Negative Correlation Properties for Matroids

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In pursuit of negatively associated measures, this thesis focuses on certain negative correlation properties in matroids. In particular, the results presented contribute to the search for matroids which satisfy

$$P(\{X : e, f \in X\}) \leq P(\{X : e \in X\})P(\{X : f \in X\})$$

for certain measures, P, on the ground set.

Let $\mathcal{M}$ be a matroid. Let $(y_g : g \in E)$ be a weighting of the ground set and let

$$Z = \sum_{X} \left(\prod_{x \in X} y_x \right)$$

be the polynomial which generates Z-sets, were $\mathsf{Z} \in \{\mathsf{B},\ \mathsf{I},\ \mathsf{S}\}$. For each of these, the sum is over bases, independent sets and spanning sets, respectively. Let e and f be distinct elements of E and let Z_e indicate partial derivative. Then $\mathcal{M}$ is Z-Rayleigh if $Z_e Z_f - Z Z_{ef} \geq 0$ for every positive evaluation of the y_gs.

The known elementary results for the B, I and S-Rayleigh properties and two special cases called negative correlation and balance are proved. Furthermore, several new results are discussed. In particular, if a matroid is binary on at most nine elements or paving or rank three, then it is I-Rayleigh if it is B-Rayleigh. Sparse paving matroids are B-Rayleigh. The I-Rayleigh difference for graphs on at most seven vertices is a sum of monomials times squares of polynomials and this same special form holds for all series parallel graphs.

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Chapter 1

Introduction

1.1 Positive and negative association

We present several results, some new, pertaining to the theory of negative association. The questions that arise here stem from the study of positive association which is more developed. To put the former into context, we give a description of positive association from [11].

A set of subsets of a set X is called an *event*. An event $\mathscr{A}$ is *increasing* if $B \supset A \in \mathscr{A}$ implies that $B \in \mathscr{A}$. A *measure* is a non-negative function,

$$\mu : 2^X \longrightarrow [0, \infty) \subset \mathbb{R},$$

for which $\mu(\mathscr{A}) = \sum_{A \in \mathscr{A}} \mu(A)$. If $\mu(A) \in \{0, 1\}$ for each $A \subseteq X$, then μ is a *counting measure*. The probability that $\mathscr{A}$ occurs is

$$P_\mu(\mathscr{A}) := \frac{\mu(\mathscr{A})}{\mu(2^X)}.$$

The subscript is omitted when μ is understood. The measure μ is *positively associated* if every pair of increasing events $(\mathscr{A}, \mathscr{B})$ satisfies

$$P(\mathscr{A} \cap \mathscr{B}) \geq P(\mathscr{A})P(\mathscr{B}).$$

The FKG theorem, due to Fortuin, Kasteleyn, and Ginibre [2], provides a locally verifiable sufficient condition for positive association. A simplified version is stated in the special case where the measure is over 2^X and the partial order is containment.

Theorem 1.1.1 (FKG)**.** *Let X be a set and let μ be a measure on 2^X. If*

$$\mu(A \cap B)\mu(A \cup B) \geq \mu(A)\mu(B)$$

for all $A, B \subseteq X$, then μ is positively associated.

Positive association has applications in a wide range of fields including physics, combinatorics, statistics, statistical mechanics and computer science, many of them surveyed in [11] and [20].

For the sake of symmetry, one might like to say that μ is negatively associated when every pair of increasing events, $(\mathscr{A}, \mathscr{B})$, satisfies $P(\mathscr{A} \cap \mathscr{B}) \leq P(\mathscr{A})P(\mathscr{B})$. However, this definition is not useful, for example, when $0 < P(\mathscr{A}) < 1$, since this gives $P(\mathscr{A} \cap \mathscr{A}) = P(\mathscr{A}) > P(\mathscr{A})P(\mathscr{A})$. To avoid this situation, we add the condition that no element of X 'affects' both $\mathscr{A}$ and $\mathscr{B}$ [11]. The element $x \in X$ *affects* $\mathscr{A}$ if there exist sets A and A' whose symmetric difference is $\{x\}$, with $A \in \mathscr{A}$ and $A' \notin \mathscr{A}$.

Definition 1.1.2 ([11])**.** *Let X be a set. A measure on 2^X is negatively associated if*

$$P(\mathscr{A} \cap \mathscr{B}) \leq P(\mathscr{A})P(\mathscr{B}),$$

for every pair of increasing events, $(\mathscr{A}, \mathscr{B})$, which are not both affected by any element $x \in X$.

Unfortunately, the analogue of the FKG condition is not sufficient in the case of negative association. In fact, there is no broad sweeping sufficient condition for negative association. Therefore it is interesting to find large classes of examples.

The research discussed here pertains to three concepts related to negative association. Let

$$Z = Z(\omega; \mathbf{y}) = \mu(\mathscr{A}), \tag{1.1.1}$$

for an event $\mathscr{A} \subseteq 2^X$ where

$$\mu(A) = \omega(A) \prod_{x \in A} y_x$$

for non-negative real y_xs and a measure ω. Let e and f be distinct elements of X and denote the *Rayleigh difference* by

$$\Delta Z\{e, f\} = Z_e Z_f - Z Z_{ef}, \tag{1.1.2}$$

where the subscripts indicate partial derivatives with respect to y_e and y_f. Notice that $\Delta Z\{e, f\} \geq 0$ is equivalent to

$$P_\mu(\{X : e, f \in X\}) \leq P_\mu(\{X : f \in X\}) P_\mu(\{X : e \in X\}).$$

Let ω be a counting measure. If $y_x = 1$ for all $x \in X$, then ω is *negatively correlated* if $\Delta Z\{e, f\} \geq 0$ for every pair of distinct elements $e, f \in X$. Suppose ω is negatively correlated and consider the Rayleigh difference, $\Delta Z\{e, f\}$, as $y_x \to \infty$, $y_x \to 0$ or $y_x = 1$ for each $x \in X - \{e, f\}$. If $\Delta Z\{e, f\} > 0$ under these conditions, then ω is *balanced*. A general measure ω is *Rayleigh* if for every pair of distinct $e, f \in X$ and every positive evaluation of the y_xs,

$$\Delta Z\{e, f\} \geq 0.$$

This thesis focuses on these properties applied to the ground set of a matroid. The properties defined above are prefixed with B, I and S when ω is the counting measure for bases, independent sets and spanning sets, respectively, of a matroid. For general measures the prefix Z is used.

When ω is a counting measure on X, the Rayleigh property implies balance which in turn implies negative correlation. Feder and Mihail [25] prove that a matroid is B-negatively correlated if and only if ω is negatively associated. That is, to show that the counting measure for bases of a matroid is negatively associated, it is enough to show that

$$P_\omega(\{X : e, f \in X\}) \leq P_\omega(\{X : f \in X\}) P_\omega(\{X : e \in X\}),$$

for each distinct pair of elements $e, f \in X$. This is conjectured for independent and spanning set counting measures but is not known to be true.

An application of balance, also brought about by Feder and Mihail, is that if a matroid is B-balanced, then there is a rapid convergence of random walks on the basis exchange graph of a matroid, to an almost uniform random sampling of the bases [25]. This can be used to estimate the number of bases, which for many matroids is hard [12].

Negative correlation, balance and the Rayleigh property are an active area of research. Thus far, the results on matroids focus mainly on bases. Several significant classes of matroids have been found to be B-Rayleigh, including regular matroids and binary matroids not containing a matroid called $\mathscr{S}_8$ as a minor. Most of our work is towards finding sufficient conditions for I-Rayleigh matroids. Before discussing the specifics of our results, these properties are illustrated with a small example.

1.2 An example

Negative correlation and the Rayleigh property are slowly reintroduced here, and formally in Chapter 2, as they apply to graphs and matroids. First, in the context of the previous section and then with a simple example. Recall that negative correlation applies only to counting measures.

Let $\mathcal{G} = (V, E)$ and let μ be the measure counting spanning trees of $\mathcal{G}$ so that $\mu(A) = 1$ if A is a spanning tree of $\mathcal{G}$ and $\mu(A) = 0$ otherwise, for $A \subseteq E$. The graph $\mathcal{G}$ is B-negatively correlated if for every pair of distinct edges e and f,

$$P(\{A : e, f \in A\}) \leq P(\{A : e \in A\})P(\{A : f \in A\}). \tag{1.2.1}$$

Written in terms of μ this gives

$$\frac{\mu(\{A : e, f \in A\})}{\mu(2^E)} \leq \frac{\mu(\{A : e \in A\})}{\mu(2^E)} \frac{\mu(\{A : f \in A\})}{\mu(2^E)},$$

or equivalently,

$$\mu(\{A : e \in A\})\mu(\{A : f \in A\}) - \mu(\{A : e, f \in A\})\mu(2^E) \geq 0.$$

This is further clarified with an example. Here and elsewhere, curly braces are dropped from small sets wherever possible. Consider the triangle, K_3, with edge labels e, f, g. Its spanning trees are fe, fg and eg, so let

$$T = |\{fe, fg, eg\}| = 3$$

be the total number of spanning trees, and let

$$T_f = |\{fe, fg\}| = 2$$

be the number of spanning trees containing f, defining T_S similarly for any $S \subseteq efg$. Notice that

$$\frac{2}{3} = \frac{T_e}{T} \geq \frac{T_{ef}}{T_f} = \frac{1}{2}.$$

This is B-negative correlation for the pair of edges e, f. By symmetry, K_3 is B-negatively correlated for any two distinct edges.

Suppose we ask whether e and f are negatively correlated relative to the counting measure on spanning forests. Let

$$F = |\{\varnothing, e, f, g, ef, eg, fg\}| = 7$$

be the total number of spanning forests, and for $S \subseteq E$ define F_S similarly to T_S. The triangle is I-negatively correlated, by symmetry, since

$$\frac{3}{7} = \frac{F_e}{F} \geq \frac{F_{ef}}{F_f} = \frac{1}{3}. \tag{1.2.2}$$

The Rayleigh property is a generalization of negative correlation. Assign weights $\mathbf{y} = (y_e, y_f, y_g)$ to the edges e, f and g, respectively, and let

$$T(\mathbf{y}) = y_f y_e + y_f y_g + y_e y_g,$$

be the generating polynomial for spanning trees of K_3, and let

$$y_f T_f(\mathbf{y}) = y_f \frac{\partial T(\mathbf{y})}{\partial y_f} = y_f(y_e + y_g),$$

generate those spanning trees containing f. The B-Rayleigh difference is

$$\begin{aligned}\Delta T(\mathbf{y})\{e,f\} =& T_e(\mathbf{y})T_f(\mathbf{y}) - T(\mathbf{y})T_{ef}(\mathbf{y}) \\ =& (y_f + y_g)(y_e + y_g) - (y_f y_e + y_f y_g + y_e y_g)(1) = (y_g)^2,\end{aligned}$$

which is non-negative for all positive evaluations of $\mathbf{y}$. This proves that K_3 is B-Rayleigh, by symmetry.

Relative to the polynomial $T(\mathbf{y})$, B-negative correlation for K_3 only required

$$\frac{T_e(\mathbf{y})}{T(\mathbf{y})} \geq \frac{T_{ef}(\mathbf{y})}{T_f(\mathbf{y})}$$

for $\mathbf{y} = (1,1,1)$, however, the B-Rayleigh property requires this to be true for all positive evaluations.

1.3 Overview

Basic definitions and notation are covered in the first two sections of Chapter 2. We give formal definitions for generating polynomials, define duality and matroid sums as well as prove several useful relations. Section 2.1.4 and Section 2.1.5 describe certain matroid constructions. The focus is on series parallel matroids, excluded minor characterizations for graphic matroids and constructions for regular and binary matriods. The latter part of Chapter 2 discusses further negative correlation, balance and the Rayleigh property as they relate to B, I and S counting measures on matroids. In Section 2.2 we prove that Z-balance is equivalent to Z-negative correlation for minors when $\mathsf{Z} \in \{\mathsf{B}, \mathsf{I}, \mathsf{S}\}$. Section 2.3 explains the motivation for the term 'Rayleigh' and describes several equivalent forms of the Rayleigh difference. Chapter 2 ends with an easy but useful proof that the **I**-Rayleigh property implies the **B**-Rayleigh property.

Chapter 3 begins with known results on the behaviour of the Rayleigh properties under minors, duality, direct sums and two-sums. Let $\mathcal{M}$ be a **Z**-Rayleigh matroid with ground set E and let $Z = Z(\omega; \mathbf{y})$ be defined as in (1.1.1) for an arbitrary measure ω on 2^E. Then for distinct elements $e, f, g \in E$,

$$Z_e^g Z_f^g - Z^g Z_{ef}^g \geq 0$$

and

$$Z_{ge} Z_{gf} - Z_g Z_{gef} \geq 0,$$

where the superscript indicates that $y_g = 0$. Equivalently, any minor of $\mathcal{M}$ is **Z**-Rayleigh. We use the fact that bases are complement to dual bases to show that a matroid is **B**-Rayleigh if and only if its dual is **B**-Rayleigh and similarly, a matroid is **I**-Rayleigh if and only if its dual is **S**-Rayleigh. These properties are used to show that if two matroids are **Z**-Rayleigh, then so are their direct sum and any two-sum, where $\mathbf{Z} \in \{\mathbf{B}, \mathbf{I}, \mathbf{S}\}$. Some but not all of these properties extend to negative correlation and balance. In particular, neither of these are closed by taking two-sums and negative correlation is not closed by taking minors.

Let $\mathcal{M}$ be a matroid with ground set E and distinct elements $e, f \in E$ so that $\{e, f\}$ is either dependent or dependent in the dual. In Section 3.3 we prove that if $y_g > 0$ for each $g \in E - \{e, f\}$, then

$$\Delta Z(\omega)\{e, f\} \geq 0,$$

whenever ω is the **B**, **I** or **S** counting measure. Furthermore, if ω_B, ω_I and ω_S are the **B**, **I** and **S** counting measures, respectively, then for dependent or dual-dependent $\{e, f\}$,

$$\Delta Z(\omega_I)\{e, f\} - \Delta Z(\omega_B)\{e, f\}$$

and

$$\Delta Z(\omega_I)\{e,f\} - \Delta Z(\omega_B)\{e,f\}$$

have positive coefficients when the y_gs are taken as indeterminates for $g \in E - \{e,f\}$.

Section 3.4 provides a comprehensive list of sufficient conditions for Z-negatively correlated, Z-balanced and Z-Rayleigh matroids, where $\mathsf{Z} \in \{\mathsf{B}, \mathsf{I}, \mathsf{S}\}$. Among these, the sixth root of unity matroids are B-Rayleigh ([30], [31]), binary B-Rayleigh matroids are characterized by the exclusion of a minor called $\mathscr{S}_8$ [31] and rank three matroids are B-Rayleigh [27]. Matroids on at most nine elements are I-negatively correlated (Royle, Wagner, private communication) and graphs on eight vertices or nine vertices and eighteen edges are I-negatively correlated [7]. Jerrum has proved that sparse paving matroids are balanced and that sparseness is necessary [13]. The list is short and it is clear that any new ideas would make up a significant proportion of the known results.

The first of these is the fact that rank three matroids are I-Rayleigh. Note that while this was shown independently by Cocks [5], we prove the stronger fact that

$$\Delta Z(\omega_I)\{e,f\} - \Delta Z(\omega_B)\{e,f\} \tag{1.3.1}$$

has positive coefficients, for distinct e, f and ω_I, ω_B defined as above. Rank three matroids are B-Rayleigh ([31]) which says that $\Delta Z(\omega_B)\{e,f\} \geq 0$ and together these imply that $\Delta Z(\omega_I)\{e,f\} \geq 0$. Thus rank three matroids satisfy the I-Rayleigh property.

Also new in Chapter 3, a binary matroid on at most nine elements is I-Rayleigh if and only if it is B-Rayleigh. This follows partly from the above result on rank three matroids and the fact, from Chapter 5, that all graphs on at most seven vertices are I-Rayleigh. To complete the proof we employ several decomposition lemmas from Section 2.1.5 and prove that three particular matroids are I-Rayleigh. These three are the dual of the Fano matroid, $\mathscr{AG}(3,2)$ (affine geometry) and the dual matroid of $K_{3,3}$. As it turns out, there are only three more obstructions to proving the result for binary matroids on ten elements. They are a regular matroid on ten elements called R_{10} (see [22]), and the duals of K_5 and $K_{3,3}$ plus an edge (not parallel to any other).

The chapter concludes with a discussion of two conjectures: regular matroids are I-Rayleigh; a binary matroid is I-Rayleigh if and only if it is B-Rayleigh. The former seems more plausible, however, we present a new proof that they are equivalent. Crucial to a positive answer to both of these is a proof of the conjecture that graphs are I-Rayleigh, in print since the early 1990s [14] and still open.

Chapter 4 stems directly from a paper on paving matroids by Mark Jerrum [13]. His results are presented, including the fact that sparse paving matroids are B-balanced and there is a non B-balanced (non-sparse) paving matroid. Jerrum's example is constructed. This result is of interest because the bases of sparse paving matroids can be hard to count and having the negative correlation property enables faster approximations of the number of bases. The main result of this chapter proves that if a paving matroid is B-Rayleigh then it is also I-Rayleigh. Jerrum's proof that sparse paving matroids are B-balanced is used to prove the new result that sparse paving matroids are indeed B-Rayleigh. As a result of the fact that B-Rayleigh paving matroids are I-Rayleigh, the proof that rank three matroids are I-Rayleigh is re-derived. Furthermore, we prove that sparse paving matroids are closed by taking duals, implying that some rank three matroids are also S-Rayleigh. This answers more than half of a conjecture by Semple and Welsh: rank three matroids are I and S-Rayleigh [4].

Since it is known that graphs are B-Rayleigh (and equivalently, B-negatively correlated [31]), it is natural to ask whether graphs are I-Rayleigh. The question appears in several papers ([20], [7], [4], [28], [14]) and in the last decade considerable evidence has been found to support a positive answer; however, little progress has been made on a proof. The efforts in [7], [4] and work by Wagner, including [28], essentially amount to proving the result for series parallel graphs and proving that graphs on at most eight vertices and nine vertices with at most 18 edges are I-negatively correlated. Chapter 5 begins with a proof, which does not appear in print, that if all graphs are I-negatively correlated, then they are all I-Rayleigh as well (published independently by Cocks [5]).

Although the work in this thesis also focuses on series parallel and small graphs, we show that for some of these graphs, the I-Rayleigh difference can be written as a sum of monomials times squares of polynomials, which proves its non-negativity. Let $\mathcal{G} = (V, E)$ be a graph with distinct edges e and f. Then for $S \subseteq E$, let $\mathbf{y}^S = \prod_{g \in S} y_g$. Wagner conjectures that if ω_I is the counting measure for (edge sets of) spanning forests of $\mathcal{G}$, then

$$\Delta Z(\omega_I)\,\{e, f\} = \sum_{S \subseteq E} \mathbf{y}^S A(S)^2,$$

where the sum is over sets S which are contained in cycles through both e and f and for each S, $A(S)$ is a polynomial. The I-Rayleigh property follows immediately from this, since it is non-negative for every positive evaluation of the y_gs. The polynomials $A(S)$ are sums of a certain form with signs which have not yet been determined. A computer program and some educated guesses for the signs shows that Wagner's conjecture holds for graphs on at most seven vertices. The conjecture is further supported by somewhat technical proofs that the special form satisfies most of the necessary conditions covered in Chapter 3. In particular, it behaves properly for minors, direct sums and series parallel graphs. The chapter concludes with considerable evidence that it holds for two-sums as well.

Summarizing the main new results presented in this thesis, if a matroid is binary on at most nine elements or paving or rank three, then it is I-Rayleigh if it is B-Rayleigh. Sparse paving matroids are B-Rayleigh.

The I-Rayleigh difference for graphs on at most seven vertices is a sum of monomials times squares of polynomials and this same special form holds for all series parallel graphs.

Chapter 2

Preliminaries

2.1 Matroid theory

For an introduction to matroid theory one may consult Oxley [17]. The reader restricted to graph theory should note that graphs are, for our purposes, a subclass of regular matroids which are a subclass of binary matroids. Chapter 4, on the other hand, is not graph theoretic.

The reader who is familiar with matroid theory and generating polynomials may go directly to Section 2.1.5 where we give several matroid constructions that are used in Chapter 3.

2.1.1 Basic definitions and notation

We summarize the notation for a matroid, $\mathcal{M}$, as follows, dropping the $\mathcal{M}$ whenever it is understood from the context.

ground set	$E(\mathcal{M})$
rank of $X \subseteq E(\mathcal{M})$	$r_{\mathcal{M}}(X)$
bases	$\mathscr{B}(\mathcal{M})$
independent sets	$\mathscr{I}(\mathcal{M})$
spanning sets	$\mathscr{S}(\mathcal{M})$
circuits	$\mathscr{C}(\mathcal{M})$
$\mathcal{M}$ contract $C \subseteq E(\mathcal{M})$	$\mathcal{M}/C$
$\mathcal{M}$ delete $D \subseteq E(\mathcal{M})$	$\mathcal{M} \setminus D$
dual	$\mathcal{M}^*$
isomorphic matroids	$\mathcal{M} \cong \mathcal{N}$

Let $\mathbf{y} = (y_e : e \in E)$ be a weighting of the ground set and let

$$\mathbf{y}^X = \prod_{e \in X} y_e,$$

where $X \subseteq E$. Write $t\mathbf{y} = (ty_e : e \in E)$ for any scalar t and $\mathbf{y}^{-1} = (1/y_e : e \in E)$. We frequently employ three generating polynomials which generate independent sets,

$$I = I(\mathcal{M}; \mathbf{y}) = \sum_{X \in \mathscr{I}} \mathbf{y}^X,$$

bases,

$$B = B(\mathcal{M}; \mathbf{y}) = \sum_{X \in \mathscr{B}} \mathbf{y}^X,$$

and spanning sets,

$$S = S(\mathcal{M}; \mathbf{y}) = \sum_{X \in \mathscr{S}} \mathbf{y}^X.$$

These are each special cases of the general weighted generating polynomial

$$Z = Z(\omega; \mathbf{y}) = \sum_{X \subseteq E} \omega(X)\mathbf{y}^X,$$

where $\omega : 2^E \longrightarrow [0,\infty)$ is a weighting of the subsets of E. Here, E may not be the ground set of a matroid, however, it is convenient to confuse the notation. Again, we drop $\mathbf{y}$ and ω when they are understood.

Define

$$Z_e = \frac{\partial Z}{\partial y_e},$$

and

$$Z^e = Z|_{y_e=0},$$

for any element $e \in E$. This notation extends naturally to subsets of E. Note that Z is *multi-affine*, since y_e occurs at most to the first power. Therefore, $y_e Z_e$ and Z^e are the terms of Z containing and not containing y_e, respectively, and we can write

$$Z = Z^e + y_e Z_e.$$

Some set theoretic notation and notational abuses are tabulated. Let X and Y be sets,

symmetric difference	$X \triangle Y$	
disjoint union	$X \dot{\cup} Y$	(X, Y disjoint)
set minus	$X - Y$	$X \setminus Y$ is reserved for matroid deletion
$\{x_1, x_2, \ldots, x_k\}$ for small k	$x_1 x_2 \cdots x_k$	whenever this is unambiguous.

For disjoint subsets $C, D \subseteq X$ and $\mathscr{A} \subseteq 2^X$, use

$$\mathscr{A}_C^D = \{A \in \mathscr{A} : C \subseteq A, D \subseteq X - A\}.$$

For $\mathscr{A}, \mathscr{B} \subseteq 2^X$, write

$$\mathscr{A} \vee \mathscr{B} = \{A \cup B : A \in \mathscr{A}, B \in \mathscr{B}\}.$$

Let P and Q be polynomials. If $P - Q$ has positive coefficients, write

$$P \gg Q.$$

Lastly, let V, W be generating polynomials for certain collections of subsets of E. Define (v, w) to be a *pair of type* VW if $\mathbf{y}^{v \dot\cup w} = \mathbf{y}^{\alpha}$, v is generated by V and w is generated by W.

As a rule of thumb, expect to see $\mathcal{M}, \mathcal{N}, \mathcal{L}, \mathcal{G}, \mathcal{H}, \mathcal{K}$ for graphs and matroids, $\mathscr{B}, \mathscr{I}, \mathscr{S}, \mathscr{F}, \mathscr{T}$ for subsets of 2^E, B, I, S, F, T for polynomials generating these, e, f, g, h, k for elements of the ground set and various other non-script letters for subsets of E.

2.1.2 Minors and duality

The *dual* of $\mathcal{M}$, denoted $\mathcal{M}^*$, is the matroid whose bases are the complements of bases of $\mathcal{M}$. Its basis generating polynomial is

$$B(\mathcal{M}^*; \mathbf{y}) = \mathbf{y}^E B(\mathcal{M}; \mathbf{y}^{-1}). \tag{2.1.1}$$

Similarly, independent sets and spanning sets are dual to each other so that

$$I(\mathcal{M}^*; \mathbf{y}) = \mathbf{y}^E S(\mathcal{M}; \mathbf{y}^{-1}). \tag{2.1.2}$$

Properties pertaining to the dual are prefixed with *co-*. It is useful to know that $(\mathcal{M}^*)^* = \mathcal{M}$ and that $(M \setminus D/C)^* = M^*/D \setminus C$, for $C \cap D = \varnothing$.

Shifting our attention to minors, the proof of the following is trivial but the fact is referred to often.

Lemma 2.1.1. *Let $\mathcal{M}$ be a matroid with $g \in E$, then*

$$I(M \setminus g) \gg I(M/g).$$

Denote the top degree terms of $Z(\omega; \mathbf{y})$ by

$$Z^{\mathsf{T}} := \lim_{\lim_{t \longrightarrow \infty}} t^{-\deg(Z)} Z(t\mathbf{y}).$$

Since B consists of the top degree terms of I,

$$B(\mathcal{M}; \mathbf{y}) = \lim_{\lim_{t \longrightarrow \infty}} t^{-r} I(\mathcal{M}; t\mathbf{y}) = I(\mathcal{M}; t\mathbf{y})^{\mathsf{T}}.$$

Let $C, D \subseteq E$ with $C \cap D = \varnothing$. Then C is dependent if and only if $\mathbf{y}^C$ does not appear as a factor in any term of B, and D is co-dependent if and only if $\mathbf{y}^D$ does not appear in any term of $B(\mathcal{M}^*; \mathbf{y})$. If D is co-dependent, by (2.1.1) every term of B contains some y_e with $e \in D$. Thus if C is dependent or D is co-dependent then $B_C^D = 0$. Otherwise $B_C^D = (I^\mathsf{T})_C^D$ but also $B_C^D = B(\mathcal{M}/C \setminus D) = I(\mathcal{M}/C \setminus D)^\mathsf{T} = (I_C^D)^\mathsf{T}$, so in general

$$B_C^D = \lim_{\lim_{t \longrightarrow \infty}} t^{-(r-|C|)} I_C^D(\mathcal{M}; t\mathbf{y}). \tag{2.1.3}$$

It is useful to have an interpretation of Z_C^D in terms of minors when Z is one of B, I, or S. Consider the situations where e is a loop, co-loop and otherwise.

