

Combination of Levene-Type Tests and
a Finite-Intersection Method for Testing
Trends in Variances

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Kimihiro Noguchi

Abstract

The problem of detecting monotonic increasing/decreasing trends in variances from k samples is widely met in many applications, e.g. financial data analysis, medical and environmental studies. However, most of the tests for equality of variances against ordered alternatives rely on the assumption of normality. Such tests are often non-robust to departures from normality, which eventually leads to unreliable conclusions. In this thesis, we propose a combination of a robust Levene-type test and a finite-intersection method, which relaxes the assumption of normality. The new combined procedure yields a more accurate estimate of sizes of the test and provides competitive powers. In addition, we discuss various modifications of the proposed test for unbalanced design cases. We present theoretical justifications of the new test and illustrate its applications by simulations and case studies.

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Chapter 1

Introduction

1.1 Thesis Introduction

A popular question that arises in studies with k samples is whether each sample has a common variance σ^2 . In many medical or biological studies, samples exhibit unequal variances, which need to be taken into account for model construction. There are many possible tests for equality of variances (Conover et al., 1981), among which the most notable ones are Bartlett's (1937) and Levene's (1960) test. Bartlett's test is designed for normal samples and is sensitive to departures from normality while Levene's test with the Brown-Forsythe (1974) modification is robust with respect to a wide range of distributions, both normal and non-normal (skewed or heavy-tailed) (Conover et al., 1981; Lim and Loh, 1996). In practice, Levene's test is recommended since samples often show non-normality. Furthermore, samples typically indicate a monotonic increasing or decreasing trend in variances after some natural ordering according to certain criteria. For example, in environmental studies, variability of pollutant concentration can be ordered depending on the proximity to the sources of pollution. In microeconomics, variability in real estate prices can be ordered spatially or according to the size. Detection of such a trend provides more useful information to researchers.

Precursory studies for developing tests for equality of variances against ordered alternatives employed the order-restricted maximum likelihood estimates of the variances applied to the classical tests for equality of variances such as Bartlett's test (Boswell and Brunk,

1969; Fujino, 1979), or based on a special regression type of assumptions (Vincent, 1961; Fujino, 1979). Such tests typically have normality assumptions, and are not robust with respect to the departures from normality. Mudholkar et al. (1993) consider a combination of the F -tests with a finite-intersection method derived from the idea of nested hypotheses suggested by Hogg (1961). In Hogg's (1961) procedure, the original null and alternative hypotheses are decomposed into a series of finite component hypotheses. The F -tests are applied at each component hypothesis, and the independent p -values from these tests are combined using a p -value combining method to calculate the p -value of the test. Such an approach is referred to as a finite-intersection method (Mudholkar et al., 1995). Mudholkar et al. (1993) show that the test based on the finite-intersection method is more powerful compared to the previously suggested tests.

To relax the normality assumptions, tests based on robust measures of scale are studied in contrast to tests based on classical variance estimates. Mudholkar et al. (1995) use Miller's (1968) pseudovalues W_{ij} , a type of measure of spread for the j th observation in the i th sample, in place of the random variables X_{ij} , $i = 1, \dots, k$, $j = 1, \dots, n_i$ to construct such tests. Neuhauser and Hothorn (2000) replace X_{ij} with Levene's transformation $Z_{ij} = |X_{ij} - \bar{X}_i|$, where \bar{X}_i is the sample mean of the i th sample, and applied it to Bartholomew's (1961) test and the double contrast test due to Bechhofer and Dunnett (1982). Hines and Hines (2000) consider preselecting contrasts which are thought to reflect directions in which the null hypothesis might fail for an ANOVA. Hines and Hines (2000) apply the idea to $Z_{50ij} = |X_{ij} - \tilde{X}_{50i}|$, the Brown-Forsythe (1974) modification (absolute deviation from its sample median), to test for a linear or quadratic trend of variances. Gastwirth et al. (2009) study a test using a simple linear regression on Z_{ij} to construct a test for ordered variances. Tests based on Levene's transformation (Levene-type tests) are generally robust, having an approximately correct size and competitive power for normal, heavy-tailed, and skewed distributions.

In this thesis, we propose a new test for equality of variances against ordered alternatives based on a combination of the two-sample t -tests and the finite-intersection method with Levene's transformation Z_{ij} . Levene's transformation robustifies the test for non-normal samples while the application of the finite-intersection method increases the power of the test. We present asymptotic normality and independence of the test statistics as

our reasoning for using nested hypotheses. Also, several modifications of the test using O'Brien's (1978) correction factor and the Hines-Hines (2000) structural zero removal method are also considered. Then, we discuss the Monte Carlo simulation study results on the size and power of the new test and its modifications. In the study, size and power are estimated under different distributions (normal, heavy-tailed, skewed), equal and unequal sample sizes, and different number of samples. The size and power are compared to those of the existing tests to show the robustness and competitiveness of the new test. Finally, we illustrate applications of the new test with real-life examples from microeconomics, engineering, environmental studies, and health studies.

1.2 Main Contributions

The main contributions of this thesis include the following:

1. development of a new powerful distribution-free test against trends in variances by combining the finite-intersection method from the nested hypotheses and Levene's transformation Z_{ij} , which can be seen as an extension of previous tests suggested by Mudholkar et al. (1995) and Gastwirth et al. (2009).
2. investigation of the asymptotic properties (asymptotic normality and independence) of the component test statistics of the new test
3. derivation of new robustification methods of the Levene-type tests for unbalanced design by extending the results of the Hines-Hines (2000) structural zero removal method and O'Brien's (1978) correction factor
4. simulation study on the size and power of the tests against trends in variances to justify the robustness and competitive power of the new test under normal, skewed, and heavy-tailed distributions and for both small and unequal sample sizes

1.3 Thesis Outline

The thesis is organized as follows. In Chapter 2, we review a brief history and concept of nested hypotheses. Nested hypotheses are linked to the idea of a finite-intersection method, which allows different ways of combining p -values from the component test statistics. Also, we describe four p -value combining techniques for the finite-intersection method.

In Chapter 3, we discuss several tests for detecting trends in variances. The tests are grouped according to the underlying concepts: 1. order-constrained maximum likelihood estimates of variances; 2. nested hypotheses; and 3. regression of variance estimates on the preselected scores for each sample.

In Chapter 4, we introduce the new test, which combines the idea of nested hypotheses and Levene's transformation. We discuss the formulation of the component test statistics and the asymptotic properties, in particular, asymptotic normality and independence. We use these asymptotic properties as a justification for using the finite-intersection method.

In Chapter 5, we propose modifications of the new test to unbalanced designs, in which the sample sizes differ. Specifically, we introduce a new correction factor for an unbalanced design based on the combination of Hines-Hines (2000) structural zero removal method and O'Brien's (1978) correction factor. We provide reasoning for the formulation of the new correction factor.

In Chapter 6, we present a Monte Carlo simulation study to check the robustness and power of the new test against previously proposed tests. The simulation study covers a wide range of distributions (normal, skewed, and heavy-tailed), different sample sizes (both equal and unequal), and different number of samples. We show that the new test with the new correction factor is robust and frequently most powerful based on the simulation study.

In Chapter 7, we discuss the applications of the tests against ordered alternatives. The real-life examples are taken from diverse disciplines, real estate prices, engineering, environmental and health studies. We show that the new test is widely applicable for both normal and non-normal distributions, and unbalanced designs.

We summarize the main results and an outline of the future work in Chapter 8.

Chapter 2

History of Nested Hypotheses

An intuitive approach to solve a complex problem is to “divide each of the difficulties under examination into as many parts as possible” as Descartes stated in *Discourse on the Method* (1637). The concept of nested hypotheses, first suggested in a formal manner by Hogg (1961), follows the philosophy of Descartes by breaking down the original hypothesis H_0 into a finite number of more manageable components.

The concept of nested hypotheses is described as follows. Let Ω be a nonempty parameter subspace of the parameter θ , and let ω be a subset of Ω . Suppose that we wish to test

$$H_0 : \theta \in \omega \text{ against } H_a : \theta \in \Omega - \omega. \quad (2.1)$$

Let us denote the nonempty subspaces of Ω by $\omega_1, \omega_2, \dots, \omega_k$ with the ordering

$$\Omega = \omega_1 \supset \omega_2 \supset \omega_3 \supset \dots \supset \omega_k = \omega. \quad (2.2)$$

We then test the hypotheses

$$H_{0(i)} : \theta \in \omega_i \text{ against } H_{a(i)} : \theta \in \Omega - \omega_i, i = 2, \dots, k. \quad (2.3)$$

The hypotheses $H_{0(i)}$ and $H_{a(i)}$, $i = 2, \dots, k$, are called nested hypotheses.

Let S_i denote the test statistic for testing

$$H_{0(i)} : \theta \in \omega_i \text{ against } H_a : \theta \in \Omega - \omega_i. \quad (2.4)$$

Given the independence of S_2, \dots, S_k , the nested hypotheses are tested iteratively. A test statistic S_i rejecting $H_{0(i)}$ at a prespecified significance level α is regarded as sufficient evidence against H_0 . If all the test statistics $S_i, i = 2, \dots, k$ do not reject the null hypotheses $H_{0(i)}, i = 2, \dots, k$, then the p -value of the test, denoted by P , is calculated as

$$P = 1 - \prod_{i=2}^k (1 - P_i), \quad (2.5)$$

where P_i is the p -value for S_i .

The concept of nested hypotheses can be seen as an application of the union-intersection principle first introduced by Roy (1953) and discussed in detail by Olkin and Tomsy (1981). The union-intersection principle is a heuristic method of test construction. Let $\{\omega_r, \omega_r^*, r \in \Gamma\}$ be a collection of sets in the parameter space of the parameter θ where Γ is an arbitrary index set. Assume that $\bigcap_{r \in \Gamma} \omega_r$ is nonempty. Define component hypotheses $H_{0(r)}$ and $H_{a(r)}$ as $H_{0(r)} : \theta \in \omega_r$ and $H_{a(r)} : \theta \in \omega_r^*$. The original null hypothesis H_0 is accepted if and only if each of the component hypotheses $H_{0(r)}, r \in \Gamma$ is accepted. If at least one of the $H_{0(r)}$ is rejected, then H_0 is rejected.

McDermott and Mudholkar (1993) consider different ways of combining tests from such component hypotheses for finite cases. In particular, McDermott and Mudholkar (1993) consider four different ways of combining independent p -values $P_i, i = 2, \dots, k$ which correspond to the test statistics $S_i, i = 2, \dots, k$ arising from nested hypotheses. The four p -value combining methods are summarized below:

1. Tippett's (1931) combination

$$\Psi_T = \min(P_2, \dots, P_k) \quad (2.6)$$

which follows a $\min(U_2, \dots, U_k)$ distribution under H_0 , where U_2, \dots, U_k are independent and $U_i \sim U(0, 1), i = 2, \dots, k$,

2. Fisher's (1932) combination

$$\Psi_F = -2 \sum_{i=2}^k \log P_i \quad (2.7)$$

which follows a χ_{2k-2}^2 distribution under H_0 ,

3. Liptak's (1958) combination

$$\Psi_N = \sum_{i=2}^k \Phi^{-1}(1 - P_i) \quad (2.8)$$

which follows a $N(0, k - 1)$ distribution under H_0 ,

4. Mudholkar-George (1979) combination

$$\Psi_L = - \left[\frac{\pi^2(k-1)(5k-3)}{3(5k-1)} \right]^{-1/2} \sum_{i=2}^k \log \left(\frac{P_i}{1 - P_i} \right) \quad (2.9)$$

which has a distribution that is well approximated by a Student's t distribution with $5k - 1$ degrees of freedom under H_0 .

Small values of Ψ_T and large values of Ψ_F , Ψ_N , and Ψ_L are seen as evidence against H_0 . Mudholkar et al. (1995) call this approach a “finite-intersection approach” in a broader sense allowing for different ways of combining tests from such finite component hypotheses.

Nested hypotheses have been applied for testing different parameters of interest:

1. equality of means and variances from k independent normal distributions against ordered alternatives based on a sequence of independent two-sample t -tests and F -tests (Hogg, 1962; Mudholkar et al., 1993; McDermott and Mudholkar, 1993),
2. equality of k independent distributions using a sequence of independent two-sample distribution-free tests (Hogg, 1962),
3. equality of the scale and location parameters of k independent exponential distributions using a two-sample likelihood ratio test considered by Epstein and Tsao (1953) (Hogg, 1963),
4. equality of variances from k independent distributions based on Miller's (1968) pseudovalues using two-sample t -tests (Mudholkar et al., 1995),
5. equality of scale-like inverse Gaussian parameters from k independent distributions against ordered alternatives using F -tests (Natarajan et al., 2005).

In practice, the null hypothesis H_0 and its nested hypotheses $H_{0(i)}$, $i = 2, \dots, k$ often take the form for a parameter of interest $\theta_i, i = 1, \dots, k$:

$$H_0 : \theta_1 = \theta_2 = \dots = \theta_k = \theta, \quad (2.10)$$

$$H_{0(i)} : [\theta_1, \theta_2, \dots, \theta_{i-1}] = \theta_i = \theta. \quad (2.11)$$

In (2.11), the parameters grouped in the brackets $[\theta_1, \theta_2, \dots, \theta_{i-1}]$ lack any order restrictions. The ordered alternatives in an increasing order and their nested hypotheses that Hogg (1962) and Mudholkar and McDermott (1993) consider in (1) are expressed as

$$H_a : \theta_1 \leq \theta_2 \leq \dots \leq \theta_k, \text{ with at least one strict inequality,} \quad (2.12)$$

$$H_{a(i)} : [\theta_1, \theta_2, \dots, \theta_{i-1}] < \theta_i. \quad (2.13)$$

Note the possibility of considering the ordered alternatives in a decreasing order. The formulation of such hypotheses is symmetric to the formulation of an increasing order. Mudholkar and McDermott (1989) also consider an extension of the ordered alternatives to more general cases of grouping the parameters.

Chapter 3

Review of Tests for Equality of Variances against Ordered Alternatives

3.1 Formulation of Null and Alternative Hypotheses

Let $\sigma_i^2, i = 1, 2, \dots, k$, be the population variances of k independent samples. We are interested in testing equality of variances against ordered alternatives (in an increasing order), which generally takes the form

$$H_0 : f(\sigma_1^2) = f(\sigma_2^2) = \dots = f(\sigma_k^2) = f(\sigma^2), \quad (3.1)$$

$$H_a : f(\sigma_1^2) \leq f(\sigma_2^2) \leq \dots \leq f(\sigma_k^2) \text{ with at least one strict inequality,} \quad (3.2)$$

where $f(x)$ is a strictly increasing smooth function for $x \in \mathfrak{R}^+$. The most popular choice is $f(x) = x$ in which case we compare the variances. When $f(x) = \sqrt{x}$ (Levene, 1960; Boos and Brownie, 1989), the hypotheses (3.1) and (3.2) correspond to testing a trend in standard deviation. Another popular choice, $f(x) = \log x$ is also widely used (Levene, 1960; Fujino, 1979; Mudholkar et al., 1995). The construction of H_a for a decreasing order is symmetric to (3.2).

3.2 Tests with Order-Constrained Maximum Likelihood Estimates

Early studies for developing tests for detecting trends in variances employ the idea of order-constrained maximum likelihood estimates of variances, $\hat{\sigma}_i^{*2}$ as proposed by Brunk (1955). Boswell and Brunk (1969) and Fujino (1979) propose replacing the unrestricted maximum likelihood estimates of variances, $\hat{\sigma}_i^2, i = 1, 2, \dots, k$, with $\hat{\sigma}_i^{*2}$. Fujino (1979) concludes that, among the tests which use $\hat{\sigma}_i^{*2}$ for normal samples with equal sample sizes n , the test based on the modified Bartlett's statistic

$$M^* = k(n-1) \log \hat{\sigma}^2 - (n-1) \sum_{i=1}^k \log \hat{\sigma}_i^{*2}, \quad (3.3)$$

where $\hat{\sigma}^2$ is the maximum likelihood estimate of σ^2 under H_0 , has the best overall performance.

3.3 Tests with the Finite-Intersection Method

Mudholkar et al. (1993) consider a combination of the F -test and the finite-intersection method for k normal populations to test the hypotheses

$$H_{0(i)} : [\sigma_1, \sigma_2, \dots, \sigma_{i-1}] = \sigma_i = \sigma^2, \quad (3.4)$$

$$H_{a(i)} : [\sigma_1, \sigma_2, \dots, \sigma_{i-1}] < \sigma_i, \quad (3.5)$$

for $i = 2, \dots, k$ where the parameters in the brackets $[\sigma_1, \sigma_2, \dots, \sigma_{i-1}]$ lack any order restrictions. The first $i-1$ samples whose corresponding parameters are in the brackets are combined and are treated as one sample. Let s_i^2 be the sample variance of the i th sample, $i = 1, \dots, k$. The proposed statistic

$$F_i = s_i^2 / s_{(i-1)}^2, \quad (3.6)$$

where

$$s_{(i-1)}^2 = \frac{\sum_{j=1}^{i-1} \nu_j s_j^2}{\sum_{j=1}^{i-1} \nu_j},$$

$$\nu_{(i-1)} = \sum_{j=1}^{i-1} \nu_j, \quad i = 2, \dots, k, \quad \nu_i = n_i - 1,$$

follows an F distribution with ν_i and $\nu_{(i-1)}$ degrees of freedom. The $k - 1$ such statistics from each nested hypothesis are combined with the four p -value combining methods, Ψ_T, Ψ_F, Ψ_N , and Ψ_L ((2.6),(2.7), (2.8), and (2.9)). Mudholkar et al. (1993) show that the test with Ψ_F is found to be superior to both the tests with the other p -value combination methods and the test based on Bartlett's (1937) statistic M^* based on the simulation study.

To robustify the test statistic (3.6), Mudholkar et al. (1995) extend the earlier approach in McDermott and Mudholkar (1993) based on Miller's (1968) pseudovalues $W'_{ij} = n_i \log s_i^2 - (n_i - 1) \log s_{i(-j)}^2$. Here $s_{i(-j)}^2$ is the sample variance of the i th sample with the j th observation omitted, $i = 1, \dots, k, j = 1, \dots, n_i$. The test statistic

$$V_i = Y_i \left[\left\{ (N - k)Q^2 + \sum_{j=1}^{i-1} Y_j^2 \right\} / (N - k + i - 1) \right]^{-1/2}, \quad (3.7)$$

where

$$Y_i = \frac{\left(\sum_{j=1}^i n_j \right) \bar{W}'_{i+1} - \sum_{j=1}^i n_j \bar{W}'_j}{\left\{ n_{i+1}^{-1} \left(\sum_{j=1}^i n_j \right) \left(\sum_{j=1}^{i+1} n_j \right) \right\}^{1/2}}, \quad \bar{W}'_i = \frac{\sum_{m=1}^i W'_m}{N},$$

$$Q^2 = \sum_{i=1}^k \sum_{j=1}^{n_i} \frac{(W'_{ij} - \bar{W}'_i)^2}{N - k}, \quad N = \sum_{j=1}^k n_j, \quad i = 1, \dots, k - 1,$$

has a distribution approximated by Student's t distribution with $N - k + i - 1$ degrees of freedom (Mudholkar et al., 1995). When $i = 1$, $\sum_{j=1}^{i-1} Y_j^2 = 0$. Mudholkar et al. (1995) conclude that the test is satisfactorily robust to samples from nonnormal symmetric distributions with moderate degree of heavy-tailedness such as Student's t distribution with 6 degrees of freedom.

3.4 Tests of Regression Type

Vincent (1961) proposes a test statistic for ordered alternatives in variances whose numerator may be viewed as an estimator of a regression slope of σ_i^2 on the linear scores i . Fujino (1979) discusses a number of tests of trends in variances (see, for example, a modified Bartlett's statistic M^* (3.3)). Among the proposed procedures is an extension of Vincent's (1961) statistic, which is based on a log transformation of σ^2 . All the aforementioned tests assume that the samples are normally distributed.

Levene (1960) proposes applications of the transformed data $Z_{ij} = |X_{ij} - \bar{X}_i|$ in place of the original observations X_{ij} for developing a procedure for testing equality of variances. The intuition behind the idea is that Z_{ij} can be viewed as an L_1 -estimator of spread subject to a constant that depends on the underlying distribution of X_{ij} . Levene's (1960) approach is shown to be robust under a broad class of distributions (Lim and Loh, 1996), and this idea eventually became very popular nowadays in many disciplines, leading to a whole family of what is so-called Levene-type tests (Gastwirth et al., 2009). Brown and Forsythe (1974) replaces the sample mean with the sample median, \tilde{X}_{50i} , to consider the Brown-Forsythe modification $Z_{50ij} = |X_{ij} - \tilde{X}_{50i}|$, which is shown to make Levene's (1960) test more robust (Conover et al, 1981, Lim and Loh, 1996).

