.153}$$

By evaluating the Killing spinor (3.139) near the boundary, ϵ can be identified with the generator of Poincare supersymmetry while ε can be identified with the generator of conformal supersymmetry of $\mathcal{N} = 4$ SYM. Thus the equations (3.152) become:

$$\begin{aligned} \gamma^{01}\epsilon &= -\epsilon, \\ \gamma^{01}\varepsilon &= \varepsilon. \end{aligned} \tag{3.154}$$

These conditions are equivalent to (3.89) and (3.90), which describe the unbroken Poincare and conformal supersymmetries respectively in the field theory. Therefore, for coincident $D7$ branes we have shown that they preserve the same half of the Poincare and conformal supersymmetries as the field theory does in the probe approximation.

D7 probe without conformal supersymmetries

The $D7$-brane embedding we have just discussed can be generalized to the case when $x^2 = \bar{x}^2$ and $x^3 = \bar{x}^3$ where $\bar{x}^2$ and $\bar{x}^3$ are arbitrary constants. The bosonic symmetry of this embedding is $ISO(1,1) \times SO(6)$. We note that the conformal and $U(1)$ symmetries are broken in the case of separated $D7$-branes just like in the field theory.

In this case, the matrix (3.145) is still given by (3.146). The supersymmetry conditions are

$$\Gamma_{D7} h(\theta, \varphi_a)\eta_1 = h(\theta, \varphi_a)\eta_1 \tag{3.155}$$

$$\Gamma_{D7} h(\theta, \varphi_a)\eta_2 = -h(\theta, \varphi_a)\eta_2 \tag{3.156}$$

$$\Gamma_{D7} h(\theta, \varphi_a)\eta_2 = h(\theta, \varphi_a)\eta_2. \tag{3.157}$$

The equations (3.156) and (3.157) imply that the conformal supersymmetries are completely broken. The equation (3.155) implies that the preserved Poincare supersymmetries satisfy:

$$\gamma^{01}\epsilon = -\epsilon. \tag{3.158}$$

When $\bar{x}^2 = \bar{x}^3 = 0$ the equation (3.157) does not have to be satisfied and half of the conformal supersymmetries are preserved. We thus recover the symmetries preserved by the field theory in the probe approximation.

The Killing Spinor

The goal of this Appendix is to construct the Killing spinor of the geometry dual to the surface operator. The geometry can be written as follows

$$\begin{aligned} &ds^2 = -H_7^{-1/2} H_3^{-1/2} dx^+ dx^- + H_7^{-1/2} H_3^{1/2} d\rho^2 + H_7^{-1/2} d\Omega_5 + H_7^{1/2} H_3^{-1/2} dz d\bar{z}, \\ &F_{0123\rho} = H_7 \partial_\rho H_3^{-1}, \end{aligned} \tag{3.159}$$

where

$$H_3 = \frac{L^4}{\rho^4} \tag{3.160}$$

and H_7 is the harmonic function of the $D7$-brane solution. To find the Killing spinor we substitute the above solution into the gravitino and dilatino variations, which in the presence of one-and five-form fluxes take the form:

$$\begin{aligned} &\delta\Psi_M = \partial_M \epsilon + \frac{1}{4}\omega_M^{AB}\Gamma_{AB}\epsilon - \frac{i}{8} e^\Phi \partial_N C \Gamma^N \Gamma_M \epsilon - \frac{i}{8 \cdot 5!} e^\Phi F_{M_1 \ldots M_5} \Gamma^{M_1 \ldots M_5} \Gamma_M \epsilon = 0, \\ &\delta\psi = (\Gamma^M \partial_M \Phi)\epsilon + i e^\Phi \partial_M C \Gamma^M \epsilon = 0. \end{aligned} \tag{3.161}$$

The dilatino variation is independent of the five-form flux and gives

$$\tau = \tau(z), \qquad \gamma_{\bar{z}} = 0, \tag{3.162}$$

as in the case of $D7$-brane solutions. When we substitute (3.159) into the gravitino variation, there will be terms proportional to ∂H_7 and terms proportional ∂H_3 which will essentially separate. The term with ∂H_7 cancel if $\epsilon \sim H_7^{-1/8}$ and (3.162) is satisfied exactly like in the case of $D7$-brane solutions. Let us concentrate on the terms proportional to ∂H_3. Let us first consider the variation $\delta\Psi_z$. We obtain:

$$\frac{H_7^{1/2}}{8H_3^{3/2}}\partial_\rho H_3 \gamma_4 \gamma_z (\epsilon + i\gamma_0\gamma_1\gamma_2\gamma_3\epsilon) = 0. \tag{3.163}$$

Note that $\partial_z \epsilon$ cancels against the terms proportional to $\partial_z H_7$ and, hence, eq. (3.163) is not a differential equation on ϵ. To satisfy (3.163) we have to require:

$$i\gamma^{0123}\epsilon = \epsilon. \tag{3.164}$$

Eqs. (3.162) and (3.164) are equivalent to (3.8) and (3.9) and, hence, ϵ has eight independent components corresponding to eight preserved supercharges. This is in agreement with our field theory discussions. Now we consider the equation $\delta\Psi_\pm = 0$. Due to the restriction (3.164), it follows that

$$\delta\Psi_\pm = \partial_\pm \epsilon = 0. \tag{3.165}$$

This means that ϵ is independent of $x^\pm$. Similarly, from the equation $\delta\Psi_\rho = 0$ we obtain

$$\partial_\rho \epsilon - \frac{1}{2\rho}\epsilon = 0, \tag{3.166}$$

which implies $\epsilon \sim \rho^{1/2}$. The last equations to consider is $\delta\Psi_a = 0$, where Ψ_a are the components of the gravitino along S^5. These equations are

$$D_a \epsilon - \frac{1}{2}\gamma_4 \Gamma_a \epsilon = 0. \tag{3.167}$$

These are the standard equations for the Killing spinor on S^5 [121], [122]. The solution is given in terms of the operator $h(\theta, \varphi_a)$ defined in (3.140). Combining the above conclusions we find that the Killing spinor is given by

$$\epsilon = h(\theta, \varphi_a) H_7^{-1/2} \rho^{1/2} \epsilon_0, \tag{3.168}$$

where both ϵ and ϵ_0 satisfy conditions (3.8) and (3.9) (note thate γ_+ and $\gamma_{\bar{z}}$ commute with $h(\theta, \varphi_a)$).

4. Supersymmetric Mass Deformation of the Bagger-Lambert Theory

The supersymmetric worldvolume theory of a single M2-brane in an arbitrary eleven dimensional supergravity background was found twenty years ago [123]. The realization that branes in eleven dimensional supergravity are related by string dualities to D-branes [16] and that the low energy effective field theory of coincident D-branes is described by non-abelian super Yang-Mills theory [124], naturally prompts the search for the worldvolume theory of coincident M2-branes.

In a recent paper [125], Bagger and Lambert have proposed a Lagrangian to describe the low energy dynamics of a stack of coincident M2-branes (see also the work by Gustavsson [126]). Their model, that incorporates insights from previous papers [127][128], includes half-BPS fuzzy 3-sphere solitons. This solutions were argued by Basu and Harvey [129] to provide the M2-branes worldvolume description of the multiple M2-branes ending on an M5-brane, generalizing a similar mechanism studied for the D1-D3 system [130]. The Bagger-Lambert theory is a 3-dimensional $\mathcal{N} = 8$ supersymmetric field theory, based on a novel algebraic structure, dubbed 3-algebra. Explicit examples of 3-algebras has been recently constructed in [131][43][44] starting from ordinary Lie algebras and considering a Lorentzian scalar product (see also [132][133]). The fact that the scalar product is not positive-definite permit to avoid a no-go theorem discussed in [37][38]. Other algebraic structures have been considered in [134][135][136][137][138]. The Bagger-Lambert theory was shown to be conformal invariant in [139] and the moduli space was discussed in [140][40][141][41][142]. In [143][144] the reduction to the theory of multiple D2-branes is discussed.

In this chapter we construct a one parameter deformation of the Bagger-Lambert theory [125] which is maximally supersymmetric [145](see also [146]). We add to their Lagrangian a mass term for all the eight scalars and fermions[46]

$$\mathcal{L}_{mass} = -\frac{1}{2}\mu^2 \,\mathrm{Tr}\left(X^I, X^I\right) + \frac{i}{2}\mu \,\mathrm{Tr}\left(\bar{\Psi}\Gamma_{3456}, \Psi\right), \tag{4.1}$$

and a Myers-like [148] flux-inducing $SO(4) \times SO(4)$ invariant potential[47] for the scalars

$$\mathcal{L}_{flux} = -\frac{1}{6}\mu\varepsilon^{ABCD}\mathrm{Tr}([X^A, X^B, X^C], X^D) - \frac{1}{6}\mu\varepsilon^{A'B'C'D'}\mathrm{Tr}([X^{A'}, X^{B'}, X^{C'}], X^{D'}) \tag{4.2}$$

[46] The deformation of the theory on multiple $M2$-branes was first considered by Bena [147].

[47] See also [149][150].

and show that the theory is supersymmetric. The possibility of adding the scalar mass term and the potential term for four of the scalars was considered in [140]. Here we show that if we give a mass to all the scalars and fermions and turn on the potential (4.2) for all the scalars that we can find a deformation of the supersymmetry transformations of the Bagger-Lambert theory [125] in such a way that the deformed field theory remains fully supersymmetric. This construction yields a novel maximally supersymmetric, Poincare invariant three dimensional field theory.

We further argue that the deformed field theory compactified on $R \times T^2$ provides the Matrix theory [151] description[48] of Type IIB string theory on the maximally supersymmetric plane wave[49] [155]. We show that the deformed field theory on $R \times T^2$ has as its algebra of symmetries the superisometry algebra of the Type IIB plane wave, as expected from a holographic dual theory. The deformed field theory on $R \times T^2$ is proposed as the nonperturbative formulation of the Type IIB string theory in the discrete lightcone quantization (DLCQ).

We show that the supersymmetric ground states of the deformed theory are given by a discrete set of states that have an interpretation[50] as a collection of fuzzy S^3's [140], where

$$[X^A, X^B, X^C] = -\mu \epsilon^{ABCD} X^D, \qquad X^{A'} = 0 \tag{4.3}$$

or alternatively:

$$[X^{A'}, X^{B'}, X^{C'}] = -\mu \epsilon^{A'B'C'D'} X^{D'}, \qquad X^A = 0. \tag{4.4}$$

We identify these states of the deformed theory with the states in the Hilbert space of the Type IIB plane wave with zero light-cone energy, which correspond to configurations of D3-brane giant gravitons in the Type IIB plane wave background [152] with fixed longitudinal momentum.

[48] In [152](see also [153]), an analogous deformation of the D0 brane Lagrangian was proposed as the Matrix theory description of the maximally supersymmetry plane wave of eleven dimensional supergravity.

[49] Given the interpretation in [154][41] for the 3-algebra $\mathcal{A}_4$, the lagrangian described in this chapter can be also thought as the Matrix theory for an orientifold projection of the maximally supersymmetric plane wave background.

[50] These states have yet another space-time interpretation as M2-branes polarizing in the presence of flux into M5-branes with S^3 topology. The supergravity description of these ground states of the deformed theory were found in [156] (see also [157]).

The Lagrangian of the deformed theory is based on the same 3-algebra structure of [125] (we review it in section 1). Even though the construction of Bagger-Lambert and in this chapter certainly provide new constructions of supersymmetric field theories, the precise connection with the worldvolume physics of coincident M2-branes still remains to be understood. Establishing in more detail the M2 brane interpretation of our deformed theory is important in understanding the deformed field theory described in this chapter as the Matrix theory description of the Type IIB plane wave.

The plan of the rest of the chapter is as follows. In section 1 we quickly review the Bagger-Lambert theory and introduce the deformation of the Lagrangian and the supersymmetry transformations that gives rise to a new maximally supersymmetric Lagrangian in three dimensions. In section 2 we argue that the deformed theory on $R \times T^2$ provides the Matrix theory description of the maximally supersymmetric Type IIB plane wave. We show that the theory on $R \times T^2$ has precisely the same symmetry algebra as the Type IIB plane wave and identify the supersymmetric grounds states of the deformed theory with the states in the Type IIB plane wave with zero light-cone energy, which correspond to configurations of D3-brane giant gravitons. In section 3 we present some details of the calculation of the deformed supersymmetry transformations and we write down the Noether charges of the deformed field theory on $R \times T^2$ showing that they satisfy the Type IIB plane wave superalgebra.

4.1. The Bagger-Lambert Theory and its Supersymmetric Deformation

The Bagger-Lambert Lagrangian

We start this section recalling basic properties of the Bagger-Lambert theory [125] that we will use in the following. The authors have proposed that this theory describes the low energy dynamics[51] of a stack of M2-branes. In this model, the transverse fluctuations of the membranes are described by eight scalar fields X^I, where $I = 3, \dots 11$ and the eight $Spin(1,2)$ worldvolume fermions are collected together in the spinor field Ψ. The Ψ is an 11-dimensional Majorana spinor satisfying the condition $\Gamma_{012}\Psi = -\Psi$ and thus it has sixteen independent real components.

