Moment Polynomials for the Riemann Zeta Function

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Mathematics
in
Pure Mathematics

Waterloo, Ontario, Canada, 2009

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In this thesis we calculated the coefficients of moment polynomials of the Riemann zeta function for $k = 4, 5, 6 \ldots 13$ using cubic acceleration, which is an improved method from quadratic acceleration used in [CFKRS3]. We then numerically verified the moment conjectures. The results we obtained appear to support the conjectures.

We also present a brief history of the moment polynomials by illustrating some of the important results of the field along with proofs for two of the classic results. The heuristics to find the integral moments of the Riemann zeta function is described in this thesis as well.

Acknowledgements

I would like to thank Dr.Rubinstein greatly for all of the help and support he has given me, which made this thesis possible. Also I would like to thank Dr.Abdol-Reza Mansouri from Queen's University for giving me the wise advise and support for me to come to Waterloo. I appreciated all the support and interest, my uncle Takayuki Yamagishi has provided for my degree and thesis. Also many thanks go to my parents Toshiyuki and Sumiko Yamagishi and my brother Ginjiro. Jason Sham and Yun Zhi Chew had helped me greatly with Latex commands. I could not have completed this thesis without corrections by Dr. Kevin G. Hare and Dr. Cameron L. Stewart. Finally, I want to thank Big R and little r who were so helpful.

Dedication

I would like to dedicate my thesis to my grand parents Teiji and Hiroko Yamagishi, and Sadao and Taka Kojima.

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Chapter 1

Introduction

In this thesis we consider the integral moments of the Riemann zeta function $\int_0^T |\zeta(1/2+it)|^{2k}dt$. Their importance lies in the fact that they can be used to estimate the maximal order of the zeta function on the critical line, $\Re s = \frac{1}{2}$. They can also be applied to the study of the distribution of prime numbers through zero density estimates and to divisor problems, see for example Chapter 12 of [T].

There are numerous results regarding the moments of the Riemann zeta function. To illustrate some of them, the first is a significant early result for $k=1$ by Hardy and Littlewood [HL1, HL2]

$$\int_0^T |\zeta(1/2+it)|^2 dt \sim T\log T. \tag{1.1}$$

In fact for case $k=1$ the lower terms have been completely determined as

$$\int_0^T |\zeta(1/2+it)|^2 dt = T\log T + (2\gamma - 1 - \log 2\pi)T + O\left(T^{\frac{1}{2}}\log T\right), \tag{1.2}$$

a result obtained by Ingham [I]. This is an improvement from Atkinson's result [A] with error term $O\left(T^{\frac{1}{2}+\varepsilon}\right)$, but it has been improved to $O\left(T^{\frac{346}{1067}+\varepsilon}\right)$ by Balasubramanian [B] and then to $O\left(T^{\frac{346}{1067}}(\log T)^c\right)$ for some constant c by Heath-Brown [H-B].

A classical result for $k=2$ is

$$\int_0^T |\zeta(1/2+it)|^4 dt = \frac{1}{2\pi^2}T(\log T)^4 + O\left(T(\log T)^3\right) \tag{1.3}$$

another result obtained by Ingham [I]. Much work has been done with this moment as well, for example Atkinson had worked with smoothed moment [A1]. Heath-Brown explicitly wrote out the first two terms with error term $O\left(T^{\frac{7}{8}+\varepsilon}\right)$ [H-B] and Motohashi obtained an explicit formula for the remainder term [Mot]. Finally, the approach in [CFKRS] is consistent with both of the works and allows one to write down all the terms, which is shown below:

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4\,dt \sim \int_0^T P_2\left(\log\tfrac{t}{2\pi}\right)\,dt \tag{1.4}$$

where

$$\begin{aligned} P_2(x) \;=\;& \frac{1}{2\pi^2}x^4 + \frac{8}{\pi^4}\left(\gamma\pi^2 - 3\zeta'(2)\right)x^3 \\ &+ \frac{6}{\pi^6}\left(-48\gamma\zeta'(2)\pi^2 - 12\zeta''(2)\pi^2 + 7\gamma^2\pi^4 + 144\zeta'(2)^2 - 2\,\gamma_1\pi^4\right)x^2 \\ &+ \frac{12}{\pi^8}\Big(6\gamma^3\pi^6 - 84\gamma^2\zeta'(2)\pi^4 + 24\gamma_1\zeta'(2)\pi^4 - 1728\zeta'(2)^3 + 576\gamma\zeta'(2)^2\pi^2 \\ &\quad + 288\zeta'(2)\zeta''(2)\pi^2 - 8\zeta'''(2)\pi^4 - 10\gamma_1\gamma\pi^6 - \gamma_2\pi^6 - 48\gamma\zeta''(2)\pi^4\Big)x \\ &+ \frac{4}{\pi^{10}}\Big(-12\zeta''''(2)\pi^6 + 36\gamma_2\zeta'(2)\pi^6 + 9\gamma^4\pi^8 + 21\gamma_1^2\pi^8 + 432\zeta''(2)^2\pi^4 \\ &\quad + 3456\gamma\zeta'(2)\zeta''(2)\pi^4 + 3024\gamma^2\zeta'(2)^2\pi^4 - 36\gamma^2\gamma_1\pi^8 - 252\gamma^2\zeta''(2)\pi^6 \\ &\quad + 3\gamma\gamma_2\pi^8 + 72\gamma_1\zeta''(2)\pi^6 + 360\gamma_1\gamma\zeta'(2)\pi^6 - 216\gamma^3\zeta'(2)\pi^6 \\ &\quad - 864\gamma_1\zeta'(2)^2\pi^4 + 5\gamma_3\pi^8 + 576\zeta'(2)\zeta'''(2)\pi^4 - 20736\gamma\zeta'(2)^3\pi^2 \\ &\quad - 15552\zeta''(2)\zeta'(2)^2\pi^2 - 96\gamma\zeta'''(2)\pi^6 + 62208\zeta'(2)^4\Big). \end{aligned} \tag{1.5}$$

Numerically,

$$\begin{aligned} P_2(x) \;=\;& 0.05066059182116888572193973160486 38\,x^4 \\ &+ 0.6988698848789799698470962842765 8502\,x^3 \\ &+ 2.4259621988466820047565753101606 63\,x^2 \\ &+ 3.2279079649012547643806898512746 68\,x \\ &+ 1.3124243859616692261684400662299 78. \end{aligned} \tag{1.6}$$

There has not yet been any analogous formulae proved for higher moments and it seems unlikely that any will be in the near future. In fact no one was even able to produce a plausible guess for the asymptotic main term for a long period of time, because the problem is so intractable. It was only recently that Conrey and Ghosh [CG] conjectured

$$\int_0^T |\zeta(1/2+it)|^6 dt \sim \frac{42}{9!}\prod_p\left(\left(1-\frac{1}{p}\right)^4\left(1+\frac{4}{p}+\frac{1}{p^2}\right)\right)T(\log T)^9$$

and Conrey and Gonek [CGo] conjectured:

$$\int_0^T |\zeta(1/2+it)|^8 dt \sim \frac{24024}{16!} \prod_p \left(\left(1-\frac{1}{p}\right)^9 \left(1+\frac{9}{p}+\frac{9}{p^2}+\frac{1}{p^3}\right)\right) T(\log T)^{16}$$

While these conjectures were formed through number theoretical arguments, J.Keating and N.Snaith had made conjectures on higher moments using random matrix theory, which is a completely different approach. There had been developments earlier which suggested the deep connection between L-functions and random matrix theory. It is believed that for any individual L-function, the zeros high up on the critical line are distributed "similarly" to the eigenvalues of random unitary matrices and this happens to be the case for the Riemann zeta-function [Mo, RS].

It is conjectured that the zeros of the characteristic polynomial of a random matrix (the eigenvalues of the matrix) and the zeros of L-function have the same statistical behavior. This suggests that the Riemann zeta function is modeled by the determinant of a random matrix whose eigenvalues have a Gaussian Unitary Ensemble (GUE) distribution. Therefore, studying the value distributions and moments of the characteristic polynomial of such random matrices may be the key to understanding the value distributions and moments of the Riemann zeta-function and other L-functions [KeS, KeS2].

Regarding the moments of the Riemann zeta-function, there is a long-standing and important conjecture that the following limit:

$$\lim_{T\to\infty} \frac{1}{(\log T)^{\lambda^2}} \frac{1}{T} \int_0^T |\zeta(1/2+it)|^{2\lambda} dt \tag{1.7}$$

exists and equals $f(\lambda)a(\lambda)$, where

$$a(\lambda) = \prod_p \left(1-\frac{1}{p}\right)^{\lambda^2} \left(\sum_{m=0}^{\infty} \left(\frac{\Gamma(\lambda+m)}{m!\Gamma(\lambda)}\right)^2 p^{-m}\right).$$

On this problem, Keating and Snaith [KeS] have conjectured that

$$f(\lambda) = \frac{(G(1+\lambda))^2}{G(1+2\lambda)}$$

where G is a Barnes G-function and this equation simplifies for integer k as

$$f(k) = \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$$

Remarkably these conjectures for cases $k = 3$ and 4, agree with conjectures by [CG] and [CGo], respectively, even though they were formed independently of each other. This strongly suggests the validity of the conjectures.

This positive incident had lead the researchers from the number theory side and the random matrix side to collaborate with hope of obtaining more results regarding these integral moments of $\zeta(s)$. Consequently more higher moments of $\zeta(s)$ and their lower terms were studied extensively in [CFKRS], [CFKRS2] and [CFKRS3] and the following conjecture was formed in their first paper.

For positive integer k, and any $\epsilon > 0$,

$$\int_0^T |\zeta(1/2+it)|^{2k} dt = \int_0^T P_k\left(\log \tfrac{t}{2\pi}\right) dt + O\left(T^{\frac{1}{2}+\epsilon}\right), \tag{1.8}$$

with the constant in the O term depending on k and ϵ. In the above equation P_k is the polynomial of degree k^2 given implicitly by the $2k$-fold residue

$$P_k(x) = \frac{(-1)^k}{k!^2}\frac{1}{(2\pi i)^{2k}} \oint \cdots \oint \frac{G(z_1,\dots,z_{2k})\Delta^2(z_1,\dots,z_{2k})}{\prod_{i=1}^{2k} z_i^{2k}} \times e^{\frac{x}{2}\sum_{i=1}^{k} z_i - z_{i+k}}\, dz_1 \dots dz_{2k},$$

with the path of integration over small circles about $z_i = 0$, where

$$\Delta(z_1,\dots,z_m) = \prod_{1\le i<j\le m}(z_j - z_i) = \left|z_i^{j-1}\right|_{m\times m}$$

denotes the Vandermonde determinant,

$$G(z_1,\dots,z_{2k}) = A_k(z_1,\dots,z_{2k})\prod_{i=1}^{k}\prod_{j=1}^{k}\zeta(1+z_i-z_{j+k}), \tag{1.9}$$

and A_k is the Euler product

$$A_k(z_1,\dots,z_{2k})$$
$$= \prod_p \prod_{i,j=1}^{k}(1-p^{-1-z_i+z_{k+j}}) \int_0^1 \prod_{j=1}^{k}\left(1-\frac{e(\theta)}{p^{\frac{1}{2}+z_j}}\right)^{-1}\left(1-\frac{e(-\theta)}{p^{\frac{1}{2}-z_{k+j}}}\right)^{-1} d\theta \tag{1.10}$$

$$= \prod_p \sum_{j=1}^{k}\prod_{i\ne j}\frac{\prod_{m=1}^{k}(1-p^{-1+z_{i+k}-z_m})}{1-p^{z_{i+k}-z_{j+k}}}. \tag{1.11}$$

Here $e(\theta) = \exp(2\pi i\theta)$.

As explained in [CFKRS3] both of these expressions for the local factor for A_k are used for different purposes. The first is used in obtaining meromorphic expressions in k for the coefficients of $P_k(x)$.

The second expression, derived in **Section 2.6** of [CFKRS], is used to numerically compute $A_k(z_1, \ldots, z_{2k})$ for specific values of $z_1, \ldots, z_{2k}$. A care must be taken to avoid poles when the terms are evaluated numerically. These individual terms in the sum over j in (1.11) have poles (though these poles cancel out when summed over j, see the paragraph following equation (2.6.16) in [CFKRS]) and they can be avoided by making sure that the z_{j+k}'s are distinct.

The main point of the conjecture is that it is believed to give the full asymptotics of the moments of zeta. The numerical results obtained in [CFKRS3] are consistent with the remainder of size $O(T^{1/2+\epsilon})$, although there is some debate regarding the error, especially in relation to moments of other families of L-functions [CFKRS, Z]. The main purpose of this thesis is to extend computations on $P_k(x)$ and test conjectures extensively by studying the polynomials numerically.

Equation (1.9) was used to obtain the meromorphic expressions in k for the coefficients of $P_k(x)$ by computing power series expansions and then the residue of the right hand side. These meromorphic expressions can also be evaluated to high precision numerically, even for non integer k. This first method of computing coefficients relied on equation (2.69) in [CFKRS3] and the authors used Maple to carry out symbolic computation. The advantage of this approach was that they were able to obtain the coefficients to many digits precision, and also to make sense of the conjecture for non-integer values of k. On the other hand, the disadvantage was that it caused difficulty implementing, even using a high level symbolic package, and required much computational power. As a result, only $c_r(k)$'s up to $r \leq 9$ were determined in this way in [CFKRS3], but this sufficed to compute all the lower terms for $k = 3$.

For this reason, the second method was developed in [CFKRS3], using the following lemma, to compute more coefficients $c_r(k)$'s.

Lemma. Suppose $F(u; v) = F(u_1, \ldots, u_k; v_1, \ldots, v_k)$ is a function of $2k$ variables, symmetric with respect to the first k variables and also symmetric with respect to the second set of k variables. Suppose also that F is regular near $(0, \ldots, 0)$, and that $f(s)$ has a simple pole of residue 1 at $s = 0$ but is otherwise analytic in a neighbourhood about $s = 0$. Let

$$H(u_1, \ldots, u_k; v_1, \ldots v_k) = F(u_1, \ldots; \ldots, v_k) \prod_{i=1}^{k} \prod_{j=1}^{k} f(u_i - v_j).$$

If for all $1 \leq i, j \leq k$, $\alpha_i - \alpha_{j+k}$ is contained in the region of analyticity of $f(s)$ then

$$\begin{aligned} &\tfrac{(-1)^k}{k!^2} \tfrac{1}{(2\pi i)^{2k}} \oint \cdots \oint \tfrac{H(z_1, \ldots, z_{2k}) \Delta(z_1, \ldots, z_{2k})^2}{\prod_{i=1}^{2k} \prod_{j=1}^{2k} (z_i - \alpha_j)} \, dz_1 \ldots dz_{2k} \\ = &\textstyle\sum_{\sigma \in \Xi} H(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(2k)}), \end{aligned} \tag{1.12}$$

where one integrates about small circles enclosing the α_j's, and where Ξ is the set of $\binom{2k}{k}$ permutations $\sigma \in S_{2k}$ such that $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(2k)$.

This second method did not suffer as much implementation issues, and allowed the authors of [CFKRS3] to obtain many more coefficients. Hence this method was chosen to be used in this thesis. The unfortunate drawbacks of this approach was that it limited the computation for integer values of k only and presented more difficulties in acceleration, yielding lower precision. The main idea of the method involves taking small shifts and using very high precision to capture cancelation amongst the high order poles of the terms in the sum, for which a more detailed explanation can be found in Chapter 3. Because this method required very little symbolically, it was implemented in `C++` using NTL [S] to carry out multiprecision arithmetic. Also, improvement was made from using quadratic acceleration as in [CFKRS3] by using cubic acceleration in order to obtain a higher level of accuracy.

In the following Chapter 2, we illustrate the heuristics for moments of $\zeta(\frac{1}{2}+it)$. In Chapter 3 as described above we give an explanation of our method for the numerical computation of the coefficients. Chapter 4 is the introduction to the Tanh-Sinh Quadrature Scheme, which is incorporated in our numerical computation of the integral moments. Chapter 5 is dedicated to verifying that our results obtained agree with the moment conjectures. Finally, we turn our attention to classical results at Chapter 6 by first presenting the proof for the approximate functional equation, one of the key tools for proving theoretical results about moments of $\zeta(s)$. Then, in the same chapter we illustrate its use by presenting the basic result (1.1) by Hardy and Littlewood.

1.1 Results

In this thesis, we have computed the coefficients of the moment polynomials up to $k = 13$, whereas it was done up to $k = 7$ in [CFKRS3]. Also, these coefficients for $r \geq 10$ were computed to greater precision here. It is worth mentioning that going to $k = 13$ is substantial, because computing the coefficients for k involves k^2 sums of $2k$ choose k terms with working precision of k^2 digits. For example, nearly 2000 digits are required for $k = 13$ with desired precision of 10 digits. The process is made even more challenging, by the fact that each term involves a complicated infinite product over primes. This computation and numerical data is detailed in Chapter 3.

In Chapter 5, we have verified the moment conjecture for various values of T up to 10^8 for $k \leq 13$ by plotting the relative conjectured error. The data required to accomplish this task was obtained by using tanh-sinh quadrature scheme described in Chapter 4. Our results seem to agree with the conjecture and support the validity of the claim. Also, we have looked at the distribution of the remainder term by constructing histograms of the normalized data for which the description and the figures can be found in Chapter 5.

Chapter 2

Heuristics

Here we illustrate the heuristics for moments of $\zeta(\frac{1}{2}+it)$. This is a procedure for conjecturing all of the main terms in the mean value of the ζ function as described in [CFKRS].

Consider

$$Z(s,\alpha) = \zeta(s+\alpha_1)\cdots\zeta(s+\alpha_k)\zeta(1-s-\alpha_{k+1})\cdots\zeta(1-s-\alpha_{2k}), \tag{2.1}$$

where $\alpha = (\alpha_1, \ldots, \alpha_{2k})$. In order to understand the structure, it is necessary to include the shifts α_j. Now we can obtain the moments of $\zeta\left(\frac{1}{2}+it\right)$ by making the shifts to tend to 0. It is because of these shifts α_j's, we get to avoid higher-order poles in our expressions. Our goal is a formula for

$$\int_{-\infty}^{\infty} Z(\tfrac{1}{2}+it,\alpha)g(t)\,dt, \tag{2.2}$$

where g is a suitable weight function. We do not define what is meant by a "suitable weight function", but we can take $g(t) = g_T(t) = f(t/T)$ for a fixed integrable function f. In particular, if we can take f to be the characteristic function of the interval $[0,1]$, to obtain the mean value $\int_0^T Z(\frac{1}{2}+it,\alpha)dt$. From this we can actually recover a fairly general weighted integral by applying partial integration.

For each ζ-function we use the approximate functional equation

$$\zeta(s) = \sum_m \frac{1}{m^s} + \chi(s)\sum_n \frac{1}{n^{1-s}} + \text{remainder} \tag{2.3}$$

and here we can ignore the range of the summation because it will just be extended to infinity in the final step. Also we ignore the remainder term and the limits on the sums. Multiplying out the resulting expression we obtain 2^{2k} terms, and we only keep those terms in which the product of χ-factors is not oscillating rapidly.

If $s = z+it$ with z bounded (but not necessarily real) then

$$\chi(s) = \left(\frac{t}{2\pi}\right)^{\frac{1}{2}-s} e^{it+\pi i/4}\left(1+O\left(\frac{1}{t}\right)\right) \tag{2.4}$$

and

$$\chi(1-s) = \left(\frac{t}{2\pi}\right)^{s-\frac{1}{2}} e^{-it-\pi i/4}\left(1+O\left(\frac{1}{t}\right)\right), \tag{2.5}$$

as $t \to +\infty$. The above two formulae will be used to determine which products of $\chi(s)$ and $\chi(1-s)$ are oscillating.

