The Graphs of Häggkvist and Hell

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

This thesis investigates Häggkvist & Hell graphs. These graphs are an extension of the idea of Kneser graphs, and as such share many attributes with them. A variety of original results on many different properties of these graphs are given.

We begin with an examination of the transitivity and structural properties of Häggkvist & Hell graphs. Capitalizing on the known results for Kneser graphs, the exact values of girth, odd girth, and diameter are derived. We also discuss subgraphs of Häggkvist & Hell graphs that are isomorphic to subgraphs of Kneser graphs. We then give some background on graph homomorphisms before giving some explicit homomorphisms of Häggkvist & Hell graphs that motivate many of our results. Using the theory of equitable partitions we compute some eigenvalues of these graphs. Moving on to independent sets we give several bounds including the ratio bound, which is computed using the least eigenvalue. A bound for the chromatic number is given using the homomorphism to the Kneser graphs, as well as a recursive bound. We then introduce the concept of fractional chromatic number and again give several bounds. Also included are tables of the computed values of these parameters for some small cases. We conclude with a discussion of the broader implications of our results, and give some interesting open problems.

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Dedication

This thesis is dedicated to my parents, Tom and Sharon Roberson, whom I love with all my heart.

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Introduction

For any two positive integers n and r, we define the *Häggkvist & Hell graph*, $HH_r(n)$, to be the graph whose vertices are the ordered pairs (h, T), where $h \in \{1, \ldots, n\}$ and $T \subseteq \{1, \ldots, n\}$ such that $|T| = r$ and T does not contain h. Vertices (h_x, T_x) and (h_y, T_y) are adjacent if and only if $T_x \cap T_y = \varnothing$, $h_x \in T_y$, and $h_y \in T_x$. For instance, $(1, \{2\})$ and $(2, \{1\})$ are the only two vertices of $HH_1(2)$ and they are adjacent. For a given vertex $v = (h, T)$ of a Häggkvist & Hell graph, we refer to h as the *head* of v, and T as the *tail* of v. Also, we use $[n]$ to refer to the set $\{1, \ldots, n\}$. For some values of n and r, the graph $HH_r(n)$ has no edges or no vertices. We take time now to remark on these degenerate cases: when $n < 2r$ no two subsets of $[n]$ are disjoint, and so $HH_r(n)$ has no edges; in the more degenerate case when $n < r + 1$ there are no vertices, and thus $HH_r(n)$ is the null graph. With this in mind, we will assume throughout this work that $n \geq 2r$ unless otherwise stated.

There are two other cases in which the structure of the Häggkvist & Hell graphs can be easily discerned. If $r = 1$, then any vertex of $HH_r(n)$ has the form $(a, \{b\})$, and its only neighbor is the vertex $(b, \{a\})$. Therefore, $HH_1(n)$ is always a matching. Because of this, we will assume that $r \geq 2$ unless otherwise stated. In the case of $n = 2r$ we see that the Häggkvist & Hell graphs are bipartite. For any vertex, (h, T), of $HH_r(2r)$, its tail will be disjoint from exactly one other r-subset of $[2r]$, say $\overline{T}$. Since the head of any vertex with tail T must be from $\overline{T}$, and vice versa, it is apparent that any vertex having T as its tail will be adjacent to any vertex with $\overline{T}$ as its tail. There are exactly r vertices with T as their tail, and the same for $\overline{T}$, so these vertices induce a $K_{r,r}$ subgraph in $HH_r(2r)$. There are $\frac{1}{2}\binom{2r}{r}$ such pairs of disjoint r-subsets, so we have that

$$HH_r(2r) = \frac{1}{2}\binom{2r}{r} K_{r,r}.$$

These graphs extend the idea of Kneser graphs $K_{n:r}$, which have the r-subsets of $[n]$ as their vertices, two of them being adjacent if they are disjoint. Though much is known about Kneser graphs (they have been studied heavily) very little is known about Häggkvist & Hell graphs. For this reason there are not a great many citations in this work, the subject being relatively new. The only two previous references to graphs of this type that we know of are [3], where they were first mentioned as a triangle-free universal graph for triangle-free cubic graphs, and [1], which proves that their chromatic number is unbounded. In both cases the authors dealt only with the $r = 3$ case, and only for very specific results.

In this work we take a comprehensive look at several properties of the Häggkvist & Hell graphs which are typically of interest to graph theorists. For some of these properties, such as diameter and odd girth, these graphs closely resemble those for the corresponding Kneser graphs. For other properties, such as independence, chromatic, and fractional chromatic numbers, these graphs seem to veer away from their natural similarity to Kneser graphs. But in either case the results found for these graphs are interesting in their own right, and this is part of the reason we studied them. Our study of them is also related to a larger work by Chris Godsil, Mike Newman, and Karen Meagher that investigates independence numbers of different Kneser-like graphs, such as the q-Kneser graphs. The *q-Kneser graph* $qK_{n:r}$ is a q-analog of the ordinary Kneser graph $K_{n:r}$. Its vertices are the r-subspaces of a vector space of dimension n over $GF(q)$, where two r-subspaces are adjacent if their intersection is the zero subspace.

It is interesting to know how well the ratio bound for independence number works for graphs like this. The ratio bound is an upper bound for the size of an independent set of a regular graph obtained using the least and greatest eigenvalues of the graph. The result was originally proven by Delsarte and is as follows:

0.0.1 Theorem (Delsarte)**.** *For a regular graph G with valency k and least eigenvalue τ,*

$$\alpha(G) \leq |V(G)|\frac{-\tau}{k-\tau}$$

where $\alpha(G)$ is the size of the largest independent set of G.

Note that the largest eigenvalue of a regular graph is simply the valency. The ratio bound is tight for all Kneser graphs, and so it is of interest to investigate how accurate a bound it is for similar graphs.

When these graphs were first introduced by Roland Häggkvist and Pavol Hell in [3] they proved the following theorem:

0.0.2 Theorem (Häggkvist & Hell)**.** *A cubic graph G admits a homomorphism to $HH_3(22)$ if and only if G is triangle-free.*

Proof. Since $HH_3(22)$ is triangle-free, it is clear that no graph containing a triangle can admit a homomorphism to it. Conversely, assume that G is a triangle-free cubic graph. We will give a homomorphism from G to $HH_3(22)$, thus proving the result. For any graph H, let H^k denote the graph with the same vertex set as H, with edge set equal to

$$E(H^k) = \{xy : \mathrm{dist}_H(x, y) \leq k\}.$$

Consider the graph G^3. This has maximum degree $3 + 6 + 12 = 21$, and thus has chromatic number at most 22. So G^3 admits a 22-coloring, say with the elements of $\{1,\ldots,22\}$. Since $V(G)=V(G^3)$, we can consider this as a coloring of G as well. Now consider the mapping f that takes a vertex x of G to the ordered pair $(a,\{b,c,d\})$, where a is the color assigned to x and $b,c,$ & d are the distinct colors of its three neighbors. Now if x and y are adjacent vertices in G, then they will be mapped to (h_x,T_x) and (h_y,T_y) respectively, where $h_x\in T_y$ and $h_y\in T_x$ since they are neighbors in G, and T_x and T_y must be disjoint since all of their neighbors are within distance 3 of each other. Thus $f(x)$ is adjacent to $f(y)$ in $HH_3(22)$, and therefore $f: G\to HH_3(22)$ is a homomorphism since it preserves adjacency. □

Upon some reflection, it is not too difficult to see how this result could be generalized to regular graphs of higher degree. Indeed, we have the following:

0.0.3 Theorem. *An r-regular graph G is triangle-free if and only if it admits a homomorphism to*

$$HH_r\left(r\frac{(r-1)^3-1}{r-2}+1\right).$$

Proof. We proceed as in the previous case, mutatis mutandis. First, let $d=r\frac{(r-1)^3-1}{r-2}$. Since $HH_r(d+1)$ is triangle-free, it is clear that no graph containing a triangle can admit a homomorphism to it. Conversely, assume that G is a triangle-free r-regular graph. We will give a homomorphism from G to $HH_r(d+1)$, thus proving the result. Consider the graph G^3. This has maximum degree

$$r+r(r-1)+r(r-1)^2=r\frac{(r-1)^3-1}{r-2}=d,$$

and thus has chromatic number at most $d+1$. So G^3 admits a $d+1$-coloring, say with the elements of $\{1,\ldots,d+1\}$. Since $V(G)=V(G^3)$, we can consider this as a coloring of G as well. Now consider the mapping f that takes a vertex x of G to the ordered pair (h,T), where h is the color assigned to x and T is the set containing the distinct colors of its r neighbors. Now if x and y are adjacent vertices in G, then they will be mapped to (h_x,T_x) and (h_y,T_y) respectively, where $h_x\in T_y$ and $h_y\in T_x$ since they are neighbors in G, and T_x and T_y must be disjoint since all of their neighbors are within distance 3 of each other. Thus $f(x)$ is adjacent to $f(y)$ in $HH_r(d+1)$, and therefore $f: G\to HH_r(d+1)$ is a homomorphism since it preserves adjacency. □

These results concern homomorphisms into Häggkvist & Hell graphs, but there are two important homomorphisms from $HH_r(n)$ that are used frequently in our work. Specifically, they are homomorphisms to the complete graph K_n, and the Kneser graph $K_{n:r}$. The first homomorphism maps vertices to their heads, while the second maps vertices to their tails. These homomorphisms are simple, but prove to be very useful. They help us find bounds for the independence number and chromatic number of $HH_r(n)$, as well as greatly aiding in understanding the overall structure of these graphs.

In this thesis we begin our investigation into the Häggkvist & Hell graphs by discussing some results about their structure, such as diameter and odd girth, whose proofs rely heavily on the known values of these parameters for Kneser graphs. In particular, we show that the odd girth of $HH_r(n)$ and $K_{n:r}$ are equal whenever $K_{n:r}$ contains no triangles, and otherwise $HH_r(n)$ has odd girth five. A similar result is shown which proves that the diameters of these two graphs are the same whenever $K_{n:r}$ has diameter at least five, otherwise $HH_r(n)$ has diameter five when $n < \frac{5}{2}r$ and four whenever n is larger. We also make mention of the arc-transitivity of Häggkvist & Hell graphs, which again mimics the Kneser graphs. This is followed by a chapter on graph homomorphisms which includes some important homomorphisms of $HH_r(n)$.

Chapter 3 introduces the subject of equitable partitions, a powerful tool for finding eigenvalues of a graph. We present two equitable partitions of $HH_r(n)$, one with 3 cells from which it is computationally tractable to find three eigenvalues of $HH_r(n)$ for any n and r, and another with $5r + 2$ cells which gives every eigenvalue of $HH_r(n)$, but is computationally difficult even for quite small r. Using this second partition, we compute all of the eigenvalues for the cases $r = 2$ and $r = 3$, we also note that the least eigenvalue from the 3-cell partition is likely the least eigenvalue of the graph in every case, as it is in the two known cases. Having the least eigenvalues for $r = 2$ and $r = 3$ we now are able to compute the ratio bound for $HH_r(n)$ for these two cases, and we compute the probable bound for all other cases, assuming that the least eigenvalue is the one we suspect. We accompany this upper bound with a lower bound that is tight in all computed cases. The value of the lower bound is $r\binom{n-1}{r}$ when $2r \leq n \leq r^2 + 1$, and $\binom{n}{r+1} + \frac{r-1}{r+1}\binom{r^2}{r}$ when $n \geq r^2$. These two values are equal for $n = r^2$ and $r^2 + 1$. We also give the following recursive bound:

$$\alpha(HH_r(n-1)) \leq \alpha(HH_r(n)) \leq \alpha(HH_r(n-1)) + \binom{n-1}{r}.$$

In Chapter 5 we investigate the chromatic number of the Häggkvist & Hell graphs. Here we show that $\chi(HH_r(n)) \leq n - 2r + 2$, which we derive from the homomorphism to the Kneser graphs. We also give a recursive bound which shows that with each increase in n, the chromatic number increases by either zero or one. These bounds are complimented by some actual chromatic numbers that we have computed using SAGE, and a proof that the chromatic number of these graphs is unbounded for all $r \geq 2$. Chapter 6 presents the notions of fractional chromatic and clique numbers, which can be viewed as dual linear optimization problems, and are therefore equal. For Häggkvist & Hell graphs it is easy to show that this parameter is bounded above by $r + 1$. We give this and a recursive bound as with the chromatic number, but here we also give a probable lower bound using our probable upper bound on the independence number of $HH_r(n)$. In our conclusion we give a summary of our results and list some open problems about these graphs and discuss possible future research.

Chapter 1

The Structure of $HH_r(n)$

Häggkvist & Hell graphs have a very algebraic and symmetric structure, like the Kneser graphs, and it is straightforward to unearth the following basic properties of these graphs:

- $HH_r(n)$ has $(r+1)\binom{n}{r+1} = (n-r)\binom{n}{r}$ vertices.
- $HH_r(n)$ is regular with valency $r\binom{n-r-1}{r-1}$
- $HH_r(n)$ is a subgraph of $HH_r(n')$ for $n \leq n'$
- Since every neighbor of the vertex (h, T) must contain h in its tail, all Häggkvist & Hell graphs are triangle-free.

In this chapter we investigate some of the structural properties of Häggkvist & Hell graphs such as connectedness, girth, odd girth, diameter, and transitivity. In particular we determine that the diameter of $HH_r(n)$ is four when $n \geq \frac{5}{2}r$ and

$$\max\left\{5, \left\lceil\frac{r-1}{n-2r}\right\rceil + 1\right\} \text{ when } 2r+1 \leq n < \frac{5}{2}r,$$

and the odd girth is

$$\max\left\{5, 2\left\lceil\frac{r}{n-2r}\right\rceil + 1\right\}.$$

To our knowledge these properties have not been previously studied, though much is known about them for the related Kneser graphs. We end the chapter by showing that Häggkvist & Hell graphs contain subgraphs which are isomorphic to some subgraphs of Kneser graphs.

1.1 Transitivity

The transitivity of a graph refers to the way in which its automorphism group acts on it. The types of transitivity we consider for Häggkvist & Hell graphs are *edge*, *vertex*, and *arc* transitivity. An arc in a graph is simply an ordered pair of adjacent vertices. A graph X is edge/vertex/arc *transitive* when its automorphism group, denoted by $\mathrm{Aut}(X)$, acts transitively on its edges/vertices/arcs respectively, meaning that for any two edges/vertices/arcs x_1 and x_2, there exists an automorphism f of X such that $f(x_1) = x_2$. Arc transitivity always implies edge transitivity, it implies vertex transitivity unless the graph has isolated vertices and an edge.

The transitivity properties of a graph can be a crucial factor in investigating its further properties. In our case we can make the strongest transitivity claim from above, that $HH_r(n)$ is always arc transitive. But before we prove this we need to identify a subgroup of $\mathrm{Aut}(HH_r(n))$:

1.1.1 Lemma. *The symmetric group on n elements, $Sym(n)$, is a subgroup of* $\mathrm{Aut}(HH_r(n))$.

Proof. Since $Sym(n)$ is a group, we only need to show that each of its elements is an automorphism of $HH_r(n)$. Let $\sigma \in Sym(n)$. Then for a vertex $u = (h_u, T_u)$ in $HH_r(n)$ we have that $\sigma(u) = (\sigma(h), \sigma(T))$. Since any element of $Sym(n)$ is injective on elements and subsets of $[n]$, they are also injective on vertices of $HH_r(n)$. Now suppose u is adjacent to a vertex $v = (h_v, T_v)$. Then $h_u \in T_v$, $h_v \in T_u$, and $T_u \cap T_v = \varnothing$. From this we see that $\sigma(h_u) \in \sigma(T_v)$, $\sigma(h_v) \in \sigma(T_u)$, and $\sigma(T_u) \cap \sigma(T_v) = \varnothing$ which shows that $\sigma(u) \sim \sigma(v)$ and thus σ preserves adjacency. Since σ preserves adjacency and is injective, it is an automorphism of $HH_r(n)$, and so the result holds. □

Now that we know that the elements of $Sym(n)$ are automorphisms of $HH_r(n)$, we have the tool we need to prove that $HH_r(n)$ is arc transitive.

1.1.2 Theorem. *$HH_r(n)$ is arc transitive and $Sym(n)$ acts arc transitively on it.*

Proof. Suppose we have two arcs, (u, v) and (u', v'), of $HH_r(n)$. We must give an automorphism that maps (u, v) to (u', v'). Let σ be an element of $Sym(n)$ such that

$$\sigma(h_u) = h_{u'}, \quad \sigma(h_v) = h_{v'},$$

and

$$\sigma(T_u \setminus \{h_v\}) = T_{u'} \setminus \{h_{v'}\}, \quad \sigma(T_v \setminus \{h_u\}) = T_{v'} \setminus \{h_{u'}\}.$$

Then clearly σ takes (u, v) to (u', v') and we have proven the theorem. □

1.1.3 Corollary. *$HH_r(n)$ is both vertex and edge transitive.*

The results of this section, though straightforward, are among the most important, because we use them throughout this thesis, oftentimes without giving them specific mention. However, some results do specifically require that the graph in question possesses a certain type of transitivity. So it is key for the reader to be very comfortable with the transitivty of the Häggkvist & Hell graphs.

1.2 Connectedness

As we have seen, in the minimal case where $n = 2r$, the graph $HH_r(n)$ is not connected, since $\frac{1}{2}\binom{2r}{r} \geq 3$ for $r \geq 2$. This is similarly true for Kneser graphs, except for when $r = 1$ which we do not consider for Häggkvist & Hell graphs. It is known that the Kneser graph $K_{n:r}$ is connected for $n \geq 2r + 1$, and in this section we will prove the analogous result for $HH_r(n)$. The following lemma, proven by Pabon and Vera in [6], gives the diameter of the Kneser graph $K_{n:r}$ for $n \geq 2r + 1$, which implies that it must be connected.

1.2.1 Lemma (Pabon & Vera)**.** *For positive integers n and r, $n \geq 2r + 1$, the Kneser graph $K_{n:r}$ has diameter* $\left\lceil \frac{r-1}{n-2r} \right\rceil + 1$.

1.2.2 Lemma. *Any two vertices in $HH_r(n)$ with the same tail are joined by a path.*

Proof. Let $x = (h_x, T)$ and $y = (h_y, T)$. If $h_x = h_y$ then x and y are joined by a path of length zero. Otherwise $h_x \neq h_y$. Then, since $n \geq 2r$ and $h_x, h_y \notin T$, there exists an r-subset of $[n] \setminus T$ that contains both h_x and h_y. Let T' be such a set, and let $t \in T$. Then $x \sim (t, T')$ and $y \sim (t, T')$, and the path $x, (t', T), y$ connects x and y. □

Now we are able to prove that the Häggkvist & Hell graphs are connected in all but the minimal case.

1.2.3 Theorem. *For $n \geq 2r + 1$, the graph $HH_r(n)$ is connected.*

Proof. Let $x = (h_x, T_x)$ and $y = (h_y, T_y)$ be two vertices of $HH_r(n)$. Since $n \geq 2r + 1$, the graph $K_{n:r}$ is connected by Lemma 1.2.1, and so there is some path $T_x = T_0, T_1, \ldots, T_k = T_y$ in $K_{n:r}$. Now let $h_{i,1} \in T_{i-1}$ and $h_{i,2} \in T_{i+1}$ for all appropriate i. We see that $(h_{i,2}, T_i) \sim (h_{i+1,1}, T_{i+1})$ for all i and by Lemma 1.2.2 $(h_{i,1}, T_i)$ and $(h_{i,2}, T_i)$ are joined by a path for all i, therefore x and y are joined by a path. □

Now that we know when $HH_r(n)$ is connected, we know when it is interesting to talk about its diameter, which is defined to be infinite for disconnected graphs. This leads us to our next section.