(i) If e is a loop then

$$\begin{array}{l|l} B^e = B = B(\mathcal{M} \setminus e; \mathbf{y}) & B_e = 0 \neq B(\mathcal{M}/e; \mathbf{y}) \\ I^e = I = I(\mathcal{M} \setminus e; \mathbf{y}) & I_e = 0 \neq B(\mathcal{M}/e; \mathbf{y}) \\ S^e = S_e = S(\mathcal{M} \setminus e; \mathbf{y}) = S(\mathcal{M}/e; \mathbf{y}) & \end{array} \tag{2.1.4}$$

(ii) If e is a co-loop then

$$\begin{array}{l|l} B^e = 0 \neq B(\mathcal{M} \setminus e; \mathbf{y}) & y_e B_e = B = y_e B(\mathcal{M}/e; \mathbf{y}) \\ & I_e = I^e = I(\mathcal{M}/e; \mathbf{y}) = I(\mathcal{M} \setminus e; \mathbf{y}) \\ S^e = 0 \neq S(\mathcal{M} \setminus e; \mathbf{y}) & S_e = S(\mathcal{M}/e; \mathbf{y}). \end{array} \tag{2.1.5}$$

(iii) If $C, D \subseteq E$ with $C \cap D = \varnothing$ and they are neither dependent nor co-dependent then

$$Z_C^D = Z(\mathcal{M} \setminus D/C; \mathbf{y}).$$

While the identities for B and I are fairly obvious, one can use the following to make the case for S. If e is a loop, then it is a co-loop of $\mathcal{M}^*$ so $(I^*)^e = (I^*)_e$. Therefore

$$S^e = (\mathbf{y}^E I^*(\mathbf{y}^{-1}))^e = (\mathbf{y}^E I^*(\mathbf{y}^{-1}))_e = S_e.$$

If e is a co-loop then it appears in every basis. A set is spanning if and only if it contains a basis, so clearly $S^e = 0$. On the other hand $B_e = B(\mathcal{M}/e;\mathbf{y})$ and $\mathbf{y}^\alpha$ is a term of S_e if and only if it contains a term of B_e, so $S_e = S(\mathcal{M}/e;\mathbf{y})$.

Analogous statements can be made about dependent and co-dependent sets.

(a) Let $X \subseteq E$ be a dependent set. Consider a maximal independent subset $X' \subset X$ and notice that every element of $X - X'$ is a loop in the matroid M/X'. Using (i), since $X' \neq X$,

$$\begin{array}{l|l} B^X = B(\mathcal{M}\setminus X;\mathbf{y}) & B_X = (B_{X'})_{X-X'} = 0 \\ I^X = I(\mathcal{M}\setminus X;\mathbf{y}) & I_X = (I_{X'})_{X-X'} = 0 \\ S^X = (S^{X'})^{X-X'} = S(\mathcal{M}\setminus X;\mathbf{y}) & S_X = (S_{X'})_{X-X'} = S(\mathcal{M}/X;\mathbf{y}). \end{array}$$

(b) Let $X \subseteq E$ be a co-dependent set and let $X' \subset X$ be a maximal co-independent subset. Notice that every element of $X - X'$ is a co-loop in the matroid $M\setminus X'$. Using (ii), since $X' \neq X$,

$$\begin{array}{l|l} B^X = (B^{X'})^{X-X'} = 0 & B_X = B(\mathcal{M}/X;\mathbf{y}) \\ I^X = (I^{X'})^{X-X'} = I(\mathcal{M}\setminus X;\mathbf{y}) & I_X = (I_{X'})_{X-X'} = I(\mathcal{M}/X;\mathbf{y}) \\ S^X = (S^{X'})^{X-X'} = 0 & S_X = (S_{X'})_{X-X'} = S(\mathcal{M}/X;\mathbf{y}). \end{array}$$

2.1.3 Matroid sums and connectivity

The *direct sum* of matroids $\mathcal{N}$ and $\mathcal{L}$ on disjoint ground sets, is the matroid whose bases are $\mathscr{B}(\mathcal{N}) \vee \mathscr{B}(\mathcal{L})$.

Let $\mathcal{N}$ and $\mathcal{L}$ be matroids, each on at least three elements, with exactly one element, g, in common which is neither a loop nor a co-loop in either of them. The *two-sum* of $\mathcal{N}$ and $\mathcal{L}$, denoted

$$\mathcal{M} = \mathcal{N} \oplus_g \mathcal{L},$$

is the matroid on the set $E(\mathcal{M}) = E(\mathcal{N}) \cup E(\mathcal{L}) - g$, whose circuits are the elements of the set

$$\mathscr{C}(\mathcal{M}) = \mathscr{C}(\mathcal{N} \setminus g) \dot\cup \mathscr{C}(\mathcal{L} \setminus g) \dot\cup (\mathscr{C}_N \vee \mathscr{C}_L),$$

in which $\mathscr{C}_N$ are circuits of $\mathcal{N}$ through g, minus g, and similarly for $\mathscr{C}_L$. The matroids $\mathcal{N}$ and $\mathcal{L}$ are called the *factors* of the two-sum. This extends to graphs with one caveat . If the edge $g = uv$ in one factor and $g = u'v'$ in the other, the factors can be two-summed either by identifying u with u' and v with v' or u with v' and v with u'. Matroids do not make this distinction; however, it does not affect the edge sets of spanning trees. The notation $\mathcal{N} \oplus_g \mathcal{L}$ implies that $\mathcal{N}$, $\mathcal{L}$ and g satisfy the necessary conditions for forming a two-sum of matroids.

Let $A \subseteq E(\mathcal{M})$ with $e \in E(\mathcal{M}) - A$. The element e is in the *closure* of A, denoted by $\mathrm{cl}(A)$, if $A \cup e$ contains a circuit containing e. Equivalently, e is in the closure of A if $r(A \cup e) = r(A)$.

In the next two lemmas $B(\mathcal{M};\mathbf{y})$ and $I(\mathcal{M};\mathbf{y})$ are characterized in terms of the generating polynomials of the factors of $\mathcal{M}$. These are particularly useful when studying the behaviour of the Rayleigh difference under two-sums.

Lemma 2.1.2. *Let $\mathcal{N}$ and $\mathcal{L}$ be matroids and let $\mathcal{M} = \mathcal{N} \oplus_g \mathcal{L}$. If $M = B(\mathcal{M};\mathbf{y}), N = B(\mathcal{N};\mathbf{y})$, and $L = B(\mathcal{L};\mathbf{y})$, then*

$$M = N_g L^g + N^g L_g. \tag{2.1.6}$$

Proof. A set X in $\mathcal{M}$ is a basis of $\mathcal{M}$ if and only if X partitions into a basis of $\mathcal{N} \backslash g$ and a basis of $\mathcal{L} \backslash g$, so that g is in the closure of exactly one of these. Such sets are generated by $N_g L^g + N^g L_g$. □

Lemma 2.1.3. *Let $\mathcal{N}$ and $\mathcal{L}$ be matroids and let $\mathcal{M} = \mathcal{N} \oplus_g \mathcal{L}$. If $M = I(\mathcal{M};\mathbf{y}), N = I(\mathcal{N};\mathbf{y})$, and $L = I(\mathcal{L};\mathbf{y})$, then*

$$M = N_g L^g + N^g L_g - N_g L_g. \tag{2.1.7}$$

Proof. A set X in $\mathcal{M}$ is independent if and only if X partitions into an independent set of $\mathcal{N} \backslash g$ and an independent set of $\mathcal{L} \backslash g$, so that g is in at most one of their closures. These sets appear in the terms of $N_g L^g + N^g L_g$, however, since N^g is termwise greater than N_g, and similarly for L, then $N_g L_g$ is generated twice and must be subtracted. □

Lemma 2.1.2 can be used to show that the dual of a two-sum is the two-sum of the duals of the factors.

Lemma 2.1.4. *If $\mathcal{M} = \mathcal{N} \oplus_g \mathcal{L}$, then*

$$\mathcal{M}^* = \mathcal{N}^* \oplus_g \mathcal{L}^*. \tag{2.1.8}$$

Proof. Let $M^* = B(\mathcal{M}^*; \mathbf{y}), N^* = B(\mathcal{N}^*; \mathbf{y})$, $L^* = B(\mathcal{L}^*; \mathbf{y})$ and M, N, L as in Lemma 2.1.2 and let $E = E(\mathcal{M})$. Since g is neither a loop nor a co-loop, $N^g = B(\mathcal{N} \setminus g; \mathbf{y})$ and $N_g = B(\mathcal{N}/g; \mathbf{y})$. Deletion and contraction are duals of each other, $(N^g)^* = (N^*)_g$ and $(N_g)^* = (N^*)^g$; thus

$$\begin{aligned}
M^* =& \mathbf{y}^E M(\mathbf{y}^{-1}) \\
=& \mathbf{y}^E (N_g L^g + N^g L_g)(\mathbf{y}^{-1}) \\
=& (N_g)^* (L^g)^* + (N^g)^* (L_g)^* \\
=& (N^*)^g (L^*)_g + (N^*)_g (L^*)^g.
\end{aligned}$$

So the bases of $\mathcal{M}^*$ are exactly those of $\mathcal{N}^* \oplus_g \mathcal{L}^*$, as required. □

The definitions of two and three-connectedness are given in terms of sums of matroids. A matroid has a *one-separation* if and only if it is the direct sum of two matroids. Seymour proves in [22] that a matroid has a *two-separation* if and only if it is expressible as a two-sum of proper minors. A matroid is *two-connected* if and only if it contains no one-separation and it is *three-connected* if and only if it contains neither a one-separation nor a two-separation.

A characterization of the behaviour of the Rayleigh properties over three-sums would allow the use of Seymour's decomposition of regular matroids [22] in a proof (or disproof) that regular matroids are **I**-Rayleigh. Unfortunately this has not yet been found.

A brief definition of three-sums is given, however, some details are overlooked. Let $\mathcal{M}$ be a binary matroid. Define the *cycles* of $\mathcal{M}$ to be the disjoint unions of circuits. It is not difficult to show that if C and C' are cycles, then $C \triangle C'$ is a cycle. Furthermore, a binary matroid is uniquely determined by its cycles. Let $\mathcal{N}$ and $\mathcal{L}$ be binary matroids whose ground sets intersect in a three-circuit, K, which does not contain a co-circuit of either of them. Require furthermore that their ground sets have at least seven elements. The *three-sum* of $\mathcal{N}$ and $\mathcal{L}$ is the matroid on ground set $E(\mathcal{N}) \triangle E(\mathcal{L})$, whose cycles are the symmetric differences of cycles in $E(\mathcal{N}) - K$ and $E(\mathcal{L}) - K$. In a graphic matroid, the cycles are the even subgraphs of the underlying graph. To visualize this for graphs, let $\mathcal{H}$ and $\mathcal{K}$ be graphs both containing K_3 as a subgraph such that it does not contain an edge cut of either $\mathcal{H}$ or $\mathcal{K}$. The graph obtained by identifying $\mathcal{H}$ and $\mathcal{K}$ on this triangle and deleting the edges of the triangle is a three-sum of $\mathcal{H}$ and $\mathcal{K}$.

2.1.4 Series parallel

Let $\mathcal{M}$ be a matroid with $g \in E$ and let $\mathcal{N}$ be a matroid with ground set $E(\mathcal{N}) = E(\mathcal{M} \setminus g) \cup \{e, f\}$. If e and f are parallel elements of $\mathcal{N}$ and both $\mathcal{N} \setminus e$ and $\mathcal{N} \setminus f$ are isomorphic to $\mathcal{M}$, then $\mathcal{N}$ is the *parallel extension* of $\mathcal{M}$ along g. If e and f are a series pair of $\mathcal{N}$ and both $\mathcal{N}/e$ and $\mathcal{N}/f$ are isomorphic to $\mathcal{M}$, then $\mathcal{N}$ is the *series extension* of $\mathcal{M}$ along g. A *series-parallel* matroid is one constructed via repeated series and parallel extensions of the free matroid on one element. Series parallel matroids are graphic.

Series and parallel extensions of $\mathcal{M}$ along g are equivalent to

$$\mathcal{M} \oplus_g \mathcal{U}_{1,3}$$

and

$$\mathcal{M} \oplus_g \mathcal{U}_{2,3},$$

respectively, where $\mathcal{U}_{r,n}$ is the uniform matroid of rank r on n elements. Note that with the given definition of two-sums, not all series parallel graphs are obtained this way (for example, various free matroids). On the other hand, the remaining ones are minors of those obtained via series and parallel extensions of $M(K_3)$ and $M(K_3)^*$.

2.1.5 Matroid decompositions

We use these constructions to make claims about the $\mathbb{I}$-Rayleigh condition on small graphic, regular and binary matroids in Section 3.6. They are well known and their proofs are outside of the scope of this thesis.

Let A be a matrix over a field, $\mathbb{F}$, and let $M(A)$ be the matroid whose elements are the columns of A and whose independent sets are those which are linearly independent over $\mathbb{F}$. We call A a representation of $M(A)$, and $\mathcal{M}$ is $\mathbb{F}$*-representable* if there is a matrix, A, over $\mathbb{F}$ such that $\mathcal{M} = M(A)$. It follows from elementary matrix manipulations that the class of $\mathbb{F}$-representable matroids is closed under duals, minors, direct sums and two-sums.

A matroid is *binary* if it is $GF(2)$-representable. This important class of matroids has the following excluded minor characterization.

Proposition 2.1.5 (Tutte (1958), [26]). *A matroid is binary if and only if it does not contain $\mathcal{U}_{2,4}$ as a minor.*

Let $\mathcal{G}$ be a graph and write $M(\mathcal{G})$ for the matroid whose elements are the edges of $\mathcal{G}$ and whose circuits are the (edge sets of) cycles of $\mathcal{G}$. The matroid $\mathcal{M}$ is called *graphic* if there is a graph $\mathcal{G}$ such that $\mathcal{M} = M(\mathcal{G})$. Graphic matroids can also be characterized by the exclusion of certain minors. Two of these are the Fano matroid and its dual, denoted F_7 and $(F_7)^*$. The bases of F_7 are illustrated in Figure 2.1.

The Fano matroid is represented over GF(2) by

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix},$$

and its dual by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

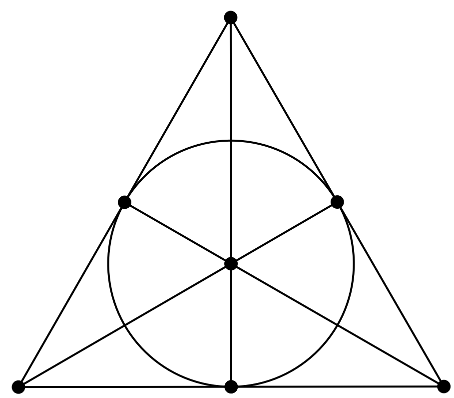


Figure 2.1: This is the Fano plane. The elements are the points in the diagram. Any three of these which do not appear together on a line form a basis.

Proposition 2.1.6. *A matroid is graphic if and only if it does not contain any of*

$$U_{2,4}, F_7, (F_7)^*, (M(K_5))^*, (M(K_{3,3}))^*$$

as a minor.

The set of graphic matroids is not closed under duality. Instead, the dual of a graphic matroid is called *co-graphic*. The matroids which are both graphic and co-graphic are the graphic matroids of planar graphs and thus $M(K_5)$ and $M(K_{3,3})$ are also excluded minors for these.

A matroid that is representable over every field is called *regular*. Seymour proves in his paper, 'Decomposition of Regular Matroids' [22], that a regular matroid can be constructed from direct sums, two-sums and three-sums of graphic and co-graphic matroids and a ten element matroid called R_{10}. A binary representation for R_{10} is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Note that graphic and co-graphic matroids are a subclass of regular matroids which are in turn contained in the class of binary matroids.

Proposition 2.1.7. *A binary matroid is regular if and only if it does not contain F_7 or $(F_7)^*$ as a minor.*

Regular and graphic matroids are a subclass of sixth root of unity matroids. For our purposes these are best characterized by Whittle.

Lemma 2.1.8 ([29])**.** *A matroid is a sixth root of unity matroid if and only if it is representable over GF(3) and GF(4).*

To speak of the decomposition of binary matroids over two-sums we introduce two matroids on eight elements. The matroids $\mathscr{S}_8$ and $\mathscr{AG}(3,2)$ have binary representations

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & b \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}, \tag{2.1.9}$$

where $b = 0$ for $\mathscr{S}_8$ and $b = 1$ for $\mathscr{AG}(3,2)$.

A matroid, $\mathcal{N}$, is a *splitter* for a class of matroids if no three-connected matroid of the class has $\mathcal{N}$ as a proper minor.

Lemma 2.1.9 (Seymour, unpublished. Appendix D [16])**.** *The matroid $\mathscr{AG}(3,2)$ is a splitter for the class of binary matroids not containing $\mathscr{S}_8$ as a minor.*

From this it follows that if a binary matroid, $\mathcal{M}$, not containing $\mathscr{S}_8$ as a minor, does contain $\mathscr{AG}(3,2)$ as a proper minor, then $\mathcal{M}$ is not three-connected. Furthermore, in the case that neither of these are proper minors of $\mathcal{M}$, then $\mathcal{M}$ can be decomposed into F_7, $(F_7)^*$ and regular matroids.

Lemma 2.1.10 ([23])**.** *If a binary matroid, $\mathcal{M}$, contains neither $\mathscr{S}_8$ nor $\mathscr{AG}(3,2)$ as a minor, then it can be constructed from direct sums and two-sums of regular matroids, F_7 and $(F_7)^*$.*

This is enough information for a useful reduction.

Corollary 2.1.11. *A binary, three-connected matroid with no $\mathscr{S}_8$ minor is regular or isomorphic to F_7, $(F_7)^*$ or $\mathscr{AG}(3,2)$.*

Proof. Let $\mathcal{M}$ be a three-connected matroid with no $\mathscr{S}_8$ minor. Since $\mathscr{AG}(3,2)$ is a splitter for binary matroids with no $\mathscr{S}_8$ minor, either $\mathscr{M} \cong \mathscr{AG}(3,2)$ or it contains no $\mathscr{AG}(3,2)$ minor. If it does not contain $\mathscr{AG}(3,2)$ as a minor, by Lemma 2.1.10, it is either F_7, $(F_7)^*$ or regular, since $\mathcal{M}$ is neither a direct sum nor a two-sum of proper minors. □

In the following fact from [22], R_{12} is a certain regular matroid on twelve elements. A precise definition of R_{12} is not needed.

Lemma 2.1.12. *Let $\mathcal{M}$ be a three-connected regular matroid which is neither graphic nor co-graphic. Then $\mathcal{M}$ contains either R_{10} or R_{12} as a minor.*

Lastly, we state the main result of [22].

Lemma 2.1.13. *Let $\mathcal{M}$ be a regular matroid. Then $\mathcal{M}$ decomposes over direct sums, two-sums and three-sums into graphic and co-graphic matroids, and R_{10}.*

2.2 Negative correlation and balance

Negative correlation and balance are restated in the context of this thesis.

Definition 2.2.1. *Let $\mathcal{M}$ be a matroid with distinct $e, f \in E$ and let $\boldsymbol{Z} \in \{\boldsymbol{B}, \boldsymbol{I}, \boldsymbol{S}\}$. Then $\mathcal{M}$ is $\boldsymbol{Z}$-negatively correlated provided that*

$$Z_e(\mathbf{1})Z_f(\mathbf{1}) - Z(\mathbf{1})Z_{ef}(\mathbf{1}) \geq 0,$$

where $\mathbf{1}$ is the all ones vector.

A stronger property, due to Feder and Mihail [25], is balance.

Definition 2.2.2. *Let $\mathcal{M}$ be a matroid and let $\boldsymbol{Z} \in \{\boldsymbol{B}, \boldsymbol{I}, \boldsymbol{S}\}$. Then $\mathcal{M}$ is $\boldsymbol{Z}$-balanced if all of its minors are $\boldsymbol{Z}$-negatively correlated.*

To see that this is equivalent to the definition given in Section 1.1, let $C, D \subseteq E - \{e, f\}$ with $C \cap D = \varnothing$ and consider

$$Z_e(\mathbf{y})Z_f(\mathbf{y}) - Z(\mathbf{y})Z_{ef}(\mathbf{y}) \geq 0,$$

where $y_g \to \infty$ for $g \in C$, $y_g = 0$ for $g \in D$ and otherwise $y_g = 1$. Equivalently

$$(\mathbf{y}^C)^{-2}\left(Z_e^D(\mathbf{y})Z_f^D(\mathbf{y}) - Z^D(\mathbf{y})Z_{ef}^D(\mathbf{y})\right) \geq 0,$$

in which every term without $(\mathbf{y}^C)^2$ disappears so that we are left with

$$(Z_C^D)_e(Z_C^D)_f - Z_C^D(Z_C^D)_{ef} \geq 0.$$

If C is dependent or D is co-dependent, then $Z_{C'}^{D'} = Z(\mathcal{M}/C' \setminus D')$, for maximal independent and co-independent sets $C' \subseteq C$ and $D' \subseteq D$, respectively. The elements of $C - C'$ are loops and those of $D - D'$ are co-loops of $\mathcal{M}/C' \setminus D'$. Their presence in $\Delta Z_C^D \{e, f\}$ can be dealt with trivially so that

$$\Delta Z(\mathcal{M}/C \setminus D)\{e, f\} \geq 0.$$

Example 2.2.3. Recall the example in Chapter 1 showing that K_3 is I-negatively correlated. To show that the triangle is I-balanced we check that its minors are I-negatively correlated. The ones which must be verified are the series pair $\mathcal{U}_{2,2}$ and the parallel pair $\mathcal{U}_{1,2}$. The spanning forests of $\mathcal{U}_{2,2}$ are $\varnothing, e, f, ef$, so that

$$\frac{|\{e, ef\}|}{|\{\varnothing, e, f, ef\}|} = \frac{2}{4} \geq \frac{1}{2} = \frac{|\{ef\}|}{|\{f, ef\}|}.$$

Those of $\mathcal{U}_{1,2}$ are $\varnothing, e, f$, so that

$$\frac{|\{e\}|}{|\{\varnothing, e, f\}|} = \frac{1}{3} \geq \frac{0}{1} = \frac{|\varnothing|}{|\{f\}|}.$$

Thus the series pair and parallel pair are I-negatively correlated. Minors with fewer than two edges are vacuously I-negatively correlated. Therefore K_3 is I-balanced.

Example 2.2.4 ([18]). To verify that $\mathscr{S}_8$ is not B-negatively correlated, consider the elements represented by columns 1 and 8 of the matrix representation given in (2.1.9). Let S be the number of bases of $\mathscr{S}_8$. Then from [18] we have $S = 48, S_1 = 28, S_8 = 20, S_{18} = 12$ so that $S_1 S_8 - S S_{18} = -16 < 0$. In fact, $\mathscr{S}_8$ is a minor-minimal B-unbalanced matroid, since its proper minors are binary matroids not containing $\mathscr{S}_8$ as a minor and are therefore B-balanced [31].

Jerrum gives an example of a B-unbalanced paving matroid [13] which is constructed in Chapter 4.

2.3 The Rayleigh condition

2.3.1 Electrical networks

Consider an electrical network represented by a connected graph $\mathcal{G} = (V, E)$, where a wire represented by the edge e, has conductance y_e. Let $\mathscr{Y}_{ab}$ be the effective conductance of the graph between distinct vertices a and b. Let $\mathscr{T}$ be the collection of spanning trees of $\mathcal{H} = \mathcal{G} \cup \{f\}$, in which f is an edge joining vertices a and b. Let T be the generating polynomial for the spanning trees of $\mathcal{H}$ so that

$$T = \sum_{X \in \mathscr{T}} \mathbf{y}^X. \tag{2.3.1}$$

Note that unless g is a cut edge (co-loop), T^g is the generating polynomial for the spanning trees of $H \setminus g$ and unless g is a loop, T_g is the generating polynomial for the spanning trees of H/g.

Since a and b are distinct and G is connected, f is neither a loop nor a cut edge. Kirchhoff proved in [15], that

$$\mathscr{Y}_{ab} = \frac{T^f}{T_f}, \tag{2.3.2}$$

The term *Rayleigh matroid* is named after Lord Rayleigh (1842-1919) and is motivated by *Rayleigh's Monotonicity Law*, which says that if the electrical conductance of any edge is increased, then the conductance of the graph between any two vertices does not decrease ([24], [31], [4], [15], [19]). In other words, the function $\mathscr{Y}_{ab}$ is an increasing function of y_e, for any edge e. Applying the quotient rule for derivatives gives

$$\frac{\partial \mathscr{Y}_{ab}}{\partial y_e} = \frac{T_e^f T_f - T^f T_{ef}}{T_f^2} \geq 0.$$

This has several equivalent forms which are presented following the formal definition of the Rayleigh condition.

There are several proofs of Rayleigh Monotonicity for graphs. Its 'classical' proof in the context of electrical networks appears in Section 1.3 of Grimmett's book [8]. In [31] it is a corollary of the fact, due to Wagner and Choe, that sixth root of unity matroids are B-Rayleigh. Possibly, the most often cited proof is due to Brooks, Smith, Stone and Tutte (BSST) [21]. They show that $T_e^f T_f - T^f T_{ef}$ is the square of a polynomial, from which the B-Rayleigh property for graphs follows immediately. A more recent, unpublished proof by Cibulka, Hladký, LaCroix, and Wagner [10], uses a combinatorial construction to prove the BSST identity.

The Rayleigh condition is defined for matroids in terms of $Z(\mathcal{M}; \omega; \mathbf{y})$, keeping in mind that the basis, independent and spanning set generating polynomials are the special cases we are interested in.

Definition 2.3.1 (The Rayleigh condition)**.** *With the above notation, a matroid $\mathcal{M}$ satisfies the Z-Rayleigh condition (ω fixed) if for every pair of distinct elements e and f,*

$$Z_e^f Z_f^e - Z_{ef} Z^{ef} \geq 0,$$

for every evaluation of $\mathbf{y}$ such that $y_g > 0$ for each $g \in E$.

Denote the Z-Rayleigh difference with respect to edges e and f by

$$\Delta Z\{e, f\} = Z_e^f Z_f^e - Z_{ef} Z^{ef}.$$

Lemma 2.3.2. *Let $\mathbf{y} = (y_g : g \in E)$ and let $Z(\mathbf{y})$ be a multi-affine polynomial. For distinct $e, f \in E$,*

$$Z_e Z_f - Z Z_{ef} = Z_e^f Z_f - Z^f Z_{ef} = Z_e^f Z_f^e - Z_{ef} Z^{ef}. \tag{2.3.3}$$

Proof. Using the fact that $Z = Z^g + y_g Z_g$ for any $g \in E$,

$$\begin{aligned} Z_e Z_f - Z Z_{ef} =& (Z_e^f + y_f Z_{ef}) Z_f - \left(Z^f + y_f Z_f\right) Z_{ef} \\ =& Z_e^f Z_f - Z^f Z_{ef} \\ =& Z_e^f (Z_f^e + y_e Z_{ef}) - (Z^{ef} + y_e Z_e^f) Z_{ef} = Z_e^f Z_f^e - Z_{ef} Z^{ef}. \end{aligned}$$ □

2.3.2 I-Rayleigh implies B-Rayleigh

In general, **B**-Rayleigh is weaker than **I**-Rayleigh; however, for some classes of matroids they have been shown to be equivalent.

Proposition 2.3.3. *If a matriod is **I**-Rayleigh, then it is also **B**-Rayleigh.*

Proof. Let $\mathcal{M}$ be an **I**-Rayleigh matroid of rank r with distinct elements $e, f \in E$. Then

$$\Delta I(\mathbf{y})\{e, f\} = I_e^f(\mathbf{y}) I_f^e(\mathbf{y}) - I_{ef}(\mathbf{y}) I^{ef}(\mathbf{y}) \geq 0,$$

whenever $y_g > 0$ for each $g \in E - \{e, f\}$. Therefore, by (2.1.3),

$$\Delta B(\mathbf{y})\{e, f\} = \lim_{t \longrightarrow \infty} t^{-(2r-2)} \Delta I(t\mathbf{y})\{e, f\} \geq 0, \tag{2.3.4}$$

for $\mathbf{y} > \mathbf{0}$. □

By considering the dual, we have the following.

Corollary 2.3.4. *If a matroid is **S**-Rayleigh, then it is also **B**-Rayleigh.*

The analogues of Proposition 2.3.3 for negative correlation and balance are not true. In Section 3.6 we show that $\mathscr{S}_8$ is a counterexample.

Chapter 3

Rayleigh Matroids

3.1 Minors and duality

The following useful formula holds for any measure. It implies that the minors of a Z-Rayleigh matroid are Z-Rayleigh when $\mathsf{Z} \in \{\mathsf{B}, \mathsf{I}, \mathsf{S}\}$.