One term we know that does not have an oscillating factor is the term, where only the "first part" of each approximate functional equation is used, for it does not have any χ-factors. With $s = \frac{1}{2} + it$, that term is

$$\begin{aligned}\sum_{m_1,\ldots,m_k,n_1,\ldots,n_k} & m_1^{-s-\alpha_1}\cdots m_k^{-s-\alpha_k} n_1^{s-1+\alpha_{k+1}}\cdots n_k^{s-1+\alpha_{2k}} \\ &= \sum_{m_1,\ldots,m_k,n_1,\ldots,n_k} m_1^{-\frac{1}{2}-\alpha_1}\cdots m_k^{-\frac{1}{2}-\alpha_k} \\ &\qquad \times n_1^{-\frac{1}{2}+\alpha_{k+1}}\cdots n_k^{-\frac{1}{2}+\alpha_{2k}}\left(\frac{n_1\cdots n_k}{m_1\cdots m_k}\right)^{it}.\end{aligned} \tag{2.6}$$

Then we only keep the diagonal from the above sum, which is

$$\sum_{m_1\cdots m_k = n_1\cdots n_k} m_1^{-\frac{1}{2}-\alpha_1}\cdots m_k^{-\frac{1}{2}-\alpha_k} n_1^{-\frac{1}{2}+\alpha_{k+1}}\cdots n_k^{-\frac{1}{2}+\alpha_{2k}}. \tag{2.7}$$

If we define

$$R(s;\alpha) = \sum_{m_1\cdots m_k = n_1\cdots n_k} \frac{1}{m_1^{s+\alpha_1}\cdots m_k^{s+\alpha_k} n_1^{s-\alpha_{k+1}}\cdots n_k^{s-\alpha_{2k}}}, \tag{2.8}$$

where the sum is over all positive $m_1,\ldots,m_k,n_1,\ldots,n_k$ such that $m_1\cdots m_k = n_1\cdots n_k$, then $R(\frac{1}{2};\alpha)$ is identified as the first piece contributing to the mean value. (The sum in equation (2.8) does not converge for $s = \frac{1}{2}$. See Theorem 2.4.1 in [CFKRS] for its analytic continuation.)

Note that the variable s in equation (2.8) is not the same as the variable $s = \frac{1}{2}+it$ from the previous equations. The trick employed here is that we begin with an expression involving s and $1-s$, noting that s will later be set to $\frac{1}{2}$. So instead we consider an expression only involving s and we set $s = \frac{1}{2}$ later.

Now consider one of the other terms, the term with the second part of the approximate functional equation from $\zeta(s+\alpha_1)$ and another second part from $\zeta(1-s-\alpha_{k+1})$. By (2.4) and (2.5),

$$\chi(s+\alpha_1)\chi(1-s-\alpha_{k+1}) \sim \left(\frac{t}{2\pi}\right)^{-\alpha_1+\alpha_{k+1}}, \tag{2.9}$$

which is not rapidly oscillating. Using this and proceeding as above, we obtain that the contribution from this term will be

$$\left(\frac{t}{2\pi}\right)^{-\alpha_1+\alpha_{k+1}} R(\tfrac{1}{2};\alpha_{k+1},\alpha_2,\ldots,\alpha_k,\alpha_1,\alpha_{k+2},\ldots,\alpha_{2k}). \tag{2.10}$$

We then consider a more general case. Note that

$$\chi(s+\beta_1)\cdots\chi(s+\beta_J)\chi(1-s-\gamma_1)\cdots\chi(1-s-\gamma_K)$$
$$\sim \left(\frac{t}{2\pi e}\right)^{-i(J-K)t} e^{i(J-K)\pi/4}\left(\frac{t}{2\pi}\right)^{-\sum\beta_j+\sum\gamma_j},$$

which is rapidly oscillating (because of the it in the exponent) unless $J = K$. Thus, we only keep those terms which involve an equal number of $\chi(s+\alpha_j)$ and $\chi(1-s-\alpha_{k+j})$ factors to keep $J = K$. Therefore, this produces a total of $\binom{2k}{k} = \sum_{j=0}^{k}\binom{k}{j}^2$ terms in the final answer.

We now describe a typical term of the conjectural formula. First note that the function $R(s;\alpha_1,\ldots,\alpha_k,\alpha_{k+1},\ldots,\alpha_{2k})$ is symmetric in $\alpha_1,\ldots,\alpha_k$ and in $\alpha_{k+1},\ldots,\alpha_{2k}$, so the entries can be rearranged in a way such that the first k are in increasing order, as are the last k. Thus, the final result will be a sum of terms indexed by the $\binom{2k}{k}$ permutations $\sigma\in S_{2k}$ with $\sigma(1)<\cdots<\sigma(k)$ and $\sigma(k+1)<\cdots<\sigma(2k)$. We denote the set of such permutations by Ξ. Second, note that the product of an equal number of $\chi(s+\alpha_j)$ and $\chi(1-s-\alpha_{k+j})$ can be written as

$$\left(\frac{t}{2\pi}\right)^{\frac{1}{2}(-\alpha_1-\cdots-\alpha_k+\alpha_{k+1}+\cdots+\alpha_{2k})}\left(\frac{t}{2\pi}\right)^{\frac{1}{2}(\alpha_{\sigma(1)}+\cdots+\alpha_{\sigma(k)}-\alpha_{\sigma(k+1)}-\cdots-\alpha_{\sigma(2k)})}, \qquad (2.11)$$

because by writing this way all the α's that don't contribute in the exponential cancel out. As a result only the α_j's that contribute remain, which is exactly the expression we want. This works because we know that α_j's that contribute from the first k will be sent to a position in the last k by the permutation in Ξ and vice versa. For example, the asymptotic equivalence (2.9) is the case $\sigma(i) = i+1$ for $1\le i\le k$, $\sigma(k+1) = 1$, and $\sigma(j) = j$ for $k+2\le j\le 2k$.

If we set

$$W(z,\alpha,\sigma) = \left(\frac{y}{2\pi}\right)^{\frac{1}{2}(\alpha_{\sigma(1)}+\cdots+\alpha_{\sigma(k)}-\alpha_{\sigma(k+1)}-\cdots-\alpha_{\sigma(2k)})} R(x;\alpha_{\sigma(1)},\ldots,\alpha_{\sigma(2k)}), \qquad (2.12)$$

for $z = x+iy$ with x and y real, then combining all terms we have

$$M(z;\alpha) := \left(\frac{y}{2\pi}\right)^{\frac{1}{2}(-\alpha_1-\cdots-\alpha_k+\alpha_{k+1}+\cdots+\alpha_{2k})}\sum_{\sigma\in\Xi} W(z,\alpha,\sigma). \qquad (2.13)$$

Then the conjecture is

$$\int_{-\infty}^{\infty} Z(\tfrac{1}{2}+it,\alpha)g(t)\,dt = \int_{-\infty}^{\infty} M(\tfrac{1}{2}+it,\alpha)(1+O(t^{-\frac{1}{2}+\varepsilon}))g(t)\,dt, \qquad (2.14)$$

with $Z(s,\alpha)$ given in (2.1) and $M(s;\alpha)$ given above. The error term of $O(t^{-\frac{1}{2}+\varepsilon})$ agrees with known examples and numerical evidence as described in Section 5 of [CFKRS].

Note that the exponent of $(t/2\pi)$ in (2.12) has half the α_j with a "+" sign and the other half with a "−" sign, and the same holds for $R(s,\alpha)$. This allows us to have an alternate interpretation of Ξ as the set of ways of choosing k elements from $\{\alpha_1,\ldots,\alpha_{2k}\}$.

The more general case of the Conjecture (1.8) is stated in terms of the Z-functions, $Z(s)=\chi(s)^{-\frac{1}{2}}\zeta(s)$ as defined in [CFKRS]. We can recover the mean value of the Z-function directly from that of the zeta function. Using the functional equation, (2.4) and (2.5) we see that

$$\begin{aligned} Z(s+\alpha_1)\cdots Z(s+\alpha_{2k}) &= \left(\frac{t}{2\pi}\right)^{\frac{1}{2}(\alpha_1+\cdots+\alpha_k-\alpha_{k+1}-\cdots-\alpha_{2k})}(1+O(1/t)) \\ &\quad\times\zeta(s+\alpha_1)\cdots\zeta(s+\alpha_k) \\ &\quad\times\zeta(1-s-\alpha_{k+1})\cdots\zeta(1-s-\alpha_{2k}). \end{aligned} \tag{2.15}$$

The factor $\left(\frac{t}{2\pi}\right)^{\frac{1}{2}(\alpha_1+\cdots+\alpha_k-\alpha_{k+1}-\cdots-\alpha_{2k})}$ can be absorbed into the weight function $g(t)$. Consequently, we obtain the conjecture

$$\int_{-\infty}^{\infty} Z(s+\alpha_1)\cdots Z(s+\alpha_{2k})g(t)\,dt=\int_{-\infty}^{\infty}\sum_{\sigma\in\Xi}W(s,\alpha,\sigma)(1+O(t^{-\frac{1}{2}+\varepsilon}))g(t)\,dt, \tag{2.16}$$

where $s=\frac{1}{2}+it$. It is described in Section 2 of [CFKRS] how to write this type of sum over permutations in a compact form involving contour integrals.

Chapter 3

Numerical Evaluation of $c_r(k)$

As described in Chapter 1 our method here of numerically computing the coefficients of the polynomials involves taking small distinct shifts and high working precision to capture cancelation amongst the order k^2 poles of the right hand side of (1.12). The authors of [CFKRS3] succeeded in obtaining many more coefficients of $P_k(x)$ this way than the first method. We will describe the method in this chapter followed by the table of the coefficients we obtained.

The idea behind this method is as follows. The polynomial $P_k(x)$ given by (1.9) can be regarded as a special case, namely when $\alpha_1 = \ldots = \alpha_{2k} = 0$, of the function $P_k(\alpha, x)$ shown below as (3.1), which the path of integration being small circles surrounding the poles α_i, and $-1/4 < \Re\alpha_j$. Thus the equation (1.12) from the lemma can be used to evaluate $P_k(\alpha, x)$. However, the terms in (1.12) have poles if the α_i's are not distinct, because otherwise it introduces $\zeta(1)$ into the expression. This is precisely the reason why we cannot simply substitute $\alpha = \mathbf{0}$ and sum the terms numerically. Instead we have to take the limit as $\alpha \to \mathbf{0}$ while making sure that all the α_i's are distinct. We also need to use very high precision to capture cancelation amongst the terms. Each individual term becomes very large when α is small and without sufficient precision the sum will not converge.

$$P_k(\alpha, x) = \frac{(-1)^k}{k!^2} \frac{1}{(2\pi i)^{2k}} \oint \cdots \oint \frac{G(z_1, \ldots, z_{2k}) \Delta(z_1, \ldots, z_{2k})^2}{\prod_{i=1}^{2k} \prod_{j=1}^{2k} (z_i - \alpha_j)} \times e^{\frac{x}{2} \sum_{i=1}^{k} z_i - z_{i+k}} \, dz_1 \ldots dz_{2k} \tag{3.1}$$

More precisely, let

$$H(z_1, \ldots, z_{2k}; x) = \exp\left(\frac{x}{2} \sum_{1}^{k} z_j - z_{j+k} \right) A_k(z_1, \ldots, z_{2k}) \prod_{i=1}^{k} \prod_{j=1}^{k} \zeta(1 + z_i - z_{j+k}),$$

and let $\epsilon_j = j\epsilon$, where $\epsilon \in \mathbb{C}$. Then, by applying (1.12) we obtain

$$P_k(x) = \lim_{\epsilon\to 0} \sum_{\sigma\in\Xi} H(\epsilon_{\sigma(1)}, \dots, \epsilon_{\sigma(2k)}; x), \tag{3.2}$$

where Ξ is the set of $\binom{2k}{k}$ permutations $\sigma \in S_{2k}$ such that $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(2k)$.

We write the coefficients of $P_k(x)$ as

$$P_k(x) = c_0(k)x^{k^2} + c_1(k)x^{k^2-1} + c_2(k)x^{k^2-2} + \cdots + c_{k^2}(k). \tag{3.3}$$

Now, expanding exp in its Taylor series, and pulling out the coefficient of x^{k^2-r}, we have

$$c_r(k) = \frac{1}{2^{k^2-r}(k^2-r)!} \lim_{\epsilon\to 0} \sum_{\sigma\in\Xi} H_r(\epsilon_{\sigma(1)}, \dots, \epsilon_{\sigma(2k)}), \tag{3.4}$$

where

$$H_r(z_1, \dots, z_{2k}) = \left(\sum_{1}^{k} z_j - z_{j+k}\right)^{k^2-r} A_k(z_1, \dots, z_{2k}) \prod_{i=1}^{k}\prod_{j=1}^{k} \zeta(1+z_i - z_{j+k}). \tag{3.5}$$

The only complication in evaluating the above for a given k and ϵ is that $A_k(z_1, \dots, z_{2k})$ is expressed as an infinite product over primes as shown in (1.10). To evaluate this $A_k(z_1, \dots, z_{2k})$, we break up the product into two portions $p \le P$ and $p > P$, where P is a large number. For the first portion $p \le P$, we use equation (1.11) to evaluate its contribution. Note that each factor only requires finitely many arithmetic steps to evaluate its value.

For the contribution of the second portion $p > P$, we use something called cubic acceleration (3.6) for each of the local factors appearing in (1.10). Note that this is an improvement from [CFKRS3] where a quadratic acceleration was used:

$$\begin{aligned} 1 - &\sum_{\substack{1\le i_1<i_2\le k\\ 1\le j_1<j_2\le k}} p^{-2-z_{i_1}-z_{i_2}+z_{k+j_1}+z_{k+j_2}} \\ &+ 4 \sum_{\substack{1\le i_1<i_2<i_3\le k\\ 1\le j_1<j_2<j_3\le k}} p^{-3-z_{i_1}-z_{i_2}-z_{i_3}+z_{k+j_1}+z_{k+j_2}+z_{k+j_3}} \\ &\quad + \sum_{\substack{1\le i_1<i_2<i_3\le k\\ 1\le j_1\ne j_2\le k}} p^{-3-z_{i_1}-z_{i_2}-z_{i_3}+2z_{k+j_1}+z_{k+j_2}} \\ &\quad\quad + \sum_{\substack{1\le i_1\ne i_2\le k\\ 1\le j_1<j_2<j_3\le k}} p^{-3-2z_{i_1}-z_{i_2}+z_{k+j_1}+z_{k+j_2}+z_{k+j_3}} \end{aligned} \tag{3.6}$$

This approximation can be obtained by first substituting $u_j = p^{-1/2-z_j}$ and $w_j = p^{-1/2+z_{k+j}}$ into the local factor of (1.10), and then working out the terms up to degree six. We do this by expanding each geometric series in the integral over θ up to degree three, multiplying them out, and collecting terms with the same number of u's and w's. Only terms of even degree appear because the integral over θ pulls out just the terms with the same number of u's and w's. The integral of any other term, which does not have the same number of u's and w's, is zero because it contains either $e(\theta)$ or $e(-\theta)$. The whole process is presented below.

$$\prod_{i,j=1}^{k}(1-p^{-1-z_i+z_{k+j}})\int_0^1 \prod_{j=1}^{k}\left(1-\frac{e(\theta)}{p^{\frac{1}{2}+z_j}}\right)^{-1}\left(1-\frac{e(-\theta)}{p^{\frac{1}{2}-z_{k+j}}}\right)^{-1} d\theta$$

$$= \prod_{i,j=1}^{k}(1-u_iw_j)\int_0^1 \prod_{j=1}^{k}(1-u_je(\theta))^{-1}(1-w_je(-\theta))^{-1}\, d\theta$$

$$\approx \quad \prod_{i,j=1}^{k}(1-u_iw_j)\int_0^1 \prod_{j=1}^{k}(1+u_je(\theta)+u_j^2e(2\theta)+u_j^3e(3\theta))^{-1} \times (1+w_je(-\theta)+w_j^2e(-2\theta)+w_j^3e(-3\theta))^{-1}\, d\theta$$

The factor in front of the integral becomes

$$1-\sum_{i,j=1}^{k}u_iw_j + \textit{terms degree } 4 \textit{ or higher}$$

and the integral becomes

$$1+\sum_{i,j=1}^{k}u_iw_j + \textit{terms degree } 4 \textit{ or higher}$$

Therefore, when we multiply them out all of the terms of degree two cancel and the result is

$$1+\sum_{i,i',j,j'}u_iu_{i'}w_jw_{j'}+\sum_{i,i',i'',j,j',j''}u_iu_{i'}u_{i''}w_jw_{j'}w_{j''}+\textit{terms degree } 8 \textit{ or higher} \tag{3.7}$$

The second factor, sum of degree four terms, in the above expression is derived in [CFKRS3] with full details and it simplifies as

$$-\sum_{\substack{1\le i_1<i_2\le k\\ 1\le j_1<j_2\le k}}u_{i_1}u_{i_2}w_{j_1}w_{j_2}. \tag{3.8}$$

Since the terms of degree two are completely canceled and all the terms of degree four are already worked out in [CFKRS3], we only need to determine what the terms of degree six are. The sum of degree six terms can be simplified using the same method as in [CFKRS3] used to determine the terms of degree four. Notice that the local factor of (1.10) is symmetric separately in u's and w's meaning even if some u_i and $u_{i'}$ are swapped the equation remains invariant and similarly for w's. Also, it is symmetric with u and w, so even if u's and w's are swapped the equation remains the same. Hence it is sufficient to look at only six terms namely, $u_1u_2u_3w_1w_2w_3$, $u_1u_2u_3w_1^2w_2$, $u_1u_2u_3w_1^3$, $u_1^2u_2w_1^2w_2$, $u_1^2u_2w_1^3$ and $u_1^3w_1^3$ instead of every possible term. In order to compute the coefficients of these terms, first set $u_j = w_j = 0$ if $4 \leq j \leq k$ and then take the partial derivatives respectively: $\frac{\partial}{\partial u_1}\frac{\partial}{\partial u_2}\frac{\partial}{\partial u_3}\frac{\partial}{\partial w_1}\frac{\partial}{\partial w_2}\frac{\partial}{\partial w_3}$, $\frac{\partial}{\partial u_1}\frac{\partial}{\partial u_2}\frac{\partial}{\partial u_3}\frac{\partial^2}{\partial w_1^2}\frac{\partial}{\partial w_2}$, $\frac{\partial}{\partial u_1}\frac{\partial}{\partial u_2}\frac{\partial}{\partial u_3}\frac{\partial^3}{\partial w_1^3}$, $\frac{\partial^2}{\partial u_1^2}\frac{\partial}{\partial u_2}\frac{\partial^2}{\partial w_1^2}\frac{\partial}{\partial w_2}$, $\frac{\partial^2}{\partial u_1^2}\frac{\partial}{\partial u_2}\frac{\partial^3}{\partial w_1^3}$, $\frac{\partial^3}{\partial u_1^3}\frac{\partial^3}{\partial w_1^3}$ evaluated at $u_1 = u_2 = u_3 = w_1 = w_2 = w_3 = 0$. Doing so gives the simplified expression for the sum of degree six terms. Putting everything together, the expression (3.7) without the term degree eight or higher becomes

$$1 - \sum_{\substack{1\leq i_1<i_2\leq k \\ 1\leq j_1<j_2\leq k}} u_{i_1}u_{i_2}w_{j_1}w_{j_2} + 4 \sum_{\substack{1\leq i_1<i_2<i_3\leq k \\ 1\leq j_1<j_2<j_3\leq k}} u_{i_1}u_{i_2}u_{i_3}w_{j_1}w_{j_2}w_{j_3}$$

$$+ \sum_{\substack{1\leq i_1<i_2<i_3\leq k \\ 1\leq j_1\neq j_2\leq k}} u_{i_1}u_{i_2}u_{i_3}w_{j_1}^2w_{j_2} + \sum_{\substack{1\leq i_1\neq i_2\leq k \\ 1\leq j_1<j_2<j_3\leq k}} u_{i_1}^2u_{i_2}w_{j_1}w_{j_2}w_{j_3} \qquad (3.9)$$