Hines and Hines (2000) and Gastwirth et al. (2009) suggest a Levene-type test statistic for testing a monotonic trend in variances by regressing Z_{ij} or Z_{50ij} on some scores ρ_i , $i = 1, \dots, k$. The test statistic takes the form

$$\beta = \frac{\sum_{i=1}^k n_i (\rho_i - \bar{\rho})(D_i - D)}{\sum_{i=1}^k n_i (\rho_i - \bar{\rho})^2}, \quad (3.8)$$

where

$$\bar{\rho} = N^{-1} \sum_{i=1}^k n_i \rho_i, \quad D_i = n_i^{-1} \sum_{m=1}^{n_i} Z_{im}, \quad D = N^{-1} \sum_{i=1}^k n_i D_i,$$

and $\rho_1 < \rho_2 < \dots < \rho_k$ are scores assigned to each sample such as $\rho_i = i$, $\rho_i = \sqrt{i}$, or $\rho_i = i^2$, $i = 1, 2, \dots, k$. If the Brown-Forsythe (1974) transformation is used, $D_i = n_i^{-1} \sum_{m=1}^{n_i} Z_{im}$. The statistic β is asymptotically normally distributed with mean 0 and variance 1. In practice,

the critical values can be approximated by Student's t distribution with $N - 2$ degrees of freedom (Gastwirth et al., 2009). Robust modifications of the test statistic β can be constructed by utilizing various estimators of location, in particular, the median or the trimmed mean (Brown and Forsythe, 1974).

By employing order-constrained sample means of Z_{ij} , Neuhauser and Hothorn (2000) propose a test based on a modification of Bartholomew's (1961) test and the Bechhofer-Dunnett (1982) double contrast test. The double contrast test has the contrast vectors $\vec{a} = (a_1, a_2, \dots, a_k)$ and $\vec{\tilde{a}} = (\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_k)$ where

$$a_i = \begin{cases} -1 & i = 1, \\ -1 + \sum_{j=k-i+1}^{k-1} j^{-1} & i = 2, \dots, k-1, \\ \sum_{j=1}^{k-1} j^{-1} & i = k, \end{cases}$$

and $\tilde{a}_i = -a_{k-i}$ for all $i = 1, \dots, k$. The test statistic for the double contrast test based on Z_{ij} ,

$$\max \left(T_c(\vec{a}), T_c(\vec{\tilde{a}}) \right), \quad (3.9)$$

where

$$T_c(\vec{a}) = \frac{\sum_{i=1}^k a_i D_i}{s \sqrt{\sum_{i=1}^k a_i^2 / n_i}},$$

$$s = \sqrt{\frac{\sum_{i=1}^k \sum_{j=1}^{n_i} (Z_{ij} - D_i)}{N - k}}, \quad N = \sum_{i=1}^k n_i, \quad i = 1, 2, \dots, k,$$

approximately follows a bivariate t -distribution with $N - k$ degrees of freedom (Neuhauser and Hothorn, 2000). Both of the modified tests are shown to be equally competitive in terms of the sizes and powers for different variance patterns from normal samples based on the simulation study (Neuhauser and Hothorn, 2000).

Chapter 4

New Test for Equality of Variances against Ordered Alternatives

Let $\{X_{i1}, \dots, X_{in_i}, i = 1, \dots, k\}$ represent k independent samples, where in each sample random variables $X_{ij}, j = 1, \dots, n_i$, are independent and identically distributed (i.i.d.) with a continuous distribution having a finite second moment m_i . Let us denote the distribution function by $G_i(x_{ij}) = G((x_{ij} - \mu_i)/\sigma_i)$ where μ_i and σ_i represent the mean and standard deviation of the i th sample respectively. We focus on tests for equality of variances against ordered alternatives in the presence of unknown and possibly unequal location:

$$H_0 : \sigma_1 = \sigma_2 = \dots = \sigma_k = \sigma, \quad (4.1)$$

$$H_a : \sigma_1 \leq \sigma_2 \leq \dots \leq \sigma_k \text{ with at least one strict inequality,} \quad (4.2)$$

The absolute deviation of X_{ij} from the i th population mean μ_i and the sample mean \bar{X}_i are denoted by $Y_{ij} = |X_{ij} - \mu_i|$ and $Z_{ij} = |X_{ij} - \bar{X}_i|$ respectively. By the finite second moment assumption of G , under the assumptions of H_0 , we can write $E(Y_{ij}) = \tau$ and $\text{Var}(Y_{ij}) = \eta^2 < \infty$. Note that $\tau = d\sigma$ where

$$d = \frac{2(\mu_i G_i(\mu_i) - H_i(\mu_i))}{\sqrt{m_i - \mu_i^2}}, \quad H_i(x) = \int_{-\infty}^x t dG_i.$$

Further, assume that $n_i/N := \lambda_i \rightarrow c_i \in (0, 1)$, a nonzero constant less than 1, where $N = \sum_{i=1}^k n_i$. In addition, let us introduce the notations $n_{(i-1)} = \sum_{m=1}^{i-1} n_m$ and $\lambda_{(i-1)} = \sum_{m=1}^{i-1} \lambda_m$,

$i = 2, \dots, k$.

The approach taken in this thesis for constructing such tests relies on the $k - 1$ nested hypotheses $H_{0(i)}, i = 2, \dots, k$, which are subsets of the original hypothesis denoted as follows:

$$H_{0(i)} : [\sigma_1, \sigma_2, \dots, \sigma_{i-1}] = \sigma_i = \sigma, \quad (4.3)$$

$$H_{a(i)} : [\sigma_1, \sigma_2, \dots, \sigma_{i-1}] < \sigma_i. \quad (4.4)$$

Let $D_i = n_i^{-1} \sum_{j=1}^{n_i} Z_{ij}$ and $D_{(i-1)} = n_{(i-1)}^{-1} \sum_{m=1}^{i-1} n_m D_m$, $i = 2, \dots, k$. Using the idea of the usual two-sample t -test considered by Hogg (1962) in the cases of unequal sample sizes, we can construct a test statistic for $H_{0(i)}$, denoted by T_i , $i = 2, \dots, k$, by comparing D_i and $D_{(i-1)}$ as follows:

$$T_i = \frac{\sqrt{N}(D_i - D_{(i-1)})}{s_{pi} \sqrt{1/\lambda_{(i-1)} + 1/\lambda_i}} \quad (4.5)$$

where

$$s_{pi} = \sqrt{\frac{(n_{(i-1)} - 1)s_{(i-1)}^2 + (n_i - 1)s_i^2}{n_{(i-1)} + n_i - 2}}, \quad (4.6)$$

$$s_i^2 = \frac{\sum_{j=1}^{n_i} (Z_{ij} - D_i)^2}{n_i - 1},$$

$$s_{(i-1)}^2 = \frac{\sum_{m=1}^{i-1} \sum_{j=1}^{n_m} (Z_{mj} - D_{(i-1)})^2}{n_{(i-1)} - 1}.$$

McDermott and Mudholkar (1993) suggest that the p -values associated with statistics from the $k - 1$ nested hypotheses (4.3), P_2, \dots, P_k , are combined to test H_0 using one of Ψ_T , Ψ_F , Ψ_N , and Ψ_L (2.6, 2.7, 2.8, and 2.9). However, the four p -value combining methods rely on the assumption that the p -values come from independent statistics. Nevertheless, asymptotic independence of T_2, \dots, T_k , stated below, establishes good approximations to the true p -value for the H_0 using such p -value combining methods.

We have the following results on the asymptotic properties of the test statistic T_i under the assumptions of H_0 .

Theorem 1 As $\min(n_1, \dots, n_k) \rightarrow \infty$, $T_i \xrightarrow{D} N(0, 1)$, $i = 2, \dots, k$.

Theorem 2 As $\min(n_1, \dots, n_k) \rightarrow \infty$, $(T_2, \dots, T_k) \xrightarrow{D} N_{k-1}(\mathbf{0}, \mathbf{I})$, where $\mathbf{0}$ denotes the zero column vector with $k - 1$ components, and \mathbf{I} is the $(k - 1) \times (k - 1)$ identity matrix.

From Theorem 2, we immediately have the following corollary.

Corollary 1 As $\min(n_1, \dots, n_k) \rightarrow \infty$, T_2, \dots, T_k are asymptotically independent.

Proofs of Theorem 1 and 2 are based on two auxiliary lemmas (See Lemma 1 and 2 in Appendix).

Remark 1 Levene's transformation $Z_{ij} = |X_{ij} - \bar{X}_i|$ can be replaced with Brown-Forsythe transformation $Z_{50ij} = |X_{ij} - \tilde{X}_{50i}|$ where \tilde{X}_{50i} denotes the sample median. Conover et al. (1981) and Lim and Loh (1996) recommend the Brown-Forsythe modification of the Levene's test (Levene(med)) as a test for equality of variances as it is robust to nonnormality and has competitive powers. Carroll and Schneider (1985) give a theoretical justification for the asymptotically correct level of test for Levene(med), which is also applicable to the new test. Theorem 1 and 2 hold for the statistics with Brown-Forsythe modification.

Remark 2 The new test can be used to test equality of variances against general alternatives $\sigma_i \neq \sigma_j, i \neq j$ using two-sided tests instead of one-sided for ordered alternatives.

Remark 3 Miller's (1968) pseudovalues $W'_{ij} = n_i \log s_i^2 - (n_i - 1) \log s_{i(-j)}^2$ in the test statistic V_i (3.7) suggested by Mudholkar and McDermott (1995) can also be replaced by Z_{ij} or Z_{50ij} . Theorem 1 and 2 hold for the statistics with such modifications.

Chapter 5

Robustifications for Unequal or Small Sample Sizes

5.1 Overview

In the real-life situations, we often encounter experiments with unbalanced designs: i.e. unequal number of observations per sample. An unbalanced design may be caused due to problems in the data collection, such as unexpected deaths of specimens, resulting in a loss of observations in what would otherwise have been a balanced design (Cabrera and McDougall, 2002). The unbalanced design could be intentional in order to reflect the population proportion of the samples (Clark-Carter, 1997). Keyes and Levy (1997) show through the simulation study that the size and power of Levene's test is affected by unbalanced designs with small sample sizes, which is of concern to our new test proposed in this thesis. The new test can be modified using a correction factor for unequal sample sizes (O'Brien, 1978; Keyes and Levy, 1997) and the Hines-Hines (2000) structural zero removal method. Each of the methods is known to improve the sizes or powers of the Levene's test. These modifications are particularly effective in the cases of unequal or small sample sizes.

5.2 Correction Factor for Unequal Sample Sizes

Keyes and Levy (1997) study the effect of unequal sample sizes on the Levene's test noting that

$$E(Z_{ij}) = \sigma_i \sqrt{\frac{2}{\pi} \left(1 - \frac{1}{n_i}\right)} \quad (5.1)$$

in a one-way ANOVA design assuming that the samples are from normal populations. When the sample sizes are small and unequal, the effect of $\sqrt{2/\pi(1-1/n_i)}$ is relatively large, which would cause the test to reject the null hypothesis when it is true (Gastwirth et al., 2009). Keyes and Levy (1997) suggest a simple remedy by replacing Z_{ij} with

$$U_{ij} = \frac{1}{\sqrt{1-1/n_i}} Z_{ij} \quad (5.2)$$

where we call $1/\sqrt{1-1/n_i}$ a correction factor for unequal sample sizes. O'Brien (1978) suggests the same correction factor for Z_{50ij} with a two-way ANOVA design. O'Neil and Mathews (2000) derive a precise expression $\kappa_{n_i}^{-1}$ for the correction factor for standard normal populations. The correction factor $\kappa_{n_i}^{-1}$ involves the expected value of a linear function of order statistics from the standard normal distribution, and $\kappa_{n_i}^{-1} \rightarrow \sqrt{2/\pi}$ as $n_i \rightarrow \infty$. However, $1/\sqrt{1-1/n_i}$ suggested by O'Brien (1978) approximates $\kappa_{n_i}^{-1}/\sqrt{2/\pi}$ well even for small sample sizes, and is less computationally expensive. Therefore, O'Brien's correction factor can be applied to Z_{ij} or Z_{50ij} for the new test.

Remark 4 The correction factor $1/\sqrt{1-1/n_i}$ should be applied to each observation based on the original sample size n_i in order to estimate the standard deviation more accurately, even after the samples are combined based on the nested hypotheses (4.3) and (4.4). This remark is important to the cases of calculating T_i (4.5) with the correction factor applied.

5.3 Hines-Hines Structural Zero Removal Method

Hines and Hines (2000) study the effects of structural zeros on the F -statistic for which the Brown-Forsythe (1974) modification Z_{50ij} is based on. When the sample size is odd,

at least one observation coincides with the median, resulting in having at least one Z_{50ij} equal to zero in each sample. Such a zero is called a structural zero. For an even sample size, by considering an orthogonal transformation, $Z_{i(1)}$ and $Z_{50i(2)}$ can be replaced by $(Z_{50i(1)} - Z_{50i(2)})/\sqrt{2}(= 0)$ and $(Z_{50i(1)} + Z_{50i(2)})/\sqrt{2}(= \sqrt{2}Z_{50i(1)})$ respectively to create a structural zero. The orthogonal transformation leaves $\sum_{i=1}^{n_i} Z_{50ij}^2$ unaffected for each sample while $\sum_{i=1}^{n_i} Z_{50ij}$ is somewhat affected. Removal of such structural zeros from each sample reduces the degrees of freedom and corrects the size of the test for small sample sizes while maintaining or improving the power of the test. This method can be applied to the new test after the Brown-Forsythe modification.

Remark 5 Hines and Hines (2000) define the orthogonal transformation based on the deviations from the sample mean, $Z_{Hij} = X_{ij} - \tilde{X}_{50i}$, i.e. $|Z_{Hij}| = Z_{50ij}$. Therefore, their expressions for the orthogonal transformation appear to be different although, mathematically, their expressions are equivalent to the ones above. Using Z_{Hij} , for an even sample size $2m$, we have $(Z_{Hi(m)} + Z_{Hi(m+1)})/\sqrt{2}(= 0)$ and $(Z_{Hi(m)} - Z_{Hi(m+1)})/\sqrt{2}(= \sqrt{2}Z_{Hi(m+1)})$ as the orthogonal transformation.

5.4 Modified Structural Zero Removal Method and a New Correction Factor

The Hines-Hines structural zero removal method can be modified and combined with the correction factor suggested by O'Brien (1978). For a sample with an even sample size, consider a transformation of $Z_{50i(1)}$ and $Z_{50i(2)}$ into $Z_{50i(1)} - Z_{50i(2)}(= 0)$ and $Z_{50i(1)} + Z_{50i(2)}(= 2Z_{50i(1)})$ respectively. This transformation leaves $\sum_{i=1}^{n_i} Z_{50ij}$ unaffected while $\sum_{i=1}^{n_i} Z_{50ij}^2$ is somewhat affected. For a sample with an odd sample size, perform the Hines-Hines structural zero method. Let D_{50i} and $D_{50i(NS)}$ be the sample means of Z_{50ij} before and after the proposed structural zero removal respectively. Then, these two

quantities are related by

$$D_{50i} = \left(1 - \frac{1}{n_i}\right) D_{50i(NS)}. \quad (5.3)$$

Therefore,

$$E(D_{50i(NS)}) = \sigma_i \left(\frac{\kappa_{n_i}}{1 - 1/n_i} \right) \quad (5.4)$$

in a one-way ANOVA design where the samples are from standard normal populations. Also, $\kappa_{n_i}^{-1}/\sqrt{2/\pi}$ is well approximated by O'Brien's correction factor $1/\sqrt{1 - 1/n_i}$. Following the idea of Keyes and Levy (1997), we introduce a new correction factor

$$\phi_{n_i} = \sqrt{1 - \frac{1}{n_i}} \quad (5.5)$$

where

$$\phi_{n_i} = \frac{1}{\sqrt{1 - 1/n_i}} \left(1 - \frac{1}{n_i}\right) \approx \frac{\kappa_{n_i}^{-1}}{\sqrt{2/\pi}} \left(1 - \frac{1}{n_i}\right), \quad (5.6)$$

for the absolute deviations from median after the modified structural zero removal, $Z_{50ij(NS)}$. This combination of the modified structural zero removal and the correction factor is applicable to the new test with the Brown-Forsythe modification. Remark 4 also applies to the new correction factor.

Remark 6 For samples with even sample sizes (even samples) with sample size $2m$, unless $X_{(m)} = X_{(m+1)}$, no structural zero is created. Thus, we can consider applying the structural zero removal method to odd samples only. This odd sample structural zero removal method leaves $\sum_{i=1}^{n_i} Z_{50ij}^2$ and $\sum_{i=1}^{n_i} Z_{50ij}$ unaffected for both even and odd samples. To further adjust for the unequal sample sizes, the correction factor ϕ_{n_i} can be applied to the odd samples after the structural zero removal, and O'Brien's correction factor $1/\sqrt{1 - 1/n_i}$ can be applied to even samples. However, the odd sample structural zero removal method has to be considered with caution. Note that Hines and Hines (2000) show that, for Levene (med) with even samples, the structural zero removal after the orthogonal transformation preserves the size and maintain or increase the power of the test well.

Chapter 6

Simulation Study

6.1 Simulation Plan

The size and power of the tests are investigated in the simulation study using six distributions and seven combinations of the sample sizes for $k = 3$ and $k = 6$. For estimating the size, nominal 5% level is used throughout the study with the variances $(\sigma_1^2, \sigma_2^2, \dots, \sigma_k^2) = (1, 1, \dots, 1)$ with 10,000 Monte Carlo simulations. The calculations of the power estimates are done in two-stages. In the first stage, exact critical values at the 5% level are computed with 10,000 Monte Carlo simulations for each configuration. Then, the empirical power is calculated for each test based on the simulated exact critical values with 10,000 Monte Carlo simulations. The empirical power based on the exact 5% level gives a fair basis for comparisons as the nominal critical values may not be good approximations to the true values especially for small sample sizes. The distributions used are: 1. normal with mean 0, 2. Student's t with 9 degrees of freedom with mean 0, 3. exponential, 4. chi-squared with 2 degrees of freedom, 5. NIG(10,0), and 6. NIG(7,2), where NIG(a,b) is the normal inverse Gaussian with excess kurtosis a , skewness b , and mean 0. Note that normal inverse Gaussian distribution is typically characterized by four parameters α' (tail heaviness), β' (asymmetry), δ' (scale), and μ' (location). However, given the mean λ , variance σ^2 , excess kurtosis κ , and skewness γ with $3\kappa > 5\gamma^2$, we can calculate the four

parameters α' , β' , δ' , and μ' by

$$\alpha' = \frac{3\sqrt{3\kappa - 4\gamma^2}}{\sigma(3\kappa - 5\gamma^2)}, \beta' = \frac{3\gamma}{\sigma(3\kappa - 5\gamma^2)}, \delta' = \frac{\sigma^2(\sqrt{\alpha^2 - \beta^2})^3}{\alpha^2}, \mu' = \lambda - \frac{\delta\beta}{\sqrt{\alpha^2 - \beta^2}}. \quad (6.1)$$

The choice of distributions covers a broad range of characteristics including symmetric to skewed and light- to heavy-tailed. Furthermore, different sample sizes, both equal and unequal, are considered in this study.

We have six different tests for the simulation study. They are: 1. L: Levene's test (Levene, 1960), 2. LT: Ltrend test (Gastwirth et al., 2009) with linear scores (1,2,3) for $k = 3$ and (1,2,3,4,5,6) for $k = 6$, 3. LN: New test presented in this thesis based on Hogg's test (1962) with Levene's transformation (Lnested test), 4. DCT: Double contrast test with Levene's transformation (Neuhauser and Hothorn, 2000), 5. M93: Modified F -test with nested hypotheses (Mudholkar et al., 1993), and 6. M95: Modified McDermott-Mudholkar test with Miller's pseudovalues (Mudholkar et al., 1995). The first four tests are named Levene-type tests as they utilize Levene's transformation. For the Levene-type tests, we consider three versions of the tests: 1. Mean-based tests which use the original Levene's transformation $Z_{ij} = |X_{ij} - \bar{X}_i|$, 2. Median-based tests which use the Brown-Forsythe modification $Z_{50ij} = |X_{ij} - \tilde{X}_{50i}|$, 3. Median-based modified tests using the combination of modified structural zero removal and the correction factor ϕ_{n_i} . M93 and M95 are classified as other tests. For the tests with nested hypotheses (LN, M93, and M95), the size and power using Ψ_N (2.8) and Ψ_L (2.9) are shown as these two methods tend to produce better estimates of the size and competitive power for small sample sizes.

6.2 Median-Based Modified and Other Tests

Among the median-based modified tests (see Tables 8.1–8.8), LN(Ψ_N) is very robust under skewed or heavy-tailed distributions and different sample sizes in general as most of the size estimates are very close to the nominal significance level of 0.05. Also, LN(Ψ_N) is often the most powerful across different distributions and sample sizes compared to the other tests. LT also possesses excellent properties of preserving the sizes and has competitive power estimates. DCT is robust, but is less powerful compared to LN(Ψ_N) and LT. M93 is also very powerful, but it is not robust under non-normal distributions. M95 tends to

have slightly inflated size estimates under non-normal distributions when $k = 3$. Also, note the low empirical power estimates for L when the sample sizes are small and have a non-decreasing pattern.

6.3 Mean- and Median-Based Tests

In general, mean-based tests are not robust under skewed distributions (see Tables 8.9–8.16). Under skewed distributions, all the tests have the size estimates of 0.10 or above for the majority of the sample sizes. For small sample sizes, mean-based tests tend to overestimate the size of the test under heavy-tailed distributions as well. Among the four mean-based tests, LT and LN show competitive empirical power estimates across different sample sizes and distributions. In particular, $\text{LN}(\Psi_N)$ seems to be the most powerful, although the power estimates are very similar among these tests.