1.3 Diameter

In order to define the diameter of a graph we must first define the distance between two vertices of a graph. For a graph X, the *distance* between two vertices x and y is denoted by $\text{dist}_X(x, y)$, or sometimes simply $\text{dist}(x, y)$, and is equal to the length of the shortest path from x to y in X. The *diameter* of a connected graph X is denoted by $\text{diam}(X)$ and is equal to

$$\max_{x,y \in V(X)} \text{dist}_X(x, y).$$

In this section we assume that $n \geq 2r+1$ since otherwise $HH_r(n)$ is not connected. The first thing we do is to prove a lower bound for the diameter of $HH_r(n)$ by applying Lemma 1.2.1:

1.3.1 Lemma. *For $n \geq 2r+1$, the diameter of $HH_r(n)$ is at least* $\left\lceil \frac{r-1}{n-2r} \right\rceil + 1$.

Proof. Let $T_x, T_y \in V(K_{n:r})$ be such that $\text{dist}(T_x, T_y) = \text{diam}(K_{n:r})$. Now let P be a shortest path from $(1, T_x)$ to $(1, T_y)$ in $HH_r(n)$. Since the tails of consecutive vertices of P must be disjoint, they represent a walk from T_x to T_y in $K_{n:r}$ of length equal to that of P. This implies that

$$\text{diam}(K_{n:r}) = \text{dist}_{K_{n:r}}(T_x, T_y) \leq \text{dist}_{HH_r(n)}((1, T_x), (1, T_y)) \leq \text{diam}(HH_r(n)).$$

Since $\text{diam}(K_{n:r}) = \left\lceil \frac{r-1}{n-2r} \right\rceil + 1$ by Lemma 1.2.1, the result is proven. □

1.3.2 Corollary. *The diameter of $HH_r(n)$ can be arbitrarily large.*

Proof. To achieve a diameter of at least M, take $r = M$ and $n = 2r+1$, then $\left\lceil \frac{r-1}{n-2r} \right\rceil + 1 = r = M$. □

Note that $\left\lceil \frac{r-1}{n-2r} \right\rceil + 1$ can be as small as 2 for $r \geq 2$, suggesting that perhaps $HH_r(n)$ may have diameter as low as 2 for some values of r and n. However, it turns out that this is not the case; $\text{diam}(HH_r(n))$ is never this small for any values of r and n.

1.3.3 Lemma. $\text{diam}(HH_r(n)) \geq 4$.

Proof. Consider the vertices $x = (1, T_x)$ and $y = (1, T_y)$ such that $T_x \cap T_y = \varnothing$. Note that two such vertices always exist for $n \geq 2r+1$. We will show that $\text{dist}(x, y) \geq 4$. Clearly x and y are not adjacent, since they have identical heads, thus they are not at distance one from each other. Now suppose that x and y share a common neighbor $z = (h_z, T_z)$. Then we have that $h_z \in T_x$ and $h_z \in T_y$, which is not possible since they are disjoint. Therefore x and y are at a distance of at least 3 from each other. Suppose that $\text{dist}(x, y) = 3$. Then there exists two vertices $z_1 = (h_1, T_1)$ and $z_2 = (h_2, T_2)$ such that $P = x, z_1, z_2, y$ is a path. However, we see that this implies that $1 \in T_1$ and $1 \in T_2$, and therefore $T_1 \cap T_2 \neq \varnothing$ and so z_1 and z_2 are not adjacent and P cannot be a path. Therefore $\text{dist}(x, y) \geq 4$. □

Now that we have this constant lower bound we would like to find out if/when it is ever achieved. Our next lemma sheds some light on just that.

1.3.4 Lemma. *For $n \geq \frac{5}{2}r$ the diameter of $HH_r(n)$ is four.*

Proof. Suppose that $n \geq \frac{5}{2}r$. Observe that this implies $n \geq \left\lceil \frac{5}{2}r \right\rceil$ since n is an integer. We must show that $\text{dist}(x, y) \leq 4$ for all vertices $x, y \in V(HH_r(n))$. Let $x = (h_x, T_x)$ and $y = (h_y, T_y)$. We have four main cases:

1. $h_x = h_y$;
2. $h_x \in T_y$ and $h_y \notin T_x$;
3. $h_x \in T_y$ and $h_y \in T_x$;
4. $h_x \notin T_y$, $h_y \notin T_x$, and $h_x \neq h_y$.

Let $C = T_x \cap T_y$ and $s = |C|$. Note that we must consider $s = 0$ to $s = r - 1$ for each of the above cases in order to prove our claim. If $s = r$, then either $x = y$ or $\text{dist}(x, y) = 2$ by Lemma 1.2.2, so we need not worry about these cases. We will prove the claim for the first case and then assure the reader that the other cases are similar.

Since $h_x = h_y$ in this case, we will refer to both as simply h. Let

$$D = [n] \setminus (T_x \cup T_y \cup \{h\}) = \{d_1, \ldots, d_{n-2r+s-1}\},$$

and $\ell = \left\lfloor \frac{r-s}{2} \right\rfloor$. Note that

$$T_x = \{x_1, \ldots, x_{r-s}\} \cup C \text{ and } T_y = \{y_1, \ldots, y_{r-s}\} \cup C$$

where $x_i \neq y_j$ for any i, j, and $r - s \geq 1$. Consider the vertices

$$z_x = (x_{r-s}, \{h, y_1, \ldots, y_\ell, d_1, \ldots, d_{r-1-\ell}\})$$

$$z_y = (y_{r-s}, \{h, x_1, \ldots, x_\ell, d_1, \ldots, d_{r-1-\ell}\})$$

$$w = (h, \{x_{r-s}, y_{r-s}, x_{\ell+1}, \ldots, x_{r-s-1}, y_{\ell+1}, \ldots, y_{r-s-1}\} \cup C)$$

Note that the tail of w has exactly $2r - s - 2\ell = 2r - s - 2\left\lfloor \frac{r-s}{2} \right\rfloor$ elements, which equals r if $r - s$ is even and equals $r + 1$ if $r - s$ is odd. But in the latter case we can just remove one of the elements from the tail that is not x_{r-s} or y_{r-s}. Now it is straightforward to see that $P = x, z_x, w, z_y, y$ is a path from x to y as long as all of the indices are valid. Upon investigation one can see that the only thing we need to check is that $d_{r-1-\ell}$ exists, i.e. that $n - 2r + s - 1 \geq r - 1 - \ell$. However, this is equivalent to

$$n \geq 3r - s - \ell \geq 3r - s - \frac{r-s}{2} = \frac{5}{2}r - \frac{s}{2}.$$

Therefore, if $n \geq \frac{5}{2}r$, then $\text{dist}(x, y) \leq 4$. The other three cases are quite similar, so we will spare you the tedium. □

Now that we know a range of values of n for which the minimum diameter is achieved, we would like to show that for values of n outside of this range, the diameter of $HH_r(n)$ is strictly greater than four. To do this we must find two vertices of $HH_r(n)$ that are always at a distance greater than four for values of n outside of the range given in Lemma 1.3.4. The next lemma shows that we are able to do just that.

1.3.5 Lemma. *For $n < \frac{5}{2}r$, the diameter of $HH_r(n)$ is strictly greater than four.*

Proof. Consider the vertices $x = (h, T_x)$ and $y = (h, T_y)$ where $T_x \cap T_y = \varnothing$. Note such a pair of vertices exists for $n \geq 2r + 1$. Let

$$T_x = \{x_1, \ldots, x_r\},\ T_y = \{y_1, \ldots, y_r\}, \text{ and } [n] \setminus (T_x \cup T_y) = D = \{d_1, \ldots, d_{k-1}\}$$

where $k = n - 2r$. From the proof of Lemma 1.3.3 we see that $\operatorname{dist}(x, y) \geq 4$, so we only need to show that there is no path of length four between x and y. Suppose that $P = x, z_x, w, z_y, y$ is a path. We will show that we need at least $\frac{5}{2}r$ elements of $[n]$ for this path to exist. Immediately we see that $h \in T_{z_x}$ and $h \in T_{z_y}$, and WLOG we can say that $z_x = (x_r, T_{z_x})$ and $z_y = (y_r, T_{z_y})$.

We have two options for the head of w, either it is h, or it is some element of D. As it turns out, this does not make a difference, but for now we will assume that it is h. At the end of the proof we will show why the other case works out to be the same. Suppose T_{z_x} and T_{z_y} contain i and j elements from D respectively. WLOG $i \leq j$.

Suppose that $d \in T_{z_x} \cap D$ and $d \notin T_{z_y}$. Since $i \leq j$, there must exist $d' \in T_{z_y} \cap D$ such that $d' \notin T_{z_x}$. Also, $d, d' \notin T_w$, but then we could simply replace the d in T_{z_x} with d', and this will still be a path from x to y and it will use fewer elements from $[n]$, so we may assume that $(T_{z_x} \cap D) \subseteq T_{z_y}$. The other $r - i - 1$ elements of T_{z_x}, and $r - j - 1$ elements of T_{z_y} come from $T_y \setminus y_r$ and $T_x \setminus x_r$ respectively.

So far, we have used $2r + 1 + j$ elements of $[n]$ in the vertices x, y, z_x, and z_y. Now we are left with the elements to be used in the tail of w. We know that $x_r, y_r \in T_w$, since these are the heads of z_x and z_y respectively. We are also able to use any of the other elements of $T_x \cup T_y$ not already used in T_{z_x} or T_{z_y}, of which there are exactly

$$2r - 2 - (r - i - 1) - (r - j - 1) = i + j.$$

This leaves $r - 2 - i - j$ elements left in the tail of w, and these must come from $D \setminus T_{z_y}$. Thus we use a total of

$$(2r + 1 + j) + (r - 2 - i - j) = 3r - 1 - i$$

elements of $[n]$. However, this does not take into account the possibility that we were able to fill the tail of w without using any elements of D, i.e. when $r - 2 - i - j \leq 0$. In this case $3r - 1 - i \leq 2r + 1 + j$, but we still use $2r + 1 + j$ elements of $[n]$ in our path P. In order to take this into account we must take the maximum of these two values. So the number of elements of $[n]$ that we use in the path P is

$$\max\{3r - 1 - i, 2r + 1 + j\}.$$

It is easy to see that letting $i = j$ can only reduce this maximum, and so we need to find the value of i for which

$$\max\{3r - 1 - i, 2r + 1 + i\}$$

is minimized. This will be minimized when $3r - 1 - i = 2r + 1 + i \Leftrightarrow i = \frac{1}{2}r - 1$. Note that i must be an integer, but this can only increase the lower bound we get on n, and so we can ignore this. Plugging in this value of i we see that we must use at least $\frac{5}{2}r$ elements of $[n]$ for the path P, thus proving the result. We see now that having an element of D as the head of w would have simply forced us to use an element of $T_{z_x} \cap D$, which only would have precluded us from having $i = 0$, and would not have reduced the number of elements of $[n]$ that we needed for the path. □

This result along with Lemma 1.3.4 immediately gives us the following corollary:

1.3.6 Corollary. *The diameter of $HH_r(n)$ is four if and only if $n \geq \frac{5}{2}r$.*

From the above result we see that the diameter of $HH_r(n)$ must be at least 5 for $n < \frac{5}{2}r$, but we also know from Lemma 1.3.1 that the diameter must be at least $\left\lceil \frac{r-1}{n-2r} \right\rceil$. Given these two lower bounds on the diameter, the best we can hope to show is that for $2r + 1 \leq n < \frac{5}{2}r$ the diameter of $HH_r(n)$ is the maximum of these two bounds. We are in fact able to do just this, but first we need a tool from the proof of another result from [6], which we will reproduce below only changing the notation in order to fit with ours.

1.3.7 Lemma (Vera & Pabon)**.** *Let $X, Y \in [n]^{(r)}$ be two different vertices in the Kneser graph $K_{n:r}$ with $2r + 1 \leq n \leq 3r - 2$, such that $|X \cap Y| = s$. Then*

$$\mathrm{dist}(X, Y) = \min\left\{2\left\lceil \frac{r-s}{n-2r} \right\rceil, 2\left\lceil \frac{s}{n-2r} \right\rceil + 1\right\}.$$

We only give the proof that this is an upper bound on $\mathrm{dist}(X, Y)$, because it is this portion that we will use for our proof of the diameter of the Häggkvist & Hell graphs.

Proof. Let $k = n - 2r$, so that $1 \leq k < n - 1$. Also, let $C = X \cap Y$, $s = |C|$, and $D = [n] \setminus (X \cup Y)$. Thus $|D| = s + k$. Assume that $X = \{a_1, \ldots, a_{r-s}\} \cup C$, and $Y = \{b_1, \ldots, b_{r-s}\} \cup C$. Let $\ell = 2\lceil (r - s)/k \rceil$. Consider the path $X = T_0, T_1, \ldots, T_\ell = Y$ between X and Y, where for $i < (r - s)/k$,

$$T_{2i-1} = \{a_1, \ldots, a_{(i-1)k}, b_{ik+1}, \ldots, b_{r-s}\} \cup D,$$
$$T_{2i} = \{b_1, \ldots, b_{ik}, a_{ik+1}, \ldots, a_{r-s}\} \cup C,$$
and
$$T_{\ell-1} = \{a_1, \ldots, a_{r-s-k}\} \cup D.$$

Also, let $D' \subseteq D$ with $|D'| = s$. Consider the vertex $X' = (Y \setminus C) \cup D'$. Note that $X \cap X' = \varnothing$, and $s' = |X' \cap Y| = r - s$. Therefore, by the previous construction, there is a path between X and Y with length equal to $2\lceil (r - s')/k \rceil = 2\lceil s/k \rceil$. Thus, there is a path between X and Y with length equal to $2\lceil s/k \rceil + 1$. So,

$$\mathrm{dist}(X, Y) \leq \min\{2\lceil (r - s)/k \rceil, 2\lceil s/k \rceil + 1\}.$$ □

We are now able to prove the following:

1.3.8 Theorem. *For $n \geq \frac{5}{2}r$ the diameter of $HH_r(n)$ is four. For $2r+1 \leq n < \frac{5}{2}r$, the diameter of $HH_r(n)$ is equal to* $\max\left\{5, \left\lceil \frac{r-1}{n-2r} \right\rceil + 1\right\}$.

Proof. The first statement has already been proven as Corollary 1.3.6. Also, Lemma 1.3.5 and Lemma 1.3.1 give the lower bound direction of the second statement. Thus we only have to show that we are able to achieve this bound for $n < \frac{5}{2}r$. We do this by showing that for any two vertices (h_x, T_x) and (h_y, T_y) in $HH_r(n)$, there is either a path between them of the same length as the shortest path between T_x and T_y in $K_{n:r}$, or there is a path between them of length at most 5. As in Lemma 1.3.4, we have four main cases:

1. $h_x = h_y$;
2. $h_x \in T_y$ and $h_y \notin T_x$;
3. $h_x \in T_y$ and $h_y \in T_x$;
4. $h_x \notin T_y$, $h_y \notin T_x$, and $h_x \neq h_y$.

We will use the same notation as in Lemma 1.3.7, so $k = n - 2r$, $C = T_x \cap T_y$, $s = |C|$, $D = [n] \setminus (T_x \cup T_y)$, and $|D| = s + k$. Note that Lemma 1.2.2 takes care of the cases in which the vertices have the same tail. So we can assume that $s \leq r - 1$. We can also immediately take care of the cases with disjoint tails, as follows. Suppose that $T_x \cap T_y = \varnothing$. Then we have the following four cases:

- If $h_x = h = h_y$, then let $t_x, t'_x \in T_x$ and $t_y, t'_y \in T_y$. Then there is a path of length five between x and y given by

$$(h, T_x), (t_x, \{h\} \cup (T_y \setminus t'_y)), (t_y, T_x), (t_x, T_y), (t_y, \{h\} \cup (T_x \setminus t'_x)), (h, T_y).$$

- If $h_x \in T_y$ and $h_y \notin T_x$, then let $t_x, t'_x \in T_x$ and $t_y \in T_y$. Then a path of length three from x to y is given by

$$(h_x, T_x), (t_x, T_y), (t_y, \{h_y\} \cup (T_x \setminus t'_x)), (h_y, T_y).$$

- If $h_x \in T_y$ and $h_y \in T_x$, then x and y are simply neighbors.

- If $h_x \notin T_y$, $h_y \notin T_x$, and $h_x \neq h_y$, then let $t_x, t'_x \in T_x$ and $t_y, t'_y \in T_y$. Then a path of length three between x and y is given by

$$(h_x, T_x), (t_x, \{h_x\} \cup (T_y \setminus t'_y)), (t_y, \{h_y\} \cup (T_x \setminus t'_x)), (h_y, T_y).$$

Thus if the tails of two vertices are disjoint, then they are at a distance of at most five. Now we consider the cases where $r - k \leq s \leq r - 1$. In this case we have that $|D| = s + k \geq r$.

1. If $h_x = h = h_y$, then let $D' \subseteq D$ be such that $h \in D'$ and $|D'| = r$, and let $t \in T_x \cap T_y$. Then a path of length two between x and y is given by

$$(h, T_x), (t, D'), (h, T_y).$$

2. If $h_x \in T_y$ and $h_y \notin T_x$, then let $D' \subseteq D$ be such that $h_y \in D'$ and $|D'| = r$, and let $d \in D' \setminus h_y$ and $t \in T_x \cap T_y$. Then there is a path of length four between x and y given by

$$(h_x, T_x), (t, \{h_x\} \cup (D \setminus h_y)), (d, T_x), (t, D'), (h_y, T_y).$$

3. If $h_x \in T_y$ and $h_y \in T_x$, let $D' \subseteq D$ be such that $|D'| = r - 1$, and let $d \in D'$, $e \in D \setminus D'$, and $t \in T_x \cap T_y$. Then a path of length four between x and y is given by

$$(h_x, T_x), (t, \{h_x\} \cup D'), (d, \{e\} \cup (T_x \setminus h_y)), (t, \{h_y\} \cup D'), (h_y, T_y).$$

4. If $h_x \notin T_y$, $h_y \notin T_x$, and $h_x \neq h_y$, then let $D' \subseteq D$ be such that $h_x, h_y \in D'$ and $|D'| = r$, and let $t \in T_x \cap T_y$. Then a path of length two between x and y is given by

$$(h, T_x), (t, D'), (h, T_y).$$

So we have taken care of all cases in which $s \geq r - k$. For the remaining cases, we will be using the two paths between T_x and T_y given in the proof of Lemma 1.3.7. We will use them to construct two paths between x and y of lengths equal to those of the paths in the Kneser graph $K_{n:r}$. In order to do this we treat each vertex in the path of the Kneser graph as a tail of a vertex in $HH_r(n)$, and then we show that we are able to pick heads for each vertex in the interior of the path such that the adjacencies are preserved. After this, all that remains to show is that in each case we are able to choose the second and second to last vertices in the paths in the Kneser graph such that they contain the heads of x and y respectively.

For vertices in the interior of the paths this is trivial. Since $n < \frac{5}{2}r$, we have that if T_1, T_2, T_3 are three consecutive vertices in the path in $K_{n:r}$, then $T_1, T_3 \subseteq [n] \setminus T_2$ which has size less than $\frac{3}{2}r$, thus there must exist some element $t \in T_1 \cap T_3$, and we can pick this as the head of T_2 in $HH_r(n)$.

So all we need to show is that we are able to choose appropriate second and second to last vertices in the paths in the Kneser graph. We have to deal with each path separately:

Recall that $T_x = \{x_1, \dots, x_{r-s}\} \cup C$ and $T_y = \{y_1, \dots, y_{r-s}\} \cup C$. For the first path given in the proof of Lemma 1.3.7, we have that $T_1 = \{y_{k+1}, \dots, y_{r-s}\} \cup D$ and $T_{\ell-1} = \{x_1, \dots, x_{r-s-k}\} \cup D$. Note that since $1 \leq s \leq r - k - 1$, we have that $|D| = s + k \leq r - 1$. Now we go through the cases:

Case 1: $h_x = h_y$. In this case $h_x, h_y \in D \subseteq T_1, T_{\ell-1}$ and so we are done.

Case 2: $h_x \in T_y$ and $h_y \notin T_x$. Here $h_y \in D$, and if we let $y_{k+1} = h_x$, then $h_x \in T_1$ and $h_y \in T_{\ell-1}$.