Proposition 3.1.1 ([28]). *Let ω be a measure on a set E. If e, f and g are distinct elements of E, then*

$$\Delta Z\{e,f\} = \Delta Z^g\{e,f\} + y_g \Theta Z\{e,f|g\} + y_g^2 \Delta Z_g\{e,f\}, \tag{3.1.1}$$

where,

$$\Theta Z\{e,f|g\} = Z_e^{fg} Z_{fg}^e + Z_f^{eg} Z_{eg}^f - Z_g^{ef} Z_{ef}^g - Z_{efg} Z^{efg} \tag{3.1.2}$$

Proof. The fact that $Z = Z^g + y_g Z_g$, for any multi-affine polynomial, yields

$$\Delta Z\{e,f\} = (Z_e^{fg} + y_g Z_{eg}^f)(Z_f^{eg} + y_g Z_{fg}^e) - (Z_{ef}^g + y_g Z_{efg})(Z^{efg} + y_g Z_g^{ef}),$$

which can be expanded to get equation (3.1.1). □

The Z-Rayleigh property for minors of $\mathcal{M}$ follows easily from this by taking limits.

Proposition 3.1.2. *Let ω be a measure on a set E. Then for distinct $e, f, g \in E$,*

$$\Delta Z^g\{e, f\} = \lim_{y_g \longrightarrow 0} \Delta Z\{e, f\},$$

and

$$\Delta Z_g\{e, f\} = \lim_{y_g \longrightarrow \infty} y_g^{-2} \Delta Z\{e, f\}.$$

Proof. This follows from Proposition 3.1.1. □

Therefore, whenever $\mathsf{Z} \in \{\mathsf{B}, \mathsf{I}, \mathsf{S}\}$, if $\mathcal{M}$ is Z-Rayleigh, then so are its minors. These special cases are proved in Proposition 3.2 of [31], Section 4.4 of [28], and Proposition 3.5 of [4].

Corollary 3.1.3. *For $\mathsf{Z} \in \{\mathsf{B}, \mathsf{I}, \mathsf{S}\}$, if a matroid is Z-Rayleigh it is also Z-balanced .*

The three basis properties are closed under taking duals.

Proposition 3.1.4 (Proposition 3.1 of [31])**.** *Let $\mathcal{M}$ be a matroid.*

(a) If $\mathcal{M}$ is B-Rayleigh, then its dual is also B-Rayleigh.

(b) If $\mathcal{M}$ is B-negatively correlated, then its dual is also B-negatively correlated.

(c) If $\mathcal{M}$ is B-balanced, then its dual is also B-balanced.

Proof. Denote the basis generating polynomial of the dual by $B^*(\mathbf{y})$. For each $g \in E$, B and B^* are related by

$$\begin{aligned} B^*(\mathbf{y}) =& \mathbf{y}^E B(\mathbf{y}^{-1}), \\ y_g (B^*)^g(\mathbf{y}) =& \mathbf{y}^E B_g(\mathbf{y}^{-1}), \text{ and} \\ y_g (B^*)_g(\mathbf{y}) =& \mathbf{y}^E B^g(\mathbf{y}^{-1}). \end{aligned}$$

Therefore

$$(y_e y_f)^2 \Delta B^*(\mathbf{y})\{e, f\} = (\mathbf{y}^E)^2 \Delta B(\mathbf{y}^{-1})\{e, f\}, \tag{3.1.3}$$

and whenever $\mathbf{y} > \mathbf{0}$, $\Delta B^*(\mathbf{y})\{e, f\} \geq 0$ is equivalent to $\Delta B(\mathbf{y})\{e, f\} \geq 0$ for all pairs of distinct elements, e and f. Setting $\mathbf{y} = \mathbf{1}$ proves (b). Suppose $\mathcal{M}$ is B-balanced. Then all of its minors are B-negatively correlated and by (b), the duals of these are B-negatively correlated, as required. □

A similar proposition relates the independent and spanning set properties.

Proposition 3.1.5 (Proposition 3.3 of [4])**.**

(a) A matroid $\mathcal{M}$ is I-Rayleigh, if and only if its dual is S-Rayleigh.

(b) A matroid $\mathcal{M}$ is I-negatively correlated, if and only if its dual is S-negatively correlated.

(c) A matroid $\mathcal{M}$ is I-balanced, if and only if its dual is S-balanced.

Proof. The proof of (a) mimics that of Proposition 3.1.4 with B and B^* replaced by I and S^*, respectively. Thus, whenever $\mathbf{y} > \mathbf{0}$, $\Delta S^*(\mathbf{y})\{e, f\} \geq 0$ is equivalent to $\Delta I(\mathbf{y})\{e, f\} \geq 0$. Once again, setting $\mathbf{y} = \mathbf{1}$ proves (b). Suppose $\mathcal{M}$ is I-balanced, then all of its minors are I-negatively correlated and by (b), this is true if and only if the duals of these are S-negatively correlated, as required. □

Taken together, these facts about minors and duals can be used to prove that the Z-Rayleigh property holds over two-sum for $\mathsf{Z} \in \{\mathsf{B}, \mathsf{I}, \mathsf{S}\}$.

3.2 Two-sums

The reader may note that these results hold in the more general case of the Potts model which is proved in [28]. It follows from this section that series and parallel extensions of Z-Rayleigh matroids are Z-Rayleigh, whenever $\mathsf{Z} \in \{\mathsf{B}, \mathsf{I}, \mathsf{S}\}$. The proof for two-sums is postponed while the case is made for direct sums.

Lemma 3.2.1. *Let $\mathcal{M}$ be the direct sum of matroids $\mathcal{N}$ and $\mathcal{L}$ and let $\mathsf{Z} \in \{\mathsf{B}, \mathsf{I}, \mathsf{S}\}$.*

(a) If $\mathcal{N}$ and $\mathcal{L}$ are Z-Rayleigh, then $\mathcal{M}$ is Z-Rayleigh.

(b) If $\mathcal{N}$ and $\mathcal{L}$ are Z-negatively correlated, then $\mathcal{M}$ is Z-negatively correlated.

(c) If $\mathcal{N}$ and $\mathcal{L}$ are Z-balanced, then $\mathcal{M}$ is Z-balanced.

Proof. Let $M = Z(\mathcal{M}), N = Z(\mathcal{N})$ and $L = Z(\mathcal{L})$. Since $\mathcal{M}$ is the direct sum of $\mathcal{L}$ and $\mathcal{N}$, then $M = NL$. If $e, f \in E(\mathcal{N})$ are distinct elements, then

$$\Delta M\{e,f\} = L^2 \Delta N\{e,f\}. \tag{3.2.1}$$

Non-negativity of $\Delta M\{e,f\}$ follows from non-negativity of $\Delta N\{e,f\}$. If $e \in E(\mathcal{N})$ and $f \in E(\mathcal{L})$ then

$$\Delta M\{e,f\} = N_e L^f N^e L_f - N_e L_f N^e L^f = 0, \tag{3.2.2}$$

which is clearly non-negative. These two cases hold when $y_g = 1$, $y_g \longrightarrow 0$ or $y_g \longrightarrow \infty$ for each $g \in E - \{e,f\}$. Therefore Z-negative correlation and Z-balance are also preserved by direct sums. $\square$

For the sake of simplifying the statement of the theorem, specific formulas for the Rayleigh differences of $\mathcal{M} = \mathcal{N} \oplus_g \mathcal{L}$ are reserved for the body of the proof.

Theorem 3.2.2 (Theorem 5.8 of [28])**.** *For $\mathsf{Z} \in \{\mathsf{B}, \mathsf{I}, \mathsf{S}\}$, two-sums of Z-Rayleigh matroids are Z-Rayleigh.*

Proof. The theorem is proved directly for two-sums of I-Rayleigh matroids and limits and duality are used to do so for B and S-Rayleigh matroids. It can be proved directly, however, for all three properties.

Let $\mathcal{M} = \mathcal{N} \oplus_g \mathcal{L}$ and suppose that $\mathcal{N}$ and $\mathcal{L}$ are I-Rayleigh. Let $M = I(\mathcal{M}; \mathbf{y}), N = I(\mathcal{N}; \mathbf{y})$, and $L = I(\mathcal{L}; \mathbf{y})$. From Lemma 2.1.3 we have

$$M = N_g L^g + N^g L_g - N_g L_g.$$

Considering $\Delta M\{e,f\}$, by symmetry there are two cases: when $e \in E(\mathcal{N})$ and $f \in E(\mathcal{L})$ or $e, f \in E(\mathcal{N})$.

Suppose $e \in E(\mathcal{N})$ and $f \in E(\mathcal{L})$. Notice that $M^e = N^e_g L^g + N^{ge} L_g - N^e_g L_g$ and similarly for M_e, M_f and M^f. We use these to expand the expression

$$\begin{aligned}
&M^f_e M^e_f - M_{ef} M^{ef} \\
&\qquad\qquad = \left(N_{eg} L^{gf} + N^g_e L^f_g - N_{eg} L^f_g\right)\left(N^e_g L^g_f + N^{eg} L_{gf} - N^e_g L_{gf}\right) \\
&\qquad\qquad - \left(N_{eg} L^g_f + N^g_e L_{gf} - N_{eg} L_{gf}\right)\left(N^e_g L^{gf} + N^{eg} L^f_g - N^e_g L^f_g\right),
\end{aligned}$$

and find that it is equal to

$$\begin{aligned}
\Delta N\{e,g\}\Delta L\{g,f\} &= \left(N^g_e N^e_g - N_{eg} N^{eg}\right)\left(L^f_g L^g_f - L_{gf} L^{gf}\right) \\
=& N^g_e N^e_g L^f_g L^g_f + N_{eg} N^{eg} L_{gf} L^{gf} - N^g_e N^e_g L_{gf} L^{gf} - N_{eg} N^{eg} L^f_g L^g_f.
\end{aligned}$$

Thus, for independent sets, when $e \in E(\mathcal{N})$ and $f \in E(\mathcal{L})$,

$$\Delta M\{e,f\} = \Delta N\{e,g\}\Delta L\{g,f\}. \tag{3.2.3}$$

Since $\Delta N\{e,g\} \geq 0$ and $\Delta L\{g,f\} \geq 0$, whenever $\mathbf{y} > \mathbf{0}$, $\Delta M\{e,f\} \geq 0$ for all $\mathbf{y} > \mathbf{0}$.

If $e, f \in E(\mathcal{N})$ we have the following from [28]. Since g is not a loop, $L_g \neq 0$, so L_g factors out of (2.1.7) and

$$M = L_g(N^g + (L^g/L_g - 1)N_g).$$

Since L is the generating polynomial for independent sets in a matroid, $L^g \gg L_g$ by Lemma 2.1.1. We assume that $y_x > 0$ for all $x \in E(\mathcal{M})$, so that $L^g/L_g - 1 \geq 0$. Defining $y_g := L^g/L_g - 1$ in the following expansion,

$$\begin{aligned}
&M^f_e M^e_f - M_{ef} M^{ef} \\
&= (L_g)^2\left(\left(N^{gf}_e + y_g N^f_{ge}\right)\left(N^{ge}_f + y_g N^e_{gf}\right) - \left(N^g_{ef} + y_g N_{gef}\right)\left(N^{gef} + y_g N^{ef}_g\right)\right)
\end{aligned}$$

yields

$$(L_g)^2\left(\Delta N^g\{e,f\} + y_g \Theta N\{e,f|g\} + y_g^2 \Delta N_g\{e,f\}\right),$$

by Proposition 3.1.1.

So, for independent sets, in the case that $e, f \in E(\mathcal{N})$,

$$\Delta M\{e, f\} = (L_g)^2 \Delta N\{e, f\}, \tag{3.2.4}$$

where $y_g = L^g / L_g - 1$. Since $\Delta N\{e, f\} \geq 0$ and $(L_g)^2 > 0$, then $\Delta M\{e, f\} \geq 0$, which proves the theorem for two-sums of I-Rayleigh matroids.

Let the ranks of $\mathcal{M}, \mathcal{N}$ and $\mathcal{L}$ be r, n and l, respectively, so that $r = l + n - 1$ and $2(r-1) = 2(l-1) + 2(n-1)$. We first prove that if $\mathcal{N}$ and $\mathcal{L}$ are B-Rayleigh, then $\Delta B(\mathcal{M})\{e, f\} \geq 0$ for $e, f \in E(\mathcal{N})$ and $\mathbf{y} > \mathbf{0}$.

Using (3.1.1) to expand $\Delta N\{e, f\}$ and redistributing $(L_g)^2$,

$$(L_g)^2 \Delta N^g\{e, f\} + (L_g L^g - (L_g)^2) \Theta N\{e, f | g\} + (L^g - L_g)^2 \Delta N_g\{e, f\}. \tag{3.2.5}$$

Let P, Q and R be polynomials. If $P = QR$, then $P^{\mathsf{T}} = Q^{\mathsf{T}} R^{\mathsf{T}}$ and if $P = Q + R$ then $P = Q^T$ whenever $\deg Q > \deg R$.

The limit

$$\Delta B(\mathcal{M}; \mathbf{y})\,\{e, f\} = \lim_{t \longrightarrow \infty} t^{-2(r-1)} \Delta I(\mathcal{M}; t\mathbf{y})\,\{e, f\}$$

is not always equal to $(\Delta I(\mathcal{M}; \mathbf{y})\,\{e, f\})^{\mathsf{T}}$; however,

$$\Delta B(\mathcal{M}; \mathbf{y})\,\{e, f\} = \lim_{t \longrightarrow \infty} t^{-2(r-1)} (\Delta I(\mathcal{M}; t\mathbf{y})\,\{e, f\})^{\mathsf{T}}.$$

Recall equation (2.1.3) which says that

$$B_C^D(\mathbf{y}) = \lim_{\lim_{t \longrightarrow \infty}} t^{-(r-|C|)} I_C^D(t\mathbf{y}), \tag{3.2.6}$$

for disjoint $C, D \subseteq E$. This formula is used to attack the summands of (3.2.5) one at a time. The first term contains no surprises,

$$(B_g(\mathcal{L}))^2 \Delta B^g(\mathcal{N})\{e, f\} = \lim_{t \longrightarrow \infty} t^{-2(r-1)} (L_g(t\mathbf{y}))^2 \Delta N^g(t\mathbf{y})\{e, f\}.$$

Note that $\deg(L^g) > \deg(L_g)$, since g is not a co-loop of $\mathcal{L}$. For the second term we use $(L_g L^g - (L_g)^2)^\mathsf{T} = (L_g L^g)^\mathsf{T}$. Therefore

$$\begin{aligned}
&B_g(\mathcal{L})B^g(\mathcal{L})\Theta B(\mathcal{N})\{e,f|g\} \\
&= \lim_{t\longrightarrow\infty} t^{-(2l-1)} L_g(t\mathbf{y})L^g(t\mathbf{y})t^{-(2n-3)}\Theta N(t\mathbf{y})\{e,f|g\} \\
&= \lim_{t\longrightarrow\infty} t^{-2(r-1)} L_g(t\mathbf{y})L^g(t\mathbf{y})\Theta N(t\mathbf{y})\{e,f|g\}.
\end{aligned}$$

In the last term we use $\left((L^g - L_g)^\mathsf{T}\right)^2 = ((L^g)^2)^\mathsf{T}$. Again, taking the limit as $t \longrightarrow \infty$,

$$\begin{aligned}
&(B^g(\mathcal{L}))^2 \Delta B_g(\mathcal{N})\{e,f\} \\
&= \lim_{t\longrightarrow\infty} t^{-2l}(L^g(t\mathbf{y}))^2 t^{-(2n-4)}\Delta N_g(t\mathbf{y})\{e,f\} \\
&= \lim_{t\longrightarrow\infty} t^{-2(r-1)}(L^g(t\mathbf{y}))^2 \Delta N_g(t\mathbf{y})\{e,f\}.
\end{aligned}$$

Therefore, when $e, f \in E(\mathcal{N})$,

$$\Delta B(\mathcal{M})\{e,f\} = (B_g(\mathcal{L}))^2 \Delta B(\mathcal{N})\{e,f\}, \tag{3.2.7}$$

where y_g is defined by $y_g := B^g(\mathcal{L})/B_g(\mathcal{L})$. Notice that $y_g > 0$ for all evaluations positive evaluations of $\mathbf{y} = (y_x : x \in E - \{e,f,g\})$. Since $\mathcal{N}$ is B-Rayleigh, $\Delta B(\mathcal{N})\{e,f\} \geq 0$ for all $\mathbf{y} > \mathbf{0}$, and thus $\Delta B(\mathcal{M})\{e,f\} \geq 0$ for all $\mathbf{y} > \mathbf{0}$.

If $e \in \mathcal{N}$ and $f \in \mathcal{L}$ then, again using equation (2.1.3),

$$\begin{aligned}
&\lim_{t\longrightarrow\infty} t^{-2(r-1)}\Delta M(t\mathbf{y})\{e,f\} \\
&= \lim_{t\longrightarrow\infty} \left(t^{-2(n-1)}\Delta N(t\mathbf{y})\{e,g\}\right)\left(t^{-2(l-1)}\Delta L(t\mathbf{y})\{g,f\}\right).
\end{aligned}$$

This is equivalent to

$$\Delta B(\mathcal{M})\{e,f\} = \Delta B(\mathcal{N})\{e,g\}\Delta B(\mathcal{L})\{g,f\}. \tag{3.2.8}$$

Since $\mathcal{N}$ and $\mathcal{L}$ are both B-Rayleigh, $\Delta B(\mathcal{N})\{e,g\}$ and $\Delta B(\mathcal{L})\{g,f\}$ are non-negative for positive evaluations of $\mathbf{y}$ and thus, so is $\Delta B(\mathcal{M})\{e,f\}$.

If $\mathcal{N}$ and $\mathcal{L}$ are B-Rayleigh, then because of the above two cases, $\mathcal{M}$ is also B-Rayleigh.

The S-Rayleigh property holds under two-sums by duality. Suppose that $\mathcal{N}$ and $\mathcal{L}$ are S-Rayleigh. Then their duals $\mathcal{N}^*$ and $\mathcal{L}^*$ are I-Rayleigh. By Lemma 2.1.4, the dual of $\mathcal{M}$ is equal to $\mathcal{M}^* = \mathcal{N}^* \oplus_g \mathcal{L}^*$, so $\mathcal{M}^*$ is I-Rayleigh and therefore $\mathcal{M}$ is S-Rayleigh. □

Corollary 3.2.3. *Let $\boldsymbol{Z} \in \{\boldsymbol{B},\ \boldsymbol{I},\ \boldsymbol{S}\}$ and let $\mathcal{M}$ be a minor-minimal, non $\boldsymbol{Z}$-Rayleigh matroid. Then $\mathcal{M}$ is three-connected.*

The following corollary is used to prove some special cases of the Rayleigh properties in the next section.

Corollary 3.2.4. *Let $\mathcal{N}$ and $\mathcal{L}$ be matroids and let $\mathcal{M} = \mathcal{N} \oplus_g \mathcal{L}$. If $e, f \in E(\mathcal{N}) - g$ and $\mathcal{N}$ is $\boldsymbol{Z}$-Rayleigh, then,*

$$\Delta Z(\mathcal{M})\,\{e, f\} \geq 0 \tag{3.2.9}$$

whenever $\mathbf{y} > \mathbf{0}$ and $\boldsymbol{Z} \in \{\boldsymbol{B},\ \boldsymbol{I},\ \boldsymbol{S}\}$.

Proof. This follows directly from the formulas for the B and I-Rayleigh differences. In the spanning set case observe that $\mathcal{N}^*$ is I-Rayleigh in which case $\mathcal{M}^*$ satisfies the corollary when Z=I, so by Proposition 3.1.5, $\Delta S(\mathcal{M})\,\{e, f\} \geq 0$. □

A lemma similar to the following can be proved for the I and S-Rayleigh differences. Instead, this is only done for the B-Rayleigh difference and used in Chapter 4.

Lemma 3.2.5. *Let $\mathcal{N}$ and $\mathcal{L}$ be matroids and let $N = B(\mathcal{N})$, $L = B(\mathcal{L})$ and $M = B(\mathcal{M})$. If $\Delta N\,\{e, f\}$ and $\Delta L\,\{e, f\}$ have positive coefficients for all pairs of distinct elements e and f of their ground sets, then $\Delta M\,\{e, f\}$ has positive coefficients when $\mathcal{M}$ is either the direct sum or two-sum of $\mathcal{N}$ and $\mathcal{L}$.*

Proof. The only case where this is not obvious from the formulas is when $\mathcal{M} = \mathcal{N} \oplus_g \mathcal{L}$ and $e, f \in E(\mathcal{N})$. By (3.2.7) this gives

$$\Delta M\{e,f\} = (L_g)^2 \Delta N\{e,f\}$$

in which $y_g = L^g/L_g$. This also has positive coefficients, since instances of y_g to the second degree are replaced with $(L^g)^2$ and y_g to the first degree with $L_g L^g$, both of which are polynomials with positive coefficients. $\square$

We prove in Section 3.3 that as a result of Corollary 3.2.4 we need not worry about cases where e and f are dependent or co-dependent.

3.3 Special cases

The results of Section 3.2 are used to make some simple observations that take care of several special cases. If e and f are in parallel or in series, then parts of the proof of Theorem 3.2.2 and two easy facts about K_3 and K_3^* imply that $\Delta Z\{e,f\}$ has positive coefficients for $\mathsf{Z} \in \{\mathsf{B}, \mathsf{I}, \mathsf{S}\}$.

Lemma 3.3.1. *Let the edge set of K_3 be $\{e,f,g\}$. Then*

$$\Delta I(K_3;\mathbf{y})\{e,f\} = (y_g+1)^2 - (y_g+1) = (y_g+1)y_g$$

and

$$\Delta I((K_3)^*;\mathbf{y})\{e,f\} = 1 - 0 = 1$$

Proof. Use $I(K_3;\mathbf{y}) = 1 + y_e + y_f + y_g + y_e y_f + y_e y_g + y_f y_g$ and $I((K_3)^*;\mathbf{y}) = 1 + y_e + y_f + y_g$. $\square$

Proposition 3.3.2. *Let $\mathcal{M}$ be a matroid with distinct elements e, f. If $\{e,f\}$ is dependent or co-dependent, then*

$$Z_e^f Z_f^e - Z_{ef} Z^{ef} \gg 0 \tag{3.3.1}$$

for $\boldsymbol{\mathsf{Z}} \in \{\boldsymbol{\mathsf{B}}, \boldsymbol{\mathsf{I}}, \boldsymbol{\mathsf{S}}\}$.

Proof. If e is a loop, then by (2.1.4), $B_e = I_e = 0$ and $S_e = S^e$, so the proposition obviously holds in this case. If e is a co-loop then by (2.1.5), $B^e = S^e = 0$ and $I_e = I^e$ and again the proposition holds. If $\{e, f\}$ is a parallel pair then $B_{ef} = I_{ef} = 0$ and if they are in series then $B^{ef} = S^{ef} = 0$.

It remains for us to show that if $\{e, f\}$ is a parallel pair, then $\Delta S\{e, f\} \gg 0$ and that if $\{e, f\}$ is a series pair, then $\Delta I\{e, f\} \gg 0$. We use two-sums and duality for this.

Let the edge set of K_3 be $\{e, f, g\}$. Suppose $\{e, f\}$ are a series pair, then let $\mathcal{N}$ be the matroid such that

$$\mathcal{M} = \mathcal{N} \oplus_g K_3.$$

By (3.2.4) and Lemma 3.3.1

$$\Delta I(\mathcal{M}; \mathbf{y})\{e, f\} = (N_g)^2((y_g + 1)y_g),$$

where $y_g = N^g/N_g - 1$. This gives

$$\Delta I(\mathcal{M}; \mathbf{y})\{e, f\} = (N^g)(N^g - N_g) \gg 0,$$

since $N^g \gg N_g$ by Lemma 2.1.1. If $\{e, f\}$ is a parallel pair then

$$\mathcal{M} = \mathcal{N} \oplus_g (K_3)^*,$$

so that

$$\mathcal{M}^* = \mathcal{N}^* \oplus_g K_3.$$

Since $\Delta I(\mathcal{M}^*; \mathbf{y})\{e, f\} \gg 0$, then $\Delta S(\mathcal{M}; \mathbf{y})\{e, f\} \gg 0$ by Proposition 3.1.5. □

Corollary 3.3.3. *Let $\mathcal{M}$ be a matroid with distinct elements e, f. If $\{e, f\}$ is dependent or co-dependent, then*

$$\Delta I\{e, f\} - \Delta B\{e, f\} \gg 0 \text{ and}$$
$$\Delta S\{e, f\} - \Delta B\{e, f\} \gg 0$$

Proof. Notice that if $\{e, f\}$ is a parallel pair, then $B_e^f = 0$ since f is a loop of $\mathcal{M}/e$. Similarly, if $\{e, f\}$ is a series pair, then $B_e^f = 0$. Since in all cases, $\Delta I\{e, f\} \gg 0$, $\Delta S\{e, f\} \gg 0$ and $\Delta B\{e, f\} = 0$, the result follows. □

This concludes our discussion of the more elementary facts regarding these polynomials. The results that follow, build on these and are more advanced.

3.4 Sufficient conditions

Considerable research has been done to classify matroids according to whether or not they have the Rayleigh properties. The known sufficient conditions for the Rayleigh properties are consolidated here; however, not all the methods which give these results are mentioned. The corollaries that follow from results of previous sections are also left out.

Theorem 3.4.1 (Cor 8.2(a) and Thm 8.9 of [30]. Cor 4.7 and Prop 5.1 of [31]). *Sixth root of unity matroids are* **B***-Rayleigh.*

By Lemma 2.1.8, this includes all graphic and regular matroids.

Wagner [27] proves that rank three matroids are **B**-Rayleigh which is used to prove in the next section that they are in fact **I**-Rayleigh.

Theorem 3.4.2. *Every matroid of at most rank three is* **B***-Rayleigh.*

Interestingly enough, the **B**-Rayleigh condition for binary matroids is characterized by the exclusion of $\mathscr{S}_8$ as a minor. The first correct proof of this appears in [31].

Theorem 3.4.3 (Theorem 3.8 of [31]). *A binary matroid is* **B***-Rayleigh if and only if it does not contain* $\mathscr{S}_8$ *as a minor.*

New in this thesis are the following facts.

(i) All graphs on seven vertices are **I**-Rayleigh.

(ii) A binary matroid on at most nine elements is I-Rayleigh if and only if it contains no $\mathscr{S}_8$ minor.

(iii) Rank three matroids are I-Rayleigh (independently by Cocks [5]).

(iv) Sparse paving matroids are B-Rayleigh

(v) A paving matroid is I-Rayleigh if and only if it is B-Rayleigh.

Previously, Wagner showed that all graphs with at most six vertices are I-Rayleigh (personal communication).

With regards to negatively correlated matroids which are possibly not Rayleigh, we have the following. By direct computation, Royle has shown that all matroids on at most nine elements are I-negatively correlated (personal communication). Grimmett and Winkler [7] show that all graphs on at most eight vertices or nine vertices and 18 edges are I-negatively correlated.

There are balanced matroids which are not known to be Rayleigh. Sparse paving matroids are B-balanced; however, there exists a B-unbalanced non-sparse paving matroid [13].

Evidently our ignorance on this subject is vast and much work needs to be done.

3.5 Rank three matroids are I-Rayleigh

Theorem 3.5.1 (E.)**.** *Let $\mathcal{M}$ be a matroid of rank at most three. Then*

$$\Delta I\{e,f\} \gg \Delta B\{e,f\}. \tag{3.5.1}$$

Proof. We may assume that $\{e,f\}$ is neither dependent nor co-dependent by Corollary 3.3.3. This also takes care of the cases where $\mathcal{M}$ has rank zero or one, so we assume $\mathcal{M}$ has rank at least two.