Median-based tests are more robust to skewed and heavy-tailed distributions, but the size estimates tend to be conservative (see Tables 8.17–8.24). The tendency seems apparent for small sample sizes such as (5,5,5) or (5,5,5,5,5,5). Among the four median-based tests, LT and LN show equally competitive empirical power estimates across different sample sizes and distributions. In particular, $\text{LN}(\Psi_N)$ seems to be slightly more powerful than LT in most of the cases tested.

Chapter 7

Case Studies

7.1 Monthly Apartment Rent in Thalwil, Switzerland

The first example is the monthly rent data ($N = 25$) for apartments in Thalwil, Switzerland, collected on October 3, 2008. The observations are grouped according to the number of rooms available in the apartment: 1. Small (less than 3 rooms) ($n_1 = 5$); 2. Medium (3 or 3.5 rooms) ($n_2 = 8$); and 3. Large (more than 3.5 rooms) ($n_3 = 12$). We conjecture an increasing pattern in variability in the monthly rent depending on the number of rooms. Since the sample sizes are small, it is difficult to judge whether the samples are normally distributed. Therefore, robust tests for non-normal samples are more favorable in order to assess the trend more accurately. The sample sizes are highly unequal. Therefore, we apply median-based modified tests for the Levene-type tests. We use Φ_N to calculate the p -values of the tests with nested hypotheses.

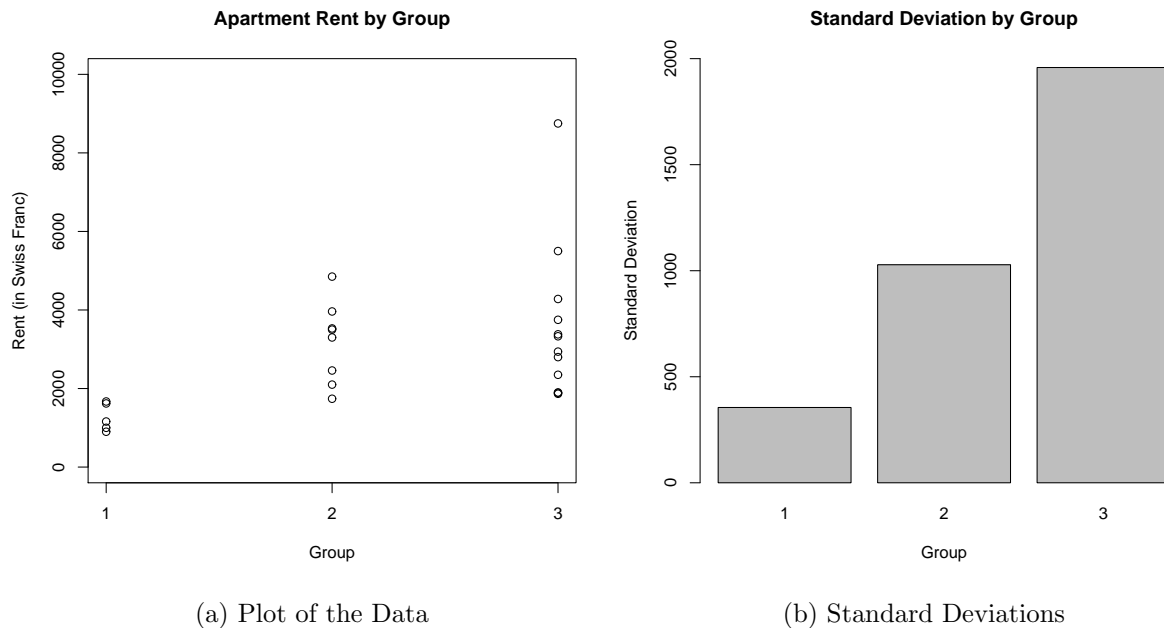


Figure 7.1: Plot of the Data and Standard Deviations by Group (1=Small, 2=Medium, 3=Large)

The plots in Figure 7.1 suggest that the standard deviations seem to have a linearly increasing pattern as the apartment size increases. Among the Levene-type tests, LN has the lowest p -value of 0.0171. The p -values provided by LT and DCT, 0.0553 and 0.0876 respectively, are on the boarder of significance. M93 and M95 have the p -values of 7.69×10^{-4} and 8.18×10^{-3} respectively. However, M93 and M95 tend to overreject the null hypothesis for non-normal samples when $k = 3$. Therefore, we can conclude that there is some evidence of an increasing trend in variability based on LN with the p -value of 0.0171. On the other hand, Levene's test with the above modifications (L) provided the p -value of 0.289, accepting the equality in variability among the samples. The phenomenon could be attributed to a weak power of Levene's test when the sample sizes are small and have a non-decreasing pattern.

7.2 Temperature-accelerated Life Test of a Type of Sheathed Tabular Heater

The second example is taken from the temperature-accelerated life test data of a type of sheathed tabular heater studied by Nelson (1972). At each of four temperatures (1708, 1660, 1620, and 1520 F°), 6 heaters were tested and the numbers of hours to failure were recorded (i.e. $n_i = 6$, $i = 1, 2, 3, 4$, and $N = 24$). We conjecture an increasing pattern in variability with a decrease in temperature.

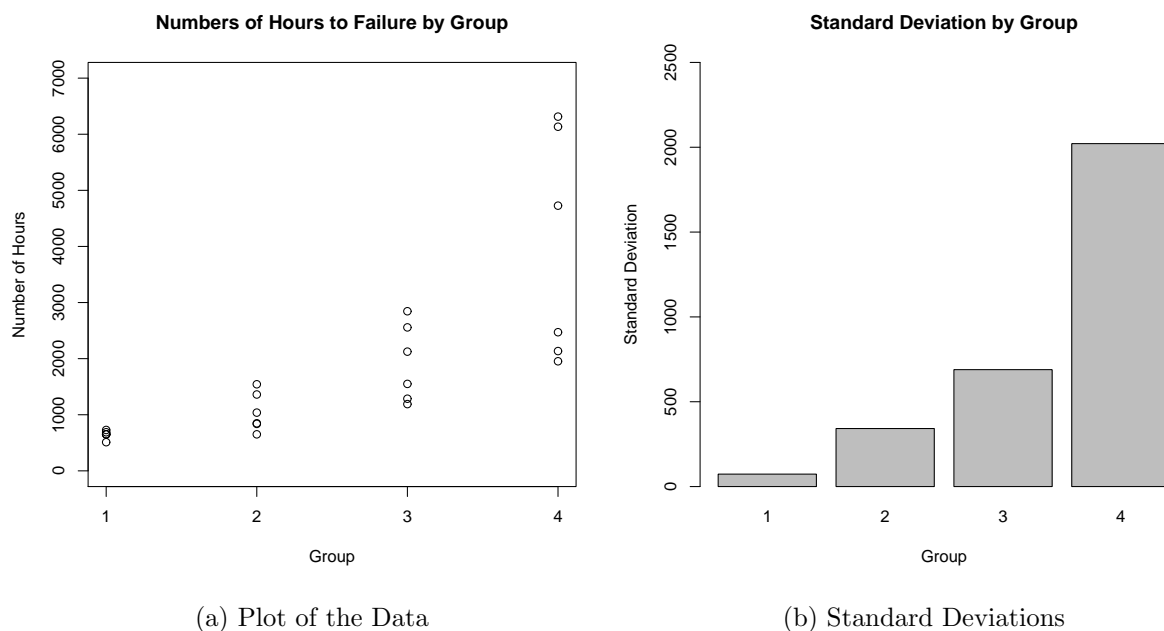


Figure 7.2: Plot of the Data and Standard Deviations by Group (1=1708 F° , 2=1660 F° , 3=1620 F° , 4=1520 F°)

The plots in Figure 7.2 display an exponentially increasing pattern in standard deviations as the temperature decreases. To determine whether there is an increasing trend in variability, robust tests for non-normal samples are preferred in this example as well since the sample sizes are small. For the Levene-type tests, the Hines-Hines structural zero

removal method with median as the center is applied to each sample. Also, for LT, linear scores (1, 2, 3, 4) and non-linear scores (-1708 , -1660 , -1620 , -1520) are applied. We use Φ_N to calculate the p -values of the tests with nested hypotheses. Among all the tests, LN has the lowest p -value of 2.34×10^{-10} , followed by M95 and LT having the non-linear score with 2.74×10^{-9} and 8.24×10^{-9} respectively. Therefore, there is strong evidence of an increasing trend in variability with a decrease in temperature. Levene's test with the same modification provided the p -value of 3.69×10^{-7} , strongly rejecting the null hypothesis of equal variability.

7.3 Soil Lead Concentration in Syracuse, New York

The next example refers to the soil lead concentration study in Syracuse, New York, conducted by Griffith (2002). The soil samples were collected at 167 locations (i.e. $N = 167$), nearly completing coverage of the city. Griffith (2002) classifies the soil samples by their location features: 1. park soil ($n_1 = 30$); 2. playground soil ($n_2 = 17$); 3. streetside soil ($n_3 = 74$); 4. house lot soil ($n_4 = 30$); and 5. vacant lot soil ($n_5 = 16$). The study concluded that the log transformed soil lead concentration level from the five location features differ significantly at the 5% level (p -value = 0.018) based on Levene's test for equality of variances. Griffith (2002) lists major sources of soil lead contamination as 1. widespread use of lead-based paints; 2. lead emissions in gasoline in earlier years; and 3. lead waste from mining/commercial/manufacturing processes. Considering the exposures to these sources, we conjecture the order of the variabilities in the soil lead concentration among the five location features as $1 < 2 < 3 < 4 < 5$ for the following reasoning.

- Park and playground soil is located away from these sources of pollution.
- Playgrounds tend to be located closer to the sources of contamination compared to parks.
- Streetside soil would probably have a comparatively lower variability due to its limited area with relatively uniform influences from these three sources of pollution.
- House lot soil would have a slightly higher variability due to paints. Presence of roofs and walls would lower the variability.

- Vacant lot soil would have the highest variability because of absence of roofs and walls, and proximity to the sources of pollution.

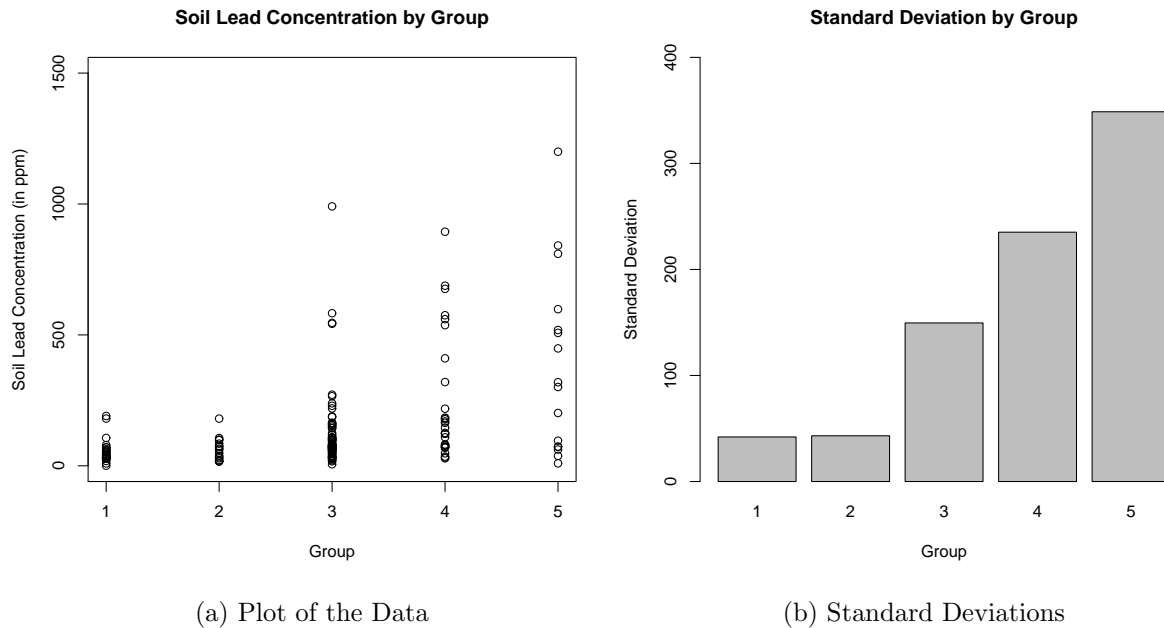


Figure 7.3: Plot of the Data and Standard Deviations by Group (1=park soil, 2=playground soil, 3=streetside soil, 4=house lot soil, 5=vacant lot soil)

The plots in Figure 7.3 suggest that, although the first two soil samples have roughly equal standard deviations, there seems to be a linearly increasing pattern in standard deviations after the first two samples. We observe that there is some evidence of non-normality in the sample distributions. For example, the streetside and house lot samples have the p -values of less than 0.01 for the Shapiro-Wilk test. The sample sizes are highly unequal as well. Thus, we follow the procedure described in the monthly rent example to test for an increasing trend in variability. Among the Levene-type tests and M95, LN has the lowest p -value of 3.03×10^{-9} , followed by DCT and LT with the p -values of 5.17×10^{-9} and 8.66×10^{-9} respectively. Therefore, there is strong evidence of an increasing trend in variability which supports the conjecture based on the tests above. M93 provided the

p -value of 4.46×10^{-21} . However, M93 is not robust to non-normal samples and the p -value is less reliable in this example. Levene's test provides the p -value of 4.16×10^{-8} , suggesting unequal variability among samples.

7.4 Testosterone Levels with Different Smoking Habits

The fourth example explores the differences in variability of the testosterone levels of men with different smoking habits. Le (1994) classifies four groups of men, 10 in each group, according to their smoking habits: 1. heavy smokers; 2. light smokers; 3. former smokers; and 4. non-smokers (i.e. $n_i = 10$, $i = 1, 2, 3, 4$, and $N = 40$). We conjecture an increasing pattern in variability with a decrease in exposure to smoking.

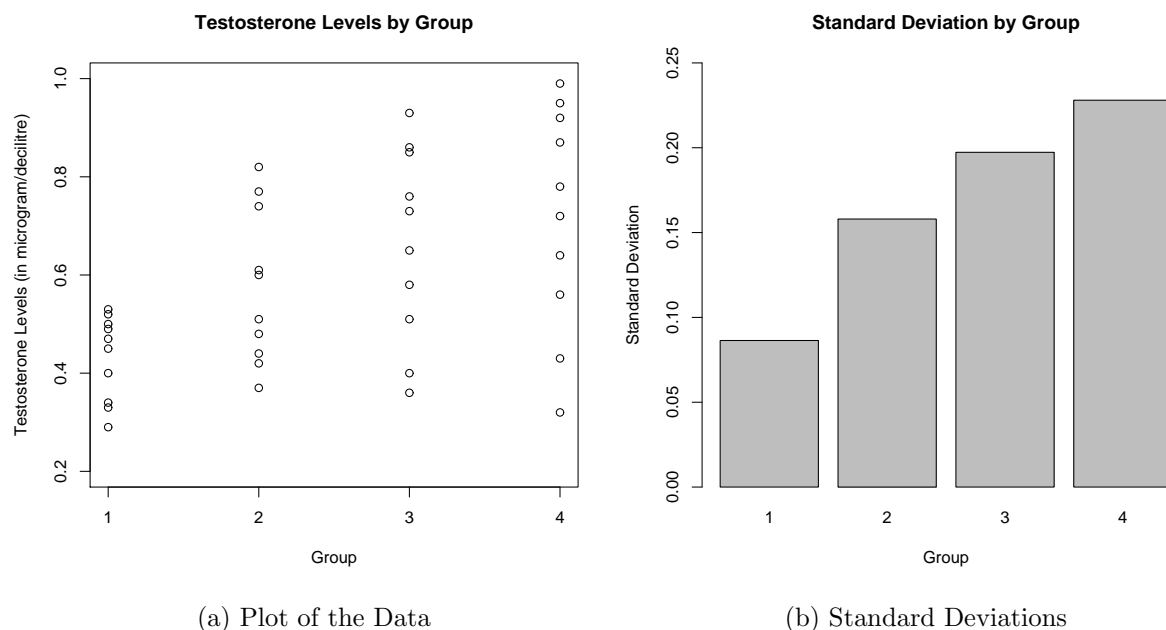


Figure 7.4: Plot of the Data and Standard Deviations by Group (1=heavy smokers, 2=light smokers, 3=former smokers, 4=non-smokers)

The plots in Figure 7.4 exhibit a linearly increasing pattern in standard deviations as

the exposure to smoking decreases. To determine whether there is an increasing trend, we prefer robust tests for non-normality in this example as the sample sizes are relatively small. Since the sample sizes are equal, we follow the procedure described in the temperature-accelerated life test example. For LT, linear scores (1, 2, 3, 4) are assigned to each group. All the tests rejected the null hypothesis strongly, indicating an increasing trend in variability with a decrease in exposure to smoking. M95 has the lowest p -value of 6.76×10^{-5} , followed by LN and LT with 3.08×10^{-4} and 1.41×10^{-3} respectively. However, M95 tends to have a slightly inflated size for $k = 3$, so the p -value should be interpreted with caution. In contrast, Levene's test with the above modification provides the p -value of 0.0266, marginally rejecting the null hypothesis of equal variability among groups.

7.5 Rapid Eye Movement (REM) Sleep Time and Ethanol Intake

The last example investigates whether rapid eye movement (REM) sleep time depends on the concentration of ethanol given in an injection (Hattan and Eacho, 1978; Devore and Peck, 1986). Four injection concentrations (in grams per kilogram weight), 0, 1, 2, and 4, were selected and twenty rats were randomly divided equally into four groups with different treatments (i.e. $n_i = 5$, $i = 1, 2, 3, 4$, and $N = 20$). The REM sleep time during a 24-hour period was recorded for each rat. The mean REM sleep time seems to increase as the concentration of ethanol decreases. We often observe that an increase in mean is associated with an increase in variability. Therefore, we conjecture an increasing trend in variability of the REM sleep time as the concentration of ethanol decreases.

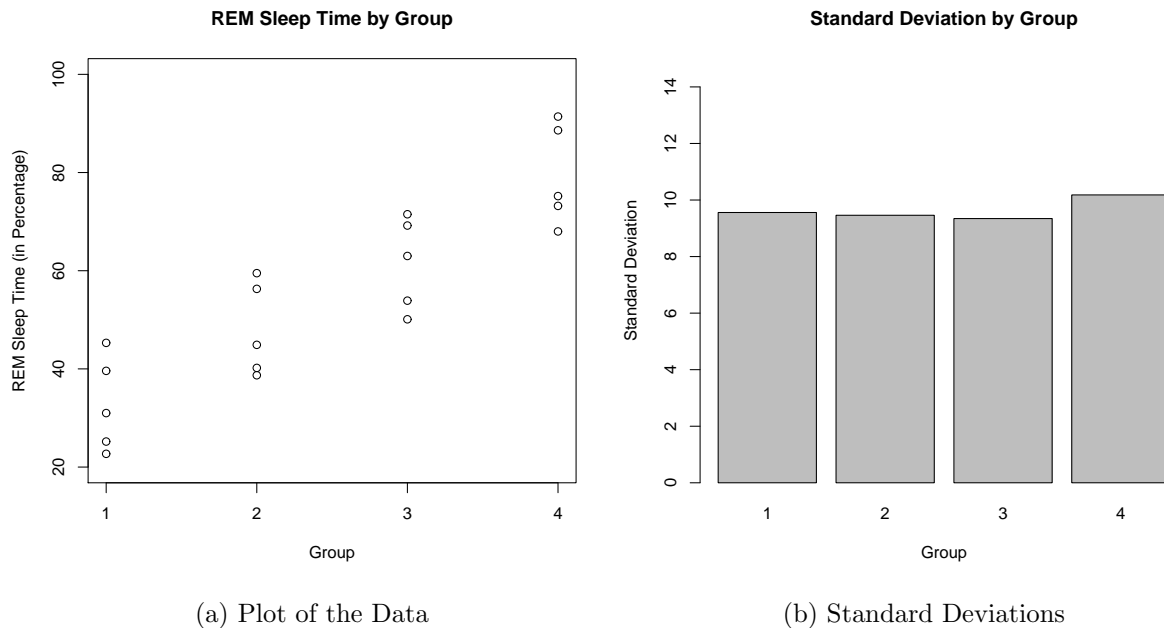


Figure 7.5: Plot of the Data and Standard Deviations by Group (1=4g/kg weight, 2=2g/kg weight, 3=1g/kg weight, 4=0g/kg weight)

The plots in Figure 7.5 suggest no obvious trend in standard deviations. To test whether there is an increasing trend in variability, We follow the procedure described in the temperature-accelerated data as the setting is quite similar. Also, for LT, linear scores (1, 2, 3, 4) and non-linear scores (-4, -2, -1, 0) are applied. All the tests have high p -values, suggesting no evidence against equality of variances. M93 has the lowest p -value of 0.419, followed by LT having linear and non-linear scores with 0.446 and 0.454 respectively. Therefore, there is no evidence of an increasing trend in variability with a decrease in concentration. Levene's test provides the p -value of 0.998, suggesting no evidence against equality of variances. In addition, there is also no evidence of a decreasing trend in variability with a decrease in concentration.