[51] In this $l_p \to 0$ limit, higher derivative corrections can be ignored.

These fields are valued in a 3-algebra $\mathcal{A}$ [125](see also [126]), i.e. $X^I = X^I_a T^a$ and $\Psi = \Psi_a T^a$ where T^a, $a = 1, \ldots, \dim\mathcal{A}$ are the generators of $\mathcal{A}$. The 3-algebra is endowed with a 3-product

$$[T^a, T^b, T^c] = f^{abc}{}_d T^d \tag{4.5}$$

where the structure constants satisfy the fundamental identity

$$f^{efg}{}_d f^{abc}{}_g = f^{efa}{}_g f^{bcg}{}_d + f^{efb}{}_g f^{cag}{}_d + f^{efc}{}_g f^{abg}{}_d. \tag{4.6}$$

The 3-algebra construction includes also a bilinear and non-degenerate scalar product $\mathrm{Tr}(\cdot,\cdot)$ that defines a non-degenarate metric h^{ab}

$$h^{ab} \equiv \mathrm{Tr}(T^a, T^b) \tag{4.7}$$

used to manipulate the algebra indices. The structure constants f^{abcd} are assumed to be totally antisymmetric in the indices. The only examples of Euclidean 3-algebras found are of the type $\mathcal{A}_4 \oplus \mathcal{A}_4 \oplus \ldots \oplus \mathcal{A}_4 \oplus C_1 \oplus \ldots C_l$, where $\mathcal{A}_4$ is defined by $f^{abcd} = \epsilon^{abcd}$ and C_i denote central elements in the algebra. The supersymmetric deformation we describe in this chapter applies to any 3-algebra with totally antisymmetric structure constants which satisfies the fundamental identity (4.6).

The Bagger-Lambert theory includes also a non-propagating gauge vector field $A_{\mu ab}$ where $\mu = 0, 1, 2$ denotes the worldvolume coordinates. The dynamics is controlled by the Lagrangian

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}(D_\mu X^{aI})(D^\mu X^I_a) + \frac{i}{2}\bar{\Psi}^a \Gamma^\mu D_\mu \Psi_a + \frac{i}{4}\bar{\Psi}_b \Gamma_{IJ} X^I_c X^J_d \Psi_a f^{abcd} \\ &- V + \frac{1}{2}\varepsilon^{\mu\nu\lambda}(f^{abcd} A_{\mu ab}\partial_\nu A_{\lambda cd} + \frac{2}{3} f^{cda}{}_g f^{efgb} A_{\mu ab} A_{\nu cd} A_{\lambda ef})\end{aligned} \tag{4.8}$$

where V is the potential

$$V = \frac{1}{12} f^{abcd} f^{efg}{}_d X^I_a X^J_b X^K_c X^I_e X^J_f X^K_g = \frac{1}{2\cdot 3!}\mathrm{Tr}([X^I, X^J, X^K], [X^I, X^J, X^K]) \tag{4.9}$$

and the covariant derivative of a field Φ is defined by

$$(D_\mu \Phi)_a = \partial_\mu \Phi_a - \tilde{A}_\mu{}^b{}_a \Phi_b \tag{4.10}$$

where $\tilde{A}_\mu{}^b{}_a \equiv f^{cdb}{}_a A_{\mu cd}$. The (4.8) is invariant under the gauge transformations

$$\begin{aligned}\delta X^I_a &= \tilde{\Lambda}^b{}_a X^I_b \\ \delta \Psi_a &= \tilde{\Lambda}^b{}_a \Psi_b \\ \delta \tilde{A}_\mu{}^b{}_a &= \partial_\mu \tilde{\Lambda}^b{}_a - \tilde{\Lambda}^b{}_c \tilde{A}_\mu{}^c{}_a + \tilde{A}_\mu{}^b{}_c \tilde{\Lambda}^c{}_a\end{aligned} \tag{4.11}$$

where $\tilde{\Lambda}^b{}_a \equiv f^{cdb}{}_a\Lambda_{cd}$ and Λ_{cd} is the gauge parameter. The Lagrangian (4.8) is also invariant under the following supersymmetry variations

$$
\begin{aligned}
\delta_\epsilon X^I_a =& i\bar{\epsilon}\Gamma^I\Psi_a \\
\delta_\epsilon \Psi_a =& D_\mu X^I_a \Gamma^\mu \Gamma^I \epsilon - \frac{1}{6} X^I_b X^J_c X^K_d f^{bcd}{}_a \Gamma^{IJK}\epsilon \\
\delta_\epsilon \tilde{A}_\mu{}^b{}_a =& i\bar{\epsilon}\Gamma_\mu \Gamma_I X^I_c \Psi_d f^{cdb}{}_a
\end{aligned}
\tag{4.12}
$$

where the supersymmetry parameter ϵ satisfies $\Gamma_{012}\epsilon = \epsilon$. The equations of motion are

$$
\begin{aligned}
\Gamma^\mu D_\mu \Psi_a + \frac{1}{2}\Gamma_{IJ} X^I_c X^J_d \Psi_b f^{cdb}{}_a =& 0 \\
D^2 X^I_a - \frac{i}{2}\bar{\Psi}_c \Gamma^I_J X^J_d \Psi_b f^{cdb}{}_a - \frac{\partial V}{\partial X^{Ia}} =& 0 \\
\tilde{F}_{\mu\nu}{}^b{}_a + \varepsilon_{\mu\nu\lambda}(X^J_c D^\lambda X^J_d + \frac{i}{2}\bar{\Psi}_c \Gamma^\lambda \Psi_d) f^{cdb}{}_a =& 0
\end{aligned}
\tag{4.13}
$$

where

$$
\tilde{F}_{\mu\nu}{}^b{}_a = \partial_\nu \tilde{A}_\mu{}^b{}_a - \partial_\mu \tilde{A}_\nu{}^b{}_a - \tilde{A}_\mu{}^b{}_c \tilde{A}_\nu{}^c{}_a + \tilde{A}_\nu{}^b{}_c \tilde{A}_\mu{}^c{}_a. \tag{4.14}
$$

The stress-energy tensor $T_{\mu\nu}$ can be computed in the usual way coupling the Bagger-Lambert theory to an external worldvolume metric and looking at the variation of the action for an infinitesimal change of the metric. In the case where the fermions are set to zero, it results

$$
T_{\mu\nu} = D_\mu X^I_a D_\nu X^{aI} - \eta_{\mu\nu}\left(\frac{1}{2} D_\rho X^{aI} D^\rho X^I_a + V\right). \tag{4.15}
$$

We note that the Chern-Simons like term in (4.8) does not contribute to the stress-energy tensor. This is because this term is topological and does not depend on the worldvolume metric.

Mass deformation

We now find a deformation of the action and supersymmetry transformations of the action of Bagger and Lambert [125] that is maximally supersymmetric. The new Lagrangian is given by

$$
\tilde{\mathcal{L}} = \mathcal{L} + \mathcal{L}_{mass} + \mathcal{L}_{flux}, \tag{4.16}
$$

where $\mathcal{L}$ is the Bagger-Lambert theory in (4.8) and:

$$
\begin{aligned}
\mathcal{L}_{mass} =& -\frac{1}{2}\mu^2 \,\mathrm{Tr}(X^I, X^I) + \frac{i}{2}\mu \,\mathrm{Tr}(\bar{\Psi}\Gamma_{3456}, \Psi) \\
\mathcal{L}_{flux} =& -\frac{1}{6}\mu \varepsilon^{ABCD}\mathrm{Tr}([X^A, X^B, X^C], X^D) - \frac{1}{6}\mu \varepsilon^{A'B'C'D'}\mathrm{Tr}([X^{A'}, X^{B'}, X^{C'}], X^{D'}).
\end{aligned}
\tag{4.17}
$$

The transverse index has been decomposed as $I = (A, A')$ where $A = 3,4,5,6$ and $A' = 7,8,9,10$ and Ψ is an eleven dimension Majorana spinor satisfying the constraint $\Gamma_{012}\Psi = -\Psi$, where the Γ-matrices satisfy the Clifford algebra in eleven dimensions. This deformation of the Lagrangian is analogous to the deformation of the Lagrangian of D0-branes considered in [152]. This deformation when restricted to only four of the scalars has been considered in [140].

The deformed Lagrangian now depends on the paramater μ. The mass term $\mathcal{L}_{mass}$ gives mass to all the scalars and fermions in the theory, while $\mathcal{L}_{flux}$ has the interpretation of the scalar potential[52] generated by a background four-form flux of eleven dimensional supergravity, of the type found by Myers [148] (see also [149][150]) in the context of D-branes in the presence of background fluxes.

The deformed theory (4.16) breaks the $SO(8)$ R-symmetry of the undeformed theory (4.8) down to $SO(4)\times SO(4)$. The deformed theory is nevertheless invariant under sixteen linearly realized supersymmetries. The supersymmetry transformations of the deformed theory are given by

$$\begin{aligned}\tilde{\delta}X^I &= i\bar{\epsilon}\Gamma^I\Psi\\ \tilde{\delta}\Psi &= D_\mu X^I\Gamma^\mu\Gamma^I\epsilon - \frac{1}{6}[X^I,X^J,X^K]\Gamma^{IJK}\epsilon - \mu\Gamma_{3456}\Gamma^I X^I\epsilon\\ \tilde{\delta}\tilde{A}_\mu{}^b{}_a &= i\bar{\epsilon}\Gamma_\mu\Gamma_I X^I_c\Psi_d f^{cdb}{}_a,\end{aligned} \tag{4.18}$$

where ϵ is a constant eleven dimensional Majorana spinor satisfying the constraint $\Gamma_{012}\epsilon = \epsilon$. By setting $\mu \to 0$ we recover the supersymmetry transformations of the undeformed theory (4.8) found by Bagger-Lambert [125]. The proof that the action (4.16) is invariant under the supersymmetry transformations is summarized in section 3.

The deformed action (4.16) is also invariant under sixteen non-linearly realized supersymmetries if the 3-algebra $\mathcal{A}_n$ has a central element $C = T^0$, so that $f^{abc0} = 0$. Then the action (4.16) is invariant under the following non-linear supersymmetry transformations[53]

$$\begin{aligned}\delta_n X^I_a &= 0\\ \delta_n\Psi &= \exp\left(-\frac{\mu}{3}\Gamma_{3456}\Gamma_\mu\sigma^\mu\right)T^0\eta\\ \delta_n\tilde{A}_\mu{}^b{}_a &= 0\end{aligned} \tag{4.19}$$

[52] We note that if we use the proposal made by Mukhi and Papageorgakis [143] to obtain by compactification the theory on D2 branes, that $\mathcal{L}_{flux}$ does indeed reduce to the known Myers term.

[53] The Bagger-Lambert theory (4.8) is also invariant under the sixteen nonlinearly realized supersymmetries obtained by setting $\mu \to 0$.

where now η is an eleven dimensional Majorana spinor satisfying the constraint $\Gamma_{012}\eta = -\eta$ and σ^μ are the three dimensional field theory coordinates.

The field theory with Lagrangian (4.16)(4.17) and with supersymmetry transformations (4.18)(4.19) defines a novel maximally supersymmetric Poincare invariant three dimensional field theory with $SO(4) \times SO(4)$ R-symmetry.

4.2. Deformed Theory as DLCQ of Type IIB Plane Wave

In [158][159], the theory of coincident M2-branes on $R \times T^2$ was argued[54] to provide the Matrix theory [151] description of Type IIB string theory in flat space, extending the Matrix string theory description in [160][161] to Type IIB string theory.

In this section we argue that the three dimensional deformed field theory (4.16) on $R \times T^2$ provides the Matrix theory[55] description of the maximally supersymmetric Type IIB plane wave background [155]:

$$\begin{aligned} ds^2 &= 2dx^+dx^- - \mu^2 x^I x^I dx^+ dx^+ + dx^I dx^I \\ F_{+1234} &= F_{+5678} = 2\mu. \end{aligned} \tag{4.20}$$

As in the case of flat space, the modular parameter τ of the torus on which the deformed field theory is defined determines the complexified coupling constant of Type IIB string theory $\tau = C_0 + i/g_s$ [167].

In this chapter we have constructed a one parameter deformation of the Bagger-Lambert field theory that preserves all the thirty-two supersymmetries. It is therefore natural to propose that the deformed theory (4.16) presented in this chapter is the Matrix theory description of the Type IIB plane wave. Also as $\mu \to 0$ the plane wave background (4.20) reduces to flat space just as the deformed field theory (4.16) goes over to the Bagger-Lambert theory (4.8), which as the candidate theory for multiple M2-branes is the Matrix theory for flat space[56].