Suppose, for an inductive proof, that $\Delta I\{e,f\} \gg \Delta B\{e,f\}$ for all proper minors of $\mathcal{M}$. Let $\mathbf{y}^\alpha$ be one of the monomials appearing in

$\Delta I\{e,f\} - \Delta B\{e,f\}$. It is enough to show that

$$[\mathbf{y}^\alpha]\Delta I\{e,f\} \geq [\mathbf{y}^\alpha]\Delta B\{e,f\}.$$

If $\alpha(g) = 2$ for some $g \in E - \{e,f\}$, then we contract g and apply the induction hypothesis to conclude that

$$\begin{aligned}&[\mathbf{y}^\alpha]\left(\Delta I(\mathcal{M})\{e,f\} - \Delta B(\mathcal{M})\{e,f\}\right)\\ =&\left[\mathbf{y}^\alpha y_g^{-2}\right]\left(\Delta I(\mathcal{M}/g)\{e,f\} - \Delta B(\mathcal{M}/g)\{e,f\}\right) \geq 0.\end{aligned}$$

If $\alpha(g) = 0$ for some $g \in E - \{e,f\}$ then we delete this element and apply the induction hypothesis to conclude that

$$\begin{aligned}&[\mathbf{y}^\alpha]\left(\Delta I(\mathcal{M})\{e,f\} - \Delta B(\mathcal{M})\{e,f\}\right)\\ =&[\mathbf{y}^\alpha]\left(\Delta I(\mathcal{M}\setminus g)\{e,f\} - \Delta B(\mathcal{M}\setminus g)\{e,f\}\right) \geq 0.\end{aligned}$$

Therefore we only need to prove (3.5.1) for the unique monomial $\mathbf{y}^\alpha$ such that $\alpha(g) = 1$ for all $g \in E - ef$.

Let $U = I - B$ and note that

$$\begin{aligned}&\Delta I\{e,f\} - \Delta B\{e,f\}\\ &= \left(B_e^f U_f^e + B_f^e U_e^f + \Delta U\{e,f\}\right) - \left(B_{ef}U^{ef} + B^{ef}U_{ef}\right).\end{aligned}$$

If $\mathcal{M}$ has rank two, then $\deg(B) = 2$ and $\deg(U) = 1$. By the above reductions we may assume that $|E| = 3$. Furthermore $B^{ef}U_{ef} = 0$ and $\Delta U\{e,f\} = 1$ so it is enough to prove that y_g appears at least once on the left hand side of

$$B_e^f U_f^e + B_f^e U_e^f \gg B_{ef}U^{ef},$$

for $g \in E - \{e,f\}$. The element g is not parallel to both e and f, since e and f are not parallel. Therefore y_g appears in one of $B_e^f U_f^e$ or $B_f^e U_e^f$, as required.

Now suppose $\mathcal{M}$ has rank three. If $\mathcal{M}$ has exactly three elements, then $\{e, f\}$ is a series pair. If it has four elements, then either it contains a co-loop or it is the uniform matroid $\mathcal{U}_{3,4}$ for which the result is easy to verify.

Since the degree of $\Delta I\{e, f\} - \Delta B\{e, f\}$ is at most three, we may assume that $E = \{e, f, g, h, k\}$ and we need only only show that

$$[\mathbf{y}^{\alpha}]\left(B_e^f U_f^e + B_f^e U_e^f\right) \geq [\mathbf{y}^{\alpha}]\left(B_{ef} U^{ef} + B^{ef}\right), \tag{3.5.2}$$

where $\mathbf{y}^{\alpha} = y_g y_h y_k$. The $\Delta U\{e, f\}$ term can be left out because $\deg(U_f^e) = \deg(U_e^f) = 1$ and $\deg(U^{ef}) = 2$. Notice that, since $\{e, f\}$ is neither dependent nor co-dependent, $\deg(B_e^f) = \deg(B_f^e) = 2$ and that U_e^f and U_f^e generate the proper subsets of B_e^f and B_f^e, respectively. The fact that $\{e, f\}$ is not co-dependent also implies that ghk is a basis, so $B^{ef} = y_g y_h y_k$. Therefore we prove that

$$[\mathbf{y}^{\alpha}]\left(B_e^f U_f^e + B_f^e U_e^f\right) > [\mathbf{y}^{\alpha}]\left(B_{ef} U^{ef}\right). \tag{3.5.3}$$

Let (A, R) be a pair of type $B_{ef}U^{ef}$. That is, $A \dot{\cup} R = E - ef$, A is a basis of $\mathcal{M}/ef$ and R is a 2-subset of ghk. Let (F, Q) be a pair of type $B_e^f U_f^e$. That is, $F \dot{\cup} Q = E - ef$, F is a basis of $\mathcal{M}/e \setminus f$ and Q contains a single element contained in a basis of $\mathcal{M}/f \setminus e$.

Recall the strong basis exchange axiom which states that if X and Y are bases, then for each $x \in X - Y$, there is $y \in Y - X$ such that $(X \cup y) - x$ is a basis. Since ghk is a basis, if gef is a basis, then at least one of ghf and gkf is a basis and also, at least one of ghe and gke is a basis. Clearly this holds for all permutations of the labels g, h, k.

Therefore each pair (A, R), of type $B_{ef}U^{ef}$ maps to at least one pair of type $B_e^f U_f^e$ and one of type $B_f^e U_e^f$. Furthermore, no two distinct pairs (A, R) and (A', R') of type $B_{ef}U^{ef}$ map to the same pairs of type $B_e^f U_f^e$ and $B_f^e U_e^f$. Assume that there are exactly three pairs of type $B_{ef}U^{ef}$ (there can be no more). Then by the pigeon hole principle, there are more than three pairs of type $B_e^f U_f^e$ and $B_f^e U_e^f$ put together. There can be at most three pairs of type $B_{ef} U^{ef}$, thus

$$[\mathbf{y}^\alpha] \left(B_e^f U_f^e + B_f^e U_e^f \right) \geq [\mathbf{y}^\alpha] \left(B_{ef} U^{ef} + B^{ef} \right),$$

in all cases, as desired. □

The above theorem provides one of two ways presented in this thesis, to arrive at the following corollary.

Corollary 3.5.2. *Rank three matroids are* **I***-Rayleigh*

Proof. This follows directly from Theorem 3.4.2 together with Theorem 3.5.1. □

In Chapter 4, Theorem 3.5.1 is re-derived as a special case of Theorem 4.4.1.

3.6 Small binary matroids

We show that a binary matroid on at most nine elements is **I**-Rayleigh if and only if it does not contain $\mathscr{S}_8$ as a minor. Furthermore, if three particular regular matroids on ten elements turn out to be **I**-Rayleigh, then the result can be strengthened to at most ten elements. The arguments draw heavily on the matroid decompositions reviewed in Section 2.1.5. We also use a result from Chapter 5, namely, that graphs on at most seven vertices are **I**-Rayleigh (Chapter 5 does not rely on results of this section).

The story of this section has a subplot. In Proposition 2.3.3 we showed that **I**-Rayleigh implies **B**-Rayleigh. When it comes to balance or negative correlation, however, the argument breaks down. The binary matroid $\mathscr{S}_8$ is an interesting counterexample because it is the only binary minor-minimal non **B**-Rayleigh matroid. Example 2.2.4 shows that it is **B**-unbalanced and the results in this section prove that its minors are **B**-balanced.

Proposition 3.6.1. *The matroid $\mathscr{S}_8$ is* **I***-balanced but not* **B***-balanced.*

The proofs we will see help show that $\mathscr{S}_8$ is **I**-balanced without manually checking that its proper minors are **I**-negatively correlated. In particular, we use the fact that its minors of at most rank three are **I**-Rayleigh, by Corollary 3.5.2, and that all binary matroids on 7 elements are **I**-Rayleigh. Verification of this depends on results from Chapter 5, the fact that most binary matroids on seven elements are graphic, and a quick check that the other ones are **I**-Rayleigh as well.

Lemma 3.6.2. *Every graph with at most eleven edges is* **I***-Rayleigh.*

Proof. By Theorem 5.3.1, every graph on at most seven vertices is **I**-Rayleigh. On the other hand let $\mathcal{G}$ be a connected graph on eight vertices with $8 \leq |E| \leq 11$. Such a graph has a vertex of degree at most two, since $2 \cdot |E| \geq 8\delta \Rightarrow 3 > \delta$, where δ is the minimum degree of $\mathcal{G}$. Therefore $\mathcal{G}$ is not three-connected and its minors are **I**-Rayleigh. Thus $\mathcal{G}$ is **I**-Rayleigh by Corollary 3.2.3. By Lemma 3.2.1 and Theorem 3.2.2, $\mathcal{G}$ is **I**-Rayleigh. Similarly, if $|V(\mathcal{G})| = 9$ and $9 \leq |E| \leq 11$ it is again not three-connected and thus **I**-Rayleigh. The same argument works for ten and eleven vertices and beyond that, the result is trivial. □

We use a computer program in Maple in the next lemma. It is available at http://www.math.uwaterloo.ca/∼atericks/

Lemma 3.6.3. *The matroids F_7, $(F_7)^*$ and $\mathscr{AG}(3,2)$ are* **I***-Rayleigh*

Proof. The Fano matroid has rank three so it is **I**-Rayleigh by Corollary 3.5.2. Since $(F_7)^*$ has co-rank three, it is **B**-Rayleigh by Theorem 3.4.2 together with Proposition 3.1.4. An easy computer test shows that

$$I((F_7)^*) - B((F_7)^*) \gg 0,$$

from which it follows that $(F_7)^*$ is **I**-Rayleigh. The matroid $\mathscr{AG}(3,2)$ does not contain $\mathscr{S}_8$ as a minor so by Theorem 3.4.3 it is **B**-Rayleigh and a similar computer test shows that

$$I(\mathscr{AG}(3,2)) - B(\mathscr{AG}(3,2)) \gg 0.$$

Note that F_7 and $(F_7)^*$ are both minors of $\mathscr{AG}(3,2)$, so by Proposition 3.1 it suffices to show that $\mathscr{AG}(3,2)$ is I-Rayleigh. On the other hand we have proved something slightly stronger. $\square$

Returning to the subplot around $\mathscr{S}_8$, the matroids F_7 and $(F_7)^*$ are the only non-graphic binary matroids on at most seven elements. Therefore, it remains only to show that $\mathscr{S}_8$ is I-negatively correlated which we check with a straightforward calculation in Maple.

We are able to show that $\Delta I((M(K_{3,3}))^*)\,\{e,f\} - \Delta B((M(K_{3,3}))^*)\,\{e,f\} \geq 0$ by an easy algebraic manipulation.

Lemma 3.6.4. *The matroid $(M(K_{3,3}))^*$ is I-Rayleigh.*

Proof. Since $(M(K_{3,3}))^*$ is co-graphic, it is B-Rayleigh by Proposition 3.1.4. Therefore it is enough to show that for distinct e and f,

$$\Delta I((M(K_{3,3}))^*)\,\{e,f\} - \Delta B((M(K_{3,3}))^*)\,\{e,f\} \geq 0.$$

By (2.1.1) and (2.1.2), this is equivalent to

$$(\mathbf{y}^{E-\{e,f\}})^2 \left(\Delta S(K_{3,3};\mathbf{y}^{-1})\,\{e,f\} - \Delta B(K_{3,3};\mathbf{y}^{-1})\right)\{e,f\} \geq 0,$$

which enables the use of the edge-symmetry of $K_{3,3}$. That is, either e and f are adjacent in $K_{3,3}$ or they are not. The calculations were done using Maple. When e and f are adjacent,

$$\Delta I((M(K_{3,3}))^*)\,\{e,f\} - \Delta B((M(K_{3,3}))^*)\,\{e,f\} \gg 0.$$

Otherwise, if e and f are not adjacent in $K_{3,3}$ and

$$\Delta I((M(K_{3,3}))^*)\,\{e,f\} - \Delta B((M(K_{3,3}))^*)\,\{e,f\}$$

contains four negative terms. Taking these, along with some positive terms we use the fact that

$$ca^2b^2 - 2abcdx + cd^2x^2$$

and

$$ca^2b^2 - 2abcdx - 2abcyz + 2dxcyz + cd^2x^2 + cy^2z^2$$

are each a monomial times the square of a polynomial to show that

$$\Delta I((M(K_{3,3}))^*)\{e,f\} - \Delta B((M(K_{3,3}))^*)\{e,f\} \geq 0$$

for every positive evaluation of $\mathbf{y}$. ◻

Theorem 3.6.5. *[E.] Every binary matroid on at most nine elements is* **I***-Rayleigh if and only if it does not contain $\mathscr{S}_8$ as a minor.*

Proof. Let $\mathcal{M}$ be a minor-minimal counterexample. If $\mathcal{M}$ contains $\mathscr{S}_8$ as a minor, then it is not **B**-Rayleigh by Theorem 3.4.3, and hence not **I**-Rayleigh, by Proposition 2.3.3. We may assume $\mathcal{M}$ is a binary matroid on at most nine elements with no $\mathscr{S}_8$ minor and is not **I**-Rayleigh. By Corollary 3.2.3, $\mathcal{M}$ is three-connected and by Corollary 2.1.11 it is regular or isomorphic to one of F_7, $(F_7)^*$ or $\mathscr{AG}(3,2)$. The only non-graphic regular matroid on at most nine elements is $(M(K_{3,3}))^*$ by Proposition 2.1.6. By Lemma 3.6.2, Lemma 3.6.3 and Lemma 3.6.4, all of these possibilities are **I**-Rayleigh which contradicts the fact that $\mathcal{M}$ is a minimal counterexample. ◻

Most matroids do not have the nice property of $\mathscr{AG}(3,2), F_7$ and $(F_7)^*$, that

$$\Delta I\{e,f\} \gg \Delta B\{e,f\}.$$

Unfortunately, among these are $(M(K_{3,3}))^*, R_{10}, (M(K_5))^*$ and the dual of $K_{3,3}$ plus an edge, not parallel to an existing edge. This is denoted by

$(M(K_{3,3}^+))^*$. On the other hand an ad-hoc method shows that $(M(K_{3,3}))^*$ is I-Rayleigh. If the last three of these regular matroids are I-Rayleigh the main result of this section can be strengthened. Note that in [4], Semple and Welsh mention that $(M(K_5))^*$ is I-Rayleigh.

Proposition 3.6.6. *If R_{10}, $(M(K_5))^*$ and $(M(K_{3,3}^+))^*$ are I-Rayleigh, then every binary matroid on ten elements is I-Rayleigh if and only if it does not contain $\mathscr{S}_8$ as a minor.*

Proof. Let $\mathcal{M}$ be a binary, non I-Rayleigh matroid on ten elements with no $\mathscr{S}_8$ minor. We may assume by the argument of Theorem 3.6.5, that $\mathcal{M}$ is regular, three-connected, and not one of the matroids mentioned in the above statement. If it is co-graphic, then it is the dual of $M(\mathcal{G})$ for a graph, $\mathcal{G}$, containing $K_{3,3}$ as a minor. The graph $\mathcal{G}$ cannot be a series or parallel extension of $K_{3,3}$ since $(M(\mathcal{G}))^*$ would not be three-connected and $\mathcal{G} \ncong K_{3,3}^+$. Thus we may assume $\mathcal{G}$ is not co-graphic. By Lemma 2.1.12 every three-connected regular matroid which is neither graphic nor co-graphic has an R_{10} or R_{12} minor. Since $\mathcal{M}$ is not R_{10} and does not have twelve elements there is no such matroid, as required. $\square$

One might be more convinced that this last proposition is useful by knowing that the matroids mentioned possess some of the necessary conditions for being I-Rayleigh. We are particularly interested in I-balance.

Proposition 3.6.7. *The matroids $(M(K_{3,3}^+))^*$, $(M(K_5))^*$ and R_{10} are I-balanced.*

Proof. The matroid $\mathscr{S}_8$ is not regular, since regular matroids are B-Rayleigh, so none of the matroids in question contain it as a minor. By Theorem 3.6.5 we need only check I-negative correlation for R_{10}, $(M(K_5))^*$ and $(M(K_{3,3}^+))^*$ which is done in Maple. $\square$

We also proved in this chapter, that $\mathscr{S}_8$ is I-balanced but not I-Rayleigh, casting a shadow on the above piece of evidence. However, one should consider that $\mathscr{S}_8$ is not regular and that no regular matroid is known not to be I-Rayleigh. The temptation to conjecture that a binary matroid is

I-Rayleigh if and only if it is B-Rayleigh is irresistible and indeed we do so in the next section.

3.7 Conjecture: Regular matroids are I-Rayleigh

Of the countless outstanding questions regarding the classification of matroids with the Rayleigh properties, we list a few interesting ones that might be answerable.

By Lemma 2.1.12, regular matroids decompose over direct sums, two and three-sums into graphic and co-graphic matroids and R_{10}. In turn, binary matroids decompose into those containing $\mathscr{S}_8$ or $\mathscr{AG}(3,2)$ minors and otherwise, a three-connected binary matroid with at least eight elements is regular. Despite the fact that we know exactly when these classes are B-Rayleigh, virtually nothing is known about which ones are I-Rayleigh. By their constructions, significant progress would be made if graphs were known to be I-Rayleigh, which is the central topic of Chapter 5. There are two more significant obstacles which will be discussed momentarily. The following theorem looks surprising but can be derived easily from the theory in Section 2.1.5.

Recall that by Corollary 2.1.11, a binary, three-connected matroid with no $\mathscr{S}_8$ minor is regular or isomorphic to $(F_7), (F_7)^*$ or $\mathscr{AG}(3,2)$. This motivates the following theorem.

Theorem 3.7.1. *[E.] The following are equivalent.*

(i) Regular matroids are I-Rayleigh.

(ii) A binary matroid is I-Rayleigh if and only if it is B-Rayleigh.

Proof. Assume (ii). By (ii) and Theorem 3.4.3, a binary matroid is I-Rayleigh if and only if it does not contain $\mathscr{S}_8$ as a minor. By Lemma 2.1.10, the class of regular matroids is contained in the class of binary matroids not containing $\mathscr{S}_8$ as a minor. This proves (i).

Conversely, let $\mathcal{M}$ be a minimal counterexample to (ii). By the argument in Theorem 3.6.5, we may assume that $\mathcal{M}$ is not I-Rayleigh, does not contain $\mathscr{S}_8$ as a minor and $|E| > 9$. By Corollary 3.2.3, $\mathcal{M}$ is three-connected and by Corollary 2.1.11 and the fact that $|E| > 9$, $\mathcal{M}$ is regular. Therefore it is regular and not I-Rayleigh, as required. □

It would seem more plausible that regular matroids are I-Rayleigh. However, Theorem 3.7.1 allows a bolder statement.

Conjecture 3.7.2. *A binary matroid is I-Rayleigh if and only if it is B-Rayleigh.*

The difficulties on the path to a proof of this lie in the decomposition of regular matroids. It is not known whether graphs are I and S-negatively correlated and the Rayleigh properties do not seem to behave nicely by taking three-sums (as they do for two-sums).

Chapter 4

Paving Matroids

A rank r matroid is *paving* if all of its circuits have at least r elements. A paving matroid is *sparse* if no two circuits of size r have a symmetric difference of exactly two elements – in some sense they are “sparse” in their circuits. In [13], Jerrum proves that sparse paving matroids are B-balanced. On the other hand, Jerrum also gives an example of a non B-balanced paving matroid. Both of Jerrum’s results are presented as well as two new results, that the B-Rayleigh and I-Rayleigh conditions are equivalent for paving matroids and that sparse paving matroids are indeed B-Rayleigh.

Throughout this chapter, $\mathcal{M}$ is a paving matroid of rank r. A few preliminary facts are given, the first three of which apply to paving matroids in general. Oxley characterizes paving matroids with the following lemma.

Lemma 4.0.3 (Proposition 1.3.10, [17]). *Let $\mathscr{C}$ be a set of r-element subsets of a finite set E. The following are equivalent.*

(a) The set $\mathscr{C}$ is the set of circuits of size r of a rank r paving matroid on ground set E.

(b) For every $C, C' \in \mathscr{C}$, if $|C \triangle C'| = 2$, then every r-subset of $C \cup C'$ is contained in $\mathscr{C}$.

Proof. Briefly, the axioms that the circuits of a matroid must satisfy are:

(i) The empty set is not a circuit.

(ii) No circuit properly contains another circuit.

(iii) Given circuits C and C' with $e \in C \cap C'$, there is a circuit contained in $(C \cup C') - e$.

If (a) is assumed, then (b) follows from (iii). Conversely, let $\mathscr{C}$ satisfy (b) and let $\mathscr{C}'$ be the collection of $(r+1)$-sets not containing members of $\mathscr{C}$. Clearly, the members of $\mathscr{C} \dot{\cup} \mathscr{C}'$ satisfy (i) and (ii).

Let $C, C' \in \mathscr{C} \cup \mathscr{C}'$. Since no circuit contains another, $|C \triangle C'| > 1$, and therefore $|C \cup C'| \geq r+2$. By (b), any set of size $r+1$ contains an element of $\mathscr{C} \cup \mathscr{C}'$. Thus $(C \cup C') - e$ contains contains an element of $\mathscr{C} \cup \mathscr{C}'$, satisfying (iii), as desired. □

The proofs in Section 4.1 and Section 4.2 use the fact that sparse paving matroids are a minor closed class. The proof of Theorem 4.4.1 requires that paving matroids be minor closed to apply induction.

Proposition 4.0.4.

(a) The class of paving matroids is closed by taking minors.

(b) The class of sparse paving matroids is closed by taking minors.

Proof. Let $\mathcal{M}$ be a paving matroid of rank r. The proof is divided into the cases where g is a loop, co-loop or otherwise. In each case $\mathcal{M}/g$ and $\mathcal{M} \setminus g$ are shown to be paving and furthermore, if either of these is not sparse, then $\mathcal{M}$ must not be sparse.

If g is a loop, then $r \leq 1$ so $\mathcal{M} \setminus g$ and $\mathcal{M}/g$ are paving. Notice that $\mathcal{M} \setminus g \cong \mathcal{M}/g$, so if $\mathcal{M} \setminus g$ is not sparse, it contains two loops which are loops of $\mathcal{M}$. Thus $\mathcal{M}$ is also not sparse, proving (b) for this case.

If g is a co-loop, then it appears in every basis and $\mathcal{M} \setminus g \cong \mathcal{M}/g$. Let $I \subseteq E(\mathcal{M}) - g$ be such that $|I| = r-1$. Then I is independent and $I \cup g$ is a basis, so $\mathcal{M} \setminus g$ and $\mathcal{M}/g$ are uniform matroids of rank $r-1$. Uniform matroids are sparse paving, so this also satisfies (b).

If g is neither a loop nor a co-loop, then $\mathcal{M} \setminus g$ is of rank r and if C' is a circuit of $\mathcal{M} \setminus g$ then $C' \in \mathscr{C}$, implying that $|C'| \geq r$. Thus $\mathcal{M} \setminus g$ is paving. Suppose $\mathcal{M} \setminus g$ is not sparse and let $C, C' \in \mathscr{C}(\mathcal{M} \setminus g)$ be such that $|C| = |C'| = r$ and $|C \triangle C'| = 2$. Then C and C' are circuits of $\mathcal{M}$ with the same properties so that $\mathcal{M}$ is not sparse. Thus (b) is proved for this case.

Let C' be a circuit of $\mathcal{M}/g$. Then $C' \cup g$ contains a circuit of $\mathcal{M}$ so that $|C'| \geq r-1$, as required for $\mathcal{M}/g$ to be paving. Suppose $\mathcal{M}/g$ is paving but not sparse and let $C, C' \in \mathscr{C}(\mathcal{M}/g)$ be such that $|C| = |C'| = r-1$ and $|C \triangle C'| = 2$. Since $\mathcal{M}$ is paving, $C \cup g$ and $C' \cup g$ are r-circuits of $\mathcal{M}$ and $|(C \cup g) \triangle (C' \cup g)| = 2$, so $\mathcal{M}$ is not sparse, proving (b) for this final case. □

The following is an application of Lemma 4.0.3 which is used in the last two steps of Theorem 4.4.1. Note that in Lemma 4.0.5 and Lemma 4.0.6, B is used to denote a basis. This should not be confused with the generating series it usually denotes.

Lemma 4.0.5. *Let $\mathcal{M}$ be a paving matroid. If B is a basis, then for any $g \in E - B$, there is at most one element $h \in B$ such that $(B \cup g) - h \notin \mathscr{B}$.*

Proof. Suppose that there are distinct $h, h' \in B$ such that for some $g \in E - B$, both $C = (B \cup g) - h$ and $C' = (B \cup g) - h'$ are dependent and, since $\mathcal{M}$ is paving, therefore circuits. Since $|C \triangle C'| = 2$, Lemma 4.0.3 implies that every r-subset of $C \cup C'$ is a circuit, which is impossible because $B \subseteq C \cup C'$. □

The following lemmas make essential use of sparseness.

Lemma 4.0.6. *Let $\mathcal{M}$ be a sparse paving matroid and let B be a basis with $g \in B$. There is at most one $h \in E - B$ such that $(B - g) \cup h$ is not a basis.*

Proof. If $(B - g) \cup h$ and $(B - g) \cup h'$ were circuits for distinct h, h', then this would violate the definition of sparseness. □

Duality helps us deduce an extra corollary from the main theorem of Section 4.4.

Lemma 4.0.7. *The class of sparse paving matroids is closed by taking duals.*

Proof. Let $\mathcal{M}$ be a rank r sparse paving matroid on n elements. We prove that the dual of $\mathcal{M}$ is paving and then that it is sparse paving.

Recall that a set $X \subseteq E$ is independent if and only if $E - X$ is co-spanning. Suppose that $\mathcal{M}^*$ is not paving. Then there is some $\mathcal{M}^*$-dependent set, $E - D$, with $|E - D| < n - r$. Therefore D, which has at least $r + 1$ elements, is not $\mathcal{M}$-spanning and thus contains no $\mathcal{M}$-basis. Since $\mathcal{M}$ is sparse paving, any $r + 1$ element set contains at most one circuit and at least one basis. Thus D contains a basis, contradicting the fact that it is not spanning. Therefore $\mathcal{M}^*$ is paving.

Suppose $\mathcal{M}^*$ is not sparse. Then it has $(n - r)$-circuits, $E - C$ and $E - C'$ such that $(E - C) - (E - C') = \{x\}$ and $(E - C') - (E - C) = \{x'\}$ where $x \neq x'$, violating sparseness.

Claim: If $|D| = r$, then D is an $\mathcal{M}$-circuit if and only if $E - D$ is an $\mathcal{M}^*$-circuit. Consider the fact that B is an $\mathcal{M}$-basis if and only if $E - B$ is an $\mathcal{M}^*$-basis. Let $D \subseteq E(\mathcal{M})$ with $|D| = r$ and $D \notin \mathscr{B}(\mathcal{M})$. Then D is dependent, and since $\mathcal{M}$ is paving, it is a circuit. Since D is not an $\mathcal{M}$-basis, then $E - D$ is not an $\mathcal{M}^*$-basis and similarly, $E - D$ is an $\mathcal{M}^*$-circuit, which proves the claim.

By this fact, C and C' are circuits of $\mathcal{M}$. Since $C - C' = \{x'\}$ and $C' - C = \{x\}$, $\mathcal{M}$ is not sparse, contrary to our assumption. Therefore $\mathcal{M}^*$ is a sparse paving matroid. ◻

Paving matroids in general are not closed under duality. Consider a matroid of rank one on three elements, with exactly two loops. If g is the non-loop, then it appears in every basis and is thus a co-loop. The dual has rank two, so it is not paving. This generalizes to the duals of all non-sparse paving matroids.

Proposition 4.0.8. *The dual of a non-sparse paving matroid is not paving.*

Proof. Let $\mathcal{M}$ be a non-sparse paving matroid of rank r on n elements. Let C and C' be r-circuits of $\mathcal{M}$ with $|C \triangle C'| = 2$. By Lemma 4.0.3, $C \cup C'$ contains no basis and is therefore not spanning. Thus, $E - (C \cup C')$ is $\mathcal{M}^*$-dependent and $|E - (C \cup C')| = n - r - 1$, so $\mathcal{M}^*$ is not paving. □

4.1 Sparse paving matroids are B-balanced

This result of Jerrum [13] is of interest because it enables polynomial time approximation of the number of bases in a class of matroids for which the bases are often hard to count exactly. Referring to [13] for details on computational complexity, we give an intuition of how the property of being B-balanced can be used to approximate the number of bases. The argument rests on two facts whose proofs are outside of the scope of this thesis.

