Homological Algebra in Operator Spaces with Applications to Harmonic Analysis

by

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A thesis
presented to the University of Waterloo
in fulfilment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Pure Mathematics

Waterloo, Ontario, Canada, 1999

0-612-38284-2

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Abstract

A homological algebra theory is developed in the category of operator spaces which closely matches the theory developed in general algebra and its extension to the Banach space setting. Using this category, we establish several results regarding the question of classifying which ideals in the Fourier algebra of a locally compact group are complemented. Furthermore we classify the groups for which the Fourier algebra is operator biprojective.

Additionally, the notion of operator weak amenability for completely contractive Banach algebras is introduced. We then study the potential operator weak amenability for the Fourier algebra and various sub-algebras of its second dual.

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Chapter 1

Introduction

The first major goal of this thesis is to construct a homological algebra theory for completely contractive Banach algebras. For our purposes, this theory commenced for general algebras by Hochschild in [43] [44] and [45], and by Johnson in [49] for Banach algebras.

Taylor in [74] and Khelemskii in [53] and [54] continued this work where they provide a complete homology theory for Banach algebras, paralleling that which existed for general algebras. Chapter 3 of this thesis builds a homological algebra theory in the category of operator spaces. We note that Paulsen in [66] presented a cohomology theory in the category of operator spaces using the Haagerup tensor product, however the presentation given here uses the operator space projective tensor product. Indeed it is one of the major themes of this thesis to demonstrate that this is the appropriate object for a homology theory in this new category, and for studying the homology of the Fourier algebra.

Using this theory, we are able to study various properties of the Fourier algebra. In fact, it is a second theme of this thesis to demonstrate that when considering questions regarding the Fourier algebra, the operator space category is most appropriate. We note that Ruan's theorem that the Fourier algebra of a locally compact group is operator amenable exactly when G is amenable is strong evidence for this perspective.

This thesis is divided into eight chapters. In Chapter 2 we introduce the important ideas in harmonic analysis, operator spaces and homology. All the basic notation and definitions can be found there.

Chapter 3 contains all the major results for the development of the homology and cohomology theory. The functors $?\otimes_{\mathcal{A}}?$ and $CB_{\mathcal{A},\mathbb{C}}(?,?)$ are introduced, and their connection between amenability, flatness, injectivity and projectivity is studied. Additionally, we recognize the derived functors of the latter functor as equivalence classes of certain extension sequences. Using this notion, we are able to provide insight into the question of which ideals in $\mathbf{A}(G)$ are complemented. Furthermore, we classify for which amenable groups we have that $\mathbf{A}(G)$ is *operator biprojective*.

In Chapter 4, we introduce the notion of operator weak amenability and we demonstrate that for large classes of groups, including all $[IN]$ groups and Hermitian groups, $\mathbf{A}(G)$ is operator weak amenable. Furthermore we study the potential operator weak amenability of several sub-algebras of the second dual of $\mathbf{A}(G)$.

We note that the category of operator spaces in indistinguishable from the category of Banach spaces whenever it is known that every bounded map from our algebra is automatically completely bounded. In Chapter 5 we classify for which groups G, the Fourier algebra has this particular property. We move on to discuss under what circumstances the Fourier algebra possesses the property that every derivation is automatically completely bounded.

As discussed earlier, we are able to use the homology theory to investigate the question of which closed ideals in the Fourier algebra are complemented. We also discuss the classification of ideals which possesses bounded approximate identities, which is related to the complementation problem. These results can be found in Chapter 6.

We complete this thesis with a summary of our results, and a discussion of open problems, as well as problems for future research.

Chapter 2

Preliminaries and Notation

2.1 Introduction

This chapter is intended to be a reference to the basic terms and notation used in this thesis. It is divided into four main sections. The first section presents the primary notation that is used throughout the thesis. The second section introduces abstract harmonic analysis and the Fourier algebra. The third section deals with operator spaces and the fourth section contains material on homological algebra.

2.2 Basic Notation

We will use the following notation consistently throughout.

G will represent an abstract group. We will generally use multiplication for the group operation, however if the group is known to be abelian, then addition will often be used.

$\mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$ denote the usual integer, real and complex groups under addition.

$\mathbb{T}$ refers to the circle group of $\mathbb{C}$ under multiplication i.e. $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$

$B(H)$ denotes the algebra of continuous linear operators on the Hilbert space H, and $\mathcal{L}(X)$ denotes the algebra of continuous linear operators on the Banach space X, under the operator norm $\|\cdot\|_{op}$.

$C(X)$ denotes the algebra of continuous complex valued functions from the topological space X, under the sup norm $\|\cdot\|_\infty$.

$C_0(X)$ denotes the subalgebra of $C(X)$ of functions which vanish at infinity, and $C_c(X)$ shall denote the subalgebra of functions with compact support.

If $(X, \mathcal{M}, \mu)$ is a measure space, then for $E \in \mathcal{M}$, 1_E shall denote the characteristic function of E.

Where X is any Banach space, X^* will denote the Banach space of continuous linear complex valued functionals, under the usual norm.

2.3 Harmonic Analysis

An abstract group G is called a ***locally compact group*** if it is endowed with a locally compact Hausdorff topology such that the group operations are continuous.

Fundamental to our study is the fact that given a locally compact group G, there exists a non-negative, non-zero regular Borel measure (called a ***Haar measure***) denoted by m_G, which is left translation invariant. That is to say

$$m_G(g^{-1}E) = m_G(E)$$

for every Borel set E and every $g \in G$.

Given a locally compact group G, and two Haar measures m_G and m'_G, it is well known that there exists a positive constant λ such that $m_G = \lambda m'_G$. By convention we shall assume that all compact groups have measure 1 and for all infinite discrete groups, the measure of a single point set is assumed to be 1.

Now suppose that we are given a Haar measure m_G, and for $s \in G$ we define a new measure m_s by $m_s(E) = m_G(Es)$ for all Borel sets E. It is easy to see that m_s is left translation invariant, and hence that there exists a constant, which we denote $\triangle(s)$, such that

$$m_s(E) = \triangle(s)m_G(E).$$

We call the function $\triangle : s \mapsto \triangle(s)$ the ***modular function*** of G. It is an easy calculation to see that $\triangle$ is a continuous group map from G onto the multiplicative group $\mathbb{R}^+$.

A group such that $\triangle(s) = 1$ for all $s \in G$ is called ***unimodular***. Since the continuous image of a compact group is compact, it follows that any compact group is unimodular. It is also easy to see that a group is unimodular exactly when the Haar measure is also right translation invariant (i.e. $m_G(Es) = m_G(E)$). Hence we conclude that all abelian groups are unimodular. (See [42] for these facts).

The following classes of locally compact groups will be of interest

[A] = Abelian groups

[K] = Compact groups

[MAP] = Maximally almost periodic groups (groups for which the finite dimensional representations separate points)

[Um] = Unimodular groups

[SIN] = Small Invariant Neighborhood groups (groups for which every neighborhood of the identity contains a compact neighborhood which is invariant under all inner automorphisms)

[IN] = Invariant Neighborhood groups (groups having a compact neighborhood of the identity which is invariant under all inner automorphisms.

[Her] = Hermitian (groups for which the group algebra (defined below) is hermitian

An excellent reference for the above groups including the relationships between the classes is the survey article by Palmer [65].

Given the measure m_G, we shall define the linear space $\mathbf{L}^1(G)$ to be space of equivalence classes of all measurable functions $f : G \mapsto \mathbb{C}$ such that the Lesbegue integral

$$\int_G |\, f(x) \,|\; dm_G(x)$$

is finite, where $f \equiv g$ for $f, g \in \mathbf{L}^1(G)$ whenever $f = g$ almost everywhere. If we provide $\mathbf{L}^1(G)$ with the norm $\| \cdot \|_1$ defined by

$$\| f \|_1 = \int_G |\, f \,|\; dm_G$$

$\mathbf{L}^1(G)$ becomes a Banach space.

Since the Haar measure is essentially unique we will usually write $\int_G f(x)\; dx$ instead of $\int_G f(x)\; dm_G(x)$ when no confusion arises.

Given $f \in \mathbf{L}^1(G)$ we can define an ***involution*** on $\mathbf{L}^1(G)$ by

$$f^*(x) = \Delta(x)^{-1}\overline{f(x^{-1})}.$$

For any two functions $f, g \in \mathbf{L}^1(G)$ we define their ***convolution***, denoted by $f * g$ by

$$f * g(s) = \int_G f(t) g(t^{-1}s)\; dt$$

It can be shown that $f * g \in \mathbf{L}^1(G)$ and the inequality $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$ holds for all $f, g \in \mathbf{L}^1(G)$. It follows that with convolution acting as multiplication, $\mathbf{L}^1(G)$ becomes an involutive Banach algebra called the ***group algebra of*** G. It is well known that the group algebra is commutative exactly when the group is commutative. In general $\mathbf{L}^1(G)$ is not a C^*-algebra.

We further define the Banach space $\mathbf{L}^\infty(G)$ to be the space of essentially bounded, m_G-measurable functions under the sup norm $\|\cdot\|_\infty$.

Given a topological group G, a positive linear functional M on $\mathbf{L}^\infty(G)$ such that

$$\begin{aligned} &(1) \qquad M(1_G) = 1 \\ \text{and } &(2) \qquad \|M\| = 1 \end{aligned}$$

is called a ***mean***. A mean is called ***left translation invariant*** if $M(f_x) = M(f)$ for all $f \in \mathbf{L}^\infty(G)$ and $x \in G$ where $f_x(s) = f(x^{-1}s)$. A locally compact group G is called ***amenable*** if there exists a left translation invariant mean on $\mathbf{L}^\infty(G)$. We shall say that G is ***amenable as a discrete group*** if the group G_d is amenable, where G_d is the abstract group G endowed with the discrete topology.

If G is compact, then the normalized Haar measure is easily seen to be a translation invariant mean, hence all compact groups are amenable. It is an easy consequence of the Markov-Kakutani fixed point theorem (see [42]) that each abelian group is amenable. The classic example of a non-amenable group is $\mathbb{F}_2$, the free group on two generators. See [39] and [67] for more on amenable groups.

Non-amenable groups can possess very interesting and sometimes pathological properties in analysis. The most famous example of this arises in the Banach-Tarski paradox (see [76]). It is important to note that a group may be amenable with respect to one topology, while not with respect to another. For example $SO(3)$ is compact hence amenable as a topological group. However it contains the subgroup $\mathbb{F}_2$ which implies that it is not amenable as a discrete group.

Let $\mathbf{M}(G)$ denote the Banach space of bounded regular Borel measures on G with the total variation norm. It is well known that $\mathbf{M}(G)$ can be identified with $C_0(G)^*$, the Banach space dual of $C_0(G)$, by the duality

$$\langle \mu, f \rangle = \int_G f(x) d\mu(x).$$

Clearly $\mathbf{L}^1(G)$ is a closed subspace of $\mathbf{M}(G)$. We can extend the convolution on $\mathbf{L}^1(G)$ to $\mathbf{M}(G)$ by the formula

$$\mu * \nu(h) = \int_G \int_G h(xy) d\mu(x) d\nu(y)$$

for each $\mu, \nu \in \mathbf{M}(G)$ and $h \in C_0(G)$. Furthermore we can extend the involution to $\mathbf{M}(G)$ by

$$d\mu^*(x) = \overline{d\mu(x^{-1})}.$$

under which $\mathbf{M}(G)$ becomes a involutive Banach algebra called the ***measure algebra of*** G , containing $\mathbf{L}^1(G)$ as a closed two sided ideal.

By a ***continuous unitary representation of*** G on the Hilbert space H (or simply a ***representation*** when no confusion arises) we mean a group morphism

$$\pi : G \mapsto \mathcal{U}(H)$$

where $\mathcal{U}(H)$ denotes the group of unitary operators on H, such that for every $\alpha \in H$, the map

$$s \mapsto \pi(s)\alpha$$

is continuous. We call H the ***space of*** π and it is often denoted H_π. Furthermore we define the ***dimension*** of π (denoted dim π) to be equal to the dimension of H_π.

Suppose we are given two representations π_1 and π_2 of the group G on the Hilbert spaces H_1 and H_2 respectively. We say that π_1 and π_2 are ***equivalent*** (and write $\pi_1 \sim \pi_2$) if there exists an isomorphism $U : H_1 \mapsto H_2$ which transforms $\pi_1(g)$ into $\pi_2(g)$ for all $g \in G$. That is

$$U(\pi_1(g)\alpha) = \pi_2(g)U(\alpha) \quad \forall g \in G,\ \alpha \in H_1.$$

Hence we obtain a ***class of representations***. We usually do not distinguish between a representation and its class. We let Σ_G denote the collection of equivalence classes of representations of G.

By far and away the most important representation for our study is the ***left regular representation*** denoted by λ and given by

$$\lambda(g)f(x) = f(g^{-1}x)$$

for all $f \in \mathbf{L}^2(G)$.

Given a *-Banach algebra $\mathcal{A}$, we define a ****-representation*** to be an involutive algebra morphism

$$\pi : \mathcal{A} \mapsto B(H)$$

such that π is continuous in the weak operator topology on H. Similar to above we can define equivalent *-representations and hence we let $\Sigma_{\mathcal{A}}$ denote the collection of equivalence classes of *-representations of $\mathcal{A}$.

It is well known that each $\pi \in \Sigma_G$ lifts to a *-representation of $\mathbf{M}(G)$ by the formula

$$\langle \pi(\mu)\alpha, \beta\rangle = \int_G \langle \pi(x)\alpha, \beta\rangle d\mu(x)$$

for each $\mu \in \mathbf{M}(G)$ and $\alpha, \beta \in H_\pi$. We call the functions of the form $\langle \pi(x)\alpha, \beta\rangle$ the ***coefficient functions*** of π.

The restriction of π to $\mathbf{L}^1(G)$ is given by the simpler formula

$$\langle \pi(f)\alpha, \beta\rangle = \int_G f(x)\langle \pi(x)\alpha, \beta\rangle d(x).$$

This is a non-degenerate *-representation of $\mathbf{L}^1(G)$ on H_π. In fact all such representations of $\mathbf{L}^1(G)$ arise in this manner. Thus there shall be no ambiguity in denoting the equivalence class of non-degenerate *-representations of $\mathbf{L}^1(G)$ by Σ_G. We observe that the left regular representation, when lifted to $\mathbf{L}^1(G)$ has the form

$$\lambda(f)(g) = f * g$$

for all $f \in \mathbf{L}^1(G)$ and $g \in \mathbf{L}^2(G)$.

We define the norm $\|\cdot\|_{C^*(G)}$ on $\mathbf{L}^1(G)$ as follows

$$\|f\|_{C^*(G)} = \sup_{\pi \in \Sigma_G} \|\pi(f)\|_{op}.$$

The completion of $\mathbf{L}^1(G)$ with respect to $\|\cdot\|_{C^*(G)}$ is a C^*- algebra called the ***group C^*-algebra of G*** and is denoted $C^*(G)$. We can define another norm $\|\cdot\|_r$ on $\mathbf{L}^1(G)$ as follows

$$\|f\|_r = \|\lambda(f)\|_{op}.$$

The completion of $\mathbf{L}^1(G)$ with respect to this norm is denoted $C^*_\lambda(G)$ and is called the ***reduced C^*-algebra of G***. It is well known that $C^*(G) = C^*_\lambda(G)$ if and only if G is amenable.

We further define $C^*_\delta(G)$ to be the C^* algebra generated by

$$\{\lambda(g) : g \in G\} \subset \mathbf{VN}(G).$$

The dual of $C^*(G)$ is denoted by $\mathbf{B}(G)$. $\mathbf{B}(G)$ may be realized as the space of coefficient functions of Σ_G. The duality is determined by the formula

$$\langle u, f\rangle = \int_G u(x)f(x)dx$$

for $u \in \mathbf{B}(G)$ and $f \in C^*(G)$. We let $\|\cdot\|_{\mathbf{B}(G)}$ be the norm on $\mathbf{B}(G)$ induced by this duality. With this norm and pointwise multiplication, $\mathbf{B}(G)$ becomes a commutative, regular semisimple Banach algebra called the ***Fourier-Stieltjes algebra*** of G .

Now let $\mathbf{A}(G)$ denote the closed subspace of $\mathbf{B}(G)$ generated by the coefficient functions of λ. Then $\mathbf{A}(G)$ is a closed two-sided ideal of $\mathbf{B}(G)$ called the ***Fourier algebra of G***. Alternatively, we can view $\mathbf{A}(G)$ as the subspace of $C_0(G)$ consisting of functions of the form

$$u(x) = \sum_{i=1}^{\infty} (f_i * \tilde{g}_i)(x)$$

where $f_i, g_i \in \mathbf{L}^2(G)$, $\tilde{g}(x) = g(x^{-1})$ and where

$$\sum_{i=1}^{\infty} \|f_i\|_2 \|g_i\|_2 < \infty.$$

The restriction of the norm $\|\cdot\|_{\mathbf{B}(G)}$ to $\mathbf{A}(G)$ is denoted $\|\cdot\|_{\mathbf{A}(G)}$ and can be given by the formula

$$\|u\|_{\mathbf{A}(G)} = \inf\left\{\sum_{i=1}^{\infty}\|f_i\|_2\|g_i\|_2 \;\middle|\; u = \sum_{i=1}^{\infty}(f_i * \tilde{g_i})\right\}.$$

It is possible to remove the "infinite sum" in the above construction. Indeed it is possible to show that $\mathbf{A}(G)$ consists of functions of the form $u = f * \tilde{g}$ for $f, g \in \mathbf{L}^2(G)$.

As a third alternative, $\mathbf{A}(G)$ can be recognized as the closure in the $\mathbf{B}(G)$ norm of the space $\mathbf{B}(G) \bigcap C_c(G)$ (which the author notes is the most transparent way to observe that $\mathbf{A}(G)$ is both an algebra and an ideal of $\mathbf{B}(G)$). The Fourier algebra was first studied for non-abelian groups by Eymard in [28].

The Banach space dual of $C^*_\lambda(G)$ can be identified with a closed ideal of $\mathbf{B}(G)$ which we denote $\mathbf{B}_\lambda(G)$. $\mathbf{B}_\lambda(G)$ is the weak-* closure of $\mathbf{A}(G)$ and moreover $\mathbf{A}(G)$ is also a closed ideal of both $\mathbf{B}_\lambda(G)$ and $\mathbf{B}(G)$.

The dual of $C^*_\delta(G)$ is denoted by $\mathbf{B}_\delta(G_d)$. It is a weak-* closed subalgebra of $\mathbf{B}(G_d)$. In general, $\mathbf{B}_\delta(G_d)$ contains $\mathbf{A}(G_d)$ and hence $\mathbf{B}_\lambda(G_d)$. Moreover $\mathbf{B}_\delta(G_d) = \mathbf{B}(G_d)$ if and only if G is amenable.[8]

All of the above facts can be found in [28] or [22].

Let $\hat{G}$ denote the subset[1] of Σ_G consisting of irreducible representations of G. If G is abelian, it is well known that each element of $\hat{G}$ is one-dimensional. In this case it possible to recognize $\hat{G}$ as an abelian topological group called the ***dual group of*** G. The Pontryagin duality theorem states that $\hat{\hat{G}}$ is both isomorphic

[1]We note that Σ_G is strictly not a set, however by restricting the size of the Hilbert spaces upon which our representations act, we can avoid this set theoretic difficulty

and homeomorphic to G. For general non-abelian groups G, we note that the set $\hat{G}$ has no natural group structure, and hence we shall call this set simply the ***dual of*** G.

If G is abelian and $\mu \in \mathbf{M}(G)$, we define the ***Fourier Stieltjes transform*** $\hat{\mu}$ by

$$\hat{\mu}(\pi) = \pi(\mu) \ \forall \, \pi \in \hat{G}.$$

One of the fundamental results in abelian harmonic analysis is that the Fourier-Stieltjes transform establishes an isometric isomorphism between $\mathbf{M}(G)$ and $\mathbf{B}(\hat{G})$ and that the restriction to the $\mathbf{L}^1(G)$ (called the ***Fourier transform***) is an isometric isomorphism onto $\mathbf{A}(\hat{G})$.

Note that in general $\mathbf{A}(G) = \mathbf{B}(G)$ if and only if G is compact.

Let $\mathbf{VN}(G)$ denote the von-Neumann subalgebra of $B(\mathbf{L}^2(G))$ generated by $\lambda(\mathbf{L}^1(G))$ or alternatively generated by $\lambda(G)$. We call $\mathbf{VN}(G)$ the ***group von-Neumann algebra*** of G. It can be shown that $\mathbf{A}(G)^* = \mathbf{VN}(G)$ and that the weak-* topology in $\mathbf{VN}(G)$ coincides with the weak operator topology. Some authors write $L(G)$ for $\mathbf{VN}(G)$ to help differentiate between the von-Neumann algebras generated by the left and right regular representations. However since we shall make no use of this latter algebra, we need not make this distinction.

Given an element $u \in \mathbf{A}(G)$ we define the ***zero set*** of u, denoted $Z(u)$ by

$$Z(u) = \{g \in G : u(g) = 0\}.$$

Note that since u is continuous, $Z(u)$ is closed. Also for a closed ideal $J \subset \mathbf{A}(G)$ we define the ***hull*** of J denoted $h(J)$ as follows

$$h(J) = \{x \in G : f(x) = 0 \ \forall f \in J\}.$$

Given a closed set $E \in G$ we define the closed ideal $\mathcal{I}(E)$ by

$$\mathcal{I}(E) = \{f \in \mathbf{A}(G) : f(x) = 0 \ \forall x \in E\}.$$

The ***coset ring*** of G denoted $\Omega(G)$ is the smallest ring of sets, which is closed under finite unions, intersections and translations containing all open subgroups, and we define the ***closed coset ring*** of G, denoted $\Omega_c(G)$ to consist of all elements of $\Omega(G_d)$ which are closed in G.

A closed set $E \subset G$ is called a ***set of spectral synthesis*** (or simply an ***S-set***) if $\mathcal{I}(E)$ is the only ideal whose hull is E. The classification of S-sets for general groups G seems impossibly difficult, however some partial results are known. In particular, if E is a discrete subset of G then E is known to be an S-set.

2.4 Operator Spaces

An *operator space* is a vector space V together with a family $\| \ \|_n$ of Banach space norms (called *operator space norms*) on $\mathbb{M}_n(V)$, the space of $n \times n$ matrices with entries in V such that

$$\text{(i)} \quad \left\| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right\|_{n+m} = \max\{\|A\|_n \ , \ \|B\|_m\}$$

for each $A \in \mathbb{M}_n(V)$, $B \in \mathbb{M}_m(V)$, and

$$\text{(ii)} \quad \|([a_{ij}])A([b_{ij}])\|_n \leq \|[a_{ij}]\|\|A\|_n\|[b_{ij}]\|$$

for each $[a_{ij}]$, $[b_{ij}] \in \mathbb{M}_n(\mathbb{C})$ and $A \in \mathbb{M}_n(V)$

We note that given operator space norms $\|\cdot\|_n$ on an operator space V, there is an induced norm on $\mathcal{K} \otimes V$ where $\mathcal{K}$ denotes the compact operators. If we let σ denote the set of matrix norms, then we shall denote the closure of this tensor product by $\mathcal{K} \otimes_\sigma V$.

Let X and Y be operator spaces and let $T : X \mapsto Y$. For each $n \in \mathbb{N}$ define

$$T^{(n)} : \mathbb{M}_n(X) \mapsto \mathbb{M}_n(Y)$$

by

$$T^{(n)}[x_{ij}] = [Tx_{ij}]$$

The map T is said to be ***completely bounded*** (or simply ***c.b.*** for short) if $\sup\{\|T^{(n)}\|\} < \infty$. In this case we let $\|T\|_{cb} = \sup\{\|T^{(n)}\|\}$. We say that T is a ***complete isometry*** if each $T^{(n)}$ is an isometry and that T is a ***complete contraction*** if each $T^{(n)}$ is a contraction. We say that two operator spaces X and Y are ***c.b. isomorphic*** if there exists a c.b. map $T : X \mapsto Y$ such that T^{-1} is also completely bounded. Furthermore we shall say that X and Y are ***c.b. isometrically isomorphic*** (or ***completely isometrically isomorphic***) if the map T can be chosen to be a complete isometry.

For the Hilbert space H, we let

$$H^{(n)} = \underbrace{H \oplus \cdots \oplus H}_{n}.$$

Since there is a canonical identification between $\mathbb{M}_n(B(H))$ and $B(H^{(n)})$, it is easy to show that $B(H)$ (and hence any closed subspace) is an operator space.

It is a fundamental result in the theory that every operator space is completely isometrically isomorphic to a norm closed subspace S of $B(H)$, the algebra of bounded operators on the Hilbert space H, where the operator space structure on S is the structure inherited from $B(H)$ ([24]).

We note that $\mathbb{M}_n \otimes X \cong \mathbb{M}_n(X)$. As such we may recognize the map $T^{(n)}$ by

$$T^{(n)} = id_n \otimes T : \mathbb{M}_n(\mathbb{C}) \otimes X \mapsto \mathbb{M}_n(\mathbb{C}) \otimes Y$$

where id_n represents the identity map on $\mathbb{M}_n(\mathbb{C})$. Thus we can conclude that

$$\|T\|_{cb} = \|id_{\mathcal{K}} \otimes T\|$$

where $id_{\mathcal{K}}$ is the identity map on the compact operators.

Given a general Banach space X, there exist two natural operator space structures called the ***maximal operator space*** and ***minimal operator space*** structures denoted $MAX(X)$ and $MIN(X)$ respectively.

We define the MAX structure as follows: for $[a_{ij}] \in \mathbb{M}_n(X)$ we let

$$\|[a_{ij}]\|_{n,MAX} = \sup\{\|\phi([a_{ij}])\| : \text{for all } \phi : X \mapsto B(H) \text{ with } \phi \text{ contractive}\}.$$

It is easy to see that given any operator space Y and any bounded linear map

$$T : MAX(X) \mapsto Y$$

it follows that T is automatically completely bounded with $\|T\|_{cb} = \|T\|$.

The MIN operator space structure can be recognized in at least three ways: first we can consider the natural embedding of X into its second dual X^{**} given by

$$x \mapsto \hat{x}$$

where

$$\hat{x}(\phi) = \phi(x) \quad \forall \phi \in X^*.$$

If we let X_1^* denote the (compact) unit ball in X^* with the weak-* topology, then it is easy to see that $\hat{x} \in C(X_1^*)$, and hence X becomes identified with a subspace of a

C^*-algebra. The restriction of the natural operator space on $C(X_1^*)$ to X becomes an operator space denoted $MIN(X)$.

Alternatively, we may consider the matrix norms given by the Banach space injective tensor norm ($\otimes_\lambda$) and set $\mathbb{M}_n(X) = X \otimes_\lambda \mathbb{M}_n(\mathbb{C})$. The final alternative is to set for $[x_{ij}] \in \mathbb{M}_n(X)$,

$$\|[x_{ij}]\|_n = \sup\{\|[\phi(x_{ij})]\|_n : \phi \in X_1^*\}.$$

It can be shown that given any operator space Y and any bounded linear map

$$S : Y \mapsto MIN(X),$$

we have that S is automatically completely bounded with $\|S\|_{cb} = \|S\|$ (see [12] and [9]).

If we let $CB(X,Y)$ denote the space of all completely bounded maps from X to Y, then $CB(X,Y)$ has a natural operator structure which can be obtained by identifying $\mathbb{M}_n(CB(X,Y))$ with $CB(X, \mathbb{M}_n(Y))$. It is important to note that continuous linear functionals are automatically completely bounded. In fact, since we can identify X^* with $CB(X,\mathbb{C})$, X^* is also an operator space called the ***standard dual*** of X (see [10]).

For operator spaces X, Y and Z, we call a bilinear map $T : X \times Y \mapsto Z$ ***jointly completely bounded***, if for $[x_{ij}] \in \mathbb{M}_n(X)$ and $[y_{kl}] \in \mathbb{M}_m(Y)$ we have that

$$\|T\|_{jcb} = \sup\{\|[T(x_{ij}, y_{kl})]\|_{mn} : \|[x_{ij}]\|_n \leq 1, \|[y_{kl}]\|_m \leq 1\}$$

is finite. Now there is an operator space analogue of the projective tensor product which we denote $X\hat{\otimes}Y$ such that

$$JCB(X,Y;Z) = CB(X\hat{\otimes}Y, Z).$$

That is to say each jointly completely bounded map extends to a unique map on this ***operator space projective tensor product.*** In particular, there is a complete isometry between $(X\hat{\otimes}Y)^*$ and $CB(X, Y^*)$.

We can define the norm of a typical element in the operator space projective tensor product with the following. Let $[x_{ij}] \in \mathbb{M}_p(X)$ and $[y_{kl}] \in \mathbb{M}_q(Y)$. We define the tensor product $x \otimes y$ to be the $pq \times pq$ matrix

$$x \otimes y = [x_{ij} \otimes y_{kl}] \in \mathbb{M}_{pq}(X \otimes Y).$$

Given any element $u \in \mathbb{M}_n(X \otimes Y)$, we can write

$$u = \alpha(x \otimes y)\beta$$

for some $\alpha \in \mathbb{M}_{n,pq}(\mathbb{C}), x \in \mathbb{M}_p(X), y \in \mathbb{M}_q(Y),$ and $\beta \in \mathbb{M}_{pq,n}(\mathbb{C})$. Now we have that the operator space projective tensor norm is given by

$$\|u\|_n = \inf\{\|\alpha\|\|x\|\|y\|\|\beta\|\}$$

where the infimum is taken over all such representations of u.