Now undoing the substitution for u_i's and w_j's, this expression becomes (3.6). Then this sum can be approximated by the following product by looking at the terms that are only multiplied by 1 when it is multiplied out, because those are precisely the terms which appear in the sum:

$$\prod_{\substack{1\leq i_1<i_2\leq k \\ 1\leq j_1<j_2\leq k}} (1 - p^{-2-z_{i_1}-z_{i_2}+z_{k+j_1}+z_{k+j_2}})$$
$$\times \prod_{\substack{1\leq i_1<i_2<i_3\leq k \\ 1\leq j_1<j_2<j_3\leq k}} (1 - p^{-3-z_{i_1}-z_{i_2}-z_{i_3}+z_{k+j_1}+z_{k+j_2}+z_{k+j_3}})^{-4}$$
$$\times \prod_{\substack{1\leq i_1<i_2<i_3\leq k \\ 1\leq j_1\neq j_2\leq k}} (1 - p^{-3-z_{i_1}-z_{i_2}-z_{i_3}+2z_{k+j_1}+z_{k+j_2}})^{-1}$$
$$\times \prod_{\substack{1\leq i_1\neq i_2\leq k \\ 1\leq j_1<j_2<j_3\leq k}} (1 - p^{-3-2z_{i_1}-z_{i_2}+z_{k+j_1}+z_{k+j_2}+z_{k+j_3}})^{-1} \qquad (3.10)$$

Thus the following approximation for the $p > P$ portion of (1.10) can be obtained by using the above expression (3.10) for each of the local factor and substituting $\zeta(s) = \prod_p (1 - \frac{1}{p^s})^{-1}$:

$$\frac{\prod_{\substack{1\le i_1<i_2\le k\\1\le j_1<j_2\le k}} \zeta(2 + z_{i_1} + z_{i_2} - z_{k+j_1} - z_{k+j_2})^{-1}}{\prod_{p\le P}\prod_{\substack{1\le i_1<i_2\le k\\1\le j_1<j_2\le k}} (1 - p^{-2-z_{i_1}-z_{i_2}+z_{k+j_1}+z_{k+j_2}})}$$

$$\times \frac{\prod_{\substack{1\le i_1<i_2<i_3\le k\\1\le j_1<j_2<j_3\le k}} \zeta(3 + z_{i_1} + z_{i_2} + z_{i_3} - z_{k+j_1} - z_{k+j_2} - z_{k+j_3})^4}{\prod_{\substack{1\le i_1<i_2<i_3\le k\\1\le j_1<j_2<j_3\le k}} (1 - p^{-3-z_{i_1}-z_{i_2}-z_{i_3}+z_{k+j_1}+z_{k+j_2}+z_{k+j_3}})^{-4}}$$

$$\times \frac{\prod_{\substack{1\le i_1<i_2<i_3\le k\\1\le j_1\ne j_2\le k}} \zeta(3 + z_{i_1} + z_{i_2} + z_{i_3} - 2z_{k+j_1} - z_{k+j_2})}{\prod_{\substack{1\le i_1<i_2<i_3\le k\\1\le j_1\ne j_2\le k}} (1 - p^{-3-z_{i_1}-z_{i_2}-z_{i_3}+2z_{k+j_1}+z_{k+j_2}})^{-1}}$$

$$\times \frac{\prod_{\substack{1\le i_1\ne i_2\le k\\1\le j_1<j_2<j_3\le k}} \zeta(3 + 2z_{i_1} + z_{i_2} - z_{k+j_1} - z_{k+j_2} - z_{k+j_3})}{\prod_{\substack{1\le i_1\ne i_2\le k\\1\le j_1<j_2<j_3\le k}} (1 - p^{-3-2z_{i_1}-z_{i_2}+z_{k+j_1}+z_{k+j_2}+z_{k+j_3}})^{-1}}$$

All the coefficients obtained are presented in the tables below. In the tables, eN means 10^N. To give a couple of examples, $2.5604415e3 = 2.5604415 \times 10^3$ and $7.08945522e - 5 = 7.08945522 \times 10^{-5}$. For $k = 4$ and 5, 25 digits of the coefficients were calculated. For $k = 6, 7, 8$ and 9 and $k = 10, 11, 12$ and 13, up to 16 digits and 12 digits were calculated respectively. The remainder of $c_r(k)$ in stopping at $p \le P$ with the cubic acceleration for the tail is $O_r(\frac{1}{P^{3-\varepsilon}})$. However the constant that depends on r in the O-term is very complicated and we had to determine visually when the numbers were stabilizing. Thus to determine if a digit of coefficients has stabilized, we set the criterion of stabilization as the digit has not changed for at least the last 20% of the data and wrote a simple C++ program to check this condition for each case. Note that the tables only contain the digits which are considered stable under this criterion. It is also worth mentioning that all the digits of the coefficients computed in this manner agree with the results by the first method of [CFKRS3], a completely different method used to compute coefficients for $k \le 7$ and $r \le 9$. This is a good indication that our results are correct.

r	$c_r(4)$	$c_r(5)$
0	$2.4650183919342273540799e-13$	$1.4160010206227312009 5e-24$
1	$5.4501405731171865593 6e-11$	$7.3804127564944513059 6e-22$
2	$5.2877296347912031138 49e-9$	$1.7797796235196529053 09e-19$
3	$2.9641143179993979459 691e-7$	$2.6358866096607247582 8e-17$
4	$1.0645950068128470513 211e-5$	$2.6840545349997485760 13e-15$
5	$2.5702983342426340235 49e-4$	$1.9936413092498971803 12e-13$
6	$4.2639216163116947218 7e-3$	$1.1184855124933629437 77e-11$
7	$4.8941424514216010271 26e-2$	$4.8427975530448041655e-10$
8	$3.8785266540195534998e-1$	$1.6398013084961560997 9e-8$
9	2.10913382864873355	$4.3749351054922463304e-7$
10	7.832535611882262357	$9.2263335029653032633e-6$
11	$1.98280681249989092e1$	$1.5376777782071079469 91e-4$
12	$3.3888932037383688e1$	$2.0190277580781319590e-3$
13	$3.82033062189019e1$	$2.0772706728484647547 5e-2$
14	$2.5604415012270e1$	$1.662505864391039365e-1$
15	$1.0618969379401e1$	1.026466777849473756
16	$7.0894645522e-1$	4.84858927834364247
17		$1.7390876090131023e1$
18		$4.704087708756173e1$
19		$9.51166179458788e1$
20		$1.41444460064317e2$
21		$1.49356949996298e2$
22		$1.0588728028421e2$
23		$4.4136230729e1$
24		$2.010650046e1$
25		-1.2701702

Table 3.1: Coefficients for $k = 4$ and 5 truncating at $P = 149643229$ and 67984901, respectively

r	$c_r(6)$	r	$c_r(6)$
0	$5.129473409149191e-40$	19	$9.351583018775044e-5$
1	$5.306732809926444e-37$	20	$1.068316421173023e-3$
2	$2.607920771148351e-34$	21	$1.018070238623615e-2$
3	$8.101613215779018e-32$	22	$8.041867930583792e-2$
4	$1.786129738009310e-29$	23	$5.229614194172495e-1$
5	$2.974316710863606e-27$	24	2.780201766519572
6	$3.887708291155868e-25$	25	$1.200111408801811e1$
7	$4.092242614068629e-23$	26	$4.179670936891263e1$
8	$3.531466385657033e-21$	27	$1.167230958294841e2$
9	$2.530637690060973e-19$	28	$2.593989729971504e2$
10	$1.519819102968592e-17$	29	$4.524908135219902e2$
11	$7.700151376092375e-16$	30	$6.0117334836509e2$
12	$3.306121041410744e-14$	31	$5.735438455312e2$
13	$1.206404151898472e-12$	32	$3.75018676133e2$
14	$3.746719254162692e-11$	33	$2.46890415604e2$
15	$9.905694285688910e-10$	34	$2.4549543694e2$
16	$2.227388576717968e-8$	35	$1.6033037688e2$
17	$4.251372866816786e-7$	36	$-3.78219665e1$
18	$6.867433576987095e-6$		

Table 3.2: Coefficients for $k = 6$ truncating at $P = 608121859$

r	$c_r(7)$	r	$c_r(7)$
0	$6.582284787600550e-60$	25	$1.289875556721964e-9$
1	$1.204143055545187e-56$	26	$2.286966740087652e-8$
2	$1.062135571749272e-53$	27	$3.568399500496953e-7$
3	$6.017265376015930e-51$	28	$4.883407104161550e-6$
4	$2.460628767324013e-48$	29	$5.839104522079822e-5$
5	$7.739012166521153e-46$	30	$6.074203732753243e-4$
6	$1.947864949495236e-43$	31	$5.471643825436489e-3$
7	$4.030768492636370e-41$	32	$4.246590340375059e-2$
8	$6.991776333723788e-39$	33	$2.824549346789606e-1$
9	$1.031401977981227e-36$	34	1.601333066518585
10	$1.308286914499358e-34$	35	7.69669961409269
11	$1.439268120143532e-32$	36	$3.1203520207257e1$
12	$1.382531215498608e-30$	37	$1.06197143546798e2$
13	$1.165775937131802e-28$	38	$3.019136554174e2$
14	$8.665247693352722e-27$	39	$7.117410357279e2$
15	$5.696222742475378e-25$	40	$1.37009445509e3$
16	$3.319764854050799e-23$	41	$2.08279874422e3$
17	$1.718397039329422e-21$	42	$2.35736353e3$
18	$7.909678889323544e-20$	43	$1.93463843e3$
19	$3.239692933574084e-18$	44	$1.7571430e3$
20	$1.180957927326837e-16$	45	$2.853378e3$
21	$3.830227005151013e-15$	46	$3.100593e3$
22	$1.104470629036126e-13$	47	$3.39399e2$
23	$2.828225823158349e-12$	48	$-1.208542e3$
24	$6.421066225760994e-11$	49	$-5.0194e2$

Table 3.3: Coefficients for $k = 7$ truncating at $P = 11015647$

r	$c_r(8)$	r	$c_r(8)$
0	$1.870442160116e-84$	33	$9.566613928613862e-15$
1	$5.570219365178e-81$	34	$2.153446226672471e-13$
2	$8.0727983790767e-78$	35	$4.402406580818171e-12$
3	$7.5876025208717e-75$	36	$8.157648469092308e-11$
4	$5.2002464291967e-72$	37	$1.367077736688555e-9$
5	$2.7705098412043e-69$	38	$2.066815459968645e-8$
6	$1.194483270804878e-66$	39	$2.811291057443101e-7$
7	$4.28399526987474e-64$	40	$3.430097716487554e-6$
8	$1.303887835529719e-61$	41	$3.741873571285350e-5$
9	$3.419015475880784e-59$	42	$3.636847919803768e-4$
10	$7.81488356430940e-57$	43	$3.137462556407600e-3$
11	$1.571639098418715e-54$	44	$2.39287182393224e-2$
12	$2.801987810577959e-52$	45	$1.60675070999268e-1$
13	$4.455999566980015e-50$	46	$9.45881317436838e-1$
14	$6.353422665784412e-48$	47	4.8616355021106
15	$8.156431845206120e-46$	48	$2.1731039865601e1$
16	$9.461598292152723e-44$	49	$8.41713345945e1$
17	$9.946939863498895e-42$	50	$2.81517268210e2$
18	$9.500575560800423e-40$	51	$8.09291773827e2$
19	$8.261074850697205e-38$	52	$1.9821841215e3$
20	$6.550580273968148e-36$	53	$4.0587357e3$
21	$4.743162490429711e-34$	54	$6.69566486e3$
22	$3.139513479717638e-32$	55	$8.420397e3$
23	$1.901113661842625e-30$	56	$8.096360e3$
24	$1.053764619203100e-28$	57	$9.49612e3$
25	$5.348197946623893e-27$	58	$1.99106e4$
26	$2.485670523907216e-25$	59	$3.09087e4$
27	$1.057803394107259e-23$	60	$1.3132e4$
28	$4.120562080123904e-22$	61	$-2.9642e4$
29	$1.468507163296662e-20$	62	$-4.058e4$
30	$4.784707767287195e-19$	63	$-8.56e3$
31	$1.423965148185230e-17$	64	$4.56e3$
32	$3.866559621216503e-16$		

Table 3.4: Coefficients for $k = 8$ truncating at $P = 1212569$

r	$c_r(9)$	r	$c_r(9)$
0	$7.920155238e-114$	41	$7.084266751890731e-23$
1	$3.6087438729e-110$	42	$2.125212780202074e-21$
2	$8.0512962717e-107$	43	$5.922419887028314e-20$
3	$1.17236240582e-103$	44	$1.531502833064513e-18$
4	$1.253005876e-100$	45	$3.67062076426678e-17$
5	$1.04814273755e-97$	46	$8.14314143690124e-16$
6	$7.1456310032e-95$	47	$1.669731514497095e-14$
7	$4.08217445964e-92$	48	$3.15948278435009e-13$
8	$1.9941925289e-89$	49	$5.50744986213134e-12$
9	$8.4594205088e-87$	50	$8.82746086945955e-11$
10	$3.1538144172e-84$	51	$1.29834184462004e-9$
11	$1.043378994182e-81$	52	$1.74846308044294e-8$
12	$3.08731306439e-79$	53	$2.15085136139740e-7$
13	$8.22414128960e-77$	54	$2.4107241046115e-6$
14	$1.9831487571e-74$	55	$2.455207175179e-5$
15	$4.34906593630e-72$	56	$2.2655721581917e-4$
16	$8.708434320967e-70$	57	$1.8883946217274e-3$
17	$1.5976187885023e-67$	58	$1.4172609520337e-2$
18	$2.69326400651095e-65$	59	$9.546112504951e-2$
19	$4.1828312947265e-63$	60	$5.751640191247e-1$
20	$5.9981084135426e-61$	61	3.08994036049
21	$7.9570365567690e-59$	62	$1.47569216973e1$
22	$9.7816383607331e-57$	63	$6.2480977622e1$
23	$1.115906365282440e-54$	64	$2.339346984e2$
24	$1.182893699629169e-52$	65	$7.72127640e2$
25	$1.16636205776320e-50$	66	$2.234306451e3$
26	$1.070749487181153e-48$	67	$5.6036535e3$
27	$9.15886309617988e-47$	68	$1.1907447e4$
28	$7.304105944731468e-45$	69	$2.0622565e4$
29	$5.433520584197941e-43$	70	$2.776900e4$
30	$3.771803089623105e-41$	71	$3.06185e4$
31	$2.443910673326659e-39$	72	$4.7186e4$
32	$1.478284920817422e-37$	73	$1.2201e5$
33	$8.34814115852834e-36$	74	$2.3136e5$
34	$4.401070135763961e-34$	75	$1.254e5$
35	$2.165700326693244e-32$	76	$-4.65e5$
36	$9.94493396339445e-31$	77	$-1.0709e6$
37	$4.260084809382352e-29$	78	$-5.79e5$
38	$1.70158659795284e-27$	79	$6.7e5$
39	$6.333925599759357e-26$	80	$8.27e5$
40	$2.195810416003295e-24$	81	$1.3e5$

Table 3.5: Coefficients for $k = 9$ truncating at $P = 170741$

r	$c_r(10)$	r	$c_r(10)$	r	$c_r(10)$
0	$3.5488849247e-148$	34	$2.18575148719e-61$	68	$1.42629661021e-8$
1	$2.35769133101e-144$	35	$2.02423634536e-59$	69	$1.62408681014e-7$
2	$7.70233663026e-141$	36	$1.77301588079e-57$	70	$1.70456051701e-6$
3	$1.64948634407e-137$	37	$1.46914275997e-55$	71	$1.64525025997e-5$
4	$2.60451944693e-134$	38	$1.15185637224e-53$	72	$1.45689671570e-4$
5	$3.23366677841e-131$	39	$8.54621001078e-52$	73	$1.18063923390e-3$
6	$3.28765141574e-128$	40	$6.00099664894e-50$	74	$8.73323673680e-3$
7	$2.81472946999e-125$	41	$3.98802497325e-48$	75	$5.88113830758e-2$
8	$2.07111222708e-122$	42	$2.50821034610e-46$	76	$3.59617996436e-1$
9	$1.33022343045e-119$	43	$1.49281437084e-44$	77	1.99170458168
10	$7.54902089850e-117$	44	$8.40669033177e-43$	78	9.96798553888
11	$3.82261070458e-114$	45	$4.47858558191e-41$	79	$4.4988103328e1$
12	$1.74111702416e-111$	46	$2.25657645231e-39$	80	$1.82759747793e2$
13	$7.18139722131e-109$	47	$1.07504536023e-37$	81	$6.668289032e2$
14	$2.69752719994e-106$	48	$4.84086016057e-36$	82	$2.177262631e3$
15	$9.27262118830e-104$	49	$2.05951825061e-34$	83	$6.31448075e3$
16	$2.92909129684e-101$	50	$8.27486511554e-33$	84	$1.60290665e4$
17	$8.53361463104e-99$	51	$3.13826128428e-31$	85	$3.4700356e4$
18	$2.30028249569e-96$	52	$1.12280970248e-29$	86	$6.1634951e4$
19	$5.75293360389e-94$	53	$3.78741215989e-28$	87	$8.726962e4$
20	$1.33822224534e-91$	54	$1.20365607466e-26$	88	$1.14723e5$
21	$2.90166426254e-89$	55	$3.60132536986e-25$	89	$2.48873e5$
22	$5.87611643423e-87$	56	$1.01360924580e-23$	90	$7.4051e5$
23	$1.11329562995e-84$	57	$2.68129989678e-22$	91	$1.4295e6$
24	$1.97642057828e-82$	58	$6.65999734832e-21$	92	$2.559e5$
25	$3.29228440661e-80$	59	$1.55171192157e-19$	93	$-6.274e6$
26	$5.15229830331e-78$	60	$3.38747993826e-18$	94	$-1.488e7$
27	$7.58349820666e-76$	61	$6.92078054809e-17$	95	$-7.972e6$
28	$1.05082543859e-73$	62	$1.32157628768e-15$	96	$2.259e7$
29	$1.37203229757e-71$	63	$2.35556047762e-14$	97	$4.021e7$
30	$1.68930527875e-69$	64	$3.91313645458e-13$	98	$1.42e7$
31	$1.96272574949e-67$	65	$6.04928242309e-12$	99	$-1.0e7$
32	$2.15317687834e-65$	66	$8.68767553827e-11$	100	$-5.21e6$
33	$2.23149114478e-63$	67	$1.15703897868e-9$		