Case 3: $h_x \in T_y$ and $h_y \in T_x$. Here if we let $y_{k+1} = h_x$ and $x_1 = h_y$, then $h_x \in T_1$ and $h_y \in T_{\ell-1}$.

Case 4: $h_x \notin T_y$, $h_y \notin T_x$, and $h_x \neq h_y$. Here we have that $h_x, h_y \in D \subseteq T_1, T_{\ell-1}$, and so we are done.

So we have shown that we can construct a path from x to y in $HH_r(n)$ with the same length as the first path between T_x and T_y given in the proof of Lemma 1.3.7. Now we must do the same for the second path.

The second vertex in the second path is $T'_x = (T_y \setminus C) \cup D'$ where $D' \subseteq D$ such that $|D'| = s \geq 1$. Then, using the same construction as for the first path, let $C' = T'_x \cap T_y = T_y \setminus C$, and $E = [n] \setminus (T'_x \cup T_y) = [n] \setminus (T_y \cup D')$, and $|E| = r + k - s \geq 2k + 1$. So $T'_x = \{d_1, \ldots, d_s\} \cup C'$ where $\{d_1, \ldots, d_s\} = D'$. Then the second to last vertex in the path is $T'_{\ell'-1} = \{d_1, \ldots, d_{s-k}\} \cup E$. Now we go through the cases:

Case 1: $h_x = h_y$. In this case $h_x, h_y \in D$, so if we let $h_x \in D'$ and $h_y \notin D'$ (possible since $|D \setminus D'| = k \geq 1$), then $h_x \in T'_x$ and $h_y \in E \subseteq T'_{\ell'-1}$ and so we are done.

Case 2: $h_x \in T_y$ and $h_y \notin T_x$. Here $h_x \in T_y \setminus C = C' \subseteq T'_x$, and we let $h_y \in D \setminus D' \subseteq E \subseteq T'_{\ell'-1}$. Thus we are done.

Case 3: $h_x \in T_y$ and $h_y \in T_x$. Here $h_x \in T_y \setminus C = C' \subseteq T'_x$, and $h_y \notin T_y$ and $h_y \notin D \supseteq D'$ and thus $h_y \in E \subseteq T'_{\ell'-1}$.

Case 4: $h_x \notin T_y$, $h_y \notin T_x$, and $h_x \neq h_y$. Here we have that $h_x, h_y \in D$ and we let $h_x \in D' \subseteq T'_x$ and $h_y \notin D'$. Thus $h_y \in E \subseteq T'_{\ell'-1}$.

Since one of these two paths must be a shortest path between T_x and T_y in $K_{n:r}$, we have now shown that for $n < \frac{5}{2}r$, any two vertices $x = (h_x, T_x)$ and $y = (h_y, T_y)$ of $HH_r(n)$ are either at a distance of at most five, or

$$\mathrm{dist}_{HH_r(n)}(x, y) \leq \mathrm{dist}_{K_{n:r}}(T_x, T_y),$$

thus

$$\mathrm{diam}(HH_r(n)) \leq \max\left\{5, \left\lceil \frac{r-1}{n-2r} \right\rceil + 1\right\}.$$

This completes the proof. □

This theorem resolves the question of the diameter of the Häggkvist & Hell graphs.

1.4 Girth and Odd Girth

The *girth* of a graph is the length of its shortest cycle, while the *odd girth* is the length of its shortest odd cycle. In many cases these parameters can give us valuable information about a graph. Such is the case of girth for planar graphs which allows one to obtain a bound on the number of edges. Whereas odd girth gives one restrictions on homomorphisms between graphs. For Häggkvist & Hell graphs, odd girth is the more interesting parameter as the girth remains constant for all values of n and r. Again, both of these values are known for Kneser graphs. We settle the question of girth immediately.

1.4.1 Theorem. *The girth of $HH_r(n)$ is four.*

Proof. Since $n \geq 2r$, there exists two disjoint r-subsets T_1 and T_2 of $[n]$. Let $h_1, h_3 \in T_2$ and $h_1 \neq h_3$, and let $h_2, h_4 \in T_1$ and $h_2 \neq h_4$. Then

$$C = (h_1, T_1), (h_2, T_2), (h_3, T_1), (h_4, T_2)$$

is a 4-cycle. As seen at the beginning of this chapter, $HH_r(n)$ is triangle-free. □

This result is actually even more trivial than finding the girth of $K_{n:r}$, though this seems to be the only case where this happens. The girth of $K_{n:r}$ is not much more difficult to work out though, and the reader is invited to do this. We have provided the actual values below to help the reader along:

The girth of $K_{n:r}$ for $n \geq 2r+1$ is three for $n \geq 3r$, four for $2r+2 \leq n \leq 3r-1$, and six for $2r+1 = n \leq 3r-1$ except for when $r = 2$ which gives the Petersen graph having girth equal to five. The exception for when $n = 2r+1$ and $r = 2$ arises because this is the only case in which the odd girth of $K_{2r+1:r}$ is less than six. Also, it is quite typical of the Petersen graph to take exception.

The odd girth of $HH_r(n)$ is closely connected to the odd girth of $K_{n:r}$ which is a result of Poljak and Tuza [5]:

1.4.2 Theorem (Poljak & Tuza)**.** *The odd girth of the Kneser graph $K_{n:r}$ is $2\left\lceil \frac{r}{n-2r} \right\rceil + 1$ for $n \geq 2r+1$.*

We use this to first prove a lower bound on the odd girth of the Häggkvist & Hell graphs. There are two methods of proving this. One way is to simply note that $HH_r(n)$ admits a homomorphism to $K_{n:r}$, and therefore must have odd girth at least as great as it. However, we will not cover this material until Chapter 2. Below we give a direct proof.

1.4.3 Lemma. *For $n \geq 2r+1$, the odd girth of $HH_r(n)$ is at least $2\left\lceil \frac{r}{n-2r} \right\rceil + 1$.*

Proof. For $n \geq \frac{5}{2}r$, we have that $2\left\lceil \frac{r}{n-2r} \right\rceil + 1 \leq 5$, but since $HH_r(n)$ is triangle-free, its odd girth must always be at least 5, which takes care of that case. So we can assume that $n \leq \frac{5}{2}r$. Suppose that C is a shortest odd cycle in $HH_r(n)$. If no tail is repeated in C, then the tails correspond to a cycle in $K_{n:r}$ and the result is proven by Theorem 1.4.2.

Otherwise, suppose that $x = (h_x, T)$ and $y = (h_y, T)$ are two vertices in C with the same tail. Let P be the path from x to y in C with odd length. Now let T_x be the tail of the unique neighbor of x in P, and let T_y be defined similarly. Since $n \leq \frac{5}{2}r$, and $T_x, T_y \subseteq [n] \setminus T$, there must be an element h of $[n]$ such that $h \in T_x \cap T_y$. Let $z = (h, T)$, and let P' be the path P with the ends, x and y, removed. The cycle $C' = z, P', z$ is a shorter odd cycle than C, which is a contradiction. □

Now we are able to give the exact value of the odd girth of the Häggkvist & Hell graphs for all values of n and r.

1.4.4 Theorem. *For $n \geq 2r+1$ the odd girth of $HH_r(n)$ is*

$$\max\left\{5, 2\left\lceil \frac{r}{n-2r} \right\rceil + 1\right\}.$$

Proof. Lemma 1.4.3 shows that the stated number is a lower bound on the odd girth of $HH_r(n)$, so we only need to show that it can be achieved. First we consider the case where $n < 3r$. In this case $2\left\lceil \frac{r}{n-2r} \right\rceil + 1 \geq 5$, so we will show that we can obtain an odd cycle with this length.

Consider a shortest odd cycle C in $K_{n:r}$: this has length $2\left\lceil \frac{r}{n-2r} \right\rceil + 1$. We will view the vertices of C as tails and show that we can pick a head for each so that the adjacencies in C are preserved. Consider a vertex T in C, with neighbors T_1 and T_2 in C. Since $n < 3r$, and $T_1, T_2 \subseteq [n] \setminus T$, we have that T_1 and T_2 cannot be disjoint. So let $h \in T_1 \cap T_2$ and let this be the head of T. If we do this for each tail then we will have a cycle C' in $HH_r(n)$ with the same length as C.

Now we still need to consider the case where $n \geq 3r$. However, for $\frac{5}{2}r \leq n \leq 3r - 1$, we have that $2\left\lceil \frac{r}{n-2r} \right\rceil + 1 = 5$, and so for $n = 3r - 1$, the odd girth of $HH_r(n)$ is five, which is as small as possible. Now for $n \geq 3r$, $HH_r(n)$ contains $HH_r(3r-1)$ as a subgraph, which means that it has odd girth at most five. But then it must have odd girth exactly five, and $2\left\lceil \frac{r}{n-2r} \right\rceil + 1 = 3$ for $n \geq 3r$, which proves the result. □

Note that we actually need to be somewhat careful in the above proof when saying that $HH_r(3r-1)$ has odd girth five, since we need that $3r - 1 \geq \frac{5}{2}r$. But this is in fact always true for $r \geq 2$.

1.5 Subgraphs

We conclude this chapter with a discussion of subgraphs of Kneser graphs that are isomorphic to subgraphs of Häggkvist & Hell graphs. In both the section on diameter and the section on girth and odd girth, we used subgraphs of $K_{n:r}$ to help us find isomorphic copies of these subgraphs in $HH_r(n)$. One such example was in Theorem 1.4.4 where we exploited the fact that two r-subsets of a set with size less than $2r$ must have an element in common. We now extend this idea to bounded degree subgraphs of Kneser graphs.

1.5.1 Theorem. *For any subgraph G of $K_{n:r}$ with maximum degree strictly less than $\frac{n-r}{n-2r}$, there is a subgraph of $HH_r(n)$ isomorphic to G.*

Proof. Let G be a subgraph of $K_{n:r}$, let Δ be the maximum degree of G, and suppose that $\Delta < \frac{n-r}{n-2r}$. Now consider a vertex X of G. X is an r-subset of $[n]$ whose neighbors in G are also r-subsets of $[n]$, and they are disjoint from X. As in previous cases we will show that we can pick a head h_X for X such that h_X is an element of every neighbor of X in G. Doing this for all vertices in G completes the proof. So we must show that the neighbors of X all share a common element, then we can pick that element and we are done. Let k be the degree of X in G. Now since the neighbors of X are all disjoint from X, they must all draw their elements from the same set of size $n - r$. Let us call this set S. Now suppose that no element of S is common to all of the neighbors of X. Then each element of S is in at most $k-1$ neighbors of X. Viewing this as a r-uniform hypergraph with S as the ground set and the k neighbors of X as the hyperedges, we know that the degree sum of the vertices (at most $(k-1)(n-r)$) is equal to the degree sum of the hyperedges (kr), thus we have the following string of inequalities:

$$\begin{aligned}(k-1)(n-r) &\geq kr\\ kn-kr-n+r &\geq kr\\ kn-2kr &\geq n-r\\ k(n-2r) &\geq n-r\end{aligned}$$

Therefore,

$$k \geq \frac{n-r}{n-2r} > \Delta$$

which is a contradiction. Therefore the neighbors of X must share a common element, and so we can pick one such element for the head, h_X, of X in $HH_r(n)$. We can do this for every vertex in G and thus the proof is complete. □

Note that in the case of the odd graphs, when $n = 2r+1$, the condition we get is $\Delta < r+1$, i.e. $\Delta \leq r$. However, in this case the valency of $K_{2r+1:r}$ is $r+1$, and since Kneser graphs are vertex transitive, they either have a perfect matching or a matching missing exactly one vertex. Therefore, $HH_r(2r+1)$ contains a copy of $K_{2r+1:r}$ minus a perfect matching whenever $\binom{2r+1}{r}$ is even, and contains a copy of $K_{2r+1:r}$ minus a matching and one edge incident to the single vertex the matching misses whenever $\binom{2r+1}{r}$ is odd.

Chapter 2

Homomorphisms

The study of homomorphisms is a rich area of graph theory. Graph homomorphisms are related to a wide range of other important topics in graph theory, such as colorings, products, and cores. We will give a brief introduction to graph homomorphisms before investigating some important homomorphisms of Häggkvist & Hell graphs. Specifically, we give homomorphisms from $HH_r(n)$ to K_n and $K_{n:r}$. All of the material in this chapter on homomorphisms in general (Sections 2.1-2.4) comes from Godsil and Royle's *Algebraic Graph Theory* [2].

2.1 Definitions

For two graphs, X and Y, a map $f : V(X) \to V(Y)$ is a *homomorphism* from X to Y if $f(x) \sim f(y)$ in Y whenever x and y are adjacent in X. Note that in the case of graphs without loops, this definition implies that if $f(x) = f(y)$, then x and y are not adjacent in X. Thus the preimage $f^{-1}(y)$ of a vertex in Y is necessarily an independent set in X. We often refer to the preimages $f^{-1}(y)$ as the *fibres* of f. If there exists a homomorphism from X to Y, then we write $X \to Y$, or say that X admits a homomorphism into Y.

There are many different types of graph homomorphisms, and we will take some time here to discuss some of the more important classes of them.

An *isomorphism* $\varphi : V(X) \to V(Y)$ is a homomorphism that is a bijection and preserves non-adjacency as well as adjacency. In other words, $\varphi(x) \sim \varphi(y)$ in Y if and only if $x \sim y$ in X. An *automorphism* is an isomorphism from a graph to itself.

An *endomorphism* is a homomorphism from a graph X into itself. A homomorphism that is both an endomorphism and an isomorphism is an automorphism.

A special type of endomorphism, known as a *retraction*, is defined as a homomorphism f from a graph X to a subgraph Y of X such that the restriction $f|Y$ of f to $V(Y)$ is the identity map. We say that a subgraph Y of X is a *retract* of X if there exists a retraction from X to Y.

2.2 Basic Properties

Though homomorphisms are not as well-behaved as isomorphisms or automorphisms, they still share some of the same properties.

2.2.1 Lemma. *If X, Y, and Z are graphs and $f : X \to Y$ and $g : Y \to Z$ are homomorphisms, then the composition $g \circ f$ is a homomorphism from X to Z.*

Proof. Consider two adjacent vertices x and y in X. We have that $f(x) \sim f(y)$ in Y, and then $g(f(x)) \sim g(f(y))$ in Z. □

Now we turn our attention to the endomorphisms of a graph X. We claim the set of endomorphisms of X form a *monoid* (a set that has an associative binary multiplication defined on it and an identity element) which we call the *endomorphism monoid* of X, and we denote $\mathrm{End}(X)$. It is easy to see that $\mathrm{End}(X)$ is indeed a monoid, since the identity map on X is an endomorphism, and Lemma 2.2.1 shows that the composition of any two endomorphisms is again an endomorphism. We also note that $\mathrm{Aut}(X) \subseteq \mathrm{End}(X)$ where $\mathrm{Aut}(X)$ is the group of automorphisms of X.

2.3 Colorings

A *proper coloring* of a graph X is a mapping of $V(X)$ into some finite set such that no two adjacent vertices are mapped to the same element. We say that X is *n-colorable* if it can be properly colored with a set of n elements. The least n for which this is possible is known as the chromatic number of X and is denoted by $\chi(X)$. The set of vertices assigned to a particular color is referred to as a *color class* of the coloring, and is necessarily an independent set. An equivalent formulation of an n coloring of a graph X is a homomorphism from X to K_n. The following lemma establishes this equivalence.

2.3.1 Lemma. *A graph X is n-colorable if and only if there exists a homomorphism from X to K_n.*

Proof. Suppose that h is a homomorphism from X to K_n. From Section 2.1 we know that the n fibres of h are independent sets, therefore they form the color classes of an n-coloring of X. Conversely, if X can be properly colored with the colors $\{1, \dots, n\}$, then the map that takes each vertex to its color must be a homomorphism, since if two vertices are adjacent, they must be of different colors, and so their images are adjacent. □

This connection between colorings and homomorphisms allows us to prove the following useful lemma.

2.3.2 Lemma. *For two graphs, X and Y, if there exists a homomorpism $h : X \to Y$, then $\chi(X) \leq \chi(Y)$.*

Proof. Suppose there exists such a homomorphism $h : X \to Y$. Let $c = \chi(Y)$. Then there exists a homomorphism $g : Y \to K_c$. By the composition of homomorphisms, $g \circ h : X \to K_c$, and thus $\chi(X) \leq c = \chi(Y)$. □

This lemma can be used to obtain bounds on the chromatic numbers of graphs, or, using the contrapositive, to disprove the existence of a homomorphism from a graph X to a graph Y with strictly smaller chromatic number.

2.4 Another Restriction

We saw in the previous section that the chromatic number can be used to show that there is no homomorphism from one graph to another, a task that is generally difficult. There is another graph parameter that we have already encountered which also can be used for this purpose: odd girth.

2.4.1 Lemma. *The homomorphic image of an odd cycle must contain an odd cycle of no greater length.*

Proof. Let C be an odd cycle and f a homomorphism of C with image Y. Since the chromatic number of an odd cycle is three, by Lemma 2.3.2 the chromatic number of Y must be at least three. This means that it cannot be bipartite and thus must contain an odd cycle. But a homomorphism is a function, so the number of vertices in Y cannot be more than that of C, thus the odd cycle in its image must be of no greater length. □

Now we can prove the desired result concerning odd girth.

2.4.2 Lemma. *If the odd girth of X is strictly less than the odd girth of Y, then there does not exist any homomorphism from X to Y.*

Proof. Suppose that f is a homomorphism from X to Y. Consider a shortest odd cycle C in X. By Lemma 2.4.1 the image of C must contain an odd cycle of no greater length than C. This is a contradiction because Y has odd girth strictly greater than X. □

These types of theorems concerning homomorphisms are nice because they can be used to disprove the existence of a homomorphism, or to obtain bounds on the parameters in question.

2.5 Homomorphisms of $HH_r(n)$

Here we present two important homomorphisms of Häggkvist & Hell graphs that will be referred to throughout this thesis.

2.5.1 Theorem. *Let $f : V(HH_r(n)) \to V(K_n)$ be defined by $f(h,T) = h$. Then f is a homomorphism from the Häggkvist & Hell graph $HH_r(n)$ to the complete graph K_n.*

Proof. Suppose that $x = (h_x, T_x)$ and $y = (h_y, T_y)$ are adjacent vertices in $HH_r(n)$. Then $h_x \in T_y$ and $h_y \notin T_y$, therefore $h_x \neq h_y$, and thus $f(x) = h_x$ and $f(y) = h_y$ are adjacent in K_n. Therefore f is a homomorphism. □

This homomorphism is of course equivalent to an n-coloring of $HH_r(n)$, however in Chapter 5 we will see that we can do better than this.

2.5.2 Theorem. *Let $g : V(HH_r(n)) \to V(K_{n:r})$ be defined by $g(h, T) = T$. Then g is a homomorphism from the Häggkvist & Hell graph $HH_r(n)$ to the Kneser graph $K_{n:r}$.*

Proof. Suppose that $x = (h_x, T_x)$ and $y = (h_y, T_y)$ are adjacent vertices in $HH_r(n)$. Then $T_x \cap T_y = \varnothing$, and therefore $g(x) = T_x$ and $g(y) = T_y$ are adjacent in $K_{n:r}$. Therefore g is a homomorphism. □

These two homomorphisms will come up in various forms frequently throughout this work, so it is important to be comfortable with both of them. Also important to us are the fibres of these homomorphisms, so we take some time now to discuss them.

Consider the fibres of f, these are the sets of vertices in $HH_r(n)$ that all have the same head, thus the fibres of f all have size $\binom{n-1}{r}$. The fibres of g on the other hand are the sets of vertices in $HH_r(n)$ that all have the same tail, therefore g has fibres all of size $n - r$.

Though there are other homomorphisms of $HH_r(n)$, these two are the ones we most often employ in the proofs of our results.

Chapter 3

Equitable Partitions

Equitable partitions are a useful and powerful tool for finding some, or all, of the eigenvalues of a graph. We will describe two equitable partitions that are useful in obtaining information about $HH_r(n)$. The first partition that we give has three cells for any Häggkvist & Hell graph, which allows us to obtain closed form expressions for three distinct eigenvalues of $HH_r(n)$. The second partition that we give has $5r+2$ cells and thus it becomes increasingly difficult to compute its eigenvalues as r increases. Before we begin finding any particular partitions, we must introduce the theory behind equitable partitions in order to understand how we can use them. The background material for this chapter comes from Godsil and Royle's *Algebraic Graph Theory* [2]; please refer to this source for a more in-depth look at equitable partitions.