There is another tensor product which we will refer to called the ***Haagerup tensor product*** denoted $\otimes_h$. (See [27])

An associative algebra $\mathcal{A}$ which is also an operator space and is such that the multiplication

$$m : \mathcal{A}\hat{\otimes}\mathcal{A} \mapsto \mathcal{A}$$

is completely contractive is called a ***completely contractive Banach algebra.*** If $\mathcal{A}$ is a completely contractive Banach algebra and $\mathcal{J}$ is a closed ideal of $\mathcal{A}$, it is easy to see that both $\mathcal{J}$ and $\mathcal{A}/\mathcal{J}$ are completely contractive Banach algebras with respect to the operator space structure each inherits from $\mathcal{A}$ (see [25]).

A Banach algebra $\mathcal{A}$ with an operator space structure such that the multiplication

$$m : \mathcal{A} \otimes_h \mathcal{A} \mapsto \mathcal{A}$$

is completely contractive will be called an ***h-algebra***. We note that if X is the predual of a von Neumann algebra $\mathcal{A}$, it inherits a natural operator space structure as follows: for $[x_{ij}] \in \mathbb{M}_n(X)$ we set

$$\|[x_{ij}]\|_n = \sup\{\|[f_{kl}(x_{ij})]\|_{nm} \ : \ [f_{kl}] \in \mathbb{M}_m(\mathcal{A}), \|[f_{kl}]\|_m \leq 1\}$$

Thus both the Fourier and Fourier-Stieltjes algebras can be given natural operator structures by virtue of their being preduals of von Neumann algebras. In each case, this operator space structure results in a completely contractive Banach algebra (see [68] and [10])

Given two operator spaces X and Y, we can consider the direct sum $X \oplus Y$ to be an operator space where

$$\|[x_{ij} \oplus y_{ij}]\|_n = \max\{\|[x_{ij}]\|_n, \|[y_{ij}]\|_n\}.$$

Unless otherwise noted, whenever we are given the direct sum of two operator spaces, we shall consider it to be an operator space in this way.

2.5 Homology and Amenable algebras

We begin with some standard definitions from homological algebra as applied to Banach spaces. The basic references are [63] and [14].

A ***left Banach $\mathcal{A}$-module*** is a left $\mathcal{A}$-module X that is itself a Banach space and for which

$$\|ax\|_X \leq \|a\|_{\mathcal{A}}\|x\|_X$$

for each $a \in \mathcal{A}$ and each $x \in X$. A ***right*** and ***two-sided*** Banach $\mathcal{A}$ module is defined analogously. We call a two sided module a ***bimodule***. If X is a left Banach $\mathcal{A}$-module, then X^* becomes a right Banach $\mathcal{A}$-module with respect to the action

$$(\phi a)(x) = \phi(ax).$$

We call X^* a ***dual right Banach $\mathcal{A}$-module***. Naturally we can define dual left and bimodules analogously.

In general, we will call an $\mathcal{A}$-module X ***symmetric*** if $ax = xa$ for every $a \in \mathcal{A}$ and $x \in X$. In particular, $\mathcal{A}^*$ is symmetric if and only if $\mathcal{A}$ is commutative.

In the category of operator spaces there are two ways to define an operator module. In this thesis we shall call an operator space X, which is a left Banach $\mathcal{A}$-module a ***left operator $\mathcal{A}$-module*** if the module map is completely contractive with respect to the projective tensor product, that is to say the module map

$$M : \mathcal{A}\hat{\otimes}X \mapsto X$$

is completely contractive. Clearly we may define operator right and bimodules analogously. Furthermore if X is a operator module, then X^* becomes a dual operator module with the dual actions defined above.

The second approach is to ask that the module action is completely contractive from the Haagerup tensor product, that is

$$M : \mathcal{A} \otimes_h X \mapsto X$$

is completely contractive. We shall call such a module X an ***h-operator*** module.

Suppose X is a left operator $\mathcal{A}$-module and Y an operator space. Then we may consider $X\hat{\otimes}Y$ as a left operator $\mathcal{A}$ module by

$$a \cdot (x \otimes y) = ax \otimes y$$

for $a \in \mathcal{A}, x \in X$ and $y \in Y$. It is clear that if Y is a right operator module, then $X\hat{\otimes}Y$ becomes a right operator module in the analogous way.

By a ***chain complex*** we mean a sequence of objects X_n with $n \in \mathbb{Z}$ and morphisms $d_n : X_{n+1} \mapsto X_n$ such that $d_n \circ d_{n+1} = 0$ for all $n \in \mathbb{Z}$. The objects could be Banach spaces, Banach algebras, Banach modules etc. and the maps naturally will be respectively Banach space maps, Banach algebra maps, Banach module maps etc.

In Chapter 3 of this thesis we will investigate certain chain complexes of operator spaces and completely bounded maps.

Typically, a chain complex is written as

$$(\Xi) : \ldots \leftarrow X_n \overset{d_n}{\leftarrow} X_{n+1} \leftarrow \ldots$$

with the arrows "pointing left". The condition that $d_n \circ d_{n+1} = 0$ is clearly equivalent to $\mathrm{im} d_{n+1} \subset \ker d_n$. We define for $n \in \mathbb{Z}$ the ***nth-homology group***[2] of the chain complex, denoted $H_n(\Xi)$ by

$$H_n(\Xi) = \frac{\ker d_n}{\mathrm{im} d_{n+1}}$$

The elements of $\mathrm{im} d_{n+1}$ are called ***n-boundaries*** and the elements of $\ker d_n$ are called ***n-cycles*** (so $H_n(\Xi)$ represents "cycles mod boundaries").

[2]We use the term "group" out of historical consistency. Indeed we have no interest in the algebraic structure of this object.

By a ***cochain complex*** we shall mean a sequence

$$(\Pi) : \ldots \to Y_n \xrightarrow{\delta_n} Y_{n+1} \to \ldots$$

with $\delta_{n+1} \circ \delta_n = 0$. Clearly there is no substantive difference between the two concepts other than the direction of the arrows, however it is standard and sometimes convenient in practice to have both concepts. It will often be the case that we will have $Y_n = X_n^*$ and $\delta_n = d_n^*$, and hence the cochain is the dual of the chain complex.

Analogous to above, we define the ***nth-cohomology group*** of the cochain complex, denoted $H^n(\Pi)$ by

$$H^n(\Pi) = \frac{\ker \delta^{n+1}}{\mathrm{im}\delta^n}.$$

Again as above, we call the elements of $\mathrm{im}\delta^n$ ***n-coboundaries*** and we call the elements of $\ker \delta^{n+1}$ ***n-cocycles*** . (Thus $H^n(\Pi)$ represents "cocycles mod coboundries). The elements d_n (or δ_n as the case may be) are called ***differentials***.

Typically a sequence may be "bounded by zeros" either on the left or right. In this case we will usually suppress these zeros. For example the sequence

$$\ldots 0 \to 0 \to 0 \to Y_0 \to Y_1 \to \ldots$$

will usually be written simply

$$0 \to Y_0 \to Y_1 \to \ldots.$$

A chain complex is called ***exact at*** X_n if $\mathrm{im} d_n = \ker d_{n-1}$, and the complex is called ***exact*** if it is exact at every term. The exactness at Y_n and the exactness of a cochain complex is defined analogously.

A linear map D from a Banach algebra $\mathcal{A}$ into a Banach $\mathcal{A}$ bimodule is called a ***derivation*** if

$$D(ab) = a \cdot D(b) + D(a) \cdot b$$

for all $a, b \in \mathcal{A}$. A derivation is called ***inner*** if there exists an element $x \in X$ such that

$$D(a) = a \cdot x - x \cdot a.$$

We say that a Banach algebra is ***Banach algebra amenable*** if every bounded derivation from $\mathcal{A}$ into any dual Banach $\mathcal{A}$ bimodule is automatically inner.

Finally if X is a space of any sort, we shall use the notation id_X (or simply id when no confusion arises) to denote the identity morphism.

Chapter 3

Homology in Operator Spaces

3.1 Introduction

In this chapter we present the homological algebra theory for completely contractive Banach Algebras. We note that Paulsen in [66] developed a cohomology theory for h-algebras which is similar to our development. However there appear to be several limitations to that category. Most noteworthy is the absence of a homology theory for which the cohomology theory is dual.

Our approach is to use the projective tensor product to develop a homology, and thus to recognize the usual cohomology theory as simply the dual of the homology.

The translation of the algebraic homological ideas to the category of Banach spaces is of course not new. See for example the work of Taylor [74], Kamowitz [51] and Kadison and Ringrose [50]. Thus it is important to note that the translation into our new category involves mainly an array of technical facts which allows us to follow in the footsteps of the algebraists. Indeed Johnson in [49] and most importantly Taylor and Khelemskii ([54] and [53]) did exactly this in the Banach space setting, generalizing the algebraic methods of Hochschild (see [43] and [44]). A good reference for the algebraic presentation is Cartan and Eilenberg [14].

The presentation of Paulsen follows the so called relative Yoneda cohomology as well as the derived functor presentation. In this sense he generalized the algebraic approaches found in MacLane [63].

The approach presented here is to extend the ideas of Taylor, Kamowitz and Khelemskii to the completely contractive algebra setting. This approach provides a complete homology theory in this category, which both reflects the dual nature of the cohomology, and explores the notion of split extensions – which for us will ultimately be the most useful application.

3.2 Extension Sequences

One of the basic concepts and tools in homological algebra is that of a short exact sequence. Recall that a short exact sequence of objects in an abelian category is a complex of the form

$$0 \to X \xrightarrow{f} A \xrightarrow{g} Y \to 0$$

where f is injective, g is surjective and $\ker g = \mathrm{im} f$. We note the expected fact that $A/f(X)$ is isomorphic in the category to Y, and naturally we write $A/f(X) \cong Y$. In this case we say that A is an extension of X by Y.

Unfortunately, this concept breaks down in the case of operator spaces. Consider the following short exact sequence:

$$0 \to 0 \to MAX(X) \xrightarrow{id} MIN(X) \to 0$$

where *id* represents the identity map. If X is any infinite dimensional Banach space, then $MAX(X)/0 = MAX(X)$ is not isomorphic in the category of operator spaces to $MIN(X)$. In this sense, one of the basic objects of homology fails to "do what we want" in our new category.

The basic way to repair this is to restrict the sequences under consideration. In [69], the authors considered cases where f was a "complete isometry" and g was a "complete quotient" map. In [79] the present author considered sequences where both f and g had inverses which were completely bounded.

In this section, we will establish a broad class of short exact sequences which avoids this isomorphism dilemma, and we will show that in some sense, this class is as broad as possible.

DEFINITION 3.2.1: Given two operator spaces X and Y, we say a c.b. map $T : X \mapsto Y$ has the ***complete isomorphism property (c.i.p.)*** if the image $T(X)$ is closed and the induced map $\tilde{T} : X/\ker T \mapsto T(X)$ is a c.b. isomorphism.

We note that any bounded map between Banach spaces satisfies the analogous property. The leads to the following:

DEFINITION 3.2.2: A chain (or cochain) complex of operator spaces is called an ***operator complex*** if each of the differential maps has the complete isomorphism property.

As discussed earlier, an important special case of a chain complex is of course the short exact sequences. In this thesis, we call any short exact operator chain complex an ***extension sequence*** or ***1-extension sequence***.

This leads to the following proposition which suggests that operator complexes are the correct tool for our category:

PROPOSITION 3.2.3: *Suppose X, Y and Z are operator spaces such that*

$$(\Xi) : 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

forms an extension sequence.Then $Y/f(X)$ is c.b. isomorphic to Z.

Proof: Since (Ξ) is a short exact sequence, we have $\mathrm{im} f = \ker g$ thus $Y/f(x) = Y/\ker g$. Since (Ξ) is an operator complex, the map g has the complete isomorphism property and hence there is a c.b. isomorphism between $Y/\ker g$ and $\mathrm{im} g = Z$ (by exactness). Thus $Y/f(X)$ is c.b. isomorphic to Z. ∎

Conversely, we have the following:

PROPOSITION 3.2.4: *Suppose X, Y and Z are operator spaces with $X \subset Y$ such that Y/X is c.b. isomorphic to Z. Then there is an extension sequence (Ξ) of the form*

$$(\Xi) : 0 \to X \xrightarrow{i} Y \xrightarrow{f} Z \to 0$$

where i represents the inclusion map $i : X \hookrightarrow Y$.

Proof: Consider the canonical quotient map $q : Y \mapsto Y/X$. By construction the short exact sequence

$$0 \to X \to Y \to Y/X \to 0$$

is an extension sequence. Let $T : Y/X \mapsto Z$ be a c.b. isomorphism. Then it is easy to see that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, equal] & Y \arrow[r, "q"] \arrow[d, equal] & Y/X \arrow[d, bend left, "T"] \\ X \arrow[r, "i"] & Y \arrow[r, "f"] & Z \arrow[u, bend left, "T^{-1}"] \end{tikzcd}$$

Hence the bottom sequence is an extension sequence.

■

In the obvious way we consider an *n-extension sequence*. Given $\mathcal{A}$ operator modules X and Y, any exact operator complex of the form

$$0 \to X \xrightarrow{\pi_1} B_1 \xrightarrow{\pi_2} B_2 \to \ldots \to B_n \xrightarrow{\pi_{n+1}} Y \to 0$$

is called an ***n-extension of X by Y***. Furthermore we call the above sequence an ***n-extension sequence***.

Unlike in algebra, it is usually necessary to consider further topological conditions on our extension sequences. An additional condition is that we will require $\mathrm{im}\pi_k$ to be complemented. The importance of this latter condition will become apparent a little later.

DEFINITION 3.2.5: An exact operator complex of $\mathcal{A}$-modules

$$\ldots \to X_{k-1} \xrightarrow{\pi_{k-1}} X_k \xrightarrow{\pi_k} X_{k+1} \xrightarrow{\pi_{k+1}} \ldots$$

is called ***admissible*** if there exist completely bounded maps (not necessarily $\mathcal{A}$-module maps) $\theta_k : X_{k+1} \mapsto X_k$ such that $\pi_k \circ \theta_k = id_{\ker \pi_{k+1}}$. An admissible complex is said to ***split*** if the maps θ_k can be chosen to be module maps. Thus a complex is admissible exactly when it splits as a complex of $\mathbb{C}$-modules.

In the special case of n-extension sequences, we note that admissibility is equivalent to the existence of a commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, equal] & B_1 \arrow[r] \arrow[d, "\tau_1"] & \cdots \arrow[r] & B_n \arrow[r] \arrow[d, "\tau_1"] & Y \arrow[d, equal] \\
X = F_0 \arrow[r, "\pi_0"] & E_1 \oplus F_1 \arrow[r, "\pi_1"] & \cdots \arrow[r] & E_n \oplus F_n \arrow[r, "\pi_n"] & E_{n+1} = Y
\end{tikzcd}$$

where the maps $\pi_k |_{F_k}: F_k \mapsto E_{k+1}$ and the maps $\tau_k : B_k \mapsto E_k \oplus F_k$ are c.b. isomorphisms. For the case $n = 1$ the present author in [79] referred to this property as *completely admissible*.

Finally, we call a map $\phi : X \mapsto Y$ ***admissible*** if there exists a map $\theta : Y \mapsto X$ such that $\phi \circ \theta = id_{\mathrm{im}\phi}$. Furthermore, we call this map θ a ***right inverse*** for ϕ.

We now have an analogue of Proposition 1.1 from [19]. See also the special case of this in [69].

LEMMA 3.2.6: *Let*

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

be an extension sequence of $\mathcal{A}$-bimodules. Then there exists a completely bounded map $F : Y \mapsto X$ such that $Ff = id_X$ if and only if there exists a map $G : Z \mapsto Y$ such that $gG = id_Z$. Furthermore F is a module map if and only if G is.

Proof: Suppose F exists. Then clearly the map fF is a completely bounded projection onto $\mathrm{im} f \subset Y$. Thus the following diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[r, "g"] \arrow[d, "fF\oplus(1-fF)"] & Z \arrow[d, equal] \\ f(X) \arrow[r, hook, "i"] & f(X)\oplus Q \arrow[r, "g"] & Z \end{tikzcd}$$

commutes, where Q is the complement of $f(X)$ in Y and the map $i : f(X) \mapsto Y$ is given by $i(x) = x \oplus 0$. Since $\mathrm{im} f = f(X) = \ker g$ and since g has the c.i.p., it follows that the induced map

$$\tilde{g} : Y/f(X) \mapsto Z$$

is a c.b. isomorphism. Since $Y/f(X) \cong Q$ it follows that $g|_Q$ is a c.b. isomorphism. Now let $G(z) = (g|_Q)^{-1}$. The fact that $gG = id_Z$ is now trivial.

Now assume that G exists. Similar to above, we see that Gg is a completely bounded projection onto a subspace P of Y which is c.b. isomorphic to Z. Let Q be the complement of P. Thus $1 - Gg$ is a completely bounded projection onto Q. Note that

$$g(1 - gG) = g - gGg = g - g = 0$$

thus $Q \subset \ker g = \mathrm{im} f$. Since $g|_P$ is an isomorphism, the reverse inclusion is obvious. Hence $Q = \ker g = \mathrm{im} f$. Since f has the c.i.p., the map $f^{-1} : Q \mapsto X$ is completely bounded. Thus we define

$$F : Y \mapsto Z$$

by

$$F(y) = f^{-1}(y - Gg(y))$$

Clearly F satisfies the desired properties. The fact that F is a module map if and only if G is a module map is strictly algebraic. Suppose that G is a module map. Note that

$$F(ay) = f^{-1}(ay - Gg(ay)) = af^{-1}(y - Gg(y)) = aF(y)$$

The right module action is similar. Conversely if F is a module map, then the subspace Q is a submodule, hence $G|_Q$ is a module map. Thus G is a module map also.

■

We note that the above Lemma fails for general short exact sequences. Consider the MAX/MIN example at the beginning of this section, see also [79].

It will arise that we will be given a completely contractive Banach algebra $\mathcal{A}$ and a left operator $\mathcal{A}$-module X, and we will wish to know when the module map

$$\pi : \mathcal{A}\hat{\otimes}X \mapsto X$$

has an inverse. Obviously this is impossible immediately whenever the map π is not onto. We call a module ***neounital*** if

$$\mathcal{A} \cdot X = \{a \cdot x : a \in \mathcal{A},\ x \in X\} = X,$$

in which case π is clearly onto. If our completely contractive Banach algebra has a bounded approximate identity $\{e_\alpha\}$, and if $x = \lim e_\alpha x$ for all $x \in X$, then we can use Cohen's Factorization theorem to guarantee X is neounital. Indeed when studying $\mathbf{L}^1(G)$ this can be a useful approach. Once again, in our setting this method will fail us. In our primary example, $\mathbf{A}(G)$ does not have a bounded approximate identity when G is not amenable, and indeed it is known that for non-amenable groups G, $\mathbf{A}(G)$ is not even a neounital $\mathbf{A}(G)$ module! (see [61]).

We solve this problem the following way: Given a Banach algebra $\mathcal{A}$ we can construct its unitization $\mathcal{A}_+$ as follows:

Let $\mathcal{A}_+ = \mathcal{A} \oplus \mathbb{C}$, and now define multiplication as follows:

$$(a, \alpha) \cdot (b, \beta) = (ab + \alpha b + \beta a, \alpha\beta).$$

Now note that if X is a left (right,bi) $\mathcal{A}$-module, then X becomes a unital left (right,bi) $\mathcal{A}_+$-module with respect to the action

$$(a, \alpha) \cdot x = a \cdot x + \alpha x$$

(a right and bimodule structure is defined analogously). In [26] Effros and Ruan showed that there is an operator space structure such that $\mathcal{A}_+$ is indeed a completely contractive Banach algebra. Using the same techniques, we can show that any operator $\mathcal{A}$ module X becomes a unital operator $\mathcal{A}_+$ module.

For completeness, we recall their construction for $\mathcal{A}_+$, then we extend this in the obvious way to show X is an operator $\mathcal{A}_+$ module.

Suppose we are given two operator spaces V and W. We construct the operator space $V^* \oplus W^*$. The induced operator space structure on the predual will be denoted $V \oplus_1 W$. We have the following important fact concerning this structure:

PROPOSITION 3.2.7: *Suppose X is an operator space and suppose that*

$$\phi : V \mapsto X \ \ \text{and} \ \ \psi : W \mapsto X$$

are completely contractive. Then the map

$$\phi \oplus_1 \psi : V \oplus_1 W \mapsto X$$

given by

$$(\phi \oplus_1 \psi)(v \oplus_1 w) = \phi(v) + \psi(w)$$

is completely contractive.

Now we give $\mathcal{A}_+$ the operator space structure $\mathcal{A} \oplus_1 \mathbb{C}$. Using the previous proposition Effros and Ruan have shown that if $\mathcal{A}$ is a completely contractive Banach algebra then so is $\mathcal{A}_+$. Using the same technique, we have

LEMMA 3.2.8: *If X is a left (right, bi) operator $\mathcal{A}$ module, then X is an essential left (right,bi) operator $\mathcal{A}_+$ module.*

Proof: The fact that X is essential is obvious. Now we simply note that the module map $(a, \alpha) \cdot x \mapsto a \cdot x + \alpha x$ is the sum of two completely contractive maps which, by Proposition 3.2.7 is clearly completely contractive. Thus X is a left operator $\mathcal{A}_+$ module (see [26]). The right and bimodule cases follow analogously.

■

3.3 Resolutions and Derived Functors

First we shall introduce the following notation

$$CB_{\mathcal{A},\mathbb{C}}(X, Z) = \{T \in CB(X, Z) \mid T(ax) = aT(x) \ \ \forall x \in X, a \in \mathcal{A}\}$$

$$CB_{\mathbb{C},\mathcal{A}}(X,Z) = \{T \in CB(X,Z) \mid T(xa) = T(x)a \ \ \forall x \in X, a \in \mathcal{A}\}$$

$$CB_{\mathcal{A},\mathcal{A}}(X,Z) = \{T \in CB(X,Z) \mid T(axb) = aT(x)b \ \ \forall x \in X, a, b \in \mathcal{A}\}$$

Naturally, these sets define respectively the morphisms in the category of left, right and two-sided operator modules.

Given X and Y left operator $\mathcal{A}$ modules, we can define a contravariant[1] functor denoted $CB_{\mathcal{A},\mathbb{C}}(?,Z)$ as follows: for any c.b. module map

$$\phi : X \mapsto Y$$

we define

$$CB_{\mathcal{A},\mathbb{C}}(\phi, Z) = \phi_* : \ CB_{\mathcal{A},\mathbb{C}}(Y,Z) \mapsto \ CB_{\mathcal{A},\mathbb{C}}(X,Z)$$

given by

$$\phi_*(T)(x) = T(\phi(x)).$$

Clearly we can define the contravariant functors $CB_{\mathbb{C},\mathcal{A}}(?,Z)$ and $CB_{\mathcal{A},\mathcal{A}}(?,Z)$ analogously. Furthermore, using the obvious changes, we can define covariant[2] functors $CB_{\mathcal{A},\mathbb{C}}(Z,?), CB_{\mathbb{C},\mathcal{A}}(Z,?)$ and $CB_{\mathcal{A},\mathcal{A}}(Z,?)$.

To see that ϕ_* is completely bounded, we have the following: for $[x_{kl}] \in \mathbb{M}_m(X)$

[1] contravariant=arrow reversing

[2] covariant=arrow preserving

and $[T_{ij}] \in \mathbb{M}_n(CB_{\mathcal{A},\mathbb{C}}(X, Z))$

$$
\begin{aligned}
\|\phi_*^{(n)}\|_n &= \sup\{\|\phi_*^{(n)}([T_{ij}])([x_{kl}])\| : \|[T_{ij}]\|_n \le 1, \|[x_{kl}]\|_m \le 1\} \\
&= \sup\{\|[T_{ij}[\phi(x_{kl})]]\|_{nm} : \|[T_{ij}]\|_n \le 1, \|[x_{kl}]\|_m \le 1\} \\
&\le \sup\{\|[T_{ij}]\|_n \|[\phi(x_{kl})]\|_m : \|[T_{ij}]\|_n \le 1, \|[x_{kl}]\|_m \le 1\} \\
&\le \sup\{\|[T_{ij}]\|_n \|\phi\|_{cb} \|[x_{kl}]\|_m : \|[T_{ij}]\|_n \le 1, \|[x_{kl}]\|_m \le 1\} \\
&\le \|\phi\|_{cb}
\end{aligned}
$$

A second functor of interest in homology theory is the tensor product functor. Suppose we are given two operator $\mathcal{A}$-bimodules X and Y. We define the tensor product $X \otimes_{\mathcal{A}} Y$ as follows:

Consider the operator subspace N of $X \hat{\otimes} Y$ given by the closed linear span of elements of the form

$$xa \otimes y - x \otimes ay$$

Now define $X \otimes_{\mathcal{A}} Y$ by

$$X \otimes_{\mathcal{A}} Y = X \hat{\otimes} Y / N$$

Similar to above, we can recognize $? \otimes_{\mathcal{A}} Z$ as a covariant functor as follows: for any c.b. module map

$$\phi : X \mapsto Y$$

we have

$$\phi \otimes_{\mathcal{A}} Z = \phi_* : X \otimes_{\mathcal{A}} Z \mapsto Y \otimes_{\mathcal{A}} Z$$

given by

$$\phi_*(x \otimes_{\mathcal{A}} z) = \phi(x) \otimes_{\mathcal{A}} z.$$

To see that ϕ_* is completely bounded we note that the map

$$\phi\hat{\otimes}id_Z : X\hat{\otimes}Z \mapsto Y\hat{\otimes}Z$$

is completely bounded with $\|\phi\hat{\otimes}id_Z\|_{cb} \leq \|\phi\|_{cb}$ (see [12]).

Also since the following diagram is commutative

$$\begin{tikzcd}
X\hat{\otimes}Z \arrow[r, "\phi\hat{\otimes}id_Z"] \arrow[d, "q_1"'] & Y\hat{\otimes}Z \arrow[d, "q_2"] \\
X \otimes_{\mathcal{A}} Z \arrow[r, "\phi_*"'] & Y \otimes_{\mathcal{A}} Z
\end{tikzcd}$$

where q_i are the canonical quotients, it follows that ϕ_* is completely bounded. Using identical arguments, it is now easy to see how to construct a covariant functor $X\otimes_{\mathcal{A}}?$.

Suppose we are given an $\mathcal{A}$-module X, a complex of the form

$$(\mathfrak{P}) : 0 \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \dots$$

and a map $\epsilon : P_0 \mapsto X$ (called an ***augmentation***) such that the resulting complex

$$0 \leftarrow X \xleftarrow{\epsilon} P_0 \leftarrow P_1 \leftarrow \dots$$

is exact and admissible. Such a complex $(\mathfrak{P})$ together with the map $\epsilon : P_0 \mapsto X$ is called a ***resolution over*** X.

A major part of the theory, will be the study of the functors $CB_{\mathcal{A},\mathbb{C}}(?, Z)$ etc. and $? \otimes_{\mathcal{A}} Z$ applied to various resolutions.

Those familiar with homological theory will recognize that for the most part we will need to confine our interest to so called *projective* resolutions. Thus we begin by introducing the notion of an injective and projective module in the operator space category.

DEFINITION 3.3.1: A left operator $\mathcal{A}$-module Y is called ***(left) injective*** if, for any admissible complex Ξ, the complex $CB_{\mathcal{A},\mathbb{C}}(\Xi, Y)$ is exact. That is to say if

$$(\Xi) : \ldots \leftarrow X_{n-1} \leftarrow X_n \leftarrow X_{n+1} \leftarrow \ldots$$

is admissible, then the complex

$$(CB_{\mathcal{A},\mathbb{C}}(\Xi, Y)) : \ldots \to CB_{\mathcal{A},\mathbb{C}}(X_{n-1}, Y) \to CB_{\mathcal{A},\mathbb{C}}(X_n, Y) \to CB_{\mathcal{A},\mathbb{C}}(X_{n+1}, Y) \to \ldots$$

is exact. If Y is a right module, we call Y ***(right) injective*** if $CB_{\mathbb{C},\mathcal{A}}(\Xi, Y)$ is exact. Finally if Y is a bimodule, we shall say Y is ***bi-injective*** or ***injective as a bimodule*** if $CB_{\mathcal{A},\mathcal{A}}(\Xi, Y)$ is exact (or equivalently, as we shall see in Section 3.5, if Y is injective as a left $\mathcal{A}\hat{\otimes}\mathcal{A}^{op}$ module). Of special note is that an object may be injective in one category, while not in another.

REMARK: This definition of injective is somewhat problematic. To see this, note that an operator space J is called *injective*, if whenever we have an operator space Y sitting as a closed subspace of Z, then any c.b. map $\theta : Y \mapsto J$ has an extension to $\bar{\theta} : Z \mapsto J$ such that $\|\theta\|_{cb} = \|\bar{\theta}\|_{cb}$. It is an important theorem of Wittstock (See [77] and [78]) that $B(H)$ is injective as an operator space. (There appears to be an array of different proofs of this result).

We note that any operator space can be considered as a $\mathbb{C}$-module. The condition of admissibility in the definition ensures that every $\mathbb{C}$-module is automatically an injective module. Since there exist non-injective operator spaces, it follows that our definition of injectivity as a $\mathbb{C}$-module does not correspond to the definition of an injective operator space. Note condition (4) of the following theorem and our definition of *projectivity* to follow.

The following theorem is the analogue of the situation in general algebra as well as Banach space theory.

THEOREM 3.3.2: *Let X be a left operator $\mathcal{A}$ module. Then the following are equivalent:*

(1) *X is injective,*

(2) *for any admissible extension sequence (Ξ), $CB_{\mathcal{A},\mathbb{C}}(\Xi, X)$ is exact,*

(3) *for any admissible c.b. module map injection $\phi : Y \mapsto Z$ and any c.b. module map $\theta : Y \mapsto X$, there is a c.b. module map $\psi : Z \mapsto X$ such that the following commutes*

$$\begin{tikzcd} Y \arrow[r, "\phi"] \arrow[d, "\theta"'] & Z \arrow[dl, dotted, "\psi"] \\ X & \end{tikzcd}$$

(4) *if Y is a complemented submodule of Z and $\theta \in CB_{\mathcal{A},\mathbb{C}}(Y, X)$ then θ has an extension to $CB_{\mathcal{A},\mathbb{C}}(Z, X)$.*

Proof: (1) $\Rightarrow$ (2) is immediate.

