Theory and Applications of Dual Asymptotic Expansions

by

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Abstract

This thesis introduces an original mathematical theory for a new kind of asymptotic expansion of real-analytic functions of two variables. The *Dual Asymptotic Expansion* (DAE) expresses a bivariate function asymptotically as a sum of products of univariate functions; the series is asymptotic in the univariate sense as each variable approaches its limiting value while the other variable remains fixed.

The DAE exists to infinitely many terms at almost every expansion point where the function is analytic; the set of exceptional points has Lebesgue measure zero. The terms of a DAE are uniquely determined by the choice of expansion point, and usually contain nonpolynomial functions. DAE's can approximate special functions by series of elementary functions with better accuracy than comparable Taylor or Padé approximations.

The thesis presents several applications and includes a small implementation of DAE methods in the MAPLE 5.4 computer algebra system.

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Dedication

I dedicate this thesis with love to the memory of my twin sister. Although our time together was brief, your influence upon my life has been nothing less than profound. Your life is continually manifested in the person I have become. Even my work bears witness to you, who taught me the meaning of duality in the very first moments of our life. I offer this thesis as my monument to you, that the world may know of you, and remember.

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Chapter 1

Introduction

This chapter begins with a brief explanation of the thesis topic. After discussing the motivation behind the thesis and giving an overview of the organization of the thesis, the chapter ends by introducing some terminology and notation that will be heavily used for the remainder of the thesis.

1.1 Thesis Topic

This thesis introduces an original mathematical theory for a new kind of asymptotic expansion of real-analytic functions of two variables. If $f(x, y)$ is a real-analytic function on some open rectangle, and (a, b) is a point in its domain, the **dual asymptotic expansion** of $f(x, y)$ at (a, b) is a series expansion of the form

$$f(x, y) \sim \sum_{n=1}^{N} \frac{g_n(x)\, h_n(y)}{c_n}, \tag{1.1}$$

where the series (1.1) satisfies *both* of the following two conditions:

1. The series is an asymptotic expansion of $f(x, y)$ as $x \to a$ for each fixed y.
2. The series is an asymptotic expansion of $f(x, y)$ as $y \to b$ for each fixed x.

This duality between the roles of the two independent variables gives dual asymptotic expansions a number of special properties that are not shared by ordinary univariate asymptotic expansions, or even by univariate expansions with a parameter. For example, in some cases, a given univariate function may have no meaningful asymptotic expansion at all with respect to a particular asymptotic sequence; in other cases, there may be several different asymptotic sequences which yield meaningful asymptotic expansions of the same function. In contrast, a given bivariate function has exactly *one* dual asymptotic expansion to any desired number of terms at almost every expansion point in its domain. In fact, the terms of the dual asymptotic expansion are completely determined by the function $f(x, y)$ and point (a, b) alone, and can always be computed explicitly by a very straightforward algorithm.

Does the characterization above sound familiar? Bivariate Taylor series are *also* completely determined by the choice of analytic function $f(x, y)$ and point (a, b), and can always be computed explicitly by a straightforward algorithm. The reader may argue that since the Taylor series at (a, b) always represents the analytic function $f(x, y)$ in a neighborhood of (a, b), and since every convergent Taylor series is also an asymptotic expansion, there is nothing to be gained by introducing a new kind of series expansion.

While it is true that Taylor series provide a canonical representation of analytic functions (by their very definition), it is not necessarily the most *desirable* representation for every conceivable purpose. For example, Padé approximations (i.e., rational function approximations which interpolate the derivatives of a function at a point up to some specified order), are better approximations of analytic functions with nearby singularities than the corresponding Taylor approximations.

Unlike Taylor series, the terms of a dual asymptotic expansion can contain *nonpolynomial* functions such as rational functions, algebraic functions, elementary transcendental functions, and even special functions. Series of nonpolynomial functions can have very different *qualitative* properties than power series and rational approximations. Due to these qualitative differences, dual asymptotic expansions will be a more favorable choice for some applications than either Taylor series or Padé approximations. For example, later in the thesis, we will see that the error function $\operatorname{erf}(x)$ can be approximated more accurately near $x = 0$ by an approximation derived from a dual asymptotic expansion than it can by a Taylor approximation — or even a Padé approximation — with a comparable number of nonzero terms.

In short, by working with series whose terms come from a more general class of functions than polynomials, we open ourselves to many new and interesting possibilities.

1.2 Thesis Motivation

A major distinguishing characteristic of dual asymptotic expansions is the considerable generality of the terms of the series. The reader may object to this, saying that there are good, historical reasons for the use of polynomial and rational function approximations; such functions can be efficiently computed using nothing more than elementary arithmetic operations. The author will agree with this observation, but will also point out that advances in computational hardware have redefined and enlarged the scope of primitive mathematical operations. For well over a decade, even desktop computers have possessed specialized numeric coprocessors which implement the standard elementary transcendental functions in hardware; consequently, these functions can be computed with the speed and accuracy of the basic arithmetic operations.

Furthermore, with the advent of sophisticated computer algebra systems such as MAPLE, it has become feasible to perform algebraic manipulations with closed-form expressions far more general than either polynomials or rational functions. Indeed the very paradigm of scientific computation has changed as a result of these developments. For example, we are no longer limited to a single, fixed numerical method for a given class of problems, but can perform symbolic preprocessing in a computer algebra system to develop a numerical method that is customized for a particular problem. The methods of the thesis embrace this new paradigm of hybrid symbolic-numerical computation.

In short, thanks to advances in both computer hardware and computer software, it is feasible to move beyond the classical approximation schemes and embrace methods of considerably greater generality.

1.3 Integral Representations

Although dual asymptotic expansions have a variety of applications, there is one application in particular which has provided the impetus for the development of much of the underlying theory: the approximation via asymptotic methods of analytic functions represented by integrals. Indeed, the asymptotic analysis of integrals is sufficiently important that entire books have been devoted to the subject — for example, [Ble-Han]. According to [Ble-Han, p. vii], the class of functions represented by integrals is much broader than it may appear at first glance. Some of the reasons for this are listed below:

1. Many large classes of functions which arise frequently in applied analysis can be represented naturally by integrals, including:

    - The *probability distribution function* for any continuous random variable X, which is defined by the indefinite integral

    $$\Pr\{X \leq x\} = \int_{-\infty}^{x} \phi(t)\, dt,$$

    where ϕ is the probability density function for X. (A list of density functions can be found in [Abr-Ste, Chapter 26] and [EDM, Volume 2, Appendix A, Table 22].)

    - The class of *linear integral transforms*

    $$L[f](s) = \int_{a}^{b} K(s,t)\, f(t)\, dt,$$

    which includes the Fourier transform, Laplace transform, sine transform, cosine transform, Hankel transform, Mellin transform, Stieltjes transform, and Hilbert transform as important special cases.

- The *Green's function representations*

$$u(x) = \int_a^b G(x, y)\, f(y)\, dy$$

of solutions to boundary-value problems, which arise naturally in many physical applications from thermodynamics, electrodynamics, classical mechanics, quantum mechanics, and fluid mechanics ([Byr-Ful, Volume 2, Chapter 7], [Stakgold, Chapter 1], [Cho-Mar]).

- The *Cauchy integrals*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{h(w)}{w - z}\, dw,$$

which are useful for solving boundary-value problems and constructing conformal mappings in the complex plane, and also have important applications to airfoils, elasticity, and digital signal processing ([Henrici86, Chapter 14]).

2. Many of the important standard functions of applied analysis have well-known integral representations; for an extensive list of specific examples, please refer to Appendix B.

3. Many of the special functions listed in Appendix B arise as eigenfunctions of Sturm-Liouville problems. The techniques used to derive integral representations for these functions are fairly systematic and can be applied to other Sturm-Liouville problems as well. (One basic idea is to find a generating function for the sequence of eigenfunctions, expand the generating function in a Laurent series, and use complex contour integrals to derive integral representations for the coefficients.)

4. In general, integral representations for many functions which satisfy a linear ordinary differential equation can be derived using systematic techniques based on linear integral transforms. For a detailed discussion of these techniques, see [Ince, Chapters 8 and 18].

Hence, the class of functions which can be represented by integrals contains a vast collection of functions of genuine interest. Since dual asymptotic expansions are extremely well-suited to approximating functions represented by integrals, the computational methods presented by the thesis have great practical value.

1.4 Organization of the Thesis

Since the thesis depends heavily on the properties of analytic functions, Chapter 2 is devoted to this topic, and contains not only standard results, but a number of original results as well. Chapter 3 consists largely of standard material on asymptotic expansions of functions of one variable, but has been rewritten from a new point of view that is better suited to the purposes of the thesis. Chapter 4 continues in this vein, treating asymptotic expansions of functions with one independent variable and one parameter. This provides a natural stepping-stone to Chapter 5, which formally introduces dual asymptotic expansions, and develops many of their fundamental properties, including uniqueness and a necessary condition for existence. Chapter 6 develops a sufficient condition for existence in terms of a nonlinear operator that is fundamental to the theory of dual asymptotic expansions; this operator also provides a straightforward algorithm for computing the terms of a dual asymptotic expansion explicitly. Chapter 7 presents several practical applications of the theory of dual asymptotic expansions, and Chapter 8 concludes the thesis with a look at what lies ahead.

1.5 Terminology and Notation

The purpose of this section is to establish some conventions for the basic mathematical terminology and notation used throughout the thesis. Most of these conventions are based on common mathematical usage, and the underlying ideas are assumed to be familiar to the reader. More specialized terminology and notation specific to the thesis will be introduced in the body of the thesis when the need arises.

1.5.1 Numbers, Intervals, and Neighborhoods

Throughout the thesis, the **natural numbers** are denoted by $\mathbb{N}$, and consist of the *nonnegative* integers; thus, $0 \in \mathbb{N}$. The **real numbers** and **complex numbers** are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. The **extended real numbers** are the elements of the set

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\},$$

and the **topology of** $\overline{\mathbb{R}}$ is the usual two-point compactification of the reals. For any subset $A \subset \overline{\mathbb{R}}$, the **closure** $\bar{A}$ and **boundary** ∂A are formed with respect to the topology of $\overline{\mathbb{R}}$. Since by construction, the extended reals *are* the closure of the reals, the notation $\overline{\mathbb{R}}$ is consistent.

Let $a, b \in \overline{\mathbb{R}}$ satisfy $a < b$. The **open interval** (a, b) is the set

$$(a, b) = \{r \in \mathbb{R} \mid a < r < b\}.$$

This definition allows open intervals to be either *bounded* or *unbounded*. The context in which the notation (a, b) is used will distinguish the open interval $(a, b) \subset \mathbb{R}$ from the point $(a, b) \in \mathbb{R}^2$. Note that every open interval is *a connected open subset of* $\mathbb{R}$; the converse is also true. If $a, b \in \mathbb{R}$ satisfy $a < b$, the **closed interval** $[a, b]$ is the set

$$[a, b] = \{r \in \mathbb{R} \mid a \leq r \leq b\}.$$

This definition forces closed intervals to be *bounded* subsets of $\mathbb{R}$, hence *compact* in $\mathbb{R}$. Note that the *strict inequality* $a < b$ ensures that intervals are always *nonempty sets.*

Let $A, B \subset \mathbb{R}$ be arbitrary sets. A **neighborhood** of a point $a \in A$ is any open interval I such that $a \in I \subset A$. Note that if A is an open interval, then A is automatically a neighborhood of every one of its points, by definition. A **deleted neighborhood** of a point $a \in A$ is a set of the form $I - \{a\}$, where I is a neighborhood of a. If I and J are open intervals, the set $I \times J \subset \mathbb{R}^2$ is called an **open rectangle.** A **rectangular neighborhood** of a point $(a, b) \in A \times B$ is any open rectangle $I \times J$ such that $(a, b) \in I \times J \subset A \times B$; note that I is therefore a neighborhood of a, and J is a neighborhood of b.

1.5.2 Functions

For the sake of clarity, the **number zero** is denoted by 0, whereas the **zero function on a set** Ω is denoted by 0_Ω. If f is an arbitrary real- or complex-valued function with domain Ω, the relation $f = 0_\Omega$ means that f is identically zero on Ω. The relation $f \neq 0_\Omega$ means that f assumes at least one nonzero value on Ω. If $W \subset \Omega$, the **restriction of f to W** is denoted by $f \mid W$. The relation $f \mid W = 0_W$ means that f is identically zero on W.

Let $\Omega \subset \mathbb{R}^n$ be an open set. The **space of real-analytic functions on** Ω is denoted by $C^\omega(\Omega, \mathbb{R})$. If $\tilde{\Omega} \subset \mathbb{C}^n$ is an open set, the **space of complex-analytic functions on** $\tilde{\Omega}$ is denoted by $C^\omega(\tilde{\Omega}, \mathbb{C})$. Since the thesis is concerned primarily with *real-valued* functions, $C^\omega(\Omega, \mathbb{R})$ will usually be shortened to $C^\omega(\Omega)$. Similarly, the **space of continuous real-valued functions on** Ω is denoted simply by $C(\Omega)$.

Remark 1.1 *Throughout the thesis, the sets A and B will always denote* open intervals*; therefore, A, B, and $A \times B$ will always be connected open sets. Furthermore, Ω will nearly always be an* open interval *in* $\mathbb{R}$ *or an* open rectangle *in* $\mathbb{R}^2$, *and will usually be one of A, B, or $A \times B$.*

Note that when Ω is an open interval (such as A or B), the elements of $C^\omega(\Omega)$ are real-analytic functions of *one* variable, but when Ω is an open rectangle (such as $A \times B$), the elements of $C^\omega(\Omega)$ are real-analytic functions of *two* variables. In the two-variable case, it is important to note that the elements of $C^\omega(\Omega)$ are *analytic in both variables jointly*, not merely analytic in each individual variable alone. Similar observations apply to the elements of the function space $C(\Omega)$ when Ω is an open interval or open rectangle.

The **space of all real-valued functions on** Ω is denoted by $\mathbb{R}^\Omega$. The elements of $\mathbb{R}^\Omega$ are *arbitrary functions*, with no restrictions; thus, $f \in \mathbb{R}^\Omega$ if and only if $f : \Omega \to \mathbb{R}$. The **set of all functions from** Ω **into the extended reals** is denoted by $\overline{\mathbb{R}}^{\Omega}$. Note that $\overline{\mathbb{R}}^{\Omega}$ is a purely set-theoretic construct, and does *not* have the structure of a linear space.

1.5.3 Operators

For purposes of the thesis, operators are defined with minimal structure in a purely set-theoretic sense: an **operator** is any mapping from a set of functions into a set of functions. It is *not* assumed that these sets of functions are linear spaces, nor that the operator itself is linear. For example, a mapping such as

$$\Psi : C^{\omega}(A \times B) \rightarrow \overline{\mathbb{R}}^{A \times B}$$

constitutes an operator under this definition.

In order to improve the readability of complicated expressions involving operators, **square brackets** are used to denote operator evaluation, and **round brackets** are used to denote function evaluation. For example, when the operator Ψ is applied to a function $f \in C^{\omega}(A \times B)$, the resulting function in $\overline{\mathbb{R}}^{A \times B}$ is denoted by $\Psi[f]$. When the function $\Psi[f]$ is applied to a point $(a, b) \in A \times B$, the resulting value in $\overline{\mathbb{R}}$ is denoted by $\Psi[f](a, b)$.

Chapter 2

Real- and Complex-Analytic Functions

This chapter begins by summarizing some useful standard results for real- and complex-analytic functions of one or several variables, and ends by developing some additional results that are specific to the thesis. All of these results will prove to be of fundamental importance in subsequent chapters.

It is assumed that the reader is familiar with the definitions and elementary properties of real- and complex-analytic functions of one or several variables. The theory of complex-analytic functions of one variable is treated extensively in [Conway], in [Knopp45], [Knopp47], and in [Henrici74], [Henrici77], [Henrici86]. A combined treatment of the theories of real- and complex-analytic functions of one or several variables can be found in [Cartan]. A concise introduction to the theories of real- and complex-analytic functions of several variables can be found in [John, pp. 61-72].

2.1 Standard Results

This section discusses two general classes of well-known properties of analytic functions: the unique continuation of real- or complex-analytic functions into overlapping domains, and the extension of real-analytic functions into the complex domain.

2.1.1 The Unique Continuation Property

Both real- and complex-analytic functions on a connected open set enjoy a number of special properties which will be used heavily throughout the thesis. The most important of these is the *Unique Continuation Property*, which has many useful consequences. The properties which concern us are summarized in the following two propositions, whose proofs can be found in [Cartan, pp. 39-41, 122], [Conway, pp. 78-79], and [Knopp45, pp. 87-90, 92-96].

For the sake of brevity, let $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$ in the two propositions below. This will allow us to state the results for both function spaces $C^\omega(\Omega, \mathbb{R})$ and $C^\omega(\Omega, \mathbb{C})$ simultaneously.

The first proposition describes properties which are specific to real- and complex-analytic functions of *one variable.*

Proposition 2.1 *Let $\Omega \subset \mathbb{F}$ be a connected open set, and let $f \in C^\omega(\Omega, \mathbb{F})$.*

1. *If the zeros of f have a limit point in Ω, then $f = 0_\Omega$.*

2. *If $f \neq 0_\Omega$ and $f(a) = 0$ for $a \in \Omega$, then a is an isolated zero of f.*

3. *If $f \neq 0_\Omega$ and $f(a) = 0$ for $a \in \Omega$, then there is a unique integer $m \geq 1$ and a unique function $g \in C^\omega(\Omega, \mathbb{F})$ such that $g(a) \neq 0$ and $f(x) = (x-a)^m g(x)$ for all $x \in \Omega$.*

Proposition 2.1 enables us to define the notion of multiplicity for *univariate* functions. Later in the chapter, we will consider a generalization of multiplicity for *bivariate* functions.

Definition 2.2 *If $f \neq 0_\Omega$ and $f(a) = 0$ for $a \in \Omega$, the positive integer m guaranteed by item 3 of Proposition 2.1 is called the* ***multiplicity of the root*** *a. If $f(a) \neq 0$, we say that a* ***has multiplicity zero****.*

Thus, multiplicity is defined at every point of Ω if $f \neq 0_\Omega$, and undefined at every point of Ω if $f = 0_\Omega$.

Remark 2.3 *As a consequence of Proposition 2.1 and Definition 2.2, every function $f \in C^\omega(\Omega, \mathbb{F})$ with $f \neq 0_\Omega$ can be written as*

$$f(x) = (x - a)^m g(x) \text{ for all } x \in \Omega$$

for some function $g \in C^\omega(\Omega, \mathbb{F})$ with $g(a) \neq 0$, where $m \in \mathbb{N}$ is the multiplicity of the point $a \in \Omega$.

The second proposition describes properties which apply to real- and complex-analytic functions of *one or several variables.*

Proposition 2.4 *Let $\Omega \subset \mathbb{F}^n$ be a connected open set, let $W \subset \Omega$ be any nonempty open subset, and let $f, g \in C^\omega(\Omega, \mathbb{F})$.*

1. *If $f \mid W = 0_W$, then $f = 0_\Omega$.*
2. *If $f \mid W = g \mid W$, then $f = g$.*
3. *If $f \cdot g = 0_\Omega$ then $f = 0_\Omega$ or $g = 0_\Omega$.*

Remark 2.5 *Items 1 and 2 of Proposition 2.4 (which are clearly equivalent) are called the **Unique Continuation Property**. Item 3 implies that the function ring $C^\omega(\Omega, \mathbb{F})$ has no divisors of zero, and is therefore an **integral domain**.*

2.1.2 Complex Extensions of Real-Analytic Functions

We can develop real-analytic counterparts for many of the results of complex-analytic function theory simply by extending real-analytic functions to the complex domain. The following proposition extends a real-analytic function of *one variable* to a complex-analytic function.

Proposition 2.6 *If $\Omega \subset \mathbb{R}$ is a connected open set and $f \in C^\omega(\Omega, \mathbb{R})$, then there exist a connected open set $\tilde{\Omega} \subset \mathbb{C}$ and a unique function $\tilde{f} \in C^\omega(\tilde{\Omega}, \mathbb{C})$ such that $\Omega \subset \tilde{\Omega}$ and $\tilde{f} \mid \Omega = f$.*

The next proposition, which is based on the material in [John, pp. 61-72], extends a real-analytic function of *several variables* to a complex-analytic function.

Proposition 2.7 *Let $\Omega \subset \mathbb{R}^n$ be a connected open set, and let $f \in C^\omega(\Omega, \mathbb{R})$. If $K \subset \Omega$ is a nonempty* compact *subset, then there exist a connected open set $\tilde{\Omega} \subset \mathbb{C}^n$ and a unique function $\tilde{f} \in C^\omega(\tilde{\Omega}, \mathbb{C})$ such that $K \subset \tilde{\Omega}$ and $\tilde{f} \mid K = f \mid K$.*

The peculiar feature of the previous result is that it actually provides a complex-analytic extension of $f \mid K$ rather than the original function f.

2.2 Thesis-Specific Results

This section presents some original results of a more specialized nature. After developing some decomposition theorems for real-analytic functions of two variables, we will apply these decomposition theorems to develop some of the mathematical machinery needed later in the thesis.

2.2.1 Decomposition Theorems

Recall that $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ always denote open intervals, and that $C^\omega(\Omega)$ is an abbreviated notation for $C^\omega(\Omega, \mathbb{R})$. Given an arbitrary point $a \in A$, Remark 2.3 implies that every function $f \in C^\omega(A)$ with $f \neq 0_A$ admits a decomposition

$$f(x) = (x - a)^m g(x) \text{ for all } x \in A,$$

where the function $g \in C^\omega(A)$ is unique and satisfies $g(a) \neq 0$, and $m \in \mathbb{N}$ is the multiplicity of the point a as defined by Definition 2.2.

The goal of this section is to develop a similar decomposition for real-analytic functions of two variables. The key to achieving this goal is to adopt a suitable bivariate generalization of zeros and their multiplicities. We begin with the following definition, which applies to *arbitrary functions* of two variables.

Definition 2.8 *Assume* $f : A \times B \to \mathbb{R}$, *and let* $(a, b) \in A \times B$. *We say that* $f(x, y)$ ***vanishes on*** $x = a$ *if*

$$f(a, y) = 0 \text{ for all } y \in B;$$

in this case, we call $x = a$ *a* ***vanishing line of*** $f(x, y)$. *Similarly, we say that* $f(x, y)$ ***vanishes on*** $y = b$ *if*

$$f(x, b) = 0 \text{ for all } x \in A,$$

and we call $y = b$ *a* ***vanishing line of*** $f(x, y)$.

These *vanishing lines* are the bivariate analogues of the *zeros* of univariate functions. Using the above terminology, we can now state the first of several decomposition theorems for analytic functions of two variables.

Theorem 2.9 *Assume* $f \in C^{\omega}(A \times B)$ *satisfies* $f \neq 0_{A \times B}$, *and let* $a \in A$. *There exist a unique* $m \in \mathbb{N}$ *and unique* $g \in C^{\omega}(A \times B)$ *such that*

$$f(x, y) = (x - a)^m g(x, y) \text{ for all } (x, y) \in A \times B,$$

subject to the restriction that $g(x, y)$ *does* not *vanish on* $x = a$.

Remark 2.10 *This result lies at the very foundation of the thesis. Since the author has not encountered the result elsewhere in the mathematical literature, a complete and rigorous proof is included in the thesis. Due to its atypically long and technical nature, the proof of Theorem 2.9 has been placed in Appendix A.*

Although *Decomposition Theorem* 2.9 is stated in terms of the variable x, an obvious corollary in terms of the variable y can be obtained simply by interchanging the roles of x and y.

Corollary 2.11 *Assume* $f \in C^{\omega}(A \times B)$ *satisfies* $f \neq 0_{A \times B}$, *and let* $b \in B$. *There exist a unique* $n \in \mathbb{N}$ *and unique* $g \in C^{\omega}(A \times B)$ *such that*

$$f(x, y) = (y - b)^n g(x, y) \text{ for all } (x, y) \in A \times B,$$

subject to the restriction that $g(x, y)$ does not *vanish on $y = b$.*

Both Theorem 2.9 and Corollary 2.11 are *asymmetric* in the sense that each result focuses on only *one* of the two variables. The following theorem combines the two previous results into a *Symmetric Decomposition Theorem* for analytic functions of two variables.

Theorem 2.12 *Assume $f \in C^\omega(A \times B)$ satisfies $f \neq 0_{A\times B}$, and let $(a, b) \in A \times B$. There exist unique $m, n \in \mathbb{N}$ and a unique $\hat{f} \in C^\omega(A \times B)$ such that*

$$f(x, y) = (x - a)^m (y - b)^n \hat{f}(x, y) \text{ for all } (x, y) \in A \times B, \tag{2.1}$$

subject to the restriction that $\hat{f}(x, y)$ does not *vanish on either $x = a$ or $y = b$.*

Proof. We will prove existence first, and then uniqueness.

Existence. By Theorem 2.9, there exist $m \in \mathbb{N}$ and $g \in C^\omega(A \times B)$ such that

$$f(x, y) = (x - a)^m g(x, y) \text{ for all } (x, y) \in A \times B, \tag{2.2}$$

with the proviso that $g(x, y)$ does not vanish on $x = a$. Consequently, $g \neq 0_{A\times B}$, and Corollary 2.11 implies there exist $n \in \mathbb{N}$ and $h \in C^\omega(A \times B)$ such that

$$g(x, y) = (y - b)^n h(x, y) \text{ for all } (x, y) \in A \times B, \tag{2.3}$$

with the proviso that $h(x, y)$ does not vanish on $y = b$. Equation (2.3) implies that $h(x, y)$ does not vanish on $x = a$ either, since otherwise $g(x, y)$ would vanish on $x = a$. Let $\hat{f} = h$, and substitute (2.3) into (2.2) to obtain (2.1).

Uniqueness. Assume that $i \in \{1, 2\}$ throughout the proof. Suppose that $m_i, n_i \in \mathbb{N}$ and $\hat{f}_i \in C^\omega(A \times B)$ are such that

$$f(x, y) = (x - a)^{m_i} (y - b)^{n_i} \hat{f}_i(x, y) \text{ for all } (x, y) \in A \times B, \tag{2.4}$$

with the proviso that $\hat{f}_i(x, y)$ does not vanish on either $x = a$ or $y = b$. Define $g_i \in C^\omega(A \times B)$ by

$$g_i(x, y) = (y - b)^{n_i} \hat{f}_i(x, y) \text{ for all } (x, y) \in A \times B, \tag{2.5}$$

and note that $g_i(x, y)$ does not vanish on $x = a$, since otherwise $\hat{f}_i(x, y)$ would vanish on $x = a$. We can rewrite (2.4) as

$$f(x, y) = (x - a)^{m_i} g_i(x, y) \text{ for all } (x, y) \in A \times B.$$

By Theorem 2.9, $m_1 = m_2$ and $g_1 = g_2$. Denote g_i simply by g, and rewrite (2.5) as

$$g(x, y) = (y - b)^{n_i} \hat{f}_i(x, y) \text{ for all } (x, y) \in A \times B.$$

Since $g \neq 0_{A \times B}$, Corollary 2.11 implies that $n_1 = n_2$ and $\hat{f}_1 = \hat{f}_2$. ∎

The existence part of the previous proof constructed suitable $m, n \in \mathbb{N}$ and $\hat{f} \in C^\omega(A \times B)$ by applying Theorem 2.9 first and Corollary 2.11 second. Another possible construction would apply Corollary 2.11 first and Theorem 2.9 second. The uniqueness portion of the proof shows that all constructions must yield the same result.

Example 2.13 *What is the symmetric decomposition of* $\sin(xy)$ *at* $(0, 0)$? *For all* $t \in \mathbb{R}$, *define*

$$\operatorname{sinc}(t) = \begin{cases} \dfrac{\sin(t)}{t} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0. \end{cases}$$

The sinc *function is real-analytic on the entire real line since* $\sin(t)/t$ *has a removable singularity at* $t = 0$. *In addition,* $\operatorname{sinc}(xy)$ *does not vanish on either* $x = 0$ *or* $y = 0$ *since* $\operatorname{sinc}(0) = 1$. *Hence, the symmetric decomposition is*

$$\sin(xy) = x^1 \cdot y^1 \cdot \operatorname{sinc}(xy).$$

We are now in a position to define the notion of multiplicity for analytic functions of two variables.

Definition 2.14 *Assume* $f \in C^{\omega}(A \times B)$ *satisfies* $f \not\equiv 0_{A\times B}$, *and let* $(a, b) \in A \times B$. *If* $m, n \in \mathbb{N}$ *denote the unique values guaranteed by* Symmetric Decomposition Theorem *2.12, we call* m *the* ***multiplicity of the line*** $x = a$, *and* n *the* ***multiplicity of the line*** $y = b$.

Example 2.15 *For* $\sin(xy)$, *Example 2.13 implies the vanishing lines* $x = 0$ *and* $y = 0$ *both have multiplicity one.*

Recall the analogy between the *zeros* of univariate functions and the *vanishing lines* of bivariate functions. A point in the domain of a univariate function is a *zero* if and only if its multiplicity is positive. In the bivariate case, $x = a$ is a *vanishing line* of $f(x, y)$ if and only if its multiplicity $m > 0$. Similarly, $y = b$ is a *vanishing line* of $f(x, y)$ if and only if its multiplicity $n > 0$. Thus, the bivariate notion of multiplicity is analogous to its univariate counterpart in every way.

2.2.2 Division Theorem

This section discusses an idea which can be informally described as "division in the limit." The essence of the idea is that an analytic function of two variables can be divided by an analytic function of the first variable to produce, in the limit, an analytic function of the second variable. The theorem presented below will make this idea more precise.

The following definition explains what it means for a limit to be uniform in a parameter.

Definition 2.16 *Assume* $f : A \times B \to \mathbb{R}$ *and* $g : B \to \mathbb{R}$ *are* arbitrary functions. *Let* $a \in A$, *and let* $[b_1, b_2] \subset B$ *be any closed subinterval. We say that the limit*

$$\lim_{x \to a} f(x, y) = g(y)$$

is uniform in y ***for*** $y \in [b_1, b_2]$ *if for every* $\varepsilon > 0$, *there is a deleted neighborhood* $I_\varepsilon - \{a\}$ *such that the inequality*

$$|f(x, y) - g(y)| < \varepsilon$$

holds for all $(x, y) \in (I_\varepsilon - \{a\}) \times [b_1, b_2]$.

The following *Division Theorem* is an important and useful consequence of *Decomposition Theorem* 2.9.

Theorem 2.17 *Assume* $f \in C^\omega(A \times B)$ *and* $g \in C^\omega(A)$, *with* $g \neq 0_A$. *Let* $a \in A$. *let* $J \subset B$ *be any open subinterval, and let* $[b_1, b_2] \subset B$ *be any closed subinterval. If the limit*

$$h(y) = \lim_{x \to a} \frac{f(x, y)}{g(x)} \tag{2.6}$$

exists for all $y \in J$, *then all of the following conclusions hold:*

1. *the limit exists* for all $y \in B$
2. *the function* $h \in C^\omega(B)$
3. *the limit is* uniform in y *for* $y \in [b_1, b_2]$.

Proof. If $f = 0_{A \times B}$, then all three conclusions follow trivially. Suppose that $f \neq 0_{A \times B}$. By *Decomposition Theorem* 2.9, there exist $m \in \mathbb{N}$ and $\hat{f} \in C^\omega(A \times B)$ such that

$$f(x, y) = (x - a)^m \hat{f}(x, y) \text{ for all } (x, y) \in A \times B, \tag{2.7}$$

with the proviso that $\hat{f}(x, y)$ does not vanish on $x = a$. Similarly, by Remark 2.3, there exist $n \in \mathbb{N}$ and $\hat{g} \in C^\omega(A)$ such that

$$g(x) = (x - a)^n \hat{g}(x) \text{ for all } x \in A, \tag{2.8}$$

where $\hat{g}(a) \neq 0$.

Define an open rectangle $R = A \times J$. Since $\hat{f}(x, y)$ does not vanish on $x = a$, the *restriction* $\hat{f} \mid R(x, y)$ also does not vanish on $x = a$ (see Proposition A.1 in Appendix A). This implies that $\hat{f}(a, b) \neq 0$ for some $b \in J$.

Since $b \in J$, limit (2.6) exists for $y = b$. If $m < n$, applying decompositions (2.7) and (2.8) to limit (2.6) for $y = b$ yields

$$\begin{aligned} h(b) &= \lim_{x \to a} \frac{f(x, b)}{g(x)} \\ &= \lim_{x \to a} \frac{(x-a)^m \hat{f}(x, b)}{(x-a)^n \hat{g}(x)} \\ &= \lim_{x \to a} \frac{\hat{f}(x, b)}{(x-a)^{n-m} \hat{g}(x)}, \end{aligned}$$

which *does not exist*, since in the limit as $x \to a$, the numerator $\hat{f}(a, b) \neq 0$, but the denominator $0^{n-m} \hat{g}(a) = 0$. Consequently, $m \geq n$.

Since $\hat{g}(a) \neq 0$, the limit

$$\begin{aligned} h(y) &= \lim_{x \to a} \frac{f(x, y)}{g(x)} \\ &= \lim_{x \to a} \frac{(x-a)^m \hat{f}(x, y)}{(x-a)^n \hat{g}(x)} \\ &= \lim_{x \to a} \frac{(x-a)^{m-n} \hat{f}(x, y)}{\hat{g}(x)} \qquad (2.9) \\ &= \frac{0^{m-n} \cdot \hat{f}(a, y)}{\hat{g}(a)} \qquad (2.10) \end{aligned}$$

exists for all $y \in B$, which proves conclusion 1. For all $y \in B$, we can write

$$h(y) = c \cdot \hat{f}(a, y),$$

where

$$c = \frac{0^{m-n}}{\hat{g}(a)} = \begin{cases} 1/\hat{g}(a) & \text{if } m = n \\ 0 & \text{if } m > n \end{cases}$$

is a constant. Since $\hat{f} \in C^\omega(A \times B)$, it follows that $h \in C^\omega(B)$, which proves conclusion 2.

Since $\hat{g}(a) \neq 0$, there is a neighborhood I of a such that $\hat{g}(x) \neq 0$ for all $x \in I$, by the continuity of $\hat{g}$. Define $\varphi \in C^{\omega}(I \times B)$ by

$$\varphi(x, y) = \frac{(x-a)^{m-n} \hat{f}(x, y)}{\hat{g}(x)} - h(y)$$

for all $(x, y) \in I \times B$. Note that $\varphi(a, y) = 0$ for all $y \in B$, by (2.10). It follows from (2.9) that limit (2.6) is uniform in y for $y \in [b_1, b_2]$ if and only if the limit

$$\lim_{x \to a} \varphi(x, y) = 0$$

is uniform in y for $y \in [b_1, b_2]$. Hence, in order to prove conclusion 3, we must show that for every $\varepsilon > 0$, there is a deleted neighborhood $I_\varepsilon - \{a\}$ such that $|\varphi(x, y)| < \varepsilon$ for all $(x, y) \in (I_\varepsilon - \{a\}) \times [b_1, b_2]$, by Definition 2.16.

Let $[a_1, a_2] \subset I$ be a closed subinterval such that $a \in (a_1, a_2)$, and define

$$K = [a_1, a_2] \times [b_1, b_2] \subset I \times B.$$

Since K is a compact subset of $\mathbb{R}^2$, and φ is continuous on K, it follows by a standard theorem of topology that φ is *uniformly continuous* on K. This means that for every $\varepsilon > 0$, there is a $\delta > 0$ such that the inequality

$$|\varphi(x, y) - \varphi(x', y')| < \varepsilon \tag{2.11}$$

holds whenever $(x, y), (x', y') \in K$ satisfy

$$\sqrt{(x - x')^2 + (y - y')^2} < \delta.$$

Let $I_\varepsilon = (a_1, a_2) \cap (a - \delta, a + \delta)$, and note that I_ε is a neighborhood of a. Let $(x, y) \in (I_\varepsilon - \{a\}) \times [b_1, b_2]$ be arbitrary, and set $(x', y') = (a, y)$. Since $x, x' \in I_\varepsilon \subset [a_1, a_2]$, it follows that $(x, y), (x', y') \in K$. Since $x \in I_\varepsilon$ implies $|x - a| < \delta$, it follows that

$$\sqrt{(x-x')^2+(y-y')^2} = \sqrt{(x-a)^2+(y-y)^2} = |x-a| < \delta.$$

Hence, (2.11) and $\varphi(a,y)=0$ imply that $|\varphi(x,y)| < \varepsilon$, which proves conclusion 3. ■

In summary, *Division Theorem* 2.17 says that if an analytic function of x and y can be divided *in the limit as* $x \to a$ by an analytic function of x for all $y \in J$, then this "division in the limit" can be carried out for all $y \in B$, and produces an analytic function of y. Furthermore, the *uniformity* of the limit in the parameter y over any *compact* interval $[b_1, b_2]$ follows automatically from the mere *existence* of the limit on the interior (b_1, b_2) of that interval.

2.2.3 Local Reduction

Let $(a,b) \in A \times B$. In this section, $\mathcal{F}(A \times B)$ denotes the set of functions $C^\omega(A \times B) - \{0_{A\times B}\}$, and $\mathcal{R}_{(a,b)}(A \times B)$ denotes the subset of $\mathcal{F}(A \times B)$ consisting of all functions $f(x,y)$ which do not vanish on either $x=a$ or $y=b$.

Reformulated in this notation, *Symmetric Decomposition Theorem* 2.12 says that for every $f \in \mathcal{F}(A \times B)$ and every $(a,b) \in A \times B$, there are a unique ordered pair $(m,n) \in \mathbb{N}^2$ and a unique function $\hat{f} \in \mathcal{R}_{(a,b)}(A \times B)$ such that

$$f(x,y) = (x-a)^m\,(y-b)^n\,\hat{f}(x,y) \text{ for all } (x,y) \in A \times B.$$

The following definition introduces some new terminology and notation based on this reformulation of the theorem.

Definition 2.18 *Given $f \in \mathcal{F}(A \times B)$ and $(a,b) \in A \times B$, let $(m,n) \in \mathbb{N}^2$ and $\hat{f} \in \mathcal{R}_{(a,b)}(A \times B)$ denote the ordered pair and function provided by* Symmetric Decomposition Theorem *2.12.*

1. *The ordered pair (m,n) is called the* ***degree of** f **at*** *(a,b), and is denoted by* $\deg_{(a,b)}(f)$.

2. *The function* $\hat{f}$ *is called the* ***reduction of*** f ***at*** (a, b).

3. *The operator* $\rho_{(a,b)} : \mathcal{F}(A \times B) \rightarrow \mathcal{R}_{(a,b)}(A \times B)$ *defined by* $\rho_{(a,b)}[f] = \hat{f}$ *is called the* ***reduction operator at*** (a, b).

4. *We say that* f ***is reduced at*** (a, b) *if* $f \in \mathcal{R}_{(a,b)}(A \times B)$.

Example 2.19 *Continuing Example 2.13, the degree of* $\sin(xy)$ *at* $(0, 0)$ *is* $(1, 1)$, *and the reduction of* $\sin(xy)$ *at* $(0, 0)$ *is* $\operatorname{sinc}(xy)$. *In addition,* $\operatorname{sinc}(xy)$ *is reduced at* $(0, 0)$.

The property that f *is reduced at* (a, b) has other equivalent formulations, which are expressed in the following proposition.

Proposition 2.20 *For all* $f \in \mathcal{F}(A \times B)$ *and* $(a, b) \in A \times B$, *the following conditions are equivalent:*

1. $f \in \mathcal{R}_{(a,b)}(A \times B)$

2. $\deg_{(a,b)}(f) = (0, 0)$

3. $\rho_{(a,b)}[f] = f$.

Proof. We will prove the equivalence of these three conditions by showing that

$$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1.$$

If $f \in \mathcal{R}_{(a,b)}(A \times B)$, then by definition, $f(x, y)$ does not vanish on either $x = a$ or $y = b$. Consequently, the lines $x = a$ and $y = b$ both have multiplicity zero, and the symmetric decomposition of f is

$$f(x, y) = (x - a)^0 (y - b)^0 \cdot \rho_{(a,b)}[f](x, y) \text{ for all } (x, y) \in A \times B. \tag{2.12}$$

By Definition 2.18, $\deg_{(a,b)}(f) = (0, 0)$, which shows that $1 \Rightarrow 2$. If $\deg_{(a,b)}(f) = (0, 0)$, it follows from symmetric decomposition (2.12) that $\rho_{(a,b)}[f] = f$, which shows that $2 \Rightarrow 3$. If $\rho_{(a,b)}[f] = f$, then

$$\rho_{(a,b)} : \mathcal{F}(A \times B) \rightarrow \mathcal{R}_{(a,b)}(A \times B) \tag{2.13}$$

implies that $f \in \mathcal{R}_{(a,b)}(A \times B)$, which shows that $3 \Rightarrow 1$. ■

Remark 2.21 *One implication of Proposition 2.20 is that $\mathcal{R}_{(a,b)}(A \times B)$ is precisely the set of* fixed-points *of the operator $\rho_{(a,b)}$. This fact, together with mapping diagram (2.13), imply that $\rho^2_{(a,b)} = \rho_{(a,b)}$. Hence, the operator $\rho_{(a,b)}$ is* idempotent.

For the sake of interpretation, let us think of the ordered pair $(m, n) \in \mathbb{N}^2$ as a *multi-index* (see [John, pp. 54-55]), and write $\mathbf{x} = (x, y)$ and $\mathbf{a} = (a, b)$ in *vector notation*. Using standard multi-index notation, we can write

$$(\mathbf{x} - \mathbf{a})^{(m,n)} = (x - a)^m (y - b)^n.$$

Letting $\mathbf{A} = A \times B$, we can now express *Symmetric Decomposition Theorem* 2.12 very concisely in the notation of Definition 2.18 as follows: for every $f \in \mathcal{F}(\mathbf{A})$ and every $\mathbf{a} \in \mathbf{A}$, the theorem implies that

$$f(\mathbf{x}) = (\mathbf{x} - \mathbf{a})^{\deg_{\mathbf{a}}(f)} \cdot \rho_{\mathbf{a}}[f](\mathbf{x}) \text{ for all } \mathbf{x} \in \mathbf{A}.$$

Thus, given a point $\mathbf{a} \in \mathbf{A}$, every nonzero analytic function $f(\mathbf{x})$ can be written as the product of a *polynomial* $(\mathbf{x} - \mathbf{a})^{\deg_{\mathbf{a}}(f)}$ and an analytic function $\rho_{\mathbf{a}}[f](\mathbf{x})$ *which is reduced at* $\mathbf{a}$.