3.1 Theory

We begin with a few basic definitions:

Definition: Let X be a graph and π a partition of $V(X)$ with cells $C_1, C_2, \ldots, C_k$. We say that π is *equitable* if there is a constant c_{ij} such that for any vertex $u \in C_i$, u has exactly c_{ij} neighbors in C_j, for all $i, j \in \{1, 2, \ldots, k\}$.

One can see that this definition is equivalent to saying that each cell of π induces a regular subgraph of X, and the edges between any two different cells form a semiregular bipartite graph. The directed multigraph whose vertices are the cells of π with c_{ij} arcs going from C_i to C_j is referred to as the quotient of X over π. If we consider the adjacency matrix of this quotient, we see that $A(X/\pi)_{ij} = c_{ij}$. It is now useful to define the characeteristic matrix of a partition.

Definition: If π is a partition of the vertex set V with k cells, then we define its *characteristic matrix* P as the $|V| \times k$ matrix whose columns are the characteristic vectors of π.

From this definition we see that P^TP is the diagonal matrix with $(P^TP)_{ii} = |C_i|$, and thus it is invertible, since the cells of π are nonempty. This allows us to prove the following lemma.

3.1.1 Lemma. *Let π be an equitable partition of the graph X, with characteristic matrix P, and let $B = A(X/\pi)$. Then $AP = PB$ and $B = (P^TP)^{-1}P^TAP$, where A is the adjacency matrix of X.*

Proof. We proceed by showing that for any vertex u and any cell C_j, we have $(AP)_{uj} = (PB)_{uj}$. The left hand side of this equality is the number of neighbors of u in C_j, which is equal to c_{ij} where $u \in C_i$. Now the only 1 in row u of P is in the i^{th} column, since $u \in C_i$. So the right hand side must also be equal to c_{ij}. Therefore we have that $AP = PB$, and multiplying on the left by $(P^TP)^{-1}P^T$, we get that $B = (P^TP)^{-1}P^TAP$. □

The next lemma characterizes equitable partitions in terms of linear algebra.

3.1.2 Lemma. *Let X be a graph with adjacency matrix A and π a partition of $V(X)$ with characteristic matrix P. Then π is equitable if and only if the column space of P is A-invariant.*

Proof. The column space of P is A-invariant if and only if there exists a matrix B such that $AP = PB$. If π is equitable, then by the previous lemma we see that such a matrix B exists, namely $A(X/\pi)$. Conversely, if $AP = PB$ for some matrix B, then any vertex in C_i must have exactly B_{ij} neighbors in C_j, and thus π is equitable. □

Now if $AP = PB$, then $A^kP = PB^k$ for any nonnegative integer k, and thus $f(A)P = Pf(B)$ for any polynomial f. Since P has linearly independent columns, if $f(A) = 0$, then $f(A)P = Pf(B)$ implies that $f(B) = 0$. Therefore, if π is an equitable partition of a graph X, then the minimum polynomial of $A(X/\pi)$ divides the minimum polynomial of A, and thus every eigenvalue of $A(X/\pi)$ is an eigenvalue of A. This is the key result that we make use of in order to find some of the eigenvalues of $HH_r(n)$.

Before we move on to finding some eigenvalues, let me remark that the orbits of any group of automorphisms of a graph X are the cells of an equitable partition. For $HH_r(n)$, this implies that any subgroup of $Sym(n)$ induces an equitable partition. However, it is not known whether $\mathrm{Aut}(HH_r(n)) = Sym(n)$.

3.2 A 3-Cell Partition

The first partition we consider is the coarser of the two. We will first give the partition for $HH_3(n)$ only, for ease of comprehension. But this partition is easy to generalize and we will present the generalization subsequently. Let π_3 be the partition of $V(HH_3(n))$ such that for a vertex $u = (a, \{b, c, d\})$ of $HH_3(n)$, we say $u \in C_1$ iff $n \notin \{a, b, c, d\}$, $u \in C_2$ iff $n \in \{b, c, d\}$, and $u \in C_3$ iff $a = n$. Clearly, C_1 induces an $HH_3(n-1)$ subgraph, while C_2 and C_3 are both independent sets. In fact C_2 is the inverse image of a maximum independent set in $K_{n:3}$ over the homomorphism $(a, \{b, c, d\}) \to \{b, c, d\}$, and C_3 is the inverse image of a vertex (which is a maximum independent set) of K_n over the homomorphism $(a, \{b, c, d\}) \to a$.

Here is a diagram to aid in comprehension:

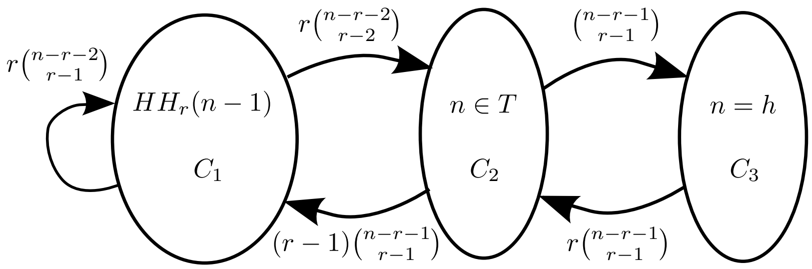

Figure 3.1: Diagram of 3-Cell Partition π_3.

Note that this is also the partition corresponding to the orbits of the subgroup of $Sym(n)$ that fixes only n, and therefore we already know it must be equitable, but we still need to compute the c_{ij}'s in order to find the eigenvalues. Since C_1 is an induced $HH_3(n-1)$ subgraph, c_{11} is simply the valency of $HH_3(n-1)$ which is $3\binom{n-5}{2}$. Now $c_{22} = c_{33} = 0$ since C_2 and C_3 are indpendent sets. Also note that $c_{13} = c_{31} = 0$ since there is not an n in the tail of any vertex in C_1 and n is the head of all the vertices in C_3.

Now consider c_{12}. For a vertex $u_1 = (a, \{b, c, d\}) \in C_1$, the neighbors of u_1 in C_2 are those vertices $(e, \{f, n, a\})$ where $e \in \{b, c, d\}$ and $f \in [n] \setminus \{a, b, c, d, n\}$. From this it is easy to see that u_1 has exactly $c_{12} = 3(n-5)$ neighbors in C_2. Now if $u_2 = (a, \{b, c, n\}) \in C_2$, then it has neighbors in C_1 of the form $(e, \{f, g, a\})$ where $e \in \{b, c\}$, and $f, g \in [n-1] \setminus \{a, b, c\}$ and $f \neq g$. This characterization of the neighbors of u_2 in C_1 makes it clear that $c_{21} = 2\binom{n-4}{2}$. Now the neighbors of u_2 in C_3 all have the form $(n, \{f, g, a\})$ where $f, g \in [n] \setminus \{a, b, c, n\}$ and $f \neq g$. From this we see that $c_{23} = \binom{n-4}{2}$. Since any vertex of C_3 has all of its neighbors in C_2, we know that $c_{32} = val(HH_3(n)) = 3\binom{n-4}{2}$.

Putting this all together we arrive at the following quotient matrix:

$$A(HH_3(n)/\pi_3) = \begin{pmatrix} 3\binom{n-5}{2} & 3(n-5) & 0 \\ 2\binom{n-4}{2} & 0 & \binom{n-4}{2} \\ 0 & 3\binom{n-4}{2} & 0 \end{pmatrix}$$

In fact, doing this for arbitrary tail size is not really that much more difficult. It is easy to work out that, analogously to the $r = 3$ case above,

$$A(HH_r(n)/\pi_3) = \begin{pmatrix} r\binom{n-r-2}{r-1} & r\binom{n-r-2}{r-2} & 0 \\ (r-1)\binom{n-r-1}{r-1} & 0 & \binom{n-r-1}{r-1} \\ 0 & r\binom{n-r-1}{r-1} & 0 \end{pmatrix}$$

$$= \binom{n-r-1}{r-1} \begin{pmatrix} r\frac{n-2r}{n-r-1} & r\frac{r-1}{n-r-1} & 0 \\ (r-1) & 0 & 1 \\ 0 & r & 0 \end{pmatrix}$$

The eigenvalues of this matrix are distinct and can be determined by elementary algebra. They are

$$val(HH_r(n)) = r\binom{n-r-1}{r-1}$$

and

$$\frac{1}{2}r\binom{n-r-2}{r-2}\left[-1 \pm \frac{1}{r-1}\sqrt{\frac{4n(n-3r-1)+r(r+3)^2}{r}}\right]$$

Which for $r = 3$, simplify to

$$3\binom{n-4}{2} \text{ and } \frac{3}{2}(n-5)\left(-1 \pm \frac{1}{3}\sqrt{3n^2 - 30n + 81}\right).$$

So now we have a closed form expression for three eigenvalues of any Häggkvist & Hell graph. In the next section we will show how to compute all of the eigenvalues for any $HH_r(n)$.

3.3 A $(5r+2)$-Cell Partition

Before we move on to our next partition of interest, we need Theorem 9.4.1 from [2]:

3.3.1 Theorem. *Let X be a vertex transitive graph, and let π be the partition of $V(X)$ into the orbits of some subgroup G of $\mathrm{Aut}(X)$. If π has a singleton cell $\{u\}$, then every eigenvalue of X is an eigenvalue of X/π.* □

In order to use the above theorem we need to find a group of automorphisms that will fix a vertex of our graph, and hopefully have as few orbits as possible. The only automorphisms of $HH_r(n)$ we know of are those from $Sym(n)$, so we will use these. If we want our group of automorphisms to fix the vertex $u =$

$(1, \{2, 3, \ldots, r+1\})$, then it must fix the number 1, and setwise fix $\{2, 3, \ldots, r+1\}$. So we simply take the largest subgroup of $Sym(n)$ that does exactly that. Clearly this group has $\{u\}$ as one of its orbits in its action on $V(HH_r(n))$ and thus this group gives us an equitable partition that satisfies the requirements of the theorem. We call this partition π_u.

We will see that the number of cells of this partition does not depend on n, which means that for a fixed r we can compute all of the eigenvalues of $HH_r(n)$ in terms of n. Unfortunately, the number of cells does increase as r increases, and in fact for $r \geq 4$ it is computationally challenging to find the closed form of the eigenvalues for $HH_r(n)$. However, for a particular value of n, it would not be so difficult.

We can divide the cells of this partition up into five varieties. The following is a description of each of these five varieties and the notation that we use for each:

$\mathcal{A}$: The vertices in cells of this variety all have 1 as their head. The different cells of this variety, and of every variety in fact, are distinguished from one another by the number of elements of $\{2, \ldots, r+1\}$ that their vertices have in their tails, in this case being any value from 0 to r. Therefore there are $r+1$ cells of this type.

$\mathcal{B}$: The vertices in these cells have the element 1 in their tail and their head is one of the elements of $\{2, \ldots, r+1\}$. Each of these vertices have $r-1$ elements other than 1 in their tail and there are $r-1$ elements in $\{2, \ldots, r+1\}$ that aren't their head and thus they can have between 0 and $r-1$ elements of $\{2, \ldots, r+1\}$ in their tail. Therefore there are r cells of this variety.

$\mathcal{C}$: The vertices in the cells of this variety all have the element 1 in their tails and they have neither 1 nor any element from $\{2, \ldots, r+1\}$ as their head. Since all of these vertices must have 1 in their tail, they can only have up to $r-1$ elements of $\{2, \ldots, r+1\}$ in their tails. Thus there are only r of this variety of cells.

$\mathcal{D}$: The vertices in cells of this variety all have an element of $\{2, \ldots, r+1\}$ as their head, and their tail does not contain the element 1. These vertices have between 0 and $r-1$ elements of $\{2, \ldots, r+1\}$ in their tails, since one element of that set is already their head. Thus there are r cells of this type.

$\mathcal{E}$: The vertices in the cells of this variety do not have 1 nor any element of $\{2, \ldots, r+1\}$ as their head, their heads come only from the set $\{r+2, \ldots, n\}$. These vertices also do not have 1 in their tails. Their tails can have from 0 to r elements in common with $\{2, \ldots, r+1\}$, thus there are $r+1$ of these cells.

We refer to the cell of the $\mathcal{A}$ variety which contains vertices with i elements of $\{2, \ldots, r+1\}$ in their tails by $\mathcal{A}_i$, and similarly for cells of the other varieties. We see that these five varieties of cells give rise to $5r+2$ cells in this partition of $HH_r(n)$. This number of cells is unfortunate, since it means that for a relatively small case, say $r = 4$, we must compute the eigenvalues of a 22×22 matrix with variable entries if we want to know the eigenvalues as a function of n in closed form. Comparatively, all of the eigenvalues of the Kneser graph $K_{n:r}$ are known in closed form for all values of n and r. However, even though there are many cells in general, separating them into the above five varieties allows us to write the quotient matrix as a 5×5 block matrix for any n and r, in a relatively simple way. To illustrate how the entries for each block are found, we will consider the case of counting the number of neighbors each vertex in the $\mathcal{A}$ variety of cells has in the $\mathcal{C}$ variety of cells:

Let u be a vertex in $\mathcal{A}_i$. Consider a hypothetical neighbor w, of u in $\mathcal{C}_j$. Since the head of w must be from the tail of u, but not from $\{2, \ldots, r+1\}$ (because of the cell it is in), there are $r - i$ choices for it. Now there are j elements in the tail of w that come from $\{2, \ldots, r+1\}$, but they cannot be the i elements from this set that are in the tail of u (since tails must be disjoint for adjacency), therefore there are $r - i$ elements of $\{2, \ldots, r+1\}$ to choose these j from, and thus there are $\binom{r-i}{j}$ possibilities for this part of the tail of w. Now since we have chosen j elements of the tail of w already, and the head of u must be in the tail of w by the adjacency condition, the only choice left for w is the choice of the $r - 1 - j$ remaining elements of the tail of w. Now these elements cannot be from anywhere in u (the tails are disjoint and we have already accounted for the head) and they can't be from $\{2, \ldots, r+1\}$, since we already chose our j from there. Since the tail of u and $\{2, \ldots, r+1\}$ intersect in i elements, and the head of u is not in $\{2, \ldots, r+1\}$, together these two restrictions keep us from choosing $2r + 1 - i$ elements of $[n]$. Thus we have $\binom{n-2r-1+i}{r-1-j}$ choices for the remaining $r - 1 - j$ elements in the tail of w. Together, all of this tells us that the number of neighbors a vertex in $\mathcal{A}_i$ has in $\mathcal{C}_j$ is

$$(r-i)\binom{r-i}{j}\binom{n-2r-1+i}{r-1-j}.$$

Now, letting i run from 0 to r, and j from 0 to $r-1$, we get all of the entries in the block corresponding to the number of neighbors the vertices in each of the cells of the $\mathcal{A}$ variety have in each of the cells of the $\mathcal{C}$ variety. The logic for the other blocks is essentially the same, just with a few numbers shifted. Below we present the whole matrix in block format. Keep in mind that the entries in this matrix are meant to be 0 if any of the factors do not make sense, i.e., if one of them is less than 0, or if one of the binomial coefficients' top argument is strictly less than its bottom argument, or if the bottom argument is less than 0. For example $\binom{-1}{1} = -1$ algebraically, but here we take this to be 0. This has to be taken into account if one wants to implement code to compute the eigenvalues for particular values of n. Because of this the expressions in this matrix have been purposefully left in the form that corresponds to the actual counting arguments used to compute them.

$(\mathbf{1},\{\mathbf{2},\ldots,\mathbf{r+1}\})$	$\mathcal{A}$ $(0\le j\le r)$	$\mathcal{B}$ $(0\le j\le r-1)$	$\mathcal{C}$ $(0\le j\le r-1)$	$\mathcal{D}$ $(0\le j\le r-1)$	$\mathcal{E}$ $(0\le j\le r)$
$\mathcal{A}$ $(0\le i\le r)$	0	$i\binom{r-i}{j}\binom{n-2r-1+i}{r-1-j}$	$(r-i)\binom{r-i}{j}\binom{n-2r-1+i}{r-1-j}$	0	0
$\mathcal{B}$ $(0\le i\le r-1)$	$\binom{r-1-i}{j-1}\binom{n-2r+i}{r-j}$	0	0	$i\binom{r-1-i}{j-1}\binom{n-2r+i}{r-j}$	$(r-1-i)\binom{r-1-i}{j-1}\binom{n-2r+i}{r-j}$
$\mathcal{C}$ $(0\le i\le r-1)$	$\binom{r-i}{j}\binom{n-2r-1+i}{r-1-j}$	0	0	$i\binom{r-i}{j}\binom{n-2r-1+i}{r-1-j}$	$(r-1-i)\binom{r-i}{j}\binom{n-2r-1+i}{r-1-j}$
$\mathcal{D}$ $(0\le i\le r-1)$	0	$i\binom{r-1-i}{j-1}\binom{n-2r-1+i}{r-1-j}$	$(r-i)\binom{r-1-i}{j-1}\binom{n-2r-1+i}{r-1-j}$	$i\binom{r-1-i}{j-1}\binom{n-2r-1+i}{r-j}$	$(r-i)\binom{r-1-i}{j-1}\binom{n-2r-1+i}{r-j}$
$\mathcal{E}$ $(0\le i\le r)$	0	$i\binom{r-i}{j}\binom{n-2r-2+i}{r-2-j}$	$(r-i)\binom{r-i}{j}\binom{n-2r-2+i}{r-2-j}$	$i\binom{r-i}{j}\binom{n-2r-2+i}{r-1-j}$	$(r-i)\binom{r-i}{j}\binom{n-2r-2+i}{r-1-j}$

Table 3.1: Adjacency matrix of the quotient $HH_r(n)/\pi_u$.

It is important to note that one of these cells, $\mathcal{E}_0$, is an induced $HH_r(n-r-1)$ subgraph. Also, this is a strictly finer partition than π_3 since if we combine all of the cells of the $\mathcal{A}$ variety, we will get C_3, all the cells of the $\mathcal{B}$ and $\mathcal{C}$ varieties, and we get C_2, and combining all of the cells of the $\mathcal{D}$ and $\mathcal{E}$ varieties and we get C_1, an $HH_r(n-1)$ induced subgraph.

3.4 Eigenvalues of $HH_2(n)$ and $HH_3(n)$

Using this block form it is easy to write code to construct the quotient matrix $A(HH_r(n)/\pi_u)$ for any value of r, and we have written such code, but we have not been able to compute the eigenvalues for $r \geq 4$. The eigenvalues for $r = 2$ are:

$$-2$$
$$1$$
$$4-n$$
$$\frac{1}{2}\left(n-4 \pm \sqrt{n^2-8n+32}\right)$$
$$-1 \pm \sqrt{2n^2-14n+25}$$
$$2n-6$$

Note that the last eigenvalue, $2n-6$, is equal to the valency of $HH_2(n)$. Now for $r = 3$, the eigenvalues are:

$$-1$$
$$3$$
$$n-6$$
$$-\frac{1}{2}(n^2-11n+30)$$
$$\frac{1}{4}\left(n^2-11n+36 \pm \sqrt{n^4-22n^3+265n^2-1584n+3456}\right)$$
$$-n+6 \pm \sqrt{n^2-12n+45}$$
$$\frac{3}{2}(n-5)\left(-1 \pm \frac{1}{3}\sqrt{3n^2-30n+81}\right)$$
$$\frac{3}{2}(n^2-9n+20)$$

Again, the last eigenvalue here is equal to the valency.

The smallest eigenvalues for $r = 2$ and $r = 3$ are

$$-1 - \sqrt{2n^2 - 14n + 25}, \text{ and}$$

$$\frac{3}{2}(n-5)\left(-1 - \frac{1}{3}\sqrt{3n^2 - 30n + 81}\right)$$

respectively. Notice that these are in fact the same as the smallest eigenvalues obtained from the coarser partition π_3. It is also easy to see that for $n = 2r$ (the least value of n for which $HH_r(n)$ is not the empty graph) the smallest eigenvalue from π_3 reduces to $-r$ which is in fact the smallest eigenvalue since $HH_r(2r) = \frac{1}{2}\binom{2r}{r}K_{r,r}$. Because of this we conjecture that this is in fact the smallest eigenvalue for all r and n.