For (2) $\Rightarrow$ (3) consider the extension sequence

$$0 \to Y \xrightarrow{\phi} Z \xrightarrow{q} Z/Y \to 0$$

where q is the canonical quotient. By (2) the sequence

$$0 \to CB_{\mathcal{A},\mathbb{C}}(Z/Y, X) \xrightarrow{q_*} CB_{\mathcal{A},\mathbb{C}}(Z, X) \xrightarrow{\phi_*} CB_{\mathcal{A},\mathbb{C}}(Y, X) \to 0$$

is exact. Thus ϕ_* is onto. (i.e. for all module maps $\theta : Y \mapsto X$ there exists a module map $\psi : Z \mapsto X$ such that $\phi_*(\psi) = \theta$ as required.)

For (3) $\Rightarrow$ (2) suppose that

$$0 \to Y \xrightarrow{\phi} Z \xrightarrow{g} Q \to 0$$

is an admissible extension sequence. Consider the sequence

$$0 \to CB_{\mathcal{A},\mathbb{C}}(Q, X) \xrightarrow{g_*} CB_{\mathcal{A},\mathbb{C}}(Z, X) \xrightarrow{\phi_*} CB_{\mathcal{A},\mathbb{C}}(Y, X) \to 0.$$

The sequence is automatically exact at $CB_{\mathcal{A},\mathbb{C}}(Q, X)$ and $CB_{\mathcal{A},\mathbb{C}}(Z, X)$, and by (3) the map ϕ_* is onto. Thus the sequence is exact at each term.

(3) $\Rightarrow$ (4) is immediate.

For (4) $\Rightarrow$ (3) we note that $\phi(Y)$ is a complemented submodule of Z and $\phi : Y \mapsto \phi(Y)$ is a c.b. isomorphism. Thus the map $\theta \circ \phi^{-1}$ has an extension to Z. Thus by (4) there exists a module map ψ such that the following commutes:

$$\begin{tikzcd} Y \arrow[r, "\phi"] \arrow[dr, "\theta"'] & \phi(Y) \arrow[r, "id"] \arrow[d, "\theta\circ\phi^{-1}"] & Z \arrow[dl, dotted, "\psi"'] \\ & X & \end{tikzcd}$$

The proof of (2) $\Rightarrow$ (1) is from standard algebra (see for example [14, II.4.1]) ■

DEFINITION 3.3.3: A left operator $\mathcal{A}$ module X is ***(left) projective*** if for any admissible complex Ξ, the complex $CB_{\mathcal{A},\mathbb{C}}(X, \Xi)$ is exact. We also have right and biprojective modules just as in the injective case.

Now we have the analogue of the previous theorem.

THEOREM 3.3.4: *Let X be a left operator $\mathcal{A}$ module. Then the following are equivalent*

(1) *X is projective*

(2) *for any admissible extension sequence (Ξ), $CB_{\mathcal{A},\mathbb{C}}(X, \Xi)$ is exact*

(3) *for any c.b. admissible surjection $\phi : Y \mapsto Z$ and any c.b. module map $\theta : X \mapsto Z$, there is a c.b. module map $\psi : X \mapsto Y$ such that the following diagram commutes*

$$\begin{tikzcd} & X \arrow[dl, dotted, "\psi"'] \arrow[d, "\theta"] \\ Y \arrow[r, "\phi"'] & Z \end{tikzcd}$$

(4) *if Z is a submodule of Y, then every $\theta \in CB_{\mathcal{A},\mathbb{C}}(X, Y/Z)$ has an extension to $CB_{\mathcal{A},\mathbb{C}}(X, Z)$.*

Proof: The proofs of these equivalences are similar to the previous theorem. ■

It is easy to see that any module is a projective and injective $\mathbb{C}$ module. As a consequence we shall see that for any module E, the module of the form $\mathcal{A}_+\hat{\otimes}E$, is projective. We require the following *reduction* formula.

PROPOSITION 3.3.5: *$CB_{\mathcal{A}_+,\mathbb{C}}(\mathcal{A}_+, X)$ is c.b. isometrically isomorphic to X and $CB_{\mathcal{A}_+,\mathbb{C}}(\mathcal{A}_+\hat{\otimes}X, Y)$ is c.b. isometrically isomorphic to $CB(X,Y)$ for all X and Y. Similarly we have complete isometries $CB_{\mathbb{C},\mathcal{A}_+}(\mathcal{A}_+, X) \cong X$ and furthermore $CB_{\mathbb{C},\mathcal{A}_+}(X\hat{\otimes}\mathcal{A}_+, Y) \cong CB(X,Y)$.*

Proof: Let $T \in CB_{\mathcal{A}_+,\mathbb{C}}(\mathcal{A}_+, X)$. Then $T(a) = aT(e)$ for all $a \in \mathcal{A}_+$ where e is the identity element. Let $x_T = T(e)$. The map $T \mapsto x_T$ is clearly a bijection. Also for $[T_{ij}] \in \mathbb{M}_n(CB_{\mathcal{A}_+,\mathbb{C}}(\mathcal{A}_+, X))$ we have

$$\begin{aligned}
\|[T_{ij}]\|_n &= \sup\{\|[T_{ij}(a_{kl})]\| : \|[a_{kl}]\| \leq 1\} \\
&= \sup\{\|a_{kl} \cdot [(x_T)_{ij}]\| : \|a_{kl}\| \leq 1\} \\
&\leq \|[(x_T)_{ij}]\|_n
\end{aligned}$$

but if we consider the element $e \in \mathbb{M}_1(\mathcal{A}_+)$ we have

$$\|T_{ij}\|_n \geq \|T_{ij}(e)\| = \|[(x_T)_{ij}]\|_n.$$

Thus the natural map is a c.b. isometric isomorphism. For the second identification we proceed similarly. As before, it is easy to see that $T \in CB_{\mathcal{A}_+,\mathbb{C}}(\mathcal{A}_+\hat{\otimes}X, Y)$

is defined on elements of the form $e \otimes x$. Thus the map $T \mapsto \bar{T}$ is an isomorphism between $CB(X,Y)$ and $CB_{\mathcal{A}_+,\mathbb{C}}(\mathcal{A}_+\hat{\otimes}X,Y)$, where $\bar{T}(x) = T(e \otimes x)$. Note that we can identify the space $CB_{\mathcal{A}_+,\mathbb{C}}(\mathcal{A}_+\hat{\otimes}X,Y)$ with $JCB(\mathcal{A}_+,X;Y)$, the space of maps which are jointly completely bounded from $\mathcal{A}_+ \times X$ to Y, such that $T(a,x) = aT(e,x)$. Thus we have

$$
\begin{aligned}
\|[T_{pq}]\|_n &= \sup\{\|[T_{pq}(a_{ij}, x_{kl})]\| : \|[a_{ij}]\| \leq 1, \|[x_{kl}]\| \leq 1\} \\
&= \sup\{\|[a_{ij} \cdot T_{pq}(e, x_{kl})] : \|[a_{ij}]\| \leq 1, \|[x_{kl}]\| \leq 1\} \\
&\leq \sup\{\|[a_{ij}]\|\|[\bar{T}_{pq}(x_{kl})]\| : \|[a_{ij}]\| \leq 1, \|[x_{kl}]\| \leq 1\} \\
&= \|[\bar{T}_{pq}]\|_n
\end{aligned}
$$

The reverse equality follows by taking $a_{ij} = e$ as before. The assertions concerning $CB_{\mathbb{C},\mathcal{A}_+}(\mathcal{A}_+,X)$ and $CB_{\mathbb{C},\mathcal{A}_+}(X\hat{\otimes}\mathcal{A}_+,Y)$ are proved in a similar manner.

∎

As a consequence of the above proposition, we have the following corollary.

COROLLARY 3.3.6: *We have $CB_{\mathcal{A},\mathbb{C}}(\mathcal{A}_+,X) \cong X$ and $CB_{\mathcal{A},\mathbb{C}}(\mathcal{A}_+\hat{\otimes}X,Y) \cong CB(X,Y)$.*

Proof: To prove the first equality, it suffices to show that $CB_{\mathcal{A},\mathbb{C}}(\mathcal{A}_+,X) = CB_{\mathcal{A}_+,\mathbb{C}}(\mathcal{A}_+,X)$, where $\mathcal{A}_+$ and X are considered as $\mathcal{A}$ modules on the left and $\mathcal{A}_+$ modules on the right. Let $T \in CB_{\mathcal{A},\mathbb{C}}(\mathcal{A}_+,X)$ and let (a,α) and $(b,\beta) \in \mathcal{A}_+$. Now

$$\begin{aligned} T[(a,\alpha)(b,\beta)] &= T(ab+\beta a+\alpha b,\alpha\beta) \\ &= T[(ab+\beta a,0)+(\alpha b,\alpha\beta)] \\ &= a\cdot T(b,\beta)+\alpha T(b,\beta) \\ &= (a,\alpha)\cdot T(b,\beta) \end{aligned}$$

A similar calculation shows $CB_{\mathcal{A},\mathbb{C}}(\mathcal{A}_+\hat{\otimes}X,Y) = CB_{\mathcal{A}_+,\mathbb{C}}(\mathcal{A}_+\hat{\otimes}X,Y)$. Now we simply apply the previous Proposition.

■

COROLLARY 3.3.7: *Any module of the form $\mathcal{A}_+\hat{\otimes}E$ for any $\mathcal{A}$-module E is projective as a left operator $\mathcal{A}$-module.*

Proof: In view of the above proposition, the complex $CB_{\mathcal{A},\mathbb{C}}(\mathcal{A}_+\hat{\otimes}E,\Xi)$ reduces to $CB(E,\Xi)$. Since any module is a projective $\mathbb{C}$ module, it follows that the complex is exact. Hence $\mathcal{A}_+\hat{\otimes}E$ is projective.

■

It is easy to see that the last four theorems have the obvious generalizations to the category of right and bimodules. In particular we can conclude:

COROLLARY 3.3.8: *Any module of the form $E\hat{\otimes}\mathcal{A}_+$ (resp. $\mathcal{A}_+\hat{\otimes}E\hat{\otimes}\mathcal{A}_+$) is a projective right (resp. bi) module for any module E.*

A module of the above form is usually called a ***free*** module.

We call the resolution over X ***projective*** if each of the P_i are projective modules. Let $\mathfrak{P}$ be a projective resolution over X. If we apply any covariant functor F to the sequence, we get a new sequence:

$$(F(\mathfrak{P})) : 0 \leftarrow F(P_0) \leftarrow F(P_1) \leftarrow F(P_2) \ldots$$

There is of course no reason to believe that the new sequence is exact, and hence it may have non-trivial homology, which we denote

$$OH_n(F(\mathfrak{P})).$$

We call this the ***nth derived functor*** of F. If F is contravariant, we obtain the sequence

$$0 \to F(P_0) \to F(P_1) \to F(P_2) \ldots$$

As before this may have non-trivial cohomology which is given by

$$OH^n(F(\mathfrak{P})).$$

Once again we call this the ***nth derived functor*** of F.[3]

As a result of the categorical properties of the resolutions, it is now an algebraic exercise to show that any two projective resolutions generate the same derived functor (up to natural isomorphism). See Appendix A.

In this thesis, we shall concentrate on the derived functors relating to $CB(X, Y)$ and $\otimes_{\mathcal{A}}$.

[3]Some authors differentiate between derived functors and derived cofunctors, however we need not make this distinction

3.4 Standard Homology and Cohomology

The basic starting point for Johnson's work on amenability, is that of the ***standard homology*** and ***standard cohomology*** (also called the Hochschild cohomology). This was generalized by Ruan to operator spaces in [68].

In this section we will recall these complexes (as described by Ruan in [68]) and study their homology and cohomology in our category.

Let $\mathcal{A}$ be a completely contractive Banach algebra and let X be an operator $\mathcal{A}$-bimodule. Consider the operator chain complex:

$$(\mathfrak{S}) : 0 \overset{d_0}{\leftarrow} X \overset{d_1}{\leftarrow} \mathcal{A}\hat{\otimes}X \overset{d_2}{\leftarrow} \mathcal{A}\hat{\otimes}\mathcal{A}\hat{\otimes}X \overset{d_3}{\leftarrow} \dots$$

where the differential maps are given by

$$\begin{aligned} d_n(a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes x) &= a_2 \otimes \cdots \otimes a_n \otimes xa_1 \\ &\quad + \sum_{i=1}^{n-1}(-1)^i a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \otimes x \\ &\quad + (-1)^n a_1 \otimes \dots a_{n-1} \otimes a_n x. \end{aligned}$$

It is clear that each of the maps d_n are completely bounded and a straightforward, albeit messy, calculation shows that $d_n \circ d_{n+1} = 0$ for all $n \in \mathbb{N}$. We denote the homology of this sequence by $OH_n(\mathcal{A}, X)$. Often this set is referred to as the *n^{th}-homology of the completely contractive Banach algebra $\mathcal{A}$ with coefficients in X.*

If $\operatorname{im} d_{n+1}$ is not closed, the space $OH_n(\mathcal{A}, X)$ is not a Banach space, much less an operator space. Fortunately this will not always be a major handicap for our purposes. Our interest will lie mainly in the case that $OH_n(\mathcal{A}, X) = 0$, in which case $\mathrm{im} d_{n+1} = \ker d_n$ which is automatically closed. However, the situation $\mathrm{im} d_k$ not closed may present a problem for us. Thus we define the following:

DEFINITION 3.4.1: An operator $\mathcal{A}$-module will be called ***differentially closed*** if the images of the differential maps d_k are closed (i.e. $\mathrm{im} d_k$ is closed for all $k > 0$.)

Later we shall see that if $\mathcal{A}$ is operator amenable, then every module is automatically differentially closed.

If we take the dual of $(\mathfrak{S})$ we get the cochain complex:

$$(\mathfrak{S}^*) : 0 \to X^* \stackrel{d_1^*}{\to} (\mathcal{A}\hat{\otimes}X)^* \stackrel{d_2^*}{\to} (\mathcal{A}\hat{\otimes}\mathcal{A}\hat{\otimes}X)^* \stackrel{d_3^*}{\to} \ldots$$

Since there is a natural c.b. isomorphism from $(\mathcal{A}\hat{\otimes}X)^*$ to $CB(\mathcal{A}, X^*)$ the above complex is isomorphic to:

$$0 \to X^* \stackrel{\delta^1}{\to} CB(\mathcal{A}, X^*) \stackrel{\delta^2}{\to} CB(\mathcal{A}\hat{\otimes}\mathcal{A}, X^*) \stackrel{\delta^3}{\to} \ldots$$

In this case the differential maps $\delta^n = d_n^*$ have a particularly nice form. For convenience we shall let $\mathcal{A}^{\hat{\otimes}^n}$ denote the n-fold projective tensor product of $\mathcal{A}$. i.e.

$$\mathcal{A}^{\hat{\otimes}^n} = \underbrace{\mathcal{A}\hat{\otimes}\ldots\hat{\otimes}\mathcal{A}}_{n}$$

Then for $T \in CB(\mathcal{A}^{\hat{\otimes}^n}, X^*)$ we have

$$\begin{aligned}\delta^n(T)(a_1 \otimes \cdots \otimes a_n) &= a_1 T(a_2 \otimes \cdots \otimes a_n) \\ &+ \sum_{i=1}^{n-1}(-1)^i T(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \\ &+ (-1)^n T(a_1 \otimes \cdots \otimes a_{n-1}) a_n.\end{aligned}$$

We shall denote the cohomology of this complex by $OH^n(\mathcal{A}, X^*)$. This set is often referred to as the *nth cohomology group of $\mathcal{A}$ with coefficients in X^**

Notice that if we replace X^* with any other operator $\mathcal{A}$-bimodule Y, the sequence ($\mathfrak{S}^*$) still makes sense, and still forms a cochain complex. In other words we are able to make sense of $OH^n(\mathcal{A}, Y)$ without Y being a dual of some other module.

Of special interest is the space $OH^1(\mathcal{A}, Y)$. Consider the following

$$\delta^2(T)(a \otimes b) = aT(b) - T(ab) + T(a)b$$

and hence $T \in \ker \delta^2$ if and only if the following identity holds

$$T(ab) = aT(b) + T(a)b.$$

These 1 cocycles are usually called ***derivations***. Also note that the 1-coboundaries are represented by the following:

$$\delta^1 x(a) = ax - xa$$

Such functions are usually called ***inner derivations***. Thus the statement $OH^1(\mathcal{A}, X) = 0$ is equivalent to saying that each completely bounded derivation is inner.

We now have the following Lemma connecting the homology and cohomology spaces:

LEMMA 3.4.2: *Let X be an operator $\mathcal{A}$-bimodule. Then X is differentially closed and $OH_n(\mathcal{A}, X) = 0$ for all $n > 0$ if and only if $OH^n(\mathcal{A}, X^*) = 0$ for all $n > 0$.*

Proof: This Lemma is similar to [49, Corollary 1.3] or [54, Proposition 5.29].

Note that $OH^n(\mathcal{A}, X^*) = 0$ is equivalent to the sequence $\mathfrak{S}^*$ being exact at every term except possibly X^*. Furthermore $OH^1(\mathcal{A}, X^*) = 0$ implies $\mathrm{im} d_1^* = \ker d_2^*$ and hence $\mathrm{im} d_1^*$ is closed. Thus $\mathrm{im} d_0$ is closed and the sequence $\mathfrak{S}$ is exact everywhere except possible at X. In any event we have $OH_n(\mathcal{A}, X) = 0$. The converse is similar.

LEMMA 3.4.3: *$OH^1(\mathcal{A}, X^*) = 0$ for all dual modules if and only if $OH^n(\mathcal{A}, X^*) = 0$ for all dual modules and $n > 0$.*

Proof: In Section 3.6 we will consider derived functors, from which we could provide an alternative proof to the above result. For now we provide the following from [68], which is the operator space version of a result of Johnson.

Using the associativity of the operator projective tensor product, it is easy to show $OH^{n+k}(\mathcal{A}, X^*) = OH^n(\mathcal{A}, CB(\mathcal{A}^{\hat{\otimes}^k}, X^*))$ from which the above follows trivially. ∎

Combining the two previous Lemmas we have the following Corollary:

COROLLARY 3.4.4: *If $OH^1(\mathcal{A}, X^*) = 0$ for all operator bimodules X, then every module is differentially closed.*

We can now prove the analogue of Lemma 3.4.3 for the homology groups.

COROLLARY 3.4.5: *Suppose every module is differentially closed. then we have*

$OH_1(\mathcal{A}, X) = 0$ for all operator bimodules X if and only if $OH_n(\mathcal{A}, X) = 0$ for all operator bimodules X and $n > 0$.

Proof: If $OH_1(\mathcal{A}, X) = 0$ holds then $OH^1(\mathcal{A}, X^*) = 0$ by [49, Proposition 1.2], so by above $OH^n(\mathcal{A}, X^*) = 0$. Now apply Lemma 3.4.2. ■

As discussed earlier, we shall often need to consider the completely contractive Banach algebra $\mathcal{A}_+$ and not just $\mathcal{A}$. However as the next Lemma shows, this will not be a restriction. See [51].

LEMMA 3.4.6: *$OH^n(\mathcal{A}, X^*) = 0$ for all dual modules and $n > 0$ if and only if $OH^n(\mathcal{A}_+, X^*) = 0$ for all dual modules and $n > 0$.*

Proof: Suppose $OH^n(\mathcal{A}, X^*) = 0$ then each derivation is inner. Let e be the adjoined identity in $\mathcal{A}_+$. Note that

$$D(e) = D(e^2) = eD(e) + D(e)e = 2D(e)$$

hence $D(e) = 0$ for all derivations. Since the operator space structure on $(\mathcal{A}, 0) \subset \mathcal{A}_+$ is precisely the structure on $\mathcal{A}$, it follows that every derivation from $\mathcal{A}$ has a unique completely bounded extension to $\mathcal{A}_+$ and hence is inner. Thus

$$OH^1(\mathcal{A}_+, X^*) = 0.$$

Hence by the above Lemma 3.4.3 we have $OH^n(\mathcal{A}_+, X^*) = 0$ for all $n > 0$.

Conversely any derivation from $\mathcal{A}_+$ is a derivation from $\mathcal{A}$ by restriction. Since the first is inner, so must the second. Again by applying Lemma 3.4.3 we have $OH^n(\mathcal{A}, X^*) = 0$ for all $n > 0$.

■

Lemma 3.4.6 shows that when studying algebras such that $OH^1(\mathcal{A}, X^*) = 0$, we may, if convenient, assume all of our completely contractive Banach algebras are unital and all modules are unital modules.

Recall that Johnson called a Banach algebra *amenable* if each bounded derivation into a dual Banach module is inner. In an analogous way, Ruan defined ***operator amenable*** by saying each completely bounded derivation into a dual operator module is inner (i.e. $OH^1(\mathcal{A}, X^*) = 0$ for all dual modules X^*.)

Combining the results of this section, we have the following

THEOREM 3.4.7: *The following are equivalent*

(1) *$\mathcal{A}$ is operator amenable (i.e. $OH^1(\mathcal{A}, X^*) = 0$ for all dual modules X^*)*

(2) *$OH^n(\mathcal{A}, X^*) = 0$ for all dual modules X^**

(3) *every module is differentially closed and $OH_1(\mathcal{A}, X) = 0$ for all modules X and $n > 0$*

(4) *every module is differentially closed and $OH_n(\mathcal{A}, X) = 0$ for all modules X and $n > 0$.*

Proof: (1) $\Rightarrow$ (2): Use Lemma 3.4.3. (3) $\Rightarrow$ (4): Use Corollary 3.4.5. (4) $\Leftrightarrow$ (2): This is Lemma 3.4.2. The rest is immediate. ∎

3.5 $\otimes_{\mathcal{A}}$ and Tor

In this section we shall investigate the functors $\otimes_{\mathcal{A}}$ and **Tor** of homological algebra, and we shall show that the standard results continue to hold in our new category.

Suppose we are given an extension sequence

$$0 \to X \to Y \to Z \to 0$$

of left operator $\mathcal{A}$-modules. We wish to know if the induced sequence

$$0 \to X \otimes_{\mathcal{A}} Q \to Y \otimes_{\mathcal{A}} Q \to Z \otimes_{\mathcal{A}} Q \to 0$$

is exact. That is to say "is the functor $? \otimes_{\mathcal{A}} Q$ exact?". In this section we will introduce the spaces **Tor** which will measure this exactness, and relate this to the standard homology of the previous section.

To make this connection, we first introduce the *bar* or *standard* resolutions in our category.

To motivate what we are eventually going to need, we first construct a simple projective resolution, which illustrates what happens in the general case.

PROPOSITION 3.5.1: *Let Y be a left $\mathcal{A}$-module. Then the complex*

$$0 \leftarrow Y \xleftarrow{\epsilon} \mathcal{A}_+ \hat{\otimes} Y \xleftarrow{d_1} \mathcal{A}_+ \hat{\otimes} \mathcal{A}_+ \hat{\otimes} Y \xleftarrow{d_2} \mathcal{A}_+ \hat{\otimes} \mathcal{A}_+ \hat{\otimes} \mathcal{A}_+ \hat{\otimes} Y \ldots$$

with augmentation

$$\epsilon(a \otimes y) = ay$$

and where the differentials are given by the following formulas for $n > 0$:

$$\begin{aligned} d_n(a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes y) &= a_1 a_2 \otimes a_3 \otimes \cdots \otimes a_n \otimes y \\ &\quad + \sum_{i=2}^{n-1} (-1)^{i-1} a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes y \\ &\quad + (-1)^n a_1 \otimes a_2 \otimes \cdots \otimes a_n y \end{aligned}$$

is a projective resolution of the left module Y.

Proof: As remarked in Corollary 3.3.7, each module of the form $\mathcal{A}_+\hat{\otimes}Y$ is projective. Thus it suffices to show that the complex is exact and admissible.

We define maps $\theta_n : \mathcal{A}_+^{\hat{\otimes}^n}\hat{\otimes}Y \mapsto \mathcal{A}_+^{\hat{\otimes}^{n+1}}\hat{\otimes}Y$ by $\theta_n(u) = e \otimes u$. Also we let $\theta_0 : Y \mapsto \mathcal{A}_+\hat{\otimes}Y$ be defined by $\theta_0(y) = e \otimes y$. It is easy to see from the definition of the matrix norms of a projective tensor product that each of these maps are completely bounded.

Now we see that the complex is exact since we have the following. Let $K = \ker d_{n-1}$ and let $k \in K$. Then $\theta_n(k) = e \otimes k$. Now $d_n(e \otimes k) = (e \cdot k) - (e \otimes d_{n-1}(k)) = k$. Hence $d_n\theta_n = id_K$. Thus $\mathrm{im} d_n = \ker d_{n-1}$ which implies the complex is admissible. ∎

To construct our standard resolution, we first need to observe the following:

PROPOSITION 3.5.2: *Let $\mathcal{A}$ be a completely contractive Banach algebra and let $\mathcal{A}_+$ be its unitization. If we let e denote the identity in $\mathcal{A}_+$, then we have a complete isometry $\mathcal{A} \cong (\mathcal{A}_+/\mathbb{C}e)$.*

Proof: For $[(a_{ij}, \mathbb{C})] \in \mathbb{M}_n(\mathcal{A}_+/\mathbb{C}e)$ we have the following equalities for $[\phi_{kl}] \in \mathbb{M}_m(\mathcal{A}^*)$, $[\tau_{kl}] \in \mathbb{M}_{m'}(\mathbb{C})$, and $n > 0$

$$\begin{aligned}
\|[(a_{ij}, \mathbb{C})]\|_n &= \inf\{\|[(a_{ij}, \lambda_{ij})]\|_{n,\mathcal{A}_+} : [\lambda_{ij}] \in \mathbb{M}_n(\mathbb{C})\} \\
&= \inf_{[\lambda_{ij}]}\{\max\{\|\phi_{kl}(a_{ij})\|_{nm}, \|\tau_{kl}(\lambda_{ij})\|_{nm'}\} : \|\phi_{kl}\| \leq 1, \|\tau_{kl}\|_{m'} \leq 1\} \\
&= \sup\{\|\phi_{kl}(a_{ij})\|_{nm} : \|\phi_{kl}\| \leq 1\} \\
&= \|a_{ij}\|_n
\end{aligned}$$

Hence the natural map is a complete isomorphism. ∎

For convenience we shall denote $\mathcal{A}_+/\mathbb{C}e$ by $\bar{\mathcal{A}}_+$. Now we consider the induced sequence

$$0 \leftarrow Y \overset{d_0}{\leftarrow} \mathcal{A}_+\hat{\otimes}Y \overset{d_1}{\leftarrow} \mathcal{A}_+\hat{\otimes}\bar{\mathcal{A}}_+\hat{\otimes}Y \overset{d_2}{\leftarrow} \mathcal{A}_+\hat{\otimes}\bar{\mathcal{A}}_+\hat{\otimes}\bar{\mathcal{A}}_+\hat{\otimes}Y \leftarrow \ldots.$$

Here the differential maps are similar but are defined on the cosets, i.e. for $n > 0$ we have

$$\begin{aligned} d_n(a_1 \otimes \bar{a_2} \otimes \cdots \otimes \bar{a_n} \otimes y) &= a_1 a_2 \otimes \bar{a_3} \otimes \cdots \otimes \bar{a_n} \otimes y \\ &\quad + \sum_{i=2}^{n-1} (-1)^{i-1} a_1 \otimes \cdots \otimes \overline{a_i a_{i+1}} \otimes \cdots \otimes y \\ &\quad + (-1)^n a_1 \otimes \bar{a_2} \otimes \cdots \otimes a_n y. \end{aligned}$$

To see that this expression is well defined, see for example [54] or [14]. The maps θ_n are given by the formula

$$\theta_n(a_1 \otimes \bar{a_2} \ldots \bar{a_n} \otimes y) = e \otimes \bar{a_1} \otimes \cdots \otimes \bar{a_n} \otimes y.$$

Proceeding in a manner similar to the proof of Proposition 3.5.1 it is easy to show that the above also generates a projective resolution of Y. In view of the isomorphism given in Proposition 3.5.2, we can identify $\bar{\mathcal{A}}_+$ with $\mathcal{A}$, and hence we have a projective resolution

$$0 \leftarrow Y \overset{d_0}{\leftarrow} \mathcal{A}_+\hat{\otimes}Y \overset{d_1}{\leftarrow} \mathcal{A}_+\hat{\otimes}\mathcal{A}\hat{\otimes}Y \overset{d_2}{\leftarrow} \mathcal{A}_+\hat{\otimes}\mathcal{A}\hat{\otimes}\mathcal{A}\hat{\otimes}Y \leftarrow \ldots.$$

Here the differentials are the same as before, except that now they are defined on elements of $\mathcal{A}$ instead of cosets. This resolution will be called the ***standard***

resolution of Y (also called the non-normalized bar resolution). It is easy to see that if X is a right module then the complex

$$0 \leftarrow X \leftarrow X\hat{\otimes}\mathcal{A}_+ \leftarrow X\hat{\otimes}\mathcal{A}\hat{\otimes}\mathcal{A}_+ \leftarrow \ldots$$

is a projective resolution of the right module X. A similar complex is available for bimodules.