Table 3.6: Coefficients for $k = 10$ truncating at $P = 675929$

r	$c_r(11)$	r	$c_r(11)$	r	$c_r(11)$
0	$1.24513138e-187$	41	$5.31266902039e-80$	82	$4.34769983325e-13$
1	$1.160572891e-183$	42	$5.83761751317e-78$	83	$6.11589981351e-12$
2	$5.33593693e-180$	43	$6.12728927464e-76$	84	$8.05538182866e-11$
3	$1.61328064e-176$	44	$6.14465744035e-74$	85	$9.91880250337e-10$
4	$3.6079689e-173$	45	$5.88837261613e-72$	86	$1.13989172286e-8$
5	$6.36556626e-170$	46	$5.39285305339e-70$	87	$1.22051628613e-7$
6	$9.22782528e-167$	47	$4.72075001991e-68$	88	$1.21535644038e-6$
7	$1.130370843e-163$	48	$3.95004795865e-66$	89	$1.12333489860e-5$
8	$1.194241563e-160$	49	$3.15943363429e-64$	90	$9.61809981820e-5$
9	$1.105316890e-157$	50	$2.41565629349e-62$	91	$7.61275464775e-4$
10	$9.07262368e-155$	51	$1.76551247604e-60$	92	$5.55832614028e-3$
11	$6.670017889e-152$	52	$1.23337238502e-58$	93	$3.73566507420e-2$
12	$4.4279593e-149$	53	$8.23513825972e-57$	94	$2.30623749600e-1$
13	$2.672503285e-146$	54	$5.25477062653e-55$	95	1.3052451738
14	$1.47494511e-143$	55	$3.20393186034e-53$	96	6.760126164
15	$7.48038632e-141$	56	$1.86632522022e-51$	97	$3.199030311e1$
16	$3.501239678e-138$	57	$1.03843954524e-49$	98	$1.3812871068e2$
17	$1.5180710761e-135$	58	$5.51781978801e-48$	99	$5.43353075e2$
18	$6.117325359e-133$	59	$2.79921252559e-46$	100	$1.94226636e3$
19	$2.29769557e-130$	60	$1.35538559507e-44$	101	$6.2767681e3$
20	$8.06505889e-128$	61	$6.26198045476e-43$	102	$1.8153278e4$
21	$2.6516126157e-125$	62	$2.75951000539e-41$	103	$4.614002e4$
22	$8.182770980e-123$	63	$1.15946929880e-39$	104	$1.001841e5$
23	$2.3745932499e-120$	64	$4.64317200771e-38$	105	$1.799009e5$
24	$6.490958842e-118$	65	$1.77135231414e-36$	106	$2.7327e5$
25	$1.6738726530e-115$	66	$6.43459047939e-35$	107	$4.8073e5$
26	$4.0778545442e-113$	67	$2.22452843485e-33$	108	$1.4164e6$
27	$9.396885825e-111$	68	$7.31501083723e-32$	109	$4.181e6$
28	$2.05058223868e-108$	69	$2.28661869914e-30$	110	$6.523e6$
29	$4.2419484994e-106$	70	$6.79044608411e-29$	111	$-7.31e6$
30	$8.3264998644e-104$	71	$1.91440617651e-27$	112	$-6.29e7$
31	$1.55219394654e-101$	72	$5.12020249162e-26$	113	$-1.263e8$
32	$2.75017404185e-99$	73	$1.29814491052e-24$	114	$-1.1e7$
33	$4.6346698212e-97$	74	$3.11735752550e-23$	115	$4.21e8$
34	$7.43374148656e-95$	75	$7.08434156622e-22$	116	$7.31e8$
35	$1.13549625358e-92$	76	$1.52216009585e-20$	117	$1.6e8$
36	$1.65268057424e-90$	77	$3.08917785224e-19$	118	$-8.1e8$
37	$2.29313645275e-88$	78	$5.91554759722e-18$	119	$-8.e8$
38	$3.03459182593e-86$	79	$1.06766921396e-16$	120	$-1.0e8$
39	$3.83152664019e-84$	80	$1.81409297789e-15$	121	$9.e7$
40	$4.61740108498e-82$	81	$2.89817262952e-14$		

Table 3.7: Coefficients for $k = 11$ truncating at $P = 85889$

r	$c_r(12)$	r	$c_r(12)$	r	$c_r(12)$
0	$2.61437e-232$	33	$1.1019465e-136$	66	$3.15659618126e-67$
1	$3.31313e-228$	34	$2.6716183e-134$	67	$2.09610369144e-65$
2	$2.07583e-224$	35	$6.196517e-132$	68	$1.33974800679e-63$
3	$8.57284e-221$	36	$1.3758253e-129$	69	$8.24107379268e-62$
4	$2.625117e-217$	37	$2.926060e-127$	70	$4.87776604103e-60$
5	$6.35697e-214$	38	$5.964171e-125$	71	$2.77749386185e-58$
6	$1.267994e-210$	39	$1.16570469e-122$	72	$1.52121625034e-56$
7	$2.142593e-207$	40	$2.18577885e-120$	73	$8.01193333444e-55$
8	$3.130612e-204$	41	$3.9336445e-118$	74	$4.05683148804e-53$
9	$4.01773e-201$	42	$6.7972368e-116$	75	$1.97436017587e-51$
10	$4.58505e-198$	43	$1.1281875e-113$	76	$9.2328042207e-50$
11	$4.699357e-195$	44	$1.79925162e-111$	77	$4.14742073168e-48$
12	$4.361351e-192$	45	$2.7580480e-109$	78	$1.78904091502e-46$
13	$3.690380e-189$	46	$4.06477872e-107$	79	$7.40817919355e-45$
14	$2.863636e-186$	47	$5.76118339e-105$	80	$2.94370013351e-43$
15	$2.048018e-183$	48	$7.85470802e-103$	81	$1.12201488838e-41$
16	$1.355825e-180$	49	$1.03035175e-100$	82	$4.10061125513e-40$
17	$8.3402e-178$	50	$1.300654119e-98$	83	$1.43633249117e-38$
18	$4.78307e-175$	51	$1.580276926e-96$	84	$4.8196818321e-37$
19	$2.564983e-172$	52	$1.848267122e-94$	85	$1.54855998651e-35$
20	$1.289616e-169$	53	$2.08119454e-92$	86	$4.76169394977e-34$
21	$6.093518e-167$	54	$2.256444648e-90$	87	$1.40050264399e-32$
22	$2.711677e-164$	55	$2.3558089495e-88$	88	$3.93774707023e-31$
23	$1.138707e-161$	56	$2.368590614e-86$	89	$1.05777224840e-29$
24	$4.520178e-159$	57	$2.2934945474e-84$	90	$2.71295150626e-28$
25	$1.6988865e-156$	58	$2.1388415121e-82$	91	$6.63907833056e-27$
26	$6.05445e-154$	59	$1.9210577630e-80$	92	$1.54911794729e-25$
27	$2.0486584e-151$	60	$1.66181757464e-78$	93	$3.44390394051e-24$
28	$6.589966e-149$	61	$1.3845226980e-76$	94	$7.28902479104e-23$
29	$2.0174718e-146$	62	$1.11090242874e-74$	95	$1.4675195821e-21$
30	$5.884306e-144$	63	$8.5839818481e-73$	96	$2.80814799362e-20$
31	$1.6366820e-141$	64	$6.38718219124e-71$	97	$5.10250621963e-19$
32	$4.345107e-139$	65	$4.57615509406e-69$	98	$8.79548303360e-18$
32	$4.345107e-139$	65	$4.57615509406e-69$	98	$8.79548303360e-18$

r	$c_r(12)$	r	$c_r(12)$	r	$c_r(12)$
99	$1.43685110544e-16$	115	$8.6990644e-1$	131	$1.41e7$
100	$2.22218436508e-15$	116	4.62298136	132	$-1.081e8$
101	$3.24998071837e-14$	117	$2.26812786e1$	133	$-4.56e8$
102	$4.4895676128e-13$	118	$1.02629298e2$	134	$-6.1e8$
103	$5.85079932579e-12$	119	$4.2781349e2$	135	$1.0e9$
104	$7.18370974878e-11$	120	$1.639916e3$	136	$5.5e9$
105	$8.2987379093e-10$	121	$5.759048e3$	137	$7.2e9$
106	$9.00703391924e-9$	122	$1.8391183e4$	138	$-6.6e9$
107	$9.1707380762e-8$	123	$5.26979e4$	139	$-3.3e10$
108	$8.7457189835e-7$	124	$1.32667e5$	140	$-3.72e10$
109	$7.7990666196e-6$	125	$2.85830e5$	141	$5.e9$
110	$6.4924284455e-5$	126	$5.2381e5$	142	$4.2e10$
111	$5.03652589379e-4$	127	$9.326e5$	143	$2.e10$
112	$3.6345151422e-3$	128	$2.342e6$	144	$1.9e9$
113	$2.435514699e-2$	129	$7.740e6$		
114	$1.512959880e-1$	130	$1.94e7$		

Table 3.8: Coefficients for $k = 12$ truncating at $P = 12979$

r	$c_r(13)$	r	$c_r(13)$	r	$c_r(13)$
0	$2.578e-282$	36	$1.290e-174$	72	$1.095238e-94$
1	$4.32e-278$	37	$3.959e-172$	73	$9.9295e-93$
2	$3.59e-274$	38	$1.168e-169$	74	$8.7199e-91$
3	$1.97e-270$	39	$3.3179e-167$	75	$7.417268e-89$
4	$8.05e-267$	40	$9.068e-165$	76	$6.110743e-87$
5	$2.60e-263$	41	$2.386e-162$	77	$4.875691e-85$
6	$6.93e-260$	42	$6.053e-160$	78	$3.767364e-83$
7	$1.569e-256$	43	$1.4801e-157$	79	$2.818777e-81$
8	$3.077e-253$	44	$3.4e-155$	80	$2.0420376e-79$
9	$5.3e-250$	45	$7.93e-153$	81	$1.4321877e-77$
10	$8.16e-247$	46	$1.74e-150$	82	$9.7234031e-76$
11	$1.129e-243$	47	$3.6945e-148$	83	$6.3894419e-74$
12	$1.417e-240$	48	$7.563e-146$	84	$4.063248e-72$
13	$1.625e-237$	49	$1.4956e-143$	85	$2.5002543e-70$
14	$1.713e-234$	50	$2.858e-141$	86	$1.48841933e-68$
15	$1.668e-231$	51	$5.2797e-139$	87	$8.5708030e-67$
16	$1.506e-228$	52	$9.429e-137$	88	$4.77301310e-65$
17	$1.267e-225$	53	$1.6284e-134$	89	$2.57011932e-63$
18	$9.96e-223$	54	$2.7202e-132$	90	$1.33786677e-61$
19	$7.337e-220$	55	$4.3960e-130$	91	$6.73093425e-60$
20	$5.079e-217$	56	$6.8740e-128$	92	$3.27219499e-58$
21	$3.312e-214$	57	$1.04024e-125$	93	$1.536728414e-56$
22	$2.039e-211$	58	$1.5236e-123$	94	$6.97005361e-55$
23	$1.187e-208$	59	$2.1604e-121$	95	$3.05236944e-53$
24	$6.55e-206$	60	$2.96582e-119$	96	$1.290252764e-51$
25	$3.434e-203$	61	$3.94221e-117$	97	$5.262808653e-50$
26	$1.710e-200$	62	$5.07426e-115$	98	$2.070747797e-48$
27	$8.112e-198$	63	$6.325e-113$	99	$7.8570339e-47$
28	$3.666e-195$	64	$7.6365e-111$	100	$2.873821165e-45$
29	$1.581e-192$	65	$8.929e-109$	101	$1.012910598e-43$
30	$6.518e-190$	66	$1.01146e-106$	102	$3.43894916e-42$
31	$2.569e-187$	67	$1.10976e-104$	103	$1.124213470e-40$
32	$9.69e-185$	68	$1.179500e-102$	104	$3.537179841e-39$
33	$3.503e-182$	69	$1.21438e-100$	105	$1.070678922e-37$
34	$1.2144e-179$	70	$1.211196e-98$	106	$3.11641068e-36$
35	$4.039e-177$	71	$1.17021e-96$	107	$8.71833920e-35$

r	$c_r(13)$	r	$c_r(13)$	r	$c_r(13)$
108	$2.34302639e-33$	129	$6.90733979e-8$	150	$3.74e6$
109	$6.045825051e-32$	130	$6.3526970e-7$	151	$1.180e7$
110	$1.497025084e-30$	131	$5.4964367e-6$	152	$3.57e7$
111	$3.5550591307e-29$	132	$4.4673281e-5$	153	$6.0e7$
112	$8.09182488e-28$	133	$3.40576381e-4$	154	$-8.9e7$
113	$1.764223585e-26$	134	$2.431847316e-3$	155	$-8.77e8$
114	$3.681978591e-25$	135	$1.62398053e-2$	156	$-2.07e9$
115	$7.35070728e-24$	136	$1.01285494e-1$	157	$1.2e9$
116	$1.402759072e-22$	137	$5.8923590e-1$	158	$2.144e10$
117	$2.5569037718e-21$	138	3.1940171	159	$5.1e10$
118	$4.448144934e-20$	139	$1.6117968e1$	160	$-9.e9$
119	$7.37932244e-19$	140	$7.566471e1$	161	$-3.5e11$
120	$1.166406237e-17$	141	$3.301742e2$	162	$-8.6e11$
121	$1.7550272264e-16$	142	$1.337418e3$	163	$-5.e11$
122	$2.51133488e-15$	143	$5.01443e3$	164	$1.6e12$
123	$3.414117120e-14$	144	$1.730246e4$	165	$3.8e12$
124	$4.40504845e-13$	145	$5.4381e4$	166	$2.5e12$
125	$5.38821792e-12$	146	$1.5319e5$	167	$-8.e11$
126	$6.241124024e-11$	147	$3.79020e5$	168	$-1.65e12$
127	$6.83724633e-10$	148	$8.14e5$	169	$-3.7e11$
128	$7.07544905e-9$	149	$1.614e6$		

Table 3.9: Coefficients for $k = 13$ truncating at $P = 1699$

Chapter 4

The Tanh-Sinh Quadrature Scheme

For numerical computation of the integral moments of $\zeta(s)$, the tanh-sinh quadrature scheme [Ba, BLJ] was used to accurately estimate each integral value between the consecutive zeros. The tanh-sinh quadrature scheme is based on the Euler-Maclaurin summation formula. It uses the fact that for a certain bell-shaped function, approximating its integral by a simple step-function summation is remarkably accurate.

The Euler-Maclaurin summation formula can be stated as follows [A2]. Let $m \geq 0$ and $n \geq 1$ be integers, and define $h = \frac{b-a}{n}$ and $x_j = a + jh$ for $0 \leq j \leq n$. Further assume that the function $f(x)$ is at least $(2m+2)$-times continuously differentiable on [a,b]. Then

$$\int_a^b f(x)dx = h\sum_{j=0}^{n} f(x_j) - \frac{h}{2}(f(a)+f(b)) - \sum_{i=1}^{m} \frac{h^{2i}B_{2i}}{(2i)!}\left(f^{(2i-1)}(b) - f^{(2i-1)}(a)\right) - E,$$

where B_{2i} denote the Bernoulli numbers, and

$$E = \frac{h^{2m+2}(b-a)B_{2m+2}f^{2m+2}(\xi)}{(2m+2)!}$$

for some $\xi \in (a,b)$.

In the circumstance where the function $f(x)$ and all of its derivatives are zero at the endpoints a and b (as in a smooth, bell-shaped function), we see that the second and third terms of the Euler-Maclaurin summation formula are zero. Therefore, in this case the error of a simple step function approximation to the integral, with interval h, is simply E. Then, it follows that the error goes to zero more rapidly than any power of h, since E is less than a constant times $h^{2m+2}/(2m+2)!$ for all m [BLJ]. For a function defined on $(-\infty,\infty)$, we can still apply the Euler-Maclaurin summation formula to the resulting doubly infinite sum approximation, as long as the function satisfies the same conditions as before. It must be that the function and all of its derivatives tend to zero rapidly for large positive and negative arguments.

The basic idea of the tanh-sinh quadrature scheme is to transform the integral of $f(x)$ on the interval $[-1, 1]$ to an integral on $(-\infty, \infty)$ using the change of variable $x = g(t)$, where $g(t) = \tanh\left(\frac{\pi}{2}\sinh t\right)$, and apply the Euler-Maclaurin summation formula. Note that $g(x)$ is a monotonic, infinitely differentiable function with the property that $g(x) \to 1$ as $x \to \infty$ and $g(x) \to -1$ as $x \to -\infty$, and all derivatives tend rapidly to zero for large positive and negative arguments. Thus one can write, for $h > 0$,

$$\begin{aligned}\int_{-1}^{1} f(x)dx \quad &= \int_{-\infty}^{\infty} f(g(t))g'(t)dt \\ &= h \sum_{j=-\infty}^{\infty} w_j f(x_j) + E, \\ &\approx h \sum_{j=-N}^{N} w_j f(x_j)\end{aligned}$$

where $x_j = g(hj)$ and $w_j = g'(hj)$, and where N is chosen large enough that $|w_j f(x_j)| < \epsilon$ for $|j| > N$. Here $\epsilon = 10^{-p}$, where p is the numeric precision level in digits. It is also important to note that if $g'(t)$ and its derivatives tend to zero fast enough for large t, positive and negative, then even in cases where $f(x)$ has an infinite derivative or an integrable singularity at one or both end points, this method still works. Meaning the resulting integrand $f(g(t))g'(t)$ will be a smooth bell-shaped function for which the Euler-Maclaurin summation argument applies and the error E of the approximation decreases faster than any power of h.

By the choice of $g(t) = \tanh(\pi/2 \cdot \sinh t)$ and $g'(t) = \pi/2 \cdot \sinh t / \cosh^2(\pi/2 \cdot \sinh t)$, the convergence to zero is very rapid. Hence the doubly infinite sum in the formula above can be approximated by a finite sum provided that we take a reasonable care to insure that the truncated tails are insignificant.

Note that the abscissas x_j and w_j can be computed for a given h, and then used for numerous problems. Also, whenever the given interval of integration is other than $[-1, 1]$, we must perform a linear scaling on the pre-computed abscissas during the quadrature computation.

Chapter 5

Checking the Moment Polynomial Conjectures

Regarding the remainder term, only for $k = 1$ it is conjectured to be $O\left(T^{\frac{1}{4}+\varepsilon}\right)$ but for other cases $k = 2, 3, 4\ldots$, the remainder term is conjectured to be $O\left(T^{\frac{1}{2}+\varepsilon}\right)$. In order to study the remainder terms, (data(T)-conjecture(T))/conjecture(T) was plotted against T as shown in Figure 5.1, where data(T) represents the values of the moment integral at height T and conjecture(T) represents the integral of the conjectured polynomial at height T.

$$\text{data(T)} = \int_0^{\text{T}} |\zeta(1/2 + it)|^{2k} dt$$

where the numerical computation of the integral was done utilizing the tanh-sinh quadrature scheme described in the previous chapter.

$$\text{conjecture(T)} = \int_0^{\text{T}} P_k(\log \frac{t}{2\pi}) dt$$

where

$$P_k(x) = c_0(k)x^{k^2} + c_1(k)x^{k^2-1} + c_2(k)x^{k^2-2} + \cdots + c_{k^2}(k) \tag{5.1}$$

and the numerical values of the coefficients obtained in Chapter 3 were used here.

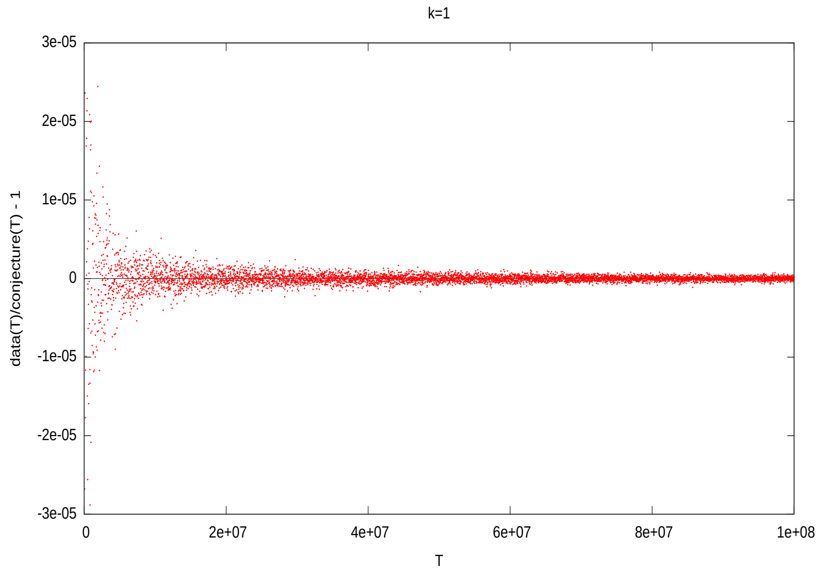
For all cases $k = 1, 2, 3\ldots, 13$, the plots seem to agree with the conjecture. We also present some of the data numerically in Table 5.3. To illustrate the values of data(T) and conjecture(T) by itself, Table 5.1 was constructed to show the two values at the largest T value we have computed.