---

[54] At that time there was no Lagrangian description of the coincident M2-brane theory.

[55] For a different proposal for the Matrix theory of the Type IIB plane wave see [162]. For the DLCQ description of the plane wave in terms of a sector of a quiver gauge theory see [163]. See also [164][165][166].

[56] See [154][41] for subtleties with this interpretation.

Matrix theory describes nonperturbatively a string/M-theory background in the discrete light cone quantization (DLCQ) [168]. In this quantization we consider a string/M-theory background with a compactified lightlike coordinate $x^- \simeq x^- + 2\pi R$ in a sector with quantized longitudinal momentum $P^+ = N/R$. The Matrix theory description of a string/M-theory background with some prescribed asymptopia must realize the same symmetries as those of the asymptotic background with the lightlike identification $x^- \simeq x^- + 2\pi R$.

If we consider the DLCQ of Type IIB string theory in $R^{1,9}$, then the $ISO(1,9)$ symmetry algebra of Minkowski space is broken by the x^- identification to the centrally extended Super-Galileo algebra $SGal(1,8)$ [168], where the central extension corresponds to P^+.

The Type IIB plane wave background (4.20) is invariant under thirty-two supersymmetries and under a thirty-dimensional bosonic symmetry algebra [155]. Unlike in flat space, the $x^- \simeq x^- + 2\pi R$ does not break any of these symmetries. It is useful to gain intuition on the action of these symmetries to notice that the bosonic symmetries of the Type IIB plane wave background (4.20) can be identified with the centrally extended Newton-Hooke algebra[57] $NH(1,8)$. This algebra of symmetries is the non-relativistic contraction[58] of the isometry algebra of AdS_9, just like the $Gal(1,8)$ symmetry algebra of Matrix string theory in flat space arises in the non-relativistic contraction of the isometry algebra of $R^{1,8}$. As in the case of flat space, the central extension corresponds to P^+. Therefore the non-central generators of $NH(1,8)$ are given by H, P^I, K^I, J^{AB} and $J^{A'B'}$, which generate time translations, spatial translations, boosts and rotations respectively, and where the transverse index has been decomposed as $I = (A, A')$.

The deformed field theory (4.16) is manifestly invariant under the action of H, J^{AB} and $J^{A'B'}$, which correspond in the deformed field theory (4.16) to the Hamiltonian and the $SO(4) \times SO(4)$ R-symmetry charges of the three dimensional field theory. The non-manifest symmetries that remain to be realized are the translations P^I and boosts K^I. We now consider the following non-linear action of these generators on the fields of the deformed field theory (4.16)

$$P^J: \qquad \begin{aligned} \delta X^I &= a^J \delta^{IJ} \cos(\mu\sigma^0) T^0 \\ \delta\Psi &= 0 \\ \delta\tilde{A}_\mu{}^b{}_a &= 0 \end{aligned} \tag{4.21}$$

[57] This algebra has appeared previously in the context of non-relativistic symmetries of string theory in e.g. [169][170][171][172].

[58] The flux in (4.20) actually breaks the $SO(8)$ rotation symmetry of the contracted algebra down to $SO(4) \times SO(4)$.

and

$$
K^J: \qquad \begin{aligned} \delta X^I &= v^J \delta^{IJ} \frac{\sin(\mu\sigma^0)}{\mu} T^0 \\ \delta\Psi &= 0 \\ \delta\tilde{A}_\mu{}^b{}_a &= 0, \end{aligned} \tag{4.22}
$$

where σ^0 is the field theory time coordinate and T^0 is a central element in the 3-algebra $\mathcal{A}$. Note that in the flat space limit $\mu \to 0$ we recover the usual Galilean transformations. Under the action of the transformations (4.21) and (4.22) the deformed Lagrangian (4.16) changes by a total derivative. This provides the field theory explanation for the existence of the central extension P^+ in $NH(1,8)$, as central extensions of symmetry algebras are always associated with symmetry transformations that result in quasi-invariant Lagrangians. The central extension appears in the commutator of translations and boosts:

$$
[P^I, K^J] = i\delta^{IJ} P^+. \tag{4.23}
$$

The original three dimensional Poincare symmetry of the field theory is broken by compactification to $R \times T^2$ to just the translation algebra. The time translation generator H is identified with the Type IIB Hamiltonian. The translation generators along the T^2 can be identified with central charges of the superalgebra [173][159]. These central charges are associated with fundamental strings and D1 strings wrapping the longitudinal direction of the Type IIB plane wave (4.20). The geometrical action of $SL(2,Z)$ on the T^2 on which the theory is defined exchanges the fundamental and D1 strings in the way expected from duality [167].

The supercharge generating the supersymmetry transformations (4.18) correspond to the dynamical supersymmetries of the Type IIB plane wave (4.20) while the supercharges generating the supersymmetry transformations (4.19) correspond to the kinematical supersymmetries of the plane wave. Thus combining the bosonic symmetries with the supersymmetry transformations found in the section 1 we conclude that the deformed field theory (4.16) is invariant under $SNH(1,8)$, or equivalenty under the superisometry algebra of the Type IIB plane background (4.20) in the DLCQ. In the last section we write the Noether charges of the deformed field theory on $R \times T^2$ and show that the commutation relations are those of the Type IIB plane wave (4.20).

Therefore the deformed field theory (4.16) has the necessary ingredients to be the Matrix theory description of the Type IIB plane wave (4.20).

Type IIB string theory on the plane wave background (4.20) contains in its Hilbert space states with zero light-cone energy – where $H = 0$ – that preserve half of the supersymmetry [152]. They correspond to configurations of giant gravitons. A giant graviton in (4.20) is a $D3$ brane which wraps S^3 or $\tilde{S}^3$ at $x^- = 0$, where S^3 ($\tilde{S}^3$) is the sphere of the first (second) R^4 in the plane wave geometry (4.20). The radius of the giant graviton is determined by the longitudinal momentum P^+ carried by the $D3$-brane [152]:

$$\frac{L^2}{\alpha'} = 2\pi g_s \mu P^+ \alpha'. \tag{4.24}$$

When considering the DLCQ of the Type IIB plane wave, the total longitudinal momentum is quantized $P^+ = N/R$. Therefore, the $H = 0$ states of the DLCQ of the plane wave are labeled by partitions of N, and each state describes a configuration of $D3$-branes whose total longitudinal momentum is $P^+ = N/R$. These $D3$-brane configurations preserve half of the supersymmetries. More precisely, they preserve all the sixteen linearly realized supersymmetries while they break all of the non-linearly supersymmetries of the plane wave background.

The deformed field theory (4.16) also contains in its Hilbert space zero energy states that preserve half of the supersymmetries of the theory. These ground states are described by constant scalar fields satisfying

$$[X^A, X^B, X^C] = -\mu\epsilon^{ABCD} X^D, \qquad X^{A'} = 0 \tag{4.25}$$

or alternatively:

$$[X^{A'}, X^{B'}, X^{C'}] = -\mu\epsilon^{A'B'C'D'} X^{D'}, \qquad X^A = 0, \tag{4.26}$$

where we have split the transverse index $I = (A, A')$, with $A = 3, 4, 5, 6$ and $A' = 7, 8, 9, 10$. These solutions automatically satisfy the supersymmetry condition[59] $\tilde{\delta}\Psi = 0$ in (4.18) and preserve all the linearly realized supersymmetries while they break the non-linearly realized supersymmetries, just like the giant gravitons in the Type IIB plane wave (4.20). It is straightforward to show that these states also have $H = 0$.

[59] The supersymmetry conditions of [162] were analyzed in [174].

We identify these states of the deformed field theory with the giant graviton configurations of the Type IIB plane wave. Further work on 3-algebras and their representation theory is important to further understand the Matrix theory proposal of this chapter.

4.3. Supplementary material for chapter 4

Supersymmetry of Deformed Field Theory

We first note that the susy variation (4.18) can be decomposed as

$$\tilde{\delta} = \delta_\epsilon + \delta_\mu, \tag{4.27}$$

where δ_ϵ are given in

$$\begin{aligned} \delta_\epsilon X^I_a =& i\bar{\epsilon}\Gamma^I\Psi_a \\ \delta_\epsilon \Psi_a =& D_\mu X^I_a \Gamma^\mu \Gamma^I \epsilon - \frac{1}{6} X^I_b X^J_c X^K_d f^{bcd}{}_a \Gamma^{IJK}\epsilon \\ \delta_\epsilon \tilde{A}_\mu{}^b{}_a =& i\bar{\epsilon}\Gamma_\mu \Gamma_I X^I_c \Psi_d f^{cdb}{}_a . \end{aligned} \tag{4.28}$$

and

$$\begin{aligned} \delta_\mu X^I_a =& 0 \\ \delta_\mu \Psi =& -\mu\Gamma_{3456}\Gamma^I X^I \epsilon\,, \\ \delta_\mu \tilde{A}_\mu{}^b{}_a =& 0 \end{aligned} \tag{4.29}$$

where ϵ is an eleven dimensional Majorana spinor subject to the constraint $\Gamma_{012}\epsilon = \epsilon$. Since $\tilde{\mathcal{L}} = \mathcal{L} + \mathcal{L}_{mass} + \mathcal{L}_{flux}$, we have that:

$$\tilde{\delta}\tilde{\mathcal{L}} = \delta_\epsilon\mathcal{L} + \delta_\epsilon\mathcal{L}_{mass} + \delta_\epsilon\mathcal{L}_{flux} + \delta_\mu\mathcal{L} + \delta_\mu\mathcal{L}_{mass} + \delta_\mu\mathcal{L}_{flux}. \tag{4.30}$$

In [125] it has already been shown that $\delta_\epsilon\mathcal{L} = 0$ up to total derivatives. It is trivial to see that $\delta_\mu\mathcal{L}_{flux} = 0$. The other terms are:

$$\begin{aligned} \delta_\epsilon\mathcal{L}_{mass} =& -\mu^2 \mathrm{Tr}(X^I, i\bar{\epsilon}\Gamma^I\Psi) + i\mu\mathrm{Tr}(D_\mu X^I, \bar{\Psi}\Gamma_{3456}\Gamma^\mu\Gamma^I\epsilon) \\ & - i\frac{1}{6}\mu\mathrm{Tr}([X^I, X^J, X^K], \bar{\Psi}\Gamma_{3456}\Gamma^{IJK}\epsilon) \end{aligned} \tag{4.31}$$

$$
\begin{aligned}
\delta_\epsilon \mathcal{L}_{flux} =& i\frac{2}{3}\mu\varepsilon^{ABCD}\mathrm{Tr}([X^A, X^B, X^C], \bar{\Psi}\Gamma^D\epsilon) \\
&+ i\frac{2}{3}\mu\varepsilon^{A'B'C'D'}\mathrm{Tr}([X^{A'}, X^{B'}, X^{C'}], \bar{\Psi}\Gamma^{D'}\epsilon) \\
=& - i\frac{2}{3}\mu\mathrm{Tr}([X^A, X^B, X^C], \bar{\Psi}\Gamma^{ABC}\Gamma_{3456}\epsilon) \\
&+ i\frac{2}{3}\mu\mathrm{Tr}([X^{A'}, X^{B'}, X^{C'}], \bar{\Psi}\Gamma^{A'B'C'}\Gamma_{3456}\epsilon)
\end{aligned} \tag{4.32}
$$

In the last step of (4.32) we have used

$$
\varepsilon^{ABCD}\Gamma^D = -\Gamma^{ABC}\Gamma_{3456}, \qquad \varepsilon^{A'B'C'D'}\Gamma^{D'} = -\Gamma^{A'B'C'}\Gamma_{789(10)}, \tag{4.33}
$$

and

$$
\Gamma_{789(10)}\epsilon = -\Gamma_{3456}\epsilon, \tag{4.34}
$$

which is implied by $\Gamma_{012}\epsilon = \epsilon$ and $\Gamma_{0123456789(10)} = -1$. We also have that

$$
\begin{aligned}
\delta_\mu \mathcal{L} =& - \frac{i}{2}\partial_\mu \mathrm{Tr}(\bar{\Psi}\Gamma^\mu, \delta_\mu \Psi) - i\mu\mathrm{Tr}(D_\mu X^I, \bar{\Psi}\Gamma_{3456}\Gamma^\mu\Gamma^I\epsilon) \\
&- i\frac{1}{2}\mu\mathrm{Tr}([X^I, X^J, X^K], \bar{\Psi}\Gamma^{IJ}\Gamma_{3456}\Gamma^K\epsilon)
\end{aligned} \tag{4.35}
$$

and that

$$
\delta_\mu \mathcal{L}_{mass} = \mu^2 \mathrm{Tr}(i\bar{\epsilon}\Gamma^I\Psi, X^I). \tag{4.36}
$$