We are now in a position to relate the derived functors of $\otimes_{\mathcal{A}}$ with standard homology. For a fixed left module Y we define $\mathbf{Tor}^n_{\mathcal{A}}(X, Y)$ to be the n^{th} derived functor of $? \otimes_{\mathcal{A}} Y$ applied to the right module X. Again we recall that the standard resolution described above is projective, so for definiteness, to calculate $\mathbf{Tor}_{\mathcal{A}}(X, Y)$ we can apply the functor $? \otimes_{\mathcal{A}} Y$ to the complex

$$0 \leftarrow X\hat{\otimes}\mathcal{A}_+ \stackrel{d_1}{\leftarrow} X\hat{\otimes}\mathcal{A}\hat{\otimes}\mathcal{A}_+ \stackrel{d_2}{\leftarrow} X\hat{\otimes}\mathcal{A}\hat{\otimes}\mathcal{A}\hat{\otimes}\mathcal{A}_+ \leftarrow \ldots$$

which results in the sequence

$$0 \leftarrow X\hat{\otimes}\mathcal{A}_+ \otimes_{\mathcal{A}} Y \stackrel{d_1\otimes_{\mathcal{A}} id}{\longleftarrow} X\hat{\otimes}\mathcal{A}\hat{\otimes}\mathcal{A}_+ \otimes_{\mathcal{A}} Y \stackrel{d_2\otimes_{\mathcal{A}} id}{\longleftarrow} X\hat{\otimes}\mathcal{A}\hat{\otimes}\mathcal{A}\hat{\otimes}\mathcal{A}_+ \otimes_{\mathcal{A}} Y \leftarrow \ldots$$

Hence we have the n^{th} ***torsion product of*** X ***and*** Y given by

$$\mathbf{Tor}^n_{\mathcal{A}}(X, Y) = \frac{\ker(d_n \otimes_{\mathcal{A}} id)}{\mathrm{im}(d_{n+1} \otimes_{\mathcal{A}} id)}.$$

In particular we see that $\mathbf{Tor}^n_{\mathcal{A}}(?, Y)$ is a functor from the category of right operator $\mathcal{A}$ modules to the category of linear spaces.

To connect the torsion product with the homology groups we require the ***enveloping algebra*** of a completely contractive Banach algebra.

DEFINITION 3.5.3: Let $\mathcal{A}$ be a completely contractive Banach algebra. Consider the algebra $\mathcal{A}^{op}$ which is the Banach space $\mathcal{A}$ with multiplication given by $a^{op} \cdot b^{op} = ba$. We call $\mathcal{A}^{op}$ the ***opposite algebra.***

If we give the opposite algebra the operator space structure given by

$$\|[a_{ij}^{op}]\|_n = \|[a_{ij}]\|_n \quad \text{for all} \quad [a_{ij}^{op}] \in \mathbb{M}_n(\mathcal{A}^{op})$$

then $\mathcal{A}^{op}$ becomes a completely contractive Banach algebra. Now we define the ***enveloping algebra*** $\mathcal{A}^e$ of $\mathcal{A}$ to be the completely contractive Banach algebra $\mathcal{A}_+\hat{\otimes}\mathcal{A}_+^{op}$. Note that if X is an operator $\mathcal{A}$-bimodule, then X becomes a left $\mathcal{A}_+\hat{\otimes}\mathcal{A}_+^{op}$ operator module where

$$(a \otimes b^{op}) \cdot x = axb$$

and a right $\mathcal{A}_+^{op}\hat{\otimes}\mathcal{A}_+$ operator module where

$$x \cdot (a^{op} \otimes b) = axb.$$

Since there is a canonical c.b. isomorphism $\mathcal{A}_+^{op}\hat{\otimes}\mathcal{A}_+ \cong \mathcal{A}_+\hat{\otimes}\mathcal{A}_+^{op}$, it follows that X can be regarded as either a left or right $\mathcal{A}^e$ operator bimodule. It is standard to consider any bimodule as a left $\mathcal{A}^e$ module and a bimodule map as a left $\mathcal{A}^e$ module map in this way. In particular we have $CB_{\mathcal{A}^e,\mathbb{C}}(X,Y) = CB_{\mathcal{A},\mathcal{A}}(X,Y)$ for operator bimodules X and Y.

There is a *reduction property* which will be useful in calculating the homology of sequences such as Ξ above. First we require the following additional technical fact:

Proposition 3.5.4: *$(X\otimes_{\mathcal{A}}Y)^*$ is c.b. isometrically isomorphic to $CB_{\mathbb{C},\mathcal{A}}(X,Y^*)$.*

Proof: Recall that the map $\phi \mapsto T_\phi$ given by

$$\langle \phi, x\otimes y\rangle = \langle T_\phi(x), y\rangle$$

is a c.b. isometric isomorphism between $(X\hat{\otimes}Y)^*$ and $CB(X,Y^*)$. Now

$$(X\otimes_{\mathcal{A}} Y)^* = \left(\frac{X\hat{\otimes}Y}{N}\right)^* \cong N^\perp$$

where $N = \overline{\text{span}}\{xa\otimes y - x\otimes ay\}$ for $a\in\mathcal{A}, x\in X$ and $y\in Y$. Now $\phi\in N^\perp$ if and only if

$$\phi(xa\otimes y - x\otimes ay) = 0$$

Thus for all $y\in Y$,

$$\langle T_\phi(xa), y\rangle = \langle T_\phi(x), ay\rangle = \langle T_\phi(x)a, y\rangle$$

Thus $T_\phi\in CB_{\mathbb{C},\mathcal{A}}(X,Y^*)$. The reverse inclusion is clear. ∎

Proposition 3.5.5: *We have the c.b. isometric isomorphism $\mathcal{A}_+\otimes_{\mathcal{A}_+} X\cong X$ and furthermore we have that $\mathcal{A}_+\hat{\otimes}M\hat{\otimes}\mathcal{A}_+\otimes_{\mathcal{A}^\bullet}X\cong M\hat{\otimes}X$, where M is any operator space.*

Proof: The map $\phi: X\mapsto \mathcal{A}_+\otimes_{\mathcal{A}_+} X$ given by

$$\phi(x) = e\otimes x$$

is easily seen to be completely contractive. Furthermore the map

$$\phi^* : (\mathcal{A}_+\otimes_{\mathcal{A}_+} X)^* \to X^*$$

is exactly the c.b. isomorphism of Proposition 3.3.5 and Proposition 3.5.4 from $(\mathcal{A}_+ \otimes_{\mathcal{A}_+} X)^*$ to X^*. Thus $(\phi^{-1})^*$ is completely contractive, hence ϕ^{-1} is completely contractive.

The fact that the map $\tau : M\hat{\otimes}X \to \mathcal{A}_+\hat{\otimes}M\hat{\otimes}\mathcal{A}_+ \otimes_{\mathcal{A}^e} X$ given by

$$\tau(m \otimes x) = e \otimes m \otimes e \otimes x$$

is a complete isometry follows similarly. ∎

COROLLARY 3.5.6: *We have the c.b. isomorphism $\mathcal{A}_+ \otimes_{\mathcal{A}} X \cong X$*

Proof: For $a \in \mathcal{A}, \lambda \in \mathbb{C}$ and $x \in X$ we have

$$\begin{aligned} (a, \lambda) \otimes_{\mathcal{A}} x &= [(a, 0) + (0, \lambda)] \otimes_{\mathcal{A}} x \\ &= a \cdot e \otimes_{\mathcal{A}} x + \lambda e \otimes_{\mathcal{A}} x \\ &= e \otimes (ax + \lambda x) = e \otimes (a, \lambda)x. \end{aligned}$$

Hence in view of the previous proposition and Corollary 3.3.6, it follows that we have a c.b. isometric isomorphism. ∎

For convenience, if $u \in \mathcal{A}_+ \otimes_{\mathcal{A}} X$ we shall denote the corresponding *reduced* element in X by $u^\flat$. Conversely, if $u \in X$ we shall let $u^\sharp = e \otimes_{\mathcal{A}} u \in \mathcal{A}_+ \otimes_{\mathcal{A}} X$. It is easy to see that $u^{\flat\sharp} = u$. Furthermore, if we have a map $T : \mathcal{A}_+\hat{\otimes}X \mapsto Y$, we let $T^\flat : X \mapsto Y$ denote the map given by $T^\flat(x) = T(x^\sharp)$ for all $x \in X$.

Now we can connect the torsion product with the standard homology via the following:

THEOREM 3.5.7: *Let X be an $\mathcal{A}$-bimodule. Then considering X as a left $\mathcal{A}^e$ module and $\mathcal{A}_+$ as a right $\mathcal{A}^e$ module, we have the following equality:*

$$\mathbf{Tor}^n_{\mathcal{A}^\bullet}(\mathcal{A}_+, X) = OH_n(\mathcal{A}, X).$$

Furthermore the equality is a c.b. isomorphism whenever X is differentially closed.

Proof: Consider the complex

$$0 \leftarrow \mathcal{A}_+\hat{\otimes}\mathcal{A}_+ \stackrel{d_1}{\leftarrow} \mathcal{A}_+\hat{\otimes}\mathcal{A}\hat{\otimes}\mathcal{A}_+ \stackrel{d_2}{\leftarrow} \dots$$

with $\epsilon : \mathcal{A}_+\hat{\otimes}\mathcal{A}_+ \mapsto \mathcal{A}_+$ given by $\epsilon(a \otimes b) = ab$, and the differentials the same as in Proposition 3.5.1. By earlier arguments, this is a resolution of the right $\mathcal{A}^e$-module $\mathcal{A}_+$, and from Corollary 3.3.8, each of these spaces is projective as a bimodule (hence as a right $\mathcal{A}^e$ module)

So to calculate $\mathbf{Tor}^n_{\mathcal{A}^\bullet}(\mathcal{A}_+, X)$, we may apply $? \otimes_{\mathcal{A}^\bullet} X$ to the above resolution to get

$$0 \leftarrow \mathcal{A}_+\hat{\otimes}\mathcal{A}_+ \otimes_{\mathcal{A}^\bullet} X \stackrel{d_1\otimes 1}{\longleftarrow} \mathcal{A}_+\hat{\otimes}\mathcal{A}\hat{\otimes}\mathcal{A}_+ \otimes_{\mathcal{A}^\bullet} X \stackrel{d_2\otimes 1}{\longleftarrow} \dots$$

In view of the Proposition 3.5.5, this complex reduces to

$$0 \leftarrow X \stackrel{d_1^\dagger}{\leftarrow} \mathcal{A}\hat{\otimes}X \stackrel{d_2^\dagger}{\leftarrow} \mathcal{A}\hat{\otimes}\mathcal{A}\hat{\otimes}X \dots$$

where

$$\begin{aligned}
{d_n}^b(a_1 \otimes \dots a_n \otimes x) &= (d_n(e \otimes a_1 \otimes \dots a_n \otimes e) \otimes_{\mathcal{A}^\bullet} x)^b \\
&= ([a_1 \otimes \cdots \otimes e + \sum_{i=1}^{n} (-1)^i e \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes e \\
&\quad + (-1)^n e \otimes \cdots \otimes a_n] \otimes_{\mathcal{A}^\bullet} x)^b \\
&= a_2 \otimes \dots a_n \otimes x a_1 + \sum_{i=1}^{n} (-1)^i a_1 \cdots \otimes a_i a_{i+1} \otimes \dots a_n \otimes x \\
&\quad + (-1)^n a_1 \otimes \cdots \otimes a_{n-1} \otimes a_n x
\end{aligned}$$

which is easily seen to be the standard homology complex. Since the complexes are c.b. isomorphic, so are the homology groups whenever $\mathrm{im} {d_n}^b$ is closed.

∎

To connect this with the exactness of the tensor product functor, we begin with the following definition:

DEFINITION 3.5.8: We call a left operator $\mathcal{A}$ module X ***operator flat*** if whenever we have an admissible complex Ξ, then $\Xi \otimes_{\mathcal{A}} X$ is exact. Furthermore, we call a completely contractive Banach algebra $\mathcal{A}$ ***operator biflat*** if $\mathcal{A}$ is a flat $\mathcal{A}\hat{\otimes}\mathcal{A}^{op}$ module.

We can connect flatness with injectivity with the following (See [53]):

PROPOSITION 3.5.9: *X is an operator flat $\mathcal{A}$ module if and only if the right module X^* is (right) injective.*

Proof: Suppose (Ξ) is an exact admissible sequence. Consider the sequences $\Xi \otimes_{\mathcal{A}} X$ and its dual $(\Xi \otimes_{\mathcal{A}} X)^* = CB_{\mathbb{C},\mathcal{A}}(\Xi, X^*)$. Clearly they are either both exact or inexact. ∎

We can relate flatness with the **Tor** functor using the following theorem from homological algebra:

THEOREM 3.5.10: *Let*

$$0 \to X \to Y \to Z \to 0$$

be an admissible sequence of operator $\mathcal{A}$ modules, and let F be an additive contravariant functor from the category of operator $\mathcal{A}$ modules into the category of algebraic $\mathcal{A}$ modules, and let F^n represent the various derived functors of F. Then there exists algebraic maps $E_n : F^{n+1}(X) \to F^n(Z)$ such that the long complex

$$0 \leftarrow F^0(X) \leftarrow F^0(Y) \leftarrow F^0(Z) \xleftarrow{E_0} F^1(X) \ldots$$

is exact. If F is covariant, we have the same long exact sequence but with arrows reversed.

The maps E_i are usually called ***connecting morphisms***. The proof of this theorem is essentially identical to the algebraic case. See [63, Chapter II.4.1].

Since $\otimes_{\mathcal{A}}$ is clearly an additive functor, we have for an admissible short exact sequence $0 \leftarrow Y_1 \leftarrow Y_2 \leftarrow Y_3 \leftarrow 0$, the sequence

$$0 \leftarrow \mathbf{Tor}^0_{\mathcal{A}}(Y_1, X) \leftarrow \mathbf{Tor}^0_{\mathcal{A}}(Y_2, X) \leftarrow \mathbf{Tor}^0_{\mathcal{A}}(Y_3, X) \xleftarrow{E_0} \mathbf{Tor}^1_{\mathcal{A}}(Y_1, X) \ldots$$

is exact. In algebra, we have that $\mathbf{Tor}^0_{\mathcal{A}}(X, Y) = X \otimes_{\mathcal{A}} Y$, however in our category we have topological problems which will affect this equality.

Now we note the following:

LEMMA 3.5.11: *For right and left operator $\mathcal{A}$ modules X and Y we have the equality*

$$\mathbf{Tor}^n_{\mathcal{A}}(X, Y) = OH_n(\mathcal{A}, Y \hat{\otimes} X).$$

Furthermore the equality is a c.b. isomorphism whenever $Y\hat{\otimes}X$ is differentially closed.

Proof: To compute the spaces $\mathbf{Tor}^n_{\mathcal{A}}(X,Y)$ we consider the complex

$$0 \leftarrow X\hat{\otimes}\mathcal{A}_+ \otimes_{\mathcal{A}} Y \xleftarrow{d_1\otimes_{\mathcal{A}} id} X\hat{\otimes}\mathcal{A}\hat{\otimes}\mathcal{A}_+ \otimes_{\mathcal{A}} Y \xleftarrow{d_2\otimes_{\mathcal{A}} id} X\hat{\otimes}\mathcal{A}\hat{\otimes}\mathcal{A}\hat{\otimes}\mathcal{A}_+ \otimes_{\mathcal{A}} Y \leftarrow \dots$$

which reduces to

$$0 \leftarrow X\hat{\otimes}Y \xleftarrow{d_1'} X\hat{\otimes}\mathcal{A}\hat{\otimes}Y \xleftarrow{d_2'} X\hat{\otimes}\mathcal{A}\hat{\otimes}\mathcal{A}\hat{\otimes}Y \leftarrow \dots$$

Using the c.b. isomorphism $X\hat{\otimes}Y \cong Y\hat{\otimes}X$ we see that the above complex is isomorphic to the standard homology complex for the bimodule $Y\hat{\otimes}X$. The fact that we have a c.b. isomorphism between the homology groups whenever $Y\hat{\otimes}X$ is differentially closed is now clear. ∎

In view of this lemma, we can make the following observation. Note that we could have considered the derived functors of $X\otimes_{\mathcal{A}}?$. However it will follow in the same fashion as the previous lemma that these functors are naturally isomorphic and the derived functors are equal to the standard homology complex for $Y\hat{\otimes}X$. (See [54, Chapter III])

To calculate $\mathbf{Tor}^0_{\mathcal{A}}(X,Y)$ we have the following:

LEMMA 3.5.12: *Let $\tau : X\hat{\otimes}\mathcal{A}\hat{\otimes}Y \mapsto X\hat{\otimes}Y$ be given by*

$$\tau(x\otimes a\otimes y) = xa\otimes y - x\otimes ay.$$

Then

$$\mathbf{Tor}^0_{\mathcal{A}}(X,Y) = \frac{X\hat{\otimes}Y}{\mathrm{im}\tau}.$$

In particular we have $\mathbf{Tor}^0_{\mathcal{A}}(X,Y) \cong X \otimes_{\mathcal{A}} Y$ *whenever* $Y\hat{\otimes}X$ *is differentially closed.*

Proof: By Lemma 3.5.11 we can calculate $\mathbf{Tor}^0_{\mathcal{A}}(X,Y)$ as the homology of the standard complex

$$(\mathfrak{S}) : 0 \overset{d_0}{\leftarrow} Y\hat{\otimes}X \overset{d_1}{\leftarrow} \mathcal{A}\hat{\otimes}Y\hat{\otimes}X \leftarrow \dots$$

with

$$\mathbf{Tor}^0_{\mathcal{A}}(X,Y) = \frac{\ker d_0}{\mathrm{im} d_1} = \frac{Y\hat{\otimes}X}{\mathrm{im} d_1}.$$

However, under the natural c.b. isomorphisms $Y\hat{\otimes}X \cong X\hat{\otimes}Y$ and

$$\phi : \mathcal{A}\hat{\otimes}Y\hat{\otimes}X \mapsto X\hat{\otimes}\mathcal{A}\hat{\otimes}Y$$

we see that $d_1 = \tau \circ \phi$. Thus

$$\mathbf{Tor}^0_{\mathcal{A}}(X,Y) = \frac{X\hat{\otimes}Y}{\mathrm{im}\tau}.$$

Clearly τ is closed if and only if $\mathrm{im} d_1$ is closed. In particular if $Y\hat{\otimes}X$ is differentially closed, then we have the c.b. isomorphism $\mathbf{Tor}^0_{\mathcal{A}}(X,Y) \cong X \otimes_{\mathcal{A}} Y$.

■

COROLLARY 3.5.13: *For a left operator* $\mathcal{A}$ *module* Y*, the following are equivalent*

(1) Y *is operator flat*

(2) $\mathbf{Tor}^1_{\mathcal{A}}(X,Y) = 0$ *and* $X\hat{\otimes}Y$ *is differentially closed, for all operator* $\mathcal{A}$*-modules* X

(3) $\mathbf{Tor}^n_{\mathcal{A}}(X,Y) = 0$ *and* $X\hat{\otimes}Y$ *is differentially closed, for all operator* $\mathcal{A}$*-modules* X *and* $n > 0$.

Proof: (1) $\Rightarrow$ (3): This follows by exactness. (3) $\Rightarrow$ (2): This is immediate. (2) $\Rightarrow$ (1): Apply Theorem 3.5.10. and Lemma 3.5.12 to conclude that the functor $? \otimes_{\mathcal{A}} Y$ is exact. ■

REMARK: Note that in Theorem 3.5.5 and Lemma 3.5.11 we noted that the c.b. isomorphism is not an isometric isomorphism. This may seem a surprise given that all of our reduction formulas were isometric isomorphisms. However it is important to note that the various derived functors are given relative to *any* projective resolution. As a result the **Tor** functors are only defined up to c.b. isomorphism. (See further discussion on this in Appendix A). We shall see the same effect in the next section when we investigate the functor **Ext**.

Now we can relate flatness to amenability.

THEOREM 3.5.14: *Let $\mathcal{A}$ be operator amenable. Then every left module is operator flat. In particular the functors $? \otimes_{\mathcal{A}} Y$ and $X \otimes_{\mathcal{A}} ?$ are exact for all left operator modules Y and right modules X.*

Proof: Suppose $\mathcal{A}$ is operator amenable. Then by Theorem 3.4.7 we have that $OH_1(\mathcal{A}, Y \hat{\otimes} X) = 0$ for all modules X and Y. By Lemma 3.5.11 we conclude that $\mathbf{Tor}^1_{\mathcal{A}}(X, Y) = 0$ and thus by Corollary 3.5.13 every left module Y is operator flat.

■

3.6 $CB_{\mathcal{A},\mathbb{C}}(X, Y)$ and Ext

In the previous section we investigated the exactness of the functor $\otimes_{\mathcal{A}}$ and in this section we shall investigate the exactness of the functor $CB_{\mathcal{A},\mathbb{C}}(?, Q)$. We shall introduce spaces **Ext** which will measure this exactness, and we shall relate this to both the standard cohomology as well as to the results of the previous section. This is to say, if $0 \leftarrow X \leftarrow Y \leftarrow Z \leftarrow 0$ is an extension sequence, is

$$0 \rightarrow CB_{\mathcal{A},\mathbb{C}}(X, Q) \rightarrow CB_{\mathcal{A},\mathbb{C}}(Y, Q) \rightarrow CB_{\mathcal{A},\mathbb{C}}(Z, Q) \rightarrow 0$$

exact?

Now for the left module Y, we define $\mathbf{Ext}^n_{\mathcal{A}}(X,Y)$ to be the nth derived functor of $CB_{\mathcal{A},\mathbb{C}}(?,Y)$ applied to the left module X.

As in the previous section we can consider the standard resolution

$$0 \leftarrow \mathcal{A}_+\hat{\otimes}X \overset{d_1}{\leftarrow} \mathcal{A}_+\hat{\otimes}\mathcal{A}\hat{\otimes}X \overset{d_2}{\leftarrow} \mathcal{A}_+\hat{\otimes}\mathcal{A}\hat{\otimes}\mathcal{A}\hat{\otimes}X \leftarrow \dots$$

Now we apply the functor $CB_{\mathcal{A},\mathbb{C}}(?,Y)$ to it to obtain:

$$0 \to CB_{\mathcal{A},\mathbb{C}}(\mathcal{A}_+\hat{\otimes}X,Y) \overset{(d_1)^*}{\to} CB_{\mathcal{A},\mathbb{C}}(\mathcal{A}_+\hat{\otimes}\mathcal{A}\hat{\otimes}X,Y) \overset{(d_2)^*}{\to} \dots$$

We calculate $\mathbf{Ext}^n_{\mathcal{A}}(X,Y)$ by finding the cohomology of the above sequence. Thus we have

$$\mathbf{Ext}^n_{\mathcal{A}}(X,Y) = \frac{\ker(d_{n+1})_*}{\mathrm{im}(d_n)_*}.$$

Thus $\mathbf{Ext}^n_{\mathcal{A}}(?,Y)$ is an additive functor from the category of left operator $\mathcal{A}$-modules into the category of linear spaces. Now wa are able to note the following special case:

THEOREM 3.6.1: *Let X be an operator $\mathcal{A}$-bimodule. Then considering X as a left operator $\mathcal{A}^e$-module we have the following equality*

$$OH^n(\mathcal{A},X) = \mathbf{Ext}^n_{\mathcal{A}^e}(\mathcal{A}_+,X).$$

Furthermore the equality is a c.b. isomorphism whenever X is differentially closed.

Proof: As we did with the Tor functor, we consider the projective resolution

$$0 \leftarrow \mathcal{A}_+ \hat{\otimes} \mathcal{A}_+ \xleftarrow{d_1} \mathcal{A}_+ \hat{\otimes} \mathcal{A} \hat{\otimes} \mathcal{A}_+ \xleftarrow{d_2} \ldots$$

of the left $\mathcal{A}^e$ module $\mathcal{A}_+$. Now we apply $CB_{\mathcal{A}_+,\mathcal{A}_+}(?, X)$ to get

$$0 \to CB_{\mathcal{A}_+,\mathcal{A}_+}(\mathcal{A}_+ \hat{\otimes} \mathcal{A}_+, X) \xrightarrow{(d_1)^*} CB_{\mathcal{A}_+,\mathcal{A}_+}(\mathcal{A}_+ \hat{\otimes} \mathcal{A} \hat{\otimes} \mathcal{A}_+, X) \xrightarrow{(d_2)^*} \ldots$$

which reduces to

$$0 \to X \xrightarrow{\delta_1} CB(\mathcal{A}, X) \xrightarrow{\delta_2} CB(\mathcal{A} \hat{\otimes} \mathcal{A}, X) \ldots$$

Calculations similar to those in the proof of Theorem 3.5.6 show that this is exactly the standard cohomology sequence.

∎

REMARK : In the case of $X = Y^*$ there is an alternative approach to the above proof using the results of the previous section. Take the standard resolution of $\mathcal{A}_+$ and apply the functor $? \otimes_{\mathcal{A}^e} X$. The dual of this sequence, by Proposition 3.5.4 and 3.3.5 is exactly the last sequence in the above proof. However, we already know that the dual of the standard homology complex, is the standard cohomology complex.

We complete this section with two important theorems. Recall that whenever X and Y are left $\mathcal{A}$-modules, we can consider the space $CB(X, Y)$ to be an operator $\mathcal{A}$-bimodule where

$$(a \cdot T)(x) = a \cdot T(x) \text{ and } (T \cdot a)(x) = T(ax).$$

The following two theorems are the direct operator space analogue of [54, III.4.12 and III.4.13]

THEOREM 3.6.2: *Let X and Y be left operator $\mathcal{A}$-modules, then we have the equality*

$$OH^n(\mathcal{A}, CB(X, Y)) = \mathbf{Ext}^n_{\mathcal{A}}(X, Y),$$

which is a c.b. isomorphism whenever $CB(X, Y)$ is differentially closed.

Proof: Once again we can calculate $\mathbf{Ext}^n_{\mathcal{A}}(X, Y)$ through the standard resolution

$$0 \leftarrow \mathcal{A}_+\hat{\otimes}X \overset{d_1}{\leftarrow} \mathcal{A}_+\hat{\otimes}\mathcal{A}\hat{\otimes}X \overset{d_2}{\leftarrow} \dots$$

which yields

$$0 \rightarrow CB_{\mathcal{A},\mathbb{C}}(\mathcal{A}_+\hat{\otimes}X, Y) \overset{(d_1)}{\rightarrow} CB_{\mathcal{A},\mathbb{C}}(\mathcal{A}_+\hat{\otimes}\mathcal{A}\hat{\otimes}X, Y) \rightarrow \dots.$$

This in turn reduces to

$$0 \rightarrow CB(X, Y) \overset{\delta^1}{\rightarrow} CB(\mathcal{A}\hat{\otimes}X, Y) \overset{\delta^2}{\rightarrow} CB(\mathcal{A}\hat{\otimes}\mathcal{A}\hat{\otimes}X, Y) \dots$$

Now using the c.b. isomorphism $CB(\mathcal{A}\hat{\otimes}X, Y) \cong CB(\mathcal{A}, CB(X, Y))$ this complex is isomorphic to the standard cohomology complex for $CB(X, Y)$. The fact that we have a cb isomorphism when $CB(X, Y)$ is differentially closed should now be clear. ■

Once we make a connection between the **Ext** groups and extension sequences, we will use the above theorem to classify all such extensions (and hence address the complemented ideal question).

We have the following additional useful result:

THEOREM 3.6.3: *Let X be a left operator $\mathcal{A}$-module, and let Y be a right operator $\mathcal{A}$-module, then $\mathbf{Ext}^n_{\mathcal{A}}(X, Y^*) = \mathbf{Ext}^n_{\mathcal{A}^{op}}(Y, X^*)$.*

Proof: By the preceding theorem we have

$$\mathbf{Ext}^n_{\mathcal{A}}(X, Y^*) = OH^n(\mathcal{A}, CB(X, Y^*)) = OH^n(\mathcal{A}, (X\hat{\otimes}Y)^*).$$

Similarly we have

$$\mathbf{Ext}^n_{\mathcal{A}^{op}}(Y, X^*) = OH^n(\mathcal{A}^{op}, CB(Y, X^*)) = OH^n(\mathcal{A}^{op}, (Y\hat{\otimes}X)^*).$$

Note the left module $(X\hat{\otimes}Y)^*$ can be identified with the right $\mathcal{A}^{op}$-module $(Y\hat{\otimes}X)^*$. By shuffling coefficients, we can see that

$$OH^n(\mathcal{A}, (X\hat{\otimes}Y)^*) = OH^n(\mathcal{A}^{op}, (Y\hat{\otimes}X)^*).$$

■

As indicated earlier one of the major reasons for studying the spaces **Ext** is that they describe the "exactness" of the functors $CB_{\mathcal{A},\mathbb{C}}(?, Y)$ and $CB_{\mathcal{A},\mathbb{C}}(X, ?)$. Indeed the same categorical properties hold here. As a result we are able to conclude the following:

THEOREM 3.6.4: *Let $0 \to X_1 \to X_2 \to X_3 \to 0$ be an admissible sequence of $\mathcal{A}$-modules. Then there exist algebraic maps*

$$E_k^* : \mathbf{Ext}^k_{\mathcal{A}}(X_3, Z) \mapsto \mathbf{Ext}^{k+1}_{\mathcal{A}}(X_1, Z)$$

such that the sequence

$$0 \to \mathbf{Ext}^0_{\mathcal{A}}(X_1, Z) \to \mathbf{Ext}^0_{\mathcal{A}}(X_2, Z) \to \mathbf{Ext}^0_{\mathcal{A}}(X_3, Z) \xrightarrow{E_0^*} \mathbf{Ext}^1_{\mathcal{A}}(X_1, Z) \to \dots$$

is exact.

Proof: Since $CB_{\mathcal{A},\mathbb{C}}(?, Z)$ is an additive functor, we may apply Theorem 3.5.9.

■

In algebra we have that $\mathbf{Ext}^0_{\mathcal{A}}(X,Y)$ equals the collection of left module morphisms from X to Y, and we have the same result here for essentially the same reason.