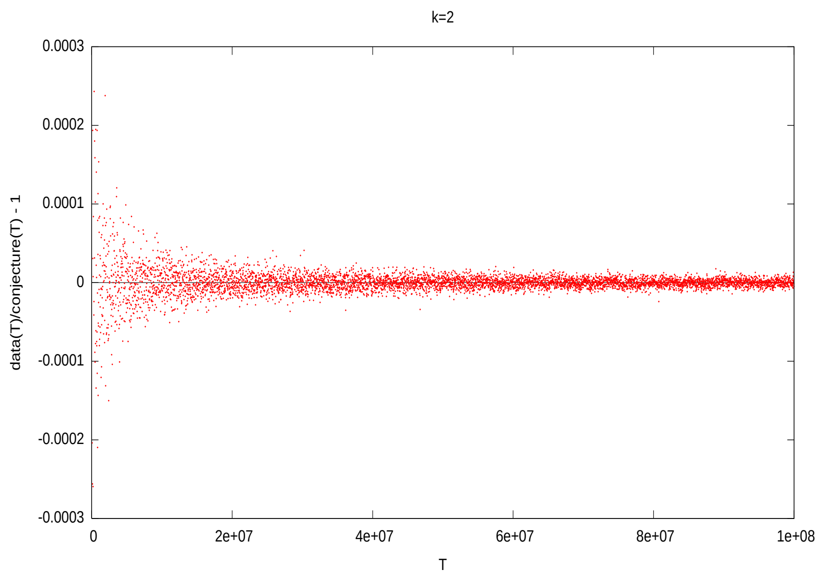
Also to have a better understanding of the distribution of the remainder term we have constructed histograms as shown in Figure 5.2. We made a guess that the distribution of the remainder term behaves similarly to conjecture(T)$\wedge$(1/2) and took the data of (data(T)-conjecture(T))/(conjecture(T)$\wedge$(1/2)) up to T= $1.0e8$.

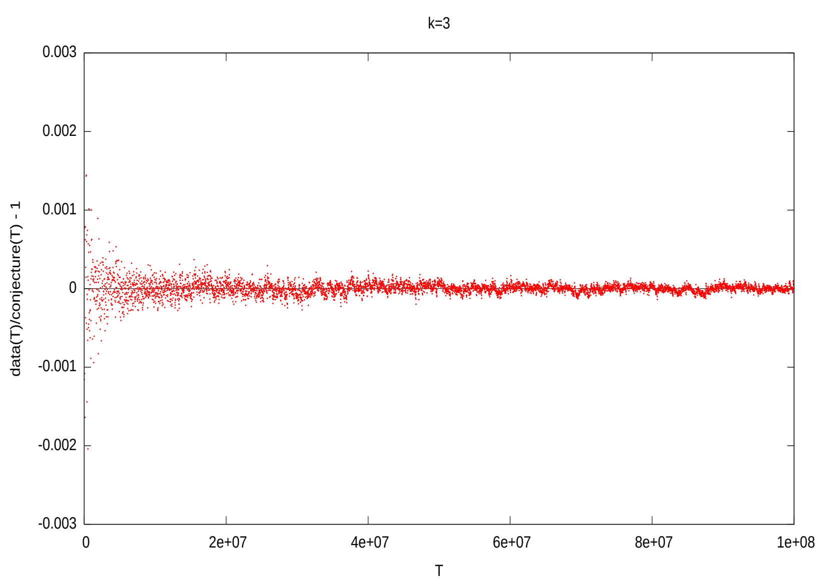
Then the data was normalized using its mean and variance shown in Table 5.2 and histograms were made from it. Our goal is to obtain a plot which resembles a normal distribution, because that will support our guess that the remainder term is highly related to conjecture(T)∧(1/2). As it can be seen in Figure 5.2 only the curve for $k = 2$ is strikingly similar to a normal distribution. For $k = 1$ and 3 to say that the curves resemble a normal distribution is an overstatement. Afterwards k magnifies the peaks in the moment integral too greatly that it distorts the smooth behavior, causing it to have large spikes. When this occurs the histogram will not have an accurate representation due to these rapid and sharp increase that occurs at the peaks.

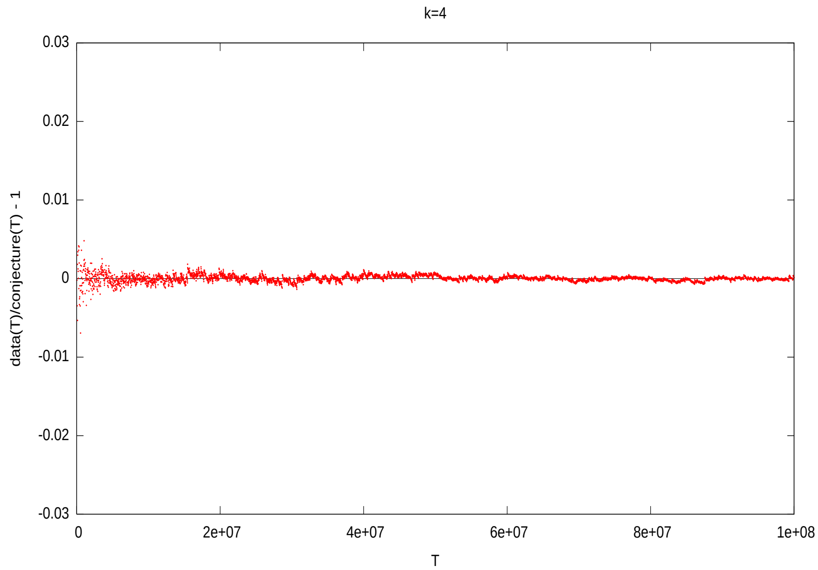
k	data(T)	conjecture(T)
1	1673723690.436009	1673723498.495434
2	637388343406.7327	637389923500.1656
3	804585314342079.9	804581403343880.5
4	$1.737648069554459e+18$	$1.737451257590349e+18$
5	$5.083767881905691e+21$	$5.08166450276854e+21$
6	$1.815301993690427e+25$	$1.813639687207906e+25$
7	$7.480512969084191e+28$	$7.468884125919685e+28$
8	$3.438511728532958e+32$	$3.430903271271959e+32$
9	$1.723885779517702e+36$	$1.719184656621986e+36$
10	$9.278504860050357e+39$	$9.251733004628787e+39$
11	$5.299108641951753e+43$	$5.286307154226168e+43$
12	$3.182548192736658e+47$	$3.179454736399122e+47$
13	$1.995624638013327e+51$	$1.999377457683425e+51$

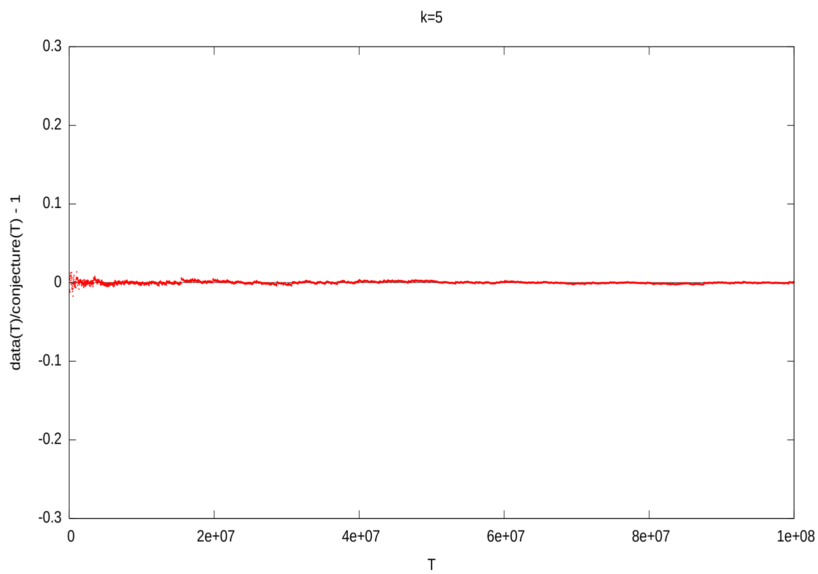
Table 5.1: The values of data(T) and conjecture(T) at T=100000000.642926, the first zero after $1.0e8$. The relative errors for various T, including the value of T for this table, are given in Table 5.3.

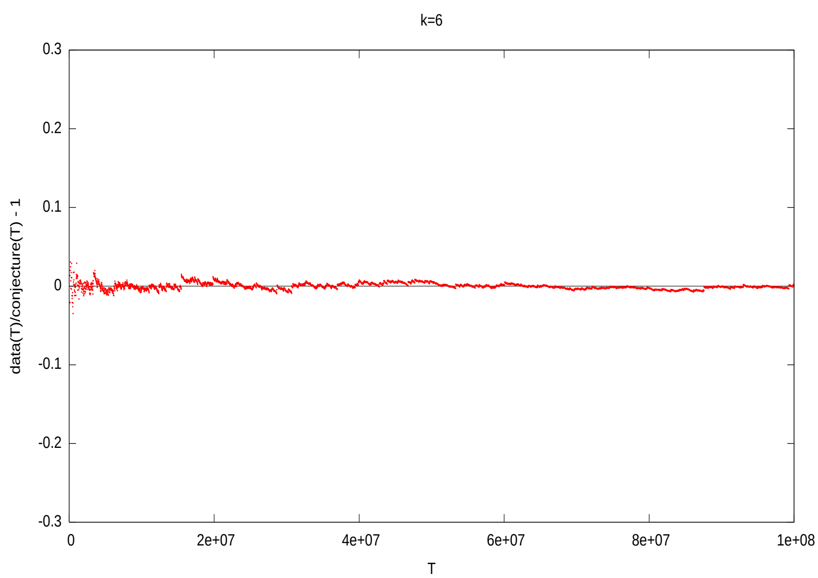

k=1
data(T)/conjecture(T) - 1
3e-05
2e-05
1e-05
0
-1e-05
-2e-05
-3e-05
0
2e+07
4e+07
6e+07
8e+07
1e+08
T

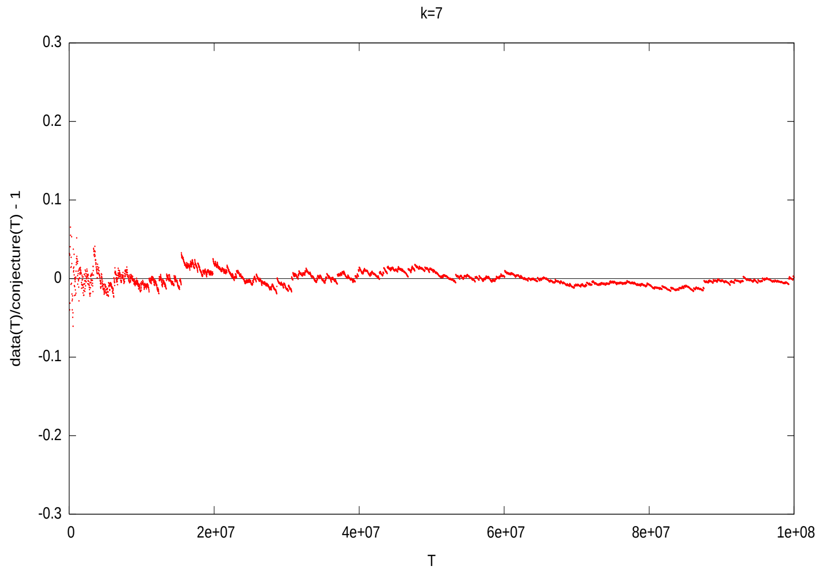

k=2
data(T)/conjecture(T) - 1
0.0003
0.0002
0.0001
0
-0.0001
-0.0002
-0.0003
0
2e+07
4e+07
6e+07
8e+07
1e+08
T

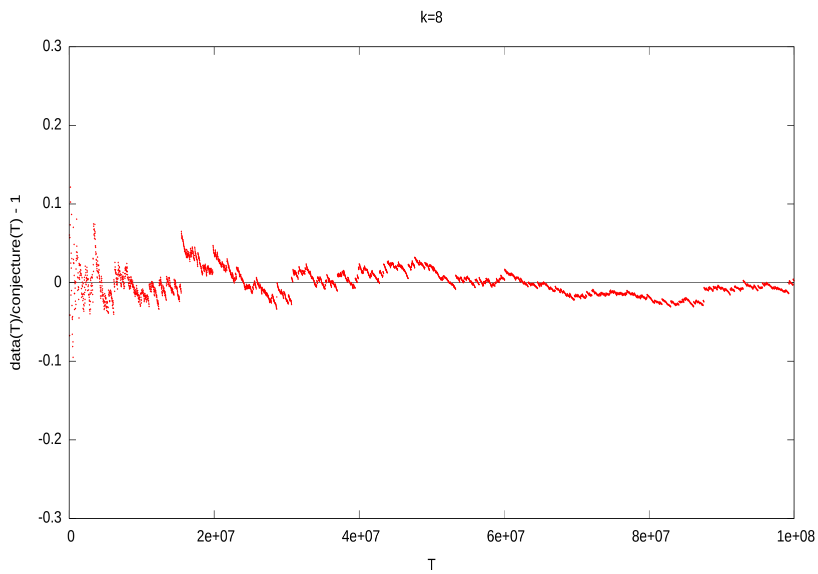

k=3
data(T)/conjecture(T) - 1
0.003
0.002
0.001
0
-0.001
-0.002
-0.003
0
2e+07
4e+07
6e+07
8e+07
1e+08
T

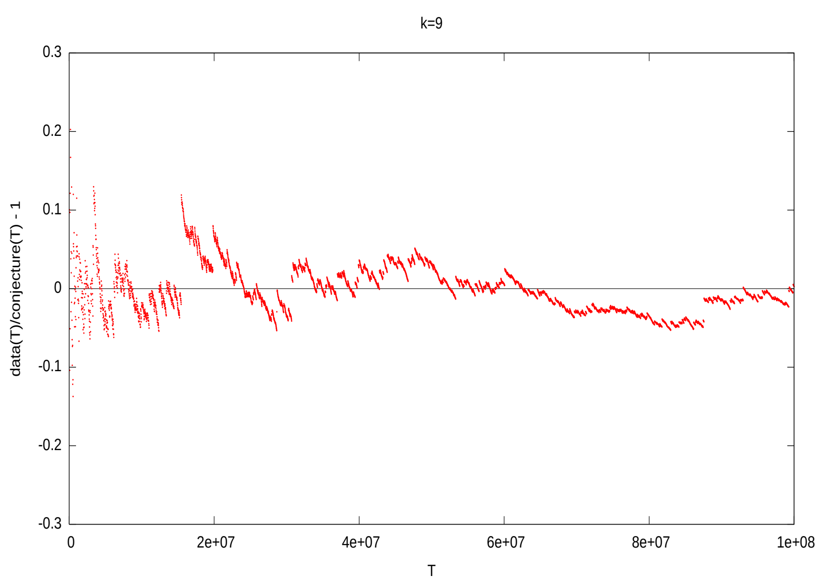

k=4
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
data(T)/conjecture(T) - 1
0
2e+07
4e+07
6e+07
8e+07
1e+08
T

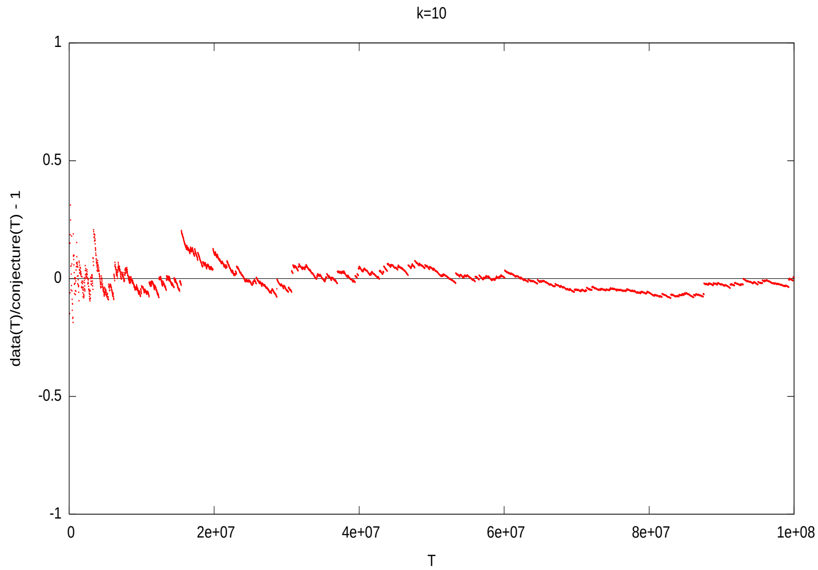

k=5
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
data(T)/conjecture(T) - 1
0
2e+07
4e+07
6e+07
8e+07
1e+08
T


k=6
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
data(T)/conjecture(T) - 1
0
2e+07
4e+07
6e+07
8e+07
1e+08
T

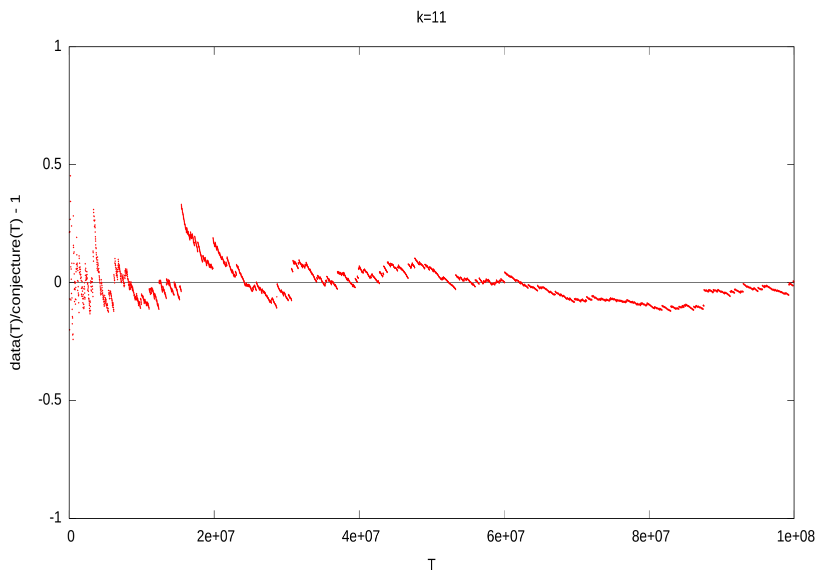

k=7
data(T)/conjecture(T) - 1
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0
2e+07
4e+07
6e+07
8e+07
1e+08
T


k=8
data(T)/conjecture(T) - 1
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0
2e+07
4e+07
6e+07
8e+07
1e+08
T