Chapter 3

Asymptotic Expansions in One Variable

The next two chapters present the elements of classical asymptotic theory which are used throughout the thesis. This chapter discusses asymptotic expansions in one independent variable, whereas the next chapter discusses asymptotic expansions which also depend on a parameter. The material in these two chapters is largely a synthesis of standard material found in [Erdelyi, Ch. 1] and [Ble-Han, Ch. 1]; however, some of this material has been modified to better suit the purposes of the thesis. The result is an original presentation of asymptotic theory which breaks with tradition in some important ways — in one sense narrowing the tradition, and in another sense, broadening it.

This chapter develops asymptotic theory over the function space $C^\omega(A)$, where $A \subset \mathbb{R}$ is an open interval. Given a function $f \in C^\omega(A)$, traditional asymptotic theory studies the limiting behavior of $f(x)$ as $x \to a$, where $a \in \bar{A}$. In many cases, a is an essential singularity of f, which can only occur when $a \in \partial A$. The asymptotic theory presented here narrows tradition by requiring that $a \in A$ rather than $\bar{A}$. Since A is open, a must be an *interior point*, and boundary points are therefore excluded from consideration. Although this restriction categorically rules out the study of asymptotic behavior near singularities, the hypothesis $a \in A$ will enable us to obtain stronger results in this chapter, and these results will prove indispensable later in the thesis.

The asymptotic theory presented here broadens tradition by carefully considering the role of the zero function. Surprisingly, it is useful to distinguish between a *finite* asymptotic expansion and an *infinite* asymptotic expansion with a finite number of nonzero terms. In the latter case, the asymptotic expansion as $x \to a$ becomes an *exact identity* for all $x \in A$. The thesis will later demonstrate how asymptotic techniques can be used to establish exact identities under suitable conditions.

3.1 The "Little Oh" Order Relation

Asymptotic theory is founded upon order relations which express the relative rates of growth or decay of two functions as $x \to a$. In order to handle finite and infinite asymptotic expansions with equal simplicity, the thesis develops asymptotic theory entirely in terms of the "little oh" order relation; the "big oh" order relation is not used here.

3.1.1 Definition and Consequences

Recall that $A \subset \mathbb{R}$ always denotes an open interval. For any two *arbitrary functions* $f, g : A \to \mathbb{R}$ and any point $a \in A$, the "little oh" order relation is defined as follows.

Definition 3.1 *We say that $f(x)$* ***is "little oh" of*** *$g(x)$* ***as*** *$x \to a$, which we denote by*

$$f(x) = o(g(x)) \text{ as } x \to a,$$

if for every $\varepsilon > 0$, there is a neighborhood I_ε of a such that the inequality

$$|f(x)| \leq \varepsilon \, |g(x)| \tag{3.1}$$

holds for all $x \in I_\varepsilon$.

The following proposition explores the properties of "little oh" order relations under addition, scalar multiplication, and composition; for brevity, the phrase "as

$x \to a$" will be omitted, but should be assumed throughout.

Proposition 3.2 *Let $f, g, h : A \to \mathbb{R}$ be arbitrary functions, and let $c \in \mathbb{R}$.*

1. *If $f(x) = o(h(x))$ and $g(x) = o(h(x))$, then $f(x) + g(x) = o(h(x))$.*

2. *If $f(x) = o(h(x))$, then $c \cdot f(x) = o(h(x))$.*

3. *If $f(x) = o(g(x))$ and $g(x) = o(h(x))$, then $f(x) = o(h(x))$.*

Proof. The proof of each property is based on a direct appeal to Definition 3.1.

1. Let $\varepsilon > 0$ be given. Since $f(x) = o(h(x))$ and $g(x) = o(h(x))$, there are neighborhoods I_1 and I_2 of a such that $|f(x)| \leq (\varepsilon/2) \cdot |h(x)|$ and $|g(x)| \leq (\varepsilon/2) \cdot |h(x)|$ for all $x \in I_1$ and all $x \in I_2$, respectively. Note that $I_1 \cap I_2$ is also a neighborhood of a. If $x \in I_1 \cap I_2$, then

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \varepsilon \ |h(x)|,$$

which proves that $f(x) + g(x) = o(h(x))$.

2. Let $\varepsilon > 0$ be given. If $c = 0$, then $|c \cdot f(x)| \leq \varepsilon \ |h(x)|$ is trivially satisfied for all $x \in A$. If $c \neq 0$, then $f(x) = o(h(x))$ implies there is a neighborhood I of a such that $|f(x)| \leq (\varepsilon/|c|) \cdot |h(x)|$, or equivalently, $|c \cdot f(x)| \leq \varepsilon \ |h(x)|$, for all $x \in I$. Hence, $c \cdot f(x) = o(h(x))$ for any constant $c \in \mathbb{R}$.

3. Let $\varepsilon > 0$ be given. Since $f(x) = o(g(x))$ and $g(x) = o(h(x))$, there are neighborhoods I_1 and I_2 of a such that $|f(x)| \leq \sqrt{\varepsilon} \ |g(x)|$ and $|g(x)| \leq \sqrt{\varepsilon} \ |h(x)|$ for all $x \in I_1$ and all $x \in I_2$, respectively. If $x \in I_1 \cap I_2$, then

$$|f(x)| \leq \sqrt{\varepsilon} \ |g(x)| \leq \varepsilon \ |h(x)|,$$

which proves that $f(x) = o(h(x))$. ■

Remark 3.3 *These three properties of the "little oh" order relation are frequently written in a more abbreviated form as:*

1. $o(h(x)) + o(h(x)) = o(h(x))$

2. $c \cdot o(h(x)) = o(h(x))$

3. $o(o(h(x))) = o(h(x))$.

Note that these short forms are not true *equations* — they are *transformation rules* which describe how to simplify various combinations of order relations into a single order relation.

3.1.2 Characterization for Analytic Functions

The two propositions of this section characterize the "little oh" order relation for *analytic functions.* We begin by proving the following lemma, which contains a simple but useful observation.

Lemma 3.4 *Let $f \in C^\omega(A)$. If $f \neq 0_A$, then for each $a \in A$, there is a deleted neighborhood $I - \{a\}$ on which f assumes only* nonzero *values.*

Proof. If $f(a) = 0$, Proposition 2.1 implies that a is an isolated zero, which means that f assumes only nonzero values on some deleted neighborhood $I - \{a\}$. If $f(a) \neq 0$, then by continuity, there is a neighborhood I of a on which f is never zero; thus, f cannot be zero on the deleted neighborhood $I - \{a\}$. ■

The first proposition explains what is necessary and sufficient for an analytic function to be "little oh" of a *nonzero* analytic function.

Proposition 3.5 *Let $f, g \in C^\omega(A)$, and suppose that $g \neq 0_A$. Then $f(x) = o(g(x))$ as $x \to a$ if and only if*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 0. \tag{3.2}$$

Proof. First, note that the hypotheses $g \in C^\omega(A)$ and $g \neq 0_A$ ensure that there is some deleted neighborhood $I_1 - \{a\}$ where g is never zero, by Lemma 3.4. Since the quotient function f/g is defined on $I_1 - \{a\}$, it is reasonable to consider the existence of the limit (3.2) as $x \to a$ in $I_1 - \{a\}$.

Suppose that $f(x) = o(g(x))$ as $x \to a$, and let $\varepsilon > 0$ be given. By Definition 3.1, there is a neighborhood I_2 of a such that

$$|f(x)| \le \frac{\varepsilon}{2}\,|g(x)| \tag{3.3}$$

for all $x \in I_2$. Since I_1 and I_2 are both neighborhoods of a, the intersection $I = I_1 \cap I_2$ is also a neighborhood of a. By construction of I, the function g is never zero on $I - \{a\}$, and inequality (3.3) holds for all $x \in I - \{a\}$; consequently, we can divide inequality (3.3) by $|g(x)|$ to obtain

$$\left|\frac{f(x)}{g(x)}\right| \le \frac{\varepsilon}{2} < \varepsilon.$$

Thus, for every $\varepsilon > 0$, there is a deleted neighborhood $I - \{a\}$ on which f/g is defined and satisfies $|f/g| < \varepsilon$, which implies that equation (3.2) holds.

Conversely, if equation (3.2) holds, then for every $\varepsilon > 0$, there is a deleted neighborhood $I - \{a\}$ such that

$$|f(x)| < \varepsilon\,|g(x)| \tag{3.4}$$

for all $x \in I - \{a\}$. Taking the limit of (3.4) as $x \to a$ in $I - \{a\}$ yields

$$|f(a)| \le \varepsilon\,|g(a)| \tag{3.5}$$

by the continuity of f and g. Inequalities (3.4) and (3.5) together imply that

$$|f(x)| \le \varepsilon\,|g(x)|$$

for all $x \in I$. By Definition 3.1, $f(x) = o(g(x))$ as $x \to a$. ■

The second proposition is also an equivalence. One of the implications of the proposition is that an *analytic function* which is "little oh" of the zero function as $x \to a$ must be zero also — not merely zero *in a neighborhood of* a, but zero *on the whole domain* A. It is interesting that the result draws a *global* conclusion from a *local* hypothesis.

Proposition 3.6 *If $f \in C^\omega(A)$ and $g = 0_A$, then $f(x) = o(g(x))$ as $x \to a$ if and only if $f = 0_A$.*

Proof. Suppose that $f(x) = o(g(x))$ as $x \to a$. Choose $\varepsilon = 1$ in Definition 3.1 and let $W = I_\varepsilon$. The inequality

$$|f(x)| \le \varepsilon \; |g(x)|$$

with $g = 0_A$ holds for all $x \in W$, which implies that $f \,|\, W = 0_W$. The *Unique Continuation Property* (Proposition 2.4) implies that $f = 0_A$. Conversely, if $f = 0_A$, then Definition 3.1 is trivially satisfied for $g = 0_A$ by setting $I_\epsilon = A$ for every $\varepsilon > 0$, since A is an open interval. ∎

3.1.3 Integration Theorems

The following two theorems demonstrate that under suitable circumstances, "little oh" order relations can be integrated to obtain new order relations. The first theorem is a standard result which establishes that order relations between *continuous functions* can be integrated with respect to the independent variable. The second theorem shows that an even stronger conclusion can be obtained for *analytic functions*.

Theorem 3.7 *Let $f, g \in C(A)$. If $f(x) = o(g(x))$ as $x \to a$, then*

$$\int_a^x f(t)\,dt = o\left(\int_a^x |g(t)|\,dt\right) \quad \text{as } x \to a. \tag{3.6}$$

Proof. Let $\varepsilon > 0$ be given. By Definition 3.1, there is a neighborhood I of a such that $|f(x)| \le \varepsilon \; |g(x)|$ for all $x \in I$. Consequently,

$$\left|\int_a^x f(t)\,dt\right| \le \int_a^x |f(t)|\,dt \le \varepsilon \int_a^x |g(t)|\,dt$$

for all $x \in I$, which establishes that order relation (3.6) holds. ■

Since $C^\omega(A) \subset C(A)$, we can certainly apply the previous theorem to analytic functions f and g; however, the theorem has one defect: the function $|g|$ need not be analytic if g is analytic. We can repair the defect by using the properties of analytic functions to eliminate the absolute value from order relation (3.6), thereby preserving the analyticity of g.

Theorem 3.8 *Let $f, g \in C^\omega(A)$. If $f(x) = o(g(x))$ as $x \to a$, then*

$$\int_a^x f(t)\,dt = o\left(\int_a^x g(t)\,dt\right) \quad \text{as } x \to a.$$

Proof. For convenience, let $F(x) = \int_a^x f(t)\,dt$ and $G(x) = \int_a^x g(t)\,dt$. Note that $F, G \in C^\omega(A)$, and $F(a) = G(a) = 0$. The proof proceeds by cases on g.

Case 1. Suppose $g \neq 0_A$. By Lemma 3.4, there is some deleted neighborhood $I - \{a\}$ on which g is never zero. Suppose that $G(b) = 0$ for some $b \in I - \{a\}$. Since $G(a) = 0$ also, Rolle's Theorem implies that $G'(c) = 0$ for some c strictly between a and b. By the Fundamental Theorem of Calculus, $G'(c) = g(c) = 0$ for $c \in I - \{a\}$, which contradicts that g is never zero on $I - \{a\}$. Hence, both g and G are never zero on $I - \{a\}$, which allows us to form the quotient functions f/g and F/G on $I - \{a\}$ *simultaneously.* As $x \to a$ in $I - \{a\}$, F/G has the indeterminate form $0/0$. By L'Hospital's Rule,

$$\lim_{x\to a} \frac{F(x)}{G(x)} = \lim_{x\to a} \frac{f(x)}{g(x)}. \tag{3.7}$$

Since $f(x) = o(g(x))$ as $x \to a$, the value of limit (3.7) is zero by Proposition 3.5. Since $F(x)/G(x) \to 0$ as $x \to a$, Proposition 3.5 implies that $F(x) = o(G(x))$ as $x \to a$.

Case 2. Suppose $g = 0_A$. By Proposition 3.6, $f = 0_A$ also. Consequently, the indefinite integrals satisfy $F = 0_A$ and $G = 0_A$, and Proposition 3.6 implies that

$F(x) = o(G(x))$ as $x \to a$. ■

3.2 Asymptotic Sequences

In this section, N is either a positive integer or infinity, and represents the number of elements in a sequence of functions $\{g_n\}_{n=1}^{N} \subset \mathbb{R}^A$. The terminology and conventions described below will enable us to handle both finite and infinite sequences with a single notation, thereby obviating the need to consider the two cases separately.

Please do not underestimate the importance of the following simple definition, which will be heavily used throughout the thesis.

Definition 3.9 *A constant n is called an* ***index*** *of the sequence $\{g_n\}_{n=1}^{N}$ if $n \in \mathbb{N}$ and satisfies $1 \le n \le N$.*

This terminology will allow us to use short, convenient phrases such as "for all *indices* $n < N$" instead of the more cumbersome "for all $n \in \mathbb{N}$ satisfying $1 \le n < N$."

Remark 3.10 *The following are important conventions for interpreting the boundary cases of Definition 3.9:*

1. *Since $N \ge 1$, it follows that $n = 1$ is always an index of $\{g_n\}_{n=1}^{N}$.*
2. *If $N < \infty$, then $n = N$ is an index of $\{g_n\}_{n=1}^{N}$ as well.*
3. *If $N = \infty$, then $n = N$ is* not *an index of $\{g_n\}_{n=1}^{N}$, since $\infty \notin \mathbb{N}$.*

3.2.1 Definition and Consequences

In the following definition, $\{g_n\}_{n=1}^{N} \subset \mathbb{R}^A$ and $a \in A$ are arbitrary.

Definition 3.11 *$\{g_n(x)\}_{n=1}^{N}$ is called an* ***asymptotic sequence as*** *$x \to a$ if the order relation*

$$g_{n+1}(x) = o(g_n(x)) \text{ as } x \to a \tag{3.8}$$

holds for all indices $n < N$.

Remark 3.12 *The following are immediate consequences of the definition:*

1. *If $N = 1$, the definition is vacuously satisfied.*

2. *If $1 < N < \infty$, order relation (3.8) must hold for $n = 1, 2, \ldots, N-1$.*

3. *If $N = \infty$, order relation (3.8) must hold for all positive integers n.*

Note that an asymptotic sequence may contain zero functions! For example, $\{0_A\}_{n=1}^{N}$ constitutes an asymptotic sequence as $x \to a$. Although it is not customary to consider asymptotic sequences containing zero functions, there is a benefit to be gained by doing so; this benefit will be fully explained at the end of the chapter.

3.2.2 Tractable Sequences and Essential Length

Although zero functions may occur in an asymptotic sequence, they cannot occur in a random, haphazard fashion — especially if the sequence consists of analytic functions. In order to describe what can actually occur, we need to introduce some new terminology, which is the purpose of this section. Note that this terminology applies to *all* sequences of functions, not merely to asymptotic sequences.

In the definitions below, $\{g_n\}_{n=1}^{N} \subset \mathbb{R}^A$ is an arbitrary sequence of functions.

Definition 3.13 *We say that $\{g_n\}_{n=1}^{N}$ is* ***tractable*** *if there is some $\bar{N} \in \mathbb{N} \cup \{\infty\}$ with $0 \leq \bar{N} \leq N$ such that the following two properties hold:*

1. *$g_n \neq 0_A$ for all indices $n \leq \bar{N}$*

2. *$g_n = 0_A$ for all indices $n > \bar{N}$.*

Recall that an index n of the sequence $\{g_n\}_{n=1}^{N}$ must satisfy $1 \leq n \leq N$. Consequently, property 1 is vacuously satisfied when $\bar{N} = 0$, and property 2 is vacuously satisfied when $\bar{N} = N$.

Proposition 3.14 *If $\{g_n\}_{n=1}^{N}$ is tractable, then $\bar{N}$ is unique.*

Proof. Suppose that N_1 and N_2 have all the properties of $\bar{N}$ in Definition 3.13. We will show that $N_1 = N_2$. The proof proceeds by cases on N_1. Note that $0 \leq N_1 \leq N$, by definition.

Case 1. Suppose $N_1 = 0$. In this case, $g_1 = 0_A$ by property 2 with $\bar{N} = N_1$. By definition, $N_2 \geq 0$. If $N_2 > 0$, then $g_1 \neq 0_A$ by property 1 with $\bar{N} = N_2$. This contradicts that $g_1 = 0_A$; therefore, $N_2 = 0 = N_1$.

Case 2. Suppose $0 < N_1 < N$. In this case, both N_1 and $N_1 + 1$ are *indices*. Consequently, $g_{N_1} \neq 0_A$ by property 1 with $\bar{N} = N_1$, and $g_{N_1+1} = 0_A$ by property 2 with $\bar{N} = N_1$. If $N_1 > N_2$, then $g_{N_1} = 0_A$ by property 2 with $\bar{N} = N_2$. This contradicts that $g_{N_1} \neq 0_A$; therefore, $N_1 \leq N_2$. If $N_1 < N_2$, then $g_{N_1+1} \neq 0_A$ by property 1 with $\bar{N} = N_2$. This contradicts that $g_{N_1+1} = 0_A$; therefore, $N_1 = N_2$.

Case 3. Suppose $N_1 = N$. By definition, $0 \leq N_2 \leq N$. We can rule out the possibility that $N_2 = 0$, since otherwise Case 1 (with the roles of N_1 and N_2 reversed) implies that $N = 0$, which contradicts that $N > 0$. Similarly, we can rule out the possibility that $0 < N_2 < N$, since otherwise Case 2 (with roles reversed) implies that $N < N$, which is self-contradictory. By process of elimination, $N_2 = N = N_1$. ■

The next definition gives us two different ways to think about the length of a tractable sequence.

Definition 3.15 *If $\{g_n\}_{n=1}^{N}$ is tractable, $\bar{N}$ is called the* ***essential length*** *of the sequence, and N is called the* ***actual length*** *of the sequence.*

The essential length of a tractable sequence is well-defined by Proposition 3.14. The following proposition characterizes tractable sequences which are also asymptotic sequences.

Proposition 3.16 *If* $\{g_n\}_{n=1}^{N} \subset \mathbb{R}^A$ *is tractable with essential length* $\bar{N}$, *then* $\{g_n(x)\}_{n=1}^{N}$ *is an asymptotic sequence as* $x \to a$ *if and only if the* truncated *sequence* $\{g_n(x)\}_{n=1}^{\bar{N}}$ *is an asymptotic sequence as* $x \to a$.

Proof. By Property 2 of Definition 3.13, $g_n = 0_A$ for all indices $n > \bar{N}$. Thus, the order relation

$$g_{n+1}(x) = o(g_n(x)) \text{ as } x \to a \tag{3.9}$$

holds trivially for all indices $n \geq \bar{N}$, since the zero function is "little oh" of *any* function by Definition 3.1. It follows that $\{g_n(x)\}_{n=1}^{N}$ is an asymptotic sequence as $x \to a$ if and only if order relation (3.9) holds for all indices $n < \bar{N}$, or equivalently, when $\{g_n(x)\}_{n=1}^{\bar{N}}$ is an asymptotic sequence as $x \to a$. ■

Remark 3.17 *The previous result is vacuously true when* $\bar{N} = 0$.

The next definition creates a scheme for classifying tractable sequences.

Definition 3.18 *Let* $\{g_n\}_{n=1}^{N}$ *be tractable with essential length* $\bar{N}$.

1. *If* $\bar{N} = N$, *then* $\{g_n\}_{n=1}^{N}$ *is said to be* ***nonterminating***.

2. *If* $0 < \bar{N} < N$, *then* $\{g_n\}_{n=1}^{N}$ *is said to be* ***terminating***.

3. *If* $\bar{N} = 0$, *then* $\{g_n\}_{n=1}^{N}$ *is said to be* ***trivial***.

By definition, every tractable sequence belongs to one of these three mutually exclusive categories.

Remark 3.19 *The meaning of Definition 3.18 is perhaps more clearly expressed in words:*

1. *A* nonterminating *sequence consists entirely of nonzero functions.*

2. *A* terminating *sequence consists of a finite number of nonzero functions followed by a countable number of zero functions; furthermore, the sequence contains at least one nonzero function* ($g_{\bar{N}}$) *and at least one zero function* ($g_{\bar{N}+1}$).

3. A trivial *sequence consists of nothing but zero functions.*

The next section shows that these three categories describe all the possibilities which we will encounter in our study of asymptotics.

3.2.3 Tractability Property for Analytic Functions

The goal of this section is to show that every asymptotic sequence of analytic functions is tractable. We begin by proving the following pair of lemmas.

The first lemma shows that if a *zero function* occurs in an asymptotic sequence of analytic functions, the remaining portion of the sequence must consist entirely of zero functions.

Lemma 3.20 *Assume* $\{g_n\}_{n=1}^{N} \subset C^{\omega}(A)$, *and let* $\{g_n(x)\}_{n=1}^{N}$ *be an asymptotic sequence as* $x \to a$. *If* $g_m = 0_A$ *for some index* m, *then* $g_n = 0_A$ *for all indices* $n \geq m$.

Proof. The proof is by mathematical induction on the index n. (The induction is *finite* if $N < \infty$, and *infinite* if $N = \infty$.) The base case $n = m$ is true by hypothesis. For the induction step, assume that the conclusion holds for some index n with $m \leq n < N$. By Definition 3.11, $g_{n+1}(x) = o(g_n(x))$ as $x \to a$. By the induction hypothesis, $g_n = 0_A$. Proposition 3.6 implies that $g_{n+1} = 0_A$, which completes the induction step. Consequently, the conclusion holds for all indices $n \geq m$. ■

The second lemma shows that if a *nonzero function* occurs in an asymptotic sequence of analytic functions, the preceding portion of the sequence must consist entirely of nonzero functions.

Lemma 3.21 *Assume* $\{g_n\}_{n=1}^{N} \subset C^{\omega}(A)$, *and let* $\{g_n(x)\}_{n=1}^{N}$ *be an asymptotic sequence as* $x \to a$. *If* $g_m \neq 0_A$ *for some index* m, *then* $g_n \neq 0_A$ *for all indices* $n \leq m$.

Proof. Suppose $g_{n'} = 0_A$ for some index $n' < m$. Lemma 3.20 implies that $g_m = 0_A$, which contradicts that $g_m \neq 0_A$. Consequently, $g_n \neq 0_A$ for all indices

$n \leq m$. ■

We are now ready to prove the main result of this section, which gives a *necessary condition* for a sequence of analytic functions to be an asymptotic sequence.

Theorem 3.22 *Assume* $\{g_n\}_{n=1}^N \subset C^\omega(A)$. *If* $\{g_n(x)\}_{n=1}^N$ *is an asymptotic sequence as* $x \to a$, *then* $\{g_n\}_{n=1}^N$ *is tractable.*

Proof. The proof proceeds by cases, with one case for each category of tractable sequences.

Case 1. If $\{g_n\}_{n=1}^N$ consists entirely of nonzero functions, then $\{g_n\}_{n=1}^N$ is tractable with essential length $\bar{N} = N$. In this case, $\{g_n\}_{n=1}^N$ is *nonterminating.*

Case 2. If $\{g_n\}_{n=1}^N$ consists entirely of zero functions, then $\{g_n\}_{n=1}^N$ is tractable with essential length $\bar{N} = 0$. In this case, $\{g_n\}_{n=1}^N$ is *trivial.*

Case 3. Assume that $\{g_n\}_{n=1}^N$ contains at least one nonzero function and at least one zero function. The latter assumption implies that $g_m = 0_A$ for some index m; by Lemma 3.20, $g_n = 0_A$ for all indices $n \geq m$. Define a special set of indices by

$$S = \{n \in \mathbb{N} \mid 1 \leq n \leq N \text{ and } g_n \neq 0_A\},$$

and note that S is nonempty by the former assumption; furthermore, every index $n \in S$ satisfies $n < m$. Since S is nonempty and bounded above, it has a largest element, which we denote by $\bar{N}$.

Since $g_{\bar{N}} \neq 0_A$, Lemma 3.21 implies that $g_n \neq 0_A$ for all indices $n \leq \bar{N}$. In addition, $g_n = 0_A$ for all indices $n > \bar{N}$, since otherwise there would be some index $n' > \bar{N}$ with $n' \in S$, which would contradict that $\bar{N}$ is the largest element of S. By Definition 3.13, $\{g_n\}_{n=1}^N$ is tractable with essential length $\bar{N}$. In this case, $\{g_n\}_{n=1}^N$ is *terminating.* ■

If we work our way down through all the intermediate results leading to Theorem 3.22, we find that the *Unique Continuation Property* (Proposition 2.4) lies at the foundation; this in turn rests upon the *analyticity* of functions and the *connectedness* of their domains. Analyticity and connectedness are *fundamental hypotheses* for the results of this section, and will play a fundamental role throughout the thesis.

3.2.4 Characterization for Analytic Functions

The following characterization of asymptotic sequences of analytic functions incorporates the *necessary condition* established by Theorem 3.22.

Proposition 3.23 *If $\{g_n\}_{n=1}^{N} \subset C^{\omega}(A)$ is tractable with essential length $\bar{N}$, then $\{g_n(x)\}_{n=1}^{N}$ is an asymptotic sequence as $x \to a$ if and only if the equation*

$$\lim_{x \to a} \frac{g_{n+1}(x)}{g_n(x)} = 0 \tag{3.10}$$

holds for all indices $n < \bar{N}$.

Proof. By Proposition 3.16, $\{g_n(x)\}_{n=1}^{N}$ is an asymptotic sequence as $x \to a$ if and only if the *truncated* sequence $\{g_n(x)\}_{n=1}^{\bar{N}}$ is an asymptotic sequence as $x \to a$. By Definition 3.11, the latter occurs if and only if the order relation

$$g_{n+1}(x) = o(g_n(x)) \text{ as } x \to a \tag{3.11}$$

holds for all indices $n < \bar{N}$. Since $g_n \neq 0_A$ for all indices $n \leq \bar{N}$ by Property 1 of Definition 3.13, Proposition 3.5 implies that order relation (3.11) holds for all indices $n < \bar{N}$ if and only if equation (3.10) holds for all indices $n < \bar{N}$. This completes the proof. ∎

3.2.5 Integration Theorems

The following theorem establishes that an asymptotic sequence of *analytic functions* can be integrated with respect to the independent variable to obtain a new asymptotic sequence.

Theorem 3.24 *Assume* $\{g_n\}_{n=1}^{N} \subset C^{\omega}(A)$, *and define*

$$G_n(x) = \int_a^x g_n(t)\,dt$$

for all $x \in A$ *and all indices* n. *If* $\{g_n(x)\}_{n=1}^{N}$ *is an asymptotic sequence as* $x \to a$, *then* $\{G_n(x)\}_{n=1}^{N}$ *is an asymptotic sequence as* $x \to a$.

Proof. By hypothesis and Definition 3.11, $g_{n+1}(x) = o(g_n(x))$ as $x \to a$ for all indices $n < N$. Since $\{g_n\}_{n=1}^{N} \subset C^{\omega}(A)$, Theorem 3.8 implies that $G_{n+1}(x) = o(G_n(x))$ as $x \to a$ for all indices $n < N$. By Definition 3.11, $\{G_n(x)\}_{n=1}^{N}$ is an asymptotic sequence as $x \to a$. ■

The following theorem shows that indefinite integration preserves the essential length of a tractable sequence of *continuous functions*, and hence whether the sequence is nonterminating, terminating, or trivial. Note that the hypotheses of this theorem are weaker than those of the previous theorem.

Theorem 3.25 *Assume* $\{g_n\}_{n=1}^{N} \subset C(A)$, *and define*

$$G_n(x) = \int_a^x g_n(t)\,dt$$

for all $x \in A$ *and all indices* n. *If* $\{g_n\}_{n=1}^{N}$ *is tractable with essential length* $\bar{N}$, *then* $\{G_n\}_{n=1}^{N}$ *is tractable with essential length* $\bar{N}$.

Proof. By elementary calculus, $g_n = 0_A$ if and only if $G_n = 0_A$. Since $g_n \neq 0_A$ for all indices $n \leq \bar{N}$, it follows that $G_n \neq 0_A$ for all indices $n \leq \bar{N}$. Similarly, since $g_n = 0_A$ for all indices $n > \bar{N}$, it follows that $G_n = 0_A$ for all indices $n > \bar{N}$. By Definition 3.13, $\{G_n\}_{n=1}^{N}$ is tractable with essential length $\bar{N}$. ■

We conclude that indefinite integration transforms any asymptotic sequence of analytic functions into another asymptotic sequence with the same essential length.

3.3 Asymptotic Expansions

Recall that N denotes either a positive integer or infinity. Let $\{g_n\}_{n=1}^{N} \subset \mathbb{R}^A$, let $\{\alpha_n\}_{n=1}^{N} \subset \mathbb{R}$, and let $a \in A$. Assume that $\{g_n(x)\}_{n=1}^{N}$ is an asymptotic sequence as $x \to a$. The formal series

$$\sum_{n=1}^{N} \alpha_n \, g_n(x) \tag{3.12}$$

is called an **asymptotic series to N terms as $x \to a$**. An asymptotic series may be finite ($N < \infty$) or infinite ($N = \infty$); in the infinite case, no *a priori* assumptions are made about the convergence of the series for particular values of $x \in A$.

If $\{g_n\}_{n=1}^{N}$ is a *tractable sequence* (with *essential length* $\bar{N}$), we say that the asymptotic series (3.12) is **nonterminating**, **terminating**, or **trivial**, respectively, if the sequence $\{g_n\}_{n=1}^{N}$ is nonterminating ($\bar{N} = N$), terminating ($0 < \bar{N} < N$), or trivial ($\bar{N} = 0$). We will also say that the asymptotic series (3.12) has **essential length** $\bar{N}$.

For any index n, the **n-th partial** sum of the asymptotic series (3.12) is defined by

$$s_n(x) = \sum_{m=1}^{n} \alpha_m \, g_m(x) \tag{3.13}$$

for all $x \in A$. Since every index n is by definition a positive integer, s_n is always a finite sum. Let $f : A \to \mathbb{R}$ be an *arbitrary function*. For any index n, the **n-th remainder** of the asymptotic series (3.12) with respect to f is defined by

$$r_n(x) = f(x) - s_n(x) \tag{3.14}$$

for all $x \in A$.

Remark 3.26 *For convenience, define $s_0 = 0_A$ and $r_0 = f$. With these definitions, equation (3.14) holds not only when n is an* index, *but when $n = 0$ as well.*

The next section defines a relationship between the function f and the asymptotic series (3.12) by stipulating the behavior of the remainders r_n.

3.3.1 Definition and Consequences

Definition 3.27 *We say that the asymptotic series (3.12) is an* ***asymptotic expansion of*** $f(x)$ ***to*** N ***terms as*** $x \to a$ *if the order relation*

$$r_n(x) = o(g_n(x)) \text{ as } x \to a \tag{3.15}$$

holds for all indices n. *We denote this relationship by*

$$f(x) \sim \sum_{n=1}^{N} \alpha_n\, g_n(x) \text{ as } x \to a. \tag{3.16}$$

Remark 3.28 *The following are immediate consequences of the definition:*

1. *If* $N < \infty$, *then order relation (3.15) must hold for* $n = 1, 2, \ldots, N$.
2. *If* $N = \infty$, *then order relation (3.15) must hold for all positive integers* n.

The following proposition shows that weaker conditions will suffice in both cases.

Proposition 3.29 *If* $r_m(x) = o(g_m(x))$ *as* $x \to a$ *holds for some index* m, *then* $r_n(x) = o(g_n(x))$ *as* $x \to a$ *holds for all indices* $n \leq m$.

Proof. The proof is by (reverse) finite induction on n. The base case $n = m$ is satisfied by hypothesis. For the induction step, suppose that $r_n(x) = o(g_n(x))$ as $x \to a$ holds for some index n with $1 < n \leq m$. The induction hypothesis, along with equations (3.13) and (3.14), the definition of an asymptotic sequence (Definition 3.11), and the transformation rules for order relations (Proposition 3.2), imply that

$$\begin{aligned} f(x) &= s_n(x) + r_n(x) \\ &= s_{n-1}(x) + \alpha_n\, g_n(x) + o(g_n(x)) \\ &= s_{n-1}(x) + \alpha_n \cdot o(g_{n-1}(x)) + o(o(g_{n-1}(x))) \\ &= s_{n-1}(x) + o(g_{n-1}(x)) + o(g_{n-1}(x)) \\ &= s_{n-1}(x) + o(g_{n-1}(x)) \end{aligned}$$

as $x \to a$. Thus, $r_{n-1}(x) = f(x) - s_{n-1}(x) = o(g_{n-1}(x))$ as $x \to a$, which completes the induction step. By induction on n, it follows that $r_n(x) = o(g_n(x))$ as $x \to a$ holds for all indices $n \leq m$. ■

Remark 3.30 *By Proposition 3.29, the following weaker conditions imply the stronger conditions of Definition 3.27:*

1. *If $N < \infty$, it suffices to show that $r_N(x) = o(g_N(x))$ as $x \to a$.*
2. *If $N = \infty$, it suffices to show that $r_n(x) = o(g_n(x))$ as $x \to a$ holds for all* sufficiently large *integers n (or more precisely, to show that there is a positive integer m such that $r_n(x) = o(g_n(x))$ as $x \to a$ holds for all integers $n \geq m$).*

The next proposition shows that asymptotic expansions are preserved by linear operations with the functions and coefficients.

Proposition 3.31 *Assume $f, g \in \mathbb{R}^A$ and $\{h_n\}_{n=1}^N \subset \mathbb{R}^A$. Let $\{\alpha_n\}_{n=1}^N \subset \mathbb{R}$, $\{\beta_n\}_{n=1}^N \subset \mathbb{R}$, and $c \in \mathbb{R}$. Assume $\{h_n(x)\}_{n=1}^N$ is an asymptotic sequence as $x \to a$. If f and g have asymptotic expansions*

$$f(x) \sim \sum_{n=1}^{N} \alpha_n h_n(x) \text{ as } x \to a \tag{3.17}$$

$$g(x) \sim \sum_{n=1}^{N} \beta_n h_n(x) \text{ as } x \to a, \tag{3.18}$$

then $f + g$ has an asymptotic expansion

$$(f+g)(x) \sim \sum_{n=1}^{N} (\alpha_n + \beta_n) h_n(x) \text{ as } x \to a, \tag{3.19}$$

and $c \cdot f$ has an asymptotic expansion

$$(c \cdot f)(x) \sim \sum_{n=1}^{N} (c \cdot \alpha_n) h_n(x) \text{ as } x \to a. \tag{3.20}$$

Proof. Let $r_n[f]$, $r_n[g]$, $r_n[f+g]$, and $r_n[c \cdot f]$ denote the n-th remainders of asymptotic expansions (3.17), (3.18), (3.19), and (3.20), respectively. The definition of an asymptotic expansion (Definition 3.27) and the transformation rules for order relations (Proposition 3.2) imply that

$$\begin{aligned} r_n[f+g](x) &= r_n[f](x) + r_n[g](x) \\ &= o(h_n(x)) + o(h_n(x)) \\ &= o(h_n(x)) \text{ as } x \to a \\ r_n[c \cdot f](x) &= c \cdot r_n[f](x) \\ &= c \cdot o(h_n(x)) \\ &= o(h_n(x)) \text{ as } x \to a \end{aligned}$$

for all indices n. Hence, asymptotic expansions (3.19) and (3.20) hold if (3.17) and (3.18) hold. ■

3.3.2 Consequences for Analytic Functions

The results of this section are specific to *analytic functions*. Suppose that $f \in C^\omega(A)$ and $\{g_n\}_{n=1}^N \subset C^\omega(A)$. Although it is customary to first establish that $\{g_n(x)\}_{n=1}^N$ is an asymptotic sequence as $x \to a$, and then to compute the coefficients $\{\alpha_n\}_{n=1}^N \subset \mathbb{R}$ of an asymptotic expansion of f with respect to $\{g_n\}_{n=1}^N$, it is sometimes useful to proceed in a different order. For instance, if we are given a *formal* series (not necessarily an *asymptotic* series)

$$\sum_{n=1}^{N} \alpha_n \, g_n(x), \tag{3.21}$$

we may be able to apply the proposition and corollary below to *simultaneously* establish that:

1. the sequence $\{g_n(x)\}_{n=1}^N$ is an asymptotic sequence as $x \to a$, and
2. the series (3.21) is an asymptotic expansion of $f(x)$ as $x \to a$.

Proposition 3.32 *Assume $f \in C^\omega(A)$, $\{g_n\}_{n=1}^N \subset C^\omega(A)$, and $\{\alpha_n\}_{n=1}^N \subset \mathbb{R}$, and let $a \in A$. Assume that $\{g_n\}_{n=1}^N$ is* nonterminating, *and suppose that the remainders of the formal series (3.21) with respect to $f(x)$ satisfy*

$$r_n(x) = o(g_n(x)) \text{ as } x \to a \tag{3.22}$$

for all indices n. If $\alpha_m \neq 0$ for some index $m > 1$, then

$$g_m(x) = o(g_{m-1}(x)) \text{ as } x \to a. \tag{3.23}$$

Proof. Order relation (3.22) with index $n = m$ implies

$$\lim_{x \to a} \frac{r_m(x)}{g_m(x)} = 0, \tag{3.24}$$

by Proposition 3.5. Since

$$\frac{r_m(x)}{g_m(x)} = \frac{r_{m-1}(x) - \alpha_m\, g_m(x)}{g_m(x)} = \frac{r_{m-1}(x)}{g_m(x)} - \alpha_m,$$

limit (3.24) can be rewritten as

$$\lim_{x \to a} \frac{r_{m-1}(x)}{g_m(x)} = \alpha_m, \tag{3.25}$$

and since $\alpha_m \neq 0$ by hypothesis, (3.25) implies

$$\lim_{x \to a} \frac{g_m(x)}{r_{m-1}(x)} = \frac{1}{\alpha_m}. \tag{3.26}$$

Order relation (3.22) with index $n = m - 1$ implies

$$\lim_{x \to a} \frac{r_{m-1}(x)}{g_{m-1}(x)} = 0. \tag{3.27}$$

Multiplying limits (3.27) and (3.26) together yields

$$\lim_{x \to a} \frac{g_m(x)}{g_{m-1}(x)} = \lim_{x \to a} \frac{r_{m-1}(x)}{g_{m-1}(x)} \cdot \lim_{x \to a} \frac{g_m(x)}{r_{m-1}(x)} = 0 \cdot \frac{1}{\alpha_m} = 0,$$

which proves order relation (3.23), by Proposition 3.5. ∎

Corollary 3.33 *Assume the hypotheses of Proposition 3.32. If* $\alpha_m \neq 0$ *for all indices* $m > 1$, *then* $\{g_n(x)\}_{n=1}^N$ *is an asymptotic sequence as* $x \to a$, *and the series (3.21) is an asymptotic expansion of* $f(x)$ *as* $x \to a$.

3.3.3 Existence and Uniqueness

This section explores the questions of existence and uniqueness of asymptotic expansions for *analytic functions*. In the course of exploring these questions, we will develop a method for computing the coefficients of an asymptotic expansion.

Assume $f \in C^\omega(A)$ and $\{g_n\}_{n=1}^N \subset C^\omega(A)$, and let $\{g_n(x)\}_{n=1}^N$ be an asymptotic sequence as $x \to a$.

Remark 3.34 *Since* $C^\omega(A)$ *is a linear space, the partial sums of an asymptotic series in* $\{g_n\}_{n=1}^N$ *must satisfy* $\{s_n\}_{n=1}^N \subset C^\omega(A)$, *and the remainders of the asymptotic series with respect to* f *must satisfy* $\{r_n\}_{n=1}^N \subset C^\omega(A)$.

Since asymptotic expansions have a very different character when $\{g_n\}_{n=1}^N$ is terminating or trivial, we will defer our study of these two cases until the end of the chapter. In this section, assume that $\{g_n\}_{n=1}^N$ is *nonterminating* (i.e., that $g_n \neq 0_A$ for all indices n).

The following theorem gives a necessary and sufficient condition for f to have a *nonterminating* asymptotic expansion

$$f(x) \sim \sum_{n=1}^{N} \alpha_n \, g_n(x) \text{ as } x \to a \tag{3.28}$$

with respect to $\{g_n\}_{n=1}^N$. In addition to answering the question of *existence*, the theorem gives a *recursive formula* for computing the coefficients $\{\alpha_n\}_{n=1}^N$. The recursion enters into the formula for α_n via the remainder r_{n-1}, which depends on the first $n-1$ coefficients $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$. The base case $n = 1$ is well-defined since $r_0 = f$, which does not depend on *any* of the coefficients.

Theorem 3.35 *Assume $f \in C^\omega(A)$ and $\{g_n\}_{n=1}^N \subset C^\omega(A)$, and let $\{g_n(x)\}_{n=1}^N$ be a* nonterminating *asymptotic sequence as $x \to a$. The function f has a* nonterminating *asymptotic expansion (3.28) with respect to $\{g_n\}_{n=1}^N$ if and only if the equation*

$$\alpha_n = \lim_{x \to a} \frac{r_{n-1}(x)}{g_n(x)} \tag{3.29}$$

holds for all indices n.

Proof. For each index n,

$$\begin{aligned} r_n(x) &= f(x) - s_n(x) \\ &= f(x) - (s_{n-1}(x) + \alpha_n\, g_n(x)) \\ &= r_{n-1}(x) - \alpha_n\, g_n(x). \end{aligned} \tag{3.30}$$

These equations hold even when $n = 1$, since we defined $s_0 = 0_A$ and $r_0 = f$. Since $g_n \in C^\omega(A)$ and $g_n \neq 0_A$, there is a deleted neighborhood $I_n - \{a\}$ on which g_n is never zero (Lemma 3.4). Dividing equation (3.30) by $g_n(x)$ yields

$$\frac{r_n(x)}{g_n(x)} = \frac{r_{n-1}(x)}{g_n(x)} - \alpha_n \tag{3.31}$$

for all $x \in I_n - \{a\}$. As $x \to a$ in $I_n - \{a\}$, the left-hand side of equation (3.31) has a limit if and only if the right-hand side has a limit; furthermore, the equation

$$\lim_{x \to a} \frac{r_n(x)}{g_n(x)} = 0 \tag{3.32}$$

holds if and only if equation (3.29) holds. Since $r_n \in C^\omega(A)$, equation (3.32) holds if and only if

$$r_n(x) = o(g_n(x)) \text{ as } x \to a, \tag{3.33}$$

by Proposition 3.5.