Chapter 8

Conclusions and Future Work

The purpose of this thesis is to present a robust test for equality of variances against ordered alternatives for k independent samples and to show its theoretical reasoning. The test is developed by combining Levene's transformation and the finite-intersection method. Levene's transformation stabilizes the size of the test for non-normal distributions while the finite-intersection method increases the power of the test. The theoretical reasoning is given by proving the asymptotic normality and independence of the component statistics $T_i, i = 2, 3, \dots, k$. To further robustify the test for unequal or small sample sizes, a combination of the modified structural zero removal method, which is a variant of the Hines-Hines structural zero removal method, and a new correction factor ϕ_{n_i} based on O'Brien's correction factor is considered. The Monte Carlo simulation study suggests that the new test with such modification ($\text{LN}(\Psi_N)$) preserves the size very well and is powerful compared to the existing tests for equality of variances against ordered alternatives for different sample sizes under normal, skewed, and heavy-tailed distributions. The new test is applicable to a wide variety of disciplines such as microeconomics, engineering, environmental studies, and health studies.

In the future, we plan to investigate the following topics:

1. Application of the bootstrap method (Boos and Brownie, 1989; Lim and Loh, 1996) to our new test, which is expected to increase the robustness and power of the test further.

2. Investigation of the optimal decomposition of H_0 and H_1 into nested hypotheses by considering more general cases of grouping the samples, such as those considered by McDermott and Mudholkar (1993).

APPENDICES

Appendix A: Lemmas for the Theorems

Lemma 1 gives convergence in probability of random variables of interest, followed by a proof of Theorem 1 which provides the asymptotic normality of T_i . Lemma 2 establishes conditions necessary to use the Cramer-Wold device (Hu and Rosenberger, 2006) which is employed in the proof of Theorem 2 to prove the asymptotic multivariate normality of the joint distribution of (T_2, \dots, T_k) with the $(k-1) \times (k-1)$ identity matrix as the variance-covariance matrix. All the lemmas and theorems are derived under the assumptions of H_0 .

Let $n_{(i-1)} = \sum_{m=1}^{i-1} n_m$, $D_i = n_i^{-1} \sum_{j=1}^{n_i} Z_{ij}$, and $D_{(i-1)} = n_{(i-1)}^{-1} \sum_{m=1}^{i-1} n_m D_m$, $i = 2, \dots, k$. Also, let s_{pi}^2 be as defined in (4.6). Then, we have the following.

Lemma 1 *As $\min(n_1, \dots, n_i) \rightarrow \infty$, $D_i \xrightarrow{p} \tau$, $D_{(i-1)} \xrightarrow{p} \tau$, and $s_{pi}^2 \xrightarrow{p} \eta^2$.*

Proof. Gastwirth (1982) shows $D_i \xrightarrow{p} \tau$. Also,

$$D_{(i-1)} = \frac{1}{n_{(i-1)}} \sum_{m=1}^{i-1} n_m D_m \xrightarrow{p} \frac{1}{n_{(i-1)}} \sum_{m=1}^{i-1} n_m \tau = \tau.$$

Similar arguments using the results $D_i \xrightarrow{p} \tau$ and $D_{(i-1)} \xrightarrow{p} \tau$ yield $s_i^2 \xrightarrow{p} \eta^2$ and $s_{(i-1)}^2 \xrightarrow{p} \eta^2$. Therefore,

$$s_{pi}^2 \xrightarrow{p} \frac{(n_{(i-1)} - 1)\eta^2 + (n_i - 1)\eta^2}{n_{(i-1)} + n_i - 2} = \eta^2.$$

Proof of Theorem 1. Let us consider the test statistic

$$T_i = \frac{\sqrt{N}(D_i - D_{(i-1)})}{s_{pi} \sqrt{1/\lambda_{(i-1)} + 1/\lambda_i}}.$$

Denote

$$Q_i = \frac{\sqrt{N}(D_i - D_{(i-1)})}{\eta \sqrt{1/\lambda_{(i-1)} + 1/\lambda_i}}.$$

Then, $T_i = \eta s_{pi}^{-1} Q_i$. By Lemma 1, $\eta s_{pi}^{-1} \xrightarrow{p} 1$.

Consider the convergence of random variable Q_i . Let

$$W_m = \frac{\sqrt{n_m}(D_m - \tau)}{\eta}, m = 1, \dots, i.$$

Then,

$$Q_i = \frac{1}{\sqrt{1/\lambda_{(i-1)} + 1/\lambda_i}} \left[\frac{1}{\sqrt{\lambda_i}} W_i - \sum_{m=1}^{i-1} \frac{\lambda_m}{\lambda_{(i-1)}} \frac{1}{\sqrt{\lambda_m}} W_m \right].$$

Gastwirth (1982) shows that $W_m \xrightarrow{D} N(0, 1)$ for $m = 1, \dots, i$. Also, by the assumption of k independent samples, D_1, \dots, D_k are independent. Thus, W_1, \dots, W_k are also independent. Therefore, $Q_i \xrightarrow{D} N(0, \xi_i^2)$, and by the Slutsky's Theorem (Lehmann, 1998), $T_i \xrightarrow{D} N(0, \xi_i^2)$ for some $\xi_i^2 > 0$, $i = 1, \dots, k$.

Now, let us calculate the value of ξ_i^2 . We have $\lambda_m \rightarrow c_m \in (0, 1)$ for $m = 1, \dots, i$. Hence,

$$\xi_i^2 = \left(\frac{1}{\sqrt{1/c_{(i-1)} + 1/c_i}} \right)^2 \left[\left(\frac{1}{\sqrt{c_i}} \right)^2 + \sum_{m=1}^{i-1} \left(\frac{c_m}{c_{(i-1)}} \frac{1}{\sqrt{c_m}} \right)^2 \right] = 1,$$

where $c_{(i-1)} := \sum_{m=1}^{i-1} c_m$. This completes the proof.

Lemma 2 proves the conditions necessary to use the Cramer-Wold device (Hu and Rosenberger, 2006), which appears in Theorem 1.

Lemma 2 As $\min(n_1, \dots, n_k) \rightarrow \infty$, $\sum_{i=2}^k b_i Q_i \xrightarrow{D} N\left(0, \sum_{i=2}^k b_i^2\right)$ for any $b_i \in \mathfrak{R}$, $i = 2, \dots, k$.

Proof. We shall prove this result by induction. For the base cases, consider $k = 2$ and $k = 3$. When $k = 2$, $b_2 Q_2 \xrightarrow{D} N(0, b_2^2)$ by Theorem 1. When $k = 3$, a straightforward calculation shows that $\sum_{i=2}^3 b_i Q_i \xrightarrow{D} N\left(0, \sum_{i=2}^3 b_i^2\right)$.

For the induction hypothesis, assume that, for $k = 2, \dots, r-1$, $r \geq 4$, $\sum_{i=2}^k b_i Q_i \xrightarrow{D} N\left(0, \sum_{i=2}^k b_i^2\right)$ holds. The induction hypothesis has the following implications.

Routine calculations show that

$$\sum_{i=2}^k b_i Q_i = \sum_{i=1}^k f_{k,i}(\lambda_1, \dots, \lambda_k) W_i,$$

where

$$\begin{aligned} f_{k,1}(\lambda_1, \dots, \lambda_k) &= -\sqrt{\lambda_1} \left(\sum_{i=2}^k \frac{b_i}{\lambda_{(i-1)} \sqrt{1/\lambda_{(i-1)} + 1/\lambda_i}} \right), \\ f_{k,i}(\lambda_1, \dots, \lambda_k) &= \frac{1}{\sqrt{\lambda_i}} \left\{ \left(\lambda_i \sum_{m=i}^{k-1} \frac{b_{m+1}}{\lambda_{(m)} \sqrt{1/\lambda_{(m)} + 1/\lambda_{m+1}}} \right) - \frac{b_i}{\sqrt{1/\lambda_{(i-1)} + 1/\lambda_i}} \right\}, \\ i &= 2, \dots, r-1, \\ f_{k,k}(\lambda_1, \dots, \lambda_k) &= \frac{1}{\sqrt{\lambda_k}} \frac{b_k}{\sqrt{1/\lambda_{(k-1)} + 1/\lambda_k}}, \end{aligned}$$

for any $k \geq 3$.

As $\min(n_1, \dots, n_k) \rightarrow \infty$, $f_{k,i}(\lambda_1, \dots, \lambda_k) \rightarrow f_{k,i}(c_1, \dots, c_k)$ for $i = 1, \dots, k$. Also, by Theorem 1, W_1, \dots, W_r are independent and $W_i \xrightarrow{D} N(0, 1)$ for $i = 1, \dots, r$. Therefore, $\sum_{i=2}^r b_i Q_i \xrightarrow{D} N\left(0, \sum_{i=1}^r \{f_{r,i}(c_1, \dots, c_r)\}^2\right)$. Thus, the induction hypothesis implies that

$$\sum_{i=1}^k \{f_{k,i}(c_1, \dots, c_k)\}^2 = \sum_{i=2}^k b_i^2 \text{ for } k = 3, \dots, r-1.$$

Consider $k = r$. The calculations above show that $\sum_{i=2}^r b_i Q_i = \sum_{i=1}^r f_{r,i}(\lambda_1, \dots, \lambda_r) W_i$. Following the argument above, we have, $\sum_{i=2}^r b_i Q_i \xrightarrow{D} N\left(0, \sum_{i=1}^r \{f_{r,i}(c_1, \dots, c_r)\}^2\right)$. Moreover,

$$\sum_{i=1}^r \{f_{r,i}(c_1, \dots, c_r)\}^2 = g_{r,1}(c_1, \dots, c_r) + g_{r,2}(c_1, \dots, c_r) + \sum_{i=1}^{r-1} \{f_{r-1,i}(c_1, \dots, c_{r-1})\}^2,$$

where

$$g_{r,1}(c_1, \dots, c_r) = \frac{b_r^2}{1/c_{(r-1)} + 1/c_r} \left[\frac{1}{c_r} + \sum_{i=1}^{r-1} \left(\frac{c_i}{c_{r-1}} \right)^2 \left(\frac{1}{c_i} \right) \right],$$

$$g_{r,2}(c_1, \dots, c_r) = 2 \left(\frac{b_r}{c_{(r-1)} \sqrt{1/c_{(r-1)} + 1/c_r}} \right) \left[\sum_{i=2}^{r-1} \frac{b_i}{\sqrt{1/c_{(i-1)} + 1/c_i}} \left(-1 + \frac{1}{c_{(i-1)}} \sum_{m=1}^{i-1} c_m \right) \right].$$

Routine calculations show that $g_{r,1}(c_1, \dots, c_r) = b_r^2$ and $g_{r,2}(c_1, \dots, c_r) = 0$. Also, by the induction hypothesis, $\sum_{i=1}^{r-1} \{f_{r-1,i}(c_1, \dots, c_{r-1})\}^2 = \sum_{i=2}^{r-1} b_i^2$. Therefore, $\sum_{i=1}^r \{f_{r,i}(c_1, \dots, c_r)\}^2 = b_r^2 + \sum_{i=2}^{r-1} b_i^2 = \sum_{i=2}^r b_i^2$. Hence,

$$\sum_{i=2}^r b_i Q_i \xrightarrow{D} N\left(0, \sum_{i=2}^r b_i^2\right).$$

This completes the induction.

Proof of Theorem 2. By the Cramer-Wold device (Corollary A.3. of Hu and Rosenberger (2006)), $(T_2, \dots, T_k) \xrightarrow{D} MVN(\mathbf{0}, \mathbf{I})$ if and only if $\sum_{i=2}^k b_i T_i \xrightarrow{D} N\left(0, \sum_{i=2}^k b_i^2\right)$ for

any nonzero $b_2, \dots, b_k \in \mathfrak{R}$. We shall prove the latter condition by using Lemma 2.

The identity $T_i = Q_i + (\eta s_{pi}^{-1} - 1) Q_i$ gives the following;

$$\begin{aligned} \sum_{i=2}^k b_i T_i &= \sum_{i=2}^k b_i [Q_i + (\eta s_{pi}^{-1} - 1) Q_i] \\ &= \sum_{i=2}^k b_i Q_i + \sum_{i=2}^k b_i (\eta s_{pi}^{-1} - 1) Q_i. \end{aligned}$$

By Lemma 2, $\sum_{i=2}^k b_i Q_i \xrightarrow{D} N\left(0, \sum_{i=2}^k b_i^2\right)$, and by Lemma 1, $\eta s_{pi}^{-1} - 1 \xrightarrow{p} 0$. The random variable Q_i converges in distribution by Theorem 1. Thus, Q_i is bounded in probability.

Therefore, $(\eta s_{pi}^{-1} - 1) Q_i \xrightarrow{p} 0$. Hence, $\sum_{i=2}^k b_i (\eta s_{pi}^{-1} - 1) Q_i \xrightarrow{p} 0$.

By the Slutsky's Theorem (Lehmann, 1998), $\sum_{i=2}^k b_i T_i \xrightarrow{D} N\left(0, \sum_{i=2}^k b_i^2\right)$ for any $b_2, \dots, b_k \in \mathfrak{R}$. Hence, by the Cramer-Wold device, $(T_2, \dots, T_k) \xrightarrow{D} MVN(\mathbf{0}, \mathbf{I})$.

Appendix B: Simulation Results

Normal Distribution									
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 1.00, 1.00)$									
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT	M93(Ψ_N)	M93(Ψ_L)	M95(Ψ_N)	M95(Ψ_L)
(5,5,5)	0.0386	0.0394	0.0401	0.0394	0.0385	0.0461	0.0465	0.0351	0.0344
(10,10,10)	0.0609	0.0526	0.0580	0.0580	0.0554	0.0512	0.0501	0.0493	0.0482
(20,20,20)	0.0522	0.0532	0.0526	0.0519	0.0523	0.0488	0.0497	0.0462	0.0466
(5,10,20)	0.0372	0.0536	0.0539	0.0510	0.0400	0.0453	0.0459	0.0642	0.0650
(20,10,5)	0.0434	0.0410	0.0429	0.0463	0.0407	0.0512	0.0515	0.0554	0.0539
(5,5,10)	0.0455	0.0564	0.0540	0.0545	0.0514	0.0458	0.0467	0.0492	0.0492
(10,5,5)	0.0438	0.0393	0.0412	0.0426	0.0398	0.0499	0.0511	0.0406	0.0385
Student's <i>t</i> Distribution with 9 Degrees of Freedom									
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 1.00, 1.00)$									
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT	M93(Ψ_N)	M93(Ψ_L)	M95(Ψ_N)	M95(Ψ_L)
(5,5,5)	0.0369	0.0476	0.0494	0.0486	0.0438	0.0632	0.0659	0.0448	0.0446
(10,10,10)	0.0535	0.0555	0.0578	0.0597	0.0567	0.0718	0.0746	0.0556	0.0554
(20,20,20)	0.0474	0.0487	0.0503	0.0508	0.0485	0.0788	0.0835	0.0532	0.0532
(5,10,20)	0.0435	0.0484	0.0545	0.0533	0.0379	0.0674	0.0724	0.0661	0.0681
(20,10,5)	0.0434	0.0454	0.0474	0.0519	0.0509	0.0704	0.0757	0.0626	0.0641
(5,5,10)	0.0431	0.0539	0.0560	0.0548	0.0457	0.0664	0.0679	0.0554	0.0555
(10,5,5)	0.0418	0.0438	0.0451	0.0494	0.0472	0.0673	0.0708	0.0445	0.0446
Exponential Distribution									
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 1.00, 1.00)$									
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT	M93(Ψ_N)	M93(Ψ_L)	M95(Ψ_N)	M95(Ψ_L)
(5,5,5)	0.0470	0.0468	0.0568	0.0583	0.0447	0.1150	0.1289	0.0643	0.0659
(10,10,10)	0.0680	0.0602	0.0679	0.0679	0.0603	0.1475	0.1684	0.0816	0.0857
(20,20,20)	0.0534	0.0522	0.0559	0.0580	0.0546	0.1719	0.1922	0.0757	0.0779
(5,10,20)	0.0582	0.0457	0.0610	0.0589	0.0316	0.1527	0.1671	0.0924	0.0990
(20,10,5)	0.0588	0.0530	0.0569	0.0620	0.0597	0.1239	0.1417	0.0844	0.0888
(5,5,10)	0.0551	0.0542	0.0619	0.0611	0.0477	0.1321	0.1451	0.0756	0.0785
(10,5,5)	0.0542	0.0526	0.0560	0.0597	0.0520	0.1175	0.1343	0.0733	0.0758

Table 8.1: Estimated Size of the Median-based Modified and Other Tests for $k = 3$ with $(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 1.00, 1.00)$. L: Levene's test, LT: Ltrend test, LN: New test presented in this thesis, DCT: Double contrast test, M93: Modified F -test with nested hypotheses, M95: Modified McDermott-Mudholkar (1995) test, Ψ_N : Liptak's p -value combination, Ψ_L : Mudholkar-George p -value combination.

Chi-squared Distribution with 2 Degrees of Freedom									
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 1.00, 1.00)$									
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT	M93(Ψ_N)	M93(Ψ_L)	M95(Ψ_N)	M95(Ψ_L)
(5,5,5)	0.0486	0.0418	0.0558	0.0548	0.0405	0.1152	0.1271	0.0643	0.0654
(10,10,10)	0.0646	0.0650	0.0696	0.0719	0.0638	0.1488	0.1683	0.0847	0.0900
(20,20,20)	0.0532	0.0521	0.0593	0.0593	0.0534	0.1703	0.1929	0.0784	0.0819
(5,10,20)	0.0545	0.0428	0.0585	0.0561	0.0281	0.1492	0.1651	0.0886	0.0931
(20,10,5)	0.0526	0.0550	0.0566	0.0617	0.0621	0.1272	0.1455	0.0850	0.0912
(5,5,10)	0.0575	0.0525	0.0629	0.0614	0.0458	0.1347	0.1489	0.0772	0.0809
(10,5,5)	0.0541	0.0553	0.0578	0.0625	0.0572	0.1194	0.1390	0.0721	0.0761
NIG(10,0) Distribution									
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 1.00, 1.00)$									
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT	M93(Ψ_N)	M93(Ψ_L)	M95(Ψ_N)	M95(Ψ_L)
(5,5,5)	0.0442	0.0453	0.0532	0.0532	0.0427	0.1130	0.1281	0.0586	0.0603
(10,10,10)	0.0480	0.0543	0.0579	0.0579	0.0506	0.1542	0.1783	0.0770	0.0805
(20,20,20)	0.0424	0.0518	0.0580	0.0573	0.0498	0.1861	0.2146	0.0759	0.0795
(5,10,20)	0.0558	0.0347	0.0412	0.0372	0.0177	0.1516	0.1691	0.0697	0.0735
(20,10,5)	0.0506	0.0622	0.0618	0.0683	0.0695	0.1339	0.1593	0.0791	0.0885
(5,5,10)	0.0487	0.0447	0.0559	0.0543	0.0340	0.1379	0.1503	0.0692	0.0749
(10,5,5)	0.0537	0.0608	0.0644	0.0689	0.0655	0.1281	0.1470	0.0777	0.0827
NIG(7,2) Distribution									
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 1.00, 1.00)$									
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT	M93(Ψ_N)	M93(Ψ_L)	M95(Ψ_N)	M95(Ψ_L)
(5,5,5)	0.0459	0.0475	0.0562	0.0574	0.0447	0.1051	0.1150	0.0599	0.0602
(10,10,10)	0.0579	0.0538	0.0620	0.0630	0.0554	0.1443	0.1640	0.0786	0.0839
(20,20,20)	0.0497	0.0507	0.0526	0.0535	0.0500	0.1588	0.1824	0.0693	0.0717
(5,10,20)	0.0519	0.0382	0.0530	0.0499	0.0246	0.1384	0.1516	0.0808	0.0834
(20,10,5)	0.0502	0.0492	0.0511	0.0562	0.0550	0.1132	0.1336	0.0736	0.0829
(5,5,10)	0.0512	0.0511	0.0618	0.0607	0.0419	0.1259	0.1363	0.0750	0.0772
(10,5,5)	0.0543	0.0583	0.0612	0.0643	0.0636	0.1162	0.1305	0.0740	0.0778

Table 8.2: Estimated Size of the Median-based Modified and Other Tests for $k = 3$ with $(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 1.00, 1.00)$. L: Levene’s test, LT: Ltrend test, LN: New test presented in this thesis, DCT: Double contrast test, M93: Modified F -test with nested hypotheses, M95: Modified McDermott-Mudholkar (1995) test, Ψ_N : Liptak’s p -value combination, Ψ_L : Mudholkar-George p -value combination. NIG(a,b): Normal inverse Gaussian distribution with mean 0, excess kurtosis a , and skewness b .