Lemma 4.1.1 (simplified from [12], Ch. 3)**.** *If the set of combinatorial objects, X, can be sampled almost uniformly at random in polynomial time, then $|X|$ can be approximated in polynomial time.*

Let $\mathcal{M}$ be a matroid. Let the basis *exchange graph*, $\mathcal{G} = G(\mathcal{M})$, be the graph whose vertices are members of $\mathscr{B}$. Connect $B, B' \in \mathscr{B}$ with an edge if $|B \triangle B'| = 2$. The idea is to sample $\mathscr{B}$ using random walks in $\mathcal{G}$. To get an almost uniformly random sample in polynomial time, the walk must diffuse rapidly throughout the graph. One way to ensure this is for $\mathcal{G}$ to have high *expansion*, which is defined as

$$\alpha(\mathcal{G}) = \min\left\{ \frac{|[A, E(\mathcal{G}) - A]|}{|A|} : A \subseteq V(\mathcal{G}), |A| \leq \frac{|V(\mathcal{G})|}{2} \right\},$$

in which $[A, E(\mathcal{G}) - A]$ is the set of edges with exactly one end in A. In other words, high expansion means that every proper subset of vertices has a large number of edges leaving it. Feder and Mihail prove that the basis exchange graph of a balanced matroid has expansion $\alpha \geq 1$ [25].

Feder and Mihail have this application in mind. However, the examples they show to be **B**-balanced are regular. Jerrum points out that linear algebra can be used to count the bases of a regular matroid exactly with little trouble. Usually this is not possible for a sparse paving matroid. We present Jerrum's proof that sparse paving matroids are **B**-balanced.

Theorem 4.1.2. *Sparse paving matroids are **B**-balanced.*

Proof. Let e and f be distinct elements of a rank r sparse paving matroid, $\mathcal{M}$, on m elements. By Proposition 3.3.2 we may assume that $\{e, f\}$ is independent and co-independent. For disjoint sets $C, D \subseteq E$, recall that $\mathscr{B}_C^D$ is the set of bases containing C but not D.

To prove Theorem 4.1.2 we show that there exist positive integers p and q such that

$$pB_e^f(\mathbf{1}) \geq qB_{ef}(\mathbf{1}) \tag{4.1.1}$$

and

$$qB_f^e(\mathbf{1}) \geq pB^{ef}(\mathbf{1}). \tag{4.1.2}$$

Let $\mathcal{G} = (\mathscr{B}_e^f \dot\cup \mathscr{B}_{ef}, \mathfrak{E})$ be the bipartite graph with bipartition $(\mathscr{B}_e^f, \mathscr{B}_{ef})$ such that $\mathcal{M}$-bases, $b_e^f \in \mathscr{B}_e^f$ and $b_{ef} \in \mathscr{B}_{ef}$, are adjacent if and only if their symmetric difference has size two. Note that $\mathcal{G}$ is a subgraph of the basis exchange graph. There are at most $r-1$ elements in b_e^f that can be exchanged to give an element of $\mathscr{B}_{ef}$ (since e must stay) so the vertex b_e^f has degree at most $r-1$. By Lemma 4.0.6 (in which we require sparseness) there is at most one $h \notin b_{ef}$ such that $(b_{ef} - f) + h$ is not a basis. Thus, the vertex b_{ef} has degree at least $m - (r+1)$. Let $p = r-1$ and $q = m - r - 1$. Counting the edges of $\mathcal{G}$, we see that

$$(r-1)|\mathscr{B}_e^f| \geq (m-r-1)|\mathscr{B}_{ef}|.$$

Similarly, let $\mathcal{H} = (\mathscr{B}_f^e \dot\cup \mathscr{B}^{ef}, \mathfrak{F})$ be the bipartite graph with bipartition $(\mathscr{B}_f^e, \mathscr{B}^{ef})$, such that the $\mathcal{M}$-bases, $b_f^e \in \mathscr{B}_f^e$ and $b^{ef} \in \mathscr{B}^{ef}$, are adjacent if their symmetric difference has size two. There are at most $m-(r+1)$ elements $h \notin b^e_f$ which can replace f in $(b^e_f - f) \cup h$, since e is not one of them. Thus the vertex b^e_f has degree at most $m-r-1$. On the other hand, by Lemma 4.0.5, there is at most one element $g \in b^{ef}$ such that $(b^{ef}-g) \cup f$ is a circuit, so the vertex b^{ef} has degree at least $r-1$ in $\mathcal{H}$. Counting the edges of $\mathcal{H}$, we see that

$$(m-r-1)|\mathscr{B}^e_f| \geq (r-1)|\mathscr{B}^{ef}|.$$

Multiplying (4.1.1) and (4.1.2) yields

$$qpB^f_e(\mathbf{1})B^e_f(\mathbf{1}) \geq qpB_{ef}(\mathbf{1})B^{ef}(\mathbf{1}),$$

so that

$$B^f_e(\mathbf{1})B^e_f(\mathbf{1}) - B_{ef}(\mathbf{1})B^{ef} \geq 0.$$

This shows that $\mathcal{M}$ is B-negatively correlated. By Proposition 4.0.4, the class of sparse paving matroids is closed by taking minors. It follows that sparse paving matroids are B-balanced. □

4.2 Sparse paving matroids are B-Rayleigh

Jerrum's argument can be applied in a different way to prove that sparse paving matroids are B-Rayleigh. In fact, the B-Rayleigh difference has positive coefficients, from which the B-Rayleigh condition follows immediately.

Theorem 4.2.1 (E.)**.** *If $\mathcal{M}$ is a sparse paving matroid, then*

$$B^f_e B^e_f \gg B_{ef}B^{ef}. \tag{4.2.1}$$

Proof. Let $\mathcal{M}$ be a sparse paving matroid of rank r on m elements and suppose the theorem holds for all proper minors of $\mathcal{M}$. By Lemma 4.0.7

and the proof of Proposition 3.1.4, the classes of sparse paving matroids and matroids satisfying (4.2.1) are closed by taking duals. Hence $\mathcal{M}$ is assumed to be simple and co-simple, by Lemma 3.2.5 and Lemma 3.3.1. By this, if $r = 2$ or $r = m - 2$, then $\mathcal{M}$ is a uniform matroid, so we also assume that $m - 2 > r > 2$.

Suppose (4.2.1) holds for all minors of $\mathcal{M}$ and let $\mathbf{y}^\alpha$ be a monomial of (4.2.1). If $\alpha(g) = 0$ for some $g \in E - \{e, f\}$, then

$$[\mathbf{y}^\alpha]B_e^f B_f^e = [\mathbf{y}^\alpha]B_e^{gf} B_f^{ge} = [\mathbf{y}^\alpha]B_e^f(\mathcal{M} \setminus g)B_f^e(\mathcal{M} \setminus g)$$

and

$$[\mathbf{y}^\alpha]B_{ef}B^{ef} = [\mathbf{y}^\alpha]B_{ef}^g B^{gef} = [\mathbf{y}^\alpha]B_{ef}(\mathcal{M} \setminus g)B^{ef}(\mathcal{M} \setminus g).$$

By induction, (4.2.1) holds for this monomial. If $\alpha(g) = 2$ for some $g \in E - \{e, f\}$, then

$$[\mathbf{y}^\alpha]B_e^f B_f^e = [\mathbf{y}^\alpha y_g^{-2}]B_{ge}^f B_{gf}^e = [\mathbf{y}^\alpha y_g^{-2}]B_e^f(\mathcal{M}/g)B_f^e(\mathcal{M}/g)$$

and

$$[\mathbf{y}^\alpha]B_{ef}B^{ef} = [\mathbf{y}^\alpha y_g^{-2}]B_{gef}B_g^{ef} = [\mathbf{y}^\alpha y_g^{-2}]B_{ef}(\mathcal{M}/g)B^{ef}(\mathcal{M}/g).$$

Again, by the induction hypothesis, (4.2.1) holds for this monomial. Thus, we only need to prove

$$[\mathbf{y}^\alpha]B_e^f B_f^e \gg [\mathbf{y}^\alpha]B_{ef}B^{ef}$$

when $\mathbf{y}^\alpha$ is the unique monomial such that $\alpha(g) = 1$ for all $g \in E - \{e, f\}$.

Let (b_e^f, b_f^e) be a pair of type $B_e^f B_f^e$. That is, $b_e^f \dot\cup b_f^e = E - \{e, f\}$, b_e^f is a basis of $\mathcal{M}/e \setminus f$ and b_f^e is a basis of $\mathcal{M}/f \setminus e$. Let (b_{ef}, b^{ef}) be a pair of type $B_{ef}B^{ef}$. That is, $b_{ef} \dot\cup b^{ef} = E - \{e, f\}$, b_{ef} is a basis of $\mathcal{M}/ef$ and b^{ef} is a basis of $\mathcal{M} \setminus ef$. Let $(X \dot\cup Y, \mathfrak{E})$ be the bipartite graph with pairs of type $B_e^f B_f^e$ in the X partition and pairs of type $B_{ef}B^{ef}$ in the Y partition. We show that $|X| \geq |Y|$. Put an edge from (b_e^f, b_f^e) to (b_{ef}, b^{ef}) if these pairs satisfy one of the following

(a) There is $g \in b_e^f$ such that $b_e^f - g = b_{ef}$.

(b) There is $g \in b^{ef}$ such that $b^{ef} - g = b_f^e$.

Note that this may give parallel edges. Let the a-degree of a vertex be the number of neighbours it has satisfying rule (a). Define b-degree similarly. The rest of the argument is similar to Jerrum's for proving that sparse paving matroids are B-balanced. The vertices in the X partition have a-degree at most p and b-degree at most q, while the vertices in the Y partition have a-degree at least q and b-degree at least p, for some integers p and q.

Since $|b_e^f| = r-1$, there at most $r-1$ elements $g \in b_e^f$ such that $b_e^f - g$ is a basis of $\mathcal{M}/ef$. Thus, the a-degree of a pair (b_e^f, b_f^e) is at most $r-1$. Now, $m - |b_e^f| - |\{e,f\}| = m-r-1$, so there are at most $m-r-1$ elements $g \notin b_f^e$ such that $b_f^e \cup g$ is a basis of $\mathcal{M} \setminus ef$. As a result of this, the pair (b_e^f, b_f^e) has b-degree at most $m-r-1$, so vertices in the X partition have degree at most $m-2$.

The arguments on the Y partition use lemmas Lemma 4.0.5 and Lemma (4.0.6). In Lemma 4.0.5, if b is a basis and some fixed f must be added, then at most one of $(b-g) \cup f$ is a circuit, for $g \in b$. In Lemma 4.0.6 some fixed f is being removed and at most one of $(b-f) \cup g$ is a circuit, for each $g \notin b$. Only the latter requires sparseness.

Notice that $b_{ef} \cup ef$ is a basis of $\mathcal{M}$, and consider $((b_{ef} \cup ef) - f) \cup g$ for $g \notin (b_{ef} \cup ef)$. By Lemma 4.0.6, at most one of these is not a circuit, so there are at least $m - |b_{ef}| - |\{e,f\}| - 1 = m-(r-2)-2-1 = m-r-1$ elements $g \notin b_{ef}$ such that $b_{ef} \cup g = b_e^f$. If g is not in any set generated by B_e^f, then g is parallel to e, but we assumed that $\mathcal{M}$ was simple, so this is not the case. Thus, a vertex in the Y partition has a-degree at least $m-r-1$. Finally, the b-degree of a Y partition vertex is at least $r-1$ for the following reason. There are r elements in b^{ef} and there is at most one $g \in b^{ef}$ such that $(b^{ef} \cup f) - g$ is a circuit. Therefore the b-degree of this vertex is at least $r-1$. Again, $(m-r-1+(r-1)) = m-2 = 2(r-1)$

From this, the number of edges is

$$(m-2)|X| \geq |\mathfrak{E}| \geq (m-2)|Y|.$$

Since $m > 4$, the theorem holds. $\square$

4.3 A B-unbalanced paving matroid

Jerrum constructs a paving matroid which is not **B**-negatively correlated (and thus not **B**-balanced). We build the r-cycles of this matroid so that they satisfy Lemma 4.0.3.

Let E be a set of 24 elements and let $\mathcal{V} \subset 2^E$ be a set of 'blocks' of E of size 8 such that each 5-subset of E is contained in exactly one block, $V \in \mathcal{V}$ (this is the Steiner system $S(5,8,24)$). Jerrum uses this to define the circuits of size 6 of a paving matroid of rank 6. Let $e, f \in E$ be distinct elements and if $V \in \mathcal{V}$ contains exactly one of e or f, then all its 6 element subsets are cycles. Since each 5-subset is contained in exactly one block, no two blocks intersect in more than four elements and since $|C| = 6 > 5$, it must be contained in a unique block. Thus, if $C, C' \subset V$, then every 6-subset of $C \cup C'$ is also in V and otherwise $C \cap C' \leq 4$, so these circuits satisfy Lemma 4.0.3.

Let $\mathcal{M}$ be the paving matroid defined above. It remains for us to count $B_e^f(\mathbf{1}), B_f^e(\mathbf{1}), B_{ef}(\mathbf{1})$ and $B^{ef}(\mathbf{1})$. Again, for disjoint sets $C, D \subseteq E$, let $\mathscr{B}_C^D$ be the bases containing C but disjoint from D and similarly for $\mathcal{V}_C^D$. In other words, $\mathscr{B}_C^D$ is the collection of sets generated by $\mathbf{y}^C B_C^D$.

Let a_{ef} be a 6-subset of E that contains both e and f. Then it is contained in a unique block containing both e and f. Thus all such sets are bases so

$$|B_{ef}(\mathbf{1})| = \binom{22}{4} = 7315.$$

To find $|\mathscr{B}_e^f|$ we count the sets containing e and not f and subtract the cycles contained in blocks, $\mathcal{V}_e^f$. To count these we count $\mathcal{V}_e$ and subtract

$\mathcal{V}_{ef}$. There are $\binom{23}{4}$ 5-subsets containing e and each block $V \in \mathcal{V}_e$ contains $\binom{7}{4}$ of these, so

$$\mathcal{V}_e = \binom{23}{4} / \binom{7}{4} = 253.$$

Similarly, there are $\binom{22}{3}$ 5-subsets containing e and f and each block $V \in \mathcal{V}_{ef}$ contains $\binom{6}{3}$ of these, so

$$\mathcal{V}_{ef} = \binom{22}{3} / \binom{6}{3} = 77,$$

so that

$$\mathcal{V}_e^f = \mathcal{V}_e - \mathcal{V}_{ef} = 253 - 77 = 176.$$

Every one of the $\binom{7}{5}$ 6-subsets containing e of a block of $\mathcal{V}_e^f$ is a circuit by definition and they are each contained in a unique block, so there $176\binom{7}{5} = 3696$ of these. The total number of sets containing e and not f is $\binom{22}{5}$ and by symmetry $|\mathscr{B}_e^f| = |\mathscr{B}_f^e|$, so

$$|\mathscr{B}_e^f| = |\mathscr{B}_f^e| = \binom{22}{5} - 3696 = 22638.$$

Finally, every 6-subset not containing e or f is a basis unless it is contained in a block of $\mathcal{V}_e^f$ or $\mathcal{V}_f^e$. In each $V \in \mathcal{V}_e^f \dot\cup \mathcal{V}_f^e$ there are $\binom{7}{6}$ such circuits and previously we showed that $|\mathcal{V}_e^f| = |\mathcal{V}_f^e| = 176$, so

$$|\mathscr{B}^{ef}| = \binom{22}{6} - 2 \cdot 176 \binom{7}{6} = 72149.$$

Evaluating the Rayleigh difference,

$$\begin{aligned}
&\Delta B(\mathbf{1})\,\{e, f\} \\
&\qquad =B_e^f(\mathbf{1}) B_f^e(\mathbf{1}) - B_{ef}(\mathbf{1}) B^{ef}(\mathbf{1}) \\
&\qquad =22638^2 - 7315 \cdot 72149 = -15290891 < 0.
\end{aligned}$$

Thus, this paving matroid is not B-negatively correlated and hence not B-balanced.

4.4 B-Rayleigh paving matroids are I-Rayleigh

Theorem 4.4.1. *If $\mathcal{M}$ is a paving matroid then*

$$\Delta I\{e,f\} \gg \Delta B\{e,f\}. \tag{4.4.1}$$

Proof. We may assume by Proposition 3.3.2 that $\{e,f\}$ is independent and co-independent. Let r be the rank of $\mathcal{M}$ and let $U = I - B$. Since $\mathcal{M}$ is paving, U generates all subsets of E with up to $r-1$ elements. Let

$$U_n = e_1 + e_2 + \cdots + e_n,$$

in which

$$e_i = \sum_X \mathbf{y}^X,$$

summing over all subsets X of size i of $E - \{e,f\}$. Thus, $U_e^f = U_f^e = U_{r-2}$, $U_{ef} = U_{r-3}$ and $U^{ef} = U_{r-1}$.

Rewriting (4.4.1) gives

$$\begin{aligned}
&\Delta I\{e,f\} - \Delta B\{e,f\} \\
&= (B_e^f + U_{r-2})(B_f^e + U_{r-2}) - (B_{ef} + U_{r-3})(B^{ef} + U_{r-1}) - (B_e^f B_f^e - B_{ef}B^{ef}) \\
&= \left(B_e^f U_{r-2} + B_f^e U_{r-2} + \Delta U\{e,f\}\right) - \left(B_{ef}U_{r-1} + B^{ef}U_{r-3}\right),
\end{aligned} \tag{4.4.2}$$

in which $\Delta U\,\{e,f\} = (U_{r-2})^2 - U_{r-1}U_{r-3}$.

Lemma 4.4.2. $\Delta U\,\{e,f\} = e_{r-2}U_{r-1} - U_{r-2}e_{r-1}$

Proof. (of Lemma 4.4.2)

$$\begin{aligned}
\Delta U\{e,f\} =& (U_{r-2})^2 - U_{r-1}U_{r-3} \\
=& (e_{r-2} + U_{r-3})U_{r-2} - (e_{r-1} + U_{r-2})U_{r-3} \\
=& e_{r-2}U_{r-2} + U_{r-3}U_{r-2} - e_{r-1}U_{r-3} - U_{r-2}U_{r-3} \\
=& e_{r-2}U_{r-2} - e_{r-1}U_{r-3} \\
=& e_{r-2}U_{r-2} - e_{r-1}U_{r-3} + (e_{r-2}e_{r-1} - e_{r-2}e_{r-1}) \\
=& e_{r-2}(U_{r-2} + e_{r-1}) - e_{r-1}(U_{r-3} + e_{r-2}) \\
=& e_{r-2}U_{r-1} - e_{r-1}U_{r-2}
\end{aligned}$$
□

Let $\mathbf{y}^X$ be a monomial of $B_e^f + B_f^e$. Then at most one of $X \cup e$ and $X \cup f$ are circuits (in particular, if its coefficient is 2 then both $X \cup e$ and $X \cup f$ are bases). Let A_{r-1} be the generating polynomial for the sets X and let D_{r-1} generate the $(r-1)$-sets, Y, such that $Y \cup e$ and $Y \cup f$ are circuits. Thus,

$$B_e^f + B_f^e = A_{r-1} - D_{r-1} + e_{r-1}.$$

Substituting into (4.4.2) and rearranging, we wish to show that

$$A_{r-1}U_{r-2} - D_{r-1}U_{r-2} + e_{r-1}U_{r-2} + \Delta U\{e,f\} \gg B_{ef}U_{r-1} + B^{ef}U_{r-3}, \tag{4.4.3}$$

By Lemma 4.4.2 this simplifies to

$$A_{r-1}U_{r-2} - D_{r-1}U_{r-2} + e_{r-2}U_{r-1} \gg B_{ef}U_{r-1} + B^{ef}U_{r-3}.$$

Let $\mathbf{y}^\alpha$ be one of the monomials appearing in (4.4.2). It is enough to show that

$$[\mathbf{y}^\alpha]\left(B_e^f U_{r-2} + B_f^e U_{r-2} + \Delta U\{e,f\}\right) \geq [\mathbf{y}^\alpha]\left(B_{ef}U_{r-1} + B^{ef}U_{r-3}\right). \tag{4.4.4}$$

Suppose that $\alpha(g) = 2$ for some $g \in E - \{e, f\}$. We contract this element and apply the induction hypothesis to conclude that,

$$\begin{aligned} &[\mathbf{y}^\alpha] \left(\Delta I(\mathcal{M})\{e,f\} - \Delta B(\mathcal{M})\{e,f\} \right) \\ = &\left[\mathbf{y}^\alpha y_g^{-2}\right] \left(\Delta I(\mathcal{M}/g)\{e,f\} - \Delta B(\mathcal{M}/g)\{e,f\} \right) \geq 0. \end{aligned}$$

If $\alpha(g) = 0$ for some $g \in E - \{e, f\}$ then we delete this element and apply the induction hypothesis to conclude that

$$\begin{aligned} &[\mathbf{y}^\alpha] \left(\Delta I(\mathcal{M})\{e,f\} - \Delta B(\mathcal{M})\{e,f\} \right) \\ = &[\mathbf{y}^\alpha] \left(\Delta I(\mathcal{M} \setminus g)\{e,f\} - \Delta B(\mathcal{M} \setminus g)\{e,f\} \right) \geq 0. \end{aligned}$$

Thus we only need to prove (4.4.4) for the unique monomial $\mathbf{y}^\alpha$ such that $\alpha(g) = 1$ for all $g \in E - \{e, f\}$.

It suffices to show that

$$(e_{r-2} - B_{ef})U_{r-1} \gg D_{r-1}U_{r-2} \tag{4.4.5}$$

and

$$A_{r-1}U_{r-2} \gg B^{ef}U_{r-3}. \tag{4.4.6}$$

Consider $\mathbf{y}^\alpha$ in (4.4.5). Let (D, Q) be a pair of type $D_{r-1}U_{r-2}$. That is, $D\dot{\cup}Q = E - \{e, f\}$, $D \cup e$ and $D \cup f$ are circuits of size r and $|Q| \leq r - 2$. Similarly, let (C, R) be a pair of type $(e_{r-2} - B_{ef})U_{r-1}$. That is $C\dot{\cup}R = E - \{e, f\}$, $C \cup \{e, f\}$ is a circuit of size r and $|R| \leq r - 1$.

Let $(X\dot{\cup}Y, \mathfrak{E})$ be a bipartite graph with pairs of type $D_{r-1}U_{r-2}$ in the set X and pairs of type $(e_{r-2} - B_{ef})U_{r-1}$ in the set Y. Put an edge from (D, Q) to (C, R) if there exists $g \in D$ so that $(C, R) = (D - g, Q \cup g)$. Suppose that $D - g$ is a basis of $\mathcal{M}/ef$. Then $(D - g) \cup ef$ is a basis of $\mathcal{M}$. Since $D \cup e$ and $D \cup f$ are circuits, Lemma 4.0.5 implies that every r-subset of $D \cup ef$ is a circuit. This contradicts $(D - g) \cup ef$ being a basis, so $D - g$ is always a circuit of M/ef. So the degree of (D, Q) must be $r - 1$. On the other hand $|R| \le r-1$ and so there at most $r-1$ edges from (C,R). Now the number of edges in $(X\dot{\cup}Y,\mathfrak{E})$ is $(r-1)|X| = |\mathfrak{E}| \le (r-1)|Y|$, so

$$[\mathbf{y}^\alpha](e_{r-2}-B_{ef})U_{r-1} \ge [\mathbf{y}^\alpha]D_{r-1}U_{r-2},$$

as required for (4.4.5).

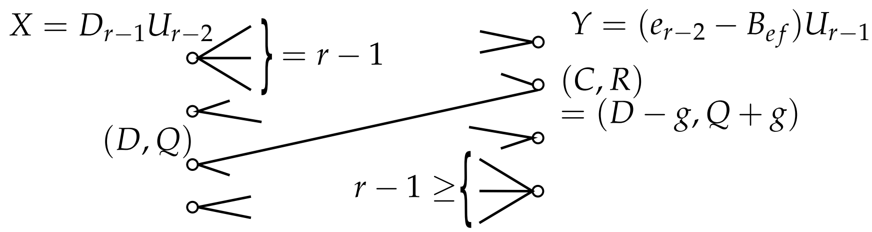


Figure 4.1: A visualization of the proof of (4.4.5)

Similarly, consider the monomial $\mathbf{y}^\alpha$ in (4.4.6). Let (J,Q) be a pair of type $B^{ef}U_{r-3}$. That is, $J\dot{\cup}Q = E-\{e,f\}$, J is a basis of $\mathcal{M}\setminus ef$ and $|Q| \le r-3$. Let (A,R) be a pair of type $A_{r-1}U_{r-2}$. That is, $A\dot{\cup}R = E-\{e,f\}$, $A\cup e$ and $A\cup f$ are bases and $|R| \le r-2$.

As before, let $(X\dot{\cup}Y,\mathfrak{E})$ be the bipartite graph with pairs of type $B^{ef}U_{r-3}$ in the set X and pairs of type $A_{r-1}U_{r-2}$ in the set Y. Put an edge from (J,Q) to (A,R) if there exists $g\in J$ so that $(A,R) = (J-g, Q\cup g)$. By Lemma 4.0.5, there is at most one $g\in J$, such that $(J-g)\cup e$ is a circuit and similarly for f. Therefore there are at least $r-2$ elements of J whose removal yields a set of type A_{r-1}. On the other hand (A,R) is adjacent to at most $|R| \le r-2$ pairs of type $B^{ef}U_{r-3}$. Thus (X,Y) has at least $(r-2)|X|$ edges but at most $(r-2)|Y|$ edges. If $r>2$ then

$$[\mathbf{y}^\alpha]A_{r-1}U_{r-2} \ge [\mathbf{y}^\alpha]B^{ef}U_{r-3},$$

and if $r\le 2$, then (4.4.6) is trivial. □

Notice that in Theorem 4.4.1, no assumptions were made on the values of the y_gs. For this reason there are corollaries for B and I-balance.

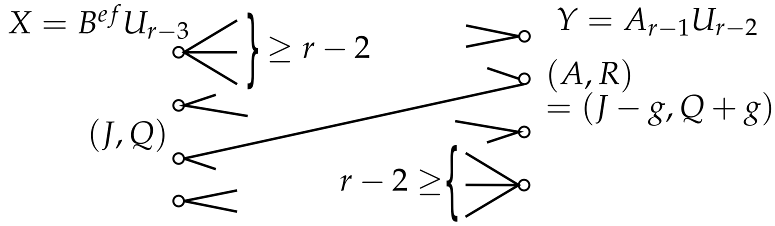


Figure 4.2: A visualization of the proof of (4.4.6)

Corollary 4.4.3. *Let $\mathcal{M}$ be a paving matroid.*

*(a) The matroid $\mathcal{M}$ is **I**-Rayleigh if and only if it is **B**-Rayleigh*

*(b) If $\mathcal{M}$ is **B**-balanced, then it is **I**-balanced*

*(c) If $\mathcal{M}$ is **B**-negatively correlated, then it is **I**-negatively correlated*

Proof. By Proposition 2.3.3 **I**-Rayleigh implies **B**-Rayleigh. For (a), (b) and (c), the other direction follows from Theorem 4.4.1. □

Theorem 4.4.1 and Theorem 4.2.1 combined with Lemma 4.0.7 and Proposition 3.1.5 yield the following.

Corollary 4.4.4. *Let $\mathcal{M}$ be a sparse paving matroid. Then $\mathcal{M}$ is **B**, **I** and **S**-Rayleigh.*

Proof. Proposition 3.1.4 implies that $\mathcal{M}$ is **B**-Rayleigh if and only if $\mathcal{M}^*$ is **B**-Rayleigh. By Lemma 4.0.7, $\mathcal{M}^*$ is sparse paving so that we can apply Corollary 4.4.3. Thus, $\mathcal{M}^*$ is **B**-Rayleigh if and only if $\mathcal{M}^*$ is **I**-Rayleigh. Proposition 3.1.5 implies that $\mathcal{M}^*$ is **I**-Rayleigh if and only if $\mathcal{M}$ is **S**-Rayleigh. Finally, $\mathcal{M}$ is indeed **B**-Rayleigh, by Theorem 4.2.1, and the three Rayleigh properties have been shown to be equivalent, as required. □

Again, Theorem 4.4.1 and Theorem 4.2.1 are combined, now to show that the **I**-Rayleigh difference for a sparse paving matroid has positive coefficients.