For each index n, we have shown that equation (3.33) holds if and only if equation (3.29) holds. It follows by this equivalence and Definition 3.27 that f has an asymptotic expansion (3.28) with respect to $\{g_n\}_{n=1}^N$ if and only if equation (3.29) holds for all indices n. ■

The next result is a consequence of the existence theorem, and lays the groundwork for uniqueness.

Proposition 3.36 *Assume $\{g_n\}_{n=1}^N \subset C^\omega(A)$, and let $\{g_n(x)\}_{n=1}^N$ be a* nonterminating *asymptotic sequence as $x \to a$. The zero function $f = 0_A$ has an asymptotic expansion*

$$f(x) \sim \sum_{n=1}^{N} \alpha_n \, g_n(x) \text{ as } x \to a \tag{3.34}$$

with respect to $\{g_n\}_{n=1}^N$ if and only if $\alpha_n = 0$ for all indices n.

Proof. Clearly, if $\alpha_n = 0$ for all indices n, then the partial sums $s_n = 0_A$ for all indices n, and the remainders $r_n = f - s_n = 0_A$ for all indices n. Consequently, the order relation $r_n(x) = o(g_n(x))$ as $x \to a$ is trivially satisfied for all indices n, which proves that asymptotic expansion (3.34) holds.

Conversely, suppose that asymptotic expansion (3.34) holds. We will show by induction on n that $\alpha_n = 0$ for all indices n. By Theorem 3.35, we can compute the coefficients via the recursive formula

$$\alpha_n = \lim_{x \to a} \frac{r_{n-1}(x)}{g_n(x)}. \tag{3.35}$$

Since $r_0 = f = 0_A$, equation (3.35) implies

$$\alpha_1 = \lim_{x \to a} \frac{r_0(x)}{g_1(x)} = 0.$$

If $\alpha_m = 0$ for all indices $m \leq n$, then $r_n = 0_A$, and equation (3.35) implies

$$\alpha_{n+1} = \lim_{x \to a} \frac{r_n(x)}{g_{n+1}(x)} = 0.$$

By induction, $\alpha_n = 0$ for all indices n. ■

The following *uniqueness* theorem is an immediate consequence of Proposition 3.36.

Theorem 3.37 *Assume $f \in C^\omega(A)$ and $\{g_n\}_{n=1}^N \subset C^\omega(A)$, and let $\{g_n(x)\}_{n=1}^N$ be a* nonterminating *asymptotic sequence as $x \to a$. If f has a* nonterminating *asymptotic expansion with respect to $\{g_n\}_{n=1}^N$, then the expansion is unique.*

Proof. Suppose that f has two asymptotic expansions

$$\begin{aligned} f(x) &\sim \sum_{n=1}^{N} \alpha_n \, g_n(x) \text{ as } x \to a \\ f(x) &\sim \sum_{n=1}^{N} \beta_n \, g_n(x) \text{ as } x \to a \end{aligned}$$

with respect to $\{g_n\}_{n=1}^N$. By Proposition 3.31, the difference of these two expansions yields

$$0_A \sim \sum_{n=1}^{N} (\alpha_n - \beta_n) \, g_n(x) \text{ as } x \to a.$$

Proposition 3.36 implies that $\alpha_n = \beta_n$ for all indices n. ■

Another consequence of Proposition 3.36 is that every *nonterminating* asymptotic sequence of analytic functions must be *linearly independent*.

Proposition 3.38 *Assume $\{g_n\}_{n=1}^N \subset C^\omega(A)$. If $\{g_n(x)\}_{n=1}^N$ is a* nonterminating *asymptotic sequence as $x \to a$, then the functions $\{g_n\}_{n=1}^N$ are* linearly independent *on any nonempty open subinterval $I \subset A$.*

Proof. The case $N = \infty$ can be reduced to the case $N < \infty$, as follows. If $N = \infty$, the *infinite sequence* of functions $\{g_n\}_{n=1}^{N}$ is linearly independent provided that every nonempty finite subset $S \subset \{g_n\}_{n=1}^{N}$ is linearly independent. Given S, let m be the largest index of the elements of S. Since $S \subset \{g_n\}_{n=1}^{m}$, it suffices to show that the *finite subsequence* $\{g_n\}_{n=1}^{m}$ is linearly independent; the subset S must also be linearly independent in order to avoid contradicting the linear independence of the superset $\{g_n\}_{n=1}^{m}$.

Assume without loss of generality that $N < \infty$. Let $\{\alpha_n\}_{n=1}^{N} \subset \mathbb{R}$, and define

$$f(x) = \sum_{n=1}^{N} \alpha_n \, g_n(x) \tag{3.36}$$

for all $x \in A$. By definition, the functions $\{g_n\}_{n=1}^{N}$ are linearly independent on I if the equation $f \,|\, I = 0_I$ implies that $\alpha_n = 0$ for all indices n.

Suppose that $f \,|\, I = 0_I$. Since $f \in C^{\omega}(A)$, the *Unique Continuation Property* (Proposition 2.4) implies that $f = 0_A$. We want to show that the asymptotic expansion

$$f(x) \sim \sum_{n=1}^{N} \alpha_n \, g_n(x) \text{ as } x \to a \tag{3.37}$$

holds. Since $N < \infty$, it suffices to show that

$$r_N(x) = o(g_N(x)) \text{ as } x \to a, \tag{3.38}$$

by Remark 3.30. Equation (3.36) means that $f = S_N$, which implies that $r_N = 0_A$. Order relation (3.38) is trivially satisfied, and asymptotic expansion (3.37) holds. Since $f = 0_A$, Proposition 3.36 implies that $\alpha_n = 0$ for all indices n. Consequently, the functions $\{g_n\}_{n=1}^{N}$ are linearly independent on I. ■

3.3.4 Integration Theorem

Recall that integrating an *asymptotic sequence* of analytic functions with respect to the independent variable produces another asymptotic sequence with the same essential length. The next theorem shows that an *asymptotic expansion* consisting of analytic functions can be integrated with respect to the independent variable to obtain another asymptotic expansion with the same essential length.

Theorem 3.39 *Assume $f \in C^{\omega}(A)$ and $\{g_n\}_{n=1}^{N} \subset C^{\omega}(A)$. Let $\{g_n(x)\}_{n=1}^{N}$ be an asymptotic sequence as $x \to a$, and let $\bar{N}$ denote its essential length. For all $x \in A$ and all indices n, define*

$$F(x) = \int_a^x f(t)\,dt, \quad G_n(x) = \int_a^x g_n(t)\,dt.$$

If f has an asymptotic expansion

$$f(x) \sim \sum_{n=1}^{N} \alpha_n\, g_n(x) \text{ as } x \to a \tag{3.39}$$

with essential length $\bar{N}$ with respect to $\{g_n\}_{n=1}^{N}$, then F has an asymptotic expansion

$$F(x) \sim \sum_{n=1}^{N} \alpha_n\, G_n(x) \text{ as } x \to a \tag{3.40}$$

with essential length $\bar{N}$ with respect to $\{G_n\}_{n=1}^{N}$.

Proof. By Theorems 3.24 and 3.25, $\{G_n(x)\}_{n=1}^{N}$ is an asymptotic sequence as $x \to a$ and has essential length $\bar{N}$. Let r_n denote the n-th remainder of asymptotic expansion (3.39), and let R_n denote the n-th remainder of asymptotic expansion (3.40). Note that

$$\begin{aligned}
R_n(x) &= F(x) - \sum_{m=1}^{n} \alpha_m\, G_m(x) \\
&= \int_a^x f(t)\,dt - \sum_{m=1}^{n} \alpha_m \int_a^x g_m(t)\,dt \\
&= \int_a^x \left(f(t) - \sum_{m=1}^{n} \alpha_m\, g_m(t)\right) dt \\
&= \int_a^x r_n(t)\,dt.
\end{aligned}$$

By Definition 3.27, asymptotic expansion (3.39) means that

$$r_n(x) = o(g_n(x)) \text{ as } x \to a$$

for all indices n. Since $\{r_n\}_{n=1}^{N} \subset C^{\omega}(A)$, Theorem 3.8 implies that

$$R_n(x) = o(G_n(x)) \text{ as } x \to a$$

for all indices n, which proves that asymptotic expansion (3.40) holds. ■

3.3.5 Exact Identity Theorem

This section discusses asymptotic expansions in the *terminating* and *trivial* cases for *analytic functions*. The following theorem shows that a terminating or trivial asymptotic expansion as $x \to a$ is equivalent to an *exact identity* for all $x \in A$.

Theorem 3.40 *Assume $f \in C^{\omega}(A)$ and $\{g_n\}_{n=1}^{N} \subset C^{\omega}(A)$, and let $\{g_n(x)\}_{n=1}^{N}$ be an asymptotic sequence as $x \to a$. If $\{g_n\}_{n=1}^{N}$ is* terminating *or* trivial *with essential length $\bar{N}$, then f has an asymptotic expansion*

$$f(x) \sim \sum_{n=1}^{N} \alpha_n \, g_n(x) \text{ as } x \to a \tag{3.41}$$

with respect to $\{g_n\}_{n=1}^{N}$ if and only if

$$f(x) = \sum_{n=1}^{\bar{N}} \alpha_n \, g_n(x) \tag{3.42}$$

for all $x \in A$.

Proof. Asymptotic expansion (3.41) holds if and only if

$$r_n(x) = o(g_n(x)) \text{ as } x \to a \tag{3.43}$$

for all indices n, by Definition 3.27. Whether $\{g_n\}_{n=1}^{N}$ is terminating ($0 < \bar{N} < N$) or trivial ($\bar{N} = 0$), it follows that $\bar{N} + 1$ is an *index*. If order relation (3.43) holds for the index $n = \bar{N} + 1$, then (3.43) holds for all indices $n \leq \bar{N} + 1$, by Proposition 3.29. It is therefore necessary and sufficient to show that (3.43) holds for all indices $n \geq \bar{N} + 1$, or equivalently, for all indices $n > \bar{N}$.

By the definition of essential length, $g_n = 0_A$ for all indices $n > \bar{N}$. Consequently, the partial sums of asymptotic series (3.41) do not change after $n = \bar{N}$, and $s_n = s_{\bar{N}}$ for all indices $n > \bar{N}$. Furthermore,

$$r_n = f - s_n = f - s_{\bar{N}} = r_{\bar{N}}$$

for all indices $n > \bar{N}$. Showing that (3.43) holds for all indices $n > \bar{N}$ therefore reduces to showing that *one* order relation holds:

$$r_{\bar{N}}(x) = o(0_A) \text{ as } x \to a. \tag{3.44}$$

We have shown that asymptotic expansion (3.41) holds if and only if order relation (3.44) holds. Since $r_{\bar{N}} \in C^{\omega}(A)$, Proposition 3.6 implies that (3.44) holds if and only if $r_{\bar{N}} = 0_A$. However, $r_{\bar{N}} = 0_A$ means that $f = s_{\bar{N}}$, or equivalently, that equation (3.42) holds for all $x \in A$. This completes the proof. ■

Remark 3.41 *Consider the following consequences of the previous theorem:*

1. *Theorem 3.40 establishes an* equivalence *between the asymptotic expansion (3.41) and the exact identity (3.42). Yet, the equivalence appears to be asymmetrical since (3.41) involves* all *the coefficients* $\{\alpha_n\}_{n=1}^{N}$, *whereas (3.42) involves only a* proper subset *of the coefficients, namely* $\{\alpha_n\}_{n=1}^{\bar{N}}$. *We can restore symmetry by noting that for all indices* $n > \bar{N}$, *the value of* α_n *in (3.41) is inconsequential since* $g_n = 0_A$. *Hence, only the coefficients* $\{\alpha_n\}_{n=1}^{\bar{N}}$ *enter into (3.41) in a significant way.*

2. *In the trivial case* ($\bar{N} = 0$), *Theorem 3.40 implies that* $f \in C^{\omega}(A)$ *has an asymptotic expansion with respect to a* trivial *asymptotic sequence if and only if* $f = s_{\bar{N}} = 0_A$. *In other words, the zero function is the one and only analytic function which has a* trivial *asymptotic expansion.*

The notion of exact identities is more interesting when the function f depends on a parameter. We will examine this generalization in the next chapter.

Chapter 4

Asymptotic Expansions with a Parameter

This chapter extends the previous chapter's results on asymptotic expansions in one variable to asymptotic expansions in one variable and one parameter. This chapter also serves as a natural intermediate step between asymptotic expansions in one variable and dual asymptotic expansions, which are discussed in the next chapter.

Although much of this chapter is inspired by standard material, many of the results presented here draw conclusions stronger than those of traditional results. This is achieved by using the full strength of analyticity for bivariate and univariate functions. In particular, *Division Theorem* 2.17 plays a central role, and gives many of the results of this chapter a distinctly different flavor than their univariate counterparts in the previous chapter.

4.1 Uniform Order Relations

This section extends many of the results of Section 3.1 to order relations which depend on a parameter.

Recall that $B \subset \mathbb{R}$ denotes an open interval, and let $[b_1, b_2] \subset B$ be any closed subinterval. If $f : A \times B \to \mathbb{R}$, we can think of $f(x, y)$ as a function of x, and treat

y as a parameter in the parameter space $[b_1, b_2]$. Let $g : A \to \mathbb{R}$, and let $a \in A$.

Suppose that for every $y \in [b_1, b_2]$, the order relation $f(x, y) = o(g(x))$ holds as $x \to a$. For every $\varepsilon > 0$, Definition 3.1 guarantees the existence of a suitable neighborhood I_ε of a, but this I_ε *may be different for different values of* y. If for every $\varepsilon > 0$, we can find a *single* I_ε *which suffices for all values of* y, then the order relation $f(x, y) = o(g(x))$ as $x \to a$ is *uniform in the parameter* y.

The next section makes this idea more precise.

4.1.1 Definition and Consequences

Definition 4.1 *Assume* $f : A \times B \to \mathbb{R}$ *and* $g : A \to \mathbb{R}$ *are* arbitrary functions. *Let* $a \in A$, *and let* $[b_1, b_2] \subset B$ *be any closed subinterval. The order relation* $f(x, y) = o(g(x))$ *as* $x \to a$ *holds* ***uniformly in*** y ***for*** $y \in [b_1, b_2]$ *if for every* $\varepsilon > 0$, *there is a neighborhood* I_ε *of* a *such that the inequality*

$$|f(x, y)| \leq \varepsilon \; |g(x)| \tag{4.1}$$

holds for all $(x, y) \in I_\varepsilon \times [b_1, b_2]$.

The following proposition states the properties of uniform order relations under addition, scalar multiplication, and composition, and is patterned after Proposition 3.2. The phrases "as $x \to a$" and "holds uniformly in y for $y \in [b_1, b_2]$" will be omitted for brevity, but should be assumed wherever the variables x and y occur, respectively.

Proposition 4.2 *Let* f, f_1, $f_2 : A \times B \to \mathbb{R}$ *and* $g, h : A \to \mathbb{R}$ *be arbitrary functions, and let* $c \in \mathbb{R}$.

1. *If* $f_1(x, y) = o(h(x))$ *and* $f_2(x, y) = o(h(x))$, *then* $f_1(x, y) + f_2(x, y) = o(h(x))$.
2. *If* $f(x, y) = o(h(x))$, *then* $c \cdot f(x, y) = o(h(x))$.
3. *If* $f(x, y) = o(g(x))$ *and* $g(x) = o(h(x))$, *then* $f(x, y) = o(h(x))$.

The proof of Proposition 4.2 is nearly identical to the proof of Proposition 3.2, and is therefore omitted.

Remark 4.3 *The short forms of these three properties remain completely unchanged by the introduction of a parameter; the transformation rules are still denoted by:*

1. $o(h(x)) + o(h(x)) = o(h(x))$

2. $c \cdot o(h(x)) = o(h(x))$

3. $o(o(h(x))) = o(h(x))$.

Although the *notation* is the exactly same, the *interpretation* is slightly different than before. If all the order relations on the left-hand side hold uniformly in y for $y \in [b_1, b_2]$, then the resulting order relation on the right-hand side *also* holds uniformly in y for $y \in [b_1, b_2]$. In short, the transformation rules *preserve the uniformity* of order relations.

4.1.2 Implications for Analytic Functions

Let $f : A \times B \longrightarrow \mathbb{R}$ and $g : A \longrightarrow \mathbb{R}$ be *arbitrary functions*, let $a \in A$, and let $[b_1, b_2] \subset B$ be any closed subinterval. If the order relation

$$f(x, y) = o(g(x)) \text{ as } x \longrightarrow a \tag{4.2}$$

holds *uniformly in* y for $y \in [b_1, b_2]$, it follows trivially from Definition 4.1 that the order relation also holds in the univariate sense for each fixed $y \in [b_1, b_2]$. The main theorem of this section shows that for *analytic functions*, the converse is also true — if order relation (4.2) holds in the univariate sense, then the order relation automatically holds *uniformly in the parameter* over any *compact interval*.

The following two propositions, which apply to *analytic functions*, give necessary and sufficient conditions for an order relation with a parameter to hold in the univariate sense.

Proposition 4.4 *Assume $f \in C^{\omega}(A \times B)$ and $g \in C^{\omega}(A)$, with $g \neq 0_A$. If $S \subset B$ is any subset, then $f(x,y) = o(g(x))$ as $x \to a$ for all $y \in S$ if and only if*

$$\lim_{x \to a} \frac{f(x,y)}{g(x)} = 0$$

for all $y \in S$.

Proof. Simply hold each $y \in S$ fixed and apply Proposition 3.5. ■

Proposition 4.5 *Assume $f \in C^{\omega}(A \times B)$ and $g = 0_A$. If $J \subset B$ is an open interval, then $f(x,y) = o(g(x))$ as $x \to a$ for all $y \in J$ if and only if $f = 0_{A \times B}$.*

Proof. For each fixed $y \in J$, define $F_y(x) = f(x,y)$ for all $x \in A$, and note that $F_y \in C^{\omega}(A)$. If $F_y(x) = o(g(x))$ as $x \to a$ for all $y \in J$, then $F_y = 0_A$ for all $y \in J$, by Proposition 3.6. This implies that $f \mid A \times J = 0_{A \times J}$. By the *Unique Continuation Property* (Proposition 2.4), $f = 0_{A \times B}$. Conversely, if $f = 0_{A \times B}$, then $F_y = 0_A$ for all $y \in J$, and Proposition 3.6 implies that $F_y(x) = o(g(x))$ as $x \to a$ for all $y \in J$. ■

We are now ready to prove the following *Uniform Order Relation Theorem*, which builds on the foundation laid by *Division Theorem* 2.17.

Theorem 4.6 *Assume $f \in C^{\omega}(A \times B)$ and $g \in C^{\omega}(A)$. Let $a \in A$, let $J \subset B$ be any open subinterval, and let $[b_1, b_2] \subset B$ be any closed subinterval. If the order relation*

$$f(x,y) = o(g(x)) \text{ as } x \to a \tag{4.3}$$

holds for all $y \in J$, we can conclude that:

1. *the order relation holds* for all $y \in B$
2. *the order relation holds* uniformly in y *for* $y \in [b_1, b_2]$.

Proof. The proof proceeds by cases on g.

Case 1. Suppose that $g \neq 0_A$. By Proposition 4.4 with $S = J$, order relation (4.3) implies that

$$\lim_{x \to a} \frac{f(x,y)}{g(x)} = 0 \tag{4.4}$$

for all $y \in J$. By *Division Theorem* 2.17, the limit

$$h(y) = \lim_{x \to a} \frac{f(x,y)}{g(x)}$$

exists for all $y \in B$, and the resulting function $h \in C^{\omega}(B)$. Since $h \mid J = 0_J$, the *Unique Continuation Property* (Proposition 2.4) implies that $h = 0_B$, or equivalently, that equation (4.4) holds for all $y \in B$. By Proposition 4.4 with $S = B$, order relation (4.3) holds for all $y \in B$, which proves conclusion 1.

Division Theorem 2.17 also implies that the limit in (4.4) is uniform in y for $y \in [b_1, b_2]$. By Definition 2.16, for every $\varepsilon > 0$, there is a deleted neighborhood $I_\varepsilon - \{a\}$ such that the inequality

$$|f(x,y)| < \varepsilon \; |g(x)| \tag{4.5}$$

holds for all $(x,y) \in (I_\varepsilon - \{a\}) \times [b_1, b_2]$. Taking the limit of (4.5) as $x \to a$ in $I_\varepsilon - \{a\}$ yields

$$|f(a,y)| \leq \varepsilon \; |g(a)| \tag{4.6}$$

for all $y \in [b_1, b_2]$, by the continuity of f and g. Inequalities (4.5) and (4.6) together imply that

$$|f(x,y)| \leq \varepsilon \; |g(x)|$$

for all $(x,y) \in I_\varepsilon \times [b_1, b_2]$. By Definition 4.1, order relation (4.3) holds uniformly in y for $y \in [b_1, b_2]$, which proves conclusion 2.

Case 2. Suppose that $g = 0_A$. By Proposition 4.5, order relation (4.3) implies that $f = 0_{A \times B}$. By Proposition 4.5 with J replaced by B, the hypothesis $f = 0_{A \times B}$ implies that order relation (4.3) holds for all $y \in B$, which proves conclusion 1. Since $f = 0_{A \times B}$, Definition 4.1 is trivially satisfied by setting $I_\varepsilon = A$ for every

$\varepsilon > 0$. Consequently, order relation (4.3) holds uniformly in y for $y \in [b_1, b_2]$, which proves conclusion 2. ■

Thus, when we are working with *analytic functions*, we never have to postulate the uniformity of an order relation as the parameter varies over a compact interval — uniformity is an automatic consequence of analyticity.

4.1.3 Integration Theorem

The following standard result establishes that an order relation which holds uniformly in a parameter can be integrated with respect to the parameter. Note that *analyticity* is not needed here — *continuity* is sufficient to establish the result.

Theorem 4.7 *Assume $f \in C(A \times B)$, and assume $g : A \to \mathbb{R}$ is an arbitrary function. Let $a \in A$, and let $[b_1, b_2] \subset B$ be any closed interval. If $f(x, y) = o(g(x))$ as $x \to a$ holds uniformly in y for $y \in [b_1, b_2]$, then*

$$\int_{b_1}^{b_2} f(x, y)\, dy = o(g(x)) \text{ as } x \to a. \tag{4.7}$$

Proof. Let $\varepsilon > 0$ be given, and let $\bar{\varepsilon} = \varepsilon/(b_2 - b_1)$. By Definition 4.1, there is a neighborhood I of a such that $|f(x, y)| \leq \bar{\varepsilon}\,|g(x)|$ for all $(x, y) \in I \times [b_1, b_2]$. For all $x \in I$,

$$\begin{aligned}
\left|\int_{b_1}^{b_2} f(x, y)\, dy\right| &\leq \int_{b_1}^{b_2} |f(x, y)|\, dy \\
&\leq \int_{b_1}^{b_2} \bar{\varepsilon}\,|g(x)|\, dy \\
&= (b_2 - b_1)\,\bar{\varepsilon}\,|g(x)| \\
&= \varepsilon\,|g(x)|,
\end{aligned}$$

which establishes that order relation (4.7) holds. ■

Notice that this theorem uses the technique of Theorem 3.7, but avoids the defect. Why does g rather than $|g|$ appear in order relation (4.7)? This occurs because the original order relation is integrated with respect to the *parameter*, and g is *independent* of the parameter.

Applying *Uniform Order Relation Theorem* 4.6 to Theorem 4.7 yields the following corollary for *analytic functions*.

Corollary 4.8 *Assume $f \in C^\omega(A \times B)$ and $g \in C^\omega(A)$. Let $a \in A$, and let $[b_1, b_2] \subset B$ be any closed subinterval. If $f(x, y) = o(g(x))$ as $x \to a$ for all $y \in (b_1, b_2)$, then*

$$\int_{b_1}^{b_2} f(x, y)\, dy = o(g(x)) \text{ as } x \to a.$$

4.2 Uniform Asymptotic Expansions

This section extends many of the results of Section 3.3 to asymptotic expansions which depend on a parameter.

Recall that N denotes either a positive integer or infinity. Let $\{g_n\}_{n=1}^N \subset \mathbb{R}^A$, let $\{\alpha_n\}_{n=1}^N \subset \mathbb{R}$, and let $a \in A$. Assume that $\{g_n(x)\}_{n=1}^N$ is an asymptotic sequence as $x \to a$. Let $f : A \times B \to \mathbb{R}$ be an *arbitrary function*, and let $S \subset B$ be an arbitrary subset. If for every fixed $y \in S$, the function $f(x, y)$ has an asymptotic expansion as $x \to a$, then each coefficient α_n of the asymptotic expansion will depend on y. Thus, the asymptotic series for $f(x, y)$ will have the form

$$\sum_{n=1}^{N} \alpha_n(y)\, g_n(x). \tag{4.8}$$

Consequently, each partial sum s_n of the asymptotic series (4.8) will depend on y, and be given by

$$s_n(x, y) = \sum_{m=1}^{n} \alpha_m(y)\, g_m(x) \tag{4.9}$$

for all $(x, y) \in A \times S$. Similarly, each remainder r_n of the asymptotic series (4.8) with respect to f will also depend on y, and be given by

$$r_n(x, y) = f(x, y) - s_n(x, y) \tag{4.10}$$

for all $(x, y) \in A \times S$.

Remark 4.9 *For convenience, define $s_0 = 0_{A \times B}$ and $r_0 = f$. With these definitions, equation (4.10) holds for $n = 0$ as well as for all* indices *n.*

The next section explains what it means for an asymptotic expansion to be *uniform in the parameter y.*

4.2.1 Definition and Consequences

Definition 4.10 *Let $[b_1, b_2] \subset B$ be any closed subinterval, and assume that $f(x, y)$ has an asymptotic expansion*

$$f(x, y) \sim \sum_{n=1}^{N} \alpha_n(y)\, g_n(x) \text{ as } x \to a \tag{4.11}$$

for every $y \in [b_1, b_2]$. We say that the asymptotic expansion holds ***uniformly in y for*** *$y \in [b_1, b_2]$ if, for all indices n, the order relation*

$$r_n(x, y) = o(g_n(x)) \text{ as } x \to a \tag{4.12}$$

holds uniformly in y for $y \in [b_1, b_2]$.

Remark 4.11 *By Proposition 4.2, the transformation rules for order relations preserve uniformity. Consequently, we can easily extend Proposition 3.29 to show that the following weaker conditions imply the stronger conditions of Definition 4.10:*

1. *If $N < \infty$, it suffices to show that $r_N(x, y) = o(g_N(x))$ as $x \to a$ holds uniformly in y for $y \in [b_1, b_2]$.*

2. *If $N = \infty$, it suffices to show that $r_n(x, y) = o(g_n(x))$ as $x \to a$ holds uniformly in y for $y \in [b_1, b_2]$ for all* sufficiently large *integers n.*

4.2.2 Implications for Analytic Functions

The main theorem of this section enables us to draw several strong conclusions about *nonterminating* asymptotic expansions involving *analytic functions* — namely, if the expansion holds for all parameter values in some open interval, then:

1. the expansion actually holds over the entire parameter domain;
2. the resulting coefficients are analytic functions of the parameter; and,
3. the expansion holds uniformly in the parameter over any compact interval.

The following theorem provides a necessary and sufficient condition for the existence of a *nonterminating* asymptotic expansion whose parameter y ranges over an arbitrary set S. The theorem also provides a useful recursive formula for computing the coefficients $\{\alpha_n\}_{n=1}^{N}$ as functions of the parameter y.

Theorem 4.12 *Assume $f \in C^{\omega}(A \times B)$ and $\{g_n\}_{n=1}^{N} \subset C^{\omega}(A)$. Let $a \in A$, and let $S \subset B$ be an arbitrary subset. Assume $\{g_n(x)\}_{n=1}^{N}$ is a* nonterminating *asymptotic sequence as $x \longrightarrow a$. The asymptotic expansion*

$$f(x, y) \sim \sum_{n=1}^{N} \alpha_n(y)\, g_n(x) \text{ as } x \longrightarrow a$$

holds for all $y \in S$ if and only if the equation

$$\alpha_n(y) = \lim_{x \to a} \frac{r_{n-1}(x, y)}{g_n(x)}$$

holds for all indices n and all $y \in S$.

Proof. Simply hold each $y \in S$ fixed and apply Theorem 3.35. ■

The framework of results constructed on the foundation of *Division Theorem* 2.17 reaches its apex in the following *Uniform Asymptotic Expansion Theorem.*

Theorem 4.13 *Assume $f \in C^\omega(A \times B)$ and $\{g_n\}_{n=1}^N \subset C^\omega(A)$. Let $a \in A$, let $J \subset B$ be any open subinterval, and let $[b_1, b_2] \subset B$ be any closed subinterval. Assume $\{g_n(x)\}_{n=1}^N$ is a* nonterminating *asymptotic sequence as $x \to a$. If the asymptotic expansion*

$$f(x, y) \sim \sum_{n=1}^{N} \alpha_n(y)\, g_n(x) \text{ as } x \to a \tag{4.13}$$

holds for all $y \in J$, we can conclude that:

1. *the asymptotic expansion holds* for all $y \in B$
2. *the coefficient functions satisfy $\{\alpha_n\}_{n=1}^N \subset C^\omega(B)$*
3. *the asymptotic expansion holds* uniformly in y *for* $y \in [b_1, b_2]$.

Proof. First, we will show by induction on n that $\alpha_n \in C^\omega(B)$ for all indices n. By Theorem 4.12 with $S = J$, the limit

$$\alpha_1(y) = \lim_{x \to a} \frac{r_0(x, y)}{g_1(x)} = \lim_{x \to a} \frac{f(x, y)}{g_1(x)}$$

exists for all $y \in J$. *Division Theorem* 2.17 implies that the limit exists for all $y \in B$, and further implies that $\alpha_1 \in C^\omega(B)$. Let $n < N$ be an index, and suppose that $\alpha_m \in C^\omega(B)$ for all indices $m \leq n$. From these assumptions and the equation

$$r_n(x, y) = f(x, y) - \sum_{m=1}^{n} \alpha_m(y)\, g_m(x),$$

it follows that $r_n \in C^\omega(A \times B)$. By Theorem 4.12 with $S = J$, the limit

$$\alpha_{n+1}(y) = \lim_{x \to a} \frac{r_n(x, y)}{g_{n+1}(x)}$$

exists for all $y \in J$. *Division Theorem* 2.17 implies that the limit exists for all $y \in B$, and further implies that $\alpha_{n+1} \in C^\omega(B)$. It follows by induction on n

that $\alpha_n \in C^\omega(B)$ for all indices n. This proves conclusion 2, and also shows that $r_n \in C^\omega(A \times B)$ for all indices n.

To say that asymptotic expansion (4.13) holds for all $y \in J$ means that for each index n, the order relation

$$r_n(x, y) = o(g_n(x)) \text{ as } x \to a \tag{4.14}$$

holds for all $y \in J$. For each index n, *Uniform Order Relation Theorem* 4.6 implies two things: that order relation (4.14) holds for all $y \in B$, and that it holds uniformly in y for $y \in [b_1, b_2]$. Thus, asymptotic expansion (4.13) holds for all $y \in B$, and it holds uniformly in y for $y \in [b_1, b_2]$. This proves conclusions 1 and 3. ■

When we are working with *analytic functions*, we never have to postulate the uniformity of an asymptotic expansion as the parameter varies over a compact interval — uniformity is an automatic consequence of analyticity, just as with order relations. Furthermore, we need not postulate the properties of the coefficients in the asymptotic expansion — their analyticity is an automatic consequence of the analyticity of the other functions.

4.2.3 Integration Theorem

The following standard result shows that an asymptotic expansion which holds *uniformly in a parameter* can be integrated with respect to the parameter to obtain another asymptotic expansion. Note that *analyticity* is not needed here — *continuity* is sufficient to establish the result.

Theorem 4.14 *Assume $f \in C(A \times B)$ and $\{g_n\}_{n=1}^N \subset C(A)$. Let $a \in A$, and let $[b_1, b_2] \subset B$ be any closed subinterval. Assume $\{g_n(x)\}_{n=1}^N$ is an asymptotic sequence as $x \to a$. If the asymptotic expansion*

$$f(x, y) \sim \sum_{n=1}^{N} \alpha_n(y)\, g_n(x) \text{ as } x \to a \tag{4.15}$$

holds uniformly in y for $y \in [b_1, b_2]$, and if $\{\alpha_n\}_{n=1}^N \subset C(B)$, then the parameter-free asymptotic expansion

$$\int_{b_1}^{b_2} f(x,y)\,dy \sim \sum_{n=1}^{N} \left(\int_{b_1}^{b_2} \alpha_n(y)\,dy \right) g_n(x) \text{ as } x \to a \tag{4.16}$$

also holds.

Proof. Let r_n denote the n-th remainder of asymptotic expansion (4.15), and let R_n denote the n-th remainder of asymptotic expansion (4.16). Note that

$$\begin{aligned} R_n(x) &= \int_{b_1}^{b_2} f(x,y)\,dy - \sum_{m=1}^{n} \left(\int_{b_1}^{b_2} \alpha_m(y)\,dy \right) g_m(x) \\ &= \int_{b_1}^{b_2} \left(f(x,y) - \sum_{m=1}^{n} \alpha_m(y)\, g_m(x) \right) dy \\ &= \int_{b_1}^{b_2} r_n(x,y)\,dy. \end{aligned}$$

By Definition 4.10, the uniformity of asymptotic expansion (4.15) means that for all indices n,

$$r_n(x,y) = o(g_n(x)) \text{ as } x \to a$$

holds uniformly in y for $y \in [b_1, b_2]$. Since

$$r_n(x,y) = f(x,y) - \sum_{m=1}^{n} \alpha_m(y)\, g_m(x),$$

it follows from the hypotheses that $r_n \in C(A \times B)$. Theorem 4.7 therefore implies that

$$R_n(x) = \int_{b_1}^{b_2} r_n(x,y)\,dy = o(g_n(x)) \text{ as } x \to a$$

for all indices n, which proves that asymptotic expansion (4.16) holds. ■

Applying *Uniform Asymptotic Expansion Theorem* 4.13 to Theorem 4.14 yields the following corollary for *analytic functions* and *nonterminating* asymptotic expansions.

Corollary 4.15 *Assume $f \in C^\omega(A \times B)$ and $\{g_n\}_{n=1}^N \subset C^\omega(A)$. Let $a \in A$, and let $[b_1, b_2] \subset B$ be any closed subinterval. Assume $\{g_n(x)\}_{n=1}^N$ is a* nonterminating *asymptotic sequence as $x \to a$. If the asymptotic expansion*

$$f(x,y) \sim \sum_{n=1}^{N} \alpha_n(y)\, g_n(x) \quad \text{as } x \to a$$

holds for all $y \in (b_1, b_2)$, then the parameter-free asymptotic expansion

$$\int_{b_1}^{b_2} f(x,y)\, dy \sim \sum_{n=1}^{N} \left(\int_{b_1}^{b_2} \alpha_n(y)\, dy \right) g_n(x) \quad \text{as } x \to a$$

also holds.

4.2.4 Exact Identity Theorem

This section discusses asymptotic expansions with a parameter in the *terminating* and *trivial* cases for *analytic functions.* The following theorem shows that a terminating or trivial asymptotic expansion with a parameter is equivalent to an *exact identity* in two variables.

Theorem 4.16 *Assume $f \in C^\omega(A \times B)$ and $\{g_n\}_{n=1}^N \subset C^\omega(A)$. Let $a \in A$, and let $J \subset B$ be any open subinterval. Assume $\{g_n(x)\}_{n=1}^N$ is a* terminating *or* trivial *asymptotic sequence as $x \to a$, and let $\bar{N}$ denote its essential length. The asymptotic expansion*

$$f(x,y) \sim \sum_{n=1}^{N} \alpha_n(y)\, g_n(x) \quad \text{as } x \to a \tag{4.17}$$

holds for all $y \in J$ if and only if $\{\alpha_n\}_{n=1}^{\bar{N}} \subset C^\omega(B)$ and the equation

$$f(x,y) = \sum_{n=1}^{\bar{N}} \alpha_n(y)\, g_n(x) \tag{4.18}$$

holds for all $(x,y) \in A \times B$.

Proof. We will consider the *terminating* and *trivial* cases separately.

Case 1. Assume that $\{g_n\}_{n=1}^N$ is *terminating* $(0 < \bar{N} < N)$. By the definition of essential length, $g_n = 0_A$ for all indices $n > \bar{N}$. Consequently, the partial sums and remainders of asymptotic series (4.17) do not change after $n = \bar{N}$, which means that $s_n = s_{\bar{N}}$ and

$$r_n = f - s_n = f - s_{\bar{N}} = r_{\bar{N}}$$

for all indices $n > \bar{N}$.

If asymptotic expansion (4.17) holds for all $y \in J$, then the order relation

$$r_n(x, y) = o(g_n(x)) \text{ as } x \to a$$

holds for all indices n and all $y \in J$; in particular, the order relation holds for all indices $n \leq \bar{N}$, which implies that the *truncated* asymptotic expansion

$$f(x, y) \sim \sum_{n=1}^{\bar{N}} \alpha_n(y)\, g_n(x) \text{ as } x \to a$$

holds for all $y \in J$.

By the definition of essential length, $g_n \neq 0_A$ for all indices $n \leq \bar{N}$. The *subsequence* $\{g_n\}_{n=1}^{\bar{N}}$ is therefore *nonterminating*. It follow by *Uniform Asymptotic Expansion Theorem* 4.13 that $\{\alpha_n\}_{n=1}^{\bar{N}} \subset C^\omega(B)$. Consequently,

$$r_{\bar{N}}(x, y) = f(x, y) - \sum_{n=1}^{\bar{N}} \alpha_n(y)\, g_n(x)$$

is defined for all $(x, y) \in A \times B$, and $r_{\bar{N}} \in C^\omega(A \times B)$.

Since $0 < \bar{N} < N$, it follows that $\bar{N} + 1$ is an *index*. This implies that order relation

$$r_{\bar{N}+1}(x, y) = o(g_{\bar{N}+1}(x)) \text{ as } x \to a \tag{4.19}$$

holds for all $y \in J$. By previous remarks, $\bar{N}+1 > \bar{N}$ implies that $r_{\bar{N}+1} = r_{\bar{N}}$ and $g_{\bar{N}+1} = 0_A$. Order relation (4.19) therefore becomes

$$r_{\bar{N}}(x,y) = o(0_A) \text{ as } x \to a$$

for all $y \in J$. Since $r_{\bar{N}} \in C^\omega(A \times B)$, Proposition 4.5 implies that $r_{\bar{N}} = 0_{A\times B}$, which means that $f = s_{\bar{N}}$, or equivalently, that equation (4.18) holds for all $(x,y) \in A \times B$.

Conversely, suppose that $\{\alpha_n\}_{n=1}^{\bar{N}} \subset C^\omega(B)$. Suppose also that equation (4.18) holds for all $(x,y) \in A \times B$, or equivalently, that $f = s_{\bar{N}}$. The analyticity of the first $\bar{N}$ coefficients implies that $s_{\bar{N}} \in C^\omega(A \times B)$; hence, the assumption $f = s_{\bar{N}}$ is consistent with the hypothesis $f \in C^\omega(A \times B)$.

To prove that asymptotic expansion (4.17) holds for all $y \in J$, we must show that the order relation

$$r_n(x,y) = o(g_n(x)) \text{ as } x \to a \tag{4.20}$$

holds for all indices n and all $y \in J$. If $n \geq \bar{N}$, then $r_n = r_{\bar{N}}$ by previous remarks, and order relation (4.20) reduces to

$$r_{\bar{N}}(x,y) = o(g_n(x)) \text{ as } x \to a$$

for all $y \in J$; this order relation is trivially satisfied, since by assumption

$$r_{\bar{N}} = f - s_{\bar{N}} = 0_{A\times B}.$$

We have shown that order relation (4.20) holds for all indices $n \geq \bar{N}$ and all $y \in J$. By Proposition 3.29 with $m = \bar{N}$, the order relation also holds for all indices $n \leq \bar{N}$ and all $y \in J$. Thus, order relation (4.20) holds for all indices n and all $y \in J$. This completes the proof of the converse, and finishes Case 1.

Case 2. Assume that $\{g_n\}_{n=1}^{N}$ is *trivial* ($\bar{N} = 0$). Since $g_n = 0_A$ for all indices n, it follows that $s_n = 0_A$ and $r_n = f - s_n = f$ for all indices n.

Asymptotic expansion (4.17) holds for all $y \in J$ if and only if the order relation

$$r_n(x,y) = o(g_n(x)) \text{ as } x \to a$$

holds for all indices n and all $y \in J$; however, previous remarks reduce this to the order relation

$$f(x,y) = o(0_A) \text{ as } x \to a,$$

which must hold for all $y \in J$.

Since $\bar{N} = 0$, the inclusion $\{\alpha_n\}_{n=1}^{\bar{N}} \subset C^\omega(B)$ is vacuously satisfied. Equation (4.18) holds for all $(x,y) \in A \times B$ if and only if

$$f = s_{\bar{N}} = s_0 = 0_{A\times B}.$$

Hence, we must show that $f(x,y) = o(0_A)$ as $x \to a$ holds for all $y \in J$ if and only if $f = 0_{A\times B}$. This follows directly from Proposition 4.5. ■

Note that the proof of Theorem 4.16 is more complicated than the proof of its univariate counterpart, Theorem 3.40. The reason is that the presence of a parameter generally causes the coefficients of the asymptotic expansion to be nonconstant; it is therefore necessary to include a condition which describes the dependence of the coefficients on the parameter. That condition is *analyticity*.

Remark 4.17 *We will work exclusively with analytic functions for the remainder of the thesis.*

Chapter 5

Dual Asymptotic Expansions

In the previous chapter, we studied the asymptotic expansion

$$f(x, y) \sim \sum_{n=1}^{N} \alpha_n(y)\, g_n(x) \text{ as } x \to a \tag{5.1}$$

in one independent variable x and one parameter y. We found that under suitable hypotheses, the coefficient functions $\{\alpha_n(y)\}_{n=1}^{N}$ are uniquely determined by the function $f(x, y)$, the real number a, and the asymptotic sequence $\{g_n(x)\}_{n=1}^{N}$. Given a function $f(x, y)$ and a point (a, b) in its domain, there are a number of interesting questions we can ask concerning asymptotic expansion (5.1):

1. Is it possible to choose the asymptotic sequence $\{g_n(x)\}_{n=1}^{N}$ as $x \to a$ in such a way that the resulting coefficients $\{\alpha_n(y)\}_{n=1}^{N}$ form an asymptotic sequence as $y \to b$?

