Algebraic characterization of multivariable dynamics

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Mathematics
in
Pure Mathematics

Waterloo, Ontario, Canada, 2009

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Let X be a locally compact Hausdorff space along with n proper continuous maps $\sigma = (\sigma_1, \cdots, \sigma_n)$. Then the pair (X, σ) is called a dynamical system. To each system one can associate a universal operator algebra called the tensor algebra $\mathcal{A}(X, \sigma)$. The central question in this theory is whether these algebras characterize dynamical systems up to some form of natural conjugacy.

In the $n = 1$ case, when there is only one self-map, we will show how this question has been completely determined. For $n \geq 2$, isomorphism of two tensor algebras implies that the two dynamical systems are piecewise conjugate. The converse was only established for $n = 2$ and 3. We introduce a new construction of the unitary group $U(n)$ that allows us to prove the algebraic characterization question in $n = 2, 3$ and 4 as well as translating this conjecture into a conjecture purely about the structure of the unitary group.

Acknowledgements

I would like to thank my supervisor Ken Davidson for his guidance, advice and many suggestions in the production of this thesis, my readers Heydar Radjavi and Jean Roydor for their proof-reading efforts, and my colleagues for providing a vibrant research environment. I would also like to thank my family and my dear fiancée Melissa for all their encouragement and support throughout my studies in mathematics.

Contents

Chapter 1

Introduction

The question of classifying dynamical systems by certain algebras has been an active area of research for over seventy years. There have been several key players in the development of this theory, a history of which will be shortly forthcoming. First however, it is good to state what our definition of a dynamical system, both one-variable and multivariable, has ended up as.

Definition 1.1 *A* one-variable dynamical system*, denoted* (X, σ)*, is a locally compact Hausdorff space* X *defined with a proper continuous map* $\sigma : X \to X$*, where proper means that the inverse image of a compact set is compact.*

And similarly defined,

Definition 1.2 *A* multivariable dynamical system*, denoted* (X, σ)*, is a locally compact Hausdorff space* X *defined with* n *proper continuous maps* $\sigma_i : X \to X$ *for* $1 \leq i \leq n$*, where* $\sigma = (\sigma_1, \cdots, \sigma_n)$*.*

The main goal of this area of study is to associate each dynamical system with a particular Banach algebra such that the algebras are isomorphic if and only if the dynamical systems are the same up to some form of natural conjugacy. In the one-variable case conjugacy takes on the form:

Definition 1.3 *Two one-variable dynamical systems,* (X_1, σ_1) *and* (X_2, σ_2)*, are* conjugate *if there exists a homeomorphism* $\tau : X_2 \to X_1$ *such that* $\tau \circ \sigma_2 = \sigma_1 \circ \tau$*.*

In the multivariable case conjugacy must be replaced by something called piecewise conjugacy which allows for an ambiguity of which maps are conjugate locally.

Chapter 2 will detail the complete solution to the one-variable dynamical system problem provided by K. Davidson and E. Katsoulis in [4], which they wrote in 2006. Along with the proof there will also be various examples of algebras that one can associate to each system.

Davidson and Katsoulis, in [3], went on to provide an almost complete proof of the multivariable case. This will be outlined in Chapter 3, along with the conjecture that would finish off this problem if proven true. The conjecture has to do with mapping an n!-simplex into the n-dimensional unitary group, $U(n)$, such that the map has a very specific partition structure. They only verified $n = 2$ and 3.

Following from this, Chapter 4 details a method of proof for this conjecture that holds promise but demonstrates how the original question turns out to be quite complicated due to the iterative structure of an n!-simplex. This permutation structure comes from the built-in ambiguity of piecewise conjugacy. Our proof works for $n = 2, 3$ and 4, extending the known cases of this result.

Now we turn to the development of this subject and how it has changed from the early days.

1.1 History

This area of theory can be considered to have originated with what is referred to as von Neumann's group-measure construction first published in 1936 by F. J. Murray and J. von Neumann [11]. Here one takes a locally compact space X with an associated positive measure m. Now let G be a countably infinite discrete group that acts on the left on X such that for $g \in G$ we have $x \mapsto gx, x \in X$ is a homeomorphism of X which transforms m into an equivalent measure.

From this situation of a space and an action, we can construct a von Neumann algebra, called the *crossed product* or *covariance von Neumann algebra*. Suppose that $L^\infty(X, m)$, an abelian von Neumann algebra, acts on a Hilbert space $\mathcal{H}$ and let $\mathfrak{H} = \mathcal{H} \otimes l^2(G)$. Now we get two representations π of $L^\infty(X, m)$ and u of G on $\mathfrak{H}$ given by

$$\pi(a)\xi(g) = (g^{-1}a)\xi(g), \quad \xi \in \mathfrak{H}, g \in G, a \in L^\infty(X, m)$$

$$u(g)\xi(h) = \xi(g^{-1}h), \quad h \in G.$$

This gives us that

$$u(g)\pi(a)u(g)^* = \pi(ga), \quad a \in L^\infty(X,m), g \in G$$

Let $L^\infty(X,m) \rtimes G$ be the crossed product algebra generated by $\pi(L^\infty(X,m))$ and $u(G)$.

Second, W. B. Arveson, in his 1967 paper [1], developed the theory into something that is more familiar in form to our area of interest. He showed that the classification, up to conjugacy, of an ergodic measure preserving automorphism of the measure algebra of the unit interval is equivalent to the classification of specific Banach algebras of operators on the Hilbert space $L^2(0,1)$.

Particularly, let M be the von Neumann algebra of all multiplications by bounded measurable functions, acting on $L^2(0,1)$. Suppose that α is a $*$-automorphism preserving the ergodic measure m, that is $m \circ \alpha = m$ on the projections of M. Let U_α be a unitary operator such that $\alpha(A) = U_\alpha A U_\alpha^*$ on M. Then since $\alpha(M)U_\alpha = U_\alpha M$ then

$$\mathcal{A}_0(\alpha) = \{\sum_{i=0}^{n} A_i U_\alpha^i : A_1, \cdots, A_n \in M, n \geq 0\}$$

forms an algebra. Let $\mathcal{A}(\alpha)$ be the closure of $\mathcal{A}_0(\alpha)$ in the operator norm. This construction gives the required theory:

Theorem (Arveson, [1, Thm. 1.8]) *Let α and β be ergodic m-preserving $*$-automorphisms of M. Then α and β are conjugate if, and only if there exists a unitary V such that $V\mathcal{A}(\alpha)V^* = \mathcal{A}(\beta)$.*

This work was generalized by Arveson and K. Josephson in their 1969 paper [2]. Instead of such a limited class of automorphisms they considered triples, (X, σ, m), where X is a locally compact Hausdorff space, $\sigma : X \to X$ a homeomorphism, and m a separable nonatomic regular Borel probability measure on X, with the conditions:

(1) $m \circ \sigma$ is mutually absolutely continuous with m (quasi-invariance),

(2) $m(U) > 0$ for every nonempty open set U,

(3) The set of periodic points of σ, $P = \cup_{n\neq 0}\{x \in X : \sigma^n(x) = x\}$, is measure

zero, that is $m(P) = 0$.

However, these are still fairly restrictive conditions as the existence of such a measure does not happen automatically, and many interesting dynamical systems are not homeomorphisms and have many periodic points.

Consider the Hilbert space $L^2(X, m)$ and the unitary operator

$$Uf = \left(\frac{dm \circ \sigma}{dm}\right)^{1/2} f \circ \sigma, \quad f \in L^2(X, m)$$

where $m \circ \sigma$ is the measure defined for $E \subset X$ to be $m \circ \sigma(E) = m(\sigma(E))$. Now let

$$\mathcal{A}(X, \sigma) = \overline{\left\{ \sum_{i=0}^{n} f_i U^i : f_1, \cdots, f_n \in C_0(X), n \geq 0 \right\}}^{\|\cdot\|}$$

They go on to show that two different separable nonatomic Borel probability measures on X that satisfy the three conditions stated previously will produce isometric isomorphic Banach algebras $\mathcal{A}(X, \sigma)$ [2, Proposition 3.3]. As well, it is shown that two algebras $\mathcal{A}(X_1, \sigma_1)$ and $\mathcal{A}(X_2, \sigma_2)$ are isomorphic as algebras if and only if (X_1, σ_1, m_1) and (X_2, σ_2, m_2) are conjugate with the additional restriction that m_1 and m_2 are both ergodic and invariant under σ_1 and σ_2 respectively [2, Theorem 3.11].

Furthermore, Arveson and Josephson prove that if X_1, X_2 are compact and the measures m_1, m_2 are invariant then a bounded isomorphism, τ, from $\mathcal{A}(X_1, \sigma_1)$ to $\mathcal{A}(X_2, \sigma_2)$ has the decomposition $\tau = \beta_1 \circ \beta_2 \circ \phi$, where β_1 is a weakly inner automorphism of $\mathcal{A}(X_2, \sigma_2)$, β_2 is an isometric automorphism of $\mathcal{A}(X_2, \sigma_2)$ and ϕ is an isometric automorphism from $\mathcal{A}(X_1, \sigma_1)$ to $\mathcal{A}(X_2, \sigma_2)$ [2, Theorem 4.10].

In 1984, J. Peters [13] associated a concrete Banach algebra to a one-variable dynamical system as defined in Definition 1.1. He originally defined this algebra to encode the information of a C*-algebra and an endomorphism but we will use the specific case when the C*-algebra is $C_0(X)$, where (X, σ) is our dynamical system.

Definition 1.4 *Let (X, σ) be a one-variable dynamical system. Let $\mathcal{H}_x = l^2(\mathbb{N})$ be the space of square summable sequences $\xi = (\xi_n)_{n=0}^{\infty}$, define for $x \in X$*

$$\pi_x(f)\xi = (f(x)\xi_0, (f \circ \sigma)(x)\xi_1, (f \circ \sigma^{(2)})(x)\xi_2, \cdots), \text{ for } f \in C_0(X),$$

and let U_x be the forward shift operator

$$U_x\xi = (0, \xi_0, \xi_1, \xi_2, \cdots).$$

The semicrossed product algebra*, denoted $C_0(X) \times_\sigma \mathbb{Z}^+$, is defined to be the norm closed operator algebra acting on $\oplus_{x\in X}\mathcal{H}_x$ and generated by*

$$\oplus_{x\in X}\pi_x(f),\ \oplus_{x\in X}U_x\pi_x(g),\ \ f, g \in C_0(X).$$

Peter's shows that two one-variable dynamical systems (X, σ) and (Y, τ) with X and Y compact and σ and τ having no fixed points, are conjugate if and only if $C(X) \times_\sigma \mathbb{Z}^+$ is isomorphic to $C(Y) \times_\tau \mathbb{Z}^+$.

This algebra will prove very useful as Davidson and Katsoulis use it to characterize the one-variable dynamical systems up to conjugation. These semi-crossed product algebras can also be defined via a universal property [9].

The last paper we will discuss in this history was written by D. W. Hadwin and T. B. Hoover in 1988 [8]. They consider one-variable dynamical systems where the space X is compact, not locally compact, but there are no longer any measure theoretic considerations making this theory much farther ranging then the previous work that was tied to the existence of a specific measure, which need not exist in many cases.

This paper marks the introduction of what are called (topological) conjugacy algebras, a class of algebras defined from a dynamical system that have a set of properties making them particularly tractable. Davidson and Katsoulis refined this idea which we will see in Section 2.1 and so their definition will be left until then.

However, they still imposed conditions on the fixed points of the map σ. Showing that $(X_1, \sigma_1), (X_2, \sigma_2)$ are conjugate dynamical systems, where X_1, X_2 are compact and the set $\{x \in X_2 : \sigma_2(x) \neq x, \sigma_2^2(x) = \sigma(x)\}$ has empty interior, if and only if there exist isomorphic conjugacy algebras for (X_1, σ_1) and (X_2, σ_2). Or, as they prove later, if and only if the semicrossed product algebras of the dynamical systems are isomorphic.

Chapter 2

One-variable case

Building on all of the work in the previous section, Davidson and Katsoulis in [4] managed to extend this algebraic characterization to one-variable dynamical systems (X, σ) where X is any locally compact Hausdorff space and σ is a proper continuous map. Gone are the measure conditions and the restrictions on the fixed point set.

2.1 Topological conjugacy algebras

Now we will define the algebra that encodes the one-variable dynamical system, called a topological conjugacy algebra.

First, we need to define some basic structures to build up what we need.

Definition 2.1 *Let X be a compact Hausdorff space and let $\sigma : X \to X$ be a continuous function. The* skew polynomial algebra*, denoted $P(X, \sigma)$, is all polynomials of the form*

$$\sum_{i=1}^{n} f_i U^i, f_i \in C(X)$$

in the variable U for which in the "skew" case we have the multiplication

$$Uf = (f \circ \sigma)U.$$

The following notion is due to Hadwin and Hoover.

Definition 2.2 *Let (X, σ) be a one-variable dynamical system where X is compact. Define $\mathcal{A}$ to be a Banach algebra that satisfies the following conditions:*

1. *$P(X, \sigma)$ is a dense subalgebra of $\mathcal{A}$ such that the units are the same, that is $1_{\mathcal{A}} = 1 \in P(X, \sigma)$,*
2. *$C(X) \subseteq P(X, \sigma) \subseteq \mathcal{A}$ is closed,*
3. *There exists an algebra homomorphism $E_0 : \mathcal{A} \to C(X)$ such that $E_0(f) = f, \forall f \in C(X)$ and $\ker E_0 = \mathcal{A}U$, and*
4. *U is not a right divisor of 0.*

For such an algebra we can define a Fourier series for each element.

Lemma 2.3 *Let $\mathcal{A}$ be defined as in the previous definition. Then we can associate a formal power series with each element of $\mathcal{A}$, that is there exist continuous maps $E_n : \mathcal{A} \to C(X)$ such that*

$$a \sim \sum_n E_n(a)U^n \in P^\infty(X, \sigma)$$

Proof. Note that E_0 has already been given in the definition of $\mathcal{A}$. Now since $C(X)$ is closed in $\mathcal{A}$ then E_0 is continuous by a classical continuity result. Hence, ker $E_0 = \mathcal{A}U$ is closed as well.

Define the map $S : \mathcal{A} \to \mathcal{A}U$ by $Sa = aU$. By the Inverse Mapping Theorem we know that S has a bounded left inverse T.

Since $a - E_0(a) \in \ker E_0 = \mathcal{A}U$, there is a unique $b \in \mathcal{A}$ such that $a = E_0(a) + bU$. So inductively define nth coefficient map, where $n \geq 1$ to be

$$E_n = E_{n-1}T(I - E_0)$$

because $E_{n-1}T(I - E_0)(a) = E_{n-1}T(b) = E_{n-1}(bU^{-1})$. Combining each step we get that

$$E_n = E_0(T(1 - E_0))^n.$$

Thus, it is clear that each E_n is continuous and we can now define an algebra homomorphism $\Delta : \mathcal{A} \to P^\infty(X, \sigma)$ given by $\Delta(a) = \sum_n E_n(a)U^n$. □

Now we can define the topological conjugacy algebra. We begin first with the case when X is compact.

Definition 2.4 *Let (X, σ) be a one-variable dynamical system where X is compact. Then $\mathcal{A}$ is a* topological conjugacy algebra *for (X, σ) if $\mathcal{A}$ is an algebra given in Definition 2.2 such that*

$$\limsup_{n} (\|E_n\| \|U^n\|)^{1/n} \leq 1.$$

From this we can define a topological conjugacy algebra for the non-compact locally compact X case. Define $\hat{X} = X \cup \{\omega\}$ to be the one-point compactification of X. Then we identify $C_0(X)$ with the continuous functions on $\hat{X}$ that vanish at ω. As well, we can extend a continuous proper map, σ, on X to a continuous map on $\hat{X}$ that has ω as a fixed point.

Definition 2.5 *Let (X, σ) be a one-variable dynamical system where X is locally compact but not compact. Let $\hat{\mathcal{A}}$ be the topological conjugacy algebra for $(\hat{X}, \hat{\sigma})$. Then the* topological conjugacy algebra *for (X, σ) is the norm closed algebra $\mathcal{A}$ generated by the polynomials with coefficients in $C_0(X)$. As well, $\hat{\mathcal{A}}$ is called the* canonical unitization *of $\mathcal{A}$.*

Now we look at several examples of topological conjugacy algebras, which shows the flexibility of the theory. The first two examples are neither Banach algebras or complete but they do satisfy Hadwin and Hoover's looser conditions for conjugacy algebras [8]. However, they serve as basic demonstrations of the theory.

