Properties of random graphs

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Combinatorics and Optimization

Waterloo, Ontario, Canada, 2008

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The thesis describes new results for several problems in random graph theory. The first problem relates to the uniform random graph model in the supercritical phase; i.e. a graph, uniformly distributed, on n vertices and $M = n/2 + s$ edges for $s = s(n)$ satisfying $n^{2/3} = o(s)$ and $s = o(n)$. The property studied is the length of the longest cycle in the graph. We give a new upper bound, which holds asymptotically almost surely, on this length. As part of our proof we establish a result about the heaviest cycle in a certain randomly-edge-weighted nearly-3-regular graph, which may be of independent interest.

Our second result is a new contiguity result for a random d-regular graph. Let $j = j(n)$ be a function that is linear in n. A $(d, d-1)$-irregular graph is a graph which is d-regular except for $2j$ vertices of degree $d-1$. A j-edge matching in a graph is a set of j independent edges. In this thesis we prove the new result that a random $(d, d-1)$-irregular graph plus a random j-edge matching is contiguous to a random d-regular graph, in the sense that in the two spaces, the same events have probability approaching 1 as $n \to \infty$. This allows one to deduce properties, such as colourability, of the random irregular graph from the corresponding properties of the random regular one. The proof applies the small subgraph conditioning method to the number of j-edge matchings in a random d-regular graph.

The third problem is about the 3-colourability of a random 5-regular graph. Call a colouring *balanced* if the number of vertices of each colour is equal, and *locally rainbow* if every vertex is adjacent to vertices of all the other colours. Using the small subgraph conditioning method, we give a condition on the variance of the number of locally rainbow balanced 3-colourings which, if satisfied, establishes that the chromatic number of the random 5-regular graph is asymptotically almost surely equal to 3. We also describe related work which provides evidence that the condition is likely to be true.

The fourth problem is about the chromatic number of a random d-regular graph for fixed d. Achlioptas and Moore recently announced a proof that a random d-regular graph asymptotically almost surely has chromatic number $k-1$, k, or $k+1$, where k is the smallest integer satisfying $d < 2(k-1)\log(k-1)$. In this thesis we prove that, asymptotically almost surely, it is not $k+1$, provided a certain second moment condition holds. The proof applies the small subgraph conditioning method to the number of balanced k-colourings, where a colouring is *balanced* if the number of vertices of each colour is equal. We also give evidence that suggests that the required second moment condition is true.

Acknowledgements

I appreciate the support and encouragement of my supervisor Nick Wormald and others in the Department of Combinatorics and Optimization at the University of Waterloo, co-authors J. Díaz, A.C. Kaporis, L.M. Kirousis, X. Pérez, as well as my friends and family.

Contents

Chapter 1

Introduction

In this thesis we present several new results about random graphs.

The theory of random graphs is about discovering the typical properties of a given class of graphs. It was originally used to prove the existence of graphs having certain properties. Today, random graphs are studied in their own right. Several different models have been developed and fruitfully explored. Random graphs have also been applied to other areas of study, including physics and computer science.

One of the earliest and most important models of random graphs is $\mathcal{G}_{n,M}$, the probability space of graphs on n labelled vertices and $M = M(n)$ edges, under the uniform probability distribution. In Chapter 2 we present a new result about the length of the longest cycle in $\mathcal{G}_{n,M}$ for a particular range of M, known as the "supercritical phase".

Another model of random graphs is $\mathcal{G}_{n,d}$, the probability space of d-regular graphs on n labelled vertices under the uniform probability distribution. Contiguity is a type of relationship between probability spaces. In Chapter 3 we prove a new result that $\mathcal{G}_{n,d}$ is contiguous to a different model of random d-regular graphs. Finally, in Chapters 4 and 5 we will present some new results about the chromatic number of $\mathcal{G}_{n,d}$.

Parts of the research in Chapters 2, 4, and 5 are joint work with Nick Wormald. The material in Chapter 4 appears in a joint paper [13] with J. Díaz, A.C. Kaporis, L.M. Kirousis, X. Pérez and N. Wormald.

In the remainder of this introductory chapter we describe our notation and give the context and significance for the problems that will be solved.

1.1 Notation and terminology

Our notation and terminology for graphs is standard; see [20]. A *pseudograph* is like a graph except that it may have loops and multiple (parallel) edges. Since most of the results about random graphs describe asymptotically what happens as the number of vertices $n \to \infty$, we next describe our notation for asymptotics. We say an event A_n holds *asymptotically almost surely* (a.a.s.) if its probability approaches 1 as $n \to \infty$. The associated adjective "asymptotically almost sure" is also abbreviated by a.a.s. To define the notation used to state asymptotic results, let $f(n)$, $g(n)$, and $\phi(n)$ be functions satisfying $|f| < \phi g$. We write $f = O(g)$ if ϕ is bounded. If $\phi \to 0$ as $n \to \infty$ then we write $f = o(g)$. We define $f \sim g$ to mean that $f(n) = (1 + o(1))g(n)$. The notation $f = \Omega(g)$ indicates that $g = O(f)$, while $f = \Theta(g)$ means that both $f = O(g)$ and $g = O(f)$ hold.

1.2 Uniform random graph $\mathcal{G}_{n,M}$

The probability space $\mathcal{G}_{n,M}$ of all n-vertex graphs with M edges under the uniform distribution is also known as the uniform random graph model. It is one of the earliest models of random graphs, originating in a simple model introduced by Erdős [14]. Much of the interest in this model comes from the study of the a.a.s. properties as the dependence of M upon n is varied. This change from a sparse graph to a dense graph, as M increases more quickly with n, is called the evolution of the random graph. One important property is the number L of vertices in the largest component of $\mathcal{G}_{n,M}$. (If there is more than one component with the maximum number of vertices, we use the lexicographically first among largest components.) When $M = cn/2$ for constant c, Erdős and Rényi [15] showed that the number of vertices in the largest component of $\mathcal{G}_{n,M}$ is a.a.s. $O(\log n)$, $\Theta(n^{2/3})$, or $\Theta(n)$ according to whether $c < 1$, $c = 1$, or $c > 1$, respectively.

Because of this dramatic change in the structure of $\mathcal{G}_{n,M}$, we often call $M = n/2$ a "phase transition". Further research showed that the phase transition extends throughout the period $M = n/2 + cn^{2/3}$ for constant c in the sense that, for this range of M, $L = c'n^{2/3}$ with a distribution over the constant c'. As a result, this range of M is known as the *critical period*. For $s = s(n)$ satisfying $n^{2/3} = o(s)$ but $s = o(n)$, the range $M = n/2 - s$ is known as the *subcritical phase* while the range $M = n/2 + s$ is known as the *supercritical phase*. For M in the supercritical phase, $\mathcal{G}_{n,M}$ a.a.s. has a unique largest component on $(4 + o(1))s$ vertices and every other component has fewer than $n^{2/3}$ vertices. A "giant component" has emerged.

1.3 Circumference of $\mathcal{G}_{n,M}$

Another important graph property is its *circumference*, the length of its longest cycle. The circumference l of $\mathcal{G}_{n,M}$ also changes dramatically during the phase transition, but it is not entirely understood. Let $\omega = \omega(n) \to \infty$. When $M = cn/2$ for fixed $c < 1$, the circumference of $\mathcal{G}_{n,M}$ is a.a.s. at most ω ([8], Corollary 5.8). In the subcritical phase, the circumference l a.a.s. satisfies $l/\omega < n/s < l\omega$ ([20], Section 5.4). During the critical period $M = n/2 + O(n^{2/3})$ it a.a.s. satisfies $l/\omega < n^{1/3} < l\omega$ ([20], Section 5.5). But for larger M researchers do not have such good estimates for the circumference. (Of course, when $M = n(\log n + \log\log n + \omega)/2$ the circumference is a.a.s. equal to n as the graph is a.a.s. Hamiltonian [21].) When $M = cn/2$ for fixed $c > 1$, there are several known a.a.s. lower bounds on the circumference of the form $(f(c) + o(1))n$ [16, 9, 17]. One of the earliest was given by Ajtai, Komlós and Szemerédi [4], who also showed an equivalence between the problems of finding paths of length $(f(c) + o(1))n$ and finding cycles of length $(f(c) + o(1))n$.

In the supercritical phase Łuczak [23] has shown that the circumference of $\mathcal{G}_{n,M}$ is a.a.s. between $(16/3+o(1))s^2/n$ and $(7.496+o(1))s^2/n$. In Chapter 2 we improve the a.a.s. upper bound to $(7 + o(1))s^2/n$.

1.4 Contiguity for random d-regular graphs

For $\mathcal{G}_{n,d}$, the space of d-regular n-vertex graphs under the uniform distribution, many important results have been established using the notion of *contiguity*. (See Section 9.5 in [20].) We say that the spaces $\mathcal{G}_n$ and $\widehat{\mathcal{G}}_n$ are *contiguous* provided that any sequence of events A_n is a.a.s. true in $\mathcal{G}_n$ if and only if it is a.a.s. true in $\widehat{\mathcal{G}}_n$. This equivalence relation is denoted $\mathcal{G}_n \approx \widehat{\mathcal{G}}_n$.

One source of contiguous spaces is probability spaces where the probabilities are altered according to the value of a random variable, as follows. We use the notation $\mathbf{P}$ for probability and $\mathbf{E}$ for expected value. If X is a random variable we define $\mathcal{G}^{(X)}$, the X-weighted space from $\mathcal{G}$, by using the rule

$$\mathbf{P}_{\mathcal{G}^{(X)}}(G) = \frac{\mathbf{P}_{\mathcal{G}}(G)X(G)}{\mathbf{E}_{\mathcal{G}}X}$$

to assign probability to each graph G. It is known (see [36]) that $\mathcal{G}_{n,d}^{(Y)} \approx \mathcal{G}_{n,d}$ for all of the following random variables Y that count subgraphs: Hamilton cycles H_n, perfect matchings M_n, 1-factorisations T_n for $d = 3$, complete Hamiltonian decompositions D_n for all even $d \geq 4$. (A 1-factorisation is a partition of the edge set into perfect matchings. A complete Hamiltonian decomposition is a partition of the edge set into Hamilton cycles.) These decompositions can be used to deduce or re-derive properties of $\mathcal{G}_{n,d}$. As a trivial example, $T_n > 0$ a.a.s. for $d = 3$ implies that $\mathcal{G}_{n,3}$ is a.a.s. 3-edge-colourable.

One example of an X-weighted space comes from combining graphs. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are sets of n-vertex graphs with the uniform distribution, define the *regular superposition* $\mathcal{G}_1 \oplus \mathcal{G}_2$ to be the union of a graph drawn at random from $\mathcal{G}_1$ with a graph drawn at random from $\mathcal{G}_2$, conditioned on the event that the result is a d-regular simple graph. Then $\mathcal{G}_1 \oplus \mathcal{G}_2 = \mathcal{G}^{(Y)}$, where $Y = Y(G)$ is the random variable counting the number of pairs (G_1, G_2) where $G_1 \in \mathcal{G}_1$, $G_2 \in \mathcal{G}_2$, $G_1 \cup G_2 = G$ and G_1 and G_2 are edge-disjoint. For example, letting $\mathcal{G}_1 = \mathcal{G}_{n,1}$, the space of perfect matchings, and $\mathcal{G}_2 = \mathcal{G}_{n,d-1}$, we see that the random variable Y defined above is simply M_n, the number of perfect matchings. Thus

$$\mathcal{G}_{n,d}^{(M_n)} = \mathcal{G}_{n,1} \oplus \mathcal{G}_{n,d-1}.$$

Since we know $\mathcal{G}_{n,d} \approx \mathcal{G}_{n,d}^{(M_n)}$, it follows that

$$\mathcal{G}_{n,d} \approx \mathcal{G}_{n,1} \oplus \mathcal{G}_{n,d-1}.$$

This type of result has been used for establishing a.a.s. properties of $\mathcal{G}_{n,d}$ by induction on d. Many other superposition results of this type are known, including results about random graphs arising from permutations [18].

This is also an area of open problems because there are other models of random d-regular graphs which are not known to be contiguous to $\mathcal{G}_{n,d}$. For example, two such models are the *random d-process* [30] and the *random star d-process* [27]. Each of these processes generates a sequence of graphs, beginning with the edgeless graph on n vertices. At each step of the random d-process, one edge is added between a pair of uniformly-chosen nonadjacent vertices whose degrees are less than d. The final graph produced by this process is a.a.s. d-regular. The random star d-process uses a different rule for adding edges. At each step, a vertex of minimum degree δ in the current graph is chosen. Edges are added between this vertex and $d-\delta$ randomly-chosen vertices of degree less than d. The resulting graph is a.a.s. d-regular. It is an open problem to determine whether these models are contiguous to $\mathcal{G}_{n,d}$.

For this thesis, we will prove a new result about a model of random graphs that is not regular. Define a j-edge matching in a graph to be a set of j independent edges; i.e. edges whose endpoints are all distinct. We are interested in j growing like a constant times n; formally we let $\gamma = j/n$ and assume that $\gamma = \gamma^* + o(1)$ for some $\gamma^* \in (0, 1/2)$. Let $\mathcal{J}_n$ be the set of j-edge matchings on n vertices. A $(d, d-1)$-irregular graph is a graph which is d-regular except for $2j$ vertices of degree $d-1$. Let $\mathcal{I}_{n,d,d-1}$ be the set of $(d, d-1)$-irregular graphs on n vertices. Let Y be the random variable counting the number of j-edge matchings in $\mathcal{G}_{n,d}$. In Chapter 3 we will prove that

$$\mathcal{G}_{n,d} \approx \mathcal{G}_{n,d}^{(Y)}. \tag{1.4.1}$$

Since $\mathcal{G}_{n,d}^{(Y)} = \mathcal{J}_n \oplus \mathcal{I}_{n,d,d-1}$, it follows that

$$\mathcal{G}_{n,d} \approx \mathcal{J}_n \oplus \mathcal{I}_{n,d,d-1}; \tag{1.4.2}$$

i.e. the regular superposition of a random j-edge matching with a random $(d, d-1)$-irregular graph is contiguous to $\mathcal{G}_{n,d}$.

This result is useful because it allows us to deduce facts about $\mathcal{I}_{n,d,d-1}$ from known results about $\mathcal{G}_{n,d}$. As one instance, if $\mathcal{G}_{n,d}$ is a.a.s. k-colourable, then we can deduce that $\mathcal{I}_{n,d,d-1}$ is a.a.s. k-colourable. For example, it is known that $\mathcal{G}_{n,4}$ is a.a.s. 3-colourable [32]. It follows that a random graph on $n/2$ degree-3 vertices and $n/2$ degree-4 vertices is also 3-colourable.

1.5 The chromatic number of random d-regular graphs

The study of the chromatic number χ of a graph has been an important challenge for graph theory and random graph theory in particular. Several new tools, such as the vertex exposure martingale [31] of Shamir and Spencer, were developed along the way. For a survey of classical and recent results, see [3]. For this thesis we are interested in $\chi(\mathcal{G}_{n,d})$, the chromatic number of random d-regular graphs.

Using the basic properties of $\mathcal{G}_{n,d}$ one can show that a.a.s. $\chi(\mathcal{G}_{n,1}) = 2$, $\chi(\mathcal{G}_{n,2}) = 3$, and $\chi(\mathcal{G}_{n,3}) = 3$. (See [32].) It required much more effort for Shi and Wormald [32] to establish that that a.a.s. $\chi(\mathcal{G}_{n,4}) = 3$. Their proof used a differential equations method to analyze a greedy colouring algorithm. The next significant question is to determine $\chi(\mathcal{G}_{n,5})$. Díaz and others [12] have shown that $\chi(\mathcal{G}_{n,5}) = 3$ with probability bounded away from 0, provided a certain four-variable function has a unique maximum at a given point in a bounded domain. They also provide extensive numerical evidence to support this "maximum hypothesis". In their proof, they study what we will call locally rainbow balanced colourings, where a colouring is *balanced* if the number of vertices of each colour is equal, and *locally rainbow* if every vertex is adjacent to at least one vertex of each of the other colours. They apply a second-moment inequality to the random variable Y counting the number of locally rainbow balanced 3-colourings. (A k-colouring is a colouring that uses at most k colours.) We noticed that we could improve their result by applying the small subgraph conditioning method (see Section 3.3). The improvement is presented in Chapter 4, where we show that the conclusion of their result can be strengthened from "with probability bounded away from 0" to "asymptotically almost surely". Thus, provided the maximum hypothesis holds, a.a.s. $\chi(\mathcal{G}_{n,5}) = 3$.

For fixed d in general, the best bounds on $\chi(\mathcal{G}_{n,d})$ are due to Achlioptas and Moore [2]. They state that if k is the smallest integer satisfying $d < 2(k-1)\log(k-1)$ then asymptotically almost surely (a.a.s.) $\chi(\mathcal{G}_{n,d})$ is $k-1$, k, or $k+1$. If, in addition, $d > (2k-3)\log(k-1)$, then a.a.s. $\chi(\mathcal{G}_{n,d})$ is k or $k+1$. In this thesis we show that $\chi(\mathcal{G}_{n,d})$ a.a.s. cannot be $k+1$, provided that a certain second moment condition holds. Therefore this would reduce the range of possibilities for $\chi(\mathcal{G}_{n,d})$ to only a.a.s. $k-1$ and k, in the first case, and would establish that $\chi(\mathcal{G}_{n,d}) = k$ a.a.s. in the second case. In particular it would provide an alternate proof of the results of Shi and Wormald [32, 33] that a.a.s. $\chi(\mathcal{G}_{n,4}) = 3$ and $\chi(\mathcal{G}_{n,6}) = 4$. It would also establish, for example, the previously-unknown result that a.a.s. $\chi(\mathcal{G}_{n,10}) = 5$.

Achlioptas and Moore begin their proof by giving the a.a.s. lower bounds on the chromatic number stated above; i.e. that the random graph is a.a.s. not $(k-2)$-colourable. (For the second case, not $(k-1)$-colourable.) Most of the difficulty is in establishing the a.a.s. upper bounds on the chromatic number. Achlioptas and Moore bound the first and second moments of the random variable counting the number of balanced k-colourings, where a colouring is *balanced* if the number of vertices of each colour is equal. These bounds are used in a second moment inequality to give a lower bound on the probability of the event that $\mathcal{G}_{n,d}$ is k-colourable. Unfortunately, this bound fails to tend to 1; it only shows that the probability of the event is bounded away from 0 as n becomes large. Achlioptas and Moore separately showed that the chromatic number is 2-point concentrated; i.e. for each d there is a $k = k(d)$ such that $\chi(\mathcal{G}_{n,d}) \in \{k, k+1\}$ a.a.s. Combining these results with the lower bounds, they obtain the specific range of two or three possible values given above, permitting $k+1$ as a possible chromatic number.

When studying random structures, this failure of the second moment inequality bound to tend to 1 can often be overcome by using the small subgraph conditioning method which we will see in Section 3.3. Using this method, we show that, given a weakened second moment condition, $\mathcal{G}_{n,d}$ is a.a.s. k-colourable. This would eliminate $k + 1$ and determine two values precisely for 2-point concentration.

A technical concern when studying balanced k-colourings of a graph on n vertices is that such a colouring necessarily requires n to be divisible by k. Nevertheless, the above results can be extended to show that k-colourings exist for all n. One method is to adjust the proof to accommodate colourings in which the sizes of the colour classes are permitted to differ by at most 1. Alternatively, one can prove that there exists a balanced k-colouring in which the endpoints of a fixed number of independent edges have colours specified in advance. To k-colour a graph whose number of vertices is $n = 2ak + b$, one removes b vertices, adds some edges so that the graph becomes d-regular again, precolours these new edges, and applies the strengthened theorem. The details are given in [13].

Chapter 2

Circumference of $\mathcal{G}_{n,M}$ in the supercritical phase

2.1 Introduction

In this chapter we present a new a.a.s. upper bound on the circumference of $\mathcal{G}_{n,M}$ for M in the supercritical phase. Recall that the circumference of a graph is the length of its longest cycle and that the supercritical phase is $M = n/2 + s$ for $s = s(n)$ satisfying $n^{2/3} = o(s)$ but $s = o(n)$. For this range, Łuczak [23] has shown that the circumference of $\mathcal{G}_{n,M}$ is a.a.s. between $(16/3 + o(1))s^2/n$ and $(7.496 + o(1))s^2/n$. In his proof, Łuczak focuses on the *core* and *kernel* of $\mathcal{G}_{n,M}$. The core of a graph is its maximal subgraph of minimum degree at least 2. The *prekernel* of a graph is obtained from the core by throwing away any cycle components. The *kernel* of a graph is obtained from the prekernel by replacing each maximal path of degree-2 vertices by a single edge. We say that a graph is a prekernel (respectively, a kernel) if it is the prekernel (respectively, kernel) of some graph.

Łuczak's insight is that, for this range of M, the kernel is much like a random 3-regular graph, and the core is much like the graph formed from the kernel by randomly subdividing its edges about $(8 + o(1))s^2/n$ times.

A random 3-regular graph a.a.s. contains a Hamilton cycle. This gives a cycle in $\mathcal{G}_{n,M}$ containing about $(2/3) \times (8+o(1))s^2/n = (16/3+o(1))s^2/n$ vertices of the core. This is Łuczak's lower bound on the circumference.

The upper bound comes from viewing the core as constructed from the kernel together with a sequence of numbers, summing to $(8 + o(1))s^2/n$, describing how many degree-2 vertices belong on each edge of the kernel. From probability theory, the sum of the largest two-thirds of the terms of such a random sequence is at most $(7.496 + o(1))s^2/n$.

In this thesis we improve upon Łuczak's upper bound by a more detailed study of how a cycle can pass through such a structure. We prove the following result.

Theorem 1 *Let $M = n/2 + s$ with $n^{2/3} = o(s)$ and $s = o(n)$. The circumference of $\mathcal{G}_{n,M}$ is a.a.s. at most $(7 + o(1))s^2/n$.*

Our main tool is the kernel configuration model, introduced in [26] to facilitate arguments like Łuczak's.

Following Łuczak's example, it is helpful to put weights on the edges of the kernel; the weight of an edge tells us how many times the edge should be subdivided to recover the core. These weights form a random sequence whose asymptotic properties we investigate in Section 2. In particular, we show that any bounded number of terms in such a sequence behave like independent random variables with exponential distribution. We also show that when a function of a bounded number of these terms is summed over many sets of such terms, the result is concentrated about its expected value. These properties are needed in Section 3 where we establish an a.a.s. upper bound on the weight of the heaviest cycle in a pseudograph with random edge weights. The upper bound is expressed in terms of a family of constants, some of which we explicitly calculate in Section 4. In Section 5 we prove an a.a.s. upper bound on the circumference of a random prekernel with a degree sequence that resembles a random 3-regular graph with subdivided edges. In Section 7 we use this result to prove Theorem 1 after, in Section 6, establishing that the degree sequence of the prekernel of $\mathcal{G}_{n,M}$ indeed shows the required resemblance.

2.2 Random sequences

Let Ω be the probability space, equipped with the uniform distribution, of all sequences of m positive integers $(X_1, X_2, \ldots, X_m)$ summing to N. We are interested in the asymptotic value of certain functions of these random variables. Letting $\omega = \omega(N) \to \infty$, our asymptotics are in terms of $N \to \infty$, uniformly over all m satisfying $\omega < m < N/\omega$. Write $\mu = N/m$.

Our first result tells us the expected value of certain functions of $X_1, X_2, \ldots, X_j$ for j bounded.

Lemma 2 *Let g be a nonnegative integrable function of a bounded number j of nonnegative variables. Suppose that for some C and d, $g(x_1, \ldots, x_j) \le C(x_1 + \cdots + x_j)^d$ for all $x_1, \ldots, x_j$. Then,*

$$\mathbf{E}\left[g\left(\frac{X_1}{\mu}, \ldots, \frac{X_j}{\mu}\right)\right] = \int_0^\infty \cdots \int_0^\infty g(x_1, \ldots, x_j) e^{-x_1 - x_2 - \cdots - x_j} dx_1 \cdots dx_j + o(1).$$

Since the X_i are identically distributed, the above theorem also holds when $(X_1, \ldots, X_j)$ is replaced by $(X_{\sigma(1)}, \ldots, X_{\sigma(j)})$ for any j distinct $\sigma(1), \ldots, \sigma(j)$ in $\{1, 2, \ldots, m\}$. Furthermore, the error represented by $o(1)$ is independent of σ.

The next result states that when such a function is summed over σ in a sufficiently rich family, the sum is asymptotically almost surely (a.a.s.) concentrated about its expected value.

Lemma 3 *Let f be a nonnegative integrable function of a bounded number k of nonnegative variables. Suppose that for some C and d, $f(x_1, \ldots, x_k) \le C(x_1 +$*

$\cdots + x_k)^d$ *for all* $x_1, \ldots, x_k$*. Define the constant*

$$E^* := \int_0^\infty \cdots \int_0^\infty f(x_1, \ldots, x_k) e^{-x_1 - x_2 - \cdots - x_k} dx_1 \cdots dx_k.$$

and assume $E^* > 0$*. Let* S *be a set of* k*-tuples with entries from* $\{1, 2, \ldots, m\}$*, with each* k*-tuple having distinct components. Let* $I = I(S) \in S \times S$ *be the pairs of tuples which intersect; that is,*

$$I = \{(\sigma, \tau) \in S \times S \mid \{\sigma(1), \ldots, \sigma(k)\} \cap \{\tau(1), \ldots, \tau(k)\} \neq \emptyset\}.$$

If $|I| = o(|S|^2)$ *then*

$$\sum_{\sigma \in S} f\left(\frac{X_{\sigma(1)}}{\mu}, \ldots, \frac{X_{\sigma(k)}}{\mu}\right) = (E^* + o(1))|S|$$

a.a.s.; that is, with probability $1 - o(1)$*. Furthermore, the* $o(1)$ *terms may be bounded independently of* S*.*

These types of concentration results are often proved using martingales or inequalities like Talagrand's; however, because we are aiming for such a coarse result, a simple application of Chebyshev's inequality will suffice for the proof.

2.2.1 Distribution of terms

In this section we establish some preliminary results about the distribution of the positive terms $X_1, X_2, \ldots, X_j$ for bounded j. It is an exercise in basic counting to show that the number of sequences in Ω is $\binom{N-1}{m-1}$. It immediately follows that for positive integers $t_1, t_2, \ldots, t_j$, the number of sequences in Ω with $X_1 = t_1$, $X_2 = t_2$, $\ldots$, $X_j = t_j$ is

$$B(t) := \binom{N-1-t}{m-j-1}$$

where $t = t_1 + t_2 + \cdots + t_j$.

Proposition 4 *Let* x *satisfy* $x < \sqrt{m}/\omega$ *and* $x < \mu/\omega$*. For positive integers* $t \leq x\mu$ *we have*

$$\frac{B(t)}{|\Omega|} = (1 + O(\omega^{-1}))\mu^{-j} e^{-t/\mu}.$$

Proof.

$$\begin{aligned}
\frac{B(t)}{|\Omega|} &= \binom{N-1-t}{m-j-1}\binom{N-1}{m-1}^{-1} \\
&= \frac{(N-1-t)!}{(N-t+j-m)!}\frac{(N-m)!}{(N-1)!}\frac{(m-1)!}{(m-j-1)!} \\
&= \prod_{i=1}^{m-j-1}(N-t-i)\prod_{i=1}^{m-1}(N-i)^{-1}\prod_{i=1}^{j}(m-i)
\end{aligned}$$

$$
\begin{aligned}
&= \left((N-t-O(m))^{-j-1}\prod_{i=1}^{m}(N-t-i)\right)\left((N-m)\prod_{i=1}^{m}(N-i)^{-1}\right) \\
&\quad \times \prod_{i=1}^{j}(m-i) \\
&= N^{-j-1}\left(1-\frac{t}{N}-O\left(\frac{m}{N}\right)\right)^{-j-1} N\left(1-\frac{m}{N}\right)m^{j}\left(1+O\left(\frac{1}{m}\right)\right) \\
&\quad \times \prod_{i=1}^{m}\left(1-\frac{t}{N-i}\right) \\
&= \mu^{-j}\left(1-O\left(\frac{x\mu}{N}\right)-O(\omega^{-1})\right)\left(1-O(\omega^{-1})\right)\left(1+O(\omega^{-1})\right) \\
&\quad \times \prod_{i=1}^{m}\left(1-\frac{t}{N-O(m)}\right) \\
&= \mu^{-j}\left(1-O\left(\frac{x}{m}\right)-O(\omega^{-1})\right)\left(1-\frac{t}{N\left(1-O(\mu^{-1})\right)}\right)^{m} \\
&= \mu^{-j}\left(1-O(\omega^{-1})\right)\left(1-\frac{t}{N}+O\left(\frac{t}{N}\mu^{-1}\right)\right)^{m} \\
&= \mu^{-j}\left(1-O(\omega^{-1})\right)\left(1-\frac{t}{N}+O\left(\frac{x}{N}\right)\right)^{m} \\
&= \mu^{-j}\left(1-O(\omega^{-1})\right)\exp\left(m\left(-\frac{t}{N}+O\left(\frac{x}{N}\right)+O\left(\frac{(t+x)^2}{N^2}\right)\right)\right) \\
&= \mu^{-j}\left(1-O(\omega^{-1})\right)e^{-t/\mu}\exp\left(O\left(\frac{x}{\mu}\right)+O\left(m\frac{x^2\mu^2}{N^2}\right)\right) \\
&= \mu^{-j}e^{-t/\mu}\left(1-O(\omega^{-1})\right)\exp\left(O(\omega^{-1})+O\left(\frac{x^2}{m}\right)\right) \\
&= \mu^{-j}e^{-t/\mu}\left(1-O(\omega^{-1})\right). \quad \blacksquare
\end{aligned}
$$

Corollary 5 *Let $x > 0$ be fixed. For any positive integers $t_1, t_2, \ldots, t_j$ summing to $t \le x\mu$ we have*

$$\mathbf{P}[X_1 = t_1, X_2 = t_2, \ldots, X_j = t_j] = (1 + O(\omega^{-1}))\mu^{-j}e^{-t/\mu}.$$

Next we bound the probability of larger terms.

Lemma 6 *Let $x > 0$ be fixed. For positive integers $t_1, t_2, \ldots, t_j$ summing to $t \ge x\mu$ we have*

$$\mathbf{P}[X_1 = t_1, X_2 = t_2, \ldots, X_j = t_j] < 2\mu^{-j}e^{-x}\left(1-\frac{1}{2\mu}\right)^{t-x\mu}$$

when N is sufficiently large.