2. If this is possible, can it be done in more than one way — or are the asymptotic sequences $\{g_n(x)\}_{n=1}^{N}$ and $\{\alpha_n(y)\}_{n=1}^{N}$ uniquely determined by the choice of function $f(x, y)$ and point (a, b)?

3. Can expansion (5.1), which is asymptotic as $x \to a$ for fixed values of y, *also* be asymptotic as $y \to b$ for fixed values of x? That is, can the expansion exhibit a *duality* between the *independent variable* and the *parameter*?

4. If asymptotic expansion (5.1) does exhibit such a duality, what properties does the expansion have? For example, is it uniform on compact intervals? Is it preserved under indefinite integration?

5. What are some of the applications of these "dual asymptotic expansions"? Does this two-variable approach to asymptotics have any advantages over traditional one-variable methods?

We will have answers to all these questions by the end of the thesis.

5.1 Definition and Examples

Our search for answers begins with the definition of a *dual asymptotic expansion.*

Definition 5.1 *Assume $f \in C^{\omega}(A \times B)$ and let $(a, b) \in A \times B$. Let N denote either a positive integer or infinity, and assume $\{g_n\}_{n=1}^{N} \subset C^{\omega}(A)$, $\{h_n\}_{n=1}^{N} \subset C^{\omega}(B)$, and $\{c_n\}_{n=1}^{N} \subset \mathbb{R} - \{0\}$. Assume also that $\{g_n(x)\}_{n=1}^{N}$ is an asymptotic sequence as $x \to a$, that $\{h_n(y)\}_{n=1}^{N}$ is an asymptotic sequence as $y \to b$, and that both sequences have the same essential length $\bar{N}$. If the asymptotic expansion*

$$f(x, y) \sim \sum_{n=1}^{N} \frac{g_n(x)\, h_n(y)}{c_n} \tag{5.2}$$

holds in the univariate sense

1. *as $x \to a$ for each fixed $y \in B$, and*

2. *as $y \to b$ for each fixed $x \in A$,*

we say that (5.2) is a ***dual asymptotic expansion of f to N terms at*** (a, b), *and we denote this relationship by*

$$f(x, y) \sim \sum_{n=1}^{N} \frac{g_n(x)\, h_n(y)}{c_n} \quad \text{as } x \to a \text{ or } y \to b. \tag{5.3}$$

Remark 5.2 *A dual asymptotic expansion can be viewed as a "twin pair" of univariate asymptotic expansions:*

1. *The dual asymptotic expansion (5.3) is a univariate asymptotic expansion in the variable x with parameter y, since*

$$f(x, y) \sim \sum_{n=1}^{N} \left(\frac{h_n(y)}{c_n}\right) g_n(x) \text{ as } x \to a$$

holds for each fixed $y \in B$.

2. *The dual asymptotic expansion (5.3) is also a univariate asymptotic expansion in the variable y with parameter x, since*

$$f(x, y) \sim \sum_{n=1}^{N} \left(\frac{g_n(x)}{c_n}\right) h_n(y) \text{ as } y \to b$$

holds for each fixed $x \in A$.

Of course, these twin asymptotic expansions are intimately related:

1. *They share a* common algebraic structure *— they can both be written in the same form as (5.3). Indeed, this shared structure makes them inseparable — each one is, by definition, the twin of the other.*

2. *Although structurally identical, they experience a* duality of purpose *— there is a duality between the* variables *and the* parameters *of the two univariate expansions, as well as a duality between the* coefficients *and the* asymptotic sequences *that appear in those expansions.*

Although the twin asymptotic expansions appear to be made of the same stuff, they behave in a complementary fashion. In short, they are identical in structure, but complementary in function.

The reader is already familiar with a large class of dual asymptotic expansions. Here is one such example.

Example 5.3 *Letting $t = -x^2y^2$ in the geometric series*

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots \qquad |t| < 1,$$

gives rise to the dual asymptotic expansion

$$\frac{1}{1+x^2y^2} \sim 1 - x^2y^2 + x^4y^4 - x^6y^6 + \cdots \text{ as } x \to 0 \text{ or } y \to 0. \tag{5.4}$$

Let us verify that all the requirements of Definition 5.1 are satisfied:

1. *The function $f(x,y) = 1/(1+x^2y^2)$ satisfies $f \in C^\omega(\mathbb{R}^2)$, and $(0,0) \in \mathbb{R}^2$; hence, we can take $A = B = \mathbb{R}$.*

2. *We have $g_n(x) = x^{2(n-1)}$, $h_n(y) = y^{2(n-1)}$, and $c_n = (-1)^{n-1}$ for all positive integers n. In addition, $\{g_n\}_{n=1}^{\infty} \subset C^\omega(\mathbb{R})$, $\{h_n\}_{n=1}^{\infty} \subset C^\omega(\mathbb{R})$, and $\{c_n\}_{n=1}^{\infty} \subset \mathbb{R} - \{0\}$; hence, we can let $N = \infty$.*

3. *Clearly, $\{x^{2(n-1)}\}_{n=1}^{\infty}$ and $\{y^{2(n-1)}\}_{n=1}^{\infty}$ are asymptotic sequences as $x \to 0$ and $y \to 0$, respectively. Since both sequences consist entirely of nonzero functions, both sequences are nonterminating with essential length $\bar{N} = N = \infty$.*

4. *If $y \neq 0$, the infinite series (5.4) converges to $1/(1+x^2y^2)$ whenever $|x| < 1/|y|$. Since convergent Taylor series are asymptotic expansions, it follows that asymptotic expansion (5.4) holds in the univariate sense as $x \to 0$ for each fixed $y \neq 0$. If $y = 0$, the expansion holds trivially as $x \to 0$. Thus, asymptotic expansion (5.4) holds as $x \to 0$ for each fixed $y \in \mathbb{R}$. By symmetry, the expansion also holds as $y \to 0$ for each fixed $x \in \mathbb{R}$.*

More generally, if $I \subset \mathbb{R}$ is a neighborhood of zero, and $F \in C^\omega(I)$ satisfies $F \neq 0_I$, the Maclaurin series

$$F(t) = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} t^n \tag{5.5}$$

gives rise to a dual asymptotic expansion in the following way. Let $\{n_k\}_{k=1}^{N}$ be the strictly increasing sequence consisting of all $n \in \mathbb{N}$ such that $F^{(n)}(0) \neq 0$. (As always, N denotes a positive integer or infinity; in this case, $N < \infty$ if and only if F is a polynomial.) We can use the sequence $\{n_k\}_{k=1}^N$ to delete exactly those terms in (5.5) whose coefficients are zero; this enables us to rewrite (5.5) as

$$F(t) = \sum_{k=1}^{N} \frac{F^{(n_k)}(0)}{(n_k)!} t^{n_k} = \sum_{k=1}^{N} \frac{t^{n_k}}{(n_k)!\,/F^{(n_k)}(0)}. \tag{5.6}$$

Let p and q be positive integers, let a, b, and $r \neq 0$ be arbitrary real constants, and let $R > 0$ be the radius of convergence of the Maclaurin series (5.5). By construction,

$$f(x,y) = F\left(\frac{(x-a)^p(y-b)^q}{r}\right)$$

is a well-defined real-analytic function in the open rectangle

$$|x-a| < \sqrt[2p]{|r|\,R}, \quad |y-b| < \sqrt[2q]{|r|\,R}.$$

Letting $t = (x-a)^p(y-b)^q/r$ in series (5.6) gives us this dual asymptotic expansion of f to N terms at (a,b):

$$f(x,y) \sim \sum_{k=1}^{N} \frac{(x-a)^{pn_k} \cdot (y-b)^{qn_k}}{r^{n_k}\,(n_k)!\,/\,F^{(n_k)}(0)} \quad \text{as } x \to a \text{ or } y \to b. \tag{5.7}$$

Example 5.4 *The dual asymptotic expansion in Example 5.3 can be obtained from (5.7) in at least two ways:*

1. *Let* $I = (-\infty, 1)$, $F(t) = 1/(1-t)$, $N = \infty$, $n_k = k-1$, $p = q = 2$, $r = -1$, *and* $a = b = 0$.

2. *Let* $I = \mathbb{R}$, $F(t) = 1/(1+t^2)$, $N = \infty$, $n_k = 2(k-1)$, $p = q = r = 1$, *and* $a = b = 0$.

Remark 5.5 *Please do not be mislead by the previous examples — not every dual asymptotic expansion is a Taylor series in disguise! Later in the thesis, we will see examples of dual asymptotic expansions whose terms contain* nonpolynomial *functions, and thus* cannot *be Taylor series.*

Although Taylor series, and more generally, univariate asymptotic expansions are allowed to have zero terms interspersed with nonzero terms, *dual asymptotic expansions are not.* Dual asymptotic expansions *can* contain zero terms, but only *after* all of the nonzero terms have appeared. This is described more precisely in the following definition, which applies our classification scheme for univariate asymptotic expansions to dual asymptotic expansions.

Definition 5.6 *We will say that dual asymptotic expansion (5.3) is* ***nonterminating, terminating****, or* ***trivial*** *if the asymptotic sequences* $\{g_n\}_{n=1}^{N}$ *and* $\{h_n\}_{n=1}^{N}$ *are nonterminating* ($\bar{N} = N$)*, terminating* ($0 < \bar{N} < N$)*, or trivial* ($\bar{N} = 0$)*, respectively. We will also say that dual asymptotic expansion (5.3) has* ***essential length*** $\bar{N}$.

The rest of this chapter explores the properties of dual asymptotic expansions, assuming that such objects exist. The next two chapters will show that such objects actually do exist, and have a number of useful applications.

5.2 Fundamental Properties

This section establishes the fundamental properties of dual asymptotic expansions with regard to uniformity in parameters, indefinite integration, and exact identities.

5.2.1 Dual Uniformity Theorem

The following theorem shows that a *nonterminating* dual asymptotic expansion is automatically uniform in each of its parameters over compact intervals.

Theorem 5.7 *Assume the conditions specified in Definition 5.1, and let* $[a_1, a_2] \subset A$ *and* $[b_1, b_2] \subset B$ *be any closed subintervals. If*

$$f(x,y) \sim \sum_{n=1}^{N} \frac{g_n(x)\, h_n(y)}{c_n} \tag{5.8}$$

is a nonterminating *dual asymptotic expansion of* f *to* N *terms at* (a, b)*, then:*

1. *as $x \to a$, asymptotic expansion (5.8) holds uniformly in y for $y \in [b_1, b_2]$*
2. *as $y \to b$, asymptotic expansion (5.8) holds uniformly in x for $x \in [a_1, a_2]$.*

Proof. Since the hypotheses are completely symmetric, conclusion 2 can be obtained from conclusion 1 simply by interchanging the roles of x and y. It therefore suffices to prove conclusion 1. By Definition 5.1, asymptotic expansion (5.8) holds as $x \to a$ for all $y \in B$, and by Definition 5.6, $\{g_n\}_{n=1}^{N}$ is a nonterminating sequence. Conclusion 1 follows immediately from *Uniform Asymptotic Expansion Theorem* 4.13 with $J = B$. ■

5.2.2 Dual Integration Theorem

The following theorem shows that a *nonterminating* dual asymptotic expansion can be integrated with respect to either of its independent variables to obtain another *nonterminating* dual asymptotic expansion.

Theorem 5.8 *Assume the conditions specified in Definition 5.1. For all $x \in A$, all $y \in B$, and all indices n, define*

$$\begin{aligned} F^{(1)}(x,y) &= \int_a^x f(t,y)\,dt, \quad G_n(x) = \int_a^x g_n(t)\,dt \\ F^{(2)}(x,y) &= \int_b^y f(x,t)\,dt, \quad H_n(y) = \int_b^y h_n(t)\,dt. \end{aligned}$$

If the expansion

$$f(x,y) \sim \sum_{n=1}^{N} \frac{g_n(x)\,h_n(y)}{c_n} \tag{5.9}$$

is a nonterminating *dual asymptotic expansion of f to N terms at (a,b), then*

$$F^{(1)}(x,y) \sim \sum_{n=1}^{N} \frac{G_n(x)\,h_n(y)}{c_n} \tag{5.10}$$

$$F^{(2)}(x,y) \sim \sum_{n=1}^{N} \frac{g_n(x)\,H_n(y)}{c_n} \tag{5.11}$$

are nonterminating *dual asymptotic expansions of $F^{(1)}$ and $F^{(2)}$ to N terms at* (a, b).

Proof. By symmetry, it suffices to prove the result for expansion (5.10). Clearly, $F^{(1)} \in C^{\omega}(A \times B)$ and $\{G_n\}_{n=1}^N \subset C^{\omega}(A)$. By Theorems 3.24 and 3.25, $\{G_n(x)\}_{n=1}^N$ is an asymptotic sequence as $x \to a$, and it essential length is $\bar{N}$. Hence, $\{G_n\}_{n=1}^N$ has the same essential length as $\{h_n\}_{n=1}^N$.

For each fixed $y \in B$, Theorem 3.39 implies that asymptotic expansion (5.10) holds as $x \to a$. Now fix $x \in A$, and consider the following three cases:

Case 1. If $x > a$, then $[a, x] \subset A$ is a compact subinterval, and we can think of (5.10) as resulting from *definite integration* from a to x with respect to the *parameter* t; the result is a *parameter-free* asymptotic expansion in y as $y \to b$. Since $\{h_n\}_{n=1}^N$ is a nonterminating sequence by hypothesis, Theorem 4.15 (with the roles of x and y reversed) implies that asymptotic expansion (5.10) holds as $y \to b$ for the specified value of x.

Case 2. If $x = a$, then expansion (5.10) reduces to

$$0_B \sim \sum_{n=1}^{N} 0 \cdot h_n(y),$$

which holds as $y \to b$ by Proposition 3.36.

Case 3. If $x < a$, then $[x, a] \subset A$ is a compact subinterval. By thinking in terms of *definite integration* from x to a with respect to the parameter t, an argument similar to Case 1 establishes that the negated asymptotic expansion

$$-F^{(1)}(x, y) \sim \sum_{n=1}^{N} \frac{-G_n(x)\, h_n(y)}{c_n}$$

holds as $y \to b$ for the specified value of x. By the linearity properties of asymptotic expansions (Proposition 3.31), the desired asymptotic expansion (5.10) also holds as $y \to b$ for the specified value of x.

We have shown that the asymptotic expansion (5.10) holds as $x \to a$ for all $y \in B$, and holds as $y \to b$ for all $x \in A$. By Definition 5.1, asymptotic expansion (5.10) is a dual asymptotic expansion at (a, b). Since $\bar{N} = N$ by hypothesis, the dual asymptotic expansion (5.10) is also *nonterminating*. ■

5.2.3 Exact Identity Theorem

The following theorem shows that a *terminating* or *trivial* dual asymptotic expansion is equivalent to an *exact identity* in two variables.

Theorem 5.9 *Assume* $f \in C^\omega(A \times B)$, $\{g_n\}_{n=1}^N \subset C^\omega(A)$, $\{h_n\}_{n=1}^N \subset C^\omega(B)$, *and* $\{c_n\}_{n=1}^N \subset \mathbb{R} - \{0\}$. *Let* $(a, b) \in A \times B$, *and assume that* $\{g_n(x)\}_{n=1}^N$ *is an asymptotic sequence as* $x \to a$, *that* $\{h_n(y)\}_{n=1}^N$ *is an asymptotic sequence as* $y \to b$, *and that both sequences have the same essential length* $\bar{N}$. *If* $\bar{N} < N$, *then the dual asymptotic expansion*

$$f(x, y) \sim \sum_{n=1}^{N} \frac{g_n(x)\, h_n(y)}{c_n} \quad \text{as } x \to a \text{ or } y \to b \tag{5.12}$$

holds if and only if the equation

$$f(x, y) = \sum_{n=1}^{\bar{N}} \frac{g_n(x)\, h_n(y)}{c_n}$$

holds for all $(x, y) \in A \times B$.

Proof. The result follows immediately from Theorem 4.16 with $J = B$. ■

5.3 Uniqueness and Normalization

This section establishes that dual asymptotic expansions are unique up to normalization, and then proposes a normalization scheme which produces true uniqueness in the *nonterminating* case.

5.3.1 Uniqueness Theorem

Assuming the conditions specified in Definition 5.1, let

$$f(x,y) \sim \sum_{n=1}^{N} \frac{g_n(x)\, h_n(y)}{c_n} \quad \text{as } x \to a \text{ or } y \to b \tag{5.13}$$

be a *nonterminating* dual asymptotic expansion. For convenience, denote the n-th term of the expansion by

$$f_n(x,y) = \frac{g_n(x)\, h_n(y)}{c_n} \tag{5.14}$$

for each index n. As usual, define the n-th partial sum by $s_n = \sum_{m=1}^{n} f_m$ and the n-th remainder by $r_n = f - s_n$ for each index n, and adopt the conventions $s_0 = 0_{A\times B}$ and $r_0 = f$.

For any $F \in C^{\omega}(A \times B)$ with $F \neq 0_{A\times B}$, define

$$\Upsilon_{(a,b)}[F](x,y) = \lim_{\hat{x}\to a^+} \left(\lim_{\hat{y}\to b^+} \frac{F(x,\hat{y})\, F(\hat{x},y)}{F(\hat{x},\hat{y})} \right)$$

for all $(x,y) \in A \times B$. In the next chapter, we will show that $\Upsilon_{(a,b)}$ is a well-defined operator, and will study its properties; however, in this chapter $\Upsilon_{(a,b)}$ merely serves as a convenient notation for the specified limit process. Note that $\Upsilon_{(a,b)}$ is defined via *one-sided* limits: we lose nothing by doing so, since *two-sided* limits, if they exists, must yield the same values.

The following theorem gives a recursive formula for computing the n-th term (5.14) of the *nonterminating* dual asymptotic expansion (5.13).

Theorem 5.10 *If (5.13) is a* nonterminating *dual asymptotic expansion, then the n-th term (5.14) is given recursively by the formula*

$$f_n(x,y) = \Upsilon_{(a,b)}[r_{n-1}](x,y) \tag{5.15}$$

for all indices n and all $(x,y) \in A \times B$.

Proof. If we think of (5.13) as an asymptotic expansion as $x \to a$ for all $y \in B$, then Theorem 4.12 with $S = B$ allows us to compute the coefficients via

$$\frac{h_n(y)}{c_n} = \lim_{\hat{x}\to a} \frac{r_{n-1}(\hat{x}, y)}{g_n(\hat{x})} \tag{5.16}$$

for all indices n and all $y \in B$. Similarly, if we think of (5.13) as an asymptotic expansion as $y \to b$ for all $x \in A$, then we can compute the coefficients via

$$\frac{g_n(x)}{c_n} = \lim_{\hat{y}\to b} \frac{r_{n-1}(x, \hat{y})}{h_n(\hat{y})} \tag{5.17}$$

for all indices n and all $x \in A$. (We have denoted the limit variables by $\hat{x}$ and $\hat{y}$ for convenience in subsequent calculations.) Solving (5.17) for c_n, and replacing x by $\hat{x}$ yields

$$c_n = \lim_{\hat{y}\to b} \frac{g_n(\hat{x})\, h_n(\hat{y})}{r_{n-1}(\hat{x}, \hat{y})}. \tag{5.18}$$

Taking the limit of (5.18) as $\hat{x} \to a$ yields

$$c_n = \lim_{\hat{x}\to a} \left(\lim_{\hat{y}\to b} \frac{g_n(\hat{x})\, h_n(\hat{y})}{r_{n-1}(\hat{x}, \hat{y})} \right). \tag{5.19}$$

Multiplying (5.19), (5.17), and (5.16) together yields the following formula for the n-th term (5.14), which holds for all indices n and all $(x, y) \in A \times B$:

$$\begin{aligned}
f_n(x, y) &= \frac{g_n(x)\, h_n(y)}{c_n} \\
&= c_n \cdot \left(\frac{g_n(x)}{c_n}\right) \cdot \left(\frac{h_n(y)}{c_n}\right) \\
&= \lim_{\hat{x}\to a} \left(\lim_{\hat{y}\to b} \frac{g_n(\hat{x})\, h_n(\hat{y})}{r_{n-1}(\hat{x}, \hat{y})} \right) \cdot \left(\lim_{\hat{y}\to b} \frac{r_{n-1}(x, \hat{y})}{h_n(\hat{y})} \right) \cdot \left(\lim_{\hat{x}\to a} \frac{r_{n-1}(\hat{x}, y)}{g_n(\hat{x})} \right) \\
&= \lim_{\hat{x}\to a} \left(\lim_{\hat{y}\to b} \frac{g_n(\hat{x})\, h_n(\hat{y})}{r_{n-1}(\hat{x}, \hat{y})} \cdot \frac{r_{n-1}(x, \hat{y})}{h_n(\hat{y})} \cdot \frac{r_{n-1}(\hat{x}, y)}{g_n(\hat{x})} \right) \\
&= \lim_{\hat{x}\to a} \left(\lim_{\hat{y}\to b} \frac{r_{n-1}(x, \hat{y})\, r_{n-1}(\hat{x}, y)}{r_{n-1}(\hat{x}, \hat{y})} \right) \\
&= \Upsilon_{(a,b)}[r_{n-1}](x, y).
\end{aligned}$$

This establishes formula (5.15). ■

Recall that Theorem 5.9 showed a function has a *trivial* dual asymptotic expansion if and only if that function is identically zero. This result does not tell us whether the zero function can have *other* kinds of dual asymptotic expansions. The following proposition shows that the zero function *cannot* have a *terminating* or *nonterminating* dual asymptotic expansion.

Proposition 5.11 *If the function $0_{A\times B}$ has a dual asymptotic expansion to N terms at the point $(a, b) \in A \times B$, then the expansion is* trivial.

Proof. Suppose that $0_{A\times B}$ has a dual asymptotic expansion

$$0_{A\times B} \sim \sum_{n=1}^{N} \frac{g_n(x)\, h_n(y)}{c_n} \tag{5.20}$$

with essential length $\bar{N}$ at the point (a, b). Consider the following two cases.

Case 1. If $\bar{N} = N$, expansion (5.20) is *nonterminating*. For any fixed $y \in B$, (5.20) is a *nonterminating* asymptotic expansion as $x \to a$. Theorem 3.36 implies that the coefficients

$$\frac{h_n(y)}{c_n} = 0$$

for all indices n. Consequently, $h_n = 0_B$ for all indices n, and $\{h_n\}_{n=1}^{N}$ is a *trivial* sequence. This contradicts the assumption that $\{h_n\}_{n=1}^{N}$ is *nonterminating*; hence, the dual asymptotic expansion (5.20) cannot be *nonterminating*.

Case 2. If $0 < \bar{N} < N$, expansion (5.20) is *terminating*. This implies that the *truncated* dual asymptotic expansion

$$0_{A\times B} \sim \sum_{n=1}^{\bar{N}} \frac{g_n(x)\, h_n(y)}{c_n}$$

holds at (a, b) and is *nonterminating*, which Case 1 showed to be impossible. Consequently, the original dual asymptotic expansion (5.20) cannot be *terminating*.

By process of elimination, dual asymptotic expansion (5.20) must be *trivial*. ■

The following *Uniqueness Theorem* establishes that *all* dual asymptotic expansions — trivial, terminating, and nonterminating — are unique up to normalization.

Theorem 5.12 *If $f \in C^\omega(A \times B)$ has a dual asymptotic expansion to N terms at the point $(a,b) \in A \times B$, then the* terms *of the expansion are unique.*

Proof. Suppose that for $i \in \{1,2\}$, the function f has two dual asymptotic expansions

$$f(x,y) \sim \sum_{n=1}^{N} f_n^{(i)}(x,y) \text{ as } x \to a \text{ or } y \to b. \tag{5.21}$$

Let $s_n^{(i)}$ and $r_n^{(i)}$ denote the n-th partial sum and remainder of expansion (5.21), respectively. Let $\bar{N}^{(i)}$ denote the essential length of expansion (5.21), and define $\bar{N} = \min(\bar{N}^{(1)}, \bar{N}^{(2)})$. It follows that the *truncated* dual asymptotic expansion

$$f(x,y) \sim \sum_{n=1}^{\bar{N}} f_n^{(i)}(x,y) \text{ as } x \to a \text{ or } y \to b$$

is *nonterminating* for $i \in \{1,2\}$.

We will show by induction on n that $f_n^{(1)} = f_n^{(2)}$ for all indices $n \leq \bar{N}$. By Theorem 5.10,

$$f_1^{(1)} = \Upsilon_{(a,b)}[r_0^{(1)}] = \Upsilon_{(a,b)}[f] = \Upsilon_{(a,b)}[r_0^{(2)}] = f_1^{(2)}.$$

Let $n < \bar{N}$ be an index, and suppose that $f_m^{(1)} = f_m^{(2)}$ for all indices $m \leq n$, in which case $r_n^{(1)} = r_n^{(2)}$. By Theorem 5.10,

$$f_{n+1}^{(1)} = \Upsilon_{(a,b)}[r_n^{(1)}] = \Upsilon_{(a,b)}[r_n^{(2)}] = f_{n+1}^{(2)}.$$

It follows by induction that $f_n^{(1)} = f_n^{(2)}$ for all indices $n \leq \bar{N}$. Note that this implies $s_{\bar{N}}^{(1)} = s_{\bar{N}}^{(2)}$.

We will now prove by contradiction that $\bar{N}^{(1)} = \bar{N}^{(2)}$. If $\bar{N}^{(1)} < \bar{N}^{(2)}$, then $\bar{N} = \bar{N}^{(1)}$ by definition of $\bar{N}$. Furthermore, $\bar{N}^{(1)} < N$, which implies the original expansion (5.21) is *terminating* or *trivial* for $i = 1$. Theorem 5.9 for $i = 1$ implies that

$$f = s^{(1)}_{\bar{N}^{(1)}} = s^{(1)}_{\bar{N}} = s^{(2)}_{\bar{N}}. \tag{5.22}$$

Since dual asymptotic expansion (5.21) holds for $i = 2$, the derived dual asymptotic expansion

$$f(x,y) - s^{(2)}_{\bar{N}}(x,y) \sim \sum_{n=\bar{N}+1}^{N} f_n^{(2)}(x,y) \text{ as } x \to a \text{ or } y \to b \tag{5.23}$$

also holds. Since $f = s^{(2)}_{\bar{N}}$ by equation (5.22), expansion (5.23) becomes

$$0_{A\times B} \sim \sum_{n=\bar{N}+1}^{N} f_n^{(2)}(x,y) \text{ as } x \to a \text{ or } y \to b. \tag{5.24}$$

Proposition 5.11 implies that expansion (5.24) is *trivial*, which means that $f_n^{(2)} = 0_{A\times B}$ for all indices $n > \bar{N}$. By construction of $\bar{N}$, it follows that $f_n^{(2)} \neq 0_{A\times B}$ for all indices $n \le \bar{N}$. These two properties establish that $\bar{N}$ is the essential length of dual asymptotic expansion (5.21) for $i = 2$. However, $\bar{N} = \bar{N}^{(1)} < \bar{N}^{(2)}$ by assumption, which contradicts that $\bar{N}^{(2)}$ is the essential length of (5.21) for $i = 2$.

Since the assumption that $\bar{N}^{(1)} < \bar{N}^{(2)}$ leads to a contradiction, we must have that $\bar{N}^{(1)} \ge \bar{N}^{(2)}$. By symmetry, we must also have $\bar{N}^{(1)} \le \bar{N}^{(2)}$, which implies that

$$\bar{N}^{(1)} = \bar{N}^{(2)} = \bar{N}.$$

By the definition of essential length,

$$f_n^{(1)} = f_n^{(2)} = 0_{A\times B}$$

for all indices $n > \bar{N}$. Since we established earlier that $f_n^{(1)} = f_n^{(2)}$ for all indices $n \le \bar{N}$, it follows that $f_n^{(1)} = f_n^{(2)}$ for all indices n. This completes the proof. ■

5.3.2 Point-Normalization

Although the *terms* of a dual asymptotic expansion

$$f(x,y) \sim \sum_{n=1}^{N} \frac{g_n(x)\,h_n(y)}{c_n} \text{ as } x \to a \text{ or } y \to b \tag{5.25}$$

are uniquely determined, the *factors* which occur in each term are *not* unique. Each term can be factored in infinitely many ways, namely

$$\frac{g_n(x)\,h_n(y)}{c_n} = \frac{(\alpha \cdot g_n(x))\,(\beta \cdot h_n(y))}{(\alpha\,\beta \cdot c_n)},$$

where $\alpha, \beta \in \mathbb{R} - \{0\}$. In the *nonterminating* case, we can force the *factors* in each term to be unique by selecting a suitable normalization scheme.

The simple normalization scheme proposed in this section is designed specifically to ensure the unique factorization of the terms. Note that other normalization schemes (e.g., normalization in the L^p spaces) will also ensure unique factorization, and may be more suitable for some applications.

Definition 5.13 *Define a mapping* $\pi_A : C^\omega(A) \times A \to \mathbb{R}$ *by*

$$\pi_A(g,a) = \begin{cases} g^{(m)}(a) & \text{if } g \neq 0_A, \text{ where } m \text{ is the multiplicity of } a. \\ 0 & \text{if } g = 0_A \end{cases}$$

Given $g \in C^\omega(A)$ *and* $a \in A$, *we call* $\pi_A(g,a)$ *the* ***point-norm of*** g ***at*** a. *We say that* g ***is point-normalized at*** a *if* $\pi_A(g,a) = 1$.

Remark 5.14 *Recall that for* $g \in C^\omega(A)$ *with* $g \neq 0_A$, *there is a unique* $\hat{g} \in C^\omega(A)$ *with* $\hat{g}(a) \neq 0$ *such that* $g(x) = (x-a)^m\,\hat{g}(x)$ *for all* $x \in A$. *In this case, the point-norm of* g *at* a *is*

$$\pi_A(g,a) = g^{(m)}(a) = m!\,\hat{g}(a) \neq 0.$$

Consequently, a nonzero analytic function g *can always be point-normalized at* a *by dividing by the nonzero constant* $\pi_A(g,a)$.

Example 5.15 *Let n be a positive integer, and consider the function*

$$g(x) = \frac{x^n}{n!}.$$

Since 0 is a root of g of multiplicity n,

$$\pi_A(g, 0) = g^{(n)}(0) = \frac{n!}{n!} = 1,$$

which means that g is point-normalized at 0 for all positive integers n. Since 1 is not a root of g,

$$\pi_A(g, 1) = g^{(0)}(1) = g(1) = \frac{1}{n!},$$

which means that g is point-normalized at 1 if and only if $n = 1$; however, the function

$$\frac{g(x)}{\pi_A(g, 1)} = n!\, g(x) = x^n$$

is point-normalized at 1 for all positive integers n. If $n = 0$, then $g(x) \equiv 1$, which is point-normalized everywhere on $\mathbb{R}$.

Definition 5.16 *We say that the dual asymptotic expansion*

$$f(x, y) \sim \sum_{n=1}^{N} \frac{g_n(x)\, h_n(y)}{c_n} \quad \text{as } x \to a \text{ or } y \to b$$

is ***point-normalized*** *if $\pi_A(g_n, a) = \pi_B(h_n, b) = 1$ for all indices n.*

Remark 5.17 *A* point-normalized *dual asymptotic expansion is automatically* nonterminating, *since by Definition 5.13, $\pi_A(0_A, a) = \pi_B(0_B, b) = 0$.*

If the dual asymptotic expansion (5.25) is *nonterminating*, then $g_n \neq 0_A$ and $h_n \neq 0_B$ for all indices n. This means that each g_n can be point-normalized at a, and each h_n can be point-normalized at b. To that end, define $\varphi_n \in C^\omega(A)$, $\psi_n \in C^\omega(B)$, and $\lambda_n \in \mathbb{R} - \{0\}$ by

$$\varphi_n = \frac{g_n}{\pi_A(g_n, a)}, \quad \psi_n = \frac{h_n}{\pi_B(h_n, b)}, \quad \lambda_n = \frac{c_n}{\pi_A(g_n, a)\, \pi_B(h_n, b)}$$

for each index n. It follows that

$$\frac{\varphi_n(x)\,\psi_n(y)}{\lambda_n} = \frac{g_n(x)\,h_n(y)}{c_n}$$

is the *unique point-normalized factorization* of the n-th term of (5.25), and that

$$f(x,y) \sim \sum_{n=1}^{N} \frac{\varphi_n(x)\,\psi_n(y)}{\lambda_n} \quad \text{as } x \to a \text{ or } y \to b$$

is the *unique point-normalized dual asymptotic expansion* of f to N terms at (a,b).

5.4 Necessary Condition for Existence

Theorem 5.10 provides a recursive formula for the terms of a *nonterminating* dual asymptotic expansion; in addition, the theorem provides a *necessary condition* for the *existence* of such an expansion. By defining $\Upsilon_{(a,b)}[0_{A\times B}] = 0_{A\times B}$, we can extend the recursive formula and the necessary condition to include *all* dual asymptotic expansions — trivial, terminating, and nonterminating — as expressed in the following theorem.

Theorem 5.18 *If $f \in C^{\omega}(A \times B)$ has a dual asymptotic expansion*

$$f(x,y) \sim \sum_{n=1}^{N} f_n(x,y) \quad \textit{as } x \to a \textit{ or } y \to b \tag{5.26}$$

at the point $(a,b) \in A \times B$, then the recursive formula

$$f_n = \Upsilon_{(a,b)}[r_{n-1}] \tag{5.27}$$

holds for all indices n.

Proof. Let $\bar{N}$ denote the essential length of dual asymptotic expansion (5.26). Since the result follows immediately by Theorem 5.10 if $\bar{N} = N$, assume that $\bar{N} < N$ for the remainder of the proof. In this case, Theorem 5.9 implies that

$f = s_{\bar{N}}$, which means that $r_{\bar{N}} = 0_{A\times B}$. Since $f_n = 0_{A\times B}$ for all indices $n > \bar{N}$, the partial sums and remainders of (5.26) do not change after $\bar{N}$. Consequently, $r_n = 0_{A\times B}$ for all indices $n \geq \bar{N}$. Equation (5.27) therefore reduces to

$$0_{A\times B} = \Upsilon_{(a,b)}[0_{A\times B}]$$

for all indices $n > \bar{N}$, which holds by the extended definition of $\Upsilon_{(a,b)}$. We need to show that (5.27) also holds for all indices $n \leq \bar{N}$. Since this is vacuously satisfied when $\bar{N} = 0$, suppose that $\bar{N} > 0$. Since (5.26) holds, the *truncated* expansion

$$f(x,y) \sim \sum_{n=1}^{\bar{N}} f_n(x,y) \text{ as } x \to a \text{ or } y \to b \tag{5.28}$$

also holds. Since (5.28) is by construction *nonterminating*, Theorem 5.10 implies that equation (5.27) holds for all indices $n \leq \bar{N}$. Hence, equation (5.27) holds for all indices n. ■

The next section reformulates the uniqueness results of this chapter in the language of dynamical systems.

5.5 Recursion, Iteration, and Orbits

In this section, we will reformulate the recursion

$$f_n = \Upsilon_{(a,b)}[r_{n-1}] \tag{5.29}$$

from Theorem 5.18 in a way that provides some additional insight into the uniqueness of dual asymptotic expansions. By definition, the terms $\{f_n\}_{n=1}^{N}$ of a series determine the partial sums $\{s_n\}_{n=1}^{N}$, which in turn determine the remainders $\{r_n\}_{n=1}^{N}$. Conversely, if the remainders with respect to a particular function f are known, it is possible to recover the partial sums via

$$s_n = f - r_n, \tag{5.30}$$

and to recover the terms via

$$f_n = s_n - s_{n-1}. \tag{5.31}$$

Thus, all of the basic information we may wish to know about a series is encoded in the sequence of remainders $\{r_n\}_{n=1}^{N}$.

Applying equation (5.30) to equation (5.31) yields

$$f_n = s_n - s_{n-1} = (f - r_n) - (f - r_{n-1}) = r_{n-1} - r_n,$$

which can be solved for r_n to obtain

$$r_n = r_{n-1} - f_n. \tag{5.32}$$

Substituting the original recursion (5.29) into (5.32) produces a new recursion

$$r_n = r_{n-1} - \Upsilon_{(a,b)}[r_{n-1}] = (I - \Upsilon_{(a,b)})\,[r_{n-1}], \tag{5.33}$$

where I denotes the identity operator. The principal advantage of (5.33) over (5.29) is that we can rewrite the *recursion* (5.33) as an *iteration*

$$r_n = (I - \Upsilon_{(a,b)})^n\,[f] \tag{5.34}$$

for all $n \in \mathbb{N}$.

Remark 5.19 *In the language of discrete dynamical systems, the sequence of remainders* $\{r_n\}_{n=1}^{N}$ *is the* ***orbit*** *of* f *under the nonlinear operator* $I - \Upsilon_{(a,b)}$.

It is clear from equation (5.34) that the sequence of remainders $\{r_n\}_{n=1}^{N}$ is completely determined by f and (a,b) alone. Since all of the basic information about the series is encoded in the remainders, equation (5.34) confirms our earlier finding that an N-term dual asymptotic expansion is completely determined by the choice of function $f \in C^{\omega}(A \times B)$ and point $(a,b) \in A \times B$.

This chapter has established the *uniqueness* of dual asymptotic expansions and developed a *necessary* condition for existence. In the next chapter, we will study the operator $\Upsilon_{(a,b)}$ in detail in order to develop a *sufficient* condition for existence as well.

Chapter 6

The Asymptotic Splitting Operator

If a function $f \in C^\omega(A \times B)$ has a dual asymptotic expansion

$$f(x,y) \sim \sum_{n=1}^{N} \frac{g_n(x)\, h_n(y)}{c_n} \text{ as } x \to a \text{ or } y \to b \tag{6.1}$$

at a point $(a,b) \in A \times B$, then the expansion (6.1) is *unique up to normalization* by the results of the previous chapter. The previous chapter also established that the n-th term of expansion (6.1) is related to the previous remainder r_{n-1} by the formula

$$\frac{g_n(x)\, h_n(y)}{c_n} = \Upsilon_{(a,b)}[r_{n-1}](x,y) \tag{6.2}$$

for all indices n and all $(x,y) \in A \times B$; this provides a *necessary condition* for the *existence* of the dual asymptotic expansion of f to N terms at (a,b).

Note that equation (6.2) implicitly requires the limit process $\Upsilon_{(a,b)}[r_{n-1}]$ to yield a *product of two analytic functions of one variable*; furthermore, $\Upsilon_{(a,b)}[r_{n-1}](x,y)$ must assume a *finite value* for all $(x,y) \in A \times B$. This chapter will determine *sufficient conditions* on r_{n-1} and (a,b) to ensure that $\Upsilon_{(a,b)}[r_{n-1}]$ exhibits these properties. In addition, the chapter will show that these properties, in conjunction with equation (6.2), provide a *sufficient condition* for the *existence* of the dual asymptotic expansion of f to N terms at (a, b).

In order to achieve the goals of this chapter, we must study the limit process denoted in the previous chapter by $\Upsilon_{(a,b)}$. We begin by formally defining the limit process $\Upsilon_{(a,b)}$ as an *operator*.

6.1 Definition and Examples

After stating the definition of $\Upsilon_{(a,b)}$ as an operator and considering some examples, we will prove that the operator $\Upsilon_{(a,b)}$ is *well-defined*.

Definition 6.1 *Given a point $(a, b) \in A \times B$, define the operator*

$$\Upsilon_{(a,b)} : C^{\omega}(A \times B) \to \overline{\mathbb{R}}^{A\times B} \tag{6.3}$$

by

$$\Upsilon_{(a,b)}[f](x, y) = \begin{cases} \lim_{\hat{x}\to a^+} \left(\lim_{\hat{y}\to b^+} \dfrac{f(x, \hat{y})\, f(\hat{x}, y)}{f(\hat{x}, \hat{y})} \right) & \text{if } f \neq 0_{A\times B} \\ 0 & \text{if } f = 0_{A\times B} \end{cases} \tag{6.4}$$

*for all $f \in C^{\omega}(A \times B)$ and all $(x, y) \in A \times B$. We call $\Upsilon_{(a,b)}$ the **asymptotic splitting operator at** (a, b).*

The following examples illustrate the action of the asymptotic splitting operator.

Example 6.2 *Let $f(x, y) = e^{-xy}$. By Definition 6.1,*

$$\Upsilon_{(a,b)}[f](x, y) = \lim_{\hat{x}\to a^+} \left(\lim_{\hat{y}\to b^+} \frac{e^{-x\hat{y}}\, e^{-\hat{x}y}}{e^{-\hat{x}\hat{y}}} \right) = \frac{e^{-xb}\, e^{-ay}}{e^{-ab}} = e^{ab-bx-ay}.$$

Example 6.3 *Let $f(x, y) = \sin(xy)$. By Definition 6.1 and L'Hospital's Rule,*

$$\Upsilon_{(0,0)}[f](x, y) = \lim_{\hat{x}\to 0^+} \left(\lim_{\hat{y}\to 0^+} \frac{\sin(x\hat{y})\sin(\hat{x}y)}{\sin(\hat{x}\hat{y})} \right)$$

$$= \lim_{\hat{x}\to 0^+}\left(\lim_{\hat{y}\to 0^+}\frac{x\cos(x\hat{y})\sin(\hat{x}y)}{\hat{x}\cos(\hat{x}\hat{y})}\right)$$
$$= \lim_{\hat{x}\to 0^+}\frac{x\sin(\hat{x}y)}{\hat{x}} = \lim_{\hat{x}\to 0^+}\frac{xy\cos(\hat{x}y)}{1} = xy.$$

In order to prove that $\Upsilon_{(a,b)}$ is *well-defined*, we must show that the limit $\Upsilon_{(a,b)}[f](x,y)$ exists in the *extended reals* $\overline{\mathbb{R}}$ for all $f \in C^{\omega}(A\times B) - \{0_{A\times B}\}$ and all $(x,y)\in A\times B$. The following two lemmas, expressed in the terminology and notation of Definition 2.18, will help us to achieve this.