Example 2.6 It is not hard to see that $P(X, \sigma)$ is a topological conjugacy algebra for (X, σ) minus the issues discussed above. We just need to have a condition on the "variable" U, that is, something like $\|U\| \leq \frac{1}{\|T\| \|I - E_0\|}$. From this we see that $P(X, \sigma)$ satisfies the conditions in Definition 2.2 and then

$$\limsup_{n} (\|E_n\| \|U^n\|)^{1/n} = \limsup_{n} \|E_0 (T(I - E_0))^n\|^{1/n} \|U^n\|^{1/n}$$

$$\leq \limsup_{n} \|E_0\|^{1/n} \|T\| \|I - E_0\| \|U\| \leq 1.$$

because $\|E_0\| = 1$.

Example 2.7 In the same way $P^\infty(X,\sigma)$ is also almost a topological conjugacy algebra for (X,σ) by imposing a similar norm condition, in this case make it $\|U^n\| \leq \frac{1}{2^{n+1}}$. Again the four conditions of Definition 2.2 are satisfied quite easily.

Besides failing to satisfy the full definition for a topological conjugacy algebra, neither $P(X,\sigma)$ or $P^\infty(X,\sigma)$ give us much information about the dynamical system. We move on to more useful examples.

Example 2.8 Let (X,σ) be a one-variable dynamical system. Let $\mathcal{H}_x = l^2(\mathbb{N})$ be the space of square summable sequences $\xi = (\xi_n)_{n=0}^\infty$ and for each $x \in X$ define

$$\pi_x(f)\xi = (f(x)\xi_0, (f\circ\sigma)(x)\xi_1, (f\circ\sigma^{(2)})(x)\xi_2, \cdots), \text{ for } f \in C_0(X),$$

and

$$V_x\xi = (\xi_1, \xi_2, \xi_3, \cdots).$$

The norm closed operator algebra $\mathcal{A}_{X,\sigma}$ acting on $\oplus_{x\in X}\mathcal{H}_x$ and generated by the operators

$$\oplus_{x\in X}\pi_x(f),\ \oplus_{x\in X}\pi_x(g)V_x,\ f,g \in C_0(X),$$

is seen to be a topological conjugacy algebra.

Hadwin and Hoover in [8] provide more examples of topological conjugacy algebras.

Example 2.9 Let W be a Banach space and let $f \to M_f$ be a faithful continuous representation of $C(X)$ as operators on W. For instance W could be $C(X)$ and M_f could be "multiplication by f". Or, if m is a Borel measure on X with $m(V) > 0$ for every non-empty open set V, and if $1 \leq q \leq \infty$, then we could let W be $L^q(X,m)$ and let M_f be multiplication by f.

Fix $p, 1 \leq p \leq \infty$, and let Y be a Banach space of all norm p-summable sequences of points in W. Let U be the backwards shift operator on Y. For f in $C(X)$, let T_f be the operator on Y defined by

$$T_f((w_n)_{n=1}^\infty) = (M_{f\cdot\sigma^n}(w_n))_{n=1}^\infty$$

It is readily verified that the norm closed algebra $\mathcal{A}$ generated by U and all the T_f's is a topological conjugacy algebra for (X,σ).

As well, note that the semi-crossed product algebras, Definition 1.4, are isomorphic to those with W a Hilbert space and $p = 2$.

Example 2.10 Suppose that the continuous map σ is freely acting on X in the sense that, for every non-empty open set V and for every positive integer n, there is a non-empty open subset V' of V such that the sets $\sigma^k(V'), 0 \le k \le n$, are pairwise disjoint. It is easily shown that σ acts freely if and only if, for each positive integer n, the set $\{x : \sigma^n(x) = x\}$ has empty interior.

Suppose m is a Borel measure on X such that $m(V) > 0$ for every non-empty open set V, and such that m and $m \circ \sigma$ are mutually continuous with Radon-Nikodym derivative $h = dm \circ \sigma / dm$. Also assume that every set with positive measure contains a set with finite positive measure.

Suppose $1 \le p \le \infty$, and let $C(X)$ act on $L^p(m)$ as multiplications. Define the operator U on $L^p(m)$ by $U(f) = (f \circ \sigma) h^{1/p}$. In the case of $p = \infty$, let the weight function $h^{1/p}$ be the constant function. Let $\mathcal{A}$ be the norm closure of $P(X, \sigma)$ in the algebra $B(L^p(m))$ of all operators on $L^p(m)$. Note that if m is counting measure and $p = \infty$, then $C(X)$ (with the supremum norm) is a norm closed subspace of $L^\infty(m)$ that is invariant for the algebra $\mathcal{A}$. In this case, once we prove that $\mathcal{A}$ is a topological conjugacy algebra, it will follow that the restriction of $\mathcal{A}$ to $C(X)$ is also a topological conjugacy algebra.

The operator U is an invertible isometry on $L^p(m)$, so $\|U\| = 1$. Suppose that $a \in P(X, \sigma)$ and $a = \sum_{i=0}^n f_i U^i$. Suppose $0 \le k \le n$ and $r > 0$, and let $V = \{x \in X : |f_k(x)| > \|f_k\| - r\}$. Choose a non-empty open subset V' of $\sigma^k(V)$ such that the sets $\sigma^i(V'), 0 \le i \le n$, are pairwise disjoint. Let f be the characteristic function of a subset E of V' with $0 < m(E) < \infty$. The functions $f \circ \sigma^i, 0 \le i \le n$, have pairwise disjoint supports and $U^k f$ vanishes off V. Thus,

$$\|af\| \ge \|(f \circ \sigma^k) af\| = \|f_k (f \circ \sigma^k) U^k f\| \ge (\|f_k\| - r)\|f\|.$$

Since r can be chosen to be arbitrarily small, it follows that $\|a\| \ge \|f_k\|$ for $0 \le k \le n$. Hence the coefficient maps $E_0, E_1, \cdots$ are continuous with norm 1 on $P(X, \sigma)$. Thus the coefficient maps can be extended to contractive linear maps on $\mathcal{A}$. It follows then that $\mathcal{A}$ is a topological conjugacy algebra for (X, σ).

2.2 Characters and representations

In this section we will develop some theory around characters and representations of the semicrossed product algebra. This turns out to be one of the key steps in the characterization of one-variable dynamical systems which will be proven in the next section.

First, a technical proposition about algebra homomorphisms and how they relate to the power series of an element in a topological conjugacy algebra.

Proposition 2.11 *Let $\mathcal{A}$ be a topological conjugacy algebra for the one-variable dynamical system (X, σ). Let $\mathcal{B}$ be an algebra and $\rho : \mathcal{A} \to \mathcal{B}$ an algebra homomorphism. If $C_0(X)U \subseteq \ker\rho$, then $\rho(a) = \rho(E_0(a))$, for all $a \in \mathcal{A}$.*

Proof. If $a \in \mathcal{A}$, then

$$a = E_0(a) + E_1(a)U + bU^2$$

for some $b \in \mathcal{A}$. We assumed that ρ annihilates the second summand above and we are done if we can show that ρ annihilates bU^2 as well.

Let (e_α) be the contractive approximate unit of $C_0(X)$ of all positive functions of norm less than 1 with compact support. The order on the net is given by, $e_\alpha \leq e_\beta$ if and only if $e_\alpha(x) \leq e_\beta(x)$, for all $x \in X$. Now (e_α) is also an approximate unit for the Banach algebra $\mathcal{A}U^2$ and so $\overline{\text{span}}(\mathcal{A}U^2C_0(X)) = \mathcal{A}U^2$. Hence the right multiplication on $\mathcal{A}U^2$ by elements of $C_0(X)$ defines an anti-representation of $C_0(X)$ on the Banach space which satisfies the hypothesis of Cohen's Factorization Theorem [12, Theorem 5.2.2]. The conclusion is that

$$\overline{\text{span}}(\mathcal{A}U^2C_0(X)) = \text{span}(\mathcal{A}U^2C_0(X)) = \mathcal{A}U^2$$

and so there exists $c_i \in \mathcal{A}$ and $g_i \in C_0(X)$ so that $bU^2 = \sum_{i=1}^n c_iU^2g_i$. But then

$$\rho(bU^2) = \sum_{i=1}^n \rho(c_iU)\rho(Ug_i) = \sum_{i=1}^n \rho(c_iU)\rho((g_i \circ \sigma)U) = 0.$$

and the conclusion follows. □

Now there is a natural partitioning of the character space, $\mathcal{M}_\mathcal{A}$, of a topological conjugacy algebra. This partitioning is important since we will show that it is preserved via homeomorphisms of topological conjugacy algebras.

Definition 2.12 *Let $\mathcal{A}$ be a topological conjugacy algebra for a one-variable dynamical system (X, σ) and let ρ be a character on $\mathcal{A}$. Then since the action of ρ on $C_0(X) \subset \mathcal{A}$ is a point evaluation on some point, we define for $x \in X$,*

$$\mathcal{M}_{\mathcal{A},x} = \{\rho \in \mathcal{M}_{\mathcal{A}} : \rho(f) = f(x), \forall f \in C_0(X) \subset \mathcal{A}\}.$$

Thus, we get the following partitioning: $\mathcal{M}_{\mathcal{A}} = \dot{\bigcup}_{x \in X} \mathcal{M}_{\mathcal{A},x}$.

From this, look at the structure of a partition based on whether it is or is not related to a fixed point of σ. First, suppose that x is not a fixed point of σ and $\rho \in \mathcal{M}_{\mathcal{A},x}$ then for all $f, g \in C_0(X)$

$$f(x)\rho(U)g(x) = \rho(fUg) = \rho((g \circ \sigma)fU) = g(\sigma(x))f(x)\rho(U) = f(x)\rho(U)g(\sigma(x))$$

which implies that $C_0(X)U \subseteq \ker \rho$. Hence, by Proposition 2.11,

$$\rho(a) = \rho(E_0(a)) = E_0(a)(x) \quad \textit{for all} \quad a \in \mathcal{A}.$$

Therefore, $\mathcal{M}_{\mathcal{A},x} = \{\theta_x\}$, a single element.

The next theorem deals with the more complicated case when x is a fixed point of σ. It turns out that in this case $\mathcal{M}_{\mathcal{A},x}$ is homeomorphic to $\sigma(U) = \mathbb{D}_r$, a closed disc. We will call a map Θ from a domain $\Omega \subset \mathbb{C}$ into $\mathcal{M}_{\mathcal{A}}$ *pointwise analytic* if $\Theta(z)(a)$ is analytic for $z \in \Omega$ for all $a \in \mathcal{A}$.

Theorem 2.13 *Let $\mathcal{A}$ be a topological conjugacy algebra for the one-variable dynamical system (X, σ) and let $x \in X$ be a fixed point for σ. Then there exists a homeomorphism*

$$\Theta_x : \sigma(U) \to \mathcal{M}_{\mathcal{A},x}$$

which is pointwise analytic on the interior $\sigma(U)^\circ$ of $\sigma(U)$ and satisfies $\Theta_x(z)(gU) = g(x)z$ for every $g \in C_0(X)$.

Proof. First suppose that X is compact. The following argument is due to Hadwin and Hoover in [8] to show that $\mathcal{M}_{\mathcal{A},x}$ is homeomorphic to the spectrum of U.

Let $r = \lim_{n\to\infty} \|U^n\|^{1/n}$. Remember that the map $Sa = aU$ from $\mathcal{A}$ to $\mathcal{A}U$ has a bounded left inverse T and is therefore bounded below by $\|T\|^{-1}$. Hence, $\|U\| \geq \|T\|^{-1}\|1\|$ and so $\|U^n\| \geq \|T\|^{-n}\|1\|$. Thus $r \geq \|T\|^{-1}$ is positive.

Now consider the power series $\sum_n E_n(a)(x)z^n$ and note that by the definition of a conjugacy algebra its radius of convergence is at least r. Hence, for any $z \in \mathbb{C}$ with $|z| < r$, the mapping

$$\mathcal{A} \ni a \longrightarrow \sum_n E_n(a)(x)z^n$$

is a well defined multiplicative functional on $\mathcal{A}$, which we denote by $\theta_{x,z}$. Since $\theta_{x,z}(U) = z$, for any z with $|z| < r$ we have $\sigma(U) = \overline{\mathbb{D}}_r$ Furthermore, the mapping $\mathcal{M}_{\mathcal{A},x} \ni \theta \to \theta(U) \in \sigma(U)$ is a continuous map between compact spaces that has dense range. Since $\theta \in \mathcal{M}_{\mathcal{A},x}$ is determined by $\theta(U)$, this map is an injection. By elementary topology the above map is a homeomorphism. Its inverse will be denoted Θ_x.

In the second case, if X is not compact, then the corresponding map $\hat{\Theta}_x$ for the canonical unitization $\hat{\mathcal{A}}$ will provide the correct map as long as it is shown that every character in $\mathcal{M}_{\mathcal{A},x}$ comes from one in $\mathcal{M}_{\hat{\mathcal{A}},x}$. This argument is contained in [4]. □

Definition 2.14 *Let* $\Theta : \mathbb{D}_s \to \mathcal{M}_{\mathcal{A}}$ *be an injection that is pointwise analytic,* $s > 0$. *Then we call the range of* Θ *an* analytic disc.

We see that for any $f \in C_0(X)$ then $\Theta(z)(\overline{f}) = \overline{\Theta(z)(f)}$. Thus $\Theta(z)(f)$ is constant by analyticity. So then an analytic disc is contained in some $\mathcal{M}_{\mathcal{A},x}$. Therefore, by the previous Theorem the set $(\mathcal{M}_{\mathcal{A},x})^\circ = \Theta_x(\sigma(U)^\circ)$ is an analytic disc for every fixed point x of σ. By the Open Mapping Theorem, $(\mathcal{M}_{\mathcal{A},x})^\circ$ is a maximal analytic disc. This maximality proves to be essential in the classification of semicrossed products.

Now we move on to our second tool for studying isomorphisms between conjugacy algebras. That is, the definition of nest representations and some accompanying lemmas. If two topological conjugacy algebras $\mathcal{A}(X,\sigma)$ and $\mathcal{A}(Y,\tau)$ are isomorphic we can prove by character theory that there is a homeomorphism between X and Y that maps the fixed points of σ to the fixed points of τ (as will be seen in the main theorem of the next section). The following theory allows us to prove that the existence of certain nest representations gives us the conjugacy of the systems that we desire.

Definition 2.15 *Let $\mathcal{A}$ be an algebra. Denote the collection of representations of $\mathcal{A}$ onto $\mathfrak{T}_2$, the upper triangular 2 × 2 matrices, as* $\mathrm{rep}_{\mathfrak{T}_2}\mathcal{A}$*. As well, let*

$$\theta_{\pi,i}(a) = \langle \pi(a), \xi_i, \xi_i \rangle, \ a \in \mathcal{A}, i = 1, 2$$

be characters that correspond to compressions on the (1,1) and (2,2) entries where $\{\xi_1, \xi_2\}$ is the canonical basis of $\mathbb{C}^2$.

Now suppose that $\mathcal{A}$ is a topological conjugacy algebra for the dynamical system (X, σ). Then define

$$\mathrm{rep}_{x_1,x_2}\mathcal{A} = \{\pi \in \mathrm{rep}_{\mathfrak{T}_2}\mathcal{A} : \theta_{\pi,1} \in \mathcal{M}_{\mathcal{A},x_i}, i = 1, 2\}.$$

Hence, $\mathrm{rep}_{\mathfrak{T}_2}\mathcal{A} = \bigcup_{x,y\in X} \mathrm{rep}_{x,y}\mathcal{A}$*.*

We will also need a few notations. Let $\gamma : \mathcal{A}_1 \to \mathcal{A}_2$ be an isomorphism of algebras. This induces isomorphisms

$$\gamma_c : \mathcal{M}_{\mathcal{A}_1} \to \mathcal{M}_{\mathcal{A}_2}, \ \ \gamma_c(\theta) = \theta \circ \gamma^{-1}, \text{and}$$

$$\gamma_r : \mathrm{rep}_{\mathfrak{T}_2}\mathcal{A}_1 \to \mathrm{rep}_{\mathfrak{T}_2}\mathcal{A}_2, \ \ \gamma_r(\pi) = \pi \circ \gamma^{-1}.$$

These isomorphisms are associated since $\gamma_c(\theta_{\pi,i}) = \theta_{\gamma_r(\pi),i}$ for $i = 1, 2$.