Proof. If $B(t) = 0$ then the required probability is zero and we are done. Otherwise, $B(i)$ is nonzero for all positive integers $i \le t$ and the probability which we must estimate is

$$\frac{B(t)}{|\Omega|} = |\Omega|^{-1} B(\lfloor x\mu \rfloor) \prod_{i=\lfloor x\mu\rfloor+1}^{t} \frac{B(i)}{B(i-1)}.$$

By Proposition 4, the product of the first two terms is $(1 + O(\omega^{-1}))\mu^{-j}e^{-\lfloor x\mu\rfloor/\mu}$. This is less than $2\mu^{-j}e^{-x}$ when N is sufficiently large. To bound the remaining product, we estimate the ratio

$$\begin{aligned}
\frac{B(i)}{B(i-1)} &= \frac{\binom{N-1-i}{m-j-1}}{\binom{N-1-i+1}{m-j-1}} \\
&= \frac{(N-1-i)!(N-i-m+j+1)!}{(N-1-i-m+j+1)!(N-i)!} \\
&= \frac{N-i-m+j+1}{N-i} \\
&= 1 - \frac{m-j-1}{N-i} \\
&< 1 - \frac{m-j-1}{N} \\
&< 1 - \frac{m/2}{N}
\end{aligned}$$

where the last inequality holds for N sufficiently large. So, for N sufficiently large,

$$\begin{aligned}
\prod_{i=\lfloor x\mu\rfloor+1}^{t} \frac{B(i)}{B(i-1)} &< \left(1 - \frac{1}{2\mu}\right)^{t-\lfloor x\mu\rfloor} \\
&\le \left(1 - \frac{1}{2\mu}\right)^{t-x\mu}
\end{aligned}$$

since decreasing the exponent makes the expression larger. The result follows. ∎

2.2.2 Proof of Lemma 2

By the definition of expected value, we have

$$\mathbf{E}\left[g\left(\frac{X_1}{\mu}, \ldots, \frac{X_j}{\mu}\right)\right] = \sum g\left(\frac{t_1}{\mu}, \ldots, \frac{t_j}{\mu}\right) \mathbf{P}[X_1 = t_1, X_2 = t_2, \ldots, X_j = t_j]$$

where the sum is over all positive integer j-tuples $t_1, t_2, \ldots, t_j$.

Fix $x > 0$. Let us split the sum into two parts, $S_1(x)$ being the sum over j-tuples where each $t_i < x\mu$, and $S_2(x)$ being the remainder. We will show that, as $N \to \infty$,

$$S_1(x) \to \int_0^x \cdots \int_0^x g(x_1, \ldots, x_j)e^{-x_1-x_2-\cdots-x_j}dx_1 \cdots dx_j,$$

while

$$|S_2(x)| < Ke^{-x/2}$$

for some constant K. As x grows, $|S_2(x)|$ approaches 0 and $S_1(x)$ is nonnegative and nondecreasing since g is nonnegative. So, taking $x \to \infty$ proves the lemma.

We begin by estimating $S_1(x)$. These terms have each $t_i \leq x\mu$, so we use Corollary 5 to estimate the probabilities as follows.

$$\begin{aligned} S_1(x) &= \sum_{t_1 < x\mu} \cdots \sum_{t_j < x\mu} g\left(\frac{t_1}{\mu}, \ldots, \frac{t_j}{\mu}\right) \mathbf{P}[X_1 = t_1, X_2 = t_2, \ldots, X_j = t_j] \\ &= \sum_{t_1 < x\mu} \cdots \sum_{t_j < x\mu} g\left(\frac{t_1}{\mu}, \ldots, \frac{t_j}{\mu}\right) (1 + O(\omega^{-1}))\mu^{-j} e^{-(t_1 + \cdots + t_j)/\mu}. \end{aligned}$$

Since $O(\omega^{-1})$ is independent of the t_i, this becomes

$$(1 + O(\omega^{-1})) \sum_{t_1 < x\mu} \cdots \sum_{t_j < x\mu} g\left(\frac{t_1}{\mu}, \ldots, \frac{t_j}{\mu}\right) \mu^{-j} e^{-(t_1 + \cdots + t_j)/\mu}.$$

Letting $M = x\mu$ we get

$$(1 + O(\omega^{-1})) \sum_{t_1 < M} \cdots \sum_{t_j < M} g\left(t_1 \frac{x}{M}, \ldots, t_j \frac{x}{M}\right) e^{-(t_1 + \cdots + t_j)x/M} \left(\frac{x}{M}\right)^j.$$

As $N \to \infty$ we have $M \to \infty$ and this expression becomes the Riemann integral

$$\int_0^x \cdots \int_0^x g(x_1, \ldots, x_j) e^{-x_1 - x_2 - \cdots - x_j} dx_1 \cdots dx_j$$

as required.

The terms of the sum $S_2(x)$ are indexed by j-tuples $t_1, t_2, \ldots, t_j$ with at least one $t_i \geq x\mu$. Consider such a term, and let $t = t_1 + t_2 + \cdots + t_j$. For N sufficiently large, the absolute value of the term is

$$g\left(\frac{t_1}{\mu}, \ldots, \frac{t_j}{\mu}\right) \mathbf{P}[X_1 = t_1, X_2 = t_2, \ldots, X_j = t_j] < C\left(\frac{t}{\mu}\right)^d 2\mu^{-j} e^{-x} \left(1 - \frac{1}{2\mu}\right)^{t - x\mu}$$

by the hypotheses about g and Lemma 6. The number of terms in $S_2(x)$ indexed by j-tuples summing to t is at most $\binom{t-1}{j-1} \leq (t+j)^{j-1} \leq (2t)^{j-1}$ for N (and hence t) sufficiently large. Thus, for N large, we have

$$\begin{aligned} |S_2(x)| &< \sum_{t \geq x\mu} (2t)^{j-1} C\left(\frac{t}{\mu}\right)^d 2\mu^{-j} e^{-x} \left(1 - \frac{1}{2\mu}\right)^{t - x\mu} \\ &= 2e^{-x} \frac{C 2^{j-1}}{\mu^{j+d}} \left(1 - \frac{1}{2\mu}\right)^{-x\mu} \sum_{t \geq x\mu} t^{j+d-1} \left(1 - \frac{1}{2\mu}\right)^t. \end{aligned}$$

The factor $(1-1/(2\mu))^{-x\mu}$ approaches $e^{x/2}$ as $N \to \infty$. The remaining sum is

$$\begin{aligned}\sum_{t \geq x\mu} t^{j+d-1}\left(1-\frac{1}{2\mu}\right)^t &\leq \sum_{t\geq 0}(t+1)(t+2)\cdots(t+j+d-1)\left(1-\frac{1}{2\mu}\right)^t \\ &= (j+d-1)!(2\mu)^{j+d}\end{aligned}$$

using the Maclaurin series expansion $k!(1-x)^{-k-1} = \sum_{t\geq 0}(t+1)(t+2)\cdots(t+k)x^t$. Combining this with the previous results, we get the desired estimate. This proves the lemma. ∎

2.2.3 Proof of Lemma 3

For each σ in S, define the random variable $Y_\sigma := f(X_{\sigma(1)}/\mu, \ldots, X_{\sigma(k)}/\mu)$. As we remarked after Lemma 2, each of these variables has the same distribution as the random variable $Y_1 := f(X_1/\mu, \ldots, X_k/\mu)$. In particular, the expected value is the constant E^*, up to an additive error of $o(1)$. We will establish the concentration of the random variable $Z := \sum_{\sigma\in S} Y_\sigma$ by showing that the variance $\mathbf{Var}[Z]$ is $o((\mathbf{E}Z)^2)$. The lemma then follows by Chebyshev's inequality.

We begin by estimating

$$\begin{aligned}(\mathbf{E}Z)^2 &= \sum_{(\sigma,\tau)\in S\times S} \mathbf{E}Y_\sigma \mathbf{E}Y_\tau \\ &= \sum_{(\sigma,\tau)\in S\times S} (E^*+o(1))(E^*+o(1)) \\ &= \sum_{(\sigma,\tau)\in S\times S} \Theta(1) \\ &= \Theta(|S|^2)\end{aligned}$$

(using the lower bound assumed on E^* in the lemma). We can write the variance as

$$\begin{aligned}\mathbf{Var}[Z] &= \mathbf{E}[Z^2]-(\mathbf{E}Z)^2 \\ &= \sum_{(\sigma,\tau)\in S\times S}(\mathbf{E}[Y_\sigma Y_\tau]-\mathbf{E}Y_\sigma\mathbf{E}Y_\tau) \\ &= \sum_{(\sigma,\tau)\in I}(\mathbf{E}[Y_\sigma Y_\tau]-\mathbf{E}Y_\sigma\mathbf{E}Y_\tau) + \sum_{(\sigma,\tau)\in(S\times S)\setminus I}(\mathbf{E}[Y_\sigma Y_\tau]-\mathbf{E}Y_\sigma\mathbf{E}Y_\tau).\end{aligned}$$

To study the terms of the second sum, let $(\sigma,\tau)\in(S\times S)\setminus I$. By Lemma 2, we have

$$\begin{aligned}&\mathbf{E}[Y_\sigma Y_\tau] \\ &= \mathbf{E}\left[f\left(\frac{X_{\sigma(1)}}{\mu},\ldots,\frac{X_{\sigma(k)}}{\mu}\right)f\left(\frac{X_{\tau(1)}}{\mu},\ldots,\frac{X_{\tau(k)}}{\mu}\right)\right]\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \cdots \int_0^\infty f(x_1, \ldots, x_k) f(x_{k+1}, \ldots, x_{2k}) e^{-x_1 - \cdots - x_{2k}} dx_1 \cdots dx_{2k} + o(1) \\
&= \left(\int_0^\infty \cdots \int_0^\infty f(x_1, \ldots, x_k) e^{-x_1 - x_2 - \cdots - x_k} dx_1 \cdots dx_k \right)^2 + o(1) \\
&= \mathbf{E} Y_\sigma \mathbf{E} Y_\tau + o(1)
\end{aligned}
$$

where $o(1)$ is independent of σ and τ. So the second sum is $o(|S|^2)$. To study the terms of the first sum, we can be more crude. By Lemma 2 and the remark following it, we know that each $\mathbf{E}[Y_\sigma Y_\tau]$ and $\mathbf{E} Y_\sigma \mathbf{E} Y_\tau$ depends only on the tuple positions where σ and τ intersect, and each value is $O(1)$. So the first sum is $O(|I|)$, which is $o(|S|^2)$ by hypothesis. Combining the two sums, we see that the variance of Z is $o(|S|^2)$, which is $o((\mathbf{E}Z)^2)$, as required. ∎

2.3 Heavy cycles in a weighted pseudograph

In the introduction we saw that the problem of bounding the circumference of $\mathcal{G}_{n,M}$ is connected to the problem of bounding the weight of the heaviest cycle in a certain edge-weighted pseudograph. In this section we study a pseudograph whose m edges are randomly weighted by positive integers summing to N. The sequence of weights is chosen uniformly at random from among all such sequences. Equivalently, we can think of the weights as being generated by the following random process applied to make a sequence of pseudographs, beginning with the given one. At each step, choose an edge uniformly at random from the current pseudograph and subdivide the edge into two edges. Repeat the procedure until the resulting pseudograph has exactly N edges. For each edge in the original pseudograph, define its weight to be the number of edges into which it has been subdivided. These weights form a sequence of m positive integers summing to N. There are exactly $(N-m)!$ ways that the process can form a given sequence, so the sequence is chosen uniformly at random from among all such sequences. Another random process for generating the weights initially gives a weight of 1 to each edge, then selects an edge at random with probability proportional to the weight of the edge and increments the weight of the selected edge by 1. The selection and incrementing is repeated until the total weight is N. It is easy to see that this process is equivalent to the previous one.

Given a subgraph of an edge-weighted pseudograph, we define the weight of the subgraph to be the sum of the weights on its edges. To establish an upper bound for the weight of a cycle in a large pseudograph, we will consider the intersection of the cycle with small trees in the pseudograph. The intersection of the cycle and the small tree will form a set of vertex-disjoint paths which begin and end at leaf vertices of the tree. We will use the maximum-weight set of such vertex-disjoint paths to bound the weight of the intersection. This motivates the following definitions.

Fix an integer $k \geq 2$. A *biased tree* T on k edges is a tree on k edges with each non-leaf vertex having degree 3 and each edge e_i having a nonnegative number b_i

called its bias. We may assume that the sum of the biases $\mathbf{b} = (b_1, b_2, \ldots, b_k)$ is 1.

Let $\mathcal{P}$ be the set of all maximal subgraphs of T which are a union of vertex-disjoint paths which begin and end at leaf vertices. Define the function

$$f_T(x_1, x_2, \ldots, x_k) = \max_{P \in \mathcal{P}} \sum_{i : e_i \in E(P)} b_i x_i$$

and the constant

$$E_T = \int_0^\infty \int_0^\infty \cdots \int_0^\infty f_T(x_1, x_2, \ldots, x_k) e^{-x_1 - x_2 - \cdots - x_k} dx_1 dx_2 \cdots dx_k. \qquad (2.3.1)$$

If $x_1, x_2, \ldots, x_k$ are weights on the edges of T, we can think of f as the maximum "biased weight" of any graph in $\mathcal{P}$.

We say that the positive constant c^* is *k-admissible* if $E_T < c^*$ for some biased tree T on k edges.

Lemma 7 *Fix an integer $k \geq 2$. Let the positive number c^* be k-admissible. Let $G = G(n)$ be a pseudograph on $v = v(n) \to \infty$ (as $n \to \infty$) vertices and $m = m(n)$ edges with minimum degree at least 3. Suppose the subgraph B of G induced by cycles of length at most k (including loops and parallel edges) and edges incident to vertices of degree greater than 3 satisfies $|E(B)| = o(v)$. Let $N = N(n)$ be a positive integer satisfying $m = o(N)$. On the edges of G put weights, a sequence chosen uniformly at random from among all sequences of m positive integers summing to N. Then, the heaviest cycle in G has weight a.a.s. at most $c^* N$.*

Proof. Denote the edges of G by $w_1, w_2, \ldots, w_m$ and their random weights by $X_1, X_2, \ldots, X_m$. We estimate m by recalling that in any graph the sum of the vertex degrees equals twice the number of edges. Since G has minimum degree at least 3, we have $2m \geq 3v$. Since G has only $o(v)$ edges incident to vertices of degree greater than 3, we have $2m \leq 3v + o(v)$. Thus $m \sim 3v/2$.

For a subgraph S of G, define its *k-neighbourhood* $\Gamma(S)$ to be the subgraph of G reachable from S by paths of length at most k. Recalling that the subgraph B of G contains all edges incident with vertices of degree greater than 3, its k-neighbourhood satisfies $|E(\Gamma(B))| \leq 2^k |E(B)| = o(v)$.

Let C be a cycle in G. Its weight $wt(C)$ is

$$\begin{aligned} wt(C) &:= \sum_{j=1}^{m} X_j I(w_j \in E(C)) \\ &= \sum_{j : w_j \in E(\Gamma(B))} X_j I(w_j \in E(C)) + \sum_{j : w_j \notin E(\Gamma(B))} X_j I(w_j \in E(C)) \end{aligned}$$

where $I(\alpha)$ is the indicator function equal to 1 if α is true and 0 otherwise. The expected value of each X_j is $\mu := N/m$, so the first sum has expected value at most $|E(\Gamma(B))| N/m = o(vN/m) = o(N)$ since $m \sim 3v/2$. It follows by Markov's inequality that the first sum is a.a.s. $o(N)$. Thus a.a.s.,

$$wt(C) = o(N) + \sum_{j : w_j \notin E(\Gamma(B))} X_j I(w_j \in E(C)). \qquad (2.3.2)$$

Since c^* is k-admissible, there is a biased tree T on k edges with $E_T < c^*$. We will study the copies of T in the graph $G \backslash B$. Let S be the set of 1-1 homomorphisms σ mapping T to $G \setminus B$. Since $k \geq 2$, each σ is uniquely defined by the mapping it induces between the edge sets. We write $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(k))$ and interpret $\sigma(i) = j$ to mean that σ maps edge e_i of T to edge w_j of G.

Consider the random variable

$$Z = \sum_{\sigma \in S} \sum_{i=1}^{k} b_i X_{\sigma(i)} I(w_{\sigma(i)} \in E(C)).$$

Expressing Z in terms of the edges of G we may write

$$\begin{aligned} Z &= \sum_{j=1}^{m} \sum_{\sigma \in S} \sum_{i=1}^{k} b_i X_j I(w_j \in E(C)) I(\sigma(i) = j) \\ &= \sum_{j=1}^{m} \sum_{i=1}^{k} b_i X_j I(w_j \in E(C)) |\{\sigma \in S \mid \sigma(i) = j\}|. \end{aligned}$$

For each edge w_j of G not in $E(\Gamma(B))$ the k-neighbourhood of w_j is the depth-k tree with internal vertices of degree 3. Thus, $|\{\sigma \in S \mid \sigma(i) = j\}|$ equals some constant independent of j. In fact, this constant is a number a, independent of i, because any σ in this set is determined by choosing one of the 2 ways to embed e_i onto w_j and then, moving outward from e_i, making one binary choice for each non-leaf vertex of T. On the other hand, for an edge $w_j \in E(\Gamma(B))$, $|\{\sigma \in S \mid \sigma(i) = j\}|$ is at most a (by the same argument, recalling that some choices are impossible because σ maps into $G \setminus B$), so we have

$$\begin{aligned} Z &= \sum_{j: w_j \in E(\Gamma(B))} O(X_j) + \sum_{j: w_j \notin E(\Gamma(B))} \sum_{i=1}^{k} a b_i X_j I(w_j \in E(C)) \\ &= \sum_{j: w_j \in E(\Gamma(B))} O(X_j) + \sum_{j: w_j \notin E(\Gamma(B))} a X_j I(w_j \in E(C)) \end{aligned}$$

since $\sum_{i=1}^{k} b_i = 1$. The first sum has $o(v)$ terms, each having expected value $O(N/m)$, so the sum is a.a.s. $o(vN/m) = o(N)$ by Markov's inequality. We now have a.a.s.

$$Z = o(N) + a \sum_{j \notin E(\Gamma(B))} X_j I(w_j \in E(C)).$$

Combining this result with (2.3.2) we get a.a.s.

$$wt(C) = \frac{1}{a} Z + o(N). \tag{2.3.3}$$

Returning to the definition of Z, we notice that the inner sum is the "biased weight" of the edges of C passing through the copy of T given by σ. These edges must form vertex-disjoint paths beginning and ending at leaves of the copy of T given by σ, so this sum is at most $f_T(X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(k)})$. So

$$Z \leq \sum_{\sigma \in S} f_T(X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(k)}).$$

We will estimate this sum by applying Lemma 3 to

$$\frac{1}{\mu} \sum_{\sigma \in S} f_T(X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(k)}) = \sum_{\sigma \in S} f_T\left(\frac{X_{\sigma(1)}}{\mu}, \frac{X_{\sigma(2)}}{\mu}, \ldots, \frac{X_{\sigma(k)}}{\mu}\right).$$

To verify the hypotheses of Lemma 3 we first note that $f_T(x_1, x_2, \ldots, x_k)$ is nonnegative, piecewise linear (and hence integrable), and bounded above by $x_1+x_2+\cdots+x_k$. We estimate $|S|$ by

$$\begin{aligned} |S| &= \sum_{j:w_j \in E(G)} |\{\sigma \in S \mid \sigma(1) = j\}| \\ &= \sum_{j:w_j \in E(\Gamma(B))} |\{\sigma \in S \mid \sigma(1) = j\}| + \sum_{j:w_j \notin E(\Gamma(B))} |\{\sigma \in S \mid \sigma(1) = j\}| \\ &= o(v) + \sum_{w_j \notin E(\Gamma(B))} a \\ &= o(v) + (1 + o(1))ma \\ &\sim ma \end{aligned}$$

using $o(v) = o(m)$. To estimate the cardinality of the set I of pairs $(\sigma, \tau) \in S \times S$ for which σ and τ represent intersecting copies of T, consider any edge f in $G \setminus B$. As we have seen previously, there are at most a copies of T using f. So, a crude upper bound for $|I|$ is $a^2|E(G \setminus B)| \leq a^2 m$, giving us $|I| = o(|S|^2)$ as required. Recalling the definition of E_T from (2.3.1) we may apply Lemma 3 and conclude a.a.s.

$$\begin{aligned} \frac{1}{\mu} Z &\leq (E_T + o(1))|S| \\ &= (E_T + o(1))am. \end{aligned}$$

Combining this with Equation (2.3.3) we get a.a.s.

$$\begin{aligned} wt(C) &\leq \frac{1}{a}\mu(E_T + o(1))am + o(N) \\ &= (E_T + o(1))N + o(N) \\ &< c^* N \end{aligned}$$

as required. ∎

Remark 1. A random 3-regular graph a.a.s. satisfies all of the hypotheses of Lemma 7. The lemma thus gives an upper bound which holds a.a.s. on the weight of the heaviest cycle in a randomly-weighted random 3-regular graph.

Remark 2. There are essentially two ingredients in the proof of Lemma 7. The first ingredient is a method for bounding the weight of a cycle in a large edge-weighted 3-regular subgraph. The second ingredient is the argument that the weight of the heaviest cycle does not change much when the remainder of the graph is included. This second ingredient is implicit in Łuczak's proof of his upper bound on the circumference of $\mathcal{G}_{n,M}$ in the supercritical phase [23]. It is the first ingredient that is the new contribution.

Remark 3. For the task of bounding the weight of a cycle in a large edge-weighted 3-regular subgraph, one might suggest investigating the weight of the lightest-weight matching. Certainly the complement of a Hamilton cycle in a 3-regular graph forms a perfect matching. But, in general, the maximum-weight cycle is not necessarily Hamiltonian. Thus, its removal from the graph does not always form a perfect matching.

2.4 Computing E_T

Recall the definitions of $f_T(\mathbf{x}) = f_T(x_1, x_2, \ldots, x_k)$ and E_T from (2.3.1). In the previous section we saw that the weight of the heaviest cycle in a certain edge-weighted pseudograph can be bounded in terms of E_T for any biased tree T. In this section we compute the value of E_T for a few small biased trees T. For some trees T we also determine the biases $\mathbf{b}$ which make E_T as small as possible. It will be convenient to assume that each $b_i > 0$; however, the results extend to nonnegative b_i by continuity.

2.4.1 One non-leaf vertex

We begin by computing E_T for the biased tree T that has one vertex of degree 3 joined to three leaf vertices. For this T the set $\mathcal{P}$ contains three subgraphs, each being simply a pair of edges, giving us

$$\begin{aligned} f_T(\mathbf{x}) = f_T(x_1, x_2, x_3) &= \max_{P \in \mathcal{P}} \sum_{i : e_i \in E(P)} b_i x_i \\ &= \max\{b_1x_1 + b_2x_2, b_1x_1 + b_3x_3, b_2x_2 + b_3x_3\} \end{aligned}$$

and

$$\begin{aligned} E_T &= \int_0^\infty \int_0^\infty \int_0^\infty f_T(\mathbf{x}) e^{-x_1 - x_2 - x_3} d\mathbf{x} \\ &= \int_0^\infty \int_0^\infty \int_0^\infty \max\{b_1x_1 + b_2x_2, b_1x_1 + b_3x_3, b_2x_2 + b_3x_3\} e^{-x_1 - x_2 - x_3} d\mathbf{x}. \end{aligned}$$

To help us compute the integral we partition the region of integration into three parts according to whichever of the arguments of the max function is the largest. We can disregard the region on which two arguments are equal because it is a set of measure zero. The part on which

$$b_1x_1 + b_2x_2 > b_1x_1 + b_3x_3 \text{ and } b_1x_1 + b_2x_2 > b_2x_2 + b_3x_3$$

is simply the set of points with positive coordinates satisfying

$$b_2 x_2 > b_3 x_3 \text{ and } b_1 x_1 > b_3 x_3$$

which may be expressed iteratively by

$$\left\{ (x_1, x_2, x_3) \mid x_3 > 0, x_2 > \frac{b_3 x_3}{b_2}, x_1 > \frac{b_3 x_3}{b_1} \right\}$$

giving us the integral

$$\int_{x_3=0}^{\infty} \int_{x_2=b_3x_3/b_2}^{\infty} \int_{x_1=b_3x_3/b_1}^{\infty} (b_1x_1 + b_2x_2) e^{-x_1-x_2-x_3} dx_1 dx_2 dx_3$$
$$= \frac{b_1 b_2 (4b_3 b_1 b_2 + b_3 b_2^2 + b_1^2 b_2 + b_1^2 b_3 + b_1 b_2^2)}{b_3^2 b_2^2 + 2b_3^2 b_2 b_1 + 2b_3 b_2^2 b_1 + b_1^2 b_3^2 + 2b_1^2 b_3 b_2 + b_1^2 b_2^2}.$$

The other two parts of the integral may be obtained by symmetry. When the three parts are added together, the result is

$$E_T = \frac{(b_1 + b_2)(b_1 + b_3)(b_2 + b_3)}{b_1 b_2 + b_1 b_3 + b_2 b_3}.$$

Recalling that biases $\mathbf{b}$ must sum to 1, we substitute $b_3 = 1 - b_1 - b_2$ to find

$$E_T = \frac{(b_1 - 1)(b_1 + b_2)(b_2 - 1)}{b_1^2 - b_1 + b_1 b_2 - b_2 + b_2^2}.$$

Since the biases $\mathbf{b}$ must also be nonnegative, we see that (b_1, b_2) ranges over the set

$$B = \{(b_1, b_2) \mid b_1 > 0, b_2 > 0, 1 - b_1 - b_2 > 0\}.$$

To minimize E_T over B we first see that the partial derivative

$$\frac{\partial}{\partial b_1} E_T = \frac{b_2^2 (b_2 - 1)(1 - 2b_1 - b_2)}{(b_1^2 - b_1 + b_1 b_2 - b_2 + b_2^2)^2}$$

equals 0 on B precisely when $2b_1 + b_2 = 1$. By symmetry, the partial derivative with respect to b_2 equals 0 on B precisely when $2b_2 + b_1 = 1$. The solution to these two equations is $b_1 = b_2 = 1/3$, giving a value of $8/9$ for E_T.

As for the boundary of B, observe that whenever b_1, b_2, or $1 - b_1 - b_2$ equals 0 we have $E_T = 1$. Thus the local extremum at $b_1 = b_2 = 1/3$ is the global minimum over B.

In summary, we have shown the following.

Proposition 8 *Let T be a biased tree on one degree-3 vertex and three leaf vertices. With $b_1 = b_2 = b_3 = 1/3$ we have*

$$E_T = \frac{8}{9}$$

and, for this tree, no other choice of biases $\mathbf{b}$ yields a lower value of E_T.

2.4.2 Reducing the number of distinct bias values

For the tree in the previous section, we saw that E_T was minimized when all of the bias values are equal. Next we describe another situation when it is best for certain bias values to be equal.

Proposition 9 *Let T be a biased tree with edges $e_1, e_2, \ldots, e_k$. Suppose that e_1 and e_2 are incident to two leaf vertices and a common non-leaf vertex. Define T' from T by replacing the biases b_1 and b_2 by their average,*

$$\mathbf{b}' = \left(\frac{b_1+b_2}{2}, \frac{b_1+b_2}{2}, b_3, b_4, \ldots, b_k\right).$$

Then

$$E_{T'} \leq E_T.$$

Proof. The integral in the definition of E_T may be expressed as the integral over nonnegative $x_1, x_2, \ldots, x_k$ satisfying $x_2 \geq x_1$ of the integrand

$$(f_T(x_1, x_2, x_3, \ldots, x_k) + f_T(x_2, x_1, x_3, \ldots, x_k))\, e^{-x_1-x_2-\cdots-x_k}.$$

Let $x_1, x_2, \ldots, x_k$ be nonnegative reals with $x_2 \geq x_1$. To establish the inequality of the integrals in the proposition, it therefore suffices to prove the corresponding inequality of the integrands,

$$\begin{aligned} f_{T'}(x_1, x_2, x_3, \ldots, x_k) &+ f_{T'}(x_2, x_1, x_3, \ldots, x_k) \\ &\leq f_T(x_1, x_2, x_3, \ldots, x_k) + f_T(x_2, x_1, x_3, \ldots, x_k). \end{aligned} \tag{2.4.1}$$

Recall that $f_T(x_1, x_2, \ldots, x_k)$ is the maximum of the biased weight of any $P \in \mathcal{P}$ when the edges are given biases $b_1, b_2, \ldots, b_k$ and weights $x_1, x_2, \ldots, x_k$. Because of the hypothesis about e_1 and e_2, the set $\mathcal{P}$ is unchanged when e_1 is exchanged with e_2. Since $b'_1 = b'_2$, it follows that if P_0 is the $P \in \mathcal{P}$ that achieves the maximum in $f_{T'}(x_1, x_2, x_3, \ldots, x_k)$, then the maximum in $f_{T'}(x_2, x_1, x_3, \ldots, x_k)$ is achieved by P_1, obtained from P_0 by exchanging e_1 with e_2, and these two maxima are equal. The left side of Equation (2.4.1) is therefore the sum of the biased weight $W_0(\mathbf{b}')$ of P_0, using weights $(x_1, x_2, x_3, \ldots, x_k)$ and biases $\mathbf{b}'$, plus the biased weight $W_1(\mathbf{b}')$ of P_1, using weights $(x_2, x_1, x_3, \ldots, x_k)$ and biases $\mathbf{b}'$. A lower bound for the right side of Equation (2.4.1) is the sum of the biased weight $W_0(\mathbf{b})$ of P_0, using weights $(x_1, x_2, x_3, \ldots, x_k)$ and biases $\mathbf{b}$, plus the biased weight $W_1(\mathbf{b})$ of P_1, using weights $(x_2, x_1, x_3, \ldots, x_k)$ and biases $\mathbf{b}$. To prove Equation (2.4.1), we will show $W_0(\mathbf{b}') - W_0(\mathbf{b}) + W_1(\mathbf{b}') - W_1(\mathbf{b}) \leq 0$.

Clearly P_0 must contain at least one of e_1 and e_2. (If not, the subgraph could be made heavier by including them.) If P_0 contains both e_1 and e_2 then we have

$$\begin{aligned} & (W_0(\mathbf{b}') - W_0(\mathbf{b})) + (W_1(\mathbf{b}') - W_1(\mathbf{b})) \\ = \; & (x_1 b'_1 + x_2 b'_2 - x_1 b_1 - x_2 b_2) + (x_2 b'_1 + x_1 b'_2 - x_2 b_1 - x_1 b_2) \\ = \; & x_1\left(2\frac{b_1+b_2}{2} - b_1 - b_2\right) + x_2\left(2\frac{b_1+b_2}{2} - b_1 - b_2\right) \\ = \; & 0. \end{aligned}$$

Otherwise, P_0 contains exactly one of e_1 and e_2. Since $x_2 \geq x_1$ and P_0 was chosen to have the greatest biased weight for T', P_0 must contain e_2. We have

$$\begin{aligned}
(W_0(\mathbf{b}') - W_0(\mathbf{b})) + (W_1(\mathbf{b}') - W_1(\mathbf{b})) &= x_2 b_2' - x_2 b_2 + x_2 b_1' - x_2 b_1 \\
&= x_2 \left(\frac{b_1 + b_2}{2} + \frac{b_1 + b_2}{2} - b_1 - b_2 \right) \\
&= 0. \quad \blacksquare
\end{aligned}$$

2.4.3 Two non-leaf vertices

In this section we study biased trees T on two non-leaf vertices, v and w, and four leaf vertices. Let e_1 and e_2 denote the two edges which are each incident to v and a leaf. Denote by e_3 the edge joining v to w, and christen the other two edges as e_4 and e_5. We want to compute the minimum value of E_T over all bias vectors $\mathbf{b}$. By Proposition 9, it suffices to consider $\mathbf{b}$ in which $b_1 = b_2$ and $b_4 = b_5$. In fact, we can simplify the problem further.