Normal Distribution									
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 2.50, 4.00)$									
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT	M93(Ψ_N)	M93(Ψ_L)	M95(Ψ_N)	M95(Ψ_L)
(5,5,5)	0.1173	0.2743	0.2735	0.2746	0.2346	0.3102	0.3175	0.2688	0.2649
(10,10,10)	0.2592	0.5135	0.5614	0.5524	0.4886	0.6327	0.6296	0.5628	0.5595
(20,20,20)	0.6260	0.8290	0.8609	0.8602	0.8200	0.9074	0.9100	0.8816	0.8821
(5,10,20)	0.1468	0.3959	0.4097	0.4061	0.3727	0.4339	0.4382	0.3560	0.3540
(20,10,5)	0.4545	0.6085	0.6204	0.6246	0.5549	0.6744	0.6820	0.6276	0.6309
(5,5,10)	0.1198	0.3027	0.3255	0.3140	0.2702	0.3731	0.3795	0.3130	0.3174
(10,5,5)	0.2197	0.4254	0.4413	0.4382	0.3847	0.4885	0.4991	0.4454	0.4437
Student's <i>t</i> Distribution with 9 Degrees of Freedom									
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 2.50, 4.00)$									
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT	M93(Ψ_N)	M93(Ψ_L)	M95(Ψ_N)	M95(Ψ_L)
(5,5,5)	0.1104	0.2480	0.2466	0.2454	0.2303	0.2700	0.2682	0.2328	0.2364
(10,10,10)	0.2476	0.4835	0.5057	0.4925	0.4577	0.5391	0.5413	0.4822	0.4760
(20,20,20)	0.5597	0.7918	0.8166	0.8130	0.7657	0.8088	0.8098	0.7688	0.7645
(5,10,20)	0.1080	0.3537	0.3708	0.3656	0.3416	0.3964	0.3951	0.3275	0.3261
(20,10,5)	0.3930	0.5530	0.5743	0.5727	0.4775	0.5789	0.5799	0.5175	0.5152
(5,5,10)	0.1004	0.3002	0.3102	0.3047	0.2762	0.3487	0.3484	0.2939	0.2890
(10,5,5)	0.2286	0.3915	0.4094	0.3998	0.3565	0.4266	0.4336	0.3732	0.3749
Exponential Distribution									
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 2.50, 4.00)$									
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT	M93(Ψ_N)	M93(Ψ_L)	M95(Ψ_N)	M95(Ψ_L)
(5,5,5)	0.0777	0.1883	0.1882	0.1867	0.1734	0.2113	0.2194	0.1729	0.1770
(10,10,10)	0.1409	0.3219	0.3254	0.3256	0.2911	0.3272	0.3292	0.2755	0.2773
(20,20,20)	0.2881	0.5278	0.5574	0.5552	0.5167	0.4948	0.4903	0.4470	0.4361
(5,10,20)	0.0573	0.2380	0.2286	0.2272	0.2230	0.2649	0.2753	0.2231	0.2197
(20,10,5)	0.2610	0.3678	0.3636	0.3687	0.3252	0.3539	0.3475	0.2872	0.2767
(5,5,10)	0.0678	0.1905	0.2006	0.2056	0.1714	0.2407	0.2443	0.2015	0.2017
(10,5,5)	0.1581	0.2626	0.2623	0.2629	0.2368	0.2708	0.2849	0.2383	0.2341

Table 8.3: Estimated Power of the Median-based Modified and Other Tests for $k = 3$ with $(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 2.50, 4.00)$. L: Levene's test, LT: Ltrend test, LN: New test presented in this thesis, DCT: Double contrast test, M93: Modified F -test with nested hypotheses, M95: Modified McDermott-Mudholkar (1995) test, Ψ_N : Liptak's p -value combination, Ψ_L : Mudholkar-George p -value combination.

Chi-squared Distribution with 2 Degrees of Freedom									
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 2.50, 4.00)$									
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT	M93(Ψ_N)	M93(Ψ_L)	M95(Ψ_N)	M95(Ψ_L)
(5,5,5)	0.0791	0.1682	0.1693	0.1656	0.1470	0.2035	0.1977	0.1673	0.1656
(10,10,10)	0.1287	0.2821	0.3039	0.3071	0.2715	0.3200	0.3113	0.2856	0.2791
(20,20,20)	0.2950	0.5254	0.5475	0.5417	0.5163	0.5072	0.4982	0.4434	0.4370
(5,10,20)	0.0496	0.2308	0.2288	0.2230	0.2150	0.2621	0.2572	0.2201	0.2165
(20,10,5)	0.2537	0.3761	0.3853	0.3798	0.3117	0.3647	0.3612	0.2891	0.2827
(5,5,10)	0.0669	0.1862	0.2050	0.1956	0.1725	0.2239	0.2222	0.1942	0.1875
(10,5,5)	0.1463	0.2541	0.2535	0.2498	0.2330	0.2604	0.2623	0.2160	0.2210
NIG(10,0) Distribution									
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 2.50, 4.00)$									
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT	M93(Ψ_N)	M93(Ψ_L)	M95(Ψ_N)	M95(Ψ_L)
(5,5,5)	0.0873	0.1905	0.2034	0.2021	0.1725	0.1975	0.1985	0.1862	0.1863
(10,10,10)	0.1558	0.3181	0.3423	0.3346	0.2965	0.2993	0.2967	0.2847	0.2807
(20,20,20)	0.3200	0.5391	0.5524	0.5579	0.5221	0.4557	0.4471	0.4298	0.4224
(5,10,20)	0.0356	0.2594	0.2683	0.2677	0.2617	0.2687	0.2528	0.2420	0.2402
(20,10,5)	0.2722	0.3538	0.3623	0.3520	0.2882	0.3184	0.3117	0.2747	0.2683
(5,5,10)	0.0448	0.2112	0.2191	0.2167	0.1898	0.2342	0.2368	0.2250	0.2167
(10,5,5)	0.1680	0.2829	0.2790	0.2776	0.2656	0.2669	0.2609	0.2382	0.2344
NIG(7,2) Distribution									
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 2.50, 4.00)$									
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT	M93(Ψ_N)	M93(Ψ_L)	M95(Ψ_N)	M95(Ψ_L)
(5,5,5)	0.0802	0.2006	0.1970	0.1964	0.1738	0.2232	0.2281	0.2094	0.2037
(10,10,10)	0.1485	0.3302	0.3441	0.3399	0.2913	0.3343	0.3303	0.3097	0.3046
(20,20,20)	0.3401	0.5862	0.6074	0.6049	0.5713	0.5208	0.5153	0.4688	0.4600
(5,10,20)	0.0550	0.2459	0.2735	0.2709	0.2410	0.2778	0.2779	0.2519	0.2505
(20,10,5)	0.2811	0.3871	0.3999	0.3980	0.3297	0.3817	0.3703	0.3264	0.3192
(5,5,10)	0.0719	0.2223	0.2177	0.2098	0.1912	0.2531	0.2559	0.2171	0.2168
(10,5,5)	0.1625	0.2750	0.2868	0.2871	0.2485	0.3001	0.3099	0.2702	0.2690

Table 8.4: Estimated Power of the Median-based Modified and Other Tests for $k = 3$ with $(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 2.50, 4.00)$. L: Levene’s test, LT: Ltrend test, LN: New test presented in this thesis, DCT: Double contrast test, M93: Modified F -test with nested hypotheses, M95: Modified McDermott-Mudholkar (1995) test, Ψ_N : Liptak’s p -value combination, Ψ_L : Mudholkar-George p -value combination. NIG(a,b): Normal inverse Gaussian distribution with mean 0, excess kurtosis a , and skewness b .

Normal Distribution									
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.00, 1.00, 1.00, 1.00, 1.00)$									
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT	M93(Ψ_N)	M93(Ψ_L)	M95(Ψ_N)	M95(Ψ_L)
(5,5,5,5,5,5)	0.0333	0.0427	0.0461	0.0494	0.0377	0.0507	0.0500	0.0371	0.0331
(10,10,10,10,10,10)	0.0615	0.0523	0.0553	0.0593	0.0527	0.0472	0.0478	0.0441	0.0426
(20,20,20,20,20,20)	0.0495	0.0511	0.0477	0.0512	0.0486	0.0454	0.0451	0.0452	0.0448
(5,5,10,10,20,20)	0.0405	0.0640	0.0699	0.0705	0.0492	0.0515	0.0525	0.0605	0.0603
(20,20,10,10,5,5)	0.0435	0.0330	0.0397	0.0419	0.0351	0.0524	0.0512	0.0585	0.0556
(10,10,10,10,10,20)	0.0525	0.0555	0.0559	0.0591	0.0504	0.0481	0.0492	0.0482	0.0456
(20,10,10,10,10,10)	0.0532	0.0527	0.0529	0.0556	0.0526	0.0530	0.0529	0.0439	0.0405
Student's <i>t</i> Distribution with 9 Degrees of Freedom									
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.00, 1.00, 1.00, 1.00, 1.00)$									
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT	M93(Ψ_N)	M93(Ψ_L)	M95(Ψ_N)	M95(Ψ_L)
(5,5,5,5,5,5)	0.0369	0.0474	0.0512	0.0553	0.0434	0.0590	0.0656	0.0415	0.0401
(10,10,10,10,10,10)	0.0613	0.0544	0.0564	0.0596	0.0550	0.0679	0.0751	0.0496	0.0496
(20,20,20,20,20,20)	0.0475	0.0523	0.0508	0.0527	0.0519	0.0759	0.0837	0.0532	0.0546
(5,5,10,10,20,20)	0.0489	0.0555	0.0628	0.0636	0.0385	0.0641	0.0711	0.0564	0.0570
(20,20,10,10,5,5)	0.0488	0.0377	0.0433	0.0476	0.0449	0.0633	0.0733	0.0604	0.0598
(10,10,10,10,10,20)	0.0566	0.0520	0.0576	0.0606	0.0497	0.0684	0.0737	0.0543	0.0538
(20,10,10,10,10,10)	0.0581	0.0516	0.0519	0.0576	0.0542	0.0665	0.0775	0.0459	0.0469
Exponential Distribution									
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.00, 1.00, 1.00, 1.00, 1.00)$									
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT	M93(Ψ_N)	M93(Ψ_L)	M95(Ψ_N)	M95(Ψ_L)
(5,5,5,5,5,5)	0.0497	0.0501	0.0585	0.0662	0.0470	0.0959	0.1206	0.0595	0.0642
(10,10,10,10,10,10)	0.0689	0.0548	0.0652	0.0720	0.0553	0.1198	0.1547	0.0687	0.0736
(20,20,20,20,20,20)	0.0574	0.0516	0.0561	0.0607	0.0514	0.1431	0.1882	0.0690	0.0756
(5,5,10,10,20,20)	0.0581	0.0542	0.0650	0.0662	0.0320	0.1256	0.1582	0.0768	0.0834
(20,20,10,10,5,5)	0.0576	0.0487	0.0508	0.0592	0.0580	0.1027	0.1386	0.0695	0.0793
(10,10,10,10,10,20)	0.0642	0.0543	0.0663	0.0728	0.0481	0.1351	0.1697	0.0723	0.0784
(20,10,10,10,10,10)	0.0647	0.0584	0.0601	0.0679	0.0654	0.1166	0.1535	0.0620	0.0686

Table 8.5: Estimated Size of the Median-based Modified and Other Tests for $k = 6$ with $(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.00, 1.00, 1.00, 1.00, 1.00)$. L: Levene's test, LT: Ltrend test, LN: New test presented in this thesis, DCT: Double contrast test, M93: Modified F -test with nested hypotheses, M95: Modified McDermott-Mudholkar (1995) test, Ψ_N : Liptak's p -value combination, Ψ_L : Mudholkar-George p -value combination.

Chi-squared Distribution with 2 Degrees of Freedom									
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.00, 1.00, 1.00, 1.00, 1.00)$									
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT	M93(Ψ_N)	M93(Ψ_L)	M95(Ψ_N)	M95(Ψ_L)
(5,5,5,5,5,5)	0.0498	0.0533	0.0662	0.0725	0.0517	0.1024	0.1275	0.0646	0.0668
(10,10,10,10,10,10)	0.0683	0.0551	0.0627	0.0694	0.0566	0.1254	0.1604	0.0688	0.0756
(20,20,20,20,20,20)	0.0574	0.0525	0.0572	0.0616	0.0544	0.1391	0.1806	0.0667	0.0688
(5,5,10,10,20,20)	0.0599	0.0534	0.0664	0.0693	0.0317	0.1308	0.1627	0.0788	0.0839
(20,20,10,10,5,5)	0.0590	0.0497	0.0532	0.0632	0.0587	0.1085	0.1463	0.0770	0.0868
(10,10,10,10,10,20)	0.0636	0.0558	0.0712	0.0783	0.0499	0.1356	0.1685	0.0749	0.0783
(20,10,10,10,10,10)	0.0648	0.0603	0.0640	0.0733	0.0669	0.1205	0.1590	0.0661	0.0713
NIG(10,0) Distribution									
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.00, 1.00, 1.00, 1.00, 1.00)$									
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT	M93(Ψ_N)	M93(Ψ_L)	M95(Ψ_N)	M95(Ψ_L)
(5,5,5,5,5,5)	0.0484	0.0549	0.0617	0.0710	0.0509	0.0994	0.1286	0.0604	0.0627
(10,10,10,10,10,10)	0.0555	0.0549	0.0567	0.0644	0.0532	0.1216	0.1621	0.0648	0.0715
(20,20,20,20,20,20)	0.0473	0.0483	0.0552	0.0594	0.0502	0.1534	0.2092	0.0655	0.0728
(5,5,10,10,20,20)	0.0622	0.0417	0.0537	0.0551	0.0238	0.1394	0.1740	0.0691	0.0767
(20,20,10,10,5,5)	0.0627	0.0564	0.0588	0.0679	0.0730	0.1174	0.1650	0.0741	0.0832
(10,10,10,10,10,20)	0.0578	0.0495	0.0611	0.0646	0.0422	0.1428	0.1895	0.0716	0.0788
(20,10,10,10,10,10)	0.0564	0.0541	0.0616	0.0686	0.0625	0.1229	0.1745	0.0617	0.0719
NIG(7,2) Distribution									
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.00, 1.00, 1.00, 1.00, 1.00)$									
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT	M93(Ψ_N)	M93(Ψ_L)	M95(Ψ_N)	M95(Ψ_L)
(5,5,5,5,5,5)	0.0496	0.0462	0.0565	0.0625	0.0467	0.0944	0.1170	0.0576	0.0599
(10,10,10,10,10,10)	0.0670	0.0589	0.0653	0.0717	0.0590	0.1162	0.1508	0.0678	0.0759
(20,20,20,20,20,20)	0.0558	0.0583	0.0614	0.0646	0.0571	0.1467	0.1938	0.0682	0.0748
(5,5,10,10,20,20)	0.0586	0.0490	0.0653	0.0674	0.0293	0.1241	0.1561	0.0688	0.0727
(20,20,10,10,5,5)	0.0575	0.0492	0.0550	0.0661	0.0589	0.1112	0.1489	0.0780	0.0901
(10,10,10,10,10,20)	0.0627	0.0540	0.0620	0.0685	0.0451	0.1233	0.1620	0.0673	0.0738
(20,10,10,10,10,10)	0.0613	0.0562	0.0620	0.0710	0.0611	0.1207	0.1632	0.0674	0.0756

Table 8.6: Estimated Size of the Median-based Modified and Other Tests for $k = 6$ with $(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.00, 1.00, 1.00, 1.00, 1.00)$. L: Levene's test, LT: Ltrend test, LN: New test presented in this thesis, DCT: Double contrast test, M93: Modified F -test with nested hypotheses, M95: Modified McDermott-Mudholkar (1995) test, Ψ_N : Liptak's p -value combination, Ψ_L : Mudholkar-George p -value combination. NIG(a,b): Normal inverse Gaussian distribution with mean 0, excess kurtosis a , and skewness b .

Normal Distribution									
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.60, 2.20, 2.80, 3.40, 4.00)$									
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT	M93(Ψ_N)	M93(Ψ_L)	M95(Ψ_N)	M95(Ψ_L)
(5,5,5,5,5,5)	0.1150	0.3633	0.3606	0.3599	0.3062	0.3820	0.3885	0.3071	0.3094
(10,10,10,10,10,10)	0.2560	0.6622	0.7052	0.7044	0.6109	0.7506	0.7570	0.6859	0.6891
(20,20,20,20,20,20)	0.6035	0.9189	0.9422	0.9429	0.9064	0.9610	0.9635	0.9509	0.9495
(5,5,10,10,20,20)	0.1394	0.5461	0.5455	0.5426	0.5072	0.5874	0.6020	0.4459	0.4322
(20,20,10,10,5,5)	0.4217	0.7305	0.7609	0.7618	0.6614	0.7936	0.8084	0.7512	0.7556
(10,10,10,10,10,20)	0.2770	0.7430	0.7571	0.7671	0.7210	0.8088	0.8198	0.7334	0.7387
(20,10,10,10,10,10)	0.4634	0.8192	0.8453	0.8493	0.7791	0.8858	0.8931	0.8578	0.8610
Student's <i>t</i> Distribution with 9 Degrees of Freedom									
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.60, 2.20, 2.80, 3.40, 4.00)$									
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT	M93(Ψ_N)	M93(Ψ_L)	M95(Ψ_N)	M95(Ψ_L)
(5,5,5,5,5,5)	0.1047	0.3549	0.3614	0.3607	0.2993	0.3693	0.3719	0.3061	0.3068
(10,10,10,10,10,10)	0.2176	0.6039	0.6323	0.6262	0.5473	0.6467	0.6484	0.5724	0.5669
(20,20,20,20,20,20)	0.5251	0.8765	0.9019	0.9054	0.8555	0.8933	0.8929	0.8533	0.8500
(5,5,10,10,20,20)	0.0986	0.4849	0.4839	0.4750	0.4493	0.5069	0.5130	0.4055	0.3990
(20,20,10,10,5,5)	0.3600	0.6463	0.6922	0.6791	0.5478	0.6857	0.6846	0.6242	0.6117
(10,10,10,10,10,20)	0.2157	0.6733	0.6795	0.6849	0.6462	0.6893	0.7005	0.6186	0.6178
(20,10,10,10,10,10)	0.3951	0.7527	0.7820	0.7771	0.7080	0.7923	0.7983	0.7366	0.7293
Exponential Distribution									
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.60, 2.20, 2.80, 3.40, 4.00)$									
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT	M93(Ψ_N)	M93(Ψ_L)	M95(Ψ_N)	M95(Ψ_L)
(5,5,5,5,5,5)	0.0853	0.2370	0.2366	0.2383	0.2054	0.2461	0.2498	0.2074	0.2045
(10,10,10,10,10,10)	0.1213	0.3854	0.4061	0.3970	0.3376	0.3923	0.3897	0.3302	0.3295
(20,20,20,20,20,20)	0.2798	0.6542	0.6515	0.6525	0.6063	0.5888	0.5752	0.5116	0.4983
(5,5,10,10,20,20)	0.0518	0.3143	0.3108	0.3121	0.3077	0.3151	0.3061	0.2537	0.2461
(20,20,10,10,5,5)	0.2380	0.4642	0.4862	0.4636	0.3706	0.4445	0.4281	0.3491	0.3378
(10,10,10,10,10,20)	0.1116	0.4520	0.4422	0.4376	0.4359	0.4242	0.4280	0.3600	0.3473
(20,10,10,10,10,10)	0.2265	0.5494	0.5283	0.5205	0.4775	0.4890	0.4906	0.4259	0.4213

Table 8.7: Estimated Power of the Median-based Modified and Other Tests for $k = 6$ with $(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.60, 2.20, 2.80, 3.40, 4.00)$. L: Levene's test, LT: Ltrend test, LN: New test presented in this thesis, DCT: Double contrast test, M93: Modified F -test with nested hypotheses, M95: Modified McDermott-Mudholkar (1995) test, Ψ_N : Liptak's p -value combination, Ψ_L : Mudholkar-George p -value combination.