Lemma 2.16 *Let $\mathcal{A}$ be a topological conjugacy algebra for the dynamical system (X, σ). If $x, y \in X$ with $\sigma(x) \neq x, \sigma(y) \neq y$ and $\pi \in \mathrm{rep}_{x,y}\mathcal{A}$, then $y = \sigma(x)$.*

Proof. We assumed $\theta_{\pi,1} = \theta_{x,0}$ and $\theta_{\pi,2} = \theta_{y,0}$, and so $\theta_{\pi,i}(gU) = 0, i = 1, 2, g \in C_0(X)$. Thus, $\pi(gU) = \begin{pmatrix} 0 & c_g \\ 0 & 0 \end{pmatrix}$ for some $c_g \in \mathbb{C}$. By Proposition 2.11 there exists at least on $g \in C_0(X)$ so that $c_g \neq 0$ or else the range of π would be commutative. Apply π to $gUf = (f \circ \sigma)gU$ for this g to get

$$\begin{pmatrix} 0 & c_g \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f(x) & t \\ 0 & f(y) \end{pmatrix} = \begin{pmatrix} f(\sigma(x)) & t' \\ 0 & f(\sigma(y)) \end{pmatrix} \begin{pmatrix} 0 & c_g \\ 0 & 0 \end{pmatrix}$$

for some $t, t' \in \mathbb{C}$, depending on f. We see that this gives $f(y) = f(\sigma(x))$ for all $f \in C_0(X)$ and so $y = \sigma(x)$. □

However, $\text{rep}_{x,\sigma(x)}\mathcal{A}$ is not empty when $x \neq \sigma(x)$. We can see this by first letting

$$\rho(f) = \begin{pmatrix} f(x) & 0 \\ 0 & f(\sigma(x)) \end{pmatrix}, \quad \rho(fU) = \begin{pmatrix} 0 & f(x) \\ 0 & 0 \end{pmatrix}$$

and $\rho(fU^n) = 0, n \geq 2, f \in C_0(X)$. We can then extend ρ by linearity to a map from the algebra of skew formal power series $P^\infty(X,\sigma)$ to the upper triangular representations $\text{rep}_{\mathfrak{T}_2}\mathcal{A}$. Then, $\rho \circ \Delta \in \text{rep}_{x,\sigma(x)}\mathcal{A}$, where $\Delta(a) = \sum_n E_n(a)U^n \in P^\infty(X,\sigma)$ is the Fourier series homomorphism.

Definition 2.17 *Let $\mathcal{A}$ be a topological conjugacy algebra for the one-variable dynamical system (X,σ). Assume that $x, y \in X$ such that $\sigma(y) = y$ but $\sigma(x) \neq x$. A* pencil of nest representations *for $\mathcal{A}$ is a set $\mathcal{P}_{x,y} \subseteq \text{rep}_{x,y}\mathcal{A}$ which satisfies*

$$\{\theta_{\pi,2} : \pi \in \mathcal{P}_{x,y}\} = (\mathcal{M}_{\mathcal{A},y})^\circ = \{\theta_{y,z} : x \in \sigma(U)^\circ\} = \Theta_y(\sigma(U)^\circ)$$

Lemma 2.18 *Let $\mathcal{A}$ be a topological conjugacy algebra for the dynamical system (X,σ) and let $\mathcal{P}_{x,y}$ be a pencil of representations for $\mathcal{A}$. Then $y = \sigma(x)$.*

Proof. Since $\mathcal{P}_{x,y}$ is a pencil, there exists a $\pi \in \mathcal{P}_{x,y}$ such that $\theta_{\pi,1} = \theta_{x,0}$ and $\theta_{\pi,2} = \theta_{y,0}$. The conclusion follows by an identical argument to that in Lemma 2.16. □

Lastly, it can be shown that for $x \in X$ if we have $x \neq \sigma(x)$ and $\sigma(x) = \sigma^{(2)}(x)$ then there exists a pencil of representations, $\mathcal{P}_{x,\sigma(x)}$, for the tensor algebra.

In particular, note that we need only consider compact spaces X. Recall that $\liminf_n \|E_n\|^{-1/n} \geq r$, where r is the spectral radius of U, so $\sigma(U) = \mathbb{D}_r$. If $|z| < r$, we define,

$$\pi_z(f) = \begin{bmatrix} f(x) & 0 \\ 0 & f(\sigma(x)) \end{bmatrix}, \qquad \pi_x(U) = \begin{bmatrix} 0 & z \\ 0 & z \end{bmatrix}.$$

For any $a \sim \sum_n E_n(a)U^n$, define

$$\begin{aligned} \pi_z(a) &= \textstyle\sum_n \pi_z(E_n(a))\pi_z(U)^n \\ &= \begin{pmatrix} E_0(a)(x) & \sum_{n\geq 1} E_n(a)(x)z^n \\ 0 & \sum_{n\geq 0} E_n(a)(\sigma(x))z^n \end{pmatrix}. \end{aligned}$$

Since $|z| < r$, $\pi_z(a)$ is well defined for all $a \in \mathcal{A}$. As well, it is easy to see that $\pi_z(U^n) = \pi_z(U)^n, n \in \mathbb{N}$, and $\pi_z(f \circ \sigma)\pi(U) = \pi_z(U)\pi_z(f)$. Consequently, it easily follows that π_z is an algebra homomorphism that maps onto $\mathfrak{T}_2$. Therefore,

$$\mathcal{P}_{x,\sigma(x)} := \{pi_z : z \in \sigma(U)^\circ\}$$

is the desired pencil of representations.

2.3 Complete characterization

Now we are ready to prove the characterization theorems.

Theorem 2.19 *Let $(X,\sigma),(Y,\tau)$ be one-variable dynamical systems. Then they are conjugate if and only if there exist topological conjugacy algebras of (X,σ) and (Y,τ) that are isomorphic as algebras.*

Proof. If the two systems are conjugate, then the algebras from Example 2.8 are clearly seen to be isomorphic.

Conversely, suppose that $\mathcal{A}, \mathcal{B}$ are conjugacy algebras for (X,σ) and (Y,τ), and that there exists an algebra homeomorphism γ_c of $\mathcal{M}_{\mathcal{A}}$ onto $\mathcal{M}_{\mathcal{B}}$ by $\gamma_c(\theta) = \theta \circ \gamma^{-1}$. It is elementary to verify that γ_c preserves analytic discs and therefore establishes a bijection between the maximal analytic discs of $\mathcal{A}$ and $\mathcal{B}$. This bijection extends to a bijection between their closures and therefore to a bijection between the collections $\{\mathcal{M}_{\mathcal{A},x} : x \in X\}$ and $\{\mathcal{M}_{\mathcal{B},y} : y \in Y\}$. In other words, for each $x \in X$ there exists a $\gamma_s(x) \in Y$ so that

$$\gamma_c(\mathcal{M}_{\mathcal{A},x}) = \mathcal{M}_{\mathcal{B},\gamma_s(x)}.$$

We have therefore defined a map $\gamma_s : X \to Y$, which maps fixed points to fixed points and satisfies

$$f(\gamma_s(x)) = (\theta_{x,0} \circ \gamma^{-1})(f)$$

for all $x \in X$ and $f \in C_0(Y)$. Notice that if $(x_i)_i$ is a net converging to some $x \in X$, then the above equation shows that $(f(\gamma_s(x_i)))_i$ converges to $f(\gamma_s(x))$, for all $f \in C_0(Y)$, and so $(\gamma_s(x_i))_i$ converges to $\gamma_s(x)$. Hence, γ_s is continuous. Repeating the above arguments with $\gamma^{-1} : \mathcal{B} \to \mathcal{A}$ in the place of γ, we obtain that $\gamma_s : X \to Y$ has a continuous inverse and is therefore a homeomorphism.

Furthermore, γ_s maps the fixed point set of σ onto the fixed point set of τ. Finally,
Claim: If $x \in X$ is not a fixed point for σ, then

$$\gamma_r(\text{rep}_{x,\sigma(x)}\mathcal{A}) \subset \text{rep}_{\gamma_s(x),\gamma_s(\sigma(x))}\mathcal{B}$$

Pick a representation $\pi \in \text{rep}_{x,\sigma(x)}\mathcal{A}$. By the first equation, we have $\theta_{\gamma_r(\pi),1} = \gamma_c(\theta_{\pi,1})$ and so $\theta_{\gamma_r(\pi),1} = \gamma_c(\theta_{x,0})$. By the second equation, $\gamma_c(\theta_{x,0}) \in \mathcal{M}_{\mathcal{B},\gamma_s(x)}$ and $\theta_{\gamma_r(\pi),1} \in \mathcal{M}_{\mathcal{B},\gamma_s(x)}$. A similar argument shows that $\theta_{\gamma_r(\pi),2} \in \mathcal{M}_{\mathcal{B},\gamma_s(\sigma(x))}$ and this proves the claim.

We now show that γ_s implements the desired conjugacy between (X, σ) and (Y, τ), i.e.

$$\gamma_s(\sigma(x)) = \tau(\gamma_s(x)), \text{for all } x \in X.$$

Since γ_s maps fixed points to fixed points, verifying the above equation becomes trivial in that case. We therefore pick an element $x \in X$ with $\sigma(x) \neq x$ and we examine two cases.

For the first case assume that $\sigma^{(2)}(x) \neq \sigma(x)$. In that case, pick a representation $\pi \in \text{rep}_{x,\sigma(x)}\mathcal{A}$. Combining the claim with Lemma 2.16, we obtain that $\tau(\gamma_s(x)) = \gamma_s(\sigma(x))$, which proves the equation.

For the second case assume that $\sigma^{(2)}(x) = \sigma(x)$ and let $\mathcal{P}_{x,\sigma(x)}$ be a pencil of representations for $\mathcal{A}$ as was constructed at then end of the last section. By the claim

$$\gamma_r(\mathcal{P}_{x,\sigma(x)}) \subset \text{rep}_{\gamma_s(x),\gamma_s(\sigma(x))}\mathcal{B}.$$

Since γ_c preserves maximal analytic discs, γ_r preserves pencils of representations and so $\gamma_r(\mathcal{P}_{x,\sigma(x)})$ is a pencil of the form $\mathcal{P}_{\gamma_s(x),\gamma_s(\sigma(x))}$. By Lemma 2.18, we have $\tau(\gamma_s(x)) = \gamma_s(\sigma(x))$, which proves the equation in the last remaining case. This proves the Theorem. □

This gives us that the semicrossed product algebras completely characterize one-variable dynamical systems. Thus, the following theorem finishes off the chapter.

Theorem 2.20 *Let $(X, \sigma), (Y, \tau)$ be one-variable dynamical systems. Then they are conjugate if and only if $C_0(X) \times_\sigma \mathbb{Z}^+$ and $C_0(Y) \times_\tau \mathbb{Z}^+$ are isomorphic as algebras.*

Proof. Peter's semicrossed products are not topological conjugacy algebras according to Definitions 2.4 and 2.5. However, there is a natural connection. Let $(C_0(X) \times_\sigma \mathbb{Z}^+)_{op}$ denote the opposite algebra of $C_0(X) \times_\sigma \mathbb{Z}^+$. That is, we define a new multiplication $\odot$ by $a \odot b = ba$ for $a, b \in C_0(X) \times_\sigma \mathbb{Z}^+$. It can easily verified that this opposite algebra is a topological conjugacy algebra. Although, $C_0(X) \times_\sigma \mathbb{Z}^+$ does not need to be isomorphic to its opposite algebra. However, it can be proven that $C_0(X) \times_\sigma \mathbb{Z}^+$ is isomorphic to $C_0(Y) \times_\tau \mathbb{Z}^+$ if and only if $(C_0(X) \times_\sigma \mathbb{Z}^+)_{op}$ is isomorphic to $(C_0(Y) \times_\tau \mathbb{Z}^+)_{op}$.

So by Theorem 2.19 if $C_0(X) \times_\sigma \mathbb{Z}^+$ is isomorphic to $C_0(Y) \times_\tau \mathbb{Z}^+$ then (X, σ) and (Y, τ) are conjugate.

Finally, if the dynamical systems are conjugate then it is clear that $C_0(X) \times_\sigma \mathbb{Z}^+$ is isomorphic to $C_0(Y) \times_\tau \mathbb{Z}^+$ by looking at their definitions. $\square$

Chapter 3

Multivariable case

The multivariable case presents many more difficulties than the previous one-variable case, though the argument has a similar structure and many parallels with the last chapter. We can define a universal algebra along the same lines as in the one-variable case:

Definition 3.1 *Let* (X, σ) *be a multivariable dynamical system. Then the* semicrossed product algebra *is defined to be the universal operator algebra* $C_0(X) \times_\sigma \mathbb{F}_n^+$ *generated by* $C_0(X)$ *and generators* $\mathfrak{s}_1, \cdots, \mathfrak{s}_n$ *satisfying the covariance relations*

$$f\mathfrak{s}_i = \mathfrak{s}_i(f \circ \sigma_i) \quad for \quad f \in C_0(X) \text{ and } 1 \leq i \leq n$$

and satisfying the contractive condition $\|\mathfrak{s}_i\| \leq 1$ *for* $1 \leq i \leq n$.

But Davidson and Katsoulis in [3] found this to be non-tractable when it came to trying to prove that the semicrossed product algebra in some way characterizes multivariable dynamical systems.

They went on to show that if one replaced the contractive condition with another norm condition, namely a row contractive condition, a much nicer object appears called the tensor algebra. This algebra also recommends itself as it is a C*-correspondence algebra as defined by Muhly and Solel [10].

Though a lot of the following theory is mirrored in the semicrossed product case we will be following only the tensor algebra. Readers wishing to pursue this semicrossed product construction as well as the full details of the tensor algebra construction and theory are directed towards [3].

3.1 The tensor algebra

As was mentioned, the tensor algebra is defined similarly to the semicrossed product algebra except for the change in the norm condition.

Definition 3.2 *Let (X,σ) be a multivariable dynamical system. Then the* tensor algebra *is defined to be the universal operator algebra $\mathcal{A}(X,\sigma)$ generated by $C_0(X)$ and generators $\mathfrak{s}_1,\cdots,\mathfrak{s}_n$ satisfying the covariance relations*

$$f\mathfrak{s}_i = \mathfrak{s}_i(f\circ\sigma_i) \quad for \quad f\in C_0(X) \text{ and } 1\le i\le n$$

and satisfying the row contractive condition

$$\|\,[\mathfrak{s}_1\ \mathfrak{s}_2\ \cdots\ \mathfrak{s}_n]\,\|\le 1.$$

Now we move on to a dilation theorem which lets us dilate to isometric representations. This in turn will allow us to use what are called boundary representations to explicitly represent the C*-envelope which is the smallest C*-algebra containing the tensor algebra as a subalgebra. Finally, we use these results to obtain a Fourier series for each element of $\mathcal{A}(X,\sigma)$ which in turn will give us that isomorphisms between tensor algebras are automatically continuous, an important result for the characterization problem.

Without further ado, the dilation theorem:

Theorem 3.3 *Let (X,σ) be a multivariable dynamical system. Let π be a $*$-representation of $C_0(X)$ on a Hilbert space $\mathcal{H}$, and let $A=[A_1\cdots A_n]$ be a row contraction satisfying the covariance relations*

$$\pi(f)A_i = A_i\pi(f\circ\sigma_i) \quad for \quad 1\le i\le n.$$

Then there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$, a $*$-representation ρ of $C_0(X)$ on $\mathcal{K}$ and a row isometry $[S_1\cdots S_n]$ such that

(i) $\rho(f)S_i = S_i\rho(f\circ\sigma_i)$ for $f\in C_0(X)$ and $1\le i\le n$.

(ii) $\mathcal{H}$ reduces ρ and $\rho(f)|_{\mathcal{H}}=\pi(f)$ for $f\in C_0(X)$.

(iii) $\mathcal{H}^\perp$ is invariant for each S_i, and $P_{\mathcal{H}}S_i|_{\mathcal{H}}=A_i$ for $1\le i\le n$.

It follows from this theorem that every row contractive representation of the covariance algebra dilates to a row isometric representation. This allows us to work with isometric representations which will allow us to produce the C*-envelope of the tensor algebra.

Now to construct the C*-envelope we first need to define some representations that Peters [13] used to define the semicrossed product of a one-variable dynamical system.

Let $\mathbb{F}_n^+$ be the free semigroup consisting of all words in the alphabet $\{1, 2, \cdots, n\}$ with the empty word $\emptyset$ as a unit. We define *Fock space* as the Hilbert space $l^2(\mathbb{F}_n^+)$ with the orthonormal basis $\{\xi_w : w \in \mathbb{F}_n^+\}$. Naturally we can define the left regular representation of the free semigroup $\mathbb{F}_n^+$ on Fock space by

$$L_v \xi_w = \xi_v w, \quad v, w \in \mathbb{F}_n^+.$$

Consider the following *orbit representations* of (X, σ). First define the orbit of an element $x \in X$ as $\mathcal{O}(x) = \{\sigma_w(x) : w \in \mathbb{F}_n^+\}$. Then we can define a $*$-representation π_x of $C_0(X)$ on the Fock space $\mathcal{F}_x = l^2(\mathbb{F}_n^+)$ by

$$\pi_x(f)\xi_w = f(\sigma_w(x))\xi_w, \quad f \in C_0(X), w \in \mathbb{F}_n^+$$

or rather we can say $\pi_x(f) = \operatorname{diag}(f(\sigma_w(x)))$. By sending the generators $\mathfrak{s}_i$ to L_i and letting $L_x = [L_1 \ \cdots \ L_n]$ it is easily seen that (π_x, L_x) is the covariant representation.