Lemma 6.4 *Assume $f \in C^{\omega}(A\times B) - \{0_{A\times B}\}$, and let $(a,b)\in A\times B$. If f is reduced at (a,b), then $\Upsilon_{(a,b)}[f](x,y)$ is finite on the lines $x = a$ and $y = b$.*

Proof. By Definition 6.1,

$$\Upsilon_{(a,b)}[f](a,y) = \lim_{\hat{x}\to a^+}\left(\lim_{\hat{y}\to b^+}\frac{f(a,\hat{y})\,f(\hat{x},y)}{f(\hat{x},\hat{y})}\right) \tag{6.5}$$
$$= \lim_{\hat{x}\to a^+}\frac{f(a,b)\,f(\hat{x},y)}{f(\hat{x},b)}$$
$$= \begin{cases} f(a,y) & \text{if } f(a,b)\neq 0 \\ 0 & \text{if } f(a,b) = 0. \end{cases}$$

for all $y\in B$, which implies that $\Upsilon_{(a,b)}[f](x,y)$ is finite on $x = a$.

How do we know that the evaluation of the inner limit in (6.5) is justified? By Definition 2.18, $f(x,y)$ does not vanish on the line $y = b$. Consequently, Lemma 3.4 implies there is a deleted neighborhood $I - \{a\}$ such that $f(\hat{x},b)\neq 0$ for all $\hat{x}\in I-\{a\}$. For each fixed $\hat{x}\in I-\{a\}$, there is a neighborhood $J_{\hat{x}}$ of b such that $f(\hat{x},\hat{y})\neq 0$ for all $\hat{y}\in J_{\hat{x}}$. Consequently, $f(a,\hat{y})/f(\hat{x},\hat{y})$ is defined and continuous for all $\hat{y}\in J_{\hat{x}}$. The evaluation of the inner limit in (6.5) is therefore justified by the continuity of $f(a,\hat{y})/f(\hat{x},\hat{y})$ at $\hat{y} = b$.

Similarly, for all $x\in A$,

$$\Upsilon_{(a,b)}[f](x,b) = \lim_{\hat{x}\to a^+}\left(\lim_{\hat{y}\to b^+}\frac{f(x,\hat{y})\,f(\hat{x},b)}{f(\hat{x},\hat{y})}\right)$$

$$\begin{aligned} &= \lim_{\hat{x}\to a^+} \frac{f(x,b)\,f(\hat{x},b)}{f(\hat{x},b)} \\ &= f(x,b). \end{aligned}$$

This shows that $\Upsilon_{(a,b)}[f](x,y)$ is finite on $y = b$, and completes the proof. ■

Lemma 6.5 *Assume $f \in C^\omega(A \times B) - \{0_{A\times B}\}$, and let $(a,b) \in A \times B$. If $(m,n) = \deg_{(a,b)}(f)$ and $\hat{f} = \rho_{(a,b)}[f]$, then*

$$\Upsilon_{(a,b)}[f](x,y) = (x-a)^m\,(y-b)^n \cdot \Upsilon_{(a,b)}[\hat{f}](x,y) \tag{6.6}$$

for all $(x,y) \in A \times B$.

Proof. By Definition 2.18,

$$f(x,y) = (x-a)^m\,(y-b)^n\,\hat{f}(x,y) \text{ for all } (x,y) \in A \times B.$$

Consequently,

$$\begin{aligned} \frac{f(x,\hat{y})\,f(\hat{x},y)}{f(\hat{x},\hat{y})} &= \frac{(x-a)^m\,(\hat{y}-b)^n\,\hat{f}(x,\hat{y})\cdot(\hat{x}-a)^m\,(y-b)^n\,\hat{f}(\hat{x},y)}{(\hat{x}-a)^m\,(\hat{y}-b)^n\,\hat{f}(\hat{x},\hat{y})} \\ &= (x-a)^m\,(y-b)^n \cdot \frac{\hat{f}(x,\hat{y})\,\hat{f}(\hat{x},y)}{\hat{f}(\hat{x},\hat{y})} \end{aligned} \tag{6.7}$$

for all $(\hat{x},\hat{y}) \in (A-\{a\}) \times (B-\{b\})$ such that $\hat{f}(\hat{x},\hat{y}) \neq 0$, and all $(x,y) \in A \times B$.

Definition 6.1 states that

$$\Upsilon_{(a,b)}[f](x,y) = \lim_{\hat{x}\to a^+}\left(\lim_{\hat{y}\to b^+} \frac{f(x,\hat{y})\,f(\hat{x},y)}{f(\hat{x},\hat{y})}\right). \tag{6.8}$$

Substituting (6.7) into (6.8) yields

$$\begin{aligned} \Upsilon_{(a,b)}[f](x,y) &= \lim_{\hat{x}\to a^+}\left(\lim_{\hat{y}\to b^+} (x-a)^m\,(y-b)^n \cdot \frac{\hat{f}(x,\hat{y})\,\hat{f}(\hat{x},y)}{\hat{f}(\hat{x},\hat{y})}\right) \qquad (6.9) \\ &= (x-a)^m\,(y-b)^n \cdot \lim_{\hat{x}\to a^+}\left(\lim_{\hat{y}\to b^+} \frac{\hat{f}(x,\hat{y})\,\hat{f}(\hat{x},y)}{\hat{f}(\hat{x},\hat{y})}\right) \end{aligned}$$

$$= \quad (x-a)^m\,(y-b)^n \cdot \Upsilon_{(a,b)}[\hat{f}](x,y)$$

for all $(x,y) \in A \times B$.

Since $0 \cdot \infty$ is undefined in the arithmetic of the extended reals, we must justify taking the factor $(x-a)^m\,(y-b)^n$ outside of the limits in equation (6.9). The factor is zero only if $x = a$ or $y = b$. Since $\hat{f} = \rho_{(a,b)}[f]$ is reduced at (a,b), Lemma 6.4 implies that $\Upsilon_{(a,b)}[\hat{f}](x,y)$ is *finite* whenever $x = a$ or $y = b$. Hence, the product $0 \cdot \infty$ cannot occur. ■

Corollary 6.6 *If $f \in C^\omega(A \times B)$ and $(a,b) \in A \times B$, then $\Upsilon_{(a,b)}[f](x,y)$ is finite on the lines $x = a$ and $y = b$.*

We are now ready to prove the main result of this section.

Proposition 6.7 *For all $(a,b) \in A \times B$, the operator $\Upsilon_{(a,b)}$ is well-defined.*

Proof. Assume $f \in C^\omega(A \times B) - \{0_{A\times B}\}$, and let $\hat{f} = \rho_{(a,b)}[f]$. By Lemma 6.5, the limit $\Upsilon_{(a,b)}[f](x,y)$ exists for all $(x,y) \in A \times B$ if and only if the limit $\Upsilon_{(a,b)}[\hat{f}](x,y)$ exists for all $(x,y) \in A \times B$. By Definition 6.1,

$$\begin{aligned}
\Upsilon_{(a,b)}[\hat{f}](x,y) &= \lim_{\hat{x}\to a^+}\left(\lim_{\hat{y}\to b^+} \frac{\hat{f}(x,\hat{y})\,\hat{f}(\hat{x},y)}{\hat{f}(\hat{x},\hat{y})}\right) \\
&= \lim_{\hat{x}\to a^+} \frac{\hat{f}(x,b)\,\hat{f}(\hat{x},y)}{\hat{f}(\hat{x},b)} \\
&= \begin{cases} \hat{f}(x,b)\,\displaystyle\lim_{\hat{x}\to a^+} \frac{\hat{f}(\hat{x},y)}{\hat{f}(\hat{x},b)} & \text{if } \hat{f}(x,b) \neq 0 \\ 0 & \text{if } \hat{f}(x,b) = 0. \end{cases}
\end{aligned} \tag{6.10}$$

Since $\hat{f}(x,y)$ does not vanish on $y = b$, there are some $x \in A$ for which $\hat{f}(x,b) \neq 0$. Consequently, equation (6.10) implies that the limit $\Upsilon_{(a,b)}[\hat{f}](x,y)$ exists for all $(x,y) \in A \times B$ if and only if the limit

$$\lim_{\hat{x}\to a^+} \frac{\hat{f}(\hat{x},y)}{\hat{f}(\hat{x},b)} \tag{6.11}$$

exists for all $y \in B$.

Fix $y \in B$, and define $F_y \in C^\omega(A)$ by $F_y(x) = \hat{f}(x, y)$ for all $x \in A$. Similarly, define $F_b \in C^\omega(A)$ by $F_b(x) = \hat{f}(x, b)$ for all $x \in A$. In this notation, limit (6.11) becomes

$$\lim_{\hat{x} \to a^+} \frac{\hat{f}(\hat{x}, y)}{\hat{f}(\hat{x}, b)} = \lim_{\hat{x} \to a^+} \frac{F_y(\hat{x})}{F_b(\hat{x})}. \tag{6.12}$$

If $F_y = 0_A$, then the limit (6.12) evaluates to zero. If $F_y \neq 0_A$, then there are $m \in \mathbb{N}$ and $\hat{F}_y \in C^\omega(A)$ with $\hat{F}_y(a) \neq 0$ such that

$$F_y(x) = (x - a)^m \hat{F}_y(x) \text{ for all } x \in A. \tag{6.13}$$

Since $\hat{f}(x, y)$ does not vanish on the line $y = b$, it follows that $F_b \neq 0_A$. Consequently, there are $n \in \mathbb{N}$ and $\hat{F}_b \in C^\omega(A)$ with $\hat{F}_b(a) \neq 0$ such that

$$F_b(x) = (x - a)^n \hat{F}_b(x) \text{ for all } x \in A. \tag{6.14}$$

Now substitute (6.14) and (6.13) into (6.12) to obtain

$$\begin{aligned} \lim_{\hat{x} \to a^+} \frac{F_y(\hat{x})}{F_b(\hat{x})} &= \lim_{\hat{x} \to a^+} \frac{(\hat{x} - a)^m \hat{F}_y(\hat{x})}{(\hat{x} - a)^n \hat{F}_b(\hat{x})} \\ &= \lim_{\hat{x} \to a^+} \left(\frac{\hat{F}_y(\hat{x})}{\hat{F}_b(\hat{x})} \cdot (\hat{x} - a)^{m-n} \right) \\ &= \begin{cases} 0 & \text{if } m > n \\ c & \text{if } m = n \\ \text{signum}(c) \cdot \infty & \text{if } m < n, \end{cases} \end{aligned} \tag{6.15}$$

where

$$c = \frac{\hat{F}_y(a)}{\hat{F}_b(a)}.$$

Since $c \neq 0$, it follows that $\text{signum}(c) \cdot \infty$ is defined, and the limit (6.15) exists in the extended reals. ■

6.2 Fundamental Properties

This section studies the fundamental properties of the asymptotic splitting operator, including: the finiteness and factorization properties of $\Upsilon_{(a,b)}$, the fixed-points of $\Upsilon_{(a,b)}$, the interpolating properties of $\Upsilon_{(a,b)}$, and the asymptotic behavior of $\Upsilon_{(a,b)}$.

6.2.1 Finiteness

It is convenient to define the following terminology and notation.

Definition 6.8 *Let $\Omega \subset A \times B$, and assume $f : \Omega \to \overline{\mathbb{R}}$.*

1. *We say that f is* ***finite*** *if $|f(x,y)| < \infty$ for all $(x,y) \in \Omega$, and denote this by $|f| < \infty$.*
2. *We say that f is* ***infinite*** *if $|f(x,y)| = \infty$ for all $(x,y) \in \Omega$, and denote this by $|f| = \infty$.*
3. *We say that f is* ***infinite almost everywhere*** *if $|f|(\Omega - E)| = \infty$ for some $E \subset \Omega$ with Lebesgue measure zero, and denote this by $|f| = \infty$* ***a.e.***

The following proposition describes the two alternative behaviors of $\Upsilon_{(a,b)}[f]$ for $f \neq 0_{A\times B}$.

Proposition 6.9 *Assume $f \in C^{\omega}(A \times B) - \{0_{A\times B}\}$. Let $(a,b) \in A \times B$, and let $\hat{f} = \rho_{(a,b)}[f]$.*

1. *If $\hat{f}(a,b) \neq 0$, then $\left|\Upsilon_{(a,b)}[f]\right| < \infty$.*
2. *If $\hat{f}(a,b) = 0$, then $\left|\Upsilon_{(a,b)}[f]\right| = \infty$ a.e.*

Proof. Note that for each point $(x,y) \in A \times B$, Lemmas 6.5 and 6.4 imply that $\left|\Upsilon_{(a,b)}[f](x,y)\right| < \infty$ if and only if $\left|\Upsilon_{(a,b)}[\hat{f}](x,y)\right| < \infty$.

1. If $\hat{f}(a,b) \neq 0$, the limit

$$\Upsilon_{(a,b)}[\hat{f}](x,y) = \lim_{\hat{x}\to a^+}\left(\lim_{\hat{y}\to b^+}\frac{\hat{f}(x,\hat{y})\,\hat{f}(\hat{x},y)}{\hat{f}(\hat{x},\hat{y})}\right) = \frac{\hat{f}(x,b)\,\hat{f}(a,y)}{\hat{f}(a,b)} \tag{6.16}$$

is easily evaluated by continuity, and is clearly finite for all $(x,y) \in A \times B$. Consequently, $\left|\Upsilon_{(a,b)}[\hat{f}]\right| < \infty$.

2. Suppose that $\hat{f}(a,b) = 0$. Define $g \in C^\omega(A)$ by $g(x) = \hat{f}(x,b)$ for all $x \in A$, and define $h \in C^\omega(B)$ by $h(y) = \hat{f}(a,y)$ for all $y \in B$. Since $\hat{f}(x,y)$ does not vanish on either $x = a$ or $y = b$, it follows that $g \neq 0_A$ and $h \neq 0_B$. Consequently, the zero-sets $g^{-1}(0) \subset A$ and $h^{-1}(0) \subset B$ are countable, and the set

$$E = \left(g^{-1}(0) \times B\right) \cup \left(A \times h^{-1}(0)\right)$$

consists of a countable union of horizontal and vertical lines in $A \times B$. Since lines in $\mathbb{R}^2$ have Lebesgue measure zero, countable subadditivity implies that E has measure zero. Now consider the complement of E in $A \times B$,

$$\begin{aligned} E^c &= \left(g^{-1}(0) \times B\right)^c \cap \left(A \times h^{-1}(0)\right)^c \\ &= \left(\left(g^{-1}(0)\right)^c \times B\right) \cap \left(A \times \left(h^{-1}(0)\right)^c\right) \\ &= \left(\left(A - g^{-1}(0)\right) \times B\right) \cap \left(A \times \left(B - h^{-1}(0)\right)\right) \\ &= \left(A - g^{-1}(0)\right) \times \left(B - h^{-1}(0)\right). \end{aligned}$$

If $(x,y) \in E^c$, the limit

$$\Upsilon_{(a,b)}[\hat{f}](x,y) = \lim_{\hat{x}\to a^+}\left(\lim_{\hat{y}\to b^+}\frac{\hat{f}(x,\hat{y})\,\hat{f}(\hat{x},y)}{\hat{f}(\hat{x},\hat{y})}\right) = \lim_{\hat{x}\to a^+}\frac{\hat{f}(x,b)\,\hat{f}(\hat{x},y)}{\hat{f}(\hat{x},b)}$$

is infinite, since the numerator tends to

$$\hat{f}(x,b)\,\hat{f}(a,y) = g(x)\,h(y) \neq 0,$$

while the denominator tends to $\hat{f}(a,b) = 0$. Consequently, $\left|\Upsilon_{(a,b)}[f] \mid E^c\right| = \infty$, and $\left|\Upsilon_{(a,b)}[f]\right| = \infty$ a.e. ■

The following corollary expresses the preceding proposition in the form of an *Alternative Theorem.*

Corollary 6.10 *If $f \in C^\omega(A \times B)$ and $(a,b) \in A \times B$, then exactly one of the following two possibilities holds:*

1. $\left|\Upsilon_{(a,b)}[f]\right| < \infty$.
2. $\left|\Upsilon_{(a,b)}[f]\right| = \infty$ *a.e.*

Since we are interested primarily in functions f for which $\Upsilon_{(a,b)}[f]$ is finite, the following definition will prove useful.

Definition 6.11 *Given $(a,b) \in A \times B$, define*

$$\mathcal{S}_{(a,b)}(A \times B) = \left\{ f \in C^\omega(A \times B) : \left|\Upsilon_{(a,b)}[f]\right| < \infty \right\}.$$

If $f \in \mathcal{S}_{(a,b)}(A \times B)$, we say that f ***splits asymptotically at*** *(a,b).*

Remark 6.12 *Whenever we write $\mathcal{S}_{(a,b)}(A \times B)$, it is understood that $(a,b) \in A \times B$.*

The following proposition characterizes the elements of $\mathcal{S}_{(a,b)}(A \times B)$.

Proposition 6.13 *If $f \in C^\omega(A \times B)$, then $f \in \mathcal{S}_{(a,b)}(A \times B)$ if and only if one of the following two possibilities holds:*

1. $f = 0_{A \times B}$,
2. $f \neq 0_{A \times B}$ *and* $\hat{f}(a,b) \neq 0$, *where* $\hat{f} = \rho_{(a,b)}[f]$.

Proof. Since $\left|\Upsilon_{(a,b)}[0_{A\times B}]\right| = |0_{A\times B}| < \infty$, it is clear that $0_{A\times B} \in \mathcal{S}_{(a,b)}(A \times B)$. If $f \neq 0_{A\times B}$, Proposition 6.9 implies that $\left|\Upsilon_{(a,b)}[f]\right| < \infty$ if and only if $\hat{f}(a,b) \neq 0$. ■

If $f \in \mathcal{S}_{(a,b)}(A \times B) - \{0_{A\times B}\}$, the following proposition shows how to compute $\Upsilon_{(a,b)}[f]$ explicitly.

Proposition 6.14 *Assume $f \in \mathcal{S}_{(a,b)}(A \times B) - \{0_{A\times B}\}$. If $(m,n) = \deg_{(a,b)}(f)$ and $\hat{f} = \rho_{(a,b)}[f]$, then*

$$\Upsilon_{(a,b)}[f](x,y) = (x-a)^m (y-b)^n \cdot \frac{\hat{f}(x,b)\,\hat{f}(a,y)}{\hat{f}(a,b)} \tag{6.17}$$

for all $(x,y) \in A \times B$.

Proof. Invoke Lemma 6.5 and apply equation (6.16) from the proof of Proposition 6.9. ■

As noted earlier, $\Upsilon_{(a,b)}[0_{A\times B}] = 0_{A\times B}$. The next proposition shows that $0_{A\times B}$ is the *only* solution of the operator equation $\Upsilon_{(a,b)}[f] = 0_{A\times B}$ over the function space $C^\omega(A \times B)$. This result is a consequence of the finiteness properties of the operator $\Upsilon_{(a,b)}$.

Proposition 6.15 *If $f \in C^\omega(A \times B)$ and $(a,b) \in A \times B$, then $\Upsilon_{(a,b)}[f] = 0_{A\times B}$ if and only if $f = 0_{A\times B}$.*

Proof. If $f = 0_{A\times B}$, then $\Upsilon_{(a,b)}[f] = 0_{A\times B}$ by definition. Inversely, suppose $f \neq 0_{A\times B}$. If $\Upsilon_{(a,b)}[f] = 0_{A\times B}$, then $\left|\Upsilon_{(a,b)}[f]\right| < \infty$, and $f \in \mathcal{S}_{(a,b)}(A\times B) - \{0_{A\times B}\}$. Proposition 6.13 implies $\hat{f}(a,b) \neq 0$, where $\hat{f} = \rho_{(a,b)}[f]$. If $(m,n) = \deg_{(a,b)}(f)$, Proposition 6.14 implies

$$\Upsilon_{(a,b)}[f](x,y) = (x-a)^m (y-b)^n \cdot \frac{\hat{f}(x,b)\,\hat{f}(a,y)}{\hat{f}(a,b)} \tag{6.18}$$

for all $(x, y) \in A \times B$. Since $\hat{f}(a, b) \neq 0$, there exist $x_0 \in A - \{a\}$ and $y_0 \in B - \{b\}$ such that $\hat{f}(x_0, b) \neq 0$ and $\hat{f}(a, y_0) \neq 0$. Equation (6.18) implies $\Upsilon_{(a,b)}[f](x_0, y_0) \neq 0$, which contradicts $\Upsilon_{(a,b)}[f] = 0_{A\times B}$. Consequently, $\Upsilon_{(a,b)}[f] \neq 0_{A\times B}$. ■

6.2.2 Factorization

Given $f \in \mathcal{S}_{(a,b)}(A \times B) - \{0_{A\times B}\}$, let $(m, n) = \deg_{(a,b)}(f)$ and $\hat{f} = \rho_{(a,b)}[f]$. Proposition 6.13 implies $\hat{f}(a, b) \neq 0$, and Proposition 6.14 shows that $\Upsilon_{(a,b)}[f]$ is the *product of univariate functions*:

$$\begin{aligned}\Upsilon_{(a,b)}[f](x, y) &= (x-a)^m (y-b)^n \cdot \frac{\hat{f}(x, b)\, \hat{f}(a, y)}{\hat{f}(a, b)} \\ &= \frac{(x-a)^m \hat{f}(x, b) \cdot (y-b)^n \hat{f}(a, y)}{\hat{f}(a, b)} \\ &= \frac{g(x)\, h(y)}{c}\end{aligned}$$

for all $(x, y) \in A \times B$, where

$$\begin{aligned}g(x) &= (x-a)^m \hat{f}(x, b) \text{ for all } x \in A, \\ h(y) &= (y-b)^n \hat{f}(a, y) \text{ for all } y \in B, \\ c &= \hat{f}(a, b).\end{aligned}$$

We can also express $\Upsilon_{(a,b)}[f]$ as a product of univariate functions when $f = 0_{A\times B}$:

$$\Upsilon_{(a,b)}[f](x, y) = 0_{A\times B} = \frac{0_A\, 0_B}{1} = \frac{g(x)\, h(y)}{c}$$

for all $(x, y) \in A \times B$, where $g = 0_A$, $h = 0_B$, and $c = 1$.

In either case, $g \in C^\omega(A)$, $h \in C^\omega(B)$, and $c \in \mathbb{R} - \{0\}$. Furthermore,

$$\Upsilon_{(a,b)}[f](x, y) = \frac{g(x)\, h(y)}{c} \tag{6.19}$$

holds by construction for all $(x, y) \in A \times B$. These properties inspire the following definition.

Definition 6.16 *Assume $f \in \mathcal{S}_{(a,b)}(A \times B)$, and let g, h, and c be as defined in the preceding discussion. We say that g, h, and c are the* ***components of the standard factorization of*** $\Upsilon_{(a,b)}[f]$.

Note that $f \neq 0_{A\times B}$ implies $g \neq 0_A$ and $h \neq 0_B$. Why is this true? Since $\hat{f}(a, b) \neq 0$, there exist $x_0 \in A - \{a\}$ and $y_0 \in B - \{b\}$ such that $\hat{f}(x_0, b) \neq 0$ and $\hat{f}(a, y_0) \neq 0$. Consequently, $g(x_0) \neq 0$ and $h(y_0) \neq 0$, which implies that $g \neq 0_A$ and $h \neq 0_B$. Inversely, $f = 0_{A\times B}$ implies that $g = 0_A$ and $h = 0_B$, by definition. These relationships are summarized in the following equivalence:

$$f = 0_{A\times B} \Longleftrightarrow g = 0_A \Longleftrightarrow h = 0_B. \tag{6.20}$$

The idea of factoring a bivariate function into the product of univariate functions plays a central role in the thesis. We can represent such factorizations concisely using the following notation.

Given two *univariate* functions $g : A \to \mathbb{R}$ and $h : B \to \mathbb{R}$, construct a new *bivariate* function

$$g \otimes h : A \times B \to \mathbb{R}$$

by setting

$$(g \otimes h)(x, y) = g(x) \cdot h(y)$$

for all $(x, y) \in A \times B$. The function $g \otimes h$ is called the **tensor product of the functions g and h** (see [Edwards, p. 242]).

If $g \in C^\omega(A)$ and $h \in C^\omega(B)$, then clearly $g \otimes h \in C^\omega(A \times B)$. It is useful to define some terminology and notation for functions of this form.

Definition 6.17 *Define*

$$\mathcal{P}(A \times B) = \{g \otimes h \mid g \in C^\omega(A) \text{ and } h \in C^\omega(B)\}.$$

If $f \in \mathcal{P}(A \times B)$, we say that f ***splits algebraically****.*

Given $f \in \mathcal{S}_{(a,b)}(A \times B)$, let $g \in C^{\omega}(A)$, $h \in C^{\omega}(B)$, and $c \in \mathbb{R} - \{0\}$ be the components of the standard factorization of $\Upsilon_{(a,b)}[f]$. Expressed in tensor product notation, the standard factorization (6.19) becomes

$$\Upsilon_{(a,b)}[f] = \frac{g \otimes h}{c},$$

which implies that

$$\Upsilon_{(a,b)} : \mathcal{S}_{(a,b)}(A \times B) \rightarrow \mathcal{P}(A \times B).$$

In words, this means that the operator $\Upsilon_{(a,b)}$ maps functions which split asymptotically at (a, b) to functions which split algebraically.

6.2.3 Fixed-Points

The following proposition characterizes the fixed-points of the operator $\Upsilon_{(a,b)}$ over the function space $C^{\omega}(A \times B)$, and states some of the consequences of this characterization.

Proposition 6.18 *Given $(a, b) \in A \times B$, all of the following are true:*

1. *The set of* fixed-points *of $\Upsilon_{(a,b)}$ in $C^{\omega}(A \times B)$ is precisely $\mathcal{P}(A \times B)$.*
2. *The inclusion $\mathcal{P}(A \times B) \subset \mathcal{S}_{(a,b)}(A \times B)$ holds.*
3. *The restriction of $\Upsilon_{(a,b)}$ to $\mathcal{S}_{(a,b)}(A \times B)$ is an* idempotent *operator.*

Proof. We will prove the three conclusions in the specified order.

1. Given $g \otimes h \in \mathcal{P}(A \times B) - \{0_{A \times B}\}$, we have

$$\begin{aligned}
\Upsilon_{(a,b)}[g \otimes h](x, y) &= \lim_{\hat{x} \to a^+} \left(\lim_{\hat{y} \to b^+} \frac{(g \otimes h)(x, \hat{y}) \cdot (g \otimes h)(\hat{x}, y)}{(g \otimes h)(\hat{x}, \hat{y})} \right) \\
&= \lim_{\hat{x} \to a^+} \left(\lim_{\hat{y} \to b^+} \frac{g(x)\, h(\hat{y}) \cdot g(\hat{x})\, h(y)}{g(\hat{x})\, h(\hat{y})} \right) \\
&= \lim_{\hat{x} \to a^+} \left(\lim_{\hat{y} \to b^+} g(x) \cdot h(y) \right)
\end{aligned}$$

$$\begin{aligned} &= g(x) \cdot h(y) \\ &= (g \otimes h)(x, y) \end{aligned}$$

for all $(x, y) \in A \times B$. As noted previously, $\Upsilon_{(a,b)}[0_{A\times B}] = 0_{A\times B}$. Thus, every element of $\mathcal{P}(A \times B)$ is a *fixed-point* of the operator $\Upsilon_{(a,b)}$. Conversely, suppose that $\Upsilon_{(a,b)}[f] = f$ for some $f \in C^{\omega}(A \times B)$. If

$$\left|\Upsilon_{(a,b)}[f](x_0, y_0)\right| = \infty$$

for some $(x_0, y_0) \in (A \times B)$, then $\Upsilon_{(a,b)}[f] = f \notin C^{\omega}(A \times B)$,which is a contradiction; hence, $\left|\Upsilon_{(a,b)}[f]\right| < \infty$ and $f \in \mathcal{S}_{(a,b)}(A \times B)$. The mapping diagram

$$\Upsilon_{(a,b)} : \mathcal{S}_{(a,b)}(A \times B) \to \mathcal{P}(A \times B)$$

implies that $\Upsilon_{(a,b)}[f] = f \in \mathcal{P}(A \times B)$. This proves conclusion 1.

2. Since $\mathcal{P}(A \times B) \subset C^{\omega}(A \times B)$, every element of $\mathcal{P}(A \times B)$ is *finite*. If $g \otimes h \in \mathcal{P}(A \times B)$, the fixed-point property of conclusion 1 implies

$$\left|\Upsilon_{(a,b)}[g \otimes h]\right| = |g \otimes h| < \infty,$$

and thus $g \otimes h \in \mathcal{S}_{(a,b)}(A \times B)$, which proves conclusion 2.

3. Conclusion 3 means that

$$(\Upsilon_{(a,b)} \mid \mathcal{S}_{(a,b)}(A \times B))^2 = \Upsilon_{(a,b)} \mid \mathcal{S}_{(a,b)}(A \times B).$$

Since

$$\Upsilon_{(a,b)} : \mathcal{S}_{(a,b)}(A \times B) \to \mathcal{P}(A \times B),$$

conclusion 3 follows from conclusions 1 and 2. ■

The set $\mathcal{P}(A\times B)$ of fixed-points of the operator $\Upsilon_{(a,b)}$ is completely independent of the choice of $(a, b) \in A \times B$, which has an interesting consequence: given any two *distinct* points $(a, b), (a', b') \in A \times B$, the *distinct* operators $\Upsilon_{(a,b)}$ and $\Upsilon_{(a',b')}$ have *exactly the same fixed-points!*

6.2.4 Interpolation

The following proposition shows that if $f \in \mathcal{S}_{(a,b)}(A \times B)$, then $\Upsilon_{(a,b)}[f](x, y)$ *interpolates* $f(x, y)$ on the lines $x = a$ and $y = b$.

Proposition 6.19 *If* $f \in \mathcal{S}_{(a,b)}(A \times B)$, *then*

$$\Upsilon_{(a,b)}[f](a, y) = f(a, y) \text{ for all } y \in B,$$
$$\Upsilon_{(a,b)}[f](x, b) = f(x, b) \text{ for all } x \in A.$$

Proof. If $f = 0_{A\times B}$, the conclusion follows trivially. If $f \neq 0_{A\times B}$, let $(m, n) = \deg_{(a,b)}(f)$ and $\hat{f} = \rho_{(a,b)}[f]$: by definition,

$$f(x, y) = (x - a)^m (y - b)^n \hat{f}(x, y)$$

for all $(x, y) \in A \times B$. Proposition 6.14 implies that

$$\Upsilon_{(a,b)}[f](x, y) = (x - a)^m (y - b)^n \cdot \frac{\hat{f}(x, b)\, \hat{f}(a, y)}{\hat{f}(a, b)}$$

for all $(x, y) \in A \times B$. By direct computation,

$$\begin{aligned} \Upsilon_{(a,b)}[f](a, y) &= 0^m (y - b)^n \hat{f}(a, y) = f(a, y) \text{ for all } y \in B, \\ \Upsilon_{(a,b)}[f](x, b) &= (x - a)^m 0^n \hat{f}(x, b) = f(x, b) \text{ for all } x \in A. \end{aligned}$$

This completes the proof. ■

The following corollary reformulates the previous result in terms of the operator $I - \Upsilon_{(a,b)}$.

Corollary 6.20 *If $f \in \mathcal{S}_{(a,b)}(A \times B)$, then $(I - \Upsilon_{(a,b)})\,[f](x, y)$ vanishes on the lines $x = a$ and $y = b$.*

6.2.5 Asymptotic Behavior

The following proposition explains why the elements of $\mathcal{S}_{(a,b)}(A \times B)$ are said to *split asymptotically* at (a, b). We will put this result to good use later in the chapter when we prove a sufficient condition for the existence of dual asymptotic expansions.

Proposition 6.21 *Assume $f \in \mathcal{S}_{(a,b)}(A \times B)$. If $g \in C^{\omega}(A)$, $h \in C^{\omega}(B)$, and $c \in \mathbb{R} - \{0\}$ are the components of the standard factorization of $\Upsilon_{(a,b)}[f]$, then*

$$f(x, y) \sim \frac{g(x)\,h(y)}{c} \quad \textit{as } x \to a \textit{ or } y \to b$$

holds as a one-term dual asymptotic expansion.

Proof. It is vacuously true that the one-term sequences $\{g\}$ and $\{h\}$ are asymptotic sequences. Furthermore, the sequences $\{g\}$ and $\{h\}$ have the same essential length since

$$f = 0_{A \times B} \iff g = 0_A \iff h = 0_B. \tag{6.21}$$

We must show that

$$f(x, y) - \frac{g(x)\,h(y)}{c} = o(g(x)) \text{ as } x \to a \text{ for all } y \in B, \tag{6.22}$$

and that

$$f(x, y) - \frac{g(x)\,h(y)}{c} = o(h(y)) \text{ as } y \to b \text{ for all } x \in A. \tag{6.23}$$

It suffices to show that (6.22) holds, since (6.23) will follow by symmetry.

If $f = 0_{A \times B}$, equivalence (6.21) implies that $g = 0_A$ and $h = 0_B$. In this case, order relation (6.22) reduces to

$$0_{A \times B} = o(0_A) \text{ as } x \to a \text{ for all } y \in B,$$

which holds by Proposition 4.5 with $J = B$.

If $f \neq 0_{A\times B}$, equivalence (6.21) implies that $g \neq 0_A$ and $h \neq 0_B$. In this case, Proposition 4.4 with $S = B$ implies that order relation (6.22) holds if and only if the equation

$$\lim_{x\to a} \frac{f(x,y) - \dfrac{g(x)\,h(y)}{c}}{g(x)} = 0 \text{ for all } y \in B \tag{6.24}$$

is satisfied; equation (6.24) is clearly equivalent to

$$\lim_{x\to a} \frac{f(x,y)}{g(x)} = \frac{h(y)}{c} \text{ for all } y \in B. \tag{6.25}$$

Let $(m,n) = \deg_{(a,b)}(f)$, let $\hat{f} = \rho_{(a,b)}[f]$, and recall that

$$f(x,y) = (x-a)^m\,(y-b)^n\,\hat{f}(x,y) \tag{6.26}$$

for all $(x,y) \in A \times B$. By Definition 6.16,

$$g(x) = (x-a)^m\,\hat{f}(x,b) \text{ for all } x \in A, \tag{6.27}$$

$$h(y) = (y-b)^n\,\hat{f}(a,y) \text{ for all } y \in B, \tag{6.28}$$

$$c = \hat{f}(a,b). \tag{6.29}$$

Applying equations (6.26) through (6.29) to limit (6.25) yields

$$\begin{aligned} \lim_{x\to a} \frac{f(x,y)}{g(x)} &= \lim_{x\to a} \frac{(x-a)^m\,(y-b)^n\,\hat{f}(x,y)}{(x-a)^m\,\hat{f}(x,b)} \\ &= \lim_{x\to a} \frac{(y-b)^n\,\hat{f}(x,y)}{\hat{f}(x,b)} \\ &= \frac{(y-b)^n\,\hat{f}(a,y)}{\hat{f}(a,b)} \\ &= \frac{h(y)}{c} \text{ for all } y \in B. \end{aligned}$$

Thus, equation (6.25) is satisfied, and order relation (6.22) holds. ∎

Remark 6.22 *Some of the elements of $S_{(a,b)}(A \times B)$ do not split algebraically; however, Proposition 6.21 shows that* every *element of $S_{(a,b)}(A \times B)$ is* asymptotic *to a function which splits algebraically. Thus, in an asymptotic sense, it is always possible to* separate the variables *of an element of $S_{(a,b)}(A \times B)$ by applying the asymptotic splitting operator $\Upsilon_{(a,b)}$.*

6.3 Dual Asymptotic Expansions Revisited

In this section, we will reap the benefits of our study of the asymptotic splitting operator $\Upsilon_{(a,b)}$. The section begins by proving a sufficient condition for the existence of dual asymptotic expansions, and ends by expressing the most significant existence and uniqueness results in the form of a *Fundamental Theorem.*

6.3.1 Sufficient Condition for Existence

The following theorem gives a *sufficient condition* for the *existence* of an N-term dual asymptotic expansion of a function f at a point (a, b).

Theorem 6.23 *Assume $f \in C^{\omega}(A \times B)$, and let $(a, b) \in A \times B$. The function f has a dual asymptotic expansion*

$$f(x, y) \sim \sum_{n=1}^{N} f_n(x, y) \text{ as } x \to a \text{ or } y \to b \tag{6.30}$$

at the point (a, b) if the equation

$$f_n = \Upsilon_{(a,b)}[r_{n-1}] \tag{6.31}$$

and inequality

$$|f_n| < \infty \tag{6.32}$$

hold for all indices n.

Proof. For simplicity, reindex the remainders by defining $R_n = r_{n-1}$ for all indices n. We will work with the sequence $\{R_n\}_{n=1}^{N}$ instead of $\{r_{n-1}\}_{n=1}^{N}$ throughout most of the proof. Note that the first term is $R_1 = r_0 = f$. Let n be an arbitrary index; in the new notation, equation (6.31) becomes

$$f_n = \Upsilon_{(a,b)}[R_n]. \tag{6.33}$$

Furthermore,

$$R_{n+1} = r_n = r_{n-1} - f_n = R_n - f_n. \tag{6.34}$$

Equations (6.34) and (6.33) together imply

$$R_{n+1} = R_n - \Upsilon_{(a,b)}[R_n] = (I - \Upsilon_{(a,b)})\,[R_n]. \tag{6.35}$$

The proof constructs the dual asymptotic expansion (6.30) in six major steps.

Step 1. The first step shows that the remainders $\{R_n\}_{n=1}^{N}$ and terms $\{f_n\}_{n=1}^{N}$ are well-defined, and proves

$$\{R_n\}_{n=1}^{N} \subset \mathcal{S}_{(a,b)}(A \times B) \text{ and } \{f_n\}_{n=1}^{N} \subset \mathcal{P}(A \times B). \tag{6.36}$$

Recall that the domain of the operator $\Upsilon_{(a,b)}$ is $C^{\omega}(A \times B)$ by definition, and that we have shown

$$\Upsilon_{(a,b)} : \mathcal{S}_{(a,b)}(A \times B) \to \mathcal{P}(A \times B).$$

The proof is by induction on the index n. In the base case,

$$R_1 = f \in C^{\omega}(A \times B) \text{ and } f_1 = \Upsilon_{(a,b)}[R_1] \in \overline{\mathbb{R}}^{A \times B}.$$

Since $|f_1| = \left|\Upsilon_{(a,b)}[R_1]\right| < \infty$, we can conclude that

$$R_1 \in \mathcal{S}_{(a,b)}(A \times B) \text{ and } f_1 \in \mathcal{P}(A \times B).$$

For the induction step, let $n < N$ be an index, and suppose that

$$R_n \in \mathcal{S}_{(a,b)}(A \times B) \text{ and } f_n \in \mathcal{P}(A \times B). \tag{6.37}$$

Since

$$\mathcal{S}_{(a,b)}(A \times B) \cup \mathcal{P}(A \times B) \subset C^{\omega}(A \times B),$$

it follows that

$$R_{n+1} = R_n - f_n \in C^{\omega}(A \times B) \text{ and } f_{n+1} = \Upsilon_{(a,b)}[R_{n+1}] \in \overline{\mathbb{R}}^{A \times B}.$$

Since $n+1$ is an index, $|f_{n+1}| = \left|\Upsilon_{(a,b)}[R_{n+1}]\right| < \infty$, and we can conclude that

$$R_{n+1} \in \mathcal{S}_{(a,b)}(A \times B) \text{ and } f_{n+1} \in \mathcal{P}(A \times B).$$

By induction, (6.37) holds for all indices n, which proves (6.36).

Step 2. The second step shows that $\{R_n\}_{n=1}^{N}$ and $\{f_n\}_{n=1}^{N}$ are both *tractable* sequences with the same *essential length* $\bar{N}$. We begin by considering $\{R_n\}_{n=1}^{N}$. If $R_n \neq 0_{A\times B}$ for all indices n, then $\{R_n\}_{n=1}^{N}$ is *nonterminating*, and $\bar{N} = N$. If $\{R_n\}_{n=1}^{N}$ is *not* nonterminating, then there is an index m such that $R_m = 0_{A\times B}$. Let M denote the smallest such index, and let $\bar{N} = M-1$. By construction, $R_n \neq 0_{A\times B}$ for all indices $n \leq \bar{N}$. Note that $R_M = 0_{A\times B}$, and that equation (6.35) implies

$$R_{n+1} = (I - \Upsilon_{(a,b)})\,[R_n] = 0_{A\times B}$$

if $R_n = 0_{A\times B}$. It follows by induction on n that $R_n = 0_{A\times B}$ for all indices $n \geq M$, or equivalently, for all indices $n > \bar{N}$. Consequently, $\{R_n\}_{n=1}^{N}$ is *terminating* or *trivial* with essential length $\bar{N}$.

We have shown that $\{R_n\}_{n=1}^{N}$ is tractable with essential length $\bar{N}$. For each index n, equation (6.33) and Proposition 6.15 imply that

$$f_n = \Upsilon_{(a,b)}[R_n] = 0_{A\times B}$$

if and only if $R_n = 0_{A\times B}$. Consequently, $\{f_n\}_{n=1}^{N}$ is also tractable, and must have essential length $\bar{N}$ as well.

Step 3. The third step constructs three sequences of factors $\{g_n\}_{n=1}^N \subset C^\omega(A)$, $\{h_n\}_{n=1}^N \subset C^\omega(B)$, and $\{c_n\}_{n=1}^N \subset \mathbb{R} - \{0\}$, and shows that $\{g_n\}_{n=1}^N$ and $\{h_n\}_{n=1}^N$ are also *tractable* with essential length $\bar{N}$. For each index n, recall that $R_n \in S_{(a,b)}(A \times B)$ by Step 1, and let $g_n \in C^\omega(A)$, $h_n \in C^\omega(B)$, and $c_n \in \mathbb{R} - \{0\}$ be the components of the standard factorization of $\Upsilon_{(a,b)}[R_n]$ as defined in Definition 6.16. For each index n,

$$f_n = \Upsilon_{(a,b)}[R_n] = \frac{g_n \otimes h_n}{c_n}, \tag{6.38}$$

and

$$f_n = 0_{A\times B} \Longleftrightarrow g_n = 0_A \Longleftrightarrow h_n = 0_B. \tag{6.39}$$

Since $\{f_n\}_{n=1}^N$ is tractable with essential length $\bar{N}$ by Step 2, equivalence (6.39) implies that $\{g_n\}_{n=1}^N$ and $\{h_n\}_{n=1}^N$ are also tractable with essential length $\bar{N}$.