Equitable partitions can be very useful when trying to find the eigenvalues of certain classes of graphs. In particular, using a method similar to what we have done here, one can take the subgroup of $Sym(n)$ that setwise fixes $\{1, 2, \ldots, r\}$ and use the equitable partition this gives of the Kneser graph $K_{n:r}$ with the singleton cell $\{\{1, 2, \ldots, r\}\}$ to find all of its eigenvalues. Though this does take a bit of algebraic trickery even after getting the quotient matrix from this partition, and it is worth noting that the size of the matrix depends on r, as in our case. Because of this we hold out some hope that perhaps we will be able to explicitly find all of the eigenvalues for $HH_r(n)$ for any r and n.

Chapter 4

Independence Number

The *independence number* of a graph G is the size of its largest independent set and is denoted $\alpha(G)$. In general, determining this value is NP-Hard. However, there are some classes of graphs that have enough usable structure or symmetry to make this problem more tractable. In this chapter we will investigate the value of $\alpha(HH_r(n))$ and give an upper and lower bound of the same order. The value of the lower bound is $r\binom{n-1}{r}$ when $2r \leq n \leq r^2+1$, and $\binom{n}{r+1} + \frac{r-1}{r+1}\binom{r^2}{r}$ when $n \geq r^2$. This lower bound depends heavily on the three cell partition given in Chapter 3. The upper bound we give is a direct application of the ratio bound for regular graphs. We suspect that our lower bound is tight.

4.1 Kneser Homomorphism Bound

The Kneser graphs are one such class of graphs whose highly symmetric structure has allowed mathematicians to not only find the exact value of its independence number, but also to characterize all of its maximum independent sets.

4.1.1 Theorem (Erdős-Ko-Rado)**.** *If $n > 2r$, then $\alpha(G) = \binom{n-1}{r-1}$, and an independent set of this size consists of the r-subsets of $\{1, \ldots, n\}$ that contain a particular element.*

Since the Häggkvist & Hell graphs are quite naturally related to the Kneser graphs, it suggests that the maximum independent sets of the latter might enlighten us as to those of the former. Indeed the first bound we discuss is one obtained from this relationship. But first we need the following lemma.

4.1.2 Lemma. *Suppose that X and Y are graphs and there exists a homomorphism $\varphi : X \to Y$. If S is an independent set in Y, then $\varphi^{-1}(S)$ is an independent set in X.*

Proof. Suppose not. Then there exist two vertices $u, w \in \varphi^{-1}(S)$ such that $u \sim w$. Since φ is a homomorphism, φ must preserve adjacency, and therefore $\varphi(u) \sim \varphi(w)$, but $\varphi(u), \varphi(w) \in S$ which is a contradiction since S is an independent set. Therefore $\varphi^{-1}(S)$ is an independent set in X. $\square$

The above lemma proves to be useful when one one is searching for a large independent set in a graph that admits a homomorphism into a more well studied graph. This is in fact exactly the situation we are in with the Häggkvist & Hell graphs.

Recall the homomorphism $\varphi : HH_r(n) \to K_{n:r}$ given by $\varphi(h,T) = T$. Using this homomorphism and the above theorem and lemma, we can obtain a large independent set of $HH_r(n)$.

4.1.3 Theorem. *For* $n \geq 2r$,

$$\alpha(HH_r(n)) \geq r\binom{n-1}{r} = (n-r)\binom{n-1}{r-1}.$$

Proof. Note that all of the fibres of φ have size $n-r$, since for each possible tail one can have the $n-r$ other elements of $\{1,\dots,n\}$ as the head. Now let S be a maximum independent set in $K_{n:r}$, say the set of all r-subsets of $\{1,\dots,n\}$ that contain the element n. Then $\varphi^{-1}(S)$ is an independent set in $HH_r(n)$ of size $(n-r)\binom{n-1}{r-1}$. □

In fact this set is exactly the set of all vertices with n in their tail. Let us refer to the set of all vertices of $HH_r(n)$ with $i \in \{1,\dots,n\}$ in their tails as $\mathcal{T}_n(i)$. Note that $(n-r)\binom{n-1}{r-1} = r\binom{n-1}{r}$ illustrated by the fact that we can fix the element k in the tail, and then pick the other r elements of $\{h\} \cup T$ from the remaining $n-1$ elements of $\{1,\dots,n\}$, and then we can choose any of those r elements to be the head. So for any $i \in [n]$ we have

$$\mathcal{T}_n(i) = r\binom{n-1}{r} = (n-r)\binom{n-1}{r-1}.$$

In some respects, this is a quite large independent set, with

$$r\binom{n-1}{r} = \frac{r}{n}|V(HH_r(n))|,$$

where for random graphs the maximum independent set has size logarithmic in the size of the graph with high probability. However, in our next section we will see an independent set that has size proportional to $|V(HH_r(n))|$ given fixed r.

It is important to note that $\mathcal{T}_n(i)$ is maximal for $n \geq 2r$. This is easily seen from the partition π_3, since the cell C_2 is simply $\mathcal{T}_n(n)$ and $c_{12} \neq 0 \neq c_{32}$, and of course we can construct an equivalent partition where C_2 is $\mathcal{T}_n(i)$ for any $i \in \{1,\dots,n\}$.

4.2 Recursive Bound

In this section we will make particular use of the 3-cell partition π_3 of $HH_r(n)$ from the chapter on equitable partitions, so you may want to take a look at its diagram again. Notice that one of the cells is an $HH_r(n-1)$ induced subgraph. In fact there are many such subgraphs, at least one for each element of $\{1,\ldots,n\}$, but we need only concern ourselves with the one for our next theorem.

4.2.1 Theorem. *For all $r \geq 2$,*

$$\alpha(HH_r(n)) \geq \alpha(HH_r(n-1)) + \binom{n-1}{r}.$$

Proof. Note that the cell C_3 of π_3, the cell of vertices whose heads are n, is both an independent set of $HH_r(n)$, and is independent from the $HH_r(n-1)$ subgraph of the same partition. So for any independent set $S \subseteq C_1$ of $HH_r(n)$, we have that $S \cup C_3$ is also an independent set of $HH_r(n)$. Now taking S to be a maximum independent set of the $HH_r(n-1)$ subgraph induced by C_1, we see that

$$\alpha(HH_r(n)) \geq \alpha(HH_r(n-1)) + |C_3| = \alpha(HH_r(n-1)) + \binom{n-1}{r}. \qquad \square$$

This recursive bound begs the question of what happens when we continually take independent sets like this until we no longer can. Is this independent set larger or smaller than the set of all vertices with a particular element in the tail? Let us refer to the set of vertices of $HH_r(i)$ with i as their head as $\mathcal{H}_i$. Then by the well-known Hockey Stick equality, we see that

$$\alpha(HH_r(n)) \geq \sum_{i=r+1}^{n} |\mathcal{H}_i| = \sum_{i=r+1}^{n} \binom{i-1}{r} = \binom{n}{r+1} = \frac{1}{r+1}|V(HH_r(n))|$$

So this set is larger than the above described set when $\frac{1}{r+1} \geq \frac{r}{n}$, that is, when $n \geq r(r+1)$.

This independent set can actually be described in a more concise manner; it is simply the set of vertices whose head is larger than every element of its tail. This is easily seen as at each step we are working in an $HH_r(i)$ subgraph and we take all those vertices with the largest possible element, i, as their head, thus all vertices in this set will have heads larger than any element in their tails. And if a vertex has a head i that is larger than any element in its tail, then it must be in $\mathcal{H}_i$, and thus is in this independent set. Describing the set in this way makes it quite a bit easier to compute its size, since for any $(r+1)$-subset of $\{1,\ldots,n\}$ there are $r+1$ vertices, there being $r+1$ choices for the head, and exactly one of these will be in the independent set, the one with the largest element as the head, and thus there must be exactly $\frac{1}{r+1}|V(HH_r(n))| = \binom{n}{r+1}$ vertices in this independent set. It is also straightforward to prove that this is an independent set from this description, since if $(h_1,T_1) \sim (h_2,T_2)$, then $h_1 \in T_2$ and $h_2 \in T_1$, and obviously it cannot be that the head of each of these is larger than anything in their respective tails, since that would imply that both $h_1 > h_2$ and $h_2 > h_1$.

Note that $\mathcal{H}_n$ is a maximal independent set in $HH_r(n) - HH_r(n-1)$, as can be easily seen by examining π_3. This implies that if S is a maximal independent set in $HH_r(n-1)$, then $S \cup \mathcal{H}_n$ is maximal in $HH_r(n)$.

4.3 How Big Can We Get?

We now have two types of large independent sets of $HH_r(n)$, and we know of an independent set in $HH_r(n)$ that is independent of $HH_r(n-1)$. We would like to find the biggest independent set we can construct using these tools for an arbitrary $HH_r(n)$. Beginning with a given $HH_r(n)$, in following with the above recursive bound on $\alpha(HH_r(n))$, we can either take $\mathcal{T}_n(n)$ and stop, since this is maximal, or we can take $\mathcal{H}_n$ and then take the largest independent set we can find in $HH_r(n-1)$. Let us make this procedure slightly more formal. Recursively define α' as follows:

$$\begin{aligned}\alpha'(HH_r(n)) &= \max\{|\mathcal{T}_n(n)|, |\mathcal{H}_n| + \alpha'(HH_r(n-1))\} \\ &= \max\{r\binom{n-1}{r}, \binom{n-1}{r} + \alpha'(HH_r(n-1))\}\end{aligned}$$

4.3.1 Theorem. *For $2r \le n \le r^2+1$, $\alpha'(HH_r(n)) = r\binom{n-1}{r}$ and $\mathcal{T}_n(n)$ is an independent set of this size. For $n \ge r^2$, $\alpha'(HH_r(n)) = \binom{n}{r+1} + \frac{r-1}{r+1}\binom{r^2}{r}$ and*

$$\left(\bigcup_{i=r^2+1}^{n} \mathcal{H}_i\right) \cup \mathcal{T}_{r^2}(r^2)$$

is an independent set of this size.

We take care of the $2r \le n \le r^2+1$ case first. To prove this we use a finite induction. For $n = 2r$ we have that $HH_r(n) = \frac{1}{2}\binom{2r}{r}K_{r,r}$ and so clearly the largest independent will simply be made up of sides of the different complete bipartite subgraphs having size equal to $\frac{1}{2}|V(HH_r(n))|$. However, the set $\mathcal{T}_n(n)$ has size $(r)\binom{2r-1}{r} = \frac{r}{2r}((r+1)\binom{2r}{r}) = \frac{1}{2}|V(HH_r(n))|$. So in fact $\mathcal{T}_n(n)$ is a maximum independent set when $n = 2r$. Now suppose that $2r < n \le r^2+1$, and that the largest independent set of $HH_r(i)$ we can construct using the method described above has size $r\binom{i-1}{r}$ for $2r \le i < n$. Then our choices for constructing a large independent set of $HH_r(n)$ are:

Choice 1: Choose $\mathcal{T}_n(n)$ which has size $r\binom{n-1}{r}$.

Choice 2: Take the set $\mathcal{H}_n$ with size $\binom{n-1}{r}$ and then the largest independent set we can construct in $HH_r(n-1)$, which inductively has size $r\binom{n-2}{r}$, giving us an independent set of total size equal to $r\binom{n-2}{r} + \binom{n-1}{r}$.

We must show that Choice 1 beats out Choice 2 for $n \le r^2+1$. We prove this by showing the ratio of the two sizes is less than unity:

$$\begin{aligned}
\frac{r\binom{n-2}{r}+\binom{n-1}{r}}{r\binom{n-1}{r}} &= \frac{\binom{n-2}{r}}{\binom{n-1}{r}}+\frac{1}{r}\\
&= \frac{n-r-1}{n-1}+\frac{1}{r}\\
&= \frac{nr-r^2-r+n-1}{(n-1)r}\\
&= 1+\frac{n-(r^2+1)}{(n-1)r}\\
&\leq 1
\end{aligned}$$

for $n \leq r^2+1$.

Thus the largest set we can construct using this method is in fact of size $r\binom{n-1}{r}$ for $n \leq r^2+1 = |\mathcal{T}_n(n)|$. Also note that when $n = r^2+1$ the above inequality becomes an equality, implying that for $n = r^2+1$ either choice will do.

We must now show that for $n \geq r^2$, the largest independent set of $HH_r(n)$ that we can construct with this method has size

$$\binom{n}{r+1}+\frac{r-1}{r+1}\binom{r^2}{r} = \left|\left(\bigcup_{i=r^2+1}^{n}\mathcal{H}_i\right)\cup\mathcal{T}_{r^2}\left(r^2\right)\right|.$$

First we will show that this holds for $n = r^2$ and $n = r^2+1$. From the above we know that $(\bigcup_{i=r^2+1}^{n}\mathcal{H}_i)\cup\mathcal{T}_{r^2}(r^2)$ has size $r\binom{n-1}{r}$ in both cases (the first union being empty for $n = r^2$). We will show that $r\binom{n-1}{r}-\binom{n}{r+1}$ is equal to $\frac{r-1}{r+1}\binom{r^2}{r}$ for $n = r^2$ and r^2+1, thus proving the desired result:

$$\begin{aligned}
r\binom{n-1}{r}-\binom{n}{r+1} &= r\binom{n-1}{r}-\frac{n}{r+1}\binom{n-1}{r}\\
&= \left(r-\frac{n}{r+1}\right)\binom{n-1}{r}\\
&= \left(\frac{r^2+r-n}{r+1}\right)\binom{n-1}{r}
\end{aligned}$$

Plugging in $n = r^2+1$ we immediately see that this is equal to $\frac{r-1}{r+1}\binom{r^2}{r}$, but for $n = r^2$ we have a bit more work to do. In this case the above quantity is equal to

$$\begin{aligned}
\left(\frac{r}{r+1}\right)\binom{r^2-1}{r} &= \left(\frac{r}{r+1}\right)\frac{r^2-r}{r^2}\binom{r^2}{r}\\
&= \left(\frac{r}{r+1}\right)\left(\frac{r-1}{r}\right)\binom{r^2}{r}\\
&= \left(\frac{r-1}{r+1}\right)\binom{r^2}{r}
\end{aligned}$$

Thus for $n = r^2$, $\mathcal{T}_n(n)$ is a largest independent set that we can construct using this method, and for $n = r^2 + 1$, both $\mathcal{T}_n(n)$ and $\mathcal{H}_n \cup \mathcal{T}_{n-1}(n-1)$ are largest independent sets that we can construct using this method, and in each case they have size equal to $r\binom{n-1}{r} = \binom{n}{r+1} + \frac{r-1}{r+1}\binom{r^2}{r}$.

Now suppose that $n > r^2+1$ and that the largest independent set of $HH_r(k)$ that we can construct using this method has size equal to

$$\binom{k}{r+1} + \frac{r-1}{r+1}\binom{r^2}{r} = \left| \left(\bigcup_{i=r^2+1}^{k} \mathcal{H}_i \right) \cup \mathcal{T}_{r^2}(r^2) \right| \text{ for } r^2 \le k < n.$$

Again, we have two choices:

Choice 1: Choose $\mathcal{T}_n(n)$ which has size $r\binom{n-1}{r}$.

Choice 2: Take the set $\mathcal{H}_n$ with size $\binom{n-1}{r}$ and then the largest independent set we can construct in $HH_r(n-1)$, which inductively has size $\binom{n-1}{r+1} + \frac{r-1}{r+1}\binom{r^2}{r}$, giving us an independent set of total size equal to $\binom{n-1}{r+1} + \frac{r-1}{r+1}\binom{r^2}{r} + \binom{n-1}{r}$.

So we must show that $\binom{n-1}{r+1} + \frac{r-1}{r+1}\binom{r^2}{r} + \binom{n-1}{r} \ge r\binom{n-1}{r}$. However, by our induction hypothesis we know that $\binom{n-1}{r+1} + \frac{r-1}{r+1}\binom{r^2}{r} \ge r\binom{n-2}{r}$, since otherwise $\mathcal{T}_{n-1}(n-1)$ would be a larger independent set than what we assumed was the largest. So now we only have left to show that

$$\binom{n-1}{r} \ge r\binom{n-1}{r} - r\binom{n-2}{r}$$

To do this we look at their ratio:

$$\begin{aligned}
\frac{r\binom{n-1}{r} - r\binom{n-2}{r}}{\binom{n-1}{r}} &= r\left(1 - \frac{n-r-1}{n-1}\right) \\
&= r\left(\frac{n-1-(n-r-1)}{n-1}\right) \\
&= \frac{r^2}{n-1} \\
&< 1
\end{aligned}$$

for $n > r^2 + 1$. This proves the result. $\square$

So we have now shown that if we want to find an independent set of $HH_r(n)$ by either picking $\mathcal{T}_n(n)$ and stopping, or by taking $\mathcal{H}_n$ and then recursively finding an independent set of $HH_r(n-1)$ by the same method, then the best we can do is

$$\mathcal{T}_n(n) \text{ with size } r\binom{n-1}{r} \text{ for } 2r \le n \le r^2 + 1$$

and

$$\left(\bigcup_{i=r^2+1}^{n} \mathcal{H}_i\right) \cup \mathcal{T}_{r^2}(r^2)$$

with size

$$\binom{n}{r+1} + \frac{r-1}{r+1}\binom{r^2}{r} = \frac{|V(HH_r(n))|}{r+1} + \frac{r-1}{r+1}\binom{r^2}{r}$$

for $n \geq r^2$.

4.3.2 Theorem. *The sets $\mathcal{T}_n(n)$ and $\left(\bigcup_{i=r^2+1}^{n} \mathcal{H}_i\right) \cup \mathcal{T}_{r^2}(r^2)$ are maximal.*

Proof. We have already seen that $\mathcal{T}_{r^2}(r^2)$ is maximal in $HH_r(r^2)$. Now we can inductively assume that $(\bigcup_{i=r^2+1}^{n-1} \mathcal{H}_i) \cup \mathcal{T}_{r^2}(r^2)$ is maximal in $HH_r(n-1)$, then $\left((\bigcup_{i=r^2+1}^{n-1} \mathcal{H}_i) \cup \mathcal{T}_{r^2}(r^2)\right) \cup \mathcal{H}_n = (\bigcup_{i=r^2+1}^{n} \mathcal{H}_i) \cup \mathcal{T}_{r^2}(r^2)$ is maximal in $HH_r(n)$. □

It is important to note that in fact we could use $\mathcal{T}_{r^2}(j)$ for any $j \in \{1, \ldots, r^2\}$ instead of $\mathcal{T}_{r^2}(r^2)$, and nothing is lost. In fact we could of course have taken the vertices with ℓ as their head and then discarded all elements with ℓ in their tail and then acted recursively on the $HH_r(n-1)$ subgraph that uses only elements from $\{1, \ldots, n\} \setminus \ell$, for any $\ell \in \{1, \ldots, n\}$, and nothing would be lost. These degrees of freedom will be put to use in the next section.

4.4 Two Large Independent Sets

Now that we have found such a large independent set of $HH_r(n)$, it is interesting to ask what is the largest independent set we can find that is disjoint from this one? It turns out that for $n \geq r^2 + 1$ we can actually find one of the same size. We do this by essentially doing the same thing we did to get the first one, but we do it backwards.

4.4.1 Theorem. *For $n \geq r^2 + 1$, there exist two disjoint independent sets in $HH_r(n)$, both having size $\binom{n}{r+1} + \frac{r-1}{r+1}\binom{r^2}{r}$.*

Proof. First, we must make one change to the large independent set described above. Instead of using $\mathcal{T}_{r^2}(r^2)$, we will use $\mathcal{T}_{r^2}(1)$. Let us call this independent set $\mathcal{L}^-$.