Lastly, we define the *full Fock representation* $(\Pi, \mathbf{L})$ where $\Pi = \sum_{x\in X}^{\oplus} \pi_x$ and $\mathbf{L} = \sum_{x\in X}^{\oplus} L_x$ on $\mathcal{F}_X = \sum_{x\in X}^{\oplus} \mathcal{F}_x$. With further work this gives us the following proposition.

Proposition 3.4 *The full Fock representation is a faithful completely isometric representation of the tensor algebra $\mathcal{A}(X, \sigma)$.*

A completely contractive representation, say ρ, of an operator algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is *maximal* if in the case when π is a completely contractive dilation of ρ on a Hilbert space $\mathcal{K} = \mathcal{H} \oplus \mathcal{K}_1$, then $\mathcal{H}$ reduces π, that is $\pi = \rho \oplus \pi_1$. In the case of the tensor algebra we can state explicitly a condition for a representation to be maximal.

Lemma 3.5 *A completely contractive representation ρ of the tensor algebra $\mathcal{A}(X, \sigma)$*

is maximal if $S_i = \rho(\mathfrak{s}_i)$ *is an isometry for* $1 \le i \le n$ *and*

$$\sum_{i=1}^{n} S_i S_i^* = E_\rho \left(\bigcup_{i=1}^{n} \sigma_i(X) \right),$$

where E_ρ *denotes the spectral measure associated to* $\rho(C_0(X))$*.*

The converse turns out to be true as well. Finally, a *boundary* representation is a maximal representation that is irreducible, that is, it has no reducing subspaces. If we can produce a completely isometric maximal representation ρ of $\mathcal{A}$ then it is known that $C^*_{env}(\mathcal{A}) = C^*(\rho(\mathcal{A}))$.

Theorem 3.6 *The* C^**-envelope of* $\mathcal{A}(X,\sigma)$ *is* $C^*(\pi(\mathcal{A}(X,\sigma)))$*, where* π *is a maximal dilation of the full Fock representation* Π*.*

Proof. The full Fock representation Π is a completely isometric representation of $\mathcal{A}(X,\sigma)$ by Proposition 3.4. Therefore, any maximal dilation π of Π will yield the C*-envelope.

In [3] this is accomplished by constructing explicit maximal dilations of the orbit representations. Taking the direct sum of the irreducible ones, that is boundary representations, yields the required maximal dilation π. □

However, this does not give an explicit description of the C*-envelope. Davidson and Katsoulis [3] also describe a second approach using what are called C*-correspondences [9] to give another description of the C*-envelope, but again this is via a representation.

In fact, Davidson and J. Roydor, in [5], showed that this C*-envelope of the tensor algebra can be explicitly described. In particular, given a dynamical system (X,σ) that is surjective (they also show what happens in the non-surjective case), that is $X = \bigcup_{i=1}^n \sigma_i(X)$, define $Y = \{1,\cdots,n\}^{\mathbb{N}} \times X^{\mathbb{N}}$ with the product topology. Let

$$\tilde{X} = \{(\mathbf{i},\mathbf{x}) \in Y : \sigma_{i_k}(x_{k+1}) = x_k \text{ for } k \ge 0\},$$

the subset of all infinite tails in Y. We also define the maps

$$\tilde{\sigma}_i(\mathbf{i},\mathbf{x}) = ((i, i_0, i_1, \cdots), (\sigma_i(x_0), x_0, x_1, \cdots)).$$

Therefore, $(\tilde{X}, \tilde{\sigma})$ is a new multivariable dynamical system which is called the *covering system* of (X, σ). Notice also that $\tilde{X}$ is the disjoint union of n sets $\tilde{X}_i = \{(\mathbf{i}, \mathbf{x}) \in \tilde{X} : i_0 = i\}$. This allows us to define an inverse map τ given by

$$\tau|_{\tilde{X}_i} = \tilde{\sigma_i}^{-1} \quad \text{for} \quad 1 \le i \le n$$

which is a local homeomorphism. They go on to show that $C^*_{env}(\mathcal{A}(X, \sigma)) \simeq C^*(\tilde{X}, \tau)$, the groupoid C*-algebra described in [6]. Finally, they use these results to show that $C^*_{env}(\mathcal{A}(X, \sigma)) \simeq \mathcal{B} \times_\alpha \mathbb{N}$ where $\mathcal{B}$ is the inductive limit of homogeneous C*-algebras. This construction mirrors how Peters in [13] defined the C*-envelope of a one-variable dynamical system.

We move on to associating to each element of the tensor algebra a Fourier series. Though similar to the one-variable case there are some distinct differences. One major difference is the necessity to pass to the C*-envelope to compute the Fourier coefficients.

Suppose π is a row contractive representation and $S = [S_1 \cdots S_n]$ satisfies the covariance relations. By the universality of the tensor algebra we have that π and λS satisfy the same conditions, for $\lambda = (\lambda_i) \in \mathbb{T}^n$. Therefore, the map that sends the generators $\mathfrak{s}_i$ to $\lambda_i \mathfrak{s}_i$ and fixes $C_0(X)$ gives us a completely isometric isomorphism of $\mathcal{A}(X, \sigma)$. Furthermore, we can extend this map uniquely to a $*$-automorphism of the C*-envelope. By letting $\lambda_i = z, 1 \le i \le n, z \in \mathbb{T}$ we obtain the gauge automorphisms γ_z.

Proposition 3.7 *The map $E(a) = \int_{\mathbb{T}} \gamma_z(a) dz$ is a completely contractive expectation of $\mathcal{A}(X, \sigma)$ onto $C_0(X)$.*

This map E also makes sense for any element of the C*-envelope. Yet in this case the range is no longer just in $C_0(X)$ but the span of words of the form $\mathfrak{s}_v f s^*_w$ for $|v| = |w|$. This extended version of the map E is used below as the Fourier series of an element of $\mathcal{A}(X, \sigma)$ is calculated within the C*-envelope of the tensor algebra.

Definition 3.8 *For every word $w \in \mathbb{F}^+_n$, the* Fourier coefficient *map $E_w : \mathcal{A}(X, \sigma) \to C_0(X)$ is defined as $E_w(a) = E(\mathfrak{s}^*_w a)$.*

Observe that

$$\mathfrak{s}_w^*\mathfrak{s}_v = \begin{cases} \mathfrak{s}_u, & \text{if } v = wu \\ \mathfrak{s}_u^*, & \text{if } w = vu \\ 0, & \text{otherwise} \end{cases}$$

Hence, $E(\mathfrak{s}_w^*\mathfrak{s}_v f) \neq 0$ only when $w = v$. Therefore, for a polynomial $a = \sum_{v\in\mathbb{F}_n^+} \mathfrak{s}_v f_v$ in $\mathcal{A}(X, \sigma)$ we have that $E_w(a) = f_w$. This extends so that we can write any element of the tensor algebra as a Fourier series:

$$a \sim \sum_{v\in\mathbb{F}_n^+} \mathfrak{s}_w f_w, \quad \forall a \in \mathcal{A}(X, \sigma).$$

However, this series does not converge in general, rather the original element is recovered by interpreting the series via the Cesaro means. Thus we have obtained a Fourier series for each element of the tensor algebra.

We need one more definition before we continue. If $\varphi : \mathcal{A} \to \mathcal{B}$ is an epimorphism between Banach algebras, then the *separating space* of φ is the two-sided closed ideal of $\mathcal{B}$ defined as

$$\mathcal{S}(\varphi) := \{b \in \mathcal{B} : \exists\{a_n\}_{n\geq 1} \subseteq \mathcal{A} \text{ so that } a_n \to 0 \text{ and } \varphi(a_n) \to b\}$$

Thus the graph of φ is closed if and only if $\mathcal{S}(\varphi) = \{0\}$ and then by the closed graph theorem we know that φ is continuous if and only if $\mathcal{S}(\varphi) = \{0\}$. This gives us the required continuity result.

Theorem 3.9 *Suppose (X, σ) and (Y, τ) are multivariable dynamical systems. Then any isomorphism γ of $\mathcal{A}(X, \sigma)$ onto $\mathcal{A}(Y, \tau)$ is automatically continuous.*

Proof. Fix one of the generating isometries of $\mathcal{A}(Y, \tau)$, say $\mathfrak{t}_1$. For any subset $\mathcal{S}$ of $\mathcal{A}(Y, \tau)$, the faithfulness of the Fourier series expansion implies that

$$\bigcap_{k\geq 0} \mathfrak{t}_1^k \mathcal{S} = \{0\}$$

Thus if $\mathcal{S}(\gamma) \neq \{0\}$, then it is known that there exists a $k_0 \in \mathbb{N}$ so that for all $k \geq k_0$ we have that $\mathfrak{t}_1^k \mathcal{S}(\gamma) = \mathfrak{t}_1^{k+1} \mathcal{S}(\gamma)$. Thus, we have

$$\mathfrak{t}_1^{k_0} \mathcal{S}(\gamma) = \bigcap_{k\geq 0} \mathfrak{t}_1^k \mathcal{S}(\gamma) = \{0\}.$$

Thus, $\mathcal{S}(\gamma) = \{0\}$ because left multiplication by $\mathfrak{t}_1$ is injective. Therefore γ is continuous. $\square$

This result will prove very useful as now we need to only consider continuous representations in the study of isomorphisms between tensor algebras.

3.2 Characters, nest representations, and piecewise conjugacy

This section will follow a similar method to that discussed in the one-variable dynamical system case in Chapter 2. Once again will we first look at characters and then study nest representations into the 2×2 upper triangular matrices. There is also an analytic structure that appears at fixed points that will be needed. Lastly, we introduce a weakened concept of conjugacy since an extension of the concept of conjugacy in the one-variable case becomes highly restrictive.

Let $\mathcal{A} = \mathcal{A}(X, \sigma)$ be a tensor algebra for the multivariable dynamical system (X, σ). First we consider $\mathcal{M}_\mathcal{A}$, the space of characters of $\mathcal{A}$ defined with the weak-$*$ topology. We will use the partitioning $\mathcal{M}_\mathcal{A} = \dot{\bigcup}_{x \in X} \mathcal{M}_{\mathcal{A},x}$ found in section 3 of Chapter 2, where $\mathcal{M}_{\mathcal{A},x}$ comprises all characters that act as the point evaluation δ_x at the point x when restricted to $C_0(X)$. Recall that we defined an expectation from $\mathcal{A}$ to $C_0(X)$ in Proposition 3.7; this allowed us to always produce the expectation $\theta_{x,0} = \delta_x E$ in $\mathcal{M}_{\mathcal{A},x}$. Lastly, because a character $\theta \in \mathcal{M}_{\mathcal{A},x}$ is always continuous then it can be determined by $z = (\theta(\mathfrak{s}_1), \cdots, \theta(\mathfrak{s}_n))$. We will denote this character as $\theta_{x,z}$ when it is defined.

Lemma 3.10 *Suppose $x \in X$ and let $I_x = \{i : 1 \leq i \leq n, \sigma_i(x) = x\}$ is the fixed point set. Then*

$$\mathcal{M}_{\mathcal{A},x} = \{\theta_{x,z} : z_i = 0 \ for \ i \notin I_x, \|z\|_2 \leq 1\} =: \overline{\mathbb{B}(I_x)}$$

Furthermore, for $a \in \mathcal{A}$, $\Theta_a(z) = \theta_{x,z}(a)$ is analytic on the ball of radius 1 in the variables $\{z_i : i \in I_x\}$ and is continuous on the closure. In the special case when x is not fixed by any σ_i then $\mathcal{M}_{\mathcal{A},x} = \{\theta_{x,0}\}$.

In parallel with the one-variable case we again define analytic sets.

Definition 3.11 *Consider a continuous bijection $\Theta : \Omega \to M \subset \mathcal{M}_{\mathcal{A}}$ for a domain $\Omega \subset \mathbb{C}^d$. Then we call M an* analytic set *if the function $\Theta(z)(a)$ is analytic on Ω for every $a \in \mathcal{A}$. It is called a* maximal analytic set *if it is maximal among all analytic subsets of $\mathcal{M}_{\mathcal{A}}$.*

Now let Θ map a domain Ω into $\mathcal{M}_{\mathcal{A}}$. For each $f \in C_0(X)$ both $\Theta(z)(f)$ and $\Theta(z)(\overline{f}) = \overline{\Theta(z)(f)}$ are analytic, and thus constant. Now continuous functions on X separate points so Θ maps into a single fibre. By Lemma 3.10 we have that $\mathcal{M}_{\mathcal{A},x}$ is homeomorphic to a closed ball. Therefore, we can conclude that the maximal analytic sets in $\mathcal{M}_{\mathcal{A}}$ are precisely the open balls $B_x = \{\theta_{x,z} : z \in \mathbb{B}(I_x)\}$ for those x fixed by at least one σ_i.

This leads us to the following conclusion:

Proposition 3.12 *The characters of $\mathcal{A}$ determine X up to homeomorphism, and identify which points are fixed and by how many of the σ_i maps.*

Proof. The above discussion shows that $\mathcal{M}_{\mathcal{A}}$ consists of a space which is fibred over X, and the fibres are determined canonically as the closures of maximal analytic sets and the remaining singletons. Thus there is a canonical quotient map of $\mathcal{M}_{\mathcal{A}}$ onto X, determining X. Now the points which are fixed by some σ_i are exactly the points with a non-trivial fibre of characters. The corresponding maximal analytic set is homeomorphic to a ball in $\mathbb{C}^d$ where $d = |I_x|$. The invariance of domain theorem shows that the dimension d is determined by the topology. □

This theory is used in [3] to show that if (X, σ) has a point that is fixed by at least two of the σ_i maps then $\mathcal{A}(X, \sigma)$ and the semicrossed product algebra $C_0(X) \times_\sigma \mathbb{F}_n^+$ are not algebraically isomorphic.

Now we come to the second part of this section, that is, the development of the theory of nest representations. It turns out that we will need a more general development of these representations than was needed in the one-variable case in Chapter 2.

First, let $\mathcal{N}_2$ denote that maximal nest $\{\{0\}, \mathbb{C}e_1, \mathbb{C}^2\}$ in $\mathbb{C}^2$. Then we can define:

Definition 3.13 *A* nest representation *ρ is a continuous representation of an operator algebra $\mathcal{A}$ on $\mathbb{C}^2$ such that Lat $\rho(\mathcal{A}) = \mathcal{N}_2$. The collection of all such representations is denoted $rep_{\mathcal{N}_2}$.*

This gives a more general definition of nest representations than in Chapter 2 because there are two unital subalgebras of $M_2(\mathbb{C})$ with $\mathcal{N}_2$ as the lattice of invariant subspaces. They are $\mathfrak{T}_2$ and the abelian algebra $\mathcal{A}(E_{12}) = \text{span}\{I, E_{12}\}$ and both have non-trivial radical.

Next, for any representation ρ of $\mathcal{A}$ into $\mathfrak{T}_2$, the compression to a diagonal entry is a homomorphism. Hence ρ gives us two characters denoted $\theta_{\rho,1}$ and $\theta_{\rho,2}$. As well, if we let ψ denote the map that gives us the (1,2)-entry of $\rho(a)$ then we can conclude that ψ is a point derivation and we have

$$\psi(ab) = \theta_{\rho,1}(a)\psi(b) + \psi(a)\theta_{\rho,2}(b), \quad a, b \in \mathcal{A}.$$

Now we define $\text{rep}_{x,y}\mathcal{A} = \{\rho \in \text{rep}_{\mathcal{N}_2} : \theta_{\rho,1} \in \mathcal{M}_{\mathcal{A},x}, \theta_{\rho,2} \in \mathcal{M}_{\mathcal{A},y}\}$ the same as in the one-variable case.

For our purposes it is enough to consider representations which restrict to $*$-representations of $C_0(X)$. This comes automatically for (completely) contractive representations of $C_0(X)$. As well, representations of $C_0(X)$ into $M_2(\mathbb{C})$ are automatically continuous, and thus diagonalizable. So, we let $\text{rep}^d_{\mathcal{N}_2}\mathcal{A}$ and $\text{rep}^d_{x,y}\mathcal{A}$ to denote the nest representations which are diagonal on $C_0(X)$.

Lastly we need two technical lemmas about nest representations, the second of which is key to recovering the dynamical system from the tensor algebra.