Chi-squared Distribution with 2 Degrees of Freedom									
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.60, 2.20, 2.80, 3.40, 4.00)$									
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT	M93(Ψ_N)	M93(Ψ_L)	M95(Ψ_N)	M95(Ψ_L)
(5,5,5,5,5,5)	0.0806	0.2247	0.2201	0.2200	0.1981	0.2275	0.2370	0.1956	0.1997
(10,10,10,10,10,10)	0.1191	0.3788	0.3914	0.3858	0.3246	0.3730	0.3695	0.3179	0.3180
(20,20,20,20,20,20)	0.2774	0.6715	0.6862	0.6776	0.6268	0.6069	0.6073	0.5296	0.5185
(5,5,10,10,20,20)	0.0482	0.3160	0.2995	0.2930	0.2951	0.3011	0.3073	0.2567	0.2535
(20,20,10,10,5,5)	0.2317	0.4597	0.4702	0.4576	0.3560	0.4361	0.4246	0.3473	0.3374
(10,10,10,10,10,20)	0.1100	0.4666	0.4419	0.4299	0.4355	0.4153	0.4216	0.3508	0.3453
(20,10,10,10,10,10)	0.2317	0.5373	0.5480	0.5369	0.4755	0.5017	0.4964	0.4309	0.4245
NIG(10,0) Distribution									
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.60, 2.20, 2.80, 3.40, 4.00)$									
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT	M93(Ψ_N)	M93(Ψ_L)	M95(Ψ_N)	M95(Ψ_L)
(5,5,5,5,5,5)	0.0768	0.2219	0.2401	0.2395	0.1848	0.2371	0.2295	0.2115	0.2061
(10,10,10,10,10,10)	0.1341	0.3993	0.4104	0.4104	0.3506	0.3598	0.3499	0.3296	0.3186
(20,20,20,20,20,20)	0.2620	0.6401	0.6475	0.6493	0.5896	0.5319	0.5001	0.4956	0.4839
(5,5,10,10,20,20)	0.0325	0.3232	0.3168	0.3216	0.3120	0.2969	0.3039	0.2759	0.2751
(20,20,10,10,5,5)	0.2472	0.4307	0.4445	0.4261	0.3366	0.4012	0.3795	0.3307	0.3151
(10,10,10,10,10,20)	0.1001	0.4493	0.4519	0.4413	0.4243	0.3965	0.3918	0.3627	0.3509
(20,10,10,10,10,10)	0.2270	0.5123	0.5165	0.5149	0.4580	0.4340	0.4308	0.3895	0.3778
NIG(7,2) Distribution									
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.60, 2.20, 2.80, 3.40, 4.00)$									
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT	M93(Ψ_N)	M93(Ψ_L)	M95(Ψ_N)	M95(Ψ_L)
(5,5,5,5,5,5)	0.0771	0.2551	0.2616	0.2566	0.1955	0.2623	0.2637	0.2443	0.2430
(10,10,10,10,10,10)	0.1395	0.4219	0.4423	0.4320	0.3679	0.4001	0.3941	0.3745	0.3678
(20,20,20,20,20,20)	0.3062	0.6816	0.7038	0.6992	0.6368	0.6016	0.5957	0.5519	0.5392
(5,5,10,10,20,20)	0.0477	0.3502	0.3494	0.3404	0.3259	0.3398	0.3351	0.3037	0.2964
(20,20,10,10,5,5)	0.2419	0.4786	0.4912	0.4748	0.3664	0.4491	0.4273	0.3658	0.3522
(10,10,10,10,10,20)	0.1142	0.4981	0.4854	0.4762	0.4686	0.4513	0.4510	0.4104	0.4038
(20,10,10,10,10,10)	0.2265	0.5533	0.5670	0.5487	0.4877	0.4952	0.4813	0.4454	0.4341

Table 8.8: Estimated Power of the Median-based Modified and Other Tests for $k = 6$ with $(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.60, 2.20, 2.80, 3.40, 4.00)$. L: Levene's test, LT: Ltrend test, LN: New test presented in this thesis, DCT: Double contrast test, M93: Modified F -test with nested hypotheses, M95: Modified McDermott-Mudholkar (1995) test, Ψ_N : Liptak's p -value combination, Ψ_L : Mudholkar-George p -value combination. NIG(a,b): Normal inverse Gaussian distribution with mean 0, excess kurtosis a , and skewness b .

Normal Distribution					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 1.00, 1.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5)	0.0842	0.0667	0.0700	0.0739	0.0710
(10,10,10)	0.0662	0.0573	0.0619	0.0620	0.0600
(20,20,20)	0.0578	0.0547	0.0540	0.0558	0.0564
(5,10,20)	0.0514	0.0764	0.0831	0.0800	0.0677
(20,10,5)	0.0590	0.0415	0.0425	0.0441	0.0458
(5,5,10)	0.0755	0.0779	0.0817	0.0841	0.0756
(10,5,5)	0.0722	0.0528	0.0513	0.0540	0.0591
Student's <i>t</i> Distribution with 9 Degrees of Freedom					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 1.00, 1.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5)	0.0831	0.0785	0.0812	0.0823	0.0786
(10,10,10)	0.0622	0.0639	0.0680	0.0678	0.0623
(20,20,20)	0.0558	0.0541	0.0540	0.0557	0.0534
(5,10,20)	0.0633	0.0727	0.0803	0.0784	0.0676
(20,10,5)	0.0608	0.0498	0.0500	0.0543	0.0580
(5,5,10)	0.0747	0.0785	0.0866	0.0864	0.0763
(10,5,5)	0.0740	0.0576	0.0565	0.0625	0.0671
Exponential Distribution					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 1.00, 1.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5)	0.1918	0.1227	0.1240	0.1358	0.1412
(10,10,10)	0.1970	0.1181	0.1194	0.1277	0.1334
(20,20,20)	0.1803	0.1126	0.1114	0.1246	0.1238
(5,10,20)	0.1881	0.1366	0.1432	0.1473	0.1336
(20,10,5)	0.1903	0.0931	0.0930	0.1082	0.1093
(5,5,10)	0.1893	0.1348	0.1390	0.1463	0.1424
(10,5,5)	0.1876	0.1068	0.1060	0.1206	0.1341

Table 8.9: Estimated Size of the Mean-based Tests for $k = 3$ with $(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 1.00, 1.00)$. L: Levene's test, LT: Ltrend test, LN: New test presented in this thesis, DCT: Double contrast test, M93: Modified F -test with nested hypotheses, M95: Modified McDermott-Mudholkar (1995) test, Ψ_N : Liptak's p -value combination, Ψ_L : Mudholkar-George p -value combination.

Chi-squared Distribution with 2 Degrees of Freedom					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 1.00, 1.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5)	0.1977	0.1284	0.1256	0.1369	0.1429
(10,10,10)	0.1983	0.1208	0.1227	0.1321	0.1357
(20,20,20)	0.1838	0.1127	0.1147	0.1234	0.1251
(5,10,20)	0.1809	0.1275	0.1416	0.1450	0.1302
(20,10,5)	0.1844	0.0983	0.0975	0.1114	0.1165
(5,5,10)	0.2005	0.1310	0.1396	0.1464	0.1431
(10,5,5)	0.1961	0.1127	0.1061	0.1226	0.1369
NIG(10,0) Distribution					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 1.00, 1.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5)	0.1200	0.1029	0.0975	0.1028	0.1051
(10,10,10)	0.0887	0.0787	0.0767	0.0813	0.0800
(20,20,20)	0.0678	0.0674	0.0692	0.0739	0.0686
(5,10,20)	0.1002	0.0595	0.0720	0.0665	0.0409
(20,10,5)	0.0937	0.0843	0.0808	0.0922	0.0972
(5,5,10)	0.1208	0.0818	0.0971	0.0981	0.0726
(10,5,5)	0.1200	0.1005	0.0943	0.1065	0.1170
NIG(7,2) Distribution					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 1.00, 1.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5)	0.1598	0.1132	0.1115	0.1211	0.1251
(10,10,10)	0.1623	0.1069	0.1060	0.1153	0.1185
(20,20,20)	0.1555	0.0978	0.0968	0.1033	0.1036
(5,10,20)	0.1476	0.1094	0.1170	0.1169	0.1005
(20,10,5)	0.1415	0.0802	0.0784	0.0913	0.0938
(5,5,10)	0.1635	0.1216	0.1305	0.1315	0.1246
(10,5,5)	0.1623	0.1093	0.1006	0.1143	0.1305

Table 8.10: Estimated Size of the Mean-based Tests for $k = 3$ with $(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 1.00, 1.00)$. L: Levene’s test, LT: Ltrend test, LN: New test presented in this thesis, DCT: Double contrast test, M93: Modified F -test with nested hypotheses, M95: Modified McDermott-Mudholkar (1995) test, Ψ_N : Liptak’s p -value combination, Ψ_L : Mudholkar-George p -value combination. NIG(a,b): Normal inverse Gaussian distribution with mean 0, excess kurtosis a , and skewness b .

Normal Distribution					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 2.50, 4.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5)	0.1294	0.2714	0.2796	0.2804	0.2551
(10,10,10)	0.2750	0.5290	0.5627	0.5672	0.5036
(20,20,20)	0.6411	0.8343	0.8637	0.8639	0.8260
(5,10,20)	0.1685	0.3993	0.4003	0.4070	0.3682
(20,10,5)	0.4247	0.6109	0.6320	0.6306	0.5494
(5,5,10)	0.1219	0.3207	0.3293	0.3305	0.2877
(10,5,5)	0.2502	0.4361	0.4435	0.4453	0.4037
Student's t Distribution with 9 Degrees of Freedom					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 2.50, 4.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5)	0.1171	0.2430	0.2511	0.2501	0.2297
(10,10,10)	0.2573	0.5010	0.5143	0.5237	0.4671
(20,20,20)	0.5656	0.7887	0.8105	0.8083	0.7697
(5,10,20)	0.1292	0.3656	0.3731	0.3723	0.3411
(20,10,5)	0.3853	0.5447	0.5702	0.5719	0.4828
(5,5,10)	0.1017	0.3126	0.3144	0.3134	0.2863
(10,5,5)	0.2307	0.3930	0.4195	0.4157	0.3660
Exponential Distribution					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 2.50, 4.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5)	0.0786	0.1750	0.1851	0.1801	0.1593
(10,10,10)	0.1415	0.3101	0.3123	0.3191	0.2933
(20,20,20)	0.2709	0.4907	0.5232	0.5143	0.4793
(5,10,20)	0.0524	0.2341	0.2453	0.2490	0.2353
(20,10,5)	0.2437	0.3490	0.3665	0.3621	0.2949
(5,5,10)	0.0593	0.1946	0.2111	0.2050	0.1753
(10,5,5)	0.1524	0.2560	0.2615	0.2604	0.2397

Table 8.11: Estimated Power of the Mean-based Tests for $k = 3$ with $(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 2.50, 4.00)$. L: Levene's test, LT: Ltrend test, LN: New test presented in this thesis, DCT: Double contrast test, M93: Modified F -test with nested hypotheses, M95: Modified McDermott-Mudholkar (1995) test, Ψ_N : Liptak's p -value combination, Ψ_L : Mudholkar-George p -value combination.

Chi-squared Distribution with 2 Degrees of Freedom					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 2.50, 4.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5)	0.0830	0.1660	0.1783	0.1747	0.1595
(10,10,10)	0.1333	0.2887	0.2993	0.2986	0.2625
(20,20,20)	0.2792	0.4957	0.5202	0.5160	0.4806
(5,10,20)	0.0488	0.2228	0.2382	0.2373	0.2155
(20,10,5)	0.2346	0.3413	0.3648	0.3561	0.2935
(5,5,10)	0.0563	0.1832	0.1987	0.1937	0.1702
(10,5,5)	0.1508	0.2400	0.2393	0.2405	0.2153
NIG(10,0) Distribution					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 2.50, 4.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5)	0.0780	0.1923	0.2020	0.1990	0.1738
(10,10,10)	0.1578	0.3096	0.3241	0.3242	0.2950
(20,20,20)	0.3096	0.5183	0.5498	0.5436	0.5016
(5,10,20)	0.0242	0.2581	0.2726	0.2686	0.2587
(20,10,5)	0.2561	0.3161	0.3315	0.3271	0.2662
(5,5,10)	0.0425	0.2150	0.2304	0.2247	0.1909
(10,5,5)	0.1825	0.2590	0.2665	0.2727	0.2479
NIG(7,2) Distribution					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 2.50, 4.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5)	0.0781	0.1823	0.1963	0.1947	0.1675
(10,10,10)	0.1479	0.3031	0.3428	0.3321	0.2794
(20,20,20)	0.3054	0.5289	0.5722	0.5674	0.5265
(5,10,20)	0.0465	0.2391	0.2681	0.2568	0.2337
(20,10,5)	0.2756	0.3586	0.3878	0.3779	0.2975
(5,5,10)	0.0649	0.2032	0.2223	0.2108	0.1814
(10,5,5)	0.1634	0.2671	0.2847	0.2807	0.2403

Table 8.12: Estimated Power of the Mean-based Tests for $k = 3$ with $(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 2.50, 4.00)$. L: Levene's test, LT: Ltrend test, LN: New test presented in this thesis, DCT: Double contrast test, M93: Modified F -test with nested hypotheses, M95: Modified McDermott-Mudholkar (1995) test, Ψ_N : Liptak's p -value combination, Ψ_L : Mudholkar-George p -value combination. NIG(a,b): Normal inverse Gaussian distribution with mean 0, excess kurtosis a , and skewness b .

Normal Distribution					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.00, 1.00, 1.00, 1.00, 1.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5,5,5,5)	0.1068	0.0672	0.0737	0.0813	0.0746
(10,10,10,10,10,10)	0.0727	0.0571	0.0598	0.0649	0.0587
(20,20,20,20,20,20)	0.0557	0.0537	0.0529	0.0562	0.0528
(5,5,10,10,20,20)	0.0652	0.0962	0.1056	0.1096	0.0821
(20,20,10,10,5,5)	0.0682	0.0314	0.0367	0.0409	0.0394
(10,10,10,10,10,20)	0.0627	0.0689	0.0657	0.0687	0.0629
(20,10,10,10,10,10)	0.0657	0.0518	0.0491	0.0532	0.0532
Student's <i>t</i> Distribution with 9 Degrees of Freedom					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.00, 1.00, 1.00, 1.00, 1.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5,5,5,5)	0.1147	0.0764	0.0787	0.0899	0.0798
(10,10,10,10,10,10)	0.0781	0.0615	0.0650	0.0705	0.0644
(20,20,20,20,20,20)	0.0577	0.0568	0.0564	0.0586	0.0562
(5,5,10,10,20,20)	0.0743	0.0836	0.0980	0.1029	0.0698
(20,20,10,10,5,5)	0.0758	0.0400	0.0411	0.0472	0.0517
(10,10,10,10,10,20)	0.0724	0.0669	0.0699	0.0727	0.0648
(20,10,10,10,10,10)	0.0713	0.0507	0.0486	0.0560	0.0583
Exponential Distribution					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.00, 1.00, 1.00, 1.00, 1.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5,5,5,5)	0.3498	0.1232	0.1257	0.1567	0.1541
(10,10,10,10,10,10)	0.3013	0.1130	0.1156	0.1408	0.1334
(20,20,20,20,20,20)	0.2949	0.1048	0.1110	0.1292	0.1234
(5,5,10,10,20,20)	0.2749	0.1437	0.1553	0.1750	0.1371
(20,20,10,10,5,5)	0.2799	0.0790	0.0841	0.1101	0.1069
(10,10,10,10,10,20)	0.2929	0.1287	0.1303	0.1523	0.1467
(20,10,10,10,10,10)	0.2910	0.0996	0.0965	0.1229	0.1279

Table 8.13: Estimated Size of the Mean-based Tests for $k = 6$ with $(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.00, 1.00, 1.00, 1.00, 1.00)$. L: Levene's test, LT: Ltrend test, LN: New test presented in this thesis, DCT: Double contrast test, M93: Modified F -test with nested hypotheses, M95: Modified McDermott-Mudholkar (1995) test, Ψ_N : Liptak's p -value combination, Ψ_L : Mudholkar-George p -value combination.

Chi-squared Distribution with 2 Degrees of Freedom					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.00, 1.00, 1.00, 1.00, 1.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5,5,5,5)	0.3528	0.1243	0.1381	0.1702	0.1602
(10,10,10,10,10,10)	0.3058	0.1148	0.1167	0.1423	0.1308
(20,20,20,20,20,20)	0.2924	0.1081	0.1123	0.1324	0.1273
(5,5,10,10,20,20)	0.2830	0.1449	0.1549	0.1766	0.1407
(20,20,10,10,5,5)	0.2776	0.0850	0.0889	0.1133	0.1111
(10,10,10,10,10,20)	0.2938	0.1291	0.1283	0.1530	0.1475
(20,10,10,10,10,10)	0.2939	0.1053	0.1046	0.1288	0.1347
NIG(10,0) Distribution					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.00, 1.00, 1.00, 1.00, 1.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5,5,5,5)	0.2304	0.1077	0.1093	0.1328	0.1206
(10,10,10,10,10,10)	0.1324	0.0801	0.0791	0.0936	0.0854
(20,20,20,20,20,20)	0.0879	0.0655	0.0694	0.0785	0.0674
(5,5,10,10,20,20)	0.1350	0.0730	0.0938	0.0993	0.0511
(20,20,10,10,5,5)	0.1420	0.0787	0.0744	0.0981	0.1017
(10,10,10,10,10,20)	0.1271	0.0762	0.0843	0.0953	0.0716
(20,10,10,10,10,10)	0.1283	0.0747	0.0749	0.0892	0.0900
NIG(7,2) Distribution					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.00, 1.00, 1.00, 1.00, 1.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5,5,5,5)	0.2948	0.1134	0.1167	0.1448	0.1411
(10,10,10,10,10,10)	0.2550	0.1075	0.1076	0.1289	0.1226
(20,20,20,20,20,20)	0.2330	0.1052	0.1042	0.1237	0.1200
(5,5,10,10,20,20)	0.2372	0.1275	0.1409	0.1595	0.1141
(20,20,10,10,5,5)	0.2386	0.0791	0.0820	0.1083	0.1034
(10,10,10,10,10,20)	0.2490	0.1135	0.1133	0.1355	0.1201
(20,10,10,10,10,10)	0.2417	0.0927	0.0963	0.1191	0.1170

Table 8.14: Estimated Size of the Mean-based Tests for $k = 6$ with $(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.00, 1.00, 1.00, 1.00, 1.00)$. L: Levene’s test, LT: Ltrend test, LN: New test presented in this thesis, DCT: Double contrast test, M93: Modified F -test with nested hypotheses, M95: Modified McDermott-Mudholkar (1995) test, Ψ_N : Liptak’s p -value combination, Ψ_L : Mudholkar-George p -value combination. NIG(a,b): Normal inverse Gaussian distribution with mean 0, excess kurtosis a , and skewness b .

Normal Distribution					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.60, 2.20, 2.80, 3.40, 4.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5,5,5,5)	0.1269	0.3667	0.3633	0.3639	0.3200
(10,10,10,10,10,10)	0.2680	0.6759	0.7099	0.7170	0.6275
(20,20,20,20,20,20)	0.6178	0.9213	0.9459	0.9445	0.9094
(5,5,10,10,20,20)	0.1694	0.5575	0.5400	0.5472	0.5138
(20,20,10,10,5,5)	0.4011	0.7284	0.7710	0.7713	0.6594
(10,10,10,10,10,20)	0.3240	0.7454	0.7619	0.7766	0.7231
(20,10,10,10,10,10)	0.4609	0.8262	0.8567	0.8568	0.7835
Student's <i>t</i> Distribution with 9 Degrees of Freedom					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.60, 2.20, 2.80, 3.40, 4.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5,5,5,5)	0.1053	0.3451	0.3584	0.3585	0.2912
(10,10,10,10,10,10)	0.2299	0.6084	0.6419	0.6389	0.5612
(20,20,20,20,20,20)	0.5286	0.8843	0.9019	0.9040	0.8621
(5,5,10,10,20,20)	0.1157	0.4927	0.4792	0.4844	0.4617
(20,20,10,10,5,5)	0.3524	0.6415	0.6884	0.6818	0.5327
(10,10,10,10,10,20)	0.2464	0.6813	0.6820	0.6956	0.6506
(20,10,10,10,10,10)	0.3915	0.7627	0.7897	0.7914	0.7062
Exponential Distribution					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.60, 2.20, 2.80, 3.40, 4.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5,5,5,5)	0.0804	0.2309	0.2443	0.2456	0.1902
(10,10,10,10,10,10)	0.1208	0.3693	0.4035	0.3901	0.3247
(20,20,20,20,20,20)	0.2473	0.6098	0.6195	0.6159	0.5550
(5,5,10,10,20,20)	0.0492	0.3078	0.3124	0.3058	0.2921
(20,20,10,10,5,5)	0.2250	0.4357	0.4605	0.4456	0.3479
(10,10,10,10,10,20)	0.1091	0.4264	0.4356	0.4253	0.4019
(20,10,10,10,10,10)	0.2082	0.5159	0.5143	0.5070	0.4388

Table 8.15: Estimated Power of the Mean-based Tests for $k = 6$ with $(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.60, 2.20, 2.80, 3.40, 4.00)$. L: Levene's test, LT: Ltrend test, LN: New test presented in this thesis, DCT: Double contrast test, M93: Modified F -test with nested hypotheses, M95: Modified McDermott-Mudholkar (1995) test, Ψ_N : Liptak's p -value combination, Ψ_L : Mudholkar-George p -value combination.

Chi-squared Distribution with 2 Degrees of Freedom					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.60, 2.20, 2.80, 3.40, 4.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5,5,5,5)	0.0754	0.2175	0.2239	0.2239	0.1875
(10,10,10,10,10,10)	0.1309	0.3632	0.3777	0.3758	0.3247
(20,20,20,20,20,20)	0.2511	0.6332	0.6473	0.6352	0.5806
(5,5,10,10,20,20)	0.0459	0.3104	0.2985	0.2993	0.2757
(20,20,10,10,5,5)	0.2153	0.4188	0.4504	0.4303	0.3234
(10,10,10,10,10,20)	0.1146	0.4450	0.4194	0.4160	0.4054
(20,10,10,10,10,10)	0.2048	0.5062	0.5241	0.5247	0.4385
NIG(10,0) Distribution					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.60, 2.20, 2.80, 3.40, 4.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5,5,5,5)	0.0787	0.2221	0.2430	0.2408	0.1737
(10,10,10,10,10,10)	0.1300	0.3744	0.3966	0.3940	0.3263
(20,20,20,20,20,20)	0.2426	0.6242	0.6384	0.6281	0.5769
(5,5,10,10,20,20)	0.0259	0.3191	0.3157	0.3144	0.2991
(20,20,10,10,5,5)	0.2222	0.3938	0.4274	0.3990	0.2999
(10,10,10,10,10,20)	0.0858	0.4328	0.4362	0.4364	0.4148
(20,10,10,10,10,10)	0.2154	0.4922	0.5074	0.5034	0.4303
NIG(7,2) Distribution					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.60, 2.20, 2.80, 3.40, 4.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5,5,5,5)	0.0701	0.2352	0.2520	0.2462	0.1896
(10,10,10,10,10,10)	0.1239	0.4029	0.4267	0.4166	0.3333
(20,20,20,20,20,20)	0.2557	0.6258	0.6646	0.6459	0.5754
(5,5,10,10,20,20)	0.0402	0.3259	0.3501	0.3432	0.3045
(20,20,10,10,5,5)	0.2078	0.4207	0.4664	0.4242	0.3200
(10,10,10,10,10,20)	0.1059	0.4601	0.4640	0.4605	0.4242
(20,10,10,10,10,10)	0.1909	0.5037	0.5317	0.5172	0.4246

Table 8.16: Estimated Power of the Mean-based Tests for $k = 6$ with $(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.60, 2.20, 2.80, 3.40, 4.00)$. L: Levene’s test, LT: Ltrend test, LN: New test presented in this thesis, DCT: Double contrast test, M93: Modified F -test with nested hypotheses, M95: Modified McDermott-Mudholkar (1995) test, Ψ_N : Liptak’s p -value combination, Ψ_L : Mudholkar-George p -value combination. NIG(a,b): Normal inverse Gaussian distribution with mean 0, excess kurtosis a , and skewness b .