Proposition 10 *Suppose T is a biased tree on two non-leaf vertices and four leaf vertices. Suppose its biases are $\mathbf{b} = (b_1, b_1, b_3, b_5, b_5)$. Define T' from T by replacing the biases by $\mathbf{b}'$ defined by*

$$\mathbf{b}' = \left(\frac{b_1 + b_5}{2}, \frac{b_1 + b_5}{2}, b_3, \frac{b_1 + b_5}{2}, \frac{b_1 + b_5}{2} \right).$$

Then

$$E_{T'} \leq E_T.$$

Proof. The proof is similar to the proof of Proposition 9. We begin by rewriting the integral in the definition of E_T, using the region X, the set of points $(x_1, x_2, \ldots, x_5)$ with nonnegative coordinates satisfying $x_1 + x_2 \leq x_4 + x_5$, as

$$E_T = \int_X \left(f_T(x_1, x_2, x_3, x_4, x_5) + f_T(x_4, x_5, x_3, x_1, x_2) \right) e^{-x_1 - x_2 - \cdots - x_5} dx_1 dx_2 \cdots dx_5.$$

Let $(x_1, x_2, \ldots, x_5) \in X$. To establish the inequality of the integrals in the proposition, it therefore suffices to prove the corresponding inequality of the integrands,

$$\begin{aligned}
f_{T'}(x_1, x_2, x_3, x_4, x_5) &+ f_{T'}(x_4, x_5, x_3, x_1, x_2) \\
&\leq f_T(x_1, x_2, x_3, x_4, x_5) + f_T(x_4, x_5, x_3, x_1, x_2).
\end{aligned} \tag{2.4.2}$$

Let P_0 be an element of $\mathcal{P}$ that attains the maximum value in the definition of $f_{T'}(x_1, x_2, x_3, x_4, x_5)$. Then by the symmetry of T' (and $\mathbf{b}'$), an element of $\mathcal{P}$ that attains the maximum in the definition of $f_{T'}(x_4, x_5, x_3, x_1, x_2)$ is P_1, formed from P_0 by exchanging e_1 with e_4 and exchanging e_2 with e_5. The left side of Equation (2.4.2) is then the biased weight $W_0(\mathbf{b}')$ of P_0 with weights $(x_1, x_2, x_3, x_4, x_5)$ and biases $\mathbf{b}'$, plus the biased weight $W_1(\mathbf{b}')$ of P_1 with weights $(x_4, x_5, x_3, x_1, x_2)$ and biases $\mathbf{b}'$. A lower bound for the right side of Equation (2.4.2) is the biased weight $W_0(\mathbf{b})$

of P_0 with weights $(x_1, x_2, x_3, x_4, x_5)$ and biases $\mathbf{b}$, plus the biased weight $W_1(\mathbf{b})$ of P_1 with weights $(x_4, x_5, x_3, x_1, x_2)$ and biases $\mathbf{b}$. Thus to prove Equation (2.4.2) it suffices to show $W_0(\mathbf{b}') + W_1(\mathbf{b}') \leq W_0(\mathbf{b}) + W_1(\mathbf{b})$. There are two cases.

If $P_0 = \{e_1, e_2, e_4, e_5\}$, then $P_0 = P_1$ and

$$\begin{aligned} W_0(\mathbf{b}) - W_0(\mathbf{b}') &= x_1(b_1 - b_1') + x_2(b_1 - b_1') + x_4(b_5 - b_1') + x_5(b_5 - b_1') \\ W_1(\mathbf{b}) - W_1(\mathbf{b}') &= x_4(b_1 - b_1') + x_5(b_1 - b_1') + x_1(b_5 - b_1') + x_2(b_5 - b_1'), \end{aligned}$$

giving us

$$W_0(\mathbf{b}) - W_0(\mathbf{b}') + W_1(\mathbf{b}) - W_1(\mathbf{b}') = 0$$

using the definition of $\mathbf{b}'$.

Otherwise, P_0 includes e_3, exactly one of $\{e_1, e_2\}$ and exactly one of $\{e_4, e_5\}$. Write $P_0 = \{e_L, e_3, e_R\}$. We have

$$\begin{aligned} W_0(\mathbf{b}') = W_1(\mathbf{b}') &= b_1'(x_L + x_R) + b_3 x_3 \\ W_0(\mathbf{b}) &= b_L x_L + b_3 x_3 + b_R x_R \\ W_1(\mathbf{b}) &= b_L x_R + b_3 x_3 + b_R x_L \end{aligned}$$

giving us

$$\begin{aligned} W_0(\mathbf{b}) - W_0(\mathbf{b}') &= x_L(b_L - b_1') + x_R(b_R - b_1') \\ W_1(\mathbf{b}) - W_1(\mathbf{b}') &= x_L(b_R - b_1') + x_R(b_L - b_1'). \end{aligned}$$

Since $b_L + b_R = b_1 + b_5 = 2b_1'$ we conclude

$$W_0(\mathbf{b}) - W_0(\mathbf{b}') + W_1(\mathbf{b}) - W_1(\mathbf{b}') = 0. \quad \blacksquare$$

So, it suffices to consider $\mathbf{b}$ of the form

$$\mathbf{b} = (b, b, 1 - 4b, b, b)$$

with $0 < b < 1/4$.

To evaluate the integral E_T we exploit some of its symmetry. It suffices to integrate over only nonnegative x_1, x_2, x_3, x_4, x_5 satisfying $x_1 \leq x_2$ and $x_4 \leq x_5$ and multiply the final result by 4. For such points, only two of the $P \in \mathcal{P}$ can attain the maximum in the definition of f_T, giving us

$$f_T(\mathbf{x}) = f_T(x_1, x_2, x_3, x_4, x_5) = \max\{bx_1 + bx_2 + bx_4 + bx_5, bx_2 + (1 - 4b)x_3 + bx_5\}.$$

We split the region of integration into two parts, according to whether

$$bx_1 + bx_2 + bx_4 + bx_5 \geq bx_2 + (1 - 4b)x_3 + bx_5,$$

i.e. $b(x_1 + x_4)/(1 - 4b) \geq x_3$. The integrals are

$$\int_{x_2=0}^{\infty} \int_{x_5=0}^{\infty} \int_{x_1=0}^{x_2} \int_{x_4=0}^{x_5} \int_{x_3=0}^{\frac{b(x_1+x_4)}{1-4b}} b(x_1 + x_2 + x_4 + x_5) e^{-x_1-x_2-x_3-x_4-x_5} dx_1 dx_2 \cdots dx_5$$

and

$$\int_{x_2=0}^{\infty}\int_{x_5=0}^{\infty}\int_{x_1=0}^{x_2}\int_{x_4=0}^{x_5}\int_{x_3=\frac{b(x_1+x_4)}{1-4b}}^{\infty}(bx_2+(1-4b)x_3+bx_5)e^{-x_1-x_2-x_3-x_4-x_5}d\mathbf{x}$$

which, when evaluated, added together, and multiplied by 4, give us

$$E_T = \frac{4(1-3b)(5b^2-5b+1)}{(7b-2)^2}.$$

For example, $b = 1/5$ puts an equal bias on every edge and gives $E_T = 8/9$, the same result we saw for the tree on one degree-3 vertex. To give the central edge e_3 twice the bias of the other edges we can put $b = 1/6$; this gives $E_T = 22/25 = 0.88$. Using calculus, one can find the best possible choice of b, described in the following proposition.

Proposition 11 *Let T be a biased tree on two degree-3 vertices and four leaf vertices. With b the unique zero of $105b^3-90b^2+24b-2$ on $0 < b < 1/4$ and equipping T with biases $\mathbf{b} = (b, b, 1-4b, b, b)$ we have*

$$E_T = \frac{4(1-3b)(5b^2-5b+1)}{(7b-2)^2} \approx 0.8797322$$

and, for this tree, no other choice of biases $\mathbf{b}$ yields a lower value of E_T.

2.4.4 Three non-leaf vertices

Let T be a biased tree on three non-leaf vertices and five leaf vertices. This is a complete binary tree on six edges with one additional edge e_1 joining the root to an additional vertex. Denote by e_2 and e_3 the other edges incident to the root. Denote by e_4 and e_5 the edges incident with e_2. Denote by e_6 and e_7 the two edges incident with e_3. We will consider biases of the form

$$\mathbf{b} = \left(b, \frac{1-5b}{2}, \frac{1-5b}{2}, b, b, b, b\right)$$

with $0 < b < 1/5$. This puts a common bias b on edges incident with a leaf and another common bias on the other two edges.

By symmetry we may compute E_T by integrating over only $x_4 \le x_5$ and $x_6 \le x_7$ and multiplying the final result by 4. In this range, f_T is the maximum of four expressions,

1. $b(x_4 + x_5 + x_6 + x_7)$,
2. $bx_1 + (1-5b)x_2/2 + b(x_5 + x_6 + x_7)$,
3. $(1-5b)(x_2 + x_3)/2 + b(x_5 + x_7)$, and
4. $bx_1 + (1-5b)x_3/2 + b(x_4 + x_5 + x_7)$.

To compute the integral, the region of integration is divided into four parts, according to which of the above expressions gives the maximum. We present the details for the first part only.

The first expression exceeds the other three if and only if $bx_4 > bx_1+(1-5b)x_2/2$ and $bx_6 > bx_1 + (1-5b)x_3/2$. To express the integral over this part as an iterated integral, we divide the part into two regions, according to whether $x_4 > x_6$ or not. The region on which $x_4 > x_6$ gives the integral

$$\int_{x_4=0}^{\infty}\int_{x_6=0}^{x_4}\int_{x_1=0}^{x_6}\int_{x_2=0}^{\frac{2b(x_4-x_1)}{1-5b}}\int_{x_3=0}^{\frac{2b(x_6-x_1)}{1-5b}}\int_{x_5=x_4}^{\infty}\int_{x_7=x_6}^{\infty} I dx_1dx_2dx_3dx_4dx_5dx_6dx_7$$

where the integrand is

$$I = b(x_4 + x_5 + x_6 + x_7)e^{-x_1-x_2-x_3-x_4-x_5-x_6-x_7}$$

which evaluates to

$$\frac{1}{100}\frac{(73b-17)b^3}{(4b-1)^3}.$$

The region on which $x_6 > x_4$ gives

$$\frac{1}{100}\frac{(73b-17)b^3}{(4b-1)(16b^2-8b+1)}.$$

The other three parts can be expressed and evaluated similarly, giving a final result of

$$E_T = \frac{726b^4 - 601b^3 + 245b^2 - 55b + 5}{5(1-b)(4b-1)^2}.$$

(See Appendix A for an implementation in Maple.) Setting $b = 1/7$ puts an equal bias on every edge and gives $E_T = 832/945 \approx 0.8804$. To give the central edges twice the bias of the other edges we can put $b = 1/9$; this gives $E_T = 328/375 \approx 0.8747$. Using calculus, one finds that the best possible choice of b is the unique zero of $-3993b^4 + 2765b^3 + 1452b^5 - 804b^2 + 105b - 5$ on $0 < b < 1/5$. This produces $E_T \in (0.8741, 0.8742)$, giving us the following proposition.

Proposition 12 *The number* $c^* = 0.8742$ *is* k*-admissible for* $k = 7$*.*

Using the same method, one can show that $c^* = 0.8697$ is k-admissible for $k = 9$. Computer simulations suggest that the value of c^* decreases only slightly as k is increased further so we do not pursue this here.

2.5 Circumference of a random prekernel with given degree sequence

In the previous sections we have established Lemma 7, an a.a.s. upper bound on the weight of the heaviest cycle in certain randomly-edge-weighted pseudographs.

In this section we use that lemma to establish an upper bound on the circumference of a random prekernel whose degree sequence satisfies certain conditions. In later sections we will see that the degree sequence of the prekernel of $\mathcal{G}_{n,M}$ a.a.s. satisfies these conditions, allowing us to use this result to establish an a.a.s. upper bound on the circumference of the prekernel of $\mathcal{G}_{n,M}$.

One of the challenges in this section arises because Lemma 7 is a statement about non-random pseudographs with random edge weightings, while we are proving a statement about random prekernels. The kernel configuration model of Pittel and Wormald, described below, allows us to rigorously make this transition. It combines a pairing model, for generating the kernel, with a random sequence of weights on the kernel edges.

Another challenge in this section is to show that the conditions on the degree sequence imply that the hypotheses of Lemma 7 are satisfied. One hypothesis requires that there are few edges incident with vertices whose degree exceeds 3. Another hypothesis requires that the number of short cycles in the kernel be small. In Łuczak's proof of his upper bound for the circumference of $\mathcal{G}_{n,M}$ in the supercritical phase, he established the first hypothesis by direct enumeration over degree sequences. We present an alternative derivation. As for the hypothesis about short cycles, Łuczak had no need for this. In [7] and [35] there are results about short cycles arising in this pairing model. However, these results apply only when the maximum degree is bounded, so they cannot be used for our application.

We are interested in studying prekernels with a given degree sequence $\mathbf{d} = (d_i)$. We say that $\mathbf{d}$ is a *prekernel degree sequence* if its number of terms $v = v(\mathbf{d})$ is finite, each term is a positive integer at least 2, and $r = r(\mathbf{d}) = \sum_i (d_i - 2)$ is even. For $j = 2, 3, \ldots$ we define

$$D_j = D_j(\mathbf{d}) = |\{i : d_i = j\}|. \tag{2.5.1}$$

The *kernel configuration model* $\mathcal{H}(\mathbf{d})$ is used to generate prekernels with degree sequence $\mathbf{d}$. It has been used successfully to calculate improved estimates for the size of the core, excess, and tree mantle [26]. We describe the model next.

For each i with $d_i \geq 3$ create a set S_i of d_i points. Let $\mathcal{P}$ be the set of perfect matchings on the union of these sets of points and choose $P \in \mathcal{P}$ uniformly at random. Then, assign the remaining numbers $\{i : d_i = 2\}$ to the edges of the perfect matching and, for each edge, choose a linear order for these numbers. The assignments and the linear ordering, denoted by f, are chosen uniformly at random. The pair (P, f) defines a random configuration in the model $\mathcal{H}(\mathbf{d})$.

Each configuration (P, f) corresponds to a prekernel $G(P, f)$ by collapsing each set S_i to a vertex (producing a kernel $K(P)$) and placing the degree-2 vertices on the edges of the kernel according to the assignment and linear orderings.

Lemma 13 *Let $\mathbf{d} = \mathbf{d}(n)$ be a prekernel degree sequence satisfying $v = v(\mathbf{d}) \to \infty$, $r = r(\mathbf{d}) \to \infty$, $r = o(v)$, $D_3 = D_3(\mathbf{d}) \sim r$, and*

$$\sum_{i:d_i \geq 3} \binom{d_i}{2} < 4r.$$

Fix a positive integer $k \geq 2$ and suppose that the positive constant c^* is k-admissible. For a random configuration (P, f) in $\mathcal{H}(\mathbf{d})$, the longest cycle in $G(P, f)$ has length a.a.s. at most c^*v as $n \to \infty$.

Proof. Define $\mathcal{P}^*$ to be the set of $P \in \mathcal{P}$ for which $K(P)$ has at most $\sqrt{r}$ edges in cycles of length at most k. We will show that a random configuration (P, f) a.a.s. has $P \in \mathcal{P}^*$. Recall that P is a random perfect matching on the points in the union of the S_i. For $j \in \{1, 2, \ldots, k\}$, the number of ways of choosing j pairs of points to form a cycle is at most

$$\frac{1}{2j}\left(\sum_{i:d_i\geq 3} 2\binom{d_i}{2}\right)^j = O\left(r^j\right).$$

The probability that j given pairs of points appear in the pairing P is asymptotic to

$$(\sum_{i:d_i\geq 3} d_i)^{-j}$$

since j is bounded. Now

$$\begin{aligned}\sum_{i:d_i\geq 3} d_i &> \sum_{i:d_i\geq 3}(d_i-2)\\ &= \sum_i (d_i-2)\\ &= r\end{aligned}$$

so the expected number of cycles of length j is $O(r^j r^{-j}) = O(1)$. Since k is fixed, the expected number of edges in such cycles is also $O(1)$. By Markov's inequality, the number of edges in cycles of length j is a.a.s. bounded above by any function $\omega = \omega(n) \to \infty$, in particular $\sqrt{r}/k$. Thus, a.a.s. $P \in \mathcal{P}^*$.

Let (P, f) be a random configuration from $\mathcal{H}(\mathbf{d})$. Define $G'(P, f)$ to be the edge-weighted pseudograph whose underlying pseudograph is $K(P)$ and whose edge-weight on e, for each edge e, is one more than the number of vertices assigned to e by f. Let A be the event that the heaviest cycle in $G'(P, f)$ has weight at most c^*v. Let P_0 be the P^* minimizing $\mathbf{P}[A \mid P = P^*]$ over $P^* \in \mathcal{P}^*$. The minimum exists because $\mathcal{P}^*$ is finite. Next we verify that, conditioned on $P = P_0$, $G'(P, f)$ satisfies the hypotheses of Lemma 7. The number of vertices v' of $G'(P, f)$ is at least $D_3 \sim r \to \infty$. The minimum degree is at least 3 because it is a kernel. The number of edges incident to cycles of length at most k (including loops and parallel edges) is at most $\sqrt{r} = o(r) = o(v')$ since $P_0 \in \mathcal{P}^*$. The number of edges incident to vertices of degree greater than 3 is at most

$$\begin{aligned}\sum_{j:d_j\geq 4} d_j &\leq 2\sum_{j:d_j\geq 4}(d_j-2)\\ &= 2\sum_j (d_j-2) - 2D_3\\ &= 2r - 2D_3\\ &= o(r)\end{aligned}$$

which is $o(v')$. The number of edges m' satisfies

$$\begin{aligned} 2m' &= \sum_{j:d_j\geq 3} d_j \\ &= 3D_3 + o(r) \end{aligned}$$

by the previous calculation, so $m' = O(D_3) = O(r) = o(v)$. To see that m' is little-oh of the sum N of the edge-weights, observe that N is the number of edges of $G(P, f)$, which is $\sum_j d_j/2 \geq v$. Next observe that the edge weights form a sequence of positive integers that is determined by the assignment f in the random configuration. There are exactly $|\{i : d_i = 2\}|!$ choices for f that produce any given sequence, so the sequence is chosen uniformly at random. We have shown that the hypotheses of Lemma 7 hold for $G'(P, f)$ conditioned on $P = P_0$, so we have $\mathbf{P}[A \mid P = P_0] = 1 - o(1)$. Now

$$\begin{aligned} \mathbf{P}[A] &\geq \sum_{P^*\in\mathcal{P}^*} \mathbf{P}[A \mid P = P^*]\mathbf{P}[P = P^*] \\ &\geq \mathbf{P}[A \mid P = P_0] \sum_{P^*\in\mathcal{P}^*} \mathbf{P}[P = P^*] \end{aligned}$$

by the choice of P_0. Since we showed $P \in \mathcal{P}^*$ a.a.s. we get $\mathbf{P}[A] = 1 - o(1)$; that is, the heaviest cycle in $G'(P, f)$ a.a.s. has weight at most c^*v. But if C is a cycle in $G(P, f)$ of some length l, C corresponds naturally to a cycle in $G'(P, f)$ of weight l. So the longest cycle in $G(P, f)$ a.a.s. has length at most c^*v. ∎

Corollary 14 *Let $\mathbf{d} = \mathbf{d}(n)$ be a prekernel degree sequence satisfying $v = v(\mathbf{d}) \to \infty$, $r = r(\mathbf{d}) \to \infty$, $r = o(v)$, $D_3 = D_3(\mathbf{d}) \sim r$, and*

$$\sum_{i:d_i\geq 3} \binom{d_i}{2} < 4r.$$

Fix $k \geq 2$ and suppose that the positive constant c^* is k-admissible. Let G be chosen uniformly at random from all prekernels with degree sequence $\mathbf{d}$. The longest cycle in G has length a.a.s. at most c^*v as $n \to \infty$.

Proof. The probability space $\mathcal{H}(\mathbf{d})$, conditioned on the event that $G(P, f)$ is a simple graph, is a uniform probability space on the prekernels with degree sequence $\mathbf{d}$ ([26], Lemma 3). By Lemma 5 in [26], $G(P, f)$ is a.a.s. a simple graph. (In fact, Lemma 5 in [26] is stated with an additional hypothesis on $\max d_i$, but this hypothesis is not used in the proof.) The result now follows from Lemma 13. ∎

2.6 Truncated multinomial distribution

In order to apply Corollary 14 to the prekernel of $\mathcal{G}_{n,M}$, we must verify the hypotheses about properties of the degree sequence. We give a new derivation of these properties, which will require some facts about the following distribution.

Let v and t be positive integers. The probability space $\mathbf{Multi}(v,t)$ consists of vectors $(d_1, d_2, \ldots, d_v)$ with distribution

$$\mathbf{P}[d_1 = j_1, d_2 = j_2, \ldots, d_v = j_v] = \frac{t!}{v^t j_1! j_2! \cdots j_v!}$$

for any vector $(j_1, j_2, \ldots, j_v)$ of nonnegative integers summing to t. This is the well-known multinomial distribution, modelling the number of balls in each bin when each of t balls is tossed into one of v bins, independently and uniformly at random. The space $\mathbf{Multi}(v,t)|_{\geq 2}$ is obtained from $\mathbf{Multi}(v,t)$ by conditioning on the event that each $d_i \geq 2$.

Lemma 15 *Let $v = v(n)$ and $r = r(n)$ satisfy $v \to \infty$, $r \to \infty$ and $r = o(v)$. If the random vector $\mathbf{d}$ is distributed as $\mathbf{Multi}(v, 2v+r)|_{\geq 2}$ then a.a.s. $D_3(\mathbf{d}) \sim r$ and*

$$\sum_{i:d_i \geq 3} \binom{d_i}{2} < 4r.$$

Proof. Define the positive number λ by

$$\frac{\lambda(e^\lambda - 1)}{e^\lambda - 1 - \lambda} = 2 + \frac{r}{v}.$$

In [11], the authors show that λ exists and they use it to define a vector of independent truncated Poisson random variables which approximate $\mathbf{Multi}(v, 2v + r)|_{\geq 2}$ as follows. Define the random variable Y taking values $j = 2, 3, \ldots$ according to the distribution

$$\mathbf{P}[Y = j] = p_j = \frac{\lambda^j}{j!(e^\lambda - 1 - \lambda)}.$$

Consider the probability space formed by vectors $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_v)$ of v independent copies of Y and let Σ be the event that their sum satisfies $\sum_i Y_i = 2v + r$. For nonnegative integers $j_1, j_2, \ldots, j_v$ summing to $2v + r$ with each $j_i \geq 2$ we have

$$\mathbf{P}[Y_1 = j_1, \ldots, Y_v = j_v] = \frac{\lambda^{2v+r}}{(e^\lambda - 1 - \lambda)^v} \prod_{i=1}^{v} \frac{1}{j_i!}$$

so this probability space, conditioned on Σ, is identical to $\mathbf{Multi}(v, 2v + r)|_{\geq 2}$. Equation (5.7) in [26] says that

$$\mathbf{P}\left[\max_i Y_i < \log v \text{ or } \sum_{i:Y_i \geq 3} \binom{Y_i}{2} \geq 4r \mid \Sigma\right] = O(r^{-1} + rv^{-1}).$$

The second claim in the lemma follows. Theorem 4(a) in [25] states that for $r \to \infty$,

$$\mathbf{P}[\Sigma] = \frac{1 + O(r^{-1})}{\sqrt{2\pi v c(1 + \bar{\eta} - c)}}$$

where $c(1+\bar{\eta}-c) \sim c-k = r/v$ by Equation (20) in [25]. It follows that

$$\mathbf{P}[\Sigma]^{-1} = O(\sqrt{r}). \tag{2.6.1}$$

To establish the first claim in the lemma, we observe that $D_3(\mathbf{Y})$ is distributed as a binomial random variable with v trials and p_3 probability of success. By Chernoff's bound,

$$\mathbf{P}\left[|D_3(\mathbf{Y}) - vp_3| > a\right] < 2\exp(-a^2/(3vp_3))$$

for $0 < a \le vp_3$. Recalling (2.6.1), in $\mathbf{Multi}(v, 2v+r)|_{\ge 2}$ we have

$$\mathbf{P}\left[|D_3(\mathbf{d}) - vp_3| > a\right] = O(\sqrt{r})\exp(-a^2/(3vp_3)).$$

Setting $a = \sqrt{vp_3}\log r$ (which satisfies $a \le vp_3$, as we will see shortly) we get $D_3(\mathbf{d}) = vp_3 + O(\sqrt{vp_3}\log r)$ with probability $1 - O(\exp(-(\log r)^2/3))$. Now $\lambda \sim 3rv^{-1}$ by Theorem 1(a) in [25], so

$$\begin{aligned} vp_3 &= v\frac{\lambda^3}{3!(e^\lambda - 1 - \lambda)} \\ &= v\frac{\lambda^3}{3!(\lambda^2/2 + O(\lambda^3))} \\ &= \frac{1}{3}v\lambda(1+O(\lambda)) \\ &= r(1+O(rv^{-1})) \end{aligned}$$

giving us $D_3(\mathbf{d}) \sim r$ a.a.s. as required. ∎

2.7 Properties of vertex degrees in $\mathcal{G}_{n,M}$

Now we proceed to establish the properties of the degree sequence of the prekernel of $\mathcal{G}_{n,M}$ that are required to apply Corollary 14. Recall that we are assuming $M = M(n) = n/2 + s$ for some $s = s(n)$ satisfying $s = o(n)$ and $n^{2/3} = o(s)$. For this range of M, it is well-known that $\mathcal{G}_{n,M}$ a.a.s. has a unique component with maximum number of vertices [6], which we call the largest component.

We begin by showing a.a.s. there are few vertices in the core that lie outside the largest component. The next result is part of the proof of Theorem 4 of [23]. Here we present a slightly more thorough proof.

Lemma 16 *Let $M = M(n) = n/2 + s$ for some $s = s(n)$ satisfying $s = o(n)$ and $n^{2/3} = o(s)$. The number of vertices in cycles of $\mathcal{G}_{n,M}$ not in the largest component is a.a.s. at most $\omega n/s$ for any $\omega = \omega(n) \to \infty$.*

Proof. Let $\bar{G}$ be the graph formed from $\mathcal{G}_{n,M}$ by removing its (lexicographically first) largest component. Let $n(\bar{G})$ and $M(\bar{G})$ represent its number of vertices and edges, respectively. Let $\epsilon > 0$ and define S to be the set of ordered pairs $(\bar{n}, \bar{M})$ satisfying

1. $(1-\epsilon)4s \le n - \bar{n} \le (1+\epsilon)4s$,
2. $(1-\epsilon)4s \le M - \bar{M} \le (1+\epsilon)4s$, and
3. $\mathbf{P}[n(\bar{G}) = \bar{n}, M(\bar{G}) = \bar{M}] > 0$.

It is known that the largest component of $\mathcal{G}_{n,M}$ has a.a.s. $4s(1+o(1))$ vertices and $4s(1+o(1))$ edges [6, 22]. So a.a.s. $(n(\bar{G}), M(\bar{G})) \in S$. For $(\bar{n}, \bar{M}) \in S$ we have $\bar{M} \le M - 4s(1-\epsilon) = n/2 + s - 4s(1-\epsilon)$ and $n \le \bar{n} + 4s(1+\epsilon)$, giving us

$$\bar{M} \le \bar{n}/2 - s(1-6\epsilon). \tag{2.7.1}$$

To estimate the number X of vertices in cycles in $\bar{G}$ we let $(\bar{n}, \bar{M}) \in S$ and condition on the non-empty event $n(\bar{G}) = \bar{n}, M(\bar{G}) = \bar{M}$. In the conditioned space, $\bar{G}$ is equally likely to be any graph on $\bar{n}$ vertices and $\bar{M}$ edges. For $3 \le k \le \bar{n}$ the number of such graphs having a cycle of length k is at most

$$\binom{\bar{n}}{k}\frac{k!}{2k}\binom{\binom{\bar{n}}{2}}{\bar{M}-k}$$

so the expected value of X in this conditioned space is

$$\begin{aligned}
\mathbf{E}[X \mid n(\bar{G}) = \bar{n}, M(\bar{G}) = \bar{M}] &\le \sum_{k=3}^{\bar{n}} k\binom{\bar{n}}{k}\frac{k!}{2k}\binom{\binom{\bar{n}}{2}}{\bar{M}-k}\binom{\binom{\bar{n}}{2}}{\bar{M}}^{-1} \\
&= \frac{1}{2}\sum_{k=3}^{\bar{n}} \frac{\bar{n}!}{(\bar{n}-k)!}\frac{\left(\binom{\bar{n}}{2} - M - k\right)!}{\left(\binom{\bar{n}}{2} - M\right)!}\frac{\bar{M}!}{(\bar{M}-k)!} \\
&< \frac{1}{2}\sum_{k=3}^{\bar{n}} \frac{\bar{n}^k \bar{M}^k}{(\binom{\bar{n}}{2} - \bar{n})^k}.
\end{aligned}$$

Using (2.7.1) this becomes

$$\begin{aligned}
\mathbf{E}[X \mid n(\bar{G}) = \bar{n}, M(\bar{G}) = \bar{M}] &< \frac{1}{2}\sum_{k=3}^{\bar{n}} \left(\frac{\frac{\bar{n}}{2} - s(1-6\epsilon)}{\frac{\bar{n}-1}{2} - 1}\right)^k \\
&< \frac{1}{2}\sum_{k=3}^{\bar{n}} \left(1 - \frac{s(1-6\epsilon)}{\bar{n}-3}\right)^k \\
&< \frac{1}{2}\sum_{k=3}^{\infty} \left(1 - \frac{s(1-6\epsilon)}{\bar{n}-3}\right)^k \\
&= \frac{1}{2} \times \frac{\bar{n}-3}{s(1-6\epsilon)}
\end{aligned}$$

which is at most n/s for n sufficiently large. By Markov's inequality,

$$\begin{aligned}
\mathbf{P}[X \ge \omega n/s \mid n(\bar{G}) = \bar{n}, M(\bar{G}) = \bar{M}] &\le \frac{\mathbf{E}[X \mid n(\bar{G}) = \bar{n}, M(\bar{G}) = \bar{M}]}{\omega n/s} \\
&< \frac{1}{\omega}.
\end{aligned}$$

So

$$
\begin{aligned}
\mathbf{P}[X < \omega n/s] &\geq \mathbf{P}[X < \omega n/s, (n(\bar{G}), M(\bar{G})) \in S] \\
&= \sum_{(\bar{n},\bar{M})\in S} (\mathbf{P}[X < \omega n/s \mid n(\bar{G}) = \bar{n}, M(\bar{G}) = \bar{M}] \\
&\quad \times \mathbf{P}[n(\bar{G}) = \bar{n}, M(\bar{G}) = \bar{M}]) \\
&\geq \left(1 - \frac{1}{\omega}\right) \mathbf{P}[(n(\bar{G}), M(\bar{G})) \in S] \\
&= \left(1 - \frac{1}{\omega}\right)(1 - o(1))
\end{aligned}
$$

since a.a.s. $(n(\bar{G}), M(\bar{G})) \in S$. Therefore a.a.s. $X < \omega n/s$. ∎

Instead of proving results about the degree sequence of the prekernel of $\mathcal{G}_{n,M}$ directly, we will actually prove results about the degree sequence of the core. The next result will allow us to transfer results about the core to the prekernel.