Combining all the pieces together we get

$$
\begin{aligned}
\tilde{\delta}\tilde{\mathcal{L}} =& - i\frac{1}{6}\mu\mathrm{Tr}([X^I, X^J, X^K], \bar{\Psi}\Gamma_{3456}\Gamma^{IJK}\epsilon) \\
&- i\frac{1}{2}\mu\mathrm{Tr}([X^I, X^J, X^K], \bar{\Psi}\Gamma^{IJ}\Gamma_{3456}\Gamma^K\epsilon) \\
&- i\frac{2}{3}\mu\mathrm{Tr}([X^A, X^B, X^C], \bar{\Psi}\Gamma^{ABC}\Gamma_{3456}\epsilon) \\
&+ i\frac{2}{3}\mu\mathrm{Tr}([X^{A'}, X^{B'}, X^{C'}], \bar{\Psi}\Gamma^{A'B'C'}\Gamma_{3456}\epsilon),
\end{aligned} \tag{4.37}
$$

where we have omitted the surface term in (4.35). Using the identities

$$
\begin{aligned}
[X^I, X^J, X^K]\Gamma_{3456}\Gamma^{IJK} =& - [X^A, X^B, X^C]\Gamma^{ABC}\Gamma_{3456} + 3[X^A, X^B, X^{A'}]\Gamma^{ABA'}\Gamma_{3456} \\
&-3[X^{A'}, X^{B'}, X^A]\Gamma^{A'B'A}\Gamma_{3456} + [X^{A'}, X^{B'}, X^{C'}]\Gamma^{A'B'C'}\Gamma_{3456}
\end{aligned} \tag{4.38}
$$

$$
\begin{aligned}
[X^I, X^J, X^K]\Gamma^{IJ}\Gamma_{3456}\Gamma^K =& - [X^A, X^B, X^C]\Gamma^{ABC}\Gamma_{3456} - [X^A, X^B, X^{A'}]\Gamma^{ABA'}\Gamma_{3456} \\
&+[X^{A'}, X^{B'}, X^A]\Gamma^{A'B'A}\Gamma_{3456} + [X^{A'}, X^{B'}, X^{C'}]\Gamma^{A'B'C'}\Gamma_{3456},
\end{aligned} \tag{4.39}
$$

one can show that the right hand side of (4.37) vanishes. This implies that the the deformed field theory is invariant under sixteen linearly realized supersymmetries.

The proposed non-linearly realized supersymmetry transformations are given by

$$\begin{aligned}\delta_n X^I_a =&0\\ \delta_n \Psi =& \exp\left(-\frac{1}{3}\mu\Gamma_{3456}\Gamma_\mu\sigma^\mu\right)T^0\eta\,,\\ \delta_n \tilde{A}_\mu{}^b{}_a =&0\end{aligned} \tag{4.40}$$

where now η is an eleven dimensional Majorana spinor subject to the constraint $\Gamma_{012}\eta = -\eta$ and T^0 is a central generator of the 3-algebra. The variation of the Lagrangian (4.16) gives

$$\begin{aligned}\delta_n\tilde{\mathcal{L}} =& i\bar{\Psi}^a\Gamma^\mu(D_\mu\delta_n\Psi)_a + \frac{i}{2}\bar{\Psi}_b\Gamma_{IJ}X^I_cX^J_d\delta_n\Psi_a f^{abcd} + i\mu\bar{\Psi}^a\Gamma_{3456}\delta_n\Psi_a\\ &-\frac{i}{2}\partial_\mu(\bar{\Psi}^a\Gamma^\mu\delta_n\Psi_a)\\ =& i\bar{\Psi}^0\Gamma^\mu\partial_\mu(e^{-\frac{1}{3}\mu\Gamma_{3456}\Gamma_\mu\sigma^\mu})\eta + i\mu\bar{\Psi}^0\Gamma_{3456}e^{-\frac{1}{3}\mu\Gamma_{3456}\Gamma_\mu\sigma^\mu}\eta\\ =&0,\end{aligned} \tag{4.41}$$

where in the second step we used that $f^{cd0}{}_b = 0$ – T^0 being central – and have ignored a total derivative. Therefore the deformed field theory (4.16) is also invariant under sixteen non-linearly realized supersymmetries.

When the deformed field theory is placed on $R\times T^2$ the three dimensional Poincare symmetry is broken. In this case the theory is invariant under the following transformations:

$$\begin{aligned}\delta_n X^I_a =&0\\ \delta_n \Psi =& \exp\left(-\mu\Gamma_{3456}\Gamma_0\sigma^0\right)T^0\eta\,.\\ \delta_n \tilde{A}_\mu{}^b{}_a =&0\end{aligned} \tag{4.42}$$

Noether Charges and Supersymmetry Algebra

The charges that generate the symmetry transformations of the deformed field theory on $R \times T^2$ are given by

$$\begin{aligned}
P^+ &= \int d^2\sigma \\
P^I &= \int d^2\sigma \left(\Pi_0^I \cos(\mu\sigma^0) + \mu X_0^I \sin(\mu\sigma^0)\right) \\
K^I &= \int d^2\sigma \left(\Pi_0^I \frac{\sin(\mu\sigma^0)}{\mu} - X_0^I \cos(\mu\sigma^0)\right) \\
J^{AB} &= -i \int d^2\sigma \left(\mathrm{Tr}(X^A, \Pi^B) - \mathrm{Tr}(X^B, \Pi^A) + \frac{i}{4}\mathrm{Tr}(\bar{\Psi}, \Gamma^{AB}\Gamma^0\Psi)\right) \\
J^{A'B'} &= -i \int d^2\sigma \left(\mathrm{Tr}(X^{A'}, \Pi^{B'}) - \mathrm{Tr}(X^{B'}, \Pi^{A'}) + \frac{i}{4}\mathrm{Tr}(\bar{\Psi}, \Gamma^{A'B'}\Gamma^0\Psi)\right) \\
Q &= \int d^2\sigma \Big(- \mathrm{Tr}(D_\mu X^I, \Gamma^\mu\Gamma^I\Gamma^0\Psi) - \frac{1}{6}\mathrm{Tr}([X^I, X^J, X^K], \Gamma^{IJK}\Gamma^0\Psi) \\
&\qquad + \mu\Gamma^I\Gamma_{3456}\Gamma^0\mathrm{Tr}(X^I, \Psi)\Big) \\
q &= -i \int d^2\sigma\Gamma^0 \exp\left(-\mu\Gamma_{3456}\Gamma_0\sigma^0\right)\Psi_0,
\end{aligned} \tag{4.43}$$

where $\int d^2\sigma$ is the integral over the T^2. The Hamiltonian of the theory is given by:

$$\begin{aligned}
H &= \Pi_a^I f^{cdba} A_{0cd} X_b^I + \frac{1}{2}\Pi_a^I \Pi^{aI} + \frac{1}{2} D_i X_a^I D_i X^{aI} \\
&+ \frac{i}{2}\bar{\Psi}^a\Gamma^0\dot{\Psi}_a - \frac{i}{2}\bar{\Psi}^a\Gamma^0 D_0\Psi_a - \frac{i}{2}\bar{\Psi}^a\Gamma^i D_i\Psi_a \\
&+ \frac{i}{4}\mathrm{Tr}([\bar{\Psi}\Gamma_{IJ}, \Psi, X^I], X^J) + V + \frac{1}{2}\mu^2\mathrm{Tr}(X^I, X^I) - \frac{i}{2}\mu\mathrm{Tr}(\bar{\Psi}, \Gamma_{3456}\Psi) \\
&+ \frac{1}{6}\mu\varepsilon^{ABCD}\mathrm{Tr}([X^A, X^B, X^C], X^D) + \frac{1}{6}\mu\varepsilon^{A'B'C'D'}\mathrm{Tr}([X^{A'}, X^{B'}, X^{C'}], X^{D'}) \\
&+ \Lambda^{cd\lambda}\dot{A}_{cd\lambda} - \frac{1}{2}\varepsilon^{\mu\nu\lambda}(f^{abcd}A_{\mu ab}\partial_\nu A_{\lambda cd} + \frac{2}{3}f^{cda}{}_g f^{efgb} A_{\mu ab}A_{\nu cd}A_{\lambda ef}).
\end{aligned} \tag{4.44}$$

Alternatively, one can write:

$$\begin{aligned}
H &= \Pi_a^I f^{cdba} A_{0cd} X_b^I + \frac{1}{2}\Pi_a^I \Pi^{aI} + \frac{1}{2} D_i X_a^I D_i X^{aI} \\
&+ \frac{i}{2}\bar{\Psi}_a\Gamma^0 f^{cdba} A_{0cd}\Psi_b - \frac{i}{2}\bar{\Psi}^a\Gamma^i D_i\Psi_a \\
&+ \frac{i}{4}\mathrm{Tr}([\bar{\Psi}\Gamma_{IJ}, \Psi, X^I], X^J) + V + \frac{1}{2}\mu^2\mathrm{Tr}(X^I, X^I) - \frac{i}{2}\mu\mathrm{Tr}(\bar{\Psi}, \Gamma_{3456}\Psi) \\
&+ \frac{1}{6}\mu\varepsilon^{ABCD}\mathrm{Tr}([X^A, X^B, X^C], X^D) + \frac{1}{6}\mu\varepsilon^{A'B'C'D'}\mathrm{Tr}([X^{A'}, X^{B'}, X^{C'}], X^{D'}) \\
&- \frac{1}{2}\varepsilon^{\mu i\lambda}(f^{abcd}A_{\mu ab}\partial_i A_{\lambda cd}) - \frac{1}{3}\epsilon^{\mu\nu\lambda} f^{cda}{}_g f^{efgb} A_{\mu ab}A_{\nu cd}A_{\lambda ef}.
\end{aligned} \tag{4.45}$$

where $i = 1, 2$.

In order to calculate the algebra generated by these charges we need the canonical momenta. Π^I is the momentum density conjugate to X^I and satisfies

$$[X_a^I(\sigma^i), \Pi_b^J(\sigma'^i)] = i\delta^2(\sigma^i - \sigma'^i)\delta_{ab}\delta^{IJ}, \tag{4.46}$$

where $i = 1, 2$ are the spatial coordinates on the membrane and $\Pi_a^I = D_0 X_a^I$. For the canonical commutation relation for the spinors, one must use Dirac brackets, which for the case of Majorana spinors results in the following commutation relation:

$$\{\Psi_a^\alpha(\sigma^i), \Psi_b^\beta(\sigma'^i)\} = -\delta^2(\sigma^i - \sigma'^i)\delta_{ab}\delta^{\alpha\beta} \tag{4.47}$$

where α, β are eleven dimensional spinor indices.

To compute the action of the symmetries on the fields, we compute the commutator of the charges with the fields. We get

$$\begin{aligned}
[P^I, X^J] &= -i\delta^{IJ}\cos(\mu\sigma^0)T^0 \\
[K^I, X^J] &= -i\delta^{IJ}\frac{\sin(\mu\sigma^0)}{\mu}T^0 \\
[J^{AB}, X^C] &= -X^A\delta^{BC} + X^B\delta^{AC} \\
[J^{AB}, \Psi] &= -\frac{1}{2}\Gamma^{AB}\Psi \\
[J^{A'B'}, X^{C'}] &= -X^{A'}\delta^{B'C'} + X^{B'}\delta^{A'C'} \\
[J^{A'B'}, \Psi] &= -\frac{1}{2}\Gamma^{A'B'}\Psi \\
[\bar\epsilon Q, X^J] &= i\bar\epsilon\Gamma^I\Psi \\
[\bar\epsilon Q, \Psi] &= D_\mu X^I\Gamma^\mu\Gamma^I\epsilon - \frac{1}{6}[X^I, X^J, X^K]\Gamma^{IJK}\epsilon - \mu\Gamma_{3456}\Gamma^I X^I\epsilon \\
[\bar\epsilon Q, A_{abi}] &= i\bar\epsilon\Gamma_i\Gamma^I X^I_{[a}\Psi_{b]} \\
[\bar\eta q, \Psi] &= \exp\left(-\mu\Gamma_{3456}\Gamma_0\sigma^0\right)T^0\eta.
\end{aligned} \tag{4.48}$$

We now show that the Noether charges (4.43) of the deformed field theory (4.16) satisfy the Type IIB plane wave superalgebra. For the even generators we get:

$$\begin{aligned}
&[P^I, H] = i\mu^2 K^I \qquad [K^I, H] = -iP^I \qquad [P^I, K^J] = i\delta^{IJ}P^+ \\
&[P^A, J^{BC}] = -\delta^{AB}P^C + \delta^{AC}P^B \qquad [P^{A'}, J^{B'C'}] = -\delta^{A'B'}P^{C'} + \delta^{A'C'}P^{B'} \\
&[K^A, J^{BC}] = -\delta^{AB}K^C + \delta^{AC}K^B \qquad [K^{A'}, J^{B'C'}] = -\delta^{A'B'}K^{C'} + \delta^{A'C'}K^{B'} \\
&[J^{AB}, J^{CD}] = -\delta^{BC}J^{AD} + \delta^{AC}J^{BD} + \delta^{BD}J^{AC} - \delta^{AD}J^{BC} \\
&[J^{A'B'}, J^{C'D'}] = -\delta^{B'C'}J^{A'D'} + \delta^{A'C'}J^{B'D'} + \delta^{B'D'}J^{A'C'} - \delta^{A'D'}J^{B'C'}.
\end{aligned} \tag{4.49}$$