LEMMA 3.6.5: *For left operator $\mathcal{A}$ modules X and Y, we have a natural c.b. isomorphism*

$$\mathbf{Ext}^0_{\mathcal{A}}(X,Y) \cong CB_{\mathcal{A},\mathbb{C}}(X,Y).$$

Proof: As in the algebraic case, the functor $CB_{\mathcal{A},\mathbb{C}}(X,?)$ is left exact. In particular, we have the sequence

$$0 \leftarrow \mathcal{A}_+\hat{\otimes}X \leftarrow \mathcal{A}_+\hat{\otimes}\mathcal{A}\hat{\otimes}X \leftarrow \dots$$

results in the complex

$$0 \to CB_{\mathcal{A},\mathbb{C}}(\mathcal{A}_+\hat{\otimes}X,Y) \to CB_{\mathcal{A},\mathbb{C}}(\mathcal{A}_+\hat{\otimes}\mathcal{A}\hat{\otimes}X,Y) \to \dots$$

which reduces to

$$0 \to CB(X,Y) \xrightarrow{d_1} CB(\mathcal{A}\hat{\otimes}X,Y) \to \dots.$$

Thus $\mathbf{Ext}^0_{\mathcal{A}}(X,Y) \cong \ker d_1$. However

$$0 \leftarrow X \xleftarrow{\epsilon} \mathcal{A}_+\hat{\otimes}X \leftarrow \dots$$

is exact. Hence

$$0 \to CB_{\mathcal{A},\mathbb{C}}(X,Y) \xrightarrow{\epsilon_*} CB(X,Y) \xrightarrow{d_1} CB(\mathcal{A}\hat{\otimes}X,Y) \to \dots$$

is exact at the term $CB(X,Y)$. Thus $\ker d_1 = \operatorname{im}\epsilon_* \cong CB_{\mathcal{A},\mathbb{C}}(X,Y)$. ∎

COROLLARY 3.6.6: *For a left module Y, the following are equivalent*

(1) *Y is injective,*

(2) $\mathbf{Ext}^1_{\mathcal{A}}(X, Y) = 0$ *for all left modules X,*

(3) $\mathbf{Ext}^n_{\mathcal{A}}(X, Y) = 0$ *for all modules X and for all $n > 0$.*

Proof: (1) $\Rightarrow$ (3): This follows from the exactness of the functor $CB_{\mathcal{A},\mathbb{C}}(?, Y)$. (3) $\Rightarrow$ (2): This is immediate. (2) $\Rightarrow$ (1): Use Theorem 3.6.4 and Lemma 3.6.5. and Theorem 3.3.2. ∎

THEOREM 3.6.7: *Let $\mathcal{A}$ be operator amenable. Then for every left operator module Y, we have that Y^* is injective.*

Proof: We note that Theorem 3.6.3 and Corollary 3.6.6 will prove this assertion. However we shall use the dual nature of the cohomology and the homological results of the previous section to prove this fact. First note that Theorem 3.5.14 shows that $\mathcal{A}$ operator amenable implies Y is flat and hence by Proposition 3.5.9 that Y^* is injective. ∎

The previous theorem shows that if $\mathcal{A}$ is operator amenable then every dual module is injective as a right module. However we can improve this to show that every dual module is in fact injective as a bimodule with the following two facts:

COROLLARY 3.6.8: *Let $\mathcal{A}$ be operator amenable. Then $\mathcal{A}\hat{\otimes}\mathcal{A}^{op}$ is operator amenable.*

Proof: We note that every derivation from $\mathcal{A}$ into any module X defines a derivation from $\mathcal{A}^{op}$ in the obvious way, and conversely. Thus if $\mathcal{A}$ is operator amenable, so is $\mathcal{A}^{op}$.

∎

PROPOSITION 3.6.9: *Let $\mathcal{A}$ be operator amenable. Then for every module Y, we have that Y^* is bi-injective.*

Proof: Applying the previous two Corollaries we have that for all operator bi-modules X, $\mathbf{Ext}_{\mathcal{A}\hat{\otimes}\mathcal{A}^{op}}(X, Y^*) = 0$. Thus by Corollary 3.6.6 we have that Y^* is injective as a $\mathcal{A}\hat{\otimes}\mathcal{A}^*$ module. Thus Y^* is bi-injective.

∎

Combining the results of this chapter to this point we have the following theorem:

THEOREM 3.6.10: *The following are equivalent*

(1) *$\mathcal{A}$ is operator amenable*

(2) *$OH^n(\mathcal{A}, X^*) = 0$ for all $n > 0$ and all dual operator $\mathcal{A}$ bimodules X^*.*

(3) *$OH_n(\mathcal{A}, X) = 0$ for all bimodules X and $n > 0$ and furthermore every bimodule is differentially closed*

(4) *$\mathbf{Ext}^n_{\mathcal{A}^e}(\mathcal{A}_+, X^*) = 0$ for all $n > 0$ and all dual operator $\mathcal{A}$ bimodules X^**

(5) *$\mathbf{Tor}^n_{\mathcal{A}^e}(X, \mathcal{A}_+) = 0$ for all bimodules X and $n > 0$ and furthermore every bimodule is differentially closed*

(6) *$\mathcal{A}_+$ is operator biflat.*

Proof: The equivalences of $(1), (2)$ and (3) are Theorem 3.4.7. The equivalence $(2) \Leftrightarrow (4)$ is Theorem 3.6.1 and $(3) \Leftrightarrow (5)$ is Theorem 3.5.6. From theorem 3.6.3 we have

$$\mathbf{Ext}^n_{\mathcal{A}^e}(\mathcal{A}_+, X^*) = \mathbf{Ext}^n_{\mathcal{A}^e}(X, \mathcal{A}_+^*).$$

Thus by Corollary 3.6.6, we conclude $\mathcal{A}_+^*$ is injective as a $\mathcal{A}^e$ module. Hence we have by Theorem 3.5.9 $\mathcal{A}_+$ is a flat $\mathcal{A}^e$ module. Conversely, if $\mathcal{A}_+^*$ is an injective

$\mathcal{A}^e$ module, then $\mathbf{Ext}^n_{\mathcal{A}^e}(X, \mathcal{A}^*_+) = 0$ by Corollary 3.6.6.

■

We note that Khelemskii defined amenability via the analogue of condition (6) in the above Theorem.

3.7 Extension Sequences and Cohomology

First we recall what we mean by an n-extension sequence. Given $\mathcal{A}$-modules X and Y, suppose we have a exact admissible operator complex of the form

$$0 \to X \to B_1 \to \ldots \to B_n \to Y \to 0.$$

for various modules B_k. We call this sequence an n-extension of X by Y. Note that in the case $n = 1$ we always have the 1-extension

$$0 \to X \to X \oplus Y \to Y \to 0$$

where the module action on $X \oplus Y$ is the diagonal action. In the case $n > 1$, we always have the sequence

$$0 \to X \xrightarrow{id} X \xrightarrow{0} 0 \ldots \to 0 \xrightarrow{0} Y \xrightarrow{id} Y \to 0.$$

Given two n-extensions

$$S : 0 \to X \to B_1 \to \ldots \to B_n \to Y \to 0.$$

and

$$T : 0 \to X \to C_1 \to \ldots \to C_n \to Y \to 0.$$

we write $S < T$ if there exist c.b. module maps $\theta_k : B_k \mapsto C_k$ such that the resulting diagram commutes (with $X \stackrel{id}{\to} X$ and $Y \stackrel{id}{\to} Y$). We shall say S and T are equivalent (and write $S \sim T$) if there exists an n-extension sequence R such that $S < R$ and $T < R$. (See McLean for the algebraic version of this or Paulsen for the h-module version).

DEFINITION 3.7.1: We let $\mathbf{Ex}^n_{\mathcal{A}}(X, Y)$ denote the set[4] of equivalence classes of n-extensions of X by Y[5].

The key fact for us is the following theorem:

THEOREM 3.7.2: *For any modules X and Y we have*

$$\mathbf{Ex}^n_{\mathcal{A}}(X, Y) = \mathbf{Ext}^n_{\mathcal{A}}(X, Y).$$

The proof of this theorem involves several steps which use primarily the categorical properties of exactness, admissibility and projectiveness. Since we have built up all the necessary categorical tools, we omit this proof. See also Paulsen's work [66].

However we have the following important case which we can prove from our work in the previous Chapter:

[4]To avoid messy set theoretic problems, we usually restrict the "size" of the modules B_k to avoid this use of "wild" set theory.

[5]Our notion is the reverse of the standard notation, however the author notes that either the definition is backwards or the fundamental theorem to follow is.

COROLLARY 3.7.3: *Let $\mathcal{A}$ be an operator amenable completely contractive Banach algebra and let*

$$(\Xi) : 0 \to X^* \to Q \to Y \to 0$$

be an admissible extension sequence of $\mathcal{A}$-bimodules. Then (Ξ) splits.

Proof: From the above we have that $\mathbf{Ex}^1_{\mathcal{A}^e}(X^*, Y) = \mathbf{Ext}^n_{\mathcal{A}^e}(Y, X^*)$, which by Theorem 3.6.2 equals $OH^1(\mathcal{A}^e, (Y\hat{\otimes}X)^*)$. Since $\mathcal{A}$ is operator amenable, it follows that $\mathcal{A}_+$ is operator amenable (Lemma 3.4.5). Hence so is $\mathcal{A}^e$ by Corollary 3.6.8. Thus $\mathbf{Ex}^1_{\mathcal{A}^e}(X^*, Y)$ is equivalent to the trivial element. Thus (Ξ) splits.

As an alternative, we can prove this result without appealing to the previous theorem. By Proposition 3.6.9 we have that X^* is injective as a bimodule. Consider the following diagram:

$$\begin{tikzcd}
X^* \arrow[r, "f"] \arrow[d, equal, "id"'] & Q \arrow[dl, dotted, "\phi"'] \\
X^* &
\end{tikzcd}$$

Since X^* is bi-injective, there exists a map $\phi \in CB_{\mathcal{A},\mathcal{A}}(Q, X^*)$ such that $f \circ \phi = id_{X^*}$. Hence by Lemma 3.2.6 the sequence splits. ■

3.8 Operator Biprojectivity and Amenability

Let $\mathcal{A}$ be a completely contractable Banach algebra and let $\pi : \mathcal{A}\hat{\otimes}\mathcal{A} \to \mathcal{A}$ be the multiplication map, and let $N = \ker \pi$. Now we consider the short complex

$$(\mathfrak{A}) : 0 \xrightarrow{i} N \to \mathcal{A}\hat{\otimes}\mathcal{A} \xrightarrow{\pi} \mathcal{A} \to 0$$

and its dual complex

$$(\mathfrak{A})^* : 0 \to \mathcal{A}^* \xrightarrow{\pi^*} (\mathcal{A}\hat{\otimes}\mathcal{A})^* \xrightarrow{i^*} N^* \to 0.$$

In this section we shall investigate the splitting of these two sequences. We shall see that the splitting of the first is related to the *operator biprojectivity* of $\mathcal{A}$ and the second is related to the operator amenability of $\mathcal{A}$.

We begin with the following lemma for modules:

LEMMA 3.8.1: *Suppose P is a left operator $\mathcal{A}$- module, and let $\pi_L : \mathcal{A}_+\hat{\otimes}P \to P$ be the module map onto P and N its kernel. Then the admissible sequence*

$$(\mathfrak{M}_L) : 0 \to N \to \mathcal{A}_+\hat{\otimes}P \xrightarrow{\pi} P \to 0$$

splits if and only if P is projective.

Proof: First note that the sequence is clearly short exact, and since the map $\tau : P \mapsto \mathcal{A}_+\hat{\otimes}P$ given by $\tau(p) = e \otimes p$ is clearly a completely bounded inverse for π_L, by Lemma 3.2.6 $(\mathfrak{M}_L)$ is admissible. Now suppose P is projective. Consider the diagram

$$\begin{tikzcd}
& P \arrow[dl, dotted, "\psi"'] \arrow[d, equal, "id"] \\
\mathcal{A}_+\hat{\otimes}P \arrow[r, "\pi_L"'] & P
\end{tikzcd}$$

By projectivity of P there exists a map ψ which extends $id : P \mapsto P$. In particular we have $\pi_L \circ \psi = id_P$, hence by Lemma 3.2.6 the sequence splits. Conversely if the sequence splits, we can consider the diagram

$$\begin{tikzcd} \mathcal{A}_+\hat{\otimes}P \arrow[rr, bend left, "\pi_L"] \arrow[d, dotted, "\psi"'] \arrow[drr, "\theta'"] & & P \arrow[ll, bend left, "\rho"] \arrow[d, "\theta"] \\ Y \arrow[rr, "\phi"'] & & Z \end{tikzcd}$$

Since $\mathcal{A}_+\hat{\otimes}P$ is projective we conclude that the module map θ' extends to ψ : $\mathcal{A}_+\hat{\otimes}P \to Y$. Since the sequence splits, π_L has a right inverse which is a module map, call it ρ. Thus the map $\psi' : P \to Y$ defined by $\psi'(p) = \rho \circ \psi(p)$ is clearly an extension of θ. ■

The last part of this proof actually proves the general result that if P is projective and $\theta : P \mapsto Q$ is a module map with a right inverse which is a module map, then Q is also projective.

To discuss the splitting of the sequences mentioned at the beginning of this section, we note that the sequence $\mathfrak{A}$ is exact only when the module map is onto. Hence we recall that if the module P is neounital, then the module map $\pi : \mathcal{A}\hat{\otimes}P \to P$ is onto. Thus we can consider the sequence

$$(\mathfrak{M}) : 0 \to N \to \mathcal{A}\hat{\otimes}P \to P \to 0.$$

which is clearly short exact. This leads to the following

PROPOSITION 3.8.2: *A neounital module P is projective if and only if the sequence*

$$(\mathfrak{M}) : 0 \to N \to \mathcal{A}\hat{\otimes}P \xrightarrow{\pi} P \to 0$$

splits.

Proof: If the sequence splits, then there exists a c.b. module map $\rho : P \to \mathcal{A}\hat{\otimes}P$ which is a right inverse for π. Clearly ρ is also a right inverse for $\pi_L : \mathcal{A}_+\hat{\otimes}P$. Thus by the previous Lemma, P is projective. Conversely, if P is projective, then by the previous lemma, there exists $\tau : P \mapsto \mathcal{A}_+\hat{\otimes}P$ which is a right inverse module map for π_L. However we note

$$\tau(P) = \tau(\mathcal{A} \cdot P) = \mathcal{A} \cdot \tau(P) \subset \mathcal{A} \cdot (\mathcal{A}_+\hat{\otimes}P) \subset \mathcal{A}\hat{\otimes}P.$$

Thus τ is an inverse for π. ■

DEFINITION 3.8.3: A completely contractive Banach algebra is called ***operator biprojective*** if it is projective as an operator $\mathcal{A}^e$ module.

To connect operator biprojectivity with splitting of certain sequences, we first note the following lemma.

LEMMA 3.8.4: *Suppose $\mathcal{A}$ is neounital. Then $\mathcal{A}$ is operator biprojective if and only if the sequence*

$$0 \to N \to (\mathcal{A}\hat{\otimes}\mathcal{A}) \xrightarrow{\pi} \mathcal{A} \to 0$$

splits as $\mathcal{A}^e$ modules.

Proof: Suppose $\mathcal{A}$ is neounital and biprojective. Then from the previous proposition we have that the sequence

$$0 \to N_e \to \mathcal{A}^e \hat{\otimes} \mathcal{A} \xrightarrow{\pi_e} \mathcal{A} \to 0$$

splits, where N_e is the kernel of the module map $\pi_e : \mathcal{A}^e \hat{\otimes} \mathcal{A} \mapsto \mathcal{A}$. Note that we have $\mathcal{A}^e \hat{\otimes} \mathcal{A} \cong \mathcal{A}_+ \hat{\otimes} \mathcal{A} \hat{\otimes} \mathcal{A}_+$. Now let $\pi_R : \mathcal{A} \hat{\otimes} \mathcal{A}_+ \mapsto \mathcal{A}$ be the "right" module map. If $\rho : \mathcal{A} \mapsto \mathcal{A}^e \hat{\otimes} \mathcal{A}$ is an $\mathcal{A}^e$ module map which is a right inverse for π_e, then it is easy to see that in particular the induced map

$$\rho : \mathcal{A} \mapsto \mathcal{A}_+ \hat{\otimes} \mathcal{A} \hat{\otimes} \mathcal{A}_+$$

is an $\mathcal{A}$-bimodule map. Clearly the map

$$id \hat{\otimes} \pi_R : \mathcal{A}_+ \hat{\otimes} \mathcal{A} \hat{\otimes} \mathcal{A}_+ \mapsto \mathcal{A}_+ \hat{\otimes} \mathcal{A}$$

is also a bimodule map, and a simple calculation shows that

$$\rho' = (id \hat{\otimes} \pi_R) \circ \rho : \mathcal{A} \mapsto \mathcal{A}_+ \hat{\otimes} \mathcal{A}$$

is a bimodule map which is an inverse for π_L. Following the idea in Proposition 3.8.2 we have that

$$\rho'(\mathcal{A}) = \rho'(\mathcal{A} \cdot \mathcal{A}) \subset \mathcal{A} \cdot (\mathcal{A}_+ \hat{\otimes} \mathcal{A}) = \mathcal{A} \hat{\otimes} \mathcal{A}$$

thus ρ' is a bimodule map which is a right inverse for π. Since ρ, π_R and id are all completely bounded, so is ρ'.

Conversely, if the sequence splits, it follows that $\mathcal{A}$ is both left projective and right projective, by Proposition 3.8.2. Hence by standard arguments $\mathcal{A} \hat{\otimes} \mathcal{A}$ is operator biprojective. Since the sequence splits as $\mathcal{A}^e$ modules, there exists a bimodule map $\rho : \mathcal{A} \mapsto \mathcal{A} \hat{\otimes} \mathcal{A}$. Thus $\mathcal{A}$ is biprojective. ∎

Since $\mathbf{A}(G)$ is neountial for all amenable groups G, we can use the previous proposition to classify for which amenable groups G, $\mathbf{A}(G)$ is operator biprojective. First note that $\mathbb{C}$ is a left operator $\mathbf{A}(G)$ module under the module action

$$u \cdot \alpha = \alpha u(e).$$

THEOREM 3.8.5: *Let G be amenable. Then the following are equivalent*

(1) $\mathbf{A}(G)$ *is operator biprojective*

(2) G *is discrete*

(3) *the left operator* $\mathbf{A}(G)$ *module* $\mathbb{C}$ *is projective.*

Proof: (3) $\Rightarrow$ (2): Let $\mathcal{I}_0$ denote the ideal of functions $u \in \mathbf{A}(G)$ which are equal to zero at e. Since $\mathcal{I}$ is cofinite dimensional, there exists a bounded projection P onto $\mathcal{I}_0$. Now $1-P : \mathbf{A}(G) \to Q$ where Q is the complement of $\mathcal{I}_0$ is $\mathbf{A}(G)$. Clearly $Q \cong \mathbf{A}(G)/\mathcal{I}_0 \cong \mathbb{C}$. Let $\gamma : \mathbf{A}(G) \to \mathbb{C}$ be given by $\gamma(u) = u(e)$. Certainly γ is completely bounded, and since there exists a c.b. map from $\mathbb{C}$ to $Q \subset \mathbf{A}(G)$, it follows that γ is admissible. Since $\mathbb{C}$ is projective, there is a right inverse module map for γ, call it τ.

Now for all $u \in \mathbf{A}(G)$ we have that $u \cdot \tau(1) = \tau(u \cdot 1) = \tau(u(e))$. Since for each $s \in G$ such that $s \neq e$ we can find an element $u \in \mathbf{A}(G)$ such that $u(s) = 0$ and $u(e) = 1$ it follows that $[\tau(1)](s) = 0$ for all $s \neq e$. Thus G is discrete.

(2) $\Rightarrow$ (1): Since G is amenable, $\mathbf{A}(G)$ is neounital. Here we can use the isomorphism $\mathbf{A}(G \times G) \cong \mathbf{A}(G)\hat{\otimes}\mathbf{A}(G)$ given by $(u \otimes v)(s,t) = u(s)v(t)$ (see [25]). The map $\tau : \mathbf{A}(G) \mapsto \mathbf{A}(G \times G)$ given by $\tau(u)(s,t) = u(s)\delta_s^t$, where δ is the Kronecker delta function, is a right inverse for the multiplication map $\pi : \mathbf{A}(G)\hat{\otimes}\mathbf{A}(G) \mapsto \mathbf{A}(G)$. It now suffices to show that this map τ is completely

bounded. Let $G_D = \{(s,s) : s \in G\}$. Clearly $\tau(\mathbf{A}(G)) \subset 1_{G_D}(\mathbf{A}(G \times G))$. But we have

$$1_{G_D}(\mathbf{A}(G \times G)) \cong \mathbf{A}(G_D) \cong \mathbf{A}(G)$$

by our future Theorem 5.2.3. Thus τ is completely bounded. Now we may apply the previous lemma to conclude that $\mathbf{A}(G)$ is biprojective.

(1) $\Rightarrow$ (3): Since G is amenable, it follows that $\mathbf{A}(G)$ has a bounded approximate identity and hence $\mathbb{C}$ is essential. Since $\mathbb{C}$ is an essential module over an operator biprojective algebra, it is projective. (See for example [54]).

■

Now going back to the sequences at the start of this section, we easily see that if the sequence $(\mathfrak{A})$ splits, then so does $(\mathfrak{A})^*$. Historically, this question goes back to Khelemskii, where in the category of Banach spaces he proved the following:

THEOREM 3.8.6: *The following are equivalent:*

(1) *The Banach algebra $\mathcal{A}$ is amenable as a Banach algebra*

(2) *$\mathcal{A}$ has a bounded approximate identity, and the sequence*

$$0 \to \mathcal{A}^* \to (\mathcal{A} \otimes_\gamma \mathcal{A})^* \to N^* \to 0$$

splits as Banach $\mathcal{A}$ bimodules.

The proof of this theorem uses the Banach space version of flatness and injectivity in the same spirit of this Chapter.

Later Curtis and Loy provided an alternative proof of this same theorem using more "traditional" Banach space methods. It is using these latter methods that Ruan and Xu were able to prove the operator space version which is as follows.

THEOREM 3.8.7: *The following are equivalent*

(1) *$\mathcal{A}$ is operator amenable*

(2) *$\mathcal{A}$ has a bounded approximate identity and the sequence $(\mathfrak{A})^*$ splits.*

We note that we could provide an alternative proof to Ruan and Xu's using the operator space versions of flatness and injectivity, however we will not do so.

Chapter 4

Operator weak Amenability

4.1 Introduction

In the previous chapter, we investigated completely contractive Banach algebras which were operator amenable. Recall that this is the case exactly when each completely bounded derivation into any dual operator module is inner. In some sense, this can be considered a rather strong condition. In particular, we are free to construct various "odd-ball" modules, into which each c.b. derivation is necessarily inner.

We note of course that given a completely contractive Banach algebra $\mathcal{A}$, its dual space $\mathcal{A}^*$ becomes a natural dual operator bimodule. In this Chapter we investigate the special case that every c.b. derivation into $\mathcal{A}^*$ is inner.

We begin with some special notation which is particular to this Chapter.

4.2 Preliminaries and Notation

Let $\odot$ represent the module action of $\mathbf{A}(G)$ acting on $\mathbf{VN}(G)$. Let $UCB(\hat{G})$ denote the closed linear span of $\mathbf{A}(G) \odot \mathbf{VN}(G)$. Then $UCB(\hat{G})$ is a topologically introverted C^*-subalgebra of $\mathbf{VN}(G)$ (See [57]). Given $\phi \in \mathbf{VN}(G)$, let $Orb(\phi) = \{u \odot \phi \mid u \in \mathbf{A}(G), \|u\| \leq 1\}$. ϕ is (weakly) almost periodic if $Orb(\phi)$ is relatively (weakly) compact. Let $WAP(\hat{G})$ and $AP(\hat{G})$ denote the spaces of weakly almost periodic and almost periodic functionals on $\mathbf{A}(G)$ respectively. $AP(\hat{G})$ and $WAP(\hat{G})$ are also topologically introverted subspaces of $\mathbf{VN}(G)$. Moreover, each of the spaces $UCB(\hat{G}), WAP(\hat{G})$, and $AP(\hat{G})$ contain $C^*_\delta(G)$ as a closed subspace, while $UCB(\hat{G})$ and $WAP(\hat{G})$ contain $C^*_\lambda(G)$.

When G is abelian, $UCB(\hat{G})$ is the Fourier transform of the C^*-algebra of uniformly continuous functions on $\hat{G}$. $WAP(\hat{G})$ and $AP(\hat{G})$ are the Fourier transforms of the C^*-algebra of weakly almost periodic functions and almost periodic functions of G respectively. In general it is not known if $AP(\hat{G})$ or $WAP(\hat{G})$ are C^*-algebras. Finally if G is compact, then $UCB(\hat{G}) = \mathbf{VN}(G)$ and if G is discrete, $C^*_\delta(G) = C^*_\lambda(G) = UCB(\hat{G})$. We refer the reader to [37] and [57] for these and other properties of the above spaces.

Let ϕ be a continuous multiplicative functional on a Banach algebra $\mathcal{A}$. A ***point derivation*** of $\mathcal{A}$ at ϕ is a linear functional $d : \mathcal{A} \mapsto \mathbb{C}$ such that $d(ab) = \phi(a)d(b) + \phi(b)d(a)$.

4.3 Operator Weak Amenability

One of the principal themes of this thesis is to show that when considering problems of cohomology for $\mathbf{A}(G)$, the operator space setting is the most appropriate. Ruan's result in [68] as well as our result on operator biprojectivity in Section 3.8 is certainly strong evidence to support this point of view. In this section, we will introduce the notion of operator weak amenability. We will then extend many of the fundamental results from the Banach algebra setting to the operator space setting. These will be used in the next Section to study operator weak amenability for $\mathbf{A}(G)$.

The following is adapted from the definition of weak amenability for Banach algebras:

DEFINITION 4.3.1: We say that a completely contractive Banach algebra $\mathcal{A}$ is ***operator weakly amenable*** if every completely bounded derivation D from $\mathcal{A}$ into $\mathcal{A}^*$ is inner. (i.e. $OH^1(\mathcal{A}, \mathcal{A}^*) = 0$)

We begin with a few simple observations which are well known for weak amenability (see [13] and [40] for analogous results).

LEMMA 4.3.2: *Let $\mathcal{A}$ be a completely contractive Banach algebra such that $\mathcal{A}^2$ is not dense in $\mathcal{A}$. Then $\mathcal{A}$ is not operator weakly amenable.*

Proof: Let $\phi \in \mathcal{A}^*$ be nonzero with $\phi(\mathcal{A}^2) = 0$. Then $D(a) = \phi(a)\phi$ is a derivation from $\mathcal{A}$ into $\mathcal{A}^*$. Moreover, D is completely bounded since $\phi \in \mathcal{A}^*$.

Assume that $D(a) = \zeta a - a\zeta$ for some $\zeta \in \mathcal{A}^*$. Let $b \in \mathcal{A}$ be such that $\phi(b) \neq 0$. Then $D(b)(b) = (\zeta b - b\zeta)(b) = 0$, whereas $D(b)(b) = \phi(b)\phi(b) \neq 0$. It follows that D cannot be inner and hence that $\mathcal{A}$ is not weakly amenable. ■

PROPOSITION 4.3.3: *Let $\mathcal{A}$ be a commutative completely contractive Banach algebra. Then $\mathcal{A}$ is operator weakly amenable if and only if every completely bounded derivation from $\mathcal{A}$ into a symmetric operator $\mathcal{A}$-module X is trivial.*

Proof: If $\mathcal{A}$ is commutative, $\mathcal{A}^*$ is symmetric. Therefore the "if" direction is trivial.