Lemma 17 *Let $M = M(n) = n/2 + s$ for some $s = s(n)$ satisfying $s = o(n)$ and $n^{2/3} = o(s)$. The core of the largest component of $\mathcal{G}_{n,M}$ is a.a.s. formed from the core of $\mathcal{G}_{n,M}$ by removing $o(s^2/n)$ vertices of degree 2. Also, the prekernel of $\mathcal{G}_{n,M}$ is a.a.s. formed from the core of $\mathcal{G}_{n,M}$ by removing $o(s^2/n)$ vertices of degree 2.*

Proof. It is well-known that the largest component of $\mathcal{G}_{n,M}$ is a.a.s. the only component that has more than one cycle. (See Theorem 5.12 in [20].) So, the core of $\mathcal{G}_{n,M}$ is a.a.s. composed of the core of the largest component together with some cycle components. By Lemma 16 the number of vertices in the cycle components is a.a.s. at most $\omega n/s = (s/n^{2/3})(n/s) = n^{1/3} = (n^{2/3})^2/n = o(s^2/n)$, since we may take $\omega = s/n^{2/3}$. Because the largest component of $\mathcal{G}_{n,M}$ a.a.s. contains more than one cycle, it follows that these cycle components are a.a.s. all of the cycle components in the core of $\mathcal{G}_{n,M}$, making the prekernel a.a.s. equal to the core of the largest component. ∎

Now we establish the required properties of the degree sequence of the prekernel of $\mathcal{G}_{n,M}$. Recall the definitions of $D_j(\mathbf{d})$, $v(\mathbf{d})$, $r(\mathbf{d})$ from (2.5.1). The results about $r(\mathbf{d})$, $v(\mathbf{d})$, and $D_3(\mathbf{d})$ in the following lemma were used by Łuczak [23]. He used the estimate of $r(\mathbf{d})$ from [22] to establish estimates for $D_j(\mathbf{d})$ and $v(\mathbf{d})$ by direct enumeration over degree sequences. Instead of studying the prekernel directly, he studied the core of the largest component. Our proof method is different, using the known estimates of $v(\mathbf{d})$ and $r(\mathbf{d})$ to establish results about the degree sequence of the core.

Lemma 18 *Let $M = M(n) = n/2 + s$ for some $s = s(n)$ satisfying $s = o(n)$ and $n^{2/3} = o(s)$. Let $\mathbf{d}$ be the degree sequence of the prekernel of $\mathcal{G}_{n,M}$. Then, a.a.s. $v(\mathbf{d}) \sim 8s^2/n$, $r(\mathbf{d}) \sim D_3(\mathbf{d}) \sim 32s^3/(3n^2)$, and*

$$\sum_{i:d_i\geq 3} 2\binom{d_i}{2} < 4r(\mathbf{d}).$$

Proof. By Lemma 17 the prekernel differs from the core a.a.s. by $o(s^2/n)$ vertices of degree 2. Thus, it suffices to prove the lemma for the degree sequence $\mathbf{d}$ of the core. Appealing to Lemma 17 again, the core differs from the core of the largest component a.a.s. by $o(s^2/n)$ vertices of degree 2. It is known [26] that the degree sequence $\mathbf{d}'$ of the core of the largest component a.a.s. has $v(\mathbf{d}') \sim 8s^2/n$ and $r(\mathbf{d}') \sim 32s^3/(3n^2)$, so we must have a.a.s. $v(\mathbf{d}) \sim 8s^2/n$ and $r(\mathbf{d}) \sim 32s^3/(3n^2)$ also. Letting $\epsilon > 0$, this means $\mathbf{d} \in S$ a.a.s. where S is the set of ordered pairs $(\bar{v}, \bar{r})$ satisfying

1. $(1-\epsilon)8s^2/n \le \bar{v} \le (1+\epsilon)8s^2/n$,
2. $(1-\epsilon)32s^3/(3n^2) \le \bar{r} \le (1+\epsilon)32s^3/(3n^2)$, and
3. $\mathbf{P}[v(\mathbf{d}) = \bar{v}, r(\mathbf{d}) = \bar{r}] > 0$.

We note that for $(\bar{v}, \bar{r}) \in S$ we have $\bar{r} = o(\bar{v})$ since $(s^3/n^2)/(s^2/n) = s/n = o(1)$ and both $\bar{v} \to \infty$ and $\bar{r} \to \infty$ since $n^{2/3} = o(s)$.

To establish the remaining properties of the degree sequence of the core of $\mathcal{G}_{n,M}$ we use Theorem 2 of [11], which proves the existence of a probability space of ordered pairs (G, I) in which

1. G, conditioned on the event $I = 1$, is distributed as the core of $\mathcal{G}_{n,M}$,
2. $\mathbf{P}[I = 1] = \Omega(1)$, and
3. the degree sequence $\mathbf{d}(G)$ of G, conditioned on $v(\mathbf{d}(G)) = \bar{v}$ and $r(\mathbf{d}(G)) = \bar{r}$, is distributed as $\mathbf{Multi}(\bar{v}, 2\bar{v} + \bar{r})|_{\ge 2}$.

(The statement of Theorem 2 of [11] actually includes the hypothesis $M \ge n$ which is not satisfied here; however, that hypothesis is not needed for their proof.)

Write $v = v(\mathbf{d}(G))$, $r = r(\mathbf{d}(G))$ and let A be the event that

1. $(1-\epsilon)32s^3/(3n^2) \le D_3(\mathbf{d}(G)) \le (1+\epsilon)32s^3/(3n^2)$, and
2. $\sum_{i: d_i(G) \ge 3} 2\binom{d_i(G)}{2} < 4r$.

To prove the lemma, we must show $\mathbf{P}[A \mid I = 1] = 1 + o(1)$ or equivalently $\mathbf{P}[A^C \mid I = 1] = o(1)$, where X^C denotes the complement of event X. We begin by writing

$$\mathbf{P}[A^C \mid I = 1] = \mathbf{P}[A^C, (v, r) \in S \mid I = 1] + \mathbf{P}[A^C, (v, r) \notin S \mid I = 1].$$

The second term is at most $\mathbf{P}[(v, r) \notin S \mid I = 1]$ which is $o(1)$ because $\mathbf{d}(G)$, conditioned on $I = 1$, is distributed like the degree sequence of the core of $\mathcal{G}_{n,M}$. We write the first term as

$$\mathbf{P}[A^C, (v, r) \in S \mid I = 1] \quad = \quad \sum_{(\bar{v}, \bar{r}) \in S} \mathbf{P}[A^C, v = \bar{v}, r = \bar{r} \mid I = 1]$$

$$
\begin{aligned}
&= \mathbf{P}[I=1]^{-1} \sum_{(\bar{v},\bar{r})\in S} \mathbf{P}[A^C, v=\bar{v}, r=\bar{r}, I=1] \\
&\leq \mathbf{P}[I=1]^{-1} \sum_{(\bar{v},\bar{r})\in S} \mathbf{P}[A^C, v=\bar{v}, r=\bar{r}] \\
&= \mathbf{P}[I=1]^{-1} \sum_{(\bar{v},\bar{r})\in S} \frac{\mathbf{P}[A^C, v=\bar{v}, r=\bar{r}]\mathbf{P}[v=\bar{v}, r=\bar{r}]}{\mathbf{P}[v=\bar{v}, r=\bar{r}]} \\
&= \mathbf{P}[I=1]^{-1} \sum_{(\bar{v},\bar{r})\in S} \mathbf{P}[A^C \mid v=\bar{v}, r=\bar{r}]\mathbf{P}[v=\bar{v}, r=\bar{r}] \\
&\leq \mathbf{P}[I=1]^{-1}\mathbf{P}[A^C \mid v=\hat{v}, r=\hat{r}]\mathbf{P}[(v,r)\in S]
\end{aligned}
$$

where $(\hat{v},\hat{r})$ is the ordered pair maximizing $\mathbf{P}[A^C \mid v=\bar{v}, r=\bar{r}]$ over all $(\bar{v},\bar{r}) \in S$. (The maximum exists because S is finite.) We now use the properties of the distribution of (G, I) to estimate each of $\mathbf{P}[I=1]^{-1}$, $\mathbf{P}[A^C \mid v=\hat{v}, r=\hat{r}]$, and $\mathbf{P}[(v,r)\in S]$. We have already noted that $\mathbf{P}[I=1]=\Omega(1)$, so we have $\mathbf{P}[I=1]^{-1}=O(1)$. Since $(\hat{v},\hat{r})\in S$ we have $\hat{v}\to\infty$, $\hat{r}\to\infty$, and $\hat{r}=o(\hat{v})$. We know that, conditioned on $v=\hat{v}$ and $r=\hat{r}$, the degree sequence of G is distributed as $\mathbf{Multi}(\hat{v}, 2\hat{v}+\hat{r})|_{\geq 2}$. Lemma 15 tells us that event A occurs a.a.s. in this model so we have $\mathbf{P}[A^C \mid v=\hat{v}, r=\hat{r}]=o(1)$. Finally, we may crudely estimate $\mathbf{P}[(v,r)\in S]=O(1)$. Combining these estimates we get $\mathbf{P}[A^C \mid I=1]=o(1)$ as required. ∎

2.8 Circumference of $\mathcal{G}_{n,M}$

Lemma 19 *Let $M=M(n)=n/2+s$ for some $s=s(n)$ satisfying $s=o(n)$ and $n^{2/3}=o(s)$. Fix k and suppose that the positive constant c^* is k-admissible. The circumference of $\mathcal{G}_{n,M}$ is a.a.s. at most $(8c^*+o(1))s^2/n$.*

Proof. Every cycle in a graph lies in the graph's core. By Lemma 17 the prekernel G of $\mathcal{G}_{n,M}$ is formed from the core of $\mathcal{G}_{n,M}$ by removing $o(s^2/n)$ vertices of degree 2. So, to prove the lemma, it suffices to show that the circumference of G is a.a.s. at most $(8c^*+o(1))s^2/n$.

By Lemma 18 there exists $\omega=\omega(n)\to\infty$ such that the degree sequence $\mathbf{d}(G)$ of G a.a.s. lies in the set D of prekernel degree sequences $\mathbf{d}$ satisfying

1. $$\sum_{i:d_i\geq 3} 2\binom{d_i}{2} < 4r(\mathbf{d}),$$

2. $(1-\omega^{-1})8s^2/n \leq v(\mathbf{d}) \leq (1+\omega^{-1})8s^2/n$,

3. $(1-\omega^{-1})32s^3/(3n^2) \leq r(\mathbf{d}) \leq (1+\omega^{-1})32s^3/(3n^2)$,

4. $(1-\omega^{-1})32s^3/(3n^2) \leq D_3(\mathbf{d}) \leq (1+\omega^{-1})32s^3/(3n^2)$, and

5. $\mathbf{P}[\mathbf{d}(G) = \mathbf{d}] > 0$.

Define A to be the event that the circumference of G is at most $c^* v(\mathbf{d}(G))$. We have

$$\mathbf{P}[A] \geq \sum_{\mathbf{d} \in D} \mathbf{P}[A \mid \mathbf{d}(G) = \mathbf{d}] \mathbf{P}[\mathbf{d}(G) = \mathbf{d}].$$

Suppose that $\mathbf{P}[A \mid \mathbf{d}(G) = \mathbf{d}]$ is minimized over $\mathbf{d} \in D$ by $\mathbf{d} = \mathbf{d}^*$. (The minimum exists since D is finite.) Then

$$\begin{aligned} \mathbf{P}[A] &\geq \mathbf{P}[A \mid \mathbf{d}(G) = \mathbf{d}^*] \sum_{\mathbf{d} \in D} \mathbf{P}[\mathbf{d}(G) = \mathbf{d}] \\ &= \mathbf{P}[A \mid \mathbf{d}(G) = \mathbf{d}^*](1 + o(1)) \end{aligned}$$

since a.a.s. $\mathbf{d} \in D$.

In general, the number of graphs on n vertices and M edges that have a given graph as their prekernel depends only on the number of vertices and edges of the given prekernel. So, conditioning on the event $\mathbf{d}(G) = \mathbf{d}^*$, G is equally likely to be each prekernel with degree sequence $\mathbf{d}^*$. The probability $P[A \mid \mathbf{d}(G) = \mathbf{d}^*]$ is thus the probability that a graph, chosen uniformly at random from all prekernels of degree sequence $\mathbf{d}^*$, has circumference a.a.s. at most $c^* v$. Since $\mathbf{d}^* \in D$, we may apply Corollary 14 to conclude that this probability is $1+o(1)$. Thus $\mathbf{P}[A] = 1+o(1)$. In other words, the circumference of G is a.a.s. at most $c^* v(\mathbf{d}(G))$. But we have seen $v(\mathbf{d}(G)) \sim 8s^2/n$ a.a.s. so the circumference is a.a.s. at most $(8c^* + o(1))s^2/n$, as required. ∎

Proof of Theorem 1. Proposition 12 tells us that $c^* = 0.8742$ is k-admissible for $k = 7$. By Lemma 19 the circumference of $\mathcal{G}_{n,M}$ is a.a.s. at most $(8c^* + o(1))s^2/n < (7 + o(1))s^2/n$. ∎

Chapter 3

A contiguity result for random d-regular graphs

3.1 Introduction

In this chapter we prove the new contiguity result (1.4.2) for random d-regular graphs $\mathcal{G}_{n,d}$. Recall from Section 1.4 that it suffices to show (1.4.1); i.e. $\mathcal{G}_{n,d} \approx \mathcal{G}_{n,d}^{(Y)}$ where Y is the random variable counting the number of j-edge matchings in $\mathcal{G}_{n,d}$.

Before beginning the proof, we introduce some of the tools that will be used. The pairing model and small subgraph conditioning method are standard tools for studying random d-regular graphs and they will be used throughout the following chapters.

3.2 Pairing model

The *pairing model* or *configuration model* $\mathcal{P}_{n,d}$ was first introduced by Bollobás [7]. A *pairing* in $\mathcal{P}_{n,d}$ is a perfect matching on a set of dn points which are grouped into n cells of d points each. A random pairing naturally corresponds in an obvious way to a random d-regular pseudograph (possibly containing loops or parallel edges), in which each cell becomes a vertex. The facts that we need about $\mathcal{P}_{n,d}$ are the following. (See [20] for proofs or more details.) The restriction of $\mathcal{P}_{n,d}$ to the set of graphs with no loops or parallel edges gives precisely the uniform distribution on $\mathcal{G}_{n,d}$. The event in $\mathcal{P}_{n,d}$ that no loop or parallel edge is formed has probability bounded away from 0 as $n \to \infty$; thus any event that holds a.a.s. in $\mathcal{P}_{n,d}$ also holds a.a.s. in $\mathcal{G}_{n,d}$. The model $\mathcal{P}_{n,d}$ is advantageous because it is usually easier to count pairings having a certain property in $\mathcal{P}_{n,d}$ rather than graphs in $\mathcal{G}_{n,d}$. For example, there is no simple expression for $|\mathcal{G}_{n,d}|$, but it is easy to see that

$$|\mathcal{P}_{n,d}| = (nd-1)(nd-3)\cdots 1$$

which we denote by $(nd-1)!!$.

3.3 Small subgraph conditioning

The small subgraph conditioning method was introduced by Robinson and Wormald [28, 29] in their proof that random 3-regular graphs are a.a.s. Hamiltonian. See [20, Chapter 9] and [36] for a full exposition.

The setting for the method is as follows. A random variable, $Y = Y_n$, counts occurrences of some structure, and depends on a parameter n which tends to ∞. The expectation $\mathbf{E}Y$ tends to infinity, and we want to show that $\mathbf{P}(Y > 0) \to 1$. The small subgraph conditioning method actually establishes a contiguity result. The method applies when the variance of Y is of the same order as $(\mathbf{E}Y)^2$. The main computation required is the asymptotic value of some joint moments of the numbers of certain small subgraphs and the random variable Y. The result which the method depends on can be stated as follows (a consequence of [36, Corollary 4.2]). (We use $[x]_m := x(x-1)\cdots(x-m+1)$ to denote falling factorials and $\wedge_{k\in K} E_k$ to denote the intersection of the events $\{E_k : k \in K\}$.)

Theorem 20 *Let $\lambda_k > 0$ and $\delta_k \geq -1$ be real numbers for $k = 1, 2, \ldots$ and suppose that for each n there are random variables $X_k = X_k(n)$, $k = 1, 2, \ldots$ and $Y = Y(n)$, all defined on the same probability space $\mathcal{G} = \mathcal{G}_n$ such that X_k is nonnegative integer valued, Y is nonnegative and $\mathbf{E}Y > 0$ (for n sufficiently large). Suppose furthermore that*

(i) For each $j \geq 1$, the variables $X_1, \ldots, X_j$ are asymptotically independent Poisson random variables with $\mathbf{E}X_k \to \lambda_k$,

(ii) if $\mu_k = \lambda_k(1 + \delta_k)$, then

$$\frac{\mathbf{E}(Y[X_1]_{m_1}\cdots[X_j]_{m_j})}{\mathbf{E}Y} \to \prod_{k=1}^{j} \mu_k^{m_k} \tag{3.3.1}$$

for every finite sequence $m_1, \ldots, m_j$ of nonnegative integers,

(iii) $\sum_k \lambda_k \, {\delta_k}^2 < \infty$,

(iv) $\mathbf{E}(Y^2)/(\mathbf{E}Y)^2 \leq \exp(\sum_k \lambda_k \, {\delta_k}^2) + o(1)$ as $n \to \infty$.

Then $\bar{\mathcal{G}} \approx \bar{\mathcal{G}}^{(Y)}$, where $\bar{\mathcal{G}}$ denotes the space $\mathcal{G}$ conditioned on the event $\mathcal{E} = \wedge_{k:\delta_k=-1}\{X_k = 0\}$. In particular, $\mathbf{P}(Y > 0 \mid \mathcal{E}) \to 1$.

3.4 Outline of proof of (1.4.1)

The proof applies the small subgraph conditioning method (Theorem 20) to the number of j-edge matchings in a random d-regular graph.

We make our calculations in the pairing model $\mathcal{P}_{n,d}$. A j-edge matching in a pairing $P \in \mathcal{P}_{n,d}$ is a set of j pairs in P that project to a j-edge matching in the pseudograph associated with P; i.e. with the property that no two points in these pairs lie in the same cell.

To apply small subgraph conditioning we will need the following asymptotic estimates.

Lemma 21 *Let Y be the random variable counting the number of j-edge matchings in $\mathcal{P}_{n,d}$. For $i = 1, 2, \ldots$ let X_i be the number of cycles of length i in $\mathcal{P}_{n,d}$. Then, letting $\gamma = j/n$,*

(a)

$$\mathbf{E}Y \sim \sqrt{\frac{1}{2\pi n\gamma(1-2\gamma)}} \left(\frac{(d-2\gamma)^{d/2-\gamma}}{d^{d/2-2\gamma}2^{\gamma}\gamma^{\gamma}(1-2\gamma)^{1-2\gamma}} \right)^{n},$$

(b) for every finite sequence of nonnegative integers $m_1, m_2, \ldots, m_k$,

$$\frac{\mathbf{E}(Y[X_1]_{m_1}\cdots[X_k]_{m_k})}{\mathbf{E}Y} \sim \prod_{i=1}^{k} \left(\lambda_i\left(1+\delta_i\right)\right)^{m_i}$$

where $\lambda_i = (d-1)^i/(2i)$ and $\delta_i = (-2\gamma)^i(d-2\gamma)^{-i}$ for $i = 1, 2, \ldots,$ and

(c)

$$\frac{\mathbf{E}(Y^2)}{(\mathbf{E}Y)^2} \sim \frac{d-2\gamma}{\sqrt{8\gamma^2 - 4\gamma^2 d - 4\gamma d + d^2}}.$$

In the remainder of this chapter we perform the calculations to verify Lemma 21 and then use small subgraph conditioning to prove (1.4.1).

3.5 Proof of Lemma 21(a) and (b)

To determine $\mathbf{E}Y$ we enumerate all possible j-edge matchings in $\mathcal{P}_{n,d}$ and the pairings that contain them. A j-edge matching is determined by choosing $2j$ of the n cells to be the cells of the matching, then choosing one of the d points in each of these cells, and finally choosing a perfect matching on these $2j$ points. This gives

$$\binom{n}{2j} d^{2j}(2j-1)!!$$

possible j-edge matchings. Once a j-edge matching has been chosen, the pairing is completed by putting a perfect matching on the remaining $dn - 2j$ points, thus

$$\mathbf{E}Y = \frac{1}{|\mathcal{P}_{n,d}|}\binom{n}{2j} d^{2j}(2j-1)!!(dn-2j-1)!!. \tag{3.5.1}$$

Substituting $|\mathcal{P}_{n,d}| = (dn-1)!!$ and using Stirling's formula establishes Lemma 21(a).

Next we prove a special case of Lemma 21(b). Let $i \in \{1, 2, \ldots\}$. We will show

$$\frac{\mathbf{E}(YX_i)}{\mathbf{E}Y} \sim \lambda_i(1+\delta_i). \tag{3.5.2}$$

To determine $\mathbf{E}(YX_i)$ we will enumerate, for each j-edge matching M, and each rooted oriented cycle C compatible with M, all of the pairings that contain M and C. Using rooted oriented cycles introduces a factor of $2i$ into the counting.

In general, at this stage in the method of small subgraph conditioning, one must enumerate pairings that contain a given cycle and some other structure. Usually it is advantageous to group the terms according to the interaction between the cycle and the other structure. This interaction is often represented by a sequence. The enumeration proceeds using generating functions or other methods. We will use a method of Janson which simplifies these computations. Janson's method uses a sequence similar to a sequence of states in a Markov chain. The enumeration proceeds by finding the eigenvalues of a related matrix. (See the note following the proof of Theorem 4.5 in [36].) It is not trivial to apply Janson's method, however, because new sequences must be defined depending on the types of structures involved.

Let M be a j-edge matching. As we saw earlier, the number of ways of choosing M is

$$\binom{n}{2j} d^{2j}(2j-1)!!.$$

Next we count all rooted oriented cycles C compatible with M. Observe that each directed edge of C is one of three types:

1. in the matching,

2. not in the matching, nor is its head incident with a matching edge, or

3. not in the matching, but its head is incident with a matching edge.

Label each edge by the number 1, 2 or 3 according to its type. Define the *intersection type* of C to be the sequence of labels on the edges of C. Suppose we are given an intersection type having a 1's, b 2's, and c 3's. To enumerate all C with this intersection type, first we choose the cells for the heads of edges of type 3. The number of ways of doing this is asymptotically $(2j)^c$. This determines all cells of the cycle except those which are the heads of edges of type 2. These remaining cells are chosen in one of asymptotically $(n-2j)^b$ ways. Next we choose the points within the cells that are used by C. The points for edges of type 1 are determined by M. There are d choices for the head of each edge of type 2 (because any point in the cell may be chosen) and $(d-1)$ choices for the head of each edge of type 3 (because one point is already used by the incident matching edge), giving a factor of $d^b(d-1)^c$. It remains to choose the points for the tails of edges of type 2 or 3. There are $(d-1)$ choices for each of the a edges that follow an edge of type 1. There are $(d-2)$ choices for each of the $c-a$ edges that follow a cell that is incident with a matching edge not in the cycle. Finally, there are $(d-1)$ choices for each of the remaining $b+c-(a+c-a)=b$ edges. To create the remainder of the pairing, we must put a perfect matching on all of the dn points except for the $2j$ that were already chosen in M and the additional $2b+2c$ points that are heads and tails of edges of type 2 or 3 in C. The number of ways of doing this is

$$(dn-2j-2b-2c-1)!! \sim \frac{(dn-2j-1)!!}{(dn-2j)^{b+c}}$$

Combining these factors, we have

$$\begin{aligned}&|\mathcal{P}_{n,d}|\mathbf{E}(YX_i)\\ \sim\ &\frac{(d-1)^i}{2i}\sum_{\mathcal{T}}\binom{n}{2j}d^{2j+b}(2j-1)!!(2j)^c(n-2j)^b(d-2)^{c-a}\frac{(dn-2j-1)!!}{(dn-2j)^{b+c}}\end{aligned}$$

where the sum is taken over the set of intersection types $\mathcal{T}$ and we have simplified $a+b+c=i$. Comparing this expression with (3.5.1) we have

$$\begin{aligned}\mathbf{E}(YX_i) &\sim \lambda_i\mathbf{E}Y\sum_{\mathcal{T}}(2j)^c(n-2j)^b d^b(d-2)^{c-a}(dn-2j)^{-b-c}\\ &\sim \lambda_i\mathbf{E}Y\sum_{\mathcal{T}}\left(\frac{1}{d-2}\right)^a\left(\frac{d(1-2\gamma)}{d-2\gamma}\right)^b\left(\frac{2\gamma(d-2)}{d-2\gamma}\right)^c.\end{aligned}$$

To evaluate the sum S in the above expression, we think of each $T\in\mathcal{T}$ as a sequence of i states. Each term in the sum is a product of the i factors corresponding to the states in T. We can think of the factors as being introduced individually during the "transitions" between the states. Transitions are allowed between all pairs of states subject to the restriction that edges of type 1 or 2 must be followed by edges of type 2 or 3. If we let M be the following "transition" matrix

$$\begin{bmatrix} 0 & \frac{1}{d-2} & \frac{1}{d-2}\\ 0 & \frac{d(1-2\gamma)}{d-2\gamma} & \frac{d(1-2\gamma)}{d-2\gamma}\\ \frac{2\gamma(d-2)}{d-2\gamma} & \frac{2\gamma(d-2)}{d-2\gamma} & \frac{2\gamma(d-2)}{d-2\gamma}\end{bmatrix}$$

then $S=\mathbf{tr}(M^i)$. One can show that M has eigenvalues 0, 1, and $(-2\gamma)(d-2\gamma)^{-1}$, thus $S=1+\delta_i$. This establishes (3.5.2).

Lemma 21(b) can be proved in a similar manner as follows. The expected value $\mathbf{E}(Y[X_1]_{m_1}\cdots[X_k]_{m_k})$ is expressed as a sum enumerating all pairings that contain a given j-edge matching and ordered set of cycles. We group the terms according to the number of vertices ν and edges μ of the isomorphism type of the ordered set of cycles. If the isomorphism type has all cycles cell-disjoint, we have $\mu=\nu=\nu_0$, defined by

$$\nu_0=\sum_j m_j.$$

In this case, the enumeration is a direct generalization of the above argument for $\mathbf{E}(YX_i)$, producing the asymptotic value for Lemma 21(b). It remains to show that the terms in the remaining cases are negligible. For the remaining cases we have $\nu<\mu$. The enumeration proceeds as in the first case, but with the following significant differences. Compared to the first case, we lose a factor of $O(n^{\nu_0-\nu})$ when choosing the cells for the cycle and we gain a factor of $O(n^{\nu_0-\mu})$ when choosing the pairs of the pairing. Thus, the net change is $O(n^{\nu-\mu})=O(n^{-1})$. As there are $O(1)$ isomorphism types, this shows that the terms in the remaining cases are negligible compared to those of the first case.