Now, to construct the second large independent set, we take the vertices with 1 as their head and then consider the $HH_r(n-1)$ subgraph that uses the elements $\{2, \ldots, n\}$. From this we take the elements with 2 as their head and then consider the $HH_r(n-2)$ subgraph that uses the elements $\{3, \ldots, n\}$, and we continue this until we get to the $HH_r(r^2)$ subgraph using only elements from $\{n - r^2 + 1, \ldots, n\}$. Then, we take the set of all vertices in this subgraph with

n in their tail. Let us call this set $\mathcal{L}^+$. Obviously both $\mathcal{L}^-$ and $\mathcal{L}^+$ have size equal to $\binom{n}{r+1} + \frac{r-1}{r+1}\binom{r^2}{r}$, it is only left to show that they are disjoint.

Of course these two sets can be described in a different manner; $\mathcal{L}^-$ is the set of vertices with heads from $\{r^2+1, \ldots, n\}$ that are greater than any element in their tails, call this $\mathcal{M}^-$, along with the vertices using only elements from $\{1, \ldots, r^2\}$ with 1 in their tails, call this $\mathcal{N}^-$. $\mathcal{L}^+$ is the set of vertices with heads from $\{1, \ldots, n-r^2\}$ that are less than any element in their tails, call this $\mathcal{M}^+$, along with the vertices using only elements from $\{n-r^2+1, \ldots, n\}$ with n in their tails, call this $\mathcal{N}^+$. Clearly $\mathcal{M}^-$ and $\mathcal{M}^+$ are disjoint since the head of a vertex cannot be both larger and smaller than everything in its tail. $\mathcal{M}^-$ and $\mathcal{N}^+$ are disjoint because if n is in the tail of a vertex, then its head cannot be larger than everything in its tail. Similarly, $\mathcal{M}^+$ and $\mathcal{N}^-$ are disjoint since if 1 is in the tail of a vertex, then its head cannot be smaller than everything in its tail. Now all that is left is to show that $\mathcal{N}^-$ and $\mathcal{N}^+$ are disjoint. But the vertices in $\mathcal{N}^-$ only use elements from $\{1, \ldots, r^2\}$, and $n \geq r^2+1$, and so they cannot have n in their tails. Therefore, $\mathcal{M}^- \sqcup \mathcal{N}^- = \mathcal{L}^-$ and $\mathcal{M}^+ \sqcup \mathcal{N}^+ = \mathcal{L}^+$ are disjoint. ◻

This implies that we are able to color $\frac{2}{r+1}|V(HH_r(n))| + 2\frac{r-1}{r+1}\binom{r^2}{r}$ vertices of $HH_r(n)$ with just two colors for $n \geq r^2+1$. For $r = 3$ this is over half of the graph, and for $r = 2$ it is more than two thirds of the graph, quite a substantial portion. It is noteworthy that for any value of n we can find two disjoint independent sets of size $\frac{1}{r+1}|V(HH_r(n))|$ each by taking one to be the set of all vertices whose head is greater than every element of its tail and the other to be the set of all vertices whose head is less than every element of its tail.

In fact we can do slightly better than this for $n \leq r(r+1)+1$, though we only care about when $n \leq r^2 < r(r+1)+1$ since otherwise we can do better by the first scheme given above. We let the first set be all of the vertices having n in their tail, which is the largest independent set we know of for $n \leq r^2+1$. Then let the second set be all of the vertices with n as their head ($\mathcal{H}_n$), and the vertices using only elements from $\{1, \ldots, n-1\}$ that have 1 in their tail, i.e. $\mathcal{T}_{n-1}(1)$. The second set having size

$$\begin{aligned}
\binom{n-1}{r} + r\binom{n-2}{r} &= \frac{r+1}{n}\binom{n}{r+1} + r\binom{n-2}{r} \\
&= \left(1 - \frac{n-r-1}{n}\right)\binom{n}{r+1} + r\binom{n-2}{r} \\
&= \binom{n}{r+1} + \left[r\binom{n-2}{r} - \frac{n-r-1}{n}\binom{n}{r+1}\right] \\
&= \binom{n}{r+1} + \left[r\binom{n-2}{r} - \binom{n-1}{r}\right] \\
&= \binom{n}{r+1} + \left[r - \frac{n-1}{r+1}\right]\binom{n-2}{r}
\end{aligned}$$

Which is at least $\binom{n}{r+1}$ for $n \leq r(r+1)+1$.

From the lower bounds on $\alpha(HH_r(n))$ given above we see that asymptotically the independence number of $HH_r(n)$ must grow at least as fast as $\frac{|V(HH_r(n))|}{r+1}$. In the next section we will present an upper bound for the size of the maximum independent set of $HH_r(n)$ and see how close this is to the lower bounds that have been presented.

4.5 The Ratio Bound

The Ratio Bound is an upper bound on the independence number of a regular graph that is derived algebraically from the least eigenvalue of the graph. For some graphs, such as the Kneser graphs, the ratio bound is known to be tight. Since the Häggkvist & Hell graphs are quite closely related to the Kneser graphs, it is an interesting question as to whether the bound is tight for them.

In order to derive the ratio bound we will need to introduce some linear algebra notions and theorems from [2]. For a real symmetric $n \times n$ matrix M, let $\theta_1(M) \geq \theta_2(M) \geq \ldots \geq \theta_n(M)$ denote its eigenvalues in nondecreasing order.

Definition: For a real symmetric $n \times n$ matrix A and a real symmetric $m \times m$ matrix B, $m \leq n$, we say that the eigenvalues of B *interlace* the eigenvalues of A if

$$\theta_{i+(n-m)}(A) \leq \theta_i(B) \leq \theta_i(A).$$

We will need the following lemma to prove the ratio bound.

4.5.1 Lemma. *If P is the characteristic matrix of a partition, not necessarily equitable, of a graph X with adjacency matrix A, then the eigenvalues of $(P^TP)^{-1}P^TAP$ interlace the eigenvalues of A.* □

We are now able to prove the ratio bound for the independence number of a regular graph.

4.5.2 Theorem (Delsarte)**.** *Let X be a k-regular graph with v vertices and least eigenvalue τ. Then*

$$\alpha(X) \leq v\frac{-\tau}{k-\tau}.$$

Proof. Suppose that P is the characteristic matrix of a partition π of $V(X)$. Consider the ij-entry of P^TAP. The ik-entry of P^TA is the number of neighbors of the vertex k in cell i of π. So the ij-entry of P^TAP is the sum of this for all vertices in cell j, i.e. the number of edges between cells i and j. Now since P^TP is a diagonal matrix with $(P^TP)_{ii}$ equal to the size of the i^{th} cell of π, the ij-entry of $(P^TP)^{-1}P^TAP$ will be the average number of neighbors in cell j of a vertex in cell i.

Now consider a partition of $V(X)$ into an independent set S and $V(X) \setminus S$. There are $|S|k$ edges between S and $V(X) \setminus S$ and so the vertices not in S have an average of $k|S|/(v - |S|)$ neighbors in S. So we have the following:

$$B = (P^T P)^{-1} P^T AP = \begin{pmatrix} 0 & k \\ \frac{k|S|}{v-|S|} & k - \frac{k|S|}{v-|S|} \end{pmatrix}$$

We see that k is an eigenvalue of B (with eigenvector $\mathbf{1}$). Also, since the sum of the eigenvalues is equal to the trace, which is $k - \frac{k|S|}{v-|S|}$ in this case, the other eigenvalue of B must be $-k|S|/(v - |S|)$. By Lemma 4.5.1 we know that the eigenvalues of $(P^T P)^{-1} P^T AP$ interlace the eigenvalues of A, which implies that

$$\tau \leq \frac{k|S|}{v - |S|}$$

Now by simply letting S be a maximum independent set of X and rearranging we get that

$$\alpha(X) \leq v \frac{-\tau}{k - \tau}.$$

□

4.6 An Upper Bound

We are now able to use the ratio bound to find an upper bound on the independence number of the Häggkvist & Hell graphs for which we know the least eigenvalue. We know the least eigenvalue for $HH_r(n)$ in the cases of $r = 2$ and $r = 3$, so we present the bound for these two cases.

For $r = 2$, the least eigenvalue is

$$\tau_2 = -1 - \sqrt{2n^2 - 14n + 25}$$

and the valency is $2n - 6$, plugging these into the ratio bound we get the following:

4.6.1 Theorem.

$$\alpha(HH_2(n)) \leq |V(HH_2(n))| \frac{\sqrt{2n^2 - 14n + 25} - (n - 5)}{n}$$ □

Comparing this to the lower bound for $r = 2$,

$$\alpha(HH_2(n)) \geq \frac{|V(HH_2(n))|}{3} + 2$$

we see that the ratio bound is strictly larger than the lower bound for most values of n, in fact as $n \to \infty$ the ratio bound approaches $(\sqrt{2} - 1)|V(HH_2(n))|$, whereas the lower bound only goes to $\frac{1}{3}|V(HH_2(n))|$. That the ratio bound is greater than our lower bound alone is of course not enough to show that it is not achieved, perhaps there is some larger independent set that we have not found yet. However, since the ratio bound has a radical in it, it cannot be achieved exactly for most n, since the independence number must be an integer. But perhaps for some large range of values of n the largest possible integer less than the ratio bound is achieved, but this seems unlikely. For a few small cases we have used the Cliquer program to find the exact values of of $\alpha(HH_2(n))$:

n	$\alpha(HH_2(n))$	$\lfloor$Ratio Bound$\rfloor$	$\lvert V(HH_2(n))\rvert$
4	6	6	12
5	12	13	30
6	22	26	60
7	37	45	105
8	58	71	168

Table 4.1: Independence numbers of $HH_2(n)$.

Note that $\alpha(HH_2(n)) = \frac{1}{3}|V(HH_2(n))| + 2$ (our lower bound) for all values of n for which the independence number is known.

For $r = 3$ the least eigenvalue is

$$\tau_3 = \frac{3}{2}(n-5)(-1-\frac{1}{3}\sqrt{3n^2-30n+81})$$

and the valency is $3\binom{n-4}{2}$, plugging these into the ratio bound we get the following:

4.6.2 Theorem.

$$\alpha(HH_3(n)) \leq |V(HH_3(n))|\frac{\sqrt{3(n^2-10n+27)}-(n-9)}{2n} \qquad \square$$

Comparing this to the lower bound for $r = 3$,

$$\alpha(HH_3(n)) \geq \frac{|V(HH_3(n))|}{4} + 42$$

we again see that the ratio bound is larger than our lower bound as it asymptotically approaches $\frac{\sqrt{3}-1}{2}|V(HH_3(n))| \approx 0.366|V(HH_3(n))|$ while the lower bound only approaches $\frac{1}{4}|V(HH_3(n))|$. We can conclude essentially the same things as the $r = 2$ case; the ratio bound cannot be reached exactly for most n since it contains a radical. Perhaps for large enough n the largest integer less than the bound is achieved, but that does not seem likely. The few small cases we were able to compute using Cliquer are listed in Table 4.2.

Note that for all values of n for which the independence number of $HH_3(n)$ is known $\alpha(HH_3(n)) = 3\binom{n-1}{3}$, which is our lower bound. In both the $r = 2$ and $r = 3$ for cases, though the bounds are disparate, they do have the same

n	$\alpha(HH_3(n))$	$\lfloor$Ratio Bound$\rfloor$	$\lvert V(HH_3(n))\rvert$
6	30	30	60
7	60	62	140
8	105	118	280

Table 4.2: Independence numbers of $HH_3(n)$.

order of magnitude in terms of n, both being linear in the number of vertices. So it seems that even though the ratio bound is probably too large in our opinion, it is only too large by a constant factor.

Now even though we don't know the least eigenvalue for general r, we have a sneaking suspicion that it is the smallest eigenvalue of X/π_3, and so it is of interest to compute the ratio bound assuming that this is in fact the case. Recall that the least eigenvalue of $HH_r(n)/\pi_3$ is

$$\tau = \frac{1}{2}r\binom{n-r-2}{r-2}\left[-1-\frac{1}{r-1}\sqrt{\frac{4n(n-3r-1)+r(r+3)^2}{r}}\right].$$

If we suppose that this is the least eigenvalue of $HH_r(n)$, which has valency $r\binom{n-r-1}{r-1}$; then the ratio bound gives us the following:

4.6.3 Theorem. *If the smallest eigenvalue of $HH_r(n)$ is*

$$\frac{1}{2}r\binom{n-r-2}{r-2}\left[-1-\frac{1}{r-1}\sqrt{\frac{4n(n-3r-1)+r(r+3)^2}{r}}\right],$$

then

$$\begin{aligned}\alpha(HH_r(n)) &\leq |V(HH_r(n))|\frac{r-1+\sqrt{\frac{q}{r}}}{2n-r-3+\sqrt{\frac{q}{r}}}\\ &= |V(HH_r(n))|\frac{\sqrt{rq}-(2n-r(r+3))}{2n(r-1)}\end{aligned}$$

in which $q = 4n(n-3r-1)+r(r+3)^2$. □

It is straightforward to see that as $n \to \infty$, this approaches

$$\frac{1}{1+\sqrt{r}}|V(HH_r(n))| > \frac{1}{1+r}|V(HH_r(n))|$$

for all values of r we are interested in (i.e. $r \geq 2$). So, tentatively speaking, the ratio bound is always asymptotically greater than our lower bound. But we need not speak so tentatively, since $\frac{-\tau}{k-\tau} = 1 - \frac{k}{k-\tau}$ increases as τ decreases. So even if we are wrong about this being the lowest eigenvalue, the bound can only be greater than this, and thus still greater than our lower bound.

Of course, there is always the possibility that for larger n there are larger independent sets than our lower bound, but we don't think that this is the case. Even though we have only computed the actual independence numbers for a few values of r and n, the values we have done it for are on both sides of the r^2 to r^2+1 dividing line. It is also worth mentioning that in the slightly degenerate case of $r=1$, which is simply a matching, both our lower and upper bounds given above simplify to $\frac{1}{2}|V(HH_1(n))|$ for all values of n, which is of course the correct value of $\alpha(HH_1(n))$. This lends some credence to our guess that the lowest eigenvalue of $HH_r(n)/\pi_3$ is always the lowest eigenvalue of $HH_r(n)$.

As with the Kneser graphs, there is a minimum value of n (dependent on r) such that for any lesser value the graph is empty. This value is $2r$ for both the Kneser graphs and the Häggkvist & Hell graphs. Crossing this dividing line drastically changes the properties of the graphs, and thus changes basically all of the interesting parameters of the graphs, including the independence number. For the Kneser graphs the independence number is $\binom{n}{r}$ (the number of vertices) for $n < 2r$ and $\binom{n-1}{r-1}$ for $n \geq 2r$.

Similarly the independence number for the Häggkvist & Hell graphs is $(r+1)\binom{n}{r+1}$ (the number of vertices) for $n < 2r$ and it seems that it is $r\binom{n-1}{r}$ for $2r \leq n \leq r^2+1$ with maximum independent sets that are inverse images of the maximum independent sets of $K_{n:r}$. However, it seems that the Häggkvist & Hell graphs have one more dividing line at $n = r^2+1$ and from that point on the maximum independent sets have size $\frac{1}{r+1}|V(HH_r(n))| + \frac{r-1}{r+1}\binom{r^2}{r}$ and are of a different type than for smaller values of n. We strongly suspect that this is the end of the story, and so we conjecture that these are the maximum independent sets of $HH_r(n)$, and thus the $n = r^2+1$ dividing line represents an interesting "phase transition" for Häggkvist & Hell graphs.

Chapter 5

Chromatic Number

As we saw earlier in Section 2.3, a *proper coloring* of a graph X is a function $f : V(X) \to S$, where S is a finite set, such that no two adjacent vertices have the same image under f. An *n-coloring* is a coloring where S is an n element set. We define the *chromatic number* of a graph X as the smallest n for which there exists a proper n-coloring of X, and we denote this value as $\chi(X)$.

In this chapter we show that the chromatic number of $HH_r(n)$ is bounded above by $n - 2r + 2$. We also show that for fixed r the chromatic number increases by either zero or one for each increase in n by one. We give the exact chromatic number of $HH_2(n)$ and $HH_3(n)$ for some values of n, and we conclude by proving that for any fixed value of r greater than or equal to two, $\chi(HH_r(n))$ is unbounded with respect to n.

5.1 Homomorphism Bound

Recall from Section 2.3 that another way of describing a coloring of a graph X is as a homomorphism from X to a complete graph. This principle was expressed in Lemma 2.3.1 which stated that X is n-colorable if and only if $X \to K_n$. This connection between colorings and homomorphisms to complete graphs allowed us to prove Lemma 2.3.2 which tells us that if $X \to Y$, then $\chi(X) \le \chi(Y)$.

This lemma enables us to obtain both lower and upper bounds on the chromatic number of a graph, and we will use it to achieve the latter. But first we need the following theorem by Lovász.

5.1.1 Theorem (Lovász)**.** *The chromatic number of the Kneser graph $K_{n:r}$ is $n - 2r + 2$.* □

A proof of this is given in [2]. Now we are able to give an upper bound for $\chi(HH_r(n))$:

5.1.2 Theorem. *The chromatic number of $HH_r(n)$ is at most $n - 2r + 2$.*

Proof. By Theorem 2.5.2 there exists a homomorphism $h : HH_r(n) \to K_{n:r}$, thus by Lemma 2.3.2 $\chi(HH_r(n)) \leq n - 2r + 2$. □

From this bound we see that the chromatic number of $HH_r(n)$ does not grow too quickly, at most linearly with n. We would like to know if this bound is always achieved, or if something more complicated occurs. The rest of this chapter will shed some light on this question.

5.2 Recursive Bound

One particularly unsavory possibility is that $\chi(HH_r(n))$ and $\chi(HH_r(n+1))$ differ by more than one for some choices of n, and are equal for others. In this section we show that the chromatic number increases by at most one for each increase in n. The equitable partition π_3 from Section 3.2 plays an important role in the proof of the following lemma, so we provide it below for the ease of the reader:

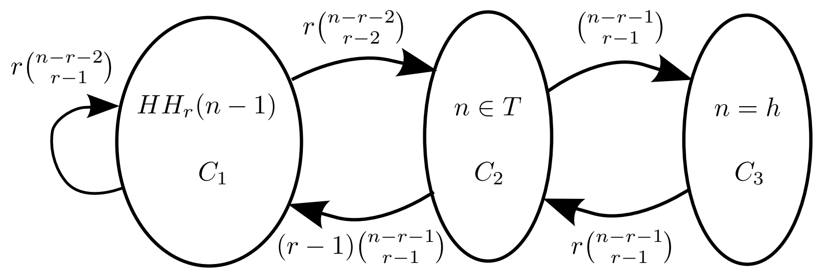

Figure 5.1: Diagram of 3-Cell Partition π_3.

5.2.1 Theorem. $\chi(HH_r(n-1)) \leq \chi(HH_r(n)) \leq \chi(HH_r(n-1)) + 1$.

Proof. The first inequality is trivially true since $HH_r(n-1)$ is a subgraph of $HH_r(n)$. For the upper bound, we color the cell C_1 with the colors $\{1, \ldots, \chi(HH_r(n-1))\}$ since it is an $HH_r(n-1)$ subgraph, then we color all vertices in C_2 with $\chi(HH_r(n-1)) + 1$ since C_2 is an independent set. Now we only need to color the vertices of C_3, but this is easy since they are an independent set and have no edges to C_1, so we can color them with the color 1. Thus we have colored $HH_r(n)$ with $\chi(HH_r(n-1)) + 1$ colors. □

So we are assured that the chromatic number of Häggkvist & Hell graphs does not behave too erratically. Also, from this result we immediately obtain the following corollary:

5.2.2 Corollary. $\chi(HH_r(n-k)) \leq \chi(HH_r(n)) \leq \chi(HH_r(n-k)) + k$. □

Given the result of Theorem 5.2.1, a natural question to ask is whether or not the chromatic number always increases by one, which is equivalent to asking if the $n - 2r + 2$ bound from Theorem 5.1.2 is always tight. If this is the case one might hope for an inductive proof of it, or perhaps a proof by showing that some graph with chromatic number $n - 2r + 2$ admits a homomorphism into $HH_r(n)$. In the next section we will look at some computed results for small values of r and n, and discuss their implications.