Corollary 4.4.5. *Let $\mathcal{M}$ be a sparse paving matroid. Then $\Delta I\{e, f\}$ has positive coefficients.*

Proof. By Theorem 4.2.1, $\Delta B\{e, f\}$ has positive coefficients and by Theorem 4.4.1, $\Delta I\{e, f\} - \Delta B\{e, f\}$ does as well, as desired. $\square$

The results of this section are used to further progress on a conjecture of Semple and Welsh addressed in the next section.

4.5 Rank three matroids revisited

Semple and Welsh [4] ask a number of questions, some of which have been answered. The last of these has been partially addressed as a result of Section 4.4.

Conjecture 4.5.1 (Semple and Welsh [4])**.** *Rank three matroids are* **I** *and* **S***-negatively correlated.*

By Theorem 3.5.1, they are **I**-Rayleigh, so only half the question remains. The following observation enables us to count the results of this chapter as further progress on Semple and Welsh's question.

Firstly, Corollary 3.5.2 can be re-derived from Corollary 4.4.3 whilst Theorem 3.5.1 is completely avoided.

Proposition 4.5.2. *Rank three matroids are* **I***-Rayleigh.*

Proof. Let $\mathcal{M}$ be a matroid of rank at most three and suppose all proper minors of the matroid $\mathcal{M}$ are **I**-Rayleigh. By Corollary 3.2.3, $\mathcal{M}$ is simple, so it has rank at least two. If $\mathcal{M}$ is rank two then it is uniform and therefore **I**-Rayleigh. If $\mathcal{M}$ is rank three then it is paving. By Theorem 3.4.2 it is **B**-Rayleigh and by Corollary 4.4.3 it is **I**-Rayleigh. $\square$

It is not yet known whether or not matroids of at most rank three are **S**-Rayleigh; however, by the same argument as Proposition 4.5.2, matroids of rank at most two are **S**-Rayleigh and otherwise, we are only concerned with simple rank three matroids. If $\mathcal{M}$ is sparse, this combined with Corollary 4.4.5 implies that $\mathcal{M}$ is also S-Rayleigh. This raises the question, when is a paving matroid not sparse? Proposition 4.5.3 provides an answer for simple rank three matroids. Note that in general, a rank r paving matroid is not sparse if and only if it contains a $\mathcal{U}_{r-1,r+1}$ restriction minor.

Proposition 4.5.3. *A simple rank three matroid is not sparse paving if and only if it contains a $\mathcal{U}_{2,4}$ restriction minor.*

Proof. Clearly a rank three matroid is simple if and only if it is paving. Let $\mathcal{M}$ be a non-sparse, rank three paving matroid with 3-circuits C and C' such that $|C\Delta C'| = 2$. By Lemma 4.0.3, every 3-subset of the four element set, $C \cup C'$, is a circuit, thus $\mathcal{M} \setminus (E - (C \cup C'))$ is $\mathcal{U}_{2,4}$.

The converse clearly holds since the three element subsets of a $\mathcal{U}_{2,4}$ submatroid violate the sparseness condition. □

By Proposition 4.5.2 and Corollary 4.4.5, a simple rank three matroid with no $\mathcal{U}_{2,4}$ restriction minor is sparse paving and thus it is both I and S-Rayleigh. The following conjecture implies a positive answer to Conjecture 4.5.1.

Conjecture 4.5.4. *Let $\mathcal{M}$ be a simple rank three matroid containing a restriction isomorphic to $\mathcal{U}_{2,4}$. Then $\mathcal{M}$ is S-Rayleigh.*

Chapter 5

The I-Rayleigh conjecture for graphs

5.1 Graph theory

For our purposes, most properties of graphs are inherited from their matroid superclass; however, there are one or two definitions and a small amount of notation that is specific to graphs.

Let $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ be a connected graph with no loops. Let the spanning forests be $\mathscr{F}(\mathcal{G}) = \mathscr{I}(M(\mathcal{G}))$ and their generating polynomial be $F(\mathcal{G}; \mathbf{y}) = I(M(\mathcal{G}); \mathbf{y})$. Similarly, we replace $\mathscr{B}$ and B by $\mathscr{T}$ and T, respectively. An edge, $e \in E$, is written uev or veu where $u, v \in V$ are the ends of e. As usual, the $\mathcal{G}$ is dropped whenever possible. We may also write $G = F(\mathcal{G}; \mathbf{y})$.

Let a cycle containing a set X be called an *X-cycle*.

In this chapter we make use of edge orientation, which is a property that the graphic matroid $M(\mathcal{G})$ does not retain.

Definition 5.1.1 (orientation). *Let $\mathcal{G}$ be a graph with an edge, uv. An* orientation *of uev designates u to be the tail and v to be the head of e (to orient the other way, write veu). A graph is* oriented *if all of its edges are oriented. Let D*

be the signed incidence matrix of $\mathcal{G}$. Then for an edge e incident with vertex v,

$$D_{ve} = \begin{cases} 1, & v \text{ is the head of } e \\ -1, & v \text{ is the tail of } e \end{cases}.$$

5.2 The conjecture

Conjecture 5.2.1. *Graphs are I-Rayleigh.*

This conjecture was written down as early as 1993 by Kahn [14] and published in 2000 [11]. Neither of these papers make a direct attempt to solve the conjecture. Wagner [28], and Semple and Welsh [4] published similar material circa 2008 where they prove the result for two-sums as in Section 3.2.

In its weaker form – graphs are I-negatively correlated – it has been mentioned in Pemantle's survey (credited as a personal communication from Winkler) [20], and Grimmett and Winkler [7]. In [7], small examples are tested for I-negative correlation. All graphs on up to eight vertices or nine vertices and 18 edges do indeed satisfy the condition. More generally, in 2008 Gordon Royle verified that all matroids on up to nine elements are I-negatively correlated (Wagner, personal communication).

Although these results provide some evidence for a positive answer to Conjecture 5.2.1, one should consider that the same things hold for matroids – two-sums of I-Rayleigh matroids are I-Rayleigh, small matroids are I-negatively correlated. It can be shown, in fact, that if Conjecture 5.2.1 is true, then I-negative correlation for graphs is not weaker than the I-Rayleigh property. This is proved for the spanning tree case by Wagner and Choe [31] and Kahn and Neiman mention that it holds for spanning forests. We present a proof of this, similar to the one in [31] (partly from conversations with Mathieu Guay-Paquet and Wagner. Also independently by Cocks [5]).

More generally, let $\mathcal{M}$ be a member of a class of matroids where I-negative correlation and the I-Rayleigh property are equivalent. The idea

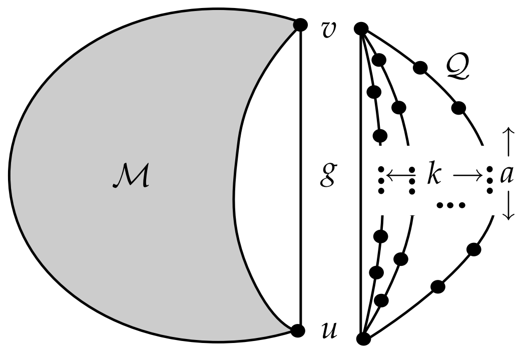


Figure 5.1: The graphs $\mathcal{M}$ and $\mathcal{Q}$ to be two-summed along g. The graph $\mathcal{Q}$ consists of the edge ugv and k edge-disjoint u,v-paths, each of length a.

is to construct a matroid $\mathcal{M}[\mathbf{y}]$ of the same class, with the property that if $\mathcal{M}[\mathbf{y}]$ is not I-negatively correlated, then $\mathcal{M}$ is not I-Rayleigh. This is done before the proof.

Let $\mathfrak{Q} = \left\{q : q = \frac{k}{2^a-1}, k \geq 0, a \geq 1, k, a \in \mathbb{Z}\right\}$ and for $q = \frac{k}{2^a-1} \in \mathfrak{Q}$, let $\mathcal{Q}$ be the graphic matroid of k edge-disjoint uv-paths (u, v distinct vertices), each of length a plus the edge $g = ugv$, see Figure 5.1. Notice that, by removing edges, there are $2^a - 1$ ways of disconnecting each uv-path. The element g is in the closure of any spanning forest of $\mathcal{Q}\backslash g$ such that exactly one uv-path is intact. There are $k(2^a-1)^{k-1}$ of these. On the other hand, there are $(2^a-1)^k$ spanning forests of $\mathcal{Q}\backslash g$ that do not contain g in their closure. Therefore $I(\mathcal{Q}\backslash g)$ has $k(2^a-1)^{k-1} + (2^a-1)^k$ terms and $I(\mathcal{Q}/g)$ has $(2^a-1)^k$ terms. Let $Q = I(\mathcal{Q})$. Then

$$\begin{aligned} q =& \frac{k}{2^a-1} = \frac{k(2^a-1)^{k-1}}{(2^a-1)^k} = \frac{Q^g(\mathbf{1}) - Q_g(\mathbf{1})}{Q_g(\mathbf{1})} \\ =& \frac{|\{F : F \text{ is a spanning forest of } \mathcal{Q}\backslash g, g \notin \mathrm{cl}(F)\}|}{|\{F : F \text{ is a spanning forest of } \mathcal{Q}\backslash g, g \in \mathrm{cl}(F)\}|}. \end{aligned}$$

Consider an element g of $\mathcal{M}$. We assume $\mathcal{M}$ has neither loops nor co-loops. Let $\mathcal{M}[\mathcal{Q}] = \mathcal{M} \oplus_g \mathcal{Q}$. Put $M = I(\mathcal{M}; \mathbf{y})$ so that by Lemma 2.1.3, we can define

$$\begin{aligned} M\left[Q\right] = I(\mathcal{M}\left[\mathcal{Q}\right]) =& Q_g M^g + Q^g M_g - Q_g M_g \\ =& Q_g M^g + (Q^g - Q_g) M_g \\ =& Q_g(M^g + y_g M_g), \end{aligned}$$

setting $y_g = \frac{Q^g - Q_g}{Q_g}$. Thus, for this choice of y_g, $M\left[Q\right] = Q_g M$.

For $X = \{g_1, g_2, \ldots, g_n\} \subseteq E$, and corresponding q_is and $\mathcal{Q}_i$s, define

$$\mathcal{M}\left[\mathbf{Q}^X\right] = \mathcal{M} \oplus_{g_1} \mathcal{Q}_1 \oplus_{g_2} \cdots \oplus_{g_n} \mathcal{Q}_n.$$

Setting $y_g = \frac{Q^g - Q_g}{Q_g}$ for each $g, \mathcal{Q}$ pair from X, let

$$M\left[\mathbf{Q}^X\right] = I\left(\mathcal{M}\left[\mathbf{Q}^X\right]\right) = \left(\prod_{g,Q} Q_g\right) M,$$

in which the product is over the pairs g, Q for each $g \in X$. Evidently, qs and $\mathcal{Q}$s can be chosen according to the choices for y_gs (assuming $y_g \in \mathfrak{Q}$) or vice versa.

Hence, if $\mathbf{y} = \mathbf{q}$ for some $\mathbf{q} \in \mathfrak{Q}^{E-\{e,f\}}$ and corresponding matroids, $\mathcal{Q}$, then

$$\left(\prod_{g,Q:g\in E-\{e,f\}} Q_g(\mathbf{1})\right)^2 \Delta M(\mathbf{y} = \mathbf{q})\left\{e,f\right\} = \Delta M\left[\mathbf{Q}^{E-\{e,f\}}\right](\mathbf{1})\left\{e,f\right\}.$$

Proposition 5.2.2. *With the above notation, if every matroid of the form* $\mathcal{M}\left[\mathbf{Q}^{E-\{e,f\}}\right]$ *is* $\mathbf{I}$*-negatively correlated, then* $\mathcal{M}$ *is* $\mathbf{I}$*-Rayleigh.*

Proof. Suppose $\mathcal{M}$ is not $\mathbf{I}$-Rayleigh. Then there exists a real evaluation $\tilde{\mathbf{y}}$ and distinct elements e and f such that $\Delta\mathcal{M}(\tilde{\mathbf{y}})\left\{e,f\right\} < 0$. Since $\mathfrak{Q}$ is dense in the positive reals, there exists $\mathbf{q} \in \mathfrak{Q}^E$ such that $\Delta\mathcal{M}(\mathbf{q})\left\{e,f\right\} < 0$. We also have

$$\Delta M\left[\mathbf{Q}^{E-\{e,f\}}\right](\mathbf{1})\left\{e,f\right\} = \left(\prod_{g,Q} Q_g(\mathbf{1})\right)^2 \Delta M(\mathbf{q})\left\{e,f\right\} < 0,$$

so that $\mathcal{M}\left[\mathbf{Q}^{E-\{e,f\}}\right]$ is not $\mathbf{I}$-negatively correlated, as desired. □

Corollary 5.2.3. *Graphs are all I-negatively correlated if and only if they are all I-Rayleigh.*

Proof. Let $\mathcal{M}$ be a graphic matroid. For distinct $e, f \in E(\mathcal{M})$, and $\mathbf{q} \in \mathfrak{Q}^{E-\{e,f\}}$, the matroids $\mathcal{Q}$ corresponding to each q are graphic and thus $\mathcal{M}\left[\mathbf{Q}^{E-\{e,f\}}\right]$ is graphic. Therefore, if for each choice of e and f, $\mathcal{M}\left[\mathbf{Q}^{E-\{e,f\}}\right]$ is I-negatively correlated for all $\mathbf{q} \in \mathfrak{Q}^{E-\{e,f\}}$, then $\mathcal{M}$ is I-Rayleigh by Proposition 5.2.2. □

Further evidence for Conjecture 5.2.1 is found by Wagner. He shows (personal communication) that the I-Rayleigh difference has a special form for small graphs, which, if true for all graphs, would imply Conjecture 5.2.1. In this chapter we show that the form holds for more graphs and also for series parallel graphs.

The fact that graphs are B-Rayleigh, goes back to Kirchhoff's law for electrical networks and Rayleigh monotonicity, as described in Section 2.3. Proofs of this have been published several times. Brooks, Smith, Stone and Tutte (BSST) [21] prove that $\Delta T\{e, f\}$ is the square of a polynomial. Wagner et. al [10] prove the same identity using elementary combinatorics. As a result, $\Delta T\{e, f\} \geq 0$ whenever the y_gs are real, implying that graphs are B-Rayleigh.

Notation is introduced in order to describe the BSST identity. Let the graph $\mathcal{G}$ have an arbitrary orientation and let e and f be distinct edges of $\mathcal{G}$. Let $\chi \subseteq \mathscr{F}(\mathcal{G} \setminus \{e, f\})$ be the collection of sets A, such that $A \cup e$ and $A \cup f$ are both spanning trees. Define

$$\chi^{+} = \sum_{A} \mathbf{y}^{A},$$

summing over members of χ where e and f have the same orientation around the unique cycle of $A \cup ef$. Define

$$\chi^{-} = \sum_{A} \mathbf{y}^{A},$$

summing over members of χ where e and f have different orientations around the unique cycle of $A \cup ef$.

Theorem 5.2.4. *With the above notation,*

$$\Delta T\{e,f\} = (\chi^{+} - \chi^{-})^2.$$

Of course, since $\Delta T\{e,f\}$ is the square of a polynomial, then $\Delta T\{e,f\} \geq 0$ for any (real) evaluation of the y_gs.

Wagner's conjecture is that for any graph $\mathcal{G}$,

$$\Delta F\{e,f\} = \sum_{S \subseteq E} \mathbf{y}^S A(S)^2,$$

in which $A(S)$ is some polynomial depending on S. We give some notation to define this formally.

Definition 5.2.5 (S-sets, A-sets and the signs $c(S,C)$)**.** *Let $\mathcal{G}$ be a graph. Let $\mathcal{S} \subseteq 2^{E-ef}$ be the collection of those sets $S \subseteq E - ef$ such that $S \cup ef$ is contained in some cycle of $\mathcal{G}$. For $S \in \mathcal{S}$, let $\mathcal{A}(S) \subseteq 2^{E-ef}$ be the collection of those spanning forests $A \subseteq E - ef$, such that $S \cup ef \subseteq C \subseteq A \cup ef$ for a unique cycle, C. Let $c(S,e,f,C) = \pm 1$, depending on $S \in \mathcal{S}$ and some cycle C containing $S \cup ef$. We write $c(S,C)$ when e and f are understood and we use a subscripted G wherever the graph $\mathcal{G}$ needs to be specified, as in $\mathcal{A}_G(S)$.*

We refer to the elements of $\mathcal{S}$ and $\mathcal{A}(S)$ as S-sets and A-sets, respectively. Throughout the rest of this chapter, given an $S \in \mathcal{S}$ and $A \in \mathcal{A}(S)$, C is the unique cycle such that $S \cup ef \subseteq C \subseteq A \cup ef$ unless otherwise noted.

Conjecture 5.2.6 (Wagner (personal communication), Sum of Squares)**.** *Let $\mathcal{G}$ be a graph with distinct edges e and f. Then for some choice of the signs $c(S,C)$,*

$$\Delta F\{e,f\} = \sum_{S \in \mathcal{S}} \mathbf{y}^S \left(\sum_{A \in \mathcal{A}(S)} c(S,C)\mathbf{y}^{A-S} \right)^2. \tag{5.2.1}$$

If $\mathcal{G}$ satisfies Conjecture 5.2.6 we say $\mathcal{G}$ (or F, or ΔF) is SOS (or Δ-SOS). Recall from Proposition 3.1.1, the formula

$$\Delta F\{e,f\} = \Delta F^g\{e,f\} + y_g \Theta F\{e,f|g\} + y_g^2 \Delta F_g\{e,f\}. \tag{5.2.2}$$

This is used again, to show that if $\mathcal{G}$ is SOS then so are its minors.

Proposition 5.2.7. *Let $\mathcal{G}$ be a graph with distinct edges e, f and g. If $\Delta F\{e,f\}$ satisfies Conjecture 5.2.6 then so do $\Delta F^g\{e,f\}$ and $\Delta F_g\{e,f\}$.*

Proof. We may assume that g is not a loop. By equation (5.2.2), $\Delta F^g\{e,f\} = \lim_{y_g \to 0} \Delta F\{e,f\}$. To show that this satisfies the SOS form for ΔF^g use

$$\lim_{y_g \longrightarrow 0} \Delta F\{e,f\} = \sum_{\substack{S \in \mathcal{S}_G \\ g \notin S}} \mathbf{y}^S \left(\sum_{\substack{A \in \mathcal{A}_G(S) \\ g \notin A}} c(S,C)\mathbf{y}^{A-S} \right)^2. \tag{5.2.3}$$

An S-set of $\mathcal{G} \setminus g$ is contained in an ef-cycle of $\mathcal{G} \setminus g$. Clearly $\mathcal{S}_{G \setminus g} \subseteq \{S : S \in \mathcal{S}_G, g \notin S\}$, the set indexing the outer sum of (5.2.3). On the other hand, given a set $\tilde{S} \in \{S : S \in \mathcal{S}_G, g \notin S\} - \mathcal{S}_{G \setminus g}$, there are no ef-cycles containing $\tilde{S}$ and not g. Thus, the set $\{A \in \mathcal{A}_G(\tilde{S}) : g \notin A\}$ is empty. Therefore, together, the sets indexing the sums in (5.2.3) are equivalent to S-sets and A-sets of $\mathcal{G} \setminus g$.

The proof for ΔF_g is slightly trickier due to the fact that when g is contracted, two cycles may be created from one. By equation (5.2.2), $\lim_{y_g \to \infty} y_g^{-2} \Delta F\{e,f\} = \Delta F_g\{e,f\}$ and we show that this satisfies the SOS form for ΔF_g. Since any term of the SOS form of ΔF which does not contain y_g^2 disappears, we are left with

$$\lim_{y_g \longrightarrow \infty} y_g^{-2} \Delta F\{e,f\} = \sum_{\substack{S \in \mathcal{S}_G \\ g \notin S}} \mathbf{y}^S \left(\sum_{\substack{A \in \mathcal{A}_G(S) \\ g \in A}} c(S,C)\mathbf{y}^{A-(S \cup g)} \right)^2. \tag{5.2.4}$$

Sets $S \in \mathcal{S}_G$ such that g is a chord of every cycle containing $S \cup ef$ are ignored. No set $A \in \mathcal{A}_G(S)$ can contain g, since otherwise $A \cup ef$ would not have a unique cycle. Thus, the set indexing the outer sum of (5.2.4) can be replaced by

$$\{S \in S_G : g \notin S, \exists C \supseteq S \cup ef \text{ such that } g \text{ is not a chord of } C\}$$

which is equal to $\mathcal{S}_{G/g}$. It remains to be shown that for $S' \in S_{G/g}$,

$$\mathcal{A}_{G/g}(S') = \left\{A \in \mathcal{A}_G(S') : g \in A\right\}. \tag{5.2.5}$$

Let $S' \in \mathcal{S}_{G/g}$ and $A' \in \mathcal{A}_{G/g}(S')$ and let $A = A' \cup g$. Since $A' \cup e$ and $A' \cup f$ are independent in G/g, $A \cup e$ and $A \cup f$ are independent in G. Furthermore there is a unique cycle $C \subseteq A \cup ef$ containing S', so $A \in \mathcal{A}_G(S')$ and $g \in A$.

Conversely, let $A \in \mathcal{A}_G(S')$ with $g \in A$. Clearly $A - g \in \mathcal{A}_{G/g}(S')$, since g cannot be a chord of $C \subseteq A \cup ef$. Therefore $\mathcal{S}_{G/g}$ and $\mathcal{A}_{G/g}(S')$ for $S' \in \mathcal{S}_{G/g}$ are exactly those sets indexed by (5.2.4). □

The SOS form also holds by taking direct sums.

Lemma 5.2.8. *Let $\mathcal{H}$ and $\mathcal{K}$ be graphs for which the SOS form holds and let $\mathcal{G}$ be their direct sum. Then $\mathcal{G}$ is SOS for some choice of signs.*

Proof. If $e \in E(\mathcal{H})$ and $f \in E(\mathcal{K})$, then $\Delta F(\mathcal{G})\{e, f\} = 0$, by Lemma 3.2.1. This is consistent with the fact that there are no cycles through e and f and hence no S-sets. Let $G = I(\mathcal{G}), H = I(\mathcal{H})$ and $K = I(\mathcal{K})$. If $e, f \in E(\mathcal{H})$ then by Lemma 3.2.1

$$\begin{aligned} \Delta G\{e,f\} =& K^2 \Delta H\{e,f\} \\ &= \sum_{S \in \mathcal{S}_H} \mathbf{y}^S \left(\sum_{A \in \mathcal{A}_H(S)} c(S,C)\mathbf{y}^{A-S} K \right)^2 . \end{aligned} \tag{5.2.6}$$

Since an ef-cycle of $\mathcal{G}$ cannot contain an edge of $\mathcal{K}$, the S-sets of $\mathcal{G}$ are contained in $\mathcal{H}$, which is consistent with (5.2.6). Given an S-set of $\mathcal{G}$,

$$\mathcal{A}_G(S) = \mathcal{A}_H(S) \vee \mathscr{F}(\mathcal{K}),$$

as required. □

Much of the evidence for Conjecture 5.2.6 comes from making educated guesses for the signs and testing the SOS form on small examples using Maple. So far we have shown that all graphs on up to seven vertices satisfy the SOS form for all distinct pairs of edges. Ad hoc modifications to the signs have enabled us to prove it for the cube ($(K_2)^3$) and the Möbius ladder (V_8) (personal communication from Wagner). Other graphs of that size for which we have not yet found signs exhibit discrepancies between the SOS form and the I-Rayleigh difference of only a few terms, on the order of tens out of tens of thousands.

Example 5.2.9. Let $\mathcal{G} = (\{a, b, c, d\}, \{1, 2, 3, 4, 5, 6\})$ so that $\mathcal{G}$ (Figure 5.2) is isomorphic to K_4 and has incidence matrix

	1	2	3	4	5	6
a	1	1	1	0	0	0
b	1	0	0	1	1	0
c	0	1	0	1	0	1
d	0	0	1	0	1	1

where we have indicated vertex and edge labels for rows and columns.

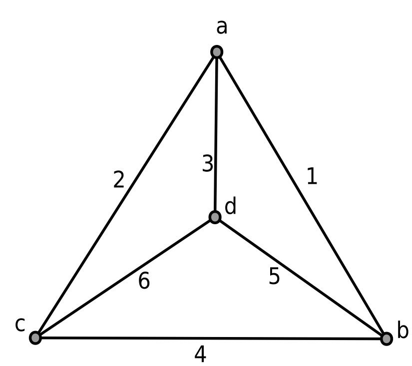


Figure 5.2: K_4 in Example 5.2.9

For the sake of brevity, commas and curly brackets are dropped for edge sets. The two cycles through edges 1 and 2 are $C_\alpha := 124$ and $C_\beta := 1256$ which means that the S-sets are the subsets of these not containing $1, 2$. That is

$$\mathcal{S} = \{\varnothing, 4, 5, 6, 56\}\,. \tag{5.2.7}$$

The A-sets depend on the cycles through 12 as well, so for the cycle C_α they are

$$4, 43, 45, 46,$$

and the A-set for the cycle C_β, is simply 56. Let $A(C_\alpha) = y_4(y_1 + y_3 + y_5 + y_6)$ and let $A(C_\beta) = y_5 y_6$. Now the SOS form for $\Delta F\{1,2\}$ is

$$\begin{aligned}
&\big(c(\varnothing, C_\alpha)A(C_\alpha) + c(\varnothing, C_\beta)A(C_\beta)\big)^2\\
+y_4 &\left(c(4, C_\alpha)y_4^{-1}A(C_\alpha)\right)^2\\
+y_5 &\left(c(5, C_\beta)y_5^{-1}A(C_\beta)\right)^2\\
+y_6 &\left(c(6, C_\beta)y_6^{-1}A(C_\beta)\right)^2\\
+y_5y_6 &\left(c(56, C_\beta)(y_5y_6)^{-1}A(C_\beta)\right)^2.
\end{aligned}$$

As it turns out, if we let $c(S, C) = 1$ for all $S, C \subseteq E$, this is indeed equal to $\Delta F\{1,2\}$. Unfortunately, it is not usually so easy.

5.3 Testing small examples

Maple was used to test Conjecture 5.2.6 by constructing the sum of squares, making an educated guess for the signs and subtracting it from the I-Rayleigh difference. To do this we produced the spanning forest generating polynomial, generated all cycles through our choice of edges e and f, and used these to find S-sets and A-sets, similar to Example 5.2.9.

The spanning forest generating polynomial must be found for not only the original graph $\mathcal{G}$, but also $\mathcal{G}/C$, for each cycle C containing e and f. The quickest way was to use a recursive algorithm employing the formula

$$F(\mathcal{G}) = F(\mathcal{G} \setminus g) + y_g F(\mathcal{G}/g), \tag{5.3.1}$$

where g is not a loop and if all edges of $\mathcal{G}$ are loops then $F(\mathcal{G}) = 1$.

To find the cycles containing e and f we used Maple's GraphTheory:-CycleBasis command to find a basis for the cycle space which is used to generate all even subgraphs of $\mathcal{G}$. From these we extracted the cycles and eliminated any which did not contain e and f. This method is adequate for K_7, but K_8 has too large a cycle space.

With these tools, finding the A-sets and S-sets is simple. The generating polynomial for all sets containing a unique cycle, C, is $\mathbf{y}^C F(\mathcal{G}/C)$, so the A-sets with $C \subseteq A \cup ef$ are

$$\mathbf{y}^C F(\mathcal{G}/C).$$

The S-sets are simply the subsets of these cycles minus e and f.

A current version of the code and the best guess for the signs is available at http://www.math.uwaterloo.ca/∼atericks/ . As a result of these tests we have the following theorem.

Theorem 5.3.1. *Every simple graph on at most seven vertices satisfies Conjecture 5.2.6*

Using Theorem 3.2.2 on two-sums we get an easy corollary.

Corollary 5.3.2. *Every graph on at most seven vertices is I-Rayleigh*

5.3.1 Signs $c(S, C)$

Wagner's signs are described here with a somewhat accidental modification which causes them to satisfy $\Delta K_7\{e, f\}$ both for adjacent and non-adjacent pairs of edges e, f. Let $\mathcal{G}$ be a graph on n vertices and let M be its signed incidence matrix in the following canonical form. Given some ordered labeling of the vertices, $V = v_1, v_2, \ldots, v_n$, orient the edge $g = v_i g v_j$ with $i < j$, v_i as the tail and v_j as the head of g. M is the incidence matrix such that its rows and columns are in lexicographic order.