3.6 Proof of Lemma 21(c)

To determine $\mathbf{E}(Y^2)$ we enumerate, for each ordered pair of (not necessarily distinct) j-edge matchings (M_1, M_2), all pairings $P \in \mathcal{P}_{n,d}$ containing M_1 and M_2. First we compute the number $N(k,l)$ of (M_1, M_2) sharing exactly k pairs and $2k + l$ cells $(0 \le k \le j, 0 \le l \le 2j)$. The pair (M_1, M_2) can be chosen as follows. First, choose the points in the k common pairs. This is done by selecting $2k$ of the n cells, then choosing one of d points in each of these selected cells, and then putting a perfect matching on the $2k$ chosen points. This gives a factor of

$$\binom{n}{2k} d^{2k}(2k-1)!!.$$

Next we choose the remaining $j-k$ pairs of M_1. By the same argument, the number of choices is

$$\binom{n-2k}{2j-2k} d^{2j-2k}(2j-2k-1)!!.$$

We subsequently choose the additional l cells for M_2 that are incident with M_1. These must be chosen from cells of M_1 that do not contain the common pairs, giving us

$$\binom{2j-2k}{l}.$$

The corresponding points can be chosen in one of $(d-1)^l$ ways. In total, $2j$ of the cells have been used so far. Next, the remaining $2j - 2k - l$ cells of M_2 and their corresponding points are chosen from among the $n-2j$ unused cells, giving a factor of

$$\binom{n-2j}{2j-2k-l} d^{2j-2k-l}.$$

Finally, there are $(2j - 2k - 1)!!$ ways to put a perfect matching on the points of M_2 not in common with points of M_1. This completes the choice of (M_1, M_2). We have shown

$$\begin{aligned} N(k,l) = & \binom{n}{2k} d^{2k}(2k-1)!! \binom{n-2k}{2j-2k} d^{2j-2k}(2j-2k-1)!! \\ \times & \binom{2j-2k}{l} (d-1)^l \binom{n-2j}{2j-2k-l} d^{2j-2k-l}(2j-2k-1)!! \end{aligned} \tag{3.6.1}$$

The chosen pair (M_1, M_2) uses $2j - k$ points, so the pairing P is completed in one of $(dn - 2j - k - 1)!!$ ways. Thus,

$$|\mathcal{P}_{n,d}|\mathbf{E}(Y^2) = \sum_{k,l} N(k,l)(dn - 2j - k - 1)!! \tag{3.6.2}$$

By considering the binomial coefficients in the expression for $N(k,l)$, we see that k and l must satisfy

$$2j - 2k - l \;\ge\; 0$$

$$\begin{aligned} n - 4j + 2k + l &\geq 0 \\ k &\geq 0 \\ l &\geq 0. \end{aligned}$$

Recall that Stirling's formula may be expressed as $x! = (x/e)^x \sqrt{2\pi\eta(x)}$ where η satisfies $\eta(x) \sim x$ if $x \to \infty$ and $\eta(x) = \Theta(x+1)$ for all x. Putting $\gamma = j/n$, $\kappa = k/n$, and $\lambda = l/n$, basic manipulations using Stirling's formula applied to (3.6.1) and (3.6.2) show that

$$\mathbf{E}(Y^2) \sim \frac{1}{2\pi^2 n^2} \sum_{\kappa,\lambda} Q(n, \kappa n, \lambda n) F(\kappa, \lambda)^n \tag{3.6.3}$$

where

$$Q(n,k,l) = \frac{n^{5/2}}{2} \frac{\eta(2j-2k)}{\eta(j-k)\eta(2j-2k-l)\sqrt{\eta(k)\eta(l)\eta(1-4j+2k+l)}}$$

and

$$\log F(\kappa, \lambda) = g_0(\kappa, \lambda) + \sum_{i=1}^{6} a_i f_i(\kappa, \lambda) \log f_i(\kappa, \lambda)$$

with the f_i and a_i defined by

$$\begin{array}{ll} a_1 = -2 & f_1(\kappa,\lambda) = 2\gamma - 2\kappa - \lambda \\ a_2 = -1 & f_2(\kappa,\lambda) = 1 - 4\gamma + 2\kappa + \lambda \\ a_3 = -1 & f_3(\kappa,\lambda) = \kappa \\ a_4 = -1 & f_4(\kappa,\lambda) = \lambda \\ a_5 = 1/2 & f_5(\kappa,\lambda) = d - 4\gamma + 2\kappa \\ a_6 = 2 & f_6(\kappa,\lambda) = \gamma - \kappa \end{array}$$

and

$$g_0(\kappa, \lambda) = \left(4\gamma - 2\kappa - \lambda - \frac{d}{2}\right) \log d + \lambda \log(d-1) + (2\gamma - 3\kappa) \log 2.$$

The sum in (3.6.3) is taken over (κ, λ) which are integer multiples of $1/n$ lying in the set R defined to be the pairs (κ, λ) satisfying

$$2\gamma - 2\kappa - \lambda \geq 0 \tag{3.6.4}$$

$$1 - 4\gamma + 2\kappa + \lambda \geq 0 \tag{3.6.5}$$

$$\kappa \geq 0 \tag{3.6.6}$$

$$\lambda \geq 0. \tag{3.6.7}$$

To estimate the sum in (3.6.3) we study the behaviour of $F(\kappa, \lambda)$ over its domain R, following the approach in [5]. To search for critical points on the interior of R

we set equal to zero the partial derivatives of $\log F$ with respect to κ and λ, then exponentiate, to get the two simultaneous equations

$$\begin{aligned}(d-4\gamma+2\kappa)(2\gamma-2\kappa-\lambda)^4-8\kappa(\gamma-\kappa)^2d^2(-1+4\gamma-2\kappa-\lambda)^2 &= 0\\ -(d-1)(2\gamma-2\kappa-\lambda)^2-d\lambda(-1+4\gamma-2\kappa-\lambda) &= 0.\end{aligned}$$

Using computer algebra software such as Maple, we find that the resultant of the above two polynomials, with respect to λ, is the following polynomial in κ,

$$-8(2d\kappa+4\gamma-4\kappa-d)^2d^4(-1+2\gamma)^4(\gamma-\kappa)^2(d\kappa-2\gamma^2),$$

whose roots are

$$\frac{d-4\gamma}{2(d-2)}, \gamma, \frac{2\gamma^2}{d}.$$

Since $\gamma < 1/2$, the first root is greater than $1/2$. In R we have $\kappa \leq \gamma < 1/2$, so κ cannot equal this first root. We discard the second root because $\kappa = \gamma$ is on the boundary of R (see (3.6.4) and (3.6.7)) and we are searching for critical points on the interior. This leaves $\kappa = \kappa^* = 2\gamma^2/d$ as the only remaining possibility. To find the corresponding value of λ we substitute (3.6.8) into (3.6.8),

$$(d-4\gamma+2\kappa)\frac{d^2\lambda^2}{(d-1)^2}(-1+4\gamma-2\kappa-\lambda)^2-8\kappa(\gamma-\kappa)^2d^2(-1+4\gamma-2\kappa-\lambda)^2=0$$

cancel the factor of $(-1+4\gamma-2\kappa-\lambda)^2$ (since this is on the boundary (3.6.5) of R),

$$(d-4\gamma+2\kappa)\frac{d^2\lambda^2}{(d-1)^2}-8\kappa(\gamma-\kappa)^2d^2=0,$$

substitute $\kappa = 2\gamma^2/d$, and solve for λ to get $\lambda = \pm 4\gamma^2(d-1)/d$. Since $\lambda \geq 0$ on R so we discard the negative solution and put $\lambda = \lambda^* = 4\gamma^2(d-1)/d$. We have shown that (κ^*, λ^*) is the only possible critical point of $\log F$ in the interior of R. Next we verify that (κ^*, λ^*) actually lies in the interior of R. Clearly $\kappa^* > 0$ and $\lambda^* > 0$. Along the other boundaries (3.6.4) and (3.6.5) we have

$$\begin{gathered}2\gamma-2\kappa^*-\lambda^* = 2\gamma(1-2\gamma) > 0\\ \lambda^*-4\gamma+2\kappa^*+1=(2\gamma-1)^2>0\end{gathered}$$

using $0 < \gamma < 1/2$. So, (κ^*, λ^*) lies in the interior of R. To show that it is indeed a critical point, one verifies that $\kappa = \kappa^*$ and $\lambda = \lambda^*$ satisfy (3.6.8) and (3.6.8). We omit the details.

Using computer algebra software such as Maple, one can determine that the Hessian of $\log F$ at (κ^*, λ^*) is a 2-by-2 matrix with upper-left entry

$$h_{1,1} = \frac{-(1-2\gamma)^2d^3-8\gamma^2d\left(2d(1-2\gamma)+(d-3)+4\gamma^2\right)-32\gamma^3(1-\gamma)}{2\gamma^2(d-2\gamma)^2(1-2\gamma)^2}$$

and determinant

$$\det H(\kappa^*, \lambda^*) = \frac{(8\gamma^2-4\gamma^2d-4\gamma d+d^2)d^2}{8(d-1)(1-2\gamma)^2(d-2\gamma)^2\gamma^4}. \tag{3.6.8}$$

Since $0 < \gamma < 1/2$ and $d \geq 3$, it is easy to see that each term in the numerator of $h_{1,1}$ is negative, and, because the denominator is nonnegative, thus $h_{1,1} < 0$. To see that $\det H(\kappa^*, \lambda^*) > 0$, we bound the numerator

$$(8\gamma^2 - 4\gamma^2 d - 4\gamma d + d^2)d^2 > (0 - d - 2d + d^2)d^2 = d^3(d-3) \geq 0$$

and observe that the denominator is nonnegative. It follows that this Hessian is negative definite. Thus, the maximum of $\log F(\kappa, \lambda)$ and hence of $F(\kappa, \lambda)$ over the interior of R is attained uniquely at (κ^*, λ^*). By substitution, one can show that the maximum value is

$$F_{max} := F(\kappa^*, \lambda^*) = \left(\frac{(d-2\gamma)^{d/2-\gamma}}{(1-2\gamma)^{1-2\gamma}\gamma^\gamma d^{d/2-2\gamma} 2^\gamma} \right)^2. \tag{3.6.9}$$

To study the behaviour of $F(\kappa, \lambda)$ on the boundary ∂R of R we use the following lemma from [5], applied to the functions F, g_0, and f_i defined after (3.6.3).

Lemma 22 (Lemma 4 in [5]) *Let R be a closed set in $\mathbb{R}^r$ and let ∂R be the boundary of R. Assume that every point in ∂R is the endpoint of an interval in $R \setminus \partial R$. For $i = 1, \ldots, m$ let $f_i(\mathbf{x})$ be an affine function such that $f_i(\mathbf{x}) > 0$ for all $\mathbf{x} \in R \setminus \partial R$. Define F to be a function on R such that*

$$F(\mathbf{x}) = g_0(\mathbf{x}) + \sum_{i=1}^{m} a_i f_i(\mathbf{x}) \log f_i(\mathbf{x})$$

with $a_i < 0$ for $i \leq m_0 \leq m$. Suppose that for every $\mathbf{x} \in R$ the directional derivative of g_0 at $\mathbf{x}$ in any direction is bounded. Let $\mathbf{x}_0 \in \partial R$ such that $f_i(\mathbf{x}_0) = 0$ for at least one $i \leq m_0$ and $f_i(\mathbf{x}_0) > 0$ for all $m_0 < i \leq m$. Then $\mathbf{x}_0$ is not a local maximum of F on R.

It is easy to check that the lemma shows that no point on ∂R is a local maximum of F on R except possibly the point $(\gamma, 0)$, where the lemma does not apply. By substitution, one can show that $F(\gamma, 0) = \sqrt{F_{max}} < F_{max}$. Therefore, (κ^*, λ^*) is the unique global maximum of $F(\kappa, \lambda)$ on R.

Let $\delta = n^{-2/5}$. Then $n\delta^2 \to \infty$ and $n\delta^3 \to 0$. Let $B(\delta)$ represent the ball of radius δ about the point (κ^*, λ^*). Write the sum in (3.6.3) as $S_1 + S_2$, where S_1 is the sum of the terms indexed by (κ, λ) in $B(\delta)$ and S_2 is sum of the other terms.

For $(\kappa, \lambda) \in B(\delta)$, one can show $Q(n, \kappa n, \lambda n) \sim \alpha(\kappa, \lambda)$ where

$$\alpha(\kappa, \lambda) = (2\gamma - 2\kappa - \lambda)^{-1}(\kappa\lambda(1 - 4\gamma + 2\kappa + \lambda))^{-1/2},$$

and in fact, $Q(n, \kappa n, \lambda n) \sim \alpha(\kappa^*, \lambda^*)$. We also estimate

$$F(\kappa, \lambda)^n = F(\kappa^*, \lambda^*)^n \exp\left(\frac{1}{2} n (\kappa - \kappa^*, \lambda - \lambda^*) H (\kappa - \kappa^*, \lambda - \lambda^*)^T + O(n\delta^3) \right) \tag{3.6.10}$$

where H is the Hessian of $\log F$ at (κ^*, λ^*). Putting $\sigma = \sqrt{n}(\kappa - \kappa^*)$ and $\tau = \sqrt{n}(\lambda - \lambda^*)$, we have

$$S_1 \sim \alpha(\kappa^*, \lambda^*) F(\kappa^*, \lambda^*)^n \sum \exp\left(\frac{1}{2}(\sigma, \tau) H (\sigma, \tau)^T\right)$$

where the sum is taken over σ, τ that are integer multiples of $n^{-1/2}$ with absolute value at most $\sqrt{n}\delta$. The above sum is a Riemann sum for

$$n \int_{-n^{1/10}}^{n^{1/10}} \int_{-n^{1/10}}^{n^{1/10}} \exp\left(\frac{1}{2}(\sigma, \tau) H (\sigma, \tau)^T\right) d\sigma d\tau$$

whose main asymptotic term is simply

$$n \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(\frac{1}{2}(\sigma, \tau) H (\sigma, \tau)^T\right) d\sigma d\tau$$

by a proof similar to the one given for Proposition 39 in the appendix. Evaluating the integral, we get that S_1 is asymptotic to

$$\alpha(\kappa^*, \lambda^*)(F_{max})^n n 2\pi |\det H(\kappa^*, \lambda^*)|^{-1/2} \sim 2\pi^2 n^2 \frac{d - 2j}{\sqrt{8j^2 - 4j^2 d - 4jd + d^2}} (\mathbf{E}Y)^2$$

using the definition of $\alpha(\kappa, \lambda)$, (3.6.8), (3.6.9), and Lemma 21(a). Recalling the expression for $\mathbf{E}(Y^2)$ given in (3.6.3), this proves Lemma 21(c), provided we can show that $S_2 = o(S_1)$. This can be shown as follows. Since (κ^*, λ^*) is the unique global maximum of F, any term of (3.6.3) indexed by $(\kappa, \lambda) \in R \setminus B(\delta)$ can be bounded above, up to a polynomial factor, by the largest of the terms indexed by $(\kappa, \lambda) \in \partial B(\delta)$. For $(\kappa, \lambda) \in \partial B(\delta)$, the exponential factor of (3.6.10) is $O(e^{-n\delta^2}) = O(e^{-n^{1/5}})$ (since we showed that H is negative definite), which produces terms that are exponentially smaller than $F(\kappa^*, \lambda^*)$. Since $Q(n, \kappa n, \lambda n)$ is bounded by a polynomial in n and the number of terms in S_2 is bounded by a polynomial in n we have $S_2 = o(S_1)$, as required.

3.7 Proof of (1.4.1)

We will apply the method of small subgraph conditioning as stated in Theorem 20.

It is well-known (e.g. see [36]) that condition (i) of the theorem holds with $\lambda_i = (d-1)^i/(2i)$. Condition (ii) holds by Lemma 21(b). Next we compute

$$\sum_{i=1}^{\infty} \lambda_i \delta_i^2 = \log\left(\frac{d - 2\gamma}{\sqrt{d^2 - 4\gamma^2 d - 4\gamma d + 8\gamma^2}}\right)$$

which establishes Condition (iii) and, in view of Lemma 21(c), Condition (iv). Noting that $\delta_i \neq -1$ for all i, we apply the theorem to conclude that $\mathcal{P}_{n,d}^{(Y)} \approx \mathcal{P}_{n,d}$. Since $\mathcal{G}_{n,d}$ is formed from $\mathcal{P}_{n,d}$ by conditioning on the event that no loops or parallel edges are formed, and this event has probability bounded away from 0, it follows that $\mathcal{G}_{n,d}^{(Y)} \approx \mathcal{G}_{n,d}$ by Proposition 9.50 in [20].

Chapter 4

Locally rainbow balanced 3-colourings of $\mathcal{G}_{n,5}$

4.1 Introduction

In [12], the authors study the number Y of locally rainbow balanced 3-colourings of $\mathcal{P}_{n,5}$. (Recall that a colouring is *balanced* if the number of vertices of each colour is equal, and *locally rainbow* if every vertex is adjacent to at least one vertex of each of the other colours.) The second moment $\mathbf{E}(Y^2)$ is expressed in the form

$$\mathbf{E}(Y^2) \sim h(n) \sum_{\mathcal{D}} q(n, \mathbf{d}) e^{nf(\mathbf{d})} \tag{4.1.1}$$

for some functions f, h, q. The sum is taken over multiples of $1/n$ in the bounded domain $\mathcal{D}$. The function f is infinitely differentiable and the function $q(n, \mathbf{d})$ is bounded by a polynomial in n, in much the same way as we expressed the second moment of the number of j-edge matchings in Chapter 3.

The authors define a four-variable function $F(w, x, y, z)$ on a bounded domain and give numerical evidence to support the following hypothesis.

Hypothesis 23 (Maximum Hypothesis) *At the point* $(1/9, 1/9, 1/9, 1/9)$ *the function* $F(w, x, y, z)$ *has a unique global maximum over its domain.*

(For a self-contained exposition of the definition of F, see Section 7 in [13].) It is shown that, under the Maximum Hypothesis, the function $f(\mathbf{d})$ in (4.1.1) has a unique global maximum at a point $\mathbf{d}^*$ in its domain. Thus, as we saw in Chapter 3, the asymptotic value for $\mathbf{E}(Y^2)$ can be determined by integrating over a region near $\mathbf{d}^*$. The result is

$$\mathbf{E}(Y^2) \sim \frac{2^2 3^{19} 5^{16}}{7^6 11^7 79^2 \sqrt{13 \cdot 17}} \frac{1}{(2\pi n)^2} \left(\frac{25}{24} \right)^n, \tag{4.1.2}$$

provided the Maximum Hypothesis holds. Using this, they prove that, under the Maximum Hypothesis, $\chi(\mathcal{G}_{n,5}) = 3$ with probability bounded away from 0. In this chapter we use small subgraph conditioning (Theorem 20) to strengthen their conclusion as follows.

Theorem 24 *Under the Maximum Hypothesis, the chromatic number of $\mathcal{G}_{n,5}$ is a.a.s. 3.*

Proof. Assume that n is divisible by 3, which is a necessary condition for balanced 3-colourings to exist. We apply Theorem 20 using the probability space $\mathcal{G}_n = \mathcal{P}_{n,5}$ with Y counting the number of locally rainbow balanced 3-colourings and X_k counting the number of k-cycles for fixed $k \geq 1$. We next discuss how conditions (i)-(iv) of Theorem 20 are verified.

It is well-known (e.g., see [36]) that condition (i) is satisfied by $\lambda_k = 4^k/(2k)$. In (4.2.1) and (4.2.2) we will see that condition (ii) holds for the function

$$\delta_k = 15^{-k} + 2(-5)^{-k} + 2(-3)^{-k}. \tag{4.1.3}$$

Substituting this function into conditions (iii) and (iv), we see that the sum is

$$\begin{aligned}\sum_k \lambda_k \delta_k^2 &= \sum_k \frac{(5-1)^k}{2k}\left(15^{-k} + 2(-5)^{-k} + 2(-3)^{-k}\right)^2 \\ &= \sum_k \frac{1}{2k}\Big(\left(\frac{4}{225}\right)^k + 4\left(\frac{-4}{45}\right)^k + 4\left(\frac{-4}{75}\right)^k \\ &\quad + 4\left(\frac{4}{9}\right)^k + 8\left(\frac{4}{15}\right)^k + 4\left(\frac{4}{25}\right)^k\Big).\end{aligned}$$

Using the identity $\sum_k \frac{1}{2k}x^k = \frac{-1}{2}\ln(1-x)$, this sum becomes

$$\begin{aligned}\sum_k \lambda_k \delta_k^2 &= \frac{-1}{2}\ln\left(\left(\frac{221}{225}\right)\left(\frac{49}{45}\right)^4\left(\frac{79}{75}\right)^4\left(\frac{5}{9}\right)^4\left(\frac{11}{15}\right)^8\left(\frac{21}{25}\right)^4\right) \\ &= \ln\left(\frac{3^{13}5^{13}}{7^6 11^4 79^2\sqrt{13\cdot 17}}\right).\end{aligned} \tag{4.1.4}$$

To verify condition (iv), we will need the asymptotic values of the first and second moments of Y. Although the asymptotic value of $\mathbf{E}Y$ was established in [12], we prefer to derive it again because it illustrates the method which we will apply in more complicated situations. The result is

$$\mathbf{E}Y \sim \sqrt{\frac{2^2 3^6 5^3}{11^3}\frac{1}{(2\pi n)^2}\left(\frac{25}{24}\right)^n}. \tag{4.1.5}$$

Under the Maximum Hypothesis, the second moment is given by (4.1.2). We compute the ratio

$$\frac{\mathbf{E}(Y^2)}{(\mathbf{E}Y)^2} \sim \frac{3^{13}5^{13}}{7^6 11^4 79^2\sqrt{13\cdot 17}},$$

which matches (4.1.4), establishing condition (iv). Having verified the four conditions, we may apply the small subgraph conditioning method to conclude $\mathbf{P}(Y > 0 \mid \mathcal{E}) \to 1$, where $\mathcal{E}$ is the event $\wedge_{\delta_k=-1}\{X_k = 0\}$.

To interpret the event $\mathcal{E}$ in the conclusion, we note that $\delta_1 = -1$ and for $k \geq 2$ we have

$$\begin{aligned} |\delta_k| &\leq 15^{-2} + 2(5)^{-2} + 2(3)^{-2} \\ &< 1. \end{aligned}$$

So the conclusion reads $\mathbf{P}(Y > 0 \mid X_1 = 0) \to 1$. Because $\mathbf{P}(X_2 = 0)$ is bounded away from 0 for large n, it follows that $Y > 0$ a.a.s. for the simple graphs $\mathcal{G}_{n,5}$. This proves Theorem 24. ∎

4.2 Joint moments

The goal of this section is to compute asymptotic values of some joint moments for the random variables which count locally rainbow balanced 3-colourings and short cycles in random regular graphs.

On the space $\mathcal{P}_{n,5}$, let Y be the random variable counting the number of locally rainbow balanced 3-colourings. We begin by computing the asymptotic value of $\mathbf{E}Y$.

Lemma 25

$$\mathbf{E}Y \sim \binom{n}{n/3, n/3, n/3} \frac{(5n/6)!^3}{|\mathcal{P}_{n,5}|} A(n)$$

where

$$A(n) = \left(\frac{3\sqrt{2}}{\sqrt{11\pi n}} 30^{n/3} \right)^3.$$

Proof. To compute this expected value we must count, for each of the $\binom{n}{n/3,n/3,n/3}$ ways to assign vertices to equal-sized colour classes, the number of pairings which make the colouring locally rainbow and balanced. All these assignments are equivalent, so fix one of them. Because the three colour classes have equal size, the number of edges between any two colours classes must be $5n/6$. In our discussion, the points in a vertex inherit the colour of that vertex.

To count the pairings which make the colouring locally rainbow and balanced we proceed in two steps. First, at each vertex v, we choose for each point in v the colour of the point it is paired with. This must be done carefully to ensure that each vertex will be adjacent to at least two colours and that the number of edges between the colour classes will be $5n/6$ as required. Then, for each pair of colour classes, we pair up the appropriate points between these classes in one of $(5n/6)!$ ways. Thus, the second step gives us a factor of $(5n/6)!^3$.

To determine the number of choices in the first step, we observe that each colour class produces an equivalent contribution. We fix one colour class, say colour 1, and construct the ordinary generating function which counts the number of ways of choosing the colour of the neighbour of each point within the class, with the indeterminate x marking one of the two possible colours. At each vertex, each of the 5 points can be assigned a mate (i.e. the other point in its pair) of either one of the two colours, provided that not all of the points are assigned to the same colour. Thus the contribution of each vertex to the generating function is $(x+1)^5 - x^5 - 1$, giving us the generating function

$$\left((x+1)^5 - x^5 - 1\right)^{n/3}.$$

Exactly $5n/6$ of these choices must be for the colour marked by x, so the total number of choices for the first step is (letting square brackets denote extraction of a coefficient)

$$N = [x^{5n/6}]\left((x+1)^5 - x^5 - 1\right)^{n/3}$$

for each colour class. Combining these results, we have

$$\mathbf{E}Y = \binom{n}{n/3, n/3, n/3} \frac{(5n/6)!^3}{|\mathcal{P}_{n,5}|} N^3.$$

Using the saddle-point method (see e.g. Section 12.1 in [24]) we will estimate N using a contour integral along the path $|z| = 1$. This is a standard method for determining the asymptotic value of the coefficient of a generating function. However, the method must be customized for each application, depending on the integrand. Often it is not trivial to show that the integrand is small when the argument is far from the saddle-point, nor is it always trivial to determine the relevant second-order approximation. It can be especially challenging to use this method in the multivariate case, as we will see in later chapters.

We begin by substituting $z = \exp(i\theta)$ and expanding in θ.

$$\begin{aligned}
N &= \frac{1}{2\pi i} \int_{|z|=1} \frac{\left((z+1)^5 - z^5 - 1\right)^{n/3}}{z^{5n/6}} dz \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\theta 5n/6} \left((e^{i\theta}+1)^5 - e^{i\theta 5} - 1\right)^{n/3} d\theta \\
&= \frac{1}{2\pi}(2^5-2)^{n/3} \int_{-\pi}^{\pi} \exp\left(-\frac{4^5 + 4(5) - (5+1)2^{1+5}}{24(2^5-2)^2} n\theta^2 + O(n\theta^3)\right) d\theta \\
&= \frac{1}{2\pi} 30^{n/3} \int_{-\pi}^{\pi} \exp\left(-\frac{11}{72} n\theta^2 + O(n\theta^3)\right) d\theta.
\end{aligned}$$

For $|\theta| \leq n^{-2/5}$, the contribution to the above is asymptotically

$$\begin{aligned}
I &= \frac{1}{2\pi} 30^{n/3} \int_{-\infty}^{\infty} \exp\left(-\frac{11}{72} n\theta^2\right) d\theta \\
&= \frac{1}{2\pi} 30^{n/3} \sqrt{\frac{72\pi}{11n}} \\
&= \frac{3\sqrt{2}}{\sqrt{11\pi n}} 30^{n/3}.
\end{aligned}$$

For $|\theta| > n^{-2/5}$, we estimate

$$\begin{aligned}
& \left|(e^{i\theta}+1)^5 - e^{5i\theta} - 1\right| \\
= & \left|5e^{i\theta} + 10e^{2i\theta} + 10e^{3i\theta} + 5e^{4i\theta}\right| \\
= & \left|e^{i\theta}\right| \left|5 + 10e^{i\theta} + 10e^{2i\theta} + 5e^{3i\theta}\right| \\
= & \left|4 + 1 + e^{i\theta} + 9e^{i\theta} + 10e^{2i\theta} + 5e^{3i\theta}\right| \\
\leq & \; 4 + \left|1 + e^{i\theta}\right| + \left|9e^{i\theta}\right| + \left|10e^{2i\theta}\right| + \left|5e^{3i\theta}\right| \\
= & \; 28 + \left|1 + e^{i\theta}\right| \\
= & \; 28 + \sqrt{(1+e^{i\theta})\overline{(1+e^{i\theta})}} \\
= & \; 28 + \sqrt{(1+e^{i\theta})(1+e^{-i\theta})} \\
= & \; 28 + \sqrt{2 + e^{i\theta} + e^{-i\theta}} \\
= & \; 28 + \sqrt{2 + 2\cos\theta} \\
\leq & \; 28 + \sqrt{2 + 2\cos(n^{-2/5})} \\
= & \; 28 + \sqrt{2 + 2\left(1 - \frac{1}{2}n^{-4/5} + O(n^{-8/5})\right)} \\
= & \; 28 + \sqrt{4 - 2\frac{1}{2}n^{-4/5} + O(n^{-8/5})} \\
= & \; 28 + 2\sqrt{1 - \frac{1}{4}n^{-4/5} + O(n^{-8/5})} \\
= & \; 28 + 2\left(1 - \frac{1}{8}n^{-4/5} + O(n^{-8/5})\right) \\
= & \; 30 - \frac{1}{4}n^{-4/5} + O(n^{-8/5})
\end{aligned}$$

and so the absolute value of $\left((e^{i\theta}+1)^5 - e^{i\theta 5} - 1\right)^{n/3}$ is at most

$$\begin{aligned}
& \left(30 - \frac{1}{4}n^{-4/5} + O\left(n^{-8/5}\right)\right)^{n/3} \\
= & \; 30^{n/3} \exp\left(\frac{n}{3}\ln\left(1 - \frac{1}{120}n^{-4/5} + O\left(n^{-8/5}\right)\right)\right) \\
= & \; 30^{n/3} \exp\left(-\frac{1}{360}n^{1/5} + O\left(n^{-3/5}\right)\right),
\end{aligned}$$

which is $o(I)$. Therefore the expression for I gives the correct asymptotic estimate for N, which is

$$N \sim \frac{3\sqrt{2}}{\sqrt{11\pi n}} 30^{n/3}.$$

Combining this with our above results, we get Lemma 25. ∎

From Lemma 25 it is easy to deduce the asymptotic value of $\mathbf{E}Y$ as stated in (4.1.5). Simply substitute $|\mathcal{P}_{n,5}| = (5n)!/(2^{5n/2}(5n/2)!)$ and apply Stirling's formula. We omit the calculations.

We now move closer to our goal of computing joint moments for locally rainbow balanced 3-colourings and short cycles. For fixed $k \geq 1$, let the random variable X_k count the number of k-cycles in $\mathcal{P}_{n,5}$. We will actually work with rooted oriented cycles, which introduces a factor of $2k$ into the counting. It will be helpful to have the following definition. For a rooted oriented cycle in a coloured graph, define its *colour type* to be the sequence of colours on its vertices. To calculate the expected value of YX_k, we will count, for each locally rainbow balanced 3-colouring and each rooted oriented k-cycle, the number of pairings which contain this cycle and respect this colouring.

As before, there are $\binom{n}{n/3,n/3,n/3}$ ways to choose the balanced 3-colouring. All are equivalent, so fix one. To enumerate the cycles and pairings which respect this colouring, we will sum over all colour types T. Once a colour type has been chosen, each vertex of the cycle can be placed in the pairing model by choosing a vertex of the correct colour and an ordered pair of points in that vertex to be used by the cycle. Hence, in total, there are asymptotically $(5 \times 4 \times n/3)^k$ ways to place the rooted oriented cycle in the pairing model. We now have

$$\mathbf{E}(YX_k) \sim \frac{1}{2k}\binom{n}{n/3, n/3, n/3}\left(\frac{20n}{3}\right)^k \frac{1}{|\mathcal{P}_{n,5}|}\sum_T f(T),$$

where $f(T)$ is the number of pairings which respect a fixed colouring and fixed rooted oriented cycle of colour type T and make the colouring locally rainbow.

To estimate the function $f(T)$, we will again fix one colour class j and construct an ordinary generating function. The generating function will count the number of ways of choosing the colour of the neighbour of each point within the class, with the indeterminate x marking one of the two possible colours.

For $j = 1, 2, 3$, let $\alpha_j(T)$ count the number of j-coloured vertices in colour type T whose two neighbours in the cycle have different colours. Let $\alpha'_j(T)$ count the number of j-coloured vertices in colour type T whose two neighbours in the cycle both have the colour marked by x. Let $\alpha''_j(T)$ count the remaining j-coloured vertices in T. We also define $\beta_j(T) = \alpha'_j(T) + \alpha''_j(T)$.