Assume that $\mathcal{A}$ is operator weakly amenable, and that X is a symmetric operator $\mathcal{A}$-module. Let $D : \mathcal{A} \mapsto X$ be a non-zero completely bounded derivation. We may by Lemma 4.3.2 assume that $\mathcal{A}^2$ is dense in $\mathcal{A}$. Hence there exists $a \in \mathcal{A}$ with $D(a^2) \neq 0$. Choose $\phi \in X^*$ such that $\phi(D(a^2)) \neq 0$. For each $x \in X$, define $R_x \in \mathcal{A}^*$ by $R_x(a) = \phi(ax)$. The map $\mathcal{R} : X \mapsto \mathcal{A}^*$ with $\mathcal{R}(x) = R_x$ is completely bounded. To see this observe that

$$\begin{aligned}
\|\mathcal{R}^{(n)}\| &= \sup\left\{\|\mathcal{R}^{(n)}([x_{ij}])\| \ : \ \|[x_{ij}]\| \leq 1,\ [x_{ij}] \in \mathbb{M}_n(X)\right\} \\
&= \sup\{\|[R_{x_{ij}}]\| \ : \ \|[x_{ij}]\| \leq 1,\ [x_{ij}] \in \mathbb{M}_n(X)\} \\
&= \sup\{\|[R_{x_{ij}}(a)]\| \ : \ \|[x_{ij}]\| \leq 1,\ [x_{ij}] \in \mathbb{M}_n(X)\ \|[a_{kl}]\| \leq 1\} \\
&= \sup\{\|\phi^{(nm)}([a_{kl}x_{ij}])\| \ : \ \|[x_{ij}]\| \leq 1,\ [x_{ij}] \in \mathbb{M}_n(X)\ \|[a_{kl}]\| \leq 1\} \\
&\leq \|\phi\|_{cb} = \|\phi\|.
\end{aligned}$$

It is now straightforward to see that the map $\tilde{D} : \mathcal{A} \mapsto \mathcal{A}^*$ given by $\tilde{D}(a) = \mathcal{R}(D(a))$ is a completely bounded derivation. Finally,

$$\tilde{D}(a)(a) = R_{D(a)}(a) = \phi(aD(a)) = 1/2\ \phi(D(a^2)) \neq 0.$$

Since $\mathcal{A}$ is operator weakly amenable, this is impossible. ∎

PROPOSITION 4.3.4: *Let $\mathcal{A}$ and $\mathcal{B}$ be commutative completely contractive Banach algebras. Let $\phi : \mathcal{A} \mapsto \mathcal{B}$ be a completely bounded homomorphism with dense* *range. If $\mathcal{A}$ is operator weakly amenable, then so is $\mathcal{B}$.*

Proof: Suppose that $\mathcal{B}$ is not operator weakly amenable. We may assume that $||\phi||_{cb} = 1$. Observe that $\mathcal{B}^*$ becomes a symmetric operator $\mathcal{A}$-module via the action:

$$(a, \Gamma) \mapsto \phi(a) \cdot \Gamma = \Gamma \cdot \phi(a)$$

for each $a \in \mathcal{A}, \Gamma \in \mathcal{B}^*$. Let $D : \mathcal{B} \mapsto \mathcal{B}^*$ be a nonzero completely bounded derivation. Then $\tilde{D}(a) = D(\phi(a))$ is a completely bounded derivation from $\mathcal{A}$ into $\mathcal{B}^*$. Moreover $\tilde{D}$ is non-zero since ϕ has dense range. Now apply Proposition 4.3.3 to conclude that $\mathcal{A}$ is not weakly amenable. ■

REMARK: The analogue of Proposition 4.3.4 may fail for Banach algebras without the assumption of commutativity (see [41]). We can show that this is also true in our setting. Assume that we have a Banach algebra $\mathcal{A}$ which is weakly amenable and a bounded homomorphism ϕ onto a dense sub-algebra of $\mathcal{B}$ where $\mathcal{B}$ is not weakly amenable. Let $D : \mathcal{B} \mapsto \mathcal{B}^*$ be bounded but not inner. If we provide both $\mathcal{A}$ and $\mathcal{B}$ with the MAX operator space structure, then ϕ is completely bounded. Moreover, the bounded derivation $D : \mathcal{B} \mapsto \mathcal{B}^*$ is completely bounded but not inner. This shows that $\mathcal{B}$ is not operator weakly amenable, while $\mathcal{A}$, being weakly amenable, will be operator weakly amenable. We conclude that Proposition 4.3.4 would fail without the assumption of commutativity.

PROPOSITION 4.3.5: *Let $\mathcal{A}$ be a completely contractive Banach algebra. If $\mathcal{A}$ has a nonzero point derivation, then $\mathcal{A}$ is not operator weakly amenable.*

Proof: Let $d : \mathcal{A} \mapsto \mathbb{C}$ be a nonzero point derivation at

ϕ. We may assume that $\mathcal{A}^2$ is dense by Lemma 4.3.2. Let $D(a) = d(a)\phi$. Then it is routine to verify that D is a nonzero derivation of $\mathcal{A}$ into $\mathcal{A}^*$. Again, since d is completely bounded, so is D.

Assume that D is inner with $D(a) = \zeta a - a\zeta$. Then $d(a)\phi(a) = D(a)(a) = (\zeta a - a\zeta)(a) = 0$ for every $a \in \mathcal{A}$. It follows that $d(a^2) = 2\ d(a)\phi(a) = 0$ for all a. Finally, since d is continuous and $\mathcal{A}^2$ is dense, we have $d(b) = 0$ for all $b \in \mathcal{A}$ which is impossible. ■

Let $\mathcal{A}$ be a commutative completely contractive Banach algebra. Let X be a symmetric operator $\mathcal{A}$-module. With respect to its natural operator space structure inherited from $CB(\mathcal{A}, X)$, and with respect to the action $(bT)(a) = (Tb)(a) = T(ab)$ for each $T \in CB_{\mathcal{A},\mathbb{C}}(\mathcal{A}, X)$, $a, b \in \mathcal{A}$, $CB_{\mathcal{A},\mathbb{C}}(\mathcal{A}, X)$ becomes a symmetric operator $\mathcal{A}$-module.

Moreover, just as is indicated in [40], if $\mathcal{I}$ is an ideal in the commutative completely contractive algebra $\mathcal{A}$, then the above action with $a \in \mathcal{A}$ makes $CB_{\mathcal{A},\mathbb{C}}(\mathcal{I}, X)$ into a symmetric operator $\mathcal{A}$-module in such a way that the restriction to $\mathcal{I}$ is the natural action of $\mathcal{I}$ on $CB_{\mathcal{A},\mathbb{C}}(\mathcal{I}, A)$. Again following [40], we define the map $j : X \mapsto CB_{\mathcal{A},\mathbb{C}}(\mathcal{A}, X)$ by $j(x)(a) = ax$. A routine calculation similar to that in the proof of Proposition 4.3.3 shows that j is a completely bounded $\mathcal{A}$-module homomorphism. We are now able to establish the following useful analog of [40, Corollary 1.3] with essentially the same proof:

PROPOSITION 4.3.6: *Let $\mathcal{A}$ be an operator weakly amenable commutative completely contractive Banach algebra. Let $\mathcal{I}$ be a closed ideal in $\mathcal{A}$. Then $\mathcal{I}$ is operator weakly amenable if and only if $\overline{\mathcal{I}^2} = \mathcal{I}$.*

Proof: The only if direction is clear from Lemma 4.3.2. Hence we shall assume that $\overline{\mathcal{I}^2} = \mathcal{I}$. Let $D : \mathcal{I} \mapsto X$ be a completely bounded derivation into a symmetric operator $\mathcal{I}$-module X. Let $j : X \mapsto CB_{\mathcal{A},\mathbb{C}}(\mathcal{I}, X)$ be as above. Then since j is a completely bounded module map $j \circ D$ is a completely bounded derivation.

Define the bilinear map $\tilde{D} : \mathcal{I} \times \mathcal{A} \mapsto CB_{\mathcal{A},\mathbb{C}}(\mathcal{I}, X)$ by

$$\tilde{D}(m, a) = j \circ D(ma) - j(aD(m)).$$

Let $m \in \mathcal{I}^2$. By [40, Theorem 1.1], the map $a \mapsto \tilde{D}(m, a)$ is a derivation of $\mathcal{A}$ into $CB_{\mathcal{A},\mathbb{C}}(\mathcal{I}, X)$. Moreover, it is clear from its definition that this map is completely bounded and thus since $\mathcal{A}$ is operator weakly amenable it is identically 0. Again from [40, Theorem 1.1 (i)], we get that $\mathcal{I}^2 D(\mathcal{I}) \subset \ker j$ and hence that $\mathcal{I}^3 D(\mathcal{I}) = 0$. However, $\overline{\mathcal{I}^2} = \mathcal{I}$ implies that $D = 0$. Hence it follows from Proposition 4.3.3 that $\mathcal{I}$ is weakly amenable.

■

4.4 Operator Weak Amenability of the Fourier Algebra

In [29], a link was established between weak amenability of $\mathbf{A}(G)$ and commutativity of the connected component of G. For example, it was shown that if G is a $[SIN]$ group, then $\mathbf{A}(G/K)$ is weakly amenable for each compact normal subgroup K of G if and only if G has an abelian connected component. Moreover, Johnson [48] has shown the existence of compact groups for which $\mathbf{A}(G)$ is not weakly amenable. In contrast, in this section, we will show that operator weak amenability of $\mathbf{A}(G)$ holds for a large class of locally compact groups which included all $[IN]$ groups.

The following lemma is related to our future Proposition 5.2.2.

LEMMA 4.4.1: *Let H be a closed subgroup of G. Then the restriction map $R : \mathbf{A}(G) \mapsto \mathbf{A}(H)$ is a completely contractive homomorphism of $\mathbf{A}(G)$ onto $\mathbf{A}(H)$.*

Proof: It is well known that R is a continuous homomorphism of $\mathbf{A}(G)$ onto $\mathbf{A}(H)$. Let $\mathbf{VN}_H(G)$ be the weak closure in $\mathbf{VN}(G)$ of span $\{\lambda_G(h) : h \in H\}$. Then $\mathbf{VN}_H(G)$ is a von Neumann subalgebra of $\mathbf{VN}(G)$. Moreover, $R^* : \mathbf{VN}(H) \mapsto \mathbf{VN}(G)$ is a *-isomorphism of $\mathbf{VN}(H)$ onto $\mathbf{VN}_H(G)$ [28]. It follows that R^* is completely contractive and hence that R is also completely contractive. ∎

PROPOSITION 4.4.2: *Let H be a closed subgroup of G. Assume that $\mathbf{A}(G)$ is operator weakly amenable. Then $\mathbf{A}(H)$ is operator weakly amenable.*

Proof: This follows immediately from Lemma 4.4.1 and Proposition 4.3.4.

∎

Since $\mathbf{A}(\{e\})$ is always operator weakly amenable, the converse to Proposition 4.4.2 can only hold if $\mathbf{A}(G)$ is operator weakly amenable for each G. We can however establish the converse if H is assumed to be open. We note that if H is open, then the complement of H, denoted H^c, is given by

$$H^c = \bigcup_{x \notin H} xH.$$

From which is immediately follows that all open subgroups are closed.

LEMMA 4.4.3: *Let H be an open subgroup of G. Then $\mathbf{A}(G)$ is operator weakly amenable if and only if $\mathbf{A}(H)$ is operator weakly amenable.*

Proof: If $\mathbf{A}(G)$ is operator weakly amenable, then by Proposition 4.4.2, so is $\mathbf{A}(H)$. Assume that $\mathbf{A}(H)$ is operator weakly amenable and that $D : \mathbf{A}(G) \mapsto \mathbf{VN}(G)$ is a completely bounded derivation. Let $u \in \mathbf{A}(G)$ be such that supp u is compact. We can find $x_1, , ..., x_n$ such that supp$u \subset \bigcup x_i H$. Let $u_i = 1_{x_i H} u$. Then $u = \sum_{i=1}^{n} u_i$. If we let D_i be the restriction of D to the algebra $1_{x_i H}\mathbf{A}(G)$, then D_i is a completely bounded derivation from the algebra $1_{x_i H}\mathbf{A}(G)$ into the symmetric operator module $\mathbf{VN}(G)$. By our future Proposition 5.2.3 and Lemma 5.2.4, $1_{x_i H}\mathbf{A}(G)$ is completely isometrically isomorphic to $\mathbf{A}(H)$. Since $\mathbf{A}(H)$ is weakly operator amenable, Proposition 4.3.3 shows that $D_i(u_i) = 0$. Thus $D(u) = 0$ since elements with compact support are dense in $\mathbf{A}(G)$. Hence $\mathbf{A}(G)$ is operator weakly amenable.

■

THEOREM 4.4.4: *Let G be a locally compact group with an amenable connected component G_0. Then $\mathbf{A}(G)$ is operator weakly amenable.*

Proof: Let $\pi : G \mapsto G/G_0$ be the canonical projection. Since G/G_0 is totally disconnected, it has an open compact subgroup $\overline{C}$. Then $H = \pi^{-1}(\overline{C})$ is an open almost connected subgroup of G which is amenable if G_0 is amenable. It follows from [68] that $\mathbf{A}(H)$ is operator amenable and hence is clearly operator weakly amenable. We now simply apply Lemma 4.4.3 to conclude that $\mathbf{A}(G)$ is operator weakly amenable. ■

This theorem provides us with a significant class of groups for which $\mathbf{A}(G)$ is operator weakly amenable. Indeed, recall the following classes of locally compact groups:

[MAP] G is Maximally Almost Periodic if the finite dimensional representations of G separate points.

[IN] G is an Invariant Neighborhood group if the identity has a compact neighborhood which is invariant under all inner automorphisms.

[Her] G is Hermitian if $\mathbf{L}^1(G)$, the group algebra of G, is a hermitian Banach $*$-algebra.

[NF] G is in the class [NF] if G has no uniformly discrete free semigroup on two generators.

COROLLARY 4.4.5: *Let G be a locally compact group such that G belongs to* [MAP], [IN], [Her] or [NF]. *Then $\mathbf{A}(G)$ is operator weakly amenable.*

Proof: In each of the classes above the connected component must be amenable (see [65]). It follows from Theorem 4.4.4 that $\mathbf{A}(G)$ is operator weakly amenable.

$\blacksquare$

Remark: We do not know if for every locally compact group $\mathbf{A}(G)$ is operator weakly amenable. However, if there is an example of a group for which $\mathbf{A}(G)$ is not operator weakly amenable, our next result shows that there must be a connected Lie group with this property.

THEOREM 4.4.6: *If* $\mathbf{A}(H)$ *is operator weakly amenable for each connected Lie group* H, *then* $\mathbf{A}(G)$ *is operator weakly amenable for each locally compact group* G.

Proof: Assume that $\mathbf{A}(H)$ is operator weakly amenable for each connected Lie group H. Let G be a locally compact group for which $\mathbf{A}(G)$ is not operator weakly amenable. Then G must contain an open almost connected subgroup G_1. Lemma 4.4.3 shows that $\mathbf{A}(G_1)$ must also fail to be operator weakly amenable. Hence there exists a non-zero completely bounded derivation $D : \mathbf{A}(G_1) \mapsto \mathbf{VN}(G_1)$.

If K is a normal subgroup of G_1, then $\mathbf{A}(G_1/K)$ is completely isometrically isomorphic to the subalgebra $\mathbf{A}(G_1 : K)$ of $\mathbf{A}(G_1)$ consisting of functions which are constant on left cosets of G. (This follows from the fact that $\mu_K * \mathbf{VN}(G_1)$ is *-isomorphic with $\mathbf{VN}(G_1/K)$ where μ_K is the Haar measure on K). It follows that if $\mathbf{A}(G_1/K)$ is operator weakly amenable, then so is $\mathbf{A}(G_1 : K)$. But $\mathbf{VN}(G_1)$ is a symmetric $\mathbf{A}(G_1 : K)$ operator module. Therefore the restriction of D to $\mathbf{A}(G_1 : K)$ must be identically 0 whenever $\mathbf{A}(G_1 : K)$ is operator weakly amenable.

The almost connected group G_1 is a projective limit of Lie groups [65]. Therefore for any $u \in \mathbf{A}(G)$ we can find a compact normal subgroup K_n such that G_1/K_n is a Lie group and such that there exists a function $u_n \in \mathbf{A}(G_1 : K_n)$ with $\|u_n - u\|_{\mathbf{A}(G)} \leq 1/n$ ([30]). Let H_n be the connected component of G_1/K_n. Then H_n is a connected Lie group, and hence by assumption $\mathbf{A}(H_n)$ is operator weakly amenable. But H_n is open in G_1/K_n, so by Lemma 4.4.3, $\mathbf{A}(G_1/K_n) \cong \mathbf{A}(G_1 : K_n)$ is also operator weakly amenable. We can conclude as above that $D(u_n) = 0$. However, since D is continuous, we get that $D(u) = 0$ for each $u \in \mathbf{A}(G)$ contradicting the assumption that D was non-zero. Hence $\mathbf{A}(G_1)$ must also have been operator weakly amenable and in turn so was $\mathbf{A}(G)$. ∎

4.5 Operator Amenability and Weak Operator Amenability of the Second Dual of A(G)

Let $\mathcal{A}$ be a completely contractive Banach algebra. Then as always $\mathcal{A}^{**}$ can be made into a Banach algebra with either of the two Arens multiplications. Observe also that $\mathcal{A}^{**}$, the standard second dual of $\mathcal{A}$, is an operator space under the dual structure inherited from $\mathcal{A}^*$ and hence from $\mathcal{A}$. We claim that $\mathcal{A}^{**}$ is a completely contractive Banach algebra with respect to either Arens product. To see this note that $\mathbb{M}_n(\mathcal{A}^{**}) = (\mathbb{M}_n(\mathcal{A}))^{**}$ and the closed unit ball of $\mathbb{M}_n(\mathcal{A})$ is weak-* dense in $\mathbb{M}_n(\mathcal{A}^{**})$. As such, every contractive element in $\mathbb{M}_n(\mathcal{A}^{**})$ is the weak-* limit of a net of contractive elements in $\mathbb{M}_n(\mathcal{A})$. From this we deduce that with respect to either Arens product, $\mathcal{A}^{**}$ is completely contractive. Finally, if X is a quotient of $\mathcal{A}^{**}$, X also becomes a completely contractive Banach algebra. Therefore, $\mathrm{A}(G)^{**}$ and any of its quotients for which we are concerned below are completely contractive Banach algebras.

THEOREM 4.5.1: *Let X be a topologically introverted subspace of* $\mathrm{VN}(G)$ *which contains $C^*_\delta(G)$. If X^* is operator weakly amenable, then every abelian subgroup H of G is finite. Moreover, G is discrete.*

Proof: It follows from the proof of [31, Theorem 3.2] that if G has an infinite abelian subgroup, then X^* has a nonzero point derivation. Proposition 4.3.5 would then imply that X^* could not be operator weakly amenable. We can therefore assume that G has no such subgroups.

Let G_0 denote the connected component of e. Then since G is periodic, so is G_0. Moreover, the same is true for any homomorphic image of G_0. Let U be a neighborhood of e in G_0. Then there exists a compact normal subgroup $N \subseteq U$ such that G_0/N is a periodic, connected Lie group. Therefore G_0/N is trivial. Since this is true for all such U, G_0 must be trivial. In particular, G is totally disconnected.

If G is nondiscrete and totally disconnected, then G contains an infinite compact subgroup K. However, by [80, Theorem 2] the infinite compact group K contains an infinite abelian subgroup which is impossible since G contains no such subgroup. It follows that G is discrete.

■

COROLLARY 4.5.2 *Let X be be any of the spaces $AP(\hat{G}), WAP(\hat{G})$, or $UCB(\hat{G})$. If X^* is operator amenable, then G is an amenable discrete group.*

If $X = \mathrm{VN}(G)$, then X^* is operator amenable if and only if G is finite.

Proof: Assume X^* is operator amenable where X is any of the spaces above. Then by Theorem 4.5.1 G is discrete. Moreover, since X^* is operator amenable it has a bounded approximate identity [68].

If $X = AP(\hat{G}), WAP(\hat{G})$ or $UCB(\hat{G})$, then $\mathbf{B}_\delta(G_d)$ is a quotient of X^* [31]. However, since G is discrete, $\mathbf{B}_\delta(G_d) = \mathbf{B}_\lambda(G)$, the reduced Fourier-Stieltjes algebra of G. In particular, $\mathbf{B}_\lambda(G)$ also has a bounded approximate identity. It is easy to see that 1_G is a weak-* cluster point of this approximate identity in $\mathbf{B}(G)$. Since $\mathbf{B}_\lambda(G)$ is weak-* closed we get $1_G \in \mathbf{B}_\lambda(G)$. Moreover, since $\mathbf{B}_\lambda(G)$ is an ideal in $\mathbf{B}(G)$, $\mathbf{B}_\lambda(G) = \mathbf{B}(G)$ and hence G is amenable.

If $X = \mathrm{VN}(G)$, then it follows from [56, Proposition 3.2 (b)] that G is compact and thus finite. Conversely if G is finite, then $\mathrm{VN}(G)^* = \mathbf{A}(G)$, which is clearly operator amenable. ■

For any of the spaces $X = AP(\hat{G}), WAP(\hat{G}), UCB(\hat{G})$ or $\mathrm{VN}(G)$, we get from Theorem 4.5.1 that if X^* is operator weakly amenable, then G is periodic with no infinite abelian subgroups. This is a severe restriction of the nature of G. The following two corollaries are obtained in the same manner as [31, Corollary 3.1 and 3.5]

COROLLARY 4.5.3: *Let G be a locally compact group which satisfies one of the following conditions: i) G is locally finite; ii) G is an elementary group; iii) G is locally solvable; or iv) G is isomorphic to a subgroup of $GL(n, \mathbb{F})$ for some n and any field $\mathbb{F}$. If X is a topologically introverted subspace of $\mathrm{VN}(G)$ which contains $C^*_\delta(G_d)$, then X is operator weakly amenable if and only if G is finite.*

Recall that a discrete group G has polynomial growth if for every finite set $F \subset G$ there exists a $p \in \mathbb{N}$ such that $|F| = O(n^p)$. It follows from the proof of [31, Corollary 3.5] that if G is infinite, it must contain an infinite abelian subgroup H. Theorem 4.5.1 implies:

COROLLARY 4.5.4: *Let G be a discrete group of polynomial growth. Let X be a topologically introverted subspace of $\mathrm{VN}(G)$ which contains $C^*_\delta(G_d)$, and such that X is operator weakly amenable, then G is finite.*

The results of this section are natural analogs of earlier results obtained in [31],[38] and [58] for the second dual of $\mathbf{A}(G)$ viewed as a Banach algebra. For example, in Corollary 4.5.2 we showed that $\mathbf{A}(G)^{**}$ is operator amenable if and only if G is finite. This was proved by Granirer in [38] for amenability in the category of Banach algebras. We note however that it is known that for the second dual $\mathcal{A}^{**}$ to be amenable, $\mathcal{A}$ must itself be amenable [36]. Since we have already stated that $\mathbf{A}(G)$ is in general not amenable even for compact groups, Granirer's result is quite natural. However, since $\mathbf{A}(G)$ is operator amenable for any amenable group, one might expect that it would be more likely that $\mathbf{A}(G)^{**}$ would be operator amenable. Corollary 4.5.2 shows that this is not the case. In fact the significance of this section is that all the known results for amenability and weak amenability for quotients of $\mathbf{A}(G)^{**}$ hold in the new category. This shows that unlike the case of $\mathbf{A}(G)$ itself, barriers for amenability and weak amenability of quotients of $\mathbf{A}(G)^{**}$ do not disappear with the addition of the operator space structure. Curiously, the reason for this could be, as suggested by the proof of Theorem 4.5.1, that the obstacles arise from the presence of abelian subgroups where our two notions of amenability agree once again.

4.6 Amenability and Weak Amenability of Ideals in $\mathbf{A}(G)$

In this section, we will take a brief look at operator amenability and operator weak amenability for ideals in $\mathbf{A}(G)$.

We know that if $\mathbf{A}(G)$ is operator amenable, then G must be amenable. We can ask if there are any structural implications of the existence of a closed ideal in $\mathbf{A}(G)$ which is operator amenable. Since the Fourier algebra of a discrete group always contains such an ideal, namely the one-dimensional ideal of functions supported on the identity, perhaps the best one could hope for would be to show that our group must contain an open amenable subgroup. In fact, we shall show that this is indeed the case. The key to our proof will be the fact that each operator amenable completely contractive Banach algebra contains a bounded approximate identity.

LEMMA 4.6.1: *Let G be a locally compact group. Assume that $\mathbf{A}(G)$ has a nonzero ideal $\mathcal{I}$ with a bounded approximate identity. Then G has an open amenable subgroup.*

Proof: Let G_1 be an open almost connected subgroup of G. Let $F = \mathcal{Z}(\mathcal{I})$. Since $\mathcal{I}$ is non-zero, there exists an $x_0 \in G \setminus \mathcal{Z}(\mathcal{I})$. By translating if necessary, we may assume that $x_0 \in G_1$. Then $\mathcal{I}_1 = 1_{G_1}\mathcal{I}$ can be viewed as a closed ideal in $\mathbf{A}(G_1)$ which is nonzero and has a bounded approximate identity. It follows from [29, Proposition 3.5] that G_1 is amenable. ∎

THEOREM 4.6.2: *$\mathbf{A}(G)$ has a nonzero closed ideal which is operator amenable if and only if G has an open amenable subgroup.*

Proof: Assume that G_1 is open and amenable. Then $\mathcal{I}(G \setminus G_1)$ is a closed ideal of $\mathbf{A}(G)$ which is completely isometrically isomorphic to $\mathbf{A}(G_1)$. Since G_1 is amenable, $\mathcal{I}(G \setminus G_1)$ is operator amenable.

Conversely, assume that $\mathcal{I}$ is a non-zero closed ideal of $\mathbf{A}(G)$ which is operator amenable. Then by [68], $\mathcal{I}$ has a bounded approximate identity. It follows from Lemma 4.6.1 that G has an open amenable subgroup. ∎

COROLLARY 4.6.3: *If $\mathbf{A}(G)$ has a non-zero ideal $\mathcal{I}$ which is operator amenable, then $\mathbf{A}(G)$ is operator weakly amenable.*

Proof: Since G has an open amenable subgroup, this follows immediately from Theorem 4.4.4, and Theorem 4.6.2. ∎

It was observed in [29], that the Fourier algebra of $SL(2,\mathbb{R})$ did not have any amenable closed ideals. Theorem 4.6.2 allows us to extend this result to the category of operator spaces.

COROLLARY 4.6.4: $\mathbf{A}(SL(2,\mathbb{R}))$ *has no non-zero closed operator amenable ideals.*

In [35], it was shown that if a closed ideal $\mathcal{I}$ in $\mathbf{A}(G)$ posses a bounded approximate identity, then there exists some $F \in \Omega_c(G)$ such that $\mathcal{I} = \mathcal{I}(F)$. Clearly, these are the only possible candidates for operator amenable ideals. To determine whether or not these ideals are in fact operator amenable, we will need the following analog of [49, Proposition 5.1]

THEOREM 4.6.5: *Let $\mathcal{A}$ be an operator amenable completely contractive Banach algebra, Let $\mathcal{I}$ be a closed ideal of $\mathcal{A}$. Then $\mathcal{I}$ is operator amenable if and only if $\mathcal{I}$ has a bounded approximate identity.*

Proof: Let $\{e_\alpha\}$ be a bounded approximate identity in $\mathcal{I}$ with $\|e_\alpha\| \leq M$ for each α.

Let X be an operator $\mathcal{I}$-module. Since $\mathcal{I}$ has a bounded approximate, in order to show that each completely bounded derivation from $\mathcal{A}$ into X^* is inner, we may assume just as in the bounded case that X is neounital. As such X^* becomes an operator $\mathcal{A}$ module with respect to the induced action of $\mathcal{A}$ on X^*. That is if $\Gamma = a\Gamma_1 b$ where $a, b \in \mathcal{I}$ and $\Gamma_1 \in X^*$, then for any $u \in \mathcal{A}$, $u\Gamma = (ua)\Gamma_1 b$ and $\Gamma u = a\Gamma_1(bu)$.

Let $D : \mathcal{I} \mapsto X^*$ be a completely bounded derivation. Then D lifts to a derivation $\tilde{D} : \mathcal{A} \mapsto X^*$, with $\tilde{D}(u) = \lim_\alpha D(e_\alpha u)$ for each $u \in \mathcal{A}$ [49]. Let $[u_{ij}] \in \mathbb{M}_n(\mathcal{A})$. Then

$$\begin{aligned}
\|\tilde{D}^{(n)}([u_{ij}])\|_n &= \lim_\alpha \|D(e_\alpha u_{ij})\|_n \\
&\leq \lim_\alpha \|D\|_{cb} \|[e_\alpha u_{ij}]\|_n \\
&\leq \sup_\alpha \|D\|_{cb} \|\|e_\alpha\|\|[u_{ij}]\|_n \\
&\leq \|D\|_{cb} M \|[u_{ij}]\|_n
\end{aligned}$$

It follows that $\tilde{D}$ is a completely bounded extension of D to $\mathcal{A}$. However, since $\mathcal{A}$ is operator amenable $\tilde{D}$ is inner, and hence D is inner. Thus $\mathcal{I}$ is also operator amenable.

The converse is obvious since each operator amenable completely contractive Banach algebra has a bounded approximate identity.

∎

We close with the following complete characterization of those ideals in the Fourier algebra of an amenable $[SIN]$ group which are operator amenable.

THEOREM 4.6.6: *Let G be an amenable $[SIN]$ group. Let $\mathcal{I}$ be a closed ideal in* $\mathrm{A}(G)$. *Then $\mathcal{I}$ is operator amenable if and only if $Z(\mathcal{I}) \in \Omega_c(G)$*

Proof: By Theorem 4.6.5, we know that $\mathcal{I}$ is operator amenable if and only if it possesses a bounded approximate identity. However in [35] it was shown that an ideal in a $[SIN]$ group posses a bounded approximate identity if and only if $Z(\mathcal{I}) \in \Omega_c(G)$.

Chapter 5

Automatic Complete Boundedness of Maps

5.1 Introduction

In [68], Ruan showed that amenability for the Banach algebra $\mathbf{L}^1(G)$ is equivalent to operator amenability when $\mathbf{L}^1(G)$ is given the operator space structure that it inherits as the predual of the von Neumann algebra $\mathbf{L}^\infty(G)$. This follows from the fact that $\mathbf{L}^1(G)$ has the MAX operator space structure and as such every bounded map from $\mathbf{L}^1(G)$ into an operator space X in completely bounded. Moreover, any Banach $\mathbf{L}^1(G)$- module can be given the MAX operator space structure which will make it into an operator $\mathbf{L}^1(G)$-module. As such any bounded derivation into a dual $\mathbf{L}^1(G)$-module can be viewed as a completely bounded derivation into a dual operator $\mathbf{L}^1(G)$-module. When considering the Fourier algebra of G the situation can be quite different.

In this Chapter we discuss for which groups G, we have that every bounded map from $\mathbf{A}(G)$ into any operator space is automatically completely bounded.