Example 5.3.3. Let $\mathcal{G} = (\{a, b, c, d\}, \{ab, ac, ad, bc, bd, cd\})$, then its signed and labelled incidence matrix is

	ab	ac	ad	bc	bd	cd
a	-1	-1	-1	0	0	0
b	1	0	0	-1	-1	0
c	0	1	0	1	0	-1
d	0	0	1	0	1	1

If S is an S-set and C is a cycle containing $S \cup ef$, the sign $c(S, C)$ is built in two steps (see Example 5.3.4 for an illustrated example). Let $R \subseteq S$ be the set of edges of a connected component of S. Let $g \in R$ be the unique representative of R, chosen with the following priority: first e, then f and followed by lexicographic order. Let

$$s = \prod_{g \in R} o(g), \tag{5.3.2}$$

where $o(g) = 1$ if g and e have the same orientation around C and otherwise $o(g) = -1$.

Next, we consider the sign of a permutation, p, of the representatives chosen above, where the ordering (e, f, lex) is the identity. The permutation p is given by the order of the components of S around C in the direction of the orientation of e. Its sign is $t = (-1)^{i(p)}$ where $i(p)$ is the number of inversions of p according to the order imposed by the identity. Put $c(S, C) = st$.

Remark. In Proposition 5.2.7 we showed that if ΔF is SOS, then ΔF_g and ΔF^g are also SOS, by making an implicit choice about the signs. Notice that in (5.2.3) and (5.2.4), the signs for $c(S, C)$ when $g \in S$ do not appear. Thus, the signs we use in our experiment satisfy this choice of signs.

Example 5.3.4. Let S be an S-set contained in the cycle C so that the components of S as they appear around C in the direction of e are $\mathbf{e}f15, \mathbf{7}8, \mathbf{2}9, \mathbf{4}$. They are illustrated in Figure 5.3, and listed so that the edges of each component are in the (e, f, lex)-order. The component representatives are $e, 7, 2, 4$ and their signs are $1, -1, -1, 1$, respectively. They form the permutation $p = (e724)$ with respect to the identity, $(e247)$. The number of inversions is 2, so in this example

$$c(S, C) = (1)(-1)(-1)(1)(-1)^2 = 1.$$

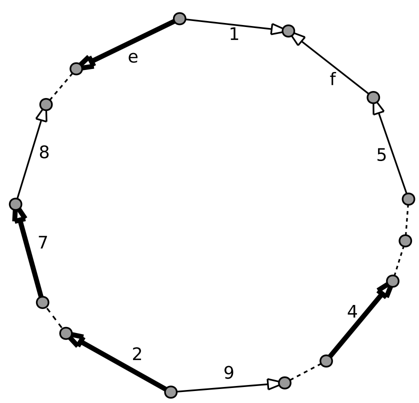


Figure 5.3: Dotted lines represent edges of $C - S$, and solid lines are edges of S. Bold lines are representative edges of connected components of S.

5.4 Series Parallel Graphs

Recall the definition of a series parallel matroid from Section 2.1.4 and note that series parallel matroids are graphic. Throughout this section let $G = F(\mathcal{G}; \mathbf{y})$, $H = F(\mathcal{H}; \mathbf{y})$, $K = F(\mathcal{K}; \mathbf{y})$, and $\mathcal{G} = \mathcal{H} \oplus_g \mathcal{K}$. The rest of our labour is dedicated to proving the following theorem.

Theorem 5.4.1. *[E.] Every series parallel graph is SOS.*

The proof contains several parts and a couple of tricky calculations. To make it more readable it is split into several lemmas which are tied together at the end. Briefly, we prove that if $\mathcal{H}$ and $\mathcal{K}$ satisfy Conjecture 5.2.6 plus a similar SOS form for $(H^g - H_g)H_g$ and $(K^g - K_g)K_g$, then G satisfies both of these SOS forms. This second form is also proved for series parallel graphs.

Let $\mathcal{H}$ and $\mathcal{K}$ be SOS graphs such that $\mathcal{G} = \mathcal{H} \oplus_g \mathcal{K}$. To prove the SOS form for $\Delta G\{e, f\}$ there are two cases: either $e \in E(\mathcal{H})$ and $f \in E(\mathcal{K})$, or both $e, f \in E(\mathcal{H})$. The former is easier to prove and holds for two-sums in general without any assumptions on $(H^g - H_g)H_g$ or $(K^g - K_g)K_g$.

Lemma 5.4.2. *Let $\mathcal{G}$ be as defined above. Then $\mathcal{G}$ satisfies the SOS form when $e \in E(\mathcal{H})$ and $f \in E(\mathcal{K})$ are edges distinct from g.*

Proof. By (3.2.3) from Theorem 3.2.2,

$$\Delta G\{e, f\} = \Delta H\{e, g\}\Delta K\{g, f\}. \tag{5.4.1}$$

We assume that

$$\Delta H\{e, g\} = \sum_{S_H \in \mathcal{S}_H} \mathbf{y}^{S_H} \left(\sum_{A_H \in \mathcal{A}_H(S_H)} c(S_H, C_H)\mathbf{y}^{A_H - S_H} \right)^2,$$

and

$$\Delta K\{g, f\} = \sum_{S_K \in \mathcal{S}_K} \mathbf{y}^{S_K} \left(\sum_{A_K \in \mathcal{A}_K(S_K)} c(S_K, C_K)\mathbf{y}^{A_K - S_K} \right)^2,$$

in which C_H is the unique cycle in $A_H \cup eg$ containing $S_H \cup eg$ and similarly for C_K.

The product of these is

$$\sum_{\substack{(S_H, S_K) \in \\ \mathcal{S}_H \times \mathcal{S}_K}} \mathbf{y}^{S_H \cup S_K} \left(\sum_{\substack{(A_H, A_K) \in \\ \mathcal{A}_H(S_H) \times \mathcal{A}_K(S_K)}} c(S_G, C_G)\mathbf{y}^{(A_H \cup A_K) - (S_H \cup S_K)} \right)^2, \tag{5.4.2}$$

where

$$c(S_G, C_G) = c(S_H \cup S_K, (C_H \cup C_K) - g) := c(S_H, C_H)c(S_K, C_K).$$

It remains for us to convince ourselves that (5.4.2) is the sum of squares we are expecting, by showing that the outer and inner sums index the S-sets and A-sets of $\mathcal{G}$. A cycle is an ef-cycle of $\mathcal{G}$ if and only if it is the symmetric difference of an eg-cycle in $\mathcal{H}$ and an gf-cycle in $\mathcal{K}$.

We begin by convincing ourselves that the outer sum of (5.4.2) indexes exactly the S-sets of $\mathcal{G}$. Let S_G be an S-set of $\mathcal{G}$, and let $S_H = S_G \cap E(\mathcal{H})$ and $S_K = S_G \cap E(\mathcal{K})$. Since S_G is contained in an ef-cycle of $\mathcal{G}$, then S_H is contained in an eg-cycle of $\mathcal{H}$ and S_K is contained in a gf-cycle of $\mathcal{K}$. Therefore $S_G = S_H \cup S_K$ is indexed by $S_H \times S_K$ and the outer sum of (5.4.2) contains at least all of the S-sets of $\mathcal{G}$. On the other hand, suppose $S_H \in \mathcal{S}_H$ and $S_K \in \mathcal{S}_K$. Then clearly $S_H \cup S_K$ is contained in an ef-cycle of $\mathcal{G}$ and is thus an S-set of $\mathcal{G}$.

Given $(S_H, S_K) \in \mathcal{S}_H \times \mathcal{S}_K$, let $(A_H, A_K) \in \mathcal{A}_H(S_H) \times \mathcal{A}_K(S_K)$. We claim that the A-sets of $\mathcal{G}$ are exactly those of the form $A_H \cup A_K$. Notice that $(A_H \cup eg) \triangle (A_K \cup gf) = (A_H \cup A_K) \cup ef$, which by our earlier observation of ef-cycles, contains a unique ef-cycle containing $S_H \cup S_K$, as required. The reverse argument holds, so the inner sum indexes exactly the A-sets of $\mathcal{G}$ for each S-set. $\square$

The proof of the case in which $e, f \in E(\mathcal{H})$ reduces to proving a sum of squares form for $(K^g - K_g)K_g$ similar to the one in Conjecture 5.2.6. For any graph $\mathcal{G}$ and an edge e let

$$\Phi G\{e\} = (G^e - G_e)G_e.$$

Recall equation (3.1.1), relating $\mathcal{G}$ with its minors,

$$\Delta G\{e, f\} = \Delta G^g\{e, f\} + y_g \Theta G\{e, f|g\} + y_g^2 \Delta G_g\{e, f\},$$

where e, f and g are distinct and

$$\Theta G\{e, f|g\} = G_e^{fg} G_{fg}^e + G_f^{eg} G_{eg}^f - G_g^{ef} G_{ef}^g - G_{efg} G^{efg}.$$

It will be necessary to derive an analogous version for $\Phi G\{e\}$ and a formula relating ΦG to ΦH and ΦK.

Lemma 5.4.3. *Let $\mathcal{G}$ be a graph with distinct edges e and f.*

(a) Then

$$\Phi G\{e\} = \Phi G^f\{e\} + y_f \Psi G\{e|f\} + y_f^2 \Phi G_f\{e\}, \tag{5.4.3}$$

where

$$\Psi G\{e|f\} = G_f^e G_e^f + G^{ef} G_{ef} - 2 G_e^f G_{ef}. \tag{5.4.4}$$

(b) If $\mathcal{G} = \mathcal{H} \oplus_g \mathcal{K}$ with $e \in E(\mathcal{H})$, then, by setting $y_g = K^g/K_g - 1$,

$$\Phi G\{e\} = (K_g)^2 \Phi H\{e\}. \tag{5.4.5}$$

Proof. For part (a) we have $G^e = G^{ef} + y_f G_f^e$ and $G_e = G_e^f + y_f G_{ef}$ so

$$\begin{aligned} \Phi G\{e\} =& (G^e - G_e) G_e \\ =& (G^{ef} + y_f G_f^e)(G_e^f + y_f G_{ef}) - (G_e^f + y_f G_{ef})^2 \\ =& (G^{ef} G_e^f - (G_e^f)^2) \\ & + y_f (G_f^e G_e^f + G^{ef} G_{ef} - 2 G_e^f G_{ef}) \\ & + y_f^2 (G_f^e G_{ef} - (G_{ef})^2) \\ =& \Phi G^f\{e\} + y_f \Psi G\{e|f\} + y_f^2 \Phi G_f\{e\}, \end{aligned} \tag{5.4.6}$$

which proves (5.4.3).

For part (b), factoring K_g out of (2.1.7), we have

$$\begin{aligned} G =& K_g H^g + K^g H_g - K_g H_g \\ =& K_g H^g + (K^g - K_g) H_g \\ =& K_g (H^g + y_g H_g), \end{aligned}$$

in which $y_g = K^g/K_g - 1$. So by (5.4.6)

$$\begin{aligned}\Phi G\{e\} =& G^e G_e - (G_e)^2 \\ =& K_g(H^{eg} + y_g H^e_g) K_g(H^g_e + y_g H_{eg}) - (K_g(H^g_e + y_g H_{eg}))^2 \\ =& (K_g)^2 \left(\Phi H^g\{e\} + y_g \Psi H\{e|g\} + y_g^2 \Phi H_g\{e\} \right) \\ =& (K_g)^2 \Phi H\{e\}. \qquad \square\end{aligned}$$

To express the SOS form for $\Phi G\{e\}$ we need some notation similar to that defined for Conjecture 5.2.6. Note, however, that Q-sets are required to be non-empty, unlike S-sets. The significance of this becomes clear later because the proof of the Φ-SOS form splits into the case where the Q-sets of $\mathcal{G}$ are contained in one factor and when they are not. The latter requires that Q-sets of each factor combine to make Q-sets of $\mathcal{G}$. The Q-sets must be non-empty because these cases are disjoint. Similarly for the Q-sets in the Δ-SOS proof.

Definition 5.4.4 (Q-sets, B-sets, signs $d(Q, D)$)**.** *Let $\mathcal{Q} \subseteq 2^{E-e}$ be the collection of sets $Q \subseteq E - e$ such that $Q \cup e$ is contained in a cycle of $\mathcal{G}$ and $Q \neq \varnothing$. For some $Q \in \mathcal{Q}$, let $\mathcal{B}(Q) \subseteq 2^{E-e}$ be the collection of those spanning forests $B \subseteq E - e$ such that $B \cup e$ contains a unique cycle, D, containing $Q \cup e$. Let $d(Q, e, D) = \pm 1$, depending on $Q \in \mathcal{Q}$ and some cycle D, containing Q. We write $d(Q, D)$ when e is understood and use a subscripted G wherever the graph $\mathcal{G}$ needs to be specified, as in $\mathcal{B}_G(Q)$.*

We refer to elements of $\mathcal{Q}$ and $B(Q)$ as Q-sets and B-sets, respectively, and throughout the rest of this chapter, given a Q-set Q and a B-set $B \in \mathcal{B}(Q)$, D is a cycle such that $Q \cup e \subseteq D \subseteq B \cup e$ unless otherwise noted. Henceforth, a graph (or its generating polynomial or I-Rayleigh difference) satisfying Conjecture 5.2.6 is called Δ-SOS and one satisfying the conclusion of the following lemma is Φ-SOS.

Lemma 5.4.5. *Let $\mathcal{G}$ be a series parallel graph with an edge, e. Then with the* *above notation*

$$\Phi G\{e\} = \sum_{Q \in \mathcal{Q}} \mathbf{y}^Q \left(\sum_{B \in \mathcal{B}(Q)} d(Q,D) \mathbf{y}^{B-Q} \right)^2 . \tag{5.4.7}$$

Proof. Series parallel graphs are constructed from iterated two-sums of K_3 and $(K_3)^*$ and taking minors of these.

In the case of K_3, let the edges be e, f, g so that

$$\Phi K_3\{e\} = (1 + y_f + y_g + y_f y_g)(1 + y_f + y_g) - (1 + y_f + y_g)^2 = y_f (y_g)^2 + y_g (y_f)^2 + y_f y_g.$$

In the case of $(K_3)^*$,

$$\Phi (K_3)^* \{e\} = (1 + y_f + y_g)(1) - (1)^2 = y_f + y_g.$$

Both K_3 and $(K_3)^*$ are Φ-SOS. The case for minors is very similar to Proposition 5.2.7. Assume that $\mathcal{G}$ is Φ-SOS. From equation (5.4.3) we have

$$\lim_{y_g \longrightarrow 0} \Phi F \{e\} = \sum_{\substack{Q \in \mathcal{Q}_G \\ g \notin Q}} \mathbf{y}^Q \left(\sum_{\substack{B \in \mathcal{B}_G(Q) \\ g \notin B}} d(Q,D) \mathbf{y}^{B-Q} \right)^2 , \tag{5.4.8}$$

and

$$\lim_{y_g \longrightarrow \infty} y_g^{-2} \Delta F \{e\} = \sum_{\substack{Q \in \mathcal{Q}_G \\ g \notin Q}} \mathbf{y}^Q \left(\sum_{\substack{B \in \mathcal{B}_G(Q) \\ g \in B}} d(Q,D) \mathbf{y}^{B-(Q \cup g)} \right)^2 . \tag{5.4.9}$$

The rest of the argument is almost identical to Proposition 5.2.7 and is not repeated here. Thus, it is enough to show that the result is preserved by taking two-sums.

Let $\mathcal{H}$ and $\mathcal{K}$ be graphs which are Φ-SOS with $e \in E(\mathcal{H})$. By Lemma 5.4.3 we have

$$\begin{aligned}\Phi G\{e\} =&(K_g)^2\left(H^e H_e - (H_e)^2\right)\\ =&(K_g)^2 \sum_{Q \in \mathcal{Q}_H} \mathbf{y}^Q \left(\sum_{B \in \mathcal{B}_H(Q)} d(Q,D)\mathbf{y}^{B-Q}\right)^2. \qquad (5.4.10)\end{aligned}$$

Using the fact that $\mathcal{Q} = \{Q \in \mathcal{Q}_H : g \in Q\} \dot\cup \{Q \in \mathcal{Q}_H : g \notin Q\}$, we split this as

$$(K_g)^2 \sum_{\substack{Q \in \mathcal{Q}_H \\ g \in Q}} \mathbf{y}^{Q-g} y_g \left(\sum_{B \in \mathcal{B}_H(Q)} d(Q,D)\mathbf{y}^{B-Q}\right)^2 \qquad (5.4.11)$$

$$+(K_g)^2 \sum_{\substack{Q \in \mathcal{Q}_H \\ g \notin Q}} \mathbf{y}^{Q} \left(\sum_{B \in \mathcal{B}_H(Q)} d(Q,D)\mathbf{y}^{B-Q}\right)^2, \qquad (5.4.12)$$

in which $y_g = K^g/K_g - 1$. We want to show that for some choice of the signs $d(Q,D)$, for $Q \in \mathcal{Q}_G$, that (5.4.10) is the Φ-SOS form for $\mathcal{G}$.

Recall that the closure of a set $X \subseteq E$ is written $\mathrm{cl}(X)$. Consider the terms in (5.4.12) and notice that given $Q \in \mathcal{Q}_H$ such that $g \notin Q$,

$$\begin{aligned}\mathcal{B}(Q) =& \{B \in \mathcal{B}_H(Q) : g \in D\}\\ &\dot\cup \{B \in \mathcal{B}_H(Q) : g \in B - D\} \qquad (5.4.13)\\ &\dot\cup \{B \in \mathcal{B}_H(Q) : g \notin B, g \in \mathrm{cl}(B)\}\\ &\dot\cup \{B \in \mathcal{B}_H(Q) : g \notin B, g \notin \mathrm{cl}(B)\} \qquad (5.4.14)\end{aligned}$$

and that the map $B \mapsto B - g$ is a bijection from (5.4.13), $\{B \in \mathcal{B}_H(Q) : g \in B - D\}$, to (5.4.14), $\{B \in \mathcal{B}_H(Q) : g \notin B, g \notin \mathrm{cl}(B)\}$. To avoid confusion we make small steps and write

$$(K_g)^2 \sum_{\substack{Q \in \mathcal{Q}_H \\ g \notin Q}} \mathbf{y}^Q \left(\sum_{B : g \in D} \wp(B) y_g + \sum_{B : g \in B-D} \wp(B) y_g + \sum_{B : g \notin B} \wp(B)\right)^2, \qquad (5.4.15)$$

where $\wp(B)$ stands for $d(Q,D)\mathbf{y}^{B-(Q\cup g)}$, to simplify notation. Making the substitution for y_g and distributing in $(K_g)^2$, the inner sum of (5.4.15) becomes

$$\begin{aligned} &\sum_{B:g\in D} \wp(B)(K^g - K_g) + \sum_{B:g\in B-D} \wp(B)(K^g - K_g) \\ &\quad + \sum_{\substack{B:g\notin B,\\ g\in \mathrm{cl}(B)}} \wp(B)K_g + \sum_{\substack{B:g\notin B\\ g\notin \mathrm{cl}(B)}} \wp(B)K_g. \end{aligned} \tag{5.4.16}$$

Since $d(Q,D)$ does not depend on edges in $B-D$,

$$\sum_{\substack{B:g\notin B\\ g\notin \mathrm{cl}(B)}} \wp(B) = \sum_{B:g\in B-D} \wp(B),$$

and these terms are combined, so (5.4.12) becomes

$$\sum_{\substack{Q\in\mathcal{Q}_H\\ g\notin Q}} \mathbf{y}^Q \left(\sum_{B:g\in D} \wp(B)(K^g - K_g) + \sum_{B:g\in B-D} \wp(B)K^g + \sum_{\substack{B:g\notin B,\\ g\in \mathrm{cl}(B)}} \wp(B)K_g \right)^2. \tag{5.4.17}$$

Since

$$\{Q \in \mathcal{Q}_G : Q \cap E(\mathcal{K}) = \varnothing\} = \{Q \in \mathcal{Q}_H : g \notin Q\},$$

we want to show that (5.4.17) is equal to

$$\sum_{\substack{Q\in\mathcal{Q}_G\\ Q\cap E(\mathcal{K})=\varnothing}} \mathbf{y}^Q \left(\sum_{B\in\mathcal{B}_G(Q)} d(Q,D)\mathbf{y}^{B-Q} \right)^2, \tag{5.4.18}$$

for some choice of signs, by showing that their inner sums are equal.

Recall from Section 2.1 that for collections of sets of edges $\mathscr{R}, \mathscr{S} \in 2^E$, we define

$$\mathscr{R} \vee \mathscr{S} = \{R \cup S : R \in \mathscr{R}, S \in \mathscr{S}\}.$$

The inner sum of (5.4.18) breaks up similarly to (5.4.17). Let $Q \in \mathcal{Q}_G$ be such that $Q \cap E(\mathcal{K}) = \varnothing$. Then any set $B \in \mathcal{B}_G(Q)$ is in exactly one of the following cases:

(i) $\mathcal{B}_1 = \{B - g : B \in B_H(Q), g \in D\} \vee (\mathscr{F}(\mathcal{K} \backslash g) - \mathscr{F}(\mathcal{K}/g))$; or

(ii) $\mathcal{B}_2 = \{B \in \mathcal{B}_H(Q) : g \notin B, g \notin \mathrm{cl}(B)\} \vee \mathscr{F}(\mathcal{K} \backslash g)$; or

(iii) $\mathcal{B}_3 = \{B \in \mathcal{B}_H(Q) : g \notin B, g \in \mathrm{cl}(B)\} \vee \mathscr{F}(\mathcal{K}/g)$.

That is, $\mathcal{B}_G(Q) = \mathcal{B}_1 \dot\cup \mathcal{B}_2 \dot\cup \mathcal{B}_3$. Figure 5.4 is an illustration of what members of these three sets might look like.

So the inner sum of (5.4.18) is equal to

$$\sum_{B \in \mathcal{B}_1} d(Q, D)\mathbf{y}^{B-Q} + \sum_{B \in \mathcal{B}_2} d(Q, D)\mathbf{y}^{B-Q} + \sum_{B \in \mathcal{B}_3} d(Q, D)\mathbf{y}^{B-Q}.$$

Clearly, if $Q \in \{Q \in \mathcal{Q}_H : g \notin Q\}$, then

$$\sum_{\substack{B_H \in \mathcal{B}_H(Q) \\ g \in D_H}} d(Q, D_H)\mathbf{y}^{B_H-(Q\cup g)}(K^g - K_g) \quad = \sum_{B_G \in \mathcal{B}_1} d(Q, D_G)\mathbf{y}^{B_G-Q}$$

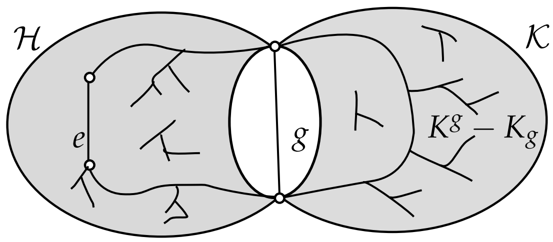
and

$$\sum_{\substack{B_H \in \mathcal{B}_H(Q) \\ g \in B_H - D_H}} d(Q, D_H)\mathbf{y}^{B_H-(Q\cup g)}K^g \quad = \sum_{B_G \in \mathcal{B}_2} d(Q, D_G)\mathbf{y}^{B_G-Q}$$

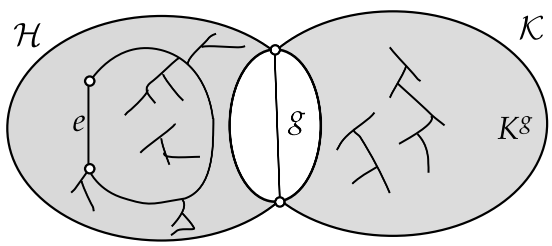
and

$$\sum_{\substack{B_H \in \mathcal{B}_H(Q) \\ g \notin B_H, \\ g \in \mathrm{cl}(B_H)}} d(Q, D_H)\mathbf{y}^{B_H-Q}K_g \quad = \sum_{B_G \in \mathcal{B}_3} d(Q, D_G)\mathbf{y}^{B_G-Q}.$$

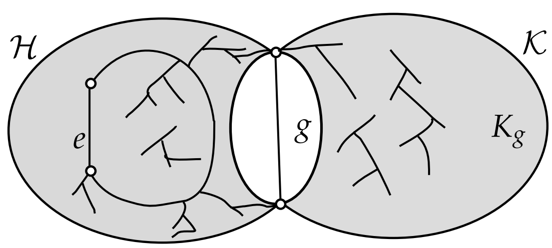
We have proved the case in which $Q \in \mathcal{Q}_G$ is entirely contained in $\mathcal{H}$. That is, (5.4.18) is equal to (5.4.12).



(a) $\mathcal{B}_1 = \{B - g : B \in B_H(Q), g \in D\} \vee (\mathscr{F}(\mathcal{K} \setminus g) - \mathscr{F}(\mathcal{K}/g))$



(b) $\mathcal{B}_2 = \{B \in \mathcal{B}_H(Q) : g \notin B, g \notin \mathrm{cl}(B)\} \vee \mathscr{F}(\mathcal{K} \setminus g)$



(c) $\mathcal{B}_3 = \{B \in \mathcal{B}_H(Q) : g \notin B, g \in \mathrm{cl}(B)\} \vee \mathscr{F}(\mathcal{K}/g)$

Figure 5.4: The sets $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$

Turning our attention now to (5.4.11), it remains for us to show that

$$\sum_{\substack{Q_G \in \mathcal{Q}_G \\ Q_G \cap E(\mathcal{K}) \neq \varnothing}} \mathbf{y}^{Q_G} \left(\sum_{B_G \in \mathcal{B}_G(Q_G)} d(Q_G, D_G) \mathbf{y}^{B_G - Q_G} \right)^2$$

$$= (K_g)^2 \sum_{\substack{Q_H \in \mathcal{Q}_H \\ g \in Q_H}} \mathbf{y}^{Q_H - g} y_g \left(\sum_{B_H \in \mathcal{B}_H(Q_H)} d(Q_H, D_H) \mathbf{y}^{B_H - Q_H} \right)^2,$$

for some choice of signs, $d(Q_G, D_G)$. To help remember that Q-sets of $\Phi K\{g\}$ are non-empty sets contained in g-cycles, rather than e-cycles, we write $\mathcal{Q}_K(g)$. Factoring y_g from the right hand side and noting that $y_g(K_g)^2 = (K^g/K_g - 1)(K_g)^2 = \Phi K\{g\}$, we have

$$\Phi K\{g\} \sum_{\substack{Q_H \in \mathcal{Q}_H \\ g \in Q_H}} \mathbf{y}^{Q_H - g} \left(\sum_{B_H \in \mathcal{B}_H(Q_H)} d(Q_H, D_H) \mathbf{y}^{B_H - D_H} \right)^2$$

$$= \sum_{Q_K \in \mathcal{Q}_K(g)} \mathbf{y}^{Q_K} \left(\sum_{B_K \in \mathcal{B}_K(Q_K)} d(Q_K, D_K) \mathbf{y}^{B_K - D_K} \right)^2$$

$$\times \sum_{\substack{Q_H \in \mathcal{Q}_H \\ g \in Q_H}} \mathbf{y}^{Q_H - g} \left(\sum_{B_H \in \mathcal{B}_H(Q_H)} d(Q_H, D_H) \mathbf{y}^{B_H - D_H} \right)^2.$$

Now it is a simple question of grouping index sets and seeing their equality, namely that

$$\{Q_G \in \mathcal{Q}_G : Q_G \cap E(\mathcal{K}) \neq \varnothing\} = \{Q_H - g : g \in Q_H \in \mathcal{Q}_H\} \vee \mathcal{Q}_K(g)$$

and given $Q_G = (Q_H - g) \cup Q_K$ from the sets above,

$$\mathcal{B}_G(Q_G) = \mathcal{B}_H(Q_H) \vee \mathcal{B}_K(Q_K).$$

This is clear when one follows the argument closely but is made clearer still by Figure 5.5 and the fact that $\varnothing \notin \mathcal{Q}_K(g)$.