5.3 Some Chromatic Numbers

In order to get a rough idea of what may be going on with the chromatic numbers of Häggkvist & Hell graphs, we have computed the exact value of $\chi(HH_r(n))$ for some small values of r and n using SAGE. We present our data below.

r	n	$\chi(HH_r(n))$	$n - 2r + 2$
2	4	2	2
2	5	3	3
2	6	4	4
2	7	4	5
3	6	2	2
3	7	3	3

Table 5.1: Chromatic numbers of $HH_r(n)$.

As one can see from the table, $\chi(HH_2(7)) = 4 < 5 = n - 2r + 2$. This eliminates any hope of proving that the homomorphism bound of Theorem 5.1.2 is always achieved, in fact this implies that $\chi(HH_2(n)) \leq n - 2r + 1$ for $n \geq 7$. So we know the bound is not achieved on a cofinite set of positive integers when $r = 2$. Perhaps the bound is always achieved for some other choices of r, but that would be an odd result. More likely is that the bound is achieved for some small values of n for each choice of r. In fact we prove here that for any choice of r, the bound is always achieved for the two smallest values of n.

5.3.1 Theorem. *For $r \geq 2$, $\chi(HH_r(2r)) = 2$ and $\chi(HH_r(2r+1)) = 3$.*

Proof. $HH_r(2r)$ is isomorphic to $\frac{1}{2}\binom{2r}{r}K_{r,r}$, which has chromatic number 2. By Theorem 1.4.4, $HH_r(2r+1)$ has an odd cycle of length $2r+1$ and thus it has chromatic number at least 3, and Theorem 5.1.2 gives that

$$\chi(HH_r(2r+1)) \leq 2r + 1 - 2r + 2 = 3,$$

which proves the result. □

We take time here to remark that since $HH_r(2r-1)$ is always empty, the above homomorphism bound is trivially met in this case as well.

So far the bounds we have given have been mostly upper bounds, except for the trivial lower bound given in Theorem 5.2.1, thus it is still an open question as to whether $\chi(HH_r(n))$ is bounded in terms of n. This matter will be investigated in this next section.

5.4 Bounded or Unbounded?

Knowing that the chromatic number of $HH_r(n)$ does not increase with every increase of n, it is natural to ask whether or not it is bounded by some finite value which depends only on r. Recall that all triangle-free r-regular graphs admit a homomorphism into $HH_r\left(r\frac{(r-1)^3-1}{r-2}+1\right)$, which implies that its chromatic number is at least r, since there exist triangle-free r-regular graphs with chromatic number r. This means that we cannot hope to bound $\chi(HH_r(n))$ for all values of r and n, but it may be possible to bound it for fixed r. However, this turns out to not be the case.

We begin with a lemma that relates the chromatic numbers of Häggkvist & Hell graphs with different tail sizes.

5.4.1 Lemma. *For $r \geq 2$, $\chi(HH_r(n)) \leq \chi(HH_{r+1}(\chi(HH_r(n)) + n))$.*

Proof. We prove this by showing that $HH_r(n)$ is isomorphic to a subgraph of $HH_{r+1}(\chi(HH_r(n)) + n)$. Let

$$f : V(HH_r(n)) \to \{n+1, \dots, n + \chi(HH_r(n))\}$$

be a proper coloring of $HH_r(n)$. Consider the map

$$g : HH_r(n) \to HH_{r+1}(\chi(HH_r(n)) + n)$$

given by

$$g(h_u, T_u) = (h_u, T_u \cup f(u)).$$

It is easy to see that this is an injective homomorphism which proves the lemma. □

This result immediately allows us to prove this next vital lemma.

5.4.2 Lemma. *For a fixed $r \geq 2$, if $\chi(HH_r(n))$ is unbounded with respect to n, then $\chi(HH_{r'}(n))$ is unbounded with respect to n for all $r' \geq r$.*

Proof. It will suffice to show that it holds for $r' = r + 1$. We will prove the contrapositve. Suppose that $\chi(HH_{r+1}(n))$ is bounded by M_{r+1} with respect to n. Then by Lemma 5.4.1, we have that

$$\chi(HH_r(n)) \leq \chi(HH_{r+1}(\chi(HH_r(n)) + n)) \leq M_{r+1}$$

for all n. Therefore $\chi(HH_r(n))$ is bounded with respect to n. □

Lemma 5.4.2 is not difficult to prove, but it is very powerful, as it turns a result for one value of r into a result for an infinite number of values of r. We would like to be able to prove that $\chi(HH_2(n))$ is unbounded, since this would take care of all possible values of r for which the chromatic number of $HH_r(n)$ is not obviously bounded. This we proceed to do.

In 2000, Gallucio, Hell, and Nešetřil showed in [1] that the chromatic number of $HH_3(n)$ could be arbitrarily large. We have reproduced their proof here because it is related to the proof of our next result.

5.4.3 Lemma. *For every c there exists a number n such that the chromatic number of $HH_3(n)$ is at least c.*

Proof. Consider the following graph S_n: the vertices of S_n are all 3-element sets of $[n]$, and two such subsets, say $\{x_1, x_2, x_3\}$ with $x_1 < x_2 < x_3$, and $\{y_1, y_2, y_3\}$ with $y_1 < y_2 < y_3$, are adjacent if $x_2 = y_1$ and $x_3 = y_2$. Note that S_n is a directed graph, but we will also call S_n its underlying undirected graph. It follows from Ramsey's theorem for the partition of triples [4] that the chromatic number of S_n may be arbitrarily large if n is large.

We now claim that S_n is isomorphic to a subgraph of some $HH_3(n')$ where $n' \geq n$. Let f be a bijection from the set of all 3-element subsets of $[n]$ to the set $\{n+1, \ldots, n+\binom{n}{3}\}$, and let $n' = n + \binom{n}{3}$. Now for $\{x_1, x_2, x_3\} \in V(S_n)$ with $x_1 < x_2 < x_3$, we let

$$g(x_1, x_2, x_3) = (x_2, \{x_1, x_3, f(x_1, x_2, x_3)\}) \in V(HH_3(n')).$$

It is easy to see that g is an injective homomorphism from $V(S_n)$ to $V(HH_3(n'))$, i.e., that S_n is isomorphic to a subgraph of $HH_3(n')$. Hence, the chromatic number of $HH_3(n')$ is at least as large as the chromatic number of S_n. (In fact, it is easy to see that g is an isomorphism onto an an induced subgraph of $HH_3(n')$.) □

Note that it was not necessary for f to be a bijection, it could have simply been a proper coloring as in our proof of Lemma 5.4.1. Along with Lemma 5.4.2, Lemma 5.4.3 immediately gives us the following corollary:

5.4.4 Corollary. *For a fixed $r \geq 3$, $\chi(HH_r(n))$ is unbounded with respect to n.* □

It is worth noting that Gallucio et al. also gave a proof in [1] that $HH_3(n)$ has chromatic number at least 4 for $n \geq 16$. However, our computation of $\chi(HH_2(6)) = 4$ along with Lemma 5.4.1 proves the same but for $n \geq 10$.

With all of the $r \geq 3$ cases taken care of, we are left with only the $HH_2(n)$ case to resolve. Using what we learned from the chapter on the independence number, we can reduce this problem somewhat. We know that the vertices of $HH_2(n)$ whose head is greater than both elements in its tail is an independent set, and we know the same is true for vertices whose head is less than both elements of its tail. So we can always color these two sets with two colors. So we only need to know if the subgraph of $HH_2(n)$ induced by the vertices not in these two sets, i.e., the vertices with heads in between the two elements of their tail, has bounded chromatic number or not.

As it turns out, this subgraph of $HH_2(n)$ and the graph S_n from the proof of Lemma 5.4.3 are very closely related: so closely related, in fact, that they are isomorphic to one another.

5.4.5 Lemma. $\chi(S_n) \leq \chi(HH_2(n)) \leq \chi(S_n) + 2$.

Proof. For a vertex $\{x_1, x_2, x_3\} \in V(S_n)$ with $x_1 < x_2 < x_3$, we let

$$f(x_1, x_2, x_3) = (x_2, \{x_1, x_3\}) \in V(HH_2(n)).$$

Clearly, this is injective. We claim that it is an injective homomorphism. Suppose that $x = \{x_1, x_2, x_3\}$ with $x_1 < x_2 < x_3$ and $y = \{y_1, y_2, y_3\}$ with $y_1 < y_2 < y_3$ are adjacent in S_n. Then WLOG $x_2 = y_1$ and $x_3 = y_2$. Now $f(x) = (x_2, \{x_1, x_3\})$ and $f(y) = (y_2, \{y_1, y_3\})$, and $x_2 \in \{y_1, y_3\}$, $y_2 \in \{y_1, y_3\}$ and $\{x_1, x_3\} \cap \{y_1, y_3\} = \varnothing$. Therefore $f(x)$ and $f(y)$ are adjacent. This proves that S_n is isomorphic to a subgraph of $HH_2(n)$ which implies the first inequality.

Now we will show that it is isomorphic to an induced subgraph of $HH_2(n)$. Suppose that $(x_2, \{x_1, x_3\})$ with $x_1 < x_2 < x_3$ and $y = (y_2, \{y_1, y_3\})$ with $y_1 < y_2 < y_3$ are adjacent in $HH_2(n)$. Either $x_2 = y_1$ or $x_2 = y_3$. Suppose that $x_2 = y_1$. Then $y_2 = x_3$ since $y_2 > y_1 = x_2$, so we have that $x_2 = y_1$ and $x_3 = y_2$. Therefore $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ are adjacent in S_n. Similarly, if $x_2 = y_3$, we deduce that $y_2 = x_1$, and thus $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ are again adjacent in S_n. Therefore, S_n is isomorphic to the subgraph of $HH_2(n)$ induced by the vertices whose heads are in between the two elements in their tails, and since we can color the rest of $HH_2(n)$ with two colors, we can color $HH_2(n)$ with $\chi(S_n) + 2$ colors. □

This result of course implies that the chromatic number of $HH_2(n)$ is unbounded, since the chromatic number of S_n is unbounded [4]. This, along with Lemma 5.4.2 imply the following theorem:

5.4.6 Theorem. *For a fixed $r \geq 2$, for any number c, there exists an integer n such that the chromatic number of $HH_r(n)$ is at least c.* □

This theorem can be strengthened with the use of Lemma 5.2.1 to achieve the following:

5.4.7 Theorem. *For any $r \geq 2$, for any positive integer k, there exists an integer n such that $\chi(HH_r(n)) = k$.* □

Even though the $\chi(HH_r(n)) \leq \chi(K_{n:r})$ bound is probably not achieved for every value of n for any fixed r, the chromatic number of the Häggkvist & Hell graphs *is* unbounded for fixed r, much like the Kneser graphs. In the next chapter we will investigate the fractional chromatic number of $HH_r(n)$ and compare it to that of $K_{n:r}$. As for exactly how fast $\chi(HH_r(n))$ grows with respect to n, this is still an open question, but intuitively we believe that it most likely grows slower and slower as n increases.

Chapter 6

Fractional Chromatic Number

In this chapter we introduce the notions of *fractional chromatic number* and *fractional clique number*, and give some background material on both. We give several bounds for the fractional chromatic number of the Häggkvist & Hell graphs including the following upper bound:

$$\chi^*(HH_r(n)) \leq \frac{n}{r}.$$

We also give a recursive upper bound and a probable lower bound using our probable upper bound on $\alpha(HH_r(n))$.

6.1 Definitions and Theorems

The theorems in this section all come from Godsil and Royle's *Algebraic Graph Theory* [2], which we follow closely here. For a more complete discussion of these concepts please refer to Chapter 7 of that reference.

We define $\mathcal{I}(X)$ to be the set of all independent sets of X, and $\mathcal{I}(X, u)$ to be the set of all independent sets of X that contain u. A *fractional coloring* of X is defined to be a non-negative real-valued function f on $\mathcal{I}(X)$ such that for any vertex x of X,

$$\sum_{S \in \mathcal{I}(X,x)} f(S) \geq 1.$$

The *weight* of a fractional coloring is the sum of all of its values, and the *fractional chromatic number* of the graph X is defined to be the minimum possible weight of a fractional coloring of X, and is denoted by $\chi^*(X)$. We call a fractional coloring regular if, for each vertex x of X, we have

$$\sum_{S \in \mathcal{I}(X,x)} f(S) = 1.$$

For any proper k-coloring of a graph X, the color classes form a collection of k pairwise disjoint independent sets $V_1, \dots, V_k$ whose union is $V(X)$. If we define a function f such that $f(V_i) = 1$ for all i, and $f(S) = 0$ for all other $S \in \mathcal{I}(X)$, then f is a fractional coloring of X with weight k. This immediately implies that

$$\chi^*(X) \leq \chi(X).$$

Conversely, suppose that f is a 01-valued fractional coloring of X of weight k. Then the support of f consists of k independent sets $V_1, \dots, V_k$ such that $\cup_{i=1}^k V_i = V(X)$. If we color each vertex x in X with the smallest i such that $x \in V_i$, then we will obtain a k-coloring of X. Therefore, the chromatic number of any graph X is equal to the minimum weight of a 01-valued fractional coloring.

A *fractional clique* of a graph X is a non-negative real-valued function on $V(X)$ such that the sum of the values of the function on the vertices of any independent set is at most 1. The *weight* of a fractional clique is the sum of all of its values. The *fractional clique number* of X is defined to be the maximum weight of a fractional clique of X, and is denoted $\omega^*(X)$.

For any clique of size k, the characteristic vector of that clique is a fractional clique of X of weight k. Thus

$$\omega(X) \leq \omega^*(X).$$

Conversely, any 01-valued fractional clique of weight k corresponds to a clique of size k, thus the clique number of a graph is equal to the maximum weight of a 01-valued fractional clique.

6.1.1 Lemma. *For any graph X,*

$$\omega^*(X) \geq \frac{|V(X)|}{\alpha(X)}.$$

Proof. Consider the function f on $V(X)$ defined as $f(x) = \alpha(X)^{-1}$ for all $x \in X$. Then for any independent set $S \in \mathcal{I}(X)$, we have

$$\sum_{x \in S} f(x) = \alpha(X)^{-1}|S| \leq \alpha(X)^{-1}\alpha(X) = 1.$$

Thus f is a fractional clique of X and it has weight $|V(X)|/\alpha(X)$, which proves the result. □

In fact, for vertex transitive graphs this bound is always met.

6.1.2 Lemma. *For a vertex transitive graph X,*

$$\omega^*(X) = \frac{|V(X)|}{\alpha(X)}$$

and $\alpha(X)^{-1}\mathbf{1}$ is a fractional clique with this weight.

Proof. Suppose that g is a nonzero fractional clique of X. Then g is a function on $V(X)$. For $\gamma \in \mathrm{Aut}(X)$, define g^γ by

$$g^\gamma(x) = g(x^\gamma).$$

So g^γ is also a fractional clique of X with the same weight as g. Now consider

$$\hat{g} := \frac{1}{|\mathrm{Aut}(X)|} \sum_{\gamma \in \mathrm{Aut}(X)} g^\gamma.$$

This must also have the same weight as g. Also, it is easily seen that $\hat{g}$ is constant on $V(X)$, since $\hat{g^\gamma} = \hat{g}$. Since $c\mathbf{1}$ is a fractional clique of X if and only if $c \leq \alpha(X)^{-1}$, the result follows. □

Note that we have not actually shown that $\chi^*(X)$ and $\omega^*(X)$ are well-defined, i.e. that the respective minimums and maximums exist. We will not give the proof of this here, but it is in [2] if the reader is interested.

In our study of homomorphisms, we have seen that both the chromatic number and odd girth can be used to rule out the possibility of a homomorphism from one graph to another. Now we would like to prove a similar result for the fractional chromatic number.

First, consider a homomorphism $\varphi : X \to Y$, and an independent set S of Y. Then $\varphi^{-1}(S)$ is an independent set in X. Let T be another independent set in Y such that $S \cap \varphi(X) = T \cap \varphi(X)$. Then $\varphi^{-1}(S) = \varphi^{-1}(T)$. Thus the preimage of an independent set in Y is determined by its intersection with $\varphi(X)$.

Now suppose that f is a fractional coloring of Y. Define the function $\hat{f}$ on $\mathcal{I}(X)$ by

$$\hat{f}(S) = \sum_{T : \varphi^{-1}(T) = S} f(T).$$

We say that $\hat{f}$ is obtained by *lifting* f. If more than one independent set of Y have the same intersection with $\varphi(X)$, then they all have the same preimage $S \in \mathcal{I}(X)$, and thus all contribute to $\hat{f}(S)$. Since every independent set T in Y contributes $f(T)$ to $\hat{f}$, the weight of $\hat{f}$ is equal to that of f. We now show that $\hat{f}$ is a fractional coloring of X.

6.1.3 Theorem. *If there is a homomorphism from X to Y and f is a fractional coloring of Y, then the lift $\hat{f}$ of f is a fractional coloring of X with the same weight as f. The support of $\hat{f}$ consist of the preimages of independent sets in the support of f.*

Proof. If $u \in V(X)$, then

$$\begin{aligned} \sum_{S \in \mathcal{I}(X,u)} \hat{f}(S) &= \sum_{u \in \varphi^{-1}(T)} f(T) \\ &= \sum_{T \in \mathcal{I}(Y, \varphi(u))} f(T). \end{aligned}$$

Thus $\hat{f}$ is a fractional coloring of X. □

This theorem immediately gives us the desired result:

6.1.4 Corollary. *If there is a homomorphism from X to Y, then $\chi^*(X) \leq \chi^*(Y)$.* □

There are a few more theorems that we will need for the results of this chapter, and we state them now without proof.

6.1.5 Theorem. *For any vertex transitive graph X, we have*

$$\omega^*(X) = \frac{V|(X)|}{\alpha(X)} = \chi^*(X).$$ □

In fact, $\omega^*(X) = \chi^*(X)$ is true for any graph X, not just vertex transitive graphs. This can be proved using strong duality by reformulating fractional colorings and fractional cliques as dual linear optimization problems.

6.1.6 Theorem. *For any graph X we have*

$$\chi^*(X) = \min\{n/r : X \to K_{n:r}\}.$$ □

6.2 Upper Bound

We are now able to start proving results about the fractional chromatic number of $HH_r(n)$. The first such result that we prove is an upper bound on $\chi^*(HH_r(n))$. By Theorem 6.1.5 we know that $\chi^*(HH_r(n)) = \frac{|V(HH_r(n))|}{\alpha(HH_r(n))}$, so we can use our lower bounds for $\alpha(HH_r(n))$ to obtain upper bounds for $\chi^*(HH_r(n))$. Using Theorem 4.1.3 we obtain the following:

6.2.1 Theorem. $\chi^*(HH_r(n)) \leq \frac{n}{r}$.

Proof. By Theorem 4.1.3, $\alpha(HH_r(n)) \geq r\binom{n-1}{r}$, thus

$$\begin{aligned}
\chi^*(HH_r(n)) &= \frac{|V(HH_r(n))|}{\alpha(HH_r(n))} \\
&\leq \frac{(r+1)\binom{n}{r+1}}{r\binom{n-1}{r}} \\
&= \frac{r+1}{r} \cdot \frac{\binom{n}{r+1}}{\binom{n-1}{r}} \\
&= \frac{r+1}{r} \cdot \frac{n}{r+1} \\
&= \frac{n}{r}
\end{aligned}$$

□

Alternatively, we could have used the fact that $HH_r(n) \to K_{n:r}$, and Theorem 6.1.6 to prove this result.