Step 4. The fourth step shows that

$$r_n(x, y) = o(g_n(x)) \text{ as } x \to a \text{ for all } y \in B \tag{6.40}$$

$$r_n(x, y) = o(h_n(y)) \text{ as } y \to b \text{ for all } x \in A \tag{6.41}$$

for all indices n. In what follows, let n be an arbitrary index. Since $R_n \in S_{(a,b)}(A \times B)$ by Step 1, and since g_n, h_n, and c_n are the components of the standard factorization of $\Upsilon_{(a,b)}[R_n]$ by Step 3, Proposition 6.21 implies the one-term dual asymptotic expansion

$$R_n(x, y) \sim \frac{g_n(x)\, h_n(y)}{c_n} \text{ as } x \to a \text{ or } y \to b \tag{6.42}$$

holds. Equations (6.34) and (6.38) imply that

$$R_{n+1}(x, y) = R_n(x, y) - \frac{g_n(x)\, h_n(y)}{c_n} \tag{6.43}$$

for all $(x, y) \in A \times B$. In the notation of equation (6.43), expansion (6.42) means that

$$R_{n+1}(x, y) = o(g_n(x)) \text{ as } x \to a \text{ for all } y \in B \tag{6.44}$$

$$R_{n+1}(x, y) = o(h_n(y)) \text{ as } y \to b \text{ for all } x \in A. \tag{6.45}$$

Since $R_{n+1} = r_n$, order relations (6.44) and (6.45) imply that (6.40) and (6.41) hold.

Step 5. The fifth step shows that the *truncated* sequence $\{g_n(x)\}_{n=1}^{\bar{N}}$ is an asymptotic sequence as $x \to a$; the *truncated* sequence $\{h_n(y)\}_{n=1}^{\bar{N}}$ is an asymptotic sequence as $y \to b$; and, the *truncated* expansion

$$f(x, y) \sim \sum_{n=1}^{\bar{N}} \frac{g_n(x)\, h_n(y)}{c_n} \text{ as } x \to a \text{ or } y \to b$$

is a *nonterminating* dual asymptotic expansion of f to $\bar{N}$ terms at (a, b). Since these results are vacuously true when $\bar{N} = 0$, assume $\bar{N} > 0$ for the remainder of this step.

By symmetry, it suffices to show that $\{g_n(x)\}_{n=1}^{\bar{N}}$ is an asymptotic sequence as $x \to a$, and to prove that the asymptotic expansion in x

$$f(x, y) \sim \sum_{n=1}^{\bar{N}} \left(\frac{h_n(y)}{c_n}\right) g_n(x) \text{ as } x \to a \tag{6.46}$$

holds for all $y \in B$. Note that $\{g_n\}_{n=1}^{\bar{N}}$ and $\{h_n\}_{n=1}^{\bar{N}}$ both *nonterminating*, and let m be a fixed index such that $1 < m \leq \bar{N}$. Since $h_m \not\equiv 0_B$, there is a point $y_m \in B$ such that $h_m(y_m) \neq 0$. Consequently, the m-th coefficient in formal series (6.46) with $y = y_m$ satisfies

$$\alpha_m = \frac{h_m(y_m)}{c_m} \neq 0.$$

By Step 4 with $y = y_m$, the order relation

$$r_n(x, y_m) = o(g_n(x)) \text{ as } x \to a$$

holds for all indices $n \leq \bar{N}$. By Proposition 3.32, the order relation

$$g_m(x) = o(g_{m-1}(x)) \text{ as } x \to a \tag{6.47}$$

also holds.

Since m was arbitrary, order relation (6.47) holds for all indices m with $1 < m \leq \bar{N}$, which proves that $\{g_n(x)\}_{n=1}^{\bar{N}}$ is an asymptotic sequence as $x \to a$. By Step 4, the order relation

$$r_n(x, y) = o(g_n(x)) \text{ as } x \to a$$

holds for all $y \in B$ and all indices $n \leq \bar{N}$, which proves that asymptotic expansion (6.46) holds for all $y \in B$.

Step 6. The sixth step shows that $\{g_n(x)\}_{n=1}^{N}$ is an asymptotic sequence as $x \to a$, that $\{h_n(y)\}_{n=1}^{N}$ is an asymptotic sequence as $y \to b$, and that

$$f(x, y) \sim \sum_{n=1}^{N} \frac{g_n(x)\, h_n(y)}{c_n} \text{ as } x \to a \text{ or } y \to b \tag{6.48}$$

is a dual asymptotic expansion of f to N terms at (a, b). If $\bar{N} = N$, all of these results follow immediately from Step 5.

Suppose $\bar{N} < N$. Since $\{g_n\}_{n=1}^{N}$ is tractable with essential length $\bar{N}$ by Step 3, and since $\{g_n(x)\}_{n=1}^{\bar{N}}$ is an asymptotic sequence as $x \to a$ by Step 5, Proposition 3.16 implies that $\{g_n(x)\}_{n=1}^{N}$ is an asymptotic sequence as $x \to a$. By symmetry, $\{h_n(y)\}_{n=1}^{N}$ is an asymptotic sequence as $y \to b$. Since $\{R_n\}_{n=1}^{N}$ has essential length $\bar{N}$ by Step 2, and since $\bar{N} < N$ by assumption, $R_{\bar{N}+1} = r_{\bar{N}} = 0_{A \times B}$, which means that $f = s_{\bar{N}}$. It follows from Theorem 5.9 that (6.48) is a dual asymptotic expansion of f to N terms at (a, b). This completes the proof. ■

The following corollary reformulates the sufficient condition for existence in terms of the operator $I - \Upsilon_{(a,b)}$.

Corollary 6.24 *Assume $f \in C^\omega(A \times B)$, and let $(a,b) \in A \times B$. The function f has a dual asymptotic expansion to N terms at the point (a,b) if the inequality*

$$\left|(I - \Upsilon_{(a,b)})^n [f]\right| < \infty$$

holds for all positive integers $n \leq N$.

Remark 6.25 *Not only does Theorem 6.23 give a sufficient condition for the existence of a dual asymptotic expansion of f to N terms at (a,b), the theorem also provides a way to compute the expansion explicitly!*

In practice, the sufficient condition is fulfilled *while computing the terms* of the dual asymptotic expansion of f at (a,b). Each new term f_n is obtained from the previous remainder r_{n-1} by applying the asymptotic splitting operator $\Upsilon_{(a,b)}$. If the resulting term is *finite*, then the term is guaranteed to split algebraically, and the term can be included in the dual asymptotic expansion. If, however, the term is *infinite* (almost everywhere), then no more terms can be computed, and the expansion of f at (a,b) must end with the previous term.

The following example illustrates this process.

Example 6.26 *Let us expand $f(x,y) = e^{-xy}$ in a three-term dual asymptotic expansion at the point (a,b). From Example 6.2, the first term is*

$$f_1(x,y) = \Upsilon_{(a,b)}[f](x,y) = e^{ab-bx-ay}.$$

and the first remainder is

$$r_1(x,y) = f(x,y) - f_1(x,y) = e^{-xy} - e^{ab-bx-ay}.$$

The second term is

$$\begin{aligned} f_2(x,y) &= \Upsilon_{(a,b)}[r_1](x,y) \\ &= \lim_{\hat{x}\to a^+}\left(\lim_{\hat{y}\to b^+} \frac{\left(e^{-x\hat{y}} - e^{ab-bx-a\hat{y}}\right)\cdot\left(e^{-\hat{x}y} - e^{ab-b\hat{x}-ay}\right)}{e^{-\hat{x}\hat{y}} - e^{ab-b\hat{x}-a\hat{y}}}\right)\end{aligned}$$

$$\begin{aligned}
&= \lim_{\hat{x}\to a^+}\left(\lim_{\hat{y}\to b^+}\frac{\left(-xe^{-x\hat{y}}+ae^{ab-bx-a\hat{y}}\right)\cdot\left(e^{-\hat{x}y}-e^{ab-b\hat{x}-ay}\right)}{-\hat{x}e^{-\hat{x}\hat{y}}+ae^{ab-b\hat{x}-a\hat{y}}}\right)\\
&= \lim_{\hat{x}\to a^+}\frac{\left(-xe^{-bx}+ae^{-bx}\right)\cdot\left(e^{-\hat{x}y}-e^{ab-b\hat{x}-ay}\right)}{-\hat{x}e^{-b\hat{x}}+ae^{-b\hat{x}}}\\
&= \lim_{\hat{x}\to a^+}\frac{(x-a)\,e^{-bx}\cdot\left(e^{-\hat{x}y}-e^{ab-b\hat{x}-ay}\right)}{(\hat{x}-a)\,e^{-b\hat{x}}}\\
&= \lim_{\hat{x}\to a^+}\frac{(x-a)\,e^{-bx}\cdot\left(-ye^{-\hat{x}y}+be^{ab-b\hat{x}-ay}\right)}{e^{-b\hat{x}}-b\,(\hat{x}-a)\,e^{-b\hat{x}}}\\
&= \frac{(x-a)\,e^{-bx}\cdot\left(-ye^{-ay}+be^{-ay}\right)}{e^{-ab}}\\
&= -\frac{(x-a)\,e^{-bx}\cdot(y-b)\,e^{-ay}}{e^{-ab}}\\
&= -(x-a)(y-b)\,e^{ab-bx-ay}.
\end{aligned}$$

by L'Hospital's Rule. With the help of the MAPLE *(version 5.4) computer algebra system, we find the third term to be*

$$f_3(x,y)=\Upsilon_{(a,b)}[r_2](x,y)=\frac{1}{2}\,(x-a)^2\,(y-b)^2\,e^{ab-bx-ay}.$$

Hence, the three-term dual asymptotic expansion of e^{-xy} *at* (a,b) *is*

$$e^{-xy}\sim e^{ab-bx-ay}-(x-a)\,(y-b)\,e^{ab-bx-ay}+\frac{1}{2}\,(x-a)^2\,(y-b)^2\,e^{ab-bx-ay} \quad (6.49)$$

as $x\to a$ *or* $y\to b$.

If the dual asymptotic expansion of f to N terms fails to exist at some point (a,b), one can simply expand f at a more suitable point (a',b'). This is illustrated later in the thesis by Examples 7.4 and 7.5.

By constructing matrices of the partial derivatives of f and analyzing their determinants, it is possible to identify those points in $A\times B$ where f admits a dual asymptotic expansion to any desired number of terms. Although this determinant criterion is beyond the scope of the thesis, it has a simple consequence which is worth mentioning. For each $N\in\mathbb{N}$, let E_N consist of all points in $A\times B$ where the dual asymptotic expansion of f to N terms fails to exist. Each exceptional set E_N has Lebesgue measure zero. Since the countable union of the exceptional sets

$$E = \bigcup_{N=0}^{\infty} E_N$$

also has measure zero, it follows that f admits a dual asymptotic expansion with *infinitely* many terms at *almost every* point in $A \times B$.

6.3.2 Fundamental Theorem

This section consolidates the most important results from both the previous and current chapters. Recall that Theorem 5.12 established the *uniqueness* of dual asymptotic expansions, and that Theorems 5.18 and 6.23 established a necessary and sufficient condition for the *existence* of dual asymptotic expansions. Taken together, these existence and uniqueness results give us the following *Fundamental Theorem*.

Theorem 6.27 *Assume $f \in C^{\omega}(A \times B)$. Let $(a, b) \in A \times B$, and let N denote a positive integer or infinity. The expansion*

$$f(x, y) \sim \sum_{n=1}^{N} f_n(x, y) \text{ as } x \to a \text{ or } y \to b \tag{6.50}$$

is a dual asymptotic expansion of f to N terms at (a, b) if and only if the equation

$$f_n = \Upsilon_{(a,b)}[r_{n-1}]$$

and inequality

$$|f_n| < \infty$$

hold for all indices n. Furthermore, when dual asymptotic expansion (6.50) exists, the terms $\{f_n\}_{n=1}^{N}$ are unique and satisfy

$$\{f_n\}_{n=1}^{N} \subset \mathcal{P}(A \times B).$$

In addition, existence implies that the remainders satisfy

$$\{r_{n-1}\}_{n=1}^{N} \subset \mathcal{S}_{(a,b)}(A \times B)$$

and are given by

$$r_n = (I - \Upsilon_{(a,b)})^n [f]$$

for all natural numbers $n \leq N$.

The following corollary to *Fundamental Theorem* 6.27 reformulates the necessary and sufficient condition for existence in terms of the operator $I - \Upsilon_{(a,b)}$.

Corollary 6.28 *Assume* $f \in C^{\omega}(A \times B)$, *and let* $(a, b) \in A \times B$. *The function* f *has a dual asymptotic expansion to* N *terms at the point* (a, b) *if and only if the inequality*

$$\left|(I - \Upsilon_{(a,b)})^n [f]\right| < \infty \tag{6.51}$$

holds for all positive integers $n \leq N$.

Remark 6.29 *The condition for existence given by Corollary 6.28 has slightly different consequences depending on whether* N *is finite or infinite.*

1. *If* $N < \infty$, *the condition that inequality (6.51) holds for* $n = N$ *presupposes that (6.51) holds for all indices* $n < N$. *In this case we can simply write*

$$\left|(I - \Upsilon_{(a,b)})^N [f]\right| < \infty.$$

2. *If* $N = \infty$, *the existence condition means that inequality (6.51) holds for all positive integers* n.

The existence condition given by Corollary 6.28 is free of extraneous notation pertaining to the terms and remainders of a series of functions. Let us use this more concise formulation to define the following classes of analytic functions.

Definition 6.30 *Let $(a, b) \in A \times B$. For each $N \in \mathbb{N}$, define*

$$\mathcal{D}^N_{(a,b)}(A \times B) = \left\{ f \in C^\omega(A \times B) : \left| (I - \Upsilon_{(a,b)})^N [f] \right| < \infty \right\}.$$

In addition, let

$$\mathcal{D}^\infty_{(a,b)}(A \times B) = \left\{ f \in C^\omega(A \times B) : \left| (I - \Upsilon_{(a,b)})^n [f] \right| < \infty \text{ for all } n \in \mathbb{N} \right\}.$$

For each $N \in \mathbb{N} \cup \{\infty\}$, the class of functions $\mathcal{D}^N_{(a,b)}(A \times B)$ is analogous to the class of N-time continuously differentiable functions $C^N(A \times B)$. For example, we have both

$$\mathcal{D}^0_{(a,b)}(A \times B) \subset \mathcal{D}^1_{(a,b)}(A \times B) \subset \cdots \subset \mathcal{D}^\infty_{(a,b)}(A \times B)$$

and

$$C^0(A \times B) \subset C^1(A \times B) \subset \cdots \subset C^\infty(A \times B),$$

as well as

$$\mathcal{D}^\infty_{(a,b)}(A \times B) = \bigcap_{N=0}^{\infty} \mathcal{D}^N_{(a,b)}(A \times B)$$

and

$$C^\infty(A \times B) = \bigcap_{N=0}^{\infty} C^N(A \times B).$$

Remark 6.31 *In the notation of Definition 6.30, Corollary 6.28 says that a function $f \in C^\omega(A \times B)$ has a dual asymptotic expansion to N terms at a point $(a, b) \in A \times B$ if and only if*

$$f \in \mathcal{D}^N_{(a,b)}(A \times B).$$

Hereafter, we will use this notation as a convenient way of invoking the existence condition.

Remark 6.32 *Whenever we write $\mathcal{D}^N_{(a,b)}(A \times B)$, it is understood that $(a, b) \in A \times B$. Since*

$$\mathcal{D}^0_{(a,b)}(A \times B) = C^\omega(A \times B),$$

we will assume that N denotes a positive integer or infinity, unless stated otherwise.

We noted in the previous chapter that the *terms* of a dual asymptotic expansion are unique, but the *factors* are not unique unless we impose some additional restrictions, such as point-normalization. If $f \in \mathcal{D}^N_{(a,b)}(A \times B)$, *Fundamental Theorem* 6.27 guarantees that $f_n = \Upsilon_{(a,b)}[r_{n-1}]$ and $r_{n-1} \in \mathcal{S}_{(a,b)}(A \times B)$ for all indices n. In conjunction with Definition 6.16, these properties provide another way to impose uniqueness on the factors of each term.

Definition 6.33 *Assume $f \in \mathcal{D}^N_{(a,b)}(A \times B)$, and let*

$$f(x,y) \sim \sum_{n=1}^{N} \frac{g_n(x)\,h_n(y)}{c_n} \quad \text{as } x \to a \text{ or } y \to b \tag{6.52}$$

be the dual asymptotic expansion of f to N terms at (a,b). If g_n, h_n, and c_n are the components of the standard factorization of $\Upsilon_{(a,b)}[r_{n-1}]$ for all indices n, we say that dual asymptotic expansion (6.52) is in ***standard form****.*

We have now finished developing the basic theory of dual asymptotic expansions. In the next chapter, we will explore some applications of this theory to problems of practical interest.

Chapter 7

Applications

This chapter explores some applications of dual asymptotic expansions, including: deriving exact identities — such as sum formulas — for analytic functions; generating univariate asymptotic expansions which are qualitatively well-behaved; and, approximating special functions by series of elementary functions.

7.1 Deriving and Validating Identities

The thesis has gone to a fair amount of trouble to develop a theoretical framework which handles asymptotic sequences and asymptotic expansions containing zero functions. This aspect of the theory reaches its culmination in the three *Exact Identity Theorems* — Theorem 3.40 for univariate asymptotic expansions, Theorem 4.16 for univariate expansions with a parameter, and Theorem 5.9 for dual asymptotic expansions. The *Exact Identity Theorems* show that an asymptotic expansion containing one or more zero functions holds if and only if the asymptotic expansion is an exact identity.

The underlying motivation for these theoretical developments is revealed in this section, which shows that dual asymptotic expansions can be used to *derive* exact identities for certain real-analytic functions of two variables. In addition, the author has developed the following *Criterion for Exactness*, which can be used to *validate* the resulting identities.

Theorem 7.1 *Let N denote a positive integer, let $f \in \mathcal{D}^N_{(a,b)}(A \times B)$, and let*

$$f(x,y) \sim \sum_{n=1}^{N} \frac{g_n(x)\, h_n(y)}{c_n} \quad \text{as } x \to a \text{ or } y \to b \tag{7.1}$$

denote the dual asymptotic expansion of f to N terms at (a,b). If expansion (7.1) is nonterminating, *then the exact identity*

$$f(x,y) \equiv \sum_{n=1}^{N} \frac{g_n(x)\, h_n(y)}{c_n} \tag{7.2}$$

holds on $A \times B$ if and only if the $(N+1) \times (N+1)$ Wronskian determinant

$$\det \begin{pmatrix} f(x,y) & g_1(x) & \cdots & g_N(x) \\ (\partial/\partial x)[f](x,y) & g_1'(x) & \cdots & g_N'(x) \\ \vdots & \vdots & \ddots & \vdots \\ (\partial/\partial x)^N[f](x,y) & g_1^{(N)}(x) & \cdots & g_N^{(N)}(x) \end{pmatrix}$$

vanishes identically on $A \times B$.

The theory of dual asymptotic expansions, in conjunction with the exactness criterion stated above, provide a *complete algorithm for the automatic generation and proof of identities of the form (7.2).* Unfortunately, the proof of the exactness criterion is beyond the scope of the thesis and is therefore omitted.

The examples presented below involve a bivariate function $f \in C^\omega(\mathbb{R}^2)$ defined in terms of a univariate function $F \in C^\omega(\mathbb{R})$ via

$$f(x,y) = F(x+y) \text{ for all } (x,y) \in \mathbb{R}^2. \tag{7.3}$$

Substituting equation (7.3) into identity (7.2) yields a more specialized identity of the form

$$F(x+y) \equiv \sum_{n=1}^{N} \frac{g_n(x)\, h_n(y)}{c_n}. \tag{7.4}$$

In this special case, the dual asymptotic expansion becomes a process which *gen-*

erates sum formulas (7.4) for certain univariate analytic functions F.

7.1.1 Elementary Functions

The first and simplest example comes from the exponential function.

Example 7.2 *The first term in the dual asymptotic expansion of* $f(x,y) = e^{x+y}$ *at the point* $(0,0)$ *is given by*

$$f_1(x,y) = \Upsilon_{(0,0)}[f](x,y) = \lim_{\hat{x}\to 0^+}\left(\lim_{\hat{y}\to 0^+}\frac{e^{x+\hat{y}}\,e^{\hat{x}+y}}{e^{\hat{x}+\hat{y}}}\right) = \frac{e^x\,e^y}{1}.$$

Consequently,

$$e^{x+y} \sim e^x\,e^y \text{ as } x \to 0 \text{ or } y \to 0 \tag{7.5}$$

holds as a one-term dual asymptotic expansion. Since the Wronskian determinant

$$\det\begin{pmatrix} e^{x+y} & e^x \\ e^{x+y} & e^x \end{pmatrix}$$

vanishes identically, the Exactness Criterion *implies that expansion (7.5) is exact. Thus, we have derived and proven the identity*

$$e^{x+y} \equiv e^x\,e^y.$$

The next four examples involve trigonometric functions. The first of these illustrates a straightforward derivation of a two-term identity.

Example 7.3 *Let us expand* $f(x,y) = \cos(x+y)$ *in a dual asymptotic expansion at the point* $(0,0)$*. The first term is*

$$\begin{aligned} f_1(x,y) &= \Upsilon_{(0,0)}[f](x,y) \\ &= \lim_{\hat{x}\to 0^+}\left(\lim_{\hat{y}\to 0^+}\frac{\cos(x+\hat{y})\,\cos(\hat{x}+y)}{\cos(\hat{x}+\hat{y})}\right) \\ &= \frac{\cos(x)\cos(y)}{1}, \end{aligned}$$

and the first remainder is

$$r_1(x,y) = f(x,y) - f_1(x,y) = \cos(x+y) - \cos(x)\cos(y).$$

The second term is

$$\begin{aligned}
f_2(x,y) &= \Upsilon_{(0,0)}[r_1](x,y) = \lim_{\hat{x}\to 0^+}\left(\lim_{\hat{y}\to 0^+}\frac{r_1(x,\hat{y})\, r_1(\hat{x},y)}{r_1(\hat{x},\hat{y})}\right)\\
&= \lim_{\hat{x}\to 0^+}\left(r_1(\hat{x},y)\lim_{\hat{y}\to 0^+}\frac{\cos(x+\hat{y})-\cos(x)\cos(\hat{y})}{\cos(\hat{x}+\hat{y})-\cos(\hat{x})\cos(\hat{y})}\right)\\
&= \lim_{\hat{x}\to 0^+}\left(r_1(\hat{x},y)\lim_{\hat{y}\to 0^+}\frac{-\sin(x+\hat{y})+\cos(x)\sin(\hat{y})}{-\sin(\hat{x}+\hat{y})+\cos(\hat{x})\sin(\hat{y})}\right)\\
&= \lim_{\hat{x}\to 0^+}\left(r_1(\hat{x},y)\frac{\sin(x)}{\sin(\hat{x})}\right)\\
&= \sin(x)\lim_{\hat{x}\to 0^+}\frac{\cos(\hat{x}+y)-\cos(\hat{x})\cos(y)}{\sin(\hat{x})}\\
&= \sin(x)\lim_{\hat{x}\to 0^+}\frac{-\sin(\hat{x}+y)+\sin(\hat{x})\cos(y)}{\cos(\hat{x})}\\
&= \sin(x)\frac{-\sin(y)}{1}.
\end{aligned}$$

by L'Hospital's Rule. Hence, the two-term dual asymptotic expansion of $\cos(x+y)$ *at* $(0,0)$ *is*

$$\cos(x+y) \sim \cos(x)\cos(y) - \sin(x)\sin(y) \text{ as } x \to 0 \text{ or } y \to 0. \tag{7.6}$$

The Wronskian determinant

$$\det\begin{pmatrix} \cos(x+y) & \cos(x) & \sin(x)\\ -\sin(x+y) & -\sin(x) & \cos(x)\\ -\cos(x+y) & -\cos(x) & -\sin(x)\end{pmatrix}$$

vanishes identically since the first and third rows are linearly dependent. By the Exactness Criterion, *expansion (7.6) is exact. Thus, we have derived and proven* *the identity*

$$\cos(x+y) \equiv \cos(x)\cos(y) - \sin(x)\sin(y). \tag{7.7}$$

The next example illustrates a difficulty which can arise when trying to derive an identity.

Example 7.4 *Let us attempt to compute the first term in the dual asymptotic expansion of* $f(x, y) = \sin(x+y)$ *at the point* $(0, 0)$*; more specifically, let us try to determine the* value *of the first term at* $(\pi/2, \pi/2)$*:*

$$\begin{aligned} f_1(\pi/2, \pi/2) &= \Upsilon_{(0,0)}[f](\pi/2, \pi/2) \\ &= \lim_{\hat{x}\to 0^+}\left(\lim_{\hat{y}\to 0^+}\frac{\sin(\pi/2+\hat{y})\,\sin(\hat{x}+\pi/2)}{\sin(\hat{x}+\hat{y})}\right) \\ &= \lim_{\hat{x}\to 0^+}\frac{\sin(\hat{x}+\pi/2)}{\sin(\hat{x})} = +\infty. \end{aligned}$$

Since the value is infinite, $\sin(x+y)$ *does not split asymptotically* $(0, 0)$*, which means that* $\sin(x+y)$ *does not have a dual asymptotic expansion at* $(0, 0)$.

We could simply differentiate identity (7.7) with respect to either x or y to obtain the sum formula for $\sin(x+y)$. It is more instructive, however, to see how dual asymptotic expansions can be used to circumvent the difficulty which arose in the previous example; the next two examples do this in two different ways.

Example 7.5 *One way around the difficulty is to expand* $f(x, y) = \sin(x+y)$ *in a dual asymptotic expansion at the point* $(\pi/2, 0)$ *instead of* $(0, 0)$. *The first term is*

$$\begin{aligned} f_1(x, y) &= \Upsilon_{(\pi/2,0)}[f](x, y) \\ &= \lim_{\hat{x}\to \pi/2^+}\left(\lim_{\hat{y}\to 0^+}\frac{\sin(x+\hat{y})\,\sin(\hat{x}+y)}{\sin(\hat{x}+\hat{y})}\right) \\ &= \frac{\sin(x)\sin(\pi/2+y)}{1}, \end{aligned}$$

and the first remainder is

$$r_1(x, y) = f(x, y) - f_1(x, y) = \sin(x+y) - \sin(x)\sin(\pi/2+y).$$

The second term is

$$\begin{aligned} f_2(x,y) &= \Upsilon_{(\pi/2,0)}[r_1](x,y) = \lim_{\hat{x}\to\pi/2^+}\left(\lim_{\hat{y}\to 0^+}\frac{r_1(x,\hat{y})\,r_1(\hat{x},y)}{r_1(\hat{x},\hat{y})}\right) \\ &= \lim_{\hat{x}\to\pi/2^+}\left(r_1(\hat{x},y)\lim_{\hat{y}\to 0^+}\frac{\sin(x+\hat{y})-\sin(x)\sin(\pi/2+\hat{y})}{\sin(\hat{x}+\hat{y})-\sin(\hat{x})\sin(\pi/2+\hat{y})}\right) \\ &= \lim_{\hat{x}\to\pi/2^+}\left(r_1(\hat{x},y)\lim_{\hat{y}\to 0^+}\frac{\cos(x+\hat{y})-\sin(x)\cos(\pi/2+\hat{y})}{\cos(\hat{x}+\hat{y})-\sin(\hat{x})\cos(\pi/2+\hat{y})}\right) \\ &= \lim_{\hat{x}\to\pi/2^+}\left(r_1(\hat{x},y)\frac{\cos(x)}{\cos(\hat{x})}\right) \\ &= \cos(x)\lim_{\hat{x}\to\pi/2^+}\frac{\sin(\hat{x}+y)-\sin(\hat{x})\sin(\pi/2+y)}{\cos(\hat{x})} \\ &= \cos(x)\lim_{\hat{x}\to\pi/2^+}\frac{\cos(\hat{x}+y)-\cos(\hat{x})\sin(\pi/2+y)}{-\sin(\hat{x})} \\ &= \cos(x)\frac{\cos(\pi/2+y)}{-1}. \end{aligned}$$

by L'Hospital's Rule. Hence, the two-term dual asymptotic expansion of $\sin(x+y)$ *at* $(\pi/2,0)$ *is*

$$\sin(x+y)\sim\sin(x)\sin(\pi/2+y)-\cos(x)\cos(\pi/2+y)\ \text{ as } x\to\pi/2 \text{ or } y\to 0. \tag{7.8}$$

The Wronskian determinant

$$\det\begin{pmatrix}\sin(x+y) & \sin(x) & \cos(x)\\ \cos(x+y) & \cos(x) & -\sin(x)\\ -\sin(x+y) & -\sin(x) & -\cos(x)\end{pmatrix}$$

vanishes identically since the first and third rows are linearly dependent. By the Exactness Criterion, *expansion (7.8) is exact. Thus, we have derived and proven the identity*

$$\sin(x+y)\equiv\sin(x)\sin(\pi/2+y)-\cos(x)\cos(\pi/2+y). \tag{7.9}$$

Setting $y = \pi/2$ in (7.9) yields

$$\sin(x + \pi/2) \equiv \cos(x), \tag{7.10}$$

and setting $x = 0$ in (7.9) yields

$$\sin(y) \equiv -\cos(\pi/2 + y). \tag{7.11}$$

Using (7.11) and (7.10), we can rewrite (7.9) in the more familiar form

$$\sin(x + y) \equiv \sin(x)\cos(y) + \cos(x)\sin(y).$$

Example 7.6 *A second way around the difficulty is to use the dual asymptotic expansion which we computed earlier for $\cos(x + y)$ at the point $(0, 0)$:*

$$\cos(x + y) \sim \cos(x)\cos(y) - \sin(x)\sin(y) \text{ as } x \to 0 \text{ or } y \to 0. \tag{7.12}$$

Taking the indefinite integral of expansion (7.12) with respect to y yields another dual asymptotic expansion, namely

$$\int_0^y \cos(x + t)\,dt \sim \cos(x)\int_0^y \cos(t)\,dt - \sin(x)\int_0^y \sin(t)\,dt \text{ as } x \to 0 \text{ or } y \to 0. \tag{7.13}$$

Evaluating the integrals in (7.13) gives us

$$\sin(x + y) - \sin(x) \sim \cos(x)\sin(y) + \sin(x)(\cos(y) - 1) \text{ as } x \to 0 \text{ or } y \to 0. \tag{7.14}$$

Since the Wronskian determinant

$$\det\begin{pmatrix} \sin(x + y) - \sin(x) & \cos(x) & \sin(x) \\ \cos(x + y) - \cos(x) & -\sin(x) & \cos(x) \\ -\sin(x + y) + \sin(x) & -\cos(x) & -\sin(x) \end{pmatrix}$$

contains linearly dependent rows and therefore vanishes identically, the Exactness Criterion *implies that expansion (7.14) is exact. Thus, we have derived and proven* *the identity*

$$\sin(x+y) - \sin(x) \equiv \cos(x)\sin(y) + \sin(x)(\cos(y) - 1),$$

which simplifies to

$$\sin(x+y) \equiv \cos(x)\sin(y) + \sin(x)\cos(y).$$

7.1.2 The Bessel Function J_0

This section is concerned with a class of special functions, namely the Bessel functions of the first kind with integer order. These functions are denoted by J_n and defined for all integers n and all real x by

$$J_n(x) = \begin{cases} \left(\frac{x}{2}\right)^n \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(n+k)!} \left(\frac{x}{2}\right)^{2k} & \text{if } n \geq 0 \\ (-1)^n J_{-n}(x) & \text{if } n < 0. \end{cases} \tag{7.15}$$

The expressions in (7.15) are actually special cases of more general formulas appearing as equations 9.1.5 and 9.1.10 in [Abr-Ste, pp. 358, 360]. It is clear from (7.15) that

$$J_n(0) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0. \end{cases}$$

The following three formulas for the derivatives of J_n hold for all integers n, all natural numbers m, and all real x:

$$J_0'(x) = -J_1(x) \tag{7.16}$$

$$J_n'(x) = \frac{J_{n-1}(x) - J_{n+1}(x)}{2} \tag{7.17}$$

$$J_n^{(m)}(x) = \frac{1}{2^m} \cdot \sum_{k=0}^{m} (-1)^k \binom{m}{k} J_{n-m+2k}(x). \tag{7.18}$$

Equations (7.16), (7.17), and (7.18) appear as equations 9.1.28, 9.1.27, and 9.1.31 in [Abr-Ste, p. 361], respectively. The above information concerning the Bessel functions J_n is sufficient to carry out the following example.

Example 7.7 *Let us compute the two-term dual asymptotic expansion of* $f(x,y) = J_0(x+y)$ *at the point* $(0,0)$. *The first term is*

$$\begin{aligned} f_1(x,y) &= \Upsilon_{(0,0)}[f](x,y) \\ &= \lim_{\hat{x}\to 0^+}\left(\lim_{\hat{y}\to 0^+} \frac{J_0(x+\hat{y})\, J_0(\hat{x}+y)}{J_0(\hat{x}+\hat{y})}\right) \\ &= \frac{J_0(x)J_0(y)}{1}. \end{aligned}$$

and the first remainder is

$$r_1(x,y) = f(x,y) - f_1(x,y) = J_0(x+y) - J_0(x)J_0(y).$$

The second term is

$$\begin{aligned} f_2(x,y) &= \Upsilon_{(0,0)}[r_1](x,y) = \lim_{\hat{x}\to 0^+}\left(\lim_{\hat{y}\to 0^+} \frac{r_1(x,\hat{y})\, r_1(\hat{x},y)}{r_1(\hat{x},\hat{y})}\right) \\ &= \lim_{\hat{x}\to 0^+}\left(r_1(\hat{x},y) \lim_{\hat{y}\to 0^+} \frac{J_0(x+\hat{y}) - J_0(x)J_0(\hat{y})}{J_0(\hat{x}+\hat{y}) - J_0(\hat{x})J_0(\hat{y})}\right) \\ &= \lim_{\hat{x}\to 0^+}\left(r_1(\hat{x},y) \lim_{\hat{y}\to 0^+} \frac{-J_1(x+\hat{y}) + J_0(x)J_1(\hat{y})}{-J_1(\hat{x}+\hat{y}) + J_0(\hat{x})J_1(\hat{y})}\right) \\ &= \lim_{\hat{x}\to 0^+}\left(r_1(\hat{x},y)\frac{J_1(x)}{J_1(\hat{x})}\right) \\ &= J_1(x) \lim_{\hat{x}\to 0^+} \frac{J_0(\hat{x}+y) - J_0(\hat{x})J_0(y)}{J_1(\hat{x})} \\ &= J_1(x) \lim_{\hat{x}\to 0^+} \frac{-J_1(\hat{x}+y) + J_1(\hat{x})J_0(y)}{(J_0(\hat{x}) - J_2(\hat{x}))/2} \\ &= J_1(x)\frac{-J_1(y)}{1/2}, \end{aligned}$$

by L'Hospital's Rule. Hence, the two-term dual asymptotic expansion of $J_0(x+y)$ at $(0,0)$ is

$$J_0(x+y) \sim J_0(x)J_0(y) - 2\,J_1(x)J_1(y) \quad \textit{as } x \to 0 \textit{ or } y \to 0. \tag{7.19}$$

The following formula is a special case of equation 9.1.75 in [Abr-Ste, p. 363]:

$$J_0(x+y) = \sum_{k=-\infty}^{\infty} J_{-k}(x)J_k(y). \tag{7.20}$$

Using $J_{-k}(x) = (-1)^k J_k(x)$, we can rewrite (7.20) as

$$J_0(x+y) = J_0(x)J_0(y) + 2\sum_{k=1}^{\infty}(-1)^k J_k(x)J_k(y), \tag{7.21}$$

which converges for all real x and y. Equation (7.21) can be regarded as an *infinite sum formula* for $J_0(x+y)$.

Notice that dual asymptotic expansion (7.19) reproduces the first two terms of sum formula (7.21). In fact, since $x = 0$ is a root of $J_k(x)$ of multiplicity k for all natural numbers k, it follows that $\{J_k(x)\}_{k=0}^{\infty}$ is an asymptotic sequence as $x \to 0$. Equation (7.21) further implies that $J_0(x+y)$ has a dual asymptotic expansion with *infinitely* many terms, namely

$$J_0(x+y) \sim J_0(x)J_0(y) + 2\sum_{k=1}^{\infty}(-1)^k J_k(x)J_k(y) \text{ as } x \to 0 \text{ or } y \to 0. \tag{7.22}$$

Expansion (7.22) shows that interesting dual asymptotic expansions occur naturally in the mathematical universe.

Let us conclude this section with the following observations concerning this example:

1. Using the method of this section, the computation of the n-th term via

$$f_n = \Upsilon_{(0,0)}[r_{n-1}]$$

requires $n-1$ applications of L'Hospital's Rule for *each* of the two limits which define the operator $\Upsilon_{(0,0)}$. Since the remainder r_{n-1} is a linear combination of n terms involving Bessel functions, and since by equation (7.18), the $(n-1)$-st derivative of *each* Bessel function will *also* be a linear combination of n Bessel functions, it is clear that this method of computing the n-th term rapidly becomes quite complicated as n grows large.

This example illustrates the need for a more efficient algorithm to compute dual asymptotic expansions. The final chapter of the thesis will describe an entirely different approach which may lead to a substantially improved algorithm.

2. The *infinite* dual asymptotic expansion (7.22) is actually a *convergent infinite series* and provides an *infinite sum formula* (7.21) for $J_0(x+y)$. An improved algorithm for computing dual asymptotic expansions would presumably enable us to determine the general term f_n in expansion (7.22), thereby *deriving* the sum formula for $J_0(x+y)$ automatically. In order to *validate* that expansion (7.22) is in fact an *exact identity* (with infinitely many terms), we need to understand the *convergence properties* of infinite dual asymptotic expansions. One possible approach to the question of convergence will be presented in the final chapter.

7.2 Generating Univariate Expansions

This section begins with a theorem that shows *univariate* asymptotic expansions can be derived from *dual* asymptotic expansions in a variety of ways. The section then explores some simple relationships between univariate and bivariate functions; these relationships facilitate the application of the theorem, and create a systematic method for generating asymptotic expansions of univariate functions. The section ends by presenting an extended example illustrating the method, and discusses important qualitative features of the asymptotic expansion produced by the method.

7.2.1 Parameterization Theorem

The following theorem shows that we can generate *univariate* asymptotic expansions from *dual* asymptotic expansions simply by parameterizing the independent variables.

Theorem 7.8 *Assume* $f \in \mathcal{D}^N_{(a,b)}(A \times B)$. *Let*

$$f(x,y) \sim \sum_{n=1}^{N} f_n(x,y) \text{ as } x \to a \text{ or } y \to b \tag{7.23}$$

denote the dual asymptotic expansion of f *to* N *terms at* (a,b), *and let* $\bar{N}$ *denote its essential length. Let* $I \subset \mathbb{R}$ *be a neighborhood of* $t_0 \in \mathbb{R}$, *and let* $X : I \to A$ *and* $Y : I \to B$ *be nonconstant real-analytic functions such that* $X(t_0) = a$ *and* $Y(t_0) = b$. *Define* $F \in C^\omega(I)$ *and* $\{F_n\}_{n=1}^N \subset C^\omega(I)$ *by*

$$F(t) = f(X(t), Y(t)) \tag{7.24}$$

$$F_n(t) = f_n(X(t), Y(t)) \tag{7.25}$$

for all $t \in I$ *and all indices* n. *It follows that:*

1. $\{F_n\}_{n=1}^N$ *has essential length* $\bar{N}$.

2. $\{F_n(t)\}_{n=1}^N$ *is an asymptotic sequence as* $t \to t_0$.

3. F *has an asymptotic expansion*

$$F(t) \sim \sum_{n=1}^{N} F_n(t) \text{ as } t \to t_0 \tag{7.26}$$

with respect to $\{F_n\}_{n=1}^N$.

Proof. *Fundamental Theorem* 6.27 implies that $f_n = \Upsilon_{(a,b)}[r_{n-1}]$ and $r_{n-1} \in \mathcal{S}_{(a,b)}(A \times B)$ for all indices n; we will invoke these properties freely throughout the proof. Let

$$f(x,y) \sim \sum_{n=1}^{N} \frac{g_n(x)\, h_n(y)}{c_n} \text{ as } x \to a \text{ or } y \to b$$

denote the *standard form* of dual asymptotic expansion (7.23), and note that $\{f_n\}_{n=1}^N$, $\{g_n\}_{n=1}^N$, $\{h_n\}_{n=1}^N$, and $\{r_{n-1}\}_{n=1}^N$ all have essential length $\bar{N}$. We will prove the three conclusions in the specified order.

1. Since $f_n = 0_{A\times B}$ for all indices $n > \bar{N}$, it follows by equation (7.25) that $F_n = 0_I$ for all indices $n > \bar{N}$. We will prove by contradiction that $F_n \neq 0_I$ for all indices $n \leq \bar{N}$. Suppose that $F_n = 0_I$ for some index $n \leq \bar{N}$, and note that
$$F_n(t) = f_n(X(t), Y(t)) = \frac{g_n(X(t))\, h_n(Y(t))}{c_n} \tag{7.27}$$
for all $t \in I$. Since $C^\omega(I)$ is an *integral domain* by Proposition 2.4, $F_n = 0_I$ implies that $g_n \circ X = 0_I$ or $h_n \circ Y = 0_I$. If $g_n \circ X = 0_I$, then $X(I) \subset g_n^{-1}(0)$. Since the continuous image of a connected set is connected, $X(I)$ is a connected subset of $g_n^{-1}(0)$. Since $g_n \in C^\omega(A)$ and $g_n \neq 0_A$, the zero-set $g_n^{-1}(0)$ consists of isolated points by Proposition 2.1; consequently, the only connected subsets of $g_n^{-1}(0)$ are singleton sets. Hence, $X(I)$ is a singleton set, which means that X is constant on I. Since this contradicts the hypotheses of the theorem, we must have $g_n \circ X \neq 0_I$. By symmetry, we must also have $h_n \circ Y \neq 0_I$, which means that $F_n = 0_I$ is impossible for any index $n \leq \bar{N}$. By definition, $\{F_n\}_{n=1}^N$ has essential length $\bar{N}$.