For any vertex through which the cycle does not pass, the contribution to the generating function is, as before, $(x+1)^5 - x^5 - 1$. For a cycle vertex whose neighbours in the cycle have different colours, we can assign the neighbour colours for the remaining points in any way, giving us $(x+1)^3$. But for a cycle vertex whose neighbours in the cycle have the same colour, we must ensure that this vertex gets at least one neighbour of a different colour so that the colouring is locally rainbow. This gives us $(x+1)^3 - x^3$ if the neighbours have the colour marked by x, and $(x+1)^3 - 1$ otherwise. Combining these functions, the number of ways of choosing the neighbour of each point within colour class j is given by the coefficient of $x^{5n/6}$ in the expression

$$\left((x+1)^5 - x^5 - 1\right)^{n/3-\alpha_j(T)-\alpha'_j(T)-\alpha''_j(T)}\left((x+1)^3\right)^{\alpha_j(T)}$$

$$\times \left((x+1)^3 - x^3\right)^{\alpha'_j(T)} \left((x+1)^3 - 1\right)^{\alpha''_j(T)}.$$

Earlier in this section we used the saddle-point method to estimate a similar coefficient. A simple comparison with that previous application makes it easy to see that the current coefficient is asymptotically

$$\frac{30^{n/3-\alpha_j(T)-\beta_j(T)} 8^{\alpha_j(T)} 7^{\beta_j(T)} 3\sqrt{2}}{\sqrt{11\pi n}}.$$

After the colour of the neighbour of each point has been chosen, it remains to pair up the points between each two colour classes. Since the k pairs in the cycle have already been chosen, the number of ways to do this is asymptotically

$$\frac{(5n/6)!^3}{(5n/6)^k}.$$

Putting $\alpha(T) = \alpha_1(T) + \alpha_2(T) + \alpha_3(T)$ and $\beta(T) = \beta_1(T) + \beta_2(T) + \beta_3(T)$, we conclude that

$$\begin{aligned} f(T) &\sim \frac{30^{n-\alpha(T)-\beta(T)} 8^{\alpha(T)} 7^{\beta(T)} 3^3 \sqrt{2}^3}{\left(\sqrt{11\pi n}\right)^3} \times \frac{(5n/6)!^3}{(5n/6)^k} \\ &= A(n) \frac{(5n/6)!^3}{(5n/6)^k} \left(\frac{8}{30}\right)^{\alpha(T)} \left(\frac{7}{30}\right)^{\beta(T)}. \end{aligned}$$

Letting $c_\alpha = 8/30$ and $c_\beta = 7/30$, it remains to estimate

$$S = \sum_T c_\alpha^{\alpha(T)} c_\beta^{\beta(T)}$$

where the sum is taken over all colour types T. In other words, we need to enumerate the colour types, introducing a factor of c_α for each cycle vertex whose neighbours have different colours, and a factor of c_β for each of the remaining cycle vertices.

It is helpful to view each colour type as a sequence of ordered pairs of colours: the colours at the endpoints of each edge, taken in the order induced by the rooted orientation of the cycle. One could consider each possible pair to be a state in a Markov chain. Number the states as follows.

state	pair of colours
1	12
2	21
3	31
4	13
5	23
6	32

Each colour type on k vertices then corresponds to a sequence of $k+1$ states where the first state equals the last state. For example, consider the colour type with colour sequence $1,2,3,2$. It corresponds to the state sequence $1,5,6,2,1$. The transition from state 1 to state 5 represents to a vertex (of colour 2) whose neighbours in the cycle have different colours (1 and 3); hence it should introduce a factor of c_α. Thus, in the matrix below, the entry at position $(1,5)$ is c_α. In this way we can construct a matrix which accounts for all possible transitions, and use it to obtain the desired enumeration. The above sum S equals $\mathrm{Tr}(M^k)$, where Tr denotes the trace, and M is the "transition" matrix

$$\begin{bmatrix} 0 & c_\beta & 0 & 0 & c_\alpha & 0 \\ c_\beta & 0 & 0 & c_\alpha & 0 & 0 \\ c_\alpha & 0 & 0 & c_\beta & 0 & 0 \\ 0 & 0 & c_\beta & 0 & 0 & c_\alpha \\ 0 & 0 & c_\alpha & 0 & 0 & c_\beta \\ 0 & c_\alpha & 0 & 0 & c_\beta & 0 \end{bmatrix}.$$

The eigenvalues of this matrix are $c_\beta + c_\alpha$, $-c_\beta + c_\alpha$, $-\frac{1}{2}c_\alpha + \frac{1}{2}\sqrt{-3c_\alpha^2 + 4c_\beta^2}$, and $-\frac{1}{2}c_\alpha - \frac{1}{2}\sqrt{-3c_\alpha^2 + 4c_\beta^2}$. The last two eigenvalues have multiplicity 2. Thus

$$\begin{aligned} S = (c_\beta + c_\alpha)^k &+ (-c_\beta + c_\alpha)^k \\ &+ 2\left(-\frac{1}{2}c_\alpha + \frac{1}{2}\sqrt{-3c_\alpha^2 + 4c_\beta^2}\right)^k \\ &+ 2\left(-\frac{1}{2}c_\alpha - \frac{1}{2}\sqrt{-3c_\alpha^2 + 4c_\beta^2}\right)^k . \end{aligned}$$

Since $c_\beta + c_\alpha = 7/30 + 8/30 = 1/2$, we may write

$$S = \frac{1}{2^k}\left(1 + \delta_k\right)$$

where

$$\delta_k = 15^{-k} + 2(-5)^{-k} + 2(-3)^{-k} \tag{4.2.1}$$

which is (4.1.3).

We conclude that

$$\mathbf{E}(YX_k) \sim \frac{1}{2k}\binom{n}{n/3, n/3, n/3}\left(\frac{20n}{3}\right)^k \frac{1}{|\mathcal{P}_{n,5}|} A(n) \frac{(5n/6)!^3}{(5n/6)^k} S,$$

and hence, combining this result with the previous lemma,

$$\begin{aligned} \frac{\mathbf{E}(YX_k)}{\mathbf{E}Y} &\sim \frac{1}{2k} 8^k S \\ &\sim \frac{4^k}{2k}\left(1 + \delta_k\right). \end{aligned}$$

The above argument is easily extended to work for higher moments, as follows. The expected value $\mathbf{E}(Y[X_1]_{m_1} \cdots [X_j]_{m_j})$ is expressed as a sum enumerating all pairings consistent with a given colouring and ordered set of cycles. We group the terms according to the number of vertices ν and edges μ of the isomorphism type of the ordered set of cycles. If the isomorphism type has all cycles cell-disjoint, then we have $\mu = \nu = \nu_0$, defined by

$$\nu_0 = \sum_k m_k.$$

In this case, the enumeration is a direct generalization of the above argument for $\mathbf{E}(YX_k)$. Next we argue that the terms in the remaining cases are negligible. For the remaining cases we have $\nu < \mu$. The enumeration proceeds as in the first case, but with the following significant differences. Compared to the first case, we lose a factor of $O(n^{\nu_0 - \nu})$ when choosing the cells for the cycle and we gain a factor of $O(n^{\nu_0 - \mu})$ when choosing the pairs of the pairing. Thus, the net change is $O(n^{\nu - \mu}) = O(n^{-1})$. As there are $O(1)$ isomorphism types, this shows that the terms in the remaining cases are negligible compared to those of the first case, and we obtain the following result, as required for (3.3.1) in accordance with (4.1.3):

$$\frac{\mathbf{E}(Y[X_1]_{m_1} \cdots [X_j]_{m_j})}{\mathbf{E}Y} \sim \prod_{k=1}^{j} \left(\frac{4^k}{2k} (1 + \delta_k) \right)^{m_k}. \tag{4.2.2}$$

Chapter 5

Balanced colourings of a random d-regular graph

5.1 Introduction

Let Y be the random variable counting the number of balanced k-colourings of $\mathcal{P}_{n,d}$. (Recall that a colouring is balanced if each colour class contains the same number of vertices.) We use the method of small subgraph conditioning (Theorem 20) to determine conditions which imply that $Y > 0$ a.a.s.

To apply the small subgraph conditioning method in this setting we need to calculate the expected number of balanced k-colourings, as well as joint moments of the number of such colourings and the number of short cycles. We assume that n is a multiple of k for feasibility.

Theorem 26 *Fix integers $d, k \geq 3$. Let Y be the number of balanced k-colourings of a random d-regular pseudograph $\mathcal{P}_{n,d}$. For $m \geq 1$, let X_m be the number of m-cycles in $\mathcal{P}_{n,d}$. Then*

$$\mathbf{E}(Y) \sim k^{k/2} \left(\frac{k-1}{2\pi(k-2)} \right)^{(k-1)/2} n^{-(k-1)/2} k^n \left(1 - \frac{1}{k} \right)^{dn/2}$$

and

$$\mathbf{E}(Y[X_1]_{p_1} \cdots [X_j]_{p_j}) \sim \prod_{m=1}^{j} \left(\lambda_m \left(1 + \delta_m \right) \right)^{p_m} \mathbf{E}(Y)$$

where

$$\begin{aligned} \lambda_m &= \frac{(d-1)^m}{2m}, \text{ and} \\ \delta_m &= \frac{(-1)^m}{(k-1)^{m-1}}. \end{aligned}$$

We next compute

$$\sum_{m \geq 1} \lambda_m \delta_m^2 = (k-1)^2 \log \left(\frac{k-1}{\sqrt{k^2 - 2k + 2 - d}} \right).$$

The small subgraph conditioning method then gives the following result.

Corollary 27 *Let Y be the number of balanced k-colourings of a random d-regular pseudograph $\mathcal{P}_{n,d}$. If*

$$\frac{\mathbf{E}(Y^2)}{(\mathbf{E}Y)^2} = \left(\frac{k-1}{\sqrt{k^2-2k+2-d}}\right)^{(k-1)^2} + o(1) \text{ as } n \to \infty$$

then $\mathcal{G}_{n,d}$ is a.a.s. k-colourable.

In this chapter we also give some evidence showing why the second moment condition in the hypothesis of the corollary is likely to be true. The small subgraph conditioning method has been used for many problems on random regular graphs where a certain value of the variance was required, and indeed it was established. For the current problem, Achlioptas and Moore [2] showed that $\mathbf{E}(Y^2)$ is within a constant factor of the value required by the condition. We express $\mathbf{E}(Y^2)$ as a sum, and show that, provided the summands have a certain form, the asymptotic value of the sum is exactly the value required by the corollary.

Let C_1 and C_2 be balanced k-colourings of a pairing $P \in \mathcal{P}_{n,d}$. The *colour count* of (C_1, C_2) is the k-by-k matrix $M = [m_{ij}]$ where $m_{ij}n/k$ is the number of cells coloured i in C_1 and coloured j in C_2. Since the colourings are balanced, we must have $M \in \mathcal{M}$, where $\mathcal{M}$ is the set of nonnegative k-by-k matrices with each row sum and column sum equal to 1. Define $T(M)$ to be the set of triples (P, C_1, C_2) where $P \in \mathcal{P}_{n,d}$ and (C_1, C_2) is a pair of balanced k-colourings of P having colour count M. Then,

$$\mathbf{E}(Y^2) = \sum_{M \in \mathcal{M} \cap \frac{k}{n}\mathbb{Z}^{k^2}} \frac{|T(M)|}{|\mathcal{P}_{n,d}|}$$

We show the following result.

Theorem 28 *Let $k \geq 3$ and $d \geq 3$ be fixed integers satisfying $k^2 - 2k - d + 2 > 0$. Suppose for $M \in \mathcal{M} \cap \frac{k}{n}\mathbb{Z}^{k^2}$ we can write*

$$\frac{|T(M)|}{|\mathcal{P}_{n,d}|} = h(n)q(n,M)e^{nf(M)}$$

for some functions $f(M)$, $g(M)$, $h(n)$ and $q(n,M)$ where, for some $\epsilon > 0$,

1. *$f(M)$ is infinitely differentiable over $M = [m_{i,j}] \in \mathcal{M}$ satisfying $\max_{i,j} |m_{i,j} - (1/k)| < \epsilon$, and $f(M)$ and has a unique global maximum over $\mathcal{M}$ at $M = (1/k)J_k$ where J_k is the k-by-k matrix of ones,*

2. *$q(n,M) = O(poly(n))$,*

3. *$q(n,M) \sim g(M)$ for $M \in \mathcal{M}$ satisfying $\max_{i,j} |m_{i,j} - (1/k)| < \epsilon$, and*

4. *$g(M)$ is infinitely differentiable for $M \in \mathcal{M}$ satisfying $\max_{i,j} |m_{i,j} - (1/k)| < \epsilon$.*

Then as $n \to \infty$

$$\frac{\mathbf{E}(Y^2)}{(\mathbf{E}Y)^2} = \left(\frac{k-1}{\sqrt{k^2-2k+2-d}}\right)^{(k-1)^2} + o(1)$$

and, in view of Corollary 27, $\mathcal{G}_{n,d}$ is a.a.s. k-colourable.

The form of the sum in the above theorem may seem esoteric but, in fact, it is the same form that was used to express the second moment in the problems of Chapters 3 and 4. Furthermore, in practically every application of small subgraph conditioning in the literature, the second moment was expressed as a sum in this form. The purpose of condition 3 is to ensure that $q(n, M)$ has no significant dependence on n near the value of M that produces the maximum of f. In this way, the dependence on n is factored out into the $h(n)$ function and we can treat $q(n, M)$ and $f(M)$ as (asymptotically) independent of n near the maximum.

The proofs of Theorems and 26 and 28 are presented in the remainder of this chapter. In each case, the relevant combinatorial objects are enumerated in terms of the coefficient of a generating function, which is estimated using the saddlepoint method. Some of the more technical results are presented as separate propositions.

5.2 Some basic combinatorial results

The enumeration of the joint factorial moments in the proof of Theorem 26 will require the following proposition about the number of sequences having certain properties.

Proposition 29 *For $m \geq 1$ let A_m be the set of m-element sequences with elements from $\{1, 2, \ldots, k\}$ in which no two consecutive elements are equal. Let $B_m \subseteq A_m$ be the set of sequences with the additional restriction that the first element is not equal to the last element. Then*

$$|B_m| = (k-1)^m + (k-1)(-1)^m.$$

Proof. Let $C_m \subseteq A_m$ be the subset containing sequences whose first element equals the last element. Since A_m is the disjoint union of B_m and C_m we have

$$|B_m| + |C_m| = |A_m| = k(k-1)^{m-1}. \tag{5.2.1}$$

For $m \geq 2$ the sequences in C_m correspond bijectively to the sequences in B_{m-1} by removing the last element, giving $|C_m| = |B_{m-1}|$. Substituting this into (5.2.1) gives the recurrence

$$|B_m| + |B_{m-1}| = k(k-1)^{m-1}.$$

With the initial condition $|B_1| = 0$, the solution of the recurrence is

$$|B_m| = (k-1)^m + (k-1)(-1)^m. \quad \blacksquare$$

In preparation for the proof of Theorem 28, the saddlepoint method will be applied in Proposition 35. The application will require the following result.

Proposition 30 *Let $\delta \in (0, 2\pi/5)$ and fix an integer $k \geq 3$. For each $1 \leq p, q \leq k$, let $-\pi \leq \theta_{p,q} \leq \pi$. Suppose $\max_{p,q} |\theta_{p,q}| > \delta$ and $\min_{p,q} |\theta_{p,q}| < \pi - \delta$. Then there exist p, q, r, and s with $p \neq r$ and $q \neq s$ such that*

$$\frac{\delta}{2} \leq |\theta_{p,q} + \theta_{r,s}| \leq 2\pi - \frac{\delta}{2}.$$

Proof. There are two cases. In the first case, suppose $\delta < |\theta_{p,q}| < \pi - \delta$ for some p and q. Let S be the set of pairs

$$S = \{(r, s) \mid 1 \leq r \leq k, r \neq p, 1 \leq s \leq k, s \neq q\}.$$

The set S is nonempty as $k \geq 2$. If there exists $(r, s) \in S$ with $|\theta_{p,q} + \theta_{r,s}| > \delta/2$ then

$$\begin{aligned} \frac{\delta}{2} < |\theta_{p,q} + \theta_{r,s}| &\leq |\theta_{p,q}| + |\theta_{r,s}| \\ &< \pi - \delta + \pi \\ &< 2\pi - \frac{\delta}{2} \end{aligned}$$

and we are finished. Otherwise, all $\theta_{r,s}$ with $(r, s) \in S$ satisfy $|\theta_{p,q} + \theta_{r,s}| \leq \delta/2$; i.e. they are all within $\delta/2$ units of $-\theta_{p,q}$, and so, because $\delta < |\theta_{p,q}| < \pi - \delta$, they all have the same sign and satisfy $\delta/2 \leq |\theta_{r,s}| \leq \pi - \delta/2$. Now let $(r, s) \in S$ and choose any (t, u) with $t \in \{1, 2, \ldots, k\} \setminus \{p, r\}$ and $u \in \{1, 2, \ldots, k\} \setminus \{q, s\}$. This is possible because $k \geq 3$. Since $(t, u) \in S$ we have, using the above observations, $\delta < |\theta_{r,s} + \theta_{t,u}| < 2\pi - \delta$, which implies the required result.

For the remaining case, we must have $|\theta_{p,q}| \in [0, \delta] \cup [\pi - \delta, \pi]$ for all $1 \leq p, q \leq k$. We claim there exist p, q, r, s with $p \neq r$, $q \neq s$, $|\theta_{p,q}| \in [0, \delta]$, and $|\theta_{r,s}| \in [\pi - \delta, \pi]$. If we prove the claim then we are finished because

$$\frac{\delta}{2} < \pi - 2\delta \leq ||\theta_{p,q}| - |\theta_{r,s}|| \leq |\theta_{p,q} + \theta_{r,s}| \leq |\theta_{p,q}| + |\theta_{r,s}| \leq \delta + \pi < 2\pi - \frac{\delta}{2}.$$

Assume for contradiction that the claim is false. By the hypothesis of the proposition there exist p and q with $|\theta_{p,q}| < \pi - \delta$. Since $|\theta_{p,q}| \in [0, \delta] \cup [\pi - \delta, \pi]$ for every $1 \leq p, q \leq k$ we must have $|\theta_{p,q}| \in [0, \delta]$. Since we are assuming that the claim is false, we must have $|\theta_{r,s}| \in [0, \delta]$ for the $(k-1)^2$ pairs (r, s) with $r \neq p$ and $s \neq q$. But the hypothesis of the proposition also gives us (t, u) with $|\theta_{t,u}| > \delta$, so an argument analogous to the previous one shows there must exist $(k-1)^2$ pairs (v, w) with $|\theta_{v,w}| \in [\pi - \delta, \pi]$. Since $(k-1)^2 + (k-1)^2$ exceeds k^2, the total number of ordered pairs, we have a contradiction, as required. ∎

5.3 Basic linear algebra results

In the statement of Theorem 28, there is a sum indexed by matrices. Several definitions and properties related to matrices will be useful in preparation for the proof of this theorem. In this section, k represents a positive integer.

We let I_k denote the k-by-k identity matrix and J_k denote the k-by-k matrix of ones. Write A^T and $\mathbf{tr}(A)$ for the transpose and trace of the matrix A, respectively. We use $\mathbf{1}$ to represent the k-dimensional vector of ones, while $\mathbf{1}^{(i)}$ represents the k-dimensional vector with entry 1 at position i and 0 elsewhere.

Doubly stochastic matrices are nonnegative matrices whose row and column sums are equal to 1.

Let $A = [a_{ij}]$ be an m-by-n matrix and $B = [b_{ij}]$ a p-by-q matrix. The *Kronecker product* $A \otimes B$ of A and B is the block matrix

$$A \otimes B \ := \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}.$$

Let α be a scalar and let A, B, and C be matrices of the same dimensions. We will freely use the following properties of the Kronecker product.

1. $\alpha(A \otimes B) = (\alpha A) \otimes B = A \otimes (\alpha B)$.
2. $(A + B) \otimes C = A \otimes C + B \otimes C$.
3. $A \otimes (B + C) = A \otimes B + A \otimes C$.

We also define $A^{\otimes 2} = A \otimes A$. See ([19], Section 4.2) for more information about the Kronecker product.

For an m-by-n matrix $A = [a_{ij}]$, define

$$\mathbf{vec}A := [a_{11}, \ldots, a_{m1}, a_{12}, \ldots, a_{m2}, \ldots, a_{1n}, \ldots, a_{mn}]^T.$$

In other words, $\mathbf{vec}A$ is formed by stacking the columns of A to form a single column vector.

The next proposition can be verified by routine matrix multiplication.

Proposition 31 *An orthonormal basis of eigenvectors for the matrix J_k is the set of vectors $f^{(1)}, f^{(2)}, \ldots, f^{(k)}$, defined by*

$$\begin{aligned} f^{(p)} &= \tfrac{\sqrt{p}}{\sqrt{p+1}} \left(\tfrac{-1}{p} \textstyle\sum_{l=1}^{p} \mathbf{1}^{(l)} + \mathbf{1}^{(p+1)} \right), \quad 1 \leq p \leq k-1 \\ f^{(k)} &= \tfrac{1}{\sqrt{k}} \mathbf{1} \end{aligned}$$

with corresponding eigenvalues

$$\begin{aligned} \lambda_p &= 0, \quad 1 \leq p \leq k-1 \\ \lambda_k &= k. \end{aligned}$$

Eigenvectors for the matrix J_k are also eigenvectors for the matrix $J_k + I_k$, giving us the next proposition.

Proposition 32 *The eigenvalues of $J_k + I_k$ are*

$$\begin{aligned} \lambda_p &= 1, \qquad 1 \leq p \leq k-1 \\ \lambda_k &= k + 1. \end{aligned}$$

Proposition 33 *(a) Let the vectors* $f^{(1)}, f^{(2)}, \ldots, f^{(k)}$ *be defined as in Proposition 31. Define* $f^{(p,q)} = f^{(p)} \otimes f^{(q)}$ *for* $1 \leq p, q \leq k$*. Then, an orthonormal basis of eigenvectors for the matrix* $(J_k - I_k)^{\otimes 2} + (k-1)^2 I_{k^2}$ *is given by* $\{f^{(p,q)}\}_{p,q=1}^{k}$ *with corresponding eigenvalues*

$$\begin{aligned}
\lambda_{p,q} &= k^2 - 2k + 2, & 1 \leq p, q \leq k-1 \\
\lambda_{p,k} &= (k-1)(k-2), & 1 \leq p \leq k-1 \\
\lambda_{k,q} &= (k-1)(k-2), & 1 \leq q \leq k-1 \\
\lambda_{k,k} &= 2(k-1)^2. &
\end{aligned}$$

The smallest of these eigenvalues is $(k-1)(k-2)$*.*

(b) The eigenvalues of $(J_k + I_k)^{\otimes 2}$ *are 1 with multiplicity* $(k-1)^2$*,* $k+1$ *with multiplicity* $2(k-1)$*, and* $(k+1)^2$ *with multiplicity 1.*

Proof. In Proposition 31 the vectors $f^{(1)}, f^{(2)}, \ldots, f^{(k)}$ are shown to be an orthonormal basis of eigenvectors for J_k with eigenvalues $0, 0, \ldots, 0, k$. Thus they also form an orthonormal basis of eigenvectors for $J_k - I_k$ with corresponding eigenvalues $-1, -1, \ldots, -1, k-1$. For any real symmetric k-by-k matrix A with orthonormal basis of eigenvectors $v_1, v_2, \ldots, v_k$ and corresponding eigenvalues $\mu_1, \mu_2, \ldots, \mu_k$, an orthonormal basis of eigenvectors for $A^{\otimes 2}$ is (see [19], Theorem 4.2.12)

$$v_p \otimes v_q, \quad 1 \leq p, q \leq k$$

with corresponding eigenvalues $\mu_p \mu_q$, $(1 \leq p, q \leq k)$. Thus an orthonormal basis of eigenvectors for $(J_k - I_k)^{\otimes 2}$ (and hence for $(J_k - I_k)^{\otimes 2} + (k-1)^2 I_{k^2}$) is given by

$$f^{(p,q)} = f^{(p)} \otimes f^{(q)}, \quad 1 \leq p, q \leq k.$$

The corresponding eigenvalues for $(J_k - I_k)^{\otimes 2} + (k-1)^2 I_{k^2}$ are

$$\begin{aligned}
\lambda_{p,q} &= (-1)^2 + (k-1)^2 = k^2 - 2k + 2, & 1 \leq p, q \leq k-1 \\
\lambda_{p,k} &= (-1)(k-1) + (k-1)^2 = (k-1)(k-2), & 1 \leq p \leq k-1 \\
\lambda_{k,q} &= (k-1)(-1) + (k-1)^2 = (k-1)(k-2), & 1 \leq q \leq k-1 \\
\lambda_{k,k} &= (k-1)^2 + (k-1)^2 = 2(k-1)^2. &
\end{aligned}$$

as required. It is easy to see that the smallest of these eigenvalues is $(k-1)(k-2)$.

The eigenvalues of $(J_k + I_k)^{\otimes 2}$ are computed from the eigenvalues of $J_k + I_k$ (given in Proposition 32) similarly. ∎

Proposition 34 *Let* $A = [a_{i,j}]$ *be a* k*-by-*k *matrix whose rows and columns each have sum* 0*. Define* $\tilde{A}$ *to be the submatrix formed from* A *by deleting the last row and column. Let* $\{f^{(i,j)}\}_{i,j=1}^{k}$ *be the orthonormal basis defined in the statement of Proposition 33. Then,*

(a) $(\mathbf{vec}A)^T f^{(i,k)} = (\mathbf{vec}A)^T f^{(k,j)} = 0$ *for* $1 \leq i, j \leq k$*,*

(b)

$$\sum_{i=1}^{k-1}\sum_{j=1}^{k-1}\left((\mathbf{vec}A)^T f^{(i,j)}\right)^2 = \sum_{i=1}^{k}\sum_{j=1}^{k} a_{i,j}^2, \text{ and}$$

(c)

$$(\mathbf{vec}\tilde{A})^T(J_{k-1}+I_{k-1})^{\otimes 2}\mathbf{vec}\tilde{A} = \sum_{i=1}^{k}\sum_{j=1}^{k} a_{i,j}^2.$$

Proof. We begin by proving (a). Let $j \in \{1,2,\ldots,k\}$. Since $f^{(k,j)} = f^{(k)} \otimes f^{(j)}$ is a linear combination of terms of the form $\mathbf{1}\otimes\mathbf{1}^{(q)} = \sum_{p=1}^{k}(\mathbf{1}^{(p)}\otimes\mathbf{1}^{(q)}), (1 \leq q \leq k)$, we have that $(\mathbf{vec}A)^T f^{(k,j)}$ is a linear combination of terms of the form $\sum_{p=1}^{k}(\mathbf{vec}A)^T(\mathbf{1}^{(p)}\otimes\mathbf{1}^{(q)}) = \sum_{p=1}^{k} a_{q,p} = 0$ since the row sums of A are 0. A similar argument shows $(\mathbf{vec}A)^T f^{(i,k)} = 0$ for $i \in \{1,2,\ldots,k\}$ using the fact that the column sums of A equal 0.

To prove (b) we apply (a) to write

$$\sum_{i=1}^{k-1}\sum_{j=1}^{k-1}\left((\mathbf{vec}A)^T f^{(i,j)}\right)^2 = \sum_{i=1}^{k}\sum_{j=1}^{k}\left((\mathbf{vec}A)^T f^{(i,j)}\right)^2,$$

which is the sum of the squares of the coordinates of $\mathbf{vec}A$ in the basis given by $\{f^{(i,j)}\}_{i,j=1}^{k}$. Since the basis is orthonormal, this expression is simply the square of the norm of $\mathbf{vec}A$ with respect to the standard basis, $\sum_{i=1}^{k}\sum_{j=1}^{k} a_{i,j}^2$.

To prove part (c) we begin by writing

$$\begin{aligned}
&\sum_{i=1}^{k}\sum_{j=1}^{k} a_{i,j}^2 \\
= \;& a_{k,k}^2 + \sum_{i=1}^{k-1} a_{i,k}^2 + \sum_{j=1}^{k-1} a_{k,j}^2 + \sum_{i=1}^{k-1}\sum_{j=1}^{k-1} a_{i,j}^2 \\
= \;& \left(\sum_{i=1}^{k-1}\sum_{j=1}^{k-1} a_{i,j}\right)^2 + \sum_{i=1}^{k-1}\left(-\sum_{j=1}^{k-1} a_{i,j}\right)^2 + \sum_{j=1}^{k-1}\left(-\sum_{i=1}^{k-1} a_{i,j}\right)^2 + \sum_{i=1}^{k-1}\sum_{j=1}^{k-1} a_{i,j}^2.
\end{aligned}$$

Since

$$\left(\sum_{i=1}^{k-1}\sum_{j=1}^{k-1} a_{i,j}\right)^2 = (\mathbf{vec}\tilde{A})^T J_{k-1}^{\otimes 2}\mathbf{vec}\tilde{A},$$

$$\sum_{i=1}^{k-1}\left(\sum_{j=1}^{k-1} a_{i,j}\right)^2 = (\mathbf{vec}\tilde{A})^T(J_{k-1}\otimes I_{k-1})\mathbf{vec}\tilde{A},$$

$$\sum_{j=1}^{k-1}\left(\sum_{i=1}^{k-1} a_{i,j}\right)^2 = (\mathbf{vec}\tilde{A})^T(I_{k-1}\otimes J_{k-1})\mathbf{vec}\tilde{A},$$

$$\sum_{i=1}^{k-1}\sum_{j=1}^{k-1} a_{i,j}^2 = (\mathbf{vec}\tilde{A})^T I_{k-1}^{\otimes 2}\mathbf{vec}\tilde{A},$$

and $(J_{k-1}+I_{k-1})^{\otimes 2} = J_{k-1}^{\otimes 2} + (J_{k-1}\otimes I_{k-1}) + (I_{k-1}\otimes J_{k-1}) + I_{k-1}^{\otimes 2}$, part (c) is proved. ∎

5.4 Asymptotic calculations for Theorem 26

In this section, we estimate the coefficient of the generating function that will be used to estimate the first moment and factorial moments in the proof of Theorem 26.