Lemma 3.14 *Let X be a locally compact space; and let $\sigma : X \to X$ be a continuous map. Let $K \subset X$, and let Ω be a domain in $\mathbb{C}^d$. Suppose there exists $\rho : K \times \Omega \to \text{rep}\mathcal{A}$ that satisfies the following*

(1) $\rho(x, z) \in \text{rep}_{x,\sigma(x)}\mathcal{A}$ for $x \in K$ and $x \in \Omega$,

(2) ρ is continuous in the point-norm topology, and

(3) $\rho(x, z)$ is analytic in $z \in \Omega$ for each fixed $x \in K$.

Then there exists $A : K \times \Omega \to \mathfrak{T}_2^{-1}$ such that

(1) $A(x, z)\rho(x, z)A(x, z)^{-1} \in \text{rep}^d\mathcal{A}$,

(2) $A(x, z)$ is continuous on $(K \setminus \{x : \sigma(x) = x\}) \times \Omega$,

(3) $A(x, z)$ is analytic in $z \in \Omega$ for each fixed $x \in K$, and

(4) $\max\{\|A(x, z)\|, \|A(x, z)^{-1}\|\} \le 1 + \|\rho(x, z)\|$.

Lemma 3.15 *If $\text{rep}_{y,x}(\mathcal{A})$ is non-empty, then there is some i such that $\sigma_i(x) = y$. As well, if $\rho \in \text{rep}^d_{y,x}(\mathcal{A})$ and $\sigma_j(x) \neq y$, then $\rho(\mathfrak{s}_j g)$ is diagonal for all $g \in C_0(X)$.*

Finally for this section, we define a natural form of conjugacy for multivariable dynamical systems. When we move to the multivariable case there is an ambiguity introduced into the construction of the universal operator algebra. Thus, isomorphism of universal algebras implies conjugacy of dynamical systems in only limited circumstances. Thus, a new concept of local conjugacy, called piecewise conjugacy, was introduced in [3].

Definition 3.16 *Two multivariable dynamical systems* (X, σ) *and* (Y, τ) *are* piecewise conjugate *if there is a homeomorphism* $\gamma : X \to Y$ *and an open cover* $\{\mathcal{V}_\alpha : \alpha \in S_n\}$ *of* X *such that*

$$\gamma^{-1}\tau_i\gamma|_{\mathcal{V}_\alpha} = \sigma_{\alpha(i)}|_{\mathcal{V}_\alpha}, \text{ for } \alpha \in S_n$$

Thus piecewise conjugacy depends upon a cover which causes the permutations to depend on each open set. So piecewise conjugacy and conjugacy only correspond under rather restrictive circumstances.

One such equivalence is given in the next proposition, though the condition on the maps σ are highly restrictive.

Proposition 3.17 *Let* (X, σ) *and* (Y, τ) *be piecewise conjugate multivariable dynamical systems. If* X *is connected and the set*

$$E := \{x \in X : |\{\sigma_1(x), \cdots, \sigma_n(x)\}| = n\}$$

is dense in X *then* (X, σ) *and* (Y, τ) *are conjugate.*

If the space X is totally disconnected it turns out that piecewise conjugacy can be explicitly stated.

Proposition 3.18 *Let* X *be a totally disconnected compact Hausdorff space and let* γ *be a homeomorphism of* X *onto a space* Y*. Then the multivariable dynamical systems* (X, σ) *and* (Y, τ) *are piecewise conjugate by* γ *if and only if* X *can be partitioned into clopen sets* $\{\mathcal{V}_\alpha : \alpha \in S_n\}$ *such that for all* $\alpha \in S_n$*,*

$$\gamma^{-1}\tau_i\gamma|_{\mathcal{V}_\alpha} = \sigma_{\alpha(i)}|_{\mathcal{V}_\alpha}.$$

3.3 The central claim

Now we are ready to state what can be proven about algebraic characterization of multivariable dynamical systems as well as the conjecture that fills in the gap in the theory.

Theorem 3.19 *Let (X,σ) and (Y,τ) be two multivariable dynamical systems. If $\mathcal{A} = \mathcal{A}(X,\sigma)$ and $\mathcal{B} = \mathcal{A}(Y,\tau)$ are isomorphic as algebras then (X,σ) and (Y,τ) are piecewise conjugate.*

Proof. Let $\gamma : \mathcal{A} \to \mathcal{B}$ be an isomorphism. This induces a bijection $\gamma_c : \mathcal{M}_{\mathcal{A}} \to \mathcal{M}_{\mathcal{B}}$ by $\gamma_c(\theta) = \theta \circ \gamma^{-1}$ and a map γ_r from $\mathrm{rep}_{\mathcal{N}_2}(\mathcal{A})$ onto $\mathrm{rep}_{\mathcal{N}_2}(\mathcal{B})$.

Since $\mathcal{M}_{\mathcal{A}}$ is endowed with the weak-$*$ topology, we see that γ_c is continuous. Because, if θ_α is a net in $\mathcal{M}_{\mathcal{A}}$ converging to θ and $b \in \mathcal{B}$ then

$$\lim_\alpha \gamma_c\theta_\alpha(b) = \lim_\alpha \theta_\alpha(\gamma^{-1}(b)) = \theta(\gamma^{-1}(b)) = \gamma_c\theta(b).$$

Similarly this holds for γ_c^{-1}. Thus, γ_c is a homeomorphism.

Observe that γ_c carries analytic sets to analytic sets. Indeed, if Θ is an analytic function of a domain Ω into $\mathcal{M}_{\mathcal{A}}$, then

$$\gamma_c\Theta(z)(b) = \Theta(z)(\gamma^{-1}(b))$$

is analytic for every $b \in \mathcal{B}$; and thus $\gamma_c\Theta$ is analytic. The same holds for γ^{-1} and so it follows that γ_c takes maximal analytic sets to maximal analytic sets. Thus it carries their closures, $\mathcal{M}_{\mathcal{A},x}$, onto sets $\mathcal{M}_{\mathcal{B},y}$. The same also holds when these sets are singletons.

It is known that the space X is the quotient of $\mathcal{M}_{\mathcal{A}}$ obtained by making each $\mathcal{M}_{\mathcal{A},x}$ a point. It follows that γ_c induces a set map γ_s of X onto Y which is a homeomorphism since both X and Y inherit the quotient topology.

Fix $x_0 \in X$, and let $y_0 = \gamma_s(x_0)$. Fix one of the maps σ_{i_0}, and consider the set

$$\mathcal{F} = \{\sigma_i, \gamma_s^{-1}\tau_j\gamma_s : [\sigma_i]_{x_0} = [\sigma_{i_0}]_{x_0} = [\gamma_s^{-1}\tau_j\gamma_s]_{x_0}\}.$$

For convenience, let us relabel so that $i_0 = 1$ and

$$\mathcal{F} = \{\sigma_1, \cdots, \sigma_k, \gamma_s^{-1}\tau_1\gamma_s, \cdots, \gamma_s^{-1}\tau_l\gamma_s\}.$$

Fix a neighbourhood $\mathcal{V}$ of x_0 on which all of these functions agree, and such that $\overline{\mathcal{V}}$ is compact. Furthermore, if $\sigma_1(x_0) \neq x_0$, then choose $\mathcal{V}$ so that $\overline{\mathcal{V}} \cap \sigma_1(\overline{\mathcal{V}}) = \emptyset$.

Now if $k = l$ for every choice of x_0 and map σ_0 then we can partition the functions into families with a common germ at x_0 that always have the same number of σ_i maps as $\gamma_s^{-1}\tau_i\gamma_s$ maps. If $\alpha \in S_n$ preserves this partition at x_0 then we define $V_{\alpha,x_0} = V$, otherwise $V_{\alpha,x_0} = \emptyset$. Finally, let $V_\alpha = \bigcup_{x\in X} V_{\alpha,x}$, for $\alpha \in S_n$, which defines the open covering of X that makes (X,σ) and (Y,τ) piecewise conjugate.

Assume then that $k \neq l$. Without loss of generality we can assume that $k > l$. Also note that $l = 0$ is a possibility.

For any $x \in \mathcal{V}$ and $z = (z_1, z_2, \cdots, z_l) \in \mathbb{C}^k$, consider the covariant representations $\rho_{x,z}$ of $\mathcal{A}_0(X,\sigma)$ into $\mathcal{M}_2$ defined by

$$\rho_{x,z}(f) = \begin{bmatrix} f(\sigma_1(x)) & 0 \\ 0 & f(x) \end{bmatrix},$$

$$\rho_{x,z}(\mathfrak{s}_i) = \begin{bmatrix} 0 & z_i \\ 0 & 0 \end{bmatrix} \quad \text{for} \ \ 1 \leq i \leq k,$$

and

$$\rho_{x,z}(\mathfrak{s}_i) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{for} \ \ k < i \leq n.$$

This extends to a well defined representation of $\mathcal{A}$, where a typical element $A \sim \sum_{w\in\mathbb{F}_n^+} \mathfrak{s}_w f_w$ is sent to

$$\rho_{x,z}(A) = \begin{bmatrix} f_0(\sigma_1(x)) & \sum_{i=1}^k f_i(x) z_i \\ 0 & f_0(x) \end{bmatrix}$$

There are no continuity problems since the Fourier coefficients are continuous.

This representation will be (completely) contractive for $\mathcal{A}(X,\sigma)$ if $z \in \overline{\mathbb{B}_k}$. For other values of z, this representation is similar to a completely contractive representation by conjugating by $\operatorname{diag}(\|z\|_2, 1)$. Thus the norm can be estimated as $\|\rho_{x,z}\| \leq \|z\|$.

The representation $\rho_{x,z}$ maps into $\mathfrak{T}_2$ and is a nest representation in $\operatorname{rep}_{\sigma_1(x),x}\mathcal{A}$ when $z \neq 0$, but is diagonal at $z = 0$. Observe that the range of $\rho_{x,z}$ for $z \neq 0$ equals $\mathfrak{T}_2$ when $\sigma_1(x) \neq x$ and equals $\mathcal{A}(E_{12})$ when $\sigma_1(x) = x$. Moreover this map is point-norm continuous, and is analytic in the second variable.

Now consider the map defined on $\overline{\mathcal{V}} \times \mathbb{C}^k$ given by

$$\Phi_0(x,z) = \gamma_r(\rho_{x,z}) \in \mathrm{rep}_{\gamma_s\sigma_1(x),\gamma_s(x)}\mathcal{B}.$$

It is known that since γ is an isomorphism between tensor algebras then it is automatically continuous; and so γ_r is also continuous. Thus Φ_0 is point-norm continuous, and is analytic in the second variable. So Φ_0 fulfils the requirements of Lemma 3.14. Hence there exists a map $A(x,z)$ of $\overline{\mathcal{V}} \times \mathbb{C}^k$ into $\mathfrak{T}_2^{-1}$, which is analytic in the second variable, so that

$$\Phi(x,z) = A(x,z)\gamma_r(\rho_{x,z})A(x,z)^{-1}$$

diagonalizes $C_0(Y)$. Moreover

$$\max\{\|A(x,z)\|, \|A(x,z)^{-1}\|\} \le 1 + \|\gamma_r\|\|z\|.$$

Recall that when $\sigma_1(x_0) \neq x_0$, we chose $\mathcal{V}$ so that $\overline{\mathcal{V}}$ is disjoint from $\sigma_1(\overline{\mathcal{B}})$. Therefore in this case, A is a continuous function.

Choose $h \in C_0(Y)$ such that $h|_{\gamma_s(\overline{\mathcal{V}})} = 1$ and $\|h\|_\infty = 1$. Define $\psi_j(z)$ to be the 1,2 entry of $\Phi(x_0,z)(\mathfrak{t}_j h)$; and set $\Psi(z) = (\psi_1(z), \cdots, \psi_n(z))$. Then Ψ is an analytic function from $\mathbb{C}^k$ into $\mathbb{C}^n$.

We can now show that $\psi_j(z) = 0$ for $j > l$.

Indeed, since $j > l$, the map $\gamma^{-1}\tau_j\gamma$ is not in $\mathcal{F}$. Hence there exists a net $(x_\lambda)_{\lambda\in\Lambda}$ in $\overline{\mathcal{V}}$ converging to x_0 so that $\gamma_s^{-1}\tau_j\gamma_s(x_\lambda) \neq \sigma_1(x_\lambda)$ for all $\lambda \in \Lambda$. By Lemma 3.15, $\Phi(x_\lambda, z)(\mathfrak{t}_j h)$ is diagonal for all λ in Λ.

First consider the case when $\sigma_1(x_0) \neq x_0$. Then $A(x,z)$ is continuous, and so $\Phi(x,z)$ is point-norm continuous. Taking limits, we conclude that $\Phi(x_0,z)(\mathfrak{t}_j h)$ is diagonal; whence $\psi_j(z) = 0$.

Now consider the case $\sigma_1(x_0) = x_0$. Recall that in this case $\Phi(x_0,z)$ has range in $\mathcal{A}(E_{12})$; so that the diagonal part consists of scalars. Fix $z \in \mathbb{C}^k$. Since

$$\max\{\|A(x_\lambda,z)\|, \|A(x_\lambda,z)^{-1}\|\} \le 1 + \|\gamma_r\|\|z\|,$$

we may pass to a subnet if necessary so that $\lim_\Lambda A(x_\lambda, z) = A(z)$ exists in $\mathfrak{T}_2^{-1}$. Since Φ_0 is point-norm continuous and $A(x_0, z) = I_2$,

$$\lim_{\lambda\in\Lambda} \Phi(x_\lambda, z)(\mathfrak{t}_j h) = \lim_{\lambda\in\Lambda} A(x_\lambda, z)\Phi_0(x_\lambda, z)(\mathfrak{t}_j h)A(x_\lambda, z)^{-1}$$

$$= A(z)\Phi_0(x_0, z)(\mathfrak{t}_j h)A(z)^{-1}$$

Therefore $A(z)\Phi_0(x_0, z)(\mathfrak{t}_j h)A(x)^{-1}$ is diagonal, and hence scalar. So $\Phi(x_0, z)(\mathfrak{t}_j h)$ is scalar and $\phi_j(z) = 0$, which proves the claim.

The function Ψ can now be considered as an analytic function from $\mathbb{C}^k$ into $\mathbb{C}^l$. Observe that $\Psi(0) = 0$. Now it is known that the zero set of Ψ has no isolated points because it is an analytic variety of dimension greater or equal to $k-l$. Thus, there exists $z_0 \neq 0$ for which $\Psi(z_0) = 0$. Then $\Phi(z_0)$ is diagonal and thus is not a nest representation. This is a contradiction which proves that $k = l$ and finishes off the proof. $\square$

Following this we have the central conjecture, namely that the tensor algebra characterizes multivariable dynamical systems up to piecewise conjugacy.

Conjecture 3.20 *Let (X, σ) and (Y, τ) be paracompact dynamical systems with $\sigma = \{\sigma_1, \cdots, \sigma_n\}$ and $\tau = \{\tau_1, \cdots, \tau_n\}$. Then (X, σ) and (Y, τ) are piecewise conjugate if and only if $\mathcal{A}(X, \sigma)$ and $\mathcal{A}(Y, \tau)$ are completely isometrically isomorphic.*

In [3] this was reduced to the following technical conjecture about the imbedding of a $n!$-simplex into the n-dimensional unitary group with certain decomposition conditions, which they proved to be true if $n \leq 3$.

Conjecture 3.21 *Let Π_n be the $n!$-simplex with vertices indexed by S_n. There is a continuous function $u : \Pi_n \to U(n)$ such that:*

(1) Every vertex is taken to its corresponding permutation matrix and,

(2) Given any two sets of partitions $A_1\dot{\cup}\cdots\dot{\cup}A_m = B_1\dot{\cup}\cdots\dot{\cup}B_m = \{1, \cdots, n\}$, with $|A_j| = |B_j|, 1 \leq j \leq m$ let $\mathcal{P}(A, B) = \{\alpha \in S_n : \alpha(A_j) = B_j, 1 \leq j \leq m\}$. Then if $x = \sum_{\alpha\in\mathcal{P}(A,B)} x_\alpha \alpha$ the non-zero matrix coefficients of $u(x)$ are supported on $\bigcup_{j=1}^m B_j \times A_j$.

This conjecture will be reformulated in Chapter 4 via a construction in the unitary group. If this is proven to be true then the central theorem follows as presented in [3].

Proof of Conjecture 3.20 modulo Conjecture 3.21

Completely isometrically isomorphic implies isomorphic as algebras. By Theorem 3.19 we know that if $\mathcal{A}(X,\sigma)$ and $\mathcal{A}(Y,\tau)$ are isomorphic as algebras then (X,σ) and (Y,τ) are piecewise conjugate.