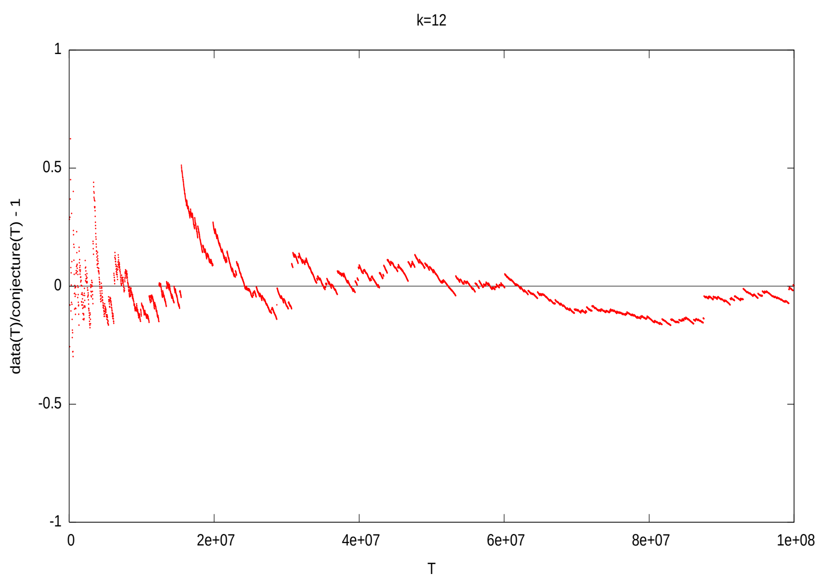

k=9
data(T)/conjecture(T) - 1
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0
2e+07
4e+07
6e+07
8e+07
1e+08
T



k=10
data(T)/conjecture(T) - 1
1
0.5
0
-0.5
-1
0
2e+07
4e+07
6e+07
8e+07
1e+08
T



k=11
data(T)/conjecture(T) - 1
1
0.5
0
-0.5
-1
0
2e+07
4e+07
6e+07
8e+07
1e+08
T



k=12
data(T)/conjecture(T) - 1
1
0.5
0
-0.5
-1
0
2e+07
4e+07
6e+07
8e+07
1e+08
T

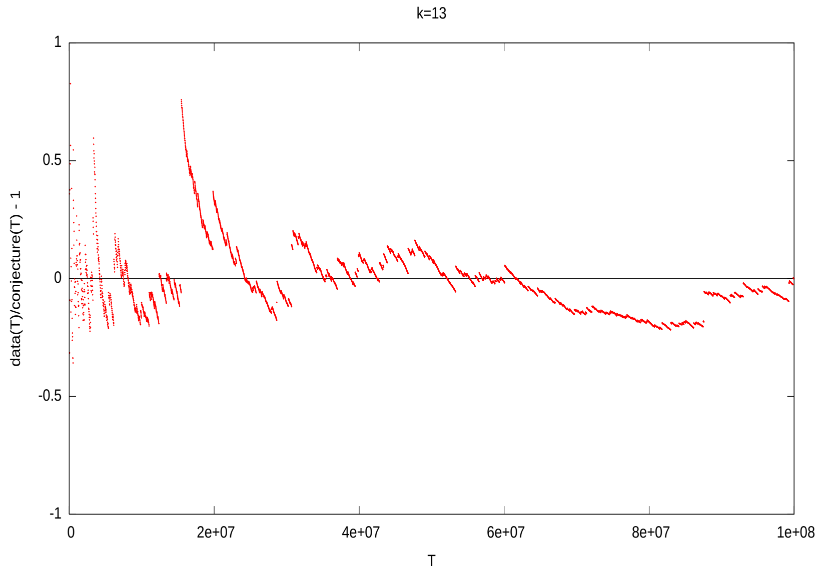
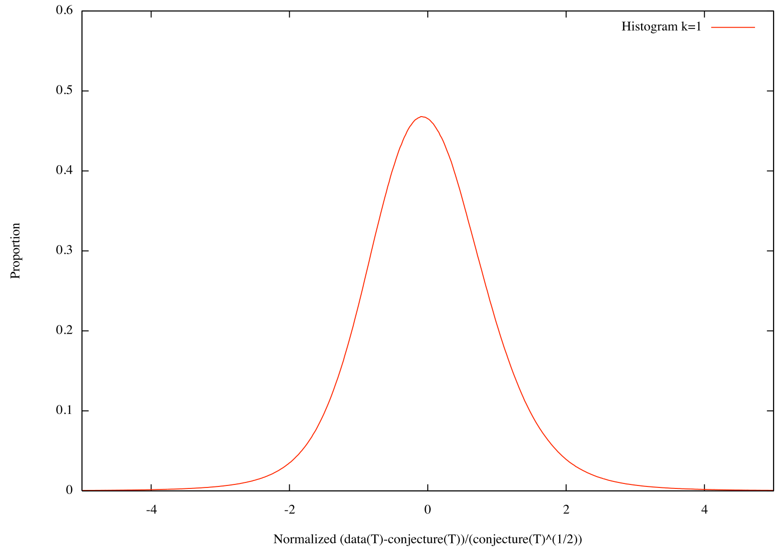
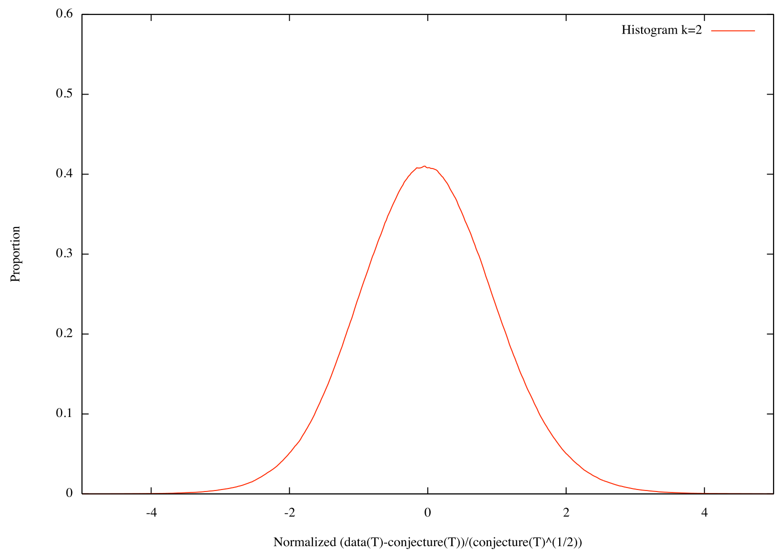


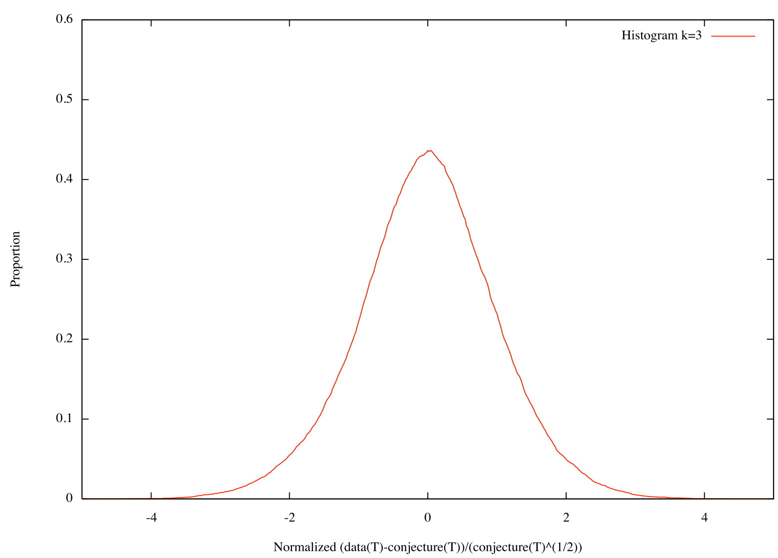
Figure 5.1: Plots of (data(T)-conjecture(T))/conjecture(T) against T for $k = 1, 2 \ldots, 13$

k	mean	variance
1	$5.381814e-05$	$1.923065e-04$
2	$2.084305e-03$	$1.317431e+01$
3	$3.813227e+01$	$8.121071e+05$
4	$-3.307175e+03$	$4.526230e+10$
5	$-5.823629e+06$	$1.821122e+15$
6	$-1.736528e+09$	$5.038794e+19$
7	$-3.337175e+11$	$1.042711e+24$
8	$-5.144205e+13$	$1.749392e+28$
9	$-6.952964e+15$	$2.518001e+32$
10	$-8.620122e+17$	$3.231620e+36$
11	$-1.005687e+20$	$3.799453e+40$
12	$-1.121710e+22$	$4.173312e+44$
13	$-1.208706e+24$	$4.345976e+48$

Table 5.2: The mean and variance of (data(T)-conjecture(T))/(conjecture(T)∧(1/2)) up to T=100000000.642926


Histogram k=1
Proportion
Normalized (data(T)-conjecture(T))/(conjecture(T)^(1/2))


Histogram k=2
Proportion
Normalized (data(T)-conjecture(T))/(conjecture(T)^(1/2))


Histogram k=3
Proportion
Normalized (data(T)-conjecture(T))/(conjecture(T)^(1/2))

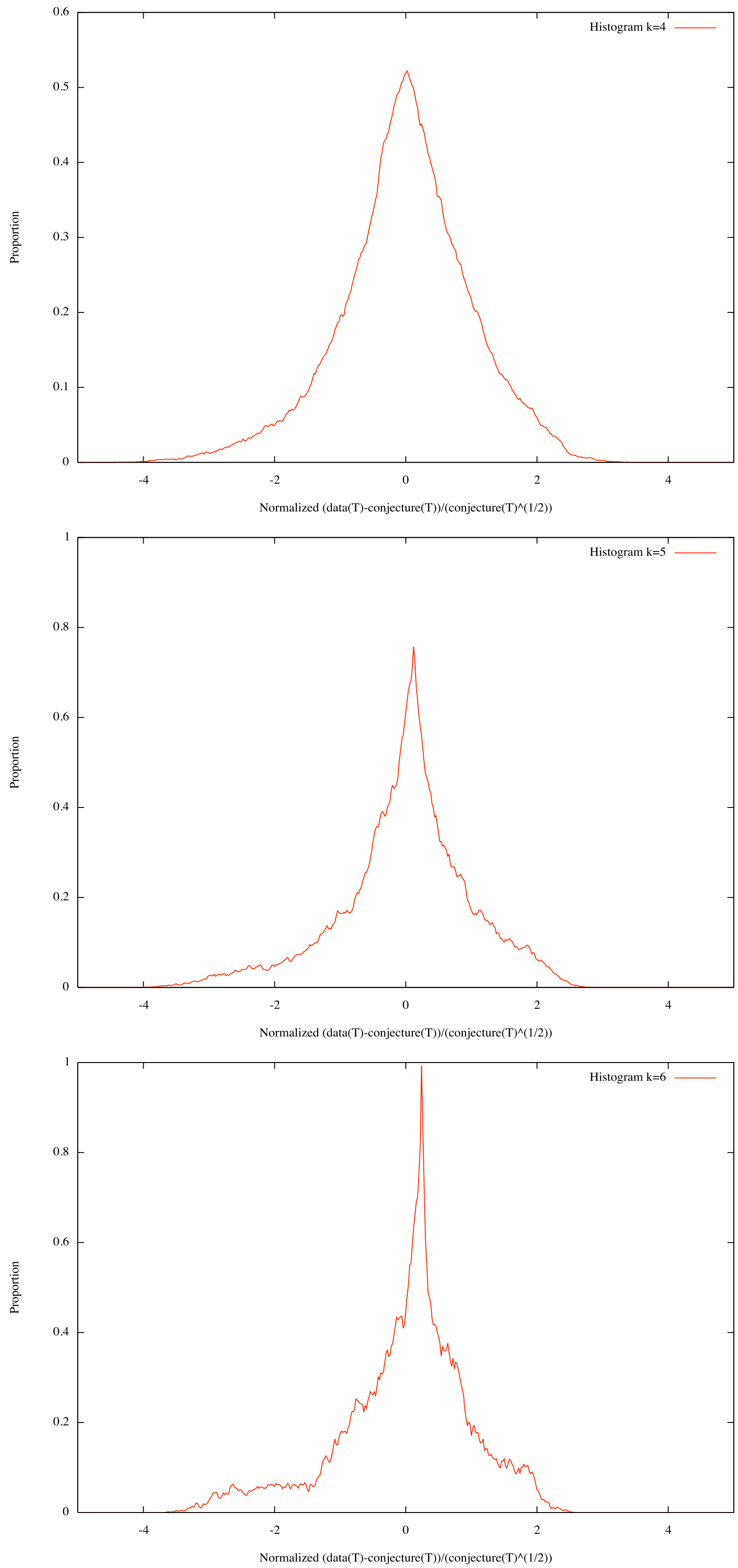


Figure 5.2: Histograms of the normalized (data(T)-conjecture(T))/(conjecture(T)∧(1/2)) for $k = 1, 2 \ldots, 6$

T	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$5.824602e+06$	$-3.478238e-07$	$1.629107e-05$	$4.397657e-05$	$-2.016111e-04$	$-1.312527e-03$
$1.108880e+07$	$-8.351679e-07$	$1.934167e-06$	$3.525066e-06$	$1.660911e-05$	$1.167072e-05$
$1.617674e+07$	$-4.828676e-07$	$-1.044762e-05$	$-3.206090e-05$	$1.292257e-04$	$1.216895e-03$
$2.115640e+07$	$4.193915e-07$	$7.823787e-06$	$4.708910e-05$	$4.139223e-04$	$1.864186e-03$
$2.605842e+07$	$1.548026e-06$	$4.829133e-06$	$2.989530e-05$	$1.491032e-04$	$3.437487e-04$
$3.090024e+07$	$9.325381e-08$	$-5.266991e-06$	$-2.255270e-06$	$4.459891e-05$	$3.902401e-04$
$3.569308e+07$	$-1.607737e-07$	$-1.255175e-06$	$5.263464e-05$	$3.360942e-04$	$1.018947e-03$
$4.044475e+07$	$-1.750449e-07$	$-5.055004e-07$	$4.798878e-05$	$4.323432e-04$	$1.652881e-03$
$4.516097e+07$	$3.310080e-07$	$6.997524e-06$	$5.781477e-05$	$4.514431e-04$	$1.867094e-03$
$4.984614e+07$	$-4.395856e-07$	$-5.613849e-06$	$4.823156e-05$	$5.321545e-04$	$2.137467e-03$
$5.450370e+07$	$5.078205e-07$	$5.231587e-06$	$2.962847e-05$	$1.702679e-04$	$6.012200e-04$
$5.913645e+07$	$-3.232702e-07$	$2.477615e-06$	$6.916365e-06$	$-2.452590e-05$	$-4.836764e-05$
$6.374667e+07$	$2.010412e-07$	$8.992210e-06$	$6.763958e-05$	$2.230587e-04$	$4.482592e-04$
$6.833630e+07$	$-3.979599e-08$	$8.428721e-06$	$5.563642e-05$	$4.310416e-05$	$-4.366804e-04$
$7.290697e+07$	$5.381202e-07$	$5.713751e-06$	$-5.371607e-06$	$-1.814539e-04$	$-9.388189e-04$
$7.746007e+07$	$-2.137517e-07$	$-4.624919e-06$	$-2.214055e-05$	$-1.674957e-05$	$-2.773094e-04$
$8.199684e+07$	$1.904663e-07$	$1.830963e-06$	$1.767216e-05$	$-6.785246e-05$	$-9.275123e-04$
$8.651832e+07$	$6.473452e-07$	$9.264351e-06$	$1.415202e-05$	$-2.052971e-04$	$-1.331397e-03$
$9.102546e+07$	$-1.421490e-07$	$5.575624e-06$	$1.489644e-05$	$-1.041422e-04$	$-7.597999e-04$
$9.551911e+07$	$1.991451e-07$	$5.461281e-06$	$1.377467e-05$	$-4.686321e-05$	$-3.772274e-04$
$1.000000e+08$	$1.153710e-07$	$-2.483873e-06$	$4.854727e-06$	$1.132690e-04$	$4.139071e-04$

T	$k=6$	$k=7$	$k=8$	$k=9$
$5.824602e+06$	$-3.951702e-03$	$-8.763773e-03$	$-1.636906e-02$	$-2.732799e-02$
$1.108880e+07$	$-2.986345e-04$	$-1.559033e-03$	$-4.645972e-03$	$-1.042512e-02$
$1.617674e+07$	$4.850759e-03$	$1.389342e-02$	$3.273049e-02$	$6.732212e-02$
$2.115640e+07$	$5.472872e-03$	$1.250979e-02$	$2.438694e-02$	$4.256763e-02$
$2.605842e+07$	$3.454207e-04$	$-3.066331e-04$	$-2.087815e-03$	$-5.373911e-03$
$3.090024e+07$	$1.811002e-03$	$5.740317e-03$	$1.418783e-02$	$2.947061e-02$
$3.569308e+07$	$2.191225e-03$	$3.908234e-03$	$6.238822e-03$	$9.237691e-03$
$4.044475e+07$	$4.211924e-03$	$8.543435e-03$	$1.499859e-02$	$2.377759e-02$
$4.516097e+07$	$5.058989e-03$	$1.061502e-02$	$1.884388e-02$	$2.968839e-02$
$4.984614e+07$	$5.623321e-03$	$1.162174e-02$	$2.048685e-02$	$3.214507e-02$
$5.450370e+07$	$1.508575e-03$	$3.086940e-03$	$5.502575e-03$	$8.790730e-03$
$5.913645e+07$	$1.138440e-04$	$6.595935e-04$	$1.617295e-03$	$2.686279e-03$
$6.374667e+07$	$5.924507e-04$	$3.528175e-04$	$-7.469733e-04$	$-3.364890e-03$
$6.833630e+07$	$-2.109233e-03$	$-5.892209e-03$	$-1.274981e-02$	$-2.358387e-02$
$7.290697e+07$	$-3.056930e-03$	$-7.588428e-03$	$-1.567410e-02$	$-2.837269e-02$
$7.746007e+07$	$-1.773504e-03$	$-6.047240e-03$	$-1.487071e-02$	$-2.988670e-02$
$8.199684e+07$	$-3.844939e-03$	$-1.043649e-02$	$-2.233577e-02$	$-4.092805e-02$
$8.651832e+07$	$-4.512358e-03$	$-1.118889e-02$	$-2.285757e-02$	$-4.082988e-02$
$9.102546e+07$	$-2.622593e-03$	$-6.486048e-03$	$-1.310675e-02$	$-2.306103e-02$
$9.551911e+07$	$-1.315361e-03$	$-3.277490e-03$	$-6.691913e-03$	$-1.194642e-02$
$1.000000e+08$	$9.165489e-04$	$1.556961e-03$	$2.217613e-03$	$2.734494e-03$

T	$k = 10$	$k = 11$	$k = 12$	$k = 13$
$5.824602e + 06$	$-4.207628e - 02$	$-6.087060e - 02$	$-8.376764e - 02$	$-1.106364e - 01$
$1.108880e + 07$	$-1.953990e - 02$	$-3.230486e - 02$	$-4.870444e - 02$	$-6.846307e - 02$
$1.617674e + 07$	$1.250140e - 01$	$2.141716e - 01$	$3.437116e - 01$	$5.225888e - 01$
$2.115640e + 07$	$6.841721e - 02$	$1.030350e - 01$	$1.471059e - 01$	$2.007898e - 01$
$2.605842e + 07$	$-1.043425e - 02$	$-1.748498e - 02$	$-2.674759e - 02$	$-3.848247e - 02$
$3.090024e + 07$	$5.385269e - 02$	$8.919783e - 02$	$1.367045e - 01$	$1.967504e - 01$
$3.569308e + 07$	$1.286702e - 02$	$1.692549e - 02$	$2.101973e - 02$	$2.458269e - 02$
$4.044475e + 07$	$3.482081e - 02$	$4.771787e - 02$	$6.167640e - 02$	$7.556159e - 02$
$4.516097e + 07$	$4.265839e - 02$	$5.681760e - 02$	$7.084172e - 02$	$8.313929e - 02$
$4.984614e + 07$	$4.599412e - 02$	$6.090126e - 02$	$7.530660e - 02$	$8.740133e - 02$
$5.450370e + 07$	$1.274722e - 02$	$1.687317e - 02$	$2.039266e - 02$	$2.233094e - 02$
$5.913645e + 07$	$3.176450e - 03$	$2.063550e - 03$	$-1.870181e - 03$	$-9.858185e - 03$
$6.374667e + 07$	$-8.310961e - 03$	$-1.647320e - 02$	$-2.871481e - 02$	$-4.576758e - 02$
$6.833630e + 07$	$-3.915427e - 02$	$-6.001588e - 02$	$-8.647148e - 02$	$-1.185457e - 01$
$7.290697e + 07$	$-4.653625e - 02$	$-7.072820e - 02$	$-1.011779e - 01$	$-1.377681e - 01$
$7.746007e + 07$	$-5.232533e - 02$	$-8.283188e - 02$	$-1.214088e - 01$	$-1.674548e - 01$
$8.199684e + 07$	$-6.714978e - 02$	$-1.013680e - 01$	$-1.433481e - 01$	$-1.923032e - 01$
$8.651832e + 07$	$-6.603103e - 02$	$-9.887172e - 02$	$-1.392058e - 01$	$-1.863661e - 01$
$9.102546e + 07$	$-3.663913e - 02$	$-5.379758e - 02$	$-7.417733e - 02$	$-9.717834e - 02$
$9.551911e + 07$	$-1.934800e - 02$	$-2.910086e - 02$	$-4.130772e - 02$	$-5.599164e - 02$
$1.000000e + 08$	$2.893698e - 03$	$2.421616e - 03$	$9.729352e - 04$	$-1.877012e - 03$

Table 5.3: (data(T)-conjecture(T))/conjecture(T) for various T for each k

Chapter 6

Classic Results

In this chapter, we present the proof for the approximate functional equation, one of the key tools regarding moments of the zeta function. We then apply it to the second moment to prove the basic result by Hardy and Littlewood. The main source of the proofs presented here is [T].