Proposition 35 *Let $k, d, a_1, a_2, \ldots, a_k$ be fixed integers with $k \geq 3$, d positive, and $s = \sum_{j=1}^k a_j$ even. As $n \to \infty$, for dn even, the coefficient of*

$$x_1^{dn/k+a_1} x_2^{dn/k+a_2} \cdots x_k^{dn/k+a_k}$$

in the generating function $\exp(\sum_{1\leq j<l\leq k} x_j x_l)$ is asymptotic to

$$\begin{aligned} C(s) &= (2\pi)^{-k}\left(\frac{k(k-1)}{dn}\right)^{(dn+s)/2} \\ &\quad \times 2e^{dn/2}(2\pi)^{k/2}\left(\frac{k(k-1)}{dn}\right)^{k/2}(2k-2)^{-1/2}(k-2)^{-(k-1)/2}. \end{aligned}$$

Proof. We will use the saddlepoint method. First we use Cauchy's formula to express the required coefficient as an integral over the product of circles $z_j = re^{i\theta_j}$, $-\pi \leq \theta_j \leq \pi$ for $j = 1, 2, \ldots, k$, where $r = \sqrt{dn/k(k-1)}$, giving us

$$\begin{aligned} &\frac{1}{(2\pi i)^k}\int_{|z_1|=r}\cdots\int_{|z_k|=r}\frac{\exp\left(\sum_{j<l} z_j z_l\right)}{z_1^{dn/k+a_1+1}\cdots z_k^{dn/k+a_k+1}}dz_1\cdots dz_k \\ &= \frac{1}{(2\pi)^k}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\frac{\exp\left(\sum_{j<l}(re^{i\theta_j})(re^{i\theta_l})\right)}{(re^{i\theta_1})^{dn/k+a_1}\cdots(re^{i\theta_k})^{dn/k+a_k}}d\theta_1 d\theta_2\cdots d\theta_k \\ &= \frac{1}{(2\pi)^k r^{dn+s}}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\frac{\exp(r^2\sum_{j<l} e^{i(\theta_j+\theta_l)})}{\exp(i\sum_j(dn/k+a_j)\theta_j)}d\theta_1\cdots d\theta_k. \end{aligned}$$

Let $g(\theta)$ denote the integrand in the last expression above. Letting $\mathbf{1}$ denote the vector of 1's, consider the image of $g(\theta)$ under the transformation $\theta \mapsto \theta + \pi\mathbf{1}$. It is clear that the numerator is fixed by this transformation. The denominator becomes

$$\begin{aligned} \exp(i\sum_j(dn/k+a_j)(\theta_j+\pi)) &= \exp(i\sum_j(dn/k+a_j)\theta_j)\exp(i(dn+s)\pi) \\ &= \exp(i\sum_j(dn/k+a_j)\theta_j) \end{aligned}$$

since dn and s are both even. So $g(\theta)$ is fixed by this transformation. Letting $\delta = \log n/\sqrt{n}$, this means that the integrals of $g(\theta)$ over regions $\{\theta : |\theta_j| \leq \delta, j = 1, 2, \ldots, k\}$ and $\{\theta : \pi - \delta \leq |\theta_j| \leq \pi, j = 1, 2, \ldots, k\}$ are equal. We will prove that the integral of $g(\theta)$ over each of these regions is asymptotically equal to

$$\begin{aligned} I &= e^{dn/2}(2\pi)^{k/2}\left(\frac{k(k-1)}{dn}\right)^{k/2}(2k-2)^{-1/2}(k-2)^{-(k-1)/2} \\ &= K\exp\left(\frac{dn}{2} - \frac{k}{2}\log n\right), \end{aligned}$$

where K is a constant, and we will show that the integral over the remaining regions is asymptotically smaller. From these results the proposition follows.

To prove that the integral over vectors θ in the remaining regions is asymptotically smaller, there are two cases: either $|\theta_{j^*}| \leq \delta$ and $\pi - \delta \leq |\theta_{l^*}| \leq \pi$ for some distinct j^* and l^*, or $\delta \leq |\theta_{j^*}| \leq \pi - \delta$ for some j^*.

In the first case, suppose that $|\theta_{j^*}| \leq \delta$ and $\pi - \delta \leq |\theta_{l^*}| \leq \pi$ for some distinct j^* and l^*. Then $\pi - 2\delta \leq |\theta_{j^*} + \theta_{l^*}| \leq \pi + 2\delta$ and hence $\cos(\theta_{j^*} + \theta_{l^*}) \leq 0$. So

$$\begin{aligned} |g(\theta)| &= \exp\left(r^2\sum_{j<l}\cos(\theta_j + \theta_l)\right) \\ &\leq \exp\left(r^2(\binom{k}{2} - 1) + r^2\cos(\theta_{j^*} + \theta_{l^*})\right) \\ &\leq \exp\left(r^2(\binom{k}{2} - 1)\right) \\ &= \exp\left(\frac{dn}{2} - \frac{dn}{k(k-1)}\right) \\ &= o(I). \end{aligned}$$

In the second case we suppose that $\delta \leq |\theta_{j^*}| \leq \pi - \delta$ for some j^*. If there is a value of l^* for which $|\theta_{j^*} + \theta_{l*}| > \delta/2$ then $\delta/2 < |\theta_{j^*} + \theta_{l*}| < 2\pi - \delta/2$. This means

$$\cos(\theta_{j^*} + \theta_{l*}) < \cos(\delta/2) = 1 - \frac{\delta^2}{8} + O(\delta^4)$$

and hence

$$\begin{aligned} |g(\theta)| &= \exp\left(r^2\sum_{j<l}\cos(\theta_j + \theta_l)\right) \\ &\leq \exp\left(r^2(\binom{k}{2} - 1) + r^2\cos(\theta_{j^*} + \theta_{l^*})\right) \\ &= \exp\left(r^2(\binom{k}{2} - 1) + r^2(1 - \frac{\delta^2}{8} + O(\delta^4))\right) \\ &= \exp\left(r^2\binom{k}{2} - r^2\frac{\delta^2}{8} + O(r^2\delta^4)\right) \end{aligned}$$

$$
\begin{aligned}
&= \exp\left(\frac{dn}{2} - \frac{d(\log n)^2}{8k(k-1)} + o(1)\right) \\
&= o(I).
\end{aligned}
$$

Otherwise, there is no such l^*. That is, for all l^* not equal to j^* we have $|\theta_{l^*} - (-\theta_{j^*})| \leq \delta/2$. This implies that all θ_l with $l \neq j^*$ have the same sign and satisfy $\delta/2 \leq |\theta_l| \leq \pi - \delta/2$. Since $k \geq 3$ we can choose two distinct such l, say l^* and l^{**}, and deduce

$$\delta \leq |\theta_{l^*} + \theta_{l^{**}}| \leq 2\pi - \delta.$$

Using the same argument as above, it follows that $|g(\theta)| = o(I)$.

This completes the proof that the integral of $g(\theta)$ over these regions is asymptotically negligible, as claimed.

It remains to show the integral of $g(\theta)$ over the region $\{\theta : |\theta_j| \leq \delta, j = 1, 2, \ldots, k\}$ is asymptotically equal to I. We begin by expanding

$$
\begin{aligned}
\log g(\theta) &= r^2\binom{k}{2} + i\left(r^2(k-1) - dn/k + O(1)\right)\sum_{j=1}^{k}\theta_j - \frac{1}{2}r^2\sum_{j<l}(\theta_j + \theta_l)^2 \\
&\quad + O\left(r^2\sum_{j=1}^{k}|\theta_j|^3\right) \\
&= r^2\binom{k}{2} - \frac{1}{2}r^2\sum_{j<l}(\theta_j + \theta_l)^2 + o(1)
\end{aligned}
$$

since $r^2 = dn/(k(k-1))$ and $|\theta_j| \leq \delta = \log n/\sqrt{n}$ for all j.

By Proposition 39 (which is stated in the appendix), for some constant $c > 0$ we have

$$
\begin{aligned}
&\int_{-\delta}^{\delta}\int_{-\delta}^{\delta}\cdots\int_{-\delta}^{\delta}\exp\left(-\frac{1}{2}r^2\sum_{j<l}(\theta_j + \theta_l)^2\right)d\theta_1 d\theta_2\cdots d\theta_k \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}r^2\sum_{j<l}(\theta_j + \theta_l)^2\right)d\theta_1 d\theta_2\cdots d\theta_k + O(e^{-c\log^2 n}) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}\theta^{\mathbf{T}}A\theta\right)d\theta_1 d\theta_2\cdots d\theta_k + O(e^{-c\log^2 n}).
\end{aligned}
$$

Here, θ^T denotes the transpose of the column vector θ and A is the matrix $A = r^2(\mathbf{1}\mathbf{1}^{\mathbf{T}} + (k-2)I_k)$, where I_k is the k-by-k identity matrix. It is well-known (see Equation 4.6.3 in [10]) that such integrals have the value $(2\pi)^{k/2}(\det A)^{-1/2}$, giving us

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}r^2\sum_{j<l}(\theta_j + \theta_l)^2\right)d\theta_1 d\theta_2\cdots d\theta_k \\
&= (2\pi)^{k/2}\left(r^{2k}(2k-2)(k-2)^{k-1}\right)^{-1/2}.
\end{aligned}
$$

We conclude

$$
\begin{aligned}
&\int_{-\delta}^{\delta}\int_{-\delta}^{\delta}\cdots\int_{-\delta}^{\delta} g(\theta)d\theta_1 d\theta_2\cdots d\theta_k \\
\sim\ & e^{r^2\binom{k}{2}}(2\pi)^{k/2}\left(r^{2k}(2k-2)(k-2)^{k-1}\right)^{-1/2} \\
\sim\ & e^{dn/2}(2\pi)^{k/2}\left(\frac{k(k-1)}{dn}\right)^{k/2}(2k-2)^{-1/2}(k-2)^{-(k-1)/2} \\
=\ & I
\end{aligned}
$$

as claimed. ∎

5.5 Proof of Theorem 26

Let Y be the number of balanced k-colourings of a random d-regular pseudograph $\mathcal{P}_{n,d}$. For $m \geq 1$, let X_m be the number of m-cycles in $\mathcal{P}_{n,d}$. We estimate the expected value of Y by enumerating all balanced k-colourings of all pseudographs in $\mathcal{P}_{n,d}$. There are $\binom{n}{n/k, n/k, \ldots, n/k}$ ways to choose the k colour classes. These choices are all equivalent so fix one. Suppose there are b_{ij} edges between colour class i and colour class j ($i, j \geq 1$). The colours of the neighbours of all of the points of colour class i can be then chosen in $\binom{dn/k}{b_{i1}, b_{i2}, \ldots, b_{ik}}$ ways. After this determination is made, edges are constructed by putting a perfect matching between the corresponding points in each pair of classes, in one of $\prod_{i<j} b_{ij}!$ ways. Thus we have

$$
\begin{aligned}
|\mathcal{P}_{n,d}|\,\mathbf{E}(Y) &= \binom{n}{n/k, n/k, \ldots, n/k}\sum_{\{b_{ij}\}}\left(\prod_i \binom{dn/k}{b_{i1}, b_{i2}, \ldots, b_{ik}}\right)\prod_{i<j} b_{ij}! \\
&= \binom{n}{n/k, n/k, \ldots, n/k}(dn/k)!^k \sum_{\{b_{ij}\}}\frac{1}{\prod_{i<j} b_{ij}!} \\
&= \binom{n}{n/k, n/k, \ldots, n/k}(dn/k)!^k \left[\prod_{l=1}^{k} x_l^{dn/k}\right]\prod_{i<j}\sum_{l\geq 0}\frac{(x_i x_j)^l}{l!} \\
&= \binom{n}{n/k, n/k, \ldots, n/k}(dn/k)!^k \left[\prod_{l=1}^{k} x_l^{dn/k}\right]\exp\left(\sum_{i<j} x_i x_j\right) \\
&\sim \binom{n}{n/k, n/k, \ldots, n/k}(dn/k)!^k C(0). \qquad (5.5.1)
\end{aligned}
$$

The last line in the above array follows from Proposition 35 where the function C is defined. After some basic manipulations using Stirling's formula we obtain the estimate for $\mathbf{E}(Y)$ stated in the theorem.

Next we estimate the expected value of YX_m where Y is the number of balanced k-colourings and X_m the number of length-m cycles. It is more convenient to count rooted oriented cycles, which introduces a factor of $2m$ into our calculations. It will be helpful to have the following definitions. For a rooted oriented cycle in a coloured graph, define its *colour type* to be the sequence T of colours on its vertices. For $j = 1, 2, \ldots, k$, let $\alpha_j(T)$ denote the number of vertices in T which have colour j. Note that the sum $\sum_j \alpha_j(T)$ is m.

To calculate the expected value of YX_m, we will count, for each balanced k-colouring and each rooted oriented m-cycle, the number of pairings which contain this cycle and respect this colouring.

As before, there are $\binom{n}{n/k, n/k, \ldots, n/k}$ ways to choose the balanced k-colouring. All are equivalent, so fix one. To enumerate the cycles and pairings which respect this colouring, we will sum over all colour types T. Once a colour type has been chosen, each vertex of the cycle can be placed in the pairing model by choosing a vertex of the correct colour and an ordered pair of points in that vertex to be used by the cycle. Hence, in total, there are asymptotically $(d(d-1)n/k)^m$ ways to place the rooted oriented cycle in the pairing model. We now have

$$\mathbf{E}(YX_m) \sim \frac{1}{2m}\binom{n}{n/k, n/k, \ldots, n/k}\left(\frac{d(d-1)n}{k}\right)^m \frac{1}{|\mathcal{P}_{n,d}|}\sum_T f(T),$$

where $f(T)$ is the number of pairings which respect a fixed balanced k-colouring and fixed rooted oriented cycle of colour type T. To count these pairings, suppose there are b_{ij} edges between colour class i and colour class j $(i, j \geq 1)$, excluding the edges of the prescribed cycle. The colours of the neighbours of all of the unmatched points of colour class i can be then chosen in $\binom{dn/k - 2\alpha_i(T)}{b_{i1}, b_{i2}, \ldots, b_{ik}}$ ways. After this determination is made, edges are constructed by putting a perfect matching between the corresponding points in each pair of classes, in one of $\prod_{i<j} b_{ij}!$ ways. Thus we have

$$\begin{aligned}
f(T) &= \sum_{\{b_{ij}\}}\left(\prod_i \binom{dn/k - 2\alpha_i(T)}{b_{i1}, b_{i2}, \ldots, b_{ik}}\right)\prod_{i<j} b_{ij}! \\
&= \sum_{\{b_{ij}\}} \frac{\prod_i (dn/k - 2\alpha_i(T))!}{\prod_{i<j} b_{ij}!} \\
&\sim \frac{(dn/k)!^k}{(dn/k)^{2m}} \sum_{\{b_ij\}} \frac{1}{\prod_{i<j} b_{ij}!} \\
&= \frac{(dn/k)!^k}{(dn/k)^{2m}} \left[\prod_{l=1}^k x_l^{dn/k - 2\alpha_l(T)}\right] \prod_{i<j}\sum_{l\geq 0} \frac{(x_i x_j)^l}{l!} \\
&= \frac{(dn/k)!^k}{(dn/k)^{2m}} \left[\prod_{l=1}^k x_l^{dn/k - 2\alpha_l(T)}\right] \exp\left(\sum_{i<j} x_i x_j\right).
\end{aligned}$$

By Proposition 35 the asymptotic value of the coefficient in the last expression is $C(-2m)$, making the entire expression independent of T. The number of colour types T is $(k-1)^m + (k-1)(-1)^m$ by Proposition 29, so we have

$$\begin{aligned}
\mathbf{E}(YX_m) \sim \frac{1}{2m}\binom{n}{n/k, n/k, \ldots, n/k}\left(\frac{d(d-1)n}{k}\right)^m \frac{1}{|\mathcal{P}_{n,d}|}\frac{(dn/k)!^k}{(dn/k)^{2m}} \\
\times ((k-1)^m + (k-1)(-1)^m)C(-2m).
\end{aligned}$$

Comparing this expression with (5.5.1) we see that

$$\begin{aligned}\frac{\mathbf{E}(YX_m)}{\mathbf{E}(Y)} &\sim \frac{1}{2m}\left(\frac{d(d-1)n}{k}\right)^m \frac{((k-1)^m+(k-1)(-1)^m)}{(dn/k)^{2m}}\frac{C(-2m)}{C(0)}\\ &\sim \frac{1}{2m}\left(\frac{d(d-1)n}{k}\right)^m \frac{((k-1)^m+(k-1)(-1)^m)}{(dn/k)^{2m}}\left(\frac{k(k-1)}{dn}\right)^{-m}\\ &\sim \frac{(d-1)^m}{2m}\left(1+\frac{(-1)^m}{(k-1)^{m-1}}\right).\\ &\sim \lambda_m(1+\delta_m).\end{aligned}$$

These arguments generalize to higher moments, as we have seen in previous chapters, giving us the theorem. ∎

5.6 Generating function for Theorem 28

In the introduction, we saw that the second moment $\mathbf{E}(Y^2)$ is related to the quantity $|T(M)|$. As the first step toward estimating this quantity, we express it as the coefficient of a generating function.

Lemma 36 *Fix positive integers d and $k \geq 3$. Assume 2 divides dn, k divides n and let $M = [m_{p,q}]$ be a k-by-k doubly stochastic matrix whose entries are integer multiples of k/n. Let $T(M)$ be the set of triples (P, C_1, C_2) where $P \in \mathcal{P}_{n,d}$ and (C_1, C_2) is a pair of balanced k-colourings of P having colour count M. Then, letting square brackets denote the extraction of a coefficient,*

$$\begin{aligned}|T(M)| &= n!\prod_{p=1}^{k}\prod_{q=1}^{k}\frac{(dm_{p,q}n/k)!}{(m_{p,q}n/k)!}\\ &\quad\times\left[\prod_{p=1}^{k}\prod_{q=1}^{k}x_{p,q}^{dm_{p,q}n/k}\right]\exp\left(\frac{1}{2}\sum_{p=1}^{k}\sum_{q=1}^{k}\sum_{\substack{r=1\\ r\neq p}}^{k}\sum_{\substack{s=1\\ s\neq q}}^{k}x_{p,q}x_{r,s}\right).\end{aligned}$$

Proof. For all $1 \leq p, q \leq k$ we must choose $m_{p,q}n/k$ cells to be assigned the colour p in the first colouring and q in the second colouring. We say that such a cell and its points have label (p, q). The number of ways of doing this is given by the multinomial coefficient

$$\frac{n!}{\prod_{p=1}^{k}\prod_{q=1}^{k}(m_{p,q}n/k)!}.$$

Suppose we know the number b_{pqrs} of edges from points labelled (p, q) to points labelled (r, s) for all $1 \leq p, q, r, s \leq k$ with $p \neq r$ and $q \neq s$. Then we choose, for each ordered pair of labels $((p, q), (r, s))$, which b_{pqrs} of the points labelled (p, q) will be paired with points labelled (r, s). The number of ways of doing this is

$$\prod_{p=1}^{k}\prod_{q=1}^{k}\frac{(dm_{p,q}n/k)!}{\prod_{\substack{r=1\\ r\neq p}}^{k}\prod_{\substack{s=1\\ s\neq q}}^{k}b_{pqrs}!}.$$

Finally, for each unordered pair of labels $\{(p,q),(r,s)\}$, we choose a bijection between the points labelled (p,q) and the points labelled (r,s) that were designated to be paired with each other. The number of ways of doing this is

$$\prod_{p=1}^{k}\prod_{q=1}^{k}\prod_{r=p+1}^{k}\prod_{\substack{s=1\\s\neq q}}^{k} b_{pqrs}!.$$

Thus, the total number of triples is

$$|T(M)| = n!\left(\prod_{p=1}^{k}\prod_{q=1}^{k}\frac{(dm_{p,q}n/k)!}{(m_{p,q}n/k)!}\right)\sum_{\{b_{pqrs}\}}\prod_{p=1}^{k}\prod_{q=1}^{k}\prod_{r=p+1}^{k}\prod_{\substack{s=1\\s\neq q}}^{k}\frac{1}{b_{pqrs}!}.$$

This is the expression used in [2]. We rewrite the inner sum in terms of the natural generating function

$$\begin{aligned}
&\sum_{\{b_{pqrs}\}}\prod_{p=1}^{k}\prod_{q=1}^{k}\prod_{r=p+1}^{k}\prod_{\substack{s=1\\s\neq q}}^{k}\frac{1}{b_{pqrs}!}\\
=&\left[\prod_{p=1}^{k}\prod_{q=1}^{k}x_{p,q}^{dm_{p,q}n/k}\right]\prod_{p=1}^{k}\prod_{q=1}^{k}\prod_{r=p+1}^{k}\prod_{\substack{s=1\\s\neq q}}^{k}\sum_{i=0}^{\infty}\frac{(x_{p,q}x_{r,s})^i}{i!}\\
=&\left[\prod_{p=1}^{k}\prod_{q=1}^{k}x_{p,q}^{dm_{p,q}n/k}\right]\prod_{p=1}^{k}\prod_{q=1}^{k}\prod_{r=p+1}^{k}\prod_{\substack{s=1\\s\neq q}}^{k}\exp\left(x_{p,q}x_{r,s}\right)\\
=&\left[\prod_{p=1}^{k}\prod_{q=1}^{k}x_{p,q}^{dm_{p,q}n/k}\right]\exp\left(\sum_{p=1}^{k}\sum_{q=1}^{k}\sum_{r=p+1}^{k}\sum_{\substack{s=1\\s\neq q}}^{k}x_{p,q}x_{r,s}\right)\\
=&\left[\prod_{p=1}^{k}\prod_{q=1}^{k}x_{p,q}^{dm_{p,q}n/k}\right]\exp\left(\frac{1}{2}\sum_{p=1}^{k}\sum_{q=1}^{k}\sum_{\substack{r=1\\r\neq p}}^{k}\sum_{\substack{s=1\\s\neq q}}^{k}x_{p,q}x_{r,s}\right).
\end{aligned}$$

This proves the result. ∎

5.7 Asymptotic calculations for Theorem 28

In this section we make an asymptotic estimate of $|T(M)|$ in preparation for the proof of Theorem 28. Recall the definition of the Kronecker product $A \otimes B$ from Section 5.3.

Lemma 37 *Fix positive integers d and $k \geq 3$. Assume 2 divides dn, k divides n and let $M = [m_{p,q}]$ be a k-by-k doubly stochastic matrix whose entries are integer multiples of k/n satisfying*

$$m_{p,q} = \frac{1}{k} + o(n^{-1/2}\log n).$$

Define the matrix $A = A(M) = [a_{p,q}]$ by $A = M - (1/k)J_k$ and let $\tilde{A}$ be the matrix obtained from A by removing its last row and column. Let $T(M)$ be the set of triples (P, C_1, C_2) where $P \in \mathcal{P}_{n,d}$ and (C_1, C_2) is a pair of balanced k-colourings of P having colour count M. Then,

$$\frac{|T(M)|}{|\mathcal{P}_{n,d}|} \sim \gamma(k)\frac{(k-1)^{dn}}{n^{k^2/2-1/2}k^{dn-2n}}e^{-n\frac{k^2-2k-d+2}{2(k^2-2k+2)}(\mathbf{vec}\tilde{A})^T(J_{k-1}+I_{k-1})^{\otimes 2}\mathbf{vec}\tilde{A}}$$

where $\gamma(k)$ is the constant

$$\gamma(k) = \frac{k^{k^2}(k-1)^{k(k-1)}}{(2\pi)^{k^2/2-1/2}(k^2-2k+2)^{(k-1)^2/2}(k-2)^{k-1}}.$$

Proof. From Lemma 36 we have

$$\frac{|T(M)|}{n!} = \prod_{p=1}^{k}\prod_{q=1}^{k}\frac{(dm_{p,q}n/k)!}{(m_{p,q}n/k)!}\left[\prod_{p=1}^{k}\prod_{q=1}^{k}x_{p,q}^{dm_{p,q}n/k}\right]\exp\left(\frac{1}{2}\sum_{p=1}^{k}\sum_{q=1}^{k}\sum_{\substack{r=1\\r\neq p}}^{k}\sum_{\substack{s=1\\s\neq q}}^{k}x_{p,q}x_{r,s}\right). \tag{5.7.1}$$

We begin by estimating the ratio of factorials. Recall that one can write Stirling's formula as $x! = \sqrt{2\pi\eta(x)}(x/e)^x$ where η is a function satisfying $\eta(x) \sim x$ as $x \to \infty$ and $\eta(x) = \Theta(x+1)$ for all $x \geq 0$. Thus,

$$\begin{aligned}
\prod_{p=1}^{k}\prod_{q=1}^{k}\frac{(dm_{p,q}n/k)!}{(m_{p,q}n/k)!} &= \prod_{p=1}^{k}\prod_{q=1}^{k}\frac{\sqrt{2\pi\eta(dm_{p,q}n/k)}(dm_{p,q}n/(ke))^{dm_{p,q}n/k}}{\sqrt{2\pi\eta(m_{p,q}n/k)}(m_{p,q}n/(ke))^{m_{p,q}n/k}}\\
&= \prod_{p=1}^{k}\prod_{q=1}^{k}\frac{\sqrt{\eta(dm_{p,q}n/k)}}{\sqrt{\eta(m_{p,q}n/k)}}d^{dm_{p,q}n/k}\left(\frac{m_{p,q}n}{ke}\right)^{(d-1)m_{p,q}n/k}\\
&= d^{dn}\left(\frac{n}{ek}\right)^{(d-1)n}\prod_{p,q}\frac{\sqrt{\eta(dm_{p,q}n/k)}}{\sqrt{\eta(m_{p,q}n/k)}}e^{(n(d-1)/k)\sum_{p,q}m_{p,q}\log m_{p,q}}
\end{aligned}$$

where in the final step we used $\sum_{p=1}^{k}\sum_{q=1}^{k}m_{p,q} = k$ which holds because M is doubly stochastic. For $m_{p,q} = 1/k + a_{p,q}$ with $a_{p,q} = o(1)$ we have

$$\frac{\sqrt{\eta(dm_{p,q}n/k)}}{\sqrt{\eta(m_{p,q}n/k)}} \sim \frac{\sqrt{dm_{p,q}n/k}}{\sqrt{m_{p,q}n/k}} \sim \sqrt{d}$$

for $1 \leq p, q \leq k$, and we expand

$$\begin{aligned}
\sum_{p,q}m_{p,q}\log m_{p,q} &= \sum_{p,q}\left(\frac{1}{k}+a_{p,q}\right)\left(\log\frac{1}{k}+\log(1+ka_{p,q})\right)\\
&= \sum_{p,q}\left(\frac{1}{k}+a_{p,q}\right)\left(-\log k + ka_{i,j} - \frac{k^2a_{p,q}^2}{2} + o(a_{p,q}^3)\right)\\
&= \sum_{p,q}\left(-\frac{1}{k}\log k + \frac{k}{2}a_{p,q}^2 + o(a_{p,q}^3)\right)\\
&= k\left(-\log k + \frac{1}{2}(\mathbf{vec}\tilde{A})^T(J_{k-1}+I_{k-1})^{\otimes 2}\mathbf{vec}\tilde{A}\right) + o(1)
\end{aligned}$$

where we used Proposition 34(c) to rewrite $\sum_{p,q} a_{p,q}^2$ in the final step. Combining these estimates we have

$$\prod_{p=1}^{k}\prod_{q=1}^{k}\frac{(dm_{p,q}n/k)!}{(m_{p,q}n/k)!} \sim d^{dn}\left(\frac{n}{ek^2}\right)^{(d-1)n} d^{\frac{k^2}{2}} e^{n\frac{d-1}{2}(\mathbf{vec}\tilde{A})^T(J_{k-1}+I_{k-1})^{\otimes 2}\mathbf{vec}\tilde{A}}. \quad (5.7.2)$$

Next we use the saddlepoint method to estimate the coefficient

$$C = \left[\prod_{p=1}^{k}\prod_{q=1}^{k} x_{p,q}^{dm_{p,q}n/k}\right] \exp\left(\frac{1}{2}\sum_{p=1}^{k}\sum_{q=1}^{k}\sum_{\substack{r=1\\ r\neq p}}^{k}\sum_{\substack{s=1\\ s\neq q}}^{k} x_{p,q}x_{r,s}\right)$$

in (5.7.1). Using Cauchy's integral formula, C can be written in terms of an integral around the product of circles $z_{p,q} = \rho_{p,q}\exp(i\theta_{p,q})$, $-\pi \leq \theta_{p,q} \leq \pi$, $(1 \leq p, q \leq k)$, as follows,

$$\begin{aligned} C &= \frac{1}{(2\pi i)^{k^2}} \int \frac{\exp\left(\frac{1}{2}\sum_{\substack{p\neq r\\ q\neq s}} z_{p,q}z_{r,s}\right)}{\prod_{p,q} z_{p,q}^{dm_{p,q}n/k+1}} \prod_{p,q} dz_{p,q} \\ &= \frac{1}{(2\pi)^{k^2}\prod_{p,q}\rho_{p,q}^{dm_{p,q}n/k}} \\ &\quad \times \int_{\theta\in[-\pi,\pi]^{k^2}} \frac{\exp\left(\frac{1}{2}\sum_{\substack{p\neq r\\ q\neq s}} \rho_{p,q}\rho_{r,s}e^{i(\theta_{p,q}+\theta_{r,s})}\right)}{\exp(i\sum_{p,q}\theta_{p,q}dm_{p,q}n/k)} \prod_{p,q} d\theta_{p,q}. \quad (5.7.3) \end{aligned}$$

Viewing $\theta = \mathbf{vec}([\theta_{p,q}])$ as a k^2-dimensional vector, let $g(\theta)$ denote the integrand in the above expression. Consider

$$\begin{aligned} g(\theta+\pi\mathbf{1}) &= \frac{\exp\left(\frac{1}{2}\sum_{\substack{p\neq r\\ q\neq s}} \rho_{p,q}\rho_{r,s}e^{i(\theta_{p,q}+\theta_{r,s}+2\pi)}\right)}{\exp(i\sum_{p,q}\theta_{p,q}dm_{p,q}n/k + i\pi\sum_{p,q}dm_{p,q}n/k)} \\ &= \frac{\exp\left(\frac{1}{2}\sum_{\substack{p\neq r\\ q\neq s}} \rho_{p,q}\rho_{r,s}e^{i(\theta_{p,q}+\theta_{r,s})}\right)}{\exp(i\sum_{p,q}\theta_{p,q}dm_{p,q}n/k + i\pi dn)} \\ &= g(\theta), \end{aligned}$$

which holds since $\sum_{p,q} m_{p,q} = k$ and dn is even. Setting $\delta = \log n/\sqrt{n}$, this tells us that the integral over the region $\{\theta \mid |\theta_{p,q}| \leq \delta \text{ for } 1 \leq p,q \leq k\}$ equals the integral over the region $\{\theta \mid \pi-\delta \leq |\theta_{p,q}| \leq \pi \text{ for } 1 \leq p,q \leq k\}$. Set each $\rho_{p,q}$ to be the common value

$$\rho_{p,q} = \rho = \frac{\sqrt{dn}}{k(k-1)}.$$

We will see that the integral over each of these regions is asymptotic to

$$\begin{aligned} I &= \frac{e^{dn/2}}{\sqrt{2}}\left(\frac{2\pi}{dn}\right)^{k^2/2} \frac{k^{k^2}(k-1)^{k(k-1)}\exp\left(\frac{-nd(k-1)^2(\mathbf{vec}\tilde{A})^T(J_{k-1}+I_{k-1})^{\otimes 2}\mathbf{vec}\tilde{A}}{2(k^2-2k+2)}\right)}{(k^2-2k+2)^{(k-1)^2/2}(k-2)^{k-1}} \\ &= K\exp\left(\frac{dn}{2} - o((\log n)^2)\right) \end{aligned}$$

(using $a_{i,j} = o(n^{-1/2}\log n)$) where K is a constant. We will also show that the integral over the remaining region is $o(I)$. The lemma then follows since by combining this with (5.7.1), (5.7.2), (5.7.3), the Stirling-formula estimate $n! \sim \sqrt{2\pi n}(n/e)^n$, and the well-known formula for the number of pairs on dn points,

$$|\mathcal{P}_{n,d}| = (dn-1)!! = \frac{(dn)!}{(dn/2)!2^{dn/2}} \sim \sqrt{2}\left(\frac{e}{dn}\right)^{dn/2},$$

we have

$$\begin{aligned}\frac{|T(M)|}{|\mathcal{P}_{n,d}|} &\sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n d^{dn}\left(\frac{n}{ek^2}\right)^{(d-1)n} d^{\frac{k^2}{2}} e^{n\frac{d-1}{2}(\mathbf{vec}\tilde{A})^T(J_{k-1}+I_{k-1})^{\otimes 2}\mathbf{vec}\tilde{A}} \\ &\quad\times \frac{1}{(2\pi)^{k^2}\prod_{p,q}\rho^{dm_{p,q}n/k}} 2I|\mathcal{P}_{n,d}|^{-1} \\ &\sim \gamma(k)\frac{(k-1)^{dn}}{n^{k^2/2-1/2}k^{dn-2n}} e^{-n\frac{k^2-2k-d+2}{2(k^2-2k+2)}(\mathbf{vec}\tilde{A})^T(J_{k-1}+I_{k-1})^{\otimes 2}\mathbf{vec}\tilde{A}}.\end{aligned}$$

To see that the integral over the remaining region is $o(I)$, let θ be any vector in this region. By the definition of this region we must have $\min_{p,q}|\theta_{p,q}| < \pi - \delta$ and $\max_{p,q}|\theta_{p,q}| > \delta$. By Proposition 30 there exist $p^*, q^*, r^*, s^* \in \{1,\dots,k\}$ with $p^* \neq r^*$ and $q^* \neq s^*$ such that

$$\frac{\delta}{2} \leq |\theta_{p^*,q^*} + \theta_{r^*,s^*}| \leq 2\pi - \frac{\delta}{2}.$$

Now

$$\cos(\theta_{p^*,q^*} + \theta_{r^*,s^*}) < \cos\left(\frac{\delta}{2}\right) = 1 - \frac{\delta^2}{8} + O(\delta^3)$$

so the absolute value of the integrand is

$$\begin{aligned}|g(\theta)| &= \frac{\left|\exp\left(\frac{1}{2}\sum_{\substack{p\neq r\\ q\neq s}}\rho^2 e^{i(\theta_{p,q}+\theta_{r,s})}\right)\right|}{\left|\exp(i\sum_{p,q}\theta_{p,q}dm_{p,q}n/k)\right|} \\ &= \exp\left(\frac{1}{2}\sum_{\substack{p\neq r\\ q\neq s}}\rho^2\cos(\theta_{p,q}+\theta_{r,s})\right) \\ &\leq \exp\left(\frac{1}{2}\rho^2\left((k^2(k-1)^2-1)1 + \cos(\theta_{p^*,q^*}+\theta_{r^*,s^*})\right)\right) \\ &= \exp\left(\frac{1}{2}\rho^2\left((k^2(k-1)^2-1)1 + 1 - \frac{\delta^2}{8} + O(\delta^3)\right)\right) \\ &= \exp\left(\frac{1}{2}\rho^2k^2(k-1)^2 - \rho^2\frac{(\log n)^2}{16n} + O(\rho^2 n^{-3/2}(\log n)^3)\right) \\ &= \exp\left(\frac{dn}{2} - \frac{d(\log n)^2}{16k^2(k-1)^2} + o(1)\right) \\ &= o(I)\end{aligned}$$

recalling that we chose $\rho = k^{-1}(k-1)^{-1}\sqrt{dn}$ and $\delta = \log n/\sqrt{n}$.