Normal Distribution					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 1.00, 1.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5)	0.0045	0.0186	0.0199	0.0172	0.0132
(10,10,10)	0.0351	0.0395	0.0433	0.0415	0.0384
(20,20,20)	0.0392	0.0451	0.0452	0.0456	0.0427
(5,10,20)	0.0230	0.0684	0.0693	0.0642	0.0526
(20,10,5)	0.0280	0.0165	0.0204	0.0214	0.0177
(5,5,10)	0.0164	0.0510	0.0393	0.0381	0.0392
(10,5,5)	0.0165	0.0130	0.0136	0.0123	0.0107
Student's t Distribution with 9 Degrees of Freedom					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 1.00, 1.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5)	0.0055	0.0220	0.0245	0.0220	0.0171
(10,10,10)	0.0307	0.0427	0.0452	0.0447	0.0413
(20,20,20)	0.0375	0.0417	0.0433	0.0436	0.0409
(5,10,20)	0.0281	0.0640	0.0679	0.0650	0.0513
(20,10,5)	0.0284	0.0227	0.0240	0.0270	0.0240
(5,5,10)	0.0151	0.0517	0.0422	0.0405	0.0395
(10,5,5)	0.0145	0.0172	0.0170	0.0168	0.0168
Exponential Distribution					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 1.00, 1.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5)	0.0131	0.0289	0.0346	0.0324	0.0218
(10,10,10)	0.0486	0.0514	0.0562	0.0572	0.0492
(20,20,20)	0.0462	0.0495	0.0515	0.0526	0.0488
(5,10,20)	0.0405	0.0588	0.0736	0.0700	0.0423
(20,10,5)	0.0416	0.0323	0.0373	0.0399	0.0367
(5,5,10)	0.0247	0.0557	0.0525	0.0494	0.0453
(10,5,5)	0.0276	0.0252	0.0289	0.0283	0.0251

Table 8.17: Estimated Size of the Median-based Tests for $k = 3$ with $(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 1.00, 1.00)$. L: Levene's test, LT: Ltrend test, LN: New test presented in this thesis, DCT: Double contrast test, M93: Modified F -test with nested hypotheses, M95: Modified McDermott-Mudholkar (1995) test, Ψ_N : Liptak's p -value combination, Ψ_L : Mudholkar-George p -value combination.

Chi-squared Distribution with 2 Degrees of Freedom					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 1.00, 1.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5)	0.0105	0.0261	0.0332	0.0305	0.0201
(10,10,10)	0.0455	0.0537	0.0590	0.0609	0.0501
(20,20,20)	0.0447	0.0482	0.0547	0.0547	0.0492
(5,10,20)	0.0394	0.0559	0.0732	0.0682	0.0398
(20,10,5)	0.0370	0.0329	0.0350	0.0376	0.0373
(5,5,10)	0.0264	0.0541	0.0542	0.0506	0.0417
(10,5,5)	0.0275	0.0284	0.0296	0.0298	0.0277
NIG(10,0) Distribution					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 1.00, 1.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5)	0.0088	0.0282	0.0331	0.0295	0.0200
(10,10,10)	0.0400	0.0457	0.0445	0.0440	0.0412
(20,20,20)	0.0418	0.0515	0.0515	0.0516	0.0482
(5,10,20)	0.0333	0.0553	0.0605	0.0576	0.0366
(20,10,5)	0.0314	0.0277	0.0301	0.0323	0.0293
(5,5,10)	0.0183	0.0472	0.0466	0.0442	0.0352
(10,5,5)	0.0195	0.0270	0.0264	0.0267	0.0264
NIG(7,2) Distribution					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 1.00, 1.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5)	0.0117	0.0279	0.0323	0.0299	0.0207
(10,10,10)	0.0418	0.0458	0.0518	0.0517	0.0437
(20,20,20)	0.0404	0.0464	0.0475	0.0483	0.0445
(5,10,20)	0.0356	0.0525	0.0653	0.0627	0.0361
(20,10,5)	0.0324	0.0282	0.0316	0.0332	0.0327
(5,5,10)	0.0228	0.0531	0.0508	0.0483	0.0387
(10,5,5)	0.0230	0.0275	0.0299	0.0296	0.0291

Table 8.18: Estimated Size of the Median-based Tests for $k = 3$ with $(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 1.00, 1.00)$. L: Levene’s test, LT: Ltrend test, LN: New test presented in this thesis, DCT: Double contrast test, M93: Modified F -test with nested hypotheses, M95: Modified McDermott-Mudholkar (1995) test, Ψ_N : Liptak’s p -value combination, Ψ_L : Mudholkar-George p -value combination. NIG(a,b): Normal inverse Gaussian distribution with mean 0, excess kurtosis a , and skewness b .

Normal Distribution					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 2.50, 4.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5)	0.1441	0.2874	0.2898	0.2978	0.2674
(10,10,10)	0.2718	0.5217	0.5664	0.5635	0.5018
(20,20,20)	0.6325	0.8322	0.8630	0.8625	0.8249
(5,10,20)	0.2257	0.3979	0.4096	0.4117	0.3778
(20,10,5)	0.3477	0.6125	0.6269	0.6295	0.5631
(5,5,10)	0.1888	0.3152	0.3374	0.3335	0.2909
(10,5,5)	0.1654	0.4447	0.4457	0.4527	0.4008
Student's <i>t</i> Distribution with 9 Degrees of Freedom					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 2.50, 4.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5)	0.1271	0.2585	0.2538	0.2571	0.2459
(10,10,10)	0.2538	0.4898	0.5131	0.5119	0.4631
(20,20,20)	0.5665	0.7926	0.8186	0.8163	0.7701
(5,10,20)	0.1831	0.3601	0.3731	0.3701	0.3460
(20,10,5)	0.3209	0.5553	0.5757	0.5842	0.4911
(5,5,10)	0.1629	0.3044	0.3232	0.3248	0.2880
(10,5,5)	0.1682	0.4041	0.4169	0.4155	0.3527
Exponential Distribution					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 2.50, 4.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5)	0.0850	0.1958	0.1991	0.2008	0.1861
(10,10,10)	0.1441	0.3252	0.3300	0.3322	0.2977
(20,20,20)	0.2902	0.5303	0.5568	0.5581	0.5184
(5,10,20)	0.0932	0.2396	0.2379	0.2304	0.2247
(20,10,5)	0.2170	0.3733	0.3743	0.3794	0.3327
(5,5,10)	0.0990	0.1966	0.2124	0.2144	0.1821
(10,5,5)	0.1322	0.2743	0.2762	0.2794	0.2556

Table 8.19: Estimated Power of the Median-based Tests for $k = 3$ with $(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 2.50, 4.00)$. L: Levene's test, LT: Ltrend test, LN: New test presented in this thesis, DCT: Double contrast test, M93: Modified F -test with nested hypotheses, M95: Modified McDermott-Mudholkar (1995) test, Ψ_N : Liptak's p -value combination, Ψ_L : Mudholkar-George p -value combination.

Chi-squared Distribution with 2 Degrees of Freedom					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 2.50, 4.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5)	0.0893	0.1787	0.1803	0.1799	0.1706
(10,10,10)	0.1346	0.2901	0.3099	0.3097	0.2797
(20,20,20)	0.2967	0.5292	0.5499	0.5446	0.5174
(5,10,20)	0.0847	0.2304	0.2323	0.2268	0.2164
(20,10,5)	0.2146	0.3785	0.3930	0.3878	0.3193
(5,5,10)	0.0911	0.1926	0.2141	0.2121	0.1790
(10,5,5)	0.1236	0.2561	0.2600	0.2660	0.2425
NIG(10,0) Distribution					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 2.50, 4.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5)	0.0920	0.1986	0.2063	0.2066	0.1793
(10,10,10)	0.1616	0.3201	0.3482	0.3419	0.3035
(20,20,20)	0.3241	0.5413	0.5556	0.5604	0.5233
(5,10,20)	0.0720	0.2587	0.2697	0.2692	0.2604
(20,10,5)	0.2459	0.3520	0.3693	0.3672	0.2962
(5,5,10)	0.0840	0.2138	0.2258	0.2309	0.2023
(10,5,5)	0.1443	0.2885	0.2825	0.2859	0.2716
NIG(7,2) Distribution					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 2.50, 4.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5)	0.0913	0.2126	0.2099	0.2104	0.1921
(10,10,10)	0.1602	0.3329	0.3495	0.3478	0.3062
(20,20,20)	0.3494	0.5872	0.6103	0.6092	0.5760
(5,10,20)	0.0982	0.2498	0.2764	0.2735	0.2435
(20,10,5)	0.2382	0.3938	0.4133	0.4111	0.3386
(5,5,10)	0.1141	0.2298	0.2327	0.2326	0.2073
(10,5,5)	0.1367	0.2898	0.3023	0.3003	0.2615

Table 8.20: Estimated Power of the Median-based Tests for $k = 3$ with $(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.00, 2.50, 4.00)$. L: Levene’s test, LT: Ltrend test, LN: New test presented in this thesis, DCT: Double contrast test, M93: Modified F -test with nested hypotheses, M95: Modified McDermott-Mudholkar (1995) test, Ψ_N : Liptak’s p -value combination, Ψ_L : Mudholkar-George p -value combination. NIG(a,b): Normal inverse Gaussian distribution with mean 0, excess kurtosis a , and skewness b .

Normal Distribution					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.00, 1.00, 1.00, 1.00, 1.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5,5,5,5)	0.0026	0.0208	0.0258	0.0228	0.0138
(10,10,10,10,10,10)	0.0289	0.0401	0.0415	0.0420	0.0366
(20,20,20,20,20,20)	0.0315	0.0430	0.0413	0.0433	0.0411
(5,5,10,10,20,20)	0.0233	0.0923	0.0866	0.0838	0.0696
(20,20,10,10,5,5)	0.0219	0.0107	0.0163	0.0177	0.0128
(10,10,10,10,10,20)	0.0274	0.0540	0.0488	0.0486	0.0450
(20,10,10,10,10,10)	0.0262	0.0311	0.0314	0.0328	0.0316
Student's <i>t</i> Distribution with 9 Degrees of Freedom					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.00, 1.00, 1.00, 1.00, 1.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5,5,5,5)	0.0044	0.0252	0.0289	0.0272	0.0214
(10,10,10,10,10,10)	0.0306	0.0420	0.0433	0.0449	0.0407
(20,20,20,20,20,20)	0.0328	0.0463	0.0440	0.0459	0.0447
(5,5,10,10,20,20)	0.0274	0.0821	0.0810	0.0781	0.0573
(20,20,10,10,5,5)	0.0258	0.0150	0.0199	0.0222	0.0222
(10,10,10,10,10,20)	0.0323	0.0521	0.0518	0.0536	0.0462
(20,10,10,10,10,10)	0.0311	0.0327	0.0308	0.0336	0.0370
Exponential Distribution					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.00, 1.00, 1.00, 1.00, 1.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5,5,5,5)	0.0120	0.0330	0.0398	0.0401	0.0278
(10,10,10,10,10,10)	0.0462	0.0475	0.0563	0.0614	0.0468
(20,20,20,20,20,20)	0.0480	0.0482	0.0530	0.0558	0.0463
(5,5,10,10,20,20)	0.0360	0.0752	0.0808	0.0827	0.0490
(20,20,10,10,5,5)	0.0377	0.0260	0.0307	0.0364	0.0346
(10,10,10,10,10,20)	0.0458	0.0571	0.0613	0.0664	0.0490
(20,10,10,10,10,10)	0.0455	0.0440	0.0449	0.0496	0.0519

Table 8.21: Estimated Size of the Median-based Tests for $k = 6$ with $(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.00, 1.00, 1.00, 1.00, 1.00)$. L: Levene's test, LT: Ltrend test, LN: New test presented in this thesis, DCT: Double contrast test, M93: Modified F -test with nested hypotheses, M95: Modified McDermott-Mudholkar (1995) test, Ψ_N : Liptak's p -value combination, Ψ_L : Mudholkar-George p -value combination.

Chi-squared Distribution with 2 Degrees of Freedom					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.00, 1.00, 1.00, 1.00, 1.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5,5,5,5)	0.0128	0.0372	0.0428	0.0447	0.0300
(10,10,10,10,10,10)	0.0458	0.0478	0.0537	0.0596	0.0472
(20,20,20,20,20,20)	0.0472	0.0490	0.0536	0.0575	0.0489
(5,5,10,10,20,20)	0.0384	0.0734	0.0825	0.0832	0.0473
(20,20,10,10,5,5)	0.0392	0.0256	0.0330	0.0387	0.0373
(10,10,10,10,10,20)	0.0450	0.0590	0.0665	0.0719	0.0505
(20,10,10,10,10,10)	0.0474	0.0454	0.0477	0.0529	0.0516
NIG(10,0) Distribution					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.00, 1.00, 1.00, 1.00, 1.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5,5,5,5)	0.0118	0.0375	0.0402	0.0426	0.0333
(10,10,10,10,10,10)	0.0359	0.0475	0.0506	0.0539	0.0448
(20,20,20,20,20,20)	0.0391	0.0454	0.0508	0.0544	0.0452
(5,5,10,10,20,20)	0.0380	0.0601	0.0701	0.0678	0.0356
(20,20,10,10,5,5)	0.0370	0.0345	0.0367	0.0435	0.0492
(10,10,10,10,10,20)	0.0399	0.0525	0.0564	0.0579	0.0430
(20,10,10,10,10,10)	0.0372	0.0417	0.0430	0.0491	0.0482
NIG(7,2) Distribution					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.00, 1.00, 1.00, 1.00, 1.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5,5,5,5)	0.0113	0.0318	0.0380	0.0383	0.0280
(10,10,10,10,10,10)	0.0436	0.0492	0.0551	0.0600	0.0491
(20,20,20,20,20,20)	0.0454	0.0535	0.0565	0.0601	0.0526
(5,5,10,10,20,20)	0.0354	0.0705	0.0834	0.0809	0.0440
(20,20,10,10,5,5)	0.0367	0.0262	0.0320	0.0393	0.0345
(10,10,10,10,10,20)	0.0458	0.0555	0.0575	0.0609	0.0453
(20,10,10,10,10,10)	0.0428	0.0411	0.0449	0.0526	0.0465

Table 8.22: Estimated Size of the Median-based Tests for $k = 6$ with $(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.00, 1.00, 1.00, 1.00, 1.00)$. L: Levene’s test, LT: Ltrend test, LN: New test presented in this thesis, DCT: Double contrast test, M93: Modified F -test with nested hypotheses, M95: Modified McDermott-Mudholkar (1995) test, Ψ_N : Liptak’s p -value combination, Ψ_L : Mudholkar-George p -value combination. NIG(a,b): Normal inverse Gaussian distribution with mean 0, excess kurtosis a , and skewness b .

Normal Distribution					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.60, 2.20, 2.80, 3.40, 4.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5,5,5,5)	0.1285	0.3789	0.3658	0.3668	0.3191
(10,10,10,10,10,10)	0.2669	0.6687	0.7064	0.7076	0.6209
(20,20,20,20,20,20)	0.6120	0.9204	0.9427	0.9434	0.9072
(5,5,10,10,20,20)	0.2268	0.5532	0.5577	0.5608	0.5124
(20,20,10,10,5,5)	0.2983	0.7305	0.7648	0.7687	0.6671
(10,10,10,10,10,20)	0.3336	0.7484	0.7652	0.7705	0.7241
(20,10,10,10,10,10)	0.4275	0.8212	0.8484	0.8500	0.7754
Student's <i>t</i> Distribution with 9 Degrees of Freedom					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.60, 2.20, 2.80, 3.40, 4.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5,5,5,5)	0.1101	0.3637	0.3670	0.3728	0.3098
(10,10,10,10,10,10)	0.2247	0.6104	0.6375	0.6308	0.5517
(20,20,20,20,20,20)	0.5312	0.8789	0.9029	0.9052	0.8578
(5,5,10,10,20,20)	0.1773	0.4936	0.4946	0.4924	0.4532
(20,20,10,10,5,5)	0.2825	0.6491	0.6986	0.6894	0.5482
(10,10,10,10,10,20)	0.2711	0.6759	0.6855	0.6904	0.6476
(20,10,10,10,10,10)	0.3643	0.7527	0.7838	0.7827	0.7039
Exponential Distribution					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.60, 2.20, 2.80, 3.40, 4.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5,5,5,5)	0.0904	0.2391	0.2453	0.2493	0.2118
(10,10,10,10,10,10)	0.1246	0.3888	0.4139	0.4055	0.3410
(20,20,20,20,20,20)	0.2821	0.6549	0.6547	0.6569	0.6073
(5,5,10,10,20,20)	0.0842	0.3175	0.3266	0.3231	0.3060
(20,20,10,10,5,5)	0.1994	0.4652	0.4910	0.4692	0.3727
(10,10,10,10,10,20)	0.1407	0.4542	0.4509	0.4464	0.4306
(20,10,10,10,10,10)	0.2134	0.5526	0.5370	0.5299	0.4768

Table 8.23: Estimated Power of the Median-based Tests for $k = 6$ with $(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.60, 2.20, 2.80, 3.40, 4.00)$. L: Levene's test, LT: Ltrend test, LN: New test presented in this thesis, DCT: Double contrast test, M93: Modified F -test with nested hypotheses, M95: Modified McDermott-Mudholkar (1995) test, Ψ_N : Liptak's p -value combination, Ψ_L : Mudholkar-George p -value combination.

Chi-squared Distribution with 2 Degrees of Freedom					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.60, 2.20, 2.80, 3.40, 4.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5,5,5,5)	0.0885	0.2284	0.2311	0.2327	0.2009
(10,10,10,10,10,10)	0.1241	0.3829	0.3941	0.3908	0.3327
(20,20,20,20,20,20)	0.2814	0.6734	0.6886	0.6805	0.6294
(5,5,10,10,20,20)	0.0768	0.3133	0.3070	0.3034	0.2903
(20,20,10,10,5,5)	0.1963	0.4600	0.4766	0.4697	0.3697
(10,10,10,10,10,20)	0.1288	0.4680	0.4437	0.4412	0.4312
(20,10,10,10,10,10)	0.2159	0.5441	0.5561	0.5485	0.4654
NIG(10,0) Distribution					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.60, 2.20, 2.80, 3.40, 4.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5,5,5,5)	0.0777	0.2278	0.2477	0.2512	0.1915
(10,10,10,10,10,10)	0.1374	0.4034	0.4145	0.4190	0.3525
(20,20,20,20,20,20)	0.2644	0.6385	0.6499	0.6508	0.5937
(5,5,10,10,20,20)	0.0585	0.3282	0.3270	0.3335	0.3056
(20,20,10,10,5,5)	0.2126	0.4309	0.4500	0.4377	0.3427
(10,10,10,10,10,20)	0.1216	0.4516	0.4570	0.4468	0.4179
(20,10,10,10,10,10)	0.2100	0.5149	0.5229	0.5237	0.4564
NIG(7,2) Distribution					
$(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.60, 2.20, 2.80, 3.40, 4.00)$					
Sample Sizes	L	LT	LN(Ψ_N)	LN(Ψ_L)	DCT
(5,5,5,5,5,5)	0.0890	0.2550	0.2734	0.2717	0.2027
(10,10,10,10,10,10)	0.1435	0.4226	0.4494	0.4429	0.3696
(20,20,20,20,20,20)	0.3103	0.6812	0.7063	0.7033	0.6397
(5,5,10,10,20,20)	0.0830	0.3516	0.3578	0.3508	0.3236
(20,20,10,10,5,5)	0.1976	0.4781	0.4991	0.4919	0.3688
(10,10,10,10,10,20)	0.1410	0.4998	0.4911	0.4902	0.4712
(20,10,10,10,10,10)	0.2108	0.5574	0.5777	0.5587	0.4851

Table 8.24: Estimated Power of the Median-based Tests for $k = 6$ with $(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2, \sigma_6^2) = (1.00, 1.60, 2.20, 2.80, 3.40, 4.00)$. L: Levene’s test, LT: Ltrend test, LN: New test presented in this thesis, DCT: Double contrast test, M93: Modified F -test with nested hypotheses, M95: Modified McDermott-Mudholkar (1995) test, Ψ_N : Liptak’s p -value combination, Ψ_L : Mudholkar-George p -value combination. NIG(a,b): Normal inverse Gaussian distribution with mean 0, excess kurtosis a , and skewness b .