The commutation relations between odd and even generators are:

$$
\begin{aligned}
[P^I, Q] &= -i\mu\Gamma^I\Gamma_{3456}q \qquad [K^I, Q] = -i\Gamma^I\Gamma^0 q \\
[H, Q] &= 0 \qquad [H, q] = -i\Gamma_{3456}\Gamma^0 q \\
[J^{AB}, Q] &= -\frac{1}{2}\Gamma^{AB}Q \qquad [J^{AB}, q] = -\frac{1}{2}\Gamma^{AB}q \\
[J^{A'B'}, Q] &= -\frac{1}{2}\Gamma^{A'B'}Q \qquad [J^{A'B'}, q] = -\frac{1}{2}\Gamma^{A'B'}q.
\end{aligned}
\tag{4.50}
$$

The anticommutators of the supercharges are:

$$
\begin{aligned}
\{q^\alpha, q^\beta\} &= i\delta^{\alpha\beta}P^+ \qquad \{q^\alpha, Q^\beta\} = -\frac{i}{2}(\Gamma^I\Gamma^0)^{\alpha\beta}P^I - \mu\frac{i}{2}(\Gamma_{3456}\Gamma^I)^{\alpha\beta}K^I \\
\{Q^\alpha, Q^\beta\} &= 2H\delta^{\alpha\beta} + i\mu(\Gamma^{AB}\Gamma_{3456}\Gamma^0)^{\alpha\beta}J^{AB} + i\mu(\Gamma^{A'B'}\Gamma_{789(10)}\Gamma^0)^{\alpha\beta}J^{A'B'}.
\end{aligned}
\tag{4.51}
$$

This is the superalgebra of the Type IIB plane wave [155](see also [175] for a useful summary of the superalgebra).

5. M2-Brane Superalgebra from Bagger-Lambert Theory

Brane intersections can be described as solitons of the worldvolume theory of one of the constituents of the intersecting system [176][177]. In particular, quarter-BPS intersections appear on the worldvolume as half-BPS solitons and the spacetime interpretation relies on the fact that the worldvolume scalars encode the brane embedding.

Many M-branes systems in M-theory have been studied using this approach. For instance, a stack of M2-branes ending on an M5-brane is associated to a self-dual string soliton on the M5-brane worldvolume [178] and the M5-M5 intersection can be described as a 3-brane vortex on the worldvolume of one of the M5-branes [179]. In a similar way, the M2-M2 intersection can be described as a 0-brane vortex on the worldvolume of one of the M2-branes [176][180]. All these examples mentioned are the worldvolume realization of previously studied quarter-BPS intersecting systems [181][182][183][184].

It was shown in [185] that the spacetime interpretation of the worldvolume solitons can be deduced also from the worldvolume supersymmetry algebra. For the case of the M2-brane the worldvolume supersymmetry algebra is given by the maximal central extension of the 3-dimensional $\mathcal{N}=8$ super-Poincare algebra [185]. The anticommutator is given by

$$\{Q^p_{\hat{\alpha}}, Q^q_{\hat{\beta}}\} = -2P_\mu(\hat{\gamma}^\mu\hat{\gamma}^0)_{\hat{\alpha}\hat{\beta}}\delta^{pq} + Z^{[pq]}\varepsilon_{\hat{\alpha}\hat{\beta}} + Z^{(pq)}_\mu(\hat{\gamma}^\mu\hat{\gamma}^0)_{\hat{\alpha}\hat{\beta}} \qquad (5.1)$$

where $Q^p_{\hat{\alpha}}$ are the eight 3-dimensional Majorana spinor supercharges and $Z^{[pq]}$,$Z^{(pq)}_\mu$ are the 0-form and the 1-form worldvolume central charges. $p, q = 1, \dots 8$ are the indices of the SO(8) automorphism group and the supercharges transform as chiral spinors of SO(8). Due to the triality relation of SO(8), we can consider the supercharges to transform in the vector representation of SO(8) and thus we can interpret the automorphism group SO(8) as the rotation group in the eight directions transverse to the M2-branes. The 0-form $Z^{[pq]}$ is in the **28** representation of SO(8) and it can be thought as a 2-form in the transverse space. This central charge is associated with M2-branes that are intersecting the original M2-branes along the time direction, a quarter-BPS system studied in [181]. The 1-form $Z^{(pq)}_\mu$ is in the $\mathbf{35}^+$ of SO(8) and it is a self-dual 4-form in the transverse space. This implies that the 1-form charge is associated to the quarter-BPS M2-M5 system [184][182].

We have seen that the M2-brane superalgebra, correctely incorporates all the possible quarter-BPS intersections between the M2-branes and the other M-branes of M-theory.[60]

[60] In the worldvolume description, these intersections are half-BPS solitons.

This implies that a complete M2-branes worldvolume theory should realize the M2-brane superalgebra (5.1), including also the central charges.

In this chapter, we verify by explicit computation that the Bagger-Lambert theory does realize the M2-brane superalgebra (5.1). The central charges that we obtain are given by

$$\begin{aligned} Z^{[pq]} &= -\int d^2\sigma \partial_i \mathrm{Tr}(X^I, D_j X^J)\varepsilon^{ij}(\gamma^{IJ})^{pq} \\ Z_\mu^{(pq)} &= -\frac{1}{12}\int d^2\sigma \partial_i \mathrm{Tr}(X^I, [X^J, X^K, X^M])\varepsilon^{0i}{}_\mu(\gamma^{IJKM})^{pq}. \end{aligned} \tag{5.2}$$

We note that the 0-form charge $Z^{[pq]}$ is the natural generalization of the charge computed in [180] using the BPS-bound for the vortex solution in the single M2-brane theory. The 1-form instead relies on the non-abelian nature of the scalar fields in the Bagger-Lambert theory and it vanishes in the limit where the stack of multiple M2-branes reduces to a single M2-brane. This is consistent with the fact that the M2-M5 intersection cannot be seen on the worldvolume of a single M2-brane. Indeed, given an intersection between branes with different dimensions, the worldvolume description of the system using the worldvolume of the lower dimensional brane is usually based on non-abelianity [130][148].

We show that a vortex solution excites the 0-form central charge and the Basu-Harvey solution [129] excites the 1-form central charge, in agreement with the interpretation of this solitons as the quarter-BPS M2-M2 intersection and the quarter-BPS M2-M5 intersection. The energy of this configurations is bounded below by the value of the corresponding central charge and the bound is saturated when the solitons are half-BPS. This is in agreement with the structure of the M2-brane superalgebra (5.2).

The rest of the chapter is organized as follows. In section 1 we write down the supercurrent associated to the supersymmetry of the Lagrangian. This enables us to express the supercharges in terms of the fields of the theory. In section 2 we use the field theory realization of the supercharges to compute the central charges. In section 3 we analyze the vortex and the Basu-Harvey solitons and show that they are associated to central charges, in agreement with the interpretation of this solutions as the worldvolume realization of intersecting systems. Section 4 summarizes our notation and includes the proof of the conservation of the supercurrent. The same section contains technical details of the computation for the central charges.

Given the invariance of the Lagrangian (4.8) under the supersymmetry variations (4.12), the Noether theorem implies the existence of a conserved supercurrent J^μ given by

$$J^\mu = -D_\nu X^I_a \Gamma^\nu \Gamma^I \Gamma^\mu \Psi^a - \frac{1}{6} X^I_a X^J_b X^K_c f^{abcd} \Gamma^{IJK} \Gamma^\mu \Psi_d. \tag{5.3}$$

In section 4 we show that $\partial_\mu J^\mu = 0$. The supercharge is thus the integral over the spatial worldvolume coordinates of the timelike component of the supercurrent, i.e.

$$\begin{aligned} Q &= \int d^2\sigma J^0 \\ &= -\int d^2\sigma (D_\nu X^I_a \Gamma^\nu \Gamma^I \Gamma^0 \Psi^a + \frac{1}{6} X^I_a X^J_b X^K_c f^{abcd} \Gamma^{IJK} \Gamma^0 \Psi_d). \end{aligned} \tag{5.4}$$

Given that the mass dimensions of the fields in the Bagger-Lambert theory are $[X] = \frac{1}{2}$ and $[A] = [\Psi] = 1$, it follows that J^0 has mass dimension $[J^0] = \frac{5}{2}$. This gives $[Q] = \frac{1}{2}$, that is the right mass dimension for the supercharge. It is easy to check that the two terms on the right hand side of (5.3) are the only gauge invariant combinations of fields with the right mass dimension and with an uncontracted spinorial index.

The supercharge Q is the generator of the supersymmetry transformation, that means that the supersymmetry variation of a field Φ is given by $\delta_\epsilon \Phi = [\bar{\epsilon} Q, \Phi]$. More in details, for Grassman-even and Grassman-odd fields Φ_E and Φ_O we have

$$\delta_\epsilon \Phi_E = \bar{\epsilon}_\alpha [Q^\alpha, \Phi_E] \qquad \delta_\epsilon \Phi^\beta_O = \bar{\epsilon}_\alpha \{Q^\alpha, \Phi^\beta_O\} \tag{5.5}$$

where we have explicitly shown the 11-dimension spinorial indices α and β. Using the canonical commutation relations, one can show that the (5.5) reproduce the supersymmetry variations of the Bagger-Lambert theory (4.12).

5.2. Central Charges

In this section, we show that the supersymmetry algebra of the Bagger-Lambert theory includes two central charge forms, as expected for a theory describing M2-branes. These central extensions are computed here explicitly using the field realization of the supercharge Q given in (5.4)[186].[61] In details, we consider the relation

$$\bar{\epsilon}_\alpha\{Q^\alpha, Q^\beta\} = \int d^2\sigma \bar{\epsilon}_\alpha \{Q^\alpha, J^{0\beta}(\sigma)\} = \int d^2\sigma \delta_\epsilon J^{0\beta}(\sigma) \tag{5.6}$$

where in the last step we used the second of the equations (5.5). The supersymmetry variation of the zeroth component of the supercurrent $\delta_\epsilon J^0$ is computed in section 4. For the case where the spinors Ψ are set to zero, it is given by

$$\delta_\epsilon J^0 = -2T^0{}_\mu \Gamma^\mu \epsilon - \partial_i(X^I_a D_j X^{aJ} \varepsilon^{ij}\Gamma^{IJ}\epsilon) - \frac{1}{12}\partial_i(X^I_a X^J_b X^K_c X^M_d f^{bcda}\varepsilon^{0i}{}_\mu \Gamma^{IJKM}\Gamma^\mu \epsilon) \tag{5.7}$$

where $i = 1, 2$ labels the spatial worldvolume coordinates. From the expression (5.7) and the relation (5.6) we get

$$\begin{aligned}\{Q^\alpha, Q^\beta\} =& -2P_\mu(\Gamma^\mu\Gamma^0)^{\alpha\beta} - \int d^2\sigma \partial_i(X^I_a D_j X^{aJ}\varepsilon^{ij})(\Gamma^{IJ}\Gamma^0)^{\alpha\beta} \\ & - \frac{1}{12}\int d^2\sigma \partial_i(X^I_a X^J_b X^K_c X^M_d f^{bcda}\varepsilon^{0i}{}_\mu)(\Gamma^{IJKM}\Gamma^\mu\Gamma^0)^{\alpha\beta}\end{aligned} \tag{5.8}$$

where P^μ is the energy momentum vector defined as $P^\mu = \int d^2\sigma T^{0\mu}$.