6.1 Approximate Functional Equation

The approximate functional equation

$$\zeta(s) = \sum_{n\le x} \frac{1}{n^s} + \chi(s) \sum_{n\le y} \frac{1}{n^{1-s}} + O(x^{-\sigma}) + O(|t|^{\frac{1}{2}-\sigma} y^{\sigma-1})$$

holds for $0 \le \sigma \le 1$ where $s = \sigma + it$, $t = 2\pi xy$, $x > h > 0$ and $y > h > 0$. [HL3, HL4, HL5, Si] It was developed to prove numerous theoretical results regarding integral moments of $\zeta(s)$ including the classical result (1.1) by Hardy and Littlewood.

In order to prove the approximate functional equation, first we want to write $\zeta(s)$ as

$$\zeta(s) \quad = \sum_{n=1}^{m} \frac{1}{n^s} + \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1} e^{-mx}}{e^x - 1} dx$$

for $\sigma > 1$.

To do this look at the integral term in the expression and sum the geometric series. Then since each term is

$$\int_0^\infty e^{-nx} x^{s-1} dx = \frac{1}{n^s} \Gamma(s)$$

and the sum is over n from $m + 1$, we recover $\zeta(s)$.

Then transform the integral into a loop-integral C, which consists of γ_1 the real axis from ∞ to $\rho (0 < \rho < 2\pi)$, γ_2 the circle $|z| = \rho$ and γ_3 the real axis from ρ to ∞. The result is

$$\zeta(s) = \sum_{n=1}^{m} \frac{1}{n^s} + \frac{e^{-i\pi s}\Gamma(1-s)}{2\pi i} \int_C \frac{z^{s-1}e^{-mz}}{e^z - 1} dz \tag{6.1}$$

where C excludes the zeros of $e^z - 1$ other than $z = 0$. The proof of this is presented below in the following section. Also note that this equation is valid for all values of s except for positive integers because $\Gamma(1-s)$ has poles at those points.

6.2 Obtaining $\frac{e^{-i\pi s}\Gamma(1-s)}{2\pi i} \int_C \frac{z^{s-1}e^{-mz}}{e^z-1} dz$

On γ_2, the circle $|z| = \rho$, each portion of the integral can be bounded as follows:

$$\begin{aligned} |z^{s-1}| &= |e^{(s-1)\log z}| \\ &\leq |z|^{\sigma-1} e^{2\pi|t|} \end{aligned} \tag{6.2}$$

$$|e^{-mz}| \leq e^{m\rho} \tag{6.3}$$

If $f : X \to \mathbb{C}$, f is continuous and X is compact then $|f(z)|$ has a minimum and a maximum for $z \in X$. Therefore by letting $f(z) = \frac{e^z-1}{z}$ and X be γ_2 and the circle enclosed, we conclude that $|f(z)|$ has a minimum $\tilde{A} > 0$. Then take A such that

$$\left| \frac{e^z - 1}{z} \right| \geq \tilde{A} > A > 0,$$

which gives

$$\frac{1}{e^z - 1} < \frac{1}{A|z|}$$

Putting all of the above together gives

$$\left| \int_{\gamma_2} \frac{z^{s-1}e^{-mz}}{e^z - 1} dz \right| \leq \frac{2\pi}{A} \rho^{\sigma-1} e^{m\rho} \tag{6.4}$$

and therefore it is clear that this integral goes to 0 as $\rho \to 0$.

Now taking in consideration the direction of the contour and that on γ_1, $z^{s-1} = e^{\log|z|(s-1)}$ whereas in γ_3, $z^{s-1} = e^{\log|z|(s-1)}e^{2\pi i}$ due to one full rotation that occurs at γ_2, the integral becomes

$$\int_C \frac{z^{s-1}e^{-mz}}{e^z - 1} dz \quad = \left(e^{2\pi i s} - 1\right) \int_0^\infty \frac{x^{s-1}e^{-mx}}{e^x - 1} dx$$

In order to get the desired coefficient we only need to manipulate the known identity of the Γ function namely

$$\Gamma(s)\Gamma(1-s) \quad = \frac{\pi}{\sin(\pi s)} \tag{6.5}$$

and as a result we obtain (6.1). $\square$

Let $t > 0$ and $x \leq y$, so that $x \leq \sqrt{\frac{t}{2\pi}}$. Then let $\sigma \leq 1$, $m = [x]$, $y = \frac{t}{2\pi x}$, $q = [y]$ and $\eta = 2\pi y$. Deform contour C of (6.1) into Γ which consists of straight lines C_0, C_1, C_2, C_3, C_4 and C_5 joining $\infty, \infty + i\eta(1+c), c\eta + i\eta(1+c), -c\eta + i\eta(1-c), -c\eta - (2q+1)\pi i, \infty - (2q+1)\pi i$ and ∞ where c is an absolute constant $0 < c \leq \frac{1}{2}$. Note that if y is an integer, a small indentation is made above the pole $z = i\eta$. By construction all the poles within Γ are $z = 0, \pm 2\pi i, \pm 4\pi i, ..., \pm 2q\pi i$ and each pole has order one. Since $z = 0$ is the only pole within C, the difference in these two integrals can be computed using Cauchy's Integral Theorem.

First we compute residues for each pole in order to use the theorem. Therefore for each $k \in \{\pm 2, \pm 4, ..., \pm 2q\}$,

$$\begin{aligned} Res(2k\pi i) \quad &= \lim_{z \to 2k\pi i} (z - 2k\pi i) \frac{z^{s-1} e^{-mz}}{e^z - 1} \\ &= (2k\pi i)^{s-1} \end{aligned} \tag{6.6}$$

and then applying Cauchy's Integral Theorem results in

$$\begin{aligned} \frac{e^{-i\pi s}\Gamma(1-s)}{2\pi i} \int_\Gamma \frac{z^{s-1}e^{-mz}}{e^z - 1} dz \quad &= \sum_{k=-q, k\neq 0}^{q} (2k\pi i)^{s-1} e^{-i\pi s} \Gamma(1-s) \\ &\quad + \frac{e^{-i\pi s}\Gamma(1-s)}{2\pi i} \int_C \frac{z^{s-1}e^{-mz}}{e^z - 1} dz. \end{aligned} \tag{6.7}$$

Simplify the summation in the formula by first pairing positive and negative terms for each k from 1 to q. Then rearranging and substituting

$$\chi(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)$$

will yield what we want as shown below.

$$\begin{aligned}\sum_{k=-q,k\neq 0}^{q}(2k\pi i)^{s-1}e^{-i\pi s}\Gamma(1-s) &= \sum_{k=1}^{q}\left((2k\pi i)^{s-1}+(-2k\pi i)^{s-1}\right)e^{-i\pi s}\Gamma(1-s)\\ &= -\sum_{k=1}^{q}k^{s-1}(2\pi)^{s}\frac{\left(e^{\frac{\pi is}{2}}-e^{\frac{-\pi is}{2}}\right)}{2\pi i}\Gamma(1-s)\\ &= -\chi(s)\sum_{k=1}^{q}\frac{1}{k^{1-s}}\end{aligned} \tag{6.8}$$

Then substituting (6.7) and (6.8) into (6.1) gives

$$\zeta(s) = \sum_{n=1}^{m}\frac{1}{n^s}+\chi(s)\sum_{n=1}^{q}\frac{1}{n^{1-s}}+\frac{e^{-i\pi s}\Gamma(1-s)}{2\pi i}\int_{\Gamma}\frac{z^{s-1}e^{-mz}}{e^z-1}dz. \tag{6.9}$$

As previously stated Γ consists of straight lines C_0, C_1, C_2, C_3, C_4 and C_5, so the only remaining task now is to bound each integral in order to get the final equation.

6.3 Bounding the integral on C_0, C_1, C_2, C_3, C_4 and C_5

In this section all of the bounds are computed in the order of C_0, C_5, C_4, C_3, C_1 and then C_2. The bound of the integral on C_2 is presented here last, because the main contribution comes from the C_2 portion and also it is the most difficult one to compute.

Let $z = u + iv = \rho e^{i\phi}(0 < \phi < 2\pi)$. Then note that

$$\left|z^{s-1}\right| = \rho^{\sigma-1}e^{-\phi t}. \tag{6.10}$$

6.3.1 Bounding the integral on C_0 and C_5

The line C_0 joins ∞ and $\infty+i\eta(1+c)$. Hence by changing the variable to $z = u+iv$ the integral over C_0 can be easily computed as 0 by looking at its absolute values.

$$\begin{aligned}\int_{C_0}\left|\frac{z^{s-1}e^{-mz}}{e^z-1}\right|dz &= \lim_{u\to\infty}\int_0^{\eta(1+c)}\left|\frac{(u+iv)e^{-m(u+iv)}}{e^{u+iv}-1}\right|dv\\ &\leq \lim_{u\to\infty}\eta(1+c)\frac{|u+i\eta(1+c)|}{e^{mu}}\\ &= 0\end{aligned} \tag{6.11}$$

Similarly the integral over C_5 is 0 as well, because C_5 is a straight line joining $\infty-(2q+1)\pi i$ and ∞

□

6.3.2 Bounding the integral on C_4

On C_4 by construction,

$$\phi \geq \tan^{-1}\left(\frac{\eta}{c\eta}\right) \geq \tan^{-1}(2) \geq \frac{5}{4}\pi.$$

Now choose positive A_4 small enough such that both $\rho > A_4\eta$ and $|e^z - 1| > A_4$ are satisfied. Note that $\rho > A_4\eta$ implies

$$\left(\frac{\rho}{A_4}\right)^{\sigma-1} = O\left(\eta^{\sigma-1}\right) \tag{6.12}$$

since $\sigma - 1 \leq 0$ and $|e^z - 1| > 0$ is easily satisfied because C_4 does not go through zeros of this function.

Then the integral on C_4 can be bounded by looking at its terms in absolute values, taking the integral and simplifying the expression by using (6.12).

$$\begin{aligned} \int_{C_4} \frac{z^{s-1}e^{-mz}}{e^z - 1} dz &= O\left(\frac{\rho^{\sigma-1}e^{-\frac{5}{4}\pi t}}{A_4}\int_{-c\eta}^{\infty} e^{-mz}dz\right) \\ &= O\left(\eta^{\sigma-1}e^{mc\eta - \frac{5}{4}\pi t}\right) \end{aligned} \tag{6.13}$$

As defined before $m = [x], y = \frac{t}{2\pi x}$ and $\eta = 2\pi y$. By substituting all these values we obtain $mc\eta = O(tc)$ and using this (6.13) can be simplified further into $O\left(\eta^{\sigma-1}e^{t(c-\frac{5}{4}\pi)}\right)$.

□

6.3.3 Bounding the integral on C_3

First notice that since

$$\tan^{-1}(\theta) = \int_0^\theta \frac{d\mu}{1+\mu^2}d\mu > \int_0^\theta \frac{d\mu}{(1+\mu)^2}d\mu = \frac{\theta}{1+\theta}$$

it follows that

$$\tan^{-1}\left(\frac{c}{1-c}\right) > c. \tag{6.14}$$

On C_3 by construction,

$$\phi \geq \frac{1}{2}\pi + \tan^{-1}\left(\frac{c}{1-c}\right) > \frac{1}{2}\pi + c + A_3,$$

where A_3 is small enough such that it satisfies $\tan^{-1}\left(\frac{c}{1-c}\right) - c > A_3 > 0$ and $|e^z - 1| > A_3$.

It is clear that on C_3, $\rho < \sqrt{(c\eta)^2 + ((1-c)\eta)^2}$ so $\rho = O(\eta)$ and $e^{-mz} = e^{-m(-c\eta+it)} = O\left(e^{mc\eta}\right)$. Also by substituting the definitions $m = [x], y = \frac{t}{2\pi x}$ and $\eta = 2\pi y$, we get $mc\eta - tc < 0$.

Thus by using these relations to bound the term inside the integral and then multiplying it by the length of the integral to bound the integral itself, we obtain

$$\begin{aligned}\int_{C_3} \frac{z^{s-1}e^{-mz}}{e^z-1}dz &= O\left(\eta^{\sigma}e^{-t(\frac{1}{2}\pi+c+A_3)}e^{mc\eta}\right)\\ &= O\left(\eta^{\sigma}e^{-t(\frac{1}{2}\pi+A_3)}\right)\end{aligned}$$

□

6.3.4 Bounding the integral on C_1

The line C_1 is represented by $z = u + i\eta(1+c)$ for $u \in [c\eta, \infty]$. Let $f(z) = \left|\frac{e^z-1}{e^z}\right| = \frac{|e^z-1|}{e^u}$, then clearly it has no zero on C_1 and also as $u \to \infty, f(z) \to 1$. Therefore there exists L such that for $u > L$, $f(z) > \frac{1}{2}$ for z on C_1. Now let $X = [c\eta + i\eta(1+c), L + i\eta(1+c)]$ a segment of C_1 from the left end point to the point with real value L. Then X is a compact set, hence $f(X)$ has a minimum say $\tilde{A}$.

Therefore for all z on C_1

$$f(z) > \tilde{A}_1 = \min\left(\frac{1}{2}, \frac{\tilde{A}}{2}\right)$$

and it follows that

$$|e^z - 1| > \tilde{A}_1 e^u.$$

As stated before $|z^{s-1}| = \rho^{\sigma-1}e^{-\phi t}$. Clearly $|e^{-mz}| = e^{-mu}$ and by construction $\phi = \arctan\left(\frac{\eta(1+c)}{u}\right)$. Take B small enough such that $\rho > B\eta$, then it follows that $\rho^{\sigma-1} = O\left(\eta^{\sigma-1}\right)$.

By substituting all these relations and simplifying, we can bound the term inside the integral as

$$\frac{z^{s-1}e^{-mz}}{e^z-1} = O\left(\eta^{\sigma-1}e^{-t\arctan\left(\frac{\eta(1+c)}{u}\right)-(m+1)u}\right) \tag{6.15}$$

In order to simplify this expression further first note that for $0 < \theta < 1$,

$$\arctan\theta = \int_0^{\theta} \frac{d\mu}{1+\mu^2} < \int_0^{\theta} \frac{d\mu}{(1-\mu)^2} = \frac{\theta}{1-\theta}.$$

Therefore by substituting $\theta = \frac{c}{1-c}$ to the inequality, we get

$$\arctan \frac{c}{1+c} < c$$

and let $A_1 = c - \arctan \frac{c}{1+c}$. Then take the derivative of $\arctan\left(\frac{(1+c)\eta}{u}\right) + \frac{u}{\eta}$ and recognize that it is an increasing function

$$\frac{d}{du}\{\arctan\left(\frac{(1+c)\eta}{u}\right) + \frac{u}{\eta}\} = \frac{-(1+c)\eta}{u^2 + (1+c)^2\eta^2} + \frac{1}{\eta} > 0.$$

Thus for all u in $[c\eta, \infty]$, $\arctan\left(\frac{(1+c)\eta}{u}\right) + \frac{u}{\eta}$ is greater than or equal to its value at $u = c\eta$. Simplify this value using A_1 and we obtain the following lower bound

$$\arctan\left(\frac{(1+c)\eta}{u}\right) + \frac{u}{\eta} \qquad \geq \frac{1}{2}\pi + A_1. \tag{6.16}$$

By using this lower bound and noticing that $m + 1 \geq x = \frac{t}{\eta}$, the exponent in the O-term in (6.15) can be further simplified to $O\left(\eta^{\sigma-1}e^{-t\left(\frac{1}{2}\pi+A_1\right)}\right)$.

Now bound the integral by splitting it up into two portions, first for interval $[0, \pi\eta]$ with the O-term obtained above and second for interval $[\pi\eta, \infty]$ with the O-term from (6.15). The second term can be simplified by recognizing that $\arctan \frac{\eta(1+c)}{u} \geq 0$ for $u > \pi\eta$ and $m + 1 \geq x$. Then we take the integrals and conclude that the first O-term is dominant, because $t\left(\frac{1}{2}\pi + A_1\right) < \pi\eta x$. This can be easily shown by substituting definitions $y = \frac{t}{2\pi x}$ and $\eta = 2\pi\eta$ into the inequality, manipulating it and remembering that $A_1 \leq c \leq \frac{1}{2} < \frac{1}{2}\pi$.

$$\begin{aligned}
\int_{C_1} \frac{z^{s-1}e^{-mz}}{e^z - 1}dz \qquad &= O\left(\eta^{\sigma-1}\int_0^{\pi\eta} e^{-t\left(\frac{1}{2}\pi+A_1\right)}du\right) \\
&\qquad + O\left(\eta^{\sigma-1}\int_{\pi\eta}^{\infty} e^{-t\arctan\left(\frac{\eta(1+c)}{u}\right)-(m+1)u}du\right) \\
&= O\left(\eta^{\sigma-1}\int_0^{\pi\eta} e^{-t\left(\frac{1}{2}\pi+A_1\right)}du\right) + O\left(\eta^{\sigma-1}\int_{\pi\eta}^{\infty} e^{-xu}du\right) \\
&= O\left(\eta^{\sigma}e^{-t\left(\frac{1}{2}\pi+A_1\right)}\right) + O\left(\eta^{\sigma-1}e^{-\pi\eta x}\right) \\
&= O\left(\eta^{\sigma}e^{-t\left(\frac{1}{2}\pi+A_1\right)}\right)
\end{aligned}$$

$\square$

6.3.5 Bounding the integral on C_2

Finally consider C_2. By construction C_2 is a line $z = i\eta + \lambda e^{\frac{1}{4}\pi i}$ where λ is real and $|\lambda| \leq \sqrt{(c\eta)^2 + (c\eta)^2} = \sqrt{2}c\eta < \eta$.

z^{s-1} on C_2 can be bounded by writing it in terms of e and looking at the real part of its exponential as follows.

$$\begin{aligned} z^{s-1} &= \left(i\eta + \lambda e^{\frac{1}{4}\pi i}\right)^{s-1} \\ &= e^{(s-1)\left(\frac{\pi}{2}i + \log\eta + \frac{\lambda}{\eta}e^{-\frac{1}{4}\pi i} - \frac{1}{2}\frac{\lambda^2}{\eta^2}e^{-\frac{1}{2}\pi i} + O\left(\frac{\lambda^3}{\eta^3}\right)\right)} \\ &= O\left(\eta^{\sigma-1} e^{\left(-\frac{\pi}{2} + \frac{\lambda}{\eta}\frac{1}{\sqrt{2}} - \frac{1}{2}\frac{\lambda^2}{\eta^2} + O\left(\frac{\lambda^3}{\eta^3}\right)\right)t}\right) \end{aligned} \tag{6.17}$$

Also,

$$\begin{aligned} \frac{e^{-mz+xz}}{e^z - 1} &= O\left(\frac{e^{(x-m-1)u}}{e^{-u} - 1}\right) \text{ for } u \geq 0 \\ &= O\left(\frac{e^{(x-m)u}}{e^{u} - 1}\right) \text{ for } u < 0. \end{aligned} \tag{6.18}$$

Notice that for $u \geq 0$, $(x-m-1)u$ is negative and for $u < 0$, $(x-m)u$ is negative and so each of the above O-term is bounded for $u \geq \frac{1}{2}\pi$ and $u < -\frac{1}{2}\pi$, respectively.