2. Since $\{F_n\}_{n=1}^N \subset C^\omega(I)$ is tractable with essential length $\bar{N}$ by conclusion 1, Proposition 3.23 implies that $\{F_n(t)\}_{n=1}^N$ is an asymptotic sequence as $t \to t_0$ if and only if the equation
$$\lim_{t\to t_0} \frac{F_{n+1}(t)}{F_n(t)} = 0 \tag{7.28}$$
holds for all indices $n < \bar{N}$. Substituting (7.27) into the limit in (7.28) yields
$$\begin{aligned}\lim_{t\to t_0} \frac{F_{n+1}(t)}{F_n(t)} &= \lim_{t\to t_0} \left(\frac{c_n}{c_{n+1}} \cdot \frac{g_{n+1}(X(t))}{g_n(X(t))} \cdot \frac{h_{n+1}(Y(t))}{h_n(Y(t))} \right) \\ &= \frac{c_n}{c_{n+1}} \cdot \lim_{t\to t_0} \frac{g_{n+1}(X(t))}{g_n(X(t))} \cdot \lim_{t\to t_0} \frac{h_{n+1}(Y(t))}{h_n(Y(t))}\end{aligned}$$

$$= \frac{c_n}{c_{n+1}} \cdot \lim_{x \to a} \frac{g_{n+1}(x)}{g_n(x)} \cdot \lim_{y \to b} \frac{h_{n+1}(y)}{h_n(y)} \tag{7.29}$$

for all indices $n < \bar{N}$. Since $\{g_n\}_{n=1}^N \subset C^\omega(A)$ and $\{h_n\}_{n=1}^N \subset C^\omega(B)$ are tractable with essential length $\bar{N}$, and since $\{g_n(x)\}_{n=1}^N$ and $\{h_n(y)\}_{n=1}^N$ are asymptotic sequences as $x \to a$ and $y \to b$, respectively, Proposition 3.23 implies that

$$\lim_{x \to a} \frac{g_{n+1}(x)}{g_n(x)} = \lim_{y \to b} \frac{h_{n+1}(y)}{h_n(y)} = 0 \tag{7.30}$$

for all indices $n < \bar{N}$. Equations (7.30) and (7.29) imply that (7.28) holds for all indices $n < \bar{N}$. This proves conclusion 2.

3. For each index n, let s_n and S_n denote the n-th partial sums of expansions (7.23) and (7.26), respectively, and let r_n and R_n denote the n-th remainders. Note that equation (7.25) implies

$$S_n(t) = \sum_{m=1}^{n} F_m(t) = \sum_{m=1}^{n} f_m(X(t), Y(t)) = s_n(X(t), Y(t)) \tag{7.31}$$

for all $t \in I$, and equation (7.24) implies

$$\begin{aligned} R_n(t) &= F(t) - S_n(t) \\ &= f(X(t), Y(t)) - s_n(X(t), Y(t)) \\ &= r_n(X(t), Y(t)) \end{aligned} \tag{7.32}$$

for all $t \in I$. We will prove conclusion 3 by considering the following two cases.

Case 1. If $\bar{N} < N$, then dual asymptotic expansion (7.23) is *terminating* or *trivial*, and *Exact Identity Theorem* 5.9 implies that $f = s_{\bar{N}}$. Equations (7.24) and (7.31) further imply that

$$F(t) = f(X(t), Y(t)) = s_{\bar{N}}(X(t), Y(t)) = S_{\bar{N}}(t)$$

for all $t \in I$. Since $F = S_{\bar{N}}$, Theorem 3.40 implies that asymptotic expansion (7.26) holds.

Case 2. If $\bar{N} = N$, then the *nonterminating* asymptotic expansion (7.26) holds if and only if the limit

$$\lim_{t \to t_0} \frac{R_{n-1}(t)}{F_n(t)} = 1 \tag{7.33}$$

holds for all indices n, by Theorem 3.35. In what follows, let n be any index. Note that equations (7.32) and (7.25), along with $f_n = \Upsilon_{(a,b)}[r_{n-1}]$, imply that

$$\frac{R_{n-1}(t)}{F_n(t)} = \frac{r_{n-1}(X(t), Y(t))}{f_n(X(t), Y(t))} = \frac{r_{n-1}(X(t), Y(t))}{\Upsilon_{(a,b)}[r_{n-1}](X(t), Y(t))} \tag{7.34}$$

for all $t \in I$. Since $\bar{N} = N$ by assumption, it follows that $\{r_{n-1}\}_{n=1}^{N}$ is *nonterminating* and $r_{n-1} \neq 0_{A \times B}$. Consequently, we can let

$$(p_n, q_n) = \deg_{(a,b)}(r_{n-1}) \text{ and } \hat{r}_{n-1} = \rho_{(a,b)}[r_{n-1}]$$

and write

$$r_{n-1}(x, y) = (x - a)^{p_n} (y - b)^{q_n} \, \hat{r}_{n-1}(x, y) \tag{7.35}$$

for all $(x, y) \in A \times B$. Since $r_{n-1} \in \mathcal{S}_{(a,b)}(A \times B) - \{0_{A \times B}\}$, Proposition 6.13 implies $\hat{r}_{n-1}(a, b) \neq 0$, and Proposition 6.14 implies

$$\Upsilon_{(a,b)}[r_{n-1}](x, y) = (x - a)^{p_n} (y - b)^{q_n} \cdot \frac{\hat{r}_{n-1}(x, b) \, \hat{r}_{n-1}(a, y)}{\hat{r}_{n-1}(a, b)} \tag{7.36}$$

for all $(x, y) \in A \times B$. Dividing equation (7.35) by equation (7.36) yields, after cancellation,

$$\frac{r_{n-1}(x, y)}{\Upsilon_{(a,b)}[r_{n-1}](x, y)} = \frac{\hat{r}_{n-1}(x, y) \, \hat{r}_{n-1}(a, b)}{\hat{r}_{n-1}(x, b) \, \hat{r}_{n-1}(a, y)} \tag{7.37}$$

for all $(x, y) \in A \times B$. Equations (7.34) and (7.37) imply

$$\frac{R_{n-1}(t)}{F_n(t)} = \frac{\hat{r}_{n-1}(X(t), Y(t))\, \hat{r}_{n-1}(a, b)}{\hat{r}_{n-1}(X(t), b)\, \hat{r}_{n-1}(a, Y(t))}. \tag{7.38}$$

Using equation (7.38), we find that limit (7.33) evaluates to

$$\lim_{t \to t_0} \frac{R_{n-1}(t)}{F_n(t)} = \frac{\hat{r}_{n-1}(a, b)\, \hat{r}_{n-1}(a, b)}{\hat{r}_{n-1}(a, b)\, \hat{r}_{n-1}(a, b)} = 1$$

by continuity, since $\hat{r}_{n-1}(a, b) \neq 0$. Since n was arbitrary, limit (7.33) holds for all indices n, and asymptotic expansion (7.26) holds.

This proves conclusion 3. ■

7.2.2 Polarization and Depolarization

When applying Theorem 7.8, it is often helpful to convert a univariate function into a bivariate function, and vice versa. The following definition introduces some terminology for describing such relationships between univariate and bivariate functions.

Definition 7.9 *Assume $I \subset \mathbb{R}$ is an open interval.*

1. *Let the univariate function $F \in C^\omega(I)$ be given, and suppose that we can find a bivariate function $f \in C^\omega(I \times I)$ such that*

   $$f(t, t) = F(t) \text{ for all } t \in I.$$

   *We say that f **is a polarization of** F.*

2. *Let the bivariate function $f \in C^\omega(I \times I)$ be given, and define the univariate function $F \in C^\omega(I)$ via*

   $$F(t) = f(t, t) \text{ for all } t \in I.$$

   *We say that F **is the depolarization of** f.*

Example 7.10 *If* $F(t) = e^{-t^2}$, *then* $f_1(x,y) = e^{-xy}$ *and* $f_2(x,y) = e^{-(x^2+y^2)/2}$ *are two different polarizations of* F. *By construction, the depolarizations of* f_1 *and* f_2 *are both* F.

Remark 7.11 *Please note the following:*

1. *A univariate function* $F \in C^\omega(I)$ *may have* many *different polarizations, but a bivariate function* $f \in C^\omega(I \times I)$ *has only* one *depolarization.*
2. *The depolarization* $f(t,t)$ *of any polarization* $f(x,y)$ *automatically reconstructs the original univariate function* $F(t)$.

The definitions of polarization and depolarization are motivated by an analogy with quadratic forms and bilinear forms on a vector space V over the reals $\mathbb{R}$. If $Q : V \to \mathbb{R}$ is a quadratic form, then there is a unique symmetric bilinear form $B : V \times V \to \mathbb{R}$ such that $B(v,v) = Q(v)$ for all $v \in V$. (If Q is positive-definite, then B is actually an inner product on V.) The bilinear form B is a *polarization* of the quadratic form Q, and the *depolarization* of B reconstructs Q.

The following corollary of Theorem 7.8 applies the polarization and depolarization processes to the problem of generating asymptotic expansions of univariate functions.

Corollary 7.12 *Assume* $I \subset \mathbb{R}$ *is an open interval. Let* $F \in C^\omega(I)$ *be given, and let* $t_0 \in I$. *Suppose there exists a polarization* f *of* F *such that* $f \in \mathcal{D}^N_{(t_0,t_0)}(I \times I)$. *Let*

$$f(x,y) \sim \sum_{n=1}^{N} f_n(x,y) \text{ as } x \to t_0 \text{ or } y \to t_0$$

denote the dual asymptotic expansion of f *to* N *terms at* (t_0,t_0), *and let* $\bar{N}$ *denote its essential length. If* F_n *denotes the depolarization of* f_n *for each index* n, *then* $\{F_n(t)\}_{n=1}^N$ *is an asymptotic sequence as* $t \to t_0$, *and*

$$F(t) \sim \sum_{n=1}^{N} F_n(t) \text{ as } t \to t_0$$

is an asymptotic expansion with essential length $\bar{N}$.

Proof. Apply Theorem 7.8 with $A = B = I$, $a = b = t_0$, and $X(t) = Y(t) = t$. ■

Given a univariate function $F \in C^{\omega}(I)$, Corollary 7.12 enables us to construct an asymptotic sequence $\{F_n(t)\}_{n=1}^{N}$ which yields a meaningful asymptotic expansion of $F(t)$ as $t \to t_0$. This is achieved by following a systematic, three-step process:

1. Find a polarization f of F such that $f \in \mathcal{D}^{N}_{(t_0,t_0)}(I \times I)$.
2. Compute the terms $\{f_n\}_{n=1}^{N}$ of the dual asymptotic expansion of f at (t_0, t_0) explicitly via $f_n = \Upsilon_{(t_0,t_0)}[r_{n-1}]$ for all indices n.
3. Depolarize the terms $\{f_n\}_{n=1}^{N}$ to obtain the asymptotic sequence $\{F_n\}_{n=1}^{N}$.

This three-step process is depicted in Figure 7.1 as a commutative diagram. The first step in the process — finding a suitable polarization — is the most difficult of the three. The next section explores some techniques designed to help us with this initial step.

	Dual Asymptotic Expansion	
$f(x, y)$	$\longrightarrow$	$\sum_{n=1}^{N} f_n(x, y)$
↑ Polarization |		| Depolarization ↓
$F(t)$	$\longrightarrow$	$\sum_{n=1}^{N} F_n(t)$
	Univariate Asymptotic Expansion	

Figure 7.1: Three-Step Process for Generating Univariate Expansions

7.2.3 Auxiliary Functions

The following definition introduces a useful device for constructing polarizations of univariate functions.

Definition 7.13 *Let $I \subset \mathbb{R}$ be an open interval. If $u \in C^\omega(I \times I)$ is such that $u(t,t) = t$ for all $t \in I$, we call u an* ***auxiliary function for*** *I. If $u(I \times I) \subset I$ also holds, we say that u is* ***well-behaved.***

An auxiliary function for I is simply a polarization of the identity function on I. The following example shows that well-behaved auxiliary functions can be obtained by means of **means**.

Example 7.14 *If I is an open interval, then each of the following functions $u \in C^\omega(I \times I)$ is a well-behaved auxiliary function for I:*

1. *The* ***arithmetic mean*** *on $I \subset \mathbb{R}$, defined by*
$$u(x,y) = \frac{x+y}{2}.$$
2. *The* ***geometric mean*** *on $I \subset (0,\infty)$, defined by*
$$u(x,y) = \sqrt{x \cdot y}.$$
3. *The* ***harmonic mean*** *on $I \subset (0,\infty)$, defined by*
$$u(x,y) = \frac{2xy}{x+y}.$$

Auxiliary functions can be used to construct polarizations in at least two different ways:

1. Let $F \in C^\omega(I)$ be given. If $u \in C^\omega(I \times I)$ is a *well-behaved* auxiliary function for I, then
$$f(x,y) = F(u(x,y))$$
defines a polarization of F.

2. Let $F \in C^\omega(I)$ be given, and let J be an open interval such that $F(I) \subset J$. If $u \in C^\omega(J \times J)$ is an auxiliary function for J, then

$$f(x, y) = u(F(x), F(y))$$

defines a polarization of F. Note that u need *not* be well-behaved in this case.

In the next section, we will apply auxiliary functions to a concrete example.

7.2.4 Example and Commentary

The following example uses an auxiliary function to generate a *univariate* asymptotic expansion via the previously described three-step process: *polarization*, computation of the *dual* asymptotic expansion, and *depolarization*.

Define $F(t) = e^{-t^2}$ for all $t \in \mathbb{R}$, and let $a \in \mathbb{R}$ be an arbitrary constant. Recall that $u(x, y) = \sqrt{x\,y}$ (the geometric mean) is a well-behaved auxiliary function for $(0, \infty)$. Define a polarization f of F by

$$f(x, y) = F(u(x, y)) = e^{-\left(\sqrt{x \cdot y}\right)^2} = e^{-x\,y},$$

and observe that

$$f(t, t) = e^{-t^2} = F(t).$$

Although $u(x, y)$ is analytic only for $x \cdot y > 0$, note that the polarization $f = F \circ u$ can be analytically continued to all of $\mathbb{R}^2$ since F is an *even* function.

In Example 6.26, we determined the three-term dual asymptotic expansion of $e^{-x\,y}$ at (a, b). The result, given by expansion (6.49), can be written in factored form as

$$e^{ab-bx-ay} \cdot \left(1 - (x-a)\,(y-b) + \frac{1}{2}\,(x-a)^2\,(y-b)^2\right). \tag{7.39}$$

Setting $b = a$ in (7.39) gives the dual asymptotic expansion of $e^{-x\,y}$ at (a, a), which is

$$e^{-a\,(x+y-a)} \cdot \left(1 - (x-a)\,(y-a) + \frac{1}{2}\,(x-a)^2\,(y-a)^2\right). \tag{7.40}$$

Letting $x = y = t$ in (7.40) depolarizes the terms and produces

$$e^{-a(2t-a)} \cdot \left(1 - (t-a)^2 + \frac{1}{2}(t-a)^4\right), \tag{7.41}$$

which is a (factored) three-term asymptotic expansion of e^{-t^2} as $t \to a$.

Let us make the following observations concerning this example:

1. If $a = 0$, expansion (7.41) reduces to

$$1 - t^2 + \frac{1}{2}t^4,$$

which is the sum of the first three terms of the Maclaurin series for e^{-t^2}. If $a \neq 0$, the presence of the exponential factor $e^{-a(2t-a)}$ in expansion (7.41) shows that (7.41) is *not* a truncated Taylor series, but a *different kind of expansion altogether.*

2. Although the original function e^{-t^2} is bounded and tends to zero as $t \to \infty$, every nonconstant Taylor polynomial $P(t)$ is unbounded and becomes infinite as $t \to \infty$. In contrast, for any fixed $a > 0$, expansion (7.41) is bounded for all $t > t_0$ and tends to zero as $t \to \infty$. Hence, *expansion (7.41) exhibits the appropriate qualitative behavior* on the semi-infinite interval (t_0, ∞) for any fixed $t_0 \in \mathbb{R}$.

3. The original function e^{-t^2} *cannot* be integrated in closed form over the class of elementary functions. In contrast, the terms in expansion (7.41) are of the form

$$e^{-a(2t-a)} \cdot \frac{1}{n!}(t-a)^{2n},$$

and *can* be integrated in closed form via repeated integration by parts. Hence, *expansion (7.41) is more amenable to integration* than the function it approximates. This property can be used to approximate the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \tag{7.42}$$

accurately near $x = a$.

In the case $a = 0$, the resulting approximation for $\operatorname{erf}(x)$ will reduce to a Maclaurin series; however, by using a somewhat different technique, we can develop a *new* approximation for $\operatorname{erf}(x)$ that is more accurate near $x = 0$ than the Maclaurin series. This technique is illustrated in the next section.

7.3 Approximating Special Functions

This section shows that dual asymptotic expansions can be used to approximate special functions by series of elementary functions. Throughout this section, we will use the MAPLE (version 5.4) computer algebra system to expedite the intermediate calculations, to compute numerical approximations when needed, and to plot the graphs of various functions.

7.3.1 A New Approximation of the Error Function

Suppose that we wish to approximate the error function $\operatorname{erf}(x)$, defined by equation (7.42), near the point $x = 0$. As a preliminary step, make the change of variable $t = x\,y$ to convert (7.42) from an *indefinite* integral of a *univariate* function into a *definite* integral of a *bivariate* function; this yields

$$\operatorname{erf}(x) = \frac{2\,x}{\sqrt{\pi}} \int_0^1 e^{-x^2 y^2} dy. \tag{7.43}$$

Now expand the integrand of (7.43) in a three-term dual asymptotic expansion at the point $(0, b)$, and extract a common factor from each term to obtain

$$e^{-x^2 y^2} \sim e^{-b^2 x^2} \cdot \left(1 - x^2\left(y^2 - b^2\right) + \frac{1}{2} x^4 \left(y^2 - b^2\right)^2\right) \tag{7.44}$$

as $x \to 0$ or $y \to b$. Taking the *definite* integral of expansion (7.44) with respect to y and multiplying by $2x\pi^{-1/2}$ produces a *univariate* asymptotic expansion in x:

$$\operatorname{erf}(x) \sim \frac{2\,x}{\sqrt{\pi}} e^{-b^2x^2} \cdot \left(1 + c_1\,x^2 + c_2\,x^4\right) \text{ as } x \to 0,$$

where

$$c_1 = b^2 - \frac{1}{3} \quad \text{and} \quad c_2 = \frac{1}{2}b^4 - \frac{1}{3}b^2 + \frac{1}{10}.$$

Hence, using the theory of dual asymptotic expansions, we have produced an approximation

$$A(x) = \frac{2\,x}{\sqrt{\pi}} e^{-b^2x^2} \left[1 + \left(b^2 - \frac{1}{3}\right) x^2 + \left(\frac{1}{2}b^4 - \frac{1}{3}b^2 + \frac{1}{10}\right) x^4\right]$$

of $\operatorname{erf}(x)$ which consists entirely of elementary functions.

7.3.2 The Classical Approximations

Let us compare the new approximation $A(x)$ to two of the best-known classical approximations. The fifth degree Taylor polynomial for $\operatorname{erf}(x)$ at $x = 0$ is

$$T(x) = \frac{2\,x}{\sqrt{\pi}} \left[1 - \frac{1}{3}x^2 + \frac{1}{10}x^4\right],$$

and a comparable Padé approximation for $\operatorname{erf}(x)$ at $x = 0$ is

$$P(x) = \frac{2\,x}{\sqrt{\pi}} \left[\frac{10 - \frac{1}{3}\,x^2}{10 + 3\,x^2}\right].$$

Note that the approximation errors for $A(x)$, $T(x)$, and $P(x)$ all have the same asymptotic order:

$$\operatorname{erf}(x) - A(x) \quad = \quad \left(\frac{1}{3}\,b^6 - \frac{1}{3}\,b^4 + \frac{1}{5}\,b^2 - \frac{1}{21}\right) \frac{1}{\sqrt{\pi}} x^7 + o(x^7) \text{ as } x \to 0,$$

$$\begin{aligned} \operatorname{erf}(x) - T(x) &= -\frac{1}{21}\frac{1}{\sqrt{\pi}}x^7 + o(x^7) \text{ as } x \to 0, \\ \operatorname{erf}(x) - P(x) &= \frac{13}{1050}\frac{1}{\sqrt{\pi}}x^7 + o(x^7) \text{ as } x \to 0. \end{aligned}$$

However, the *coefficients* of the x^7 terms in these asymptotic error formulas are *not* the same. In particular, the coefficient of x^7 in the error for $A(x)$ is

$$\left(\frac{1}{3}b^6 - \frac{1}{3}\,b^4 + \frac{1}{5}\,b^2 - \frac{1}{21}\right)\frac{1}{\sqrt{\pi}}, \tag{7.45}$$

which depends on the free parameter b.

7.3.3 Choosing the Parameter

We can *increase* the asymptotic order of the approximation error for $A(x)$ by finding values of b which make the coefficient (7.45) equal to zero. Since the polynomial (7.45) is cubic in b^2, it is possible to solve for the roots of (7.45) in terms of radicals. MAPLE is able to compute these roots exactly; however, as often happens with solution by radicals, the results are too unwieldy to display here.

Polynomial (7.45) has exactly one positive real root, whose *exact* value we will denote by b_0. Numerical computation in MAPLE shows that

$$b_0 \approx 0.6292111285.$$

Upon setting $b = b_0$, the approximation error for $A(x)$ becomes

$$\operatorname{erf}(x) - A(x) = c \cdot x^9 + o(x^9) \text{ as } x \to 0,$$

where the *exact* value of the constant c is also unwieldy; the approximate value is

$$c \approx 0.0006956928575.$$

Compare the approximation error for $A(x)$ to the approximation errors for $T(x)$

and $P(x)$, which are roughly

$$\begin{aligned} \operatorname{erf}(x) - T(x) &\approx -0.0269\,x^7 + 0.00522\,x^9 + o(x^9) \text{ as } x \to 0, \\ \operatorname{erf}(x) - P(x) &\approx 0.00699\,x^7 - 0.00493\,x^9 + o(x^9) \text{ as } x \to 0. \end{aligned}$$

Not only do the errors for $T(x)$ and $P(x)$ have nonvanishing x^7 terms, but the coefficients of the x^9 terms are nearly an order of magnitude larger than the coefficient c.

7.3.4 Comparing the Accuracy

In light of our analysis of the errors, we expect $A(x)$ to approximate $\operatorname{erf}(x)$ noticeably better than both $T(x)$ and $P(x)$ near $x = 0$. In order to test this hypothesis, let us compare the accuracy of the approximations $T(x)$, $P(x)$, and $A(x)$ using the plots in Figures 7.2, 7.3, and 7.4.

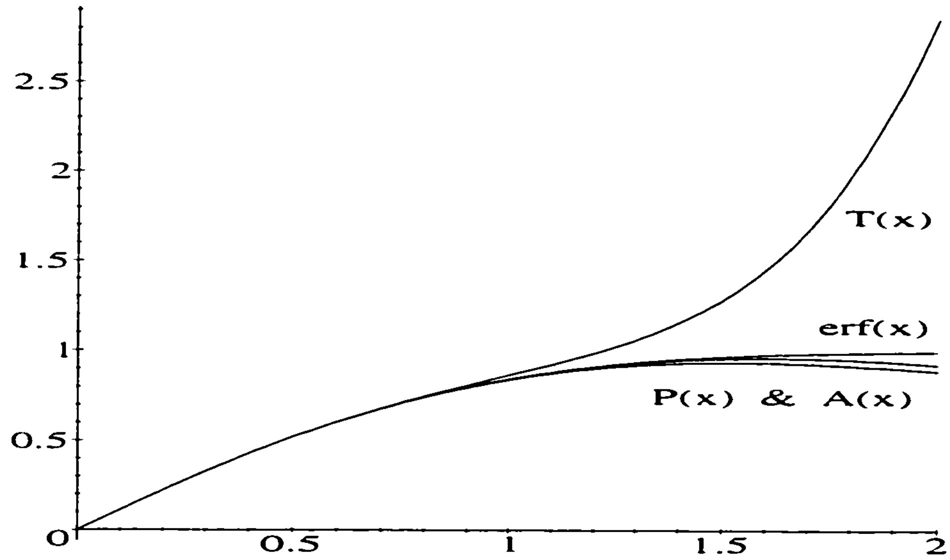


Figure 7.2: Plot of $T(x)$, $P(x)$, $A(x)$, and $\operatorname{erf}(x)$ on $[0, 2]$.

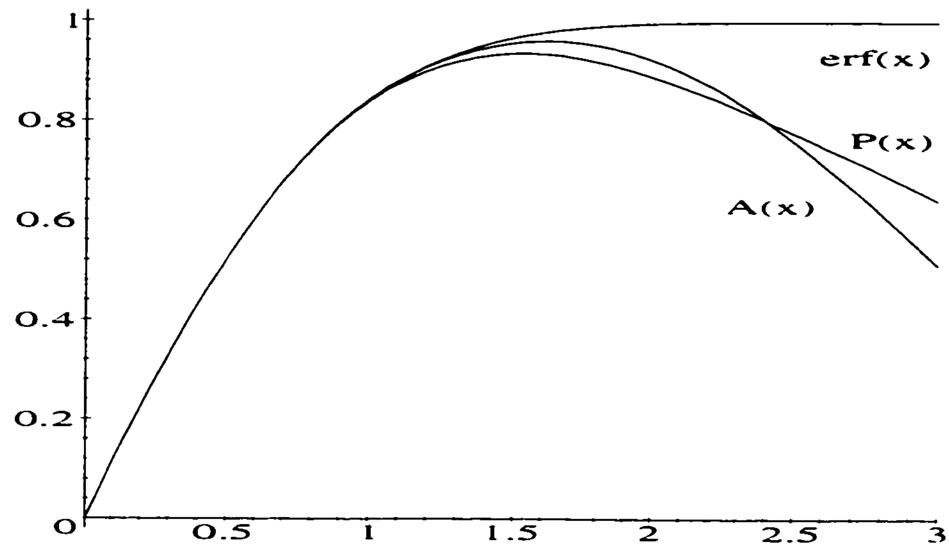


Figure 7.3: Plot of $P(x)$, $A(x)$, and $\operatorname{erf}(x)$ on $[0, 3]$.

Figure 7.2 shows a plot of $T(x)$, $P(x)$, $A(x)$, and $\operatorname{erf}(x)$ on the interval $[0, 2]$. The wildly misbehaved curve in this plot corresponds to the Taylor polynomial $T(x)$. Figure 7.3 omits $T(x)$ and shows a plot of $P(x)$, $A(x)$, and $\operatorname{erf}(x)$ on the interval $[0, 3]$.

How do we know which curve belongs to which function in these plots? (The curves were labeled *manually*, not by the software which produced them.) The asymptotic behavior of $P(x)$, $A(x)$, and $\operatorname{erf}(x)$ identifies each of these curves uniquely since $P(x) \to -\infty$, $A(x) \to 0$, and $\operatorname{erf}(x) \to 1$ as $x \to \infty$. To see this more clearly, consider Figure 7.4, which shows a plot of $P(x)$, $A(x)$, and $\operatorname{erf}(x)$ on the interval $[0, 8]$; the longer interval accentuates the behavior of the three functions for "large" values of x, and makes it easier to identify the three curves.

Together, Figures 7.3 and 7.4 make clear that $A(x)$ is a better approximation of $\operatorname{erf}(x)$ than $P(x)$ as x varies from 0 to 2.4. This conclusion is corroborated by Table 7.1, which lists the numerical values of the approximation errors for $T(x)$, $P(x)$, and $A(x)$ on the interval $[0, 3]$.

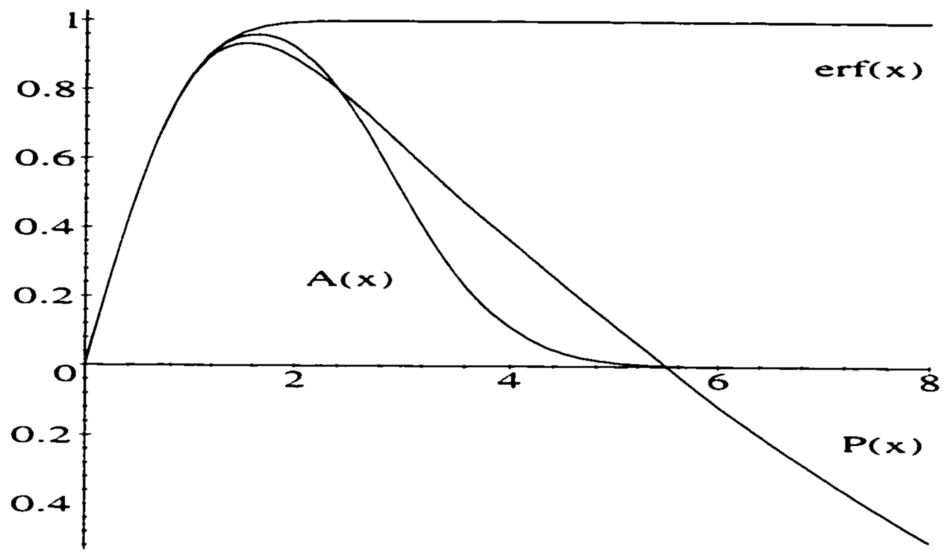


Figure 7.4: Plot of $P(x)$, $A(x)$, and $\mathrm{erf}(x)$ on $[0, 8]$.

On the basis of these comparisons, we can conclude that the best approximation to $\mathrm{erf}(x)$ on the interval $[0, 2.4]$ is the new approximation $A(x)$; the second-best approximation is the Padé approximation $P(x)$; and, the worst approximation of the three is the Taylor polynomial $T(x)$. Hence, in this example, the new approximation $A(x)$ out-performs *both* of the classical approximations.

x	$\text{erf}(x) - T(x)$	$\text{erf}(x) - P(x)$	$\text{erf}(x) - A(x)$
0	0	0	0
0.1000000000	-0.0000000027	0.0000000007	0
0.2000000000	-0.0000003412	0.0000000869	0.0000000004
0.3000000000	-0.0000057742	0.0000014344	0.0000000133
0.4000000000	-0.0000426832	0.0000102386	0.0000001704
0.5000000000	-0.0002000920	0.0000459210	0.0000012227
0.6000000000	-0.0007023858	0.0001528686	0.0000060264
0.7000000000	-0.0020175401	0.0004129796	0.0000228559
0.8000000000	-0.0050003865	0.0009552744	0.0000714256
0.9000000000	-0.0110665618	0.0019591884	0.0001921704
1.0	-0.0223899016	0.0036496175	0.0004587312
1.100000000	-0.0421143838	0.0062838213	0.0009926575
1.200000000	-0.0745714669	0.0101321624	0.0019784257
1.300000000	-0.1254945811	0.0154558884	0.0036761160
1.400000000	-0.2022243002	0.0224855275	0.0064287224
1.500000000	-0.3138999715	0.0314030007	0.0106613046
1.600000000	-0.4716359077	0.0423295495	0.0168699650
1.700000000	-0.6886822354	0.0553203520	0.0255999223
1.800000000	-0.9805720644	0.0703655571	0.0374134945
1.900000000	-1.365257613	0.0873966092	0.0528503643
2.0	-1.863238291	0.1062962545	0.0723838331
2.100000000	-2.497683713	0.1269105025	0.0963775129
2.200000000	-3.294554206	0.1490609760	0.1250471659
2.300000000	-4.282720839	0.1725564342	0.1584318854
2.400000000	-5.494086350	0.1972026588	0.1963778290
2.500000000	-6.963707842	0.2228102882	0.2385363047
2.600000000	-8.729921611	0.2492005266	0.2843763980
2.700000000	-10.83447009	0.2762088803	0.3332108376
2.800000000	-13.32263089	0.3036872127	0.3842324868
2.900000000	-16.24334741	0.3315044787	0.4365579527
3.0	-19.64936085	0.3595464905	0.4892743772

Table 7.1: Approximation Errors for $T(x)$, $P(x)$, and $A(x)$ on $[0, 3]$

Chapter 8

Conclusions

This final chapter strives to impart some perspective on the thesis. Let us reflect on our recent journey through the world of dual asymptotic expansions, and ponder what we may encounter on the road ahead.

8.1 Potential New Applications

This section discusses two broad application areas which may prove well-suited to the methods of the thesis.

8.1.1 Hermite Interpolation

The previous chapter presented an example showing that dual asymptotic expansions can be used successfully to approximate special functions in terms of elementary functions, and that the resulting approximation can be made more accurate than both the Taylor and Padé approximations. All of these approximations share a common property: they reproduce the *exact values* of the derivatives of the original function at the point in question up to some specified order. Approximations with this property are examples of a scheme called **generalized Taylor interpolation.** Technically, this is a method of *interpolation* rather than approximation, since some characteristics of the original function are reproduced *exactly* by the interpolating function; for our purposes, this is a minor distinction since well-chosen interpolating functions can yield good approximations.

Generalized Taylor interpolation is a special case of a much more general scheme called **Hermite interpolation**. In this scheme, the interpolating function reproduces the exact values of the derivatives of the original function *at a number of different points* up to some specified order, where the orders need not be the same for all points. Note that taking only zero-order derivatives reduces the method to ordinary interpolation at discrete points, whereas interpolating at only one point returns us to generalized Taylor interpolation.

By the interpolating properties of the asymptotic splitting operator, the expression

$$(I - \Upsilon_{(a,b)})[f](x, y)$$

vanishes on the lines $x = a$ and $y = b$. This implies there is some function $g \in C^{\omega}(A \times B)$ such that

$$(I - \Upsilon_{(a,b)})[f](x, y) \equiv (x - a)\,(y - b) \cdot g(x, y).$$

Applying this idea repeatedly to the remainder formula for dual asymptotic expansions yields

$$r_n(x, y) \equiv (I - \Upsilon_{(a,b)})^n\,[f](x, y) \equiv (x - a)^n (y - b)^n \cdot g_n(x, y),$$

from which it can be seen that dual asymptotic expansions are indeed generalized Taylor interpolants.

Presumably, a *different kind of expansion* could be generated by applying the asymptotic splitting operator at a number of *different* points (a_n, b_n), thereby producing a remainder of the form

$$r_n(x, y) \equiv \prod_{m=1}^{n} (I - \Upsilon_{(a_m,b_m)})\,[f](x, y) \equiv \prod_{m=1}^{n} (x - a_m)\,(y - b_m) \cdot g_n(x, y).$$

This method can accommodate any Hermite interpolation scheme by a suitable choice of points (a_n, b_n). In fact, when interpolating by this method at n *specified* points $a_1, \ldots, a_n \in A$, there are still n parameters $b_1, \ldots, b_n \in B$ which can be *freely chosen.* These extra n degrees of freedom may dramatically decrease the approximation error of an interpolation scheme produced in this way. For example, a generalized Taylor interpolant produced by this method may be able to interpolate the first $2n$ nonzero derivatives of a function using a series with only n nonzero terms.

8.1.2 Singular Perturbation Problems

In the previous chapter, we noted that asymptotic expansions produced by the methods of the thesis can be qualitatively well-behaved for large arguments when the corresponding Taylor approximation is not. In fact, it is precisely this drawback of Taylor approximations which results in the failure of direct Taylor series methods to produce satisfactory solutions $u(t, \varepsilon)$ to singular perturbation problems; the solutions are unsatisfactory because they fail to be uniformly valid at fixed values of the small parameter ε for large values of the independent variable t.

In contrast, dual asymptotic expansions may be ideally suited for singular perturbation problems, for two reasons:

1. The exact solution to such problems is (generally assumed to be) an *analytic function $u(t, \varepsilon)$ of two variables*, and the methods of the thesis are *specifically designed for analytic functions of two variables.*

2. Any *good* approximation to the exact solution must exhibit the appropriate qualitative behavior, and dual asymptotic expansions appear to possess the qualitative behavior of the functions from which they are derived.

The major challenge in this application is to determine how to compute the terms of the dual asymptotic expansion of a function which is defined indirectly — for example, by an initial-value problem involving an ordinary differential equation. The only hint the author can provide at this time is to expand at a point (t_0, ε_0), where the initial data is specified at t_0, and ε_0 is small but *positive.* When $\varepsilon_0 =$

0, the dual asymptotic expansion may in some cases reduce to a Taylor series, thereby losing the qualitative advantage offered by the method of dual asymptotic expansions.

8.2 Further Theoretical Developments

This section proposes some natural extensions to the theoretical results presented in the thesis.

8.2.1 Convergence Properties

The example of the infinite sum formula for the Bessel function $J_0(x+y)$ underscored the need to understand the convergence properties of *infinite* dual asymptotic expansions. The author proposes to study this question by the following approach.

If $K \subset A \times B$ is a compact rectangle, we can define a norm on the function space $C^\omega(A \times B)$ by

$$\|f\|_K = \max_{(x,y)\in K} |f(x,y)|.$$

Note that $\|f\|_K = 0$ implies $f = 0_{A\times B}$ by *Unique Continuation*. Let the subset

$$S \subset C^\omega(A \times B) - \{0_{A\times B}\}$$

be arbitrary, and suppose that

$$\Phi : S \to C^\omega(A \times B)$$

is an operator (though not necessarily a *linear* operator). We can define the operator norm of Φ by

$$\|\Phi\|_K = \sup_{f\in S} \frac{\|\Phi[f]\|_K}{\|f\|_K}.$$

Let $f \in \mathcal{D}^\infty_{(a,b)}(A \times B)$, and define the subset

$$S_{(a,b)}(f) = \{(I - \Upsilon_{(a,b)})^n[f] \mid n \in \mathbb{N}\},$$

which is the *orbit* of the function f under the operator $I - \Upsilon_{(a,b)}$. Assume that $0_{A\times B} \notin S_{(a,b)}(f)$, and let

$$\alpha = \left\|(I - \Upsilon_{(a,b)}) \mid S_{(a,b)}(f)\right\|_K .$$

If we can determine suitable hypotheses on the point (a, b), the function f, and the rectangle K to ensure that $\alpha < 1$, it will follow immediately from the inequality

$$\|r_n\|_K = \left\|(I - \Upsilon_{(a,b)})^n [f]\right\|_K \leq \alpha^n \ \|f\|_K$$

that the infinite series

$$\sum_{n=1}^{\infty} \frac{g_n(x)\, h_n(y)}{c_n}$$

converges uniformly to $f(x, y)$ for all $(x, y) \in K$. Hence, the question of uniform convergence on compact rectangles K can be reduced to a study of the operator norm

$$\left\|(I - \Upsilon_{(a,b)}) \mid S_{(a,b)}(f)\right\|_K .$$

8.2.2 Asymptotic Inner-Product Spaces

The infinite sum formula for $J_0(x + y)$ also called attention to the need for a more efficient algorithm to compute dual asymptotic expansions. Let $f \in C^\omega(A \times B)$ and $(a, b) \in A \times B$ be given. The author contends that a much better algorithm can be developed by applying an *Asymptotic Gram-Schmidt* process to the univariate sequences of partial derivatives

$$\left\{\left(\frac{\partial}{\partial y}\right)^n [f](x, b)\right\}_{n=0}^{\infty} \quad \text{and} \quad \left\{\left(\frac{\partial}{\partial x}\right)^n [f](a, y)\right\}_{n=0}^{\infty} .$$

The *Asymptotic Gram-Schmidt* process — which is analogous to the well-known Gram-Schmidt process of classical linear algebra — is an algorithm developed by the author to transform a given sequence of functions into an asymptotic sequence with the same linear span. The analogy between the classical Gram-Schmidt algorithm and its asymptotic counterpart is made possible by defining an *Asymptotic Inner*

Product

$$[f,g]_a = \lim_{x\to a^+} \frac{f(x)}{g(x)}, \tag{8.1}$$

which endows a linear space of analytic functions with the structure of an *Asymptotic Inner-Product Space.*

The astute reader will recognize that limits of the form (8.1) have appeared numerous times throughout the thesis — for example, in the characterization of the Landau "little oh" order relation, and in the computation of the coefficients of univariate asymptotic expansions. Both of these operations apply the asymptotic inner product implicitly. Even the asymptotic splitting operator itself can be written in terms of the asymptotic inner product, which indicates that there is a strong natural connection between these two bodies of ideas.

The author has developed an abstract theory of *Nonlinear Inner-Product Spaces*, which shows that the theories of classical inner-product spaces and *Asymptotic Inner-Product Spaces* and are both special cases of a single more general theory. The author hopes to make more information about these developments available in the near future, and plans to use these ideas to reformulate the univariate and bivariate theory of the thesis into a single, more unified framework.

8.2.3 Expansions at Boundary Points

The thesis has confined itself to studying asymptotic behavior at points in the *domain of analyticity* of the function in question. However, it is the behavior at *boundary points* which is often of greatest interest in applications. How might we extend the theory to develop dual asymptotic expansions at boundary points? Before we can answer this question, we need to understand what lies at the foundation of the current theory.

The ubiquitous *Unique Continuation Property*, despite its numerous cameo appearances throughout the thesis, is not the fundamental property which makes things work. The role of *Unique Continuation* has been to extend various local results to produce a global theory. If extending the theory to boundary points requires us to sacrifice *Unique Continuation*, we may have to be content with local rather than global results; however, since asymptotic analysis is by definition a study of local properties, this is not a serious loss.

The fundamental existence and uniqueness theory for dual asymptotic expansions is based upon the properties of the asymptotic splitting operator. The asymptotic splitting operator in turn depends heavily upon the *Symmetric Decomposition Theorem*. At the heart of the *Symmetric Decomposition Theorem* lies the defining property of analytic functions, namely: that every analytic function can be represented locally by a convergent power series. Hence, it is *local series representations* which lie at the foundation of the theory of dual asymptotic expansions. The question of how to extend the theory to boundary points thus becomes a question of how to represent analytic functions by more general types of series expansions at boundary points.

Certain classes of boundary points appear to offer ready-made solutions to this problem. For example, an analytic function with an isolated singularity can be expanded in a Laurent series, and an analytic function with an algebraic branch point can be expanded in a Puiseux series. Consequently, the *Symmetric Decomposition Theorem* should easily generalize when the Laurent or Puiseux series contains only a finite number of terms with negative powers — for example, as the Laurent series does when the singularity is a pole. Note that the degree of a function at a *pole* will be a pair of *integers* (rather than a pair of natural numbers, as for points of analyticity), and the degree of a function at an algebraic branch point will be a pair of *rational numbers*.

Extending the theory to poles and algebraic branch points would substantially enlarge the scope of the methods presented in the thesis. For example, this would allow us to develop dual asymptotic expansions at the poles of meromorphic functions, as well as at the branch points of algebraic functions.

Even greater breadth might be achieved by employing yet more general kinds of series expansions. Some of these generalized series have already been studied and implemented by various researchers in computer algebra, such as Keith Geddes and Gaston Gonnet ([Ged-Gon]), as well as Bruno Salvy ([Salvy]). By using the univariate methods of these researchers as a new foundation, it may be possible to significantly extend the bivariate methods presented in the thesis. Indeed, a successful synergy between these univariate and bivariate methods may well open new vistas for both.

Appendix A

Proof of Decomposition Theorem

This appendix contains a proof of *Decomposition Theorem* 2.9, which is restated below. Before we can prove this theorem, we need to understand how a function's vanishing or not vanishing on the line $x = a$ is affected by restricting the function to a subset of its domain.

Suppose that $f : A \times B \longrightarrow \mathbb{R}$ is an *arbitrary function*. Let $I \times J \subset A \times B$ be an open rectangle such that $a \in I$, and let $g = f \mid I \times J$. To say that $f(x, y)$ vanishes on $x = a$ means that f vanishes on the set $\{a\} \times B$, by Definition 2.8. However, to say that $g(x, y)$ vanishes on $x = a$ means that f vanishes on the *subset* $\{a\} \times J$, which is an ostensibly weaker condition. The next proposition shows that for *analytic functions*, the two conditions are actually equivalent.