Spinors Decomposition

In order to better analyze the structure of the $\mathcal{N} = 8$ superalgebra, we need to write the anticommutator (5.8) in terms of 3-dimensional spinors. To this end, we decompose the $Spin(1, 10)$ Dirac matrices in terms of $Spin(1, 2) \otimes Spin(8)$ Dirac matrices. In details, we take

$$\Gamma^\mu = \hat{\gamma}^\mu \otimes \bar{\gamma}_9 \qquad \text{and} \qquad \Gamma^I = 1 \otimes \bar{\gamma}^I \tag{5.9}$$

where

$$\{\hat{\gamma}^\mu, \hat{\gamma}^\nu\} = 2\eta^{\mu\nu}, \qquad \{\bar{\gamma}^I, \bar{\gamma}^J\} = 2\delta^{IJ}, \qquad \bar{\gamma}_9 = \bar{\gamma}^3 \ldots \bar{\gamma}^{10} \tag{5.10}$$

and it is easy to check that the matrices (5.9) satisfies the 11-dimensinal Clifford algebra. The $\hat{\gamma}^\mu$ are 2×2 real matrices. Explicitly

$$\hat{\gamma}^0 = i\sigma^2_{\hat{\alpha}\hat{\beta}} = \varepsilon_{\hat{\alpha}\hat{\beta}} \qquad \hat{\gamma}^1 = \sigma^1_{\hat{\alpha}\hat{\beta}} \qquad \hat{\gamma}^2 = \sigma^3_{\hat{\alpha}\hat{\beta}} \tag{5.11}$$

[61] For a review, see for instance [187].

where the σ's are Pauli matrices and $\hat{\alpha}, \hat{\beta} = 1, 2$ are 3-dimensional spinorial indices. The $\bar{\gamma}^I$ are 16×16 real matrices given by

$$\bar{\gamma}^I = \begin{pmatrix} 0 & \gamma^I_{\dot{p}p} \\ \gamma^I_{q\dot{q}} & 0 \end{pmatrix} \tag{5.12}$$

where $(\gamma^I_{p\dot{p}})^T = \gamma^I_{\dot{p}p}$ are 8×8 real gamma matrices satisfying

$$\gamma^I_{p\dot{p}}\gamma^J_{\dot{p}q} + \gamma^J_{p\dot{p}}\gamma^I_{\dot{p}q} = 2\delta^{IJ}\delta_{pq} \qquad \gamma^I_{\dot{p}p}\gamma^J_{p\dot{q}} + \gamma^J_{\dot{p}p}\gamma^I_{p\dot{q}} = 2\delta^{IJ}\delta_{\dot{p}\dot{q}}. \tag{5.13}$$

Given that $\Gamma_{012} = -\hat{\gamma}_{012} \otimes \bar{\gamma}_9 = -1 \otimes \bar{\gamma}_9$, spinors with definite Γ_{012} chirality, have a definite $\bar{\gamma}_9$ chirality.[62]

Using the matrices decomposition just described and the fact that $\Gamma_{012}Q = Q$, the equation (5.8) can be written as

$$\begin{aligned} \{Q^p_{\hat{\alpha}}, Q^q_{\hat{\beta}}\} =& -2P_\mu(\hat{\gamma}^\mu\hat{\gamma}^0)_{\hat{\alpha}\hat{\beta}}\delta^{pq} - \int d^2\sigma\partial_i(X^I_a D_j X^{aJ}\varepsilon^{ij})(\gamma^{IJ})^{pq}\varepsilon_{\hat{\alpha}\hat{\beta}} \\ & - \frac{1}{12}\int d^2\sigma\partial_i(X^I_a X^J_b X^K_c X^M_d f^{bcda}\varepsilon^{0i}{}_\mu)(\gamma^{IJKM})^{pq}(\hat{\gamma}^\mu\hat{\gamma}^0)_{\hat{\alpha}\hat{\beta}} \end{aligned} \tag{5.14}$$

where $(\gamma^{IJ})_{pq} = \gamma^{[I}_{p\dot{r}}\gamma^{J]}_{\dot{r}q}$ and $(\gamma^{IJKM})_{pq} = \gamma^{[I}_{p\dot{r}}\gamma^J_{\dot{r}r}\gamma^K_{r\dot{t}}\gamma^{M]}_{\dot{t}q}$.

Thus, we conclude that the Bagger-Lambert Lagrangian realizes the centrally extended 3-dimensional $\mathcal{N} = 8$ superalgebra

$$\{Q^p_{\hat{\alpha}}, Q^q_{\hat{\beta}}\} = -2P_\mu(\hat{\gamma}^\mu\hat{\gamma}^0)_{\hat{\alpha}\hat{\beta}}\delta^{pq} + Z^{[pq]}\varepsilon_{\hat{\alpha}\hat{\beta}} + Z^{(pq)}_\mu(\hat{\gamma}^\mu\hat{\gamma}^0)_{\hat{\alpha}\hat{\beta}} \tag{5.15}$$

where the central charges are given by

$$\begin{aligned} Z^{[pq]} &= -\int d^2\sigma\partial_i \mathrm{Tr}(X^I, D_j X^J)\varepsilon^{ij}(\gamma^{IJ})^{pq} \\ Z^{(pq)}_\mu &= -\frac{1}{12}\int d^2\sigma\partial_i \mathrm{Tr}(X^I, [X^J, X^K, X^M])\varepsilon^{0i}{}_\mu(\gamma^{IJKM})^{pq}. \end{aligned} \tag{5.16}$$

Using the property $(\gamma^{IJ})^{qp} = -(\gamma^{IJ})^{pq}$, $(\gamma^{IJKM})^{qp} = (\gamma^{IJKM})^{pq}$ it follows that the 0-form central charge is antisymmetric in p, q indices and the 1-form central charge is symmetric in p, q. Given that $\varepsilon_{\hat{\alpha}\hat{\beta}} = -\varepsilon_{\hat{\beta}\hat{\alpha}}$ and $(\hat{\gamma}^\mu\hat{\gamma}^0)_{\hat{\alpha}\hat{\beta}} = (\hat{\gamma}^\mu\hat{\gamma}^0)_{\hat{\beta}\hat{\alpha}}$ the right hand side of the (5.16) is correctly symmetric under the exchange $(p, \hat{\alpha}) \leftrightarrow (q, \hat{\beta})$.

---

[62] In this representation $\bar{\gamma}_9 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

The equations (5.16) give the field realization of the central charges of the extended 3-dimensional $\mathcal{N}=8$ superalgebra. They are boundary terms and they are equal to zero for field configurations that are non-singular and topologically trivial. In the next section we will discuss half-BPS configurations that excite the central charges of the Bagger-Lambert theory.

5.3. Solitons of the Bagger-Lambert Theory

Vortices

We consider vortex configurations [176][180] where only the scalars X^3, X^4 and the gauge vector $\tilde{A}_\nu{}^b{}_a$ are excited. Given the interpretation of the Bagger-Lambert theory as a theory of coincident M2-branes, these configurations describe two stacks of membranes intersecting along the time direction[63]

$$\begin{array}{cccccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ M2 & X & X & X & & & & & & & & \\ M2 & X & & & X & X & & & & & & \end{array} \tag{5.17}$$

It is convenient to introduce the complex worldvolume coordinates z and $\bar{z}$

$$z = \sigma^1 + i\sigma^2 \qquad \bar{z} = \sigma^1 - i\sigma^2 \tag{5.18}$$

and the complex scalars Φ and $\bar{\Phi}$

$$\Phi = \frac{1}{2}(X_3 - iX_4) \qquad \bar{\Phi} = \frac{1}{2}(X_3 + iX_4). \tag{5.19}$$

Thus, considering a configuration where only Φ, $\bar{\Phi}$ and $\tilde{A}_\mu{}^b{}_a$ are switched on, and such that $D_0\Phi = D_0\bar{\Phi} = 0$, the BPS conditions that follow from the supersymmetry variations (4.12) reduce to

$$D_z\Phi\Gamma^z\Gamma^\Phi\epsilon + D_{\bar{z}}\Phi\Gamma^{\bar{z}}\Gamma^\Phi\epsilon + D_z\bar{\Phi}\Gamma^z\Gamma^{\bar{\Phi}}\epsilon + D_{\bar{z}}\bar{\Phi}\Gamma^{\bar{z}}\Gamma^{\bar{\Phi}}\epsilon = 0 \tag{5.20}$$

[63] This is the analog of the vortex like solution for $\mathcal{N}=4$ SYM describing a surface operator interpreted as the intersection D3$\cap$D3$= R^2$ [188][189].

where

$$\Gamma^{\Phi} = \Gamma^3 + i\Gamma^4 \qquad \Gamma^{\bar{\Phi}} = \Gamma^3 - i\Gamma^4 \qquad \Gamma^z = \Gamma^1 + i\Gamma^2 \qquad \Gamma^{\bar{z}} = \Gamma^1 - i\Gamma^2. \tag{5.21}$$

For this configuration, the energy density is given by

$$\mathcal{H} = 4\mathrm{Tr}(D_z\Phi, D_{\bar{z}}\bar{\Phi}) + 4\mathrm{Tr}(D_{\bar{z}}\Phi, D_z\bar{\Phi}) = \frac{\mathcal{Z}^0}{2} + 8\mathrm{Tr}(D_{\bar{z}}\Phi, D_z\bar{\Phi}) \tag{5.22}$$

where $\mathcal{Z}^0$ is the density of the 0-form central charge $Z^{[pq]}$ evaluated for this field configuration. Thus, considering a positive definite scalar product $\mathrm{Tr}(\cdot,\cdot)$, it results $\mathcal{H} \geq \frac{\mathcal{Z}^0}{2}$ and the bound is saturated when

$$D_{\bar{z}}\Phi = D_z\bar{\Phi} = 0. \tag{5.23}$$

When this last condition is satisfied, it follows from the BPS equation (5.20) that the solution preserve the supersymmetries satisfying $\Gamma^z\Gamma^{\Phi}\epsilon = 0$ or equivalently $\Gamma^{1234}\epsilon = \epsilon$. Thus, for the case where the gauge field is equal to zero, i.e. $\tilde{A}_{\mu}{}^b{}_a = 0$ the vortex configuration given by

$$\Phi = \frac{c_a T^a}{z} \tag{5.24}$$

where c_a are arbitrary constants is a half-BPS state.[64] The singularity in $z = 0$ excite the 0-form central charge $Z^{[pq]}$ (5.16), in agreement with the interpretation of the vortex solution as the brane intersection (5.17).

We now discuss the case where also the gauge vector is excited and to analyze this configuration we use the Lorenzean 3-algebra constructed in [131]. In this model, the 3-algebra indices a are split into $a = (0, \tilde{a}, \varphi)$ and the structure constants are given by

$$f^{0\tilde{a}\tilde{b}\tilde{c}} = f^{\varphi\tilde{a}\tilde{b}\tilde{c}} = C^{\tilde{a}\tilde{b}\tilde{c}}\ , \qquad f^{0\varphi\tilde{a}\tilde{b}} = f^{\tilde{a}\tilde{b}\tilde{c}\tilde{d}} = 0 \tag{5.25}$$

where $C^{\tilde{a}\tilde{b}\tilde{c}}$ are the structure constants of a compact semi-simple Lie algebra satisfying the usual Jacobi identity. The structure constants (5.25) solve the fundamental identity (4.6) and they are totally antisymmetric. Following [131], we introduce null generators on the 3-algebra

$$T^{\pm} = \pm T^0 + T^{\phi} \tag{5.26}$$

and in this basis the structure constants become

$$f^{+\tilde{a}\tilde{b}\tilde{c}} = 2C^{\tilde{a}\tilde{b}\tilde{c}}\ , \qquad f_{-\tilde{a}\tilde{b}\tilde{c}} = C_{\tilde{a}\tilde{b}\tilde{c}}\ , \qquad f^{-\tilde{a}\tilde{b}\tilde{c}} = f_{+\tilde{a}\tilde{b}\tilde{c}} = 0. \tag{5.27}$$

64 In the sense that it preserves half of the supersymmetries (4.12).

The gauge vector A^{ab}_μ is decomposed as

$$A^{\tilde a}_\mu \equiv A^{-\tilde a}_\mu \ , \qquad B^{\tilde a}_\mu \equiv \frac{1}{2} C^{\tilde a \tilde b \tilde c} A_{\mu \tilde b \tilde c}. \tag{5.28}$$

We consider a configuration where only the $\tilde a$ components of the scalar field are excited, we call this field $\tilde\Phi$. Thus

$$\tilde\Phi = \frac{c_{\tilde a} T^{\tilde a}}{z}. \tag{5.29}$$

Taking $B_\mu = 0$, the equation (5.20) reduce to

$$\tilde D_z \tilde\Phi \Gamma^z \Gamma^\Phi \epsilon + \tilde D_{\bar z} \tilde\Phi \Gamma^{\bar z} \Gamma^\Phi \epsilon + \tilde D_z \bar{\tilde\Phi} \Gamma^z \Gamma^{\bar\Phi} \epsilon + \tilde D_{\bar z} \bar{\tilde\Phi} \Gamma^{\bar z} \Gamma^{\bar\Phi} \epsilon = 0 \tag{5.30}$$

where

$$\tilde D_\mu \tilde\Phi^{\tilde a} \equiv \partial_\mu \tilde\Phi^{\tilde a} + 2 C^{\tilde a}{}_{\tilde b \tilde c} A^{\tilde c}_\mu \tilde\Phi^{\tilde b} \tag{5.31}$$

is the covariant derivative for a field in the adjoint representation of the Lie algebra with structure constants $C^{\tilde a \tilde b \tilde c}$. The energy density now is

$$\mathcal{H} = 4\mathrm{Tr}(\tilde D_z \tilde\Phi, \tilde D_{\bar z} \bar{\tilde\Phi}) + 4\mathrm{Tr}(\tilde D_{\bar z} \tilde\Phi, \tilde D_z \bar{\tilde\Phi}) = \frac{\mathcal{Z}^0}{2} + 8\mathrm{Tr}(\tilde D_{\bar z} \tilde\Phi, \tilde D_z \bar{\tilde\Phi}) \tag{5.32}$$

and given that [131] $\mathrm{Tr}(T^{\tilde a}, T^{\tilde b}) = \delta^{\tilde a \tilde b}$, it results $\mathcal{H} \geq \frac{\mathcal{Z}^0}{2}$. The $\mathcal{Z}^0$ is the 0-form central charge evaluated for this solution and the BPS-bound is saturated when $\tilde D_{\bar z} \tilde\Phi = \tilde D_z \bar{\tilde\Phi} = 0$. Thus, it follows that the configuration where only $\tilde\Phi$ and $\mathcal{A}_\mu = A_{\tilde a \mu} T^{\tilde a}$ are excited, is half-BPS if

$$[\tilde\Phi, \mathcal{A}_{\bar z}] = [\bar{\tilde\Phi}, \mathcal{A}_z] = 0 \tag{5.33}$$

where $[\cdot,\cdot]$ is the usual Lie commutator. Also in this case, the preserved supersymmetries satisfy $\Gamma^z \Gamma^\Phi \epsilon = 0$ and this configuration excites the the 0-form central charge. This implies that with respect to the single M2-brane theory, the vortex solutions of the Bagger-Lambert theory includes extra degrees of freedom, given by the the components of the gauge vector that commute with the scalar fields.