So conversely assume that (X,σ) and (Y,τ) are piecewise conjugate. Namely, there is a homeomorphism γ and an open cover $\{\mathcal{V}_\alpha : \alpha \in S_n\}$. From this simplify so $Y = X$ and $\gamma = id$, for ease of notation. Let $\{g_\alpha : \alpha \in S_n\}$ be a partition of unity relative to the cover. Now we can define a map $g : X \to \Pi_n$ given by

$$g(x) = (g_\alpha(x))_{\alpha \in S_n}$$

By Conjecture 3.21 there exists a map u from Π_n to the unitary group that satisfies the block decomposition condition. Let $v = g \circ u$ which maps X to $U(n)$. For $x \in X$ there is a minimum partition (A, B) and an open neighbourhood $\mathcal{V}$ of x such that

$$\sigma_i|_{\mathcal{V}} = \tau_j|_{\mathcal{V}} \text{ for } i \in A_s \text{ and } j \in B_s,\ 1 \leq s \leq m.$$

Because the permutations α respect this block structure when $g_\alpha(x) \neq 0$ then so does the map v.

Define operators in $\mathcal{A}(X,\sigma)$ by $T_i = \sum_{j=1}^n \mathfrak{s}_j v_{ij}$, where v_{ij} are the matrix coefficients of v. Since the $\mathfrak{s}_j$ have pairwise orthogonal ranges then

$$T_k^* T_i = \sum_{j=1}^n \overline{v_{kj}} v_{ij} = \delta_{ki} I$$

which implies that the T_i are isometries with pairwise orthogonal ranges. Now if $v_{ij}(x) \neq 0$, then σ_j and τ_i agree on a neighbourhood of x and thus $v_{ij}(f \circ \sigma_j) = v_{ij}(f \circ \tau_i)$ for all i, j. Hence,

$$fT_i = f \sum_{j=1}^n \mathfrak{s}_j v_{ij} = \sum_{j=1}^n \mathfrak{s}_j v_{ij}(f \circ \sigma_j) = \sum_{j=1}^n \mathfrak{s}_j v_{ij}(f \circ \tau_i) = T_i(f \circ \tau_i).$$

Next we know that $\mathcal{A}(X,\sigma)$ is generated by $C_0(X)$ and $T_iC_0(X)$ for $1 \le i \le n$ because for $1 \le k \le n$ we have

$$\sum_{i=1}^{n} T_i \overline{v_{ik}} f = \sum_{i=1}^{n} \sum_{j=1}^{n} \mathfrak{s}_j v_{ij} \overline{v_{ik}} f = \sum_{j=1}^{n} \mathfrak{s}_j f \sum_{i=1}^{n} v_{ij} \overline{v_{ik}} = \mathfrak{s}_k f.$$

Therefore, there is a completely contractive homomorphism of $\mathcal{A}(Y,\tau)$ onto $\mathcal{A}(X,\sigma)$ sending $\mathfrak{t}_i$ to T_i for $1 \le i \le n$ and which is the identity on $C_0(Y) = C_0(X)$. Similarly there is a completely contractive homomorphism of $\mathcal{A}(X,\sigma)$ onto $\mathcal{A}(Y,\tau)$ which is the inverse on the generators $\mathfrak{s}_j$. Hence, these maps are completely isometric isomorphisms. $\square$

Chapter 4

The reformulation of the conjecture

In Chapter 3 we saw that the characterization of multivariable dynamical systems rests on a question about mapping a simplex in a particular way into the unitary group. We will use this fact to restate the conjecture as a question purely about the structure of the unitary group.

4.1 Construction and reformulation

From algebraic topology it is known that the unitary group $U(n)$ is a CW-complex with a largest simplex of size $n!$. However, this is a general theory for all Lie groups and it does not give us the block decomposition condition of the conjecture. The following inductive construction gives a very "nice" subset of $U(n)$ that contains all the permutation matrices and is contractible at every level.

It was pointed out to the author in a private communication from R. Kane that neither $SU(n)$ or $O(n)$ can be contained in such a contractible subset of $\mathcal{U} \subset U(n)$ since the inclusion map $SU(n) \to U(n)$ induces an injective map from $H_*(SU(n))$ into $H_*(U(n))$, but because $H^*(\mathcal{U}) = \begin{cases} 0, & * > 0 \\ \mathbb{Z}, & * = 0 \end{cases}$, since $\mathcal{U}$ is connected and contractible, then the induced homology map $H_*(SU(n)) \to H_*(U(n))$ is trivial for $* > 0$, a contradiction. A similar contradiction occurs when we consider $O(n)$ instead of $SU(n)$.

Therefore, the structure of such a contractible subset of $U(n)$ will be quite "twisted" as can be seen in the following construction.

Proposition 4.1 *There exists a copy of the complex unit sphere missing one point in n dimensions in the unitary group $U(n)$.*

Proof. Consider the inverse stereographic projection $\mathfrak{p} : \mathbb{R}^{n-1} \to \mathrm{S}^{n-1} \setminus \{-e_1\}$ given by

$$v_1 = \mathfrak{p}(y_2, \cdots, y_n) = \frac{2}{1 + y_2^2 + \cdots + y_n^2} \begin{pmatrix} \frac{1-y_2^2-\cdots-y_n^2}{2} \\ y_2 \\ \vdots \\ y_n \end{pmatrix}.$$

Next consider $\frac{d}{dt}_{|t=0}\ \mathfrak{p}(y_2, \cdots, y_k + t, \cdots, y_n)$ for $2 \leq k \leq n$ to get an orthogonal basis for the tangent plane of $\mathfrak{p}(y_2, \cdots, p_n)$. Now,

$$v_k = \frac{d}{dt}_{|t=0}\ \mathfrak{p}(y_2, \cdots, y_k + t, \cdots, y_n) = \frac{4}{(1 + y_2^2 + \cdots + y_n^2)^2} \begin{pmatrix} -y_k \\ -y_2 y_k \\ \vdots \\ \frac{1+y_2^2+\cdots-y_k^2+\cdots+y_n^2}{2} \\ \vdots \\ -y_n y_k \end{pmatrix}$$

and $\|v_k\| = \frac{2}{1+y_2^2+\cdots+y_n^2}$ for $k \in \{2, \cdots, n\}$. Thus for $n \geq 2$ let $\tilde{\mathfrak{S}}(n)$

$$= \left\{ \begin{bmatrix} 1 & & & \\ & z_2 & & \\ & & \ddots & \\ & & & z_n \end{bmatrix} \times \begin{bmatrix} v_1 & \frac{v_2}{\|v_2\|} & \cdots & \frac{v_n}{\|v_n\|} \end{bmatrix} \times \begin{bmatrix} 1 & & & \\ & \overline{z_2} & & \\ & & \ddots & \\ & & & \overline{z_n} \end{bmatrix} \;\middle|\; \begin{matrix} z_2, \cdots, z_n \in \mathrm{S}^1, \\ y_2, \cdots, y_n \geq 0 \end{matrix} \right\}.$$

Note that $\tilde{\mathfrak{S}}(n) \subset U(n)$. However, the (1,1)-entry can only be real. To allow for the first column to be anything but $-e_1$ let $y_1 \in \mathbb{R}$ and consider the following variation to the matrices in $\tilde{\mathfrak{S}}(n)$:

$$\frac{2}{1 + \sum_{i=1}^n y_i^2} \begin{bmatrix} \frac{1-y_1^2-\cdots-y_n^2}{2} + iy_1 & -\overline{z_2} y_2 & \cdots & -\overline{z_n} y_n \\ z_2 y_2 & \frac{1-y_2^2+\cdots+y_n^2}{2} & & -z_2 \overline{z_n} y_2 y_n \\ \vdots & & \ddots & \vdots \\ z_n y_n & -z_n \overline{z_2} y_2 y_n & \cdots & \frac{1+y_2^2+\cdots+y_{n-1}^2-y_n^2}{2} \end{bmatrix},$$

which have nonzero determinants because the determinant is always real when $y_1 = 0$ (just look at the structure in $\tilde{\mathfrak{S}}(n)$). Now we need to introduce a small twist to ensure that all the permutation matrices show up in our construction. Let $0 < \theta < 1/2$ be irrational. We can rewrite $\tilde{\mathfrak{S}}(n)$ with this scaling into a simpler form. Namely:

$$\frac{2}{1+\sum_{i=1}^{n}|y_i|^2}\begin{bmatrix} e^{\theta\pi i}(\frac{1-|y_1|^2-\cdots-|y_n|^2}{2}+iy_1) & -\overline{y_2} & \cdots & -\overline{y_n} \\ e^{\theta\pi i}y_2 & \frac{1-|y_2|^2+\cdots+|y_n|^2}{2} & & -y_2\overline{y_n} \\ \vdots & & \ddots & \vdots \\ e^{\theta\pi i}y_n & -y_n\overline{y_2} & \cdots & \frac{1+|y_2|^2+\cdots+|y_{n-1}|^2-|y_n|^2}{2} \end{bmatrix}$$

with $y_1 \in \mathbb{R}$ and $y_2, \cdots, y_n \in \mathbb{C}$. By applying the Gram-Schmidt orthonormalization to the above matrices we get a subset $\mathfrak{S}(n)$ of $U(n)$ which is a copy of the complex unit sphere missing one point, specifically the point $-e^{\theta\pi i}e_1$. Note also that $\mathfrak{S}(n)$ is contractible. $\square$

The above matrix identification of the complex n-sphere missing one point (in its pre-Gram-Schmidt form) is central to the following theory and will be used many times.

Now we can define a large subset of the unitary group that is a lot nicer in its properties. This definition is motivated by the well known fact that

$$U(n)/U(n-1) \simeq S^{2n-1}.$$

Definition 4.2 *Let* $\mathcal{U}_1 = \mathfrak{S}(1) = \mathrm{S}^1 \setminus \{-e^{\theta\pi i}\}$ *and*

$$\mathcal{U}_n = \mathfrak{S}(n) \times \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{U}_{n-1} \end{bmatrix} \subset U(n),\ n \geq 2.$$

Observe that $\mathcal{U}_n$ is a locally trivial bundle and thus the following sequence of homotopy groups is exact [7, pp. 77-84]:

$$\cdots \to \pi_{k+1}(\mathfrak{S}(n)) \to \pi_k(\mathcal{U}_{n-1}) \to \pi_k(\mathcal{U}_n) \to \pi_k(\mathfrak{S}(n)) \to \cdots$$

By induction on n, this gives us that $\mathcal{U}_n$ is contractible because $\mathcal{U}_1$ is contractible.

Next we must check that $\mathcal{U}_n$ contains all of the necessary elements, specifically the permutation matrices.

Proposition 4.3 *The permutation matrices, which we will denote by S_n, are contained in $\mathcal{U}_n$.*

Proof. Because of the inductive structure of $\mathcal{U}_n$, namely $\mathcal{U}_n = \mathfrak{S}(n) \times \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{U}_{n-1} \end{bmatrix}$, we can prove inductively that it contains the permutation matrices.

First, for $k_1, \cdots, k_n \geq 0$ and $\alpha = \{\alpha_1, \cdots, \alpha_n\} \in \{-1, 1\}^n$, define the set

$$S_n^{\alpha}(k_1, \cdots, k_n) = \begin{bmatrix} \alpha_1 e^{-k_1\theta\pi i} & & \\ & \ddots & \\ & & \alpha_n e^{-k_n\theta\pi i} \end{bmatrix} \times S_n$$

of all permutation type matrices that have a $\alpha_j e^{-k_j\theta\pi i}$ in the jth row, where θ is the same irrational number chosen in Proposition 4.1.

We begin with looking at the case $n = 2$. Let $k_1, k_2 \geq 0$ and $\alpha \in \{-1, 1\}^2$. Since θ is irrational, $\pm e^{-k\theta\pi i} \neq -e^{\theta\pi i}$ for $k \geq 0$. Thus we have the following multiplications:

$S \in \mathfrak{S}(2)$	$\times$	$\begin{bmatrix} 1 & 0 \\ 0 & U \end{bmatrix}, U \in \mathcal{U}_1$	$\in \mathcal{U}_2$
$\begin{bmatrix} \alpha_1 e^{-k_1\theta\pi i} & 0 \\ 0 & 1 \end{bmatrix}$	$\times$	$\begin{bmatrix} 1 & 0 \\ 0 & \alpha_2 e^{-k_2\theta\pi i} \end{bmatrix}$	$\in S_2^{\alpha}(k_1, k_2)$
$\begin{bmatrix} 0 & -\alpha_2 e^{(k_2+1)\theta\pi i} \\ \alpha_2 e^{-k_2\theta\pi i} & 0 \end{bmatrix}$	$\times$	$\begin{bmatrix} 1 & 0 \\ 0 & -\alpha_1\alpha_2 e^{-(k_1+k_2+1)\theta\pi i} \end{bmatrix}$	$\in S_2^{\alpha}(k_1, k_2)$

Hence, from the above table we can see that $S_2^{\alpha}(k_1, k_2) \subset \mathcal{U}_2$ for all $k_1, k_2 \geq 0$ and $\alpha \in \{-1, 1\}^2$.

Now assume that $S_{n-1}^{\alpha'}(k_1, \cdots, k_{n-1}) \subset \mathcal{U}_{n-1}$ for $k_1, \cdots, k_{n-1} \geq 0$ and $\alpha' \in \{-1, 1\}^{n-1}$. Then we can construct the following matrices when $k_1, \cdots, k_n \geq 0$,

$\alpha \in \{-1,1\}^n$ with $\alpha_i = \alpha'_i, 1 \le i \le n-1$ and when $2 \le m \le n$:

$$\frac{S \in \mathfrak{S}(n) \qquad \times \qquad \begin{bmatrix} 1 & 0 \\ 0 & U \end{bmatrix}, U \in \mathcal{U}_{n-1} \quad \in \qquad \mathcal{U}_n}{\begin{gathered}\begin{bmatrix} \alpha_1 e^{-k_1\theta\pi i} & 0 \\ 0 & I_{n-1} \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & S^{\beta}_{n-1}(k_2,\cdots,k_n) \end{bmatrix} \in \ S^{\alpha}_n(k_1,\cdots,k_n) \\ \begin{bmatrix} 0 & 0 & -\alpha_m e^{(k_m+1)\theta\pi i} & 0 \\ 0 & I_{m-2} & 0 & 0 \\ \alpha_m e^{-k_m\theta\pi i} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n-m} \end{bmatrix} \\ \times \begin{bmatrix} 1 & 0 \\ 0 & S^{\gamma}_{n-1}(\cdots,k_{m-1},k_1+k_m+1,k_{m+1},\cdots) \end{bmatrix} \in S^{\alpha}_n(k_1,\cdots,k_n)\end{gathered}}$$

where $\beta, \gamma \in \{-1,1\}^{n-1}$ with $\beta_i = \alpha_i, 1 \le i \le m-1$ and $\beta_i = \alpha_{i+1}, m \le i \le n-1$ as well as $\gamma_i = \alpha_{i+1}, 1 \le i \le m-2$, $\gamma_{m-1} = \alpha_1\alpha_m$ and $\gamma_i = \alpha_{i+1}, m \le i \le n-1$.

Thus by induction $S^{\alpha}_n(k_1,\cdots,k_n) \subset \mathcal{U}_n$ for $k_1,\cdots,k_n \ge 0$ and $\alpha \in \{-1,1\}^n$. Therefore, $S_n = S^{\{1\}^n}_n(0,\cdots,0) \subset \mathcal{U}_n$. So all permutation matrices are contained in $\mathcal{U}_n$. $\square$

Now we are ready to reformulate the Davidson-Katsoulis conjecture into a conjecture about the structure of the Unitary group. First, consider a partial ordering on the set of partitions of $\{1,\cdots,n\}$.

Definition 4.4 *Suppose $A = \{A_1,\cdots,A_m\}$ and $A' = \{A'_1,\cdots,A'_{m'}\}$ are partitions of $\{1,\cdots,n\}$. Then A' is* finer *than A if for every $1 \le i \le m$ there is a partition of A_i in A'. Similarly a pair of partitions (A',B') is finer than another pair (A,B) if A' is finer than A and B' is finer than B.*

Conjecture 4.5 *For any two partitions $A = \{A_1,\cdots,A_m\}$ and $B = \{B_1,\cdots,B_m\}$ where $A_1\dot\cup\cdots\dot\cup A_m = B_1\dot\cup\cdots\dot\cup B_m = \{1,\cdots,n\}$, with $|A_j| = |B_j|$ for $1 \le j \le m$, there exists a subset of $\mathcal{U}_n$, denoted $\mathcal{U}_{(A,B)}$, that has the following properties:*

(1) It is contractible.

(2) If $U \in \mathcal{U}_{(A,B)}$ then the nonzero entries of U are supported on $\bigcup_{i=1}^m B_i \times A_i$.

(3) If $S \in S_n$ and its nonzero entries are supported on $\bigcup_{i=1}^m B_i \times A_i$, then $S \in \mathcal{U}_{(A,B)}$.