5.2 Automatic c.b. of Maps on $\mathbf{A}(G)$ and $\mathbf{B}(G)$

We begin with the following:

PROPOSITION 5.2.1 *Let G be a locally compact group. Then $\mathbf{A}(G) = \mathbf{A}(G)_{MAX}$ if and only if G is abelian.*

Proof: The standard dual of $\mathbf{A}(G)$ is the von Neumann algebra $\mathbf{VN}(G)$ and $\mathbf{A}(G) = \mathbf{A}(G)_{MAX}$ if and only if $\mathbf{VN}(G) = \mathbf{VN}(G)_{MIN}$ (see [10]). Since $\mathbf{VN}(G)$ is a unital operator algebra, $\mathbf{VN}(G) = \mathbf{VN}(G)_{MIN}$ if and only if $\mathbf{VN}(G)$ is abelian [11]. However $\mathbf{VN}(G)$ is abelian if and only if G is abelian. ■

Clearly if G is abelian, then since $\mathbf{A}(G) = \mathbf{A}(G)_{MAX}$ and $\mathbf{A}(G)$ is completely isometrically isomorphic with $L^1(\hat{G})$, there is little to be gained by looking at $\mathbf{A}(G)$ as a completely contractive Banach algebra rather than simply as a Banach algebra. We will now show however, that if G does not contain an abelian subgroup of finite index, then the nature of $\mathbf{A}(G)$ when viewed as a completely contractive Banach algebra is fundamentally different from that of $\mathbf{A}(G)$ viewed as a Banach algebra. This we believe explains why it is necessary to retain the operator space structure when studying the cohomology of the Fourier algebra.

Before providing the main result of this section, we require a few useful observations.

PROPOSITION 5.2.2: *Let H be a closed subgroup of G. Then $\mathbf{A}(G)/I(H)$ is completely isometrically isomorphic with $\mathbf{A}(H)$.*

Proof: Let $\tilde{u} \in \mathbf{A}(G)/I(H)$. Define $\Gamma : \mathbf{A}(G)/I(H) \mapsto \mathbf{A}(H)$ by $\Gamma(\tilde{u}) = v|_H$ where v is chosen so that $Q(v) = \tilde{u}$ and $Q : \mathbf{A}(G) \mapsto \mathbf{A}(G)/I(H)$ is the quotient map. It is known that Γ is an isometric isomorphism of $\mathbf{A}(G)/I(H)$ onto $\mathbf{A}(H)$ [34]. To see that Γ is a complete isometry observe that $I(H)^{\perp} = VN_H(G) = \{T \in \mathbf{VN}(G) : \mathrm{supp}(T) \subset H\}$. Also $VN_H(G)$ is a von Neumann subalgebra of $\mathbf{VN}(G)$ which is *-isomorphic with $\mathbf{VN}(H)$ [28]. It follows from [10] that $(\mathbf{A}(G)/I(H))^*$ is completely isometrically isomorphic to $VN_H(G)$ and hence to $\mathbf{VN}(H)$. Thus $(\mathbf{A}(G)/I(H))$ is completely isometrically isomorphic to $\mathbf{A}(H)$. ∎

PROPOSITION 5.2.3: *Let H be an open subgroup. Then $1_H\mathbf{A}(G)$ is completely isometrically isomorphic with $\mathbf{A}(H)$.*

Proof: Let $u \in \mathbf{A}(H)$. Let $\underline{u} \in \mathbf{A}(G)$ be such that $\underline{u}(x) = u(x)$ if $x \in H$ and $\underline{u}(x) = 0$ otherwise. It is well known that $\Gamma : \mathbf{A}(H) \mapsto \mathbf{A}(G)$ defined by $\Gamma(u) = \underline{u}$ is an isometric isomorphism of $\mathbf{A}(H)$ into $1_H\mathbf{A}(G)$ [28].

Let $[u_{ij}] \in \mathbb{M}_n(\mathbf{A}(H))$ with $\|[u_{ij}]\|_n = 1$. It follows from Proposition 5.2.2 that $\|[\underline{\tilde{u}_{ij}}]\|_n = 1$. Let $\epsilon > 0$. We can find $[v_{ij}] \in \mathbb{M}_n(\mathbf{A}(G))$ such that $[\tilde{v}_{ij}] = [\underline{\tilde{u}_{ij}}]$ and $\|[v_{ij}]\| \leq 1 + \epsilon$. Now

$$\|\Gamma^{(n)}([u_{ij}])\| = \|[\underline{u_{ij}}]\|_n = \|P([v_{ij}])\|_n$$

where $P(v) = 1_H v$. However $1_H \in \mathbf{B}(G)$ and $\|1_H\| = 1$. It follows that $\|P\|_{cb} = 1$. Hence

$$\|\Gamma^{(n)}([u_{ij}])\|_n \leq 1 + \epsilon$$

Therefore we can conclude that $\|\Gamma\|_{cb} \leq 1$.

To complete the proof, observe that $\Gamma^{-1} : 1_H\mathbf{A}(G) \mapsto \mathbf{A}(H)$ is simply the restriction of the quotient map $Q : \mathbf{A}(G) \mapsto \mathbf{A}(G)/I(H)$ composed with the complete isometry of Proposition 5.2.2. It follows that $\|\Gamma^{-1}\|_{cb} = 1$ and hence that Γ is a complete isometry. ∎

LEMMA 5.2.4: *Let $t \in G$. The map $L_t : \mathbf{A}(G) \mapsto \mathbf{A}(G)$ defined by $L_t u(x) = u(tx)$ is a complete isometry of $\mathbf{A}(G)$.*

Proof: Let $L_t^* : \mathbf{VN}(G) \mapsto \mathbf{VN}(G)$. Then $L_t^*(T) = \delta_t * T$ where $*$ is the product in $\mathbf{VN}(G)$. It follows that L_t^* and hence L_t is completely bounded with $\|L_t\|_{cb} = 1$. Clearly we also have that $\|L_t^{-1}\|_{cb} = \|L_{t^{-1}}\|_{cb} = 1$, and hence L_t is a complete isometry.

∎

This leads to the major result of this section.

THEOREM 5.2.5 *The following are equivalent*

(i) Every bounded map from $\mathbf{A}(G)$ into any operator space is completely bounded.

(ii) Every bounded map from $\mathbf{A}(G)$ into $\mathbf{VN}(G)$ is completely bounded.

(iii) G has an abelian subgroup of finite index.

Proof: (i) $\Rightarrow$ (ii) immediate.

(ii) $\Rightarrow$ (iii): Recall that

$$CB(\mathbf{A}(G), \mathbf{VN}(G)) = (\mathbf{A}(G) \hat{\otimes} \mathbf{A}(G))^*$$

and

$$B(\mathbf{A}(G), \mathbf{VN}(G)) = (\mathbf{A}(G) \otimes_\gamma \mathbf{A}(G))^*$$

where $\hat{\otimes}$ and $\otimes_\gamma$ denote the operator space tensor product and Banach space tensor product respectively. Since from (ii) we have that

$$B(\mathbf{A}(G), \mathbf{VN}(G)) = CB(\mathbf{A}(G), \mathbf{VN}(G)),$$

we conclude from the open mapping theorem that there exists some number $N \in \mathbb{N}$ such that $\|\Phi\|_{cb} \leq N\|\Phi\|$ for each $\Phi \in B(\mathbf{A}(G), \mathbf{VN}(G)) = CB(\mathbf{A}(G), \mathbf{VN}(G))$. Let $u \in \mathbf{A}(G) \otimes \mathbf{A}(G)$. Then

$$\begin{aligned} \|u\|_{\mathbf{A}(G)\hat{\otimes}\mathbf{A}(G)} &= \sup\{\Phi(u) : \Phi \in CB(\mathbf{A}(G), \mathbf{VN}(G)),\ \|\Phi\|_{cb} \leq 1\} \\ &\geq \sup\{\Phi(u) : \Phi \in B(\mathbf{A}(G), \mathbf{VN}(G)),\ \|\Phi\| \leq 1/N\} \\ &= 1/N\|u\|_{\mathbf{A}(G)\otimes_\gamma\mathbf{A}(G)} \end{aligned}$$

This shows that the natural injection

$$inj : (\mathbf{A}(G) \otimes \mathbf{A}(G), \|\cdot\|_\gamma) \to (\mathbf{A}(G) \otimes \mathbf{A}(G), \|\cdot\|_\wedge)$$

extends to an isomorphism of $\mathbf{A}(G) \otimes_\gamma \mathbf{A}(G)$ onto $\mathbf{A}(G)\hat{\otimes}\mathbf{A}(G)$. To complete the proof, observe that the canonical injection of $\mathbf{A}(G)\hat{\otimes}\mathbf{A}(G)$ into $\mathbf{A}(G \times G)$ is known to be surjective ([25]). It follows that the same will be true of the canonical injection of $\mathbf{A}(G) \otimes_\gamma \mathbf{A}(G)$ into $\mathbf{A}(G \times G)$. Using a result of Losert [62], we conclude that G has an abelian subgroup of finite index.

(iii) $\Rightarrow$ (i): Assume that G has an abelian subgroup H of finite index in G. Let $\{x_iH\}$ be a complete set of cosets of G. Then 1_H is a completely bounded projection of $\mathbf{A}(G)$ onto a sub-algebra of $\mathbf{A}(G)$ which is completely isometrically isomorphic to $\mathbf{A}(H)$. Since H is abelian, $\mathbf{A}(H)$ has the MAX operator structure. It follows that every bounded linear map from $\mathbf{A}(H)$ into any operator space is completely bounded. Thus if T is a bounded linear map from $\mathbf{A}(G)$ to any operator space, the above argument and Lemma 5.2.4 shows that $T \circ 1_{x_iH}$ is completely bounded. However, since $T = \sum_{i=1}^n T \circ 1_{x_iH}$ it follows that T is also completely bounded. ■

We can extend this result to $\mathbf{B}(G)$ with the following:

COROLLARY 5.2.6: *Let G be a locally compact group. Then every bounded map from $\mathbf{B}(G)$ into an arbitrary operator space X is completely bounded if and only if G has an abelian subgroup of finite index.*

Proof: If H is abelian, we note that $\mathbf{B}(H)$ is again the predual of an abelian von Neumann algebra and is hence a MAX operator space. We can proceed as before to show that $1_H\mathbf{B}(G)$ is c.b. isomorphic to $\mathbf{B}(H)$ when H is an open subgroup of G. Thus when G has an abelian subgroup of finite index, that every bounded map from $\mathbf{B}(G)$ into an arbitrary operator space is completely bounded follows in the same manner as for $\mathbf{A}(G)$ above.

To prove the converse, first note that $\mathbf{A}(G)$ is a complemented ideal in $\mathbf{B}(G)$ and the projection P is induced by the central projection in $W^*(G)$ corresponding to the left regular representation ([5]). As such P is completely bounded. If G does not have an abelian subgroup of finite index, then by Theorem 5.2.5 there is an operator space X and a linear map $\Gamma : \mathbf{A}(G) \mapsto X$ which is bounded but not completely bounded. Then $\Gamma \circ P$ is the desired map.

∎

It is worthwhile to shed further light on the previous two results. Let π be a continuous unitary representation of G. Let A_π denote the closed subspace of $\mathbf{B}(G)$ generated by the coefficient functions of π. Then A_π inherits an operator space structure from $\mathbf{B}(G)$. Moreover, the central projection P_π in the von Neumann algebra $\mathbf{B}(G)^* = W^*(G)$ associated with π is such that $A_\pi = P_\pi\mathbf{B}(G)$ and $A_\pi^* = VN_\pi = P_\pi W^*(G)$ (see [5]). If π is irreducible, then $VN_\pi = B(H_\pi)$ and hence $A_\pi = TC(H_\pi)$, the trace class operators on the Hilbert space H_π. Moreover, since the operator space structure on $\mathbf{B}(G)$ is the standard predual operator space structure, the induced structure on A_π agrees with the standard operator space structure $TC(H_\pi)$ inherits from $B(H_\pi)$.

Let G be a locally compact group with an infinite dimensional irreducible representation π. From the remark above, we see that $A_\pi = TC(H_\pi)$. Let $\Gamma : B(H_\pi) \mapsto B(H_\pi)$ be the transpose map. Since Γ is weak-* to weak-* continuous and has norm 1 , the preadjoint map $\Gamma_* : TC(H_\pi) \mapsto TC(H_\pi)$ also has norm 1. However, since it is well known that Γ is not completely bounded, neither is Γ_* which we view as a bounded map from A_π onto itself. It follows that the map $\Gamma \circ P_\pi$ is a bounded map from $\mathbf{B}(G)$ into $\mathbf{B}(G)$ which is not completely bounded.

Now let G be a locally compact group which has irreducible finite dimensional representations of arbitrarily large degree. Then we can construct in a similar manner as above a bounded map from $\mathbf{B}(G)$ into $\mathbf{B}(G)$ which is not completely bounded. Let $\{\pi_n\}$ be a sequence of irreducible finite dimensional representations with dim $\pi_n >$ dim π_m whenever $n > m$. Let $\pi = \bigoplus \pi_n$ be the direct sum of the π_n's. Then by [5], the space $A_\pi = \bigoplus_{L^1} A_{\pi_n} = \bigoplus_{L^1} TC(H_{\pi_n})$. It follows from [10] that as an operator space A_π is the operator predual of $VN_\pi = \bigoplus_{L^\infty} VN_{\pi_n} = \bigoplus_{L^\infty} \mathbb{M}_{\dim \pi_n}$. If we define $\Gamma : A_\pi \mapsto A_\pi$ by $\Gamma = \bigoplus(\Gamma_n)_*$ where $\Gamma_n : \mathbb{M}_{\dim \pi_n} \mapsto \mathbb{M}_{\dim \pi_n}$ is the transpose map and $(\Gamma_n)_*$ is its preadjoint, then once again Γ is a map of norm one. However, since $\|\Gamma_n\|_{cb} \geq \dim \pi_n$, Γ is not completely bounded. If we again let P_π denote the central projection in $W(G)$ associated with π, then the $\Gamma \circ P_\pi$ is once more a bounded map from $\mathbf{B}(G)$ onto A_π which is not completely bounded.

Finally, since G has finite dimensional irreducible representations of bounded degree if and only if G has an abelian subgroup of finite index, we have established an alternate proof of Corollary 5.2.6. We have also provided a proof for the following proposition:

PROPOSITION 5.2.7: *Let π be a continuous unitary representation of G. If either π contains an infinite dimensional irreducible subrepresentation or if π contains finite dimensional subrepresentations of arbitrarily large degree, then there exists a contractive linear map $\Gamma : A_\pi \mapsto A_\pi$ which is not completely bounded.*

A locally compact group is said to belong to the class [AR] if the left regular representation λ is completely reducible. It is well known that for $G \in$ [AR], $\mathbf{A}(G)$ has the Radon-Nikodym Property (in fact this characterizes the groups in [AR] [75]) and hence that the unit ball of $\mathbf{A}(G)$ is the closed convex hull of its extreme points. [AR] contains not only all compact groups but also a variety of noncompact groups (see [7]). The $ax + b$ group is in [AR] as its left regular representation is the direct sum of two infinite dimensional irreducible representations.

PROPOSITION 5.2.8: *Let G be either a noncompact [AR] group or a compact group which does not contain an open abelian subgroup. Then there exists an isometry from $\mathbf{A}(G)$ onto $\mathbf{A}(G)$ which is not completely bounded.*

Proof: In either case, the map Γ constructed as in the remark preceding Proposition 5.2.7 is an isometry of $\mathbf{A}(G)$ onto itself which is not completely bounded.

■

We can make a case for the significance of the results of this section for our study if we recall what is currently known concerning the bounded cohomology of $\mathbf{A}(G)$. At present, the only known groups for which $\mathbf{A}(G)$ is amenable are those which contain an abelian subgroup of finite index. That the Fourier algebra of such a group is amenable was originally proved for compact groups by Johnson [48]. The same result was later established for general locally compact groups by Lau, Loy and Willis in [59] and independently by Forrest in [29]. However, Johnson showed that for a finite group, the minimal norm for a virtual diagonal for $\mathbf{A}(G)$ is given by

$$\frac{\Sigma_{\pi\in\hat{G}}(\dim\,\pi)^3}{\Sigma_{\pi\in\hat{G}}(\dim\,\pi)^2}.$$

This provided the first concrete evidence to suggest that in the presence of either an infinite dimensional irreducible representation or of finite dimensional representations of arbitrarily large degree, (precisely the setting of Proposition 5.2.7) $\mathbf{A}(G)$ would fail to be amenable. In [59], Johnson's work was refined considerably and the case for the link between amenability of $\mathbf{A}(G)$ and "approximate commutativity" of G was strengthened.

Finally, note that the fact that $\mathbf{A}(G)\hat{\otimes}\mathbf{A}(G)$, the operator space tensor product of $\mathbf{A}(G)$ with itself, is completely isometrically isomorphic to $\mathbf{A}(G\times G)$ is at the heart of Ruan's theorem that $\mathbf{A}(G)$ is operator amenable precisely when G is amenable. Moreover, the failure of the corresponding result for the Banach algebra projective tensor product in the absence of an abelian subgroup of finite index (as demonstrated by Losert [62]) is the essence of our proof of Theorem 4.5. Indeed, one might view Theorem 5.2.5 as a restatement within our current context of Losert's theorem. Once again, we are led to conclude that Banach algebra amenability for $\mathbf{A}(G)$ is a commutative phenomenon, whereas the category of operator spaces with its richer structure provides the appropriate setting for studying the cohomology of the Fourier algebra of a typical locally compact group.

5.3 Automatic Complete Boundedness and Derivations

In section 5.2 we showed that unless G contained an abelian subgroup of finite index, there are bounded linear maps from $\mathbf{A}(G)$ into $\mathbf{VN}(G)$ which are not completely bounded. Conversely, if G does contain such a subgroup, then it is clear that each bounded derivation from $\mathbf{A}(G)$ into $\mathbf{VN}(G)$ is completely bounded. Moreover, since any such group is amenable, every derivation from $\mathbf{A}(G)$ into any Banach $\mathbf{A}(G)$ module is automatically continuous [32] and hence is automatically completely bounded.

Since one of our main interests in this Chapter is to study the potential operator weak amenability of $\mathbf{A}(G)$ as compared with the usual notion of weak amenability of $\mathbf{A}(G)$, a natural question arises: For which locally compact groups are all (bounded) derivations from $\mathbf{A}(G)$ into $\mathbf{VN}(G)$ automatically completely bounded? More generally, when is any derivation of $\mathbf{A}(G)$ into any arbitrary $\mathbf{A}(G)$-module automatically completely bounded?

Recall that in [29] Forrest showed that any locally compact group with an abelian connected component is such that $\mathbf{A}(G)$ is weakly amenable. It follows that in this case the only bounded derivation from $\mathbf{A}(G)$ to $\mathbf{VN}(G)$ is zero, which is trivially completely bounded. We can speculate about the possibility that these are the only such groups without bounded derivations into $\mathbf{VN}(G)$ that are not completely bounded.

We begin with a proposition which is somewhat analogous to [32, Theorem 1].

PROPOSITION 5.3.1: *Let G be a locally compact group such that every derivation from $\mathbf{A}(G)$ into a finite dimensional, symmetric operator $\mathbf{A}(G)$-module is completely bounded, then G is amenable.*

Proof: Assume that G is not amenable. Then $\mathbf{A}(G)^2$ is not closed in $\mathbf{A}(G)$ [61]. We follow the construction in [21]. In particular, we can find a closed cofinite ideal $\mathcal{I}$ in $\mathbf{A}(G)$ and another cofinite ideal $\mathcal{K}$ which is not closed and satisfies $\mathcal{I}^2 \subset \mathcal{K} \subset \mathcal{I}$. Let X be the radical of the finite dimensional algebra $\mathbf{A}(G)/\mathcal{K}$. Then $X = \mathcal{I}/\mathcal{K}$. By Wedderburn's Theorem there exists a subalgebra $\mathcal{B}$ of $\mathbf{A}(G)/\mathcal{K}$ such that

$$\mathbf{A}(G)/\mathcal{K} = \mathcal{B} \oplus X$$

with multiplication in $\mathcal{B} \oplus X$ given by

$$(b_1, x_1)(b_2, x_2) = (b_1 b_2, b_1 x_2 + x_1 b_2).$$

Let $\Gamma : \mathbf{A}(G) \mapsto \mathbf{A}(G)/\mathcal{K}$ be the quotient map and define a symmetric $\mathbf{A}(G)$-module structure on X by

$$a \cdot x = \Gamma(a)x = x \cdot a$$

We wish to recognize X as an operator $\mathbf{A}(G)$-module. First we fix any norm $\|\cdot\|_X$ on X. Now we define a new norm $\|\cdot\|'$ on X by setting for all $x \in X$

$$\|x\|' = \sup\{\|a \cdot x\|_X : \|a\| \leq 1\}.$$

Since the kernel of the map $a \mapsto a \cdot x$ is closed for each $x \in X$, it is now easy to see that the finite dimensional module X is a Banach $\mathbf{A}(G)$-module with respect to the norm $\|\cdot\|'$ on X. Now we can give X the MIN operator structure. Since X is finite dimensional, it is easy to see that the module action extends to a bounded map

$$m : \mathbf{A}(G) \hat{\otimes} MIN(X) \mapsto MIN(X)$$

which is clearly completely bounded. The small difficulty is that this map is not necessarily completely contractive. To correct this we define a new collection of operator space norms $\|\cdot\|''_n$ on $\mathbb{M}(X)$ by the formula

$$\|[x_{ij}]\|''_n = \sup\{\|[a_{kl}x_{ij}]\|'_{nm,MIN} : [a_{kl}] \in \mathbb{M}_m(\mathbf{A}(G)), \|[a_{kl}]\|_m \leq 1\}.$$

It is now straightforward to verify that with this new structure, the module map is completely contractive and hence X is a symmetric operator $\mathbf{A}(G)$ bimodule.

Finally, if Π is the projection of $\mathbf{A}(G)/\mathcal{K}$ onto its second coordinate, then $D = \Pi \circ \Gamma$ is a derivation from $\mathbf{A}(G)$ into X. Moreover, since $(\ker D) \cap \mathcal{J} = \mathcal{K}$ is not closed, D is not continuous and hence is not completely bounded.

■

We can obtain a converse to the previous proposition for the class of locally compact groups with an abelian subgroup of finite index.

PROPOSITION 5.3.2: *Let G be a locally compact group with an abelian subgroup of finite index. Then every derivation from $\mathbf{A}(G)$ into an operator $\mathbf{A}(G)$ module is completely bounded.*

Proof: Since G is amenable, every derivation from $\mathbf{A}(G)$ into any Banach $\mathbf{A}(G)$ module is automatically bounded [32]. It now follows from Theorem 5.2.5 that every derivation into an operator $\mathbf{A}(G)$ module is automatically completely bounded.

■

While for amenable groups, derivations from $\mathbf{A}(G)$ into $\mathbf{VN}(G)$ are automatically bounded even for groups such as $SO(3)$ it is not the case that they are always completely bounded. In fact, it turns out that for compact Lie groups, Proposition 5.3.2 was the best we could do.

COROLLARY 5.3.3: *Let G be a compact Lie group. Then every derivation from* $\mathbf{A}(G)$ *into* $\mathbf{VN}(G)$ *is completely bounded if and only if G has an abelian subgroup of finite index.*

Proof: The "if" direction follows immediately form the previous proposition.

Conversely, since G is amenable, $\mathbf{A}(G)$ is operator amenable, and hence clearly operator weakly amenable. It follows that the only completely bounded derivation from $\mathbf{A}(G)$ into $\mathbf{VN}(G)$ is the zero map. However, $\mathbf{A}(G)$ is not weakly amenable [48]. This means there must exist a nonzero bounded derivation D from $\mathbf{A}(G)$ into $\mathbf{VN}(G)$ which cannot be completely bounded. ∎

For noncompact Lie groups we have the following:

PROPOSITION 5.3.4: i) *Let G be a Lie group. Assume that every bounded derivation from* $\mathbf{A}(G)$ *into a symmetric operator* $\mathbf{A}(G)$*-module is completely bounded. Then every compact subgroup K of G has an abelian subgroup of finite index.*

ii) Let G be a semisimple Lie group with a compact connected component K. Then every bounded derivation from $\mathbf{A}(G)$ *into* $\mathbf{VN}(G)$ *is completely bounded if and only if G is discrete.*

iii) Let G be a semisimple Lie group with a compact connected component. Then every derivation from $\mathbf{A}(G)$ *into a symmetric operator* $\mathbf{A}(G)$*-module is completely bounded if and only if G is amenable and discrete.*

Proof: By Lemma 4.4.1 the restriction map R establishes a completely contractive homomorphism from $\mathbf{A}(G)$ onto $\mathbf{A}(K)$. This allows us to view $\mathbf{VN}(K)$ as a symmetric operator $\mathbf{A}(G)$-module with respect to the action defined by $u \cdot T = R(u) \odot T = T \odot R(u) = T \cdot u$ for each $u \in \mathbf{A}(G), T \in \mathbf{VN}(K)$ (here $\odot$ denotes the usual action of $\mathbf{A}(K)$ on $\mathbf{VN}(K)$). If K does not have an abelian subgroup of finite index, then Corollary 5.3.3 shows that there exists a derivation $D : \mathbf{A}(K) \mapsto \mathbf{VN}(K)$ which is bounded but not completely bounded. Let $\tilde{D}(u) = D(R(u))$ for each $u \in \mathbf{A}(G)$. Then it is easy to see that $\tilde{D}$ defines a bounded derivation from $\mathbf{A}(G)$ into $\mathbf{VN}(K)$. We claim that $\tilde{D}$ is not completely bounded.

Observe that $R = \Gamma \circ Q$ where $Q : \mathbf{A}(G) \mapsto \mathbf{A}(G)/I(K)$ is the quotient map and $\Gamma : \mathbf{A}(G)/I(K) \mapsto \mathbf{A}(K)$ is the map defined in Proposition 5.2.2. Let $M > 0$. Since D is not completely bounded, there exists $[u_{ij}] \in \mathbb{M}_n(\mathbf{A}(K))$ with $\|[u_{ij}]\|_n \leq 1$ but $\|D([u_{ij}]\|_n > M$. However since Q is a complete quotient map and Γ is a complete isometry there exists $[v_{ij}] \in \mathbb{M}_n(\mathbf{A}(G))$ with $\|[v_{ij}]\|_n \leq 2$ and $R^{(n)}([v_{ij}]) = (\Gamma \circ Q)^{(n)}([v_{ij}]) = [u_{ij}]$. Thus $\|\tilde{D}^{(n)}([v_{ij}])\|_n = \|D^{(n)}[u_{ij}]\| > M$. Hence $\tilde{D}$ is not completely bounded.

ii) Assume that K is not abelian. Then as before $\mathbf{A}(K)$ is not weakly amenable and hence by [29, Lemma 2.1] $\mathbf{A}(G)$ cannot be weakly amenable. However K is open since G is a Lie group. Furthermore $\mathbf{A}(K)$ is operator weakly amenable since K is compact. Thus by Lemma 4.4.3, $\mathbf{A}(G)$ is operator weakly amenable. This means there exists a nonzero bounded derivation from $\mathbf{A}(G)$ into $\mathbf{VN}(G)$ which is not completely bounded. Since this is impossible, K must be abelian. However, since G is semisimple, this means that K is trivial and hence that G is discrete.

For the converse, we observe that if G is discrete, then $\mathbf{A}(G)$ is weakly amenable [32] and the only bounded derivation of $\mathbf{A}(G)$ into $\mathbf{VN}(G)$ is the zero map which is obviously completely bounded.

iii) Assume that every derivation from $\mathbf{A}(G)$ into a symmetric operator $\mathbf{A}(G)$-module is completely bounded, then by (ii) G is discrete. Moreover, by Proposition 5.3.1, G must be amenable.

The converse follows trivially since, if G is discrete $\mathbf{A}(G)$ is operator weakly amenable and if G is amenable then every derivation is bounded [32]. As such the only derivation from $\mathbf{A}(G)$ into a symmetric operator $\mathbf{A}(G)$-module is once again the zero map.

∎

We can apply the previous Corollary to the family $\mathbb{R}^n \ltimes_\rho SO(n)$ of Euclidean motion groups to conclude that:

COROLLARY 5.3.5: *Let G be the semidirect product $\mathbb{R}^n \ltimes_\rho SO(n)$ where the action of $SO(n)$ on $\mathbb{R}^n$ is the natural action. If $n \geq 3$, there exists a bounded derivation $D : \mathbf{A}(G) \mapsto \mathbf{VN}(G)$ which is not completely bounded.*

The various types of amenability which we have mentioned have one major structural flaw in common – they are not inherited by subalgebras. In particular, we do not know if given a compact normal subgroup K, $\mathbf{A}(G)$ being (operator) weakly amenable implies that $\mathbf{A}(G : K) \cong \mathbf{A}(G/K)$ is also (operator) weakly amenable. Were this the case, we would be able to make significant progress in classifying those groups for which $\mathbf{A}(G)$ is (operator) weakly amenable. For example, if we knew that $\mathbf{A}(G/K)$ always inherited weak amenability from $\mathbf{A}(G)$, we could conclude that for $G \in [SIN]$, weak amenability of $\mathbf{A}(G)$ would be equivalent to G having an abelian connected component [29, Theorem 2.7]. In this case, it would be true that for any $[SIN]$-group with a nonabelian connected component there would be a bounded derivation from $\mathbf{A}(G)$ into $\mathbf{VN}(G)$ which was not completely bounded. At present the best we can do is the following related result:

PROPOSITION 5.3.6: *Let G be a* [SIN] *group for which G_0 is not abelian. Then for every neighborhood U of the identity, there exists a compact normal $K_U \subset U$ and a bounded map $\Gamma_U : \mathbf{A}(G) \mapsto \mathbf{VN}(G/K_U)$ which is such that the restriction of Γ_U to $\mathbf{A}(G : K_U)$ is a derivation that is not completely bounded.*

Proof: Let G_0 be the connected component of G. Then $G_0 = V \times K$ where V is a vector group and K is a nonabelian compact connected group (see [65]). Let U be a neighborhood of $e \in G$. Then, since G is a projective limit of Lie groups, there exists a compact normal subgroup $K_U \subset U$ such that G/K_U is a Lie group and if $\phi : G \mapsto G/K_U$ is the quotient map, then $\phi(K)$ is a nonabelian compact connected subgroup of G. It follows from [48] and [29] that $\mathbf{A}(G/K_U)$ is not weakly amenable. Hence there exists a nonzero bounded derivation D_U from $\mathbf{A}(G/K_U)$ into $\mathbf{VN}(G/K_U)$.