Basu-Harvey Solitons

To describe a stack of M2-branes ending on an M5-brane

	0	1	2	3	4	5	6	7	8	9	10
$M2$	X	X	X								
$M5$	X	X		X	X	X	X				

(5.34)

it is necessary to switch on the X^3,X^4,X^5,X^6 scalar fields [129]. Given that these fields depend only on the worldvolume coordinate σ^2, the BPS condition is [127]

$$\frac{dX^A}{d\sigma^2}\Gamma^A\Gamma^2\epsilon - \frac{1}{6}\varepsilon^{BCDA}\Gamma^A[X^B, X^C, X^D]\Gamma^{3456}\epsilon = 0 \tag{5.35}$$

where $A, B, C, D = 3, 4, 5, 6$ and we used $\varepsilon^{ABCD}\Gamma^D = -\Gamma^{ABC}\Gamma_{3456}$. For this field configuration the energy density is given by

$$\mathcal{H} = \frac{1}{2}\text{Tr}(\partial_2 X^A, \partial_2 X^A) + \frac{1}{12}\text{Tr}([X^A, X^B, X^C], [X^A, X^B, X^C]). \tag{5.36}$$

Following [140], we write the potential as

$$V(X) = \frac{1}{2}\text{Tr}\left(\frac{\partial W}{\partial X^A}, \frac{\partial W}{\partial X^A}\right) \tag{5.37}$$

where

$$W = \frac{1}{24}\varepsilon^{ABCD}\text{Tr}(X^A, [X^B, X^C, X^D]). \tag{5.38}$$

Thus

$$\begin{aligned}\mathcal{H} &= \frac{1}{2}\text{Tr}\left(\partial_2 X^A + \frac{\partial W}{\partial X^A}, \partial_2 X^A + \frac{\partial W}{\partial X^A}\right) - \text{Tr}\left(\partial_2 X^A, \frac{\partial W}{\partial X^A}\right) \\ &= \frac{1}{2}\text{Tr}\left(\partial_2 X^A + \frac{\partial W}{\partial X^A}, \partial_2 X^A + \frac{\partial W}{\partial X^A}\right) + \frac{\mathcal{Z}^1}{2}\end{aligned} \tag{5.39}$$

where $\mathcal{Z}^1$ is the density of $Z_\mu^{(pq)}$, the 1-form central charge. Thus, for this field configuration $\mathcal{H} \geq \frac{\mathcal{Z}^1}{2}$ and the bound is saturated when

$$\frac{dX^A}{d\sigma^2} - \frac{1}{6}\varepsilon^{BCDA}[X^B, X^C, X^D] = 0. \tag{5.40}$$

When the (5.40) is satisfied, it follows from (5.35) that the field configuration is half-BPS and the preserved supersymmetries satisfy $\Gamma^2\epsilon = \Gamma^{3456}\epsilon$. This is the configuration proposed by Basu and Harvey as the M2-brane worldvolume soliton describing the branes system (5.34). In this section we have verified that the central charge associated to this state is the 1-form central charge, i.e. the central charge associated to the M2-M5 intersection.

5.4. Supplementary material for chapter 5

Notation

We summarize here our notation. The indices are

$$
\begin{aligned}
\text{worldvolume coordinates}: &\quad \mu,\nu = 0,1,2 \\
\text{spatial worldvolume coordinates}: &\quad i,j = 1,2 \\
\text{transverse space coordinates}: &\quad I,J = 3,\dots 10 \\
Spin(1,10)\ \text{spinorial indices}: &\quad \alpha,\beta = 1,\dots 32 \\
Spin(1,2)\ \text{spinorial indices}: &\quad \hat{\alpha},\hat{\beta} = 1,2 \\
Spin(8)\ \text{chiral spinorial indices}: &\quad p,q,\dot{p},\dot{q} = 1,\dots 8 \\
\mathcal{A}\ \text{algebra indices}: &\quad a,b = 1,\dots,\text{dim}\mathcal{A}
\end{aligned}
\tag{5.41}
$$

The Dirac matrices Γ are a representation of the 11-dimensional Clifford algebra, i.e. given $m,n = 0,\dots,10$ it results

$$\{\Gamma^m,\Gamma^n\} = 2\eta^{mn} \tag{5.42}$$

and

$$C^T = -C \qquad \Gamma_m^T = -C\Gamma_m C^{-1}. \tag{5.43}$$

We take Γ_m to be real matrices and $C = \Gamma^0$. The 11-dimensional spinors are Majorana (real) spinors with definite chirality respect to Γ_{012}. Thus, they have 16 independent real components.

Supercurrent Conservation

We now show that the supercurrent (5.3) is conserved. An easy computation gives

$$
\begin{aligned}
\partial_\mu J^\mu = &-\partial_\mu(D_\nu X_a^I)\Gamma^\nu\Gamma^I\Gamma^\mu\Psi^a - D_\nu X_a^I\Gamma^\nu\Gamma^I\Gamma^\mu\partial_\mu\Psi^a \\
&-\frac{1}{2}\partial_\mu X_a^I X_b^J X_c^K f^{abcd}\Gamma^{IJK}\Gamma^\mu\Psi_d \\
&-\frac{1}{6}X_a^I X_b^J X_c^K f^{abcd}\Gamma^{IJK}\Gamma^\mu\partial_\mu\Psi_d.
\end{aligned}
\tag{5.44}
$$

Using the fundamental identity (4.6) the previous equation can be rewritten as

$$
\begin{aligned}
\partial_\mu J^\mu = &-(D_\mu D_\nu X_a^I)\Gamma^\nu\Gamma^I\Gamma^\mu\Psi^a - D_\nu X_a^I\Gamma^\nu\Gamma^I\Gamma^\mu D_\mu\Psi^a \\
&-\frac{1}{2}D_\mu X_a^I X_b^J X_c^K f^{abcd}\Gamma^{IJK}\Gamma^\mu\Psi_d \\
&-\frac{1}{6}X_a^I X_b^J X_c^K f^{abcd}\Gamma^{IJK}\Gamma^\mu D_\mu\Psi_d.
\end{aligned}
\tag{5.45}
$$

Inserting the equations of motion (4.13) and using the identity

$$\bar{\Psi}_c\Gamma^{IJ}\Psi_b\Gamma^I\Psi_a X^J_d f^{cdba} = -\bar{\Psi}_c\Gamma_\mu\Psi_b\Gamma^\mu\Gamma^J\Psi_a X^J_d f^{cdba}, \tag{5.46}$$

the right hand side of the (5.45) results to be equal to zero.

Supersymmetry Variation of J^0

In this appendix we compute the supersymmetry variation of J^0, the zeroth component of the supercurrent (5.3). Considering the ansatz $\Psi = 0$ we get

$$\begin{aligned}\delta_\epsilon J^0 = &- D_\mu X^I_a D_\nu X^{aJ}\Gamma^\mu\Gamma^I\Gamma^0\Gamma^\nu\Gamma^J\epsilon + \frac{1}{6}D_\mu X^I_a X^J_b X^K_c X^M_d f^{bcda}\Gamma^\mu\Gamma^I\Gamma^0\Gamma^{JKM}\epsilon\\ &- \frac{1}{6}D_\mu X^I_a X^J_b X^K_c X^M_d f^{bcda}\Gamma^{JKM}\Gamma^0\Gamma^\mu\Gamma^I\epsilon\\ &+ \frac{1}{36}X^I_a X^J_b X^K_c X^L_e X^M_f X^N_g f^{abcd} f^{efg}{}_d\Gamma^{IJK}\Gamma^0\Gamma^{LMN}\epsilon.\end{aligned} \tag{5.47}$$

We note that the right hand side of (5.47) contains one term with two covariant derivatives, two terms with one covariant derivative and one term without covariant derivatives. Let's look first at the term with two covariant derivatives. Using the identity

$$\begin{aligned}-\Gamma^\mu\Gamma^I\Gamma^0\Gamma^\nu\Gamma^J =&\Gamma^0\Gamma^{\mu\nu}\Gamma^{IJ} + \Gamma^0\Gamma^{\mu\nu}\delta^{IJ} + \Gamma^0\eta^{\mu\nu}\Gamma^{IJ}\\ &+ \Gamma^0\eta^{\mu\nu}\delta^{IJ} - 2\eta^{\mu 0}\Gamma^\nu\delta^{IJ} - 2\eta^{\mu 0}\Gamma^\nu\Gamma^{IJ}\end{aligned} \tag{5.48}$$

we have

$$\begin{aligned}-D_\mu X^I_a D_\nu X^{aJ}\Gamma^\mu\Gamma^I\Gamma^0\Gamma^\nu\Gamma^J\epsilon =&(D_0 X^I_a D_0 X^{aI} + D_i X^I_a D_i X^{aI})\Gamma^0\epsilon + 2D_0 X^I_a D_i X^{aI}\Gamma^i\epsilon\\ &+ D_i X^I_a D_j X^{aJ}\Gamma^0\Gamma^{ij}\Gamma^{IJ}\epsilon.\end{aligned} \tag{5.49}$$

The two terms with one covariant derivative can be rearranged using the identity

$$-\Gamma^\mu\Gamma^0\Gamma^I\Gamma^{JKM} - \Gamma^0\Gamma^\mu\Gamma^{JKM}\Gamma^I = 2\eta^{\mu i}\Gamma^0\Gamma^i\Gamma^{IJKM} - 6\eta^{\mu 0}\delta^{I[J}\Gamma^{KM]} \tag{5.50}$$

and the last of the equations of motion (4.13). Thus we get

$$\begin{aligned}&+\frac{1}{6}D_\mu X^I_a X^J_b X^K_c X^M_d f^{bcda}\Gamma^\mu\Gamma^I\Gamma^0\Gamma^{JKM}\epsilon - \frac{1}{6}D_\mu X^I_a X^J_b X^K_c X^M_d f^{bcda}\Gamma^{JKM}\Gamma^0\Gamma^\mu\Gamma^I\epsilon =\\ &+\frac{1}{3}D_i X^I_a X^J_b X^K_c X^M_d f^{bcda}\Gamma^0\Gamma^i\Gamma^{IJKM}\epsilon + \frac{1}{2}\varepsilon_{ij}\tilde{F}^{ijcd}X^I_c X^J_d\Gamma^{IJ}\epsilon.\end{aligned} \tag{5.51}$$

Using the fundamental identity (4.6) one can show that

$$\tilde{A}_i{}^g{}_a X^I_g X^J_b X^K_c X^M_d f^{bcda} \Gamma^{IJKM} = 0 \tag{5.52}$$

thus the (5.51) can be rewritten as

$$\begin{aligned}
&+\frac{1}{6} D_\mu X^I_a X^J_b X^K_c X^M_d f^{bcda} \Gamma^\mu \Gamma^I \Gamma^0 \Gamma^{JKM} \epsilon - \frac{1}{6} D_\mu X^I_a X^J_b X^K_c X^M_d f^{bcda} \Gamma^{JKM} \Gamma^0 \Gamma^\mu \Gamma^I \epsilon = \\
&+\frac{1}{12} \partial_i (X^I_a X^J_b X^K_c X^M_d f^{bcda} \Gamma^0 \Gamma^i \Gamma^{IJKM} \epsilon) + \frac{1}{2} \varepsilon_{ij} \tilde{F}^{ijcd} X^I_c X^J_d \Gamma^{IJ} \epsilon.
\end{aligned} \tag{5.53}$$

The term without covariant derivatives can be simplified using the expression

$$\Gamma^{IJK} \Gamma^{LMN} = \Gamma^{IJKLMN} + 9\Gamma^{[IJ}{}_{[MN} \delta^{K]}_{L]} + 18 \Gamma^{[I}{}_{[N} \delta^K_L \delta^{J]}_{M]} + 6 \delta^{[I}_{[N} \delta^J_M \delta^{K]}_{L]} \tag{5.54}$$

and the property of the f^{abcd} structure constants. We get

$$\begin{aligned}
&\frac{1}{36} X^I_a X^J_b X^K_c X^L_e X^M_f X^N_g f^{abcd} f^{efg}{}_d \Gamma^{IJK} \Gamma^0 \Gamma^{LMN} \epsilon = \\
&\frac{1}{6} \Gamma^0 \epsilon X^I_a X^J_b X^K_c X^I_e X^J_f X^K_g f^{abcd} f^{efg}{}_d = 2\Gamma^0 \epsilon V
\end{aligned} \tag{5.55}$$

where V is the potential defined in (4.9).