It remains to show that

$$\int_{\theta \in [-\delta,\delta]^{k^2}} g(\theta) d\theta \sim I.$$

For $\theta \in [-\delta,\delta]^{k^2}$ we have

$$\log g(\theta) = \rho^2 \frac{1}{2} \sum_{\substack{p\neq r\\ q\neq s}} \left(1 + i(\theta_{p,q} + \theta_{r,s}) - \frac{(\theta_{p,q}+\theta_{r,s})^2}{2} + O(|\theta|^3)\right) - i\frac{dn}{k}\sum_{p,q}\theta_{p,q}m_{p,q}.$$

Regrouping the terms and substituting $m_{p,q} = k^{-1} + a_{p,q}$ this becomes

$$\begin{aligned}
\log g(\theta) = \quad & \frac{\rho^2}{2}k^2(k-1)^2 \\
+ \quad & \sum_{p,q}\theta_{p,q}\left(2i(k-1)^2\frac{\rho^2}{2} - i\frac{dn}{k}\left(\frac{1}{k} + a_{p,q}\right)\right) \\
- \quad & \frac{\rho^2}{2}\left((k-1)^2\sum_{p,q}\theta_{p,q}^2 + \sum_{\substack{p\neq r\\ q\neq s}}\theta_{p,q}\theta_{r,s}\right) \\
+ \quad & O(\rho^2|\theta|^3)
\end{aligned}$$

Let c be the constant $c = d/(2k^2(k-1)^2)$. Recalling $\rho = k^{-1}(k-1)^{-1}\sqrt{dn}$ we find

$$\log g(\theta) = \frac{dn}{2} - i\frac{dn}{k}(\mathbf{vec}A)^T\theta - cn\theta^T B\theta + O(n^{-1/2}(\log n)^3) \tag{5.7.4}$$

where B is the matrix

$$B = (k-1)^2 I_{k^2} + (J_k - I_k)^{\otimes 2}.$$

Define $h(\theta) = -i(dn/k)(\mathbf{vec}A)^T\theta - cn\theta^T B\theta$. Proposition 33 gives us an orthonormal basis $\{f^{(p,q)}\}_{p,q=1}^k$ of eigenvectors for B and corresponding sequence of eigenvalues $(\lambda_{p,q})_{p,q=1}^k$. Introduce the new variables $(\tau_{p,q})_{p,q=1}^k$ to perform the change of basis $\theta = \sum_{p,q} f^{(p,q)}\tau_{p,q}$. This gives

$$\begin{aligned}
h(\theta) &= -i\frac{dn}{k}(\mathbf{vec}A)^T\sum_{p,q} f^{(p,q)}\tau_{p,q} - cn\sum_{p,q}\lambda_{p,q}\tau_{p,q}^2 \\
&= \sum_{p,q}\left(-i(dn/k)(\mathbf{vec}A)^T f^{(p,q)}\tau_{p,q} - cn\lambda_{p,q}\tau_{p,q}^2\right).
\end{aligned}$$

Let $p,q \in \{1,\ldots,k\}$. Using the identity

$$\int_{-\infty}^{\infty} e^{ax-bx^2}dx = \sqrt{\frac{\pi}{b}}\exp\left(\frac{a^2}{4b}\right)$$

(for $b > 0$), we have that $\int_{[-\infty,\infty]^{k^2}} \exp(h(\theta))d\theta$ is a product of terms of the form

$$\begin{aligned}
&\int_{-\infty}^{\infty} \exp\left(-i\frac{dn}{k}(\mathbf{vec}A)^T f^{(p,q)}\tau_{p,q} - cn\lambda_{p,q}\tau_{p,q}^2\right) d\tau_{p,q} \\
= &\sqrt{\frac{\pi}{cn\lambda_{p,q}}} \exp\left(\frac{-d^2 n((\mathbf{vec}A)^T f^{(p,q)})^2}{4ck^2\lambda_{p,q}}\right).
\end{aligned}$$

So by Proposition 39, for some constant $c' > 0$ we have

$$\begin{aligned}
\int_{[-\delta,\delta]^{k^2}} e^{h(\theta)}d\theta &= \int_{[-\infty,\infty]^{k^2}} e^{h(\theta)}d\theta + O(e^{-c'(\log n)^2}) \\
&= \prod_{p,q} \sqrt{\frac{\pi}{cn\lambda_{p,q}}} \exp\left(\frac{-d^2 n((\mathbf{vec}A)^T f^{(p,q)})^2}{4ck^2\lambda_{p,q}}\right) + O(e^{-c'(\log n)^2}) \\
&\sim \prod_{p,q} \sqrt{\frac{\pi}{cn\lambda_{p,q}}} \exp\left(\frac{-d^2 n((\mathbf{vec}A)^T f^{(p,q)})^2}{4ck^2\lambda_{p,q}}\right)
\end{aligned}$$

since the entries of A are $o(n^{-1/2}\log n)$. Recalling (5.7.4) we now have

$$\int_{[-\delta,\delta]^{k^2}} g(\theta)d\theta \sim e^{dn/2} \prod_{p,q} \sqrt{\frac{\pi}{cn\lambda_{p,q}}} \exp\left(\frac{-d^2 n((\mathbf{vec}A)^T f^{(p,q)})^2}{4ck^2\lambda_{p,q}}\right) \tag{5.7.5}$$

We will simplify the above product using the values of $\lambda_{p,q}$ given in Proposition 33. First, the contribution to the product from $1 \le p, q \le k-1$ is

$$\begin{aligned}
&\prod_{p=1}^{k-1}\prod_{q=1}^{k-1} \sqrt{\frac{\pi}{cn\lambda_{p,q}}} \exp\left(\frac{-d^2 n((\mathbf{vec}A)^T f^{(p,q)})^2}{4ck^2\lambda_{p,q}}\right) \\
\sim &\left(\sqrt{\frac{\pi}{cn(k^2-2k+2)}}\right)^{(k-1)^2} \prod_{p=1}^{k-1}\prod_{q=1}^{k-1} \exp\left(\frac{-d^2 n((\mathbf{vec}A)^T f^{(p,q)})^2}{4ck^2(k^2-2k+2)}\right) \\
= &\left(\sqrt{\frac{\pi}{cn(k^2-2k+2)}}\right)^{(k-1)^2} \exp\left(\frac{-d^2 n(\sum_{p=1}^{k-1}\sum_{q=1}^{k-1}(\mathbf{vec}A)^T f^{(p,q)})^2}{4ck^2(k^2-2k+2)}\right) \\
= &\left(\sqrt{\frac{\pi}{cn(k^2-2k+2)}}\right)^{(k-1)^2} \exp\left(\frac{-d^2 n(\mathbf{vec}\tilde{A})^T (J_{k-1}+I_{k-1})^{\otimes 2}\mathbf{vec}\tilde{A}}{4ck^2(k^2-2k+2)}\right)
\end{aligned}$$

where the last step used Propositions 34(b) and 34(c). The contribution to the product when exactly one of p or q equals k is

$$\left(\sqrt{\frac{\pi}{cn(k-1)(k-2)}}\right)^{2(k-1)}$$

since Proposition 34(a) tells us that $((\mathbf{vec}A)^T f^{(p,q)})^2 = 0$ when $p = k$ or $q = k$. When $p = q = k$ the contribution to the product is

$$\sqrt{\frac{\pi}{2cn(k-1)^2}}$$

Substituting these contributions into (5.7.5) we get

$$
\begin{aligned}
&\int_{[-\delta,\delta]^{k^2}} g(\theta)d\theta \\
\sim \quad & e^{dn/2}\left(\frac{\pi}{cn}\right)^{k^2/2} \frac{\exp\left(\frac{-nd^2(\mathbf{vec}\tilde{A})^T(J_{k-1}+I_{k-1})^{\otimes 2}\mathbf{vec}\tilde{A}}{4ck^2(k^2-2k+2)}\right)}{(k^2-2k+2)^{(k-1)^2/2}\left((k-1)(k-2)\right)^{k-1}\sqrt{2}(k-1)} \\
= \quad & e^{dn/2}\left(\frac{2\pi k^2(k-1)^2}{dn}\right)^{k^2/2} \frac{\exp\left(\frac{-nd(k-1)^2(\mathbf{vec}\tilde{A})^T(J_{k-1}+I_{k-1})^{\otimes 2}\mathbf{vec}\tilde{A}}{2(k^2-2k+2)}\right)}{(k^2-2k+2)^{(k-1)^2/2}\left((k-1)(k-2)\right)^{k-1}\sqrt{2}(k-1)} \\
= \quad & \frac{e^{dn/2}}{\sqrt{2}}\left(\frac{2\pi}{dn}\right)^{k^2/2} \frac{k^{k^2}(k-1)^{k(k-1)}\exp\left(\frac{-nd(k-1)^2(\mathbf{vec}\tilde{A})^T(J_{k-1}+I_{k-1})^{\otimes 2}\mathbf{vec}\tilde{A}}{2(k^2-2k+2)}\right)}{(k^2-2k+2)^{(k-1)^2/2}(k-2)^{k-1}} \\
= \quad & I,
\end{aligned}
$$

as required. ∎

5.8 Proof of Theorem 28

We want to estimate the sum

$$\mathbf{E}(Y^2) = \sum_{M\in\mathcal{M}\cap\frac{k}{n}\mathbb{Z}^{k^2}} \frac{|T(M)|}{|\mathcal{P}_{n,d}|}. \tag{5.8.1}$$

We begin by estimating the sum of the terms near $M = (1/k)J_k$. For $\delta > 0$ and any positive integer p, let $B_p(\delta)$ be the set of p-by-p matrices $M = [m_{ij}]$ for which $\max_{i,j}|m_{ij} - (1/k)| < \delta$. For a $(k-1)$-by-$(k-1)$ matrix M define $\overline{M}$ to be the k-by-k matrix formed from M by adding a new row and column so that every row sum and column sum is 1. Define $\overline{B_p(\delta)} = \{\overline{M} \mid M \in B_p(\delta)\}$.

Proposition 38 *If $M \in B_{k-1}(\delta)$ then $\overline{M} \in B_k((k-1)^2\delta)$.*

Proof. Write $M = [m_{i,j}]$ and $\overline{M} = [\overline{m}_{i,j}]$. We must verify that $|\overline{m}_{i,j} - 1/k| < (k-1)^2\delta$ for $1 \le i,j \le k$. For $1 \le i,j \le k-1$ we have $m_{i,j} = \overline{m}_{i,j}$, making the result immediate. For $1 \le i \le k-1$ and $j = k$ we have

$$
\begin{aligned}
\left|\overline{m}_{i,k} - \frac{1}{k}\right| &= \left|1 - m_{i,1} - m_{i,2} - \cdots - m_{i,k-1} - \frac{1}{k}\right| \\
&= \left|\left(\frac{1}{k} - m_{i,1}\right) + \left(\frac{1}{k} - m_{i,2}\right) + \cdots + \left(\frac{1}{k} - m_{i,k-1}\right) + \frac{1}{k} - \frac{1}{k}\right| \\
&< (k-1)\delta
\end{aligned}
$$

since $M \in B_{k-1}(\delta)$. The result for $i = k$ and $1 \le j \le k-1$ follows by symmetry. For $i = j = k$ we have

$$\left|\overline{m}_{k,k} - \frac{1}{k}\right| = \left|1 - \overline{m}_{1,k} - \overline{m}_{2,k} - \cdots - \overline{m}_{k-1,k} - \frac{1}{k}\right|$$

$$\begin{aligned} &= \left|\left(\frac{1}{k} - \overline{m}_{1,k}\right) + \left(\frac{1}{k} - \overline{m}_{2,k}\right) + \cdots + \left(\frac{1}{k} - \overline{m}_{k-1,k}\right) + \frac{1}{k} - \frac{1}{k}\right| \\ &< (k-1)(k-1)\delta \end{aligned}$$

by the above result. ∎

Now we set

$$\delta = \frac{1}{(k-1)^2} \min\left(\epsilon, \frac{1}{2k}\right)$$

and consider $\overline{M}$ for $M \in B_{k-1}(\delta) \cap \frac{k}{n}\mathbb{Z}^{(k-1)^2}$. By Proposition 38, $\overline{M} \in B_k(1/(2k))$, so $\overline{M}$ is nonnegative and hence $\overline{M} \in \mathcal{M}$. Furthermore, the entries of $\overline{M}$ are in $\frac{k}{n}\mathbb{Z}$ because they are integer linear combinations of k/n and $1 = \frac{k}{n} \times \frac{n}{k}$, using the fact that k divides n. This shows that

$$\frac{|T(\overline{M})|}{|\mathcal{P}_{n,d}|}$$

is a term in the sum (5.8.1), suggesting that we express (5.8.1) as $\mathbf{E}(Y^2) = S_1 + S_2$ where

$$S_1 := \sum_{M \in B_{k-1}(\delta) \cap \frac{k}{n}\mathbb{Z}^{(k-1)^2}} \frac{|T(\overline{M})|}{|\mathcal{P}_{n,d}|}$$

and S_2 is the sum of the remaining terms. By the hypothesis of the theorem, each term in S_1 has the form

$$\frac{|T(\overline{M})|}{|\mathcal{P}_{n,d}|} = h(n) q(n, \overline{M}) e^{n f(\overline{M})}.$$

By Proposition 38 we have $\overline{M} \in B_k(\epsilon)$, so $q(n, \overline{M}) \sim g(\overline{M})$, and hence

$$S_1 \sim h(n) \sum_{M \in B_{k-1}(\delta) \cap \frac{k}{n}\mathbb{Z}^{(k-1)^2}} g(\overline{M}) e^{n f(\overline{M})}$$

By iterating the Euler-Maclaurin summation formula (see [1], p. 806),

$$S_1 \sim \left(\frac{n}{k}\right)^{(k-1)^2} h(n) \int_{M \in B_{k-1}(\delta)} g(\overline{M}) e^{n f(\overline{M})} dM.$$

We make some observations about this integrand in preparation for using Laplace's method. This integrand is infinitely differentiable on the region of integration. Since f has a unique global maximum over $\mathcal{M}$ at $(1/k)J_k$, it follows that $f(\overline{M})$ has a unique global maximum over $B_{k-1}(\delta)$ at $M = M^* = (1/k)J_{k-1}$. Lemma 37 gives us the expansion, valid for $A = o(n^{-1/2} \log n) J_{k-1}$,

$$\frac{|T(\overline{M^* + A})|}{|\mathcal{P}_{n,d}|} \sim \gamma(k) \frac{(k-1)^{dn}}{n^{k^2/2 - 1/2} k^{dn-2n}} e^{-n \frac{k^2 - 2k - d + 2}{2(k^2 - 2k + 2)} (\mathbf{vec} A)^T (J_{k-1} + I_{k-1})^{\otimes 2} \mathbf{vec} A}$$

but we also know

$$\frac{|T(\overline{M^*+A})|}{|\mathcal{P}_{n,d}|} \sim h(n)g(\overline{M^*+A})e^{nf(\overline{M^*+A})}$$

so we must have

$$h(n)g(\overline{M^*})e^{nf(\overline{M^*})} \sim \gamma(k)\frac{(k-1)^{dn}}{n^{k^2/2-1/2}k^{dn-2n}}$$

and the Hessian H of $f(\overline{M})$ at $M = M^*$ must be

$$H \quad = \quad -\frac{k^2-2k-d+2}{2(k^2-2k+2)}(J_{k-1}+I_{k-1})^{\otimes 2}.$$

Using Proposition 33 for the eigenvalues of $(J_{k-1}+I_{k-1})^{\otimes 2}$ we deduce that H is negative definite and has determinant

$$-k^{2k-2}\left(\frac{k^2-2k-d+2}{k^2-2k+2}\right)^{(k-1)^2}. \tag{5.8.2}$$

(Here we used the assumption that $k^2-2k-d+2>0$.) Now we may apply the multidimensional Laplace method (see [34, Theorem IX.5.3]) to conclude

$$\begin{aligned}
& S_1 && (5.8.3)\\
\sim\ & \left(\frac{n}{k}\right)^{(k-1)^2}\frac{(2\pi/n)^{(k-1)^2/2}}{|\det H|^{1/2}}h(n)g(\overline{M^*})e^{nf(\overline{M^*})} && (5.8.4)\\
\sim\ & \frac{n^{(k-1)^2}}{k^{(k-1)^2}}\frac{(2\pi/n)^{(k-1)^2/2}}{|\det H|^{1/2}}\gamma(k)\frac{(k-1)^{dn}}{n^{k^2/2-1/2}k^{dn-2n}}\\
=\ & \frac{(2\pi)^{-k+1}n^{-k+1}k^{2k-1-dn+2n}(k-1)^{k^2-k+dn}}{|\det H|^{1/2}(k^2-2k+2)^{(k-1)^2/2}(k-2)^{k-1}}\\
=\ & \frac{k^{k-1}(k-1)^{k^2-2k+1}}{|\det H|^{1/2}(k^2-2k+2)^{(k-1)^2/2}}\frac{k^k(k-1)^{k-1}n^{-(k-1)}k^{2n}(k-1)^{dn}}{(2\pi(k-2))^{k-1}k^{dn}}.
\end{aligned}$$

Comparing the above expression to $\mathbf{E}(Y)$ given in Theorem 26 and substituting (5.8.2) we have

$$\begin{aligned}
S_1 \ &\sim\ k^{1-k}\left(\frac{k^2-2k-d+2}{k^2-2k+2}\right)^{-(k-1)^2/2}\frac{k^{k-1}(k-1)^{k^2-2k+1}}{(k^2-2k+2)^{(k-1)^2/2}}\mathbf{E}(Y)^2\\
&=\ (k^2-2k-d+2)^{-(k-1)^2/2}(k-1)^{k^2-2k+1}\mathbf{E}(Y)^2\\
&=\ \left(\frac{k-1}{\sqrt{k^2-2k+2-d}}\right)^{(k-1)^2}\mathbf{E}(Y)^2.
\end{aligned}$$

To prove the theorem it suffices to show $S_2 = o(S_1)$. Let M be an index of any term of S_2. This implies $M \notin \overline{B_{k-1}(\delta)}$, so we must have $M \in \mathcal{M} \setminus B_k(\delta)$ since $B_k(\delta) \cap \mathcal{M} \subseteq \overline{B_{k-1}(\delta)}$. But $\mathcal{M} \setminus B_k(\delta)$ is a compact set, so the continuous function f must attain its maximum value, say α, on this set. Since $\overline{M^*} \in B_k(\delta)$ is the unique global maximum for f on $\mathcal{M}$, we must have $\alpha < f(\overline{M^*})$ and hence $\beta := (\alpha + f(\overline{M^*}))/2 < f(\overline{M^*})$. Now

$$\begin{aligned}\frac{|T(M)|}{|\mathcal{P}_{n,d}|} &= h(n)q(n,M)e^{nf(M)} \\ &\leq h(n)q(n,M)e^{n\alpha} \\ &= h(n)o(e^{n\beta}) \qquad (5.8.5)\end{aligned}$$

since $q(n,M) = O(poly(n))$. The number of terms in S_2 is bounded by a polynomial in n, so by comparing (5.8.5) to (5.8.4) we see that $S_2 = o(S_1)$. ∎

Chapter 6

Conclusion

In this thesis we have studied several properties of various models of random graphs. New results were obtained, and many more questions were raised.

For the circumference of the random graph $\mathcal{G}_{n,M}$ in the supercritical phase, a new a.a.s. upper bound was obtained. It seems likely that this technique can be used to obtain slightly better upper bounds. It would be interesting to know what is the best upper bound that can be produced by this technique. It would also be interesting to extend this technique to the range $M = cn$ for constant c.

The contiguity result for $\mathcal{G}_{n,d}$ was established for j-edge matchings where j grows linearly with n. It should be possible to extend this work to other functions j.

The results on the chromatic number of $\mathcal{G}_{n,d}$ and $\mathcal{G}_{n,5}$ are conditional on hypotheses. It may be possible to slightly weaken them, but it seems that a new technique will be required to prove them completely.

It is significant that the method of small subgraph conditioning was used to establish these results about colouring. Previously, this method was usually used for problems about spanning subgraphs. This novel use suggests that the small subgraph conditioning method might be applicable to other problems, opening new avenues of inquiry for many other important problems in random graph theory and beyond.

Appendices

Appendix A

Maple code for Section 2.4.4

In this appendix we present Maple code for evaluating the integral in Section 2.4.4. The region of integration is split into four parts according to the description in Section 2.4.4. Some of the parts are split into smaller regions in order to express them as iterated integrals.

Instead of using the bias values

$$(b, (1-5b)/2, (1-5b)/2, b, b, b, b),$$

we use the simpler set

$$(1, \alpha, \alpha, 1, 1, 1, 1).$$

This introduces a scaling factor of $5 + 2\alpha$ into the computation.

```
restart;
> assume(alpha,posint);
> pd:=exp(-x1-x2-x3-x4-x5-x6-x7);
>
> # PART 1
> igrnd:=x4+x5+x6+x7;
> I1 := int(int(int(int(int(int(int(igrnd*pd,
> x7=x6..infinity),
> x5=x4..infinity),
> x3=0..(x6-x1)/alpha),
> x2=0..(x4-x1)/alpha),
> x1=0..x6),
> x4=x6..infinity),
> x6=0..infinity)
> +int(int(int(int(int(int(int(igrnd*pd,
> x7=x6..infinity),
> x5=x4..infinity),
> x3=0..(x6-x1)/alpha),
> x2=0..(x4-x1)/alpha),
```

```
> x1=0..x4),
> x4=0..x6),
> x6=0..infinity);
>
> # PART 2
> igrnd:=x1+alpha*x2+x5+x6+x7;
> I2 := int(int(int(int(int(int(int(igrnd*pd,
> x7=x6..infinity),
> x5=x4..infinity),
> x3=0..(x6+alpha*x2-x4)/alpha),
> x4=0..x6+alpha*x2),
> x6=0..x1),
> x2=0..x1/alpha),
> x1=0..infinity)
> +int(int(int(int(int(int(int(igrnd*pd,
> x7=x6..infinity),
> x5=x4..infinity),
> x3=0..(x6+alpha*x2-x4)/alpha),
> x4=0..x1+alpha*x2),
> x6=x1..infinity),
> x2=0..x1/alpha),
> x1=0..infinity)
> +int(int(int(int(int(int(int(igrnd*pd,
> x7=x6..infinity),
> x5=x4..infinity),
> x4=0..x6+alpha*x2-alpha*x3),
> x3=0..(x6+x1)/alpha),
> x6=0..x1),
> x2=x1/alpha..infinity),
> x1=0..infinity)
> +int(int(int(int(int(int(int(igrnd*pd,
> x7=x6..infinity),
> x5=x4..infinity),
> x4=0..alpha*x2-x1),
> x3=0..(x6+x1)/alpha),
> x6=x1..infinity),
> x2=x1/alpha..infinity),
> x1=0..infinity)
> +int(int(int(int(int(int(int(igrnd*pd,
> x7=x6..infinity),
> x5=x4..infinity),
> x3=0..(x6+alpha*x2-x4)/alpha),
> x4=alpha*x2-x1..x1+alpha*x2),
> x6=x1..infinity),
> x2=x1/alpha..infinity),
```

```
> x1=0..infinity)
> ;
>
> # PART 3
> igrnd:=alpha*x2+alpha*x3+x5+x7;
> I3 := int(int(int(int(int(int(int(igrnd*pd,
> x7=x6..infinity),
> x5=x4..infinity),
> x6=0..alpha*x3-x1),
> x4=0..alpha*x2-x1),
> x1=0..alpha*x3),
> x2=alpha*x3/alpha..infinity),
> x3=0..infinity)
> +int(int(int(int(int(int(int(igrnd*pd,
> x7=x6..infinity),
> x5=x4..infinity),
> x6=0..alpha*x3-x1),
> x4=0..alpha*x2-x1),
> x1=0..alpha*x2),
> x2=0..alpha*x3/alpha),
> x3=0..infinity);
>
> # PART 4
> igrnd:=x1+alpha*x3+x4+x5+x7;
> I4 := int(int(int(int(int(int(int(igrnd*pd,
> x7=x6..infinity),
> x5=x4..infinity),
> x6=0..alpha*x3+x4-alpha*x2),
> x2=0..(alpha*x3+x4)/alpha),
> x3=0..x1/alpha),
> x4=0..x1),
> x1=0..infinity)
> +int(int(int(int(int(int(int(igrnd*pd,
> x7=x6..infinity),
> x5=x4..infinity),
> x6=0..alpha*x3+x4-alpha*x2),
> x2=0..(x1+x4)/alpha),
> x3=x1/alpha..infinity),
> x4=0..x1),
> x1=0..infinity)
> +int(int(int(int(int(int(int(igrnd*pd,
> x7=x6..infinity),
> x5=x4..infinity),
> x2=0..(alpha*x3+x4-x6)/alpha),
> x6=0..alpha*x3+x1),
```

```
> x3=0..x1/alpha),
> x4=x1..infinity),
> x1=0..infinity)
> +int(int(int(int(int(int(int(igrnd*pd,
> x7=x6..infinity),
> x5=x4..infinity),
> x2=0..(x4-x1)/alpha),
> x6=0..alpha*x3+x1),
> x3=x1/alpha..infinity),
> x4=x1..infinity),
> x1=0..infinity)
> +int(int(int(int(int(int(int(igrnd*pd,
> x7=x6..infinity),
> x5=x4..infinity),
> x6=0..alpha*x3+x4-alpha*x2),
> x2=(x4-x1)/alpha..(x1+x4)/alpha),
> x3=x1/alpha..infinity),
> x4=x1..infinity),
> x1=0..infinity)
> ;
```

Appendix B

A Gaussian-like integral

The following integral often arises when we integrate a function whose logarithm we have approximated by its second-order Taylor expansion.

Proposition 39 *Let k be a positive integer. Define the function $f : \mathbb{R}^k \to \mathbb{C}$ by*

$$f(\theta) = -ic_1 n a^T \theta - c_2 n \theta^T B \theta$$

where i is the imaginary unit, a is a fixed k-dimensional vector, B is a fixed k-by-k positive definite matrix, and c_1 and $c_2 > 0$ are constants. Let $\delta = c_3 n^{-1/2} \log n$ for some constant $c_3 > 0$. Then, as $n \to \infty$,

$$\int_{[-\delta,\delta]^k} e^{f(\theta)} d\theta = \int_{[-\infty,\infty]^k} e^{f(\theta)} d\theta + O(e^{-c(\log n)^2})$$

for some constant $c > 0$.

Proof. The difference between the integrals is

$$\begin{aligned} \left| \int_{[-\infty,\infty]^k \setminus [-\delta,\delta]^k} e^{f(\theta)} d\theta \right| &\leq \int_{[-\infty,\infty]^k \setminus [-\delta,\delta]^k} \left| e^{f(\theta)} \right| d\theta \\ &= \int_{[-\infty,\infty]^k \setminus [-\delta,\delta]^k} e^{-c_2 n \theta^T B \theta} d\theta \\ &\leq \int_{\theta : |\theta| > \delta} e^{-c_2 n \theta^T B \theta} d\theta. \end{aligned}$$

Since B is positive definite we have $\theta^T B \theta \geq \lambda |\theta|^2$ where $\lambda > 0$ is the smallest eigenvalue of B. Thus,

$$\begin{aligned} \int_{\theta : |\theta| > \delta} e^{-c_2 n \theta^T B \theta} d\theta &\leq \int_{\theta : |\theta| > \delta} e^{-c_2 n \lambda |\theta|^2} d\theta \\ &= \int_{r > \delta} e^{-c_2 n \lambda r^2} O(r^{k-1}) dr \\ &= O(e^{-c_4 \lambda n \delta^2}) \end{aligned}$$

for some $c_4 > 0$. This final expression is $O(e^{-c(\log n)^2})$, as required. ∎

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