Appendix C: R Code for the New Test in Chapter 4

Appendix C contains R code for the new test (LN) statistic (4.5) as well as its robustifications suggested in Chapter 5. The code for LN invokes the code for Ltrend test (LT) when no robustification is applied, which is also included in this appendix.

```
#####
#####LN TEST#####

ltrend.combined<-function(y,group,option=c("median","mean","trim.mean"),
tail=c("right","left","both"),trim.alpha=0.25, cor.tech = c("none", "zero.removal",
"zero.removal2", "zero.removal3", "cor.zero1", "cor.zero2","correction.factor"))
{

### none: usual Levene
### zero.removal: Hines-Hines structural zero removal
### zero.removal2: Odd sample structural zero removal
### zero.removal3: Modified structural zero removal
### correction.factor: O'Brien correction factor
### cor.zero1: Odd sample structural zero removal and O'Brien correction factor
### cor.zero2: Modified structural zero removal and O'Brien correction factor

option <- match.arg(option)
tail <- match.arg(tail)
```

```
cor.tech <- match.arg(cor.tech)

if (length(y)!=length(group))
{
  stop("the length of the data (y) does not match the length of the group")
}

### sort the order just in case the input is not sorted by group

reorder <- order(group)
group <- group[reorder]
y<-y[reorder]
gr<-group
group<-as.factor(group)

if (option == "mean")
{
  means<-tapply(y, group, mean)
}
else if (option == "median")
{
  means<-tapply(y, group, median)
}
else
{option = "trim.mean"
  trimmed.mean <- function(y) mean(y, trim = trim.alpha)
  means <- tapply(y, group, trimmed.mean)
}

z<-y-means[group]
n<-tapply(z,group,length)
```

```
ngroup<-n[group]
if(cor.tech == "correction.factor")
{
  correction<-sqrt(ngroup/(ngroup-1))
  z<-z*correction
}
k<-length(n)
m<-k-1

p<-double(m)
q<-double(m)

if(option != "median" && cor.tech != "correction.factor")
{
  cor.tech = "none"
}

if(cor.tech == "cor.zero2" || cor.tech == "zero.removal" || cor.tech == "zero.removal3")
{
  resp.mean <-z
  k<-length(n)
  temp<-double()
  endpos<-double()
  startpos<-double()
  for(i in 1:k)
  {
    group.size<-n[i]
    j<-i-1
    if(i==1)start<-1
    else start<-sum(n[1:j])+1
    startpos<-c(startpos,start)
```



```

end<-sum(n[1:i])
endpos<-c(endpos,end)
sub.resp.mean<-resp.mean[start:end]
sub.resp.mean<-sub.resp.mean[order(sub.resp.mean)]
if(group.size%%2==1)
{
mid<-(group.size+1)/2
temp2<-sub.resp.mean[-mid]
}
if(group.size%%2==0)
{
mid<-group.size/2
denom<-1
if(cor.tech == "zero.removal") denom<-sqrt(2)
replace1<-(sub.resp.mean[mid+1]-sub.resp.mean[mid])/denom
temp2<-sub.resp.mean[c(-mid,-mid-1)]
temp2<-c(temp2,replace1)
}
temp<-c(temp,temp2)
}
resp.mean<-abs(temp)
if(cor.tech == "cor.zero2")
{
correction<-sqrt((ngroup-1)/ngroup)
correction<-correction[-endpos]
resp.mean<-resp.mean*correction
}

for(i in 1:m)
{
n1<-n[1:i]-1

```

```
j<-i+1
n2<-n[j]-1
sum1<-sum(n1)
sum2<-sum(n2)
sum3<-sum1+sum2
subgroup<-c(rep(1,sum1),rep(2,sum2))
sub.resp.mean<-resp.mean[1:sum3]
mu<-mean(sub.resp.mean)

z <- as.vector(sub.resp.mean - mu)
d <- subgroup
statistic = summary(lm(z ~ d))$coefficients[2, 3]
df = summary(lm(z ~ d))$df[2]

if (tail == "left")
{
p.value = pt(statistic,df,lower.tail=TRUE)
log.p.value = pt(statistic,df,lower.tail=TRUE,log.p=T)
log.q.value = pt(statistic,df,lower.tail=FALSE,log.p=T)
}
else if (tail == "right")
{
p.value = pt(statistic,df,lower.tail=FALSE)
log.p.value = pt(statistic,df,lower.tail=FALSE,log.p=T)
log.q.value = pt(statistic,df,lower.tail=TRUE,log.p=T)
}
else
{tail = "both";
p.value = pt(statistic,df,lower.tail=TRUE)
log.p.value = pt(statistic,df,lower.tail=TRUE,log.p=T)
log.q.value = pt(statistic,df,lower.tail=FALSE,log.p=T)
```

```
    if(p.value >= 0.5)
    {
      p.value = pt(statistic,df,lower.tail=FALSE)
      log.p.value = pt(statistic,df,lower.tail=FALSE,log.p=T)
      log.q.value = pt(statistic,df,lower.tail=TRUE,log.p=T)
    }
    p.value = p.value*2
    log.p.value = log.p.value+log(2)
    log.q.value = log(1-p.value)
  }
  correlation = cor(d,z)
p[i]<-log.p.value
q[i]<-log.q.value
}

}

# start here

else if(cor.tech == "cor.zero1" || cor.tech == "zero.removal2")
{
  resp.mean <-z
  k<-length(n)
  temp<-double()
  endpos<-double()
  endpos2<-double()
  startpos<-double()
  for(i in 1:k)
  {
    group.size<-n[i]
    j<-i-1
```

```

if(i==1)start<-1
else start<-sum(n[1:j])+1
startpos<-c(startpos,start)
end<-sum(n[1:i])
endpos<-c(endpos,end)
if(group.size%%2==1)
{
endpos2<-c(endpos2,end)
}
sub.resp.mean<-resp.mean[start:end]
sub.resp.mean<-sub.resp.mean[order(sub.resp.mean)]
if(group.size%%2==1)
{
mid<-(group.size+1)/2
temp2<-sub.resp.mean[-mid]
}
if(group.size%%2==0)
{
temp2<-sub.resp.mean
}
temp<-c(temp,temp2)
}
resp.mean<-abs(temp)
if(cor.tech == "cor.zero1")
{
correction.contrast<-ngroup%%2
correction.contrast2<-abs(ngroup%%2-1)
correction<-correction.contrast*sqrt((ngroup-1)/ngroup)+
correction.contrast2*sqrt(ngroup/(ngroup-1))
if(length(endpos2)>0)
{

```

```

correction<-correction[-endpos2]
}
resp.mean<-resp.mean*correction
}

for(i in 1:m)
{
n1orig<-n[1:i]
n1<-n1orig-n1orig%%2
j<-i+1
n2orig<-n[j]
n2<-n2orig-n2orig%%2
sum1<-sum(n1)
sum2<-sum(n2)
sum3<-sum1+sum2
subgroup<-c(rep(1,sum1),rep(2,sum2))
sub.resp.mean<-resp.mean[1:sum3]
mu<-mean(sub.resp.mean)

  z <- as.vector(sub.resp.mean - mu)
  d <- subgroup
  statistic = summary(lm(z ~ d))$coefficients[2, 3]
  df = summary(lm(z ~ d))$df[2]

  if (tail == "left")
  {
p.value = pt(statistic,df,lower.tail=TRUE)
log.p.value = pt(statistic,df,lower.tail=TRUE,log.p=T)
log.q.value = pt(statistic,df,lower.tail=FALSE,log.p=T)
}
  else if (tail == "right")

```

```
{
  p.value = pt(statistic,df,lower.tail=FALSE)
  log.p.value = pt(statistic,df,lower.tail=FALSE,log.p=T)
  log.q.value = pt(statistic,df,lower.tail=TRUE,log.p=T)
}
else
  {tail = "both";
  p.value = pt(statistic,df,lower.tail=TRUE)
  log.p.value = pt(statistic,df,lower.tail=TRUE,log.p=T)
  log.q.value = pt(statistic,df,lower.tail=FALSE,log.p=T)
  if(p.value >= 0.5)
  {
    p.value = pt(statistic,df,lower.tail=FALSE)
    log.p.value = pt(statistic,df,lower.tail=FALSE,log.p=T)
    log.q.value = pt(statistic,df,lower.tail=TRUE,log.p=T)
  }
  p.value = p.value*2
  log.p.value = log.p.value+log(2)
  log.q.value = log(1-p.value)
  }
  correlation = cor(d,z)
  p[i]<-log.p.value
  q[i]<-log.q.value
}

}

# end here

else
```

```

{
  for(i in 1:m)
  {
    n1<-n[1:i]
    j<-i+1
    n2<-n[j]
    sum1<-sum(n1)
    sum2<-sum(n2)
    subgroup<-c(rep(1,sum1),rep(2,sum2))
    a<-length(subgroup)
    subz<-z[1:a]
    test<-ltrend.test(subz,subgroup,option=option,tail=tail,
    trim.alpha=trim.alpha,cor.tech="none")
    p[i]<-test$log.p.value
    q[i]<-test$log.q.value
  }
}

psiT<-exp(min(p))
psiF<--2*sum(p)
psiN<--sum(qnorm(p,lower.tail=TRUE,log.p=TRUE))
A<-pi^2*m*(5*m+2)/(15*m+12)
psiL<--A^(-1/2)*sum(p-q)
psiT.pvalue<-1-(1-psiT)^m
psiF.pvalue<-pchisq(psiF,df=2*m,lower.tail=FALSE)
psiN.pvalue<-pnorm(psiN,mean=0,sd=sqrt(m),lower.tail=FALSE)
psiL.pvalue<-pt(psiL,df=5*m+4,lower.tail=FALSE)

return(list(T=c(list(statistic=psiT,p.value=psiT.pvalue)),
F=c(list(statistic=psiF,p.value=psiF.pvalue)),
N=c(list(statistic=psiN,p.value=psiN.pvalue)),

```

```
L=c(list(Statistic=psiL,p.value=psiL.pvalue)))  
}
```



```
#####
#####LT TEST#####

ltrend.test<-function (y, group, score = NULL,
option = c("median", "mean", "trim.mean"),tail = c("both","right","left"),
trim.alpha = 0.25, bootstrap = FALSE, num.bootstrap = 1000,
cor.tech = c("none","zero.removal", "zero.removal2", "zero.removal3"
,"correction.factor", "cor.zero1", "cor.zero2"))
{
  if (is.null(score))
  {
    score <- group
  }

  if (length(y)!=length(group))
  {
    stop("the length of the data (y) does not match the length of the group")
  }

  option <- match.arg(option)
  tail <- match.arg(tail)
  cor.tech <- match.arg(cor.tech)

  DNAME = deparse(substitute(y))
  y <- na.omit(y)
  if ((option == "trim.mean") & (trim.alpha == 1))
  {
    stop("trim.alpha value of 0 to 0.5 should be provided for the trim.mean option")
  }

  ### sort the order just in case the input is not sorted by group
```

```
reorder <- order(group)
group <- group[reorder]
y<-y[reorder]
score <- score[reorder]

gr <- score
group <- as.factor(group)
if (option == "mean")
{
  means <- tapply(y, group, mean)
  METHOD = "ltrend test based on classical Levene's
  procedure using the group means"
}
else if (option == "median")
{
  means <- tapply(y, group, median)
  METHOD = "ltrend test based on the modified Brown-Forsythe
  Levene-type procedure using the group medians"
}
else
{
  option = "trim.mean"
  trimmed.mean <- function(y) mean(y, trim = trim.alpha)
  means <- tapply(y, group, trimmed.mean)
  METHOD = "ltrend test based on the modified Brown-Forsythe
  Levene-type procedure using the group trimmed means"
}
n <- tapply(y, group, length)
ngroup <- n[group]
resp.mean <- abs(y - means[group])
```

```

if(cor.tech == "correction.factor")
{
METHOD = paste(METHOD,"with correction factor applied")
correction <- 1/sqrt(1-1/ngroup)
resp.mean <- resp.mean*correction
}

### zero.removal is only applicable for those with median as center
if(option != "median" && cor.tech != "correction.factor")
{
cor.tech = "none"
}

### structural zero removal
if(cor.tech == "zero.removal" || cor.tech == "zero.removal3" || cor.tech == "cor.zero2")
{
METHOD = paste(METHOD,"with structural zeros removed")
resp.mean <- y - means[group]
k<-length(n)
temp<-double()
endpos<-double()
startpos<-double()
for(i in 1:k){
group.size<-n[i]
j<-i-1
if(i==1)start<-1
else start<-sum(n[1:j])+1
startpos<-c(startpos,start)
end<-sum(n[1:i])
endpos<-c(endpos,end)
}
}

```

```
sub.resp.mean<-resp.mean[start:end]
sub.resp.mean<-sub.resp.mean[order(sub.resp.mean)]
if(group.size%%2==1)
{
mid<-(group.size+1)/2
temp2<-sub.resp.mean[-mid]
if(cor.tech=="cor.zero2")
{
ntemp<-length(temp2)+1
correction<-sqrt((ntemp-1)/ntemp)
temp2<-correction*temp2
}
}
if(group.size%%2==0)
{
mid<-group.size/2
if(cor.tech=="zero.removal")
{
denom<-sqrt(2)
}
else
{
denom<-1
}
replace1<-(sub.resp.mean[mid+1]-sub.resp.mean[mid])/denom
temp2<-sub.resp.mean[c(-mid,-mid-1)]
temp2<-c(temp2,replace1)
if(cor.tech=="cor.zero2")
{
ntemp<-length(temp2)+1
correction<-sqrt((ntemp-1)/ntemp)
```

```

    temp2<-correction*temp2
  }
}
temp<-c(temp,temp2)
}
ngroup2 <- ngroup[-endpos]-1
resp.mean<-abs(temp)
zero.removal.gr<-gr[-endpos]
}

### correction technique (zero.removal2)
else if(cor.tech=="zero.removal2" || cor.tech=="cor.zero1")
{
METHOD = paste(METHOD,"with structural zeros removed2")
resp.mean <- y - means[group]
k<-length(n)
temp<-double()
endpos<-double()
endpos2<-double()
startpos<-double()
for(i in 1:k)
{
group.size<-n[i]
j<-i-1
if(i==1)start<-1
else start<-sum(n[1:j])+1
startpos<-c(startpos,start)
end<-sum(n[1:i])
endpos<-c(endpos,end)
  if(group.size%%2==1)
  {

```

```

    endpos2<-c(endpos2,end)
  }
sub.resp.mean<-resp.mean[start:end]
sub.resp.mean<-sub.resp.mean[order(sub.resp.mean)]
if(group.size%%2==1)
{
mid<-(group.size+1)/2
temp2<-sub.resp.mean[-mid]
  if(cor.tech=="cor.zero1")
  {
ntemp<-length(temp2)+1
correction<-sqrt((ntemp-1)/ntemp)
temp2<-correction*temp2
  }
}
if(group.size%%2==0)
{
temp2<-sub.resp.mean
  if(cor.tech=="cor.zero1")
  {
ntemp<-length(temp2)
correction<-sqrt(ntemp/(ntemp-1))
temp2<-correction*temp2
  }
}
temp<-c(temp,temp2)
}
resp.mean<-abs(temp)
if(length(endpos2)>0)
{
zero.removal.gr<-gr[-endpos2]

```

```
}
else
{
zero.removal.gr<-gr
}
}

else
{
cor.tech="none"
}

mu <- mean(resp.mean)
z <- as.vector(resp.mean - mu)
if(cor.tech=="zero.removal" || cor.tech=="zero.removal2"
||cor.tech=="cor.zero1" || cor.tech=="zero.removal3"
||cor.tech=="cor.zero2")
{
d <- as.numeric(zero.removal.gr)
}
else
{
d <- as.numeric(gr)
}
statistic = summary(lm(z ~ d))$coefficients[2, 3]
df = summary(lm(z ~ d))$df[2]

if (tail == "left")
{
METHOD = paste(METHOD, "(left-tailed)")
p.value = pt(statistic,df,lower.tail=TRUE)
```

```
log.p.value = pt(statistic,df,lower.tail=TRUE,log.p=TRUE)
log.q.value = pt(statistic,df,lower.tail=FALSE,log.p=TRUE)
}
else if (tail == "right")
{
METHOD = paste(METHOD, "(right-tailed)")
p.value = pt(statistic,df,lower.tail=FALSE)
log.p.value = pt(statistic,df,lower.tail=FALSE,log.p=TRUE)
log.q.value = pt(statistic,df,lower.tail=TRUE,log.p=TRUE)
}
else
{tail = "both";
METHOD = paste(METHOD, "(two-tailed)")
p.value = pt(statistic,df,lower.tail=TRUE)
log.p.value = pt(statistic,df,lower.tail=TRUE,log.p=TRUE)
log.q.value = pt(statistic,df,lower.tail=FALSE,log.p=TRUE)
if(p.value >= 0.5)
{
p.value = pt(statistic,df,lower.tail=FALSE)
log.p.value = pt(statistic,df,lower.tail=FALSE,log.p=TRUE)
log.q.value = pt(statistic,df,lower.tail=TRUE,log.p=TRUE)
}
p.value = p.value*2
log.p.value = log.p.value+log(2)
log.q.value = log(1-p.value)
}
correlation = cor(d,z)

### store the non-bootstrap p-value.
non.bootstrap.p.value <- p.value
```



```
### bootstrapping (followed Lim and Loh pg.291)
if(bootstrap == TRUE)
{
METHOD = paste("bootstrap", METHOD)

### step 2
R<-0
N<-length(y)

### step 3. calculation of the fractional trimmed mean
frac.trim.alpha = 0.2
b.trimmed.mean<-function(y)
{
nn<-length(y)
wt<-rep(0,nn)
y2<-y[order(y)]
lower<-ceiling(nn*frac.trim.alpha)+1
upper<-floor(nn*(1-frac.trim.alpha))
if(lower>upper) stop("frac.trim.alpha value is too large")
m<-upper-lower+1
frac<-(nn*(1-2*frac.trim.alpha)-m)/2
wt[lower-1]<-frac
wt[upper+1]<-frac
wt[lower:upper]<-1
return(weighted.mean(y2,wt))
}
b.trim.means <- tapply(y, group, b.trimmed.mean)
rm<-y-b.trim.means[group]

### enters a loop, as specified in step 7
for (j in 1:num.bootstrap)
```

```
{

### step 4
sam<-sample(rm,replace=TRUE)
boot.sample <- sam

### step 5. if n_i < 10 for at least one sample size,
### then enters the following if-branch.
if(min(n) < 10)
{
U<-runif(1)-0.5
means <- tapply(y, group, mean)
v <- sqrt(sum((y - means[group])^2)/N)
boot.sample <- ((12/13)^(0.5))*(sam + v*U)
}

### step 6. compute the bootstrap statistic,
### and increment R to R + 1 if necessary.

if(option=="mean")
{
boot.means <- tapply(boot.sample, group, mean)
}

else if(option=="median")
{
boot.means <- tapply(boot.sample, group, median)
}

else
{option="trim.mean";
```

```

trimmed.mean<-function(boot.sample) mean(boot.sample, trim=trim.alpha)
boot.means <- tapply(boot.sample, group, trimmed.mean)
}

if(option!="median"||zero.removal!=TRUE)
{
resp.boot.mean <- abs(boot.sample - boot.means[group])*correction
}

### structural zero removal
if(option == "median" && zero.removal == TRUE)
{
resp.boot.mean <- boot.sample - boot.means[group]
boot.temp<-double()
for(i in 1:k)
{
group.size<-n[i]
start<-startpos[i]
end<-endpos[i]
sub.resp.boot.mean<-resp.boot.mean[start:end]
sub.resp.boot.mean<-sub.resp.boot.mean[order(sub.resp.boot.mean)]
if(group.size%%2==1)
{
mid<-(group.size+1)/2
boot.temp2<-sub.resp.boot.mean[-mid]
}
if(group.size%%2==0)
{
mid<-group.size/2
replace1<-(sub.resp.boot.mean[mid+1]-sub.resp.boot.mean[mid])/sqrt(2)
boot.temp2<-sub.resp.boot.mean[c(-mid,-mid-1)]
}
}
}

```

```
boot.temp2<-c(temp2,replace1)
}
boot.temp<-c(boot.temp,boot.temp2)
}
resp.boot.mean<-abs(boot.temp)
}

boot.mu <- mean(resp.boot.mean)
boot.z <- as.vector(resp.boot.mean - boot.mu)
statistic2 = summary(lm(boot.z ~ d))$coefficients[2, 3]

if(tail=="right")
{
if(statistic2 > statistic) R<-R+1
}
else if(tail=="left")
{
if(statistic2 < statistic) R<-R+1
}
else
{tail="both";
if(abs(statistic2) > abs(statistic)) R<-R+1
}
}

### step 8. the bootstrap p-value calculation.
### num.bootstrap = number of bootstraps.
p.value <- R/num.bootstrap
}

STATISTIC = correlation ### statistic is changed to correlation.
```

```
names(STATISTIC) = "Correlation"  
structure(list(statistic = STATISTIC, p.value = p.value,  
t.statistic = statistic, method = METHOD, data.name = DNAME,  
non.bootstrap.p.value = non.bootstrap.p.value,  
log.p.value = log.p.value, log.q.value = log.q.value), class = "htest")  
}
```

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