Collecting all the pieces together we have

$$\begin{aligned}
\delta_\epsilon J^0 = {} & (D_0 X^I_a D_0 X^{aI} + D_i X^I_a D_i X^{aI} + 2V) \Gamma^0 \epsilon + 2 D_0 X^I_a D_i X^{aI} \Gamma^i \epsilon + \\
& - \partial_i (X^I_a D_j X^{aJ} \varepsilon^{ij} \Gamma^{IJ} \epsilon) + \frac{1}{12} \partial_i (X^I_a X^J_b X^K_c X^M_d f^{bcda} \Gamma^0 \Gamma^i \Gamma^{IJKM} \epsilon)
\end{aligned} \tag{5.56}$$

Considering the ansatz $\Psi = 0$, the components of the stress-energy tensor (4.15) are

$$\begin{aligned}
T_{00} = {} & \frac{1}{2} D_0 X^I_a D_0 X^{aI} + \frac{1}{2} D_i X^I_a D_i X^{aI} + V \\
T_{0i} = {} & D_0 X^I_a D_i X^{aI}
\end{aligned} \tag{5.57}$$

Using the (5.57) and the identity $\Gamma^0 \Gamma^i = -\epsilon^{ij} \Gamma^j \Gamma_{012}$ the (5.56) can be rewritten as

$$\delta_\epsilon J^0 = -2 T^0{}_\mu \Gamma^\mu \epsilon - \partial_i (X^I_a D_j X^{aJ} \varepsilon^{ij} \Gamma^{IJ} \epsilon) - \frac{1}{12} \partial_i (X^I_a X^J_b X^K_c X^M_d f^{bcda} \varepsilon^{0i}{}_\mu \Gamma^{IJKM} \Gamma^\mu \epsilon). \tag{5.58}$$

6. Conclusions

During the past ten years, advances in string theory have revealed interesting interconnections between string models and ordinary gauge theories. The paradigm of these dualities is that some closed string theory on a certain background is associated to a particular gauge theory, defined on the boundary of the spacetime where the string theory lives. Since these relations are strong-weak dualities, they provide significant theoretical insights in the study of string theory, gauge theory and gravity and represents a promising new technique to analyze many aspects of phenomenological physics that cannot be described with a more traditional approach.

This thesis presents studies on different aspects of the duality. In the first chapters, we have analyzed the holographic description of non-local operators defined in supersymmetric gauge theories. In the last two chapters, we have considered few aspects of the Bagger-Lambert multiple M2-branes theory. This theory provides the holographic dual of M-theory on a certain background.

In chapter 2 we have focused our attention on half-BPS Wilson loops in $\mathcal{N} = 4$ SYM. We have shown that the Wilson loops in a higher representation of the gauge group correspond to D-branes on the string theory side [76][190]. This analysis, generalizes the proposal in [25][26] where Wilson loops in the fundamental representation are associated to fundamental strings on the string theory side. It follows that operators in the symmetric representation of the gauge group are associated to D3-branes and operators in the antisymmetric representations are associated to D5-branes. A Wilson loop in a generic representation can be realized either as a stack of D3-branes, or equivalently as a stack of D5-branes. We have proved this correspondence considering flat space D-brane systems that besides the stack of N D3-branes giving rise to the standard AdS/CFT, includes also some extra D3-branes or D5-branes. In the near horizon, when the backreaction can be neglected, the extra D-branes become probes in $\mathrm{AdS}_5\times\mathrm{S}^5$. On the gauge theory side, the extra D-branes introduce degrees of freedom localized on a one dimensional subspace of the spacetime. The SYM theory thus is deformed with a defect field theory leaving on the spacetime curve where the non-local operator is defined. Integrating out the physics associated to this extra D-branes, introduces in the theory Wilson loops in a representation that is encoded by the characteristic of the branes.

This procedure can be extended to other D-brane systems to obtain a correspondence between non-perturbative objects in string theory and operators in the gauge theory side. For instance, it would be very interesting to prove that the giant gravitons [191] in string theory correspond to determinant operators in the gauge theory side as proposed few years ago [192][78].

The intersecting systems we have analyzed to study the Wilson loops in higher order representations present an interesting feature. The defect field theories associated to the extra D3-branes and D5-branes are exactly the same, except that the fields on the defect have different statistic (bosonic for the D3 case, fermionic for the D5 case). Thus, we can switch to an intersecting D-brane model to the other just switching the statistics of the defect fields. It would be interesting to see if there is any first principles explanation of this fact.

One of outstanding issues in the gauge-gravity duality is to exhibit the origin of the loop equation of gauge theory in the gravitational description. This important problem has thus far remained elusive. Having shown that Wilson loops are more naturally described in the bulk by D-branes instead of by fundamental strings, it is natural to search for the origin of the loop equation of gauge theory in the D-brane picture instead of the fundamental string picture.

The description of Wilson loops as defect field theories studied in chapter 2, has been extended to a particular class of surface operators in chapter 3 [193]. In particular we have studied a supersymmetric D3-D7 intersection, where the D7-branes intersect the stack of D3-branes along two spacetime coordinates. An interesting result of this analysis is that in the low energy field theory description, the gauge theory is defined in a curved spacetime. This means that the backreaction of the D7-branes cannot be neglected, not even in the decoupling limit. In this way we have enlarged the holographic duality to gauge theories defined in a non-trivial background. It would be interesting to explore in more detail the dictionary relating bulk and gauge theory computations for this new class of dualities.

Integrating out the physics on the defect field theory associated to the D7-branes, we find the explicit form of this class of surface operators, which is given by a WZW model supported on the surface. Thus, differently from the previously studied surface operators, the operators described in chapter 3 are expressed in terms of the fields in the gauge theory, i.e. they are order-type operators.

An important problem for the future is to understand the physics encoded in the expectation value of surface operators and to determine whether they can be useful probes of new phases of gauge theory. For the surface operators in this thesis it would be interesting to compute their expectation value in perturbation theory. Given that these operators are supersymmetric it is conceivable that the computation of their expectation value can be performed in a reduced model, just like the expectation value of supersymetric circular Wilson loops can be computed by a matrix integral [113][114]. Also, it would be interesting to compute the expectation value of the surface operator by calculating the on-shell action of the corresponding supergravity solutions.

In the rest of the thesis we have studied multiple M2-branes low energy theories. In Chapter 4 [145] we have constructed a one parameter mass deformation of the Bagger-Lambert Lagrangian that preserves all the supersymmetries. This model represents a novel example of a maximally supersymmetric 3-dimensional gauge theory. We have shown that when it is compactified on $R \times T^2$, the theory presents the superisometries of the Type IIB pp-wave. Given the M2-branes interpretation of the Bagger-Lambert theory, the compactified theory can be thought as the Matrix theory for strings on Type IIB pp-wave. It would be interesting to study a similar deformation for the $\mathcal{N} = 6$ Bagger-Lambert theories introduced in [35]. It would be nice to discuss their possible connection to the Matrix theories.

The Bagger-Lambert Lagrangian possesses another nice feature. It realizes the full M2-brane superalgebra, including also two central charges related to higher dimensional objects (chapter 5 [194]). These charges are associated to the intersections between the M2-branes and other M-branes and they should be realized by a Lagrangian describing the low energy physics of M2-branes. We have performed this analysis considering the formulation of the Bagger-Lambert theory that is maximally supersymmetric ($\mathcal{N} = 8$), i.e. considering fields valued in a 3-algebra with an antisymmetric 3-product. In the future, it would be interesting to extend this analysis to the $\mathcal{N} = 6$ formulation of the Bagger-Lambert theory [35]. This is based on a 3-algebra with a 3-product that is not antisymmetric and for a particular 3-algebra it reduces to the ABJM theory [36]. The analysis of the superalgebra can be useful to consolidate the M-theory interpretation of the ABJM theory and can shed light on the physical meaning of the other $\mathcal{N} = 6$ theories.

Given a theory of multiple M2-branes, it is possible to perform a dimensional reduction along one of the direction parallel to the M2-branes to get an action for multiple fundamental strings. In the future it would be interesting to apply this procedure to the candidate multiple M2-branes theories. It would be very interesting to understand how the multiple strings couple with the RR-fields in a non-trivial background. The expectation is that there will be Myers-like terms, because as we have seen in chapter 2, there is some evidence that coincident fundamental strings in the presence of an RR-field, polarize into higher dimensional D-branes.

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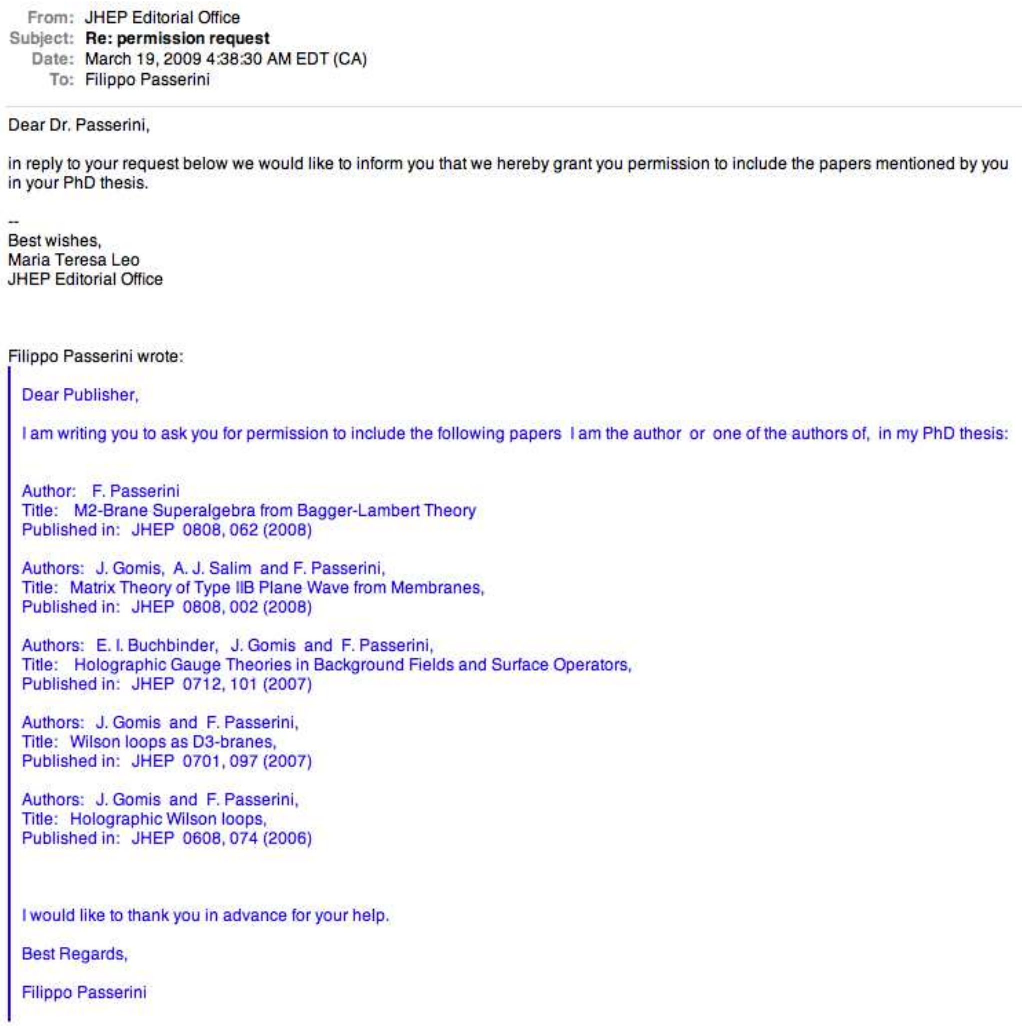

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