Since $\mathbf{A}(G/K_U)$ is completely isometrically isomorphic to $\mathbf{A}(G : K_U)$, we may assume that D_U maps $\mathbf{A}(G : K_U)$ into $\mathbf{VN}(G/K_U)$. Finally, let P_{K_U} be the projection of $\mathbf{A}(G)$ onto $\mathbf{A}(G : K_U)$. Then $\Gamma_U = D_U \circ P_{K_U}$ is the desired map. ■

Chapter 6

Complemented Ideals in $\mathbf{A}(G)$

6.1 Introduction

The contents of this Chapter investigates which closed ideals of the Fourier algebra of a locally compact group are complemented. Indeed, it is this question which served as the author's original motivation for studying the homological properties of completely contractive algebras.

To understand the history of this question, we consider the group algebra of the circle group $G = \mathbb{T}$. We recall the ***Hardy space*** $\mathcal{H}^1(\mathbb{T})$ which is defined by

$$\mathcal{H}^1(\mathbb{T}) = \{f \in \mathbf{L}^1(\mathbb{T}) : \hat{f}(n) = 0 \ \text{ for all } \ n < 0\}$$

Note that the function $g(z) = z^n$ satisfies $\hat{g}(k) = 0$ if $k \neq n$ and $\hat{g}(k) = 1$ if $k = n$. From this it is easy to see that $\mathcal{H}^1(\mathbb{T})$ is simply the closure of the analytic polynomials in the $\mathbf{L}^1(G)$ norm. (see [23]).

Note that $\widehat{f * g}(n) = \hat{f}(n) \cdot \hat{g}(n)$ from which it is easy to see that $\mathcal{H}^1(\mathbb{T})$ is a closed two sided ideal of $\mathbf{L}^1(\mathbb{T})$.

For some time it was questioned whether $\mathcal{H}^1(\mathbb{T})$ was complemented in $\mathbf{L}^1(\mathbb{T})$. The answer to this question goes back to D.J. Newmans result [64], and more generally Walter Rudin's paper ([71]) in 1962, where he classified the complemented ideals in $\mathbf{L}^1(G)$ for compact abelian groups. To begin this chapter, we shall review his construction, which will serve as our motivation for the general question.

For $x \in G$ we define the translation map

$$\tau_x : \mathbf{L}^1(G) \mapsto \mathbf{L}^1(G)$$

given by

$$\tau_x(f(y)) = f(x^{-1}y).$$

A subspace X of $\mathbf{L}^1(G)$ is called ***translation invariant*** if $\tau_x(f) \in X$ for all $x \in G$ and $f \in X$.

It is well known (See [72, p. 157]) that $\mathcal{I}$ is an ideal of $\mathbf{L}^1(G)$ if and only if $\mathcal{I}$ is translation invariant.

Suppose $\mathcal{I}$ is complemented, and let $P : \mathbf{L}^1(G) \mapsto \mathcal{I}$ be a continuous projection onto the ideal. Now let

$$Tf = \int_G \tau_{x^{-1}} P(\tau_x f) dx$$

then $Tf \in \mathcal{I}$, $Tf = f \ \forall f \in \mathcal{I}$, and $T\tau_x = \tau_x T$ and thus we get

$$T(f * g) = f * T(g)$$

which implies $Tf = f * \mu$ for some $\mu \in M(G)$. Since T is a projection, μ is idempotent and hence by Cohen's Idempotent Theorem we have

$$h(\mathcal{I}) = \{\gamma \in \hat{G} : \hat{f}(\gamma) = 0 \ \forall f \in \mathcal{I}\} \in \Omega(\hat{G})$$

where $\Omega(\hat{G})$ represents the closed coset ring of $(\hat{G})$

Conversely if $h(\mathcal{I}) \in \Omega(\hat{G})$, then there is an idempotent measure μ with $Z(\mu) = h(\mathcal{I})$. Therefore the projection defined by

$$P(f) = f * \mu$$

is a projection onto $\mathcal{I}$, whence it is complemented.

Thus we conclude that the ideal $\mathcal{I}$ is complemented if and only if $h(\mathcal{I}) \in \Omega(\hat{G})$. In particular, since $\mathbb{Z}^- \notin \Omega(\hat{G})$ it follows that $\mathcal{H}^1(\mathbb{T})$ is not complemented in $\mathbf{L}^1(\mathbb{T})$.

We note that the idea in the proof was to "average away" the effects of the group to produce a projection which commutes with translations, and hence commutes with convolution.

If G is non-compact the previous construction does not work since $m(G) = \infty$. In [70] H. Rosenthal extended this result to the non-compact case by using the amenability of the group in the same spirit as Rudin.

Consider $1 - P^* : \mathbf{L}^\infty(G) \to \mathcal{I}^\perp$. We can use the invariant mean on $\mathbf{L}^\infty(G)$ to get a map $T : \mathbf{L}^\infty(G) \to \mathcal{I}^\perp$ such that $T(f * \phi) = f * T(\phi)$ (i.e. T commutes with the $L^1(G)$- module action). Thus $\mathcal{I}^\perp$ is invariantly complemented and hence $\mathcal{I}$ has a bounded approximate identity. From this it can be shown that $h(\mathcal{I}) \in \Omega(\hat{G}_d)$ the coset ring of $\hat{G}$ with the discrete topology.

Unfortunately the converse to the above result is false. Indeed the question of identifying sufficient conditions for an ideal to be complemented is extremely difficult. For example see [2], for the case $G = \mathbb{R}$, [1] for $G = \mathbb{R}^2$ and the papers [3] and [4] for further discussion on this question.

To understand our current approach, we consider the following sequences of $\mathbf{L}^1(G)$-modules

$$\Sigma : 0 \mapsto \mathcal{I} \overset{i}{\mapsto} \mathbf{L}^1(G) \overset{v}{\mapsto} \mathbf{L}^1(G)/\mathcal{I} \mapsto 0$$

and its dual sequence

$$\Sigma^* : 0 \mapsto \mathcal{I}^\perp \overset{v^*}{\mapsto} \mathbf{L}^\infty(G) \overset{i^*}{\mapsto} \mathcal{I}^* \mapsto 0,$$

where i is the inclusion map, v is the canonical quotient map, and i^*, v^* are their adjoints.

The essence of Rosenthal's result was to show that if $\mathcal{I}$ is complemented then Σ^* splits (i.e. i^* and v^* have inverses which are module maps).

So in general we wish to consider sequences of $\mathcal{A}$-modules

$$\Sigma : 0 \mapsto X^* \overset{f}{\mapsto} Y \overset{g}{\mapsto} Z \mapsto 0$$

and determine when such sequences split. Presumably the reader can recognize the homological flavour of this question.

6.2 Splitting of Exact Sequences

We begin with the following theorem, which follows from our work in Chapter 3. (See [79] for an alternate proof).

THEOREM 6.2.1: *Let $\mathcal{A}$ be an operator amenable Banach algebra, and let*

$$\Sigma : 0 \mapsto X^* \overset{f}{\mapsto} Y \overset{g}{\mapsto} Z \mapsto 0$$

be an admissible short exact sequence of operator $\mathcal{A}$-bimodules with X^* a dual operator $\mathcal{A}$-bimodule. Then Σ splits.

Proof: This is Corollary 3.7.3. ■

Suppose $\mathcal{A}$ is an operator amenable Banach algebra, and let $\mathcal{J}$ be a closed ideal. We wish to consider the sequence

$$\Sigma : 0 \mapsto \mathcal{J} \overset{i}{\mapsto} \mathcal{A} \overset{v}{\mapsto} \mathcal{A}/\mathcal{J} \mapsto 0$$

and its dual sequence

$$\Sigma^* : 0 \mapsto \mathcal{J}^\perp \overset{v^*}{\mapsto} \mathcal{A}^* \overset{i^*}{\mapsto} \mathcal{J}^* \mapsto 0,$$

where i is the inclusion map, v is the canonical quotient map, and i^*, v^* are their adjoints. Clearly i and v are completely bounded module maps and thus so are i^* and v^*. (Note that $(\mathcal{A}/\mathcal{J})^* \cong \mathcal{J}^\perp$).

Now suppose that $\mathcal{J}$ is completely complemented by a projection $P : \mathcal{A} \mapsto \mathcal{J}$. We define a map $Q : \mathcal{A}/\mathcal{J} \mapsto \mathcal{A}$ by

$$Q(a + J) = a - P(a).$$

Note that Q is well defined since if $a + J = b + J$ then $a - b \in J$ so

$$Q(a - b + J) = a - b - P(a - b) = 0.$$

Furthermore Q is completely bounded and a left inverse for v. Thus we conclude the exact sequence Σ^* is admissible. We shall say that a submodule Y of an $\mathcal{A}$-module X is called ***invariantly complemented*** if there exists a projection T onto X which commutes with the module action, i.e. $T : X \mapsto Y$ such that $T(ax) = aT(x)$ for all $a \in \mathcal{A}$ and $x \in X$. Applying Theorem 1 we have:

THEOREM 6.2.2: *Let $\mathcal{J}$ be a closed ideal in an operator amenable Banach algebra. If $\mathcal{J}$ is complemented, then $\mathcal{J}^\perp$ is invariantly complemented.*

Now we are ready to provide the connection between complete invariant complementation and bounded approximate identities in the operator space category.

THEOREM 6.2.3: *Let $\mathcal{A}$ be an operator amenable Banach algebra, and $\mathcal{J}$ a closed ideal. Then $\mathcal{J}^{\perp}$ is complemented if and only if $\mathcal{J}$ has a bounded approximate identity.*

Proof: Assume $\mathcal{J}^{\perp}$ is complemented by a completely bounded projection. Then by Theorem 6.2.1, Σ^* splits. In particular Σ^* splits as Banach algebras , and hence by standard Banach algebra arguments, $\mathcal{J}$ has a bounded approximate identity. (See [19]).

For the converse we follow [19, Proposition 3.5]. If $\mathcal{J}$ has a bounded approximate identity, let Φ be a weak-* limit point in $\mathcal{J}^{**}$. Define $S : \mathcal{J}^* \mapsto \mathcal{A}^*$ by

$$\langle S(\phi), a\rangle = \langle \Phi, \phi \cdot a\rangle \quad \forall \phi \in \mathcal{J}^*.$$

Clearly $i^*S = id_{\mathcal{J}^*}$. Now define an operator T on $\mathcal{A}^*$ by

$$T(\phi) = \phi - Si^*(\phi).$$

Routine calculations show that if $x \in \mathcal{J}$ we have

$$\langle T\phi, x\rangle = \langle \phi, x\rangle - \langle Si^*\phi, x\rangle = 0$$

and if $\phi \in \mathcal{J}^{\perp}$ then $T\phi = \phi$. In particular we see that T is a projection onto $\mathcal{J}^{\perp}$. It now suffices to show that S is completely bounded (whence T is). Now

$$\begin{aligned}
\| S^{(n)} \| &= \sup_{\|[\phi_{ij}]\| \le 1} \{\| [S(\phi_{ij})] \| \quad [\phi_{ij}] \in M_n(\mathcal{J}^*)\} \\
&= \sup_{\|[\phi_{ij}]\| \le 1} \sup_{\|[a_{kl}]\| \le 1} \{\| \langle S(\phi_{ij}), a_{kl}\rangle \|\} = \sup_{\|[\phi_{ij}]\| \le 1} \sup_{\|[a_{kl}]\| \le 1} \{\| \langle \Phi, \phi_{ij} \cdot a_{kl}\rangle \|\} \\
&= \sup_{\|[\phi_{ij}]\| \le 1} \sup_{\|[a_{kl}]\| \le 1} \| \Phi^{(nm)}[\phi_{ij} \cdot a_{kl}] \|\} \\
&\le \| \Phi \|
\end{aligned}$$

To see the last inequality, note that Φ is a linear functional on $\mathcal{J}^*$ and hence is completely bounded with $\| \Phi \|_{cb} = \| \Phi \|$ and we also have that the module action is completely contractive. Hence S is completely bounded and the result is proven.

■

Remark: Notice that the operator amenability of the algebra was not necessary in the construction of the map T. Since T is a module map we can conclude in general that if $\mathcal{J}$ possesses a bounded approximate identity, then $\mathcal{J}^\perp$ is invariantly complemented by a c.b. map.

The next corollary now follows easily.

COROLLARY 6.2.4: *If $\mathcal{J}$ is complemented closed ideal in an operator amenable Banach algebra, then $\mathcal{J}$ has a bounded approximate identity.*

6.3 Ideals in the Fourier Algebra

Applying the results of the last section, we are able to provide conditions for a closed ideal in $\mathbf{A}(G)$ to be complemented by a completely bounded projection.

THEOREM 6.3.1: *Let G be an amenable group. If $\mathcal{J}$ is a closed ideal such that $\mathcal{J}^\perp$ is complemented in $\mathbf{A}(G)^*$, then $h(\mathcal{J}) \in \Omega_c(G)$. In particular if $\mathcal{J}$ is complemented, then $h(\mathcal{J}) \in \Omega_c(G)$.*

Proof: Since G is amenable, $\mathbf{A}(G)$ is operator amenable. Therefore by Corollary 6.2.4, $\mathcal{J}^\perp$ complemented by a completely bounded projection implies that $\mathcal{J}$ has a bounded approximate identity $\{u_\alpha\}$. It follows from [28] that $u_\alpha \in B(G_d)$ and

$\|u_\alpha\|_{B(G_d)}=\|u_\alpha\|_{\mathbf{A}(G)}$. Let u be a weak-* limit point of this bounded approximate identity in $B(G_d)$. It is routine to show that u is an idempotent in $B(G_d)$ with $Z(u) = h(\mathcal{J})$. By Host's Idempotent Theorem [46] we can conclude that $Z(u) \in \Omega(G_d)$ and hence $h(J) \in \Omega_c(G)$. (See also [34]) ∎

Remark: We note that Rosenthal's result ([70]) is for bounded projections, as opposed to completely bounded projections. However if G is abelian then $\mathbf{A}(G) \cong \mathbf{L}^1(\hat{G})$ which is known to have the MAX operator space structure. In this case it follows that every bounded projection is automatically completely bounded, and hence in particular we see that Theorem 6.3.1 is a true generalization of Rosenthal's result to the non-abelian case. Unfortunately, for the Fourier algebra to have the MAX operator space structure its dual space $\mathbf{VN}(G)$ will have the MIN operator space structure, (see [10]), from which it follows that $\mathbf{VN}(G)$ is a commutative operator algebra and hence G is abelian (see [11]). Thus we cannot conclude that every bounded projection is automatically completely bounded for arbitrary G.

Also note that as discussed earlier the converse of Theorem 6.3.1 is false in general — even in the abelian case. (See [2] for the case $G = \mathbb{R}$). However, in the discrete case the converse does hold and we have the following characterization of ideals complemented by completely bounded maps.

Corollary 6.3.2: *Let G be an amenable discrete group. Then $\mathcal{J}$ is complemented if and only if $h(\mathcal{J}) \in \Omega_c(G)$.*

Proof:($\Rightarrow$) This follows immediately from Theorem 6.3.1.

($\Leftarrow$) If $h(\mathcal{J}) = E \subset \Omega_c(G)$ then the characteristic function of E, denoted 1_E is an element of $B(G)$([28]) . Thus the map $P(u) = u \cdot 1_E$ from $\mathbf{A}(G)$ onto $\mathcal{J}$ is a completely bounded projection of $\mathbf{A}(G)$ onto $\mathcal{I}(h(\mathcal{J}))$. Since this is a set of spectral synthesis it follows that $\mathcal{I}(h(\mathcal{J})) = \mathcal{J}$. ∎

The following example due to Leinert shows that the condition on the amenability of the group is necessary for the previous corollary. Let $\mathbb{F}_2$ be the free group on $\{a, b\}$. It is well known that $\mathbb{F}_2$ is not an amenable group. Let $E = \{a^n b^n : n = 1, 2, ...\}$, then the characteristic function 1_E of E is completely bounded (see [60] for details), however E is clearly not an element of the coset ring. In particular, $\mathcal{I}(E)$ provides an example of a complemented ideal in $A(\mathbb{F}_2)$ whose hull is not in the coset ring.

Chapter 7

Summary and Open Problems

QUESTION 1: One of our main objectives in Chapter 3 was to show that questions and constructions relating to the homology and cohomology in operator spaces are most naturally realized using the operator space projective tensor product as the natural product. Many authors who studied the cohomology of operator spaces and von Neumann algebras (see [66] and [73]) use the Haagerup tensor product as the basic object. This of course leads to the question:

When do the tensor products $\otimes_h$ and $\hat{\otimes}$ agree?

In the case that the operator spaces are indeed C^*-algebras, then Kumar and Sinclair have shown in [55] that the tensor products (and hence the cohomology) agree exactly when either one of the algebras is finite dimensional, or when both of the algebras possess a collection of irreducible representations of bounded degree (C^* algebras of this type are called ***subhomogeneous*** – note this corrects [47]). However in the case of the Fourier algebra, the solution is not so clear. Should it be the case that the tensor products agree when the group has irreducible representations of bounded degree, then our results in Chapter 5 show that this is the case exactly when every map from $\mathbf{A}(G)$ into any operator space in automatically completely bounded. In this case we have that the "operator space" homology corresponds exactly to the " Banach space" homology.

QUESTION 2: In Chapter 4 we discussed the weak operator amenability of $\mathbf{A}(G)$ and showed that for a large class of groups the Fourier algebra is weak operator amenable. This leads to the natural question:

Do there exist groups G for which $\mathbf{A}(G)$ is not weak operator amenable?

QUESTION 3: In Chapter 6 we gave a complete classification of the ideals in the Fourier algebra of a discrete group which are complemented by a completely bounded projection. Furthermore we gave necessary conditions for complementation for general amenable groups. We note that providing explicit sufficient conditions for non discrete groups is extremely difficult and using present techniques is out of reach. (See [52] for a discussion of the groups $\mathbb{R}^3$ and $\mathbb{R}^4$). However for non-amenable groups, we still lack any sort of necessary conditions. Our current approach using the cohomology groups may provide more detailed insight. Since the sets $\mathbf{Ext}^1$ represent equivalence classes of extension sequences, perhaps we can find conditions on the zero set of an ideal in $\mathbf{A}(G)$ to be complemented for all groups with reference to the contents of $\mathbf{Ext}^1$. Thus we have the question:

Given an arbitrary non-amenable group G, are there reasonable necessary conditions for an ideal in $\mathbf{A}(G)$ to be complemented?

QUESTION 4: Our major focus has been on the so called Hochschild cohomology in the operator space category. However there is another cohomology called the *cyclic cohomology* introduced by Connes (See [16]) in the algebraic category. A Banach space version of this was developed by Christian and Sinclair in [15]. It is possible to connect this new cohomology with the spaces $H^n(\mathcal{A}, \mathcal{A}^*)$, and hence to the notion of weak amenability. Connes has found interesting applications of this cyclic cohomology in the study of differential geometry (see [18], [17]). The ideas of differential geometry are rather complex, but relate to the notion of replacing usual "scalar" ideas with that of operators on a Hilbert space. Given that the morphisms of completely bounded maps are somewhat natural for the category of operators on Hilbert spaces, it may be interesting to develop a "completely bounded cyclic cohomology", which leads to the question

What connections exist between operator weak amenability and the natural notion of completely bounded cyclic cohomology?

Given the application of cyclic cohomology and noncommutative geometry to quantum mechanics, (see for example [17]) it would be interesting to see if the operator space category can find any applications there.

QUESTION 5: Given a Banach algebra $\mathcal{A}$, there is a natural $\mathcal{A}$-module structure on $\mathcal{A}^*$ which we exploited in our study of weak amenability. Using these same ideas, it is easy to recognize the second dual $\mathcal{A}^{**}$ of $\mathcal{A}$ as an $\mathcal{A}$- module, and so forth. Let us denote the nth dual of $\mathcal{A}$ by $\mathcal{A}^{[n]}$. In [20] Dales, Ghahramani and Grønbæk defined the Banach algebra $\mathcal{A}$ to be ***n-weak amenable*** if each bounded derivation from $\mathcal{A}$ into $\mathcal{A}^{[n]}$ is inner. In [20] it was shown that $L^1(G)$ is $2n+1$ weak amenable for all groups G. The classification of which groups G are $2n$-weak amenable remains open. These notions appear to have natural generalizations to the operator space category. Hence we shall say a completely contractive Banach algebra $\mathcal{A}$ is ***n-operator weak amenable*** if $OH^1(\mathcal{A}, \mathcal{A}^{[n]}) = 0$. This leads us to the question:

For which groups G is $\mathbf{A}(G)$ n-operator weak amenable.

QUESTION 6: Fundamental to this thesis is the observation that the Fourier algebra has better "amenability properties" when considered in the operator space category. In particular we note that the class of groups for which $MAX(\mathbf{A}(G))$ is operator amenable is considerably smaller than the class of groups for which $\mathbf{A}(G)$ with its natural structure is operator amenable. We wonder how altering the operator space structure can affect the potential operator amenability of a completely contractive Banach algebra. This leads to the following two questions:

Are $MIN(\mathbf{A}(G))$ and $MIN(\mathbf{L}^1(G))$ completely contractive Banach algebras? Assuming the answer to the above question is yes, then we have:

For what groups G are $MIN(\mathbf{A}(G))$ and $MIN(\mathbf{L}^1(G))$ operator amenable?

Chapter 8

Appendix A

In this appendix, we provide the equivalent of a *Comparison Theorem* in our category. Perhaps we should first make a few observations. Note that in general we will be given an additive functor from the category of operator spaces into the category of operator spaces. The various derived functors of this functor become functors from the category of operator spaces into the category of linear spaces. The key is that the various quotients are not necessarily Banach spaces, hence we cannot guarantee that the "image" of our functor is inside the category of operator spaces.

The usual comparison theorem for resolutions now asserts that given an additive functor F and two projective resolutions of X, the functors derived from the two resolutions are naturally isomorphic (in the target category). However we wish to show that in the instances that $F^n(X)$ are operator spaces, then the derived functors are in fact c.b. isomorphic.

We begin with the following. Let

$$\ldots \leftarrow K_{n-1} \overset{d_n}{\leftarrow} K_n \overset{d_{n+1}}{\leftarrow} K_{n+1} \leftarrow \ldots \quad (\mathfrak{K})$$

$$\dots \leftarrow k'_{n-1} \overset{d'_n}{\leftarrow} K'_n \overset{d'_{n+1}}{\leftarrow} K'_{n+1} \leftarrow \dots \quad (\mathfrak{K}')$$

be two admissible operator complexes of $\mathcal{A}$ modules. A ***chain transformation*** $f : \mathfrak{K} \mapsto \mathfrak{K}'$ is a family of c.b. module maps $f_n : K_n \mapsto K'_n$ such that the resulting diagram

$$\begin{tikzcd}
\cdots & K_{n-1} \arrow[l] \arrow[d, "f_{n-1}"] & K_n \arrow[l, "d_n"'] \arrow[d, "f_n"] & K_{n+1} \arrow[l, "d_{n+1}"'] \arrow[d, "f_{n+1}"] & \cdots \arrow[l] \\
\cdots & K'_{n-1} \arrow[l] & K'_n \arrow[l, "d'_n"'] & K'_{n+1} \arrow[l, "d'_{n+1}"'] & \cdots \arrow[l]
\end{tikzcd}$$

commutes. Note that each function f_n defines a function

$$(f_n)_* : OH_n(\mathfrak{K}) \mapsto OH_n(\mathfrak{K}')$$

given by

$$(f_n)_*(k + d_{n+1}(K_{n+1})) = f_n(k) + d'_{n+1}(K'_{n+1}).$$

This leads to the following:

PROPOSITION A1: *If Imd_{n+1} and Imd'_{n+1} are both closed, then $(f_n)_*$ is completely bounded.*

Proof: Let $s \in \mathbb{N}$ and $[k_{ij}] \in \mathbb{M}_s(K_n)$, and $[\bar{k}_{ij}]$ the corresponding element in $OH_n(\mathfrak{K})$. Now

$$\begin{aligned}
\|(f_n)_*^{(s)}(\bar{k}_{ij})\| &= \inf\{\|f_n(k_{ij}) + y_{ij}\| : y_{ij} \in Imd'_{n+1}\} \\
&\leq \inf\{\|f_n(k_{ij}) + f_n x_{ij}\| : x_{ij} \in Imd_{n+1}\} \\
&\leq \inf\{\|f_n\|_{cb}\|k_{ij} + x_{ij}\| : x_{ij} \in Imd_{n+1}\} \\
&= \|f_n\|_{cb}\|\bar{k}_{ij}\|
\end{aligned}$$

Thus $(f_n)_*$ is completely bounded. ■

If $f, g : \mathfrak{K} \mapsto \mathfrak{K}'$ are chain transformations, we call a family of c.b. module maps $s_n : K_n \mapsto K'_{n+1}$ a ***chain homotopy*** and say f and g are ***chain equivalent*** (and write $s : f \sim g$) if

$$d'_{n+1}s_n + s_{n-1}d_n = f_n + g_n.$$

PROPOSITION A2: *If $s : f \sim g : \mathfrak{K} \mapsto \mathfrak{K}'$ then*

$$(f_n)_* = (g_n)_* : OH_n(\mathfrak{K}) \mapsto OH_n(\mathfrak{K}').$$

Proof: The proof of this claim is identical to the algebraic case. (See [63, p. 40].

■

We shall say that a chain transformation $f : \mathfrak{K} \mapsto \mathfrak{K}'$ is a ***chain equivalence*** if there exists a chain transformation $h : \mathfrak{K}' \mapsto \mathfrak{K}$ and chain homotopies $s : hf \sim 1_{\mathfrak{K}}$ and $t : fh \sim 1_{\mathfrak{K}'}$.

COROLLARY A3: *If $f : \mathfrak{K} \mapsto \mathfrak{K}'$ is a chain equivalence then*

$$(f_n)_* : OH_n(\mathfrak{K}) \mapsto OH_n(\mathfrak{K}')$$

is an algebraic isomorphism. If $OH_n(\mathfrak{K})$ and $OH_n(\mathfrak{K}')$ are both operator spaces, then $(f_n)_*$ is a c.b. isomorphism.

Proof: Let $h : \mathfrak{K}' \mapsto \mathfrak{K}$ be a chain transformation such that there is a homotopy such that $s : hf \sim 1_{\mathfrak{K}}$. Clearly we have that $(f_n \circ h_n)_* = (f_n)_*(h_n)_*$. Since $(id_{K_n})_* = id_{OH_n}$ it follows that $(f_n)_*(h_n)_* = (h_n)_*(f_n)_* = id$. Thus $(f_n)_*$ is an algebraic isomorphism. Now by Proposition A1, if Imd_{n+1} and Imd'_{n+1} are closed, then both $(f_n)_*$ and $(h_n)_*$ are completely bounded. Hence we conclude that $(f_n)_*$ is a c.b. isomorphism.

Now we may turn our attention to projective resolutions.

THEOREM A4: *Let*

$$\to P_1 \to P_0 \xrightarrow{\epsilon} X : (\mathfrak{P})$$

be a projective resolution of X*, and let*

$$\to Y_1 \to Y_0 \xrightarrow{\epsilon'} X : (\mathfrak{Y})$$

be any resolution. Then there exists a chain transformation $f_n : P_n \mapsto Y_n$ *such that* $\epsilon' \circ f_0 = \epsilon$*. Furthermore if* $g_n : P_n \mapsto Y_n$ *is another such chain transformation, then* f *and* g *are chain homotopic.*

Proof: The proof of this Theorem requires only categorical properties of projectivity and exactness. See [63, p. 88] for the algebraic case. ∎

THEOREM A5: *Let* $\mathfrak{P}$ *and* $\mathfrak{P}'$ *be two projective resolutions of* X*. Let* F *be an additive functor from the category of operator* $\mathcal{A}$*-modules to operator* $\mathcal{A}$*-modules. Then there exists a chain equivalence* $f : F(\mathfrak{P}) \mapsto F(\mathfrak{P}')$*. In particular, if* $OH_n(F(\mathfrak{P}))$ *and* $OH_n(F(\mathfrak{P}'))$ *are operator spaces, then they are c.b. isomorphic.*

Proof: By Theorem A4, we can find a chain transformation $f : \mathfrak{P} \mapsto \mathfrak{P}'$ and another chain transformation $g : \mathfrak{P}' \mapsto \mathfrak{P}$ such that there exists a chain homotopy with $s : gf \sim id$. Since F is additive, it follows that $F(f)$ and $F(g)$ are chain homotopic such that $F(s) : F(g)F(f) \sim id$. By applying Proposition A3, we have that $OH_n(F(\mathfrak{P}))$ is c.b. isomorphic to $OH_n(F(\mathfrak{P}))$ whenever they are both operator spaces.

∎

We observe that this theorem asserts that any two projective resolutions define the same derived functors up to c.b. isomorphism whenever they produce operator spaces. However we note that we obtain only c.b. isomorphisms, not completely isometric isomorphisms. The reason for this appears to be in our choice of operator complexes and our definition of projectivity in this category.

As an alternative, we could study the operator space category under the following assumptions:

(1) All maps in an admissible complex must be complete isometries or complete quotients.

(2) A module P is *projective* only if whenever $\phi : P \mapsto X/Y$ is contractive, then there exists a contractive extension $\psi : P \mapsto X$.

One can now verify that under the above assumptions (along with the obvious necessary changes to injectivity etc.) that our derived functors are defined up to complete isometric isomorphism. There is however a difficulty in using this notion of equivalence of operator spaces. The above definition of projectivity concentrates on the problem of extension of maps, while our primary interest in projectivity relates to the exactness of functors – which ignores the "norm preserving" properties of the extension.

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