Proposition A.1 *Assume $f \in C^\omega(A \times B)$. Let $I \times J \subset A \times B$ be an open rectangle such that $a \in I$, and let $g = f \mid I \times J$. The function $f(x, y)$ vanishes on $x = a$ if and only if the restriction $g(x, y)$ vanishes on $x = a$.*

Proof. Clearly, if $f(a, y) = 0$ for all $y \in B$, then $g(a, y) = 0$ for all $y \in J$, since $J \subset B$. To show the converse, define $F_a(y) = f(a, y)$ for all $y \in B$, and note that $F_a \in C^\omega(B)$. If $g(x, y)$ vanishes on $x = a$, then $F_a \mid J = 0_J$. The *Unique Continuation Property* of Theorem 2.4 implies that $F_a = 0_B$, which means that $f(x, y)$ vanishes on $x = a$. ■

We are now ready to prove *Decomposition Theorem* 2.9, which is restated here as Theorem A.2

Theorem A.2 *Assume $f \in C^\omega(A \times B)$ satisfies $f \neq 0_{A\times B}$, and let $a \in A$. There exist a unique $m \in \mathbb{N}$ and unique $g \in C^\omega(A \times B)$ such that*

$$f(x,y) = (x-a)^m g(x,y) \text{ for all } (x,y) \in A \times B, \tag{A.1}$$

subject to the restriction that $g(x,y)$ does not *vanish on $x = a$.*

Proof. We will prove uniqueness first, and then existence.

Uniqueness. Suppose that $m, n \in \mathbb{N}$ and $g, h \in C^\omega(A \times B)$ are such that

$$f(x,y) = (x-a)^m g(x,y) = (x-a)^n h(x,y) \tag{A.2}$$

for all $(x,y) \in A \times B$, subject to the restriction that $g(x,y)$ and $h(x,y)$ do not vanish on $x = a$. If $m < n$, we can divide (A.2) by $(x-a)^m$ to obtain

$$g(x,y) = (x-a)^{n-m} h(x,y) \tag{A.3}$$

for all $x \in A - \{a\}$ and all $y \in B$. Since $n - m > 0$, taking the limit of (A.3) as $x \to a$ in $A - \{a\}$ yields

$$g(a,y) = 0 \text{ for all } y \in B,$$

by the continuity of g. This contradicts the assumption that $g(x,y)$ does not vanish on $x = a$. Consequently, $m \geq n$. Similarly, if $m > n$, we can show that $h(x,y)$ vanishes on $x = a$, which again contradicts our assumptions. Consequently, $m = n$.

Since $m = n$, equation (A.2) implies that

$$g(x,y) = h(x,y) \tag{A.4}$$

for all $x \in A - \{a\}$ and all $y \in B$. By the continuity of g and h, equation (A.4) implies that

$$g(a,y) = h(a,y) \tag{A.5}$$

for all $y \in B$. Equations (A.4) and (A.5) together imply that $g = h$.

We have shown that $m = n$ and $g = h$, which means that decomposition (A.1) is unique.

Existence. We will prove the existence of decomposition (A.1) in four major steps.

Step 1. This step will show via power series that a decomposition exists locally on some rectangular neighborhood. Let $b \in B$ be arbitrary. By definition of analyticity, $f \in C^\omega(A \times B)$ has a convergent power series representation

$$f(x, y) = \sum_{i+j=0}^{\infty} c_{ij} (x-a)^i (y-b)^j \tag{A.6}$$

on some rectangular neighborhood $I_b \times J_b$ of $(a, b) \in A \times B$. (The neighborhood $I_b \times J_b$ is indexed by b to reflect that the neighborhood depends upon the choice of b.)

By definition, "convergence" for multivariate power series means *absolute convergence* (see [John, p. 62]). Since every rearrangement of an absolutely convergent series converges to the same sum, the order of summation is therefore inconsequential. In fact, it is even permissible to rearrange the absolutely convergent series (A.6) into a double infinite sum

$$f(x, y) = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} c_{ij} (x-a)^i (y-b)^j \right) \tag{A.7}$$

(see [Knopp56, pp. 83-87]).

For each $i \in \mathbb{N}$, the absolute convergence of the inner subseries

$$\sum_{j=0}^{\infty} c_{ij} (x-a)^i (y-b)^j \tag{A.8}$$

follows from the absolute convergence of the original series (A.6), by comparison.

The convergence of (A.8) for all $(x, y) \in I_b \times J_b$ allows us to define an analytic function $h_i \in C^\omega(J_b)$ for each $i \in \mathbb{N}$ by

$$h_i(y) = \sum_{j=0}^{\infty} c_{ij}\,(y-b)^j \tag{A.9}$$

for all $y \in J_b$. We can now write (A.7) as

$$f(x, y) = \sum_{i=0}^{\infty} (x-a)^i\, h_i(y). \tag{A.10}$$

If $h_i = 0_{J_b}$ for all $i \in \mathbb{N}$, then (A.10) implies that $f \mid I_b \times J_b = 0_{I_b \times J_b}$. By the *Unique Continuation Property* (2.4), $f = 0_{A \times B}$, which contradicts our assumption that $f \neq 0_{A\times B}$. Consequently, $h_i \neq 0_{J_b}$ for at least one $i \in \mathbb{N}$. Let m_b denote the smallest $i \in \mathbb{N}$ such that $h_i \neq 0_{J_b}$.

Using m_b, we can rewrite (A.10) as

$$\begin{aligned} f(x, y) &= \sum_{i=m_b}^{\infty} (x-a)^i\, h_i(y) \\ &= (x-a)^{m_b} \sum_{i=0}^{\infty} (x-a)^i\, h_{i+m_b}(y) \\ &= (x-a)^{m_b}\, g_b(x, y), \end{aligned}$$

where $g_b \in C^\omega(I_b \times J_b)$ is defined by

$$g_b(x, y) = \sum_{i=0}^{\infty} (x-a)^i\, h_{i+m_b}(y)$$

for all $(x, y) \in I_b \times J_b$. Note that $g_b(a, y) = h_{m_b}(y)$ for all $y \in J_b$. It follows by definition of m_b that $h_{m_b} \neq 0_{J_b}$, which implies that $g_b(x, y)$ does not vanish on $x = a$.

In summary, for every $b \in B$, we have produced a rectangular neighborhood $I_b \times J_b$ of (a, b) and shown that there exist $m_b \in \mathbb{N}$ and $g_b \in C^\omega(I_b \times J_b)$ such that

$$f(x, y) = (x - a)^{m_b} g_b(x, y) \tag{A.11}$$

for all $(x, y) \in I_b \times J_b$, subject to the restriction that $g_b(x, y)$ does not vanish on $x = a$. This completes Step 1.

Step 2. For each $b \in B$, Step 1 produces a neighborhood J_b of b. Step 2 will show that any closed subinterval $[c, d] \subset B$ can be covered by a *finite* sequence of *overlapping* neighborhoods from $\{J_b\}_{b \in B}$, beginning with J_c and ending with J_d. More precisely, we will construct a finite sequence $\{b_k\}_{k=0}^{n} \subset B$ with $n \geq 1$ such that

$$J_{b_k} \cap J_{b_{k+1}} \neq \varnothing$$

for $k = 0, \ldots, n-1$, and such that $b_0 = c$ and $b_n = d$.

Since $b \in J_b$ for each $b \in B$, it follows that

$$[c, d] \subset \bigcup_{b \in [c,d]} J_b.$$

Thus, $\{J_b\}_{b \in [c,d]}$ is a covering of $[c, d]$ by open sets. Since $[c, d]$ is compact, there is a *finite* subset $\mathcal{B}_0 \subset [c, d]$ such that $\{J_b\}_{b \in \mathcal{B}_0}$ covers $[c, d]$. Define $\mathcal{B} = \mathcal{B}_0 \cup \{c, d\}$ to construct an enlarged subcover $\{J_b\}_{b \in \mathcal{B}}$ which is guaranteed to contain J_c and J_d.

We will construct the desired sequence $\{b_k\}_{k=0}^{n}$ by iterating a *successor function* $S : \mathcal{B} \to \mathcal{B}$, which is defined using the following notation. For each $b \in B$, let $\underline{b}$ and $\overline{b}$ denote the left and right endpoints, respectively, of the open interval J_b. In this notation, $J_b = (\underline{b}, \overline{b})$, with $\underline{b} < b < \overline{b}$. Given any $b \in \mathcal{B} \subset [c, d]$, the function value $S(b)$ is determined by one of these two rules:

Case 1. If $J_b \cap J_d = (\underline{b}, \overline{b}) \cap (\underline{d}, \overline{d}) = \varnothing$, then $c \leq b < \overline{b} \leq \underline{d} < d$, which implies that $\overline{b} \in [c, d]$. Since $[c, d]$ is covered by $\{J_{b'}\}_{b' \in \mathcal{B}}$, there exists at least one $b' \in \mathcal{B}$ such that $\overline{b} \in J_{b'}$. Let $S(b)$ denote the largest such b'.

Case 2. If $J_b \cap J_d \neq \varnothing$, let $S(b) = d$.

Note that for every $b \in \mathcal{B}$, the interval J_b and its successor $J_{S(b)}$ satisfy

$$J_b \cap J_{S(b)} \neq \varnothing, \tag{A.12}$$

which means that the two intervals *overlap*. This is easily proven by cases from the definition of S. In Case 1, $J_b \cap J_d = \varnothing$. By definition, $\overline{b} \in J_{S(b)} = (\underline{S(b)}, \overline{S(b)})$, which implies that $\underline{S(b)} < \overline{b} < \overline{S(b)}$. Since $J_b = (\underline{b}, \overline{b})$, it follows that $J_b \cap J_{S(b)} \neq \varnothing$. In Case 2, $J_b \cap J_d \neq \varnothing$. By definition, $S(b) = d$, and $J_b \cap J_{S(b)} = J_b \cap J_d \neq \varnothing$.

Next, we define an *infinite* sequence $\{b_k\}_{k=0}^{\infty} \subset \mathcal{B}$ recursively via

$$b_0 = c \tag{A.13}$$

$$b_{k+1} = S(b_k), \text{ for all } k \in \mathbb{N}. \tag{A.14}$$

It follows immediately from (A.14) and (A.12) that

$$J_{b_k} \cap J_{b_{k+1}} = J_{b_k} \cap J_{S(b_k)} \neq \varnothing, \text{ for all } k \in \mathbb{N}.$$

Since we defined $b_0 = c$ in equation (A.13), all that remains is to prove $b_n = d$ for some $n \geq 1$.

Suppose that $J_{b_k} \cap J_d = \varnothing$ for all $k \in \mathbb{N}$. Each successor $b_{k+1} = S(b_k)$ is therefore computed via Case 1 of the definition of S. In Case 1, $\overline{b} \in J_{S(b)}$ for all $b \in \mathcal{B}$. This implies that

$$\overline{b_k} \in J_{S(b_k)} = J_{b_{k+1}} = (\underline{b_{k+1}}, \overline{b_{k+1}})$$

for all $k \in \mathbb{N}$. Consequently, $\overline{b_k} < \overline{b_{k+1}}$ for all $k \in \mathbb{N}$, which means the infinite sequence $\{\overline{b_k}\}_{k=0}^{\infty}$ is *strictly increasing* and therefore consists of *infinitely many distinct values*; this cannot occur, because the sequence $\{\overline{b_k}\}_{k=0}^{\infty}$ is the image under the mapping $b \mapsto \overline{b}$ of the sequence $\{b_k\}_{k=0}^{\infty} \subset \mathcal{B}$, which consists of *finitely many distinct values* since $\mathcal{B}$ is a *finite* set.

We conclude that there must be at *least* one $k \in \mathbb{N}$ such that $J_{b_k} \cap J_d \neq \varnothing$. Assume that N is the smallest such k, and let $n = N + 1 \geq 1$. By definition, $b_n = b_{N+1} = S(b_N) = d$. Hence, the *finite subsequence* $\{b_k\}_{k=0}^{n}$ has all the desired properties, and Step 2 is complete.

Step 3. This step will show that the m_b from Step 1 is independent of b. It suffices to show that $m_c = m_d$ for every $c, d \in B$ with $c < d$. Note that $[c, d] \subset B$. By Step 2, there is a finite sequence $\{b_k\}_{k=0}^{n} \subset [c, d]$, with $n \geq 1$, such that

$$J_{b_k} \cap J_{b_{k+1}} \neq \varnothing$$

for $k = 0, \ldots, n-1$, and such that $b_0 = c$ and $b_n = d$.

Recall that for each $b \in B$, Step 1 produces a rectangular neighborhood $I_b \times J_b$ of (a, b), along with $m_b \in \mathbb{N}$ and $g_b \in C^{\omega}(I_b \times J_b)$ such that

$$f(x, y) = (x-a)^{m_b} g_b(x, y)$$

for all $(x, y) \in I_b \times J_b$, with the proviso that $g_b(x, y)$ does not vanish on $x = a$. For simplicity, define

$$\begin{aligned} R_k &= I_{b_k} \times J_{b_k} \\ M_k &= m_{b_k} \\ G_k &= g_{b_k} \end{aligned}$$

for $k = 0, 1, \ldots, n$. Thus, $M_k \in \mathbb{N}$ and $G_k \in C^{\omega}(R_k)$ satisfy

$$f(x, y) = (x-a)^{M_k} G_k(x, y)$$

for all $(x, y) \in R_k$, and $G_k(x, y)$ does not vanish on $x = a$, for $k = 0, 1, \ldots, n$.

For $k = 0, \ldots, n-1$, consider the intersection of two consecutive open rectangles

$$\begin{aligned} R_k \cap R_{k+1} &= (I_{b_k} \times J_{b_k}) \cap \left(I_{b_{k+1}} \times J_{b_{k+1}}\right) \\ &= \left(I_{b_k} \cap I_{b_{k+1}}\right) \times \left(J_{b_k} \cap J_{b_{k+1}}\right) \\ &= A_k \times B_k, \end{aligned}$$

where

$$A_k = I_{b_k} \cap I_{b_{k+1}} \text{ and } B_k = J_{b_k} \cap J_{b_{k+1}}.$$

Since $a \in I_b$ for all $b \in B$, it follows that $a \in A_k$, hence $A_k \neq \varnothing$. By construction, $B_k \neq \varnothing$ (Step 2). Since the nonempty intersection of two open intervals is also an open interval, we conclude that $R_k \cap R_{k+1} = A_k \times B_k$ is an open rectangle for $k = 0, \ldots, n-1$.

The following two decompositions

$$f(x, y) = (x - a)^{M_k} G_k(x, y) = (x - a)^{M_{k+1}} G_{k+1}(x, y)$$

hold for all $(x, y) \in R_k \cap R_{k+1} = A_k \times B_k$. The *restrictions* of the functions G_k and G_{k+1} to $A_k \times B_k$ are obviously analytic. Recall that $a \in A_k$, and note that the *restrictions*

$$[G_k \mid A_k \times B_k]\,(x, y) \text{ and } [G_{k+1} \mid A_k \times B_k]\,(x, y)$$

do not vanish on $x = a$, by Proposition A.1. The uniqueness portion of this theorem, proven earlier, implies that $M_k = M_{k+1}$ for $k = 0, \ldots, n-1$, and thus

$$m_c = m_{b_0} = M_0 = M_n = m_{b_n} = m_d,$$

which completes Step 3.

Step 4. This step will construct the desired analytic function $g \in C^\omega(A \times B)$ from a compatible family of analytic functions whose domains cover $A \times B$. Since m_b is independent of b by Step 3, we can denote m_b simply by m for the remainder of the proof. Thus, for each $b \in B$, Step 1 provides a rectangular neighborhood $I_b \times J_b$ of (a, b) and a function $g_b \in C^\omega(I_b \times J_b)$ such that

$$f(x, y) = (x - a)^m g_b(x, y) \tag{A.15}$$

for all $(x, y) \in I_b \times J_b$, with the proviso that $g_b(x, y)$ does not vanish on $x = a$. Let

$U = (A - \{a\}) \times B$, and define

$$G(x, y) = \frac{f(x, y)}{(x - a)^m} \tag{A.16}$$

for all $(x, y) \in U$. Note that $G \in C^\omega(U)$. Our immediate goal is to show that the function

$$g(x, y) = \begin{cases} G(x, y) & \text{if } (x, y) \in U, \\ g_b(x, y) & \text{if } (x, y) \in I_b \times J_b \text{ for some } b \in B \end{cases}$$

is well-defined for all $(x, y) \in A \times B$. After achieving this goal, we will show that g has all the properties required by the theorem.

In order to show that g is well-defined, we must prove two things — that the family of functions

$$\Gamma = \{G\} \cup \{g_b\}_{b \in B}$$

is compatible, and that family of domains

$$\Delta = \{U\} \cup \{I_b \times J_b\}_{b \in B}$$

covers $A \times B$. By definition, Γ is a **compatible family of functions** if every pair of functions in Γ agree on their common domain. If the pair consists of G and g_b for some $b \in B$, then

$$G(x, y) = \frac{f(x, y)}{(x - a)^m} = g_b(x, y)$$

for all $(x, y) \in U \cap (I_b \times J_b)$, by (A.16) and (A.15). If the pair consists of g_c and g_d for some $c, d \in B$, then

$$g_c(x, y) = g_d(x, y)$$

for all $(x, y) \in (I_c \times J_c) \cap (I_d \times J_d)$ by a uniqueness argument similar to the one at the end of Step 3. Thus, Γ is a compatible family of functions. The family of domains Δ covers $A \times B$ if every point of $A \times B$ belongs to at least one domain $D \in \Delta$. Recall that $(a, b) \in I_b \times J_b$ for each $b \in B$. Consequently,

$$\{a\} \times B \subset \bigcup_{b \in B} I_b \times J_b$$

and

$$A \times B = U \cup (\{a\} \times B) \subset \bigcup_{D \in \Delta} D.$$

Hence, Δ covers $A \times B$, and g is well-defined.

We will now show that g satisfies all the requirements of the theorem. Since analyticity is a *local* property, $G \in C^\omega(U)$ and $g_b \in C^\omega(I_b \times J_b)$ for all $b \in B$ imply that $g \in C^\omega(A \times B)$. From the definition of g, it follows that

$$f(x, y) = (x - a)^m g(x, y) \tag{A.17}$$

for all $(x, y) \in U = (A - \{a\}) \times B$. By the continuity of f and g on $A \times B$, (A.17) also holds for all $(x, y) \in \{a\} \times B$, and thus for all $(x, y) \in A \times B$. By the definition of g,

$$g \mid I_b \times J_b = g_b$$

for any $b \in B$. Since $g_b(x, y)$ does not vanish on $x = a$, Proposition A.1 implies that $g(x, y)$ does not vanish on $x = a$. This completes Step 4, and the proof of existence. ■

Appendix B

Integral Representations

This appendix contains a list of integral representations for many of the standard functions of applied analysis. A more comprehensive collection of integral representations can be found in the references cited below.

In keeping with the orientation of the thesis, this appendix focuses primarily on integrals over the real line. Complex contour integrals have generally been omitted; however, when z is complex, the *proper indefinite integral*

$$f(z) = \int_a^z g(t)\,dt, \quad |a| < \infty$$

can be interpreted as a contour integral over the line segment joining the points a and z in the complex plane, and the *improper indefinite integral*

$$f(z) = \int_z^\infty g(t)\,dt$$

can be interpreted as a contour integral over some ray joining z to the point at infinity.

Throughout this appendix, z and w denote complex variables, x denotes a real variable, and n denotes an integer, *unless stated otherwise*. In the absence of explicit restrictions, a formula is valid over the entire domain implicitly specified by its arguments.

B.1 Elementary Transcendental Functions

Most of the material in this section can be found in [Abr-Ste, Chapter 4].

B.1.1 Logarithm Function

$$\ln(z) = \int_1^z \frac{dt}{t}, \quad |\arg(z)| < \pi \tag{B.1}$$

B.1.2 Inverse Trigonometric Functions

$$\sin^{-1}(z) = \int_0^z \frac{dt}{\sqrt{1-t^2}}, \quad \left|\arg(1-z^2)\right| < \pi \tag{B.2}$$

$$\tan^{-1}(z) = \int_0^z \frac{dt}{1+t^2}, \quad \left|\arg(1+z^2)\right| < \pi \tag{B.3}$$

B.1.3 Inverse Hyperbolic Functions

$$\sinh^{-1}(z) = \int_0^z \frac{dt}{\sqrt{1+t^2}}, \quad \left|\arg(1+z^2)\right| < \pi \tag{B.4}$$

$$\cosh^{-1}(z) = \int_1^z \frac{dt}{\sqrt{t^2-1}}, \quad |\arg(z-1)| < \pi \tag{B.5}$$

$$\tanh^{-1}(z) = \int_0^z \frac{dt}{1-t^2}, \quad \left|\arg(1-z^2)\right| < \pi \tag{B.6}$$

B.1.4 Sinc Function

$$\operatorname{sinc}(z) = \frac{\sin(z)}{z} = \int_0^1 \cos(z\,t)\,dt \tag{B.7}$$

B.2 Special Functions

The material in this section was compiled from [Abr-Ste, Chapters 5–7 and 17], [EDM, Volume 2, Appendix A, Tables 16.I, 17.I, and 19.II], and [Arfken, Chapters 5 and 10].

B.2.1 Gamma and Beta Functions

These formulas are valid for Re $z > 0$ and Re $w > 0$:

Gamma Function

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt \tag{B.8}$$

Beta Function

$$B(z, w) = \int_0^1 t^{z-1}(1-t)^{w-1} \, dt \tag{B.9}$$

$$= \int_0^\infty \frac{t^{z-1}}{(1+t)^{z+w}} \, dt \tag{B.10}$$

$$= 2\int_0^{\pi/2} (\sin\theta)^{2z-1} (\cos\theta)^{2w-1} \, d\theta \tag{B.11}$$

Incomplete Gamma Function

$$\gamma(z, x) = \int_0^x e^{-t} t^{z-1} \, dt, \quad x \geq 0 \tag{B.12}$$

Incomplete Beta Function

$$B_x(z, w) = \int_0^x t^{z-1}(1-t)^{w-1} \, dt, \quad 0 \leq x \leq 1 \tag{B.13}$$

B.2.2 Log-Gamma, Digamma, and Polygamma Functions

These formulas are valid for Re $z > 0$:

Log-Gamma Function

$$\ln\Gamma(z) = \int_0^\infty \left[(z-1)\, e^{-t} - \frac{e^{-t} - e^{-zt}}{1 - e^{-t}} \right] \frac{dt}{t} \tag{B.14}$$

Digamma Function

$$\psi(z) = \frac{d}{dz} \ln\Gamma(z) = \int_0^\infty \left[\frac{e^{-t}}{t} - \frac{e^{-zt}}{1 - e^{-t}} \right] dt \tag{B.15}$$

Polygamma Functions

$$\psi^{(n)}(z) = \left[\frac{d}{dz}\right]^n \psi(z) = (-1)^{n+1} \int_0^\infty \frac{t^n e^{-zt}}{1-e^{-t}}\, dt, \quad n \geq 1 \tag{B.16}$$

B.2.3 Error Functions

Error Function

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt \tag{B.17}$$

Complementary Error Function

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\, dt \tag{B.18}$$

B.2.4 Exponential and Logarithmic Integrals

In these formulas, PV denotes the Cauchy principal value integral:

Exponential Integrals

$$\mathrm{Ei}(x) = \mathrm{PV} \int_{-\infty}^x \frac{e^t}{t}\, dt, \quad x \neq 0 \tag{B.19}$$

$$E_1(z) = \int_z^\infty \frac{e^{-t}}{t}\, dt, \quad |\arg z| < \pi \tag{B.20}$$

$$E_n(z) = \int_1^\infty \frac{e^{-zt}}{t^n}\, dt, \quad \mathrm{Re}\ z > 0,\ n \geq 0 \tag{B.21}$$

Logarithmic Integral

$$\mathrm{Li}(x) = \mathrm{PV} \int_0^x \frac{dt}{\ln t}, \quad x \geq 0,\ x \neq 1 \tag{B.22}$$

B.2.5 Sine and Cosine Integrals

In these formulas, γ denotes Euler's constant:

Sine Integrals

$$\mathrm{Si}(z) = \int_0^z \frac{\sin t}{t}\, dt \tag{B.23}$$

$$\mathrm{si}(z) = -\int_z^\infty \frac{\sin t}{t}\, dt \tag{B.24}$$

Cosine Integral

$$\mathrm{Ci}(z) = \gamma + \ln z + \int_0^z \frac{\cos t - 1}{t}\, dt, \quad |\arg z| < \pi \tag{B.25}$$

$$= -\int_z^\infty \frac{\cos t}{t}\, dt, \quad |\arg z| < \pi \tag{B.26}$$

Hyperbolic Sine Integral

$$\mathrm{Shi}(z) = \int_0^z \frac{\sinh t}{t}\, dt \tag{B.27}$$

Hyperbolic Cosine Integral

$$\mathrm{Chi}(z) = \gamma + \ln z + \int_0^z \frac{\cosh t - 1}{t}\, dt, \quad |\arg z| < \pi \tag{B.28}$$

$$= -\int_z^\infty \frac{\cosh t}{t}\, dt, \quad |\arg z| < \pi \tag{B.29}$$

B.2.6 Fresnel Integrals

Fresnel Sine Integral

$$S(z) = \int_0^z \sin\left(\frac{\pi}{2}t^2\right)\, dt \tag{B.30}$$

Fresnel Cosine Integral

$$C(z) = \int_0^z \cos\left(\frac{\pi}{2}t^2\right)\, dt \tag{B.31}$$

B.2.7 Elliptic Integrals

In these formulas, φ, α, x, m, and n all denote real variables. Many of these symbols have special names, are related to one another in some way, and have restrictions on their range of values; this information is summarized in the following table:

Symbol	Name	Relation	Range
φ	amplitude		$0 \le \varphi \le \pi/2$
α	modular angle		$0 \le \alpha \le \pi/2$
x		$x = \sin\varphi$	$0 \le x \le 1$
m	parameter	$m = \sin^2\alpha$	$0 \le m \le 1$
n	characteristic		all reals

The precise form of the definitions of the elliptic integrals depends upon the symbols which are chosen as arguments. The most common forms are listed below.

Elliptic Integral of the First Kind

$$F(\varphi\backslash\alpha) = \int_0^{\varphi} \left(1 - \sin^2\alpha \, \sin^2\theta\right)^{-1/2} d\theta, \quad \alpha \neq \pi/2 \tag{B.32}$$

$$F(x|m) = \int_0^{x} \left[\left(1 - t^2\right)\left(1 - m\,t^2\right)\right]^{-1/2} dt, \quad m \neq 1 \tag{B.33}$$

Elliptic Integral of the Second Kind

$$E(\varphi\backslash\alpha) = \int_0^{\varphi} \left(1 - \sin^2\alpha \, \sin^2\theta\right)^{1/2} d\theta \tag{B.34}$$

$$E(x|m) = \int_0^{x} \left(1 - t^2\right)^{-1/2} \left(1 - m\,t^2\right)^{1/2} dt \tag{B.35}$$

Elliptic Integral of the Third Kind

$$\Pi(n;\varphi\backslash\alpha) = \int_0^{\varphi} \left(1 - n\sin^2\theta\right)^{-1} \left(1 - \sin^2\alpha \, \sin^2\theta\right)^{-1/2} d\theta \tag{B.36}$$

$$\Pi(n;x|m) = \int_0^{x} \left(1 - n\,t^2\right)^{-1} \left[\left(1 - t^2\right)\left(1 - m\,t^2\right)\right]^{-1/2} dt \tag{B.37}$$

Complete Elliptic Integral of the First Kind

$$K(m) = \int_0^{\pi/2} \left(1 - m\sin^2\theta\right)^{-1/2} d\theta, \quad m \neq 1 \tag{B.38}$$

$$= \int_0^1 \left[\left(1-t^2\right)\left(1 - m\,t^2\right)\right]^{-1/2} dt, \quad m \neq 1 \tag{B.39}$$

Complete Elliptic Integral of the Second Kind

$$E(m) = \int_0^{\pi/2} \left(1 - m\sin^2\theta\right)^{1/2} d\theta \tag{B.40}$$

$$= \int_0^1 \left(1-t^2\right)^{-1/2}\left(1 - m\,t^2\right)^{1/2} dt \tag{B.41}$$

Appendix C

Maple Routines

This appendix demonstrates a rudimentary implementation of the asymptotic splitting operator (ASO) and the basic algorithm for computing a dual asymptotic expansion (DAE) in the MAPLE (version 5.4) computer algebra system. The first section contains the MAPLE source code for the two routines ASO and DAE. The second section illustrates the use of these routines with examples taken from the body of the thesis.

C.1 Maple Source Code

```
macro(ASO=Asymptotic_Splitting_Operator):

Asymptotic_Splitting_Operator :=
proc(expr::algebraic, eq1::name=algebraic, eq2::name=algebraic)
  local x, y, a, b, f, X, Y;
  x := lhs(eq1); a := rhs(eq1);
  y := lhs(eq2); b := rhs(eq2);
  if normal(expr)=0 then RETURN(0) fi;
  f := unapply(expr, x, y);
  limit(limit(f(x,Y)*f(X,y)/f(X,Y), Y=b, right), X=a, right);
end:
```

```
macro(DAE=Dual_Asymptotic_Expansion):

Dual_Asymptotic_Expansion :=
proc(expr::algebraic, eq1::name=algebraic, eq2::name=algebraic,
ord::posint)
  local remainder, term, n;
  remainder := expr;  # remainder with 0 terms
  for n from 1 to ord do
    term := Asymptotic_Splitting_Operator(remainder, eq1, eq2);
    remainder := remainder - factor(term);  # with 'n' terms
  od;
  expr - remainder;  # expansion with 'ord' terms
end:
```

C.2 Examples of Use

Asymptotic Splitting Operator

```
> ASO(exp(-x*y), x=a, y=b);
```

$$e^{(-b\,x-y\,a+b\,a)}$$

```
> ASO(sin(x*y), x=0, y=0);
```

$$x\,y$$

Dual Asymptotic Expansion

```
> DAE(1/(1+x^2*y^2), x=0, y=0, 4);
```

$$1-x^2\,y^2+x^4\,y^4-x^6\,y^6$$

```
> DAE(exp(-x*y), x=a, y=b, 3);
```

$$e^{(-b\,x-y\,a+b\,a)} - (-b+y)\,(x-a)\,e^{(-b\,x-y\,a+b\,a)} + \frac{1}{2}\,(-b+y)^2\,(x-a)^2\,e^{(-b\,x-y\,a+b\,a)}$$

```
> DAE(cos(x+y), x=0, y=0, 2);
```

$$\cos(x)\cos(y) - \sin(x)\sin(y)$$

```
> DAE(sin(x+y), x=Pi/2, y=0, 2);
```

$$\sin(x)\cos(y) + \cos(x)\sin(y)$$

```
> DAE(BesselJ(0,x+y), x=0, y=0, 2);
```

$$\mathrm{BesselJ}(0,\,y)\,\mathrm{BesselJ}(0,\,x) - 2\,\mathrm{BesselJ}(1,\,y)\,\mathrm{BesselJ}(1,\,x)$$

Bibliography

[Abr-Ste] Abramowitz, M. and Stegun, I.A., editors (1964). *Handbook of Mathematical Functions: with Formulas, Graphs, and Mathematical Tables.* Applied Mathematics Series, no. 55, United States National Bureau of Standards. Republished in 1965 by Dover Publications, New York. Ninth Dover printing includes corrections to previous editions.

[Arfken] Arfken, G. (1985). *Mathematical Methods for Physicists*, 3rd ed. Academic Press, San Diego.

[Ble-Han] Bleistein, N. and Handelsman, R.A. (1975). *Asymptotic Expansions of Integrals.* Holt, Rinehart and Winston, New York. Reprinted with corrections in 1986 by Dover Publications, New York.

[Byr-Ful] Byron, F.W., Jr. and Fuller, R.W. (1969, 1970). *Mathematics of Classical and Quantum Physics, Volumes 1 & 2.* Addison-Wesley Series in Advanced Physics, Addison-Wesley, Reading, Massachusetts. Reprinted in one volume with corrections in 1992 by Dover Publications, New York.

[Cartan] Cartan, H. (1973). *Elementary Theory of Analytic Functions of One or Several Complex Variables.* ADIWES International Series in Mathematics, Addison-Wesley Publishing Company, Reading, Massachusetts. Reprinted in 1995 by Dover Publications, New York.

[Cho-Mar] Chorin, A.J. and Marsden, J.E. (1992). *A Mathematical Introduction to Fluid Mechanics*, 3rd ed. Texts in Applied Mathematics, no. 4, Springer-Verlag, New York.

[Coddington] Coddington, E.A. (1961). *An Introduction to Ordinary Differential Equations.* Prentice-Hall Mathematics Series, Prentice-Hall, Englewood Cliffs, New Jersey. Reprinted with corrections in 1989 by Dover Publications, New York.

[Conway] Conway, J.B. (1978). *Functions of One Complex Variable*, 2nd ed. Graduate Texts in Mathematics, no. 11, Springer-Verlag, New York.

[Davis] Davis, P.J. (1963). *Interpolation and Approximation.* Blaisdell Publishing Company, Waltham, Massachusetts. Reprinted with corrections in 1975 by Dover Publications, New York.

[Den-Krz] Dennery, P. and Krzywicki, A. (1967). *Mathematics for Physicists.* Harper's Physics Series, Harper & Row, New York. Reprinted with corrections in 1996 by Dover Publications, Mineola, New York.

[EDM] Itô, K., ed. (1986). *Encyclopedic Dictionary of Mathematics, Volumes 1 & 2*, second English edition. Translated from *Iwanami Sūgaku Ziten*, third Japanese edition, by the original contributors under the auspices of the Mathematical Society of Japan. Paperback edition published in 1993 by The MIT Press, Cambridge, Massachusetts.

[Edwards] Edwards, R.E. (1965). *Functional Analysis: Theory and Applications.* Holt, Rinehart and Winston, New York. Reprinted with corrections in 1995 by Dover Publications, New York.

[Epstein] Epstein, B. (1962). *Partial Differential Equations: An Introduction.* McGraw-Hill, New York. Reprinted in 1983 by Robert E. Krieger Publishing Company, Malabar, Florida.

[Erdelyi] Erdélyi, A. (1956). *Asymptotic Expansions.* Dover Publications, New York.

[Friedman] Friedman, B. (1956). *Principles and Techniques of Applied Mathematics.* John Wiley & Sons, New York. Reprinted in 1990 by Dover Publications, New York.

[Ge-Cz-La] Geddes. K.O., Czapor, S.R. and Labahn, G. (1992). *Algorithms for Computer Algebra.* Kluwer Academic Publishers, Boston.

[Ged-Gon] Geddes. K.O. and Gonnet. G.H. (1989). A New Algorithm for Computing Symbolic Limits Using Hierarchical Series. *Symbolic and Algebraic Computation*, 490–495. Lecture Notes in Computer Science, vol. 358. ed. P. Gianni, Springer-Verlag, New York. Proceedings ISSAC'88, Rome.

[Go-VaLo] Golub, G.H. and Van Loan, C.F. (1996). *Matrix Computations*, 3rd ed. The Johns Hopkins University Press, Baltimore.

[Henrici74] Henrici. P. (1974). *Applied and Computational Complex Analysis, Volume 1: Power Series — Integration — Conformal Mapping — Location of Zeros.* John Wiley & Sons, New York. Wiley Classics Library edition published in 1988 by Wiley-Interscience, New York.

[Henrici77] Henrici, P. (1977). *Applied and Computational Complex Analysis, Volume 2: Special Functions — Integral Transforms — Asymptotics — Continued Fractions.* John Wiley & Sons, New York. Wiley Classics Library edition published in 1991 by Wiley-Interscience, New York.

[Henrici86] Henrici, P. (1986). *Applied and Computational Complex Analysis, Volume 3: Discrete Fourier Analysis — Cauchy Integrals — Construction of Conformal Maps — Univalent Functions.* John Wiley & Sons, New York. Wiley Classics Library edition published in 1993 by Wiley-Interscience, New York.

[Hor-Joh] Horn, R.A. and Johnson, C.R. (1985). *Matrix Analysis.* Cambridge University Press, Cambridge.

[Ince] Ince, E.L. (1926). *Ordinary Differential Equations.* Longmans, Green and Co. Reprinted in 1956 by Dover Publications, New York.

[John] John, F. (1982). *Partial Differential Equations*, 4th ed. Applied Mathematical Sciences, no. 1, Springer-Verlag, New York.

[Knopp45] Knopp, K. (1945). *Theory of Functions, Part I: Elements of the General Theory of Analytic Functions.* Translated from the fifth edition of *Funktionentheorie* by F. Bagemihl. Dover Publications, New York.

[Knopp47] Knopp, K. (1947). *Theory of Functions, Part II: Applications and Continuation of the General Theory.* Translated from the fourth edition of *Funktionentheorie. II Teil* by F. Bagemihl. Dover Publications, New York.

[Knopp56] Knopp, K. (1956). *Infinite Sequences and Series.* Translated by F. Bagemihl. Dover Publications, New York.

[Maple96a] Heal, K.M., Hansen, M.L. and Rickard, K.M. (1996). *Maple V Learning Guide.* Springer-Verlag, New York.

[Maple96b] Monagan, M.B., Geddes, K.O., Heal, K.M., Labahn, G. and Vorkoetter, S. (1996). *Maple V Programming Guide.* Springer-Verlag, New York.

[Maple96c] Redfern, D. (1996). *The Maple Handbook: Maple V Release 4.* Springer-Verlag, New York.

[Salvy] Salvy, B. (1991). Examples of Automatic Asymptotic Expansions. *SIGSAM Bulletin 25*, 4–17.

[Sal-Zim] Salvy, B. and Zimmermann, P. (1994). Gfun: A Maple Package for the Manipulation of Generating and Holonomic Functions in One Variable. *ACM Transactions on Mathematical Software 20*, 163–177.

[Sansone] Sansone, G. (1959). *Orthogonal Functions*, revised English edition. Translated from the Italian by A.H. Diamond, with a foreword by E. Hille. Pure and Applied Mathematics, vol. 9, Interscience Publishers, New York. Reprinted in 1991 by Dover Publications, New York.

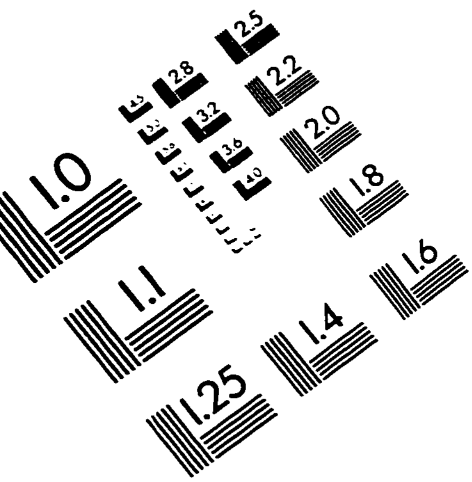
[Sobolev] Sobolev, S.L. (1964). *Partial Differential Equations of Mathematical Physics*. Translated from the third Russian edition by E.R. Dawson. English translation edited by T.A.A. Broadbent. Pergamon Press, London. Reprinted in 1989 by Dover Publications, New York.

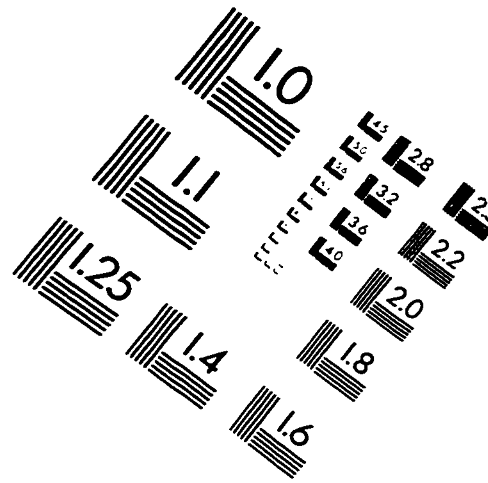
[Stakgold] Stakgold, I. (1979). *Green's Functions and Boundary Value Problems*. Pure and Applied Mathematics, Wiley-Interscience, New York.

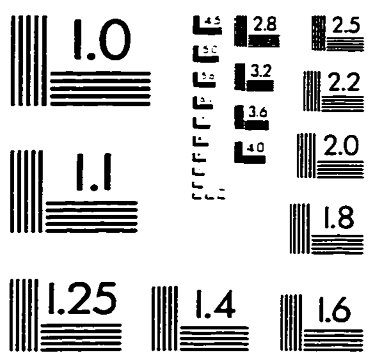
[Tricomi] Tricomi, F.G. (1957). *Integral Equations*. Interscience Publishers, New York. Reprinted in 1985 by Dover Publications, New York.

[Yosida] Yosida, K. (1960). *Lectures on Differential and Integral Equations*. Interscience Publishers, New York. Reprinted in 1991 by Dover Publications, New York.

IMAGE EVALUATION TEST TARGET (QA-3)







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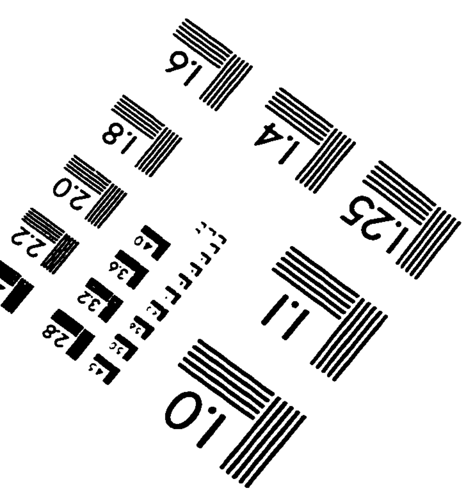

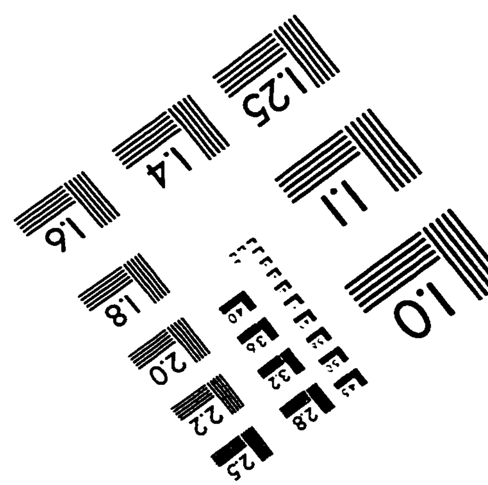

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