Recall that this was not our only lower bound on the independence number of $HH_r(n)$. Theorem 4.3.1 states that for $n \geq r^2$,

$$\alpha(HH_r(n)) \geq \frac{|V(HH_r(n))|}{r+1} + \frac{r-1}{r+1}\binom{r^2}{r}.$$

Using this bound along with Theorem 6.1.5 we arrive at the following:

6.2.2 Theorem. *For* $n \geq r^2$,

$$\chi^*(HH_r(n)) \leq (r+1)\left(1 - \frac{(r-1)\binom{r^2}{r}}{(r+1)\binom{n}{r+1} + (r-1)\binom{r^2}{r}}\right) < r+1$$

Proof.

$$\begin{aligned}
\chi^*(HH_r(n)) &= \frac{|V(HH_r(n))|}{\alpha(HH_r(n))} \\
&\leq \frac{|V(HH_r(n))|}{\frac{|V(HH_r(n))|}{r+1} + \frac{r-1}{r+1}\binom{r^2}{r}} \\
&= (r+1)\left(\frac{|V(HH_r(n))|}{|V(HH_r(n))| + (r-1)\binom{r^2}{r}}\right) \\
&= (r+1)\left(1 - \frac{(r-1)\binom{r^2}{r}}{|V(HH_r(n))| + (r-1)\binom{r^2}{r}}\right) \\
&= (r+1)\left(1 - \frac{(r-1)\binom{r^2}{r}}{(r+1)\binom{n}{r+1} + (r-1)\binom{r^2}{r}}\right)
\end{aligned}$$

□

Note that this bound is smaller than the previous bound for $n > r^2 + 1$, because the bound for $\alpha(HH_r(n))$ we used to compute this bound is greater than the other for these values of n.

So unlike the chromatic number of the Häggkvist & Hell graphs, the fractional chromatic number is bounded for fixed r. This differs from the Kneser graphs which have fractional chromatic number $\chi^*(K_{n:r}) = n/r$ which is of course unbounded for fixed r.

6.3 Recursive Bound Using Fractional Cliques

In this section we find a recursive bound for $\chi^*(HH_r(n))$ using its equivalence with $\omega^*(HH_r(n))$ from Theorem 6.1.5. The recursive bound given here is akin to the recursive bound for the chromatic number in that they both rely heavily on the 3-cell partition π_3.

6.3.1 Theorem. *For $r \geq 2$, $n \geq 2r$, we have*

$$\omega^*(HH_r(n-1)) \leq \omega^*(HH_r(n))$$
$$\leq \frac{n}{n-r-1}\left(1 - \frac{\binom{n-1}{r}}{\binom{n-1}{r} + \alpha(HH_r(n-1))}\right)\omega^*(HH_r(n-1)).$$

Proof. The first inequality is trivially true by Theorem 6.1.4 since the inclusion map is a homomorphism. Recall the partition π_3 and its three cells: C_1 which is an $HH_r(n-1)$ subgraph, C_2 which contains all vertices with n in their tail, and C_3 which contains all vertices that have n as their head. Now consider a fractional clique f of $HH_r(n)$ with maximum weight $\omega^*(HH_r(n))$, that is constant on the vertices. We will use f to construct a fractional clique f' of $HH_r(n-1)$. Note that for all independent sets $S \subseteq C_1$, we have that $S \cup C_3$ is also independent. This implies that

$$\sum_{x \in S} f(x) + \sum_{x \in C_3} f(x) \leq 1.$$

Since f is constant on the vertices, and $|C_3| = \binom{n-1}{r}$, this gives

$$\sum_{x \in S} f(x) + \frac{\omega^*(HH_r(n))}{|V(HH_r(n))|}\binom{n-1}{r} \leq 1.$$

Now consider the fractional clique of $HH_r(n-1)$ given by

$$f'(x) = f(x) + \frac{\binom{n-1}{r}(\omega^*(HH_r(n))/|V(HH_r(n))|)}{\alpha(HH_r(n-1))}$$

for all $x \in V(HH_r(n-1))$. For any $S \in \mathcal{I}(HH_r(n-1))$, we have

$$\sum_{x \in S} f'(x)$$
$$= \sum_{x \in S} f(x) + \frac{\binom{n-1}{r}(\omega^*(HH_r(n))/|V(HH_r(n))|)}{\alpha(HH_r(n-1))}$$
$$\leq \left(\sum_{x \in S} f(x)\right) + \alpha(HH_r(n-1))\left[\frac{\binom{n-1}{r}(\omega^*(HH_r(n))/|V(HH_r(n))|)}{\alpha(HH_r(n-1))}\right]$$
$$= \left(\sum_{x \in S} f(x)\right) + \binom{n-1}{r}\frac{\omega^*(HH_r(n))}{|V(HH_r(n))|}$$
$$\leq 1$$

by the above. Therefore f' is a fractional clique of $HH_r(n-1)$, and it has weight equal to

$$
\begin{aligned}
&f'(x)|V(HH_r(n-1))| \\
&= \left(f(x) + \frac{\binom{n-1}{r}(\omega^*(HH_r(n))/|V(HH_r(n))|)}{\alpha(HH_r(n-1))} \right) |V(HH_r(n-1))| \\
&= \left(\frac{\omega^*(HH_r(n))}{|V(HH_r(n))|} + \frac{\binom{n-1}{r}\omega^*(HH_r(n))}{\alpha(HH_r(n-1))|V(HH_r(n))|} \right) |V(HH_r(n-1))| \\
&= \omega^*(HH_r(n)) \left(\frac{|V(HH_r(n-1))|}{|V(HH_r(n))|} \right) \left(1 + \frac{\binom{n-1}{r}}{\alpha(HH_r(n-1))} \right) \\
&= \omega^*(HH_r(n)) \left(\frac{n-r-1}{n} \right) \left(1 + \frac{\binom{n-1}{r}}{\alpha(HH_r(n-1))} \right).
\end{aligned}
$$

Therefore

$$
\omega^*(HH_r(n-1)) \geq \omega^*(HH_r(n)) \left(\frac{n-r-1}{n} \right) \left(1 + \frac{\binom{n-1}{r}}{\alpha(HH_r(n-1))} \right)
$$

and upon rearranging we obtain the desired inequality. □

It is worth noting that if we consider the outer inequality from above:

$$
\omega^*(HH_r(n-1)) \leq \frac{n}{n-r-1} \left(1 - \frac{\binom{n-1}{r}}{\binom{n-1}{r} + \alpha(HH_r(n-1))} \right) \omega^*(HH_r(n-1))
$$

we can solve for $\alpha(HH_r(n-1))$ and we get the following:

$$
\alpha(HH_r(n-1)) \geq \binom{n-1}{r}
$$

which is the size of the independent set we get by taking the set of vertices with the greatest element as their head. In other words, by recursively using the recursive bound from Theorem 4.2.1. This proof is more difficult, but still of interest because it exemplifies the similarities between the proofs of these two recursive bounds. Namely that both make particular use of the fact that C_3 is not just an independent set, but is also independent of C_1.

As it turns out, the above recursive upper bound on $\omega^*(HH_r(n))$ can be obtained from Theorem 4.2.1 algebraically with a shorter proof. However, this proof is of interest because it gives the bound directly using only fractional clique ideas.

6.4 Lower Bound

As we did in the case of the upper bound, we can use bounds for $\alpha(HH_r(n))$ to obtain bounds for $\chi^*(HH_r(n))$. However, this time we must use the upper bounds on $\alpha(HH_r(n))$.

Though we have only proven an upper bound for $r = 2, 3$, we will again assume that the smallest eigenvalue of π_3 is the smallest eigenvalue of $HH_r(n)$ for all r, and use the upper bound obtained with that hypothesis. Again, applying Theorem 6.1.5, we get the following:

6.4.1 Theorem. *If the smallest eigenvalue of $HH_r(n)$ is*

$$\frac{1}{2}r\binom{n-r-2}{r-2}\left[-1-\frac{1}{r-1}\sqrt{\frac{4n(n-3r-1)+r(r+3)^2}{r}}\right],$$

then

$$\chi^*(HH_r(n)) \geq \frac{2n-r-3+\sqrt{q/r}}{r-1+\sqrt{q/r}}$$

in which $q = r(r+3)^2 + 4n(n-3r-1)$.

The proof is just simple algebra. Note that if the least eigenvalue is not the one we suspect then the right hand side of the above inequality will get smaller.

We see that for fixed r, this upper bound tends to $\sqrt{r}+1$ as $n \to \infty$. So we can tentatively say that for fixed r,

$$\sqrt{r}+1 \leq \lim_{n\to\infty} \chi^*(HH_r(n)) \leq r+1.$$

We know this limit exists because $\chi^*(HH_r(n))$ is increasing (Theorem 6.3.1) and bounded above (Theorem 6.2.2).

One of the more interesting consequences of these results is that for $n > r^2 + 1$, there is some other Kneser graph $K_{n':r'}$ that $HH_r(n)$ must admit a homomorphism to. This is because the bound we gave for $\chi^*(HH_r(n))$ for these values of n is strictly less than n/r, so $K_{n:r}$ cannot be the Kneser graph for which the minimum in Theorem 6.1.6 is obtained. Which other Kneser graph $HH_r(n)$ admits a homomorphism into is still an open, and very interesting, question.

To end this chapter, we will leave you with a table of fractional chromatic numbers that we have computed.

r	n	$\chi^*(HH_r(n))$	Bound 6.2.1	Bound 6.2.2	Bound 6.4.1
2	4	2	2	2	2
2	5	5/2	5/2	5/2	$\sqrt{5}$
2	6	30/11	3	30/11	$\frac{1+\sqrt{13}}{2}$
2	7	105/37	7/2	105/37	7/3
2	8	84/29	4	84/29	$\frac{3+\sqrt{41}}{4}$
3	6	2	2	N/A	2
3	7	7/3	7/3	N/A	$3\sqrt{2}-2$
3	8	8/3	8/3	N/A	$\frac{\sqrt{33}-1}{2}$

Table 6.1: Fractional chromatic numbers of $HH_2(n)$ and $HH_3(n)$.

Chapter 7

Conclusion

We summarize our results and give some interesting open problems.

7.1 Discussion of Results

Structurally speaking we have seen many similarities between Kneser graphs and Häggkvist & Hell graphs. The three main structural parameters we investigated: girth, odd girth, and diameter, are the same for $K_{n:r}$ and $HH_r(n)$ for

$$2r + 2 \leq n \leq 3r - 1,$$

$$n < \frac{7}{3}r - \frac{1}{3}, \text{ and}$$

$$n < 3r \text{ respectively.}$$

The differences between the two graphs are explained by the heads of vertices in $HH_r(n)$, which impose lower bounds on these parameters for the larger values of n that the Kneser graphs are able to avoid. Below we have restated the significant results concerning the structure of Häggkvist & Hell graphs:

- $HH_r(n)$ is connected for $n \geq 2r + 1$.
- $diam(HH_r(n)) = \begin{cases} \infty, & \text{if } n \leq 2r \\ \max\left\{5, \left\lceil \frac{r-1}{n-2r} \right\rceil + 1\right\}, & \text{if } 2r + 1 \leq n < \frac{5}{2}r \\ 4, & \text{if } n \geq \frac{5}{2}r \end{cases}$
- $odd\ girth(HH_r(n)) = \begin{cases} \infty, & \text{if } n \leq 2r \\ \max\left\{5, 2\left\lceil \frac{r}{n-2r} \right\rceil + 1\right\}, & \text{if } n \geq 2r + 1 \end{cases}$
- $girth(HH_r(n)) = 4$
- For any subgraph G of $K_{n:r}$ with maximum degree strictly less than $\frac{n-r}{n-2r}$, there is a subgraph of $HH_r(n)$ isomorphic to G.

With respect to the automorphism groups of $K_{n:r}$ and $HH_r(n)$, both are acted on arc transitively by $Sym(n)$, and neither is distance transitive in general.

Perhaps the most significant connection between between the Kneser graphs and the Häggkvist & Hell graphs is the homomorphism that takes vertices of $HH_r(n)$ to their tails. This homomorphism gives rise to many of the results of this thesis, both structural and algebraic. It is this relation that gives us bounds on the algebraic properties of the Häggkvist & Hell graphs such as the independence, chromatic, and fractional chromatic numbers. However, these bounds do seem to drift away from the actual values of these parameters for larger values of n, resembling the above mentioned structural results. Though this specific homomorphism did not come up in the study of the structural properties of the Häggkvist & Hell graphs, the more general analogy between the vertices of $K_{n:r}$ and the tails of vertices of $HH_r(n)$ was used extensively here.

In general our results for the algebraic properties of the Häggkvist & Hell graphs are inequalities rather than equalities. This is similar to known results for other variations on Kneser graphs, such as the q-Kneser graphs, for which bounds are known for many of their algebraic properties such as clique number, chromatic number, and independence number, though there are few results that give the exact values of these parameters. This state of things can likely be ascribed to the increased complexity of these graphs, and the decreased amount of research directed at them, as compared to the Kneser graphs.

One of the major differences between the Kneser graphs and Häggkvist & Hell graphs is their clique numbers. For the Kneser graphs it is not hard to see that $\omega(K_{n:r}) = \lfloor n/r \rfloor$, whereas for the Häggkvist & Hell graphs $\omega(HH_r(n)) = 2$ for all $n \geq 2r$. This implies that for $n \geq 3r$, the clique number of $K_{n:r}$ is strictly greater than that for $HH_r(n)$. Another major difference between these two classes of graphs is that for fixed r, the fractional chromatic number of $HH_r(n)$ is bounded above by $r+1$, while it is unbounded for $K_{n:r}$. Of course this stems from the existence of an independent set of $HH_r(n)$ that is larger than the one obtained from the preimage of a maximum independent set in $K_{n:r}$ via the aforementioned homomorphism. In particular, the independent set in question has size

$$\binom{n}{r+1} + \frac{r-1}{r+1}\binom{r^2}{r}$$

which is on the order of the size of the graph.

Despite these differences, there are also some striking similarities in the algebraic properties of these graphs. One of the more notable ones being that for any fixed r, the chromatic numbers of both the Kneser and Häggkvist & Hell graphs are unbounded. Also, though the eigenvalues of these two graphs do not appear to be closely related, it is interesting to note that the number of distinct eigenvalues do not depend on n for either graph, but in fact depend only on r.

When we take a broad look at all of our results, it becomes clear that the larger n gets, the stronger the differences in $K_{n:r}$ and $HH_r(n)$ become. There could be many reasons for this trend, but we believe an important factor to be the sparsity of the graphs. We have that

$$|V(HH_r(n))| = (n-r)|V(K_{n:r})|$$

and

$$|E(HH_r(n))| = r^2|E(K_{n:r})|.$$

From this we see that as n increases, $HH_r(n)$ gets more and more sparse in comparison to $K_{n:r}$. Interestingly enough, the proportion of edges of the two graphs is equal to the proportion of vertices of the two graphs at the point $n = r^2 + r$, which is exactly the point when the independent set of vertices whose head is larger than any element in their tail is the same size as the independent set given by the homomorphism to the Kneser graph. Also interesting is that, for the smallest value of n such that the edge proportion becomes smaller than the vertex proportion, i.e. $n = r^2 + r + 1$, the chromatic number of $HH_r(n)$ did not increase from that of $HH_r(n-1)$ for the first time in the one case we computed (though this is hardly much to go on). In any case, we believe that this decreasing trend in density of Häggkvist & Hell graphs relative to Kneser graphs may be largely responsible for the decreasing similarity between the two. This explanation is particularly satisfying since decreased density is typically correlated with increased girth/odd girth, diameter, and independence number, as well as with decreased chromatic and fractional chromatic number, which coincides perfectly with the differences in these graphs for large n.

7.2 Open Problems

This research into Häggkvist & Hell graphs has produced many results, but also left us with many interesting questions. We leave you with a discussion of some such questions.

We have shown that subgraphs of $K_{n:r}$ with maximum degree strictly less than $\frac{n-r}{n-2r}$ are isomorphic to some subgraph of $HH_r(n)$; it would be interesting to see which Häggkvist & Hell graphs contain subgraphs isomorphic to whole Kneser graphs.

We know that $HH_3(n)$ contains isomorphic copies of the Petersen graph $K_{5:2}$ if and only if $n \geq 10$. This is because no two vertices in such a subgraph can have the same head, and we can construct such a subgraph by assigning a different element of $\{1, \ldots, 10\}$ as the head of each vertex in the Petersen graph, and then letting the tail of a vertex be the set of three elements that are the heads of its neighbors. In fact this must be an induced subgraph since adding any edge to the Petersen graph creates a triangle.

In fact, we can generalize this result somewhat; for any triangle-free Kneser graph $K_{n:r}$ (i.e. $n < 3r$), let $s = \binom{n-r}{r}$ and $m = \binom{n}{r}$. We can construct an isomorphic copy of $K_{n:r}$ in $HH_s(m)$ by assigning a different element of

$\{1, \ldots, \binom{n}{r}\}$ as the head of each vertex, and then filling in the tail of each vertex with the heads of its neighbors. In particular this implies that the odd graphs $K_{2r+1:r}$, with $r \geq 2$, are isomorphic to subgraphs of $HH_{r+1}\left(\binom{2r+1}{r}\right)$. However, we don't know if these are induced subgraphs in general, and we don't know if these are the only Häggkvist & Hell graphs that have subgraphs isomorphic to these Kneser graphs. A more complete study of this would be interesting.

There are quite a few results regarding homomorphisms between different Kneser graphs; we would be curious to see which of these are able to be extended to Häggkvist & Hell graphs. In particular, it is quite easy to see that $HH_r(n)$ is an induced subgraph of $HH_{tr}(tn)$ for any positive integer t, as is analogously true for Kneser graphs. However, it is not so clear as to whether there is an analogous homomorphism to the one from $K_{n:r}$ to $K_{n-2:r-1}$ [2] for Häggkvist & Hell graphs.

Of course we would like to know more eigenvalues of $HH_r(n)$-all of them, if possible. Perhaps there are some linear algebra techniques that can transform the adjacency matrix of $HH_r(n)/\pi_u$ into a matrix whose eigenvalues are easier to compute, as is the case with Kneser graphs. This of course would allow us to compute the correct ratio bound for all r.

Any improvement on bounds for the independence number, chromatic number, or fractional chromatic number, would be of interest, especially if one were able to pin down the exact value of any of them. The independence number seems likely to be the easiest candidate for the latter, though of course this immediately gives us the fractional chromatic number.

One the most striking open questions is actually one we already mentioned. Since the fractional chromatic number of $HH_r(n)$ is strictly less than that of $K_{n:r}$ for $n \geq r^2+2$, there must be some other Kneser graph $K_{n':r'}$ that $HH_r(n)$ admits a homomorphism to. We have no idea what Kneser graph this may be, or even if there is a nice form for the values of n' and r'. Also, this homomorphism may give us better bounds on the independence number or chromatic number of $HH_r(n)$, so it is a very enticing question for future study.

Also of some interest is whether it may be possible to find exact values for some of the algebraic parameters of $HH_r(2r+1)$. These graphs are the Häggkvist & Hell analogs of the odd graphs $K_{2r+1:r}$, which have been studied extensively. These graphs may be simpler to deal with since they are the minimal nontrivial examples of Häggkvist & Hell graphs. We have already been able to compute their chromatic number exactly, namely three. Perhaps the independence number of these graphs is a more tractable problem than the general case, and of course this would also give us the fractional chromatic number.

Though we did not go into the study of them, we are interested in whether or not Häggkvist & Hell graphs are cores. Cores are graphs with no proper endomorphisms, and they are the minimal elements of the equivalence classes of homomorphic equivalence. Two graphs X and Y are homomorphically equivalent if $X \to Y$ and $Y \to X$. It is known that $K_{n:r}$ is a core for $n \geq 2r+1$. We would like to see if the same is true for Häggkvist & Hell graphs. We suspect it is.

Finally, there are some quite natural generalizations of Häggkvist & Hell graphs that are of potential interest. The most obvious generalization is to let the heads of vertices be of sizes other than one. In other words, the vertices are all ordered pairs (α, β) of subsets of $[n]$ where

$$|\alpha| = r_1, \quad |\beta| = r_2, \quad \alpha \cap \beta = \varnothing.$$

Two vertices (α, β) and (α', β') are adjacent if

$$\alpha \subseteq \beta', \quad \alpha' \subseteq \beta, \quad \beta \cap \beta' = \varnothing.$$

We can also consider the q-analogs of these graphs, using subspaces as with the q-Kneser graphs, but we may be getting ahead of ourselves here.

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