(4) If (A', B') is another pair of partitions satisfying the same conditions as above and (A', B') is finer than (A, B) then $\mathcal{U}_{(A',B')} \subset \mathcal{U}_{(A,B)}$.

From this we shall prove that the Davidson-Katsoulis Conjecture follows. Here we state the conjecture again.

Conjecture 3.21 *Let Π_n be the $n!$-simplex with vertices indexed by S_n. There is a continuous function $u : \Pi_n \to U(n)$ such that:*

(1) Every vertex is taken to its corresponding permutation matrix and,

(2) Given any two partitions $A_1 \dot{\cup} \cdots \dot{\cup} A_m = B_1 \dot{\cup} \cdots \dot{\cup} B_m = \{1, \cdots, n\}$, with $|A_j| = |B_j|, 1 \leq j \leq m$, let $\mathcal{P}(A, B) = \{\alpha \in S_n : \alpha(A_j) = B_j, 1 \leq j \leq m\}$. Then if $x = \sum_{\alpha \in \mathcal{P}(A,B)} x_\alpha \alpha$ the non-zero matrix coefficients of $u(x)$ are supported on $\bigcup_{j=1}^m B_j \times A_j$.

Proposition 4.6 *If Conjecture 4.5 is true then Conjecture 3.21 is true as well.*

Proof. First map each vertex to its corresponding permutation matrix, $\alpha \mapsto u(\alpha)$. Next, given two vertices α_1, α_2, there exists a minimum partition (A, B) such that $u(\alpha_1)$ and $u(\alpha_2)$ are supported on $\bigcup_{j=1}^m B_j \times A_j$ and there is no other partition (A', B') such that $u(\alpha_1)$ and $u(\alpha_2)$ are supported on $\bigcup_{j=1}^m B'_j \times A'_j \subset \bigcup_{j=1}^m B_j \times A_j$.

Then since $\mathcal{U}_{(A,B)} \subset \mathcal{U}_n$ is contractible we can fill in the map $u(x_1\alpha_1 + x_2\alpha_2) \subset \mathcal{U}_{(A,B)}$, where $x_1 + x_2 = 1, x_1, x_2 \geq 0$ or specifically that 1-cell of the simplex. Note that the minimum partition for two vertices may be (A, B) where $A = B = \{\{1, \cdots, n\}\}$.

Next for k vertices $\alpha_1, \cdots, \alpha_k$, with $3 \leq k \leq n!$, again there exists a minimum partition (A, B) with $u(\alpha_i) \in \mathcal{U}_{(A,B)}$, $1 \leq i \leq k$ and thus also contains $u(\sum_{j=1, j\neq i}^k x_j\alpha_j)$, for $1 \leq i \leq k$.

Now the boundary (or skeleton, where at least one of the $x_1, \cdots, x_k$ is zero) of a k-simplex can be associated with the $k-2$ dimensional sphere S^{k-2}. Namely, there exist k equidistant points on S^{k-2} which are associated with the vertices of the k-simplex and from there connect the vertices by geodesics and so on until there is a homeomorphic map, f, from the boundary of the k-simplex to S^{k-2}. By induction we can conclude that u has been defined already for this boundary or skeleton.

Because $\mathcal{U}_{(A,B)}$ is contractible then there is a homotopy, $H(t, \mathrm{S}^{k-2})$, between $u(f^{-1}(S^{k-2}))$ and and the point $u(\sum_{j=1}^{k} \frac{1}{k}\alpha_j)$ which is chosen to be in $\mathcal{U}_{(A,B)}$. This homotopy continuously fills in the required face or $(k-1)$-cell of the simplex.

In this way the function $u : \Pi_n \to \mathcal{U}_n \subset U(n)$ is defined with the required conditions. □

4.2 Partial answers

There is no particular difficulty in defining such a $\mathcal{U}_{(A,B)} \subset \mathcal{U}_n$ for some set of partitions (A, B). However, defining them so that condition (4) of Conjecture 4.5 holds, that is these sets must contain all finer $\mathcal{U}_{(A',B')}$, is where the difficulty lies. However, we can give partial results that easily prove the $n = 2$ and 3 cases and also let us prove the $n = 4$ case which was until now unproven. Hopefully this method will give an idea how a proof of the Conjecture may follow.

First when A and B are made up of n 1-element sets, define $\mathcal{U}_{(A,B)}$ to be the single permutation matrix that has its nonzero entries supported on $\bigcup_{i=1}^{n} B_i \times A_i$.

Next we define a few necessary concepts:

Definition 4.7 *From Proposition 4.1 consider all matrices in $\mathfrak{S}(n)$ such that $y_k = 0$ for some $2 \le k \le n$. They will have the following pre-Gram-Schmidt orthonormalization form:*

$$\begin{bmatrix} e^{\theta\pi i}(\frac{1-\sum_{l=1,l\neq k}^{n}|y_l|^2}{2} + iy_1) & \cdots & -\overline{y_{k-1}} & 0 & -\overline{y_{k+1}} & \cdots \\ \vdots & & \vdots & \vdots & \vdots & \\ e^{\theta\pi i}y_{k-1} & \cdots & \frac{1+\sum_{l=1,l\neq k}^{n}|y_l|^2-2|y_{k-1}|^2}{2} & 0 & -y_{k-1}\overline{y_{k+1}} & \cdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots \\ e^{\theta\pi i}y_{k+1} & \cdots & -y_{k+1}\overline{y_{k-1}} & 0 & \frac{1+\sum_{l=1,l\neq k}^{n}|y_l|^2-2|y_{k+1}|^2}{2} & \cdots \\ \vdots & & \vdots & \vdots & \vdots & \end{bmatrix}.$$

This is easily seen to be a copy of $\mathfrak{S}(n-1)$ imbedded in $\mathfrak{S}(n)$ with a 1 in entry (k, k). We will label this imbedding $\mathfrak{S}(n-1)^{k,k}$ and thus we have $\mathfrak{S}(n-1)^{k,k}$ homeomorphic to $\mathfrak{S}(n-1)$.

Definition 4.8 *For $1 \leq j \leq n$ define the map $\Psi_{j,n}(U) : \mathcal{U}_n \to \mathfrak{S}(n+1)$ by*

$$\Psi_{j,n}(U) = \begin{bmatrix} 0 & \overline{u_{1,j}} & \cdots & \overline{u_{n,j}} \\ -e^{\theta\pi i}u_{1,j} & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ -e^{\theta\pi i}u_{n,j} & * & \cdots & * \end{bmatrix}$$

which is in $\mathfrak{S}(n+1)$ determined by setting $y_1 = 0$ and $y_i = -u_{i-1,j}$ for $2 \leq i \leq n+1$. Then set $\Phi_{j,n}(U) = \Psi_{j,n}(U) \times \begin{bmatrix} 1 & 0 \\ 0 & U \end{bmatrix} \in \mathcal{U}_{n+1}$.

The map $\Phi_{j,n}$ is bijective and is easily seen to be a homeomorphism. Observe that $\Phi_{j,n}(U)e_{j+1} = e_1$. Thus, $\Phi_{j,n}$ should be thought of as taking

$$U \quad \text{to} \quad \begin{bmatrix} & & & 1 & & \\ -e^{\theta\pi i}u_{1,j} & u'_{11} & \cdots & 0 & \cdots & u'_{1n} \\ \vdots & \vdots & & \vdots & & \vdots \\ -e^{\theta\pi i}u_{n,j} & u'_{n1} & \cdots & 0 & \cdots & u'_{nn} \end{bmatrix},$$

that is, an imbedded copy of U in the bottom left corner of $\mathcal{U}_{n+1}$ with the first and $j+1$ columns interchanged, but still maintaining the necessary structure of $\mathcal{U}_n$.

We can move on to define $\mathcal{U}_{(A,B)}$ in the case when there is an $(n-1) \times (n-1)$ block in the partition structure.

Proposition 4.9 *Let $1 \leq i, j \leq n, n \geq 3$, then for*

$$A = \{\{j\}, \{1, \cdots, j-1, j+1, \cdots n\}\} \quad \textit{and} \quad B = \{\{i\}, \{1, \cdots, i-1, i+1, \cdots, n\}\},$$

$\mathcal{U}_{(A,B)}$ can be defined in $\mathcal{U}_n$.

Proof. The goal of this proof is not only to define such a set but to define the largest set in $\mathcal{U}_n$ that satisfies the required conditions for $\mathcal{U}_{(A,B)}$ hopefully allowing for ease of construction in the other types of partitions. To this end we want $\mathcal{U}_{(A,B)}$ homeomorphic to $\mathcal{U}_{n-1}$.

First, for ease of notation let $\mathcal{U}_m^{i,j}$ denote a subset of $\mathcal{U}_{m+1}$ with its (i,j) entry equal to 1. Inductively we will define these sets such that $\mathcal{U}_m^{i,j}$ is homeomorphic to

$\mathcal{U}_m$ and then we can define $\mathcal{U}_{(A,B)} = \mathcal{U}_{n-1}^{i,j}$. It also needs to be shown that these $\mathcal{U}_m^{i,j}$ preserve the structure of the $\mathcal{U}_m$, that is, they must contain all of the permutation matrices that have a 1 at the (i,j)-entry.

We begin with when $i = j$. If $i = 1$ then just define $\mathcal{U}_m^{1,1} = \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{U}_m \end{bmatrix}, m \geq 1$. Now using Definition 4.7 it is easy to set up the inductive step. We define $\mathcal{U}_1^{2,2} = \mathfrak{S}(1)^{2,2}$ and then

$$\mathcal{U}_m^{i,i} = \mathfrak{S}(m)^{i,i} \times \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{U}_{m-1}^{i-1,i-1} \end{bmatrix},$$

which is exactly the same way we defined the sets $\mathcal{U}_m$. Therefore, we see that $\mathcal{U}_m^{i,i}$ is homeomorphic to $\mathcal{U}_m$ by taking out the ith row and jth column. Thus, $\mathcal{U}_m^{i,i}$ still has the same structure as $\mathcal{U}_m$. In correspondence with this inductive definition, for $i, j \geq 2$, define

$$\mathcal{U}_m^{i,j} = \mathfrak{S}(m)^{i,i} \times \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{U}_{m-1}^{i-1,j-1} \end{bmatrix}.$$

We can see that the structure of $\mathcal{U}_m^{i,j}$ is then dependent on that of $\mathcal{U}_{m-1}^{i-1,j-1}$. Hence, the only sets left to define are $\mathcal{U}_m^{i,1}$ and $\mathcal{U}_m^{1,j}$ for $2 \leq i, j \leq m$.

The first case is simple enough, for $2 \leq i \leq m+1$, just take $S \in \mathfrak{S}(m)$ such that $y_i = e^{-\theta\pi i}$ then we can define

$$\mathcal{U}_m^{i,1} = S \cdot \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{U}_m \end{bmatrix} = \begin{bmatrix} & & -e^{\theta\pi i} & \\ & I_{i-2} & & \\ 1 & & & \\ & & & I_{n-i} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{U}_m \end{bmatrix}.$$

Then from Proposition 4.3 we know that $S_m^\alpha(k_1, \cdots, k_m) \subset \mathcal{U}_m$ for $k_1, \cdots, k_m \geq 0$ and $\alpha \in \{-1, 1\}^m$, then it follows that the imbedded copy $S_m^\alpha(k_1, \cdots, k_m)^{i,1} \subset \mathcal{U}_m^{i,1}$.

In the second case assume that $2 \leq j \leq m+1$ and $U = [u_{k,l}]_{k,l=1}^m \in \mathcal{U}_m$. Then using Definition 4.8 we can define

$$\mathcal{U}_m^{1,j} = \Phi_{j-1,m}(\mathcal{U}_m) = \left\{ \Psi_{j-1,m}(U) \cdot \begin{bmatrix} 1 & 0 \\ 0 & U \end{bmatrix} : U \in \mathcal{U}_m \right\}.$$

In particular this construction still insures that $\mathcal{U}_m^{1,j}$ is homeomorphic to $\mathcal{U}_m$. Lastly, we can see that if $T \in S_m^\alpha(k_1, \cdots, k_m) \subset \mathcal{U}_m$, then

$$\Psi_{j,m}(T) = \begin{bmatrix} 0 & 0 & \alpha_j e^{k_j \theta\pi i} & 0 \\ 0 & I_{j-1} & 0 & 0 \\ -\alpha_j e^{(1-k_j)\theta\pi i} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{m-j} \end{bmatrix}.$$

So if $k_j \geq 1$ then $\Psi_{j,m}(T) \cdot T \in S_m^\alpha(k_1, \cdots, k_m)^{1,j+1} \subset \mathcal{U}_m^{1,j+1}$ and from this we can see that all of these permutation type matrices are achieved. $\square$

We can extend this definition so that $\mathcal{U}_{(A,B)}$ in the above context is *maximal contractible*, that is, one cannot add any more elements without making it fail to be contractible.

Corollary 4.10 *Let (A, B) be as in the previous proposition. Then we can define $\tilde{\mathcal{U}}_{(A,B)}$ containing $\mathcal{U}_{(A,B)}$ to be maximal contractible, that is, such that the $B_1 \times A_1$ entry is homeomorphic to $\mathcal{U}_1$ and the $B_2 \times A_2$ entries are homeomorphic to $\mathcal{U}_{n-1}$.*

Proof. We can use the iterative definitions of the $\mathcal{U}_m^{i,j}$ in the previous proposition to get this result.

In particular, define

$$\tilde{\mathcal{U}}_m^{1,1} = \begin{bmatrix} \mathfrak{S}(1) & \\ & I_m \end{bmatrix} \times \begin{bmatrix} 1 & \\ & \mathcal{U}_m \end{bmatrix} = \begin{bmatrix} \mathcal{U}_1 & \\ & \mathcal{U}_m \end{bmatrix} \in \mathcal{U}_{m+1},$$

$$\tilde{\mathcal{U}}_1^{2,2} = \mathfrak{S}(1)^{2,2} \times \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{U}_1 \end{bmatrix} = \begin{bmatrix} \mathcal{U}_1 & \\ & \mathcal{U}_1 \end{bmatrix} \in \mathcal{U}_2,$$

$$\tilde{\mathcal{U}}_m^{i,1} = \begin{bmatrix} & & -e^{\theta\pi i}\bar{\phi} & \\ & I_{i-2} & & \\ \phi \in \mathcal{U}_1 & & & \\ & & & I_{m-i} \end{bmatrix} \times \begin{bmatrix} 1 & \\ & \mathcal{U}_m \end{bmatrix} \in \mathcal{U}_{m+1}, \quad \text{and}$$

$$\tilde{\mathcal{U}}_m^{1,j} = \left\{ \Psi_{j-1,m}(\mathcal{U}_1 U) \cdot \begin{bmatrix} 1 & 0 \\ 0 & U \end{bmatrix} : U \in \mathcal{U}_m \right\} \in \mathcal{U}_{m+1}.$$

Then we can define the rest in the same way as in the proposition, that is

$$\tilde{\mathcal{U}}_m^{i,j} = \mathfrak{S}(m)^{i,i} \times \begin{bmatrix} 1 & 0 \\ 0 & \tilde{\mathcal{U}}_{m-1}^{i-1,j-1} \end{bmatrix}.$$

All of these sets $\tilde{\mathcal{U}}_m^{i,j}$ are easily seen to be maximal contractible. □

Proposition 4.9 also allows us to define $\mathcal{U}_{(A,B)}$ when there is one large partition block and the rest is just a permutation matrix.

Corollary 4.11 *Let A, B be two partitions such that $|A_i| = |B_i| = 1, 1 \leq i \leq m-1$ and so $|A_m| = |B_m| = n - m + 1$. Then $\mathcal{U}_{(A,B)}$ can be defined in $\mathcal{U}_n$ such that it is homeomorphic to $\mathcal{U}_{n-m+1}$.*

Proof. From the proof of the previous proposition we see that $\mathcal{U}_{n-1}^{i,j}$ was defined in such a way that any matrix in $\mathcal{U}_n$ that has a 1 at the (i, j)-entry is contained in $\mathcal{U}_{n-1}^{i,j}$. Thus, any matrix that has a 1 at both the (i, j) and (k, l)-entries will be contained in both $\mathcal{U}_{n-1}^{i,j}$ and $\mathcal{U}_{n-1}^{k,l}$. But then since $\mathcal{U}_{n-1}^{i,j}$ still has the same structure as $\mathcal{U}_n$ we see that $\mathcal{U}_{n-2}^{(i,j),(k,l)} \subset \mathcal{U}_{n-1}^{i,j}$, is all matrices in $\mathcal{U}_{n-1}^{i,j}$ that have a 1 in their (k, l)-entry, but then it follows that this $\mathcal{U}_{n-2}^{(i,j),(k,l)} = \mathcal{U}_{n-1}^{i,j} \bigcap \mathcal{U}_{n-1}^{k,l}$ which is homeomorphic to $\mathcal{U}_{n-2}$.