Then bound e^{-xz} by its real part using $x = \frac{t}{\eta}$ obtained by putting the two definitions, $y = \frac{t}{2\pi x}$ and $\eta = 2\pi y$, together.

$$\begin{aligned} \left|e^{-xz}\right| &= \left|e^{-\frac{t}{\eta}(i\eta + \lambda e^{\frac{1}{4}\pi i})}\right| \\ &= e^{-\frac{\lambda t}{\eta\sqrt{2}}} \end{aligned} \tag{6.19}$$

Hence the integral for the portion $|u| > \frac{1}{2}\pi$ can be bounded by first substituting all the bounds obtained above and taking the integral with respect to λ. Here note that initially the integral is from $-\sqrt{2}c\eta$ to $\sqrt{2}c\eta$, because this is exactly the range of λ. Up until now the only restriction so far on c had been that $0 < c < \frac{1}{2}$, thus we can choose c to be as close to as 0 as we need. Remembering that $\left|\frac{\lambda}{\eta}\right| \leq \sqrt{2}c$, by choosing c small enough $\left|\frac{\lambda^3}{\eta^3}\right|$ will go to 0 much faster than $\left|\frac{\lambda^2}{\eta^2}\right|$. By using this fact, the integral in the O-term can be bounded by a simple integral involving some $A > 0$ and extended the range of integration. Then, we take the integral to obtain the bound we desire.

$$\begin{aligned}
\int_{C_2,|u|>\frac{1}{2}\pi} \frac{z^{s-1}e^{-mz}}{e^z-1}dz &= \int_{C_2,|u|>\frac{1}{2}\pi} \frac{z^{s-1}e^{(x-m)z}}{e^z-1}e^{-xz}dz \\
&= O\left(\eta^{\sigma-1}e^{-\frac{1}{2}\pi t}\int_{-\sqrt{2}c\eta}^{\sqrt{2}c\eta} e^{\left(-\frac{1}{2}\frac{\lambda^2}{\eta^2}+O\left(\frac{\lambda^3}{\eta^3}\right)\right)t}d\lambda\right) \\
&= O\left(\eta^{\sigma-1}e^{-\frac{1}{2}\pi t}\int_{-\infty}^{\infty} e^{-A\frac{\lambda^2}{\eta^2}t}d\lambda\right) \\
&= O\left(\eta^{\sigma}t^{-\frac{1}{2}}e^{-\frac{1}{2}\pi t}\right)
\end{aligned}$$

The same argument applies to the portion $|u| \le \frac{1}{2}\pi$, if $|e^z-1| > A_2$ in this region for some constant A_2. Suppose it doesn't, for example, because the contour gets too close to the pole at $z = 2q\pi i$. In this case, we take the contour around arc of a circle $|z-2q\pi i| = \frac{1}{2}\pi$. Now $|e^z-1| > A_2$ is satisfied in this region. Then we look at $\log(z^{s-1}e^{-mz})$ to obtain a bound for $z^{s-1}e^{-mz}$. On this circle

$$z = 2q\pi i + \frac{1}{2}\pi e^{i\theta}$$

thus it follows that

$$z^{s-1} = e^{(s-1)\left(\frac{\pi}{2}i+\log\left(2q\pi+\frac{1}{2i}\pi e^{i\theta}\right)\right)}$$

and

$$e^{-mz} = e^{-\frac{1}{2}m\pi e^{i\theta}}.$$

We substitute these values into $\log(z^{s-1}e^{-mz})$ to get an expression to work with. To simplify one of the terms in the expression, we look at the series expansion of log as follows.

$$(s-1)\log\left(1+\frac{e^{i\theta}}{4qi}\right) = -(\sigma-1+it)\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{e^{i\theta}}{4qi}\right)^k = \frac{te^{i\theta}}{4q} + O(1).$$

Then we combine two of the terms together

$$-\frac{1}{2}m\pi e^{i\theta} + \frac{te^{i\theta}}{4q} = -\frac{1}{2}e^{i\theta}\left(\frac{2mq\pi - t}{2q}\right)$$

and obtain that this is $O(1)$ by substituting the definitions $m = [x], q = [y]$ and $y = \frac{t}{2\pi x}$ and remembering that $x \le y$.

$$\begin{aligned}
\log\left(z^{s-1}e^{-mz}\right) &= -\frac{1}{2}m\pi e^{i\theta} + (\sigma-1)\frac{\pi}{2}i - \frac{\pi}{2}t \\
&\qquad + (s-1)\log(2q\pi) + (s-1)\log\left(1+\frac{e^{i\theta}}{4qi}\right) \\
&= -\frac{1}{2}m\pi e^{i\theta} - \frac{\pi}{2}t + (s-1)\log(2q\pi) + \frac{te^{i\theta}}{4q} + O(1) \\
&= -\frac{\pi}{2}t + (s-1)\log(2q\pi) + O(1) \qquad (6.20)
\end{aligned}$$

Hence by taking the exponential we get what we need, which is

$$\begin{aligned} \left|z^{s-1}e^{-mz}\right| &= O\left(e^{-\frac{\pi}{2}t+(s-1)\log(2q\pi)}\right) \\ &= O\left(\eta^{\sigma-1}e^{-\frac{\pi}{2}t}\right). \end{aligned} \tag{6.21}$$

Therefore, the contribution of this portion of the integral is

$$\int_{C_2,|u|\leq\frac{1}{2}\pi} \frac{z^{s-1}e^{-mz}}{e^z-1}dz \qquad = O\left(\eta^{\sigma-1}e^{-\frac{\pi}{2}t}\right).$$

Putting the two portions together, we obtain the overall bound

$$\int_{C_2} \frac{z^{s-1}e^{-mz}}{e^z-1}dz = O\left(\eta^{\sigma}t^{-\frac{1}{2}}e^{-\frac{\pi}{2}t}\right) + O\left(\eta^{\sigma-1}e^{-\frac{\pi}{2}t}\right).$$

□

Now that all the bounds for the integral are found, we look at the coefficient $\frac{e^{-i\pi s}\Gamma(1-s)}{2\pi i}$.

6.4 Bound of the coefficient $\frac{e^{-i\pi s}\Gamma(1-s)}{2\pi i}$

We find the bound for $\Gamma(1-s)$ using the following known formula

$$\log\Gamma(z) = \left(z-\frac{1}{2}\right)\log z - z + \frac{1}{2}\log 2\pi + \int_0^\infty \frac{[u]-u+\frac{1}{2}}{u+z}du.$$

First we substitute $z = 1-s$ into the formula, simplify it and look at the real part.

$$\begin{aligned} \Re\log\Gamma(1-s) \quad &= \left(\frac{1}{2}-\sigma\right)\log\sqrt{(1-\sigma)^2+t^2} + t\arg(1-\sigma-it) + (\sigma-1) \\ &\quad + \frac{1}{2}\log 2\pi + \int_0^\infty \frac{[u]-u+\frac{1}{2}}{(u+1-\sigma)^2+t^2}(u+1-\sigma)du. \end{aligned} \tag{6.22}$$

Then we simplify one of the terms using properties of tan,

$$t\arg(1-\sigma-it) = \frac{-\pi}{2}t + t\tan\left(\frac{1-\sigma}{t}\right) = \frac{-\pi}{2}t + O(1).$$

To simplify the integral term let $f(u) = \frac{u+1-\sigma}{(u+1-\sigma)^2+t^2}$. Then $f(u)$ is a decreasing function on $[0, \infty]$ and $\lim_{u\to\infty} f(u) = 0$. Define $S_n = \int_{\frac{n}{2}}^{\frac{n+1}{2}} \left([u] - u + \frac{1}{2}\right) f(u)du$ for $n \geq 0$. Then by construction

$$\int_0^\infty \left([u] - u + \frac{1}{2}\right) f(u)du = \sum_{n=0}^{\infty} S_n = \sum_{n=0}^{\infty} (-1)^n |S_n|$$

because S_n is positive for n even and negative for n odd. For each S_n, contribution of $[u] - u + \frac{1}{2}$ portion of the integral from $\frac{n}{2}$ to $\frac{n+1}{2}$ is in fact the same apart from the parity. Therefore, since $f(u)$ is decreasing it follows that $|S_{n+1}| \leq |S_n|$. Also, because $\lim_{u\to\infty} f(u) = 0$, it is easy to see that $\lim_{n\to\infty} |Sn| = 0$ as well. By applying the Alternating Series Test we obtain

$$\int_0^\infty \left([u] - u + \frac{1}{2}\right) f(u)du = O(1).$$

Thus (6.22) becomes

$$\begin{aligned} \Re \log \Gamma(1-s) \quad &= \left(\frac{1}{2} - \sigma\right) \log \sqrt{(1-\sigma)^2 + t^2} + \frac{-\pi}{2} t \\ &\qquad + (\sigma - 1) + \frac{1}{2} \log 2\pi + O(1). \end{aligned}$$

By taking the exponential, remembering that $\sigma \leq 1$ and simplifying it we get

$$\begin{aligned} |\Gamma(1-s)| \quad &= e^{\Re \log \Gamma(1-s)} \\ &= O\left(\left(\sqrt{(1-\sigma)^2 + t^2}\right)^{\frac{1}{2}-\sigma} e^{\frac{-\pi}{2}t} e^{(\sigma-1)} \sqrt{2\pi}\right) \\ &= O\left(t^{\frac{1}{2}-\sigma} e^{\frac{-\pi}{2}t}\right). \end{aligned}$$

Because $|e^{-i\pi s}| = e^{\pi t}$, we get that

$$\frac{e^{-i\pi s}\Gamma(1-s)}{2\pi i} = O\left(t^{\frac{1}{2}-\sigma} e^{\frac{\pi}{2}t}\right).$$

□

Now combine all the bounds for the integral we have obtained so far together with the bound on the coefficient. By letting $A = \min\{A_1, A_3\}$, we can combine the bounds for C_1 and C_2 into one O-term. Then simplify the remaining O-terms by substituting $\eta = \frac{t}{x}$ and $t^{-\frac{1}{2}}x = O(1)$, the latter equation is obtained by manipulating the restriction $x \leq \sqrt{\frac{t}{2\pi}}$.

$$\frac{e^{-i\pi s}\Gamma(1-s)}{2\pi i}\int_{\Gamma}\frac{z^{s-1}e^{-mz}}{e^z-1}dz \qquad = O\left(t^{-\frac{1}{2}}e^{-t(\frac{3}{4}\pi-c)}x^{1-\sigma}\right)+O\left(t^{\frac{1}{2}}e^{-At}x^{-\sigma}\right) + O\left(x^{-\sigma}\right)+O\left(t^{-\frac{1}{2}}x^{1-\sigma}\right)$$

$$= O\left(e^{-t(\frac{3}{4}\pi-c)}x^{-\sigma}\right)+O\left(t^{\frac{1}{2}}e^{-At}x^{-\sigma}\right)+O\left(x^{-\sigma}\right)$$

The second O-term here can be simplified using the following result, if $f(t) = O\left(e^{-\tilde{c}t}\right)$ for any $\tilde{c}$ such that $0\le\alpha<\tilde{c}<\beta$, then $t^r f(t) = O\left(e^{-\tilde{c}t}\right)$ for $0\le\alpha<\tilde{c}<\beta$ where r is any real number. Note that this result applies to the second O-term, because α and β can be constructed by utilizing the freedom to make A as small as we want.

To prove this result we need to show that as $t\to\infty$, $\left|\frac{t^r f(t)}{e^{-\tilde{c}t}}\right| \le \tilde{c_0}$ for some constant $\tilde{c_0}$. First it is clear that there exists ε such that $\alpha<\tilde{c}+\varepsilon<\beta$, so $f(t) = O\left(e^{-(\tilde{c}+\varepsilon)t}\right)$. In other words, there exists $\tilde{c_0}$ such that $\left|\frac{f(t)}{e^{-(\tilde{c}+\varepsilon)t}}\right| \le \tilde{c_0}$ as $t\to\infty$. Since $\left|\frac{t^r}{e^{\varepsilon t}}\right|\le 1$ as $t\to\infty$, replace $f(t)$ with $t^r f(t)$ and bound it as follows

$$\left|\frac{t^r f(t)}{e^{-\tilde{c}t}}\right| = \left|\frac{t^r}{e^{\varepsilon t}}\right|\left|\frac{f(t)}{e^{-(\tilde{c}+\varepsilon)t}}\right| \le \tilde{c_0}.$$

□

After applying this result to the second O-term, by defining $B=\min\{(\frac{3}{4}\pi-c),A\}$ it can be combined with the first O-term into one simple O-term, $O(e^{-Bt}x^{-\sigma})$. Since $t=2\pi yx$, which is obtained from the definitions, we can apply the same result again with respect to x this time. Then by recognizing that e^{-Bt} goes to 0 faster than $x^{-\sigma}$ for any $0\le\sigma\le 1$ we can simplify the bound further into $O\left(x^{-\sigma}\right)$.

Thus the equation obtained is

$$\zeta(s) \qquad = \sum_{n\le x}\frac{1}{n^s}+\chi(s)\sum_{n\le y}\frac{1}{n^{1-s}}+O\left(x^{-\sigma}\right)$$

for $x\le y$. Now in order to deduce $x\ge y$ case change s into $1-s$ and multiply both sides of the equation by $\chi(s)$. Then use $\chi(s)\chi(1-s)=1$ and the functional equation $\zeta(s)=\chi(s)\zeta(1-s)$ to simplify.

$$\zeta(s) \qquad = \chi(s)\sum_{n\le x}\frac{1}{n^{1-s}}+\sum_{n\le y}\frac{1}{n^s}+|\chi(s)|O\left(x^{\sigma-1}\right)$$

By definition

$$\chi(s) = 2^s\pi^{s-1}\sin(\frac{1}{2}\pi s)\Gamma(1-s)$$

and it was found in **Section 6.4** that $\Gamma(1-s) = O\left(t^{\frac{1}{2}-\sigma}e^{t\frac{-\pi}{2}}\right)$. Clearly 2^s and π^{s-1} are both $O(1)$ and by using the definition of sin, it is a straightforward computation to get $\sin(\frac{1}{2}\pi s) = O(e^{t\frac{\pi}{2}})$. Therefore $\chi(s) = O\left(t^{\frac{1}{2}-\sigma}\right)$. By using this O-term and interchanging x and y so that the equation is for $x \geq y$, the result is

$$\zeta(s) \qquad = \sum_{n\leq x} \frac{1}{n^s} + \chi(s) \sum_{n\leq y} \frac{1}{n^{1-s}} + O\left(t^{\frac{1}{2}-\sigma} y^{\sigma-1}\right).$$

By combining the formulas for both cases, it finally gives us the approximate functional equation

$$\zeta(s) = \sum_{n\leq x} \frac{1}{n^s} + \chi(s) \sum_{n\leq y} \frac{1}{n^{1-s}} + O(x^{-\sigma}) + O(|t|^{\frac{1}{2}-\sigma} y^{\sigma-1})$$

for $0 \leq \sigma \leq 1$ where $s = \sigma + it$, $t = 2\pi xy$, $x > h > 0$ and $y > h > 0$.

Note that it was assumed that $t > 0$ in this proof, but if this is not the case then take $\bar{s}$ so that the imaginary portion will be positive and substitute into the above formula. The same result is obtained in the end because $\zeta(\bar{s}) = \overline{\zeta(s)}$ and $\chi(\bar{s}) = \overline{\chi(s)}$.

□

6.5 Hardy and Littlewood

To illustrate a use of the approximate functional equation we prove the basic result (1.1) by Hardy and Littlewood. In order to do so we first prove the following lemma.

Lemma.

$$\sum_{0<m<n<T} \frac{1}{m^\sigma n^\sigma \log \frac{n}{m}} = O\left(T^{2\sigma-2} \log T\right)$$

for $\frac{1}{2} \leq \sigma < 1$, and uniformly for $\frac{1}{2} \leq \sigma \leq \sigma_0 < 1$.

The proof of it is as follows. Let Σ_1 denote the sum of the terms for which $m < \frac{1}{2}n$, Σ_2 the remainder. In Σ_1, $\log \frac{n}{m} > \log 2$, so that

$$\Sigma_1 < \frac{1}{\log 2} \sum_{0<m<n<T} m^{-\sigma} n^{-\sigma} < \frac{1}{\log 2} \left(\sum_{0<n<T} n^{-\sigma} \right)^2 < \frac{1}{\log 2} T^{2-2\sigma}.$$

In Σ_2 we write $m = n - r$, where $1 \leq r \leq \frac{1}{2}n$, and then

$$\log \frac{n}{m} = -\log\left(1 - \frac{r}{n}\right) > \frac{r}{n}.$$

Hence

$$\Sigma_2 < \sum_{0<n<T} \sum_{1\le r\le \frac{1}{2}n} \frac{(n-r)^{-\sigma} n^{-\sigma}}{\frac{r}{n}} < \sum_{0<n<T} n^{1-2\sigma} \sum_{1\le r\le \frac{1}{2}n} \frac{1}{r} < T^{2-2\sigma}\log T$$

and the result follows.

□

In the approximate functional equation from **Section 6.1**, take $\sigma = \frac{1}{2}$, $t > 2$, and $x = \frac{t}{2\pi\sqrt{\log t}}$, $y = \sqrt{\log t}$. Then, since $\left|\chi\left(\frac{1}{2}+it\right)\right| = 1$ and the first O-term can be bounded by switching the summation into integration,

$$\begin{aligned}
\zeta(1/2+it) &= \sum_{n<x} n^{-\frac{1}{2}-it} + O\left(\sum_{n<y} n^{-\frac{1}{2}}\right) + O\left(t^{-\frac{1}{2}}(\log t)^{\frac{1}{4}}\right) + O\left((\log t)^{-\frac{1}{4}}\right) \\
&= \sum_{n<x} n^{-\frac{1}{2}-it} + O\left((\log t)^{\frac{1}{4}}\right) \\
&= Z + O\left((\log t)^{\frac{1}{4}}\right) \qquad (6.23)
\end{aligned}$$

say. Now

$$\begin{aligned}
\int_0^T |Z|^2\,dt &= \int_0^T \left(\sum_{m<x} m^{-\frac{1}{2}-it}\right)\left(\sum_{n<x} n^{-\frac{1}{2}+it}\right) dt \\
&= \int_0^T \sum_{m,n<x} \frac{1}{\sqrt{mn}} \left(\frac{n}{m}\right)^{it} dt \qquad (6.24)
\end{aligned}$$

We then invert the order of integration and summation remembering that x is a function of t. The term in (m,n) occurs if

$$\frac{t}{2\pi\sqrt{\log t}} = x > \max(m,n) = \frac{T_1}{2\pi\sqrt{\log T_1}}$$

say, where $T_1 = T_1(m,n)$. Thus each term in (m,n) will be integrated for $T_1 \le t \le T$. By writing $X = \frac{T}{2\pi\sqrt{\log T}}$, we get

$$\begin{aligned}
&= \sum_{m,n<X} \frac{1}{\sqrt{mn}} \int_{T_1}^T \left(\frac{n}{m}\right)^{it} dt \\
&= \sum_{n<X} \frac{T-T_1(n,n)}{n} + \sum_{\substack{m,n<X \\ m\ne n}} \frac{1}{\sqrt{mn}} \frac{\left(\frac{n}{m}\right)^{iT} - \left(\frac{n}{m}\right)^{iT_1}}{i\log\left(\frac{n}{m}\right)} \\
&= \sum_{n<X} \frac{1}{n} - \sum_{n<X} \frac{T_1(n,n)}{n} + O\left(\sum_{0<m<n<X} \frac{1}{\sqrt{mn}\log\left(\frac{n}{m}\right)}\right) \qquad (6.25)
\end{aligned}$$

The first term is

$$T \log X + O(T) = T \log T + o(T \log T)$$

obtained by using summation by parts. For the second term we know that by the definition of T_1

$$\frac{T_1(n,n)}{n} = 2\pi\sqrt{\log T_1(n,n)} \leq 2\pi\sqrt{\log T}.$$

Therefore the sum of these terms for $n < X$ can be bounded by $X 2\pi\sqrt{\log T} = T$ and hence it is $O(T)$.

By applying the lemma for $\sigma = \frac{1}{2}$, the last term is

$$O(X \log X) = O(T\sqrt{\log T})$$

and we have obtained

$$\int_0^T |Z|^2 \, dt \sim T \log T.$$

Now we simplify the expression as below. We obtain the first O-term by remembering that $\zeta(1/2+it)$ is analytic and we apply the Cauchy Schwartz Inequality to the second O-term.

$$\begin{aligned}
\int_0^T |\zeta(1/2+it)|^2 \, dt &= \int_0^T |Z|^2 \, dt + O(1) \\
&\qquad + O\left(\int_2^T |Z| \, (\log t)^{\frac{1}{4}} dt\right) + O\left(\int_2^T (\log t)^{\frac{1}{2}} dt\right) \\
&= \int_0^T |Z|^2 \, dt \\
&\qquad + O\left((T \log T)^{\frac{1}{2}} (T (\log T)^{\frac{1}{2}})^{\frac{1}{2}}\right) + O\left(T (\log T)^{\frac{1}{2}}\right) \\
&= T \log T + o(T \log T)
\end{aligned}$$

This proves the theorem.

□

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