Summarizing the two cases we have

$$\sum_{\substack{Q_G \in \mathcal{Q}_G \\ Q_G \cap E(\mathcal{K}) = \varnothing}} \mathbf{y}^{Q_G} \left(\sum_{B \in \mathcal{B}_G(Q_G)} d(Q_G, D_G) \mathbf{y}^{B - Q_G} \right)^2,$$

$$= (K_g)^2 \sum_{\substack{Q_H \in \mathcal{Q}_H \\ g \notin Q_H}} \mathbf{y}^{Q_H - g} \left(\sum_{B \in \mathcal{B}_H(Q_H)} d(Q_H, D_H) \mathbf{y}^{B - Q_H} \right)^2,$$

when $d(Q_G, D_G) = d(Q_H, D_H)$ for $Q_G = Q_H$ and $D_G = D_H$. Let $d(Q_K, g, D_K)$ be the signs for $\Phi K\{g\}$, then from the second case,

$$\sum_{\substack{Q_G \in \mathcal{Q}_G \\ Q_G \cap E(\mathcal{K}) \neq \varnothing}} \mathbf{y}^{Q_G} \left(\sum_{B_G \in \mathcal{B}_G(Q_G)} d(Q_G, D_G) \mathbf{y}^{B_G - Q_G} \right)^2$$

$$= \Phi K\{g\} \sum_{\substack{Q_H \in \mathcal{Q}_H \\ g \in Q_H}} \mathbf{y}^{Q_H - g} \left(\sum_{B \in \mathcal{B}_H(Q_H)} d(Q_H, D_H) \mathbf{y}^{B - Q_H} \right)^2,$$

where the signs $d(Q_G, D_G) = d(Q_K, g, D_K) d(Q_H, D_H)$ for $Q_G = Q_K \cup (Q_H - g)$ and $D_G = (D_K \cup D_H) - g$. ◻

We use Lemma 5.4.5 to prove that the SOS conjecture holds over two-sums when both $e, f \in E(\mathcal{H})$. It resembles the proof of Lemma 5.4.5 and it is recommended that the reader understand that one first.

Lemma 5.4.6. *Let $\mathcal{H}$ and $\mathcal{K}$ be graphs which are both Φ-SOS and Δ-SOS with $e, f \in E(\mathcal{H})$, distinct from g. Then $\mathcal{H} \oplus_g \mathcal{K}$ is Δ-SOS.*

Proof. Let $e, f \in E(\mathcal{H})$. By (3.2.4) in Theorem 3.2.2,

$$\Delta G\{e, f\} = (K_g)^2 \Delta H\{e, f\},$$

where $y_g = K^g / K_g - 1$.

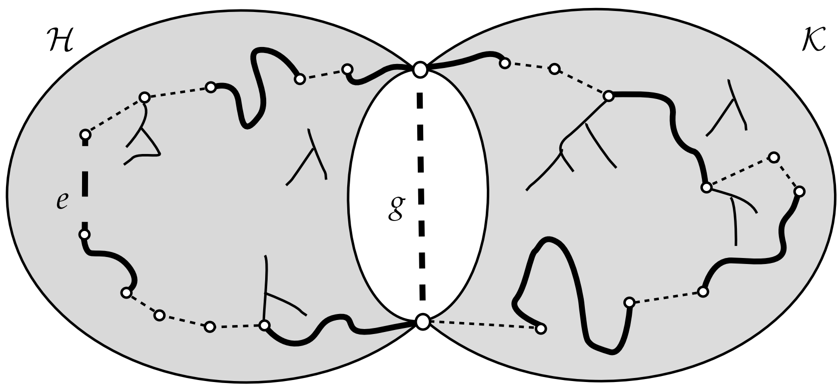


Figure 5.5: An element of $\mathcal{B}_G(Q_G)$ may look like this. Thin dotted lines represent edges of $Q_G = (Q_H - g) \cup Q_K$, thick solid lines complete a cycle $D_G \supseteq Q_G + e$.

Recall the notation given in Definition 5.2.5 for S-sets, A-sets, the cycles C, and the signs $c(S, C)$. Note that wherever necessary, sets pertaining to a graph are marked by a subscript, for example $\mathcal{S}_H$ for the S-sets of $\mathcal{H}$. By the induction hypothesis, we have

$$\begin{aligned} \Delta G\{e,f\} =& (K_g)^2 \Delta H\{e,f\} \\ &= (K_g)^2 \sum_{S \in \mathcal{S}_H} \mathbf{y}^S \left(\sum_{A \in \mathcal{A}_H(S)} c(S,C) \mathbf{y}^{A-S} \right)^2. \end{aligned} \tag{5.4.19}$$

Using the fact that $\mathcal{S}_H = \{S \in \mathcal{S}_H : g \in S\} \dot\cup \{S \in \mathcal{S}_H : g \notin S\}$, (5.4.19) splits as follows,

$$= (K_g)^2 \sum_{\substack{S \in \mathcal{S}_H \\ g \in S}} \mathbf{y}^S \left(\sum_{A \in \mathcal{A}_H(S)} c(S,C) \mathbf{y}^{A-S} \right)^2 \tag{5.4.20}$$

$$+ (K_g)^2 \sum_{\substack{S \in \mathcal{S}_H \\ g \notin S}} \mathbf{y}^S \left(\sum_{A \in \mathcal{A}_H(S)} c(S,C) \mathbf{y}^{A-S} \right)^2. \tag{5.4.21}$$

The goal of this is to show that for some choice of signs, $c(S, D)$, where $S \in \mathcal{S}_G$, that (5.4.19) is the Δ-SOS form for G. Similarly to the argument in Lemma 5.4.5, we develop the inner sum (5.4.21), which is indexed by S-sets not containing g. Given $S \in \{S \in \mathcal{S}_H : g \notin S\}$,

$$\begin{aligned}\mathcal{A}_H(S) = & \{A \in \mathcal{A}_H(S) : g \in C\} \\ & \dot\cup \{A \in \mathcal{A}_H(S) : g \in A - C\} \qquad (5.4.22)\\ & \dot\cup \{A \in \mathcal{A}_H(S) : g \notin A, g \in \mathrm{cl}(A)\} \\ & \dot\cup \{A \in \mathcal{A}_H(S) : g \notin A, g \notin \mathrm{cl}(A)\}, \qquad (5.4.23)\end{aligned}$$

and $A \mapsto A - g$ is a bijection from (5.4.22), $\{A \in \mathcal{A}(S) : g \in A - C\}$, to (5.4.23), $\{A \in \mathcal{A}(S) : g \notin A, g \notin \mathrm{cl}(A)\}$. Since the signs, $c(S, C)$, do not depend on edges not in C, we can combine these terms in the sum. First, split (5.4.21) into

$$(K_g)^2 \sum_{\substack{S \in \mathcal{S}_H \\ g \notin S}} \mathbf{y}^S \left(\sum_{A : g \in C} \wp(A) y_g + \sum_{A : g \in A - C} \wp(A) y_g + \sum_{A : g \notin C} \wp(A) \right)^2. \tag{5.4.24}$$

where $\wp(A)$ stands for $c(S, C)\mathbf{y}^{A-(S \cup g)}$. Making the substitution for y_g and distributing $(K_g)^2$, the inner sum of (5.4.24) becomes

$$\begin{aligned}&\sum_{A : g \in C} \wp(A)(K^g - K_g) + \sum_{A : g \in A - C} \wp(A)(K^g - K_g) \\ &\quad + \sum_{\substack{A : g \notin A, \\ g \in \mathrm{cl}(A)}} \wp(A) K_g + \sum_{\substack{A : g \notin A \\ g \notin \mathrm{cl}(A)}} \wp(A) K_g.\end{aligned} \tag{5.4.25}$$

Since $c(S, C)$ does not depend on edges of $A - C$,

$$\sum_{A : g \in A - C} \wp(A) = \sum_{\substack{A : g \notin A \\ g \notin \mathrm{cl}(A)}} \wp(A), \tag{5.4.26}$$

and we combine these terms so that (5.4.21) becomes

$$\sum_{\substack{S\in\mathcal{S}_H\\ g\notin S}} \mathbf{y}^S \left(\sum_{A:g\in C} \wp(A)(K^g - K_g) + \sum_{A:g\in A-C} \wp(A)K^g + \sum_{\substack{A:g\notin A,\\ g\in \mathrm{cl}(A)}} \wp(A)K_g \right)^2 . \tag{5.4.27}$$

Now we use the fact that

$$\{S \in \mathcal{S}_G : S \cap E(\mathcal{K}) = \varnothing\} = \{S \in \mathcal{S}_H : g \notin S\}$$

to show that (5.4.27) is equal to

$$\sum_{\substack{S\in\mathcal{S}_G\\ S\cap E(\mathcal{K})=\varnothing}} \mathbf{y}^S \left(\sum_{A\in\mathcal{A}_G(S)} c(S,C)\mathbf{y}^{A-S} \right)^2 , \tag{5.4.28}$$

by showing equality of their inner sums, for a certain choice of signs. Given $S \in \{S \in \mathcal{S}_H : g \notin S\}$, $\mathcal{A}_G(S) = \mathcal{A}_1 \dot\cup \mathcal{A}_2 \dot\cup \mathcal{A}_3$, where

$$\begin{aligned}
\mathcal{A}_1 &= \{A - g : A \in A_H(S), g \in C\} \vee (\mathscr{F}(\mathcal{K}\setminus g) - \mathscr{F}(\mathcal{K}/g)),\\
\mathcal{A}_2 &= \{A \in \mathcal{A}_H(S) : g \notin A, g \notin \mathrm{cl}(A)\} \vee \mathscr{F}(\mathcal{K}\setminus g),\\
\mathcal{A}_3 &= \{A \in \mathcal{A}_H(S) : g \notin A, g \in \mathrm{cl}(A)\} \vee \mathscr{F}(\mathcal{K}/g).
\end{aligned}$$

Notice that up to labeling, Figure 5.4 is useful here.

So the inner sum of (5.4.28) is equal to

$$\sum_{A\in\mathcal{A}_1} c(S,C)\mathbf{y}^{A-S} + \sum_{A\in\mathcal{A}_2} c(S,C)\mathbf{y}^{A-S} + \sum_{A\in\mathcal{A}_3} c(S,C)\mathbf{y}^{A-S}. \tag{5.4.29}$$

Clearly, if $S \in \{S \in \mathcal{S}_H : g \notin S\}$,

$$\sum_{\substack{A_H\in\mathcal{A}_H(S)\\ g\in C_H}} c(S,C_H)\mathbf{y}^{A_H-(S\cup g)}(K^g - K_g) = \sum_{A_G\in\mathcal{A}_1} c(S,C_G)\mathbf{y}^{A_G-S},$$

and

$$\sum_{\substack{A_H \in \mathcal{A}_H(S) \\ g \in A_H - C_H}} c(S, C_H) \mathbf{y}^{A_H - (S \cup g)} K^g = \sum_{A_G \in \mathcal{A}_2} c(S, C_G) \mathbf{y}^{A_G - S}, \tag{5.4.30}$$

and

$$\sum_{\substack{A_H \in \mathcal{A}_H(S) \\ g \notin A_H \\ g \in \mathrm{cl}(A_H)}} c(S, C_H) \mathbf{y}^{A_H - S} K_g = \sum_{A_G \in \mathcal{A}_1} c(S, C_G) \mathbf{y}^{A_G - S}, \tag{5.4.31}$$

whenever $c(S, C_G) = c(S, C_H)$ for $C_G \cap E(\mathcal{H}) = C_H - g$. This proves the case in which $S \in \mathcal{S}_G$ is entirely contained in $\mathcal{H}$. That is, (5.4.28) is equal to (5.4.21).

The terms indexed by S-sets which contain edges of $\mathcal{K}$, (5.4.20), remain and we wish to show that

$$\begin{aligned} &\sum_{\substack{S_G \in \mathcal{S}_G \\ S_G \cap E(\mathcal{K}) \neq \emptyset}} \mathbf{y}^{S_G} \left(\sum_{A_G \in \mathcal{A}_G(S_G)} c(S_G, C_G) \mathbf{y}^{A_G - S_G} \right)^2 \\ =&(K_g)^2 \sum_{\substack{S_H \in \mathcal{S}_H \\ g \in S_H}} \mathbf{y}^{S_H - g} y_g \left(\sum_{A_H \in \mathcal{A}_H(S_H)} c(S_H, C_H) \mathbf{y}^{A_H - S_H} \right)^2, \end{aligned}$$

for some choice of the signs, $c(S_G, C_G)$.

Factoring y_g from the right hand side and noting that $y_g (K_g)^2 =$

$(K^g/K_g - 1)(K_g)^2 = \Phi K\{g\}$, we have

$$\begin{aligned}
&\Phi K\{g\} \sum_{\substack{S_H \in \mathcal{S}_H \\ g \in S_H}} \mathbf{y}^{S_H - g} \left(\sum_{A_H \in \mathcal{A}_H(S_H)} c(S_H, C_H) \mathbf{y}^{A_H - C_H} \right)^2 \\
&= \sum_{Q \in \mathcal{Q}_K} \mathbf{y}^Q \left(\sum_{B \in \mathcal{B}_K(Q)} d(Q, D) \mathbf{y}^{B-D} \right)^2 \\
&\times \sum_{\substack{S_H \in \mathcal{S}_H \\ g \in S_H}} \mathbf{y}^{S_H - g} \left(\sum_{A_H \in \mathcal{A}_H(S_H)} c(S_H, C_H) \mathbf{y}^{A_H - C_H} \right)^2 .
\end{aligned}$$

Once again, we look at the index sets. Clearly

$$\{S_G \in \mathcal{S}_G : S_G \cap E(\mathcal{K}) \neq \varnothing\} = \{S_H - g : g \in S_H \in \mathcal{S}_H\} \vee \mathcal{Q}_K(g).$$

As for the inner sum, given $S_G = (S_H - g) \cup Q$ from the above sets and recalling that $Q \neq \varnothing$,

$$\mathcal{A}_G(S_G) = \{A - g : A \in \mathcal{A}_H(S_H)\} \vee \mathcal{B}_K(Q).$$

Summarizing the two cases we have

$$\begin{aligned}
&\sum_{\substack{S \in \mathcal{S}_G \\ S \cap E(\mathcal{K}) = \varnothing}} \mathbf{y}^S \left(\sum_{A \in \mathcal{A}_G(S)} c_G(S, C_G) \mathbf{y}^{A-S} \right)^2 \\
=&(K_g)^2 \sum_{\substack{S \in \mathcal{S}_H \\ g \notin S}} \mathbf{y}^S \left(\sum_{A \in \mathcal{A}_H(S)} c_H(S, C_H) \mathbf{y}^{A-S} \right)^2 ,
\end{aligned}$$

when $c_G(S, C_G) = c_H(S, C_H)$ for $C_G \cap E(\mathcal{H}) = C_H$. Let $d(Q, g, D)$ be the signs for $\Phi K\{g\}$. From the second case,

$$\sum_{\substack{S_G \in \mathcal{S}_G \\ S_G \cap E(\mathcal{K}) \neq \varnothing}} \mathbf{y}^{S_G} \left(\sum_{A_G \in \mathcal{A}_G(S_G)} c_G(S_G, C_G) \mathbf{y}^{A_G - S_G} \right)^2$$

$$= \Phi K\{g\} \sum_{\substack{S_H \in \mathcal{S}_H \\ g \in S_H}} \mathbf{y}^{S_H - g} \left(\sum_{A_H \in \mathcal{A}_H(S_H)} c_H(S_H, C_H) \mathbf{y}^{A_H - S_H} \right)^2,$$

in which the signs $c_G(S_G, C_G) = d(Q, g, D) c_H(S_H, C_H)$ for $S_G = (S_H - g) \cup Q$ and $C_G = (C_H \cup D) - g$. □

We are finally in a position to prove Theorem 5.4.1 which states that series parallel graphs satisfy the Δ-SOS form.

Proof. (of Theorem 5.4.1) Let $\mathcal{G}$ be a series parallel graph. Then either there is a series parallel graph $\mathcal{H}$ which is a proper minor of $\mathcal{G}$ such that $\mathcal{G} = \mathcal{H} \oplus_g \mathcal{K}$ where $\mathcal{K} = K_3$ or $(K_3)^*$, or $\mathcal{G}$ is a minor of a series parallel graph obtained this way. By Proposition 5.2.7 we need only prove the theorem for the first case. By Theorem 5.3.1 the base cases are Δ-SOS. Suppose that all series parallel graphs which are minors of $\mathcal{G}$ are Δ-SOS. If $e \in E(\mathcal{H})$ and $f \in E(\mathcal{K})$ then by Lemma 5.4.2, $\Delta G\{e, f\}$ is SOS. If $e, f \in E(\mathcal{H})$ or $e, f \in E(\mathcal{K})$ then Lemma 5.4.6 is applicable, since $\mathcal{H}$ and $\mathcal{K}$ are Φ-SOS by Lemma 5.4.5. Thus, $\Delta G\{e, f\}$ is SOS. □

5.4.1 Two-sums of Δ-SOS graphs

One might have hoped to prove, more generally, that if $\mathcal{H}$ and $\mathcal{K}$ satisfy Conjecture 5.2.6, that $\mathcal{G} = \mathcal{H} \oplus_g \mathcal{K}$ does as well. The problem lies in the fact that we are not assuming $\mathcal{H}$ and $\mathcal{K}$ are Φ-SOS. To get around this we might try to bootstrap this assumption by showing that it follows from the induction hypothesis. In fact, this looks promising and it is given as the following conjecture.

Conjecture 5.4.7 (E.). *If $\mathcal{G}$ is Δ-SOS, then $\mathcal{G}$ is Φ-SOS.*

Let $\mathcal{G}$ be a Δ-SOS graph and suppose that its proper minors are also Φ-SOS. By (5.4.3) we have

$$\Phi G\{e\} = \Phi G^f\{e\} + y_f \Psi G\{e|f\} + y_f^2 \Phi G_f\{e\},$$

where

$$\begin{aligned}\Psi G\,\{e|f\} =& G_f^e G_e^f + G^{ef} G_{ef} - 2G_e^f G_{ef} + (G^{ef} G_{ef} - G^{ef} G_{ef}) \\ =& \Delta G\,\{e,f\} + 2(G^{ef} G_{ef} - G_e^f G_{ef}).\end{aligned} \tag{5.4.32}$$

Just as in Lemma 5.4.5, we want to show that

$$\Phi G\{e\} = \sum_{Q \in \mathcal{Q}} \mathbf{y}^Q \left(\sum_{B \in \mathcal{B}(Q)} d(Q,D) \mathbf{y}^{B-Q} \right)^2, \tag{5.4.33}$$

using slightly different assumptions. They are that

(i)

$$\Phi G^f\{e\} = \sum_{Q \in \mathcal{Q}_{G \setminus f}} \mathbf{y}^Q \left(\sum_{B \in \mathcal{B}_{G \setminus f}(Q)} d(Q,D) \mathbf{y}^{B-Q} \right)^2,$$

(ii)

$$\Phi G_f\{e\} = \sum_{Q \in \mathcal{Q}_{G/f}} \mathbf{y}^Q \left(\sum_{B \in \mathcal{B}_{G/f}(Q)} d(Q,D) \mathbf{y}^{B-Q} \right)^2, \tag{5.4.34}$$

and

(iii)

$$\Delta G\,\{e,f\} = \sum_{S \in \mathcal{S}_G} \mathbf{y}^S \left(\sum_{A \in \mathcal{A}_G(S)} c(S,C) \mathbf{y}^{A-S} \right)^2.$$

We divide (5.4.33) into two cases where Q-sets contain and do not contain f. In the latter case (5.4.33) becomes

$$\sum_{\substack{Q\in\mathcal{Q}_G\\ f\notin Q}} \mathbf{y}^Q \left(y_f \left(\sum_{B:f\in D} \wp(B) + \sum_{B:f\in B-D} \wp(B) \right) + \sum_{\substack{B:f\notin B\\ f\notin \mathrm{cl}(B)}} \wp(B) + \sum_{\substack{B:f\notin B\\ f\in \mathrm{cl}(B)}} \wp(B) \right)^2, \tag{5.4.35}$$

where $\wp(B)$ stands for $d(Q,D)\mathbf{y}^{B-(Q\cup f)}$. The Q-sets with f give

$$y_f \left(\sum_{\substack{Q\in\mathcal{Q}\\ f\in Q}} \mathbf{y}^{Q-f} \left(\sum_{B\in\mathcal{B}(Q)} d(Q,D)\mathbf{y}^{B-Q} \right)^2 \right). \tag{5.4.36}$$

We are left with comparing the coefficients of two polynomials in y_f. The degree 0 and 2 terms come from (5.4.35), for y_f^0,

$$\sum_{\substack{Q\in\mathcal{Q}_G\\ f\notin Q}} \mathbf{y}^Q \left(\sum_{\substack{B:f\notin B\\ f\notin \mathrm{cl}(B)}} \wp(B) + \sum_{\substack{B:f\notin B\\ f\in \mathrm{cl}(B)}} \wp(B) \right)^2,$$

for y_f^2,

$$y_f^2 \left(\sum_{\substack{Q\in\mathcal{Q}_G\\ f\notin Q}} \mathbf{y}^Q \left(\sum_{B:f\in D} \wp(B) + \sum_{B:f\in B-D} \wp(B) \right)^2 \right).$$

The degree 1 terms are a little more tricky because they involve both

(5.4.36) and the cross terms of (5.4.35),

$$y_f \left(\sum_{\substack{Q \in \mathcal{Q} \\ f \in Q}} \mathbf{y}^{Q-f} \left(\sum_{B \in \mathcal{B}(Q)} d(Q,D) \mathbf{y}^{B-Q} \right)^2 \right)$$
$$+ y_f \left(2 \sum_{\substack{Q \in \mathcal{Q}_G \\ f \notin Q}} \mathbf{y}^Q \left(\sum_{B: f \in D} \wp(B) + \sum_{B: f \in B - D} \wp(B) \right) \left(\sum_{\substack{B: f \notin B \\ f \notin \mathrm{cl}(B)}} \wp(B) + \sum_{\substack{B: f \notin B \\ f \in \mathrm{cl}(B)}} \wp(B) \right) \right) \tag{5.4.37}$$

Having read the SP proofs of the previous section, this probably already looks promising. It can be shown in a similar manner to the proofs of the last section that the degree 0 and 2 terms are the SOS forms of ΦG^f and ΦG_f, and that the first term of (5.4.37) is the SOS form of ΔG, which accounts for the first part of (5.4.32). What is less fortunate, however, is that we have a seemingly new generating polynomial identity to prove. The proof of the following proposition draws on the arguments of this section and is very similar to several cumbersome proofs already explained in detail. Thus it is omitted.

Proposition 5.4.8. *If*

$$G^{ef} G_{ef} - G_e^f G_{ef}$$
$$= \sum_{\substack{Q \in \mathcal{Q}_G \\ f \notin Q}} \mathbf{y}^Q \left(\sum_{B: f \in D} \wp(B) + \sum_{B: f \in B - D} \wp(B) \right) \left(\sum_{\substack{B: f \notin B \\ f \notin cl(B)}} \wp(B) + \sum_{\substack{B: f \notin B \\ f \in cl(B)}} \wp(B) \right),$$

for a certain choice of signs, then Conjecture 5.4.7 is true.

Bibliography

[1] Nobel lectures, physics 1901-1921, 1967.

[2] J. Ginibre C. M. Fortuin, P. W. Kasteleyn. Correlation inequalities on some partially ordered sets. *Comm. Math. Phys.*, 22(2):89–103, 1971.

[3] P. W. Kasteleyn C. M. Fortuin. On the random-cluster model. i. introduction and relation to other models. *Physica*, 57:536–564, 1972.

[4] D. Welsh C. Semple. Negative correlation in graphs and matroids. *Comb. Probab. Comput.*, 17(3):423–435, 2008.

[5] C. C. Cocks. Correlated matroids. *Combin. Probab. Comput.*, 17(4):511–518, 2008.

[6] David Eppstein. Parallel recognition of series-parallel graphs. *Inf. Comput.*, 98(1):41–55, 1992.

[7] S. N. Winkler G. R. Grimmett. Negative association in uniform forests and connected graphs. *Random Structures Algorithms*, 24(4):444–460, 2004.

[8] G. Grimmett. Probability on graphs, `http://www.statslab.cam.ac.uk/~grg/books/pgs.html`.

[9] G. Grimmett. *The random-cluster model*, volume 333. Springer-Verlag, Berlin, 2006.

[10] M. A. LaCroix D. G. Wagner J. Cibulka, J. Hladky. A combinatorial proof of rayleigh monotonicity for graphs, 2008.

[11] M. Neiman J. Kahn. Negative correlation and log-concavity. *ArXiv e-prints*, December 2007.

[12] M. Jerrum. *Counting, sampling and integrating: algorithms and complexity*. Birkhäuser Verlag, Basel, 2003.

[13] M. Jerrum. Two remarks concerning balanced matroids. *Combinatorica*, 26(6):733–742, 2006.

[14] J. Kahn. A normal law for matchings. *Combinatorica*, 20(3):339–391, 2000.

[15] G. Kirchhoff. Uber die auflösung der gleichungen, auf welche man bei der untersuchungen der linearen vertheilung galvanischer ströme geführt wird. *Ann. Phys. Chem.*, 72:497–508, 1847.

[16] C. Merino. *Matroids, the Tutte polynomial and the chip firing game*. PhD thesis, Somerville College, University of Oxford, 1999.

[17] J. G. Oxley. *Matroid theory*. Oxford Science Publications. The Clarendon Press Oxford University Press, New York, 1992.

[18] D. J. A. Welsh P. D. Seymour. Combinatorial applications of an inequality from statistical mechanics. *Math. Proc. Cambridge Philos. Soc.*, 77:485–495, 1975.

[19] J. L. Snell P. G. Doyle. *Random walks and electric networks*, volume 22 of *Carus Mathematical Monographs*. Mathematical Association of America, Washington, DC, 1984.

[20] R. Pemantle. Towards a theory of negative dependence. *J. Math. Phys.*, 41(0022-2488):1371–1390, 2000.

[21] A. H. Stone W. T. Tutte R. L. Brooks, C. A. B. Smith. The dissection of rectangles into squares. *Duke Math. J.*, 7:312–340, 1940.

[22] P. D. Seymour. 217. *Journal of Combinatorial Theory Series B*, 28(A1980KC17200005):305–359, 1980.

[23] P. D. Seymour. Matroids and multicommodity flows. *European J. Combin.*, 2(3):257–290, 1981.

[24] A. D. Sokal. The multivariate Tutte polynomial (alias Potts model) for graphs and matroids. In *Surveys in combinatorics 2005*, volume 327 of *London Math. Soc. Lecture Note Ser.*, pages 173–226. Cambridge Univ. Press, Cambridge, 2005.

[25] M. Mihail T. Feder. Balanced matroids. In *STOC '92: Proceedings of the twenty-fourth annual ACM symposium on Theory of computing*, pages 26–38, New York, NY, USA, 1992. ACM.

[26] W. T. Tutte. A homotopy theorem for matroids. I, II. *Trans. Amer. Math. Soc.*, 88:144–174, 1958.

[27] D. G. Wagner. 1. *Electronic Journal of Combinatorics*, 12(000229076400003), May 16 2005.

[28] D. G. Wagner. Negatively correlated random variables and mason's conjecture for independent sets in matroids. *Ann. Comb.*, 12(2):211–239, 2008.

[29] G. Whittle. 15. *Transactions of the American Mathematical Society*, 349(A1997WM52600005):579–603, FEB 1997.

[30] A. D. Sokal D. G. Wagner Y. B. Choe, J. G. Oxley. Homogeneous multivariate polynomials with the half-plane property. *Adv. in Appl. Math.*, 32(1-2):88–187, 2004.

[31] D. G. Wagner Y. B. Choe. Rayleigh matroids. *Comb. Probab. Comput.*, 15(5):765–781, 2006.

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