Therefore we can use this argument repeatedly to see that $\mathcal{U}_{(A,B)} = \bigcap_{i=1}^{m-1} \mathcal{U}_{n-1}^{B_i,A_i}$ is homeomorphic to $\mathcal{U}_{n-m+1}$ and still contains the appropriate permutation matrices and structure. □

Corollary 4.12 *Let A, B be two partitions as in the previous corollary. Then we can define a maximum contractible set $\tilde{\mathcal{U}}_{(A,B)}$ that contains $\mathcal{U}_{(A,B)}$.*

Proof. As in Corollary 4.11 we get this set by showing that $\tilde{\mathcal{U}}_{(A,B)} = \bigcap_{i=1}^{m-1} \tilde{\mathcal{U}}_{n-1}^{B_i,A_i}$.

We know that $\mathcal{U}_{(A,B)} = \bigcap_{i=1}^{m-1} \mathcal{U}_{n-1}^{B_i,A_i}$. Then we can simply add in what we need as in Corollary 4.10.

Let $\mathcal{U}_{(A,B)}^{(1)} \subset \tilde{\mathcal{U}}_{n-1}^{B_1,A_1}$ be defined such that the $B_1 \times A_1$ entry is homeomorphic to $\mathcal{U}_1$ and the other $n-1 \times n-1$ partition block is $\mathcal{U}_{(\{A_2,\cdots,A_m\},\{B_2,\cdots,B_m\})}$. Then $\mathcal{U}_{(A,B)}^{(1)} = \tilde{\mathcal{U}}_{n-1}^{B_1,A_1} \cap \bigcap_{i=2}^{m-1} \mathcal{U}_{n-1}^{B_i,A_i}$.

Continue in this way defining $\mathcal{U}^{(j)}_{(A,B)} \subset \tilde{\mathcal{U}}^{B_j,A_j}_{n-1}$ such that its $B_j \times A_j$ entry is homeomorphic to $\mathcal{U}_1$, one partition block is $\tilde{\mathcal{U}}_{(\{A_1,\cdots,A_{j-1}\},\{B_1,\cdots,B_{j-1}\})}$ and the remaining partition block is $\mathcal{U}_{(\{A_{j+1},\cdots,A_m\},\{B_{j+1},\cdots,B_m\})}$. Then

$$\mathcal{U}^{(j)}_{(A,B)} = \bigcap_{i=1}^{j} \tilde{\mathcal{U}}^{B_i,A_i}_{n-1} \cap \bigcap_{i=j+1}^{m-1} \mathcal{U}^{B_i,A_i}_{n-1}.$$

This process takes $\mathcal{U}^{(j-1)}_{(A,B)}$ as a subset of $\mathcal{U}^{B_j,A_j}_{n-1}$ into $\tilde{\mathcal{U}}^{B_j,A_j}_{n-1}$ and then calls it $\mathcal{U}^{(j)}_{(A,B)}$.

Thus, let $\tilde{\mathcal{U}}_{(A,B)} = \mathcal{U}^{(m-1)}_{(A,B)} = \bigcap_{i=1}^{m-1} \tilde{\mathcal{U}}^{B_i,A_i}_{n-1}$. This set is maximal contractible because each $B_i \times A_i$ entry is homeomorphic to $\mathcal{U}_1$ for $1 \leq i \leq m-1$ and the $B_m \times A_m$ entries are homeomorphic to $\mathcal{U}_{n-m+1}$. □

From this we can now prove the $n = 2$ and 3 cases by this new method. Note that Davidson and Katsoulis had already proven these cases in a different way.

Example 4.13 For $n = 2$ if we let $A = \{\{s_1(1)\}, \{s_1(2)\}$ and $B = \{\{s_2(1)\}, \{s_2(2)\}\}$ where $s_1, s_2 \in S_2$ then $\mathcal{U}_{(A,B)} \subset S_2 \subset \mathcal{U}_2$. Therefore, Conjecture 4.5 holds which implies that Theorem 3.20 is true.

For $n = 3$, partitions (A, B) come only in two types:

- $A = \{A_1, A_2, A_3\}$ where $|A_i| = 1, i = 1, 2, 3$, in which case $\mathcal{U}_{(A,B)}$ is just a permutation matrix.

- $A = \{A_1, A_2\}$ where $|A_i| = i, i = 1, 2$, which is the case proven in Proposition 4.9 and so $\mathcal{U}_{(A,B)}$ exists with the required properties.

Therefore, the conditions of Conjecture 4.5 are satisfied and the theorem applies.

Lastly, this theory allows us to prove the $n = 4$ case by brute force methods. This was as yet unproven and will hopefully shine a light on the proof of Conjecture 4.5.

Theorem 4.14 *For $n = 4$, dynamical systems are completely characterized by their tensor algebras up to piecewise conjugacy.*

Proof. So far we know that $S_4 \subset \mathcal{U}_4$ and by Corollary 4.11 if $m = 2, 3$ and (A, B) are two partitions such that $|A_i| = |B_i| = 1$ for $1 \leq i \leq m-1$ and $|A_m| = |B_m| = 5-m$ then $\mathcal{U}_{(A,B)} \subset \mathcal{U}_4$.

Thus, the only partitions (A, B) where $\mathcal{U}_{(A,B)}$ is undefined is the case when $m = 2$ and $|A_1| = |A_2| = 2$, of which there are eighteen such partitions.

In the light of Corollary 4.10 we have all of the 3×3 partitions with $\mathcal{U}_1$ and $\mathcal{U}_2$ blocks, that is the sets $\tilde{\mathcal{U}}_2^{i,j}$. Hence we have the following sets:

$$\begin{bmatrix} \mathcal{U}_2 & \\ & \mathcal{U}_2 \end{bmatrix} = \begin{bmatrix} \mathfrak{S}(2) & \\ & I_2 \end{bmatrix} \times \begin{bmatrix} 1 & \\ & \tilde{\mathcal{U}}_2^{1,1} \end{bmatrix}$$

$$\begin{bmatrix} (\mathcal{U}_2)_{11} & & (\mathcal{U}_2)_{12} & \\ (\mathcal{U}_2)_{21} & & (\mathcal{U}_2)_{22} & \\ & (\mathcal{U}_2)_{11} & & (\mathcal{U}_2)_{12} \\ & (\mathcal{U}_2)_{21} & & (\mathcal{U}_2)_{22} \end{bmatrix} \text{``}=\text{''} \begin{bmatrix} \mathfrak{S}(2) & \\ & I_2 \end{bmatrix} \times \begin{bmatrix} 1 & \\ & \tilde{\mathcal{U}}_2^{1,2} \end{bmatrix}$$

$$\begin{bmatrix} (\mathcal{U}_2)_{11} & & (\mathcal{U}_2)_{12} \\ (\mathcal{U}_2)_{21} & & (\mathcal{U}_2)_{22} \\ & \mathcal{U}_2 & \end{bmatrix} \text{``}=\text{''} \begin{bmatrix} \mathfrak{S}(2) & \\ & I_2 \end{bmatrix} \times \begin{bmatrix} 1 & \\ & \tilde{\mathcal{U}}_2^{1,3} \end{bmatrix}$$

We say "=" since the right hand side is not equal to the left hand side but rather is homeomorphic in each partition element. There are six more partitions that follow like this, two of them being shown below. Note that $(\mathcal{U}_2)_{i,j}$ refers to the (i, j) entry of $\mathcal{U}_2 \subset M_2(\mathbb{C})$ so that we can identify $\mathcal{U}_2$ imbedded into larger matrices.

$$\begin{bmatrix} (\mathcal{U}_2)_{11} & & & (\mathcal{U}_2)_{12} \\ & (\mathcal{U}_2)_{11} & (\mathcal{U}_2)_{12} & \\ (\mathcal{U}_2)_{21} & & & (\mathcal{U}_2)_{22} \\ & (\mathcal{U}_2)_{21} & (\mathcal{U}_2)_{22} & \end{bmatrix} \text{``}=\text{''} \begin{bmatrix} \mathfrak{S}(2)_{11} & & \mathfrak{S}(2)_{12} & \\ & 1 & & \\ \mathfrak{S}(2)_{21} & & \mathfrak{S}(2)_{22} & \\ & & & 1 \end{bmatrix} \times \begin{bmatrix} 1 & \\ & \tilde{\mathcal{U}}_2^{2,3} \end{bmatrix}$$

$$\begin{bmatrix} (\mathcal{U}_2)_{11} & (\mathcal{U}_2)_{12} & \\ & & \mathcal{U}_2 \\ (\mathcal{U}_2)_{21} & (\mathcal{U}_2)_{22} & \end{bmatrix} \text{``}=\text{''} \begin{bmatrix} \mathfrak{S}(2)_{11} & & & \mathfrak{S}(2)_{12} \\ & 1 & & \\ & & 1 & \\ \mathfrak{S}(2)_{21} & & & \mathfrak{S}(2)_{22} \end{bmatrix} \times \begin{bmatrix} 1 & \\ & \tilde{\mathcal{U}}_2^{3,1} \end{bmatrix}$$

Thus, nine partitions of the eighteen have been accounted for. Next we use the map $\Psi_{j,m}$ that was given in Definition 4.8. We have:

$$\begin{bmatrix} & (\mathcal{U}_2)_{11} & (\mathcal{U}_2)_{12} \\ \mathcal{U}_2 & & \\ & (\mathcal{U}_2)_{21} & (\mathcal{U}_2)_{22} \end{bmatrix} \text{``}=\text{''} \Psi_{2,3}(\mathfrak{S}(1) \cdot S) \times \begin{bmatrix} 1 & \\ & S \in \mathfrak{S}(2)^{3,3} \end{bmatrix} \times \begin{bmatrix} 1_2 & \\ & \mathcal{U}_2 \end{bmatrix}$$

$$\begin{bmatrix} & & (\mathcal{U}_2)_{11} & (\mathcal{U}_2)_{12} \\ (\mathcal{U}_2)_{11} & (\mathcal{U}_2)_{12} & & \\ & & (\mathcal{U}_2)_{21} & (\mathcal{U}_2)_{22} \\ (\mathcal{U}_2)_{21} & (\mathcal{U}_2)_{22} & & \end{bmatrix} \text{“}=\text{”} \Psi_{3,3}(\mathfrak{S}(1)\cdot S)\times \begin{bmatrix} 1 & \\ & \mathfrak{S}(2)^{2,2} \end{bmatrix} \times \begin{bmatrix} 1_2 & \\ & \mathcal{U}_2 \end{bmatrix}$$

$$\begin{bmatrix} & \mathcal{U}_2 \\ \mathcal{U}_2 & \end{bmatrix} \text{“}=\text{”} \Psi_{2,3}(\mathfrak{S}(1)\cdot S \in \mathfrak{S}(2)^{1,1}) \times \begin{bmatrix} 1 & & & \\ & & * & * \\ & S_{11} & * & * \\ & S_{21} & * & * \end{bmatrix} \times \begin{bmatrix} 1_2 & \\ & \mathcal{U}_2 \end{bmatrix}$$

Finally consider the partition $(A, B) = (\{\{2,3\},\{1,4\}\},\{\{1,4\},\{2,3\}\})$. This construction is where brute force must be used. For the $n > 4$ case this must be generalized in order for this method of proof to work.

Let $r_1, r_2, s_1, s_2 \in [0,1]$ with $r_1^2 + r_2^2 = s_1^2 + s_2^2 = 1$ then we have the following matrix in $\mathfrak{S}(4) \times \mathfrak{S}(3)$:

$$\begin{bmatrix} & r_1 & r_2 & \\ e^{\theta\pi i} r_1 & * & * & \\ e^{\theta\pi i} r_2 & * & * & \\ & & & 1 \end{bmatrix} \times \begin{bmatrix} 1 & & & \\ & e^{\theta\pi i} s_1 r_1 & * & * \\ & e^{\theta\pi i} s_1 r_2 & * & * \\ & e^{\theta\pi i} s_2 & * & * \end{bmatrix} = \begin{bmatrix} 0 & -e^{\theta\pi i} s_1 & row_1 \\ e^{\theta\pi i} r_1 & 0 & row_2 \\ e^{\theta\pi i} r_2 & 0 & row_3 \\ 0 & e^{\theta\pi i} s_2 & row_4 \end{bmatrix}$$

By the structure of $\mathfrak{S}(3)$ as seen in Proposition 4.1 we know that $row_i^t \in \mathbb{R}^2$. Since the above matrix is unitary then $\dim(\mathrm{Span}\{row_1^t, row_4^t\}) = 1$ and because $0 < \theta/2 < \pi/2$ then for all choices of r_1, r_2, s_1, s_2

$$\varphi(r_1, r_2, s_1, s_2) = \frac{row_1^t + e^{\theta/2\pi i} row_4^t}{\|row_1^t + e^{\theta/2\pi i} \cdot row_4^t\|}$$

is defined and is a continuous function into $\{z \in \mathbb{C}^2 : |z| = 1, z \neq -e^{\theta\pi i} e_1\}$. Then we have the set $\mathcal{V}$ in $\mathfrak{S}(4) \times \mathfrak{S}(3) \times \mathfrak{S}(2) \subset \mathcal{U}_4$ consisting of the following matrices:

$$\begin{bmatrix} 0 & -e^{\theta\pi i} s_1 & row_1 \\ e^{\theta\pi i} r_1 & 0 & row_2 \\ e^{\theta\pi i} r_2 & 0 & row_3 \\ 0 & e^{\theta\pi i} s_2 & row_4 \end{bmatrix} \times \begin{bmatrix} I_2 & & \\ & \varphi(r_1, r_2, s_1, s_2) & * \end{bmatrix} = \begin{bmatrix} 0 & -e^{\theta\pi i} s_1 & * & 0 \\ e^{\theta\pi i} r_1 & 0 & 0 & * \\ e^{\theta\pi i} r_2 & 0 & 0 & * \\ 0 & e^{\theta\pi i} s_2 & * & 0 \end{bmatrix}$$

which is in the right (A, B) partition form, is contractible and contains four permutation type matrices, though they are not the ones in S_4. Let $(A^i, B^i), 1 \leq i \leq 4$ be the four finer partitions of (A, B) that contain a 2×2 partition block. Finally define

$$\mathcal{U}_{(A,B)} = \mathcal{V} \ \cup \ (\bigcup_{i=1}^{4} \tilde{\mathcal{U}}_{(A^i,B^i)}).$$

Visually this is the following union:

$$\begin{array}{ccccc}
 & & \begin{bmatrix} & * & * & \\ * & & & \\ & & & * \\ & * & * & \end{bmatrix} & & \\
 & & \cup & & \\
\begin{bmatrix} & * & & \\ * & & & * \\ * & & & * \\ & & * & \end{bmatrix} & \cup & \begin{bmatrix} & * & * & \\ * & & & * \\ * & & & * \\ & * & * & \end{bmatrix} & \cup & \begin{bmatrix} & & * & \\ * & & & * \\ * & & & * \\ & * & & \end{bmatrix} \\
 & & \cup & & \\
 & & \begin{bmatrix} & * & * & \\ & & & * \\ * & & & \\ & * & * & \end{bmatrix} & &
\end{array}$$

where the outside 4 are the maximal contractible sets $\tilde{\mathcal{U}}_{(A^i,B^i)}$ and the center is the contractible set $\mathcal{V}$. The most important point is that any two sets that adjoin in the diagram (including diagonally) have non-empty contractible intersection. Hence, $\mathcal{U}_{(A,B)}$ is contractible and contains all finer partitions.

The construction of the remaining 5 partitions follows exactly as above. Therefore, we have satisfied the conditions of Conjecture 4.5 for $n = 4$ and thus dynamical systems are completely characterized by their tensor algebras up to piecewise conjugacy. □

We feel fairly confident in the truth of Conjecture 4.5. As we have seen the $n = 4$ case above contains another level of complexity over that of the $n = 2$ or 3 cases. Thus the truth of it should lead to a proof of the aforementioned Conjecture, and hence the complete characterization of multivariable dynamical systems up to piecewise conjugacy.

Another possibility for proof of the characterization result is to examine what happens to the covering systems $(\tilde{X}, \tilde{\sigma})$, introduced in [5] and outlined in Section 1 of Chapter 3, when two dynamical systems are piecewise conjugate.

Finally, we can ask some questions about further directions of research stemming from this study of dynamical systems. An important point to remember is that here we assumed that a multivariable dynamical system had unrelated maps. A further avenue of research is to examine what happens when there are relations among these maps, for instance when all the maps commute. Is there a similar characterization up to some natural conjugacy? Are all such dynamical systems equivalent to our unrelated map case when one quotients by the center?

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